MATH135 S115
Mathematics IA
Assignment 2

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Tutorial Group: D2, Wed 15:00, C5C 238

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**Due 14:00, 07/05 2015**

*Please sign the declaration below, and staple this sheet to the front of your solutions. Your assignment must be submitted at the Science Centre, E7A Level 1.*

**Your assignment must be STAPLED, please do not put it in a plastic sleeve.**

**PLAGIARISM** Plagiarism involves using the work of another person and presenting it as one's own. For this assignment, the following acts constitute plagiarism:

a) Copying or summarizing another person's work.
b) Where there was collaborative preparatory work, submitting substantially the same final version of any material as another student.

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**STATEMENT TO BE SIGNED BY STUDENT**

1. I have read the definition of plagiarism that appears above.
2. In my assignment I have carefully acknowledged the source of any material which is not my own work.
3. I am aware that the penalties for plagiarism can be very severe.
4. If I have discussed the assignment with another student, I have written the solutions independently.

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| FEEDBACK | | |
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| Work | Presentation | Total |
| | | |

1. Given is the function $f$ determined by $f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$.

   (a) Find the natural domain $D$ of the given functional expression.

   (b) Look at potential symmetry properties of the function.

   (c) Find the roots of the function, that is, the points $x \in D$ where $f(x) = 0$.

   (d) Asymptotic analysis: check for the existence of horizontal and vertical asymptotes.

   (e) Investigate the behaviour of the first derivative.

   (f) Investigate the behaviour of the second derivative.

   (g) Collect the results in a table, and determine the type of the critical points.

   (h) Finally, sketch the graph of $f$, using the previous table as a guideline.

2. The function arcsinh is the inverse function of sinh.

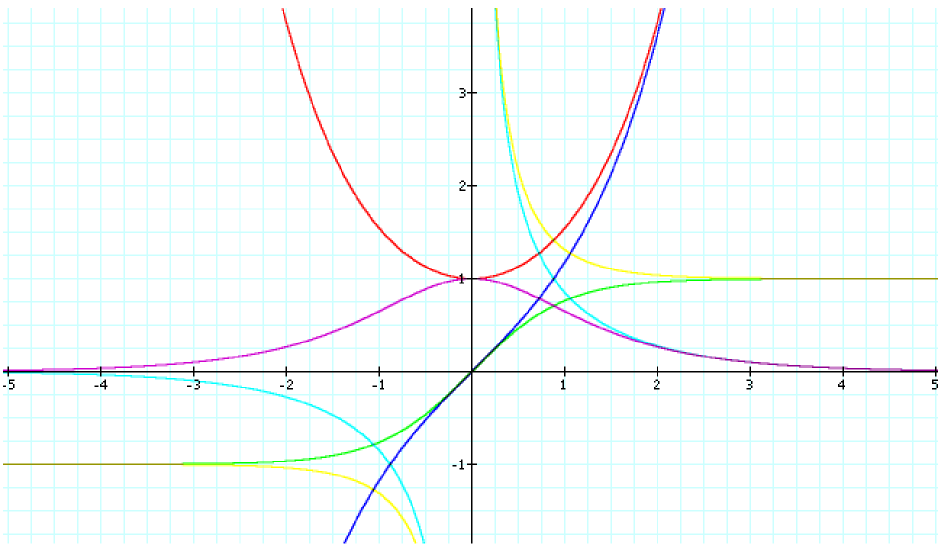
   (a) Find the derivative of the function given by $x \mapsto \sinh x = \dfrac{e^x - e^{-x}}{2}$. Show that the derivative is everywhere (strictly) positive. Infer that sinh is injective. What is the range of this function?

   (b) Use the definition of $\cosh x$ and $\sinh x$ to show that $\cosh^2 x - \sinh^2 x = 1$.

   (c) Use the chain rule to find the derivative of the function determined by $x \mapsto \text{arcsinh}x$.

3. Hughes-Hallett *et al*, 2013. Chapter 3, Section 9, Problem 20.

4. Hughes-Hallett *et al*, 2013. Chapter 4, Section 3, Problem 42.



CHAPTER 1

# QUESTION 1

## 1.1 Questions

Given is the function $f$ determined by $f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$.

(a) Find the natural domain $D$ of the given functional expression.

