Simulation of SDEs

A Small Analysis

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Contents

1 Introduction

This small project has the objective of making Stochastic Differential Equations seem more intuitive by plotting them on a graph and playing with the parameters. Simulations for the following processes have been included:

- Ornstein-Uhelnbeck Process
- Gompertz/Fox Process
- Geometrical Brownian Motion (Black-Scholes)

In addition, I wanted to experiment with simulation methods. For that, I chose the Euler-Maruyama Method.

The code was written in a Jupyter Notebook, available in my GitHub repository `https://www.github.com/carmojudicemota/simulations-of-SDEs`.

2 Euler-Maruyama Method

The Euler-Maruyama Method is the stochastic version of the Euler Method (for ordinary differential equations).

From a general SDE:

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t$$

We compute for a small increment step Δt:

$$X_{t+\Delta t} = X_t + a(X_t, t)\, \Delta t + b(X_t, t)\, \sqrt{\Delta t}\, \xi_t$$

where $\xi_t \sim \mathcal{N}(0, 1)$ are independent standard normal random variables. The step size implemented was 0.01, over a time period of 12.

3 Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process can be generally defined as:

$$dX_t = -\theta(X_t - \mu)\, dt + \sigma, dW_t$$

$$X_0 = x_0$$

where:

- θ: speed of mean reversion,

- μ: long-term mean,
- σ: volatility (noise intensity),
- W_t: standard Wiener process (Brownian motion).

Intuitively, the drift term $-\theta(X_t - \mu)\,dt$ will pull X_t towards the mean and the diffusion term σ, dW_t will add random noise.

The simulation was run with four times, each time with different parameters. The parameters were the following, with lines colored accordingly:

- Blue: with parameters $\theta = 10, \mu = 0, \sigma = 5, x_0 = 10$
- Orange: with parameters $\theta = 1, \mu = 0, \sigma = 1, x_0 = 0$
- Green: with parameters $\theta = 1, \mu = 0, \sigma = 1, x_0 = -10$
- Red: with parameters $\theta = 1, \mu = -10, \sigma = 1, x_0 = 0$

Therefore, in theory, we would expect that the Blue line, with highest θ compared to others, would drop towards its mean much faster. As for the μ we would expect that the empiric average of the first three lines to tend towards 0, and for the last line to tend towards -10 as t increases. With a higher σ we would expect the Blue line to show a lot more noise (variation). Finally, the Blue line will begin, at time 0, in $X_t = 10$, the Orange and Red lines in 0, and the Green line in -10.

The results obtained after applying the Euler-Maruyama Method for 12 time periods, with a step size of 0.01 were:

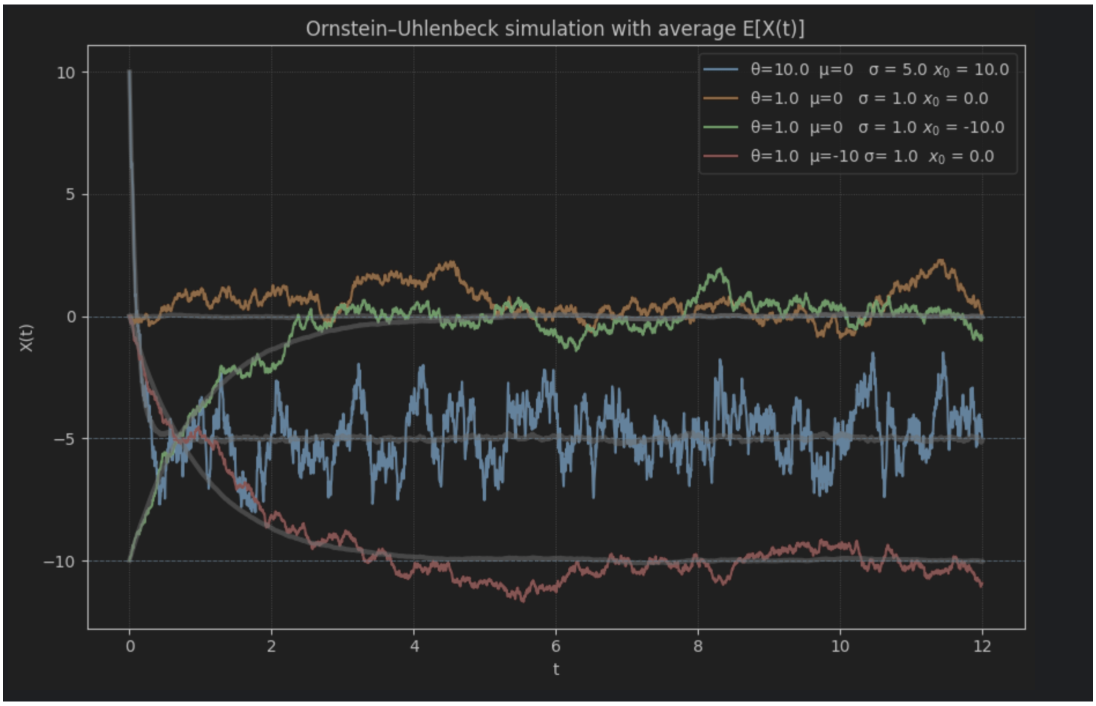

Figure 1: Results for a simulation of four Ornstein-Uhlenbeck paths

Here we can see that the four lines show exactly as we predicted.

