Exam 2
CSc 75010: Theoretical Computer Science
Graduate Center of CUNY
22 November 2002
(Sample Exam)

Do five of the following six problems. Write each answer on a separate piece of paper.

1. (a) Define *decidable set* and give an example of a set that is not decidable.

   (b) Assume that the alphabet and tape alphabet are: $\Sigma = \Gamma = \{0, 1\}$. Give an implementation level description of a Turing machine that decides the language:

   $$\{w \mid w \text{ contains an equal number of 0's and 1's}\}$$

   (c) Define *decidable function* and give an example of a function that is not decidable.

   (d) Assume that the alphabet and tape alphabet are: $\Sigma = \Gamma = \{0, 1\}$. Give an implementation level description of a Turing machine that copies the input string on the tape. That is, if the input to the machine is the string w, the output is ww.

   (e) Define *decidable set* and give an example of a set that is not decidable.

   (f) Assume that the alphabet and tape alphabet are: $\Sigma = \Gamma = \{0, 1\}$. Give an implementation level description of a Turing machine that decides the language:

   $$\{w \mid w \text{ is a palindrome}\}$$

   (g) Define *decidable function* and give an example of a function that is not decidable.

   (h) Assume that the alphabet and tape alphabet are: $\Sigma = \Gamma = \{0, 1\}$. Give an implementation level description of a Turing machine that takes as input a number in binary representation and mulitplies it by 2 (that is, shifts it to the right).

2. (a) Show that for any finite set Σ, the set of all finite strings of Σ, Σ^* is countable.

   (b) Let $\Sigma = \{0, 1\}$. Show that the set of all infinite strings over Σ is uncountable.

   (c) Show that the set of all positve rational numbers is countable.

   (d) Show that the set of all positve real numbers is uncountable.

3. (a) Show that the Halting Problem is undecidable, using the diagonalization method.

   (b) Show that the set of all finitely defined computable functions (that is, the set of all functions that diverge for all but a finis not decidable.

   (c) Show that the Halting Problem is undecidable, using the diagonalization method.

   (d) Show that the set of all constant computable functions is not decidable.

   (e) Show that the Halting Problem is undecidable, using the diagonalization method.

   (f) Show that the set of all totally undefined computable functions (that is the set of all functions that diverge for all inputs) is not decidable.

   (g) Show that the Halting Problem is undecidable, using the diagonalization method.

(h) Show that the set of all totally defined computable functions is not decidable.

4. (a) Show that the set of decidable languages is closed under concatenation.
   (b) Show that if A and $\bar{A}$ are Turing-recognizable, then A is decidable.
   (c) Show that the set of Turing-recognizable languages is closed under star.
   (d) Show that if A is decidable, then A and $\bar{A}$ are Turing-recognizable.
   (e) Show that the set of decidable languages is closed under intersection.
   (f) Show that if A and $\bar{A}$ are Turing-recognizable, then A is decidable.
   (g) Show that the set of decidable languages is closed under union.
   (h) Show that if A and $\bar{A}$ are Turing-recognizable, then A is decidable.
   (i) Show that the set of Turing-recognizable languages is closed under concatenation.
   (j) Show that if A is decidable, then A and $\bar{A}$ are Turing-recognizable.

5. (a) State the Post Correspondence Problem (PCP).
   (b) Show that PCP is decidable over the alphabet $\Sigma = \{1\}$.
   (c) State Rice's Theorem.
   (d) Prove Rice's Theorem.

6. (a) If $A \leq_m B$ and B regular does that imple A is regular? Why or why not? Justify your answer.
   (b) Show for all A, B, there exists a set J such that $A \leq_T J$ and $B \leq_T J$.
   (c) Show that $\leq_m$ is transitive.
   (d) Show for all A, B, there exists a set J such that $A \leq_T J$ and $B \leq_T J$.
   (e) Show that if A Turing-recognizable and $A \leq_m \bar{A}$, then A is decidable.
   (f) Show that if $A \leq_T B$ and $B \leq_T C$ implies $A \leq_T C$.
   (g) Show that for all A and B, $A \leq_m B$ implies $\bar{A} \leq_m \bar{B}$.
   (h) Show for all A, B, there exists a set J such that $A \leq_T J$ and $B \leq_T J$.