(b) Look at potential symmetry properties of the function.

(c) Find the roots of the function, that is, the points $x \in D$ where $f(x) = 0$.

(d) Asymptotic analysis: check for the existence of horizontal and vertical asymptotes.

(e) Investigate the behaviour of the first derivative.

(f) Investigate the behaviour of the second derivative.

(g) Collect the results in a table, and determine the type of the critical points.

(h) Finally, sketch the graph of $f$, using the previous table as a guideline.

## 1.2 Solutions

(a) Natural domain $D$ is goverend by $e^x$ and $e^{-x}$. As these can take any real number, $D : x \in \mathbb{R}$.

(b) Symmetrical properties:

(i) Function never dips below $x$ axis, so any symmetry (if present) *must* be about the $y$ axis.

(ii) For vertical symmetry, $f(x) = f(-x)$:

$$f(x) \stackrel{?}{=} f(-x) \tag{1.1}$$
$$\frac{e^x + e^{-x}}{2} \stackrel{?}{=} \frac{e^{-x} + e^x}{2} \tag{1.2}$$
$$e^x + e^{-x} \stackrel{?}{=} e^{-x} + e^x \tag{1.3}$$
$$\tag{1.4}$$

$\therefore f$ *is* symmetrical about the $y$ axis.

(iii) $f(0) = \frac{e^0 + e^{-0}}{2} = 1$, and is likely a minimum because throwing larger values for $x$ in produces larger numbers.

(c) $f(x) = 0$ iff $e^x + e^{-x} = 0$. As both $e^x$ and $e^{-x}$ are strictly positive), they will always sum to a strictly positive number, therefore $f(x) \neq 0$, so no real roots. This ties closely with the previous part, $f(x) \geq 1$, never touching the $x$ axis.

(d) $e^x$ dominates the function as $x \to \pm\infty$, as $x \to +\infty$, $f(x) \to \infty$ even faster. This is also true of $x \to -\infty$ due to the symmetry of $f$, so there are no asymptotes.

(e) First derivative of $\cosh x$:

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{1.5}$$
$$= \frac{1}{2}e^x + e^{-x} \tag{1.6}$$
$$(\cosh x)' = \frac{1}{2}e^x - e^{-x} \tag{1.7}$$

from looking on Wikipedia, this is $\sinh x$

$$(\cosh x)' = \sinh x \tag{1.8}$$

Roots of $\sinh x$

$$0 = \frac{1}{2}e^x - e^{-x} \tag{1.9}$$
$$= e^x - e^{-x} \tag{1.10}$$

Let $x = 0$

$$0 = 1 - 1 \tag{1.11}$$

Therefore one root exists at $x = 0$. As seen in (b)(iii), this is a minimum because of the "larger number test". This can be demonstrated in the second derivative test below.

(f) Second derivative of $\cosh x$.

This is the same as the first derivative of $\sinh x$:

$$\sinh x = \frac{1}{2}e^x - e^{-x} \tag{1.12}$$

$$(\sinh x)' = \frac{e^x + e^{-x}}{2} \tag{1.13}$$

$$= \cosh x \tag{1.14}$$

Ah! Some interesting ring behaviours! With no roots for $(\cosh x)''$, and that $(\cosh x)'' \geq 1$ we can infer that $\cosh x$ has a minimum at $x = 0$ given in (1.11).

(g) Tabulating this sort of stuff in LaTeXreally sucks. Please accept the logical ordering of the investigations in lieu of a table.

(h) Plotting of $\cosh x$ in Mathematica with the following code:

`Plot[Sinh[x], {x,-10,10}, ImageSize → Large]` gives the graph in figure 1.1 on page 6.