4 Gompertz/Fox process

The Gompertz (or Fox) process can be generally defined as:

$$dX_t = aX_t \ln\left(\frac{b}{X_t}\right) dt + \sigma X_t \, dW_t$$

where:

- a: speed of mean reversion,
- b: long-term equilibrium,
- σ: volatility (noise intensity),
- W_t: standard Wiener process (Brownian motion).

The simulation was run with four times, each time with different parameters. The parameters were the following, with lines colored accordingly:

- Blue: with parameters $a = 1, b = 10, \sigma = 0.3, x_0 = 1$
- Orange: with parameters $a = 1, b = 5, \sigma = 0.3, x_0 = 0.5$
- Green: with parameters $a = 0.5, b = 10, \sigma = 0.3, x_0 = 2$
- Red: with parameters $a = 1, b = 20, \sigma = 0.4, x_0 = 5$

Therefore, in theory, we would expect that the empirical average of the Green line would tend to the mean slightly slower than the others, since it has a smaller a. As for the long term equilibrium, we would expect Blue's and Green's empirical averages to tend to 10, Orange's to 5 and Red's to 20. As for volatility, σ, we would expect the Red line to show a larger volatility in the beginning, moreover, since volatility is proportional to X_t we would expect the volatility to increase more for Blue and Green than for Orange, even though they have the same σ value, once, on average, they will show higher X_t. Lastly, the process will start at time 0 with the Blue line in $X_t = 1$, the Orange in 0.5, Green in 2 and Red in 5.

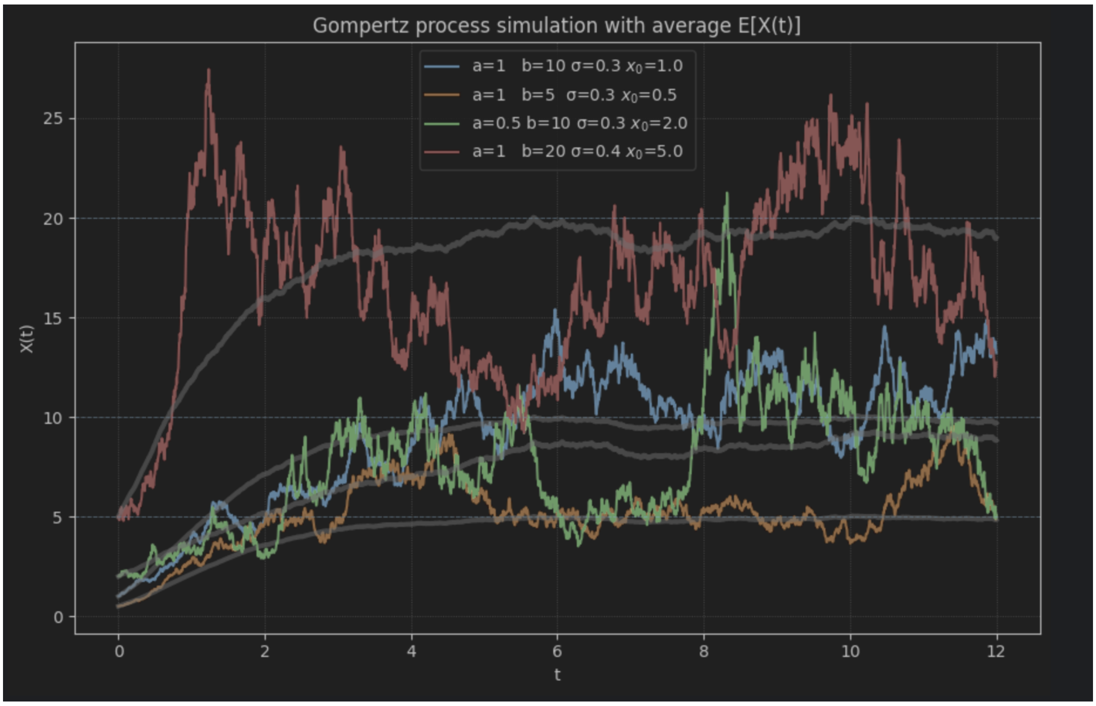

Figure 2: Results for a simulation of four Gompertz paths

The simulated results were as expected. We can observe that although there might be periods where X_t increases or decreases a lot, it's path tends to end up around the empirical average.

