SECTION 8.3

11, (11). $y' + (x+2)y = 0$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + (x+2) \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

Performing a shift so that the exponents of the first and second terms are equal to n we obtain

$$\sum_{n=-1}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

Explicitly writing the first two terms from the first series and the first term from the third series yields

$$0(a_0)x^{-1} + 1(a_1) + \sum_{n=1}^{\infty} (n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n + 2a_0 + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$2a_0 + a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} + a_{n-1} + 2a_n] x^n = 0.$$

From this last equation we obtain the relations

$$2a_0 + a_1 = 0$$

$$a_{n+1} = -a_{n-1}/(n+1) - 2a_n/(n+1), \;\; n \geq 0.$$

From these equations we can write

$$a_1 = -2a_0$$

$$n = 1, \quad a_2 = -a_0/2 - a_1 = 3a_0/2$$

$$n = 2, \quad a_3 = -a_1/3 - 2a_2/3 = 2a_0/3 - a_0 = -a_0/3.$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = a_0(1 - 2x + 3x^2/2 - x^3/3 + \cdots).$$

13, (13). $z'' - x^2 z = 0$

We proceed directly by writing

$$z = \sum_{n=0}^{\infty} a_n x^n$$

$$z' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$z'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Performing a shift so that the exponents are equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

Explicitly writing the first four terms from the first series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-2}] x^n = 0.$$

From this last equation we obtain the relations

$$a_2 = 0$$

$$a_3 = 0$$

$$a_{n+2} = a_{n-2}/[(n+2)(n+1)],\ n \geq 2.$$

From these equations we can write

$$n = 2, \quad a_4 = a_0/12$$

$$n = 3, \quad a_5 = a_1/20.$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y = a_0(1 + x^4/12 + \cdots) + a_1(x + x^5/20 + \cdots).$$

15, (15). $y'' + (x-1)y' + y = 0$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + (x-1) \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponents of the first and third series are equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_nx^n - \sum_{n=-1}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_nx^n = 0.$$

Explicitly writing the first two terms from the first series and the first term from the third series yields

$$0(a_0)x^{-2}+0(a_1)x^{-1}+\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty} na_nx^n-0(a_0)x^{-1}-\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n+\sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} na_nx^n - \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n - (n+1)a_{n+1} + a_n]x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + (n+1)a_n]x^n = 0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = a_{n+1}/(n+2) - a_n/(n+2),\ n \geq 0.$$

From these equations we can write

$$n = 0,\quad a_2 = a_1/2 - a_0/2$$

$$n = 1,\quad a_3 = a_2/3 - a_1/3 = -a_1/6 - a_0/6.$$

Thus we can write

$$y = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$y = a_0(1 - x^2/2 - x^3/6 + \cdots) + a_1(x + x^2/2 - x^3/6 + \cdots).$$

17, (17). $w'' - x^2w' + w = 0$

We proceed directly by writing

$$w = \sum_{n=0}^{\infty} a_nx^n$$

$$w' = \sum_{n=0}^{\infty} na_n x^{n-1}$$

$$w'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponents of the first and second series are equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n-1)a_{n-1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Explicitly writing the first three terms from the first series and the first term from the third series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n-1)a_{n-1}x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$a_0 + 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n-1)a_{n-1}x^n + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$a_0 + 2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n-1)a_{n-1} + a_n]x^n = 0$$

From this last equation we obtain the relations

$$a_2 = -a_0/2$$

$$a_{n+2} = -a_n/[(n+2)(n+1)] + (n-1)a_{n-1}/[(n+2)(n+1)], \;\; n \geq 1.$$

From these equations we can write

$$n = 1, \quad a_3 = -a_1/6 + 0(a_0) = -a_1/6$$

Thus we can write

$$w = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$w = a_0(1 - x^2/2 + \cdots) + a_1(x - x^3/6 + \cdots).$$

19, (19). $y' - 2xy = 0$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - 2x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0$$

Performing a shift so that the exponents of the first and second series are equal to n we obtain

$$\sum_{n=-1}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} 2 a_{n-1} x^n = 0.$$