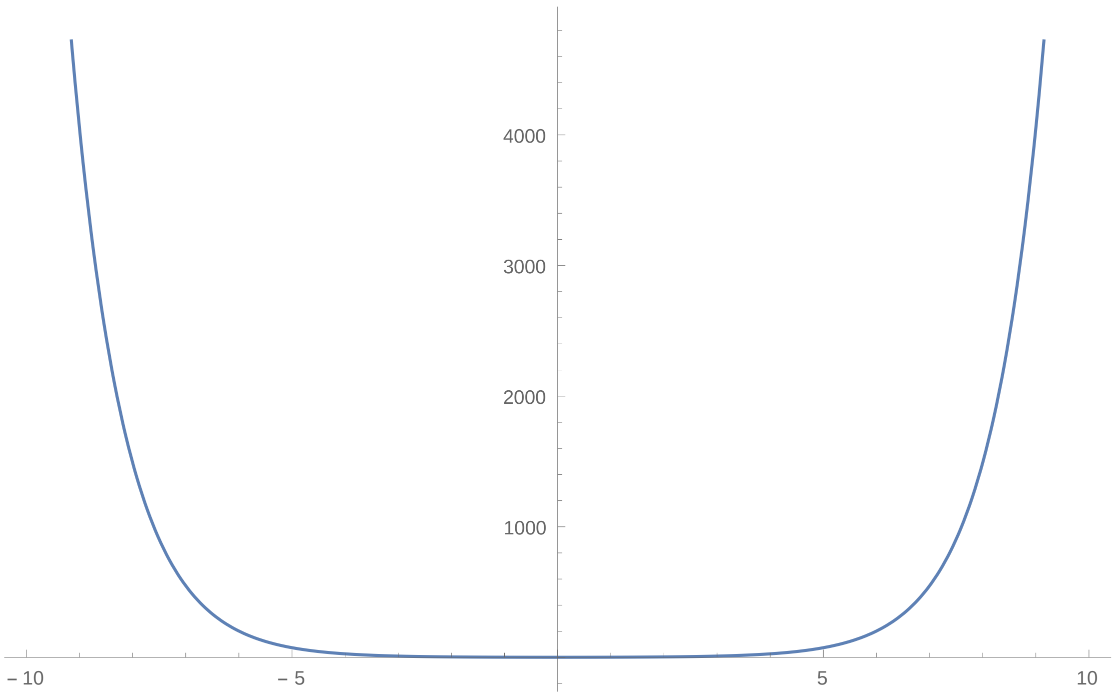



Figure 1.1: Plot of $\cosh x$

CHAPTER 2

# Question 2

## 2.1 Questions

The function arcsinh is the inverse function of sinh.

(a) Find the derivative of the function given by $x \mapsto \sinh = \frac{e^x - e^{-x}}{2}$. Show that the derivative is everywhere (strictly) positive. Infer that sinh is injective. What is the range of this function?

(b) Use the definition of $\cosh x$ and $\sinh x$ to show that $\cosh^2 x - \sinh^2 x = 1$.

(c) Use the chain rule to find the derivative of the function determined by $x \mapsto \operatorname{arcsinh} x$.

## 2.2 Solutions

(a) From question 1, part (f):

$$\sinh x = \frac{1}{2} e^x - e^{-x} \tag{2.1}$$

$$(\sinh x)' = \frac{1}{2} e^x + e^{-x} \tag{2.2}$$

$$= \cosh x \tag{2.3}$$

Also from Q1, part (b) section(iii) and part (c):

Given that $(\sinh x)' = \cosh x = \frac{1}{2} e^x + e^{-x}$ and that both $e^x$ and $e^{-x}$ are strictly positive, they will sum to a strictly positive number. Halving a positive number does not change its sign. Therefore $(\sinh x)'$ is strictly positive. The minimum of $\cosh x$ was shown to be 1 in Question 1, therefore, the derivitive is *always* equal to or greater than 1.

$\sinh x$ is injective because it is one-to-one and can be mapped back from a value in its range back to the original $x$ input by way of its inverse function, $\operatorname{arcsinh} x = \ln(x + \sqrt{1+x^2})$.

"What is the range of this function?" Unsure if asking for range of $\cosh x$ or $\sinh x$, so have both:
Range($\cosh x$): $x \in \mathbb{R} | x \geq 1$.
Range($\sinh x$): $x \in \mathbb{R}$.

