SECTION 8.4

7, (7). $y' + 2(x-1)y = 0,\ x_0 = 1$

For these problems it is necessary that the coefficient functions are written as power series about x_0; the coefficient function for y is already in this form, hence we proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n$$

$$y' = \sum_{n=0}^{\infty} na_n(x-1)^{n-1}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} na_n(x-1)^{n-1} + 2(x-1)\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

or after simplification

$$\sum_{n=0}^{\infty} na_n(x-1)^{n-1} + \sum_{n=0}^{\infty} 2a_n(x-1)^{n+1} = 0.$$

Performing a shift so that the exponents of both series is equal to n we obtain

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}(x-1)^n + \sum_{n=1}^{\infty} 2a_{n-1}(x-1)^n = 0$$

$$a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1}(x-1)^n + \sum_{n=1}^{\infty} 2a_{n-1}(x-1)^n = 0$$

$$a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + 2a_{n-1}](x-1)^n = 0.$$

From this last equation we obtain the recurrence relation

$$a_{n+1} = -2a_{n-1}/(n+1),\ n \geq 1$$

as well as the fact

$$a_1 = 0.$$

From these two pieces of information we can write

$$n = 1, \quad a_2 = -a_0$$

$$n = 2, \quad a_3 = -2a_1/3 = 0$$
$$n = 3, \quad a_4 = -a_2/2 = a_0/2$$
$$n = 4, \quad a_5 = -2a_3/5 = 0$$
$$n = 5, \quad a_6 = -a_4/3 = -a_0/6.$$

Thus we have

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n = a_0+a_1(x-1)+a_2(x-1)^2+a_3(x-1)^3+a_4(x-1)^4+a_5(x-1)^5+a_6(x-1)^6+\cdots$$

$$y = a_0[1 - (x-1)^2 + (x-1)^4/2 - (x-1)^6/6 + \cdots].$$

9, (9). $(x^2 - 2x)y'' + 2y = 0, \; x_0 = 1$

For these problems it is necessary that the coefficient functions are written as power series about x_0; hence we must express the function $f(x) = x^2 - 2x$ as a Taylor series about $x_0 = 1$. Following the standard approach we write

$f(x) = x^2 - 2x \quad f(1) = -1$

$f'(x) = 2x - 2 \quad f'(1) = 0$

$f''(x) = 2 \qquad f''(1) = 2$

$f^n(x) = 0$ for all n ≥ 3

so that

$$f(x) = \sum_{n=0}^{\infty} f^n(1)(x-1)^n/n! = -1 + 0(x-1) + 2(x-1)^2/2 = -1 + (x-1)^2.$$

Next we proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n$$

$$y' = \sum_{n=0}^{\infty} na_n(x-1)^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2}$$

and substituting into the above equation obtain

$$[-1+(x-1)^2]\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + 2\sum_{n=0}^{\infty} a_n(x-1)^n = 0$$

or after simplification

$$-\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=0}^{\infty} 2a_n(x-1)^n = 0.$$

Performing a shift so that the exponent of the first series is equal to n we have

$$-\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n + \sum_{n=0}^{\infty} 2a_n(x-1)^n = 0$$

$$\sum_{n=0}^{\infty} [-(n+2)(n+1)a_{n+2} + n(n-1)a_n + 2a_n](x-1)^n = 0$$

$$\sum_{n=0}^{\infty} [-(n+2)(n+1)a_{n+2} + (n^2-n+2)a_n](x-1)^n = 0$$

From this last equation we obtain the recurrence relation

$$a_{n+2} = (n^2-n+2)a_n/[(n+2)(n+1)],\ n \geq 0.$$

From this relation we can write

$$n=0, \quad a_2 = a_0$$

$$n=1, \quad a_3 = a_1/3$$

Thus we have

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$y = a_0[1+(x-1)^2+\cdots] + a_1[(x-1)+(x-1)^3/3+\cdots].$$

11, (11). $x^2y'' - y' + y = 0,\ x_0 = 2$

For these problems it is necessary that the coefficient functions are written as power series about x_0; hence we must express the function $f(x) = x^2$ as a Taylor series about $x_0 = 2$. Following the standard approach we write

