SECTION 4.6

1, (2). $y'' + 4y = \tan 2t$

The characteristic equation is given by

$$r^2 + 4 = 0$$

so there exists two imaginary roots given by $r = \pm 2i$ and the homogeneous solution is given by

$$y_h = c_1 \cos 2t + c_2 \sin 2t.$$

Hence we choose $y_1 = \cos 2t$ and $y_2 = \sin 2t$ so that the equations

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = \tan 2t$$

can be written as

$$v_1' \cos 2t + v_2' \sin 2t = 0$$

$$-2v_1' \sin 2t + 2v_2' \cos 2t = \tan 2t.$$

Multiplying the first equation by $2\sin 2t$ and the second by $\cos 2t$ and adding the equations together yields

$$2v_2'(\sin^2 2t + \cos^2 2t) = 2v_2' = \sin 2t$$

so that

$$v_2 = (1/2)\int \sin 2t \ dt = -(1/4)\cos 2t.$$

Multiplying the first equation by $2\cos 2t$ and the second by $-\sin 2t$ and adding the equations together yields

$$2v_1'(\sin^2 2t + \cos^2 2t) = 2v_1' = -\sin 2t \tan 2t$$

so that

$$v_1 = -(1/2)\int \sin 2t \tan 2t \ dt.$$

Using integration by parts the second integral becomes

$$v_1 = -(1/2)[-(1/2)\cos 2t \tan 2t + \int \sec^2 2t \cos 2t \ dt] = -(1/2)[-(1/2)\cos 2t \tan 2t + \int \sec 2t \ dt]$$

$$v_1 = -(1/2)[-(1/2)\cos 2t \tan 2t + (1/2)\ln|\sec 2t + \tan 2t|].$$

Hence a particular solution is given by

$$y_p = (1/4)[\cos 2t \tan 2t - \ln|\sec 2t + \tan 2t|]\cos 2t + -(1/4)\cos 2t \sin 2t = -(1/4)\cos 2t \ln|\sec 2t + \tan 2t|$$

and the general solution is given by

$$y_g = y_h + y_p = c_1 \cos 2t + c_2 \sin 2t - (1/4)\cos 2t \ln|\sec 2t + \tan 2t|.$$

3, (-). $2x'' - 2x' - 4x = 2e^{3t}$

The characteristic equation is given by

$$2r^2 - 2r - 4 = 2(r+1)(r-2) = 0$$

so there exists two real, distinct roots given by 2 and -1 and the homogeneous solution is given by

$$x_h = c_1 e^{2t} + c_2 e^{-t}.$$

Hence we choose $x_1 = e^{2t}$ and $y_2 = e^{-t}$ so that the equations

$$v_1' x_1 + v_2' x_2 = 0$$

$$v_1' x_1' + v_2' x_2' = e^{3t}$$

can be written as

$$v_1' e^{2t} + v_2' e^{-t} = 0$$
$$2v_1' e^{2t} - v_2' e^{-t} = e^{3t}.$$

Adding the equations together yields

$$3v_1' e^{2t} = e^{3t}$$

so that

$$v_1 = (1/3)\int e^t dt = (1/3)e^t.$$

Multiplying the the first equation by -2 and adding the equations together yields

$$-3v_2'e^{-t} = e^{3t}$$

so that

$$v_2 = -(1/3)\int e^{4t}\, dt = -(1/12)e^{4t}.$$

Hence a particular solution is given by

$$x_p = (1/3)e^t e^{2t} - (1/12)e^{4t}e^{-t} = (1/4)e^{3t}$$

and the general solution is given by

$$x_g = x_h + x_p = c_1e^{2t} + c_2e^{-t} + (1/4)e^{3t}.$$

5, (4). $y'' - 2y' + y = t^{-1}e^t$

The characteristic equation is given by

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

so there exists one real, repeated root given by 1 and the homogeneous solution is given by