(b) Show $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x = (\cosh x)(\cosh x) \tag{2.4}$$
$$\sinh^2 x = (\sinh x)(\sinh x) \tag{2.5}$$

So

$$1 \stackrel{?}{=} \cosh^2 x - \sinh^2 x \tag{2.6}$$
$$\stackrel{?}{=} (\cosh x)(\cosh x) - (\sinh x)(\sinh x) \tag{2.7}$$
$$\stackrel{?}{=} \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x + e^{-x}}{2}\right)^2 \tag{2.8}$$
$$\stackrel{?}{=} \left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right) \tag{2.9}$$
$$4 \stackrel{?}{=} \left(e^{2x} + 2 + e^{-2x}\right) - \left(e^{2x} - 2 + e^{-2x}\right) \tag{2.10}$$
$$\stackrel{?}{=} 2 + 2 \tag{2.11}$$
$$4 = 4 \tag{2.12}$$

LHS = RHS □

(c) Seek to differentiate $\operatorname{arcsinh} x$ using chain rule.

$$(\operatorname{arcsinh} x)' = \left(\ln\left(x + \sqrt{1+x^2}\right)\right)' \tag{2.13}$$

Let

$$f = \ln(x) \tag{2.14}$$
$$g = x + \sqrt{1+x^2} \tag{2.15}$$

Such that

$$f' = \frac{1}{x} \tag{2.16}$$

$$g' = 1 + \left(\sqrt{1+x^2}\right)' \quad \text{Suddenly, a nested chain rule appears!} \tag{2.17}$$

Let

$$h(x) = \sqrt{x} \tag{2.18}$$

$$j(x) = 1 + x^2 \tag{2.19}$$

Such that

$$\left(\sqrt{1+x^2}\right)' = (h(j(x)))' = h'(j(x)) \cdot j'(x) \tag{2.20}$$

$$h'(x) = \frac{1}{2\sqrt{x}} \cdot j'(x) \tag{2.21}$$

$$j'(x) = 2x \tag{2.22}$$

So

$$h'(j(x)) \cdot j'(x) = \frac{1}{2\sqrt{1+x^2}} \cdot 2x \tag{2.23}$$

$$h'(j(x)) \cdot j'(x) = \frac{x}{\sqrt{1+x^2}} \tag{2.24}$$

Apply back in (2.17)

$$g' = 1 + \frac{x}{\sqrt{1+x^2}} \tag{2.25}$$

Apply this to original problem:

$$\left(\ln\left(x+\sqrt{1+x^2}\right)\right)' = f'(g(x)) \cdot g'(x) \tag{2.26}$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{x}{\sqrt{1+x^2}}\right) \tag{2.27}$$

TODO: some magic happens here... I know it happens because Wolfram|Alpha confirmed the above line to equal the next line... which is the final line!

$$= \frac{\left(1+\frac{x}{\sqrt{1+x^2}}\right)}{x+\sqrt{1+x^2}} \qquad = \frac{1}{\sqrt{x^2+1}} \tag{2.28}$$

□

# CHAPTER 3

# QUESTION 3

## 3.1 Questions

Hughes-Hallett *et al*, 2013. Chapter 3, Section 9, Problem 20:

The speed of sound in dry air is: $f(T) = 331.3\sqrt{1+\frac{T}{273.15}}$ meters/second where $T$ is the temperature in degrees Celsius. Find a linear function that approximates the speed of sound for temperatures near 0°C.

## 3.2 Solutions

For a linear approximation, we need to know the gradient near $T = 0$. In this case, the gradient represents the change in speed of sound in metres per second as the temperature changes. A handy point of reference will be to know how the speed of sound at $T = 0$:

$$f(T) = 331.3\sqrt{1+\frac{T}{273.15}} \tag{3.1}$$

$$f(0) = 331.3\sqrt{1+\frac{0}{273.15}} \tag{3.2}$$

$$= 331.3 \tag{3.3}$$

Next we need the gradient at $T = 0$, or, $\frac{df}{dT}$.

$$f(T) = 331.3\sqrt{1 + \frac{T}{273.15}} \tag{3.4}$$

$$\frac{df}{dT} \quad \text{product rule: } u'v + v'u \tag{3.5}$$

Let

$$u = 331.3 \tag{3.6}$$

$$v = \sqrt{1 + \frac{T}{273.15}} \tag{3.7}$$

Such that

$$u' = 0 \tag{3.8}$$

$$v' \quad \text{chain rule, } a'(b(x)) \cdot b'(x) \tag{3.9}$$

Let

$$a(x) = \sqrt{x} \tag{3.10}$$

$$b(x) = 1 + \frac{x}{273.15} \tag{3.11}$$

Such that

$$a'(x) = \frac{1}{2\sqrt{x}} \tag{3.12}$$

$$b'(x) = \frac{1}{273.15} \quad \text{by quotient rule} \tag{3.13}$$

Apply to chain rule from from (3.9)

