$$s = 0: \quad 14 = 2A + 2C = 2(3) + 2C \longrightarrow C = 4$$
$$s = 1: \quad 36 = A + 3B + 3C = 3 + 3B + 3(4) \longrightarrow B = 7.$$

Therefore we have

$$L^{-1}\Big(\frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)}\Big) = L^{-1}\Big(\frac{3}{s+2} + \frac{7s+4}{s^2 - 2s + 2}\Big)$$

$$L^{-1}\Big(\frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)}\Big) = L^{-1}\Big(\frac{3}{s+2}\Big) + L^{-1}\Big(\frac{7s+4}{(s-1)^2 + 1}\Big)$$

$$L^{-1}\Big(\frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)}\Big) = L^{-1}\Big(\frac{3}{s+2}\Big) + L^{-1}\Big(\frac{7(s-1)}{(s-1)^2 + 1}\Big) + L^{-1}\Big(\frac{11}{(s-1)^2 + 1}\Big)$$

$$f(t) = 3e^{-2t} + 7e^t \cos t + 11e^t \sin t.$$

SECTION 7.5

1. $y'' - 2y' + 5y = 0, \ y(0) = 2, \ y'(0) = 4$

Taking the Laplace transform of both sides of the equation gives

$$(s^2Y(s) - s(2) - 4) - 2(sY(s) - 2) + 5Y(s) = 0$$

$$(s^2 - 2s + 5)Y(s) - 2s = 0$$

$$Y(s) = \frac{2s}{s^2 - 2s + 5}$$

$$Y(s) = \frac{2s}{(s-1)^2 + 2^2}$$

$$Y(s) = \frac{2(s-1) - 2}{(s-1)^2 + 2^2}$$

$$Y(s) = 2\Big(\frac{(s-1)}{(s-1)^2 + 2^2}\Big) + \frac{2}{(s-1)^2 + 2^2}.$$

3. $y'' + 6y' + 9y = 0,\ y(0) = -1,\ y'(0) = 6$

Taking the Laplace transform of both sides of the equation gives

$$(s^2Y(s) - s(-1) - 6) + 6(sY(s) - (-1)) + 9Y(s) = 0$$

$$\begin{aligned}(s^2 + 6s + 9)Y(s) + s &= 0\\ Y(s) &= -\frac{s}{s^2 + 6s + 9}\\ Y(s) &= -\frac{s}{(s+3)^2}.\end{aligned}$$

Since the denominator is completely factored, we proceed to partial fraction expansion:

$$\frac{s}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}.$$

After obtaining common denominators and equating numerators we have

$$s = A(s+3) + B.$$

Choosing appropriate values of s gives:

$$s = -3: \quad -3 = B$$

$$s = 0: \quad 0 = 3A + B = 3A - 3 \ \longrightarrow\ A = 1.$$

Therefore we have

$$\begin{aligned}\frac{s}{(s+3)^2} &= \frac{1}{s+3} - \frac{3}{(s+3)^2}\\ -\frac{s}{(s+3)^2} &= -\frac{1}{s+3} + \frac{3}{(s+3)^2}.\end{aligned}$$

Taking inverse transforms yields

$$y(t) = -e^{-3t} + 3te^{-3t}.$$

5. $w'' + w = t^2 + 2,\ w(0) = 1,\ w'(0) = -1$

Taking the Laplace transform of both sides of the equation gives

$$
\begin{aligned}
(s^2W(s) - s(1) - (-1)) + W(s) &= \frac{2}{s^3} + \frac{2}{s} \\
(s^2+1)W(s) - s + 1 &= \frac{2}{s^3} + \frac{2}{s} \\
(s^2+1)W(s) &= s - 1 + \frac{2}{s^3} + \frac{2}{s} \\
(s^2+1)W(s) &= \frac{2 + 2s^2 + (s-1)s^3}{s^3} \\
W(s) &= \frac{2 + 2s^2 - s^3 + s^4}{s^3(s^2+1)}.
\end{aligned}
$$

Since the denominator is completely factored, we proceed to partial fraction expansion:

$$\frac{2 + 2s^2 - s^3 + s^4}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+1}.$$

