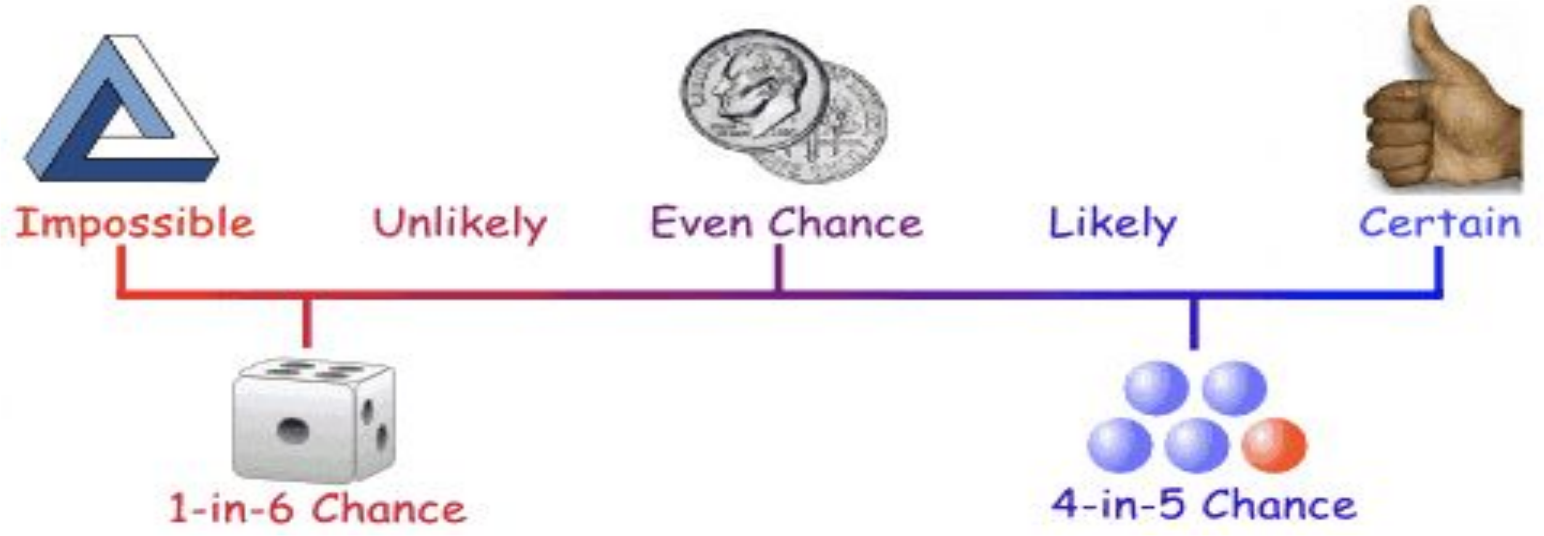
Probability

What is Probability?

Probability Outcome



Subjective Probability

- We associate a real number P(E) between 0 and 1 in a subjective manner to the event E. Numbers near 0 are interpreted as less likely and numbers near 1 are interpreted as highly likely

Random Experiment

- A **random experiment** is an experiment or a process for which the outcome cannot be predicted with certainty.
- The **sample space** (denoted S) of a random experiment is the set of all possible outcomes (**Universal Set**).
- An **event (space)** is a set of outcomes of a random experiment (subset of sample space).

Ellipses

We use ellipses to simplify things:

- {1, 2, 3, …, 10}
- {1, 2, 3, …}
- {…, -2, -1, 0, 1, 2, …}

Random Experiments, Sample Space, Event

Random Experiment	**Sample Space**	**Event**
Toss a coin once	S={H, T}	Head={H}
Toss a coin twice	S={(H,H),(H,T),(T,H),(T,T)}	Exactly one head={(H,T), (T,H)}
Roll a single dice	S={1,2,3,4,5,6}	Outcome Even={2,4,6}
Roll two dice	S={(1,1), (1,2),..., (5,6),(6,6)}	Sum of two dice is five={(1,4), (2,3), (3,2), (4,1)}
Lifetime of a car battery	$S=[0,\infty)$	Battery fails before 12 months = [0,12)

Sample Spaces

- In our example the first four examples are discrete sample spaces while the last is an example of a continuous sample space.
- **Discrete Sample Space:** Outcomes in sample space can be counted (1,2,3,..).
- **Continuous Sample Space:** Outcomes in sample space are cannot be counted and lie in an interval.