$$v' = a'(b(T)) \cdot b'(T) \tag{3.14}$$

$$= \frac{1}{2\sqrt{1 + \frac{x}{273.15}}} \cdot \frac{1}{273.15} \tag{3.15}$$

Apply to product rule from (3.5)

$$\frac{df}{dT} = (u'v) + (v'u) \tag{3.16}$$

$$= (0) + \left( \frac{1}{2\sqrt{1 + \frac{T}{273.15}}} \cdot \frac{1}{273.15} \cdot 331.3 \right) \tag{3.17}$$

Evaluate the gradient at $T = 0$ (because it makes a horrible square root go away), and collect like terms

$$= \frac{1}{2} \cdot \frac{331.3}{273.15} \tag{3.18}$$

$$\approx 0.60644\ldots \tag{3.19}$$

We can take the speed at 0°C (from 3.3) and sum with $T$ multiplied by the gradient as a scaling factor:

$$f(T) \approx 331.3 + T \cdot (0.60644) \tag{3.20}$$

Such that the speed of sound in dry air with temperature near 0°C is approximated by:

$$f(T) \approx 331.3 + T \cdot (0.60644) \text{ metres} \cdot \text{s}^{-1}.$$

CHAPTER 4

# QUESTION 4

## 4.1 Questions

Hughes-Hallett *et al*, 2013. Chapter 4, Section 3, Problem 42:

On the same side of a straight river are two towns, and the townspeaople want to build a pumping station, $S$. See figure 4.1. The pumping station is to be at the river's edge with pipes extending straight to the two towns. Where should the pumping station be located to minimize the total length of pipe?

## 4.2 Solutions

County needs to order about 6.4 miles of pipe. An exact value is presented following verbose working to solutions.

S, Town 1 and Town 2 are awful names. Let Town 1 be North Drysdale and Town 2 be South Drysdale. Pumping Station will be named the Richard Nixon Pumphouse[1].

[1] because it's a watergate.

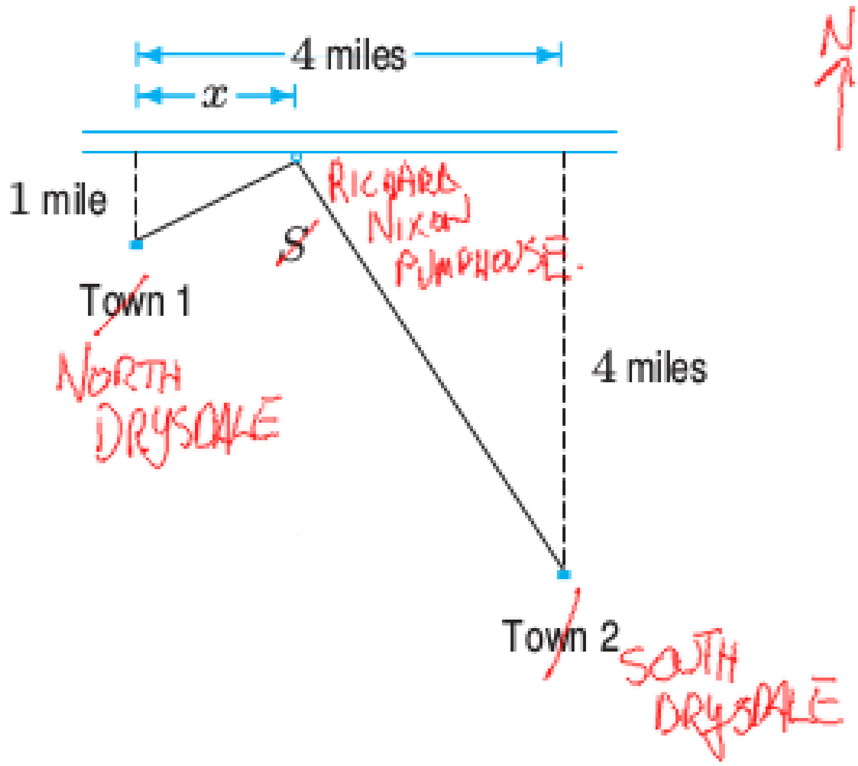



Figure 4.1: Ye olde map

Recognise there are two Pythaorean triads to be formed where the pipe forms hypotenuse of both triangles to the Watergate Pumphouse. As given in the diagram, $x$ is the distance from the intersection of a due North bearing from North Drysdale to the river. This yields two distances.