After obtaining common denominators and equating numerators we have

$$2 + 2s^2 - s^3 + s^4 = As^2(s^2+1) + Bs(s^2+1) + C(s^2+1) + (Ds+E)s^3$$

$$2 + 2s^2 - s^3 + s^4 = (A+D)s^4 + (B+E)s^3 + (A+C)s^2 + Bs + C.$$

Comparing similar powers of s we obtain:

$$
\begin{aligned}
s^0 &:\ 2 = C \\
s^1 &:\ 0 = B \\
s^2 &:\ 2 = A + C = A + 2 \ \longrightarrow\ A = 0 \\
s^3 &:\ -1 = B + E = 0 + E \ \longrightarrow\ E = -1 \\
s^4 &:\ 1 = A + D = 0 + D \ \longrightarrow\ D = 1
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\frac{s}{(s+3)^2} &= \frac{1}{s+3} - \frac{3}{(s+3)^2} \\
-\frac{s}{(s+3)^2} &= -\frac{1}{s+3} + \frac{3}{(s+3)^2}.
\end{aligned}
$$

$$\frac{2+2s^2-s^3+s^4}{s^3(s^2+1)} = \frac{0}{s}+\frac{0}{s^2}+\frac{2}{s^3}+\frac{s-1}{s^2+1}$$
$$\frac{2+2s^2-s^3+s^4}{s^3(s^2+1)} = \frac{2}{s^3}+\frac{s-1}{s^2+1}$$
$$\frac{2+2s^2-s^3+s^4}{s^3(s^2+1)} = \frac{2}{s^3}+\frac{s}{s^2+1}-\frac{1}{s^2+1}.$$

Taking inverse transforms yields

$$w(t) = t^2+\cos t-\sin t.$$

7. $y''-7y'+10y = 9\cos t+7\sin t,\ y(0)=5\ y'(0)=-4$

Taking the Laplace transform of both sides of the equation gives

$$(s^2Y(s)-s(5)-(-4))-7(sY(s)-5)+10Y(s) = \frac{9s}{s^2+1}+\frac{7}{s^2+1}$$
$$(s^2-7s+10)Y(s)-5s+39 = \frac{9s+7}{s^2+1}$$
$$(s^2-7s+10)Y(s) = \frac{9s+7}{s^2+1}+5s-39$$
$$(s-5)(s-2)Y(s) = \frac{9s+7+(5s-39)(s^2+1)}{s^2+1}$$
$$Y(s) = \frac{-32+14s-39s^2+5s^3}{(s^2+1)(s-5)(s-2)}.$$

Since the denominator is completely factored, we proceed to partial fraction expansion:

$$\frac{-32+14s-39s^2+5s^3}{(s^2+1)(s-5)(s-2)} = \frac{A}{s-5}+\frac{B}{s-2}+\frac{Cs+D}{s^2+1}.$$

After obtaining common denominators and equating numerators we have

$$-32+14s-39s^2+5s^3 = A(s-2)(s^2+1)+B(s-5)(s^2+1)+(Cs+D)(s-2)(s-5).$$

Choosing appropriate values of s gives:

$$s = 2: \quad -120 = -15B \longrightarrow B = 8$$

$$s = 5: \quad -312 = 78A = \longrightarrow A = -4$$

$$s = 0: \quad -32 = -2A - 5B + 10D = -2(-4) - 5(8) + 10D \longrightarrow D = 0$$

$$s = 1: \quad -52 = -2A - 8B + 4C + 4D = -2(-4) - 8(8) + 4C + 4(0) \longrightarrow C = 1$$

Therefore we have

$$\frac{-32 + 14s - 39s^2 + 5s^3}{(s^2+1)(s-5)(s-2)} = \frac{-4}{s-5} + \frac{8}{s-2} + \frac{s}{s^2+1}.$$

Taking inverse transforms yields

$$y(t) = -4e^{5t} + 8e^{2t} + \cos t.$$

9. $z(t)'' + 5z(t)' - 6z(t) = 21e^{t-1},\ z(1) = -1\ z'(1) = 9$

We begin by substituting $t+1$ for t in the original equation which yields

$$z(t+1)'' + 5z(t+1)' - 6z(t+1) = 21e^{(t+1)-1} = 21e^t.$$

Next we substitute $y(t) = z(t+1)$ into the modified equation and obtain

$$y(t)'' + 5y(t)' - 6y(t) = 21e^t, y(0) = -1,\ y'(0) = 9.$$

Taking the Laplace transform of both sides of the equation gives

$$(s^2Y(s) - s(-1) - (9)) + 5(sY(s) - (-1)) - 6Y(s) = \frac{21}{s-1}$$

$$(s^2 + 5s - 6)Y(s) + s - 4 = \frac{21}{s-1}$$

$$(s+6)(s-1)Y(s) = \frac{21}{s-1} + 4 - s$$

$$(s+6)(s-1)Y(s) = \frac{21 + (4-s)(s-1)}{s-1}$$

$$(s+6)(s-1)Y(s) = \frac{-s^2+5s+17}{(s+6)(s-1)^2}.$$

Since the denominator is completely factored, we proceed to partial fraction expansion:

$$\frac{-s^2+5s+17}{(s+6)(s-1)^2} = \frac{A}{s+6} + \frac{B}{s-1} + \frac{C}{(s-1)^2}.$$

