Portfolio Optimization
The Mean-Variance model

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Foundation of Portfolio Optimization

- **Harry Markowitz** presented in 1952 the basics of portfolio selection:
  - *finding a combination of assets that in a given period of time produces the highest possible return at the least possible risk*
- Two main steps:
  - identifying the combinations of assets, or portfolios, that are optimal with respect to their expected **return** and **risk**
  - choosing that portfolio that best suits the **investor's utility function**
- Utility function concerns investor's personal constraints, such as, tolerance to risk, type and number of assets desired, and total capital for the investment

The Mean-Variance Model

- The departing hypothesis of Markowitz for his portfolio selection model is that investors should consider expected return a desirable thing and abhor the variance of return

- A clear mathematical explanation to the widely accepted belief among investors on the importance of **diversification in portfolio selection**

Portfolio returns (just a remind…)

- In general, for a portfolio of *n* assets with investment shares x_i such that $x_1 + \ldots + x_n = 1$

$$1 + R_{p,t} = \sum_{i=1}^{n} x_i (1 + R_{i,t})$$

$$R_{p,t} = \sum_{i=1}^{n} x_i R_{i,t}$$

$$= x_1 R_{1t} + \cdots + x_n R_{nt}$$

Mean and Variance of a portfolio

- **Mean value** (or expected return) at time *t* of the portfolio ***w***:

$$\mu_{\mathbf{w}} = E(R_t^{\mathbf{w}}) = \sum_{i=1}^{N} w_i E(R_{i,t})$$

- **Variance** of portfolio ***w***:

$$\sigma_{\mathbf{w}}^2 = Var(R_t^{\mathbf{w}}) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij} = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

Let’s consider two opposite cases

- **Case 1**: The returns of the *N* assets in the portfolio are pairwise uncorrelated
  - *the greater the number of pairwise uncorrelated assets, the smaller the risk of the portfolio*
- **Case 2**: The returns of the assets in the portfolio are similarly correlated.
  - *the greater the presence of similarly correlated assets, the closer is the risk of the portfolio to a risk common to all assets* (e.g., some weighted average of individual assets’ risks)
- Therefore, **diversification** in a mean-variance world is accomplished by considering **highly uncorrelated assets** in some reasonable number
- By plotting the variance of the portfolio as a function of *N*, given by either of the above cases, one sees that from 15 to 20 are reasonable numbers for the size of a portfolio.

Minimum Risk Mean-Variance Portfolio

- According to the Markowitz's mean of returns versus variance of returns rule, investors' main concern is to obtain a certain level of benefits under the smallest possible amount of risk

- Therefore, the Markowitz portfolio selection problem amounts to:
  - Find weights $\boldsymbol{w} = (w_1, \ldots, w_N)$ such that, for a given expected rate of return r^*, the expected return of the portfolio determined by $\boldsymbol{w}$ is r^* while its variance is minimal

Min variance as an optimization problem

- This minimum variance for given return portfolio optimization problem can be mathematically formulated as follows:

$$
\begin{aligned}
&\min_{w} \; w'Cw \\
\text{subject to:}\quad & w'\mu = r^*, \\
\text{and}\quad & \sum_{i=1}^{N} w_i = 1
\end{aligned}
$$

- This is a **Quadratic Programming** (QP) problem, since the objective is quadratic and the constraints are linear, which can be reduced to a linear system of equations and solve (via Lagrangian relaxation)

- Constraint means that the investor uses all his budget for the *N* assets

Some considerations

- Observe that the model imposes no restrictions on the values of the weights (i.e., there could be either long or short positions)

- Under these relaxed conditions on weights (and assuming extended no arbitrage) this portfolio optimization problem can be solved analytically using *Lagrange multipliers*

- Solving a linear system, one obtains the weights for a portfolio with mean r^* and the smallest possible variance

- This solution is termed *efficient* in the sense of being the portfolio with that expected return r^* and minimum variance, and such that any other portfolio on the same set of securities with same expected return r^* must have a higher variance

The Efficient Frontier and the Minimum Variance Portfolio

- For a set of *N* assets for constituting a portfolio, consider different values of *r** for the expected return of the portfolio, and for each of these values solve the QP to get the efficient solution of weights ***w***

- From these weights obtain the corresponding standard deviation of the portfolio ***w:***