Updated Definition of Probability

$$P(A) = \frac{Event\ outcomes\ favorable\ to\ A}{Sample\ space}$$

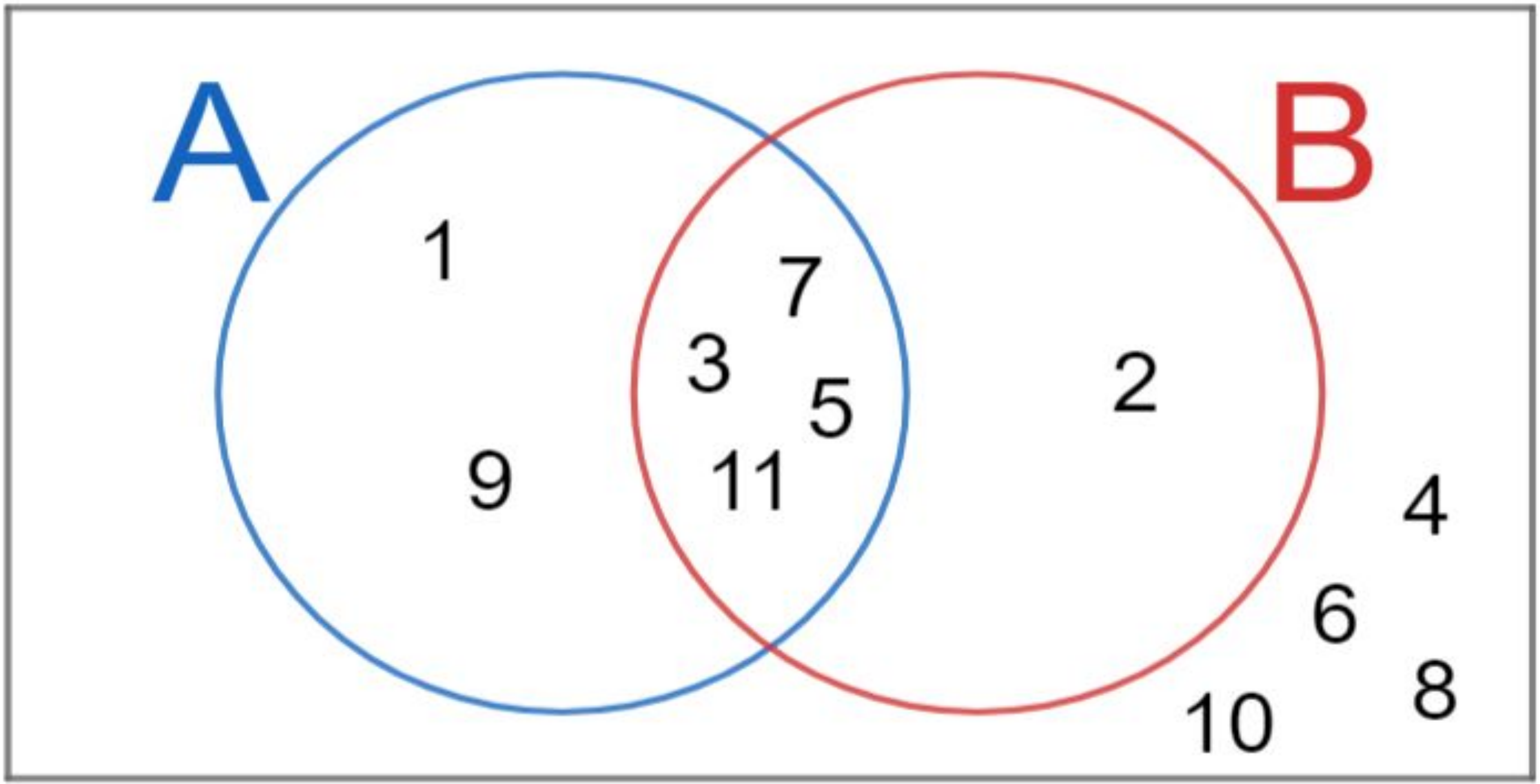
Set Theory

- A **Set** is a collection of objects called elements or members of the set.
  - S = {Hello, Hi, Hola}
  - A = {1,2,3,4,5,9}
- We relate an item in the set with the set using the "∈ (element of)" relation.
  - Hi $\in$ S.
  - 1 $\in$ A.

Venn Diagram

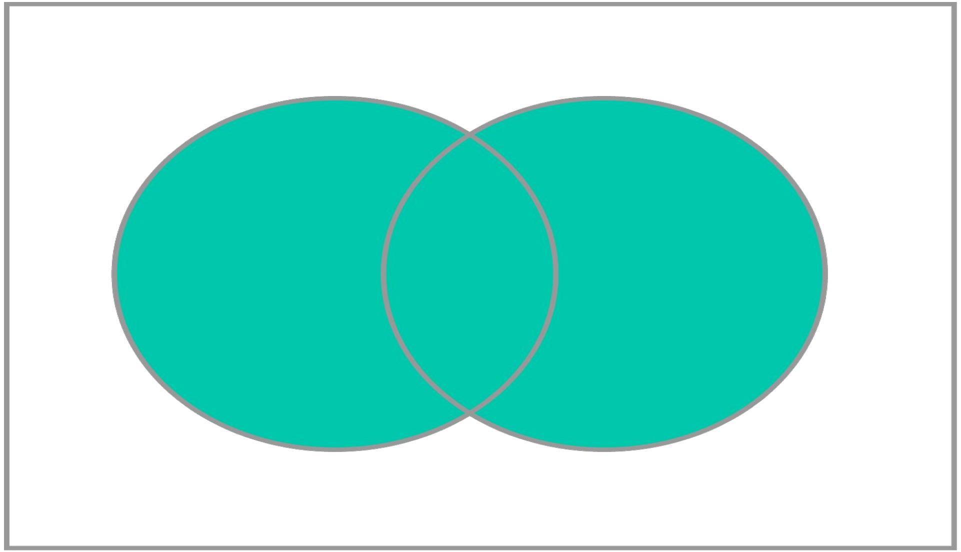
