CS103X Handout 11
Autumn 2003 October 10, 2003

Introduction to Induction

A Childhood Analogy

Imagine a semi-infinite line of dominoes, each precariously standing so that the toppling of any one of them is sure to propagate into the toppling of several (perhaps an infinite number of) other dominoes. In fact, our stream of dominoes is such that tipping the first domino (that which is next to only one other domino, not two) will tip the second, which in turn will topple the third, and so forth, so that eventually, each and every one of them will plummet to its death. But—how do we know that every domino will fall if we don't have an infinite amount of time to confirm it? Of course, we **know** (even children would believe this), but **why** do we know?

Now imagine that it's 1995 (sixth grade for most of you) and that your mathematics teacher (mine was Miss Lattimore—very strange woman) spray paints some formula on the blackboard which looks like this:

$$\sum_{1 \le j \le n} j = \frac{n(n+1)}{2}$$

Clearly this is true for $n = 1$, right? But is it true for $n = 1573$? for $n = 1996$? for $n = 2^{1000}$? Well, even if you've never seen this formula before, experimenting with small values of n certainly makes it look like it is true.

$$n = 1 \qquad \sum_{1 \le j \le 1} j = 1 = 1 = \frac{1(1+1)}{2} = \frac{n(n+1)}{2}$$

$$n = 2 \qquad \sum_{1 \le j \le 2} j = 1 + 2 = 3 = \frac{2(2+1)}{2} = \frac{n(n+1)}{2}$$

$$n = 3 \qquad \sum_{1 \le j \le 3} j = 1 + 2 + 3 = 6 = \frac{3(3+1)}{2} = \frac{n(n+1)}{2}$$

$$n = 4 \qquad \sum_{1 \le j \le 4} j = 1 + 2 + 3 + 4 = 10 = \frac{4(4+1)}{2} = \frac{n(n+1)}{2}$$

Perhaps the most magical of observations: look at the difference between $\sum_{1 \le j \le m} j$ and $\sum_{1 \le j \le m-1} j$ — in fact, look at this difference from two perspectives.

$$\text{Way 1:} \quad \sum_{1 \le j \le m} j - \sum_{1 \le j \le m-1} j = \sum_{m \le j \le m} j = m$$

$$\text{Way 2:} \quad \frac{m(m+1)}{2} - \frac{(m-1)((m-1)+1)}{2} = (m+1)\frac{m}{2} - (m-1)\frac{m}{2} = 2\,\frac{m}{2} = m$$

I think the second statement above is a touch more enlightening. Regardless of the value of m, it seems that the difference between the two closed formulas is always m. Is this just a cool coincidence? Or does it support Miss Lattimore's claim that $\sum_{1 \le j \le n} j = \frac{n(n+1)}{2}$? We'll see, young student… we'll see.

Relevance to CS?

All right, all right.. pipe down.. it's coming.

Finally, consider the following procedure (written here in C) which **presumably** shuffles a deck of 52 cards. Forgive the intrusion of real C code, but I think seeing the code is a refreshing reminder that this is in fact a computer science class. The following implementation is largely grabbed from Don Knuth's <u>Searching and Sorting</u> textbook. For those of you who haven't programmed yet, don't worry about this. I'll explain the general sorting algorithm in lecture outside the programming domain, so you'll be able to follow this example there.

```
typedef enum {
    Heart, Diamond, Spade, Club
} suit;

enum {
    Ace = 1, Jack = 11, Queen, King
}; // only the integer binding is important, so there's no need for a new datatype

typedef struct {
    int cardRank;
    suit cardSuit;
} card;

void main(void) {
    card deck[52];
    int cardIndex;
    int cardToSwap;

    for (cardIndex = 0; cardIndex < 52; cardIndex++) {
        deck[cardIndex].cardRank = (cardIndex % 13) + 1;
        deck[cardIndex].cardSuit = cardIndex / 12;
    }

    // we have a brand new, unshuffled deck

    Randomize();
    for (cardIndex = 0; cardIndex < 52; cardIndex++) {
        cardToSwap = RandomInteger(cardIndex, 51);      // assume this is written
        SwapCards(deck + cardIndex, deck + cardToSwap); // assume this is written, too
    }

    // the question of the page: is any one of the 52! different shuffled decks equally
    // likely to be generated? Put another way, is the Ace of Spades (or any other
    // card, for that matter) likely to end up in any of the 52 possible positions
    // with equal probability? Not quite as obvious as our domino problem, huh?
}
```

