

$$L\{f(t)\} = -(2/s)e^{-2s} - (1/s^2)e^{-2s} + (1/s^2) + (5/s)e^{-2s}, \quad s > 0$$

$$L\{f(t)\} = (3/s)e^{-2s} - (1/s^2)e^{-2s} + (1/s^2), \quad s > 0$$

$$L\{f(t)\} = \frac{(3s-1)e^{-2s} + 1}{s^2}, \quad s > 0.$$

Therefore we have

$$(s^2 + 4)Y(s) + s = \frac{(3s-1)e^{-2s} + 1}{s^2}$$

$$(s^2 + 4)Y(s) = \frac{(3s-1)e^{-2s} + 1}{s^2} - s$$

$$(s^2 + 4)Y(s) = \frac{(3s-1)e^{-2s} + 1 - s(s^2)}{s^2}$$

$$Y(s) = \frac{(3s-1)e^{-2s} + 1 - s^3}{s^2(s^2 + 4)}.$$

SECTION 7.6

$$5. \quad g(t) = \begin{cases} 0, & 0 < t < 1 \\ 2, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ 3, & 3 < t \end{cases}$$

Using the unit step function we can write

$$g(t) = 0 + (2-0)u(t-1) + (1-2)u(t-2) + (3-1)u(t-3)$$

$$g(t) = 2u(t-1) - u(t-2) + 2u(t-3).$$

Hence we have

$$L\{g(t)\} = \frac{2e^{-s}}{s} - \frac{e^{-2s}}{s} + \frac{2e^{-3s}}{s}$$

$$L\{g(t)\} = \frac{2e^{-s} - e^{-2s} + 2e^{-3s}}{s}.$$

7. The function, given graphically in this problem, can be written piecewise as

$$g(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 1, & 2 < t \end{cases}$$

and hence using the unit step function as

$$g(t) = 0 + (t - 0)u(t - 1) + (1 - t)u(t - 2)$$

$$g(t) = tu(t - 1) + (1 - t)u(t - 2).$$

Hence we have

$$L\{g(t)\} = e^{-s}L\{t + 1\} + e^{-2s}L\{1 - (t + 2)\} = e^{-s}L\{t + 1\} - e^{-2s}L\{1 + t\}$$

$$L\{g(t)\} = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$L\{g(t)\} = e^{-s}\left(\frac{1 + s}{s^2}\right) - e^{-2s}\left(\frac{1 + s}{s^2}\right)$$

$$L\{g(t)\} = \left(\frac{(1 + s)(e^{-s} - e^{-2s})}{s^2}\right).$$

9. The function, given graphically in this problem, can be written piecewise as

$$g(t) = \begin{cases} 0, & 0 < t < 1 \\ t - 1, & 1 \leq t < 2 \\ -t + 3, & 2 \leq t < 3 \\ 0, & 3 \leq t \end{cases}$$

and hence using the unit step function as

$$g(t) = 0 + ((t - 1) - 0)u(t - 1) + ((-t + 3) - (t - 1))u(t - 2) + (0 - (-t + 3))u(t - 3)$$

$$g(t) = (t - 1)u(t - 1) + (-2t + 4)u(t - 2) + (t - 3)u(t - 3).$$

Hence we have

$$L\{g(t)\} = e^{-s}L\{(t + 1) - 1\} + e^{-2s}L\{-2(t + 2) + 4\} + e^{-3s}L\{(t + 3) - 3\}$$

$$L\{g(t)\} = e^{-s}L\{t\} + e^{-2s}L\{-2t\} + e^{-3s}L\{t\}$$

$$\begin{aligned}
L\{g(t)\} &= e^{-s}L\{t\} - 2e^{-2s}L\{t\} + e^{-3s}L\{t\} \\
L\{g(t)\} &= (e^{-s} - 2e^{-2s} + e^{-3s})L\{t\} \\
L\{g(t)\} &= \left(e^{-s} - 2e^{-2s} + e^{-3s}\right)\frac{1}{s^2} \\
L\{g(t)\} &= \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s^2}.
\end{aligned}$$