$f(x) = x^2 \quad f(2) = 4$

$f'(x) = 2x \quad f'(2) = 4$

$f''(x) = 2 \quad f''(2) = 2$

$f^n(x) = 0$ for all n ≥ 3

so that

$$f(x) = \sum_{n=0}^{\infty} f^n(1)(x-2)^n/n! = 4+4(x-2)+2(x-2)^2/2 = 4+4(x-2)+(x-2)^2.$$

Next we proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n(x-2)^n$$

$$y' = \sum_{n=0}^{\infty} na_n(x-2)^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2}$$

and substituting into the above equation obtain

$$[4+4(x-2)+(x-2)^2]\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2}-\sum_{n=0}^{\infty} na_n(x-2)^{n-1}+\sum_{n=0}^{\infty} a_n(x-2)^n = 0$$

or after simplification

$$\sum_{n=0}^{\infty} 4n(n-1)a_n(x-2)^{n-2}+\sum_{n=0}^{\infty} 4n(n-1)a_n(x-2)^{n-1}+\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^n$$

$$-\sum_{n=0}^{\infty} na_n(x-2)^{n-1} + \sum_{n=0}^{\infty} a_n(x-2)^n = 0.$$

Performing a shift so that the exponents of the first, second, and fourth series are equal to n we have

$$\sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2}(x-2)^n+\sum_{n=0}^{\infty} 4n(n+1)a_{n+1}(x-2)^n+\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^n-\sum_{n=0}^{\infty} (n+1)a_{n+1}(x-2)^n$$

$$+ \sum_{n=0}^{\infty} a_n(x-2)^n = 0.$$

$$\sum_{n=0}^{\infty}[4(n{+}2)(n{+}1)a_{n+2}{+}4n(n{+}1)a_{n+1}{+}n(n{-}1)a_n{-}(n{+}1)a_{n+1}{+}a_n](x{-}2)^n = 0$$

$$\sum_{n=0}^{\infty}[4(n+2)(n+1)a_{n+2} + (4n-1)(n+1)a_{n+1} + (n^2-n+1)a_n](x-2)^n = 0$$

From this last equation we obtain the recurrence relation

$$a_{n+2} = -(4n-1)a_{n+1}/[4(n+2)] - (n^2-n+1)a_n/[4(n+2)(n+1)],\ n \geq 0.$$

From this relation we can write

$$n = 0, \quad a_2 = a_1/8 - a_0/8$$

$$n = 1, \quad a_3 = -a_2/4 - a_1/24 = -7a_1/96 + a_0/32.$$

Thus we have

$$y = \sum_{n=0}^{\infty} a_n(x-2)^n = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \cdots$$

$$y = a_0[1{-}(x{-}2)^2/8{+}(x{-}2)^3/32{+}\cdots]{+}a_1[(x{-}2){+}(x{-}2)^2/8{+}{-}7(x{-}2)^3/96{+}\cdots].$$

13, (13). $x' + (\sin t)x = 0,\ x(0) = 1$

For this problem it is not practical to develop a recursion formula so we determine coefficients in the power series expansion about zero by explicitly writing out the first several terms in the expansion:

$$x = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + \cdots.$$

The equation $x(0) = 1$ tells us that $a_0 = 1$ so we can write

$$x = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + \cdots \quad (1)$$

and after differentiating with respect to t

$$x' = a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + 6a_6t^5 + \cdots.$$

The Taylor expansion about zero for $\sin t$ is given by

$$\sin t = \sum_{n=0}^{\infty}(-1)^n t^{2n+1}/(2n+1)! = t - t^3/6 + t^5/120 + \cdots \quad (2)$$

hence multiplying (1) by (2) we obtain after simplification

$$(\sin t)x = t + a_1 t^2 + (a_2 - \frac{1}{6})t^3 + (a_3 - \frac{a_1}{6})t^4 + (a_4 - \frac{a_2}{6} + \frac{1}{120})t^5 + \cdots.$$

Thus we can write

$$x' + (\sin t)x = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + \cdots$$

$$+ t + a_1 t^2 + (a_2 - \frac{1}{6})t^3 + (a_3 - \frac{a_1}{6})t^4 + (a_4 - \frac{a_2}{6} + \frac{1}{120})t^5 + \cdots$$

or after collecting similar terms

$$x' + (\sin t)x = a_1 + (2a_2 + 1)t + (3a_3 + a_1)t^2 + (4a_4 + a_2 - \frac{1}{6})t^3$$

$$+ (5a_5 + a_3 - \frac{a_1}{6})t^4 + (6a_6 + a_4 - \frac{a_2}{6} + \frac{1}{120})t^5 + \cdots.$$

