17, (17). $y'' - y = -11t + 1$

The characteristic equation is given by

$$r^2 - 1 = (r+1)(r-1) = 0$$

so there exists two, real, distinct roots given by 1 and -1 and the homogeneous solution is given by

$$y_h = c_1 e^t + c_2 e^{-t}.$$

The right hand side of the equation is a polynomial of degree one and multiplies an exponential of the form $e^{0\cdot t}$. Since $-1, 1 \neq 0$, $s = 0$ and the particular solution has the form

$$y_p = t^s(At + B)e^{0\cdot t} = At + B.$$

Therefore we have

$$\begin{aligned} y_p &= At + B \\ y_p' &= A \\ y_p'' &= 0 \end{aligned}$$

so that the ODE becomes

$$0 - (At + B) = -11t + 1.$$

From the final equation we have $A = 11$ and $B = -1$ so that a particular solution is given by

$$y_p = 11t - 1$$

and the general solution is given by

$$y_g = y_h + y_p = c_1 e^t + c_2 e^{-t} + 11t - 1.$$

19, (19). $y'' - 3y' + 2y = e^x \sin x$

The characteristic equation is given by

$$r^2 - 3r + 2 = (r-2)(r-1) = 0$$

so there exists two, real, distinct roots given by 1 and 2 and the homogeneous solution is given by

$$y_h = c_1 e^x + c_2 e^{2x}.$$

The right hand side of the equation is a polynomial of degree zero and multiplies an exponential of the form $e^{1 \cdot x}$ and a sine function of the form $\sin 1 \cdot x$. Since $1, 2 \neq 1 \pm i$, $s = 0$ and the particular solution has the form

$$y_p = x^s A e^x \sin x + x^s B e^x \cos x = A e^x \sin x + B e^x \cos x.$$

Therefore we have

$$y_p = A e^x \sin x + B e^x \cos x = e^x (A \sin x + B \cos x)$$

$$y_p' = e^x (A \sin x + B \cos x) + e^x (A \cos x - B \sin x) = e^x ((A - B) \sin x + (A + B) \cos x)$$

$$y_p'' = e^x ((A - B) \sin x + (A + B) \cos x) + e^x ((A - B) \cos x - (A + B) \sin x) = e^x (-2B \sin x + 2A \cos x)$$

so that the ODE becomes

$$e^x (-2B \sin x + 2A \cos x) - 3e^x ((A - B) \sin x + (A + B) \cos x) + 2e^x (A \sin x + B \cos x) = e^x \sin x.$$

After division by e^x and rearranging we obtain

$$(B - A) \sin x + (-A - B) \cos x = \sin x.$$

Equating similar trig terms we obtain the system of equations

$$B - A = 1$$

$$-A - B = 0$$

which yields $A = -1/2$ and $B = 1/2$ so that a particular solution is given by

$$y_p = e^x (-\sin x + \cos x)/2$$

and the general solution is given by

$$y_g = y_h + y_p = c_1 e^x + c_2 e^{2x} + e^x (-\sin x + \cos x)/2.$$

21, (21). $y'' + 2y' + 2y = e^{-\theta} \cos \theta$

The characteristic equation is given by

$$r^2 + 2r + 2 = 0$$

so there exists two complex conjugate roots given by the quadratic equation

$$r = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i.$$

and the homogeneous solution is given by

$$y_h = c_1 e^{-\theta} \cos\theta + c_2 e^{-\theta} \sin\theta.$$

The right hand side of the equation is a polynomial of degree zero and multiplies an exponential of the form $e^{-1\cdot\theta}$ and a cosine function of the form $\cos 1 \cdot \theta$. Since $-1 \pm i = -1 \pm i$, $s = 1$ and the particular solution has the form

$$y_p = \theta^s A e^{-\theta} \sin\theta + \theta^s B e^{-\theta} \cos\theta = A\theta e^{-\theta} \sin\theta + B\theta e^{-\theta} \cos\theta.$$

