

CHAPTER 8 HOMEWORK SOLUTIONS

Note: the first number corresponds to the homework problem from the 5th edition while the second number in parentheses corresponds to the homework problem in the 6th edition. If the problem does not occur in the 6th edition, the symbol $(-)$ is present.

SECTION 8.1

$$1, (1). \quad y' = x^2 + y^2, \quad y(0) = 1$$

The Taylor series about zero has the form

$$y(x) = \sum_{n=0}^{\infty} y^{(n)}(0)x^n/n! = y(0) + y'(0)x + y''(0)x^2/2 + y'''(0)x^3/6 + \cdots.$$

To compute the first three nonzero terms we need the first three derivatives of y which do not vanish when evaluated at zero:

$$y(0) = 1$$

$$y' = x^2 + y^2 \quad \longrightarrow \quad y'(0) = 0^2 + 1^2 = 1$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(x^2 + y^2) = 2x + 2yy' \quad \longrightarrow \quad y''(0) = 2(0) + 2(1)(1) = 2.$$

Hence

$$y(x) = 1 + x + 2x^2/2 + \cdots$$

$$y(x) = 1 + x + x^2 + \cdots.$$

$$3, (3). \quad y' = \sin y + e^x, \quad y(0) = 0$$

The Taylor series about zero has the form

$$y(x) = \sum_{n=0}^{\infty} y^{(n)}(0)x^n/n! = y(0) + y'(0)x + y''(0)x^2/2 + y'''(0)x^3/6 + \cdots.$$

To compute the first three nonzero terms we need the first three derivatives of y which do not vanish when evaluated at zero:

$$y(0) = 0$$

$$y' = \sin y + e^x \longrightarrow y'(0) = \sin 0 + e^0 = 1$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(\sin y + e^x) = y' \cos y + e^x \longrightarrow y''(0) = 1 \cdot \cos 0 + e^0 = 2$$

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(y' \cos y + e^x) = y'' \cos y - (y')^2 \sin y + e^x \longrightarrow y'''(0) = 2 \cdot \cos 0 - (1)^2 \cdot \sin 0 + e^0 = 3.$$

Hence

$$y(x) = x + 2x^2/2 + 3x^3/6 + \dots$$

$$y(x) = x + x^2 + x^3/2 + \dots$$

$$5, (5). \quad x'' + tx = 0, \quad x(0) = 1, \quad x'(0) = 0$$

The Taylor series about zero has the form

$$x(t) = \sum_{n=0}^{\infty} x^{(n)}(0)t^n/n! = x(0) + x'(0)t + x''(0)t^2/2 + x'''(0)t^3/6 + \dots$$

To compute the first three nonzero terms we need the first three derivatives of x which do not vanish when evaluated at zero:

$$x(0) = 1$$

$$x'(0) = 0$$

$$x'' = -tx \longrightarrow x''(0) = 0(1) = 0$$

$$x''' = \frac{d}{dt}(x'') = \frac{d}{dt}(-tx) = -x - tx' \longrightarrow x'''(0) = -1 - 0(0) = -1$$

$$x^{iv} = \frac{d}{dt}(x''') = \frac{d}{dt}(-x - tx') = -2x' - tx'' \longrightarrow x^{iv}(0) = -2(0) - 0(0) = 0$$

$$x^v = \frac{d}{dt}(x^{iv}) = \frac{d}{dt}(-2x' - tx'') = -3x'' - tx''' \longrightarrow x^v(0) = -3(0) - 0(-1) = 0$$

$$x^{vi} = \frac{d}{dt}(x^v) = \frac{d}{dt}(-3x'' - tx''') = -4x''' - tx^{iv} \longrightarrow x^{vi}(0) = -4(-1) - 0(0) = 4$$

Hence

$$x(t) = 1 + 0(t) + 0(t^2)/2 - 1(t^3)/6 + 0(t^4)/24 + 0(t^5)/120 + 4(t^6)/720 + \dots$$

$$x(t) = 1 - t^3/6 + t^6/180 + \dots$$

$$7, (7). \quad y''(\theta) + y(\theta)^3 = \sin \theta, \quad y(0) = 0, \quad y'(0) = 0$$

The Taylor series about zero has the form

$$y(\theta) = \sum_{n=0}^{\infty} y^{(n)}(0)\theta^n/n! = y(0) + y'(0)\theta + y''(0)\theta^2/2 + y'''(0)\theta^3/6 + \dots$$