$$\text{North Drysdale pipe} = \sqrt{1+x^2} \tag{4.1}$$

$$\text{South Drysdale pipe} = \sqrt{4^2+(4-x)^2} \tag{4.2}$$

$$= \sqrt{16+(16-8x+x^2)} \tag{4.3}$$

$$= \sqrt{x^2-8x+32} \tag{4.4}$$

Let Total Pipe be given by the function $p(x)$ such that:

$$p(x) = \text{Total Pipe} \tag{4.5}$$
$$= \text{North Drysdale pipe} + \text{South Drysdale pipe} \tag{4.6}$$
$$= \sqrt{1+x^2} + \sqrt{x^2 - 8x + 32} \tag{4.7}$$

Now we seek global minima of $p(x)$. Using Mathematica, $p(x)$ can be plotted (figure 4.2) using the code:

```
Plot[Sqrt[1 + x^2] + Sqrt[32 - 8 x + x^2], {x, -2, 4},PlotRange → {0,10}, ImageSize
→ Large].
```

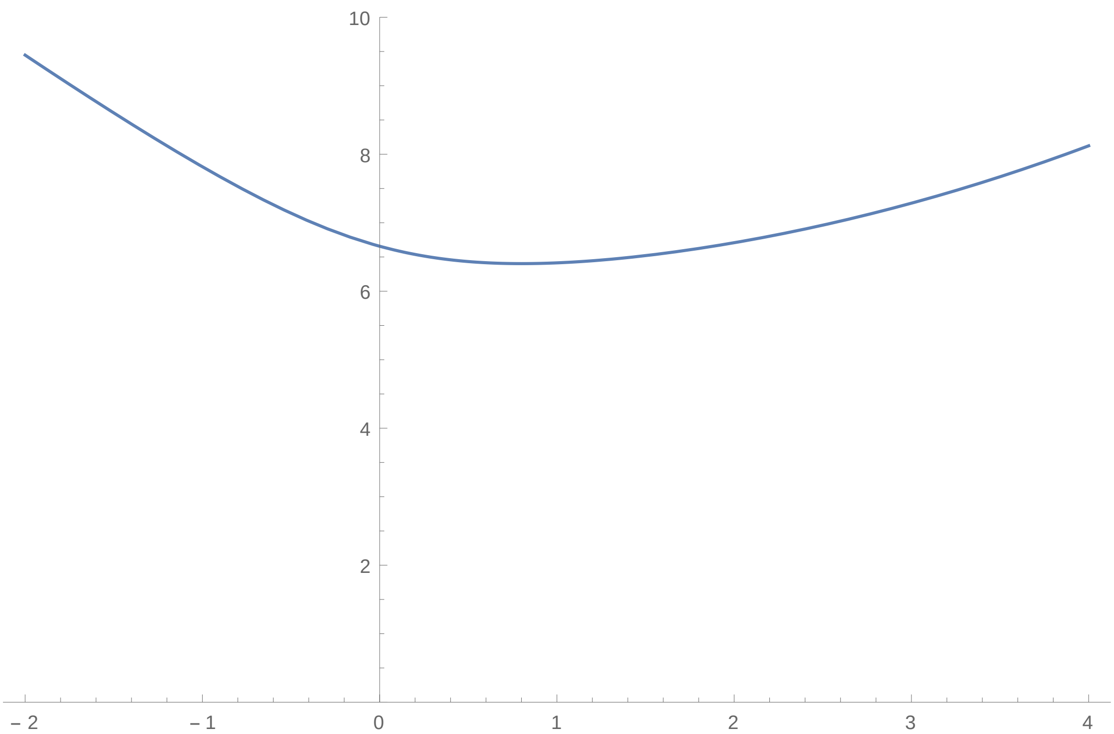



Figure 4.2: Plot of $p(x) = \sqrt{1+x^2} + \sqrt{x^2 - 8x + 32}$

The minima is somewhere near $x = 1$ indicating around 1 mile is good guess, but a more precise answer can be found if we take find the derivative of $p(x)$, and then the roots of $p'(x)$ should give us the exact distance for $x$. From here we can also work out how much pipe is required.[2]

$$\begin{aligned} p(x) &= \sqrt{1+x^2} + \sqrt{x^2-8x+32} \\ p'(x) &= (\sqrt{1+x^2})' + (\sqrt{x^2-8x+32})' \end{aligned} \tag{4.8}$$