The last inline comment of the above program presents a very interesting question: we surmise that the shuffle algorithm does its job, but can we **prove** to someone who needs a shuffling routine for her blackjack game that this tiny little `for` loop really does its job correctly? And can we **prove** that it does its job for 52 cards, as well as 104, 208, 416, 832, or 52^{100} cards? Bottom line: does it really shuffle the deck regardless of what the deck looks like?

Perhaps you see how each of the three problems are similar, perhaps not. The intrinsic feature that each of these statements share is that each is a statement about some set of objects—a stream of dominoes, the set of positive integers, a deck of cards. Even more crucial is that each collection can be recursively defined: an array of cards is defined as a card followed by another array of cards; a line of dominoes is defined as a single domino aside another line of dominoes. You see where this is going? <you politely nod yes, even if it's not quite clear yet>

Good.

The Principle of Mathematical Induction

The principle of mathematical induction is used to prove statement about sets of objects—most often the set of integers, but not always. As we'll see in this and forthcoming handouts, induction can be used to prove results about a large variety of discrete objects—the complexity of algorithms, the correctness of sorting, searching, or shuffling routines, theorems about graphs, trees, positive integers, prime numbers, lists, and so forth. Its only shortcoming is that it is rarely, if at all, used to **discover** new identities or theorems; rather, induction is used to prove that some speculation (obtained some other way—perhaps by testing the validity of some algebraic identity by examining small cases… $n = 1$, $n = 2$, $n = 3$, and $n = 4$) is true.

For example, many algebraic identities state that $P(n)$ is true for all positive integers n, where $P(n)$ is some shorthand notation for a propositional statement about some integer n. **Mathematical induction is a technique for proving theorems of this kind.** In fact, I can generalize this a tad. Induction should be seriously considered as a method of proof whenever the statement to be proved is of the form:

For all positive integers n, something involving n is true.

In particular, when considering mathematical induction, the key phrases to look for are "**for all**" and "**integers greater than**".

A proof which uses mathematical induction to prove that $P(n)$ is true for every positive integer n consists of two crucial subproofs:

1. **A Proof That The Base Case Holds**: Showing that $P(1)$ is true.
2. **A Proof That The Induction Step Holds**: Showing the implication $P(k) \Rightarrow P(k+1)$ is true for all positive integers k. Restated, proving that the truth of some property for an integer k **implies** that truth of that same property for $k + 1$.

Here, $P(k)$ is called the **inductive hypothesis**. When we complete both steps of a proof by mathematical induction, we have shown that $P(n)$ is true for all positive integers n.

Since mathematical induction is such an important technique, it is worthwhile to gab in detail about the steps of a proof using this technique. The first step is to show that P(1) is true, which amounts to nothing more than replacing n by 1 and showing that the statement which results from the substitution is valid. Then we must show that $P(k) \Rightarrow P(k + 1)$ is true for every positive integer k. That $P(k) \Rightarrow P(k + 1)$ is true can be shown **by assuming that P(k) is true** and, under that assumption, showing that P(k + 1) must also be true.

I'll make it a point to make the examples that follow of publishable quality, but when reading through the examples, really focus on the mathematical details involved in the two crucial subproofs. More often than not, the math involved in establishing a base case is trivial; albeit, the mathematics needed to prove the induction step to be valid is usually more complex. You'll also notice that my examples are primped and polished to begin with a restatement of the statement to be proved, and that it usually ends with some touchy-fuzzy-feel-really-good remark like "By the principle of mathematical induction we see that …`blah blah blah`." These opening and closing remarks are more syntactic sugar than is really necessary, but I'm going to initially require that you be as formal as possible in your presentation of induction proofs, and I'll relax that requirement after I'm sure y'all understand how induction works.

