The MATH130 Student's Guide to Chen and Duong's MATH130 notes.

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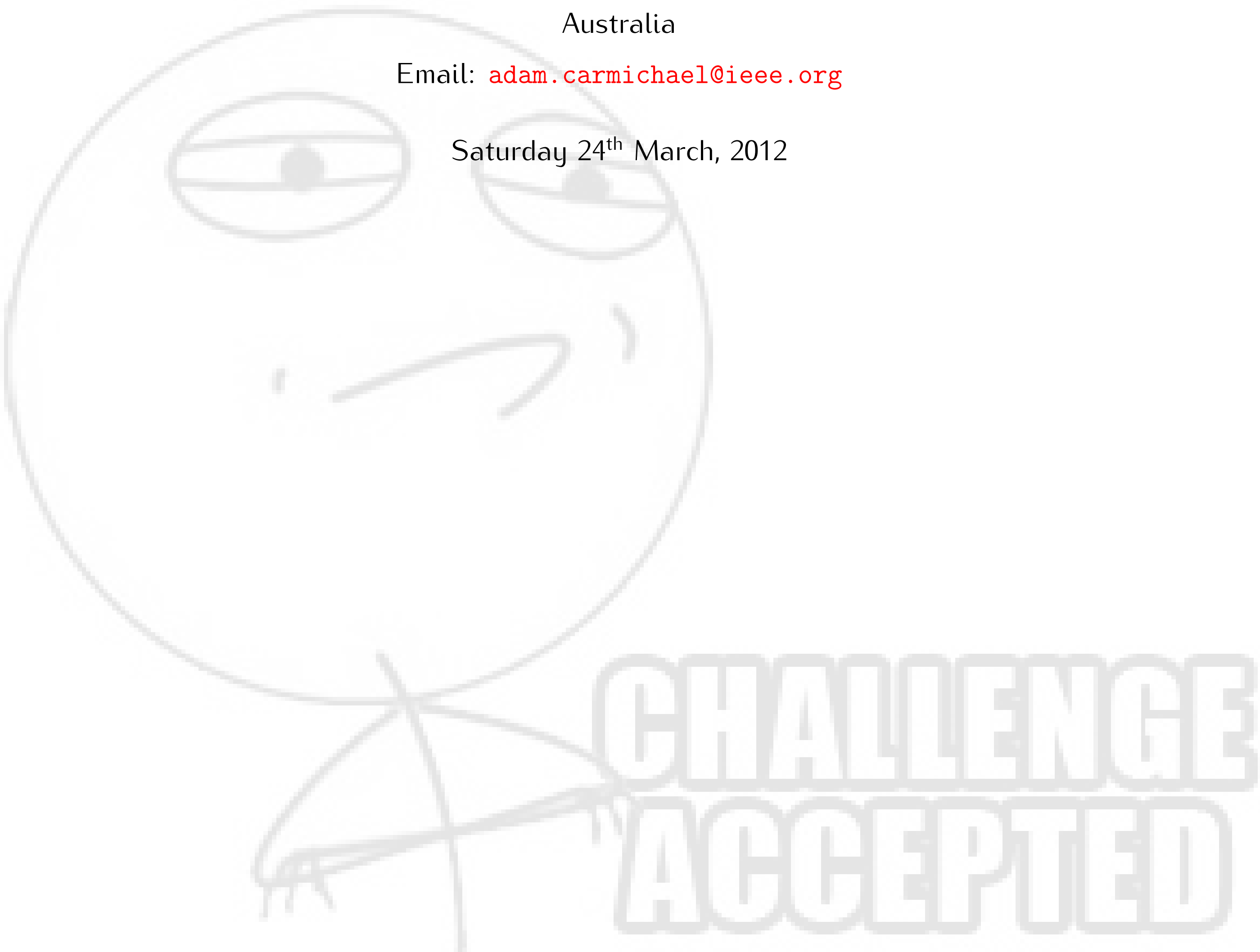

Chapter 1

Copyright and Terms of Use, and Other Bits

1.1 Copyright

I chose a NoDerivs suffix as it is trivial to take this work, and push "compile" button in any LaTeX editor and generate a textbook to take to a publisher. It would be unfair (on myself, as well as everyone who helped) to allow that to happen. Should you wish to generate a textbook, please contact the author to discuss textbook licensing.

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1.2 Terms and conditions of use

This work is an open source project. It strives to be technically correct and accurate, though, it may not be.

1.3 A cautionary word

I've failed MATH130 at least twice. I've been pissed with the Maths department. I've written most of this book whilst ropable at them for not doing it themselves. This may not be the best resource, but they haven't really done a good enough job of organizing their resources, so resources are limited. Please attend lectures, tutorials and practicals. Go ask your teachers, tutors, professors, lecturers, and friends questions and be active in classes. It is said the only way to learn math is to do math. If you say "screw math", it *will* screw you[1].

1.4 Typesetting

This book has been typeset in LaTeX using a Lenovo X201 Tablet running Microsoft Windows 7, Eclipse Indigo, TeXlipse and MikTex 2.9 64-bit beta. I've been drinking copious amounts of energy drinks and coffee; if you would like to become a sponsor and have your name mentioned here, let me know.

The source is presently available from https://github.com/carneeki/Grokking-MATH130. You should be able to find updates of the PDF at http://goo.gl/X14C4. The PDF may not always match the latest source, but you can always compiled the PDF yourself.

Additionally, most URLs in the PDF should be clickable. URLs in footnotes may not be; this is something that needs to be debugged.

1.5 Introduction

In 1999 a pair of elite mathematicians[2] decided to write some notes for a subject they knew plenty about. The notes proved difficult to understand by some and could have been made easier by means of an introduction in plain simple English. Today they survive as reference documents on Rutherglen. If you can find them, and you read them, and you have this guide,

[1] And it might screw you anyway...

[2] Chen and Duong

then maybe you can pass MATH130.
(To be read whilst playing the introduction to the A-Team).

Typically the syllabus is broken into two streams, calculus and algebra. This gives rise to certain problems if algebra falls behind calculus because there are prerequisites in algebra to solving some calculus problems. As such, these notes will be arranged such that the algebra material is covered first.

Contents

Chapter 2

Types of Numbers & Symbols Used

Maths is pretty much a written language used to convey what people want to do to numbers, variables, or other bits of information. There are various types of numbers, and sometimes special symbols are used to denote what conditions must be placed on those numbers.

2.1 Types of Numbers

Table 16.1.2 outlines the types of numbers encountered in MATH130, followed by a few additional types of numbers that are handy to bear in mind.

2.2 Types of Symbols

Table 2.2 outlines the types of symbols you are likely to encounter in MATH130. This list is partially built from Wikipedia [3].

Symbol	Name	In MATH130	Description & Example
$\mathbb{N}$	Natural Number	Yes	Any whole number greater than zero. $1, 2, 3$
$\mathbb{R}$	Real Numbers	Yes	Any number along a continuum. $-1, 0, 1, \pi$. Note: $\mathbb{R} = \mathbb{Q} + \mathbb{I}$
$\mathbb{Z}$	Integers	Yes	Any whole number. $-1, 0, 1$
$\mathbb{I}$	Irrational Numbers	Yes	Numbers which cannot be expressed as a fraction. $e, \pi, \sqrt[2]{2}$. Most real numbers are irrational.
$\mathbb{Q}$	Rational Numbers	Yes	Numbers which can be represented as a fraction. $\frac{1}{2}, 1, \frac{0}{4}$
$\mathbb{C}$	Complex Numbers	Yes [1]	Numbers which have both a real part, and an imaginary part. $\frac{2}{-1} = i$
$\mathbb{P}$	Prime Numbers	No [2]	Numbers which are divisible only by themselves and one. $1, 2, 3, 5, 7, 11, 13$

Table 2.1: Types Of Numbers

2.3 A Brief Interlude on Language

An expression and an equation are *different* things, despite the fact that they look similar. An expression might be something like $3x - 8$, however, x has no value. By the fact that $3x - 8$ has no value *assigned* to it, it is an expression. If we were to assign a value to it, $3x - 8 = 0$ then we can call it an equation.

$$3x - 8 \qquad \text{an expression} \tag{2.1}$$

$$3x - 8 = 0 \qquad \text{an equation} \tag{2.2}$$

(2.3)

Symbol	Example	Read as
$=$	$x = y$	x is equal to y
$\neq$	$x \neq y$	x is not equal to y
$\approx$	$x \approx y$	x is approximately equal to y
$\equiv$	$x \equiv y$	x is equivalent to y
$<$	$x < y$	x is less than y
$>$	$x > y$	x is greater than y
$\leq$	$x \leq y$	x is equal to or less than y
$\geq$	$x \geq y$	x is equal to or greater than y
!	$n!$	n factorial. $(n * (n-1) * \ldots * 3 * 2 * 1)$
$f(x)$	$f(x) = mx + b$	Function of x is equal to $mx + b$
$\sum$	$\sum_{i=1}^{10} t_i$	Sum of terms t for values 1 to 10
$\int$	$\int_0^\infty e^{-x}\ \mathrm{d}x$	Integrate e^{-x} from 0 to ∞ with respect to x
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{\mathrm{d}y}{\mathrm{d}x}$	Differentiate y with respect to x
$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$	$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$	Differentiate $f(x)$ with respect to x
$\frac{\mathrm{d}}{\mathrm{d}x}(f(x))$	$\frac{\mathrm{d}}{\mathrm{d}x}(f(x))$	
$y'\ \mathrm{d}x$	$y'\ \mathrm{d}x$	Differentiate y with respect to x. This is called "Newton's notation". If no variable is listed, it safe to assume it is the variable supplied to the function.
$f'(x)$	$f'(x)$	
$f''(x)$	$f''(x)$	Differentiate $f(x)$ twice.
$\in$	$x \in \mathbb{R}$	x is an element of $\mathbb{R}$
$\notin$	$x \notin \mathbb{R}$	x is not an element of $\mathbb{R}$
$\cup$	$\mathbb{Z} \cup \mathbb{N} \in \mathbb{R}$	The union of $\mathbb{Z}$ and $\mathbb{N}$ are in set $\mathbb{R}$.
$\cap$	$\mathbb{Z} \cap \mathbb{N} \in \mathbb{N}$	Common elements of $\mathbb{Z}$ and $\mathbb{N}$ are in set $\mathbb{N}$.

Table 2.2: Types of mathematical symbols

Chapter 3

Number Systems & Factorization

The following are the laws of *Associativity*:

$$a + (b + c) = (a + b) + c \tag{3.1}$$

$$a * (b * c) = (a * b) * c \tag{3.2}$$

The following are laws of *Commutativity*:

$$a + b = b + a \tag{3.3}$$

$$a * b = b * a \tag{3.4}$$

By combining these rules we get the *Distributite Laws*:

$$a * (b + c) = a * b + a * c \tag{3.5}$$

$$(b + c) * a = (b * a) + (c * a) \tag{3.6}$$

$$= (a * b) + (a * c) \tag{3.7}$$

These laws are not compatible with subtraction or division.

These give rise to special cases called a quadratic which will be introduced in section 3.4, Introduction to Polynomials, and in further detail in chapter 7, Polynomials. A brief summary is as follows:

$$(a+b)^2 = (a+b)*(a+b) \tag{3.8}$$
$$= (a+b)*a + (a+b)*b \tag{3.9}$$
$$= a^2 + ba + ab + b^2 \tag{3.10}$$
$$= a^2 + 2ab + b^2 \tag{3.11}$$

In turn:

$$(a-b)^2 = a^2 - 2ab - +b^2 \tag{3.12}$$

And

$$a^2 - b^2 = (a-b)(a+b) \tag{3.13}$$

$$a^3 - b^3 = (a-b)(a^2 + 2ab + b^2) \tag{3.14}$$

There are some basic laws that need to be understood to manipulate numbers The first group of laws are called the "Distributive laws".

The distributive laws are all about expanding brackets, that is to say, in 3.6, first we multiply a with the first term inside the brackets (b) to give us ab, then we multply a with the second term, c to give us ac. When we add them (the + symbol in the brackets) we get $ab + ac$. Another way to think about it is a is distributed to each term inside the brackets. This also applies in 3.7 with c, and it yields the same result as in 3.6.

An integer is *even* iff [1] its square is *even*. An integer is *odd* iff its square is *odd*.

Why? If x is even, then $x = 2k$ for some other integer k then $x^2 = 4k^2$ so $\frac{x^2}{2} = 2k$ and so $x^2 is even$.

Similarly, if x is odd, then $x - 1 = 2l$ for some integer l.

Then $x = 2l + 1$, hence $x^2 = (2l+1)^2 = 4l^2 + 4l + 1$.

So $x^2 - 1 = 4l^2 + 4l$. Given $4l^2 + 4l$ is even, so $x^2 = 1+$ even so x^2 is odd.

This proves that some numbers in reality are not fractions, such as *surds* (irrational numbers) eg $\sqrt{2}$. An alternate way to prove this, is to assume $\sqrt{2} \in \mathbb{Q}$ and derive a contradiction.

Fractions ($\mathbb{Q}$) take the form $\frac{a}{b}$ where a and b are integers. We can always assume that a

[1] and only if = iff

and b are not even (because they are divisible by 2 and cancel out).

$$\text{Then}\sqrt{2} * b = a\text{hence} \tag{3.15}$$
$$2 * b^2 = a^2\text{even} \tag{3.16}$$
$$a^2 = 2k\text{for another integer}k \tag{3.17}$$
$$\text{so}a^2 = 4k^2 \tag{3.18}$$
$$4k^2 = 2b^2 \tag{3.19}$$
$$2k^2 = b^2 \tag{3.20}$$
$$\text{so } b^2\text{even} \tag{3.21}$$
$$\text{so } b\text{even} \tag{3.22}$$

3.1 Introduction to Fractions

It turns out that any repeating decimal can be written as a fraction:

$$2.1 \times 100000 = \tag{3.23}$$
$$= 2.1 \times \frac{10}{10} \tag{3.24}$$
$$= \frac{21}{10} \tag{3.25}$$

What about $1.33333\ldots$?

$$1.33333 = \frac{4}{3} \tag{3.26}$$

Or $1.373737\ldots$??

$$\text{Let } x = 1.373737\ldots \tag{3.27}$$
$$100x = 137.3737\ldots \tag{3.28}$$
$$100x - x = 137.373737\ldots - 1.373737\ldots \tag{3.29}$$
$$99x = 136 \tag{3.30}$$
$$so$$
$$x = \frac{136}{99} \tag{3.31}$$
$$\therefore 1.373737\ldots = x = \frac{136}{99} \tag{3.32}$$

For longer decimal places we need to create 2 numbers using x with the same decimal part so that when we subtract the decimal part we get a whole number.

$$\text{Let } x = 36.2593593593\ldots \tag{3.33}$$

$$so10x = 362.593593\ldots \tag{3.34}$$

$$10000x = 362593.593\ldots \tag{3.35}$$

$$10000x - 10x = 362593.593\ldots - 362.593\ldots \tag{3.36}$$

$$9990x = \ldots \tag{3.37}$$

(we have now subtracted the recurring decimal component from the fraction)

$$\therefore x = \frac{362593 - 362}{9990} \tag{3.38}$$

$$\tag{3.39}$$

As it turns out, fractions will fall into one of three categories:

(a) recurring decimals such as $\frac{99}{101} = 0.9801\ldots$.

(b) non-recurring decimals such as $\frac{1}{2} = 0.5$, these first two are called $\mathbb{Q}$ or rational numbers.

(c) non-recurring decimals such as $\pi = 3.141592653\ldots$ or $\sqrt{2} = 1.41421356\ldots$, these are denoted by the symbol $\mathbb{I}$, and are called irrational numbers. [2]

3.1.1 Fractional Operations - Adding

A good reason why we *don't* add fractions in the following way:

> add the tops (to give the numerator), add the bottoms (to give the denominator (or quotient). example: $\frac{1}{2} + \frac{1}{2} \neq \frac{2}{4} = \frac{1}{2}$
>
> The *correct* way of adding fractions is to use a common quotient then add the numerator and keep the denominator the same.[3]

[2] These numbers require some complex calculus to prove they are non-recurring, a much simpler number is $0.1234567890111121314\ldots$ which has a pattern, that is not recurring, it is to simply "add one" to the previous number.

[3] a simpler (but only partial) answer could be "you're adding like with like" – a MATH130 student from the audience of Chris Gordon's lecture at 2011-08-08 10:31AM

Consider you order 3 slices of pizza from Hot Momma's pizza at the MQ bar, and you are given 2 more for being a regular customer:

$$\frac{3}{8} = \ldots \tag{3.40}$$

$$\frac{3}{8} + \frac{2}{8} = \frac{5}{8} \tag{3.41}$$

$$\tag{3.42}$$

You have $\frac{5}{8}$ or "five eights" of a whole pizza.[4].