Setting coefficients equal to zero yields

$$a_1 = 0$$

$$2a_2 + 1 = 0 \longrightarrow a_2 = -1/2$$

$$3a_3 + a_1 = 0 \longrightarrow a_3 = 0$$

$$4a_4 + a_2 - \frac{1}{6} = 0 \longrightarrow a_4 = 1/6$$

$$5a_5 + a_3 - \frac{a_1}{6} = 0 \longrightarrow a_5 = 0$$

$$6a_6 + a_4 - \frac{a_2}{6} + \frac{1}{120} = 0 \longrightarrow a_6 = -31/120.$$

Placing the value of these coefficients into (1) gives

$$x = 1 - t^2/2 + t^4/6 - 31t^6/120 + \cdots.$$

15, (15). $(x^2 + 1)y'' - e^x y' + y = 0,\ y(0) = 1,\ y'(0) = 1$

For this problem it is not practical to develop a recursion formula so we determine coefficients in the power series expansion about zero by explicitly writing out the first several terms in the expansion:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots.$$

The equation $y(0) = 1$ tells us that $a_0 = 1$ and $y'(0) = 1$ implies $a_1 = 1$ so we can write

$$y = 1 + x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

and after differentiating with respect to x

$$y' = 1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots \quad (1),$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots.$$

The Taylor expansion about zero for e^x is given by

$$e^x = \sum_{n=0}^{\infty} x^n/n! = 1 + x + x^2/2 + x^3/6 + x^4/24 + \cdots \quad (2)$$

hence multiplying (1) by (2) we obtain after simplification

$$e^x y' = 1 + (2a_2 + 1)x + (3a_3 + 2a_2 + \frac{1}{2})x^2 + (4a_4 + 3a_3 + a_2 + \frac{1}{6})x^3 + \cdots.$$

In addition multiplying x^2 by y'' yields

$$x^2y'' = 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 + \cdots.$$

Thus we can write

$$\begin{aligned}(x^2+1)y'' - e^x y' + y &= 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 + \cdots \\ &+ 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots \\ &- 1 - (2a_2 + 1)x - (3a_2 + 2a_2 + \frac{1}{2})x^2 - (4a_4 + 3a_3 + a_2 + \frac{1}{6})x^3 + \cdots \\ &+ 1 + x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots\end{aligned}$$

or after collecting similar terms up to the third power of x

$$\begin{aligned}(x^2+1)y'' - e^x y' + y &= 2a_2 + (6a_3 - 2a_2)x + (12a_4 - 3a_3 - a_2 - \frac{1}{2})x^2 \\ &+ (20a_5 - 4a_4 + 4a_3 - a_2 - \frac{1}{6})x^3 + \cdots.\end{aligned}$$

Setting coefficients equal to zero yields

$$2a_2 = 0 \longrightarrow a_2 = 0$$

$$6a_3 - 2a_2 = 0 \longrightarrow a_3 = 0$$

$$12a_4 - 3a_3 - a_2 - \frac{1}{2} = 0 \longrightarrow a_4 = 1/24$$

$$20a_5 - 4a_4 + 4a_3 - a_2 - \frac{1}{6} = 0 \longrightarrow a_5 = 1/60$$

Placing the value of these coefficients into the expansion for $y(x)$ gives

$$y = 1 + x + x^4/24 + x^5/60 + \cdots.$$

17, (17). $y'' - (\sin x)y = 0,\ y(\pi) = 1,\ y'(\pi) = 0$

For this problem it is not practical to develop a recursion formula so we determine coefficients in the power series expansion about π by explicitly writing out the first several terms in the expansion:

$$y = a_0 + a_1(x-\pi) + a_2(x-\pi)^2 + a_3(x-\pi)^3 + a_4(x-\pi)^4 + a_5(x-\pi)^5 + a_6(x-\pi)^6 + \cdots.$$