Let's tackle Miss Lattimore's math identity and prove that, for all positive integers n,

$$\sum_{1 \le j \le n} j = \frac{n(n+1)}{2}$$

Affix your thinking cap and let's go for it. The next pages provide a few examples. Examples are a good thing, so we'll look at a lot of them.

Proposition: Prove that the sum of the first n positive integers is given by n(n+1)/2. Stated mathematically, prove that $\sum_{1 \le j \le n} j = \frac{n(n+1)}{2}$.

Let P(n) denote the fact that for any positive integer n, $\sum_{1 \le j \le n} j = \frac{n(n+1)}{2}$.

Always restate what you are trying to prove in the form P(n).

P(1): $\sum_{1 \le j \le 1} j = 1$, clearly.

$\frac{1(1+1)}{2} = \frac{2}{2} = 1$, as well.

Therefore, P(1) is true, and the base case holds.

P(k)$\Rightarrow$ P(k+1): Assume that P(k) is true—that is, assume that $\sum_{1 \le j \le k} j = \frac{k(k+1)}{2}$. We can massage this statement by adding k + 1 to each side of the equality, and look what happens:

$$\sum_{1 \le j \le k} j + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

See, this isn't all that much math, but it's definitely the heart of the proof.

$$\sum_{1 \le j \le k+1} j = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= k\frac{(k+1)}{2} + 2\frac{(k+1)}{2}$$

$$= (k+2)\frac{(k+1)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

Therefore, $\sum_{1 \le j \le k} j = \frac{k(k+1)}{2}$ implies that $\sum_{1 \le j \le k+1} j = \frac{(k+1)((k+1)+1)}{2}$; that is, P(k) $\Rightarrow$ P(k+1). Therefore, by the principle of mathematical induction, we have shown that P(n) is true for all n—specifically, that $\sum_{1 \le j \le n} j = \frac{n(n+1)}{2}$. Cool! Now let's move on to more creative induction examples.

An Example

Proposition: Prove that for every positive integer n,

$$\sum_{1 \le j \le n} j(j!) = (n+1)! - 1$$

Let P(n) denote the proposition that $\sum_{1 \le j \le n} j(j!) = (n+1)! - 1$.

Base case: Substitute 1 for n in P(n), and prove that P(1) is true.

$$\sum_{1 \le j \le 1} j(j!) = 1(1!) = 1 \cdot 1 = 1, \text{ and } (1+1)! - 1 = 2 \cdot 1 - 1 = 2 - 1 = 1.$$

Left side equals right side when n = 1, so our base case certainly holds.

Induction: As usual, assume that P(k) is true—namely, assume that $\sum_{1 \le j \le k} j(j!) = (k+1)! - 1$.

As usual, we manipulate the inductive hypothesis (which we are assuming to be true) just enough so that we arrive at the truth of P(k+1).

$\sum_{1 \le j \le k} j(j!) = (k+1)! - 1$ is assumed to be true, and we notice that the left-hand side differs from the left-hand side of our goal by one term—namely, $(k+1)(k+1)!$. Here's the sequence of mathematical steps connecting P(k) to P(k+1):

$$\sum_{1\le j\le k} j(j!) = (k+1)! - 1$$

$$\sum_{1\le j\le k} j(j!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$$

$$\sum_{1\le j\le k+1} j(j!) = (k+1)! + (k+1)(k+1)! - 1$$

$$\sum_{1\le j\le k+1} j(j!) = ((k+1)+1)(k+1)! - 1$$

$$= (k+2)(k+1)! - 1 = (k+2)! - 1$$

Amazing how that just kind of works, isn't it? **By assuming the truth of P(k)** and working a little mathematical magic (with the help of Mickey the Rodent), we see that P(k+1) can be shown to be true as well.

Finally, since we have shown that P(1) is true and that P(k+1) is true whenever P(k) is true, we have shown that P(n) is true for all n greater than or equal to 1.

An Example

Proposition: Every convex polygon with $n \geq 3$ vertices has exactly $\frac{n(n-3)}{2}$ diagonals.