To compute the first three nonzero terms we need the first three derivatives of y which do not vanish when evaluated at zero:

$$y(0) = 0$$

$$y'(0) = 0$$

$$y(\theta)'' = -y(\theta)^3 + \sin \theta \longrightarrow y''(0) = -0^3 + 0 = 0$$

$$y(\theta)''' = \frac{d}{d\theta}(y''(\theta)) = \frac{d}{d\theta}(-y(\theta)^3 + \sin \theta) = -3y(\theta)^2 y'(\theta) + \cos(\theta) \longrightarrow y'''(0) = -3(0)(0) + 1 = 1$$

$$y^{iv}(\theta) = \frac{d}{d\theta}(y'''(\theta)) = \frac{d}{d\theta}(-3y(\theta)^2 y'(\theta) + \cos(\theta)) = -6y(\theta)(y'(\theta))^2 - 3y(\theta)^2 y''(\theta) - \sin \theta$$

$$\longrightarrow y^{iv}(0) = -6(0)(0)^2 - 3(0)^2(0) - 0 = 0$$

$$y^v(\theta) = \frac{d}{dt}(y^{iv}(\theta)) = \frac{d}{d\theta}(-6y(\theta)(y'(\theta))^2 - 3y(\theta)^2 y''(\theta) - \sin \theta)$$

$$\begin{aligned}
&= -6(y'(\theta))^3 - 12y(\theta)y'(\theta)y''(\theta) - 6y(\theta)y'(\theta)y''(\theta) - 3y(\theta)^2y'''(\theta) - \cos \theta \\
&= -6(y'(\theta))^3 - 18y(\theta)y'(\theta)y''(\theta) - 3y(\theta)^2y'''(\theta) - \cos \theta \\
&\longrightarrow y^v(0) = -6(0)^3 - 18(0)(0)(0) - 3(0)(1) - 1 = -1
\end{aligned}$$

$$\begin{aligned}
y^{vi}(\theta) &= \frac{d}{dt}(y^v(\theta)) = \frac{d}{d\theta}(-6(y'(\theta))^3 - 18y(\theta)y'(\theta)y''(\theta) - 3y(\theta)^2y'''(\theta) - \cos \theta) \\
&= -18(y'(\theta))^2y''(\theta) - 18(y'(\theta))^2y''(\theta) - 18y(\theta)(y''(\theta))^2 - 18y(\theta)y'(\theta)y'''(\theta) - 6y(\theta)y'(\theta)y'''(\theta) \\
&\quad - 3y(\theta)^2y^{iv}(\theta) + \sin \theta \\
&= -36(y'(\theta))^2y''(\theta) - 18y(\theta)(y''(\theta))^2 - 24y(\theta)y'(\theta)y'''(\theta) - 3y(\theta)^2y^{iv}(\theta) + \sin \theta \\
&\longrightarrow y^{vi}(0) = -36(0)^2(0) - 18(0)(0)^2 - 24(0)(0)(1) - 3(0)^2(0) + 0 = 0
\end{aligned}$$

$$\begin{aligned}
y^{vii}(\theta) &= \frac{d}{dt}(y^{vi}(\theta)) = \frac{d}{d\theta}(-36(y'(\theta))^2y''(\theta) - 18y(\theta)(y''(\theta))^2 - 24y(\theta)y'(\theta)y'''(\theta) - 3y(\theta)^2y^{iv}(\theta) + \sin \theta) \\
&= -72y'(\theta)(y''(\theta))^2 - 36(y'(\theta))^2y'''(\theta) - 18y'(\theta)(y''(\theta))^2 - 36y(\theta)y''(\theta)y'''(\theta) - 24(y'(\theta))^2y'''(\theta) \\
&\quad - 24y(\theta)y''(\theta)y'''(\theta) - 24y(\theta)y'(\theta)y^{iv}(\theta) - 6y(\theta)y'(\theta)y^{iv}(\theta) - 3y(\theta)^2y^v(\theta) + \cos \theta \\
&= -90y'(\theta)(y''(\theta))^2 - 60(y'(\theta))^2y'''(\theta) - 60y(\theta)y''(\theta)y'''(\theta) - 30y(\theta)y'(\theta)y^{iv}(\theta) - 3y(\theta)^2y^v(\theta) + \cos \theta \\
&\longrightarrow y^{vii}(0) = -90(0)(0)^2 - 60(0)^2(1) - 60(0)(0)(0) - 30(0)(0)(0) - 3(0)^2(0) + 1 = 1
\end{aligned}$$

Hence

$$\begin{aligned}
y(\theta) &= 0 + 0(\theta) + 0(\theta^2)/2 + 1(\theta^3)/6 + 0(\theta^4)/24 - 1(\theta^5)/120 + 0(\theta^6)/720 + 1(\theta^7)/5040 + \dots \\
y(\theta) &= \theta^3/6 - \theta^5/120 + \theta^7/5040 + \dots
\end{aligned}$$