The equation $y(\pi) = 1$ tells us that $a_0 = 1$ and $y'(\pi) = 0$ implies $a_1 = 0$ so we can write

$$y = 1 + a_2(x-\pi)^2 + a_3(x-\pi)^3 + a_4(x-\pi)^4 + a_5(x-\pi)^5 + a_6(x-\pi)^6 + \cdots \quad (1)$$

and after differentiating with respect to x

$$y' = 2a_2(x-\pi) + 3a_3(x-\pi)^2 + 4a_4(x-\pi)^3 + 5a_5(x-\pi)^4 + 6a_6(x-\pi)^5 + \cdots ,$$

$$y'' = 2a_2 + 6a_3(x-\pi) + 12a_4(x-\pi)^2 + 20a_5(x-\pi)^3 + 30a_6(x-\pi)^4 + \cdots.$$

The Taylor expansion about π for $\sin x$ is given by

$$\sin x = \sum_{n=0}^{\infty} (-1)^{n+1}(x-\pi)^{2n+1}/(2n+1)! = -(x-\pi) + (x-\pi)^3/6 - (x-\pi)^5/120 + \cdots \quad (2)$$

hence multiplying (1) by (2) we obtain after simplification

$$(\sin x)y = -(x-\pi) + (-a_2 + \frac{1}{6})(x-\pi)^3 - a_3(x-\pi)^4 + (-a_4 + \frac{a_2}{6} - \frac{1}{120})(x-\pi)^5 + \cdots.$$

Thus we can write

$$y'' - (\sin x)y = 2a_2 + 6a_3(x-\pi) + 12a_4(x-\pi)^2 + 20a_5(x-\pi)^3 + 30a_6(x-\pi)^4 + \cdots$$

$$+ (x-\pi) - (-a_2 + \frac{1}{6})(x-\pi)^3 + a_3(x-\pi)^4 - (-a_4 + \frac{a_2}{6} - \frac{1}{120})(x-\pi)^5 + \cdots$$

or after collecting similar terms up to the fourth power of $(x-\pi)$

$$y'' - (\sin x)y = 2a_2 + (6a_3+1)(x-\pi) + (12a_4)(x-\pi)^2 + (20a_5 + a_2 - \frac{1}{6})(x-\pi)^3$$

$$+\ (30a_6 + a_3)(x-\pi)^4 + \cdots.$$

Setting coefficients equal to zero yields

$$2a_2 = 0 \ \longrightarrow\ a_2 = 0$$

$$6a_3 + 1 = 0 \ \longrightarrow\ a_3 = 1/6$$

$$12a_4\ = 0 \longrightarrow\ a_4 = 0$$

$$20a_5 + a_2 - \frac{1}{6} = 0 \longrightarrow\ a_5 = 1/120$$

$$30a_6 + a_3 = 0 \longrightarrow\ a_6 = 1/180$$

Placing the value of these coefficients into the expansion for $y(x)$ gives

$$y = 1 - (x-\pi)^3/6 + (x-\pi)^5/120 + (x-\pi)^6/(180) + \cdots.$$

19, (19). $y'' - e^{2x}y' + (\cos x)y = 0,\ y(0) = -1,\ y'(0) = 1$

For this problem it is not practical to develop a recursion formula so we determine coefficients in the power series expansion about zero by explicitly writing out the first several terms in the expansion:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots.$$

The equation $y(0) = -1$ tells us that $a_0 = -1$ and $y'(0) = 1$ implies $a_1 = 1$ so we can write

$$y = -1 + x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots \quad (1)$$

and after differentiating with respect to x

$$y' = 1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots \quad (2),$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots.$$

The Taylor expansion about zero for e^{2x} is given by

$$e^{2x} = \sum_{n=0}^{\infty}(2^n x^n)/n! = 1 + 2x + 2x^2 + 4x^3/3 + 2x^4/3 + \cdots \quad (3)$$

hence multiplying (2) by (3) we obtain after simplification

$$e^{2x}y' = 1 + (2a_2 + 2)x + (3a_2 + 4a_2 + 2)x^2 + (4a_4 + 6a_3 + 4a_2 + \frac{4}{3})x^3 + \cdots.$$

The Taylor expansion about zero for $\cos x$ is given by

$$\cos x = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)! = 1 - x^2/2 + x^4/24 + \cdots \quad (4)$$

hence multiplying (1) by (4) we obtain after simplification

$$(\cos x)y = -1 + x + (a_2 + \frac{1}{2})x^2 + (a_3 - \frac{1}{2})x^3 + \cdots.$$