You shouldn't be discouraged that our statement concerns only those integers greater than or equal to 3. Our dominoes analogy simply suggests that we explicitly tip the third domino toward the fourth, and rest assured that if a domino can be shown to tip its neighbor, all dominoes from numero 3 onward will certainly topple. Let's begin, as usual, by formulating P(n):

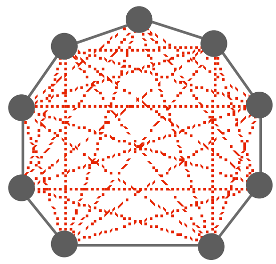
Let P(n) denote that fact that any polygon with n vertices, where $n \geq 3$, has exactly $\frac{n(n-3)}{2}$ diagonals.

P(3): Well, when a convex polygon has exactly **three** vertices, that convex polygon is a triangle. Clearly the number of diagonals in this case is the big fat zero, and that's exactly what our proposed formula tells us:

$$\left.\frac{n(n-3)}{2}\right|_{n=3} = \frac{3(3-3)}{2} = 0$$

That means that our base case holds.

P(k)$\Rightarrow$ P(k+1): As always, we assume that P(k) is true for k greater than or equal to 3. We want to see if the truth of P(k) mandates the truth of P(k+1). We begin by drawing a convex polygon with k vertices:

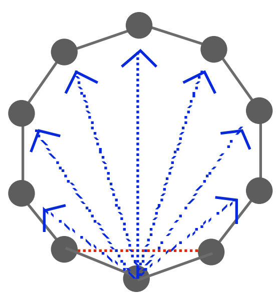


Imagine that the convex polygon drawn to the left sports k vertices, not just 9. Everything in my argument will concern a polygon of k vertices, but it's hard to draw such a polygon.

How many diagonals does this k-vertex convex polygon have? Well, the induction hypothesis tells us that the number diagonals—or, as the drawing puts it, the number of dashed lines—is given by:

$$\frac{k(k-3)}{2}$$

So what happens when we add a (k+1)st vertex? We need a new picture (where I omit all of the previously drawn diagonals to spare us the clutter):



First off, one of the previously drawn edges (an edge is a fancy word for a line that connects two adjacent vertices) has become a diagonal for a new and improved convex polygon. Also note that our brand new vertex forms a diagonal with all but two of the previously existing vertices (the two directly on either side are excluded). That's $1+(k-2)=k-1$ new diagonals on top of the ones that were previously present. Therefore, the new number of diagonals for this (k+1)-vertex convex polygon is given as:

$$\frac{k(k-3)}{2}+(k-1)=\frac{k^2-3k}{2}+\frac{2k-2}{2}=$$
$$\frac{k^2-k-2}{2}=\frac{(k+1)(k-2)}{2}=\frac{(k+1)((k+1)-3)}{2}$$

Ach, und Himmel! That what P(k+1) says it should be! So, it appears that the truth of P(k) implies the truth of P(k+1). You know that that means:

By the principle of mathematical induction, P(n) is true for all integers n greater than or equal to 3.

Why does induction work?

Most of you probably see why induction works, but some of you probably don't, so it's certainly not a bad idea to at least formalize our understanding of why induction is valid. Formally, the validity of mathematical induction follows from the following fundamental axiom about the set of integers.

The Well-Ordering Property: Every nonempty set of nonnegative integers has a least element. Sounds kind of obvious, no? Well, the fact that it is obvious pretty much explains why it's an **axiom** and not a **theorem**.

Suppose we know that P(1) is true and that the implication $P(k) \rightarrow P(k+1)$ is true for all positive integers k. In order to prove then that P(n) is true for all positive integers n, we simply assume that there exists some positive integer b for which P(b) is false. (We're using a **proof** by **contradiction** argument here to explain why induction works. You'll see in a second.)

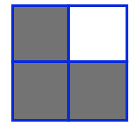
So we're assuming that there is an integer b such that P(b) is false—not true, a fib, one huge lie. Well, that means that there is a nonempty set of positive integers for which P(n) is false, and that this particular set of integers (just like all nonempty sets) must contain a least element. (Hey! The Well-Ordering Principle says so.) So we simply assign a name to this least element—say we call it s (for **s**mallest). We know that s cannot possibly be 1, because we know for sure that P(1) is true. Since s is positive and greater than 1, s - 1 is a positive integer. Furthermore, since s - 1 is less than s, we must have that **P(s-1) is true**, since s itself is the least such integer for which P(n) is false.