After obtaining common denominators and equating numerators we have

$$-s^2+5s+17 = A(s-1)^2 + B(s+6)(s-1) + C(s+6).$$

Choosing appropriate values of s gives:

$$s=1: \quad 21 = 7C \ \longrightarrow \ C = 3$$

$$s=-6: \quad -49 = 49A = \longrightarrow \ A = -1$$

$$s=0: \quad 17 = A - 6B + 6C = -1 - 6B + 6(3) \ \longrightarrow \ B = 0$$

Therefore we have

$$\frac{-s^2+5s+17}{(s+6)(s-1)^2} = -\frac{1}{s+6} + \frac{3}{(s-1)^2}.$$

Taking inverse transforms yields

$$y(t) = -e^{-6t} + 3te^{t}$$

and since $y(t-1) = w(t)$, we have

$$w(t) = -e^{-6(t-1)} + 3(t-1)e^{t-1}.$$

11. $y(t)'' - y(t) = t - 2,\ y(2) = 3\ y'(2) = 0$

We begin by substituting $t+2$ for t in the original equation which yields

$$y(t+2)'' - y(t+2) = (t+2) - 2 = t.$$

Next we substitute $w(t) = y(t+2)$ into the modified equation and obtain

$$w(t)'' - w(t) = t, w(0) = 3,\ w'(0) = 0.$$

Taking the Laplace transform of both sides of the equation gives

$$(s^2W(s) - s(3) - (0)) - W(s) = \frac{1}{s^2}$$

$$(s^2 - 1)W(s) - 3s = \frac{1}{s^2}$$

$$(s-1)(s+1)W(s) = \frac{1}{s^2} + 3s$$

$$(s-1)(s+1)W(s) = \frac{1 + 3s(s^2)}{s^2}$$

$$W(s) = \frac{1 + 3s^3}{s^2(s-1)(s+1)}$$

Since the denominator is completely factored, we proceed to partial fraction expansion:

$$\frac{1+3s^3}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1}.$$

After obtaining common denominators and equating numerators we have

$$1 + 3s^3 = As(s-1)(s+1) + B(s+1)(s-1) + Cs^2(s+1) + Ds^2(s-1).$$

Choosing appropriate values of s gives:

$$s = 1: \quad 4 = 2C \quad \longrightarrow \quad C = 2$$

$$s = -1: \quad -2 = -2D = \longrightarrow \quad D = 1$$

$$s = 0: \quad 1 = -B \quad \longrightarrow \quad B = -1$$

$$s = 2: \quad 25 = 2A + 3B + 12C + 4D = 2A + 3(-1) + 12(2) + 4(1) \quad \longrightarrow \quad A = 0$$

Therefore we have

$$\frac{1+3s^3}{s^2(s-1)(s+1)} = -\frac{1}{s^2} + \frac{2}{s-1} + \frac{1}{s+1}.$$

Taking inverse transforms yields

$$w(t) = -t + 2e^t + e^{-t}$$

and since $w(t-2) = y(t)$, we have

$$y(t) = -(t-2) + 2e^{t-2} + e^{-(t-2)}.$$

13. $y(t)'' - y'(t) - 2y(t) = -8\cos t - 2\sin t,\ y(\pi/2) = 1\ y'(\pi/2) = 0$

We begin by substituting $t + \pi/2$ for t in the original equation which yields

$$y(t+\pi/2)'' - y'(t+\pi/2) - 2y(t+\pi/2) = -8\cos(t+\pi/2) - 2\sin(t+\pi/2) = 8\sin t - 2\cos t$$

where we have made use of the facts $\cos(t+\pi/2) = -\sin t$ and $\sin(t+\pi/2) = \cos t$. Next we substitute $w(t) = y(t+\pi/2)$ into the modified equation and obtain