$$y_h = c_1e^t + c_2te^t.$$

Hence we choose $y_1 = e^t$ and $y_2 = te^t$ so that the equations

$$v_1'y_1 + v_2'y_2 = 0$$

$$v_1'y_1' + v_2'y_2' = t^{-1}e^t$$

can be written as

$$v_1'e^t + v_2'te^t = 0$$

$$v_1'e^t + v_2'(t+1)e^t = t^{-1}e^t$$

Multiplying the first equation by $t+1$ and the second by $-t$ and adding the equations together yields

$$v_1'e^t = -e^t$$

so that

$$v_1 = \int -1\, dt = -t.$$

Subtracting the first equation form the second yields

$$v_2'e^t = t^{-1}e^t$$

so that

$$v_2 = \int t^{-1}\, dt = \ln|t|.$$

Hence a particular solution is given by

$$y_p = -te^t + te^t \ln|t|$$

and the general solution is given by

$$y_g = y_h + y_p = c_1e^t + c_2te^t - te^t + te^t \ln|t| = c_1e^t + c_2te^t + te^t \ln|t|.$$

7, (6). $y'' + 16y = \sec 4\theta$

The characteristic equation is given by

$$r^2 + 16 = 0$$

so there exists two imaginary roots given by $r = \pm 4i$ and the homogeneous solution is given by

$$y_h = c_1 \cos 4\theta + c_2 \sin 4\theta.$$

Hence we choose $y_1 = \cos 4\theta$ and $y_2 = \sin 4\theta$ so that the equations

$$v_1'y_1 + v_2'y_2 = 0$$

$$v_1'y_1' + v_2'y_2' = \sec 4\theta$$

can be written as

$$v_1' \cos 4\theta + v_2' \sin 4\theta = 0$$

$$-4v_1' \sin 4\theta + 4v_2' \cos 4\theta = \sec 4\theta$$

Multiplying the first equation by $4\sin 4\theta$ and the second by $\cos 4\theta$ and adding the equations together yields

$$4v_2'(\cos^2 4\theta + \sin^2 4\theta) = 4v_2' = 1$$

so that

$$v_2 = \int 1/4 \; d\theta = \theta/4.$$

Multiplying the first equation by $4\cos 4\theta$ and the second by $-\sin 4\theta$ and adding the equations together yields

$$4v_1'(\cos^2 4\theta + \sin^2 4\theta) = 4v_1' = -\tan 4\theta$$

so that

$$v_1 = (1/4)\int -\tan 4\theta \; d\theta = (1/16)\ln|\cos 4\theta|.$$

Hence a particular solution is given by

$$y_p = (1/16)\ln|\cos 4\theta|\cos 4\theta + (\theta/4)\sin 4\theta$$

and the general solution is given by

$$y_g = y_h + y_p = c_1\cos 4\theta + c_2\sin 4\theta + (1/16)\ln|\cos 4\theta|\cos 4\theta + (\theta/4)\sin 4\theta.$$

9, (8). $y'' + 4y = \csc^2 2t$

The characteristic equation is given by

$$r^2 + 4 = 0$$

so there exists two imaginary roots given by $r = \pm 2i$ and the homogeneous solution is given by

$$y_h = c_1\cos 2t + c_2\sin 2t.$$

Hence we choose $y_1 = \cos 2t$ and $y_2 = \sin 2t$ so that the equations

$$v_1'y_1 + v_2'y_2 = 0$$

$$v_1'y_1' + v_2'y_2' = \csc^2 2t$$

can be written as

$$v_1'\cos 2t + v_2'\sin 2t = 0$$

$$-2v_1'\sin 2t + 2v_2'\cos 2t = \csc^2 2t.$$

Multiplying the first equation by $2\sin 2t$ and the second by $\cos 2t$ and adding the equations together yields