But! We know that $P(k) \rightarrow P(k+1)$ is true for all positive integers k—in particular, for k = s - 1. But we know that P(k+1) = P(s) is false by assumption, so the only way that the implication $P(k) \rightarrow P(k+1)$ can be true when P(k+1) is false is for **P(k) = P(s - 1) itself to be false**.

Whoa! Wait just a cotton-picking minute there! From one angle we are **assured** that P(s-1) is **true**, and from another angle we are told that P(s-1) is **false**. I'd call that a contradiction! So the assumption that P(s) is false must be incorrect, since it was the only assumption we made and, therefore, **must** be the source of the contradiction. **Ergo, P(n) must be true for all positive integers n**. Wow! Even Pooh rolls his head back in disbelief!

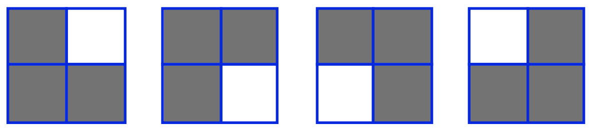
An Example

Proposition: Let n be a positive integer. Show that any 2^n by 2^n checkerboard, with one square removed, can be tiled using L-shaped pieces, where each such piece covers three squares, as shown right here:



Let P(n) be the proposition that any 2^n by 2^n checkerboard with one square removed can be tiled using L-shaped pieces. We can use mathematical induction to prove that P(n) is true for all integers greater than or equal to 1.

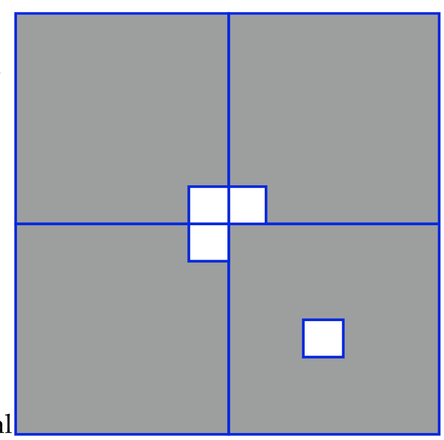
P(1): Clearly any 2 by 2 checkerboard can be tiled using only one L-shape. We are free to rotate the L-shape any which way we like, and each of the four different checkerboards can be tiled like-a so:



Therefore, P(1) is true, so our base case holds.

P(k)→ P(k+1): Assume that P(k) is true; that is, assume that **ANY** (note the emphasis) 2^k by 2^k checkerboard with one square removed can be tiled using L-shaped pieces. It must be shown that under this assumption that P(k + 1) is also true; that is, any 2^{k+1} by 2^{k+1} checkerboard with one square removed can be tiled using L-shaped pieces.

To see this, consider a 2^{k+1} by 2^{k+1} checkerboard with one square removed. Split this checkerboard into 4 checkerboards of size 2^k by 2^k by dividing it in half in both directions. Notice that no square has been removed from three of the four checkerboards, but the fourth checkerboard does have one square removed, so by the induction hypothesis, it can be covered with our infamous L-shaped pieces.



Now temporarily remove the square from each of the other three boards that has the center of the original, undivided checkerboard as one of its corners, as shown to the right. By the inductive hypothesis, each of these three checkerboards can also be tiled using our L-shaped pieces. Furthermore, the three squares that were temporarily removed can now be covered by one additional L-shaped piece.

So, by the principle of mathematical induction, we have shown that P(n) is true for all positive integers n.

An Example

Prove that the following identity about Harmonic numbers holds for all nonnegative integers n:

$$\sum_{0 \le k < n} kH_k = \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4}$$

Let $P(n)$ represent the belief that $\sum_{0 \le k < n} kH_k = \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4}$ for all nonnegative integers n. We prove our conjecture using induction by first showing that $P(0)$ holds and that $P(m+1)$ holds whenever $P(m)$ holds.

First, prove P(0):

That's actually quite easy, because both sides are zero when n is 0. Therefore P(0) holds.