- A diagram that shows all possible logical relations between a finite collection of different sets.
- Depict elements as points in the plane, and sets as regions inside closed curves.
- Consists of multiple overlapping closed curves, usually circles, each representing a set.

Venn Diagram Example



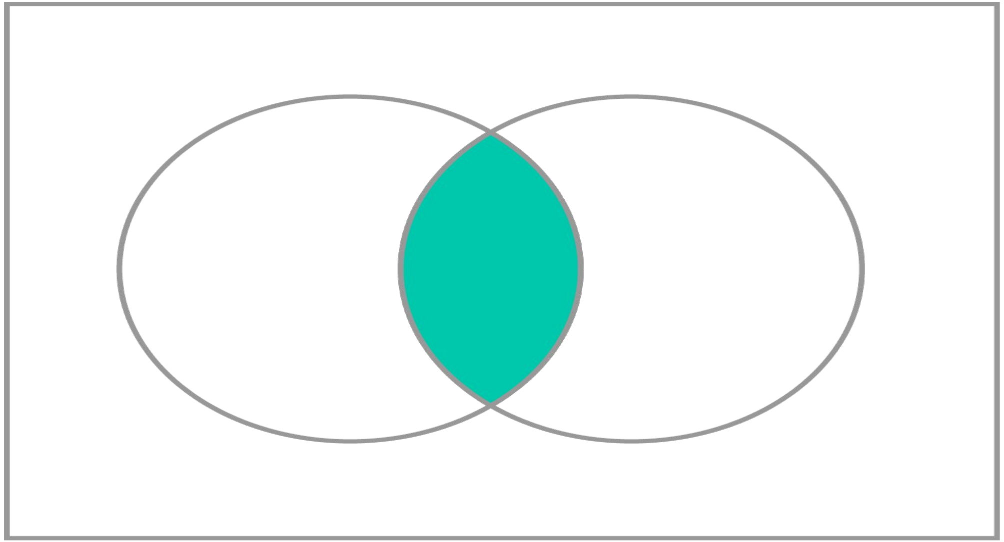
Union of Sets

- The **union** of two events A and B (denoted A U B) is the event consiting of all outcomes that belong to A or B or both.



