

## Review Packet for Exam 1

The **order** of a DE is the order of the highest-order derivative in the equation.

A DEQ is **linear** if it can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

**Theorem.** (Existence and Uniqueness of Solution) Given the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

if  $f$  and  $\frac{\partial f}{\partial y}$  are defined for  $(x_0, y_0)$ , then the IVP has a **unique** solution  $\phi(x)$  in some rectangle containing  $(x_0, y_0)$ .

**ex.** The initial value problem  $y' = \sqrt{y^2 - 16}$ ,  $y(x_0) = y_0$  has

$$f(x, y) = \sqrt{y^2 - 16} \text{ and } \frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 16}}$$

$f$  exists only for  $y \geq 4$  or  $y \leq -4$  and  $\frac{\partial f}{\partial y}$  exists only for  $y > 4$  or  $y < -4$

so a unique solution to the IVP is guaranteed only where  $y > 4$  or  $y < -4$ .

So for example, we are not guaranteed a solution for  $y_0 = 3$  but we are guaranteed a solution for  $y_0 = 7$

### Verifying or finding particular solutions

**ex.** Show that  $\phi(x) = \sin x - \cos x$  is a particular solution to the IVP

$$y'' + y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

$$\phi'(x) = \cos x + \sin x$$

$$\phi''(x) = -\sin x + \cos x$$

Check to see that  $\phi(x)$  satisfies all 3 equations:

1. Plug in  $\frac{d^2 y}{dx^2} + y = 0$   
 $-\sin x + \cos x + \sin x - \cos x = 0 \checkmark$
2.  $y(0) = -1 \checkmark$
3.  $y'(0) = 1 \checkmark$

ex. Find the values of  $m$  and  $n$  for which  $e^{mx}$  is a solution to  $y'' - 4y' - 5y = 0$

$$\begin{aligned}\phi(x) &= e^{mx} \\ \phi'(x) &= me^{mx} \\ \phi''(x) &= m^2e^{mx}\end{aligned}$$

Plug into  $y'' - 4y' - 5y = 0$

$$m^2e^{mx} - 4me^{mx} - 5e^{mx} = e^{mx}(m^2 - 4m - 5) = 0$$

$$e^{mx}(m-5)(m+1) = 0$$

$$m = 5, -1$$

**Def.** a DE is **separable** if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y).$$

ex. Solve  $\frac{dy}{dx} = x^2(1+y)$ ,  $y(0) = 3$

$$\begin{aligned}\frac{1}{1+y} dy &= x^2 dx \\ \int \frac{1}{1+y} dy &= \int x^2 dx \\ \ln|1+y| &= \frac{x^3}{3} + C \\ e^{\ln|1+y|} &= e^{\frac{x^3}{3} + C} \\ |1+y| &= e^{\frac{x^3}{3}} e^C \\ |1+y| &= Ce^{\frac{x^3}{3}} \\ 1+y &= Ce^{\frac{x^3}{3}} \\ y &= Ce^{\frac{x^3}{3}} - 1\end{aligned}$$

**Insert initial conditions to get**

$$3 = Ce^0 - 1$$

$$3 = C - 1$$

$$C = 4$$

$$y = 4e^{\frac{x^3}{3}} - 1$$

Def. A first-order linear DE has the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x), \quad (1)$$

where  $a_1(x) \neq 0$ .

Method for solving linear equations:

1. Write equation in standard form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2. Find the integrating factor  $\mu(x) = e^{\int P(x) dx}$

3. Multiply (1) by  $\mu(x)$  and rewrite the DE as

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

4. Integrate to solve for  $y = \frac{1}{\mu(x)} \left[ \int \mu(x)Q(x) dx + C \right]$

ex. Solve  $(x^2 + 4) \frac{dy}{dx} + xy = x$

Put in standard form:  $\frac{dy}{dx} + \frac{x}{(x^2+4)}y = \frac{x}{(x^2+4)}$

$$P(x) = \frac{x}{x^2+4}$$

$$\mu(x) = e^{\int \frac{x}{x^2+4} dx}$$

Let  $u = x^2 + 4$  then  $du = 2xdx$ ,  $dx = du/2x$ , and we get

$$e^{\frac{1}{2} \int \frac{du}{u}} = e^{\frac{1}{2} \ln|u|}$$

$$= e^{\ln|u|^{\frac{1}{2}}} = \sqrt{u} = \sqrt{x^2 + 4}$$

Then the DE becomes

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

$$\frac{d}{dx}[\sqrt{x^2 + 4}y] = \frac{x}{x^2+4}\sqrt{x^2 + 4} = \frac{x}{\sqrt{x^2+4}}$$

$$[\sqrt{x^2 + 4}y] = \int \frac{x}{\sqrt{x^2+4}} dx$$

Let  $u = x^2 + 4$ , then  $du = 2xdx$ ,  $dx = du/2x$ , and we get

$$\frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = u^{\frac{1}{2}} + C = \sqrt{x^2 + 4} + C$$

$$[\sqrt{x^2 + 4}y] = \sqrt{x^2 + 4} + C$$

$$\text{Then } y = \sqrt{x^2 + 4} \frac{1}{\sqrt{x^2+4}} + \frac{C}{\sqrt{x^2+4}} = 1 + \frac{C}{\sqrt{x^2+4}}$$