$$\sigma^* = std(R^{w^*}) = \sqrt{(w^*)'\mathbf{C}w^*}.$$

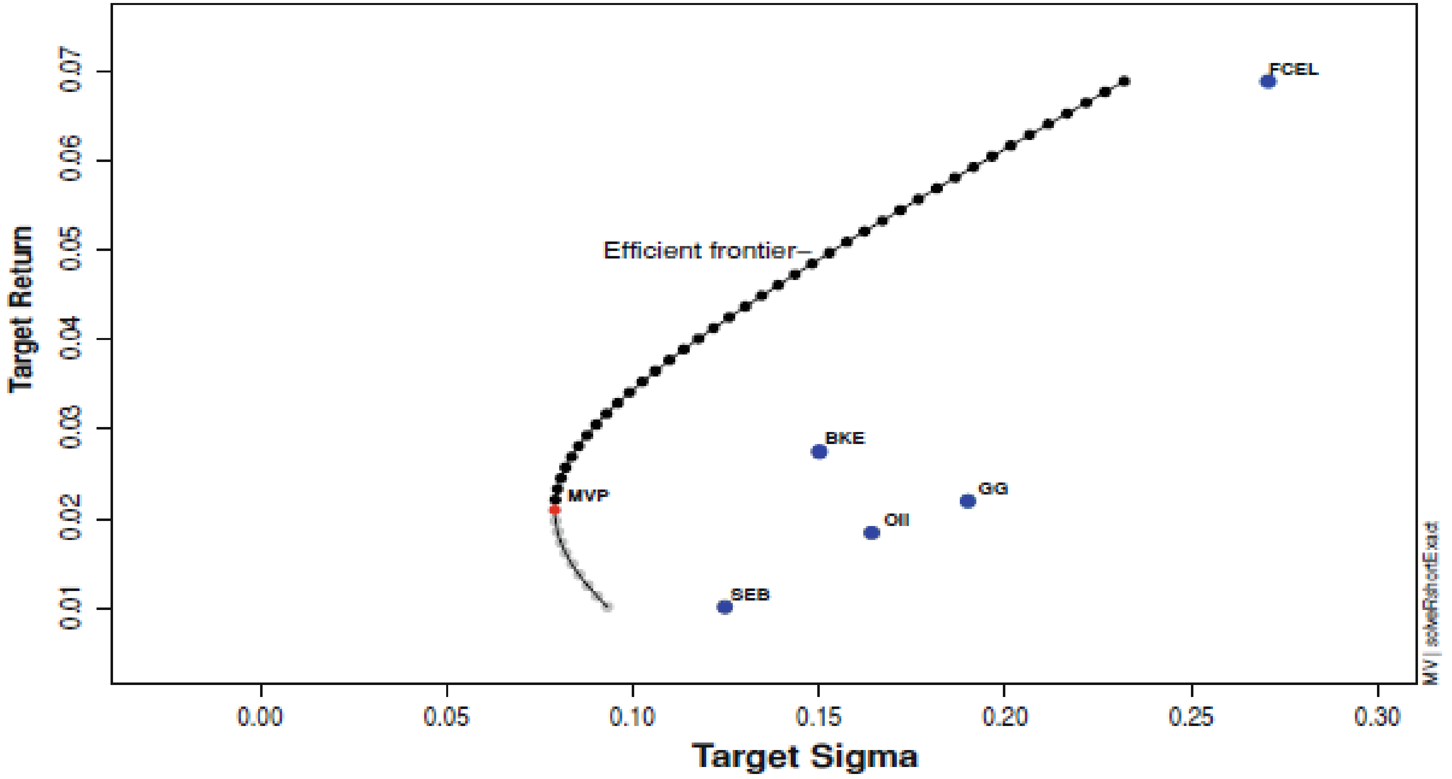
Efficient frontier

The points *(σ*,r*)* in the *σ-μ* plane, or the *risk-mean* plane, is the right branch of a hyperbola

The vertex coordinates of this hyperbola (σ_{MVP} , r_{MVP}) are the standard deviation and mean of the *minimum variance portfolio* (**MVP**)

The part of the curve above the MVP's point ($r > r_{MVP}$) is called the ***efficient frontier***

*T*he symmetric reflexion curve below MVP's point is called the ***minimum variance locus***



Portfolios with a risk-free asset

Portfolio with a risk-free asset

- We have so far assumed that the assets available for constituting a portfolio are all risky
- Now we deal with the case of adding a risk-free asset and the pricing model obtained as consequence
- Adding a risk free asset to a portfolio corresponds to lending or borrowing cash at a known interest rate r_0 and with zero risk:
  - Lending corresponds to the risk free asset having a positive weight
  - borrowing corresponds to its having a negative weight

Portfolio with a risk-free asset

- Let $r_f = r_0\,\tau$ be the risk free rate, or return, over the time period τ of the risk free asset

- A portfolio consisting solely of this risk free asset has mean value r_f and variance 0

- This risk free portfolio is represented in the *risk-mean* plane by the point $(0, r_f)$

- Now consider a portfolio consisting of the risk free asset, with mean value r_f plus N risky assets with aggregated mean and variance given by

$$\mu_w = E(R_t^w) = \sum_{i=1}^{N} w_i E(R_{i,t}) \qquad \sigma_w^2 = Var(R_t^w) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij} = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

Portfolio with a risk-free asset

- Let w_f be the weight of the risk free asset in the portfolio; therefore:

$$1 - w_f = w_1 + \ldots + w_N$$

- which implies that the total wealth has been distributed in two parts: one for investing in the risk free asset and the rest for investing among N risky assets

- We can then view the portfolio $(\mathbf{w}, w_f) = (w_1, \ldots, w_N, w_f)$ as consisting of one risk free asset, with weight w_f, mean r_f and zero standard deviation, together with a risky asset, which is the aggregation of the N risky assets, with weight $1 - w_f$, mean μ_w and standard deviation σ_w

- Note that this pair of risky and a risk free asset has covariance equal to zero

Portfolio with a risk-free asset

- Then the expected return and standard deviation of this combined portfolio $\omega = (\boldsymbol{w}, w_f)$ are

$$\mu_\omega = w_f\, r_f + (1 - w_f)\mu_{\boldsymbol{w}}$$
$$\sigma_\omega = (1 - w_f)\sigma_{\boldsymbol{w}}$$

- We see in these equations that the mean and the standard deviation of the portfolio ω depend linearly on w_f . Therefore, varying w_f we get that the different portfolios represented by the two previous equations trace a straight line from the point $(0, r_f)$ and passing through the point $(\sigma_{\boldsymbol{w}}, \mu_{\boldsymbol{w}})$ in the risk-mean plane

- Furthermore, varying the weights of the N risky assets, i.e. building another risky portfolio ***w'***, and combining it with the risk-free asset, we get another straight line describing all possible portfolios which are combination of these two

- We conclude that the feasible region of mean-variance portfolios obtained from *N* risky assets and one risk free asset is a triangle with a vertex in the point representing the risk-free asset and enveloping the hyperbola containing all feasible portfolios on the *N* risky assets.