Thus we can write

$$y'' - e^{2x}y' + (\cos x)y = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots$$

$$-\ 1 - (2a_2 + 2)x - (3a_2 + 4a_2 + 2)x^2 - (4a_4 + 6a_3 + 4a_2 + \frac{4}{3})x^3 + \cdots$$

$$+\ \ -1 + x + (a_2 + \frac{1}{2})x^2 + (a_3 - \frac{1}{2})x^3 + \cdots$$

or after collecting similar terms up to the first power of x

$$y'' - e^{2x}y' + (\cos x)y = (2a_2 - 2) + (6a_3 - 2a_2 - 1)x + \cdots.$$

Setting coefficients equal to zero yields

$$2a_2 - 2 = 0 \ \longrightarrow\ a_2 = 1$$

$$6a_3 - 2a_2 - 1 = 0 \ \longrightarrow\ a_3 = 1/2$$

Placing the value of these coefficients into the expansion for $y(x)$ gives

$$y = -1 + x + x^2 + x^3/2 + \cdots.$$

21, (21). $y' - xy = \sin x$.

We begin with the expansions for $y,\ y'$, and $\sin x$:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - (1/6)x^3 + (1/120)x^5 - (1/4200)x^7 + \cdots$$

and

$$-xy = -x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots) = -a_0 x - a_1 x^2 - a_2 x^3 - a_3 x^4 - a_4 x^5 - a_5 x^6 - a_6 x^7 + \cdots.$$

Substituting these expansions back into the ODE yields

$$(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots) + (-a_0 x - a_1 x^2 - a_2 x^3 - a_3 x^4 - a_4 x^5 - a_5 x^6 - a_6 x^7 + \cdots)$$

$$= x - (1/6)x^3 + (1/120)x^5 - (1/4200)x^7 + \cdots.$$

Equating coefficients for like powers of x gives

$x^0: \quad a_1 = 0$

$x^1: \quad 2a_2 - a_0 = 1 \quad \rightarrow \quad a_2 = (1/2)(a_0 + 1)$

$x^2: \quad 3a_3 - a_1 = 0 \quad \rightarrow \quad a_3 = 0$

$x^3: \quad 4a_4 - a_2 = -(1/6) \quad \rightarrow \quad a_4 = (1/4)(a_2 - (1/6))$

$a_4 = (1/4)[(1/2)(a_0 + 1) - (1/6)] = (1/8)a_0 + (1/12)$

$x^4: \quad 5a_5 - a_3 = 0 \quad \rightarrow \quad a_5 = 0$

$x^5: \quad 6a_6 - a_4 = (1/120) \quad \rightarrow \quad a_6 = (1/6)(a_4 + (1/120))$

$a_6 = (1/6)[(1/8)a_0 + (1/12) + (1/120)] = (1/48)a_0 + (11/720).$

Thus

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

$$= a_0+(0)x+[(1/2)(a_0+1)]x^2+(0)x^3+[(1/8)a_0+(1/12)]x^4+(0)x^5+[(1/48)a_0+(11/720)]x^6$$

or after regrouping

$$= a_0[1+(1/2)x^2+(1/8)x^4+(1/48)x^6+\cdots]+[(1/2)x^2+(1/12)x^4+(11/720)x^6].$$

23, (23). $z'' + xz' + z = x^2 + 2x + 1$

We begin with the expansions for z, z', and z'':

$$z = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

$$z' = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$

$$z'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots$$

and

$$xz' = x(a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5+\cdots) = a_1x+2a_2x^2+3a_3x^3+4a_4x^4+5a_5x^5+6a_6x^6+\cdots.$$

Substituting these expansions back into the ODE yields

$$(2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\cdots)+(a_1x+2a_2x^2+3a_3x^3+4a_4x^4+5a_5x^5+6a_6x^6+\cdots)$$

$$+(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots) = x^2 + 2x + 1.$$

Equating coefficients for like powers of x gives

x^0 : $2a_2 + a_0 = 1 \quad \rightarrow \quad a_2 = -(1/2)a_0 + (1/2)$

x^1 : $6a_3 + a_1 + a_1 = 2 \quad \rightarrow \quad a_3 = (1/6)(-2a_1 + 2) = -(1/3)a_1 + (1/3)$.