5 Geometric Brownian Motion (Black-Scholes Model)

A Geometric Brownian Motion can be generally defined as:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

where:

- μ: drift coefficient (average growth rate),
- σ: volatility or diffusion coefficient (noise intensity),
- W_t: standard Wiener process (Brownian motion).

In the Black-Scholes Model $\mu S_t \, dt$ determines the deterministic trend and $\sigma S_t \, dW_t$ the random component associated with the process. This model makes very strong assumptions about the nature of stocks and markets which may not always mirror reality. Some assumptions are:

- the prices in the market S_t follow a Geometric Browninan Motion (what we modeled);

- markets are frictionless (no arbitrage, continuous trading, constant interest rate r, etc.)
- the stock does not pay any dividends

With these assumptions we are able to derive the Black Scholes model formulation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

The simulation was run with four times, each time with different parameters. Additionally, both the empirical and true expectations were computed. The parameters were the following, with lines colored accordingly:

- Blue: with parameters $\mu = 0.1, \sigma = 0.2, s_0 = 100$
- Orange: with parameters $\mu = 0, \sigma = 0.2, s_0 = 100$
- Green: with parameters $\mu = 0.1, \sigma = 0.4, s_0 = 100$
- Red: with parameters $\mu = -0.1, \sigma = 0.2, s_0 = 100$

So, theoretically, we would expect each path to start at $S_t = 100$ when $t = 0$, but they should diverge, each varying around their true mean, the Green line with slightly more variations than the others.

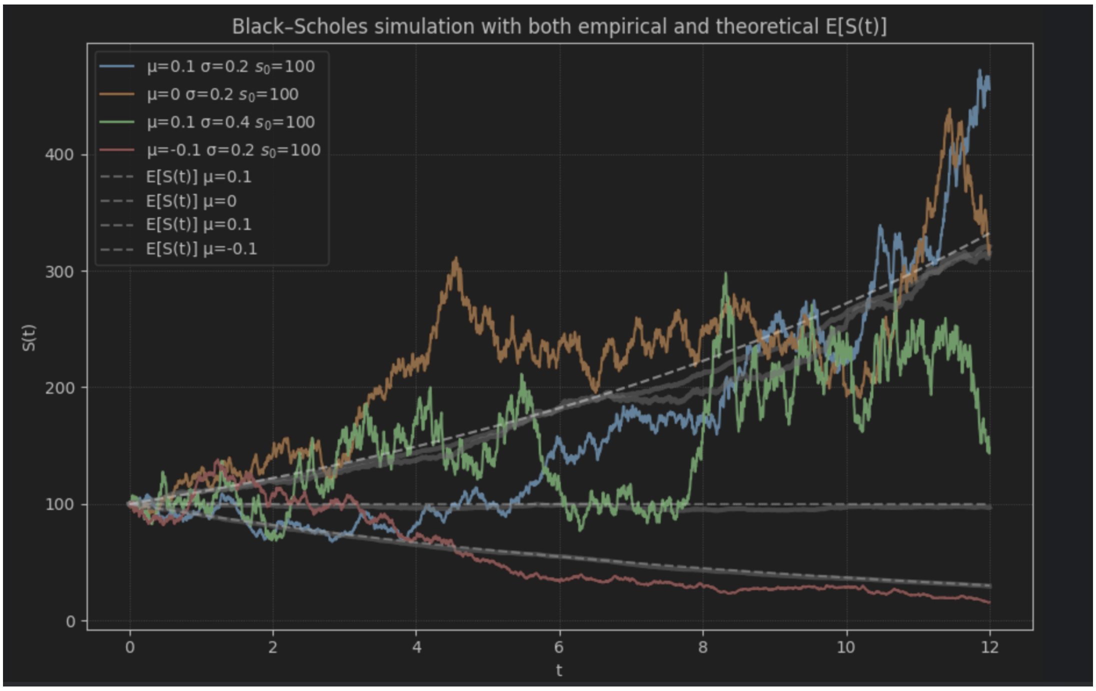

Figure 3: Results for a simulation of four GBM paths

What we see is that, although the Orange, Blue and Red lines follow as we predicted, the Green line seems to deviate a bit from its true mean, 0. However, if we look carefully at the empirical mean, computed for 200 paths, we can see that it coincides almost perfectly with the theoretical mean, for we can conclude that this path was just one specific random path, but it does not represent a true ascending trend as it seems to.

6 Conclusion

In this small analysis we showed that we can effectively simulate SDE with the Euler-Maruyama Method. The results expected theoretically are met empirically. After this analysis the formulation of SDE's has become a lot more intuitive and manageable. Additionally, writing the code for the simulation of these processes has aided in a more technical understanding of the Euler-Maruyama Method.