SECTION 2.4

9, (10). $(2xy+3)dx+(x^2-1)dy=0$

We begin by testing the equation. Since

$$\frac{\partial}{\partial y}(2xy+3)=2x=\frac{\partial}{\partial x}(x^2-1)$$

the ODE is exact.

Hence, the function $F(x,y)$ is given by

$$F(x,y)=\int (2xy+3)\ dx+g(y)$$

$$F(x,y)=x^2y+3x+g(y).$$

To determine $g(y)$, we proceed as follows

$$\frac{\partial}{\partial y}F(x,y)=\frac{\partial}{\partial y}(x^2y+3x+g(y))=x^2+g'(y)=x^2-1.$$

Thus we have

$$g'(y)=-1$$

$$g(y)=-y$$

so that

$$F(x,y)=x^2y+3x-y$$

and a solution is given by

$$x^2y+3x-y=C$$

$$y(x^2-1)=C-3x$$

$$y=\frac{C-3x}{x^2-1}.$$

11, (11). $(\cos x\cos y+2x)dx-(\sin x\sin y+2y)dy=0$

We begin by testing the equation. Since

$$\frac{\partial}{\partial y}(\cos x\cos y+2x)=-\cos x\sin y=\frac{\partial}{\partial x}-(\sin x\sin y+2y)$$

the ODE is exact.

Hence, the function $F(x, y)$ is given by

$$F(x, y) = \int (\cos x \cos y + 2x) \ dx + g(y)$$

$$F(x, y) = \sin x \cos y + x^2 + g(y).$$

To determine $g(y)$, we proceed as follows

$$\frac{\partial}{\partial y} F(x, y) = \frac{\partial}{\partial y} (\sin x \cos y + x^2 + g(y)) = -\sin x \sin y + g'(y) = -(\sin x \sin y + 2y).$$

Thus we have

$$g'(y) = -2y$$

$$g(y) = -y^2$$

so that

$$F(x, y) = \sin x \cos y + x^2 - y^2$$

and a solution is given by

$$\sin x \cos y + x^2 - y^2 = C.$$

13, (13). $(t/y)dy + (1 + \ln y)dt = 0$

We begin by testing the equation. Since

$$\frac{\partial}{\partial t}(t/y) = 1/y = \frac{\partial}{\partial y}(1 + \ln y)$$

the ODE is exact.

Hence, the function $F(y, t)$ is given by

$$F(y, t) = \int (t/y) \ dy + g(t)$$

$$F(y, t) = t \ln |y| + g(t).$$

Since the function in the original ODE was $\ln y$, this assumes we are working only with positive values of y so that we write

$$F(y, t) = t \ln y + g(t).$$

To determine $g(t)$, we proceed as follows

$$\frac{\partial}{\partial t}F(y,t) = \frac{\partial}{\partial t}(t\ln y + g(t)) = \ln y + g'(t) = (1+\ln y).$$

Thus we have

$$g'(t) = 1$$
$$g(t) = t$$

so that

$$F(y,t) = t\ln y + t$$

and a solution is given by

$$t\ln y + t = C.$$

15, (15). $(\cos\theta)dr - (r\sin\theta - e^{\theta})d\theta = 0$

We begin by testing the equation. Since

$$\frac{\partial}{\partial \theta}(\cos\theta) = -\sin\theta = \frac{\partial}{\partial r} - (r\sin\theta - e^{\theta})$$

the ODE is exact.

Hence, the function $F(r,\theta)$ is given by

$$F(r,\theta) = \int \cos\theta \ dr + g(\theta)$$
$$F(r,\theta) = r\cos\theta + g(\theta).$$

To determine $g(\theta)$, we proceed as follows

$$\frac{\partial}{\partial \theta}F(r,\theta) = \frac{\partial}{\partial \theta}(r\cos\theta + g(\theta)) = -r\sin\theta + g'(\theta) = -(r\sin\theta - e^{\theta}).$$

Thus we have

$$g'(\theta) = e^{\theta}$$
$$g(\theta) = e^{\theta}$$

so that

$$F(r,\theta) = r\cos\theta + e^{\theta}$$

and a solution is given by

$$r\cos\theta + e^\theta = C$$

$$r = \frac{C - e^\theta}{\cos\theta}$$

$$r = (C - e^\theta)\sec\theta.$$

17, (17). $(1/y)dx - (3y - x/y^2)dy = 0$

We begin by testing the equation. Since

$$\frac{\partial}{\partial y}(1/y) = -y^{-2} \neq y^{-2} = \frac{\partial}{\partial x} - (3y - x/y^2)$$

the ODE is not exact.