**Method for solving exact equations:**

$$M(x, y)dx + N(x, y)dy = 0$$

1. Check exactness:  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

2. If exact, write  $\frac{\partial F}{\partial x} = M(x, y)$  and then

$$f(x, y) = \int M dx + g(y)$$

3. Determine  $g(y)$ :

- take the partial derivative w.r.t  $y$  of both sides of  $f(x, y)$

$$\frac{\partial f}{\partial y} = (d/dy)(\int M dx) + g'(y)$$

- substitute  $N(x, y) = \frac{\partial f}{\partial y}$  to determine  $g'(y)$

- find  $g(y)$  through integration

$$g(y) = \int g'(y) dy$$

4. Substitute  $g(y)$  into (\*) to get  $f(x, y)$ .

5. Solution:  $F(x, y) = C$

**ex Solve**  $(6x^2 - y + 3)dx + (3y^2 - x - 2)dy = 0$

$$M = 6x^2 - y + 3, \quad N = 3y^2 - x - 2$$

Check for exactness:

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} = -1$$

Construct  $F(x, y)$ :

$$F(x, y) = \int M dx + g(y) = \int (6x^2 - y + 3) dx + g(y) = 2x^3 - xy + 3x + g(y)$$

Find  $g(y)$ :

$$\frac{\partial F}{\partial y} = -x + g'(y) = N = 3y^2 - x - 2$$

$$g'(y) = 3y^2 - 2$$

$$g(y) = \int (3y^2 - 2) dy = y^3 - 2y + C$$

Then

$$F(x, y) = 2x^3 - xy + 3x + y^3 - 2y + C$$

$$2x^3 - xy + 3x + y^3 - 2y = C$$

## Method for solving equations that require integrating factors to be exact

If  $M(x, y)dx + N(x, y)dy = 0$  is not exact.

(a) If  $\frac{\partial M/\partial y - \partial N/\partial x}{N}$  depends only on  $x$ , then use

$$\mu(x) = \exp \left[ \int \left( \frac{\partial M/\partial y - \partial N/\partial x}{N} \right) dx \right]$$

as an integrating factor.

(b) If  $\frac{\partial N/\partial x - \partial M/\partial y}{M}$  depends only on  $y$ , then use

$$\mu(y) = \exp \left[ \int \left( \frac{\partial N/\partial x - \partial M/\partial y}{M} \right) dy \right]$$

an integrating factor.

(c) Proceed with the method for exact equations described previously.

ex Solve  $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$

Check for exactness:

$$\frac{\partial M}{\partial y} = 4y + 2$$

$$\frac{\partial N}{\partial x} = 2y + 1$$

The DE is not exact so we must find the integrating factor:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y+2-2y-1}{x(2y+1)} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Multiply the DE by the integrating factor:

$$(2xy^2 + 2xy + 4x^3)dx + (2x^2y + x^2)dy = 0$$

The equation is now exact (Check for yourself!)

Construct  $F(x, y)$ :

$$f(x, y) = \int (2xy^2 + 2xy + 4x^3)dx = x^2y^2 + x^2y + x^4 + g(y)$$

Find  $g(y)$ :

$$\frac{\partial f}{\partial y} = 2x^2y + x^2 + g'(y) = N = 2x^2y + x^2$$

$$g'(y) = 0$$

$$g(y) = C$$

Then

$$f(x, y) = x^2y^2 + x^2y + x^4 = C$$

### Method for Solving Homogeneous Equations:

Using substitution  $v = y/x$  (or  $u = x/y$ ) will reduce a homogeneous DE

$$M(x, y)dx + N(x, y)dy = 0$$

to a separable first-order DE.

1. Make the substitution  $v = \frac{y}{x}$ . Now our DE is

$$\frac{dy}{dx} = G(v) \quad (*)$$

2. Our new variables are  $v$  and  $x$ , so we need to make  $\frac{dy}{dx}$  in terms of  $v$  and  $x$ .

$$v = \frac{y}{x} \rightarrow y = vx$$

$$\frac{dy}{dx} = x \frac{dv}{dx} + v$$

3. Sub this into (\*), separate variables and integrate

4. The last step after integration is to put the equation back in terms of  $x$  and  $y$ .

ex  $(y^2 + yx)dx - x^2dy = 0$

Let  $y = vx$ , then  $dy = xdv + vdx$  and we get  
 $(v^2x^2 + vx^2)dx - x^2(xdv + vdx) = 0$

Group the  $dx$  and the  $dy$  terms separately:

$$(v^2x^2 + vx^2 - vx^2)dx - x^3dv = 0$$

$$x^2v^2dx = x^3dv$$

$$x^3dv = x^2v^2dx$$

$$xdv = v^2dx$$

Separate variables:

$$\frac{dv}{v^2} = \frac{dx}{x}$$

Integrate:

$$-\frac{1}{v} = \ln|x| + c$$

Replace  $v$  with  $\frac{y}{x}$

$$-\frac{x}{y} = \ln|x| + c$$

Solve for  $y$

$$-\frac{y}{x} = \frac{1}{\ln|x| + c}$$

$$y = -\frac{x}{\ln|x| + c}$$

## Method for solving Bernoulli Equations

**Def.** A first-order equation that can be written as

$$(*) \quad \frac{dy}{dx} + P(x)y = Q(x)y^n,$$

$P(x)$  and  $Q(x)$  continuous on an interval  $(a, b)$  and  $n$  is a real number, is called a Bernoulli Equation.

