SECTION 7.2

1. t

By definition, we have

$$L\{t\} = \int_0^\infty \ e^{-st} \ t \ dt$$

$$L\{t\} = \lim_{u \longrightarrow \infty} \int_0^u \ e^{-st} \ t \ dt.$$

Using integration by parts

$$L\{t\} = \lim_{u \longrightarrow \infty} [-(t/s)e^{-st}|_0^u + (1/s) \lim_{u \longrightarrow \infty} \int_0^u \ e^{-st} \ dt]$$

$$L\{t\} = \lim_{u \longrightarrow \infty} [-(t/s)e^{-st}|_0^u - (1/s^2)e^{-st}|_0^u]$$

$$L\{t\} = \lim_{u \longrightarrow \infty} [-(u/s)e^{-su} - (1/s^2)e^{-su} + (1/s^2)].$$

We know $\lim_{u \longrightarrow \infty} \ e^{-su} = 0, \ s > 0$ and using L'Hospital's Rule

$$\lim_{u \longrightarrow \infty} \ ue^{-su} = \lim_{u \longrightarrow \infty} \ \frac{u}{e^{su}} = \lim_{u \longrightarrow \infty} \ \frac{1}{se^{su}} = 0, \ s > 0.$$

Hence we have

$$L\{t\} = (1/s^2), \ s > 0.$$

3. e^{6t}

By definition, we have

$$L\{e^{6t}\} = \int_0^\infty \ e^{-st}e^{6t} \ dt = \int_0^\infty \ e^{(6-s)t} \ dt$$

$$L\{e^{6t}\} = \lim_{u \longrightarrow \infty} \int_0^u \ e^{(6-s)t} \ dt$$

$$L\{e^{6t}\} = \lim_{u \longrightarrow \infty} \ \frac{1}{6-s} e^{(6-s)t}|_0^u$$

$$L\{e^{6t}\} = \lim_{u \longrightarrow \infty} \frac{1}{6-s}(e^{(6-s)u} - 1)$$

$$L\{e^{6t}\} = \frac{1}{s-6}, \ s > 6.$$

5. $\cos 2t$

By definition, we have

$$L\{\cos 2t\} = \int_0^\infty e^{-st} \cos 2t \ dt$$

$$L\{\cos 2t\} = \lim_{u \longrightarrow \infty} \int_0^u e^{-st} \cos 2t \ dt.$$

Using integration by parts

$$\int e^{-st} \cos 2t \ dt = (1/2)e^{-st} \sin 2t + \int (s/2)e^{-st} \sin 2t \ dt.$$

Again applying integration by parts

$$\int e^{-st} \cos 2t \ dt = (1/2)e^{-st} \sin 2t + (s/2)[-(1/2)e^{-st} \cos 2t - \int (s/2)e^{-st} \cos 2t \ dt]$$

$$\int e^{-st} \cos 2t \ dt = (1/2)e^{-st} \sin 2t - (s/4)e^{-st} \cos 2t - (s^2/4) \int e^{-st} \cos 2t \ dt]$$

$$(1 + s^2/4) \int e^{-st} \cos 2t \ dt = (1/2)e^{-st} \sin 2t - (s/4)e^{-st} \cos 2t$$

$$\frac{4+s^2}{4} \int e^{-st} \cos 2t \ dt = (1/2)e^{-st} \sin 2t - (s/4)e^{-st} \cos 2t$$

$$\int e^{-st} \cos 2t \ dt = \frac{4}{4+s^2}[(1/2)e^{-st} \sin 2t - (s/4)e^{-st} \cos 2t]$$

so that

$$L\{\cos 2t\} = \lim_{u \longrightarrow \infty} \frac{4}{4+s^2}[(1/2)e^{-st} \sin 2t - (s/4)e^{-st} \cos 2t]|_0^u$$

$$L\{\cos 2t\} = \lim_{u \longrightarrow \infty} \frac{4}{4+s^2}[((1/2)e^{-su} \sin 2u - (s/4)e^{-su} \cos 2u) - (-(s/4))]$$

$$L\{\cos 2t\} = \frac{4}{4+s^2} \frac{s}{4} = \frac{s}{s^2+4}, \ s > 0.$$

7. $e^{2t}\cos 3t$

By definition, we have

$$L\{e^{2t}\cos 3t\} = \int_0^\infty \ e^{-st}e^{2t}\cos 3t \ dt$$

$$L\{e^{2t}\cos 3t\} = \lim_{u\longrightarrow\infty}\int_0^u \ e^{(2-s)t} \ \cos 3t \ dt.$$