What if you have different quotients? You *need* to convert to a common quotient:

$$\frac{1}{2} + \frac{7}{10} = \tag{3.43}$$

$$= \frac{5}{5} \times \frac{1}{2} + \frac{7}{10} \tag{3.44}$$

$$= \frac{5}{10} \times \frac{7}{10} \tag{3.45}$$

$$= \frac{12}{10} \tag{3.46}$$

$$= \frac{2 \times 6}{2 \times 5} \tag{3.47}$$

the two's divide out, which simplifies the fraction

$$= \frac{6}{5} \tag{3.48}$$

$$\tag{3.49}$$

Equation 3.45 is where the important heavy lifting of the operation of converting to a common quotient comes into play.

A general case:

$$\frac{a}{b} + \frac{c}{d} = \tag{3.50}$$

$$= [\frac{a}{b}\ times \frac{d}{d}] + [\frac{c}{d}\ times \frac{b}{b}] \qquad = \frac{ad + cd}{db} \tag{3.51}$$

$$\tag{3.52}$$

3.1.2 Fractional Operations - Division

Dividing fractions has a reasonably simply rule to remember: Multiply by the inverse of one fraction:

[4] I want a slice of that pizza if it's the supreme

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{bd}{bd} \tag{3.53}$$

$$= \frac{\frac{a}{b} \times b \times d}{\frac{c}{d}\ timesb \times d} \tag{3.54}$$

divide out common terms

$$= \frac{ad}{cb} \tag{3.55}$$

3.2 Introduction to Irrational Numbers (Surds)

A *surd* is *an archaic*[5] *term for an irrational number.* This is basically a number which cannot be written as a decimal because, if you tried

(a) you'd go on forever as it has an infinite number of decimal places and

(b) it has no repeating parts to the decimal places.

For this reason it cannot be expressed as a fraction in the form $\frac{p}{q}$ where p and q are integers. Examples of irrational numbers are $e, \pi, \sqrt[2]{2}$
A more formal[6] definition is:

$$\mathbb{I} \ni \left.\frac{p}{q}\right\} \qquad \text{p,q} \in \mathbb{Z}$$

There are some handy things we can do with irrational numbers. Consider $\sqrt[2]{8} = 2.828427124$[7]. It can be rewritten like this:

$$\sqrt[2]{8} = \sqrt[2]{2 \times 4} \tag{3.56}$$

$$= \sqrt[2]{2} \times \sqrt[2]{4} \tag{3.57}$$

$$= \sqrt[2]{2} \times 2 \tag{3.58}$$

$$= 2\sqrt[2]{2} \tag{3.59}$$

[5] almost as old as the Maths Deptartment

[6] poorly worded, but means the same thing as above

[7] It's even longer than 2.8284271247461900976033774484193961571393437507538961463533594759814649..., it's infinite remember!

3.2.1 Rationalizing the Denominator

Often examiners will give us a fraction and say "rationalize the denominator"[8]. I don't know why - they just do. In order to get the marks in the exam, we can rationalize the denominator by multiplying that fraction by 1.

While the notion of multiplying by 1 sounds silly, consider that $1 \in \mathbb{R} = \frac{p}{q}$ where {p,q} can be a surd, x: $\frac{x}{x} = 1$. This gives rise to the following possibility of:

$$\frac{5}{\sqrt[2]{2}} = \tag{3.60}$$

$$= \frac{5}{\sqrt[2]{2}} \times \frac{\sqrt[2]{2}}{\sqrt[2]{2}} \tag{3.61}$$

The next part is where the useful stuff happens, if we square a square-root then they "undo" each other, and we are left over with the bit inside the square-root

$$= \frac{5 \times \sqrt[2]{2}}{(\sqrt[2]{2}) \times (\sqrt[2]{2})} \tag{3.62}$$

$$= \frac{5\sqrt[2]{2}}{2} \tag{3.63}$$

The denominator might not always be a square-root, University of North Texas' next example includes a cube-root, so we must multiply by 1 again.

$$\frac{2}{\sqrt[3]{5}} = \tag{3.64}$$

sometimes it's nicer to lay things out to see what's going on:

$$= \frac{2}{\sqrt[3]{5}} \times \frac{\sqrt[3]{5}}{\sqrt[3]{5}} \times \frac{\sqrt[3]{5}}{\sqrt[3]{5}} \tag{3.65}$$

but we still condense it into the root symbol:

$$= \frac{2}{\sqrt[3]{5}} \times \frac{\sqrt[3]{5 \times 5}}{\sqrt[3]{5 \times 5}} \tag{3.66}$$

$$= \frac{2\sqrt[3]{5 \times 5}}{(\sqrt[3]{5})(\sqrt[3]{5 \times 5})} \tag{3.67}$$

$$= \frac{2\sqrt[3]{25}}{5} \tag{3.68}$$

$$\tag{3.69}$$

8 "sudo rationalize the denominator" if you want to be a troll

The last example involves more than one term on the denominator. In this particular case, we will still multiply by 1, however we are using some trickery from an upcoming section 3.4, "Introduction to Polynomials" specifically equation 3.84.

$$\frac{2}{1+\sqrt[2]{3}} = \tag{3.70}$$

$$= \frac{2}{1+\sqrt[2]{3}} \times \frac{1-\sqrt[2]{3}}{1-\sqrt[2]{3}} \tag{3.71}$$

$$= \frac{2(1-\sqrt[2]{3})}{1-3} \tag{3.72}$$

if the denominator part of above step does not make sense, then please refer to equation 3.84 in section 3.4, "Introduction to Polynomials"

$$= \frac{2(1-\sqrt[2]{3})}{-2} \tag{3.73}$$

$$= -\frac{2(1-\sqrt[2]{3})}{2} \tag{3.74}$$

$$= -\frac{1(1-\sqrt[2]{3})}{1} \tag{3.75}$$

$$= -(1-\sqrt[2]{3}) \tag{3.76}$$

The examples for rationalizing the denominator come from University of Northern Texas: http://www.math.unt.edu/mathlab/emathlab/How%20to%20Rationalize%20the%20Denominator%20of%20a%20Fraction.htm

3.3 Factorization

Factorization and simplification go hand in hand. Factorization is the decomposition of expressions into the product of simpler terms called *factors*.
Example of simplification:

$$2x^2 + 3x + 1 = (2x+1)(x+1) \tag{3.77}$$

$$\because = 2x^2 + x + 2x + 1 \tag{3.78}$$

Example of factorization:

$$a^4 - b^4 = (a^2)^2 - (b^2)^2 \text{by power laws} \tag{3.79}$$

$$= (a^2 - b^2)(a^2 + b^2) \tag{3.80}$$

$$= (a+b)(a+b)(a^2 + b^2) \tag{3.81}$$

3.4 Introduction to Polynomials

This section forms only an introduction to polynomials. More detail on polynomials is covered in chapter 7, "Polynomials".

Polynomials are a way of packing certain types of long equations into neater, more compact forms. The following equations show how the distributive laws can be applied to 3 polynomial equations. These 3 equations form the basic 3 rules of polynomials and their form should be memorised to make solving more complex problems easier down the track.

$$\begin{aligned}
(a+b)^2 &= (a+b)(a+b) \\
&= a^2 + 2ab + b^2 \qquad (3.82) \\
(a-b)^2 &= (a-b)(a-b) \\
&= a^2 - 2ab + b^2 \qquad (3.83) \\
(a+b)(a-b) &= a^3 + ab - ab - b^2 \\
&= a^2 - b^2 \qquad (3.84) \\
(a+b)(a^2 - ab + b^2) &= a^3 + b^3 \qquad (3.85) \\
(a-b)(a^2 + ab + b^2) &= a^3 - b^3 \qquad (3.86)
\end{aligned}$$

Equation 3.84 is often called the difference of two squares where a^2 and b^2 represent both squares.

Equation 3.86 is often called the difference of two cubes.

3.5 Quadratics

Quadratics are an important type of the distributive law. They represent 3 coefficients and a variable. Equations 3.82 through to 3.84 are the classic 3 ways in which quadratics are introduced in textbooks. A more formal definition has been provided by the table 3.87 (from [5]).

$$ax^2 + bx + c = 0 \tag{3.87}$$

These particular quadratics are often called squares which are covered in more

Where:	
x	is the indeterminate variable
a	is the quadratic coefficient
b	is the linear coefficient
c	is the constant coefficient

Table 3.1: Components to a quadratic

detail in section 3.6, Completing the Square.

3.6 Completing the Square

Completing the square is useful for solving quadratic equations as well as graphing quadratic functions, as well as evaluating integrals in calculus. The key concept behind completing the square in MATH130 is that we want to convert a quadratic polynomial like:

$$ax^2 + bx + c$$

into the form

$$a(x-h)^2 + k$$

To do this we must find h and k. There are two ways about doing this depending on whether the value of $a = 1$.

3.6.1 General Case, when a = 1

If given an expression like:

$$x^2 + bx + c \tag{3.88}$$

we can form a square like this:

$$(x + \frac{1}{2}b)^2 = x^2 + bx + \frac{1}{4}b^2 \tag{3.89}$$

however, we have not taken into account the constant c so, we should really write the following to take it into account

$$x^2 + bx + c = (x + \frac{1}{2}b)^2 + k \tag{3.90}$$

$$ax^2 + bx + c = 0 \text{ (iff } a \neq 0) \tag{3.91}$$
$$= a(x^2 + \frac{bx}{a} + \frac{c}{a}) \tag{3.92}$$
$$= x^2 + \frac{bx}{a} + \frac{c}{a} \tag{3.93}$$
$$-\frac{c}{a} = x^2 + \frac{2b}{2a}x \tag{3.94}$$
$$x^2 + \frac{2b}{2a} + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2 \tag{3.95}$$
$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a} * \frac{4a}{4a} \tag{3.96}$$
$$= \frac{b^2 - 4ac}{4a^2} \text{ iff } b^2 \geq 4ac \tag{3.97}$$
$$\text{then} \tag{3.98}$$
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{3.99}$$
$$\text{case 1: } x \geq 0 \tag{3.100}$$
$$x = -\frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{3.101}$$
$$= -\frac{b}{2a} + \frac{sqrtb^2 - 4ac}{\sqrt{4a^2}} \tag{3.102}$$
$$= \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ (if } a \geq 0) \tag{3.103}$$
$$= \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ (if } a < 0) \tag{3.104}$$
$$\text{case 2: } x < 0 \tag{3.105}$$
$$x = -\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}} \tag{3.106}$$
$$= -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \text{ (if } a \geq 0) \tag{3.107}$$
$$\text{or } = \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ (if } a < 0) \tag{3.108}$$

The following examples come courtesy of Wikipedia [4] (with intermediary working provided by author):

$$x^2 + 6x + 11 = \tag{3.109}$$

First: halve b to give us 3, and then put it into the complete square form (remembering the k-value):

$$= (x+3)^2 + k \tag{3.110}$$

Next calculate k

$$(x+3)^2 = x^2 + 6x + 9 \tag{3.111}$$
$$11 - 9 = 2 \tag{3.112}$$
$$\therefore k = 2 \tag{3.113}$$

Substitute $k = 2$ back into equation

$$x^2 + 6x + 11 = (x+3)^2 + 2 \tag{3.114}$$

Another example:

$$x^2 + 14x + 30 = \tag{3.115}$$

First: halve b to give us 3, and then put it into the complete square form (remembering the k-value):

$$= (x+7)^2 + k \tag{3.116}$$

Next calculate k

$$30 - 7^2 = -30 - 49 = -19 \tag{3.117}$$

Substitute $k = -19$ back into equation

$$x^2 + 14x + 30 = (x+7)^2 - 19 \tag{3.118}$$

Another example:

$$x^2 - 2x + 7 = \tag{3.119}$$

First: halve b to give us 3, and then put it into the complete square form (remembering the k-value):

$$= (x-1)^2 + k \tag{3.120}$$

calculate k

$$7 - (-1^2) = 7 - 1 = 6 \tag{3.121}$$

Substitute $k = 6$ back into equation

$$x^2 - 2x + 7 = (x - 1)^2 + 6 \tag{3.122}$$

From these 3 examples, the pattern should become evident as a 3 stage process:

(a) Halve b and put into the complete square form

(b) Calculate $c - h^2 = k$

(c) Substitute k back into equation and rewrite in full.

3.6.2 Non-monic Case, when a != 1

If given an equation like

$$3x^2 + 12x + 27 = \tag{3.123}$$

we can factor out the coefficient a and the complete the square as in a general case

$$\begin{aligned} 3x^2 + 12x + 27 &= 3(x^2 + 4x + 9) && (3.124)\\ &= 3((x+2)^2 + 5) && (3.125)\\ &= 3((x+2)^2) + 15 && (3.126) \end{aligned}$$

this gives rise to the form:

$$a(x-h)^2 + k \tag{3.127}$$

Another example:

$$\begin{aligned} 2x^2 + 7x + 6 &= && (3.128)\\ \text{Let: } a &= 2, b = 7, c = 6 && (3.129)\\ \text{Substitute quadratic formula:} & && (3.130)\\ b^2 - 4ac &= 49 - 48 && (3.131)\\ x_1 &= -\frac{7}{4} + \frac{\sqrt{1}}{4} = -\frac{6}{4} = -\frac{3}{2} && (3.132)\\ x_2 &= -\frac{7}{4} + \frac{\sqrt{1}}{4} = -\frac{8}{4} = -2 && (3.133)\\ 2x^2 + 7x + 6 &= 2(x - (-\frac{3}{2}))(x - (-2)) && (3.134) \end{aligned}$$

Example:

$$x^3 + 2x - 5x - 6 \tag{3.135}$$

$$= (x - R)(\text{some quadratic}) \tag{3.136}$$

by trial and error we can work out if

$$x^3 + 2x - 5x = 6 \tag{3.137}$$

$$x = 0 = 0 + 0 - 0 - 6 \neq 0 \tag{3.138}$$

$$x = 1 = 1 + 2 - 5 - 6 = -8 \neq 0 \tag{3.139}$$

$$x = -1 = -1 + 2 + 5 - 6 = 0 \tag{3.140}$$

$$\text{so } (x - 1) \text{ is a root}$$

$$x^3 + 2x - 5x - 6 = \tag{3.141}$$

$$= (x - (-1))(\text{some quadratic}) \tag{3.142}$$

$$= (x + 1)(x^2 + x - 6) \tag{3.143}$$

$$= (x + 1)(x - 2)(x + 3) \tag{3.144}$$

3.6.3 Completing The Square Formulae

When $a = 1$

$$x^2 + bx + c = (x - \frac{-b}{2})^2 + k \tag{3.145}$$

$$= (x + \frac{b}{2})^2 + (c - \frac{b^2}{4}) \tag{3.146}$$

When $\neq 1$

$$ax^2 + bx + c = a(x - \frac{-b}{2})^2 + k \tag{3.147}$$

$$= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a}) \tag{3.148}$$

3.7 Simplification

Suppose we wish to simplify

$$\frac{x+1}{x^2-x} - \frac{x-1}{x^2+x} = \tag{3.149}$$

1. we want to change to the same denominator

$$= \frac{x+1}{x(x-1)} - \frac{x-1}{x(x+1)} \tag{3.150}$$

halfway there; multiple both terms by the denominator of each other

$$= \frac{x+1}{x(x-1)}\frac{x+1}{x+1} - \frac{x-1}{x(x+1)}\frac{x-1}{x-1} \tag{3.151}$$

$$= \frac{(x+1)^2-(x-1)^2}{x(x-1)(x+1)} \tag{3.152}$$

we can go a bit further

$$= \frac{(x^2+2x+1)-(x^2-2x+1)}{x(x-1)(x+1)} \tag{3.153}$$

$$= \frac{x^2+2x+1-x^2+2x-1}{x(x-1)(x+1)} \tag{3.154}$$

$$= \frac{2x+2x}{x(x-1)(x+1)} \tag{3.155}$$

$$= \frac{4}{x^2-1} \tag{3.156}$$

Note: The expression is only defined when $x \neq 0$ and $x \neq \pm 1$

Simplify:

$$(\frac{4}{x} - \frac{3}{y}) \div (\frac{5}{x} + \frac{6}{y}) = \tag{3.157}$$

$$= \frac{\frac{4}{x} - \frac{3}{y}}{\frac{5}{x} + \frac{6}{y}} \tag{3.158}$$

$$\frac{4y}{xy} - \frac{3x}{yx} = \frac{4y-3x}{xy} \tag{3.159}$$

$$\frac{5y}{xy} + \frac{6c}{yx} = \frac{5y+6x}{xy} \tag{3.160}$$

$$(\frac{4}{x} - \frac{3}{y}) \div (\frac{5}{x} + \frac{6}{y}) = \frac{\frac{4y-3x}{xy}}{\frac{5y+6x}{xy}} \tag{3.161}$$

$$= \left(\frac{4y-3x}{xy}\right) \cdot \left(\frac{xy}{5y+6x}\right) \tag{3.162}$$

$$= \frac{4y-3x}{5y+6x} \tag{3.163}$$

Simplify:

$$\frac{x^6+y^6}{x^2+y^2} = \tag{3.164}$$

$$\text{notice that } x^6+y^6 = (x^2)^3+(y^2)^3 \tag{3.165}$$

$$\frac{x^6+y^6}{x^2+y^2} = \frac{(x^2+y^2)(x^4-x^2y^2+y^4)}{x^2+y^2} \tag{3.166}$$

$$= x^4-x^2y^2+y^4 \tag{3.167}$$

This is defined only when $x^2+y^2 \neq 0$, which ocurrs when both $x=y=0$, which would be bad[9].

