1. $y'' - 2y' + y = g(t),\ y(0) = -1,\ y'(0) = 1$

Taking the Laplace transform of both sides of the equation gives

$$s^2 Y(s) - s(-1) - 1 - 2(sY(s) - (-1)) + Y(s) = L\{g(t)\}$$

$$(s^2 - 2s + 1)Y(s) + s - 3 = L\{g(t)\}$$
$$(s-1)^2 Y(s) = 3 - s + L\{g(t)\}$$
$$Y(s) = -\frac{s-3}{(s-1)^2} + L\{g(t)\}\left(\frac{1}{(s-1)^2}\right).$$

We could perform a partial fraction expansion on the first term but we note

$$-\frac{s-3}{(s-1)^2} = -\frac{(s-1)-2}{(s-1)^2}$$

$$-\frac{s-3}{(s-1)^2} = -\frac{s-1}{(s-1)^2} + \frac{2}{(s-1)^2}$$
$$-\frac{s-3}{(s-1)^2} = -\frac{1}{s-1} + \frac{2}{(s-1)^2}$$

so that

$$Y(s) = -\frac{1}{s-1} + \frac{2}{(s-1)^2} + L\{g(t)\}\left(\frac{1}{(s-1)^2}\right)$$

and taking inverse transforms yields

$$y(t) = -e^t + 2te^t + g(t) * te^t$$

$$y(t) = -e^t + 2te^t + \int_0^t e^{t-v}(t-v)g(v)\ dv.$$

3. $y'' + 4y' + 5y = g(t),\ y(0) = 1,\ y'(0) = 1$

Taking the Laplace transform of both sides of the equation gives

$$s^2 Y(s) - s(1) - 1 + 4(sY(s) - (1)) + 5Y(s) = L\{g(t)\}$$

$$(s^2+4s+5)Y(s) - s - 5 = L\{g(t)\}$$
$$((s+2)^2+1)Y(s) = s + 5 + L\{g(t)\}$$
$$Y(s) = \frac{s+5}{(s+2)^2+1} + L\{g(t)\}\Big(\frac{1}{(s+2)^2+1}\Big)$$
$$Y(s) = \frac{s+2}{(s+2)^2+1} + \frac{3}{(s+2)^2+1} + L\{g(t)\}\Big(\frac{1}{(s+2)^2+1}\Big).$$

Taking inverse transforms yields

$$y(t) = e^{-2t}\cos t + 3e^{-2t}\sin t + g(t) * e^{t-2}\sin t$$
$$y(t) = e^{-2t}\cos t + 3e^{-2t}\sin t + \int_0^t e^{-2(t-v)}\sin(t-v)g(v)\ dv.$$

5. $\frac{1}{s(s^2+1)}$

The function can be written as

$$\frac{1}{s(s^2+1)} = \Big(\frac{1}{s}\Big)\Big(\frac{1}{s^2+1}\Big)$$

so that

$$L^{-1}\Big\{\frac{1}{s(s^2+1)}\Big\} = 1 * \sin t$$
$$L^{-1}\Big\{\frac{1}{s(s^2+1)}\Big\} = \int_0^t \sin v\ dv$$
$$L^{-1}\Big\{\frac{1}{s(s^2+1)}\Big\} = -\cos v\big|_0^t$$
$$L^{-1}\Big\{\frac{1}{s(s^2+1)}\Big\} = -\cos t + 1.$$

7. $\frac{14}{(s+2)(s-5)}$

The function can be written as

$$\frac{1}{(s+2)(s-5)} = 14\Big(\frac{1}{s+2}\Big)\Big(\frac{1}{s-5}\Big)$$

so that

$$L^{-1}\Big\{\frac{1}{(s+2)(s-5)}\Big\} = 14(e^{-2t} * e^{5t})$$

$$L^{-1}\Big\{\frac{1}{(s+2)(s-5)}\Big\} = 14\int_0^t \ e^{-2(t-v)}e^{5v} \ dv$$

$$L^{-1}\Big\{\frac{1}{(s+2)(s-5)}\Big\} = 14\int_0^t \ e^{-2t}e^{2v}e^{5v} \ dv$$