$$2v_2'(\sin^2 2t + \cos^2 2t) = 2v_2' = \csc^2 2t \cos 2t = \csc 2t \cot 2t$$

so that

$$v_2 = (1/2)\int \csc 2t \cot 2t \; dt = -(1/4)\csc 2t.$$

Multiplying the first equation by $2\cos 2t$ and the second by $-\sin 2t$ and adding the equations together yields

$$2v_1'(\sin^2 2t + \cos^2 2t) = 2v_1' = -\sin 2t \csc^2 2t = -\csc 2t$$

so that

$$v_1 = -(1/2)\int \csc 2t \; dt = (-1/4)\ln|\csc 2t - \cot 2t|.$$

Hence a particular solution is given by

$$y_p = (-1/4)\ln|\csc 2t - \cot 2t|\cos 2t + -(1/4)\csc 2t \sin 2t = (-1/4)\ln|\csc 2t - \cot 2t|\cos 2t - (1/4)$$

and the general solution is given by

$$y_g = y_h + y_p = c_1\cos 2t + c_2 \sin 2t - (1/4)\ln|\csc 2t - \cot 2t|\cos 2t - (1/4).$$

11, (11). $y'' + y = \tan^2 t$

The characteristic equation is given by

$$r^2 + 1 = 0$$

so there exists two imaginary roots given by $r = \pm i$ and the homogeneous solution is given by

$$y_h = c_1 \cos t + c_2 \sin t.$$

Hence we choose $y_1 = \cos t$ and $y_2 = \sin t$ so that the equations

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = \tan^2 t$$

can be written as

$$\begin{aligned} v_1' \cos t + v_2' \sin t &= 0 \\ -v_1' \sin t + v_2' \cos t &= \tan^2 t. \end{aligned}$$

Multiplying the first equation by $\sin t$ and the second by $\cos t$ and adding the equations together yields

$$v_2'(\sin^2 t + \cos^2 t) = v_2' = \tan^2 t \cos t = \sin t \tan t$$

so that

$$v_2 = \int \sin t \tan t \; dt.$$

Using integration by parts we obtain

$$v_2 = -\cos t \tan t + \int \cos t \sec^2 t \, dt = -\sin t + \int \sec t \, dt = -\sin t + \ln|\sec t + \tan t|.$$

Multiplying the first equation by $\cos t$ and the second by $-\sin t$ and adding the equations together yields

$$v_1'(\sin^2 t + \cos^2 t) = v_1' = -\sin t \tan^2 t$$

so that

$$v_1 = -\int \sin t \tan^2 t \; dt.$$

Using integration by parts we obtain

$$v_1 = \cos t \tan^2 t - \int 2 \tan t \sec t \; dt = \sin t \tan t - 2 \sec t.$$

Hence a particular solution is given by

$$\begin{aligned} y_p = (\sin t \tan t - 2\sec t)\cos t + (-\sin t + \ln|\sec t + \tan t|)\sin t &= \sin^2 t - 2 - \sin^2 t + (\ln|\sec t + \tan t|)\sin t \\ &= (\ln|\sec t + \tan t|)\sin t - 2. \end{aligned}$$

and the general solution is given by

$$y_g = y_h + y_p = c_1 \cos t + c_2 \sin t + (\ln|\sec t + \tan t|)\sin t - 2.$$

13, (13). $y'' + 4y = \sec^4 2t$

The characteristic equation is given by

$$r^2 + 4 = 0$$

so there exists two imaginary roots given by $r = \pm 2i$ and the homogeneous solution is given by

$$y_h = c_1 \cos 2t + c_2 \sin 2t.$$

Hence we choose $y_1 = \cos 2t$ and $y_2 = \sin 2t$ so that the equations

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = \sec^4 2t$$

can be written as

$$v_1' \cos 2t + v_2' \sin 2t = 0$$

$$-2v_1' \sin 2t + 2v_2' \cos 2t = \sec^4 2t.$$

Multiplying the first equation by $2\sin 2t$ and the second by $\cos 2t$ and adding the equations together yields