Apply chain rule, let $f(x) = \sqrt{x}$ and $g(x) = (1+x^2)$ st

$$(f(x))' = f'(g(x)) \cdot g'(x) \tag{4.9}$$

$$= \frac{1}{2\sqrt{1+x^2}} \cdot 2x \tag{4.10}$$

Apply chain rule, let $f(x) = \sqrt{x}$ and $g(x) = (x^2-8x+32)$ st

$$(f(x))' = f'(g(x)) \cdot g'(x) \tag{4.11}$$

$$= \frac{1}{2\sqrt{x^2-8x+32}} \cdot (2x-8) \tag{4.12}$$

Sum both derivatives

$$p'(x) = \frac{1}{2\sqrt{1+x^2}} \cdot 2x + \frac{1}{2\sqrt{x^2-8x+32}} \cdot (2x-8) \tag{4.13}$$

Collect like terms

$$= \frac{2x}{2\sqrt{1+x^2}} + \frac{2x-8}{2\sqrt{x^2-8x+32}} \tag{4.14}$$

Divide out the twos

$$= \frac{x}{\sqrt{1+x^2}} + \frac{x-4}{\sqrt{x^2-8x+32}} \tag{4.15}$$

Cross multiply to make a single fraction:

$$= \frac{\left(x\sqrt{x^2-8x+32}\right) + \left((x-4)\sqrt{1+x^2}\right)}{\sqrt{1+x^2}\sqrt{x^2-8x+32}} \tag{4.16}$$

[2] Apologies if this appears highly verbose, but I don't understand my own working unless I annotate it.

Since we want the roots, we need to find when $p'(x) = 0$

$$
\begin{aligned}
0 &= \frac{\left(x\sqrt{x^2-8x+32}\right) + \left((x-4)\sqrt{1+x^2}\right)}{\sqrt{1+x^2}\sqrt{x^2-8x+32}} && (4.17)\\
&= \left(x\sqrt{x^2-8x+32}\right) + \left((x-4)\sqrt{1+x^2}\right) && (4.18)\\
\left(x\sqrt{x^2-8x+32}\right) &= -(x-4)\sqrt{1+x^2} && (4.19)\\
\left(x\sqrt{x^2-8x+32}\right)^2 &= \left(-(x-4)\sqrt{1+x^2}\right)^2 && (4.20)\\
(x^2)(x^2-8x+32) &= (-(x-4))^2(1+x^2) && (4.21)\\
x^4-8x^3+32x^2 &= (x^2-8x+16)(1+x^2) && (4.22)\\
&= x^2-8x+16+x^4-8x^3+16x^2 && (4.23)\\
15x^2+8x-16 &= 0 && (4.24)
\end{aligned}
$$

Apply quadratic formula

$$
\begin{aligned}
(3x+4)(5x-4) &= 0 && (4.25)\\
x &= \frac{4}{5} \quad \text{or} \quad \frac{-4}{3} && (4.26)
\end{aligned}
$$

$\frac{4}{5} = 0.8$ miles. The negative solution is discounted because it is west of North Drysdale and South Drysdale is south-east resulting in more pipe than necessary.

Finally, the total amount of pipe necessary can be found substituting $x = \frac{4}{5}$ into the original equation,

$$
\begin{aligned}
p\left(\frac{4}{5}\right) &= \sqrt{1+\frac{4}{5}^2} + \sqrt{\frac{4}{5}^2 - 8\frac{4}{5} + 32} && (4.27)\\
&= \sqrt{41} && (4.28)\\
&\approx 6.4 && (4.29)
\end{aligned}
$$

Therefore, about 6.4 miles of pipe needs to be ordered. □

**The Unique Solution**

In discussion with colleagues, one unique and interesting solution was recognised, however it is rather impractical.