Asset & Portfolio returns

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One-Period Simple Return

- Most financial studies involve returns—instead of prices—of assets:
  - Asset returns is a complete and scale-free summary of the investment opportunity for an average investor
  - Return series have more attractive statistical properties than price series
  - Several definitions of asset returns
- Define, P_t = price of an asset in period t (assume no dividends)

Holding period and holding period return

- Consider purchasing an asset (e.g., stock, bond, ETF, mutual fund, option, etc.) at time t_0 for the price P_0 and then selling the asset at time t_1 for the price P_1
- If there are no intermediate cash flows (e.g., dividends) between t_0 and t_1 the rate of return over the period t_0 to t_1 is the percentage change in price:

$$R(t_0, t_1) = \frac{P_1 - P_0}{P_0}$$

- The time between t_0 to t_1 1 is called the ***holding period*** and ***$R(t_0, t_1)$*** is called the ***holding period return***
- the holding period can be any amount of time: one second; five minutes; eight hours; two days, six minutes, and two seconds; fifteen years

Simple Returns

Gross Return:
The total rate of return on an investment before the deduction of any fees or expenses. The gross rate of return is quoted over a specific period of time, such as a month, quarter or year. It is often quoted as the rate of return on an investment in advertising flyers and commercials.

- P_t => the nominal price of the asset at time t, the price at the end of month t on an asset that pays no dividends
- P_{t-1} => the price at the end of month $t-1$

one-month simple net return on an investment in the asset between months $t-1$ and t

$$R_{t-1,t} = \frac{P_t - P_{t-1}}{P_{t-1}} = \%\Delta Pt,$$

Writing $\frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$ we can define the **simple gross return** as $1 + R_{t-1,t} = \frac{P_t}{P_{t-1}}$

The one-month gross return has the interpretation of the future value of \$1 invested in the asset for one-month

Simple return calculation

- Consider a one-month investment in Microsoft stock. Suppose you buy the stock in month $t-1$ at P_{t-1} = \$85 and sell the stock the next month for P_t = \$90
- Further assume that Microsoft does not pay dividends between months t-1 and t
- The ***one-month simple net*** and ***gross returns*** are then

$$R_t = \frac{90-85}{85} = \frac{90}{85} - 1 = 1{,}0588 - 1 = 0{,}0588$$

$$1+R_t = 1{,}0588$$

- The one-month investment in Microsoft yielded a 5,88% per month return
- Alternatively, \$1 invested in Microsoft stock in month $t-1$ grew to \$1,0588 in month t

Simple and compound return

- We will prove that ***k*-period simple gross return** is just the product of the k one-period simple gross returns

- ***k-period simple net return***: $R_t[k] = \dfrac{P_t - P_{t-k}}{P_{t-k}}$

- The simple two-month return on an investment in an asset between months $t-2$ and t is defined as

$$R_t(2) = \frac{P_t - P_{t-2}}{P_{t-2}} = \frac{P_t}{P_{t-2}} - 1$$

Multi-period returns

$$R_t(2) = \frac{P_t - P_{t-2}}{P_{t-2}} = \frac{P_t}{P_{t-2}} - 1$$

- Writing $\frac{P_t}{P_{t-2}} = \frac{P_t}{P_{t-1}} * \frac{P_{t-1}}{P_{t-2}}$ the *simple two-month return* can be expressed as:

$$R_t(2) = \frac{P_t}{P_{t-1}} * \frac{P_{t-1}}{P_{t-2}} - 1 = (1 + R_t)(1+R_{t-1}) - 1$$

- Then the *simple two-month **gross** return* becomes:

$$1 + R_t(2) = (1 + R_t)(1+R_{t-1}) = 1 + R_{t-1}+R_t + R_{t-1}R_t$$

which is a product of the two simple one-month gross returns and not one plus the sum of the two one-month returns.