1. Divide  $(*)$  by  $y^n$ :  $y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$
2. Substitute  $v = y^{1-n}$  (this will transform the Bernoulli equation into a linear equation).
3. To find  $\frac{dy}{dx}$  in terms of  $w$ , differentiate  $v = y^{1-n}$  :
 
$$\frac{dv}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx}$$
4. Substitute this into the above equation (1):
 
$$y^{-n} \left( \frac{1}{1-n}y^n \frac{dv}{dx} \right) + P(x)y^{1-n} = Q(x)$$

$$\left( \frac{1}{1-n} \frac{dv}{dx} \right) + P(x)v = Q(x)$$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

This is now a linear equation in  $v$  and can be solved as one.

**ex**  $\frac{dy}{dx} + y = xy^{-2}$

$n = -2$ , then  $v = y^{1-(-2)} = y^3$  and we get

$$\frac{dv}{dx} = 3y^2 \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2} \frac{dv}{dx}$$

Our equation becomes

$$\frac{1}{3y^2} \frac{dv}{dx} + y = xy^{-2}$$

Multiply by  $3y^2$

$$\frac{dv}{dx} + 3y^3 = 3x$$

$$\frac{dv}{dx} + 3v = 3x$$

Now our equation is linear with  $P(x) = 3$  and  $Q(x) = 3x$

$$\mu(x) = e^{\int 3dx} = e^{3x}$$

$$\frac{d}{dx}[e^{3x}v] = 3xe^{3x}$$

$$e^{3x}v = 3 \int xe^{3x}$$

Integrating by parts gives  $e^{3x}v = 3[e^{3x}(\frac{x}{3} - \frac{1}{9}) + C]$

$$v = x - \frac{1}{3} + 3Ce^{-3x}$$

$$y^3 = x - \frac{1}{3} + 3Ce^{-3x}$$

$$y = \sqrt[3]{x - \frac{1}{3} + 3Ce^{-3x}}$$

## Classification by type

**A DE is linear if it can be written in the form**

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

**Def.** a DE is separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y).$$

**Def.** A first-order linear DE has the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x), \quad a_1(x) \neq 0.$$

**Def.** A DE is exact if it is in the form  $M(x, y)dx + N(x, y)dy = 0$  and  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

**Def.** A DE  $\frac{dy}{dx} = f(x, y)$  is homogeneous if the right hand side can be expressed as a function of  $y/x$  alone.

**Def.** A first-order equation that can be written as  $\frac{dy}{dx} + P(x)y = Q(x)y^n$ ,  $P(x)$  and  $Q(x)$  continuous on an interval  $(a, b)$  and  $n$  is a real number, is called a Bernoulli Equation. Note: Linear equations are always Bernoulli.

**ex** Classify  $\frac{dy}{dx} = 5y + y^3$

**Separable?**

$$\frac{dy}{5y+y^3} = dx; \text{ so it is separable}$$

**Linear in  $x$ ?**

$$\frac{dx}{dy} = \frac{1}{5y+y^3} \text{ so it is linear in } x$$

**Bernoulli?**

$$\frac{dy}{dx} - 5y = y^3 \text{ so it is a Bernoulli equation}$$

**Exact?**

$$dy = (5y + y^3)dx$$

$$(5y + y^3)dx - dy = 0$$

$$M_y = 5 + 3y^2; \quad N_x = 0; \text{ so it is not exact}$$

**Linear in  $y$ ?**

$y^3$  term it so not linear in  $y$

**Misc. Integration tips and identities:**

$$e^{x+y} = e^x e^y$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$1. \int u \, dv = uv - \int v \, du$$

$$2. \int u^n \, du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$3. \int \frac{1}{u} \, du = \ln|u| + C \quad (n = -1)$$

$$4. \int e^u \, du = e^u + C$$

$$5. \int a^u \, du = \frac{a^u}{\ln a} + C$$

$$6. \int \sin u \, du = -\cos u + C$$

$$7. \int \cos u \, du = \sin u + C$$

$$8. \int \sec^2 u \, du = \tan u + C$$

$$9. \int \sec u \tan u \, du = \sec u + C$$

$$10. \int \tan u \, du = \ln|\sec u| + C$$

$$11. \int \cot u \, du = \ln|\sin u| + C$$

$$12. \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$13. \int \csc u \, du = \ln|\csc u - \cot u| + C$$

$$14. \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left( \frac{u}{a} \right) + C$$

$$15. \int \frac{du}{1+u^2} = \tan^{-1} u + C$$