11. $\frac{e^{-2s}}{s-1}$

$$L^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} = e^{t-2}u(t-2).$$

13. $\frac{e^{-2s}-3e^{-4s}}{s+2}$

$$\begin{aligned}
L^{-1}\left\{\frac{e^{-2s}-3e^{-4s}}{s+2}\right\} &= L^{-1}\left\{\frac{e^{-2s}}{s+2}\right\} - 3L^{-1}\left\{\frac{e^{-4s}}{s+2}\right\} \\
L^{-1}\left\{\frac{e^{-2s}-3e^{-4s}}{s+2}\right\} &= e^{-2(t-2)}u(t-2) - 3e^{-2(t-4)}u(t-4).
\end{aligned}$$

15. $\frac{se^{-3s}}{s^2+4s+5}$

$$\begin{aligned}
L^{-1}\left\{\frac{se^{-3s}}{s^2+4s+5}\right\} &= L^{-1}\left\{\frac{se^{-3s}}{(s+2)^2+1}\right\} \\
L^{-1}\left\{\frac{se^{-3s}}{s^2+4s+5}\right\} &= L^{-1}\left\{\frac{(s+2)e^{-3s}}{(s+2)^2+1}\right\} - 2L^{-1}\left\{\frac{e^{-3s}}{(s+2)^2+1}\right\} \\
L^{-1}\left\{\frac{se^{-3s}}{s^2+4s+5}\right\} &= e^{-2(t-3)}\cos(t-3)u(t-3) - 2e^{-2(t-3)}\sin(t-3)u(t-3)
\end{aligned}$$

$$L^{-1}\left\{\frac{se^{-3s}}{s^2 + 4s + 5}\right\} = e^{-2(t-3)}[\cos(t-3) - 2e^{-2(t-3)}\sin(t-3)]u(t-3).$$

17. $\frac{e^{-3s}(s-5)}{(s+1)(s+2)}$

Before taking inverse transforms we need to do a partial fraction expansion on the function:

$$\frac{(s-5)}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

After obtaining common denominators and equating numerators we have

$$s-5 = A(s+2) + B(s+1).$$

Choosing appropriate values of s gives:

$$s = -1 : -6 = A$$

$$s = -2 : -7 = -B \longrightarrow B = 7.$$

Therefore we have

$$\begin{aligned} L^{-1}\left\{\frac{e^{-3s}(s-5)}{(s+1)(s+2)}\right\} &= L^{-1}\left\{\frac{-6e^{-3s}}{s+1}\right\} + L^{-1}\left\{\frac{7e^{-3s}}{s+2}\right\} \\ L^{-1}\left\{\frac{e^{-3s}(s-5)}{(s+1)(s+2)}\right\} &= -6e^{-(t-3)}u(t-3) + 7e^{-2(t-3)}u(t-3) \\ L^{-1}\left\{\frac{e^{-3s}(s-5)}{(s+1)(s+2)}\right\} &= [-6e^{-(t-3)} + 7e^{-2(t-3)}]u(t-3). \end{aligned}$$

21. $f(t) = t$, $0 < t < 2$, and $f(t)$ has period 2.

The windowed version of the periodic function is given above

$$f_T(t) = \begin{cases} t, & 0 \leq t < 2 \\ 0, & 2 < t \end{cases}$$

so that by definition we have

$$F_T(s) = \int_0^\infty e^{-st} f_T(t) dt$$

$$F_T(s) = \int_0^2 e^{-st} t dt.$$

Using integration by parts gives

$$F_T(s) = -\frac{te^{-st}}{s} \Big|_0^2 + \int_0^2 \frac{te^{-st}}{s} dt$$

$$F_T(s) = -\frac{te^{-st}}{s} \Big|_0^2 - \frac{e^{-st}}{s^2} \Big|_0^2$$

$$F_T(s) = -\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}$$

$$F_T(s) = \frac{1 - (2s + 1)e^{-2s}}{s^2}.$$

Hence since $F(s) = \frac{F_T(s)}{1 - e^{-sT}}$ we find

$$F(s) = \frac{1 - (2s + 1)e^{-2s}}{s^2(1 - e^{-2s})}.$$

23. $f(t) = \begin{cases} e^{-t}, & 0 \leq t < 1 \\ 1, & 1 < t < 2 \end{cases}$ and $f(t)$ has period 2.

