CHAPTER 2 HOMEWORK SOLUTIONS

Note: the first number corresponds to the homework problem from the 5th edition while the second number in parentheses corresponds to the homework problem in the 6th edition. If the problem does not occur in the 6th edition, the symbol $(-)$ is present.

SECTION 2.2

7, (-). $\frac{dy}{dx} = y(2 + \sin x)$

Using the technique for separation of variables we obtain:

$$\frac{1}{y}\ dy = (2 + \sin x)\ dx$$

$$\int \frac{1}{y}\ dy = \int (2 + \sin x)\ dx + C$$

$$\ln|y| = 2x - \cos x + C$$

$$e^{\ln|y|} = e^{(2x - \cos x + C)}$$

$$|y| = e^{(2x - \cos x)} e^{C}$$

$$|y| = Ce^{(2x - \cos x)}, C > 0$$

$$y = Ce^{(2x - \cos x)}, |C| > 0.$$

Note that $y = 0$ is a solution to the ODE which can obtained without the separation of variables technique.

9, (-). $\frac{dy}{dx} = \frac{1-x^2}{y^2}$

Using the technique for separation of variables we obtain:

$$y^2\ dy = (1 - x^2)\ dx$$

$$\int y^2\ dy = \int (1 - x^2)\ dx + C$$

$$y^3/3 = x - x^3/3 + C$$
$$y^3 = 3x - x^3 + C$$
$$y = (3x - x^3 + C)^{\frac{1}{3}}.$$

11, (11). $\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2}$

Using the technique for separation of variables we obtain:

$$\cos^2 y \; dy = \frac{1}{1+x^2} \; dx$$
$$\int \cos^2 y \; dy = \int \frac{1}{1+x^2} \; dx + C$$
$$\int \frac{1}{2}(1 + \cos 2y) \; dy = \tan^{-1} x + C$$
$$\frac{y}{2} + \frac{\sin 2y}{4} = \tan^{-1} x + C$$
$$2y + \sin 2y = 4 \tan^{-1} x + C.$$

13, (-). $\frac{dx}{dt} + x^2 = x$

Using the technique for separation of variables we obtain:

$$\frac{dx}{dt} = -x^2 + x = x(1-x)$$
$$\frac{1}{x(1-x)} \; dx = dt$$
$$\int \frac{1}{x(1-x)} \; dx = \int \; dt + C.$$

To evaluate the integral requires partial fraction expansion:

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$
$$\frac{1}{x(1-x)} = \frac{A(1-x)}{x(1-x)} + \frac{Bx}{x(1-x)}.$$

Equating the numerators gives:

$$1 = A(1-x) + Bx.$$

Let x = 1, $\longrightarrow 1 = A(0) + B(1) \longrightarrow B = 1.$
Let x = 0, $\longrightarrow 1 = A(1) + B(0) \longrightarrow A = 1.$

Therefore the equation may be written as:

$$\int \frac{1}{x} + \frac{1}{1-x}\ dx = \int\ dt + C$$

$$\ln|x| - \ln|1-x| = t + C$$

$$\ln\left|\frac{x}{1-x}\right| = t + C$$

$$e^{\ln|\frac{x}{1-x}|} = e^{(t+C)}$$

$$\left|\frac{x}{1-x}\right| = e^t e^C$$

$$\left|\frac{x}{1-x}\right| = Ce^t,\ \ C > 0$$

$$\frac{x}{1-x} = Ce^t,\ \ |C| > 0$$

$$x = (1-x)Ce^t$$

$$x = Ce^t - xCe^t$$

$$x + xCe^t = Ce^t$$

$$x(1 + Ce^t) = Ce^t$$

$$x = \frac{Ce^t}{1+Ce^t}$$

$$x = \frac{Ce^t}{1+Ce^t} \times \frac{-1}{-1}$$

$$x = \frac{-Ce^t}{-1-Ce^t}$$

$$x = \frac{Ce^t}{Ce^t - 1}.$$

Note also that $x = 0$ and $x = 1$ are solutions since, for example, $\frac{d}{dt}(1) = 0$ and the right hand side of the original ODE is equal to zero if $x = 1$. The solution $x = 0$ may be obtained from $x = \frac{Ce^t}{Ce^t-1}$ by setting $C = 0$; there does not exist a real number C, however, such that $x = \frac{Ce^t}{Ce^t-1} = 1$ for all t.

