

Review Packet for Exam 1

The **order** of a DE is the order of the highest-order derivative in the equation.

A DEQ is **linear** if it can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

Theorem. (Existence and Uniqueness of Solution) Given the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

if f and $\frac{\partial f}{\partial y}$ are defined for (x_0, y_0) , then the IVP has a **unique** solution $\phi(x)$ in some rectangle containing (x_0, y_0) .

ex. The initial value problem $y' = \sqrt{y^2 - 16}$, $y(x_0) = y_0$ has $f(x, y) = \sqrt{y^2 - 16}$ and $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{y^2 - 16}}$

f exists only for $y \geq 4$ or $y \leq -4$ and $\frac{\partial f}{\partial y}$ exists only for $y > 4$ or $y < -4$
so a unique solution to the IVP is guaranteed only where $y > 4$ or $y < -4$.

So for example, we are not guaranteed a solution for $y_0 = 3$ but we are guaranteed a solution for $y_0 = 7$

Verifying or finding particular solutions

ex. Show that $\phi(x) = \sin x - \cos x$ is a particular solution to the IVP

$$y'' + y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

$$\phi'(x) = \cos x + \sin x$$

$$\phi''(x) = -\sin x + \cos x$$

Check to see that $\phi(x)$ satisfies all 3 equations:

1. Plug in $\frac{d^2 y}{dx^2} + y = 0$

$$-\sin x + \cos x + \sin x - \cos x = 0 \quad \checkmark$$

2. $y(0) = -1 \quad \checkmark$

3. $y'(0) = 1 \quad \checkmark$

ex. Find the values of m and n for which e^{mx} is a solution to $y'' - 4y' - 5y = 0$

$$\phi(x) = e^{mx}$$

$$\phi'(x) = me^{mx}$$

$$\phi''(x) = m^2e^{mx}$$

$$\text{Plug into } y'' - 4y' - 5y = 0$$

$$m^2e^{mx} - 4me^{mx} - 5e^{mx} = e^{mx}(m^2 - 4m - 5) = 0$$

$$e^{mx}(m - 5)(m + 1) = 0$$

$$m = 5, -1$$

Def. a DE is **separable** if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y).$$

ex. Solve $\frac{dy}{dx} = x^2(1 + y)$, $y(0) = 3$

$$\frac{1}{1+y} dy = x^2 dx$$

$$\int \frac{1}{1+y} dy = \int x^2 dx$$

$$\ln|1+y| = \frac{x^3}{3} + C$$

$$e^{\ln|1+y|} = e^{\frac{x^3}{3} + C}$$

$$|1+y| = e^{\frac{x^3}{3}} e^C$$

$$|1+y| = Ce^{\frac{x^3}{3}}$$

$$1+y = Ce^{\frac{x^3}{3}}$$

$$y = Ce^{\frac{x^3}{3}} - 1$$

Insert initial conditions to get

$$3 = Ce^0 - 1$$

$$3 = C - 1$$

$$C = 4$$

$$y = 4e^{\frac{x^3}{3}} - 1$$

Def. A first-order linear DE has the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x), \quad (1)$$

where $a_1(x) \neq 0$.

Method for solving linear equations:

1. Write equation in standard form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2. Find the integrating factor $\mu(x) = e^{\int P(x) dx}$

3. Multiply (1) by $\mu(x)$ and rewrite the DE as

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

4. Integrate to solve for $y = \frac{1}{\mu(x)} [\int \mu(x)Q(x) dx + C]$

ex. Solve $(x^2 + 4) \frac{dy}{dx} + xy = x$

Put in standard form: $\frac{dy}{dx} + \frac{x}{(x^2+4)}y = \frac{x}{(x^2+4)}$

$$P(x) = \frac{x}{x^2+4}$$

$$\mu(x) = e^{\int \frac{x}{x^2+4} dx}$$

Let $u = x^2 + 4$ **then** $du = 2xdx$, $dx = du/2x$, **and we get**

$$e^{\frac{1}{2} \int \frac{du}{u}} = e^{\frac{1}{2} \ln |u|}$$

$$= e^{\ln |u|^{\frac{1}{2}}} = \sqrt{u} = \sqrt{x^2 + 4}$$

Then the DE becomes $\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$

$$\frac{d}{dx}[\sqrt{x^2 + 4}y] = \frac{x}{x^2+4} \sqrt{x^2 + 4} = \frac{x}{\sqrt{x^2+4}}$$

$$[\sqrt{x^2 + 4}y] = \int \frac{x}{\sqrt{x^2+4}} dx$$

Let $u = x^2 + 4$, **then** $du = 2xdx$, $dx = du/2x$, **and we get**

$$\frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = u^{\frac{1}{2}} + C = \sqrt{x^2 + 4} + C$$

$$[\sqrt{x^2 + 4}y] = \sqrt{x^2 + 4} + C$$

Then $y = \sqrt{x^2 + 4} \frac{1}{\sqrt{x^2+4}} + \frac{C}{\sqrt{x^2+4}} = 1 + \frac{C}{\sqrt{x^2+4}}$