The windowed version of the periodic function is given above so that by definition we have

$$F_T(s) = \int_0^\infty e^{-st} f_T(t) dt$$

$$F_T(s) = \int_0^1 e^{-st} e^{-t} dt + \int_1^2 e^{-st} \cdot 1 dt$$

$$F_T(s) = \int_0^1 e^{-(s+1)t} dt + \int_1^2 e^{-st} dt$$

$$F_T(s) = -\frac{e^{-(s+1)t}}{s+1} \Big|_0^1 - \frac{e^{-st}}{s} \Big|_1^2$$

$$\begin{aligned}
F_T(s) &= -\frac{e^{-(s+1)}}{s+1} + \frac{1}{s+1} - \frac{e^{-2s}}{s} + \frac{e^{-s}}{s} \\
F_T(s) &= \frac{1 - e^{-(s+1)}}{s+1} + \frac{e^{-s} - e^{-2s}}{s} \\
F_T(s) &= \frac{s(1 - e^{-(s+1)}) + (s+1)(e^{-s} - e^{-2s})}{s(s+1)}
\end{aligned}$$

Hence since $F(s) = \frac{F_T(s)}{1 - e^{-sT}}$ we find

$$F(s) = \frac{s(1 - e^{-(s+1)}) + (s+1)(e^{-s} - e^{-2s})}{s(s+1)(1 - e^{-2s})}.$$

25. The function given graphically in problem 25 has a windowed version given by

$$f(t) = \begin{cases} 1, & 0 < t < a \\ 0, & a < t < 2a \end{cases}$$

and has period $2a$. Therefore we find

$$\begin{aligned}
F_T(s) &= \int_0^\infty e^{-st} f_T(t) dt \\
F_T(s) &= \int_0^a e^{-st} \cdot 1 dt + \int_a^{2a} e^{-st} \cdot 0 dt \\
F_T(s) &= \int_0^a e^{-st} dt \\
F_T(s) &= -\frac{e^{-st}}{s} \Big|_0^a \\
F_T(s) &= -\frac{e^{-as}}{s} + \frac{1}{s} \\
F_T(s) &= \frac{1 - e^{-as}}{s}.
\end{aligned}$$

Hence since $F(s) = \frac{F_T(s)}{1 - e^{-sT}}$ we find

$$F(s) = \frac{1 - e^{-as}}{s(1 - e^{-2as})}$$

$$F(s) = \frac{1 - e^{-as}}{s(1 - e^{-as})(1 + e^{-as})}$$

$$F(s) = \frac{1}{s(1 + e^{-as})}.$$

27. The function given graphically in problem 25 has a windowed version given by

$$f(t) = \begin{cases} t/a, & 0 \leq t \leq a \\ -(t/a) + 2, & a \leq t \leq 2a \end{cases}$$

and has period $2a$. Therefore we find

$$F_T(s) = \int_0^\infty e^{-st} f_T(t) dt$$

$$F_T(s) = \int_0^a e^{-st} (t/a) dt + \int_a^{2a} e^{-st} [-(t/a) + 2] dt$$

$$F_T(s) = \frac{1}{a} \left[-\frac{te^{-st}}{s} \Big|_0^a + \int_0^a \frac{e^{-st}}{s} dt \right] - \frac{e^{-st} [-(t/a) + 2]}{s} \Big|_a^{2a} - \int_a^{2a} \frac{e^{-st}}{sa} dt$$

$$F_T(s) = \frac{1}{a} \left[-\frac{te^{-st}}{s} \Big|_0^a - \frac{e^{-st}}{s^2} \Big|_0^a \right] - \frac{e^{-st} [-(t/a) + 2]}{s} \Big|_a^{2a} + \frac{e^{-st}}{s^2 a} \Big|_a^{2a}$$

$$F_T(s) = \frac{1}{a} \left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} \right] + \frac{e^{-a}}{s} + \frac{e^{-2as}}{s^2 a} - \frac{e^{-as}}{s^2 a}$$

