$$L(t^2\cos bt) = \frac{2s^3 - 6sb^2}{(s^2+b^2)^3}.$$

SECTION 7.4

1. $\frac{6}{(s-1)^4}$

Using the formulas $L(t^n) = \frac{n!}{s^{n+1}}$ and $L(e^{at}f(t)) = F(s-a)$ we have

$$L^{-1}\Big(\frac{6}{(s-1)^4}\Big) = L^{-1}\Big(\frac{6}{3!}\frac{3!}{(s-1)^4}\Big)$$

$$L^{-1}\Big(\frac{6}{(s-1)^4}\Big) = \frac{6}{3!}L^{-1}\Big(\frac{3!}{(s-1)^4}\Big)$$

$$L^{-1}\Big(\frac{6}{(s-1)^4}\Big) = \frac{6}{6}t^3e^t$$

$$L^{-1}\Big(\frac{6}{(s-1)^4}\Big) = t^3e^t.$$

3. $\frac{s+1}{s^2+2s+10}$

After completing the square we have

$$L^{-1}\Big(\frac{s+1}{s^2+2s+10}\Big) = L^{-1}\Big(\frac{s+1}{(s+1)^2+3^2}\Big)$$

$$L^{-1}\Big(\frac{s+1}{s^2+2s+10}\Big) = e^{-t}\cos 3t.$$

5. $\frac{1}{s^2+4s+8}$

After completing the square we have

$$L^{-1}\left(\frac{1}{s^2+4s+8}\right) = L^{-1}\left(\frac{1}{(s+2)^2+2^2}\right)$$

$$L^{-1}\left(\frac{1}{s^2+4s+8}\right) = L^{-1}\left(\frac{2}{2}\frac{1}{(s+2)^2+2^2}\right)$$

$$L^{-1}\left(\frac{1}{s^2+4s+8}\right) = L^{-1}\left(\frac{1}{2}\frac{2}{(s+2)^2+2^2}\right)$$

$$L^{-1}\left(\frac{1}{s^2+4s+8}\right) = \frac{1}{2}L^{-1}\left(\frac{2}{(s+2)^2+2^2}\right)$$

$$L^{-1}\left(\frac{1}{s^2+4s+8}\right) = \frac{1}{2}e^{-2t}\sin 2t.$$

7. $\frac{2s+16}{s^2+4s+13}$

After completing the square we have

$$L^{-1}\left(\frac{2s+16}{s^2+4s+13}\right) = L^{-1}\left(\frac{2s+16}{(s+2)^2+3^2}\right)$$

$$L^{-1}\left(\frac{2s+16}{s^2+4s+13}\right) = L^{-1}\left(\frac{2s+4+12}{(s+2)^2+3^2}\right)$$

$$L^{-1}\left(\frac{2s+16}{s^2+4s+13}\right) = L^{-1}\left(\frac{2(s+2)}{(s+2)^2+3^2}+\frac{12}{(s+2)^2+3^2}\right)$$

$$L^{-1}\left(\frac{2s+16}{s^2+4s+13}\right) = 2L^{-1}\left(\frac{s+2}{(s+2)^2+3^2}\right)+L^{-1}\left(\frac{3}{3}\frac{12}{(s+2)^2+3^2}\right)$$

$$L^{-1}\left(\frac{2s+16}{s^2+4s+13}\right) = 2L^{-1}\left(\frac{s+2}{(s+2)^2+3^2}\right)+L^{-1}\left(\frac{12}{3}\frac{3}{(s+2)^2+3^2}\right)$$

$$L^{-1}\left(\frac{2s+16}{s^2+4s+13}\right) = 2L^{-1}\left(\frac{s+2}{(s+2)^2+3^2}\right)+4L^{-1}\left(\frac{3}{(s+2)^2+3^2}\right)$$

$$L^{-1}\left(\frac{2s+16}{s^2+4s+13}\right) = 2e^{-2t}\cos 3t+4e^{-2t}\sin 3t.$$

9. $\frac{3s-15}{2s^2-4s+10}$

After completing the square we have

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = L^{-1}\Big(\frac{3s-15}{2[(s-1)^2+2^2]}\Big)$$

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = L^{-1}\Big(\frac{1}{2}\frac{3s-15}{(s-1)^2+2^2}\Big)$$

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = \frac{1}{2}L^{-1}\Big(\frac{3s-3-12}{(s-1)^2+2^2}\Big)$$

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = \frac{1}{2}L^{-1}\Big(\frac{3(s-1)-12}{(s-1)^2+2^2}\Big)$$

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = \frac{1}{2}L^{-1}\Big(\frac{3(s-1)}{(s-1)^2+2^2}\Big) + \frac{1}{2}L^{-1}\Big(\frac{-12}{(s-1)^2+2^2}\Big)$$