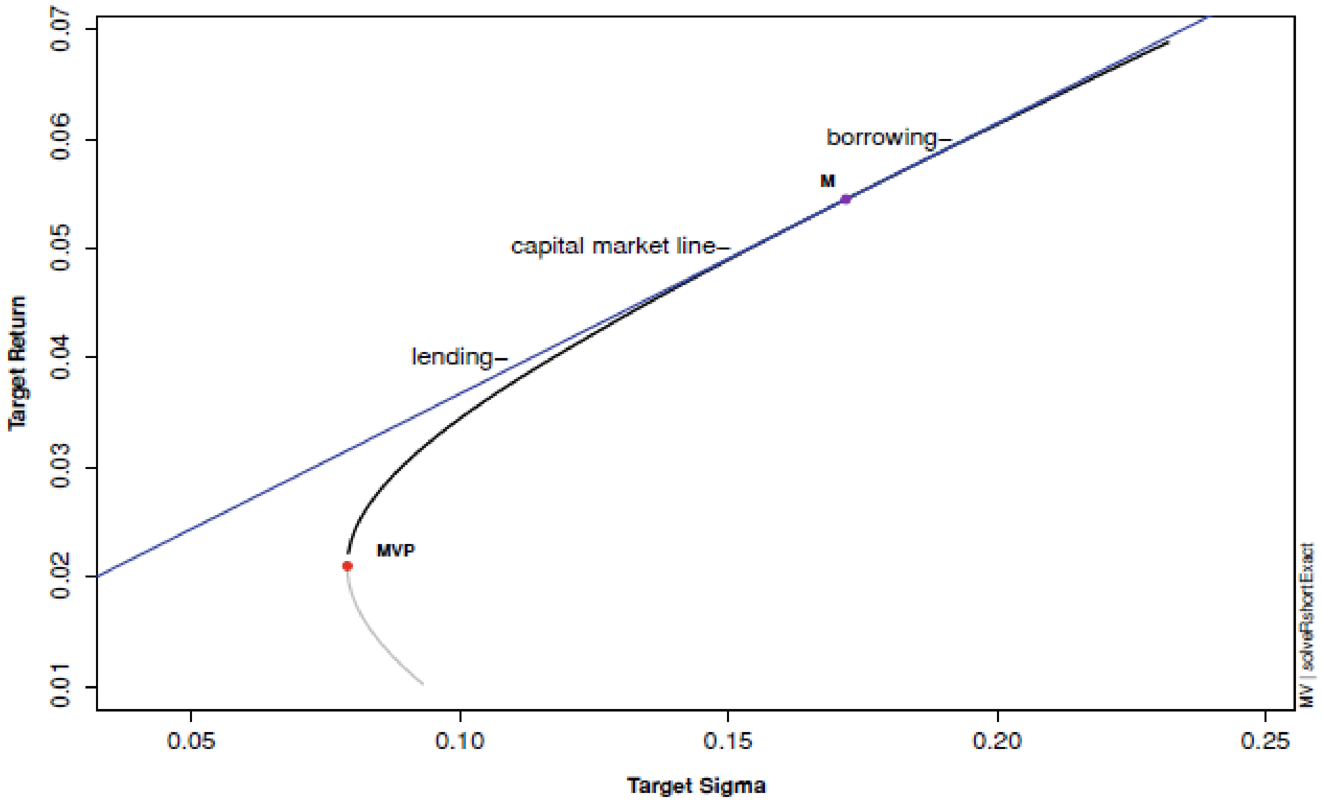


Fig. 8.3 Feasible region and efficient frontier of portfolios with a risk free asset with $r_f = 2\,\%$. Capital Market Line (*blue*); Market Portfolio (M). Efficient portfolios above (resp. *below*) M need borrowing (resp. *lending*) at the risk free rate

The Capital Market Line and the Market Portfolio

- The efficient frontier for a portfolio of risky assets and one risk-free asset is now a straight line with intercept point $(0, r_f)$ and tangent to the efficient frontier of risky portfolios (denoted from now on EF_r) in the risk-mean plane

- This tangent line describing all efficient portfolios is named the *Capital Market Line* (CML), and the point where the CML makes contact with the curve EF_r has as coordinates the standard deviation and expected return of a particular portfolio named the *Market Portfolio*

- The Market Portfolio is the best portfolio with respect to the (excess return)/(risk) ratio, and it is the best representative of the market for it contains shares of every stock in proportion to the stock's weight in the market

The Capital Market Line and the Market Portfolio

- Let θ be the angle between the horizontal axis and a line passing through $(0, r_f)$ and a point $(std(R^w), E(R^w))$ corresponding to some feasible portfolio of risky assets only. Then

$$\tan\theta = \frac{E(R^w) - r_f}{std(R^w)}$$

- The Market Portfolio is the point that maximizes $\tan(\theta)$, for this gives the slope of the CML computed at the point tangent to the risky efficient frontier (and this is the reason why the Market Portfolio is also known as *the tangency portfolio*)

- From the equation above it is clear that the Market Portfolio gives the maximum (excess return)/(risk) ratio; and the weights ***w*** that are solution of the problem of maximizing $\tan(\theta)$ are in proportions to the stocks' market weights