Using integration by parts

$$\int \ e^{(2-s)t} \ \cos 3t \ dt = \frac{1}{3}e^{(2-s)t} \ \sin 3t - \int \ \frac{2-s}{3}e^{(2-s)t} \ \sin 3t \ dt.$$

Again applying integration by parts

$$\int \ e^{(2-s)t} \ \cos 3t \ dt = \frac{1}{3}e^{(2-s)t} \ \sin 3t - \frac{2-s}{3}[-\frac{1}{3}e^{(2-s)t} \ \cos 3t + \int \ \frac{2-s}{3}e^{(2-s)t} \ \cos 3t \ dt]$$

$$\int \ e^{(2-s)t} \ \cos 3t \ dt = \frac{1}{3}e^{(2-s)t} \ \sin 3t + \frac{2-s}{9}e^{(2-s)t} \ \cos 3t - (\frac{2-s}{3})^2\int \ e^{(2-s)t} \ \cos 3t \ dt$$

$$(1+(\frac{2-s}{3})^2)\int \ e^{(2-s)t} \ \cos 3t \ dt = \frac{1}{3}e^{(2-s)t} \ \sin 3t + \frac{2-s}{9}e^{(2-s)t} \ \cos 3t$$

$$\frac{9+(s-2)^2}{9}\int \ e^{(2-s)t} \ \cos 3t \ dt = \frac{1}{3}e^{(2-s)t} \ \sin 3t + \frac{2-s}{9}e^{(2-s)t} \ \cos 3t$$

$$\int \ e^{(2-s)t} \ \cos 3t \ dt = \frac{3}{9+(s-2)^2}e^{(2-s)t} \ \sin 3t + \frac{2-s}{9+(s-2)^2}e^{(2-s)t} \ \cos 3t$$

so that

$$L\{e^{2t}\cos 3t\} = \lim_{u\longrightarrow\infty} \ \frac{3}{9+(s-2)^2}e^{(2-s)t} \ \sin 3t + \frac{2-s}{9+(s-2)^2}e^{(2-s)t} \ \cos 3t|_0^u$$

$$L\{e^{2t}\cos 3t\} = \lim_{u\longrightarrow\infty} \ (\frac{3}{9+(s-2)^2}e^{(2-s)u} \ \sin 3u + \frac{2-s}{9+(s-2)^2}e^{(2-s)u} \ \cos 3u) - \frac{2-s}{9+(s-2)^2}$$

$$L\{e^{2t}\cos 3t\} = \frac{s-2}{9+(s-2)^2}, \ s > 2.$$

9. $f(t) = \begin{cases} 0 & 0 < t < 2 \\ t & 2 < t \end{cases}$

By definition, we have

$$L\{f(t)\} = \int_0^2 e^{-st} \cdot 0 \ dt + \int_2^\infty e^{-st} t \ dt = \int_2^\infty e^{-st} t \ dt$$

$$L\{f(t)\} = \lim_{u \longrightarrow \infty} \int_2^u e^{-st} t \ dt.$$

Using integration by parts

$$L\{f(t)\} = \lim_{u \longrightarrow \infty} [-(t/s)e^{-st}|_2^u + (1/s) \lim_{u \longrightarrow \infty} \int_2^u e^{-st} \ dt]$$

$$L\{f(t)\} = \lim_{u \longrightarrow \infty} [-(t/s)e^{-st}|_2^u - (1/s^2)e^{-st}|_2^u]$$

$$L\{f(t)\} = \lim_{u \longrightarrow \infty} [-(u/s)e^{-su} + (2/s)e^{-2s} - (1/s^2)e^{-su} + (1/s^2)e^{-2s}].$$

We know $\lim_{u \longrightarrow \infty} e^{-su} = 0, \ s > 0$ and using L'Hospital's Rule

$$\lim_{u \longrightarrow \infty} ue^{-su} = \lim_{u \longrightarrow \infty} \frac{u}{e^{su}} = \lim_{u \longrightarrow \infty} \frac{1}{se^{su}} = 0, \ s > 0.$$

Hence we have

$$L\{f(t)\} = [2/s + (1/s^2)]e^{-2s} = \frac{(2s+1)e^{-2s}}{s^2}, \ s > 0.$$

11. $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t \end{cases}$

By definition, we have

$$L\{f(t)\} = \int_0^\pi e^{-st} \sin t \ dt + \int_2^\infty e^{-st} \cdot 0 \ dt = \int_0^\pi e^{-st} \sin t \ dt$$

