SECTION 7.3

1. $t^2 + e^t \sin 2t$

$$\begin{aligned} L(t^2 + e^t \sin 2t) &= L(t^2) + L(e^t \sin 2t) \\ L(t^2 + e^t \sin 2t) &= \frac{2!}{s^3} + \frac{2}{(s-1)^2 + 2^2} \\ L(t^2 + e^t \sin 2t) &= \frac{2}{s^3} + \frac{2}{(s-1)^2 + 4} \end{aligned}$$

3. $e^{-t} \cos 3t + e^{6t} - 1$

$$\begin{aligned} L(e^{-t} \cos 3t + e^{6t} - 1) &= L(e^{-t} \cos 3t) + L(e^{6t}) - L(1) \\ L(e^{-t} \cos 3t + e^{6t} - 1) &= \frac{s+1}{(s+1)^2 + 3^2} + \frac{1}{s-6} - \frac{1}{s} \\ L(e^{-t} \cos 3t + e^{6t} - 1) &= \frac{s+1}{(s+1)^2 + 9} + \frac{1}{s-6} - \frac{1}{s}. \end{aligned}$$

5. $2t^2 e^{-t} - t + \cos 4t$

$$\begin{aligned} L(2t^2 e^{-t} - t + \cos 4t) &= 2L(t^2 e^{-t}) - L(t) + L(\cos 4t) \\ L(2t^2 e^{-t} - t + \cos 4t) &= 2\frac{2!}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2 + 4^2} \\ L(2t^2 e^{-t} - t + \cos 4t) &= \frac{4}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2 + 16}. \end{aligned}$$

7. $(t-1)^4$

$$L((t-1)^4) = L(t^4 - 4t^3 + 6t^2 - 4t + 1)$$
$$L((t-1)^4) = L(t^4) - 4L(t^3) + 6L(t^2) - 4L(t) + L(1)$$
$$L((t-1)^4) = \frac{4!}{s^5} - 4\frac{3!}{s^4} + 6\frac{2!}{s^5} - 4\frac{1!}{s^2} + \frac{1}{s}$$
$$L((t-1)^4) = \frac{24}{s^5} - \frac{24}{s^4} + \frac{12}{s^5} - \frac{4}{s^2} + \frac{1}{s}.$$

9. $e^{-t} t \sin 2t$

$$L(e^{-t} t \sin 2t) = -\frac{d}{ds} L(e^{-t} \sin 2t)$$
$$L(e^{-t} t \sin 2t) = -\frac{d}{ds}\ \frac{2}{(s+1)^2+4}$$
$$L(e^{-t} t \sin 2t) = -\frac{0 \cdot ((s+1)^2+4) - 2(2(s+1))}{[(s+1)^2+4]^2}$$
$$L(e^{-t} t \sin 2t) = \frac{4(s+1)}{[(s+1)^2+4]^2}.$$

11. $\cosh bt$

$$L(\cosh bt) = L(\frac{e^{bt}+e^{-bt}}{2})$$
$$L(\cosh bt) = L(\frac{e^{bt}}{2}) + L(\frac{e^{-bt}}{2})$$
$$L(\cosh bt) = (1/2)L(e^{bt}) + (1/2)L(e^{-bt})$$
$$L(\cosh bt) = (1/2)\frac{1}{(s-b)} + (1/2)\frac{1}{(s+b)}$$

$$L(\cosh bt) = \frac{1}{2(s-b)} + \frac{1}{2(s+b)}$$
$$L(\cosh bt) = \frac{s+b}{2(s-b)(s+b)} + \frac{s-b}{2(s+b)(s-b)}$$
$$L(\cosh bt) = \frac{s}{s^2-b^2}.$$

13. $\sin^2 t$

$$L(\sin^2 t) = L((1/2)(1-\cos 2t))$$
$$L(\sin^2 t) = (1/2)L(1) - (1/2)L(\cos 2t)$$
$$L(\sin^2 t) = (1/2)\frac{1}{s} - (1/2)\frac{s}{(s^2+4)}$$
$$L(\sin^2 t) = \frac{1}{2s} - \frac{s}{2(s^2+4)}.$$