The Sharpe Ratio

- Let *($std(R_M)$, $E(R_M)$)* be the point describing the Market Portfolio, i.e. the tangency portfolio or contact point between the CML and the EF_r

- As before, $(0, r_f)$ represents the risk-free asset. Then any point $(std(R^w), E(R^w))$ on the CML, that is, any efficient portfolio, is such that

$$E(R^w) = w_f\, r_f + (1 - w_f)E(R_M)$$

$$std(R^w) = (1 - w_f)\ \ std(R_M)$$

- where w_f is the fraction of wealth invested in the risk-free asset:
  - If $w_f \geq 0$ we are lending at the risk-free rate; whereas if $w_f < 0$ we are borrowing at the risk-free rate, in order to increase our investment in risky assets

The Sharpe Ratio

$E(R^w) = w_f r_f + (1 - w_f)E(R_M)$
$= r_f - r_f + w_f r_f + (1 - w_f)E(R_M)$
$= r_f - r_f(1 - w_f) + (1 - w_f)E(R_M)$
$= r_f - (1 - w_f)(E(R_M) - r_f)$

From $std(R^w) = (1 - w_f)\ std(R_M)$, *replace* $(1 - w_f)$ *with* $std(R^w) / std(R_M)$

- We can rewrite the CML as

$$
\begin{aligned}
E(R^w) &= r_f + (1 - w_f)(E(R_M) - r_f) \\
&= r_f + \frac{(E(R_M) - r_f)}{std(R_M)} std(R^w)
\end{aligned}
$$

- The quantity $SR_M = (E(R_M) - r_f)/std(R_M)$ is known as the *Sharpe ratio* of the Market Portfolio
- In general, the Sharpe ratio of any portfolio is the number:

$$SR^W = (E(R^W) - r_f)/std(R^W)$$

- which gives a measure of the portfolio reward to variability ratio, and has become standard for portfolio evaluation

The Sharpe Ratio

- The higher the Sharpe ratio the better the investments, with the upper bound being the Sharpe ratio of the Market Portfolio
- Thus, all efficient portfolios, with risky assets and one risk-free asset, should have same Sharpe ratio and equal to the Sharpe ratio of the market:

$$SR^W = SR_M$$

- Therefore, summarizing, the best an investor can do in a mean-variance world is to allocate a proportion of his investment money in a risk-free asset and the rest in the Market Portfolio. This guarantees the best possible ratio of excess return to variability

The Capital Asset Pricing Model and the Beta of a Security

- The Capital Market Line shows an equilibrium between the expected return and the standard deviation of an efficient portfolio consisting of a risk-free asset and a basket of risky assets

- It would be desirable to have a similar equilibrium relation between risk and reward of an individual risky asset with respect to an efficient risky portfolio, where it could be included

- This risk-reward equilibrium for individual assets and a generic efficient portfolio (i.e., the Market Portfolio), is given by the *Capital Asset Pricing Model* (CAPM)

The Capital Asset Pricing Model and the Beta of a Security

- **Theorem** (**CAPM**) *Let $E(R_M)$ be the expected return of the Market Portfolio, $\sigma_M = std(R_M)$ its standard deviation, and r_f the risk-free return in a certain period of time τ*
- *Let $E(R_i)$ be the expected return of an individual asset i, σ_i its standard deviation, and $\sigma_{iM} = Cov(R_i, R_M)$ be the covariance of the returns R_i and R_M*
- *Then:*

$$E(R_i) = r_f + \beta_i(E(R_M) - r_f)$$

- where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{Cov(R_i, R_M)}{Var(R_M)}.$$

The Capital Asset Pricing Model and the Beta of a Security

- The CAPM states that the expected excess rate of return of asset *i*, $E(R_i)-r_f$, also known as the asset's *risk premium*, is proportional by a factor of β_i to the expected excess rate of return of the Market Portfolio, $E(R_M) - r_f$, or the *market premium*

- The coefficient β_i, known as the *beta* of asset *i*, is then the degree of the asset's risk premium relative to the market premium (we will discuss more about beta…)

CAPM as pricing model

- Suppose that we want to know the price P of an asset whose payoff after a period of time τ is set to be some random value P_τ
- Then the rate of return of the asset through the period τ is $R_\tau = (P_\tau/P) - 1$, and by the CAPM the expected value of R_τ relates to the expected rate of return of the market, on the same period of time, as follows:

$$E(R_\tau) = \frac{E(P_\tau)}{P} - 1 = r_f + \beta(E(R_M) - r_f)$$

- Where ß is the beta of the asset

CAPM as pricing model

- Solving for *P* we get the pricing formula:

$$P = \frac{E(P_\tau)}{1 + r_f + \beta(E(R_M) - r_f)}$$

- This equation is a natural generalization of the discounted cash flow formula for risk free assets to include the case of risky assets, where the present value of a risky asset is its expected future value discounted back by a risk-adjusted interest rate given by $r_f + \beta(E(R_M) - r_f)$

- Indeed, observe that if the asset is risk free, e.g. a bond, then $E(P_\tau) = P_\tau$ and $\beta=0$, since the covariance with any constant return is null; hence, $P = P_\tau / (1 + r_f)$

On the meaning of beta

- Formally, the beta of an asset measures the linear dependence of the asset's return and the return of the market in proportion to the asset to market volatility ratio

- To better see this rewrite β_i in terms of correlations:

$$\beta_i = \frac{Cov(R_i, R_M)}{Var(R_M)} = \rho(R_i, R_M)\frac{\sigma_i}{\sigma_M}$$