$$F_T(s) = \frac{e^{-2as} - 2e^{-as} + 1}{as^2}$$

$$F_T(s) = \frac{(e^{-as} - 1)^2}{as^2}$$

Hence since $F(s) = \frac{F_T(s)}{1 - e^{-sT}}$ we find

$$F(s) = \frac{(e^{-as} - 1)^2}{as^2(1 - e^{-2as})}$$

$$F(s) = \frac{(e^{-as} - 1)^2}{as^2(1 - e^{-as})(1 + e^{-as})}$$

$$F(s) = \frac{(1 - e^{-as})^2}{as^2(1 - e^{-as})(1 + e^{-as})}$$

$$F(s) = \frac{1 - e^{-as}}{as^2(1 + e^{-as})}.$$

29. $y'' + y = u(t - 3), \quad y(0) = 0, \quad y'(0) = 1$

Taking the Laplace transform of both sides of the equation gives

$$s^2 Y(s) - s(0) - 1 + Y(s) = \frac{e^{-3s}}{s}$$

$$(s^2 + 1)Y(s) - 1 = \frac{e^{-3s}}{s}$$

$$(s^2 + 1)Y(s) = \frac{e^{-3s}}{s} + 1$$

$$Y(s) = \frac{e^{-3s}}{s(s^2 + 1)} + \frac{1}{s^2 + 1}.$$

To find the inverse transform we must perform a partial fraction expansion on the first term.

Expanding $\frac{1}{s(s^2+1)}$ gives

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

After obtaining common denominators and equating numerators we have

$$1 = A(s^2 + 1) + (Bs + C)s.$$

Choosing appropriate values of s gives:

$$s = 0 : \quad 1 = A$$

$$s = 1 : \quad 1 = 2A + B + C = 2 + B + C$$

$$s = -1 : \quad 1 = 2A + B - C = 2 + B - C \longrightarrow C = 0, \quad B = -1.$$

Hence

$$\begin{aligned} \frac{1}{s(s^2 + 1)} &= \frac{1}{s} - \frac{s}{s^2 + 1} \\ \frac{e^{-3s}}{s(s^2 + 1)} &= \frac{e^{-3s}}{s} - \frac{se^{-3s}}{s^2 + 1}. \end{aligned}$$

Thus the solution to the IVP is given by

$$y(t) = L^{-1}\left\{\frac{1}{s^2+1} + \frac{e^{-3s}}{s} - \frac{se^{-3s}}{s^2+1}\right\}$$

$$y(t) = \sin t + u(t-3) - u(t-3)\cos(t-3)$$

$$y(t) = \sin t + [1 - \cos(t-3)]u(t-3).$$

31. $y'' + y = t - (t-4)u(t-2)$, $y(0) = 0$, $y'(0) = 1$

Taking the Laplace transform of both sides of the equation gives

$$s^2Y(s) - s(0) - 1 + Y(s) = \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} - \frac{2}{s}\right)$$

$$(s^2+1)Y(s) - 1 = \frac{1}{s^2} - \frac{e^{-2s}(1-2s)}{s^2}$$

$$(s^2+1)Y(s) = \frac{1}{s^2} + 1 - \frac{e^{-2s}(1-2s)}{s^2}$$

$$(s^2+1)Y(s) = \frac{s^2+1}{s^2} - \frac{e^{-2s}(1-2s)}{s^2}$$

$$Y(s) = \frac{s^2+1}{s^2(s^2+1)} - \frac{e^{-2s}(1-2s)}{s^2(s^2+1)}$$

$$Y(s) = \frac{1}{s^2} - \frac{e^{-2s}(1-2s)}{s^2(s^2+1)}.$$

To find the inverse transform we must do a partial fraction expansion on the second term.

Expanding $\frac{1-2s}{s^2(s^2+1)}$;

$$\frac{1-2s}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}.$$

After obtaining common denominators and equating numerators we have

$$1-2s = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$1-2s = (A+C)s^3 + (B+D)s^2 + As + B.$$

Equating like powers of s yields

$$s^0 : 1 = B$$

$$s^1 : -2 = A$$

$$s^2 : 0 = B + D \longrightarrow D = -1$$

$$s^3 : 0 = A + C \longrightarrow C = 2$$

Hence

$$\begin{aligned} \frac{1-2s}{s^2(s^2+1)} &= -\frac{2}{s} + \frac{1}{s^2} + \frac{2s-1}{s^2+1}. \\ -\frac{(1-2s)e^{-2s}}{s^2(s^2+1)} &= \frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} - \frac{(2s-1)e^{-2s}}{s^2+1}. \end{aligned}$$

