Equations exact with integrating factor:

ex. Solve $(2y^2 + 3x)\,dx + 2xy\,dy = 0$

$M = 2y^2 + 3x, \qquad N = 2xy$

$M_y = 4y, \qquad N_x = 2y$ the equation is not exact

$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{2y}{2xy} = \frac{1}{x}$ this is a function of x alone.

$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

multiply the original equation by x :

$(2xy^2 + 3x^2)\,dx + 2x^2y\,dy = 0$

$M = 2xy^2 + 3x^2, \qquad N = 2x^2y$

$M_y = 4xy, \qquad N_x = 4xy$ the equation is now exact

$F(x,y) = \int M\,dx + g(y) = \int (2xy^2 + 3x^2)\,dx + g(y)$

$F(x,y) = x^2y^2 + x^3 + g(y)$

$F_y = 2x^2y + g'(y) = N = 2x^2y$

$g'(y) = 0$

$g(y)$ is a constant.

Then

$F(x,y) = x^2y^2 + x^3 = C$

ex. Solve $(x + 3x^3 \sin y)\, dx + x^4 \cos y\, dy = 0$

$M_y = 3x^3 \cos y, \qquad N_x = 4x^3 \cos y \qquad$ the equation is not exact

$\frac{M_y - N_x}{N} = \frac{3x^3 \cos y - 4x^3 \cos y}{x^4 \cos y} == -\frac{1}{x} \qquad$ this is a function of x alone.

$\mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = x^{-1}$

multiply the original equation by x^{-1} :

$(1 + 3x^2 \sin y)\, dx + x^3 \cos y\, dy = 0$

$M_y = 3x^3 \cos y, \qquad N_x = 3x^2 \cos y \qquad$ the equation is now exact

$F(x, y) = \int M\, dx + g(y) = \int (1 + 3x^2 \sin y)\, dx + g(y)$

$F(x, y) = x + x^3 \sin y + g(y)$

$F_y = x^3 \cos y + g'(y) = N = x^3 \cos y$

$g'(y) = 0$

$g(y)$ is a constant.

Then

$F(x, y) = x + x^3 \sin y = C$

ex. Solve $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$

$M_y = x, \qquad N_x = 4x \qquad$ the equation is not exact

$\frac{M_y - N_x}{N} = -\frac{3x}{2x^2+3y^2-20} \qquad$ this is not a function of x alone.

$\frac{N_x - M_y}{M} = \frac{4x-x}{xy} = \frac{3}{y} \qquad$ this is a function of y alone.

$\mu(x) = e^{\int \frac{3}{y} dx} = e^{3\ln y} = y^3$

multiply the original equation by y^3 :

$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$

$M_y = 4xy^3, \qquad N_x = 4xy^3 \qquad$ the equation is now exact

$F(x, y) = \int M\,dx + g(y) = \int xy^4\,dx + g(y)$

$F(x, y) = \frac{x^2y^4}{2} + g(y)$

$F_y = 2x^2y^3 + g'(y) = N = 2x^2y^3 + 3y^5 - 20y^3$

$g'(y) = 3y^5 - 20y^3$

$g(y) = \frac{y^6}{2} - 5y^4 + C$

Then

$F(x, y) = \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4 = C$