On the meaning of beta

- Using the previous equation, we can provide interpretations of the values of an asset's beta as a measure of its co-movement with the market
- Let $\beta = \beta_i$ and $\rho = \rho(R_i, R_M)$ the correlation coefficient of the asset and the market

Value of beta	Effect on correlation and volatility ratio	Interpretation
$\beta < 0$	$\rho < 0$, $\frac{\sigma_i}{\sigma_M} > 0$	Asset moves in the opposite direction of the movement of the market
$\beta = 0$	$\rho = 0$	Movements of the asset and the market are uncorrelated
$0 < \beta \leq 1$	$\rho > 0$, $0 < \frac{\sigma_i}{\sigma_M} \leq 1/\rho$	Asset moves in the same direction as the market, volatility of asset can be $<$ or $>$ volatility of market
$\beta > 1$	$\rho > 0$, $\frac{\sigma_i}{\sigma_M} > 1/\rho > 1$	Asset moves in the same direction as the market but with greater volatility

On the meaning of beta

- For example, a $\beta = 0$ means that the asset and the market are uncorrelated, since the standard deviation ratio σ_i/σ_M is always positive

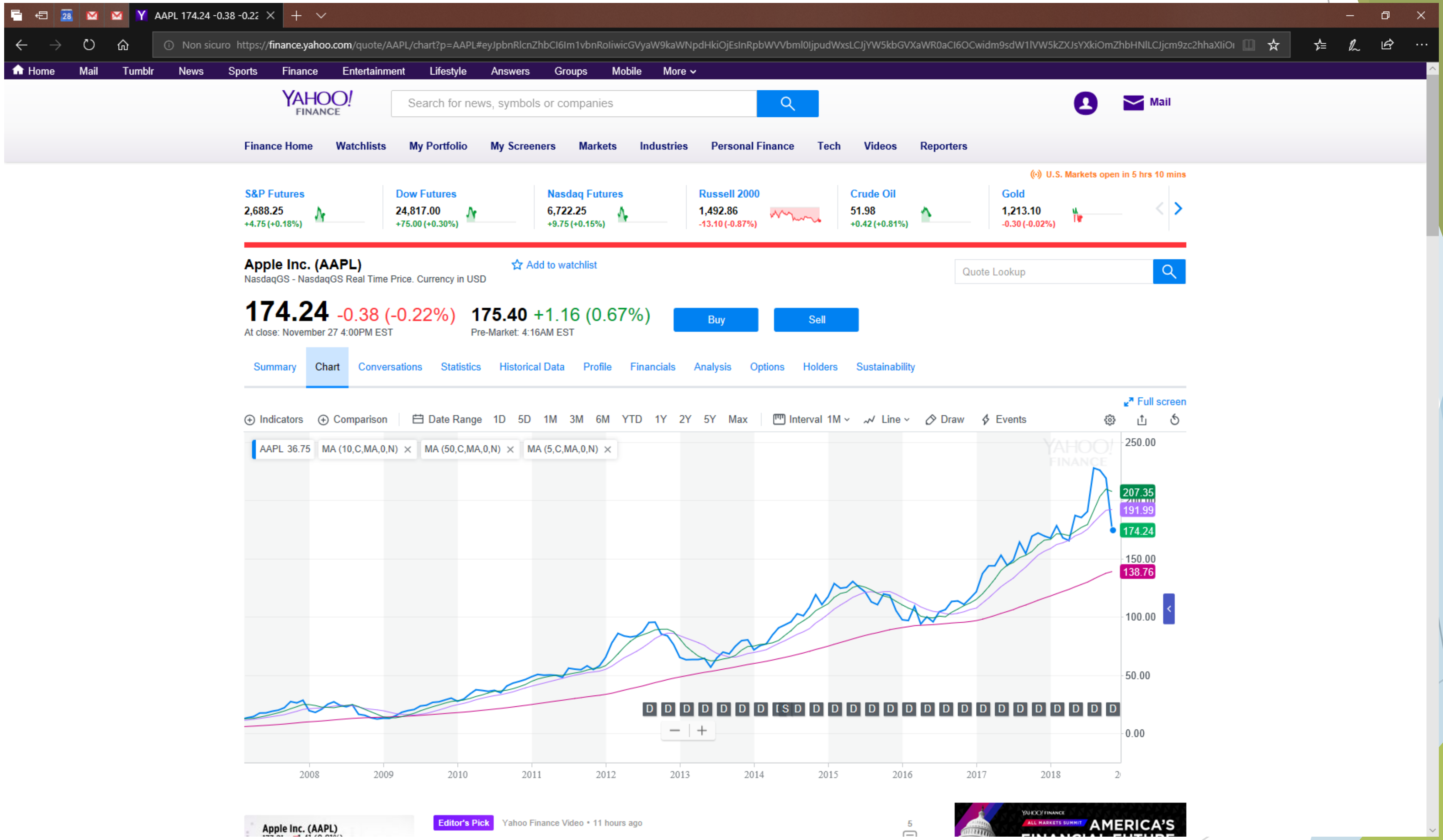
- In this case, the CAPM states that $E(R_i) = r_f$, or that the risk premium is zero

- The explanation for this surprising conclusion is that the risk of an asset which is uncorrelated with an efficient portfolio gets neutralize by the risk compounded from the different positions of the portfolio, and hence, one should not expect greater benefits than those that could be obtained at the risk-free rate