19, (19). $(2x + \frac{y}{1+x^2y^2})dx + (\frac{x}{1+x^2y^2} - 2y)dy = 0$

We begin by testing the equation. Using the quotient rule we obtain

$$\frac{\partial}{\partial y}\left(2x + \frac{y}{1+x^2y^2}\right) = \frac{\partial}{\partial y}\ \frac{y}{1+x^2y^2}$$

$$\frac{\partial}{\partial y}\left(2x + \frac{y}{1+x^2y^2}\right) = \frac{(1)(1+x^2y^2) - (y)(2x^2y)}{(1+x^2y^2)^2}$$

$$\frac{\partial}{\partial y}\left(2x + \frac{y}{1+x^2y^2}\right) = \frac{1 - x^2y^2}{(1+x^2y^2)^2}.$$

Similarly

$$\frac{\partial}{\partial x}\left(\frac{x}{1+x^2y^2} - 2y\right) = \frac{\partial}{\partial x}\ \frac{x}{1+x^2y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{1+x^2y^2} - 2y\right) = \frac{(1)(1+x^2y^2) - (x)(2xy^2)}{(1+x^2y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{1+x^2y^2} - 2y\right) = \frac{1 - x^2y^2}{(1+x^2y^2)^2}$$

so that the ODE is exact.

Hence, the function $F(x, y)$ is given by

$$F(x, y) = \int \left(2x + \frac{y}{1+x^2y^2}\right)\, dx + g(y)$$

$$F(x,y) = x^2 + \arctan(xy) + g(y).$$

(Note: the integral $\int \frac{y}{1+x^2y^2}\, dx$ can be evaluated using the u substitution $u = xy$.)

To determine $g(y)$, we proceed as follows

$$\frac{\partial}{\partial y}F(x,y) = \frac{\partial}{\partial y}(x^2 + \arctan(xy) + g(y)) = \frac{x}{1+x^2y^2} + g'(y) = \frac{x}{1+x^2y^2} - 2y.$$

Thus we have

$$g'(y) = -2y$$
$$g(y) = -y^2$$

so that

$$F(x,y) = \arctan(xy) + x^2 - y^2$$

and a solution is given by

$$\arctan(xy) + x^2 - y^2 = C.$$

21, (21). $(1/x + 2y^2x)dx + (2yx^2 - \cos y)dy = 0,\ y(1) = \pi$

We begin by testing the equation. Since

$$\frac{\partial}{\partial y}(1/x + 2y^2x) = 4xy = \frac{\partial}{\partial x}(2yx^2 - \cos y)$$

the ODE is exact.

Hence, the function $F(x,y)$ is given by

$$F(x,y) = \int (1/x + 2y^2x)\ dx + g(y)$$

$$F(x,y) = \ln|x| + x^2y^2 + g(y).$$

(Note: since the initial conditions have $x = 1 > 0$ and the ODE is not defined for $x = 0$, we can assume $x > 0$ and write $F(x,y) = \ln x + x^2y^2 + g(y)$.)

To determine $g(y)$, we proceed as follows

$$\frac{\partial}{\partial y}F(x,y) = \frac{\partial}{\partial y}(\ln x + x^2y^2 + g(y)) = 2yx^2 + g'(y) = 2yx^2 - \cos y.$$

Thus we have

$$g'(y) = -\cos y$$
$$g(y) = -\sin y$$

so that

$$F(x, y) = \ln x + x^2y^2 - \sin y$$

and a solution is given by

$$\ln x + x^2y^2 - \sin y = C.$$

Inserting the initial conditions

$$\ln 1 + (1)^2(\pi)^2 - \sin \pi = C$$
$$\pi^2 = C.$$

Hence, the solution is given by

$$\ln x + x^2y^2 - \sin y = \pi^2.$$

23, (23). $(e^t y + te^t y)dt + (te^t + 2)dy = 0, \; y(0) = -1$

We begin by testing the equation. Since

$$\frac{\partial}{\partial y}(e^t y + te^t y) = e^t + te^t = \frac{\partial}{\partial t}(te^t + 2)$$

the ODE is exact.

Hence, the function $F(t, y)$ is given by

$$F(t, y) = \int (te^t + 2) \; dy + g(t)$$
$$F(t, y) = (te^t + 2)y + g(t).$$

To determine $g(t)$, we proceed as follows

$$\frac{\partial}{\partial t}F(x, y) = \frac{\partial}{\partial t}((te^t + 2)y + g(t)) = (t + 1)e^t y + g'(t) = e^t y + te^t y.$$

Thus we have

$$g'(t) = 0$$

$$g(t) = K$$

so that

$$F(t, y) = (te^t + 2)y + K$$

and a solution is given by

$$(te^t + 2)y + K = C$$

$$(te^t + 2)y = C.$$

Inserting the initial conditions

$$((0)e^0 + 2)(-1) = C$$

$$-2 = C.$$

Hence, the solution is given by

$$(te^t + 2)y = -2$$

$$y = -2/(te^t + 2).$$

25, (25). $(y^2 \sin x)dx + (1/x - y/x)dy = 0,\ y(\pi) = 1$

We begin by testing the equation. Since

$$\frac{\partial}{\partial y}(y^2 \sin x) = 2y \sin x \neq -x^{-2} + yx^{-2} = \frac{\partial}{\partial x}(1/x - y/x)$$

the ODE is not exact.