$$R_t(2) = R_{t-1}+R_t + R_{t-1}R_t$$

Multi-period returns

- In general, the k-month gross return is defined as the product of k one-month gross returns:

$$
\begin{aligned}
1 + R_t[k] = \frac{P_t}{P_{t-k}} &= \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}} \\
&= (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k})(1 + R_{t-k+1}) \\
&= \prod_{j=0}^{k-1} (1 + R_{t-j})
\end{aligned}
$$

Continuously Compounded Returns

- The natural logarithm of the simple gross return of an asset is called the **continuously compounded return** or **log return**:

$$r_t = \ln(1 + R_t) = \ln \frac{P_t}{P_{t-1}} = p_t - p_{t-1} \quad \text{where} \quad p_t = \ln P_t$$

- Advantages of log returns:
  - Easy to compute multi-period returns

$$\begin{aligned} r_t[k] &= \ln(1 + R_t[k]) = \ln\left[(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})\right] \\ &= r_t + r_{t-1} + \cdots + r_{t-k} \end{aligned}$$

  - More tractable statistical properties

Example

- Suppose that the price of Microsoft in month $t-2$ is \$80 and no dividend is paid between months $t-2$ and t. The two-month return is

$$R_t(2) = \frac{90-80}{80} = \frac{90}{80} - 1 = 1{,}1250 - 1 = 0{,}1250$$

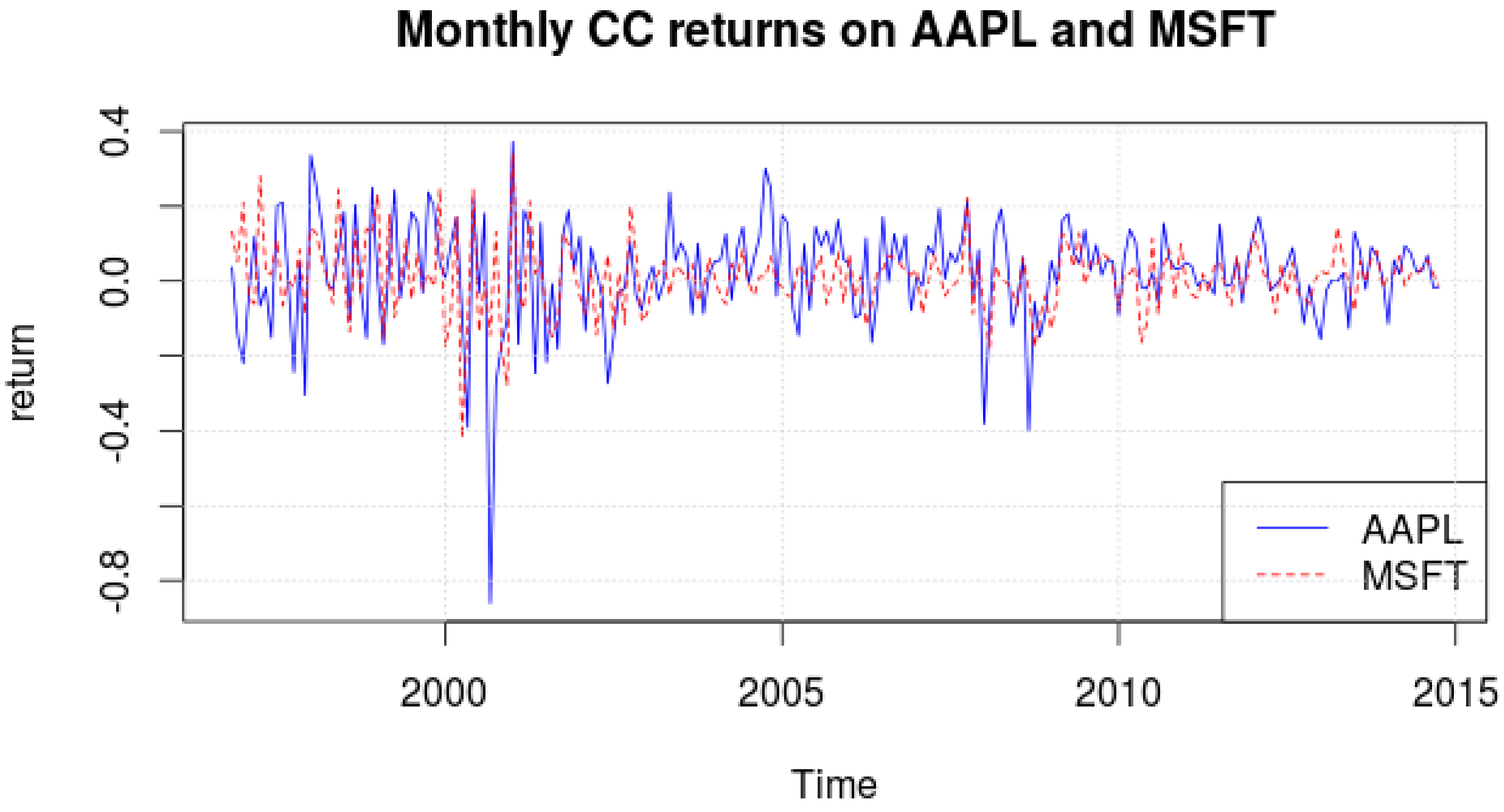
or 12.50% per two months. The two one-month returns are

