3.5 Rational functions and Asymptotes.

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0}$$

$\deg N = n$

$\deg D = m$

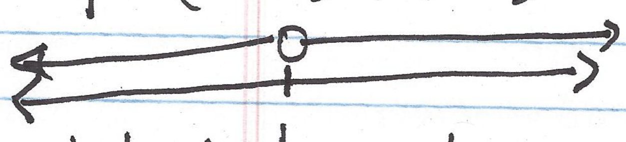
Domain: $D(x) \neq 0$

$$f(x) = \frac{1}{x-1} \qquad D_f = \{x \mid x \neq 1\}.$$

$D_f = (-\infty, 1) \cup (1, \infty)$.

$x - 1 \neq 0$

$x \neq 1$



Vertical Asymptote(s).

line $x = a$ is a Vertical Asymptote

$f(x) \to \pm\infty$ as $x \to a$ or $x \to a^-$ or $x \to a^+$

$f(x) = \frac{1}{x-1}$ $\quad x = 1$ is a Vertical Asymptote?

as $x \to 1^- = 0.999999$ $\quad f(x) = \frac{1}{0^-} = -\infty$

Vertical and Horizontal Asymptotes.

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + \dots + a_1 x + a_0}{b_m x^m + \dots + b_1 x + b_0}$$

Where $N(x)$ and $D(x)$ have no common factors.

1. f has Vertical Asymptotes at the Zeros of $D(x)$

2. f has a horizontal Asymptote.

   if a) $\deg N < \deg D$

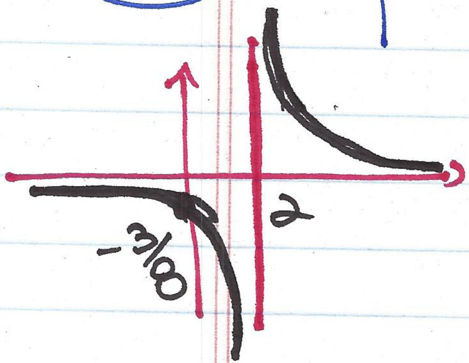
   $y = 0$ is a horizontal Asymptote.

   b) if $\deg N = \deg D$.

   $y = \dfrac{a_n}{b_n}$ is a Horizontal Asymptote.

   c) $\deg N > \deg D$ No Horizontal Asymptote.

#18 $f(x) = \dfrac{3}{(x-2)^3}$ $D_f = (-\infty, 2) \cup (2, \infty)$



as $x \to 2$ $f(x) = \dfrac{3}{0}$

$x = 2$ is VA

as $x \to \infty$ $f(x) \to \dfrac{3}{\infty} = \boxed{0}$ $y = 0$ is Horizontal Asymptotal.

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as $x \to -\infty$ $f(x) \to \dfrac{3}{-\infty} = \boxed{0}$

#20 $f(x) = \dfrac{x^2 - 4x}{x^2 - 4}$ $D_f = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

$x \neq \pm 2$

as $x \to 2$ $f(x) = \dfrac{-4}{0}$

$x = 2$ is VA ✓

as $x \to -2$ $f(x) = \dfrac{12}{0}$ $\nearrow \infty$, $\to -\infty$, $\searrow$ DNE

$x = -2$ is VA

as $x \to \infty$ $f(x) \sim \dfrac{x^2}{x^2} = \boxed{1}$ $y = 1$ is Horizontal Asymptote.

as $x \to -\infty$ $f(x) \sim \dfrac{x^2}{x^2} = \boxed{1}$ $y = 1$ is HA as $x \to \infty$