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = \frac{3}{2}L^{-1}\Big(\frac{(s-1)}{(s-1)^2+2^2}\Big) - \frac{1}{2}L^{-1}\Big(\frac{2}{2}\frac{12}{(s-1)^2+2^2}\Big)$$

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = \frac{3}{2}L^{-1}\Big(\frac{(s-1)}{(s-1)^2+2^2}\Big) - \frac{1}{2}L^{-1}\Big(\frac{12}{2}\frac{2}{(s-1)^2+2^2}\Big)$$

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = \frac{3}{2}L^{-1}\Big(\frac{(s-1)}{(s-1)^2+2^2}\Big) - 3\sqrt{3}L^{-1}\Big(\frac{2}{(s-1)^2+2^2}\Big)$$

$$L^{-1}\Big(\frac{3s-15}{2s^2-4s+10}\Big) = \frac{3}{2}e^t\cos 2t - 3e^t\sin 2t.$$

11. $\frac{s^2-26s-47}{(s-1)(s+2)(s+5)}$

Since the denominator is already factored, we proceed to partial fraction expansion:

$$\frac{s^2-26s-47}{(s-1)(s+2)(s+5)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+5}.$$

After obtaining common denominators and equating numerators we have

$$s^2-26s-47 = A(s+2)(s+5) + B(s-1)(s+5) + C(s-1)(s+2).$$

Choosing appropriate values of s gives:

$$s = 1: \quad -72 = 18A \longrightarrow A = -4$$

$$s = -2: \quad 9 = -9B \longrightarrow B = -1$$

$$s = -5: \quad 108 = 18C \longrightarrow C = 6.$$

Therefore we have

$$\frac{s^2 - 26s - 47}{(s-1)(s+2)(s+5)} = \frac{-4}{s-1} + \frac{-1}{s+2} + \frac{6}{s+5}.$$

13. $\frac{-2s^2-3s-2}{s(s+1)^2}$

Since the denominator is already factored, we proceed to partial fraction expansion:

$$\frac{-2s^2 - 3s - 2}{s(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s}.$$

After obtaining common denominators and equating numerators we have

$$-2s^2 - 3s - 2 = A(s+1)s + Bs + C(s+1)^2.$$

Choosing appropriate values of s gives:

$$s = 0: \quad -2 = C$$

$$s = -1: \quad -1 = -B \longrightarrow B = 1$$

$$s = 1: \quad -7 = 2A + B + 4C = 2A + 1 + 4(-2) \longrightarrow A = 0.$$

Therefore we have

$$\frac{-2s^2 - 3s - 2}{s(s+1)^2} = \frac{1}{(s+1)^2} + \frac{-2}{s}.$$

15. $\frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)}$

Since the denominator is already factored, we proceed to partial fraction expansion:

$$\frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+5}.$$

After obtaining common denominators and equating numerators we have

$$-2s^2+8s-14 = A(s^2-2s+5) + (Bs+C)(s+1).$$

Choosing appropriate values of s gives:

$$s=-1: \quad -24 = 8A \longrightarrow A=-3$$

$$s=0: \quad -14 = 5A + C = 5(-3) + C \longrightarrow C = 1$$

$$s=1: \quad -8 = 4A + 2B + 2C = 4(-3) + 2B + 2(1) \longrightarrow B = 1.$$

Therefore we have

$$\frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)} = \frac{-3}{s+1} + \frac{s+1}{s^2-2s+5}.$$

Completing the squares in the last term yields

$$\frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)} = \frac{-3}{s+1} + \frac{s+1}{(s-1)^{+}24}$$

$$\frac{-2s^2+8s-14}{(s+1)(s^2-2s+5)} = \frac{-3}{s+1} + \frac{(s-1)+2}{(s-1)^{+}24}.$$

17. $\frac{3s+5}{s(s^2+s-6)}$

Since the denominator is not completely factored, we write

$$\frac{3s+5}{s(s^2+s-6)} = \frac{3s+5}{s(s+3)(s-2)}$$

and then proceed to partial fraction expansion:

$$\frac{3s+5}{s(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}.$$

After obtaining common denominators and equating numerators we have

$$3s + 5 = A(s+3)(s-2) + Bs(s-2) + Cs(s+3).$$

Choosing appropriate values of s gives:

$$s = 0: \quad 5 = -6A \quad \longrightarrow \quad A = -5/6$$

$$s = -3: \quad -4 = 15B \quad \longrightarrow \quad B = -4/15$$

$$s = 2: \quad 11 = 10C \quad \longrightarrow \quad C = 11/10.$$

Therefore we have

$$\frac{3s+5}{s(s+3)(s-2)} = \frac{-5}{6s} + \frac{-4}{15(s+3)} + \frac{11}{10(s-2)}.$$