Prove that $P(m) \Rightarrow P(m+1)$ holds provided that m is nonnegative.

Assume that $P(m)$ is true—that is, assume that

$$\sum_{0 \le k < m} kH_k = \frac{m(m-1)}{2} H_m - \frac{m(m-1)}{4}$$

It then follows that

$$\begin{aligned}
\sum_{0 \le k < m+1} kH_k &= \sum_{0 \le k < m} kH_k + mH_m \\
&= \frac{m(m-1)}{2} H_m - \frac{m(m-1)}{4} + mH_m \quad (\text{because } P(m) \text{ is true}) \\
&= \frac{m(m+1)}{2} H_m - \frac{m(m-1)}{4} \\
&= \frac{m(m+1)}{2} \left(H_{m+1} - \frac{1}{m+1} \right) - \frac{m(m-1)}{4} \\
&= \frac{m(m+1)}{2} H_{m+1} - \frac{m}{2} - \frac{m(m-1)}{4} = \frac{m(m+1)}{2} H_{m+1} - \frac{m(m+1)}{4}
\end{aligned}$$

Therefore, $P(m) \Rightarrow P(m+1)$ is true for all nonnegative m. Therefore, by the principle of mathematical induction, $P(n)$ holds for all nonnegative integers n.

I did a pretty good job of making the math all work out. The above answer is so neat and clean and tidy and lovely that it implies that it was terribly obvious what mathematical path to follow. That's hogwash. I gave this problem on a midterm once before, and it took me a while to figure out how to connect P(m) to P(m + 1), and I'm the frigging teacher! Here's what I do to look all impressive and smart even though I'm not all that (oh, go on… you really think I'm smart? Ohmygodyou'rethebest!)

Jerry's Induction Tip #1

If you know where you're headed in an induction proof, 'cheat' a little bit by computing the difference between the expressions relevant to P(m+1) and P(m). In the above example, I knew that the difference between

$$\frac{m(m-1)}{2}H_m - \frac{m(m-1)}{4} \text{ and } \frac{m(m+1)}{2}H_{m+1} - \frac{m(m+1)}{4}$$

was supposed to be the term that gets added in the induction step: mH_m. It also makes it clear that an H_m somehow needs to become a H_{m+1}. By computing the difference between the two expressions above, we will witness the algebra used to cross and cancel, and in the process understand how $\frac{m(m-1)}{2}H_m - \frac{m(m-1)}{4}$ and mH_m need to combine in order to give us $\frac{m(m+1)}{2}H_{m+1} - \frac{m(m+1)}{4}$.

Here's another example where the above tip will help you make progress much more efficiently.

$$\sum_{1\le k\le n} H_k^2 = (n+1)H_n^2 - (2n+1)H_n + 2n \text{, for all integers } n > 0.$$

This is another straightforward induction proof, although the algebra is messier. Let P(n) denote the belief that $\sum_{1\le k\le n} H_k^2 = (n+1)H_n^2 - (2n+1)H_n + 2n$ for all integers n > 0. Establish P(1), and establish that P(m) implies P(m+1) for all m greater than or equal to 1.

Prove P(1): $\sum_{1\le k\le 1} H_k^2 = H_1^2 = 1$. $(n+1)H_n^2 - (2n+1)H_n + 2n\Big|_{n=1} = 2(1) - 3(1) + 2(1) = 1$.
Therefore, the base case holds.

Prove that P(m) implies P(m+1):
Assume that P(m) is true for m greater than or equal to 1, and consider the sum $\sum_{1\le k\le m+1} H_k^2$.

