Portfolio Selection

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Portfolio Selection

- When a portfolio deviates from the investor's expected return, its positions must be revised by either increasing or decreasing (possibly down to zero) their weights in the portfolio, in order to align the portfolio back to the investor's financial objectives

- The problem we are posed with is how to do this reallocation of wealth effectively, increasing future gains, through a finite sequence of trading periods, that conforms an investor's schedule for revising the portfolio positions

- To model solutions to this problem, we focus on a simple situation of an investor who is long in a number *m* of securities and with a determined investment horizon. We assume that there are no transaction costs, and that the only limitation for an investor to trade any amount of each security at any time is the amount of money he possesses at the time

Portfolio Selection

- The time between the present and the ending date of the life of the portfolio is divided into n trading periods, limited by time instants $t = 0, \ldots, n$, with $t = 0$ representing the initial time and $t = n$ the final time

- At the end of each period $[t-1, t]$, $t = 1, \ldots, n$, the proportion of wealth invested in each position is revised, being the overall goal to maximize the total wealth at the final date

- The change in the market price for a given period is represented by the *market vector* $\mathbf{x}_t = (x_{t1}, \ldots, x_{tm})$, a sequence of price relatives (i.e., simple gross returns) for the given trading period of the m securities:
  - each $x_{ti} = P_i(t)/P_i(t-1)$ is the ratio of closing to opening price (price relative) of the i-th security for the period $[t-1, t]$

Portfolio Selection

- The distribution of wealth among the *m* securities for the given period is represented by the portfolio vector $\boldsymbol{w}_t = (w_{t1}, \ldots, w_{tm})$ with non-negative entries summing to one; that is $w \in \mathcal{W}$, where $\mathcal{W} = \left\{w \in \mathbb{R}_+^m, \sum_{j=1}^{m} w_j = 1\right\}$

- Thus, an investment according to portfolio $\boldsymbol{w}_t$ produces an increase of wealth, with respect to market vector $\boldsymbol{x}_t$ for the period $[t-1, t]$, by a factor of

$$\boldsymbol{w}_t' \cdot \boldsymbol{x}_t = \sum_{j=1}^{m} w_{tj} x_{tj}.$$

Portfolio Selection

- A sequence of n investments according to a selection of n portfolio vectors $\boldsymbol{w}^n=(\boldsymbol{w}_1, \ldots,\boldsymbol{w}_n)$ results in a wealth increase of

$$S_n(w^n, x^n) = \prod_{t=1}^{n} w_t' \cdot x_t = \prod_{t=1}^{n} \sum_{j=1}^{m} w_{tj} x_{tj}$$

- where $\boldsymbol{x}^n = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is the sequence of price relative vectors corresponding to the n trading periods considered

- The quantity $S_n(\boldsymbol{w}^n, \boldsymbol{x}^n)$ is the *wealth factor* achieved by $\boldsymbol{w}^n$ with respect to the sequence of market vectors $\boldsymbol{x}^n$

An example

- At all times of investment, a *j*-th stock (1 ≥ *j* ≥ *m*) is represented by the portfolio vector $e_j = (0, \dots, 1, \dots 0) \in \mathcal{W}$ (where 1 is in the *j*-th coordinate)

- It has a wealth factor for *n* trading periods equal to the *n*-period simple gross return:

$$S_n(e_j, x^n) = \prod_{t=1}^{n} x_{tj} = \prod_{t=1}^{n} \frac{P_j(t)}{P_j(t-1)} = R_j(n) + 1.$$

Portfolio Selection Strategy

- **Definition** A portfolio selection strategy for n trading periods is a sequence of n choices of portfolio vectors $\boldsymbol{w}^n = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$, where each $w_t \in \mathcal{W}$. A portfolio selection algorithm is an algorithm that produces a portfolio selection strategy

- If *ALG* is a portfolio selection algorithm, by identifying it with its output (a selection strategy $\boldsymbol{w}^n$), we can also use $S_n(ALG, \boldsymbol{x}^n)$ to denote the wealth factor of *ALG* with respect to a sequence of market vectors $\boldsymbol{x}^n$

Evaluating Performances of Portfolio Selection Strategies

- To evaluate the performance of a portfolio selection strategy (or algorithm), independently of any statistical property of returns, the common procedure is to compare its wealth factor against the wealth factor achieved by the best strategy in a class of reference investment strategies (a benchmark strategy)

- An alternative is to compare their *exponential growth rate*, which for a selection algorithm *ALG* is defined as

$$W_n(ALG, x^n) = \frac{1}{n} \ln S_n(ALG, x^n)$$

- We can rewrite the wealth factor $S_n(ALG, x^n)$ in terms of the exponential growth rate as

$$S_n(ALG, x^n) = \exp(nW_n(ALG, x^n)).$$

Buy and Hold example

- This is the simplest strategy where an investor initially allocates all his wealth among *m* securities according to portfolio vector $w_1 \in \mathcal{W}$, and does not trade anymore

- The wealth factor of this strategy after *n* trading periods is

$$S_n(w_1, x^n) = \sum_{j=1}^{m} w_{1j} \prod_{t=1}^{n} x_{tj}$$

Constant Rebalanced Portfolios

- A constant rebalanced portfolio (CRP) is a market timing strategy that uses the same distribution of weights throughout all trading periods

- Let CRP_w be the CRP strategy with fixed weights $\boldsymbol{w} = (w_1, \ldots, w_m)$

- The wealth factor achieved by applying this strategy for n trading periods is

$$S_n(\mathrm{CRP}_{\boldsymbol{w}}, \boldsymbol{x}^n) = \prod_{t=1}^{n} \sum_{j=1}^{m} w_j x_{tj}.$$

Constant Rebalanced Portfolios

- For a sequence of market vectors $\boldsymbol{x}^n$, the best constant rebalanced portfolio is given by the solution $\boldsymbol{w^*}$ of the optimization problem:

$$\max_{\boldsymbol{w}\in\mathcal{W}} S_n(\mathrm{CRP}_{\boldsymbol{w}}, x^n)$$

- that is, $\boldsymbol{w^*}$ maximizes the wealth $S_n(\mathrm{CRP}_w, \boldsymbol{x}^n)$ over all portfolios $\boldsymbol{w}$ of m fixed real positive values applied to the sequence of n market vectors $\boldsymbol{x}^n$

Constant Rebalanced Portfolios

- The optimal strategy CRP_{w*} outperforms the following common portfolio strategies:
  - Buy and Hold
  - Arithmetic mean of stocks
  - Geometric mean of stocks

- The best constant rebalanced portfolio strategy CRP_{w*} is an extraordinary profitable strategy by the above properties; however, it is unrealistic in practice because it can only be computed with complete knowledge of future market performance

Volatility

Volatility

- One of the most important features of financial assets, and possibly the most relevant for professional investors, is the asset volatility
- Volatility refers to a degree of fluctuation of the asset returns. However, it is not something that can be directly observed

Volatility

- One can observe the return of a stock every day, by comparing the change of price from the previous to the current day, but one can not observe how the return fluctuates in a specific day

- We need to make further observations of returns (and of the price) at different times on the same day to make an estimate of the way returns vary daily (so that we can talk about *daily volatility*), but these might not be sufficient to know precisely how returns will fluctuate.

Volatility

- Therefore, volatility can not be observed but estimated from some model of the asset returns

- A general perspective, useful as a framework for volatility models, is to consider volatility as the conditional standard deviation of the asset returns.