Thus the solution to the IVP is given by

$$\begin{aligned} y(t) &= L^{-1} \left\{ \frac{1}{s^2} + \frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} - \frac{(2s-1)e^{-2s}}{s^2+1} \right\} \\ y(t) &= L^{-1} \left\{ \frac{1}{s^2} + \frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} - 2 \left(\frac{se^{-2s}}{s^2+1} \right) + \frac{e^{-2s}}{s^2+1} \right\} \\ y(t) &= t + 2u(t-2) - (t-2)u(t-2) - 2u(t-2)\cos(t-2) + u(t-2)\sin(t-2) \\ y(t) &= t + [2 - (t-2) - 2\cos(t-2) + \sin(t-2)]u(t-2) \\ y(t) &= t + [4 - t - 2\cos(t-2) + \sin(t-2)]u(t-2) \end{aligned}$$

$$33. y'' + 2y' + 2y = u(t-2\pi) - u(t-4\pi), \quad y(0) = 1, \quad y'(0) = 1$$

Taking the Laplace transform of both sides of the equation gives

$$\begin{aligned} (s^2Y(s) - s(1) - 1) + 2(sY(s) - 1) + 2Y(s) &= \frac{e^{-2\pi s}}{s} - \frac{e^{-4\pi s}}{s} \\ (s^2 + 2s + 2)Y(s) - s - 3 &= \frac{e^{-2\pi s}}{s} - \frac{e^{-4\pi s}}{s} \\ (s^2 + 2s + 2)Y(s) &= \frac{e^{-2\pi s}}{s} - \frac{e^{-4\pi s}}{s} + s + 3 \end{aligned}$$

$$Y(s) = \frac{e^{-2\pi s}}{s(s^2 + 2s + 2)} - \frac{e^{-4\pi s}}{s(s^2 + 2s + 2)} + \frac{s + 3}{s^2 + 2s + 2}$$

We note that the denominators are completely factored and in order to find the inverse transform we must do a partial fraction expansion on $\frac{1}{s(s^2+2s+2)}$.

Hence we write

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}.$$

After obtaining common denominators and equating numerators we have

$$1 = A(s^2 + 2s + 2) + (Bs + C)s.$$

Choosing appropriate values of s gives:

$$s = 0 : \quad 1 = 2A \longrightarrow A = 1/2$$

$$s = 1 : \quad 1 = 5A + B + C = 5/2 + B + C$$

$$s = -1 : \quad 1 = A + B - C = 1/2 + B - C \longrightarrow C = -1, \quad B = -1/2.$$

Hence

$$\begin{aligned} \frac{1}{s(s^2 + 2s + 2)} &= \frac{1}{2s} - \frac{s + 2}{2(s^2 + 2s + 2)} \\ \frac{1}{s(s^2 + 2s + 2)} &= \frac{1}{2s} - \frac{s + 2}{2((s + 1)^2 + 1)} \\ \frac{1}{s(s^2 + 2s + 2)} &= \frac{1}{2s} - \frac{s + 1}{2((s + 1)^2 + 1)} - \frac{1}{2((s + 1)^2 + 1)}. \end{aligned}$$

Thus the solution to the IVP is given by

$$\begin{aligned} y(t) &= L^{-1} \left\{ e^{-2\pi s} \left(\frac{1}{2s} - \frac{s + 1}{2((s + 1)^2 + 1)} - \frac{1}{2((s + 1)^2 + 1)} \right) \right\} \\ &+ L^{-1} \left\{ -e^{-4\pi s} \left(\frac{1}{2s} - \frac{s + 1}{2((s + 1)^2 + 1)} - \frac{1}{2((s + 1)^2 + 1)} \right) + \frac{s + 3}{(s + 1)^2 + 1} \right\} \end{aligned}$$