$$L^{-1}\Big\{\frac{1}{(s+2)(s-5)}\Big\} = 14e^{-2t}\int_0^t \ e^{7v} \ dv$$

$$L^{-1}\Big\{\frac{1}{(s+2)(s-5)}\Big\} = 2e^{-2t}[e^{7v}]|_0^t$$

$$L^{-1}\Big\{\frac{1}{(s+2)(s-5)}\Big\} = 2e^{-2t}[e^{7t}-1]$$

$$L^{-1}\Big\{\frac{1}{(s+2)(s-5)}\Big\} = 2e^{5t}-2e^{-2t}.$$

9. $\frac{s}{(s^2+1)^2}$

The function can be written as

$$\frac{s}{(s^2+1)^2} = \Big(\frac{s}{s^2+1}\Big)\Big(\frac{1}{s^2+1}\Big)$$

so that

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \cos t * \sin t$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \int_0^t \ \cos(t-v)\sin v \ dv$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \int_0^t \ [\cos t\cos v + \sin t \sin v]\sin v \ dv$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \int_0^t \cos t\cos v\sin v + \sin t\sin^2 v\ dv$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \cos t\int_0^t \cos v\sin v\ dv + \sin t\int_0^t \frac{1}{2}(1-\cos 2v)\ dv$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \cos t\Big[\frac{1}{2}\sin^2 v\Big]|_0^t + \sin t\Big[\frac{1}{2}v - \frac{1}{4}\sin 2v\Big]|_0^t$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \frac{1}{2}\cos t\sin^2 t + \frac{1}{2}t\sin t - \frac{1}{4}\sin t\sin 2t$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \frac{1}{2}\cos t\sin^2 t + \frac{1}{2}t\sin t - \frac{1}{4}\sin t(2\sin t\cos t)$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \frac{1}{2}\cos t\sin^2 t + \frac{1}{2}t\sin t - \frac{1}{2}\sin^2 t\cos t$$

$$L^{-1}\Big\{\frac{s}{(s^2+1)^2}\Big\} = \frac{1}{2}t\sin t.$$

11. $\frac{s}{(s-1)(s+2)}$

Using the hint we write

$$\frac{s}{(s-1)(s+2)} = \Big(\frac{s}{s-1}\Big)\Big(\frac{1}{s+2}\Big) = \Big(1+\frac{1}{s-1}\Big)\Big(\frac{1}{s+2}\Big) = \frac{1}{(s-1)(s+2)} + \frac{1}{s+2}$$

so that

$$L^{-1}\Big\{\frac{s}{(s-1)(s+2)}\Big\} = e^{-2t} + e^t * e^{-2t}$$

$$L^{-1}\Big\{\frac{s}{(s-1)(s+2)}\Big\} = e^{-2t} + \int_0^t e^{(t-v)}e^{-2v}\ dv$$

$$L^{-1}\Big\{\frac{s}{(s-1)(s+2)}\Big\} = e^{-2t} + \int_0^t e^t e^{-v}e^{-2v}\ dv$$

$$L^{-1}\Big\{\frac{s}{(s-1)(s+2)}\Big\} = e^{-2t} + e^t\int_0^t e^{-3v}\ dv$$

$$L^{-1}\Big\{\frac{s}{(s-1)(s+2)}\Big\} = e^{-2t} + e^t\Big[-\frac{e^{-3v}}{3}\Big]|_0^t$$

$$L^{-1}\left\{\frac{s}{(s-1)(s+2)}\right\} = e^{-2t} + e^{t}\left[-\frac{e^{-3t}}{3} + \frac{1}{3}\right]$$

$$L^{-1}\left\{\frac{s}{(s-1)(s+2)}\right\} = e^{-2t} - \frac{e^{-2t}}{3} + \frac{e^{t}}{3}$$

$$L^{-1}\left\{\frac{s}{(s-1)(s+2)}\right\} = \frac{2e^{-2t}}{3} + \frac{e^{t}}{3}.$$

13. $L\left\{\int_0^t (t-v)e^{3v}\ dv\right\}$

Using the definition of the convolution we have

$$L\left\{\int_0^t (t-v)e^{3v}\ dv\right\} = L\{t * e^{3t}\}$$

$$L\left\{\int_0^t (t-v)e^{3v}\ dv\right\} = \frac{1}{s^2(s-3)}.$$

15. $y(t) + 3\int_0^t y(v)\sin(t-v)\ dv = t$

Taking the Laplace transform of both sides of the equation yields

$$L\{y(t)\} + 3L\{y(t) * \sin t\} = L\{t\}$$

$$Y(s) + \frac{3Y(s)}{s^2+1} = \frac{1}{s^2}$$

$$\frac{3Y(s) + (s^2+1)Y(s)}{s^2+1} = \frac{1}{s^2}$$

$$\frac{(s^2+4)Y(s)}{s^2+1} = \frac{1}{s^2}$$

$$Y(s) = \frac{s^2+1}{s^2(s^2+4)}.$$

At this stage we could do a partial fractions expansion, but since we are in the section on convolution we will proceed as follows:

$$Y(s) = \frac{s^2}{s^2(s^2+4)} + \frac{1}{s^2(s^2+4)}$$
$$Y(s) = \frac{1}{s^2+4} + \left(\frac{1}{s^2}\right)\left(\frac{1}{s^2+4}\right)$$
$$Y(s) = \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{1}{s^2}\right)\left(\frac{2}{s^2+4}\right).$$