Find all solutions of

$$x^4-3x^2+2=0 \tag{3.168}$$

$$\text{Let } t=x^2 \tag{3.169}$$

$$t^2-3t+2=0 \tag{3.170}$$

$$(t-2)(t-1)=0 \tag{3.171}$$

$$\text{4 cases}$$

$$x^2=1 \to x=\pm 1 \tag{3.172}$$

$$x^2=2 \to x=\pm 1 \tag{3.173}$$

$$x^4-3x^2+2=(x^2-2)(x^2-1) \tag{3.174}$$

$$=(x-\sqrt(2))(x+\sqrt(2))(x-1)(x+1) \tag{3.175}$$

[9]"Try to imagine every molecule inside your body exploding at the speed of light." – Egon Spengler, Ghostbusters

Chapter 4

Exponentials & Logarithms

Exponents, powers or indices and logarithms are ways of expressing numbers that have been multiplied or divided a number of times.

If we were to plot an exponential equation on a graph, we would notice that the graph has a constant doubling time, ie for every unit we double on the y axis, the number of units on the horizontal axis are constant.

If the graph increases at an increasing rate, it does not necessarily mean it the graph is exponential.

4.1 Powers, Exponentials and Indices

Indices are putting a power or index in the top right corner of a number, which indicates how many times we must multiply or divide that number by itself to give us a total number. With this information in mind, we need to define some parts of the language behind how logs and indices should be used.

$$b^x = y \tag{4.1}$$

$$= b * b * \ldots * b \text{ (x times)} \tag{4.2}$$

When we want to perform manipulations to several numbers, there are various power

The convention I will use here is:	
b	is the base or *radix*
x	is the index
y	is the output

Table 4.1: Parts of an exponential function

laws we must bear in mind... They are summarised as follows:

$$b^0 = 1 \tag{4.3}$$

$$b^1 = b \tag{4.4}$$

$$b^x * b^y = b^{x+y} \tag{4.5}$$

$$b^x - b^y = b^x * \frac{1}{b^y} \tag{4.6}$$

$$= b^{x-y} \tag{4.7}$$

$$b^{-(x+y)} = (b^{-1})^x * (a^{-1})^y \tag{4.8}$$

$$= (b^{-x}) * (b^{-y}) \tag{4.9}$$

$$(b^x)^d = b^{xd} \tag{4.10}$$

$$bx^{-y} = \frac{b}{x^y} \tag{4.11}$$

similarly

$$bx^{-1} = \frac{1}{b} \tag{4.12}$$

$$b^{(\frac{x}{y})} = (b^x)^{\frac{1}{y}} \tag{4.13}$$

$$= \sqrt[y]{b^x} \tag{4.14}$$

$$db^{(-\frac{x}{y})} = \frac{d}{\sqrt[y]{b^x}} \text{ combining (4.11) and (4.13)} \tag{4.15}$$

$$\text{Also note that } b^x = 0 \text{ is impossible} \tag{4.16}$$

Converting notation between the forms exhibited in 4.13 and 4.14 is often extremely useful in Calculus, in particular, differentiation.

4.2 Logarithms

Until this point, only exponentials where we want to do something to a base number have been covered. When we want to see how a base number has been affected by it's power a logarithm is the way to undo the power.

Think of a log as an undoing function to an exponential.[1]

4.2.1 Log Laws

Many of these laws can be derived from the index laws, and have been included in a way to preserve order with those laws for easier reference.

$$\log_b(1) = 0 \quad \text{by (4.3)} \tag{4.17}$$

$$\log_b(b) = 1 \quad \text{by (4.4)} \tag{4.18}$$

$$\log_b(xy) = \log_b(x) + \log_b(y) \quad \text{by (4.5)} \tag{4.19}$$

$$\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y) \quad \text{by (4.7)} \tag{4.20}$$

$$\log_b(x^d) = d * \log_b(x) \quad \text{by (4.10)} \tag{4.21}$$

$$\log_b(\sqrt[y]{x}) = \frac{\log_b(x)}{y} \quad \text{by (4.13)} \tag{4.22}$$

$$\log_b(0) = \text{undefined} \quad \text{by (4.16)} \tag{4.23}$$

Here's where some new stuff is introduced:

$$b^{\log_b(x)} = x \quad \text{Logs of the same base cancel as an index} \tag{4.24}$$

$$\log_b(b^x) = x \quad \text{Logs of the same base cancel as an index} \tag{4.25}$$

$$\log_b(a) = \frac{1}{\log_a(b)} \quad \text{Remember back to the initial statement}^2 \tag{4.26}$$

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)} \quad \text{Changing log bases} \tag{4.27}$$

$$(\log_a(b))(\log_b(x)) = \log_a(x) \tag{4.28}$$

Referring back to 4.1, but replace the variable b with a gives rise to the easy to remember translation between the logarithms and exponentials: "logs are gay" [3]

$$\log_a(y) = x \iff a^x = y \tag{4.29}$$

[1]Chris Gordon, MATH130 lecturer for algebra stream, evening session on logarithms, 2011.

[3]Elizabeth Camilleri, advanced mathematics student, on my whiteboard at home. Though, this really needs her picture (included) to accompany the text for full effect.

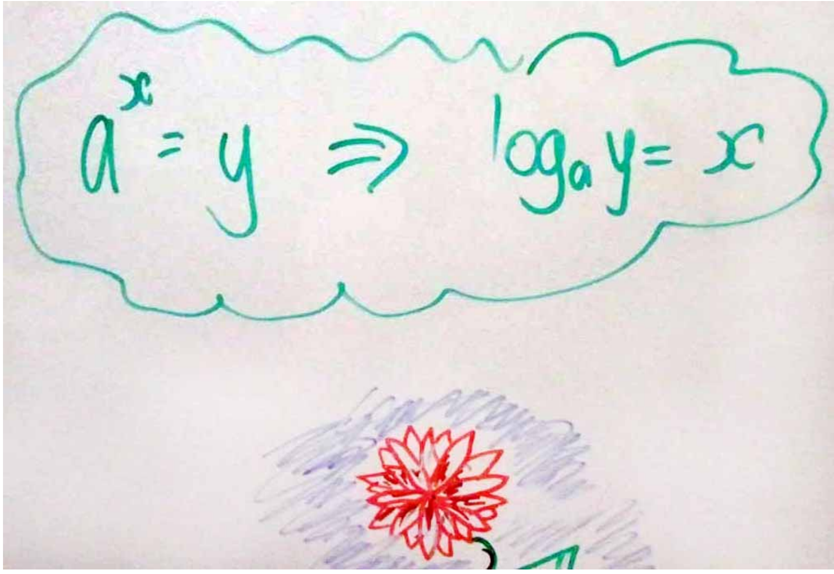

Figure 4.1: LogsAreGay

4.2.2 Converting Logarithm Bases

Suppose there are two logarithms, of different bases. It is often nice to use logarithms of the same base as it keeps the maths simpler (and sometimes things will divide or cancel out). To do this, we apply the change of base formula (4.27). Consider the following:

$$b^{log_b(x)} = x \tag{4.30}$$

$$log_a(b^{b*log_b(x)}) = log_a(x) \text{ take log base-a of both sides} \tag{4.31}$$

$$(log_a(b))(log_b(x)) = \text{by (4.10)} \tag{4.32}$$

There is an alternate way using division, and can be memorised as wrote:

$$log_b(x) = \frac{log_a(x)}{log_a(b)} \quad (4.27)$$

4.2.3 Problem solving with logs

The golden rule to remember when dealing with indices and powers is the Svensson-Cranbrookian Log Method, to be sung to "When you're happy and you know it":

When you're having trouble with a power take a log.

Suppose we have to solve the following equation:

$$2^x = \frac{5}{3^{x+1}} \tag{4.33}$$

We use logarithms:

$$\begin{aligned}
2^x &= \frac{5}{3^{x+1}} &(4.34)\\
log(2^x) &= log(\frac{5}{3^{x+1}}) &(4.35)\\
xlog(2) &= log(5) - log(3^{x+1}) &(4.36)\\
xlog(2) &= log(5) - (x+1) * log(3) &(4.37)\\
xlog(2) + xlog(3) &= log - log(3) &(4.38)\\
x(log(2) + xlog(3)) &= log(5) - log(3) &(4.39)\\
x &= \frac{log(5) - log(3)}{log(2) + log(3)} &(4.40)\\
&= \frac{log(\frac{5}{3})}{log(6)} &(4.41)\\
&= \ldots \text{some decimal} \ldots &(4.42)
\end{aligned}$$

4.3 Euler Constant: Base *e*

Most calculations involving logarithms will be to various bases depending on the topic. In COMP it is often base 2 (binary), in other fields it is often base 10. In MATH and most of the sciences it is often base *e*, which is approximately 2.71828.[1] Numerically it is an interesting number as it crops up all over the place, sometimes in the most bizarre of locations; however we do not need to worry about that for MATH130.

The base e is used on most calculators with a button that looks like e^x. To use the logarithm with base e it is usually a button that looks like "ln", or $log_e(x)$.

An important thing to note about e, which will be raised further in calculus chapters is when you differentiate e, it is the only function where it's integral and derivative are both the same.

Chapter 5

Trigonometry

Trigonometry is about ratios of angles and sides. Specifically, trigonometry is about the ratios of angles and sides in triangles with respect to each other when drawn inside a special circle called *the unit circle*.

A brief interlude on language. Most of our trigonometric measurements are *NOT* going to be measured in degrees. Most of our measurements of angles are measured in radians. A *radian* is the ratio of angles with respect pi.
To convert between radians and degrees use the following formulae:

$$\theta^\circ = r \times \frac{180^\circ}{\pi} \tag{5.1}$$

$$r = \theta^\circ \times \frac{\pi}{180^\circ} \tag{5.2}$$

A more formal definition of the radian: *The radian measure of an angle θ is the arclength subtended by the angle in a unit circle.* This definition requires we define what a unit circle is. [1]

A full circle:

$$r = \theta^\circ \times \frac{\pi}{180^\circ} \tag{5.3}$$

$$r = 360^\circ \times \frac{\pi}{180^\circ} \tag{5.4}$$

$$r = 2 \tag{5.5}$$

[1] This works fine for *anti-clockwise* angles, however, a radian can never be negative because a length (the arclength in this case) can never be less than zero.

5.1 The Unit Circle

The unit circle is such an important part of trigonometry that it is worth looking it up on Wikipedia *AND* as many other places as possible.

A semi-formal definition of the unit circle:
$\cos\theta$ is the x-value of the intersection of the unit circle with the angle θ.
$\sin\theta$ is the y-value of the intersection of the unit circle with the angle θ[2].

UTUCDTCTV:[3]

$$\cos(\frac{-2\pi}{3}) = \ldots \tag{5.6}$$

π is $\frac{1}{2}$ circle.
$\frac{\pi}{3}$ is $\frac{1}{3}$ of $\frac{1}{2}$ a circle.
At this point, there has been no trigonometry...
The rest of this is high school geometry to determine angles involving triangles (such as Pythagoras).

5.1.1 Properties of The Unit Circle

$$-1 \leq \sin\theta \leq \quad 1 \tag{5.7}$$
$$\tag{5.8}$$

$\sin\theta$ is the y-value on the unit circle which is bound by -1 and 1.

$$\sin -\theta = -\sin\theta \tag{5.9}$$
$$\tag{5.10}$$

[2]These words are not quite complete, and would require the unit circle picture
[3]Use the Unit Circle Definition to Calculate The Value

5.1.2 SOH CAH TOA

$$\sin\theta = \frac{o}{h} \tag{5.11}$$

$$\cos\theta = \frac{a}{h} \tag{5.12}$$

$$\tan\theta = \frac{o}{a} \tag{5.13}$$

5.1.3 Angle Sum Formula

There are 3 angle sum formulae to learn, and while the 3rd can be deduced from the first two, it is easier to just memorise it as the derivation is beyond the scope of MATH130.

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{5.14}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\sin(\beta) \mp \cos(\beta)\sin(\alpha) \tag{5.15}$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \tag{5.16}$$

$$\tag{5.17}$$

Chapter 6

Trig Identities

Inverse, quotient, angle sum, and the first primitive identity are the trig identities are required to memorize. Everything else can be derived.[1]

$$\cos\theta = \frac{y}{1} \qquad = x \tag{6.1}$$
$$\sin\theta = \frac{x}{1} \qquad = y \tag{6.2}$$
$$\tan\theta = \frac{y}{x} \tag{6.3}$$

6.1 Reciprocal Identities

It should be worth noting the temptation to call these identities the "inverse identities", however, this is technically not true. Inverse identities are not a part of MATH130 in any of the semesters I have done MATH130, however they will be included for completeness at the end of this chapter in section 6.9. These are the

[1]However, if you can memorize all of these identities, it may save crucial seconds in test and exam situations.

reciprocal identities because we take the *reciprocal* of an identity.

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x} \tag{6.4}$$
$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y} \tag{6.5}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{x}{y} \tag{6.6}$$
$$\tag{6.7}$$

6.2 Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{6.8}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{6.9}$$

6.3 Primitive Identities

Only the first primitive identity needs to eb memorized, remaining primitives can be derived from the first.

$$\sin^2\theta + \cos^2\theta = 1 \tag{6.10}$$

Divide (6.10) by $\sin^2$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \tag{6.11}$$
$$1 + \cot^2\theta = \csc^2\theta \tag{6.12}$$

Divide (6.10) by $\cos^2$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \tag{6.13}$$
$$\tan^2\theta + 1 = \sec^2\theta \tag{6.14}$$

6.4 Angle Sum Identities

Angle sum identities need to be memorized.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{6.15}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{6.16}$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \tag{6.17}$$

Angle sum identities form the basis for the double angle identities. There should be no need to memorize the double angle identities as they can be derived from the angle sum identities as $A = B = 2A$.

6.5 Double Angle Identity: sin(2A)

$$\sin(A + A) = \sin(A)\cos(A) + \sin(A)\cos(A) \tag{6.18}$$

$$= 2\sin(A)\cos(A) \tag{6.19}$$

6.6 Double Angle Identity: cos(2A)

$\cos(2A)$ has 3 solutions we need to concern ourselves with in MATH130. The 2nd and 3rd solutions combine rearrangements of the primitive identity (6.10).

$$\cos(A+A) = \cos(A)\cos(A) - \sin(A)\sin(A) \tag{6.20}$$
$$= \cos^2(A) - \sin^2(A) \tag{6.21}$$

Rearrange (6.10) to make $\sin^2(A)$ the subject

$$\sin^2(A) + \cos^2(A) = 1$$
$$\sin^2(A) = 1 - \cos^2(A) \tag{6.22}$$

sub into (6.21)

$$= \cos^2(A) - \sin^2(A)$$
$$= \cos^2(A) - (1 - \cos^2(A)) \tag{6.23}$$
$$= 2\cos^2(A) - 1 \tag{6.24}$$

Rearrange (6.10) to make $\cos^2(A)$ the subject

$$\sin^2(A) + \cos^2(A) = 1$$
$$\cos^2(A) = 1 - \sin^2(A) \tag{6.25}$$

sub into (6.21)

$$= \cos^2(A) - \sin^2(A)$$
$$= (1 - \sin^2(A)) - \sin^2(A) \tag{6.26}$$
$$= 1 - 2\sin^2(A) \tag{6.27}$$

6.7 Double Angle Identity: tan(2A)

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} \tag{6.28}$$
$$= \frac{2\tan A}{1 - \tan^2 A} \tag{6.29}$$

6.8 Trigonometric Calculus Identities

Think of differentiation of trig functions as a loop with 4 steps, and each time you differentiate, you have a different value which is fed into the next step of differentiation. When you reach the last point, you are back where you started.

$$\frac{\mathrm{d}\sin x}{\mathrm{d}x} = \cos x \tag{6.30}$$
$$\frac{\mathrm{d}\cos x}{\mathrm{d}x} = -\sin x \tag{6.31}$$
$$\frac{\mathrm{d}(-\sin x)}{\mathrm{d}x} = -\cos x \tag{6.32}$$
$$\frac{\mathrm{d}(-\cos x)}{\mathrm{d}x} = \sin x \tag{6.33}$$

There are 4 quadrants to the unit circle, and differentiation is the process where by we find the gradient of a function at a point θ. Keep referring back to figure 6.1 for this next part:

(a) Consider the gradient of $\sin(0)$. $\sin(x)$ is at $(0, 0)$ and it has a gradient that is at it's steepest: 1. Note how $\cos(0) = 1$.