(22) $f(x) = \dfrac{x^2+2x+1}{2x^2-x-3} = \dfrac{(x+1)^2}{(2x-3)(x+1)}$

$g(x) = \dfrac{x+1}{2x-3}$ $\quad D_f = \{x \mid x \neq \tfrac{3}{2}, -1\}$

as $x \to \tfrac{3}{2}$ $\quad f(x) = \dfrac{25/4}{0}$

$x = \tfrac{3}{2}$ is VA

as $x \to -1$ $\quad f(x) \sim \dfrac{0}{0}$

$\dfrac{3.\cancel{0}}{5.\cancel{0}}$

Nooo_

$x \to -1^-$ $\quad f(x) \sim \dfrac{(x+1)^{\cancel{2}}}{(2x-3)\cancel{(x+1)}} \sim \dfrac{x+1}{(2x-3)}$

$x = -1$ is Hole. $\quad \sim \dfrac{0}{-5} = 0$

The line $y = b$ is a Horizontal Asymptote.

as $x \to \infty$ or $x \to -\infty$.

$f(x) \longrightarrow b$.

(24) $f(x) = \dfrac{3-14x-5x^2}{3+7x+2x^2}$

find any asymptotes and holes:

$$f(x) = \frac{-5x^2-14x+3}{2x^2+7x+3} = \frac{-5x^2-14x+3}{(2x+1)(x+3)}$$

$$D_f = \{x \mid x \neq -\tfrac{1}{2}, -3\}.$$

as $x \to -\frac{1}{2}$ $\quad f(x) \to \dfrac{-\frac{5}{4}+7+3}{0}$

$\boxed{x = -\frac{1}{2}}$ is VA ✓

as $x \to -3$ $\quad f(x) \to \dfrac{-45+42+3}{0}$

at $x=-3$ is Hole.

$$f(x) \sim \frac{\cancel{(x+3)}(-5x+1)}{(2x+1)\cancel{(x+3)}} \sim \boxed{\frac{16}{-5}}$$

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{\cancel{(x+3)}(-5x+1)}{(2x+1)\cancel{(x+3)}} = \boxed{\frac{16}{-5}} \quad \underline{\underline{\text{Hole}}}$$

as $x \to \pm\infty$ $\quad f(x) \sim \dfrac{-5x^2}{2x^2} = -\dfrac{5}{2}$ $\underline{\underline{\text{HA}}}$

(27) $f(x) = \dfrac{x^2+3x-4}{-x^3+27} = \dfrac{(x+4)(x-1)}{(3-x)(9+3x+x^2)}$

$D_f = \{x \mid x \neq 3\}$ $\qquad a^3-b^3 = (a-b)(a^2+ab+b^2)$

irreducible

$x=3$ is VA

f is continuous on $(-\infty,3)\cup(3,\infty)$.

Horizontal Asymptote. $f(x) \sim \dfrac{x^2}{-x^3} = \dfrac{-1}{x}$

as $x \to \pm\infty$ $f(x) \to 0$

$y=0$ is Horizontal Asymptote.

44 $f(x) = \dfrac{2x^2+3x-2}{x^2+x-2} = \dfrac{2x^2+3x-2}{(x+2)(x-1)}$

Find the zeros of $f(x)$

$x=-2$ Hole $\therefore$ Hole $\left(-2, \frac{5}{3}\right)$

$D_f = \{x \mid x \neq -2, 1\}$ ✓

$\lim_{x\to -2} f(x) = \lim_{x\to -2} \dfrac{(2x-1)\cancel{(x+2)}}{\cancel{(x+2)}(x-1)} = \dfrac{-5}{-3} = \boxed{\dfrac{5}{3}}$

$f(x)=0 \therefore \dfrac{N(x)}{D(x)} = 0 \therefore N(x)=0$,

$\boxed{VA \therefore x=1}$ $\boxed{HA \therefore y=2}$ $D(x) \neq 0$ ✓

$f(x)=0 \therefore 2x^2+3x-2=0$

$\boxed{x \neq -2, 1}$ ✓

$(2x-1)(x+2)=0$

$\boxed{x=\frac{1}{2}}$; $\cancel{x=-2}$

Zero ↓ Hole

x-intercept