Therefore we have

$$y_p = A\theta e^{-\theta} \sin\theta + B\theta e^{-\theta} \cos\theta = e^{-\theta}(A\theta \sin\theta + B\theta \cos\theta)$$

$$y_p' = -e^{-\theta}(A\theta \sin\theta + B\theta \cos\theta) + e^{-\theta}(A \sin\theta + B \cos\theta + A\theta \cos\theta - B\theta \sin\theta)$$

$$= e^{-\theta}[(-(A+B)\theta + A) \sin\theta + ((A-B)\theta + B) \cos\theta]$$

$$y_p'' = -e^{-\theta}[(-(A+B)\theta + A) \sin\theta + ((A-B)\theta + B) \cos\theta]$$

$$+e^{-\theta}[-(A+B) \sin\theta + (A-B) \cos\theta + (-(A+B)\theta + A) \cos\theta - ((A-B)\theta + B) \sin\theta]$$

$$= e^{-\theta}[2(B\theta - (A+B)) \sin\theta + 2(-A\theta + (A-B)) \cos\theta]$$

so that the ODE becomes

$$e^{-\theta}[2(B\theta - (A+B)) \sin\theta + 2(-A\theta + (A-B)) \cos\theta] + 2e^{-\theta}[(-(A+B)\theta + A) \sin\theta + ((A-B)\theta + B) \cos\theta]$$

$$+2e^{-\theta}(A\theta \sin\theta + B\theta \cos\theta) = e^{-\theta} \cos\theta.$$

After division by $2e^{-\theta}$ and rearranging we obtain

$$-B \sin\theta + A \cos\theta = (1/2) \cos\theta.$$

Equating similar trig terms we obtain the system of equations

$$B = 0$$

$$A = 1/2$$

so that a particular solution is given by

$$y_p = (1/2)\theta e^{-\theta}\sin\theta$$

and the general solution is given by

$$y_g = y_h + y_p = c_1 e^{-\theta}\cos\theta + c_2 e^{-\theta}\sin\theta + (1/2)\theta e^{-\theta}\sin\theta.$$

23, (23). $y' - y = 1,\ y(0) = 0$

The characteristic equation is given by

$$r - 1 = 0$$

so there exists one real, distinct root given by 1 and the homogeneous solution is given by

$$y_h = c_1 e^t.$$

The right hand side of the equation is a polynomial of degree zero and multiplies an exponential of the form $e^{0\cdot t}$. Since $1 \neq 0$, $s = 0$ and the particular solution has the form

$$y_p = t^s A e^{0\cdot t} = A.$$

Therefore we have

$$y_p = A$$

$$y_p' = 0$$

so that the ODE becomes

$$-A = 1$$

and a particular solution is given by

$$y_p = -1$$

and the general solution is given by

$$y_g = y_h + y_p = c_1 e^t - 1.$$

Inserting the initial values yields

$$0 = c_1 e^0 - 1 = c_1 - 1$$

so that $c_1 = 1$; hence the solution is given by

$$y = e^t - 1.$$

25, (25). $z'' + z = 2e^{-x},\ z(0) = 0,\ z'(0) = 0$

The characteristic equation is given by

$$r^2 + 1 = 0$$

so there exists two imaginary roots given by $r = \pm i$ and the homogeneous solution is given by

$$z_h = c_1 \cos x + c_2 \sin x.$$

The right hand side of the equation is a polynomial of degree zero and multiplies an exponential of the form $e^{-1\cdot x}$. Since $\pm i \neq -1$, $s = 0$ and the particular solution has the form

$$z_p = t^s A e^{-1\cdot x} = A e^{-x}.$$

Therefore we have

$$z_p = A e^{-x}$$

$$z_p' = -A e^{-x}$$

$$z_p'' = A e^{-x}$$

so that the ODE becomes

$$A e^{-x} + A e^{-x} = 2e^{-x}.$$

From this equation it is obvious that $A = 1$, so that a particular solution is given by