(b) Consider the gradient of $\sin(\frac{\pi}{2})$. $sin(x)$ is at a maximum at point $(\frac{\pi}{2}, 1)$, and the gradient is 0. Note how $\cos(x)$ is plotted and when evaluated $\cos(\frac{\pi}{2}) = 0$.

(c) Consider the gradient of $\sin(\pi)$. $sin(x)$ is zero, and is at it's steepest "downward slope". Note how $\cos(x)$ is plotted and evaluated $\cos(\pi) = -1$.

(d) Consider the gradient of $\sin(\frac{3\pi}{2})$. $\sin(x)$ is at a minimum at point $(\frac{3\pi}{2}, -1)$, and $\cos(x)$ is plotted and evaluated as $\cos(\frac{3\pi}{2}) = 0$.

The next step would be to analyse the gradient of $\cos(x)$. One would find that at any arbitrary point, the gradient of $\cos(x)$ will equal the value of $\sin(x) \cdot -1$.

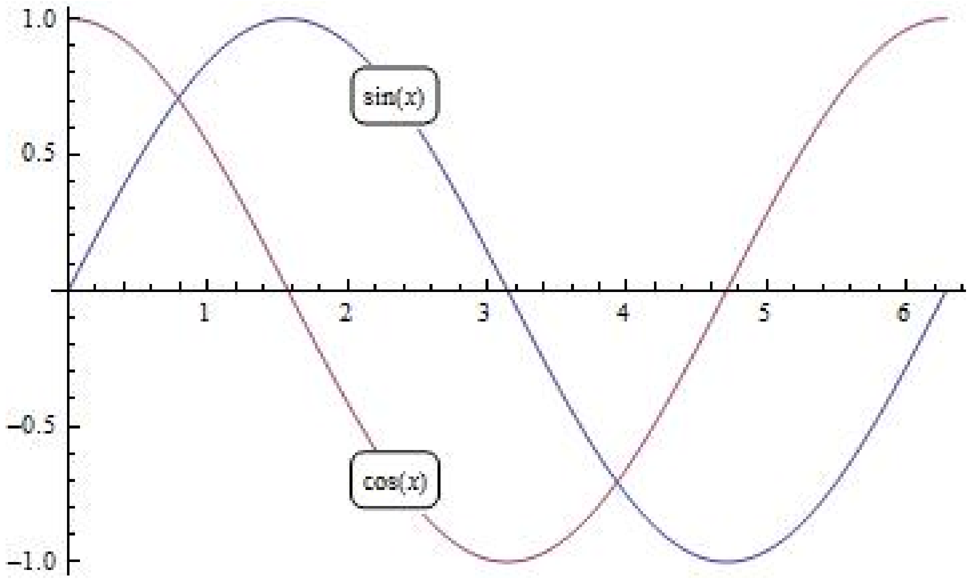

Figure 6.1: $\sin(x)$ and $\cos(x)$

6.9 Inverse Identities

An inverse trig function can be though of as an "undoing function" for a trig function.[2] All of the inverse trig functions are called "arc<function>". The inverse functions hold true iff $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\sin(\arcsin\theta) = \theta \quad \Longrightarrow \quad \arcsin(\sin\theta) = \theta \tag{6.34}$$

$$\cos(\arccos\theta) = \theta \quad \Longrightarrow \quad \arccos(\cos\theta) = \theta \tag{6.35}$$

$$\tan(\arctan\theta) = \theta \quad \Longrightarrow \quad \arctan(\tan\theta) = \theta \tag{6.36}$$

$$\sec(\operatorname{arcsec}\theta) = \theta \quad \Longrightarrow \quad \operatorname{arcsec}(\sec\theta) = \theta \tag{6.37}$$

$$\csc(\operatorname{arccsc}\theta) = \theta \quad \Longrightarrow \quad \operatorname{arccsc}(\csc\theta) = \theta \tag{6.38}$$

$$\cot(\operatorname{arccot}\theta) = \theta \quad \Longrightarrow \quad \operatorname{arccot}(\cot\theta) = \theta \tag{6.39}$$

[2] This is much like how a logarithm is the undoing function for an exponent.

Chapter 7

TODO: Polynomials

Definition: Sum of natural powers of x, scaled and added.

Example:

$$6x^4 + 20x^3 + \sqrt{2}x^2 + 13x - 2 \tag{7.1}$$

$$\tag{7.2}$$

Two important terms are *coefficient*; which is a number in front of the x, and *degree* which is the highest power of x in the expression.

Polynomials are useful and common for approximating other functions. Here we approximate sin(x):

$$x - \frac{x^3}{6} \tag{7.3}$$

$$x - \frac{x^3}{6} + \frac{x^5}{120} \tag{7.4}$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{7.5}$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \tag{7.6}$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \tag{7.7}$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^1 1}{11!} \tag{7.8}$$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^1 1}{11!} \tag{7.9}$$

$$\tag{7.10}$$

The main polynomials studied in MATH130 are:

quadratics

$$y = ax^2 + bx + c \qquad \text{General Form} \tag{7.11}$$

$$y = a(x-h)^2 + k \qquad \text{Standard Form} \tag{7.12}$$

$$y = a(x-x_1)(x-x_2) \qquad \text{Factored Form} \tag{7.13}$$

cubics

$$y = ax^3 + bx^2 + cx + d \tag{7.14}$$

7.1 Quadratics

7.1.1 General Form

$$y = ax^2 + bx + c$$

This form is perhaps the most common form of an equation - it is completely unfactored and appears the most natural (at least to me).
Key advantages of general form:

- $c = y$-intercept and is easily read straight off the equation.
- a tells us a lot about the shape of the parabola:
  - A "happy" parabola has a positive value
  - A "sad" parabola has a negative value
  - Larger values of a give steeper parabolas

7.1.2 Standard Form

$$y = a(x-h)^2 + k$$

Key advantages of standard form (sometimes called the *vertex form*):

- a tells us the same information about the shape of the parabolas as in the general form.

- h tells us the x coordinate of the minimum/maximum point, however, to get the coordinate, we must multiply h by -1.
- k tells us the y coordinate of the minimum/maximum point.
- Useful for finding the roots / zeros (where $y = 0$) of a quadratic

Consider solving:

$$8(x-7)^2 - 41 = 0 \tag{7.15}$$

Because there is only one x value it makes it easier to solve:

$$8(x-7)^2 = 41 \tag{7.16}$$

$$(x-7)^2 = \frac{41}{8} \tag{7.17}$$

$$x - 7 = \pm\sqrt{\frac{41}{8}} \tag{7.18}$$

$$x = \pm\sqrt{\frac{41}{8}} + 7 \tag{7.19}$$

It is also useful for sketching. Consider:

$$y = 2(x-1)^2 + 3 \tag{7.20}$$

(a) Start with $y = x^2$

(b) Because there is an $(x-1)$ we shift the parabola $y = x^2$ by 1 to the right.
$y = (x-1)^2$

(c) Because there is a 2 in front of the $(x-1)$, we make the parabola steeper by a factor of two.[1]
$y = 2(x-1)^2$

(d) Because there is a $+3$ on the end, it lifts the parabola up by 3.
$y = 2(x-1)^2 + 3$

7.1.3 Factored Form

$$y = a(x - x_1)(x - x_2)$$

Key advantages of standard form:

[1] called “stretching vertically”

- a tells us the same information about the shape of the parabolas as in the general form.

- x_1 and x_2 give us the x-intercepts (when multiplied by -1) of the equation.

Factored form is useful for finding the equation given a parabola. Suppose we know the x-intercepts of a parabola are -3 and -1, and it has a y-intercept of 6:

$$\begin{aligned}
f(x) &= a(x - x_1)(x - x_2) && (7.21)\\
&= a(x - -3)(x - -1) && (7.22)\\
&= a(x + 3)(x + 1) && (7.23)\\
6 &= a(0 + 3)(0 + 1) \text{ substitute } x = 0 \text{ to get } y\text{-int} && (7.24)\\
6 &= a(3) && (7.25)\\
2 &= a && (7.26)\\
\therefore f(x) &= 2(x + 3)(x + 1) && (7.27)\\
&= 2x^2 + 8x + 6 && (7.28)
\end{aligned}$$

7.1.4 Quadratic Formula

To get the roots of a quadratic (to factorize it), there is a long[2] formula we can use called the *quadratic formula* to get values of x. If given the general form of the quadratic, we can substitute the values of a, b, and c into this formula to get values for x which can then be used in the factored form.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7.29}$$

[2] horrible

By applying the quadratic formula to our previous example from the factored form, we can show all 3 forms of the quadratic:

$$= 2x^2 + 8x + 6$$

$$\text{where } a = 2, b = 8, c = 6$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7.30}$$

$$= \frac{-8 \pm \sqrt{8^2 - 4(2)(6}}{2(2)} \tag{7.31}$$

$$= \frac{-8 \pm \sqrt{56 - 24}}{4} \tag{7.32}$$

$$= \frac{-8 \pm \sqrt{32}}{4} \tag{7.33}$$

$$= \frac{-8 \pm 4\sqrt{2}}{4} \tag{7.34}$$

$$= \pm 8\sqrt{2} \tag{7.35}$$

$$\text{substitute } x \text{ into vertex form} \tag{7.36}$$

$$f(x) = a()^2 + k \tag{7.37}$$

7.2 Cubics

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Chapter 8

TODO: Absolute Values

Chapter 9

TODO:Inequalities

When we multiply or divide by a negative, change the sign of the inequality. Eg

$$-3x > 9 \tag{9.1}$$

$\div$both sides by -3

$$x > -3 \text{is NOT correct}$$

$$x < -3 \text{is correct} \tag{9.2}$$

Consider

$$\frac{3}{x-1} > 4 \tag{9.3}$$

$$3 > 4(x-1) \tag{9.4}$$

$$3 > 4x - 4 \tag{9.5}$$

$$3 + 4 > 4x \tag{9.6}$$

$$7 > 4x \tag{9.7}$$

$$\frac{7}{4} > x \tag{9.8}$$

$$\therefore x < \frac{7}{4} \tag{9.9}$$

However, this presents a problem; if we substitute $x = 1$ in, we have a “disaster”, with a quotient of 0.

$$\frac{3}{1-1} > 4 \text{ cannot possibly be a solution} \tag{9.10}$$

Try $x = 0$

$$\frac{3}{0-1} > 4 \tag{9.11}$$
$$\frac{3}{-1} > 4 \tag{9.12}$$
$$-3 > 4 \text{ which we know to be false} \tag{9.13}$$

Let's ask ourselves "Why didn't this work?". What went wrong was that we don't know whether $x - 1$ is positive or negative. There are several resolutions to the problem.

(a) Take cases: consider

  (i) $x - 1 > 0$

  (ii) $x - 1 < 0$

  And prove whether they are logically true.

(b) Multiply by a non-negative. In this case $(x-1)^2$ because a square can never be a negative number.

(c) Rearrange to a form such as $\frac{A}{B} < 0$ and consider signs.[1]

(d) Plot the inequality: $y = \frac{3}{x-1}$ and see where the y value is bigger than 4.

[1] it could also be $\frac{A}{B} > 0$

First way with cases:

$$\frac{3}{x-1} > 4 \tag{9.14}$$

$$\text{Assume } x-1 > 0 \tag{9.15}$$

$$3 > 4(x-1) \tag{9.16}$$

$$> 4x-4 \tag{9.17}$$

$$7 > 4x \tag{9.18}$$

$$\frac{7}{4} > x \tag{9.19}$$

$$x < \frac{7}{4} \tag{9.20}$$

We only keep x's where $x-1>0$, $x>1$

$$\text{Assume } x-1 < 0 \tag{9.21}$$

$$\frac{3}{x-1} < 4 \tag{9.22}$$

$$x > \frac{7}{4} \tag{9.23}$$

We only keep x's where $x-1<0$, $x<1$

So we combine our answer is when we take the answers that are logically acceptable.

Second way with squares:

$$\frac{3}{x-1} > 4 \tag{9.24}$$

$$(x-1)^2 \times \frac{3}{x-1} > 4(x-1)^2 \tag{9.25}$$

$$(x-1)3 > 4(x-1)^2 \tag{9.26}$$

$$0 > 4(x-1)^2 - 3(x-1) \tag{9.27}$$

$$> (x-1)(4(x-1)-3) \tag{9.28}$$

$$> (x-1)(4x-7) \tag{9.29}$$

$$\text{we can plot this} \tag{9.30}$$

$$\therefore 1 < x < \frac{7}{4} \tag{9.31}$$

$$\tag{9.32}$$

3rd Way: (was skipped for tutorial exercise)

Graphical Way: Sketch $y = \frac{3}{x-1}$
Vertical asymptote $x = 1$.

$$x = 1 + \text{a small number} \tag{9.33}$$
$$y = \text{BIG} \tag{9.34}$$

A second problem:

$$\frac{4x}{2x+3} > 2 \tag{9.35}$$
$$\tag{9.36}$$

By squares (quadratic) method:

$$\frac{4x}{2x+3} > 2 \tag{9.37}$$
$$(2x+3)^2 \times \frac{4x}{2x+3} > 2(2x+3)^2 \tag{9.38}$$
$$4x \times (2x+3) > 2(2x+3)^2 \tag{9.39}$$
$$0 > 2(2x+3)^2 - 4x(2x+3) \tag{9.40}$$
$$> (2x+3)(2(2x+3) - 4x) \tag{9.41}$$
$$> (2x+3)(4x+6-4x) \tag{9.42}$$
$$> (2x+3)(6) \tag{9.43}$$
$$> 12x + 18 \tag{9.44}$$
$$-12x > 18 \tag{9.45}$$
$$12x < -18 \tag{9.46}$$
$$x < -\frac{3}{2} \therefore x < -\frac{3}{2} \tag{9.47}$$

By signs:

$$\frac{4x}{2x+3} - 2 > 0 \tag{9.48}$$

$$\frac{4x}{2x+3} - \frac{2x+3}{2x+3} > 0 \tag{9.49}$$

$$\frac{-6}{2x+3} > 0 \tag{9.50}$$

$$-6\text{always -ve} \tag{9.51}$$

$$2x+3 < 0 \tag{9.52}$$

$$2x < -3 \tag{9.53}$$

$$x < \frac{-3}{2} \tag{9.54}$$

$$\tag{9.55}$$

Chapter 10

TODO: Simultaneous Equations

Chapter 11

TODO: Matrices

Addition

$$\mathbb{Z}_{M,N} = \mathbb{A} + \mathbb{B}$$
$$z_{i,j} = a_{i,j} + b_{i,j} \tag{11.1}$$

Subtraction

$$\mathbb{Z}_{M,N} = \mathbb{A} - \mathbb{B}$$
$$z_{i,j} = a_{i,j} - b_{i,j} \tag{11.2}$$

Multiplication

$$\mathbb{Z}_{M,N} = \mathbb{A} \times \mathbb{B}$$
$$z_{i,j} = \sum_{k=1}^{N} a_{i,k} \times b_{k,j} \tag{11.3}$$

11.1 Language of Matrices

A matrix is nothing more than a bunch of numbers arranged into a grid of M rows and N columns and is represented usually by $\mathbb{A}$. To represent a specific element inside the matrix $\mathbb{A}$, it is usually referred to as $a_{i,j}$ [1]

[1] I don't know why we change to lowercase, but it is the convention. Maybe it is to prevent confusion between referring to a whole matrix (of particular dimensions) and a specific element of a matrix.

An example of a “3 by 4” matrix called $\mathbb{A}$ is as follows:

$$\mathbb{A}_{2,3} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix} \tag{11.4}$$

Or more generally:

$$\mathbb{A}_{M,N} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,j} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & a_{i,3} & \dots & a_{i,j} \end{pmatrix} \tag{11.5}$$

In matrix algebra we go vertically describing the rows first then horizontally we describe the columns.[2]

There is also a special kind of matrix called the *identity matrix* often represented by $\mathbb{I}$. This is a special matrix, which must be square in shape (ie for $\mathbb{A}_{M,N}, M = N$), and consists of all zeroes except for a diagonal row of ones from the top left corner to the bottom right corner.

A 4x4 identity matrix:

$$\mathbb{I}_{4,4} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11.6}$$

An important property of the identity matrix is that when a matrix is multiplied by it’s identity matrix, the product is the original matrix.

$$\mathbb{A} \;\times\; \mathbb{I} \;=\; \mathbb{A} \tag{11.7}$$

Another impotant property is that a matrix can be multiplied by it’s identity matrix in either order (we will see later in section 11.4, Multiplying Matrices that in matrix algebra, order is important[3].

$$\mathbb{I} \;\times\; \mathbb{A} \;=\; \mathbb{A} \tag{11.8}$$

[2]This is in contrast to the cartesian plane e.g. map reading where we go horizontally then vertically.