$$R_{t-1} = \frac{85-80}{80} = \frac{5}{80} = 1{,}0625 - 1 = 0{,}0625$$

$$R_t = \frac{90-85}{85} = \frac{5}{85} = 1{,}0588 - 1 = 0{,}0588$$

And the two-month return is

$$1+R_t(2) = 1{,}0625*1{,}0588=1{,}1250$$


Monthly CC returns on AAPL and MSFT
return
0.4
0.0
-0.4
-0.8
2000
2005
2010
2015
Time
AAPL
MSFT

Portfolio returns

- Consider an investment of $V\$$ in two assets, named asset A and asset B
- Let x_A denote the fraction or share of wealth invested in asset A, and let x_B denote the remaining fraction invested in asset B
- The dollar amounts invested in assets A and B are $V\$ \times x_A$ and $V\$ \times x_B$, respectively
- We assume that the investment shares add up to 1, so that $x_A + x_B = 1$
- The collection of investment shares (x_A , x_B) defines a ***portfolio***
- Negative values for x_A or x_B represent ***short sales*** *("vendite allo scoperto")*

Portfolio returns

$$\$V \times [x_A(1 + R_{A,t}) + x_B(1 + R_{B,t})]$$

- Let $R_{A,t}$ and $R_{B,t}$ denote the ***simple one-period returns*** on assets A and B
- We wish to determine the simple one-period return on the portfolio defined by (x_A , x_B)
- At the end of period t, the investments in assets A and B are worth $V\$ \times x_A (1+ R_{A,t})$ and $V\$ \times x_B (1+ R_{B,t})$, respectively
- Hence, at the end of period the portfolio is worth $V\$ \times [x_A(1 + R_{A,t}) + x_B(1 + R_{B,t})]$

$$\$V(1 + R_{p,t}) = \$V\,[x_A(1 + R_{A,t}) + x_B(1 + R_{B,t})]$$

$$= \$V\,[x_A + x_B + x_A R_{A,t} + x_B R_{B,t}]$$

$$= \$V\,[1 + x_A R_{A,t} + x_B R_{B,t}]$$

$$\boxed{\Rightarrow R_{p,t} = x_A R_{A,t} + x_B R_{B,t}}$$

The simple portfolio return is a weighted average of the simple returns on assets A and B, where the weights are the portfolio shares x_A and x_B