19. $\frac{1}{(s-3)(s^2+2s+2)}$

Since the denominator is already factored, we proceed to partial fraction expansion:

$$\frac{1}{(s-3)(s^2+2s+2)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+2s+2}.$$

After obtaining common denominators and equating numerators we have

$$1 = A(s^2+2s+2) + (Bs+C)(s-3).$$

Choosing appropriate values of s gives:

$$s = 3: \quad 1 = 17A \quad \longrightarrow \quad A = 1/17$$

$$s = 0: \quad 1 = 2A - 3C = 2(1/17) - 3C \quad \longrightarrow \quad C = -5/17$$

$$s = 1: \quad 1 = 5A - 2B - 2C = 5(1/17) - 2B - 2(-5/17) \quad \longrightarrow \quad B = -1/17.$$

Therefore we have

$$\frac{1}{(s-3)(s^2+2s+2)} = \frac{1}{17(s-3)} + \frac{-s+-5}{17(s^2+2s+2)}.$$

Completing the squares in the last term yields

$$\frac{1}{(s-3)(s^2+2s+2)} = \frac{1}{17(s-3)} + \frac{-s+-5}{17((s+1)^2+1)}.$$

$$\frac{1}{(s-3)(s^2+2s+2)} = \frac{1}{17(s-3)} + \frac{-(s+1)+-4}{17((s+1)^2+1)}$$

$$\frac{1}{(s-3)(s^2+2s+2)} = (1/17)\Big[\frac{1}{s-3} - \frac{s+1}{(s+1)^2+1} - \frac{4}{(s+1)^2+1}\Big].$$

21. $\frac{6s^2-13s+2}{s(s-1)(s-6)}$

Since the denominator is already factored, we proceed to partial fraction expansion:

$$\frac{6s^2-13s+2}{s(s-1)(s-6)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-6}.$$

After obtaining common denominators and equating numerators yields

$$6s^2-13s+2 = A(s-1)(s-6) + Bs(s-6) + Cs(s-1).$$

Choosing appropriate values of s gives:

$$s=1: \quad -5 = -5B \ \longrightarrow\ B = 1$$

$$s=0: \quad 2 = 6A \ \longrightarrow\ A = 1/3$$

$$s=6: \quad 140 = 30C \ \longrightarrow\ C = 14/3.$$

Taking inverse transforms we obtain

$$L^{-1}\Big(\frac{6s^2-13s+2}{s(s-1)(s-6)}\Big) = L^{-1}\Big(\frac{1/3}{s} + \frac{1}{s-1} + \frac{14/3}{s-6}\Big)$$

$$L^{-1}\Big(\frac{6s^2-13s+2}{s(s-1)(s-6)}\Big) = L^{-1}\Big(\frac{1/3}{s}\Big) + L^{-1}\Big(\frac{1}{s-1}\Big) + L^{-1}\Big(\frac{14/3}{s-6}\Big)$$

$$L^{-1}\Big(\frac{6s^2-13s+2}{s(s-1)(s-6)}\Big) = 1/3 + e^{t} + (14/3)e^{6t}.$$

23. $\frac{5s^2+34s+53}{(s+3)^2(s+1)}$

Since the denominator is already factored, we proceed to partial fraction expansion:

$$\frac{5s^2+34s+53}{(s+3)^2(s+1)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+1}.$$

After obtaining common denominators and equating numerators yields

$$5s^2+34s+53 = A(s+1)(s+3) + B(s+1) + C(s+3)^2.$$

Choosing appropriate values of s gives:

$$s=-1: \quad 24 = 4C \longrightarrow C = 6$$

$$s=-3: \quad -4 = -2B \longrightarrow B = 2$$

$$s=0: \quad 53 = 3A + B + 9C = 3A + 2 + 9(6) \longrightarrow A = -1.$$

Taking inverse transforms we obtain

$$L^{-1}\left(\frac{5s^2+34s+53}{(s+3)^2(s+1)}\right) = L^{-1}\left(\frac{-1}{s+3} + \frac{2}{(s+3)^2} + \frac{6}{s+1}\right)$$

$$L^{-1}\left(\frac{5s^2+34s+53}{(s+3)^2(s+1)}\right) = L^{-1}\left(\frac{-1}{s+3}\right) + L^{-1}\left(\frac{2}{(s+3)^2}\right) + L^{-1}\left(\frac{6}{s+1}\right)$$

$$L^{-1}\left(\frac{5s^2+34s+53}{(s+3)^2(s+1)}\right) = -e^{-3t} + 2te^{-3t} + 6e^{-t}.$$