[3]as opposed to normal algebra where you can multiply in any which way and it doesn’t matter

11.2 Adding Matrices

Two matrices, $\mathbb{A}$ and $\mathbb{B}$ can be summed if and only if the matrices have equal rows and columns. To add $\mathbb{A}$ and $\mathbb{B}$, sum the elements $a_{i,j}$ with $b_{i,j}$ such that:

$$\mathbb{Z} = \mathbb{A} + \mathbb{B} \tag{11.9}$$

where

$$z_{i,j} = a_{i,j} + b_{i,j} \tag{11.10}$$

An example:

$$\mathbb{A}_{2,2} = \begin{pmatrix} 1 & 5 \\ 3 & -4 \end{pmatrix} \tag{11.11}$$

$$\mathbb{B}_{2,2} = \begin{pmatrix} 4 & -8 \\ 2 & 10 \end{pmatrix} \tag{11.12}$$

$$\mathbb{Z} = \mathbb{A} + \mathbb{B} \tag{11.13}$$

$$= \begin{pmatrix} 1 & 5 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} 4 & -8 \\ 2 & 10 \end{pmatrix} \tag{11.14}$$

$$= \begin{pmatrix} (1+4) & (5+-8) \\ (3+2) & (-4+10) \end{pmatrix} \tag{11.15}$$

$$= \begin{pmatrix} 5 & -3 \\ 5 & 6 \end{pmatrix} \tag{11.16}$$

11.3 Subtracting Matrices

Two matrices, $\mathbb{A}$ and $\mathbb{B}$ can be subtracted if and only if the matrices have matching rows and columns. To subtract $\mathbb{B}$ from $\mathbb{A}$, subtract the elements $b_{i,j}$ from $a_{i,j}$ such that:

$$\mathbb{Z} = \mathbb{A} - \mathbb{B} \tag{11.17}$$

where

$$z_{i,j} = a_{i,j} - b_{i,j} \tag{11.18}$$

An example:

$$\mathbb{A}_{2,2} = \begin{pmatrix} 1 & 5 \\ 3 & -4 \end{pmatrix} \tag{11.19}$$

$$\mathbb{B}_{2,2} = \begin{pmatrix} 4 & -8 \\ 2 & 10 \end{pmatrix} \tag{11.20}$$

$$\mathbb{A} - \mathbb{B} = \begin{pmatrix} 1 & 5 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} 4 & -8 \\ 2 & 10 \end{pmatrix} \tag{11.21}$$

$$= \begin{pmatrix} (1-4) & (5--8) \\ (3-2) & (-4-10) \end{pmatrix} \tag{11.22}$$

$$= \begin{pmatrix} -3 & 13 \\ 1 & -14 \end{pmatrix} \tag{11.23}$$

11.4 Multiplying Matrices

Formal definitions for the multiplication of matrices like the one mentioned at the start of this chapter appear scary, but really, it is just *a lot*[4] of simple additions.

(a) The first rule for multiplying matrices is that the number of columns in $\mathbb{A}$ matches the number of rows in $\mathbb{B}$ (that is to say $\mathbb{A}_N = \mathbb{B}_M$).

(b) Remember that $\mathbb{Z}_{i,j}$ is a merely sum of several multiplication operations.

Formally, $z_{i,j} = \sum_{k=1}^{N} a_{i,k} \times b_{k,j}$

Broken down, this means for each element in a given row in $\mathbb{A}$, we must multiply that element by a matching element in it's corresponding column in $\mathbb{B}$ and sum all the results.

For the matrices $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{P}$:

$$\begin{pmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{pmatrix} \tag{11.24}$$

[4] Because of this, expect to use up *a lot more* paper.

where:

$$
\begin{aligned}
p_{1,1} &= a_{1,1} \times b_{1,1} + a_{1,2} \times b_{2,1} + a_{1,3} \times b_{3,1} \\
p_{2,1} &= a_{2,1} \times b_{1,1} + a_{2,2} \times b_{2,1} + a_{2,3} \times b_{3,1} \\
p_{1,2} &= a_{1,1} \times b_{1,2} + a_{1,2} \times b_{2,2} + a_{1,3} \times b_{3,2} \\
p_{2,2} &= a_{2,1} \times b_{1,2} + a_{2,2} \times b_{2,2} + a_{2,3} \times b_{3,2}
\end{aligned}
$$

11.5 Identity Matrix

For the identity matrix:

$$\mathbb{A} \times \mathbb{I} = \mathbb{A}$$

Proof by general case:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} [(1 \times a) + (0 \times c)] & [(1 \times b) + (0 \times c)] \\ [(0 \times a) + (1 \times c)] & [(0 \times b) + (1 \times d)] \end{pmatrix} \tag{11.25}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{11.26}$$

This is the matrix equivalent of the number 1[5]

$$1 \times a = a \tag{11.27}$$

$$\frac{1}{a} \times a = 1 \tag{11.28}$$

in matrix terms:

$$\mathbb{A}^{-1} \times \mathbb{A} = \mathbb{I} \tag{11.29}$$

This leads into the topic of calculating the inverse of a matrix.

[5]NOTE: this is only equivalent to one, it is not actually the value of one (to my knowledge).

11.6 Inverse of Matrices

For a 2x2 matrix:

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{11.30}$$

$$\mathbb{A}^{-1} = \frac{1}{ad - bc} \times \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{11.31}$$

For a 3x3 matrix: TODO THIS...

For a 4x4 matrix, in the words of Sal Kahn, "You'll be there all day, and for a 5x5 you're almost bound to make a mistake and is best left for a computer."[6]

11.7 TODO: Dividing Matrices

11.8 Summary of Matrix Operations

[6] "In my mind, the only thing less pleasant than inverting a 3 by 3 matrix is inverting a 4 by 4 matrix." – both quotes from http://www.khanacademy.org/video/inverting-matrices–part-2?playlist=Linear%20Algebra

Chapter 12

Parabolas

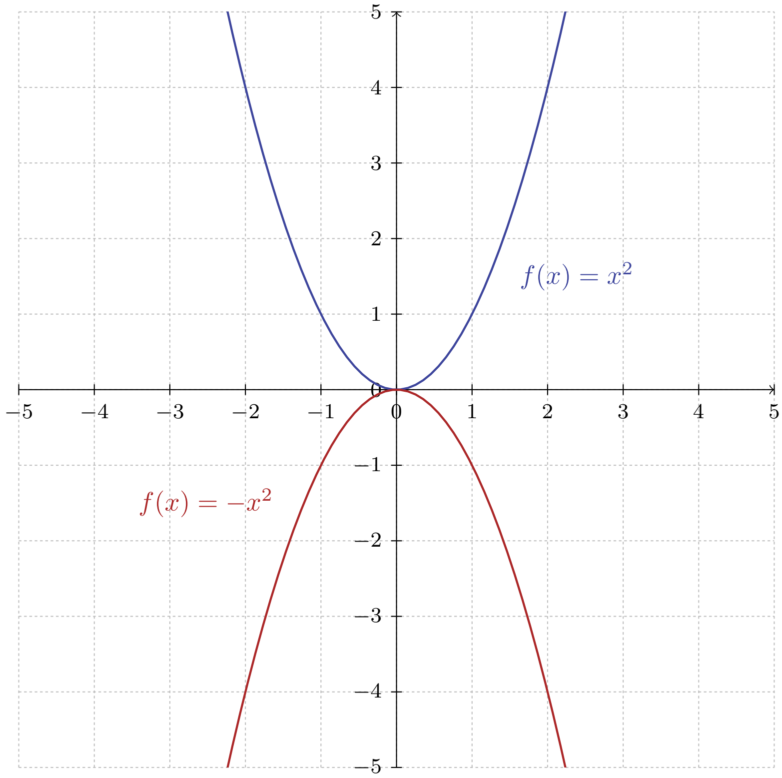

Figure 12.1: A parabolic function: $f(x) = x^2$

Where:	
m	represents a component of the gradient (covered more in section ?? "Differentiation").
x	is the independent variable
y	is the dependent variable

Chapter 13

Progressions

Progressions are when you have a bunch of numbers in a group, and they are related to eachother in an order, such as "add one to the last one", or "divide by one plus the last one." They fall into two types which resemble the form:[1]

Arithmetic Progressions $a_1 + (1-1)d, a_2 + (2-1)d, a_3 + (3-1)d, \dots$ Essentially, a bunch of numbers separated by a common delta. They sometimes also go by the term *arithmetic sequence*. I shall use the term progression however, as it feels less confusing to me when we enter the topic of sums.

Geometric Progressions $a, ar, ar^2, ar^3, ar^4, \dots$

If you cannot recognise a form of numbers, you can perform what I call the "double difference test". We can taking any 2 consecutive numbers, and compare their difference with any other 2 consecutive numbers.

(a) If the delta is identical between both pairs, you have an arithmetic progression.

(b) If the delta is different between both pairs, you have a geometric progression.

13.1 Arithmetic Progressions

Arithmetic progressions are really nothing more than a set of numbers which have a difference from one number to the next of a common number, (d), as seen in the

[1] again, there are other types, but they fall out of the scope of MATH130. Some examples include Harmonic Progressions, and a Taylor Series

double difference test. An arithmetic progression takes the form:

$$a_n = a_1 + (n-1)d \tag{13.1}$$

or more generally:

$$a_n = a_m + (n-m)d \tag{13.2}$$

Sometimes knowing the in the form of $n =$ is handy, this is derived in the subsection the sum of arithmetic progressions:

$$n = \frac{1}{d}(a_n - a_m) + m \tag{13.3}$$

Where:	
a_n	is an arbitrary term we wish to determine
n	is the n^{th} arbitrary value to index the term
m	is an arbitrary term we are given
a	is the number itself
d	is the delta between numbers

Table 13.1: Components to an arithmetic progression.

13.1.1 Determining the n-th term of an arithmetic progression

Given some sequence of numbers, $1, 3, 5, 7, 9, \ldots$, we can determine the delta, $d = 2$ (as the difference of any pair of numbers is 2), and $a_1 = 1$. Armed with this knowledge we can determine any term. For example the 1000^{th} term can be obtained as follows:

$$\begin{aligned}
a_n &= a_m + (n-m)d \\
a_{1000} &= a_1 + (1000-1) * 2 && (13.4)\\
&= 1 + 999 * 2 && (13.5)\\
&= 1 + 1998 && (13.6)\\
&= 1999 && (13.7)
\end{aligned}$$

It also works backwards for numbers before first term, suppose given the same sequence as above we wanted to determine what the -1000^{th} term[2],[3] is.

$$
\begin{aligned}
a_n &= a_m + (n-m)d \\
a_{-1000} &= a_1 + (-1000+1) * 2 \qquad (13.8)\\
&= 1 + (-999) * 2 \qquad (13.9)\\
&= 1 - 1998 \qquad (13.10)\\
&= -1997 \qquad (13.11)\\
& \qquad (13.12)
\end{aligned}
$$

Now you might be saying "whoa hold on a sec, we had 1999 before, shouldn't we expect -1999?". The answer is no. The reason: we must traverse across "zero" to get to -1 as the term before 1, so our negative number is always going to be d less than than the positive number (and then multiplied by -1).

[2] putting my poor language aside, let's pretend that a preceeding term could take place before the first term.

[3] this is partly why math is better than English, we can describe things and even poor grammar can convey the meaning better in math than poor grammar in English.

13.1.2 Sum of an Arithmetic Progression - aka Arithmetic Series

For some reason[4], we call a sum of subset of elements in an arithmetic progression an *arithmetic series*. To determine the sum:

$$S_n = \frac{n}{2}(a_1 + a_n) \tag{13.13}$$

$$\text{or} = \frac{n}{2}(2a_1 + (n-1)d) \tag{13.14}$$

Where:	
Sn	is the sum of an arbitrary term
a_n	is the arbitrary term
a_1	is the first term
d	is the delta between numbers

Table 13.2: Components to an arithmetic series.

Let's take our odd number progression from before: $1, 3, 5, 7, 9, \ldots, 1999$ and determine what the sum of these numbers are.

First we need to find what the value of n in the progression is, pretending we don't know it from before). We can rearrange (13.2) to make n the subject of the equation:

$$a_n = a_m + (n-m)d$$

$$a_n = a_1 + (n-1)d$$

$$a_1 + (n-1)d = a_n \qquad \text{swap sides} \tag{13.15}$$

$$(n-1)d = a_n - a_1 \qquad \text{subtract } a_1 \tag{13.16}$$

$$n - 1 = \frac{1}{d}(a_n - a_1) \qquad \text{divide by } d \tag{13.17}$$

$$n = \frac{1}{d}(a_n - a_1) + m \qquad \text{add } m \tag{13.18}$$

Substitute values:

$$n = \frac{1}{2}(1999 - 1) + 1 \tag{13.19}$$

$$= \frac{1}{2}(1998) + 1 \tag{13.20}$$

$$= 999 + 1 \tag{13.21}$$

$$= 1000 \tag{13.22}$$

[4] no pun intended

Now we can substitute into the sum equation (13.13)

$$
\begin{aligned}
S_n &= \frac{n}{2}(a_1 + a_n) \\
S_{1000} &= \frac{1000}{2}(1 + 1999) && (13.23) \\
&= 500 \cdot 2000 && (13.24) \\
&= 1000000 && (13.25)
\end{aligned}
$$

13.2 Geometric Progressions

Geometric progressions are really nothing more than a set of numbers which have a ratio-based difference from one number to the next of a common ration, (r). If you apply the double difference test, a geometric progression will fail, however if you either divide two pairs of numbers by their previous or next consecutive number (a "double quotient test"), you should have the same ratio.

A finite geometric progression of numbers will look like $ar^0, ar^1, ar^2, \ldots ar^{n-1}$

$$a_n = ar^{n-1} \tag{13.26}$$

Sometimes knowing the in the form of $n =$ is handy, we can do this with logarithms (see section 4.2.1 for a list of log laws - many of which should be committed to memory)

$$\frac{a_n}{a} = r^{n-1} \tag{13.27}$$

$$\log_r(\frac{a_n}{a}) = \log_r(n-1) \tag{13.28}$$

$$\log_r(a_n) - \log_r(a) = \log_r(n-1) \tag{13.29}$$

$$a_n - a = n - 1 \tag{13.30}$$

$$a_n - a + 1 = n \tag{13.31}$$

From this form, getting a is a matter of rearranging:

$$a_n - n = a \tag{13.32}$$

Where:	
a	is the first term of the geometric progression
a_n	is an arbitrary term we wish to determine
r	is the ratio between numbers
n	is the n^{th} arbitrary value to index the term

Table 13.3: Components to a Geometric Progression.

13.2.1 Determining the n-th term of an geometric progression

Given some sequence of numbers, $1, 2, 4, 8, 16, 32, \ldots$, we can see the formula follows: $1 = ar^{(1-1)} = ar^0 = a * 1 = a = 1$, and we can see that the ratio is that of doubling ($r = 2$), such that $1 = a2^{(1-1)} = a2^0 = a * 1 = a = 1$. Armed with this knowledge we can determine any term. For example the 10^{th} term can be obtained as follows:

$$a_n = ar^{n-1} \qquad \text{by (13.26)}$$

$$a_{10} = 1 * 2^{10-1} \tag{13.33}$$

$$= 1 * 2^{10-1} \tag{13.34}$$

$$= 1 * 2^9 \tag{13.35}$$

$$= 512 \tag{13.36}$$

Another example that is easily worked out using finders and no calculator.

$$a_n = ar^{n-1} \qquad \text{by (13.26)}$$

$$a_5 = 1 * 2^{5-1} \tag{13.37}$$

$$= 1 * 2^{5-1} \tag{13.38}$$

$$= 1 * 2^4 \tag{13.39}$$

$$= 16 \tag{13.40}$$

Simply raise one finger[5], and each time you double, raise another finger. If you have the normal number of digits per hand, the 5^{th} will match the result.

[5]ideally your thumb or pinky!

13.2.2 Sum of an Geometric Progression - aka Geometric Series

We call a sum of subset of elements in an geometric progression an *geometric series*.

To determine the sum:

$$S_n = ar^0 + ar^1 + ar^2 + \ldots + ar^{n-1} \tag{13.41}$$

If we multiply both sides by r

$$rS_n = ar^1 + ar^2 + \ldots + ar^{n-1} + ar^n \tag{13.42}$$

and we take the difference of these two equations:

$$S_n - rS_n = a - ar^n \tag{13.43}$$

S_n can also be expressed as:

$$S_n = \frac{a - ar^n}{1 - r} \tag{13.44}$$

Because (and it's easier to understand why when you see this working):

$$S_n = \frac{S_n - rS_n}{x} \tag{13.45}$$

Let $S_n = y$

$$y = \frac{y - (ry)}{x} \tag{13.46}$$

$$yx = y - yr \tag{13.47}$$

$$x = 1 - r \tag{13.48}$$

Where:	
Sn	is the sum of an arbitrary term
n	is the n^{th} index of the term we are interested in
a	is the first term
r	is the ratio between numbers

Table 13.4: Components to a geometric series.