On the meaning of beta

- A different situation is presented by $\beta > 1$
- This implies that the asset and the market are positively correlated; but since $1/\rho \geq 1$ always, the risk of the asset is greater than the risk of the market
- This extra risk might have its compensation since from the CAPM follows that $E(R_i) > E(R_M)$, i.e. the asset could beat the market

Beta is a standard indicator provided by Financial Data Providers



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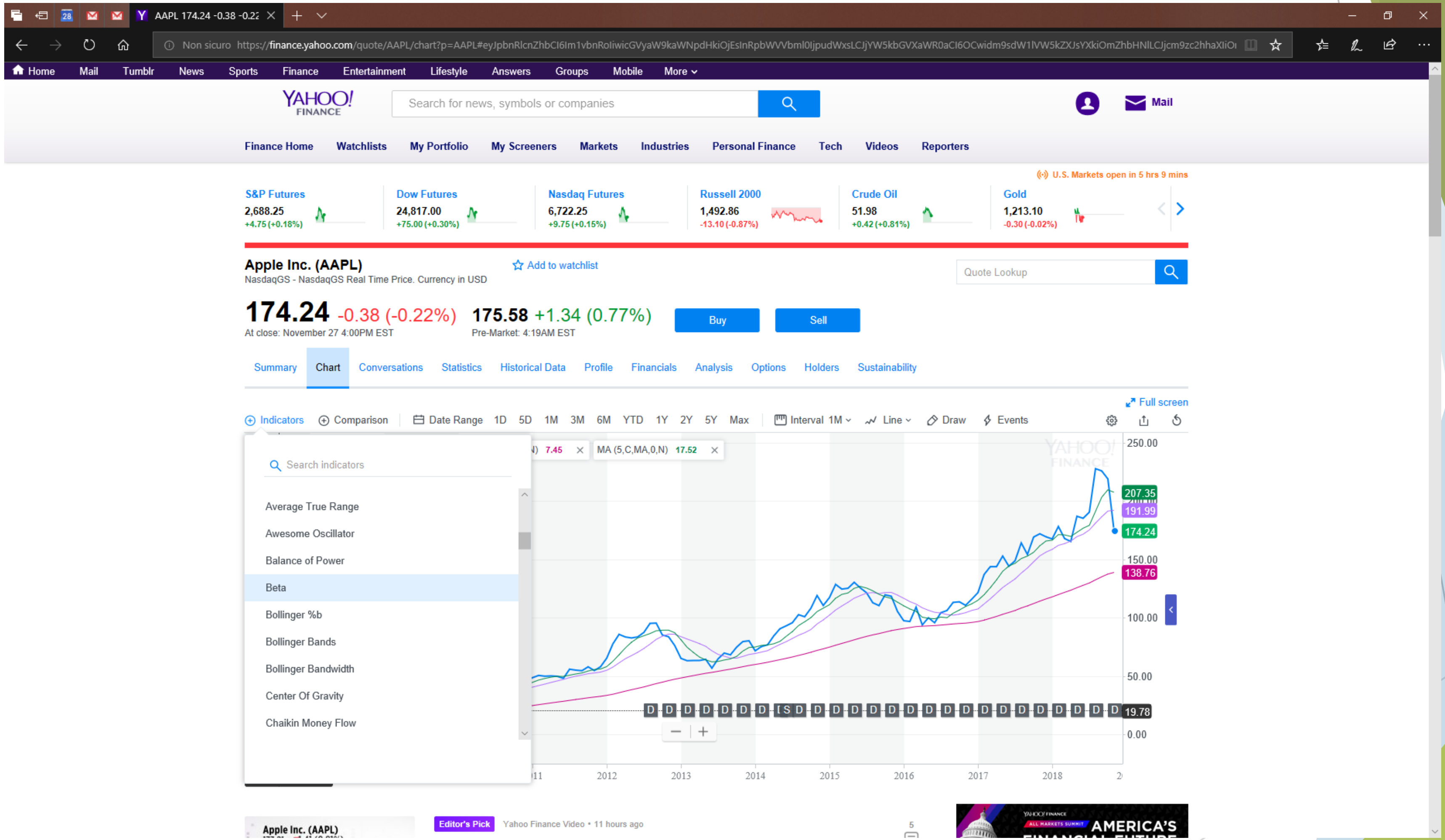