$$
\begin{aligned}
\sum_{1 \le k \le m+1} H_k^2 &= \sum_{1 \le k \le m} H_k^2 + H_{m+1}^2 \\
&= (m+1)H_m^2 - (2m+1)H_m + 2m + H_{m+1}^2 \\
&= (m+1)\left(H_{m+1} - \frac{1}{m+1}\right)^2 - (2m+1)\left(H_{m+1} - \frac{1}{m+1}\right) + 2m + H_{m+1}^2 \\
&= (m+1)\left(H_{m+1}^2 - \frac{2H_{m+1}}{m+1} + \frac{1}{(m+1)^2}\right) - (2m+1)H_{m+1} + \frac{2m+1}{m+1} + 2m + H_{m+1}^2 \\
&= (m+1)H_{m+1}^2 - 2H_{m+1} + \frac{1}{m+1} - (2m+1)H_{m+1} + \frac{2m+1}{m+1} + 2m + H_{m+1}^2 \\
&= (m+1)H_{m+1}^2 + H_{m+1}^2 - (2m+1)H_{m+1} - 2H_{m+1} + 2m + \frac{1}{m+1} + \frac{2m+1}{m+1} \\
&= (m+2)H_{m+1}^2 - (2m+3)H_{m+1} + 2m + \frac{2m+2}{m+1} \\
&= (m+2)H_{m+1}^2 - (2m+3)H_{m+1} + 2m + 2
\end{aligned}
$$

To be more explicit:

$$
\begin{aligned}
\sum_{1 \le k \le m+1} H_k^2 &= (m+2)H_{m+1}^2 - (2m+3)H_{m+1} + 2m + 2 \\
&= ((m+1)+1)H_{m+1}^2 - (2(m+1)+1)H_{m+1} + 2(m+1)
\end{aligned}
$$

That means that P(m+1) is true whenever P(m) is true, so that P(n) is true for all positive integers n.

Variations On Mathematical Induction

From the discussion above, you know that part of a proof by induction has you assume that P(k) is true and use that assumption to drive a proof that P(k + 1) is true as well. From a notational point of view, sometimes you'll prefer to assume that P(k - 1) is true and prove that P(k) is true. These two approaches can be used, depending on your notational preference. What is important is that you establish that if one of the statements on the infinite number line is true, then the next one must also be true.

Also, when using induction, it is not necessary that the first value for n be 1. We've already seen an example of this above, but here's another: For all integers $n \geq 5$,, $2^n > n^2$; this baby can be proved by induction as well. The only modification is that, in order to start your proof, you must verify that P(n) is true for the first given value of n—n = 3 in the case of the convex polygon example, and n = 5 in the example just presented. The second part of the induction proof remains the same.

In fact, the only restriction on your choice of a base case (or even multiple base cases) and your inductive steps is that every integer in the range of interest get included, i.e. that you somehow guarantee that every domino will fall if you push one or more of them. Some variations on induction that are as legitimate as the original version we've been working with throughout the handout thus far:

Variation 1: Establish P(1) and P(2), and establish that $P(k) \Rightarrow P(k+2)$.
Variation 2: Establish P(1), P(2), P(3), and P(4), and establish $P(k) \Rightarrow P(k+4)$.
Variation 3: Establish P(1), and establish $P(k) \Rightarrow P(2k)$ and $P(k) \Rightarrow P(2k+1)$.

These are just three of the theoretically unlimited number of variations. You'll be sentenced to working with one or two variations on your second problem set, so brace yourself.

Backward Induction

Sometimes it's possible to use induction backward, proving things from n to $n - 1$ instead of the usual forward manner. For example, consider the statement $P(n)$:

$$x_1 x_2 \ldots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^n, \text{ if } x_1, x_2, \cdots x_n \geq 0$$

i) Prove that $P(2)$ is true.

Not all that obvious, actually.

$$(x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 \geq 0$$
$$(x_1 + x_2)^2 \geq 4x_1x_2$$
$$x_1x_2 \leq \frac{1}{4}(x_1 + x_2)^2$$
$$x_1x_2 \leq \left(\frac{x_1 + x_2}{2} \right)^2$$

ii) By setting $x_k = \dfrac{x_1 + x_2 + \cdots + x_k}{k - 1}$, prove that $P(k)$ implies $P(k - 1)$ whenever $k \geq 2$.

Assuming that $P(k)$ is true, we are permitted to work with the inequality

$$x_1x_2\ldots x_k \le \left(\frac{x_1+x_2+\cdots+x_k}{k}\right)^k,$$

which we assume to be true for any and all nonnegative choices of $x_1, x_2, \ldots, x_k$.
In particular, it is true for those choices where x_k just so happens to be given by:

$$x_k = \frac{x_1+x_2+\cdots+x_{k-1}}{k-1}.$$

Don't get tripped up on this one. If by assumption $P(k)$ is true for all possible x_k, then it's certainly true for the subset of points where x_k just so happens to equal the average of all the other coordinates.