$$2v_2'(\sin^2 2t + \cos^2 2t) = 2v_2' = \sec^4 2t \cos 2t = \sec^3 2t$$

so that

$$v_2 = (1/2) \int \sec^3 2t \ dt.$$

Using integration by parts we obtain

$$\int \sec^3 2t \ dt = (1/2) \sec 2t \tan 2t - \int \sec 2t \tan^2 2t \ dt$$

$$\int \sec^3 2t \ dt = (1/2) \sec 2t \tan 2t - \int \sec 2t (\sec^2 2t - 1) \ dt$$

$$\int \sec^3 2t \ dt = (1/2) \sec 2t \tan 2t - \int \sec^3 2t \ dt + \int \sec 2t \ dt$$

$$2 \int \sec^3 2t \ dt = (1/2) \sec 2t \tan 2t + (1/2) \ln |\sec 2t + \tan 2t|$$

$$v_2 = (1/2) \int \sec^3 2t \ dt = (1/8) \sec 2t \tan 2t + (1/8) \ln |\sec 2t + \tan 2t|.$$

Multiplying the first equation by $2\cos 2t$ and the second by $-\sin 2t$ and adding the equations together yields

$$2v_1'(\sin^2 2t + \cos^2 2t) = 2v_1' = -\tan 2t \sec^3 2t$$

so that

$$v_1 = -(1/2)\int \tan 2t \sec^3 2t \; dt = -(1/12)\sec^3 2t.$$

Hence a particular solution is given by

$$y_p = -(1/12)\sec^3 2t \cos 2t + (1/8)(\sec 2t \tan 2t + \ln|\sec 2t + \tan 2t|)\sin 2t$$

$$= -(1/12)\sec^2 2t + (1/8)\tan^2 2t + (1/8)(\ln|\sec 2t + \tan 2t|)\sin 2t$$

$$= -(1/12)\sec^2 2t + (1/8)(\sec^2 2t - 1) + (1/8)(\ln|\sec 2t + \tan 2t|)\sin 2t$$

$$= (1/24)\sec^2 2t - (1/8) + (1/8)(\ln|\sec 2t + \tan 2t|)\sin 2t.$$

and the general solution is given by

$$y_g = y_h + y_p = c_1\cos 2t + c_2 \sin 2t + (1/24)\sec^2 2t - (1/8) + (1/8)(\ln|\sec 2t + \tan 2t|)\sin 2t.$$

15, (15). $y'' + y = 3\sec t - t^2 + 1$
From the form of the right hand side of the equation, it is best to solve this problem using the Superposition Principle; as a result we separately solve the problems

$$y'' + y = 3\sec t$$

$$y'' + y = -t^2 + 1.$$

The characteristic equation is given by

$$r^2 + 1 = 0$$

so there exists two imaginary roots given by $r = \pm i$ and the homogeneous solution is given by

$$y_h = c_1 \cos t + c_2 \sin t.$$

We solve the first equation using variation of parameters. Hence we choose $y_1 = \cos t$ and $y_2 = \sin t$ so that the equations

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = 3\sec t$$

can be written as

$$v_1' \cos t + v_2' \sin t = 0$$

$$-v_1' \sin t + v_2' \cos t = 3\sec t.$$

Multiplying the first equation by $\sin t$ and the second by $\cos t$ and adding the equations together yields

$$v_2'(\sin^2 t + \cos^2 t) = v_2' = 3 \sec t \cos t = 3$$

so that

$$v_2 = \int 3 \; dt = 3t.$$

Multiplying the first equation by $\cos t$ and the second by $-\sin t$ and adding the equations together yields