15. $\cos^3 t$

Note first that
$$\cos^3 t = \cos^2 t \cos t = (1/2)(1+\cos 2t)\cos t$$
$$\cos^3 t = (1/2)[\cos t + \cos 2t \cos t] = (1/2)[\cos t + (1/2)\cos(2t+t) + (1/2)\cos(2t-t)]$$
$$\cos^3 t = (1/2)\cos t + (1/4)\cos 3t + (1/4)\cos t = (3/4)\cos t + (1/4)\cos 3t.$$
So that
$$L(\cos^3 t) = L((3/4)\cos t + (1/4)\cos 3t)$$
$$L(\cos^3 t) = L((3/4)\cos t) + L((1/4)\cos 3t)$$
$$L(\cos^3 t) = (3/4)L(\cos t) + (1/4)L(\cos 3t)$$
$$L(\cos^3 t) = (3/4)\frac{s}{(s^2+1)} + (1/4)\frac{s}{(s^2+9)}$$

$$L(\cos^3 t) = \frac{3s}{4(s^2+1)} + \frac{s}{4(s^2+9)}.$$

17. $\sin 2t \sin 5t$

$$L(\sin 2t \sin 5t) = L((1/2)\cos(2t-5t) - (1/2)\cos(2t+5t))$$
$$L(\sin 2t \sin 5t) = L((1/2)\cos(-3t)) - L((1/2)\cos(2t+5t))$$
$$L(\sin 2t \sin 5t) = (1/2)L(\cos(3t)) - (1/2)L(\cos(7t))$$
$$L(\sin 2t \sin 5t) = (1/2)\frac{s}{s^2+9} - (1/2)\frac{s}{s^2+49}$$
$$L(\sin 2t \sin 5t) = \frac{s}{2(s^2+9)} - \frac{s}{2(s^2+49)}$$

19. $\cos nt \sin mt$

$$L(\cos nt \sin mt) = L((1/2)\sin(nt+mt) + (1/2)\sin(mt-nt))$$
$$L(\cos nt \sin mt) = (1/2)L(\sin(n+m)t + (1/2)L(\sin(m-n)t)$$
$$L(\cos nt \sin mt) = (1/2)\frac{n+m}{s^2+(n+m)^2} + (1/2)\frac{m-n}{s^2+(m-n)^2}$$
$$L(\cos nt \sin mt) = \frac{n+m}{2(s^2+(n+m)^2)} + \frac{m-n}{2(s^2+(m-n)^2)}.$$

21. Since $L(\cos bt) = F(s) = \frac{s}{s^2+b^2}$ the translation property states that $L(e^{at}\cos bt) = F(s-a) = \frac{(s-a)}{(s-a)^2+b^2}$.

23. Theorem 4 tell us that if $f(t)$ and $f'(t)$ are piecewise continuous on $[0,\infty)$ and both are of exponential order α, then for $s > \alpha$, $L(f'(t)) = sL(f(t)) - f(0)$. The functions $f(t) = t\sin bt$ and

$f'(t) = \sin bt + bt\cos bt$ are continuous for all real t and are of exponential order $\alpha < 1$ (see exercise 7.2 # 29). Hence

$$\frac{2bs^2}{(s^2+b^2)^2} = L(\sin bt + bt\cos bt) = L(\frac{d}{dt}\, t\sin bt) = sL(t\sin bt) - f(0) = s\Big(\frac{2bs^2}{(s^2+b^2)^2}\Big) - 0 = \frac{2bs^2}{(s^2+b^2)^2}.$$

25. $a.L(t\cos bt)$ $\quad$ $b.Lt^2\cos bt)$

$$L(t\cos bt) = -\frac{d}{ds}\Big(\frac{s}{s^2+b^2}\Big)$$

$$L(t\cos bt) = -\frac{1\cdot(s^2+b^2) - s(2s)}{(s^2+b^2)^2}$$

$$L(t\cos bt) = -\frac{s^2+b^2-2s^2}{(s^2+b^2)^2}$$

$$L(t\cos bt) = \frac{s^2-b^2}{(s^2+b^2)^2}.$$

$$L(t^2\cos bt) = \frac{d^2}{ds^2}\Big(\frac{s}{s^2+b^2}\Big) = -\frac{d}{ds}\Big(-\frac{d}{ds}\Big(\frac{s}{s^2+b^2}\Big)\Big)$$

$$L(t^2\cos bt) = -\frac{d}{ds}\Big(\frac{s^2-b^2}{(s^2+b^2)^2}\Big)$$

$$L(t^2\cos bt) = -\frac{((s^2+b^2)^2 - (s^2-b^2)(2)(s^2+b^2)(2s)}{(s^2+b^2)^4}$$

$$L(t^2\cos bt) = -\frac{(s^2+b^2)[2s(s^2+b^2) - (s^2-b^2)(2)(2s)}{(s^2+b^2)^4}$$

$$L(t^2\cos bt) = -\frac{(s^2+b^2)[2s^3+2sb^2-4s^3+4sb^2]}{(s^2+b^2)^4}$$

$$L(t^2\cos bt) = -\frac{-2s^3+6sb^2}{(s^2+b^2)^3}$$