Chapter 14

Introduction to Calculus

In a nutshell calculus is about measuring small changes. There are 4 major topics in calculus, with the first two of interest to MATH130 students; the latter two being more advanced topics for down the road.

Differential Calculus Studying the rates at which things change at specific points (most often, the rate at which something changes with respect to time - eg acceleration can be differentiated to determine speed of an object at a given time). This is concerned with the instantaneous rate of change. The derivative (gradient) is able to show this.

Integral Calculus The reverse of differentiation: taking a series of changes and turning this into a different metric (eg, taking a series of points of speed and turning this into the acceleration of an object). This is concerned with the accumulation of metrics. The area between a curve and the x-axis is able to show this.

Multivariable Calculus Extends the previous two types of calculus by allowing one to differentiate or integrate with respect to multiple variables.

Vector Caclulus Concerns itself with differentiating and integrating things called vector fields in 3D space. It is a subset of multivariable calculus in that a vector is a set of points in 3D that define a ray (kind of like a line, but with 2 end points) starting at 0,0,0 and ending at some x,y,z coordinate in 3D space.

There are several principles which are required for differential and integral calculus:

Limits A limit is basically a really small value that represents the difference between the two inputs of a function. We say it is *"sufficiently close"* with the result is arbitrarily close enough to be deemed the closest we can measure.

Derivatives Consider a non-linear function, $f(x)$. The derivative is the gradient of the function at a given point. The gradient changes depending on the x-value we supply; so we find a derivative at a certain point. This is done by the process of differentiation; and we then substitute the value of x in for that point to determine[1] the gradient.

Fundamental theorem States that differentiation and integration are inverse operations - that is, one will undo the other. There are two parts to integration; definite integrals and indefinite integrals (sometimes called *antiderivatives*). These will be covered in whole sections unto themselves.

Let's leave this all in the back of our minds and jump into the mechanics by looking at functions and graphs as it provides a nice graphical way to provide meaning to the concepts presented.

[1] derive??

Chapter 15

Functions & Graphs

If you are a computer programmer, the best way to think of a function in maths is the same way you think of a function in a functional programming language. If you are not a computer programmer, perhaps the best way to think of a function is like a little machine that takes a number, it does something to that number, and it displays an output.[1] Here is an example of a function that simply "doubles" the input:

$$f(x) = 2x \tag{15.1}$$

Here is what happens when we input the number 5 into our function...

$$f(5) = 2(5) \tag{15.2}$$
$$= 10 \tag{15.3}$$

And now the number -5:

$$f(-5) = 2(-5) \tag{15.4}$$
$$= -10 \tag{15.5}$$

Functions can be (and often are) more complex, here's a quadratic function:

$$f(a) = (a+3)^2 \tag{15.6}$$
$$= (a+3)(a+3) \tag{15.7}$$
$$= a^2 + 6a + 9 \tag{15.8}$$

[1] Congratulations, you are now thinking like a programmer as well as a mathematician!

And if we substitute 5 in for a we get:

$$f(5) = 5^2 + 6(5) + 9 \tag{15.9}$$
$$= 25 + 30 + 9 \tag{15.10}$$
$$= 64 \tag{15.11}$$

For completeness, if we substitute -5 in for a we get:

$$f(-5) = -5^2 + 6(-5) + 9 \tag{15.12}$$
$$= 25 - 6(5) + 9 \tag{15.13}$$
$$= 4 \tag{15.14}$$

When it comes to graphing functions, you can rename your y axis to equal $f(x)$, so graphing your function is now the same as before graphs with one additional bonus: now you can let x be an abitrary number[2] as part of the function. Figure 15.2 shows a function, $f(x) = x$.
Types of graphs that can be encountered in MATH130:

(a) Line 15.2

(b) Parabola 15.3

(c) Hyperbola 15.4

(d) Cubic 15.5

(e) Absolute 15.6

A function has a *domain* which is all the acceptable values of x as inputs that the function can use. Some examples:

[2]This is called a *independent variable*

$$f(x) = x^2$$

$$domain : 0 \le x \ge 0 \tag{15.15}$$

$$\tag{15.16}$$

$$f(x) = \frac{1}{x}$$

$$domain : x \neq 0 \tag{15.17}$$

$$\tag{15.18}$$

$$f(x) = \sqrt{x}$$

$$domain : x \ge 0 \tag{15.19}$$

$$\tag{15.20}$$

$$f(x) = \sqrt{x-2} + \sqrt{5+x}$$

$$domain : (x-2) \ge 0 \tag{15.21}$$

$$\therefore \ge 2 \tag{15.22}$$

$$\tag{15.23}$$

$$domain : 5 + x \ge 0 \tag{15.24}$$

$$\therefore \ge -5 \tag{15.25}$$

$$andso : \tag{15.26}$$

$$x : x \ge 2 \tag{15.27}$$

The last equation demonstrates that the domain is the more restrictive of the conditions of the two "sub-domains" of each square root portion of the function.

An important aspect of functions is that for any given element of the domain, there can only be one output. That is to say, for every input there is only one output. For every output, there may be any number of inputs.

15.1 Arithmetic With Functions

Suppose we have:

$$f(x) = x^2 + 1 \tag{15.28}$$
$$g(x) = \frac{1}{1-x} \tag{15.29}$$

and

$$h = f(x) + g(x) \tag{15.30}$$

then

$$= x^2 + 1 + \frac{1}{1-x} \tag{15.31}$$

however, if

$$i = f(x) \times g(x) \tag{15.32}$$

then

$$= (x^2 + 1) \times (\frac{1}{1-x}) \tag{15.33}$$

however, if

$$j = f \circ g \tag{15.34}$$

then

$$= j(x) = f(g(x)) \tag{15.35}$$
$$= (\frac{1}{1-x})^2 + 1 \tag{15.36}$$

> "Remember to "algebra" the function to minimise them and to see if they equal a simpler equation"[3]

[3] according to Gareth Richardson

15.2 Linear Functions

Linear functions are functions with a *constant rate of change*. What this means is that over the entire range of the function, the *gradient* or slope remains the same.

The gradient can be determined by:

$$\text{Rate of change} = \frac{\Delta\text{function}}{\Delta\text{input}} \tag{15.37}$$

Often in high school this is called "rise over run" and takes the form:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{15.38}$$

(or the "final y minus the initial y divided by the final x minus the initial x").

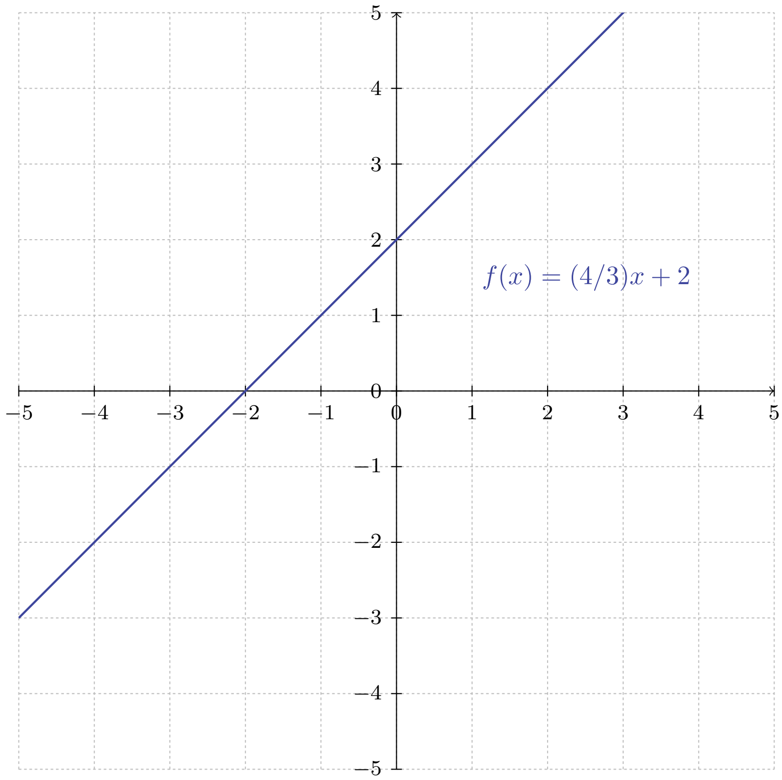

Figure 15.1: A linear function: $f(x) = mx + b$

Linear function in the *General Form* $Ax + By + C = 0$. It may also take the handy *Slope-intercept form* $f(x) = mx + b$. This is useful because the gradient of the line cane be read straight from the equation, and is just a rearrangement of the general form.

Another useful form of the linear function is:

$$y = m(x - x_1) + y_1 \tag{15.39}$$

If given only 2 points and we are to find the equation of the line:

(a) we need to determine the gradient

(b) we need to substitute the x,y values in to form the equation

Determining the gradient is done using the formula:

$$\frac{y_2 - y_1}{x_2 - x_1} = m$$

Example: we are given the points (7,1), (2,5) and need to find the equation of the line connecting these points.

$$\frac{5-1}{2-7} = \frac{4}{-5} \tag{15.40}$$

$$= -\frac{4}{5} \tag{15.41}$$

If we want to find a parallel line passing through specific points, remember that the gradient (m) must be the same in both equations, and we must subtitute the x and y values for the specific points into the new arbitrary equation to solve for the new equation.

Example: find an equation for the line through $(3, 4)$ and parallel to the line through $(7, 1)$ and $(2, 5)$ from out previous example:

$$\text{let } m = -\frac{4}{5}$$

Now substitute x & y values into the slope-intercept form:

$$y = mx + b$$

$$4 = -\frac{4}{5} \times 3 + b \tag{15.42}$$

and solve for b

$$4 + \frac{4}{5} \times 3 = b \tag{15.43}$$

$$4 + \frac{12}{5} = b \tag{15.44}$$

$$\frac{32}{5} = b \tag{15.45}$$

$$\therefore y = -\frac{4}{5}x + \frac{32}{5} \tag{15.46}$$

15.2.1 Perpendicular Lines

Perpendicular lines has a particular special property.

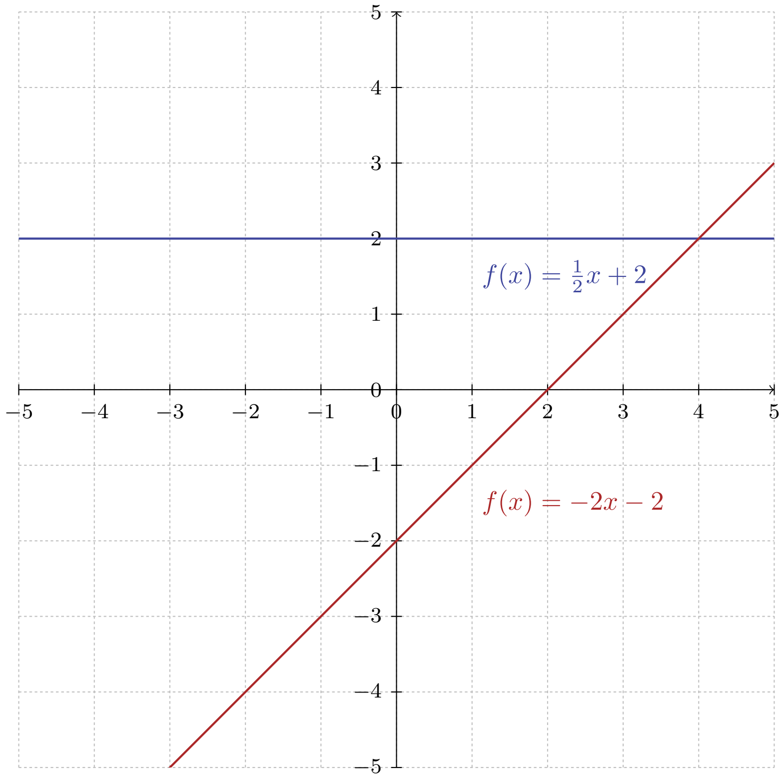

Figure 15.2: A linear function: $x = 2$

15.2.2 Vertical lines

There is a special case where you cannot work out the gradient, suppose there was no change in an x value.

$$m = \frac{\Delta y}{\Delta x} \tag{15.47}$$

BUT we cannot divide by zero! So we could say the the line has no slope.

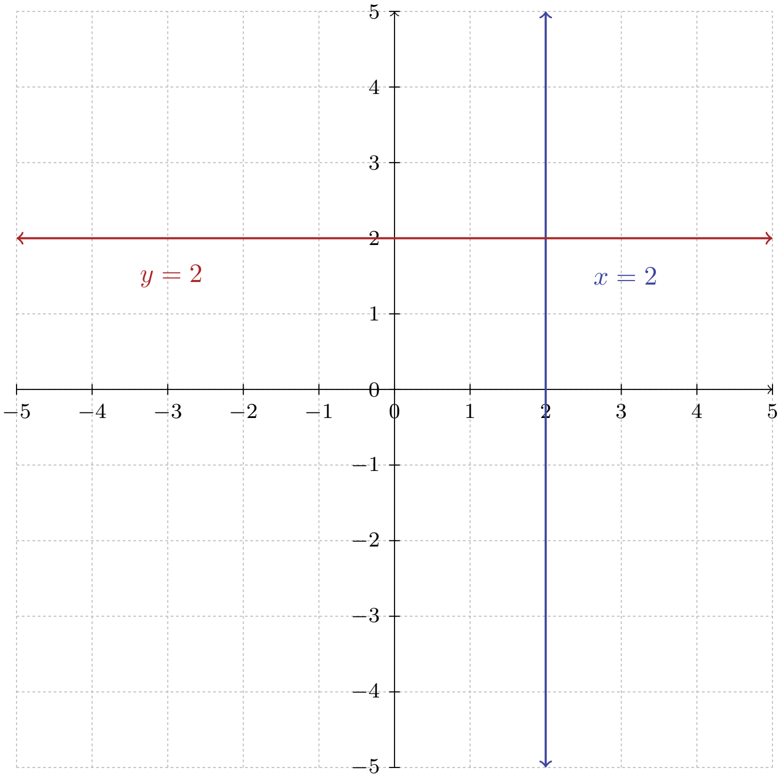

Figure 15.3: A linear function: $x = 2$

15.3 Parabolic Functions

For an introduction to parabolas, it is highly recommended to read chapter 12, “Parabolas”.