Using integration by parts

$$\int_0^\pi e^{-st} \sin t \ dt = -e^{-st} \cos t|_0^\pi - \int_0^\pi se^{-st} \cos t \ dt.$$

Using integration by parts a second time gives

$$\int_0^\pi e^{-st} \sin t \ dt = -e^{-st} \cos t|_0^\pi - se^{-st} \sin t|_0^\pi - \int_0^\pi s^2 e^{-st} \sin t \ dt$$

$$(1+s^2)\int_0^{\pi} e^{-st}\sin t\ dt = -e^{-st}\cos t|_0^{\pi} - se^{-st}\sin t|_0^{\pi}$$

$$\int_0^{\pi} e^{-st}\sin t\ dt = \frac{1}{1+s^2}[-e^{-st}\cos t|_0^{\pi} - se^{-st}\sin t|_0^{\pi}]$$

$$L\{f(t)\} = \frac{1}{1+s^2}[e^{-\pi s}+1] = \frac{e^{-\pi s}+1}{s^2+1}$$

for all values of s.

29. a. $t^3 \sin t$

In this problem we use the fact that $t < e^t$ for $t \geq 0$.

$$|t^3 \sin t| \leq |t^3| = |t|^3 < (e^t)^3 = e^{3t},\ t \geq 0.$$

b. $100e^{49t}$

This problem is already in the correct form $|f(t)| \leq Me^{\alpha t}$, for $t > n$ since we can take $M = 100$ and $\alpha = 49$.

c. e^{t^3}

This function is not of exponential order since e^x is increasing and there does not exist a constant α and a real number n such that $t^3 < \alpha t$ for $t > n$.

d. $t \ln t$

In this problem we use the readily established facts that fact that $t < e^t$ for $t \geq 0$ and $t > \ln t$ for $t \geq 0$.

$$|t \ln t| < |t \cdot t| = |t^2| = |t|^2 \leq (e^t)^2 = e^{2t},\ t \geq 0.$$

e. $\cosh t^2$

By definition $\cosh t^2 = (1/2)(e^{t^2} + e^{-t^2})$; hence $(1/2)e^{t^2} < \cosh t^2$ and e^{t^2} is not of exponential order by the argument similar to that given in part c.). Therefore, $\cosh t^2$ is not of exponential order.

f. $\frac{1}{t^2+1}$

We know $t^2 + 1 > 1$ for all t so that $\frac{1}{t^2+1} < 1$; therefore,

$$|\frac{1}{t^2+1}| < 1 = 1 \cdot e^{0 \cdot t}.$$

g. $\sin(t^2) + t^4 e^{6t}$

We know $t^4 e^{6t} > 1$ for $t > 1$ and $|\sin(t^2)| \leq 1$ for all t. Combining these facts and the previously developed result that $t^n < e^{nt}$ for $t > 1$ yields

$$|\sin(t^2) + t^4 e^{6t}| \leq |t^4 e^{6t} + t^4 e^{6t}| \leq 2|t^4 e^{6t}| < 2e^{4t}e^{6t} = 2e^{10t}, \ t > 1.$$

h. $3 - e^{t^2} + \cos 4t$

We know that $2 \leq 3 + \cos 4t$ so that

$$(1/2)e^{t^2} < |e^{t^2} - 2| = |2 - e^{t^2}| \leq |3 - e^{t^2} + \cos 4t|, \ \sqrt{\ln 4} < t.$$

By an argument similar to that given in c.), e^{t^2} is not of exponential order. By the above inequality this implies $f(t) = 3 - e^{t^2} + \cos 4t$ is also not of exponential order.

i. $e^{\frac{t^2}{t+1}}$

We know that $t + 1 > t$ so that for $t > -1$, $\frac{1}{t} > \frac{1}{t+1}$. This implies that for $t > -1$, $t = \frac{t^2}{t} > \frac{t^2}{t+1}$. Therefore since e^x is an increasing function

$$|e^{\frac{t^2}{t+1}}| < e^t.$$

j. $\sin(e^{t^2}) + e^{\sin t}$

We know that $|\sin x| \leq 1$ and $0 < e^{\sin t} \leq e$; therefore

$$|\sin(e^{t^2}) + e^{\sin t}| \leq ||\sin(e^{t^2})| + |e^{\sin t}|| \leq (1 + e) = (1 + e)e^{0 \cdot t}.$$