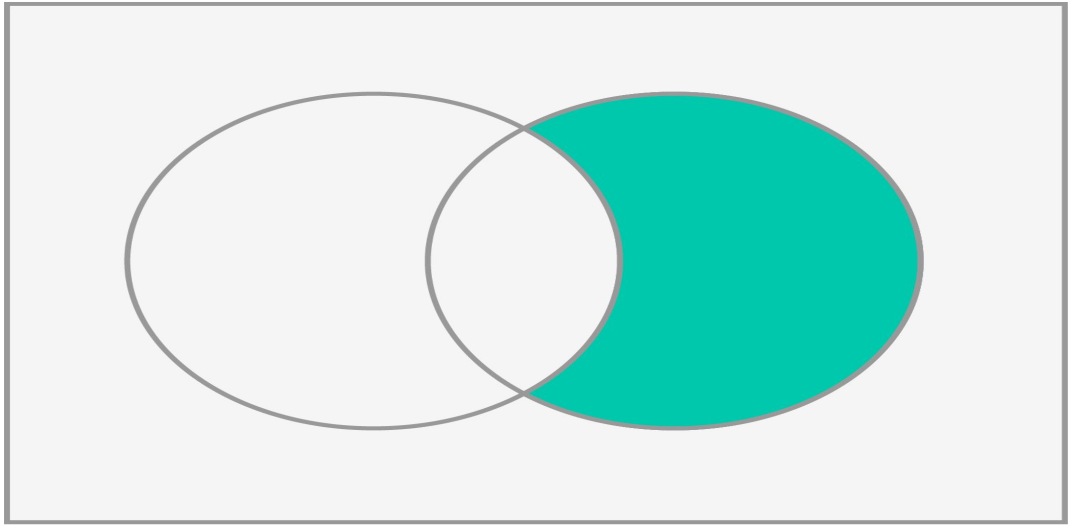
Intersection

- The **intersection** of two events A and B (denoted A ∩ B) is the even consisting of all outcomes common to both A and B



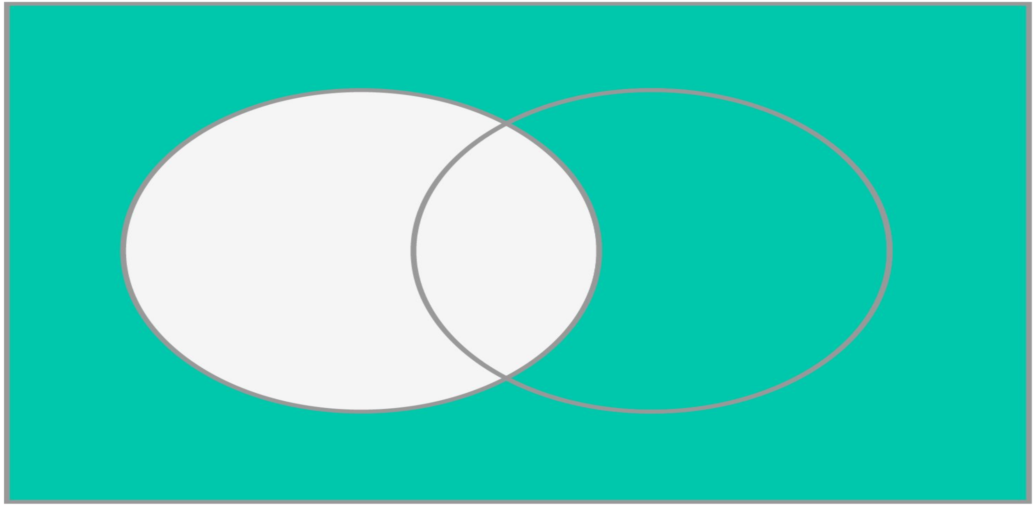
Difference(Relative Complement)

- The **difference**(relative complement) of A with respect to a set B, written B \ A, is the set of elements in B but not in A.