Therefore taking inverse transforms yields

$$y(t) = (1/2)\sin 2t + (1/2)(t * \sin 2t)$$
$$y(t) = (1/2)\sin 2t + (1/2)\int_0^v (t-v)\sin 2v\ dv$$
$$y(t) = (1/2)\sin 2t + (1/2)[-(1/2)(t-v)\cos 2v|_0^t - (1/2)\int_0^v \cos 2v\ dv]$$
$$y(t) = (1/2)\sin 2t + (1/2)[-(1/2)(t-v)\cos 2v|_0^t - (1/4)(\sin 2v)|_0^t]$$
$$y(t) = (1/2)\sin 2t + (1/2)[(1/2)t - (1/4)(\sin 2t)]$$
$$y(t) = (1/2)\sin 2t + (1/4)t - (1/8)(\sin 2t)$$
$$y(t) = (3/8)\sin 2t + (1/4)t.$$

17. $y(t) + \int_0^t (t-v)y(v)\ dv = 1$

Taking the Laplace transform of both sides of the equation yields

$$L\{y(t)\} + L\{t * y(t)\} = L\{1\}$$
$$Y(s) + \frac{Y(s)}{s^2} = \frac{1}{s}$$
$$\frac{s^2Y(s) + Y(s)}{s^2} = \frac{1}{s}$$
$$\frac{(s^2+1)Y(s)}{s^2} = \frac{1}{s}$$

$$Y(s) = \frac{s^2}{s(s^2+1)}$$

$$Y(s) = \frac{s}{s^2+1}.$$

Therefore taking inverse transforms yields

$$y(t) = \cos t.$$

19. $y(t) + \int_0^t (t-v)^2 y(v)\ dv = t^3 + 3$

Taking the Laplace transform of both sides of the equation yields

$$L\{y(t)\} + L\{t^2 * y(t)\} = \frac{6}{s^4} + \frac{3}{s}$$

$$Y(s) + Y(s)\left(\frac{2}{s^3}\right) = \frac{6+3s^3}{s^4}$$

$$Y(s)\left(\frac{2+s^3}{s^3}\right) = \frac{6+3s^3}{s^4}$$

$$Y(s) = \frac{s^3(6+3s^3)}{(2+s^3)s^4}$$

$$Y(s) = \frac{3}{s}.$$

Therefore taking inverse transforms yields

$$y(t) = 3.$$

21. $y'(t) + y(t) - \int_0^t y(v)\sin(t-v)\ dv = -\sin t,\ y(0) = 1$

Taking the Laplace transform of both sides of the equation yields

$$L\{y'(t)\} + L\{y(t)\} - L\{y(t) * \sin t\} = -\frac{1}{s^2+1}$$

$$\begin{aligned}
sY(s) - (1) + Y(s) - Y(s)\Big(\frac{1}{s^2+1}\Big) &= -\frac{1}{s^2+1} \\
Y(s)\Big(\frac{-1+(s+1)(s^2+1)}{s^2+1}\Big) &= -\frac{1}{s^2+1} + 1 \\
Y(s)\Big(\frac{s^3+s^2+s}{s^2+1}\Big) &= \frac{s^2}{s^2+1} \\
Y(s) &= \frac{s^2(s^2+1)}{(s^2+1)s(s^2+s+1)} \\
Y(s) &= \frac{s}{s^2+s+1} \\
Y(s) &= \frac{s}{(s+1/2)^2+3/4} \\
Y(s) &= \frac{s+1/2}{(s+1/2)^2+3/4} - \frac{1/2}{(s+1/2)^2+3/4}.
\end{aligned}$$