Method for solving exact equations:

$$M(x, y)dx + N(x, y)dy = 0$$

1. Check exactness: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

2. If exact, write $\frac{\partial F}{\partial x} = M(x, y)$ and then

$$f(x, y) = \int M dx + g(y)$$

3. Determine $g(y)$:

- take the partial derivative w.r.t y of both sides of $f(x, y)$

$$\frac{\partial f}{\partial y} = (d/dy)(\int M dx) + g'(y)$$

- substitute $N(x, y) = \frac{\partial f}{\partial y}$ to determine $g'(y)$

- find $g(y)$ through integration

$$g(y) = \int g'(y) dy$$

4. Substitute $g(y)$ into (*) to get $f(x, y)$.

5. Solution: $F(x, y) = C$

ex Solve $(6x^2 - y + 3)dx + (3y^2 - x - 2)dy = 0$

$$M = 6x^2 - y - 3, \quad N = 3y^2 - x - 2$$

Check for exactness:

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x} = -1$$

Construct $F(x, y)$:

$$F(x, y) = \int M dx + g(y) = \int (6x^2 - y + 3) dx + g(y) = 2x^3 - xy + 3x + g(y)$$

Find $g(y)$:

$$\frac{\partial F}{\partial y} = -x + g'(y) = N = 3y^2 - x - 2$$

$$g'(y) = 3y^2 - 2$$

$$g(y) = \int (3y^2 - 2) dy = y^3 - 2y + C$$

Then

$$F(x, y) = 2x^3 - xy + 3x + y^3 - 2y + C$$

$$2x^3 - xy + 3x + y^3 - 2y = C$$

Method for solving equations that require integrating factors to be exact

If $M(x, y)dx + N(x, y)dy = 0$ is not exact.

(a) If $\frac{\partial M/\partial y - \partial N/\partial x}{N}$ depends only on x , then use

$$\mu(x) = \exp \left[\int \left(\frac{\partial M/\partial y - \partial N/\partial x}{N} \right) dx \right]$$

as an integrating factor.

(b) If $\frac{\partial N/\partial x - \partial M/\partial y}{M}$ depends only on y , then use

$$\mu(y) = \exp \left[\int \left(\frac{\partial N/\partial x - \partial M/\partial y}{M} \right) dy \right]$$

an integrating factor.

(c) Proceed with the method for exact equations described previously.

ex Solve $(2y^2 + 2y + 4x^2)dx + (2xy + x)dy = 0$

Check for exactness:

$$\frac{\partial M}{\partial y} = 4y + 2$$

$$\frac{\partial N}{\partial x} = 2y + 1$$

The DE is not exact so we must find the integrating factor:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y+2-2y-1}{x(2y+1)} = \frac{2y+1}{x(2y+1)} = \frac{1}{x}$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Multiply the DE by the integrating factor:

$$(2xy^2 + 2xy + 4x^3)dx + (2x^2y + x^2)dy = 0$$

The equation is now exact (Check for yourself!)

Construct $F(x, y)$:

$$f(x, y) = \int (2xy^2 + 2xy + 4x^3)dx = x^2y^2 + x^2y + x^4 + g(y)$$

Find $g(y)$:

$$\frac{\partial f}{\partial y} = 2x^2y + x^2 + g'(y) = N = 2x^2y + x^2$$

$$g'(y) = 0$$

$$g(y) = C$$

Then

$$f(x, y) = x^2y^2 + x^2y + x^4 = C$$

Method for Solving Homogeneous Equations:

Using substitution $v = y/x$ (or $u = x/y$) will reduce a homogeneous DE

$$M(x, y)dx + N(x, y)dy = 0$$

to a separable first-order DE.

1. Make the substitution $v = \frac{y}{x}$. Now our DE is

$$\frac{dy}{dx} = G(v) \quad (*)$$

2. Our new variables are v and x , so we need to make $\frac{dy}{dx}$ in terms of v and x .

$$v = \frac{y}{x} \rightarrow y = vx$$
$$\frac{dy}{dx} = x \frac{dv}{dx} + v$$

3. Sub this into (*), separate variables and integrate

4. The last step after integration is to put the equation back in terms of x and y .

ex $(y^2 + yx)dx - x^2dy = 0$

Let $y = vx$, then $dy = xdv + vdx$ and we get
 $(v^2x^2 + vx^2)dx - x^2(xdv + vdx) = 0$

Group the dx and the dy terms separately:

$$(v^2x^2 + vx^2 - vx^2)dx - x^3dv = 0$$

$$x^2v^2dx = x^3dv$$

$$x^3dv = x^2v^2dx$$

$$xdv = v^2dx$$

Separate variables:

$$\frac{dv}{v^2} = \frac{dx}{x}$$

Integrate:

$$-\frac{1}{v} = \ln|x| + c$$

Replace v with $\frac{y}{x}$

$$-\frac{x}{y} = \ln|x| + c$$

Solve for y

$$-\frac{y}{x} = \frac{1}{\ln|x| + c}$$

$$y = -\frac{x}{\ln|x| + c}$$

Method for solving Bernoulli Equations

Def. A first-order equation that can be written as

$$(*) \quad \frac{dy}{dx} + P(x)y = Q(x)y^n,$$

$P(x)$ and $Q(x)$ continuous on an interval (a, b) and n is a real number, is called a Bernoulli Equation.