$$w(t)'' - w'(t) - 2w(t) = 8\sin t - 2\cos t, w(0) = 1,\ w'(0) = 0.$$

Taking the Laplace transform of both sides of the equation gives

$$(s^2W(s) - s(1) - (0)) - (sW(s) - 1) - 2W(s) = \frac{8}{s^2+1} - \frac{2s}{s^2+1}$$

$$(s^2 - s - 2)W(s) - s + 1 = \frac{-2s+8}{s^2+1}$$

$$(s-2)(s+1)W(s) = \frac{-2s+8}{s^2+1} + s - 1$$

$$(s-2)(s+1)W(s) = \frac{-2s+8+(s-1)(s^2+1)}{s^2+1}$$

$$W(s) = \frac{s^3 - s^2 - s + 7}{(s^2+1)(s-2)(s+1)}$$

Since the denominator is completely factored, we proceed to partial fraction expansion:

$$\frac{s^3 - s^2 - s + 7}{(s^2+1)(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}.$$

After obtaining common denominators and equating numerators we have

$$s^3 - s^2 - s + 7 = A(s^2+1)(s+1) + B(s^2+1)(s-2) + (Cs+D)(s-2)(s+1).$$

Choosing appropriate values of s gives:

$$s = -1: \quad 6 = -6B \longrightarrow B = -1$$

$$s = 2: \quad 9 = 15A \longrightarrow A = 3/5$$

$$s = 0: \quad 7 = A - 2B - 2D = 3/5 - 2(-1) - 2D \longrightarrow D = -11/5$$

$$s = 1: \quad 6 = 4A - 2B - 2C - 2D = 4(3/5) - 2(-1) - 2C - 2(-11/5) \longrightarrow C = 7/5$$

Therefore we have

$$\frac{s^3 - s^2 - s + 7}{(s^2+1)(s-2)(s+1)} = \frac{3}{5(s-2)} - \frac{1}{s+1} + \frac{7s - 11}{5(s^2+1)} = \frac{3}{5(s-2)} - \frac{1}{s+1} + \frac{7s}{5(s^2+1)} - \frac{11}{5(s^2+1)}.$$

Taking inverse transforms yields

$$w(t) = (3/5)e^{2t} - e^{-t} + (7/5)\cos t - (11/5)\sin t$$

and since $w(t - \pi/2) = y(t)$, we have

$$y(t) = (3/5)e^{2(t-\pi/2)} - e^{-(t-\pi/2)} + (7/5)\cos(t - \pi/2) - (11/5)\sin(t - \pi/2)$$

$$y(t) = (3/5)e^{2(t-\pi/2)} - e^{\pi/2 - t} + (7/5)\sin t + (11/5)\cos t$$

where we have made use of the facts $\cos(t - \pi/2) = \sin t$ and $\sin(t - \pi/2) = -\cos t$.

15. $y'' - 3y' + 2y = \cos t, \ y(0) = 0, \ y'(0) = -1$

Taking the Laplace transform of both sides of the equation gives

$$(s^2 Y(s) - s(0) - (-1)) - 3(sY(s) - 0) + 2Y(s) = \frac{s}{s^2+1}$$

$$(s^2 - 3s + 2)Y(s) + 1 = \frac{s}{s^2+1}$$

$$(s^2 - 3s + 2)Y(s) = \frac{s}{s^2+1} - 1$$

$$(s-1)(s-2)Y(s) = \frac{s - (s^2+1)}{s^2+1}$$

$$Y(s) = \frac{-s^2 + s - 1}{(s^2+1)(s-1)(s-2)}.$$

17. $y'' + y' - y = t^3, \ y(0) = 1, \ y'(0) = 0$

Taking the Laplace transform of both sides of the equation gives

$$
\begin{aligned}
(s^2Y(s) - s(1) - 0) + (sY(s) - 1) - Y(s) &= \frac{6}{s^4} \\
(s^2 + s - 1)Y(s) - s - 1 &= \frac{6}{s^4} \\
(s^2 + s - 1)Y(s) &= \frac{6}{s^4} + 1 + s \\
(s^2 + s - 1)Y(s) &= \frac{6 + (1 + s)s^4}{s^4} \\
Y(s) &= \frac{s^5 + s^4 + 6}{s^4(s^2 + s - 1)}.
\end{aligned}
$$