Note that $\frac{s+3}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + 2\left(\frac{1}{(s+1)^2+1}\right)$ so we have

$$y(t) = (1/2)[1 - e^{-(t-2\pi)} \cos(t - 2\pi) - e^{-(t-2\pi)} \sin(t - 2\pi)]u(t - 2\pi)$$

$$\begin{aligned}
& -(1/2)[1 - e^{-(t-4\pi)} \cos(t-4\pi) - e^{-(t-4\pi)} \sin(t-4\pi)]u(t-4\pi) + 2e^{-t} \sin t + e^{-t} \cos t. \\
& y(t) = (1/2)[1 - e^{-(t-2\pi)} \cos t - e^{-(t-2\pi)} \sin t]u(t-2\pi) \\
& -(1/2)[1 - e^{-(t-4\pi)} \cos t - e^{-(t-4\pi)} \sin t]u(t-4\pi) + 2e^{-t} \sin t + e^{-t} \cos t.
\end{aligned}$$

35. $z'' + 3z' + 2z = e^{-3t}u(t-2)$, $z(0) = 2$, $z'(0) = -3$

Taking the Laplace transform of both sides of the equation gives

$$\begin{aligned}
(s^2 Z(s) - s(2) - (-3)) + 3(sZ(s) - 2) + 2Z(s) &= e^{-2s}L(e^{-3(t+2)}) \\
(s^2 + 3s + 2)Z(s) - 2s - 3 &= e^{-2s}e^{-6}L(e^{-3t}) \\
(s+2)(s+1)Z(s) &= e^{-2s}e^{-6}\left(\frac{1}{s+3}\right) + 2s+3 \\
Z(s) &= e^{-2s-6}\left(\frac{1}{(s+3)(s+2)(s+1)}\right) + \frac{2s+3}{(s+2)(s+1)}.
\end{aligned}$$

We note that the denominators are completely factored and in order to find the inverse transform we must do two partial fraction expansions.

For the first expansion we write

$$\frac{1}{(s+3)(s+2)(s+1)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}.$$

After obtaining common denominators and equating numerators we have

$$1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2).$$

Choosing appropriate values of s gives:

$$s = -1 : 1 = 2A \longrightarrow A = 1/2$$

$$s = -2 : 1 = -B \longrightarrow B = -1$$

$$s = -3 : 1 = 2C \longrightarrow C = 1/2.$$

Hence

$$\frac{1}{(s+3)(s+2)(s+1)} = \frac{1}{2(s+1)} - \frac{1}{s+2} + \frac{1}{2(s+3)}.$$

Next we expand $\frac{2s+3}{(s+2)(s+1)}$;

$$\frac{2s+3}{(s+2)(s+1)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

After obtaining common denominators and equating numerators we have

$$2s+3 = A(s+2) + B(s+1).$$

Choosing appropriate values of s gives:

$$s = -1 : 0 = A$$

$$s = -2 : -1 = -B \longrightarrow B = 1.$$

Hence

$$\frac{2s+3}{(s+2)(s+1)} = \frac{1}{s+1} + \frac{1}{s+2}.$$

Thus the solution to the IVP is given by

$$y(t) = L^{-1} \left\{ e^{-2s-6} \left(\frac{1}{2(s+1)} - \frac{1}{s+2} + \frac{1}{2(s+3)} \right) + \frac{1}{s+1} + \frac{1}{s+2} \right\}.$$

$$y(t) = e^{-6} [(1/2)e^{-(t-2)} - e^{-2(t-2)} + (1/2)e^{-3(t-2)}] u(t-2) + e^{-t} + e^{-2t}$$

$$y(t) = (1/2)[e^{-(t+4)} - 2e^{-2(t+1)} + e^{-3t}] u(t-2) + e^{-t} + e^{-2t}.$$

$$37. y'' + 4y = g(t), \quad y(0) = 1, \quad y'(0) = 3, \quad g(t) = \begin{cases} \sin t, & 0 \leq t \leq 2\pi \\ 0, & 2\pi < t \end{cases}$$