Substituting for x_k, we find that:

$$x_1x_2\ldots x_{k-1}\frac{x_1+x_2+\cdots+x_{k-1}}{k-1} \le \left(\frac{x_1+x_2+\cdots+x_{k-1}+\dfrac{x_1+x_2+\cdots+x_{k-1}}{k-1}}{k}\right)^k,$$

$$= \left(\frac{(k-1)(x_1+x_2+\cdots+x_{k-1})+(x_1+x_2+\cdots+x_{k-1})}{k(k-1)}\right)^k$$

$$= \left(\frac{kx_1+kx_2+\cdots+kx_{k-1}}{k(k-1)}\right)^k$$

$$= \left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)^k$$

$$x_1x_2\ldots x_{k-1}\frac{x_1+x_2+\cdots+x_{k-1}}{k-1} \le \left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)^k$$

$$x_1x_2\ldots x_{k-1} \le \left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)^{k-1}$$

Now we should take care to not divide by zero in that last step, but division by zero is only a threat when all of the nonnegative x_i are identically zero. But we can verify the truth of that particular case directly, without having to do the icky math.

iii) Show that $P(k)$ and $P(2)$ imply $P(2k)$.

Well, if we assume that $P(k)$ is true, then we know that:

$$x_1x_2\ldots x_k \le \left(\frac{x_1+x_2+\cdots+x_k}{k}\right)^k$$

$$x_{k+1}x_{k+2}\ldots x_{2k} \le \left(\frac{x_{k+1}+x_{k+2}+\cdots+x_{2k}}{k}\right)^k$$

$$x_1x_2\ldots x_kx_{k+1}x_{k+2}\ldots x_{2k} \le \left(\frac{x_1+x_2+\cdots+x_k}{k}\right)^k\left(\frac{x_{k+1}+x_{k+2}+\cdots+x_{2k}}{k}\right)^k$$

All I did above was write down $P(k)$'s inequality two times, using different variable names for the second. The third line is the product of the first two.

Assuming $P(2)$ is true allows me to use the following inequality as well (note the variable name changes again):

$$y_1y_2 \le \left(\frac{y_1+y_2}{2}\right)^2$$

The truth of $P(2)$ justifies the second-to-last step of the following:

$$\begin{aligned} x_1x_2\ldots x_kx_{k+1}x_{k+2}\ldots x_{2k} &\le \left(\left(\frac{x_1+x_2+\cdots+x_k}{k}\right)\left(\frac{x_{k+1}+x_{k+2}+\cdots+x_{2k}}{k}\right)\right)^k \\ &\le \left(\frac{\frac{x_1+x_2+\cdots+x_k}{k}+\frac{x_{k+1}+x_{k+2}+\cdots+x_{2k}}{k}}{2}\right)^{2k} \\ &= \left(\frac{x_1+x_2+\cdots+x_k+x_{k+1}+x_{k+2}+\cdots+x_{2k}}{2k}\right)^{2k} \end{aligned}$$

But this implies that $P(2k)$ is true as well. Wow!

iv) Explain why this implies the truth of $P(n)$ for all positive n.

Well, all we need to be guaranteed is that every integer greater than or equal to two gets covered. We explicitly covered the $n = 2$ case, $P(2)$ and $P(2)$ give us $P(4)$, and then $P(2)$ and $P(4)$ give us $P(8)$, and then $P(2)$ and $P(8)$ give us $P(16)$, etc. For any integer that doesn't happen to be a perfect power of two, you can simply establish the truth of $P(k)$ for k equal to the least perfect power of two greater than it, and then use the $P(k) \Rightarrow P(k-1)$ induction to step down to and cover the integer of interest.

Admittedly, the mathematics of the proof is a bit much if you're predisposed to whining about math. What the problem does quite beautifully is immerse you in a perfectly valid induction technique, regardless of whether the variation is unusual or not. All long as the induction maps over all elements of interest, it's a valid version. That's the important thing to get here.