19. $y'' + 5y' - y = e^t - 1,\ y(0) = 1,\ y'(0) = 1$

Taking the Laplace transform of both sides of the equation gives

$$
\begin{aligned}
(s^2Y(s) - s(1) - 1) + 5(sY(s) - 1) - Y(s) &= \frac{1}{s-1} - \frac{1}{s} \\
(s^2 + 5s - 1)Y(s) - s - 6 &= \frac{s - (s-1)}{s(s-1)} \\
(s^2 + 5s - 1)Y(s) &= \frac{1}{s(s-1)} + s + 6 \\
(s^2 + 5s - 1)Y(s) &= \frac{1 + (s+6)s(s-1)}{s(s-1)} \\
Y(s) &= \frac{s^3 + 5s^2 - 6s + 1}{s(s-1)(s^2 + 5s - 1)}.
\end{aligned}
$$

21. $y'' - 2y' + y = \cos t - \sin t,\ y(0) = 1,\ y'(0) = 3$

Taking the Laplace transform of both sides of the equation gives

$$(s^2Y(s) - s(1) - 3) - 2(sY(s) - 1) + Y(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$(s^2 - 2s + 1)Y(s) - s - 1 = \frac{s-1}{s^2+1}$$

$$(s^2 - 2s + 1)Y(s) = \frac{s-1}{s^2+1} + s + 1$$

$$(s-1)^2Y(s) = \frac{s - 1 + (s+1)(s^2+1)}{s^2+1}$$

$$Y(s) = \frac{s^3 + s^2 + 2s}{(s^2+1)(s-1)^2}.$$

23. $y'' + 4y = g(t),\ y(0) = -1,\ y'(0) = 0,\ g(t) = \begin{cases} t & t < 2 \\ 5 & 2 < t \end{cases}$

Taking the Laplace transform of both sides of the equation gives

$$(s^2Y(s) - s(-1) - 0) + 4Y(s) = L\{g(t)\}$$

$$(s^2 + 4)Y(s) + s = L(g(t)).$$

To calculate $L\{g(t)\}$ we use the definition of the Laplace Transform

$$L\{g(t)\} = \int_0^2 e^{-st}t\ dt + \int_2^\infty e^{-st}5\ dt$$

$$L\{f(t)\} = \int_0^2 e^{-st}t\ dt + \lim_{u\longrightarrow\infty} 5\int_2^u e^{-st}\ dt.$$

Using integration by parts on the first integral we obtain

$$L\{f(t)\} = -(t/s)e^{-st}|_0^2 + (1/s)\int_0^2 e^{-st}\ dt + \lim_{u\longrightarrow\infty} -(5/s)e^{-st}|_2^u$$

$$L\{f(t)\} = -(t/s)e^{-st}|_0^2 - (1/s^2)e^{-st}|_0^2 + \lim_{u\longrightarrow\infty} -(5/s)e^{-st}|_2^u$$

$$L\{f(t)\} = -(2/s)e^{-2s} - (1/s^2)e^{-2s} + (1/s^2) + \lim_{u\longrightarrow\infty}[-(5/s)e^{-us} + (5/s)e^{-2s}]$$

$$L\{f(t)\} = -(2/s)e^{-2s} - (1/s^2)e^{-2s} + (1/s^2) + (5/s)e^{-2s},\ s > 0$$
$$L\{f(t)\} = (3/s)e^{-2s} - (1/s^2)e^{-2s} + (1/s^2),\ s > 0$$
$$L\{f(t)\} = \frac{(3s-1)e^{-2s}+1}{s^2},\ s > 0.$$

Therefore we have

$$(s^2+4)Y(s) + s = \frac{(3s-1)e^{-2s}+1}{s^2}$$
$$(s^2+4)Y(s) = \frac{(3s-1)e^{-2s}+1}{s^2} - s$$
$$(s^2+4)Y(s) = \frac{(3s-1)e^{-2s}+1-s(s^2)}{s^2}$$
$$Y(s) = \frac{(3s-1)e^{-2s}+1-s^3}{s^2(s^2+4)}.$$

SECTION 7.6

5. $g(t) = \begin{cases} 0, & 0 < t < 1 \\ 2, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ 3, & 3 < t \end{cases}$

Using the unit step function we can write

$$g(t) = 0 + (2-0)u(t-1) + (1-2)u(t-2) + (3-1)u(t-3)$$
$$g(t) = 2u(t-1) - u(t-2) + 2u(t-3).$$

Hence we have

$$L\{g(t)\} = \frac{2e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{2e^{-3s}}{s}$$
$$L\{g(t)\} = \frac{2e^{-s} - e^{-2s} + 2e^{-3s}}{s}.$$

7. The function, given graphically in this problem, can be written piecewise as