15, (15). $y^{-1}\ dy + ye^{\cos x}\sin x\ dx = 0$

Using the technique for separation of variables we obtain:

$$y^{-1}\ dy = -ye^{\cos x}\sin x\ dx$$
$$y^{-2}\ dy = -e^{\cos x}\sin x\ dx$$
$$\int y^{-2}\ dy = \int -e^{\cos x}\sin x\ dx + C$$
$$-y^{-1} = e^{\cos x} + C$$
$$y^{-1} = -e^{\cos x} + C$$
$$y = \frac{1}{C - e^{\cos x}}.$$

17, (17). $y' = x^3(1-y),\ y(0) = 3$

Using the technique for separation of variables we obtain:

$$\frac{1}{1-y}\ dy = x^3\ dx$$
$$\int \frac{1}{1-y}\ dy = \int x^3\ dx + C$$
$$\int \frac{1}{1-y}\ dy = \int x^3\ dx + C$$
$$-\ln|1-y| = \frac{x^4}{4} + C$$
$$\ln|1-y| = -\frac{x^4}{4} + C$$
$$e^{\ln|1-y|} = e^{-\frac{x^4}{4}+C}$$
$$|1-y| = e^{-\frac{x^4}{4}}e^C$$
$$|1-y| = Ce^{-\frac{x^4}{4}},\ C > 0$$
$$1-y = Ce^{-\frac{x^4}{4}},\ |C| > 0.$$

Substituting the initial conditions yields:

$$1-(3) = Ce^{-\frac{0^4}{4}}$$

$$-2 = C.$$

Hence, we have

$$1 - y = -2e^{-\frac{x^4}{4}}$$
$$y - 1 = 2e^{-\frac{x^4}{4}}$$
$$y - 1 = 2e^{-\frac{x^4}{4}}$$
$$y = 1 + 2e^{-\frac{x^4}{4}}.$$

19, (19). $\frac{dy}{dx} = 2\sqrt{y+1}\cos x, \; y(\pi) = 0$

Using the technique for separation of variables we obtain:

$$\frac{1}{\sqrt{y+1}}\; dy = 2\cos x\; dx$$

$$\int \frac{1}{\sqrt{y+1}}\; dy = \int 2\cos x\; dx + C.$$

Using u substitution $u = \sqrt{y+1}$ or integrating directly yields

$$2\sqrt{y+1} = 2\sin x + C$$

$$\sqrt{y+1} = \sin x + C.$$

Substituting the initial conditions yields:

$$\sqrt{0+1} = \sin\pi + C$$

$$1 = 0 + C$$
$$1 = C.$$

Hence, we have

$$\sqrt{y+1} = \sin x + 1$$
$$y + 1 = (\sin x + 1)^2$$
$$y = (\sin x + 1)^2 - 1$$
$$y = \sin^2 x + 2\sin x + 1 - 1$$
$$y = \sin^2 x + 2\sin x.$$

21, (-). $\frac{dy}{d\theta} = y \sin\theta,\ y(\pi) = -3$

Using the technique for separation of variables we obtain:

$$\frac{1}{y}\ dy = \sin\theta\ d\theta$$

$$\begin{aligned}
\int \frac{1}{y}\ dy &= \int \sin\theta\ d\theta + C \\
\ln|y| &= -\cos\theta + C \\
|y| &= e^{-\cos\theta + C} \\
|y| &= e^{-\cos\theta} e^{C} \\
|y| &= Ce^{-\cos\theta},\ C > 0 \\
y &= Ce^{-\cos\theta},\ |C| > 0.
\end{aligned}$$

Substituting the initial conditions yields:

$$-3 = Ce^{-\cos\pi} = Ce^{1} = Ce$$

$$C = -3e^{-1}.$$

Hence, we have

$$\begin{aligned}
y &= -3e^{-1}e^{-\cos\theta} \\
y &= -3e^{-1-\cos\theta}.
\end{aligned}$$

23, (23). $\frac{dy}{dt} = 2t\cos^2 y,\ y(0) = \pi/4$

Using the technique for separation of variables we obtain:

$$\frac{1}{\cos^2 y}\ dy = 2t\ dt$$

$$\begin{aligned}
\int \frac{1}{\cos^2 y}\ dy &= \int 2t\ dt + C \\
\int \sec^2 y\ dy &= \int 2t\ dt + C \\
\tan y &= t^2 + C.
\end{aligned}$$

Substituting the initial conditions yields:

$$\tan(\pi/4) = 0^2 + C$$

$$1 = C.$$

Hence, we have

$$\tan y = t^2 + 1$$

$$y = \tan^{-1}(1 + t^2).$$

25, (25). $\frac{dy}{dx} = x^2(1 + y),\ y(0) = 3$

Using the technique for separation of variables we obtain:

$$\frac{1}{1+y}\ dy = x^2\ dx$$

$$\int \frac{1}{1+y}\ dy = \int x^2\ dx + C$$

$$\ln|1 + y| = \frac{1}{3}x^3 + C$$

$$e^{\ln|1+y|} = e^{\frac{1}{3}x^3 + C}$$

$$|1 + y| = e^{\frac{1}{3}x^3} e^C$$

$$|1 + y| = Ce^{\frac{1}{3}x^3},\ C > 0$$

$$1 + y = Ce^{\frac{1}{3}x^3},\ |C| > 0$$

$$y = Ce^{\frac{1}{3}x^3} - 1.$$

Substituting the initial conditions yields:

$$3 = Ce^{\frac{1}{3}0^3} - 1$$

$$3 = C - 1$$

$$C = 4.$$

Hence, we have

$$y = 4e^{\frac{1}{3}x^3} - 1.$$