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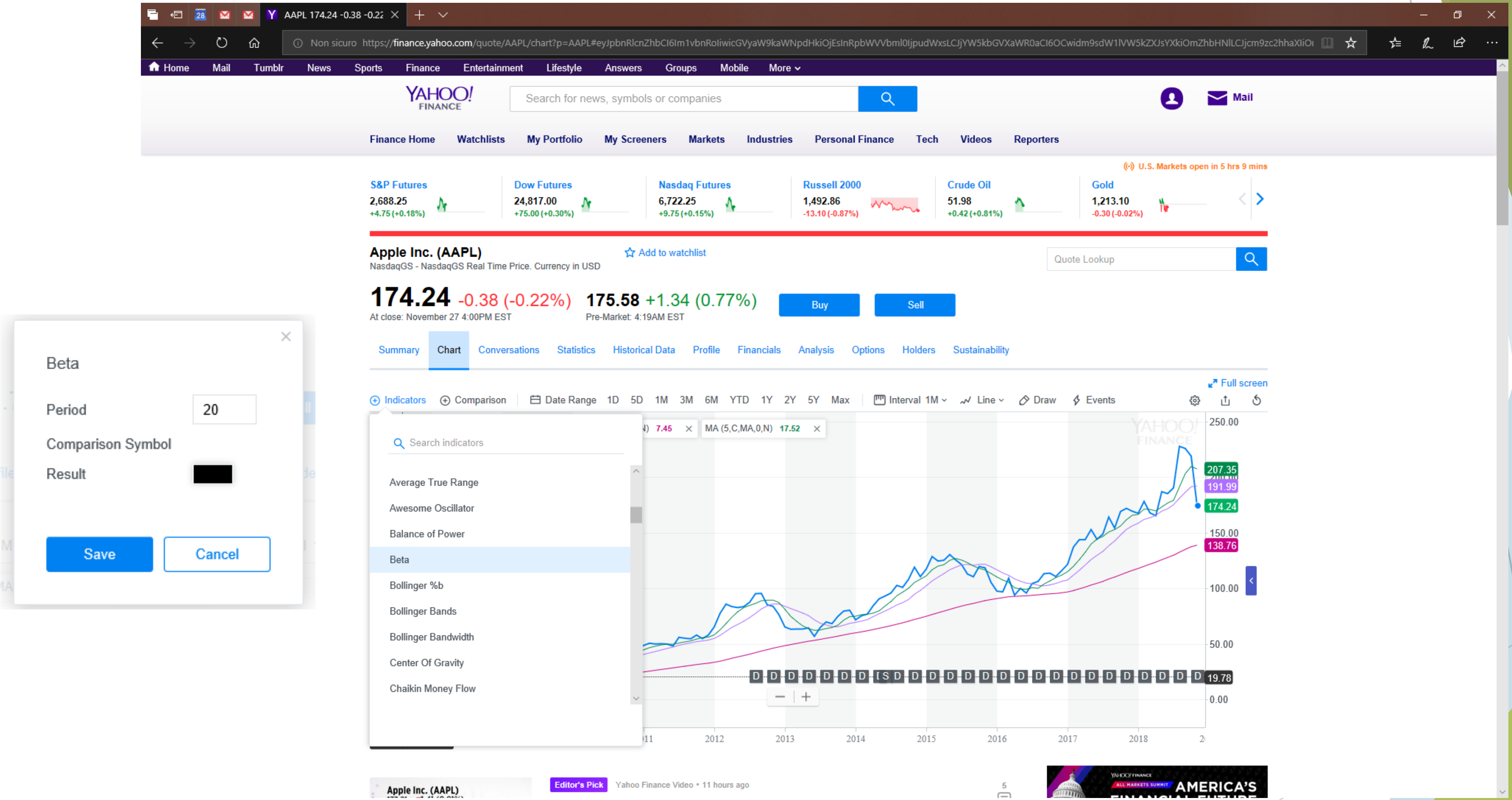

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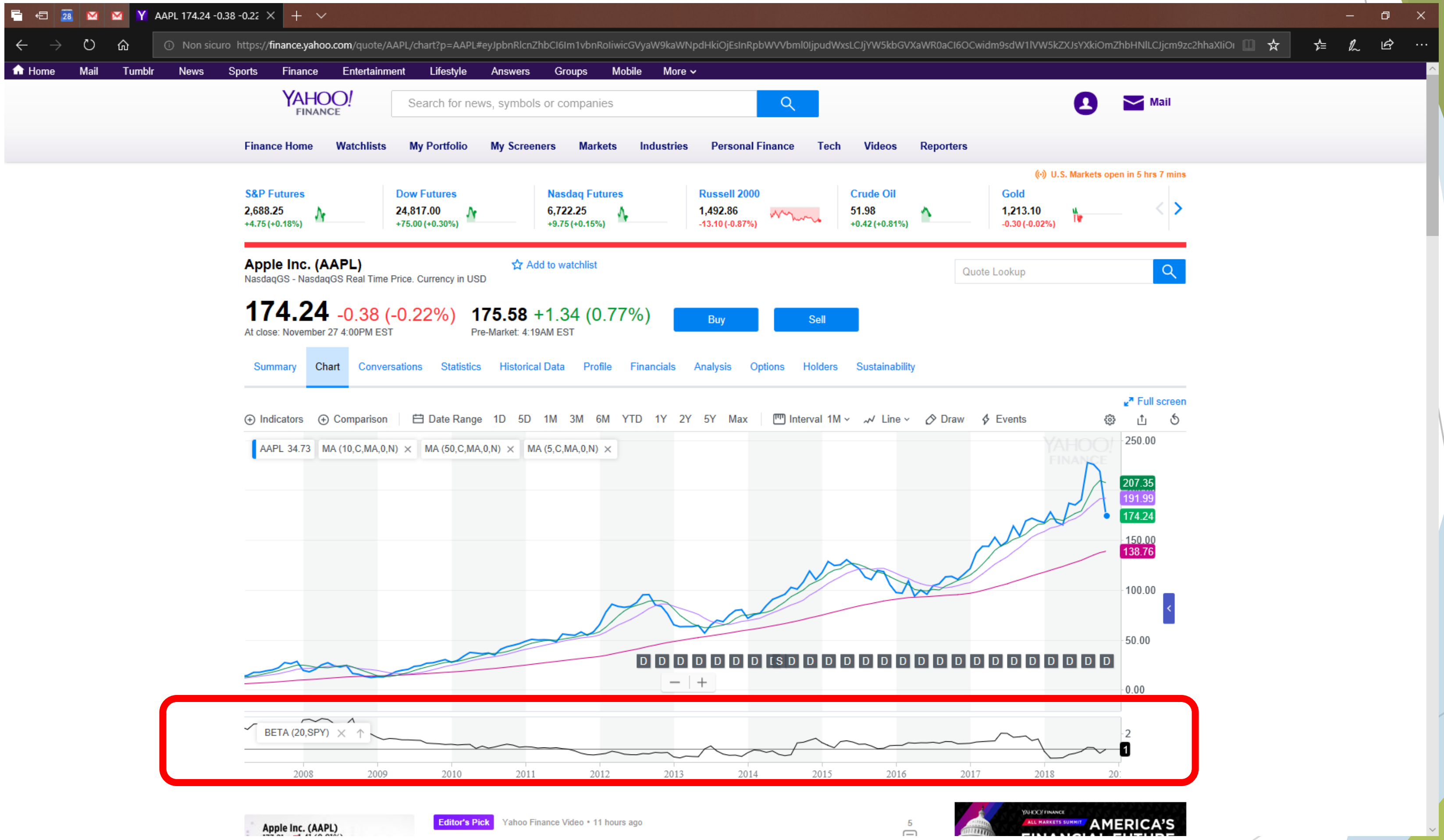