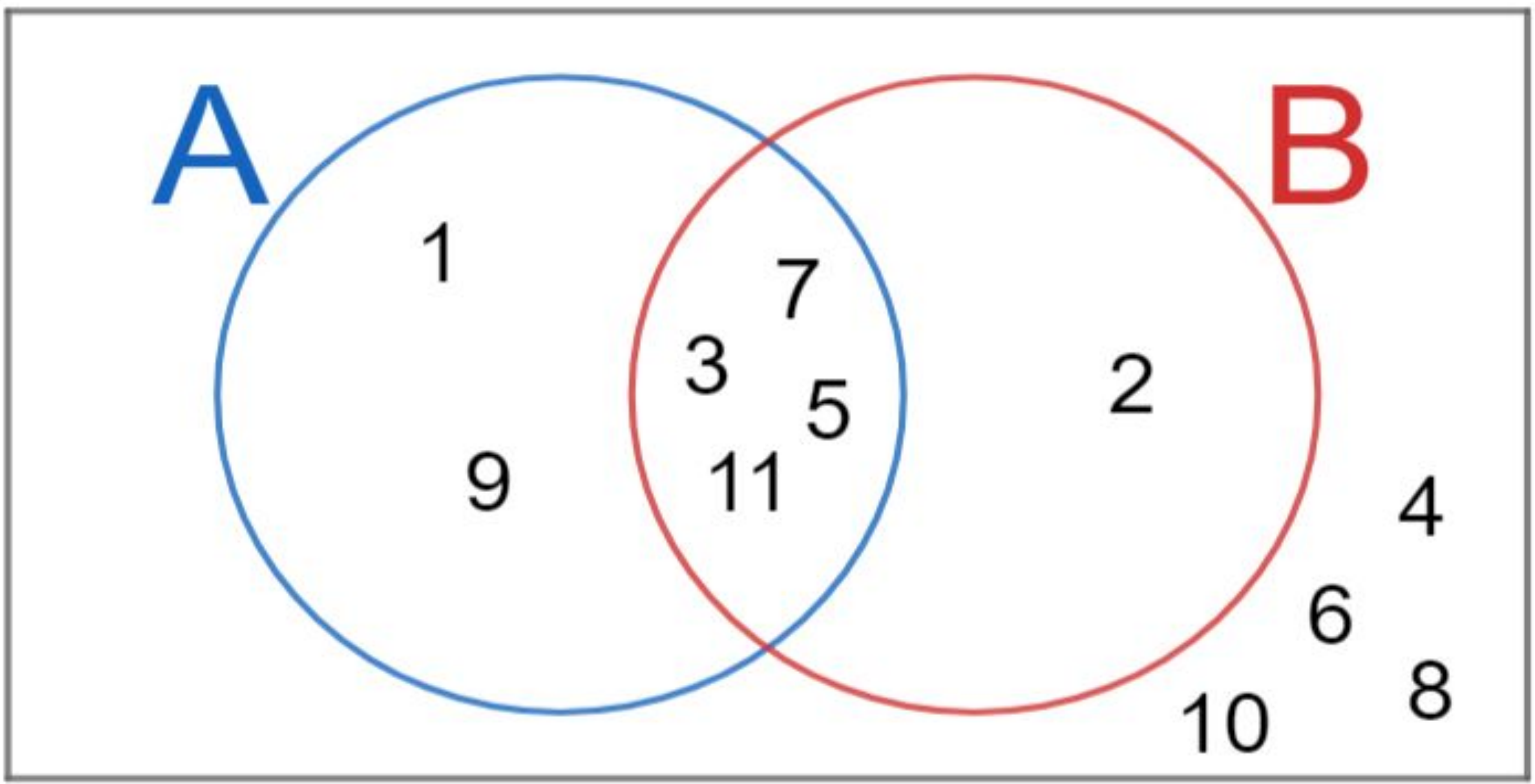


Absolute Complement

- The **absolute complement** of A (or simply the complement of A) is the set of elements not in A.