Therefore taking inverse transforms yields

$$y(t) = e^{-t/2}\cos(\sqrt{3}t/2) - e^{-t/2}\sin(\sqrt{3}t/2).$$

23. $y'' + 9y = g(t),\ y(0) = 2,\ y'(0) = -3$

The transfer function is given by

$$H(s) = \frac{1}{s^2+9}$$

and the impulse response function is given by

$$h(t) = L^{-1}\{H(s)\} = (1/3)\sin 3t.$$

For the homogeneous system, the characteristic equation is given by

$$r^2 + 9 = 0$$

so that

$$y_h = c_1\cos 3t + c_2\sin 3t.$$

To satisfy the initial conditions we write

$$y_h = c_1 \cos 3t + c_2 \sin 3t$$

$$y_h' = -3c_1 \sin 3t + 3c_2 \cos 3t$$

from which we obtain the system

$$2 = c_1$$

$$-3 = 3c_2.$$

Hence $c_1 = 2$ and $c_2 = -1$ and a solution to the ODE is given by

$$y(t) = \int_0^t (1/3) \sin 3(t-v) g(v)\ dv + 2\cos 3t - \sin 3t.$$

25. $y'' - y' - 6y = g(t),\ y(0) = 1,\ y'(0) = 8$

The transfer function is given by

$$H(s) = \frac{1}{s^2 - s - 6} = \frac{1}{(s-3)(s+2)}$$

and the impulse response function is given by
(note at this stage we could do partial fraction expansion, but in the spirit of the section choose to use convolution.)

$$h(t) = e^{3t} * e^{-2t}$$

$$h(t) = \int_0^v e^{3(t-v)} e^{-2v}\ dv$$

$$h(t) = e^{3t} \int_0^v e^{-5v}\ dv$$

$$h(t) = e^{3t}[(-1/5)e^{-5v}]|_0^t$$

$$h(t) = (-1/5)e^{3t}[e^{-5t} - 1]$$

$$h(t) = (1/5)[e^{3t} - e^{-2t}].$$

For the homogeneous system, the characteristic equation is given by

$$r^2 - r - 6 = (r-3)(r+2) = 0$$

so that

$$y_h = c_1 e^{3t} + c_2 e^{-2t}.$$

To satisfy the initial conditions we write

$$y_h = c_1 e^{3t} + c_2 e^{-2t}$$

$$y_h' = 3c_1 e^{3t} - 2c_2 e^{-2t}$$

from which we obtain the system

$$1 = c_1 + c_2$$

$$8 = 3c_1 - 2c_2.$$

Hence $c_1 = 2$ and $c_2 = -1$ and a solution to the ODE is given by

$$y(t) = \int_0^t (1/5)[e^{3(t-v)} - e^{-2(t-v)}]g(v)\ dv + 2e^{3t} - e^{-2t}.$$

27. $y'' - 2y' + 5y = g(t),\ y(0) = 0,\ y'(0) = 2$

The transfer function is given by

$$H(s) = \frac{1}{s^2 - 2s + 5} = \frac{1}{(s-1)^2 + 2^2}$$

and the impulse response function is given by
(note at this stage we could do partial fraction expansion, but in the spirit of the section choose to use convolution.)

$$h(t) = (1/2)e^t \sin t.$$

For the homogeneous system, the characteristic equation is given by

$$r^2 - 2r + 5 = 0$$

so that $r = (1/2)(-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}) = 1 \pm 2i$ and

$$y_h = c_1 e^t \cos 2t + c_2 e^t \sin 2t.$$

To satisfy the initial conditions we write

$$y_h = c_1 e^t \cos 2t + c_2 e^t \sin 2t$$

$$y_h' = c_1 e^t(\cos 2t - 2\sin 2t) + c_2 e^t(\sin 2t + 2\cos 2t)$$

from which we obtain the system

$$0 = c_1$$

$$2 = c_1 + 2c_2.$$

Hence $c_1 = 0$ and $c_2 = 1$ and a solution to the ODE is given by

$$y(t) = \int_0^t (1/2)e^{t-v}\sin(t-v)g(v)\ dv + e^t \sin 2t.$$

SECTION 7.8

1. $\int_{-\infty}^{\infty} (t^2 - 1)\delta(t)\ dt$

To evaluate this integral we use the fact that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\ dt = f(0).$$

Therefore

$$\int_{-\infty}^{\infty} (t^2 - 1)\delta(t)\ dt = (0^2 - 1) = -1.$$

3. $\int_{-\infty}^{\infty} (\sin 3t)\delta\Big(t - \frac{\pi}{2}\Big)\ dt$

To evaluate this integral we use the fact that

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\ dt = f(a).$$

Therefore

$$\int_{-\infty}^{\infty} (\sin 3t)\delta\Big(t - \frac{\pi}{2}\Big)\ dt = \sin 3\Big(\frac{\pi}{2}\Big) = -1.$$