Compute portfolio return

- Consider a portfolio of Microsoft and Starbucks stocks in which you initially purchase ten shares of each stock at the end of month *t−1* at the prices

$$P_{msft,t-1} = \$85, \;\; P_{sbux,t-1} = \$30,$$

- The initial value of the portfolio is

$$V_{t-1} = 10 \times \$85 + 10 \times 30 = \$1,150.$$

- The portfolio shares are

$$x_{msft} = 850/1150 = 0.7391, \;\; x_{sbux} = 300/1150 = 0.2609.$$

- The end of month *t* prices are

$$P_{msft,t} = \$90 \text{ and } P_{sbux,t} = \$28.$$

Compute portfolio return

- Assuming Microsoft and Starbucks do not pay a dividend between periods *t − 1* and *t*, the one-period returns are

$$R_{msft,t} = \frac{\$90 - \$85}{\$85} = 0.0588$$

$$R_{sbux,t} = \frac{\$28 - \$30}{\$30} = -0.0667$$

- The return on the portfolio is

$$R_{p,t} = (0.7391)(0.0588) + (0.2609)(-0.0667) = 0.02609$$

- and the value at the end of month t is

$$V_t = \$1,150 \times (1.02609) = \$1,180$$

Portfolio returns

- In general, for a portfolio of *n* assets with investment shares x_i such that $x_1 + \ldots + x_n = 1$

$$1 + R_{p,t} = \sum_{i=1}^{n} x_i (1 + R_{i,t})$$

$$R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t}$$

$$= x_1 R_{1t} + \cdots + x_n R_{nt}$$

Adjusting for dividends

Capital Gain
guadagno in conto capitale. Utilizzato in ambito borsistico e finanziario per identificare il guadagno ottenuto dalla compravendita di azioni, di obbligazioni, e di altri strumenti finanziari. E' la differenza tra il prezzo di acquisto e quello di vendita di uno strumento finanziario.
In Europa attualmente il **capital gain** è soggetto a tassazione con aliquote differenziate da Paese a Paese.

- If an asset pays a dividend, D_t, sometime between months $t-1$ and t, the total net return calculation becomes

$$R_t^{total} = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \underbrace{\boxed{\frac{P_t - P_{t-1}}{P_{t-1}}}}_{\text{capital gain}} + \underbrace{\boxed{\frac{D_t}{P_{t-1}},}}_{\text{dividend yield}}$$

- If an asset pays dividends periodically, the definition of asset returns must be modified:
  - D_t = dividend payment of an asset between periods $t-1$ and t
  - P_t = price of the asset at the end of period t
- The total gross return is:

$$1 + R_t^{total} = \frac{P_t + D_t}{P_{t-1}}$$

Adjusting for dividends

Dividend yeld

rapporto dividendo-prezzo corrisponde al rapporto tra l'ultimo dividendo annuo per azione corrisposto agli azionisti o annunciato e il prezzo in chiusura dell'anno di un'azione ordinaria. Esso è utilizzato come indicatore del rendimento immediato indipendentemente dal corso del titolo azionario.

- If an asset pays a dividend, D_t, sometime between months $t-1$ and t, the total net return calculation becomes

$$R_t^{total} = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \underbrace{\boxed{\frac{P_t - P_{t-1}}{P_{t-1}}}}_{\text{capital gain}} + \underbrace{\boxed{\frac{D_t}{P_{t-1}}}}_{\text{dividend yield}},$$

- If an asset pays dividends periodically, the definition of asset returns must be modified:
  - D_t = dividend payment of an asset between periods $t-1$ and t
  - P_t = price of the asset at the end of period t
- The total gross return is:

$$1 + R_t^{total} = \frac{P_t + D_t}{P_{t-1}}$$

Total return on Microsoft stock when dividends are paid

- Consider a one-month investment in Microsoft stock
- Suppose you buy the stock in month *t−1* at P_{t-1}= *85$* and sell the stock the next month for P_t = *90$*
- Assume Microsoft pays a *1$* dividend between months *t−1* and *t*
- The capital gain, dividend yield and total return are then

$$
\begin{aligned}
R_t &= \frac{\$90 + \$1 - \$85}{\$85} = \frac{\$90 - \$85}{\$85} + \frac{\$1}{\$85} \\
&= 0.0588 + 0.0118 \\
&= 0.0707
\end{aligned}
$$

- The one-month investment in Microsoft yields a 7.07% per month total return
- The **capital gain** component is 5.88%, and the **dividend yield** component is 1.18%.