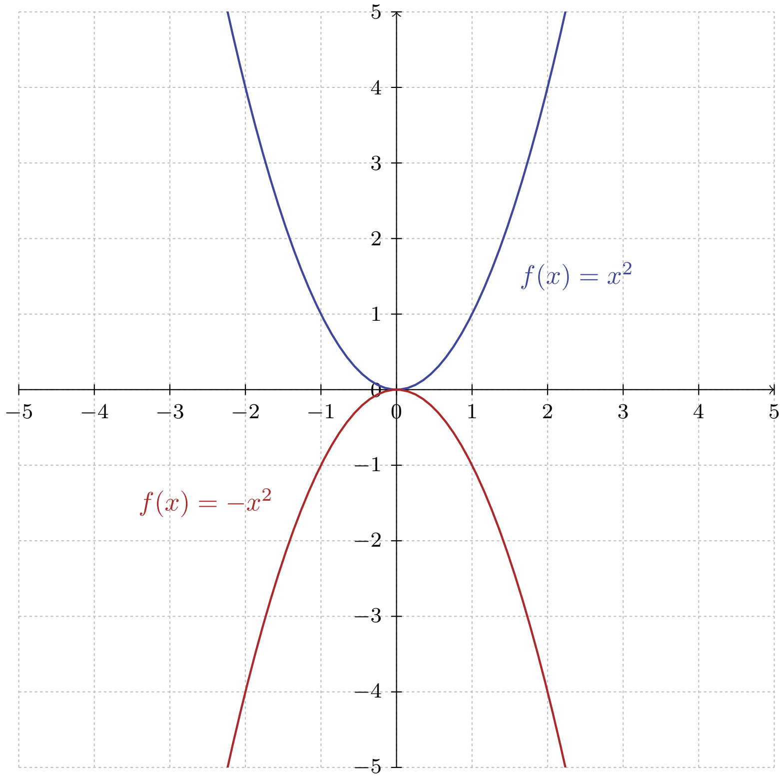

Figure 15.4: A parabolic function: $f(x) = x^2$

15.4 Hyperbolic Functions

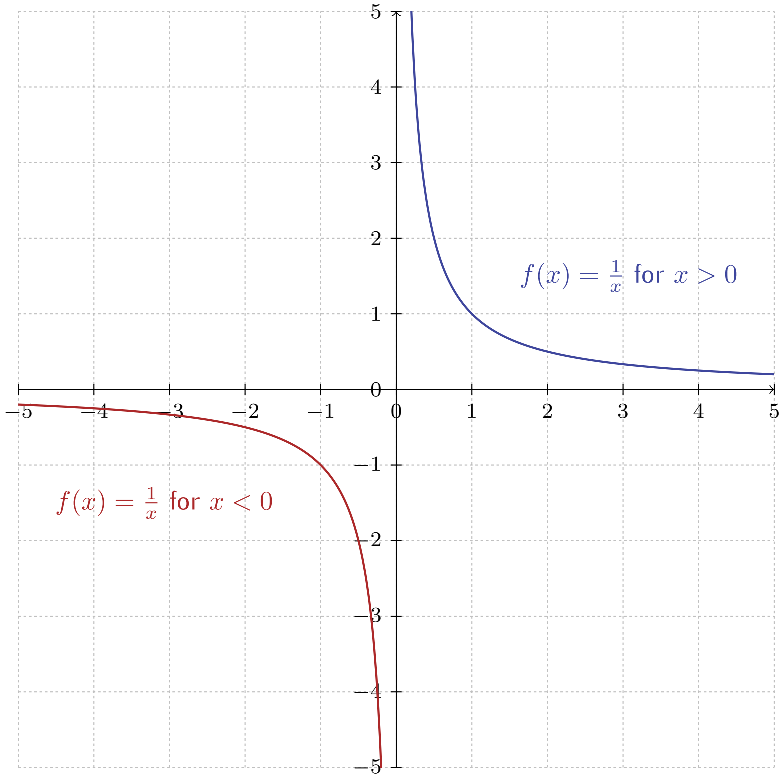

Figure 15.5: Hyperbolic function: $f(x) = \frac{1}{x}$

Where:	
x	is the independent variable
y	is the dependent variable

15.5 Cubic Functions

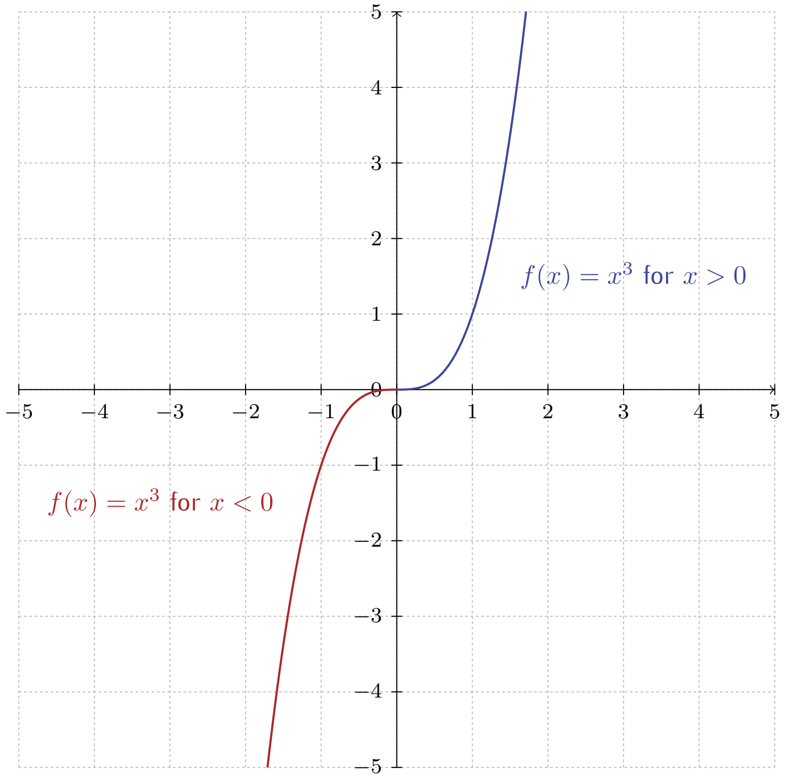

Figure 15.6: A cubic function: $f(x) = x^3$

Where:	
m	represents a component of the gradient (covered more in Chapter 16 "Differentiation").
x	is the independent variable
y	is the dependent variable
x	is the independent variable
y	is the dependent variable

15.6 Absolute Value Functions

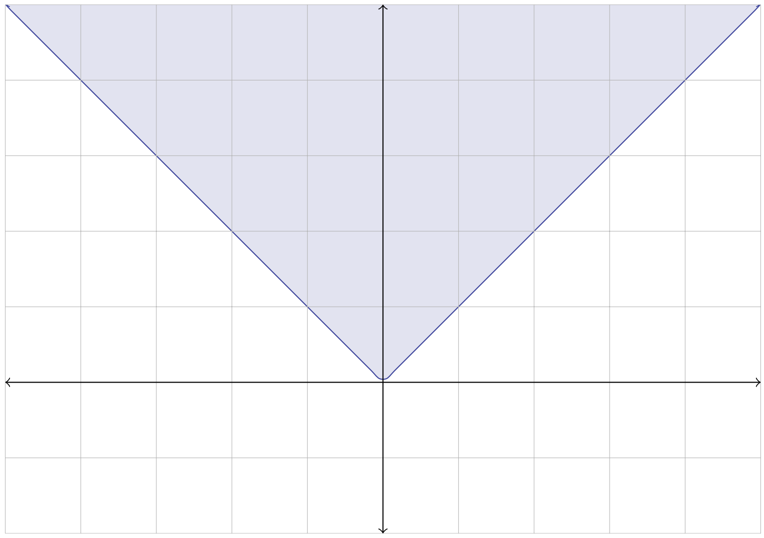

Figure 15.7: A function of absolute value: $f(x) = |x|$

An absolute function in the form $f(x) = |x|$

Where:	
m	represents a component of the gradient (covered more in Chapter 16 “Differentiation”).
x	is the independent variable
y	is the dependent variable
x	is the independent variable
y	is the dependent variable

15.7 Limits

There are some types of functions you may be asked to evaluate for various values of x which are unreasonable. One such example is the hyperbolic function $f(x) = \frac{1}{x}$ where $x = 0$. In MATH130, we consider this value an illegal or "undefined" value - but there is still a way to evaluate it. Consider taking a table of values15.7: Note how as x gets smaller, y

x	5	4	3	2	1	0.8	0.6	0.4	0.2	0.1	...
y	0.20	0.25	0.33	0.50	1.00	1.25	1.67	2.50	5.00	10.00	...

Table 15.1: Table of values for hyperbolic function $f(x) = \frac{1}{x}$

gets bigger. The way this is formally worded is "as x approaches 0, y approaches infinity, and written:

$$x \to 0 \; f(x) \to \infty \tag{15.48}$$

From this point, we can see that x cannot be zero, however all other $\mathbb{R}$ are acceptable. Building on this we can define it as a set:

$$\begin{aligned} & x \to 0 \; f(x) \to \infty \\ & x \in \{\mathbb{R}, x \neq 0\} \end{aligned} \tag{15.49}$$

The key to understanding how limits work is to identify what x values are undefined or otherwise illegal. Key indicators of this phenomena are when you see x inside a squareroot sign, or as a divisor in a quotient:

Chapter 16

Differentiation

In a nutshell, differentiation is the process of finding the rate of change at an instant point in time. Often, we are looking at data or formulae that has been plotted on a graph to form a curve.[1]

An example of this would be if we throw a ball in the air. The velocity of the ball changes from second to second, however, the rate at which the velocity changes (called acceleration) also changes. At first, the ball slows down quite quickly, and at the maximum heigh, the velocity plateaus, reaches zero, and then starts falling, slowly at first then faster and faster until it hits the ground.

In reality, cannot get an exact value for instantaneous points in time, however, we can measure the height of the ball at sufficiently fast intervals to get sufficiently accurate average velocities. At some point, the value of what you measure will not change by very much with respect to the change in time. This is what we call the *limit* when the measured data is constant.

There are 5 methods of differentiation which are useful for MATH130. These are:

- Fundamental theorem of Calculus

[1] We say curve, and we can do it on a straight line too, but given a straight line is, well, straight, there is no change in the rate of change

- Power Method (well, that's what I call it)
- Product Rule
- Quotient Rule
- Chain Rule

16.1 Fundamental Theorem of Calculus

The *fundamental theorem of calculus* is the basis for the other 4 rules. We measure the change in the y value of a function divided by the change in the x value. What this boils down to is:

$$\frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{16.1}$$

It is sometimes called the limit of the difference quotient.

If this quantity makes sense as $h \to 0$ (h approaches zero), we end up with the exact rate of change of f at x. This is called *the derivative of f at x*.

It is worth noting at this point that we are finding the *tangent* of a point along a curve which is a fancy way of saying that as we "zoom in", the gradient of the line starts to resemble the curve itself.[2]

16.1.1 Language of Calculus

Because the notion of calculus was a joint effort between several mathematicians[3] operating in secret from each other, and without a project manager there are two important notations.[4]

The first notation is the "dash", "prime", or "Lagrange's" notation and appears as such:

$$f(x) = \dots$$
$$f'(x) = \dots$$

[2] This is hardly a mathematical definition, but it describes the process of what is going on.

[3] Gottfried Leibniz woke up one day and thought "I'm going to invent a whole new branch of mathematics to annoy students for the next few hundred years." Approximately 10 years earlier, Sir Isaac Newton thought "I know what will really get Leibniz's goat... I'll get the drop on him with this idea I have." Consequently, the two never became friends.

[4] there are more, but we don't need to know about them for MATH130

Secondly is Leibniz's notation:

$$
\begin{aligned}
y &= \dots \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \dots
\end{aligned}
$$

Euler's notation: (not so common in MATH130)

$$
\begin{aligned}
f(x) &= \dots \\
Df(x) &= \dots
\end{aligned}
$$

Each notation has their merits and usefulness; Lagrange's form is neat and compact for simple derivatives, however Leibniz's notation describe what is being differentiated and what is in respect to[5].

It is important to be familiar with both Lagrange's and Leibniz's as they will often be used interchangeably for brevity, neatness and ease of memorising them. In terms of how you answer a question - if there is no stated style of notation, go for what "looks" like it works [6] and is clear and neat. Clear and neat usually results in the marker understanding what you are on about, so even if you are wrong, you might get partial marks.

A useful tip, when you are first getting used to differentiation, it may be handy to use Leibniz's notation and say in your mind what you are differentiating, and what it is in respect to.

16.1.2 Using the Fundamental Theorem of Calculus

Suppose we drop a ball from a height ($s = 0$). The formula is given by

$$
\begin{aligned}
s &= f(t) \\
&= 5t^2 \\
&= \frac{1}{2}gt^2 \qquad (16.2)
\end{aligned}
$$

Height at 1 second:

$$
= 5(1)^2 \qquad (16.3)
$$

[5] which is useful for the chain rule discussed shortly

[6] Munner's Law: If it looks wrong it probably is. – Cliff Munro, 1996 Cranbrook School, Design and Technology teacher. Author's corollary: If it doesn't look wrong, hopefully it's right.

Height at 1.2 seconds

$$= 5(1.2)^2 \tag{16.4}$$

The difference is given by:

$$\frac{\Delta s}{\Delta t} = \lim_{t\to 0} \tag{16.5}$$
$$= \lim_{t\to 0} \frac{f(1.2) - f(1)}{0.2} \tag{16.6}$$
$$= \lim_{t\to 0} \frac{5(1.2)^2 - 5(1)^2}{0.2} \tag{16.7}$$
$$= 11 \tag{16.8}$$

If we try successively smaller values for h, we get a table that looks like this:

h	$\frac{f(1+h)-f(1)}{h}$	value
0.2	$\frac{s(1.2)^2-5(1)^2}{0.2}$	11 m/s
0.1	$\frac{s(1.1)^2-5(1)^2}{0.1}$	10.5 m/s
0.05	$\frac{s(1.05)^2-5(1)^2}{0.05}$	10.25 m/s
0.01	$\ldots$	10.05 m/s
0.001	$\ldots$	10.005 m/s

Table 16.1: The ball falling

We have shown $f'(1) = 10$.

16.2 Approximating Functions

One use of tangents is to approximate functions. By finding the tangent of a line at a point, you get a rough idea of the function itself. This becomes useful for very complicated functions[7], given that the complexity of a function may make computing the function itself prohibitive, and "near enough is good enough" as we are taking an arbitrarily close value of h in the fundamental theorem of calculus.

[7] probably beyond the realm of MATH130

A different method for approximation is covered in significantly more detail in Chapter 21, "Newton's Method". Newton's method provides increasingly higher levels of precision in an estimation, and can be applied ad infinitum to many functions.

For example: Suppose we are to approximate the function of $f(x) = \sqrt{x}$.

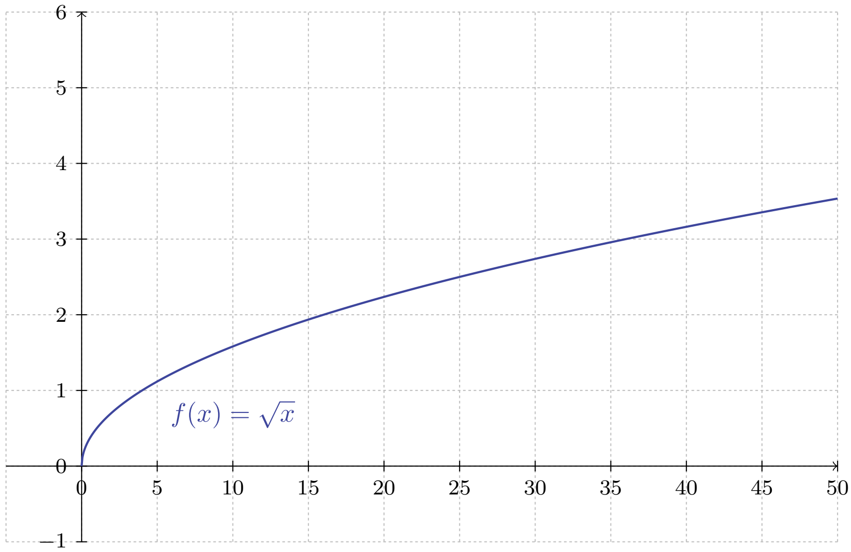

Figure 16.1: Function: $f(x) = \sqrt{x}$

$$f(x) = \sqrt{x} \tag{16.9}$$

$$\text{Then } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{16.10}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \tag{16.11}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{(\sqrt{x+h} - \sqrt{x})}{(\sqrt{x+h} - \sqrt{x})} \tag{16.12}$$

identify difference of two squares

$$a^2 - b^2 = (a-b)(a+b)$$

$$\text{Let } a = \sqrt{x+h} \tag{16.13}$$

$$b = \sqrt{x} \tag{16.14}$$

$$\text{So } f'(x) = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \tag{16.15}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \tag{16.16}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \tag{16.17}$$

it is now safe to take a limit of $h = 0$

$$= \frac{1}{2\sqrt{x}} \tag{16.18}$$

We have found that if

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \tag{16.19}$$

then

$$f'(x) = \frac{1}{2\sqrt{x}} \tag{16.20}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} \tag{16.21}$$

The slope of the tangent to $\sqrt{x}$ at $x = 36$ is

$$f'(36) = \frac{1}{2\sqrt{36}} \tag{16.22}$$

$$= \frac{1}{12} \tag{16.23}$$

The equation of the tangent is found using $y = m(x-a)+b$ using the slope in point form (a, b) so

$$T(x) = \frac{1}{12} \cdot (x-36) + 6 \tag{16.24}$$

$\sqrt{37}$ from our diagram is approximately $T(37)$

$$T(37) = \frac{1}{12} + 6 \tag{16.25}$$

$$= 6\frac{1}{12} \tag{16.26}$$

$$\sqrt{37} \approx 6.083\ldots \tag{16.27}$$

16.3 Power Method

The power method is by far the easiest to understand of all 3 methods, and if possible, it may be easier to rearrange a part of an equation into index notation and differentiate that way. This method is not always possible, but by using the power laws from the series of equations starting with 4.3, sometimes a shortcut can be made, which is why Chapter 4 is important to know very well.

Put simply, the power method can be understood as "multiply the base by the power and subtract one from the power", and is demonstrated in equation 16.29 below.

$$f(x) = x^a + k \tag{16.28}$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = ax^{a-1} \tag{16.29}$$

The value k represents a constant, often just a plain number, though it doesn't have to be. The important thing about the k-value in this example is that there is no x component. Hence it "disappears".

Functions may have more than one term, consider the following quadratic:

$$f(x) = x^2 + 2xb + b^2 \tag{16.30}$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 2x^1 + 2b \tag{16.31}$$

$$= 2(x + b) \tag{16.32}$$

Here we are asked to differentiate with respect to x (denoted by the symbol $\frac{\mathrm{d}}{\mathrm{d}x}$). To do this, we bring the power of x^2 to the front, and subtract 1 to give $2x^1$, and we do the same for the term $2xb$ by looking at the invisible power (it's there, we just don't write it out of laziness![8]): $2 * x^1 * b$

[8] Or convenience, neatness, brevity.