$$z_p = e^{-x}$$

and the general solution is given by

$$z_g = z_h + z_p = c_1 \cos x + c_2 \sin x + e^{-x}$$

with the derivative

$$z_g' = -c_1 \sin x + c_2 \cos x - e^{-x}.$$

Inserting the initial values yields

$$0 = c_1 \cos 0 + c_2 \sin 0 + e^0 = c_1 + 1$$

$$0 = -c_1 \sin 0 + c_2 \cos 0 - e^0 = c_2 - 1$$

so that $c_1 = -1$ and $c_2 = 1$; hence the solution is given by

$$z = -\cos x + \sin x + e^{-x}.$$

27, (27). $y'' - y' - 2y = \cos x - \sin 2x, \ y(0) = -7/20, \ y'(0) = 1/5$

The characteristic equation is given by

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

so there exists two real, distinct roots given by 2 and -1 and the homogeneous solution is given by

$$y_h = c_1 e^{2x} + c_2 e^{-x}.$$

The right hand side of the equation consists of two trig terms with different arguments so that principle of superposition must be used; hence we find particular solutions for the two separate equations

$$y'' - y' - 2y = \cos x$$

$$y'' - y' - 2y = -\sin 2x.$$

For the first equation, the right hand side is a polynomial of degree zero and multiplies an exponential of the form $e^{0 \cdot x}$ and a cosine function of the form $\cos(1 \cdot x)$. Since $2, -1 \neq 0 \pm i$, $s = 0$ and the particular solution has the form

$$y_p = x^s A e^{0 \cdot x} \cos x + x^s B e^{0 \cdot x} \sin x = A \cos x + B \sin x.$$

Therefore we have

$$y_p = A \cos x + B \sin x$$

$$y_p' = -A \sin x + B \cos x$$

$$y_p'' = -A\cos x - B\sin x$$

so that the ODE becomes

$$(-A\cos x - B\sin x) - (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = \cos x$$

$$(-3A - B)\cos x + (A - 3B)\sin x = \cos x.$$

Comparing similar trig functions we obtain the simultaneous equations

$$-3A - B = 1$$

$$A - 3B = 0.$$

From these equations we find $A = -3/10$ and $B = -1/10$, so that a particular solution is given by

$$y_p = (-3/10)\cos x + (-1/10)\sin x.$$

For the second equation the right hand side is a polynomial of degree zero and multiplies an exponential of the form $e^{0\cdot x}$ and a sine function of the form $\sin(2\cdot x)$. Since $2, -1 \neq 0 \pm 2i$, $s = 0$ and the particular solution has the form

$$y_p = x^s A e^{0\cdot x}\cos 2x + x^s B e^{0\cdot x}\sin 2x = A\cos 2x + B\sin 2x.$$

Therefore we have

$$y_p = A\cos 2x + B\sin 2x$$

$$y_p' = -2A\sin 2x + B\cos 2x$$

$$y_p'' = -4A\cos 2x - 4B\sin 2x$$

so that the ODE becomes

$$(-4A\cos 2x - 4B\sin 2x) - (-2A\sin 2x + 2B\cos 2x) - 2(A\cos 2x + B\sin 2x) = -\sin 2x$$

$$(-6A - 2B)\cos 2x + (2A - 6B)\sin 2x = -\sin 2x.$$

Comparing similar trig functions we obtain the simultaneous equations

$$-6A - 2B = 0$$

$$2A - 6B = -1.$$

From these equations we find $A = -1/20$ and $B = 3/20$, so that a particular solution is given by

$$y_p = (-1/20)\cos 2x + (3/20)\sin 2x.$$

Therefore the general solution is given by

$$y_g = y_h + y_p = c_1 e^{2x} + c_2 e^{-x} + (-3/10)\cos x + (-1/10)\sin x + (-1/20)\cos 2x + (3/20)\sin 2x$$

with the derivative

$$y_g' = 2c_1 e^{2x} - c_2 e^{-x} + (3/10)\sin x + (-1/10)\cos x + (2/20)\sin 2x + (6/20)\cos 2x.$$