Adjusting for Inflation

- The return calculations considered are based on the *nominal or current prices of assets*. Returns computed from nominal prices are ***nominal returns***
- The real return on an asset over a particular horizon takes into account the growth rate of the general price level over the horizon
  - If the nominal price of the asset grows faster than the general price level then the nominal return will be greater than the inflation rate and the real return will be positive
  - Conversely, if the nominal price of the asset increases less than the general price level then the nominal return will be less than the inflation rate and the real return will be negative
- The ***computation of real returns*** on an asset is a two step process:
  - Deflate the nominal price of the asset by the general price level
  - Compute returns in the usual way using the deflated prices

Adjusting for Inflation

- Consider computing the **real simple one-period return** on an asset. Let P_t denote the nominal price of the asset at time t and let CPI_t denote an index of the general price level (e.g. Consumer Price Index) at time t
- The deflated or real price at time t is

$$P_t^{\text{Real}} = \frac{P_t}{CPI_t},$$

- Compute returns in the usual way using the deflated prices

$$\begin{aligned} R_t^{\text{Real}} &= \frac{P_t^{\text{Real}} - P_{t-1}^{\text{Real}}}{P_{t-1}^{\text{Real}}} = \frac{\frac{P_t}{CPI_t} - \frac{P_{t-1}}{CPI_{t-1}}}{\frac{P_{t-1}}{CPI_{t-1}}} \\ &= \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t} - 1. \end{aligned}$$

- Alternatively, define inflation as

$$\pi_t = \%\Delta CPI_t = \frac{CPI_t - CPI_{t-1}}{CPI_{t-1}}$$

$$R_t^{\text{Real}} = \frac{1 + R_t}{1 + \pi_t} - 1$$

Example

- Compute real return on Microsoft stock Suppose the CPI in months *t−1* and *t* is 1 and 1.01, respectively, representing a 1% monthly growth rate in the overall price level. The real prices of Microsoft stock are

$$P_{t-1}^{\text{Real}} = \frac{\$85}{1} = \$85, \;\; P_t^{\text{Real}} = \frac{\$90}{1.01} = \$89.1089$$

- The real monthly return is $$R_t^{\text{Real}} = \frac{\$89.10891 - \$85}{\$85} = 0.0483$$

- The nominal return and inflation over the month are $R_t = \frac{\$90 - \$85}{\$85} = 0.0588, \;\; \pi_t = \frac{1.01 - 1}{1} = 0.01$

- Then the real return is $$R_t^{\text{Real}} = \frac{1.0588}{1.01} - 1 = 0.0483$$

- Notice that simple real return is almost, but not quite, equal to the simple nominal return minus the inflation rate

Annualize returns

- Returns are often converted to an annual return to establish a standard for comparison
- Assume same monthly return R_m for 12 months:

$$\text{Compound annual gross return (CAGR)} = 1+R_A = 1+R_t(12) = (1+R_m)^{12}$$

$$\text{Compound annual net return} = R_A = (1+R_m)^{12} - 1$$

Note: We don't use $R_A = 12R_m$ because this ignores compounding.

Annualized return on Microsoft

- Suppose the one-month return, R_t , on Microsoft stock is 5.88%. If we assume that we can get this return for 12 months then the compounded annualized return is

$$R_A = (1.0588)^{12} - 1 = 1.9850 - 1 = 0.9850$$

- or 98.50% per year

Average returns

- For investments over a given horizon, it is often of interest to compute a measure of average return over the horizon.
- Consider a sequence of monthly investments over the year with monthly returns

Two possibilites:

1. Arithmetic average (can be misleading)

$$\bar{R} = \frac{1}{12}(R_1 + \cdots + R_{12})$$

2. Geometric average (better measure of average return)

$$(1+\bar{R})^{12} = (1+R_A) = (1+R_1)(1+R_2)\cdots(1+R_{12})$$

$$\Rightarrow \bar{R} = (1+R_A)^{1/12} - 1$$

$$= [(1+R_1)(1+R_2)\cdots(1+R_{12})]^{1/12} - 1$$

Average returns

Consider a two period invesment with returns

$$R_1 = 0.5, \;\; R_2 = -0.5$$

\$1 invested over two periods grows to

$$FV = \$1 \times (1 + R_1)(1 + R_2) = (1.5)(0.5) = 0.75$$

for a 2-period return of

$$R(2) = 0.75 - 1 = -0.25$$

Hence, the 2-period investment loses 25%

Average returns

The arithmetic average return is

$$\bar{R} = \frac{1}{2}(0.5 + -0.5) = 0$$

This is misleading becuase the actual invesment lost money over the 2 period horizon. The compound 2-period return based on the arithmetic average is

$$(1 + \bar{R})^2 - 1 = 1^2 - 1 = 0$$

The geometric average is

$$[(1.5)(0.5)]^{1/2} - 1 = (0.75)^{1/2} - 1 = -0.1340$$

This is a better measure because it indicates that the investment eventually lost money. The compound 2-period return is

$$(1 + \bar{R})^2 - 1 = (0.867)^2 - 1 = -0.25$$

Annualizing returns

- Very often returns over different horizons are annualized, i.e., converted to an annual return, to facilitate comparisons with other investments.
- The annualization process depends on the holding period of the investment and an implicit assumption about compounding
- If our investment horizon is one year then the annual gross and net returns are just

$$
\begin{aligned}
1 + R_A &= 1 + R_t(12) = \frac{P_t}{P_{t-12}} = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-11}), \\
R_A &= R_t(12).
\end{aligned}
$$

- Consider a one-month investment in an asset with return R_t. What is the annualized return on this investment?
- If we assume that we receive the same return R = R_t every month for the year, then the gross annual return is $1 + R_A = 1 + R_t(12) = (1 + R)^{12}$.
- The net annual return is then $R_A = (1 + R)^{12} - 1$.

Compute annualized return from one-month return

- In the first example, the one-month return, R_t on Microsoft stock was 5,88%
- If we assume that we can get this return for 12 months, what is the annualized return ?

$$R_A = (1.0588)^{12} - 1 = 1.9850 - 1 = 0.9850,$$

98,50% per year!

- Consider a two-month investment with return $R_t(2)$. If we assume that we receive the same two-month return $R(2) = R_t(2)$ for the next six two-month periods, then the gross and net annual returns are

$$1 + R_A = (1 + R(2))^6,$$
$$R_A = (1 + R(2))^6 - 1.$$

The annual gross return is defined as the two-month return compounded for 6 months.

Compute annualized return from two-month return

- Suppose the two-month return, $R_t(2)$ on Microsoft stock is 12,5%
- If we assume that we can get this two-month return for the next 6 two-month periods then the annualized return is

$$R_A = (1.1250)^6 - 1 = 2.0273 - 1 = 1.0273$$

102.73% per year

- Now suppose that our ***investment horizon is two years***. That is, we start our investment at time *t – 24* and cash out at time *t*
- The two-year gross return is $1 + R_t(24) = \frac{P_t}{P_{t-24}}$
- What is the annual return on this two-year investment?
- To determine the annual return we solve the following relationship for R_A: the annual return is compounded twice to get the two-year return and the relationship is then solved for the annual return.

$$(1 + R_A)^2 = 1 + R_t(24) \Longrightarrow$$
$$R_A = (1 + R_t(24))^{1/2} - 1.$$

Compute annualized return from two-year return

- Suppose that the price of Microsoft stock 24 months ago is P_{t-24} = \$50 and the price today is P_t = \$90
- The two-year gross return is 1+ $R_t(24)$ = \$90 /\$50 = 1.8000 which yields a two-year net return of $R_t(24)$ = 0.80 = 80%
- The annual return for this investment is defined as

$$R_A = (1.800)^{1/2} - 1 = 1.3416 - 1 = 0.3416,$$