SECTION 4.4

9, (10). $y'' + 3y = -9$

The characteristic equation is given by

$$r^2 + 3 = 0$$

so there exists two imaginary roots given by $r = \pm\sqrt{3}i$.

The right hand side of the equation is polynomial of degree zero and an exponential of the form $e^{0\cdot x}$. Since $0 \neq \pm\sqrt{3}i$, $s = 0$ and the particular solution has the form

$$y_p = t^s A e^{rx} = t^0 A e^{0\cdot x} = A.$$

Therefore we have

$$\begin{aligned} y_p &= A \\ y_p' &= 0 \\ y_p'' &= 0 \end{aligned}$$

so that the ODE becomes

$$0 + 3(A) = -9$$

or $A = -3$. Thus a particular solution is given by

$$y_p = -3.$$

11, (14). $2z'' + z = 9e^{2t}$

The characteristic equation is given by

$$2r^2 + 1 = 0$$

so there exists two imaginary roots given by $r = \pm\frac{1}{\sqrt{2}}i$.

The right hand side of the equation is a polynomial of degree zero and an exponential of the form e^{2t}. Since $2 \neq \pm\frac{1}{\sqrt{2}}i$, $s = 0$ and the particular solution has the form

$$z_p = t^s A e^{rx} = t^0 A e^{2t} = A e^{2t}.$$

Therefore we have

$$z_p = Ae^{2t}$$
$$z_p' = 2Ae^{2t}$$
$$z_p'' = 4Ae^{2t}$$

so that the ODE becomes

$$2(4Ae^{2t}) + Ae^{2t} = 9e^{2t}$$

$$9Ae^{2t} = 9e^{2t}$$

and hence $A = 1$. Thus a particular solution is given by

$$z_p = e^{2t}.$$

13, (13). $y'' - y' + 9y = 3\sin 3t$

The characteristic equation is given by

$$r^2 - r + 9 = 0$$

so there exists two complex conjugate roots given by the quadratic equation

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(9)}}{2(1)} = \frac{1 \pm \sqrt{-35}}{2} = \frac{1}{2} \pm \frac{\sqrt{35}}{2}i.$$

The right hand side of the equation is a polynomial of degree zero and an exponential of the form $e^{\alpha t}$ where $\alpha = 0$ and a term $\sin \beta t$ with $\beta = 3$. Since $0 \pm 3 \neq \frac{1}{2} \pm \frac{\sqrt{35}}{2}i$, $s = 0$ and the particular solution has the form

$$y_p = t^s[Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t] = A\cos 3t + B\sin 3t.$$

Therefore we have

$$y_p = A\cos 3t + B\sin 3t$$
$$y_p' = -3A\sin 3t + 3B\cos 3t$$
$$y_p'' = -9A\cos 3t - 9B\sin 3t$$

so that the ODE becomes

$$(-9A\cos 3t - 9B\sin 3t) - (-3A\sin 3t + 3B\cos 3t) + 9(A\cos 3t + B\sin 3t) = 3\sin 3t$$

$$(-9A - 3B + 9A)\cos 3t + (-9B + 3A + 9B)\sin 3t = 3\sin 3t$$

$$(-3B)\cos 3t + (3A)\sin 3t = 3\sin 3t.$$

Equating similar terms yields

$$(-3B)\cos 3t = 0 \cdot \cos 3t$$

so that $B = 0$ and

$$(3A)\sin 3t = 3\sin 3t$$

so that $A = 1$. Thus a particular solution is given by

$$y_p = \cos 3t.$$

15, (15). $y'' - 5y' + 6y = xe^x$

The characteristic equation is given by

$$r^2 - 5r + 6 = (r-2)(r-3) = 0$$

so there exists two real distinct roots given by 2 and 3.