On the meaning of beta

- However, our interpretation of beta differs from mainstream interpretation of this statistic, and as it is frequently used by most financial institutions and investors, and as explained in various financial reports, textbooks, and many sources of popular knowledge (e.g. *Wikipedia*)

- The most common usage of beta is as some measure of risk relative to the market. But this is only one factor in the equation, the volatility ratio, which is then confounded with the correlation factor

- Then so often it seems that mainstream interpretation wrongly considers the product of these two statistics as if it were a conjunction, as it is the case when $0 < \beta \leq 1$ which is taken to mean that *jointly* $0 < \rho \leq 1$ *and* $0 < \sigma_i / \sigma_M \leq 1$

On the meaning of beta

- This leads to interpretations such as "*movement of the asset is in the same direction as, but less than the movement of the benchmark*", or equivalently that "*the asset and the market are correlated but the asset is less volatile*"

- This is not only wrong but dangerous, for it could lead to bad investment decisions (and surely have done so in the past)

- The conclusion that can be drawn for a positive beta but less than one is that there is some correlation between the return of the asset and the return of the market, but the relative volatility can be any value in the interval $(0, 1/\rho)$,which leaves open the possibility for the volatility of the asset being greater than the volatility of the market

On the meaning of beta

- The widespread wrong use and abuse of beta have been reported decades ago (see, References in the textbook “Computational Finance: An introductory course with R”), but beta is still today interpreted in a manner inconsistent with its mathematical formula

- There are several academic proposals for adjusting beta to fit with the desire use as a measure of market risk

- One such alternative estimator is precisely to consider a beta as the relative volatility with the sign for correlation; that is, $\beta = (\text{sign } \rho)\sigma_i/\sigma_M$. This β has the computational advantage of being easier to calculate, as it is the quotient of two standard deviations, and it does measure the risk of a stock against a portfolio, or the market

Estimating beta from sample

- Given n pairs of sample returns of stock i, R_i, and the market, R_M, over the same time period, the beta of stock i with respect to the market can be estimated by using the unbiased estimators for the covariance and the variance statistics:

$$\widehat{\beta}_i = \frac{\sum_{t=1}^{n} (R_{i,t} - \widehat{\mu}(R_i))(R_{M,t} - \widehat{\mu}(R_M))}{\sum_{t=1}^{n} (R_{M,t} - \widehat{\mu}(R_M))^2}$$

- For this estimator we have assumed that the risk-free rate r_f remains constant through the time period considered. If we have a variable risk-free rate, then instead of the returns $R_{i,t}$ and $R_{M,t}$, we should consider the excess returns $R_{i,t} - r_f$ and $R_{M,t} - r_f$

Optimization of Portfolios Under Different Constraint Sets

- Let us go back to our general mean-variance portfolio model…

- We have N assets with expected return vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)$, where $\mu_i = E(R_i)$ is the expected return of asset i, and covariance matrix $\mathbf{C} = [\sigma_{ij}]_{1 \geq i,j \geq N}$, where $\sigma_{ij} = Cov(R_i, R_j)$

- We choose a level of risk $\gamma \subseteq [0, 1]$, and the problem is to find a vector of weights $\boldsymbol{w} = (w_1, \ldots, w_N)$ that maximizes

$$\gamma w'\mu - (1-\gamma)w'\mathbf{C}w$$

$$\text{subject to:} \quad \sum_{i=1}^{N} w_i = 1$$

Optimization of Portfolios Under Different Constraint Sets

- The constraint is related to the budget and is a necessary restriction to norm the solution to the mean-variance problem

- There are several other constraints that can be added to the model to adapt it to more realistic scenarios

- In any case, the addition of these constraints turns the optimization problem computationally harder to resolve, and for that we would have to recur to optimization heuristics

Optimization of Portfolios Under Different Constraint Sets

- ***Upper and lower bounds in holdings***. These constraints limit the proportions of each asset that can be held in the portfolio, and model situations where investors require to have only long positions ($w_i \geq 0$), or specific proportions of certain assets
- ***Turnover limits in purchase or sale***. These constraints impose upper bounds on the variation of the holdings from one period to the next (e.g. to limit proportions added or subtracted when rebalancing, or to model periodic administrative commissions and taxes)
- ***Trading limits***. These constraints impose lower bounds on the variation of the holdings from one period to the next (e.g., if modifications to the portfolio should be enforced to trade a minimum volume for each position)
- ***Size of portfolio***. This refers to imposing a limit in the number of assets that can be included in the portfolio