We begin by writing $g(t) = \sin t - (\sin t)u(t-2\pi)$ and taking the Laplace transform of both sides of the equation find

$$(s^2 Y(s) - s(1) - (3)) + 4Y(s) = \frac{1}{s^2 + 1} - e^{-2\pi s} L(\sin(t + 2\pi))$$

$$(s^2 + 4)Y(s) - s - 3 = \frac{1}{s^2 + 1} - e^{-2\pi s} L(\sin t)$$

$$\begin{aligned}
(s^2 + 4)Y(s) &= \frac{1}{s^2 + 1} + s + 3 - e^{-2\pi s} L(\sin t) \\
(s^2 + 4)Y(s) &= \frac{1 + (s + 3)(s^2 + 1)}{s^2 + 1} - e^{-2\pi s} \left(\frac{1}{s^2 + 1} \right) \\
Y(s) &= \frac{4 + s + 3s^2 + s^3}{(s^2 + 1)(s^2 + 4)} - e^{-2\pi s} \left(\frac{1}{(s^2 + 1)(s^2 + 4)} \right).
\end{aligned}$$

We note that the denominators are completely factored and in order to find the inverse transform we must do two partial fraction expansions.

For the first expansion we write

$$\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}.$$

After obtaining common denominators and equating numerators we have

$$\begin{aligned}
1 &= (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1) \\
1 &= (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D).
\end{aligned}$$

Equating like values of s gives:

$$\begin{aligned}
s^0 : \quad 1 &= 4B + D \\
s^2 : \quad 0 &= B + D \longrightarrow B = 1/3, D = -1/3 \\
s^1 : \quad 0 &= 4A + C \\
s^3 : \quad 0 &= A + C \longrightarrow A = 0, C = 0.
\end{aligned}$$

Hence

$$\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3(s^2 + 1)} - \frac{1}{3(s^2 + 4)}.$$

Next we expand $\frac{4+s+3s^2+s^3}{(s^2+1)(s^2+4)}$;

$$\frac{4 + s + 3s^2 + s^3}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4}.$$

After obtaining common denominators and equating numerators we have

$$4 + s + 3s^2 + s^3 = (A + C)s^3 + (B + D)s^2 + (4A + C)s + (4B + D).$$

Equating like values of s gives:

$$s^0 : 4 = 4B + D$$

$$s^2 : 3 = B + D \longrightarrow B = 1/3, D = 8/3$$

$$s^1 : 1 = 4A + C$$

$$s^3 : 1 = A + C \longrightarrow A = 0, C = 1.$$

Hence

$$\frac{4 + s + 3s^2 + s^3}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3(s^2 + 1)} + \frac{s + 8/3}{s^2 + 4}$$

$$\frac{4 + s + 3s^2 + s^3}{(s^2 + 1)(s^2 + 4)} = \frac{1}{3(s^2 + 1)} + \frac{s}{s^2 + 4} + \frac{8}{3(s^2 + 4)}.$$

Thus the solution to the IVP is given by

$$y(t) = L^{-1} \left\{ \frac{1}{3(s^2 + 1)} + \frac{s}{s^2 + 4} + \frac{8}{3(s^2 + 4)} - e^{-2\pi s} \left(\frac{1}{3(s^2 + 1)} - \frac{1}{3(s^2 + 4)} \right) \right\}.$$

$$y(t) = (1/3) \sin t + \cos 2t + (4/3) \sin 2t - [(1/3) \sin(t - 2\pi) - (1/6) \sin(2(t - 2\pi))]u(t - 2\pi)$$

$$y(t) = (1/3) \sin t + \cos 2t + (4/3) \sin 2t - [(1/3) \sin t - (1/6) \sin 2t]u(t - 2\pi)$$

$$y(t) = (1/3)[1 - u(t - 2\pi)] \sin t + \cos 2t + (1/6)[8 + u(t - 2\pi)] \sin 2t.$$

$$39. y'' + 5y' + 6y = g(t), \quad y(0) = 0, \quad y'(0) = 2, \quad g(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t, & 1 < t < 5 \\ 1, & 5 < t \end{cases}$$

We begin by writing $g(t) = tu(t - 1) + (1 - t)u(t - 5)$ and taking the Laplace transform of both sides of the equation find

$$(s^2 Y(s) - s(0) - (2)) + 5(sY(s) - (0)) + 6Y(s) = e^{-s}L(t + 1) + e^{-5s}L(1 - (t + 5))$$

$$(s^2 + 5s + 6)Y(s) - 2 = e^{-s}L(t + 1) - e^{-5s}L(t + 4)$$

$$(s + 2)(s + 3)Y(s) = 2 + e^{-s} \left(\frac{1}{s^2} + \frac{1}{s} \right) - e^{-5s} \left(\frac{1}{s^2} + \frac{4}{s} \right)$$

$$(s + 2)(s + 3)Y(s) = 2 + e^{-s} \left(\frac{1 + s}{s^2} \right) - e^{-5s} \left(\frac{1 + 4s}{s^2} \right)$$

$$Y(s) = \frac{2}{(s+2)(s+3)} + e^{-s} \left(\frac{1+s}{s^2(s+2)(s+3)} \right) - e^{-5s} \left(\frac{1+4s}{s^2(s+2)(s+3)} \right).$$