Thus

$$z = a_0+a_1x+a_2x^2+a_3x^3+\cdots = a_0+a_1x+[-(1/2)a_0+(1/2)]x^2+[-(1/3)a_1+(1/3)]x^3+\cdots$$

or after regrouping

$$z = a_0[1-(1/2)x^2+\cdots]+a_1[x-(1/3)x^3+\cdots]+[(1/2)x^2+(1/3)x^3+\cdots].$$

25, (25). $(1+x^2)y'' - xy' + y = e^{-x}$

We begin with the expansions for y, y', y'', and e^{-x}:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots$$

$$e^{-x} = \sum_{n=0}^{\infty}(-1)^n x^n/(n!) = 1-x+(1/2)x^2-(1/6)x^3+(1/24)x^4-(1/120)x^5+\cdots$$

and

$$-xy' = -x(a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5+\cdots) = -a_1x-2a_2x^2-3a_3x^3-4a_4x^4-5a_5x^5-6a_6x^6+\cdots$$

$$x^2y'' = x^2(2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\cdots) = 2a_2x^2+6a_3x^3+12a_4x^4+20a_5x^5+30a_6x^6+\cdots.$$

Substituting these expansions back into the ODE yields

$$(2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\cdots)+(2a_2x^2+6a_3x^3+12a_4x^4+20a_5x^5+30a_6x^6+\cdots)$$

$$+(-a_1x-2a_2x^2-3a_3x^3-4a_4x^4-5a_5x^5-6a_6x^6+\cdots)+(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\cdots)$$

$$= 1 - x + (1/2)x^2 - (1/6)x^3 + (1/24)x^4 - (1/120)x^5 + \cdots.$$

Equating coefficients for like powers of x gives

$$x^0: \quad 2a_2 + a_0 = 1 \quad \rightarrow \quad a_2 = -(1/2)a_0 + (1/2)$$

$$x^1: \quad 6a_3 - a_1 + a_1 = -1 \quad \rightarrow \quad a_3 = -(1/6).$$

Thus

$$y = a_0+a_1x+a_2x^2+a_3x^3+\cdots = a_0+a_1x+[-(1/2)a_0+(1/2)]x^2+[-(1/6)]x^3+\cdots$$

or after regrouping

$$y = a_0[1-(1/2)x^2+\cdots] + a_1[x+\cdots] + [(1/2)x^2 - (1/6)x^3 + \cdots].$$

27, (27). $(1-x^2)y'' - y' + y = \tan x$

We begin with the expansions for y, y', y'', and e^{-x}:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots$$

$$\tan x = x + (1/3)x^3 + (2/15)x^5 + \cdots$$

and

$$-y' = -(a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5+\cdots) = -a_1-2a_2x-3a_3x^2-4a_4x^3-5a_5x^4-6a_6x^5+\cdots$$

$$-x^2y'' = -x^2(2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\cdots) = -2a_2x^2-6a_3x^3-12a_4x^4-20a_5x^5-30a_6x^6+\cdots.$$

Substituting these expansions back into the ODE yields

$$(2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\cdots)+(-2a_2x^2-6a_3x^3-12a_4x^4-20a_5x^5-30a_6x^6+\cdots)$$

$$+(-a_1-2a_2x-3a_3x^2-4a_4x^3-5a_5x^4-6a_6x^5+\cdots)+(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\cdots)$$

$$= x + (1/3)x^3 + (2/15)x^5 + \cdots.$$

Equating coefficients for like powers of x gives

$x^0: \quad 2a_2 - a_1 + a_0 = 0 \quad \to \quad a_2 = (1/2)a_1 - (1/2)a_0$

$x^1: \quad 6a_3 - 2a_2 + a_1 = 1 \quad \to \quad a_3 = (1/3)a_2 - (1/6)a_1 + (1/6)$

$a_3 = (1/3)[(1/2)a_1 - (1/2)a_0] - (1/6)a_1 - (1/6) = -(1/6)a_0 + (1/6).$

Thus

$$y = a_0+a_1x+a_2x^2+a_3x^3+\cdots = a_0+a_1x+[(1/2)a_1-(1/2)a_0]x^2+[-(1/6)a_0+(1/6)]x^3+\cdots$$

or after regrouping

$$y = a_0[1-(1/2)x^2-(1/6)x^3+\cdots] + a_1[x+(1/2)x^2+\cdots] + [(1/6)x^3+\cdots].$$