The right hand side of the equation is a polynomial of degree one and an exponential of the form $e^{1 \cdot x}$. Since 1 is not a root of the characteristic equation, $s = 0$ and the particular solution has the form

$$y_p = x^s(Ax+B)e^x = x^0(Ax+B)e^x = (Ax+B)e^x.$$

Therefore we have

$$\begin{aligned} y_p &= (Ax+B)e^x \\ y_p' &= [Ax + (A+B)]e^x \\ y_p'' &= [Ax + (2A+B)]e^x \end{aligned}$$

so that the ODE becomes

$$[Ax + (2A+B)]e^x - 5[Ax + (A+B)]e^x + 6(Ax+B)e^x = xe^x$$

$$[2Ax + (-3A+2B)]e^x = xe^x.$$

Equating similar terms yields

$$2Axe^x = xe^x$$

$$(-3A+2B)e^x = 0e^x$$

so that $A = 1/2$ and $B = 3/4$. Thus a particular solution is given by

$$y_p = (x/2 + 3/4)e^x.$$

17, (17). $y'' - 2y' + y = 8e^t$

The characteristic equation is given by

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

so there exists one real, repeated root given by 1.

The right hand side of the equation is a polynomial of degree zero and an exponential of the form $e^{1 \cdot t}$. Since 1 is a repeated root of the characteristic equation, $s = 2$ and the particular solution has the form

$$y_p = t^s(A)e^t = t^2(A)e^t = At^2e^t.$$

Therefore we have

$$\begin{aligned} y_p &= At^2e^t \\ y_p' &= A(t^2 + 2t)e^t \\ y_p'' &= A(t^2 + 4t + 2)e^t \end{aligned}$$

so that the ODE becomes

$$A(t^2 + 4t + 2)e^t - 2A(t^2 + 2t)e^t + At^2e^t = 8e^t$$

$$2Ae^t = 8e^t$$

and $A = 4$. Thus a particular solution is given by

$$y_p = 4t^2e^t.$$

19, (19). $4y'' + 11y' - 3y = -2te^{-3t}$

The characteristic equation is given by

$$4r^2 + 11r + -3 = (4r - 1)(r + 3) = 0$$

so there exists two real, distinct roots given by -3 and 1/4.

The right hand side of the equation is a polynomial of degree one and an exponential of the form $e^{-3 \cdot t}$. Since -3 is the coefficient of t in the exponential and a single root of the characteristic equation, $s = 1$ and the particular solution has the form

$$y_p = t^s(At + B)e^{-3t} = t(At + B)e^{-3t} = (At^2 + Bt)e^{-3t}.$$

Therefore we have

$$y_p = (At^2 + Bt)e^{-3t}$$

$$y_p' = -3(At^2 + Bt)e^{-3t} + (2At + B)e^{-3t} = (-3At^2 + (2A - 3B)t + B)e^{-3t}$$

$$y_p'' = -3(-3At^2+(2A-3B)t+B)e^{-3t}+(-6At+2A-3B)e^{-3t} = (9At^2+(-12A+9B)t+2A-6B)e^{-3t}$$

so that the ODE becomes

$$4(9At^2+(-12A+9B)t+2A-6B)e^{-3t}+11(-3At^2+(2A-3B)t+B)e^{-3t}-3(At^2+Bt)e^{-3t} = -2te^{-3t}$$

$$0 \cdot At^2 - 26At + 8A + 0 \cdot Bt - 13B = -2t$$

$$-26At + 8A - 13B = -2t.$$

From the final equation we have $A = 1/13$ and $B = 8/169$ so that a particular solution is given by

$$y_p = (t^2/13 + 8t/169)e^{-3t}.$$

21, (21). $x'' - 4x' + 4x = te^{2t}$

The characteristic equation is given by

$$r^2 - 4r + 4 = (r - 2)^2 = 0$$

so there exists one real, repeated roots given by 2.

The right hand side of the equation is a polynomial of degree one and an exponential of the form $e^{2 \cdot t}$. Since 2 is the coefficient of t in the exponential and a double root of the characteristic equation, $s = 2$ and the particular solution has the form

$$y_p = t^s(At + B)e^{2t} = t^2(At + B)e^{2t} = (At^3 + Bt^2)e^{2t}.$$

Therefore we have

$$x_p = (At^3 + Bt^2)e^{2t}$$

$$x_p' = 2(At^3 + Bt^2)e^{2t} + (3At^2 + 2Bt)e^{2t} = (2At^3 + (3A + 2B)t^2 + 2Bt)e^{2t}$$

$$x_p'' = 2(2At^3+(3A+2B)t^2+2Bt)e^{2t}+(6At^2+2(3A+2B)t+2B)e^{2t} = (4At^3+(12A+4B)t^2+(6A+8B)t+2B)e^{2t}$$