Explicitly writing the first two term from the first series yields

$$0(a_0)x^{-1} + a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} 2 a_{n-1} x^n = 0$$

$$a_1 + \sum_{n=1}^{\infty} [(n+1)a_{n+1} - 2a_{n-1}] x^n = 0.$$

From this last equation we obtain the relations

$$a_1 = 0$$

$$a_{n+1} = 2a_{n-1}/(n+1), \;\; n \geq 1.$$

From these relations we can write

$$n = 1, \quad a_2 = 2a_0/2 = a_0 = a_0/1!$$

$$n = 2, \quad a_3 = 2a_1/3 = 0$$

$$n = 3, \quad a_4 = 2a_2/4 = 2^2 a_0/[2 \cdot 4] = 2^2 a_0/[2^2(1 \cdot 2)] = a_0/2!$$

$$n = 4, \quad a_5 = 2a_3/5 = 0$$

$$n = 5, \quad a_6 = 2a_4/6 = 2^3 a_0/[2 \cdot 4 \cdot 6] = 2^3 a_0/[2^3(1 \cdot 2 \cdot 3)] = a_0/3!.$$

Hence for n odd, $n = 2k+1$, $a_{2k+1} = 0$; for n even, $n = 2k$, $a_{2k} = a_0/k!$.
Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{k=0}^{\infty} x^{2k}/k!.$$

21, (21). $y'' - xy' + 4y = 0$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} n a_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0.$$

Performing a shift so that the exponent of the first series is equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0.$$

Explicitly writing the first two terms from the first series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n + 4 a_n] x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (4-n)a_n]x^n = 0$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = (n-4)a_n/[(n+2)(n+1)], \ n \geq 0.$$

From these relations we can write

$$n=0, \quad a_2 = -4a_0/[2 \cdot 1]$$

$$n=1, \quad a_3 = -3a_1/[3 \cdot 2]$$

$$n=2, \quad a_4 = -2a_2/[4 \cdot 3] = (2 \cdot 4)a_0/[4 \cdot 3 \cdot 2 \cdot 1]$$

$$n=3, \quad a_5 = -a_3/[5 \cdot 4] = (-3 \cdot -1)a_1/[5 \cdot 4 \cdot 3 \cdot 2]$$

$$n=4, \quad a_6 = 0a_4/[6 \cdot 5] = 0$$

$$n=5, \quad a_7 = a_5/[7 \cdot 6] = (-3 \cdot -1 \cdot 1)a_1/[7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2]$$

$$n=6, \quad a_8 = 2a_6/[8 \cdot 7] = 0$$

$$n=7, \quad a_9 = 3a_7/[9 \cdot 8] = (-3 \cdot -1 \cdot 1 \cdot 3)a_1/[9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2]$$

$$n=8, \quad a_{10} = 4a_8/[10 \cdot 9] = 0$$

$$n=9, \quad a_{11} = 5a_9/[11 \cdot 10] = (-3 \cdot -1 \cdot 1 \cdot 3 \cdot 5)a_1/[11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2]$$

Hence, $a_0 = a_0$, $a_2 = -2a_0$, $a_4 = a_0/3$ and $a_n = 0$ for all other even subscripts;

for n odd, $n = 2k+1$, $a_{2k+1} = (-3 \cdot -1 \cdot 1 \cdots (2k-5))a_1/(2k+1)!$.

Thus we can write

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0(1 - 2x^2 + x^4/3) + a_1[x + \sum_{k=1}^{\infty} (-3 \cdot -1 \cdot 1 \ \cdots (2k-5))x^{2k+1}/(2k+1)!].$$

23, (23). $z'' - x^2 z' - xz = 0$

We proceed directly by writing

$$z = \sum_{n=0}^{\infty} a_n x^n$$

$$z' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$z'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} n a_n x^{n-1} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Performing a shift so that the exponents of all the series are equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} (n-1)a_{n-1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

Explicitly writing the first three terms from the first series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n-1)a_{n-1}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n-1)a_{n-1} - a_{n-1}]x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - n a_{n-1}]x^n = 0.$$

From this last equation we obtain the relations

$$a_2 = 0$$

$$a_{n+2} = n a_{n-1}/[(n+2)(n+1)], \ n \geq 1.$$

From these relations we can write

$$n = 1, \quad a_3 = 1 \cdot a_0/[3 \cdot 2]$$

$$n = 2, \quad a_4 = 2a_1/[4 \cdot 3]$$

$$n = 3, \quad a_5 = 3a_2/[5 \cdot 4] = 0$$

$$n = 4, \quad a_6 = 4a_3/[6 \cdot 5] = (1 \cdot 4)a_0/[6 \cdot 5 \cdot 3 \cdot 2]$$

$$n = 5, \quad a_7 = 5a_4/[7 \cdot 6] = (2 \cdot 5)a_1/[7 \cdot 6 \cdot 4 \cdot 3]$$

$$n = 6, \quad a_8 = 6a_5/[8 \cdot 7] = 0$$

$$n = 7, \quad a_9 = 7a_6/[9 \cdot 8] = (1 \cdot 4 \cdot 7)a_0/[9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2]$$