16.4 Product Rule

A function $f(x)$ is a product of two functions, $u(x) * v(x)$. For example:

$$f(x) = x^3 \sin(x) \tag{16.33}$$
$$= u(x)v(x) \tag{16.34}$$

In this case, we can see that $u(x) = x^3$ and $v(x) = \sin(x)$. While there is a mathematical proof[9] it is not necessary for MATH130. All we need to know is:

$$f(x) = u(x) * v(x) \tag{16.35}$$
$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = u'(x) * v(x) + v'(x) * u(x) \tag{16.36}$$

So for 16.33, to find the derivative:

$$f(x) = x^3 \, \sin(x)$$
$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}((x^3)) * \sin(x) + x^3 * \frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) \tag{16.37}$$
$$= 3x^2 * \sin(x) + x^3 * \cos(x) \tag{16.38}$$

This example makes use of a derivative of a trigonometric function. This will be explored in chapter 19, “Differentiation of Trig Functions” - until then, just ignore the trigonometric part.

16.5 Quotient Rule

A quotient is a division - just like from primary school: $\frac{u}{v}$. Previously we may have called the u part the numerator, nowadays it is called the *dividend*, and the v denominator previously called the denominator is now called the *divisor*, with the *quotient* being the result.

To do quotient rule differentiation, we are actually using a modified version of the product rule, however for MATH130 it is only required to think of it

[9] refer to section 29.1 of appendix

as a separate rule.[10] The rule takes the form of:

$$f(x) = \frac{u}{v} \tag{16.39}$$

$$f'(x) = \frac{\frac{\mathrm{d}u}{\mathrm{d}x} \cdot v - u \cdot \frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \tag{16.40}$$

OR

$$f'(x) = \frac{u'v - uv'}{v^2} \tag{16.41}$$

Although both 16.40 and 16.41 are identical, 16.41 is tidier, and may be easier to remember. Consider the following example:

$$f(x) = \frac{x^2}{4x} \tag{16.42}$$

$$f'(x) = \frac{\mathrm{d}(4x)x^2}{\mathrm{d}x} - x^2(\frac{\mathrm{d}4x}{\mathrm{d}x}) \tag{16.43}$$

$$f'(x) = (4x)2x - x^2\,(4) \tag{16.44}$$

$$f'(x) = 8x^2 - 4x^2 \tag{16.45}$$

$$f'(x) = 4x^2 \tag{16.46}$$

[10] Mathematical proof of this is in section 29.2 of the appendix.

16.6 Chain Rule

The chain rule is useful for differentiation a function $f(x)$ where there are functions inside of functions (such as $\ln(\sin(x))$). To do this, we break the function up into it's components, give them some names [11] and apply the chain rule.

The rule takes the form as follows[12]:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \tag{16.47}$$

$$\tag{16.48}$$

Here is an example of where we might use the chain rule:[13][14]

$$f(x) = (x^2)^3 \tag{16.49}$$

Let $u = x^2$ such that

$$f(x) = u^3 \tag{16.50}$$

The first step in 16.50 is to identify the chain rule, and to name the function inside u. It just makes things easier this way for most of the time.

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \tag{16.51}$$

$$= 3u^2 \cdot 2x \tag{16.52}$$

[11]" *Let*" is possibly the most important word you will come across in mathematics. You can use it to redefine stuff if it's too complex and break it into smaller manageable pieces and put it back together again." – Chris Gordon, MATH130 lecturer, Macquarie University, Semester 1 2011

[12]using this notation makes it clear as to what is actually going on

[13]Note that the v function above in this example is simply u^3.

[14]While we could use the power method to solve this particular problem in 2 steps, we will demonstrate chain rule first, and then the power method.

16.50 It often helps to rewrite the equation in terms of $f(x)$ and u, and then to write out the chain rule. Normally we are differentiating $f(x)$ with respect to x. Here we differentiate $f(x)$ with respect to u (ie $\frac{\mathrm{d}f}{\mathrm{d}u}$). *THEN* we multiply by $\frac{\mathrm{d}u}{\mathrm{d}x}$. The rest is plain old algebra. So.. substitue values back for $\frac{\mathrm{d}u}{\mathrm{d}x}$.

$$\begin{aligned} &= 3u^2 \cdot 2x \\ &= 3(x^2)^2 \cdot 2x && (16.53) \\ &= 3x^4 \cdot 2x && (16.54) \\ &= 6x^5 && (16.55) \end{aligned}$$

As a point of exercise, here's how much faster it is using the power method:

$$\begin{aligned} f(x) &= \left(x^2\right)^3 \\ f(x) &= x^6 && (16.56) \end{aligned}$$

Now we use the differentiation power method, bring the power out the front and reduce the power by one

$$f'(x) = 6x^5 \tag{16.57}$$

Although the power method here took only two steps, it should be noted that power method cannot be used for all chain rule problems[15] – but for easy ones like this, it is far faster to use power method). The following

[15] in fact, most of the time the power method won't work because maths teachers are these hideous evil monsters who despise free time and kittens

example cannot use power method, and we *should* use the chain rule:

$$f(x) = \ln(\sin(x)) \tag{16.58}$$

$$\text{Let } u(x) = \sin(x) \tag{16.59}$$

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \tag{16.60}$$

$$\frac{\mathrm{d}\sin(x)}{\mathrm{d}x} = \cos(x) \tag{16.61}$$

$$\frac{\mathrm{d}\ln(u)}{\mathrm{d}u} = \frac{1}{u} \tag{16.62}$$

$$\text{substitute values back:} \tag{16.63}$$

$$f'(x) = \frac{1}{\sin(x)} \cdot \cos(x) \tag{16.64}$$

$$= \csc(x) \cdot \cos(x) \tag{16.65}$$

$$\text{or} \tag{16.66}$$

$$= \frac{\cos(x)}{\sin(x)} \tag{16.67}$$

$$= \cot(x) \tag{16.68}$$

Chapter 17

Maxima and Minima

If the function has a negative slope, then $f'(x) < 0$. If the function has a positive slope, then $f'(x) > 0$. If the function has a slope of 0, then $f'(x) = 0$.[1]

$$f \downarrow \Leftrightarrow f' < 0 \tag{17.1}$$

$$f \uparrow \Leftrightarrow f' > 0 \tag{17.2}$$

(17.3)

Suppose we were to take the derivative of a derivative. This is called the *second derivative* and can be written as:

$$f''(x) = \dots$$

$$\frac{d^2 f}{dx^2} = \dots$$

[1] However it is only instantaneously flat. If you think about it, 0 is slightly bigger than -0.00001 and slightly smaller than 0.00001 (and you can find arbitrarily smaller numbers that zero fits between)

Chapter 18

TODO: Differentiation of Exponents and Logs

Chapter 19

Differentiation of Trigonometric Functions

This section combines aspects of the unit circle with aspects of differentiation. A solid foundation in both of these may not be necessary, but it will certainly help. Consider the plot of the function $f(x) = \sin(x)$. While doing so, recall what the derivative actually is[1]. In particular, pay attention to *how the slope changes as you move back and forth along the graph.*

[1] It is the rate of change of something, and it is represented by the slope of the graph you are looking at. If this is at all foreign, consider looking at Chapter 16.

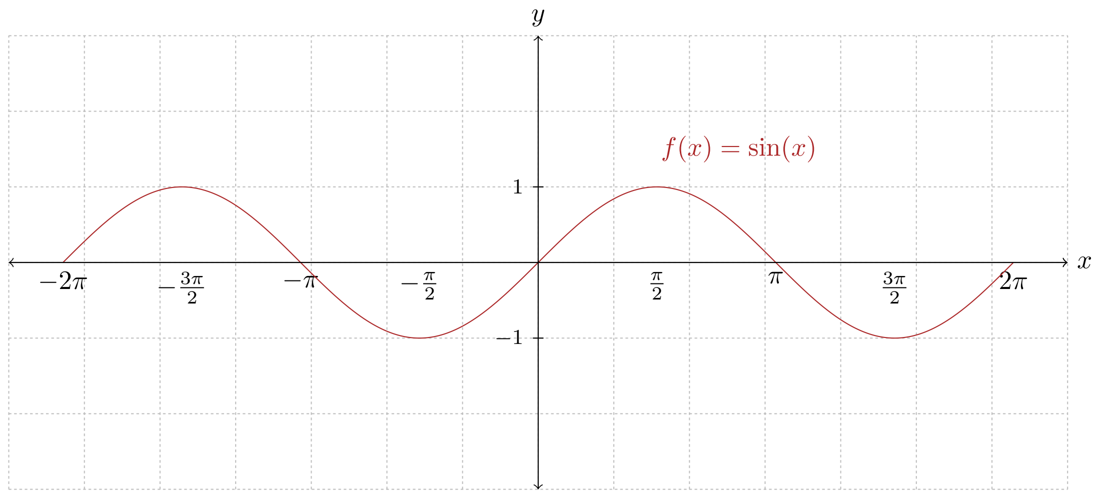

Figure 19.1: $\sin(x)$

Here are some equations that are handy to remember. Source [2].

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1} \tag{19.1}$$

$$\frac{\mathrm{d}e^{ax}}{\mathrm{d}x} = ae^{ax} \tag{19.2}$$

$$\frac{\mathrm{d}\ln(x)}{\mathrm{d}x} = \frac{1}{x} \tag{19.3}$$

$$\frac{\mathrm{d}\sin(ax)}{\mathrm{d}x} = (a)\cos(ax) \tag{19.4}$$

$$\frac{\mathrm{d}\cos(ax)}{\mathrm{d}x} = (-a)\sin(ax) \tag{19.5}$$

$$\frac{\mathrm{d}\tan(ax)}{\mathrm{d}x} = (a)\sec^2(ax) \tag{19.6}$$

$$\frac{\mathrm{d}\sec(ax)}{\mathrm{d}x} = (a)\sec(ax)\tan(ax) \tag{19.7}$$

$$\frac{\mathrm{d}\csc(ax)}{\mathrm{d}x} = (-a)\csc(ax)\cot(ax) \tag{19.8}$$

$$\frac{\mathrm{d}\cot(ax)}{\mathrm{d}x} = (-a)\csc^2(ax) \tag{19.9}$$

$$\frac{\mathrm{d}\sin^{-1}(\frac{x}{a})}{\mathrm{d}x} = \frac{1}{\sqrt[2]{a^2 - x^2}} \tag{19.10}$$

$$\frac{\mathrm{d}\sin^{-1}(\frac{x}{a})}{\mathrm{d}x} = \frac{-1}{\sqrt[2]{a^2 - x^2}} \tag{19.11}$$

$$\frac{\mathrm{d}\sin^{-1}(\frac{x}{a})}{\mathrm{d}x} = \frac{a}{\sqrt[2]{a^2 - x^2}} \tag{19.12}$$

For a proof, refer to appendix section 30, "Differentiation of Trig Functions Proof"

Chapter 20

TODO: Grokking Word Problems

Chapter 21

TODO: Newton's Method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{21.1}$$

Newton's Method is used to find successively better approximations of a function.

We take a value of x, x_0 and subtract the function at x_0 divided by the derivative of that function at x_0. The result, x_1 is an answer close to the roots of that function.

What we can do next, is put x_1 back into the same process and get a better approximation of the function. If we repeat this over and over, we will either get an exact value of the roots of the function, or we will get arbitrarily close to it.

Chapter 22

TODO: Practical Uses of Differentiation

Chapter 23

TODO: Integration

tl;dr: Integration is the reverse process of Differentiation.

If we were to delve more into what integration is, then we could describe it as finding the area "underneath a curve" (more accurately, between the curve and the x-axis).

There are many different ways of doing this, and MATH130 requires we know 2 of them:

Simpson's Rule

Trapezoidal Rule

There are two types of integration:

Anti-derivatives or *indefinite integrals*

Definite Integrals

The principles behind indefinite and definite integrals are the same, however, with definite integrals, you have a range for which you are integrating your function, where as indefinite integrals have no range (beyond the complete range of the function itself).

Chapter 24

TODO: Trapezoidal Rule

Chapter 25

TODO: Simpson's Rule

Chapter 26

TODO: Average values of a function

Chapter 27

TODO: Practical Uses of Integration

Chapter 28

Glossary

Symbol	Name	LaTeX
$\mathbb{A}$	Blackboard A	\mathbb{A}
$\mathbb{B}$	Blackboard B	\mathbb{B}
$\mathbb{C}$	Blackboard C	\mathbb{C}
$\mathbb{D}$	Blackboard D	\mathbb{D}
$\mathbb{E}$	Blackboard E	\mathbb{E}
$\mathbb{F}$	Blackboard F	\mathbb{F}
$\mathbb{G}$	Blackboard G	\mathbb{G}
$\mathbb{H}$	Blackboard H	\mathbb{H}
$\mathbb{I}$	Blackboard I	\mathbb{I}
$\mathbb{J}$	Blackboard J	\mathbb{J}
$\mathbb{K}$	Blackboard K	\mathbb{K}
$\mathbb{L}$	Blackboard L	\mathbb{L}
$\mathbb{M}$	Blackboard M	\mathbb{M}
$\mathbb{N}$	Blackboard N	\mathbb{N}
$\mathbb{O}$	Blackboard O	\mathbb{O}
$\mathbb{P}$	Blackboard P	\mathbb{P}
$\mathbb{Q}$	Blackboard Q	\mathbb{Q}
$\mathbb{R}$	Blackboard R	\mathbb{R}
$\mathbb{S}$	Blackboard S	\mathbb{S}
$\mathbb{T}$	Blackboard T	\mathbb{T}
$\mathbb{U}$	Blackboard U	\mathbb{U}
$\mathbb{V}$	Blackboard V	\mathbb{V}
$\mathbb{W}$	Blackboard W	\mathbb{W}
$\mathbb{X}$	Blackboard X	\mathbb{X}
$\mathbb{Y}$	Blackboard Y	\mathbb{Y}
$\mathbb{Z}$	Blackboard Z	\mathbb{Z}

Table 28.1: Blackboard letters. Blackboard notation must be inside math mode.

Greek Uppercase	Lowercase	Name	LaTeX(Upper)	LaTeX(Lower)
	α	Alpha		\alpha
	β	Beta		\beta
Γ	γ	Gamma	\Gamma	\gamma
	ϵ	Epsilon		\epsilon
	ζ	Zeta		\zeta
	η	Eta		\eta
Θ	θ	Theta	\Theta	\theta
	ι	Iota		\iota
	κ	Kappa		\kappa
Λ	λ	Lambda	\Lambda	\lambda
	μ	Mu		\mu
	ν	Nu		\nu
Ξ	ξ	Xi	\Xi	\xi
Π	π	Mu		\mu

Table 28.2: Blackboard letters. Blackboard notation must be inside math mode.

Chapter 29

Proofs of Calculus

29.1 ProofofProductRule

$$x = 1 \tag{29.1}$$

29.2 Proof of Quotient Rule

$$x = 1 \tag{29.2}$$

29.3 Proof of Chain Rule

$$x = 1 \tag{29.3}$$

Chapter 30

Differentiation of Trig Functions Proof

If we have the function $f(x) = \sin(x)$, and take two points, P and Q where $P = (x, f(x))$ and $Q = (x + h, f(x + h)$ where $h \neq 0$ we can construct a line joining P and Q, and the gradient of this line is given by the "rise over run" formula:

$$\frac{\Delta Y}{\Delta X} = \frac{f(x+h) - f(x)}{(x+h) - x} \tag{30.1}$$

$$= \frac{\sin(x+h) - \sin(x)}{h} \tag{30.2}$$

Chapter 31

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Dom Verity Macquarie University, for being a source of inspiration and convincing me to try{} just one more time before I threw an exception.

[1] Giggity

[2] blunt

[3] ie, if it was "crap", you'd call it "crap"

[4] and not as one of the "ugly cousins"

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[5] Gracious Professionalism is a common law trademark of the United States Foundation for Inspiration and Recognition of Science and Technology (US FIRST).

[6] alwayz!

[7] Put a donk on it!

Bibliography

[1] N. J. A. S. <njas@research.att.com>. A001113 - oeis, 1995. URL http://oeis.org/A001113. [Online; accessed 9-June-2011].

[2] K. Riley, M. Hobson, and S. Bence. *Mathematical Methods for Physics and Engineering*. Cambridge University Press, third edition, 2006. ISBN 978-0-521-67971-8.

[3] Wikipedia. List of mathematical symbols — wikipedia, the free encyclopedia, 2011. URL http://en.wikipedia.org/w/index.php?title=List_of_mathematical_symbols&oldid=431875373. [Online; accessed 7-June-2011].

[4] Wikipedia. Completing the square — wikipedia, the free encyclopedia, 2011. URL http://en.wikipedia.org/w/index.php?title=Completing_the_square&oldid=429461631. [Online; accessed 14-June-2011].

[5] WolframAlpha. quadratic formula - wolframalpha, 2011. URL http://www.wolframalpha.com/input/?i=quadratic+formula. [Online; accessed 7-June-2011].