Venn Diagram Example

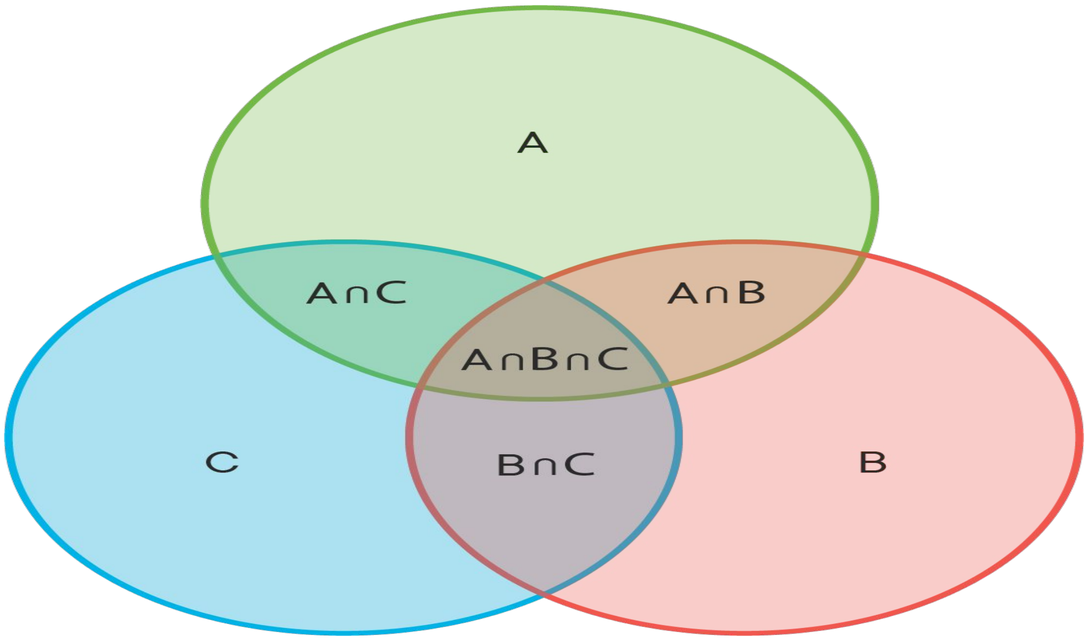


In-Class Problems

- Find the union of A and B
- Find the intersection of A and B
- Find the difference(relative complement) of A with respect to a set B.
- Find the absolute complement of A

Inclusion Exclusion Principle

- The name comes from the idea that the principle is based on over-generous inclusion, followed by compensating exclusion.



Example Inclusion Exclusion

- In a recent survey on pet ownership 40 had a dog, 60 had a cat, and 50 had a bird.
- 25 owned a dog and a cat, 30 owned a cat and a bird, and 35 owned a dog and a bird. In the survey 10 households had all three pets.
- **How many households had at least one of the three?**

Axioms of Probability

- Positivity

$$\mathbf{0 \leq \boldsymbol{P(A)} \leq 1}$$

- Probability of Sample Space

$$\boldsymbol{P(S) = 1}$$

- Additivity

$$if\ A \cap B = \emptyset, then\ P(A \cup B) = P(A) + P(B)$$

Addition law of probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Relative Frequency

$$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$$

Permutation

- An ordered arrangement of distinct terms. The total number of permutations of n distinct items is given by:

$$n(n-1)\cdots(2)(1) = n!$$

- Where the symbol n! is read as “n factorial” and “0!” is defined to be 1.

Example

- An ATM card requires a 4 digit pin how many different pins are possible?
- How many different social security personal identification numbers are possible?
- How many different 7-place license plates are possible if the first two place are letters and the other five are for numbers?