We note that the denominators are completely factored and in order to find the inverse transform we must do three partial fraction expansions.

For the first expansion we write

$$\frac{2}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}.$$

After obtaining common denominators and equating numerators we have

$$2 = A(s+3) + B(s+2).$$

Choosing appropriate values of s gives:

$$s = -2 : \quad 2 = A$$

$$s = -3 : \quad 2 = -B \longrightarrow B = -2.$$

Hence

$$\frac{2}{(s+2)(s+3)} = \frac{2}{s+2} - \frac{2}{s+3}.$$

Next we expand $\frac{1+s}{s^2(s+2)(s+3)}$;

$$\frac{1+s}{s^2(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}.$$

After obtaining common denominators and equating numerators we have

$$1+s = As(s+2)(s+3) + B(s+2)(s+3) + Cs^2(s+3) +Ds^2(s+2).$$

Equating like values of s gives:

$$s = 0 : \quad 1 = 6B \longrightarrow B = 1/6$$

$$s = -2 : \quad -1 = 4C \longrightarrow C = -1/4$$

$$s = -3 : \quad -2 = -9D \longrightarrow D = 2/9$$

$$s = 1 : \quad 2 = 12A + 12B + 4C + 3D = 12A + 12(1/6) + 4(-1/4) + 3(2/9) \longrightarrow A = 1/36.$$

Hence

$$\frac{1+s}{s^2(s+2)(s+3)} = \frac{1}{36s} + \frac{1}{6s^2} - \frac{1}{4(s+2)} + \frac{2}{9(s+3)}.$$

Finally we expand $\frac{1+4s}{s^2(s+2)(s+3)}$;

$$\frac{1+4s}{s^2(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+3}.$$

After obtaining common denominators and equating numerators we have

$$1+4s = As(s+2)(s+3) + B(s+2)(s+3) + Cs^2(s+3) +Ds^2(s+2).$$

Equating like values of s gives:

$$s = 0 : \quad 1 = 6B \quad \longrightarrow \quad B = 1/6$$

$$s = -2 : \quad -7 = 4C \quad \longrightarrow \quad C = -7/4$$

$$s = -3 : \quad -11 = -9D \quad \longrightarrow \quad D = 11/9$$

$$s = 1 : \quad 5 = 12A + 12B + 4C + 3D = 3A + 12(1/6) + 4(-7/4) + 3(11/9) \quad \longrightarrow \quad A = 19/36.$$

Hence

$$\frac{1+4s}{s^2(s+2)(s+3)} = \frac{19}{36s} + \frac{1}{6s^2} - \frac{7}{4(s+2)} + \frac{11}{9(s+3)}.$$

Thus the solution to the IVP is given by

$$\begin{aligned} y(t) &= L^{-1} \left\{ \frac{2}{s+2} - \frac{2}{s+3} + e^{-s} \left(\frac{1}{36s} + \frac{1}{6s^2} - \frac{1}{4(s+2)} + \frac{2}{9(s+3)} \right) \right\} \\ &\quad - L^{-1} \left\{ e^{-5s} \left(\frac{19}{36s} + \frac{1}{6s^2} - \frac{7}{4(s+2)} + \frac{11}{9(s+3)} \right) \right\} \\ y(t) &= 2e^{-2t} - 2e^{-3t} + [(1/36) + (1/6)(t-1) - (1/4)e^{-2(t-1)} + (2/9)e^{-3(t-1)}]u(t-1) \\ &\quad - [(19/36) + (1/6)(t-5) - (7/4)e^{-2(t-5)} + (11/9)e^{-3(t-5)}]u(t-5). \end{aligned}$$

SECTION 7.7