1. Divide $(*)$ by y^n : $y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x)$
2. Substitute $v = y^{1-n}$ (this will transform the Bernoulli equation into a linear equation).
3. To find $\frac{dy}{dx}$ in terms of w , differentiate $v = y^{1-n}$:

$$\frac{dv}{dx} = (1-n)y^{1-n-1} \frac{dy}{dx}$$

4. Substitute this into the above equation (1):

$$y^{-n} \left(\frac{1}{1-n} y^n \frac{dv}{dx} \right) + P(x)y^{1-n} = Q(x)$$

$$\left(\frac{1}{1-n} \frac{dv}{dx} \right) + P(x)v = Q(x)$$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

This is now a linear equation in v and can be solved as one.

ex $\frac{dy}{dx} + y = xy^{-2}$

$n = -2$, then $v = y^{1-(-2)} = y^3$ and we get

$$\frac{dv}{dx} = 3y^2 \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3y^2} \frac{dv}{dx}$$

Our equation becomes

$$\frac{1}{3y^2} \frac{dv}{dx} + y = xy^{-2}$$

Multiply by $3y^2$

$$\frac{dv}{dx} + 3y^3 = 3x$$

$$\frac{dv}{dx} + 3v = 3x$$

Now our equation is linear with $P(x) = 3$ and $Q(x) = 3x$

$$\mu(x) = e^{\int 3dx} = e^{3x}$$

$$\frac{d}{dx}[e^{3x}v] = 3xe^x$$

$$e^{3x}v = 3 \int xe^{3x}$$

Integrating by parts gives $e^{3x}v = 3[e^{3x}(\frac{x}{3} - \frac{1}{9}) + C]$

$$v = x - \frac{1}{3} + 3Ce^{-3x}$$

$$y^3 = x - \frac{1}{3} + 3Ce^{-3x}$$

$$y = \sqrt[3]{x - \frac{1}{3} + 3Ce^{-3x}}$$

Classification by type

A DE is linear if it can be written in the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x).$$

Def. a DE is separable if it can be written in the form

$$\frac{dy}{dx} = g(x)p(y).$$

Def. A first-order linear DE has the form

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x), \quad a_1(x) \neq 0.$$

Def. A DE is exact if it is in the form $M(x, y)dx + N(x, y)dy = 0$ and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Def. A DE $\frac{dy}{dx} = f(x, y)$ is homogeneous if the right hand side can be expressed as a function of y/x alone.

Def. A first-order equation that can be written as $\frac{dy}{dx} + P(x)y = Q(x)y^n$, $P(x)$ and $Q(x)$ continuous on an interval (a, b) and n is a real number, is called a Bernoulli Equation. Note: Linear equations are always Bernoulli.

ex Classify $\frac{dy}{dx} = 5y + y^3$

Separable?

$$\frac{dy}{5y+y^3} = dx; \text{ so it is separable}$$

Linear in x ?

$$\frac{dx}{dy} = \frac{1}{5y+y^3} \text{ so it is linear in } x$$

Bernoulli?

$$\frac{dy}{dx} - 5y = y^3 \text{ so it is a Bernoulli equation}$$

Exact?

$$dy = (5y + y^3)dx$$

$$(5y + y^3)dx - dy = 0$$

$$M_y = 5 + 3y^2; N_x = 0; \text{ so it is not exact}$$

Linear in y ?

$$y^3 \text{ term it so not linear in } y$$

Misc. Integration tips and identities:

$$e^{x+y} = e^x e^y$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$1. \int u \, dv = uv - \int v \, du$$

$$2. \int u^n \, du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$3. \int \frac{1}{u} \, du = \ln |u| + C \quad (n = -1)$$

$$4. \int e^u \, du = e^u + C$$

$$5. \int a^u \, du = \frac{a^u}{\ln a} + C$$

$$6. \int \sin u \, du = -\cos u + C$$

$$7. \int \cos u \, du = \sin u + C$$

$$8. \int \sec^2 u \, du = \tan u + C$$

$$9. \int \sec u \tan u \, du = \sec u + C$$

$$10. \int \tan u \, du = \ln |\sec u| + C$$

$$11. \int \cot u \, du = \ln |\sin u| + C$$

$$12. \int \sec u \, du = \ln |\sec u + \tan u| + C$$

$$13. \int \csc u \, du = \ln |\csc u - \cot u| + C$$

$$14. \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$$

$$15. \int \frac{du}{1 + u^2} = \tan^{-1} u + C$$