Permutation (without repetition)

- The number of permutations of k items out of n distinct items is:

$$P_k^n = \frac{n!}{(n-k)!}$$

Combination

- The number of unordered arrangements of k items out of n is:

$$\binom{N}{k} = \frac{P_k^n}{k!} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)!k!}$$

- The extra k! accounts for the fact that combinations do not distinguish between the different orders that the k objects can appear in. We are just selecting (or choosing) the k objects, not arranging them. (Think Baskin-Robbins)
- Order of selection is not considered relevant

Example

- For the letters ABC
- How many combinations of two letters are there?
- How many two letter “words” ways can we make from ABC?

More interesting Example

A 5-card poker hand is a full house if it consists of 3 cards of the same denomination and 2 cards of the same denomination (That is a full house is three of a kind plus a pair). What is the **probability** of a full house?

Permutation (with repetition)

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Example

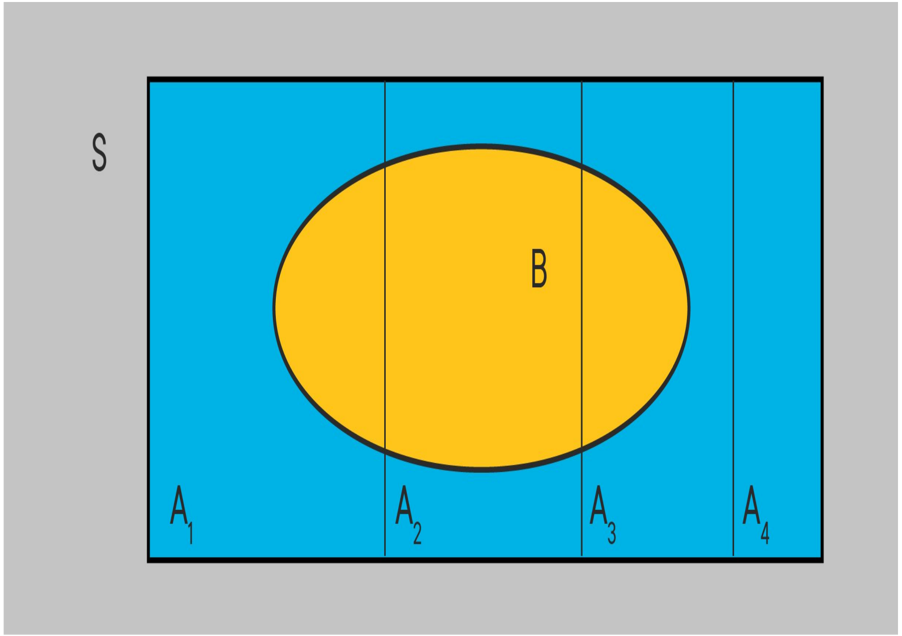
How many different letter arrangements can be formed using the letters Pepper?

Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Law of Total Probability

$$
\begin{aligned}
P(B) &= P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) \\
&\quad + P(B \cap A_4) \\
&= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) \\
&\quad + P(B \mid A_3)P(A_3) + P(B \mid A_4)P(A_4)
\end{aligned}
$$



Example

In a certain county, 60% of registered voters are Republicans, 30% are Democrats and 10% are Independents.

When those voters were asked about increasing military spending

40% of Republicans opposed it

65% of the Democrats opposed it

55% of the Independents opposed it.

What is the probability that a randomly selected voter in this county opposes increased military spending?

Example Cont

R = {registered republicans}, $P(R) = 0.6$

D = {registered democrats}, $P(D) = 0.3$

I= {registered independents}, $P(I) = 0.1$

B = {registered voters opposing increased military spending}

Example Continued

You also know that:

$P(B \mid R) = 0.4$

$P(B \mid D) = 0.65$

$P(B \mid I) = 0.55$

Using Total Probability

$$Pr(B) = Pr(B \mid R)Pr(R) + Pr(B \mid D)Pr(D) + Pr(B \mid I)Pr(I)$$
$$= (0.4 * 0.6) + (0.65 * 0.3) + (0.55 * 0.1) = 0.49$$

Questions