Inserting the initial values yields

$$-7/20 = c_1 + c_2 - (3/10) - (1/20)$$

$$1/5 = 2c_1 - c_2 - (1/10) + (6/20)$$

or after rearrangement

$$0 = c_1 + c_2$$

$$0 = 2c_1 - c_2$$

so that $c_1 = 0$ and $c_2 = 0$; hence the solution is given by

$$y = (-3/10)\cos x + (-1/10)\sin x + (-1/20)\cos 2x + (3/20)\sin 2x.$$

29, (29). $y'' - y = \sin\theta - e^{2\theta},\ y(0) = 1,\ y'(0) = -1$

The characteristic equation is given by

$$r^2 - 1 = (r-1)(r+1) = 0$$

so there exists two real, distinct roots given by 1 and -1 and the homogeneous solution is given by

$$y_h = c_1 e^{\theta} + c_2 e^{-\theta}.$$

The right hand side of the equation consists of the sum of a trig terms and an exponential so that principle of superposition must be used; hence we find particular solutions for the two separate equations

$$y'' - y = \sin\theta$$

$$y'' - y = -e^{2\theta}.$$

For the first equation, the right hand side is the product of a polynomial of degree zero, an exponential of the form $e^{0\cdot\theta}$, and a sine function of the form $\sin(1\cdot\theta)$. Since $1, -1 \neq 0 \pm i$, $s = 0$ and the particular solution has the form

$$y_p = \theta^s A e^{0\cdot\theta}\cos\theta + \theta^s B e^{0\cdot\theta}\sin\theta = A\cos\theta + B\sin\theta.$$

Therefore we have

$$y_p = A\cos\theta + B\sin\theta$$

$$y_p' = -A\sin\theta + B\cos\theta$$

$$y_p'' = -A\cos\theta - B\sin\theta$$

so that the ODE becomes

$$(-A\cos\theta - B\sin\theta) - (A\cos\theta + B\sin\theta) = \sin\theta$$

$$-2A\cos\theta + -2B\sin\theta = \sin\theta.$$

Comparing similar trig functions we obtain the simultaneous equations

$$-2A = 0$$

$$-2B = 1.$$

From these equations we find $A = 0$ and $B = -1/2$, so that a particular solution is given by

$$y_p = -\sin\theta/2.$$

For the second equation the right hand side is a polynomial of degree zero and multiplies an exponential of the form $e^{2\cdot\theta}$. Since $1, -1 \neq 2$, $s = 0$ and the particular solution has the form

$$y_p = \theta^s A e^{2\cdot\theta} = A e^{2\theta}.$$

Therefore we have

$$y_p = Ae^{2\theta}$$

$$y_p' = 2Ae^{2\theta}$$

$$y_p'' = 4Ae^{2\theta}$$

so that the ODE becomes

$$4Ae^{2\theta} - Ae^{2\theta} = -e^{2\theta}$$

$$3Ae^{2\theta} = -e^{2\theta}.$$

From this equation we find $A = -1/3$ so that a particular solution is given by

$$y_p = -e^{2\theta}/3.$$

Therefore the general solution is given by

$$y_g = y_h + y_p = c_1e^{\theta} + c_2e^{-\theta} - \sin\theta/2 - e^{2\theta}/3$$

with the derivative

$$y_g' = c_1e^{\theta} - c_2e^{-\theta} - \cos\theta/2 - 2e^{2\theta}/3.$$

Inserting the initial values yields

$$1 = c_1 + c_2 - 1/3$$

$$-1 = c_1 - c_2 - (1/2) - 2/3$$

or after rearrangement

$$4/3 = c_1 + c_2$$

$$1/6 = c_1 - c_2$$

so that $c_1 = 3/4$ and $c_2 = 7/12$; hence the solution is given by

$$y = 3e^{\theta}/4 + 7e^{-\theta}/12 - \sin\theta/2 - e^{2\theta}/3.$$