$$v_1'(\sin^2 t + \cos^2 t) = v_1' = -3 \sec t \sin t = -3 \tan t$$

so that

$$v_1 = -3 \int \tan t \; dt = 3 \ln |\cos t|.$$

Hence a particular solution is given by

$$y_p = 3 \cos t \ln |\cos t| + 3t \sin t.$$

The second equation is solved using the method of undetermined coefficients. Since the right hand side is a 2nd degree polynomial multiplying an exponential of the form $e^{0 \cdot t}$, we know $s = 0$ and $y_p = At^2 + Bt + C$ so that

$$y_p = At^2 + Bt + C$$
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

and the substituting into the ODE gives

$$2A + At^2 + Bt + C = -t^2 + 1.$$

Comparing coefficients of similar powers of t yields $A = -1, B = 0$ and $C = 3$ so that

$$y_p = -t^2 + 3.$$

Combining all of the above gives

$$y_g = y_h + y_{p1} + y_{p2} = c_1 \cos t + c_2 \sin t + 3 \cos t \ln|\cos t| + 3t \sin t - t^2 + 3.$$

17, (17). $(1/2)y'' + 2y = \tan 2t - (1/2)e^t$
From the form of the right hand side of the equation, it is best to solve this problem using the Superposition Principle; as a result we separately solve the problems

$$y'' + 4y = 2\tan 2t$$

$$y'' + 4y = -e^t.$$

The characteristic equation is given by

$$r^2 + 4 = 0$$

so there exists two imaginary roots given by $r = \pm 2i$ and the homogeneous solution is given by

$$y_h = c_1 \cos 2t + c_2 \sin 2t.$$

We solve the first equation using variation of parameters. Hence we choose $y_1 = \cos 2t$ and $y_2 = \sin 2t$ so that the equations

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = 2\tan 2t$$

can be written as

$$v_1' \cos 2t + v_2' \sin 2t = 0$$

$$-v_1' \sin 2t + v_2' \cos 2t = 2\tan 2t.$$

Multiplying the first equation by $2\sin 2t$ and the second by $\cos 2t$ and adding the equations together yields

$$2v_2'(\sin^2 2t + \cos^2 2t) = 2v_2' = 2\sin 2t$$

so that

$$v_2 = \int \sin 2t \; dt = -(1/2)\cos 2t.$$

Multiplying the first equation by $2\cos 2t$ and the second by $-\sin 2t$ and adding the equations together yields

$$2v_1'(\sin^2 2t + \cos^2 2t) = 2v_1' = -2\sin 2t \tan 2t$$

so that

$$v_1 = -\int \sin 2t \tan 2t \; dt.$$

Using integration by parts the second integral becomes

$$v_1 = -[-(1/2)\cos 2t \tan 2t + \int \sec^2 2t \cos 2t \; dt] = -(1/2)[-(1/2)\cos 2t \tan 2t + \int \sec 2t \; dt]$$

$$v_1 = -[-(1/2)\cos 2t \tan 2t + (1/2)\ln|\sec 2t + \tan 2t|].$$

Hence a particular solution is given by

$$y_p = -[-(1/2)\cos 2t \tan 2t + (1/2)\ln|\sec 2t + \tan 2t|]\cos 2t + -(1/2)\cos 2t \sin 2t.$$

$$y_p = -(1/2)\ln|\sec 2t + \tan 2t|\cos 2t$$

The second equation is solved using the method of undetermined coefficients. Since the right hand side is a zero degree polynomial multiplying an exponential of the form $e^{1\cdot t}$, we know $s = 0$ and $y_p = Ae^t$ so that

$$y_p = Ae^t$$

$$y_p' = Ae^t$$

$$y_p'' = Ae^t$$

and the substituting into the ODE gives

$$Ae^t + 4Ae^t = -Ae^t.$$

Hence $A = -1/5$ and

$$y_p = -e^t/5.$$

Combining all of the above gives

$$y_g = y_h + y_{p1} + y_{p2} = c_1 \cos 2t + c_2 \sin 2t - (1/2)\ln|\sec 2t + \tan 2t|\cos 2t - e^t/5.$$