$n = 8, \quad a_{10} = 8a_7/[10 \cdot 9] = (2 \cdot 5 \cdot 8)a_1/[10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3].$

From this table. we can subdivide the coefficients into three categories:

for $n = 3k + 2, \quad a_{3k+2} = 0;$

for $n = 3k$, let's examine, for example, a_9; if we multiply the denominator by the numerator for this term, the product is 9!; hence we can write $a_9 = (1 \cdot 4 \cdot 7)^2 a_0/9!$; in general this pattern is followed so that we obtain $a_{3k} = (1 \cdot 4 \cdots (3k-2))^2 a_0/(3k)!$;

for $n = 3k + 1$, let's examine, for example, a_{10}; if we multiply the denominator by the numerator for this term, the product is 10!; hence we can write $a_{10} = (2 \cdot 5 \cdot 8)^2 a_0/10!$; in general this pattern is followed so that we obtain $a_{3k+1} = (2 \cdot 5 \cdots (3k-1))^2 a_1/(3k+1)!$;

Thus we have

$$z = \sum_{n=0}^{\infty} a_n x^n = a_0(1+\sum_{k=1}^{\infty} (1 \cdot 4 \cdots (3k-2))^2 x^{3k}/(3k)!)+a_1(x+\sum_{k=1}^{\infty} (2 \cdot 5 \cdots (3k-1))^2 x^{3k+1}/(3k+1)!).$$

25, (25). $w'' + 3xw' - w = 0, \; w(0) = 2, \; w'(0) = 0$

We proceed directly by writing

$$w = \sum_{n=0}^{\infty} a_n x^n$$

$$w' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$w'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 3x \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 3n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponent of the first series is equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 3na_nx^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

Explicitly writing the first two terms from the first series yields

$$0(a_0)x^{-2} + 0(a_1)x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 3na_nx^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 3na_nx^n - \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + 3na_n - a_n]x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (3n-1)a_n]x^n = 0$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = -(3n-1)a_n/[(n+2)(n+1)], \; n \geq 0;$$

in addition the initial values give

$$a_0 = 2$$

$$a_1 = 0.$$

From these equations we can write

$$n = 0, \quad a_2 = a_0/2 = 1$$

$$n = 1, \quad a_3 = -2a_1/6 = 0$$

$$n = 2, \quad a_4 = -5a_2/12 = -5/12$$

$$n = 3, \quad a_5 = 8a_3/20 = 0$$

$$n = 4, \quad a_6 = -11a_4/30 = 11/72$$

Thus we can write

$$w = \sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

$$w = 2 + x^2 - 5x^4/12 + 11x^6/72 + \cdots.$$

27, (27). $(x+1)y'' - y = 0,\ y(0) = 0,\ y'(0) = 1$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$(x+1)\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponents of the first and second series are equal to n we obtain

$$\sum_{n=-1}^{\infty} n(n+1) a_{n+1} x^n + \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Explicitly writing the first two term from the first series and the first two terms form the second series yields

$$0(a_0)x^{-1} + \sum_{n=0}^{\infty} n(n+1) a_{n+1} x^n + 0(a_0) + 0(a_1)x + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [n(n+1) a_{n+1} + (n+2)(n+1) a_{n+2} - a_n] x^n = 0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = -n a_{n+1}/(n+2) + a_n/[(n+2)(n+1)],\ n \geq 0;$$

in addition the initial values give

$$a_0 = 0$$

$$a_1 = 1.$$

From these equations we can write

$$n = 0, \quad a_2 = a_0/2 = 0$$
$$n = 1, \quad a_3 = -a_2/3 + a_1/6 = 1/6$$
$$n = 2, \quad a_4 = -a_3/2 + a_2/12 = -1/12$$
$$n = 3, \quad a_5 = -3a_4/5 + a_3/20 = 1/20 + 1/120 = 7/120.$$

Thus we have

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y = x + x^3/6 - x^4/12 + 7x^5/120 + \cdots.$$