so that the ODE becomes

$$(4At^3+(12A+4B)t^2+(6A+8B)t+2B)e^{2t}-4(2At^3+(3A+2B)t^2+2Bt)e^{2t}+4(At^3+Bt^2)e^{2t} = te^{2t}$$

$$0 \cdot At^3 + 0 \cdot At^2 + 6At + 0 \cdot Bt^2 + 0 \cdot Bt + 2B = t.$$

From the final equation we have $A = 1/6$ and $B = 0$ so that a particular solution is given by

$$x_p = t^3e^{2t}/6.$$

23, (23). $y'' - 7y' = \theta^2$

The characteristic equation is given by

$$r^2 - 7r = r(r - 7) = 0$$

so there exists two real, distinct roots given by 0 and 7.

The right hand side of the equation is a polynomial of degree two and an exponential of the form $e^{0 \cdot \theta}$. Since 0 is the coefficient of θ in the exponential and a single root of the characteristic equation, $s = 1$ and the particular solution has the form

$$y_p = t^s(A\theta^2 + B\theta + C)e^{0\cdot\theta} = \theta(A\theta^2 + B\theta + C) = A\theta^3 + B\theta^2 + C\theta.$$

Therefore we have

$$y_p = A\theta^3 + B\theta^2 + C\theta$$

$$y_p' = 3A\theta^2 + 2B\theta + C$$

$$y_p'' = 6A\theta + 2B$$

so that the ODE becomes

$$(6A\theta + 2B) - 7(3A\theta^2 + 2B\theta + C) = \theta^2$$

$$-21A\theta^2 + (6A - 14B)\theta + (2B - 7C) = \theta^2.$$

From the final equation we have $A = -1/21$, $B = -1/49$, $C = -2/343$ so that a particular solution is given by

$$y_p = -\theta^3/21 - \theta^2/49 - 2\theta/343.$$

25, (25). $y'' + 2y' + 4y = 111e^{2t}\cos 3t$

The characteristic equation is given by

$$r^2 + 2r + 4 = 0$$

so there exists two complex conjugate roots given by the quadratic equation

$$r = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)} = \frac{-2 \pm \sqrt{-12}}{2} = -1 \pm \sqrt{3}i.$$

The right hand side of the equation is a polynomial of degree zero and an exponential of the form $e^{2\cdot t}$ and a cosine function of the form $\cos 3 \cdot t$. Since $-1 \pm \sqrt{3}i \neq 2 \pm 3i$, $s = 0$ and the particular solution has the form

$$y_p = t^s Ae^{2t}\cos 3t + t^s Be^{2t}\sin 3t = Ae^{2t}\cos 3t + Be^{2t}\sin 3t.$$

Therefore we have

$$y_p = Ae^{2t}\cos 3t + Be^{2t}\sin 3t$$

$$y_p' = 2Ae^{2t}\cos 3t - 3Ae^{2t}\sin 3t + 2Be^{2t}\sin 3t + 3Be^{2t}\cos 3t = e^{2t}((2A+3B)\cos 3t + (-3A+2B)\sin 3t)$$

$$y_p'' = 2e^{2t}((2A+3B)\cos 3t + (-3A+2B)\sin 3t) + e^{2t}(-(6A+9B)\sin 3t + (-9A+6B)\cos 3t)$$

$$= e^{2t}((-5A + 12B)\cos 3t + (-12A - 5B)\sin 3t)$$

so that the ODE becomes

$$e^{2t}((-5A+12B)\cos 3t + (-12A-5B)\sin 3t) + 2e^{2t}((2A+3B)\cos 3t + (-3A+2B)\sin 3t) + 4e^{2t}(A\cos 3t + B\sin 3t)$$

$$= 111e^{2t}\cos 3t$$

or after rearrangement and division by e^{2t}

$$(3A + 18B)\cos 3t + (-18A + 3B)\sin 3t = 111\cos 3t$$

which after equating like trig terms yield the simultaneous equations

$$3A + 18B = 111$$

$$-18A + 3B = 0.$$

From the final equations we have $A = 1$ and $B = 6$ so that a particular solution is given by

$$y_p = e^{2t}\cos 3t + 6e^{2t}\sin 3t.$$