SECTION 4.3

21, (21). $y'' + 2y' + 2y = 0,\ y(0) = 2,\ y'(0) = 1$

The characteristic equation is given by

$$r^2 + 2r + 2 = 0$$

so there exists two complex conjugate roots given by the quadratic equation

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i.$$

Hence the general solution and its derivative are given by

$$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$$

$$y' = -c_1 e^{-x} \cos x - c_1 e^{-x} \sin x - c_2 e^{-x} \sin x + c_2 e^{-x} \cos x.$$

Inserting the initial conditions yields

$$2 = c_1 e^0 \cos 0 + c_2 e^0 \sin 0 = c_1$$

$$1 = -c_1 e^0 \cos 0 - c_1 e^0 \sin 0 - c_2 e^0 \sin 0 + c_2 e^0 \cos 0 = -c_1 + c_2.$$

The linear system has the solution $c_1 = 2,\ c_2 = 3$ so that the answer is given by

$$y = 2e^{-x} \cos x + 3e^{-x} \sin x.$$

23, (23). $w'' - 4w' + 2w = 0,\ w(0) = 0,\ w'(0) = 1$

The characteristic equation is given by

$$r^2 - 4r + 2 = 0$$

so there exists two real, distinct roots given by the quadratic equation

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}.$$

Hence the general solution and its derivative are given by

$$w = c_1 e^{(2+\sqrt{2})x} + c_2 e^{(2-\sqrt{2})x}$$

$$w' = c_1(2+\sqrt{2})e^{(2+\sqrt{2})x} + c_2(2-\sqrt{2})e^{(2-\sqrt{2})x}.$$

Inserting the initial conditions yields

$$0 = c_1e^0 + c_2e^0 = c_1 + c_2$$

$$1 = c_1(2+\sqrt{2})e^0 + c_2(2-\sqrt{2})e^0 = c_1(2+\sqrt{2}) + c_2(2-\sqrt{2}).$$

The linear system has the solution $c_1 = \sqrt{2}/4,\ c_2 = -\sqrt{2}/4$ so that the answer is given by

$$w = (\sqrt{2}/4)(e^{(2+\sqrt{2})x} - e^{(2-\sqrt{2})x}).$$

25, (25). $y'' - 2y' + 2y = 0,\ y(\pi) = e^{\pi},\ y'(\pi) = 0$

The characteristic equation is given by

$$r^2 - 2r + 2 = 0$$

so there exists two complex conjugate roots given by the quadratic equation

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

Hence the general solution and its derivative are given by

$$y = c_1e^x \cos x + c_2e^x \sin x$$

$$y' = c_1e^x \cos x - c_1e^x \sin x + c_2e^x \sin x + c_2e^x \cos x.$$

Inserting the initial conditions yields

$$e^{\pi} = c_1e^{\pi} \cos \pi + c_2e^{\pi} \sin \pi = -c_1e^{\pi}$$

$$0 = c_1e^{\pi} \cos \pi - c_1e^{\pi} \sin \pi + c_2e^{\pi} \sin \pi + c_2e^{\pi} \cos \pi = -c_1e^{\pi} - c_2e^{\pi}.$$

The linear system has the solution $c_1 = -1,\ c_2 = 1$ so that the answer is given by

$$y = -e^x \cos x + e^x \sin x.$$