25. $\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$

Since the denominator is already factored, we proceed to partial fraction expansion:

$$\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+2s+5}.$$

After obtaining common denominators and equating numerators yields

$$7s^2+23s+30 = A(s^2+2s+5) + (Bs+C)(s-2)\ .$$

Choosing appropriate values of s gives:

$$s=2: \quad 104 = 13A \longrightarrow A = 8$$

$$s=0: \quad 30 = 5A - 2C = 5(8) - 2C \;\longrightarrow\; C = 5$$

$$s=1: \quad 60 = 8A - B - C = 8(8) - B - 5 \;\longrightarrow\; B = -1.$$

Taking inverse transforms we obtain

$$L^{-1}\left(\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}\right) = L^{-1}\left(\frac{8}{s-2} + \frac{-s+5}{s^2+2s+5}\right)$$

$$L^{-1}\left(\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}\right) = L^{-1}\left(\frac{8}{s-2}\right) + L^{-1}\left(\frac{-s+5}{(s+1)^2+2^2}\right)$$

$$L^{-1}\left(\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}\right) = L^{-1}\left(\frac{8}{s-2}\right) + L^{-1}\left(\frac{-(s+1)+6}{(s+1)^2+2^2}\right)$$

$$L^{-1}\left(\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}\right) = L^{-1}\left(\frac{8}{s-2}\right) - L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + 3L^{-1}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$L^{-1}\left(\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}\right) = 8e^{2t} - e^{-t}\cos 2t + 3e^{-t}\sin 2t.$$

27. $s^2F(s) - 4F(s) = \frac{5}{s+1}$

Solving the above equation for $F(s)$ we obtain

$$F(s)(s^2-4) = \frac{5}{s+1}$$

$$F(s) = \frac{5}{(s+1)(s^2-4)}$$

$$F(s) = \frac{5}{(s+1)(s-2)(s+2)}.$$

Since the denominator is now factored, we proceed to partial fraction expansion:

$$\frac{5}{(s+1)(s-2)(s+2)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2}.$$

After obtaining common denominators and equating numerators we have

$$5 = A(s+2)(s-2) + B(s+1)(s+2) + C(s+1)(s-2).$$

Choosing appropriate values of s gives:

$$s = -1: \quad 5 = -3A \longrightarrow A = -5/3$$

$$s = -2: \quad 5 = 4C \longrightarrow C = 5/4$$

$$s = 2: \quad 5 = 12B \longrightarrow B = 5/12.$$

Therefore we have

$$L^{-1}\Big(\frac{5}{(s+1)(s-2)(s+2)}\Big) = L^{-1}\Big(\frac{-5/3}{s+1} + \frac{5/12}{s-2} + \frac{5/4}{s+2}\Big)$$

$$L^{-1}\Big(\frac{5}{(s+1)(s-2)(s+2)}\Big) = L^{-1}\Big(\frac{-5/3}{s+1}\Big) + L^{-1}\Big(\frac{5/12}{s-2}\Big) + L^{-1}\Big(\frac{5/4}{s+2}\Big)$$

$$f(t) = -(5/3)e^{-t} + (5/12)e^{2t} + (5/4)e^{-2t}.$$

29. $sF(s) + 2F(s) = \frac{10s^2+12s+14}{s^2-2s+2}$

Solving the above equation for $F(s)$ we obtain

$$F(s)(s+2) = \frac{10s^2 + 12s + 14}{s^2 - 2s + 2}$$

$$F(s) = \frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)}.$$

Since the denominator is now factored, we proceed to partial fraction expansion:

$$\frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)} = \frac{A}{s+2} + \frac{Bs + C}{s^2 - 2s + 2}.$$

After obtaining common denominators and equating numerators we have

$$10s^2 + 12s + 14 = A(s^2 - 2s + 2) + (Bs + C)(s + 2).$$

Choosing appropriate values of s gives:

$$s = -2: \quad 30 = 10A \longrightarrow A = 3$$

$$s = 0: \quad 14 = 2A + 2C = 2(3) + 2C \longrightarrow C = 4$$
$$s = 1: \quad 36 = A + 3B + 3C = 3 + 3B + 3(4) \longrightarrow B = 7.$$

Therefore we have

$$L^{-1}\Big(\frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)}\Big) = L^{-1}\Big(\frac{3}{s+2} + \frac{7s+4}{s^2 - 2s + 2}\Big)$$

$$L^{-1}\Big(\frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)}\Big) = L^{-1}\Big(\frac{3}{s+2}\Big) + L^{-1}\Big(\frac{7s+4}{(s-1)^2 + 1}\Big)$$

$$L^{-1}\Big(\frac{10s^2 + 12s + 14}{(s+2)(s^2 - 2s + 2)}\Big) = L^{-1}\Big(\frac{3}{s+2}\Big) + L^{-1}\Big(\frac{7(s-1)}{(s-1)^2 + 1}\Big) + L^{-1}\Big(\frac{11}{(s-1)^2 + 1}\Big)$$

$$f(t) = 3e^{-2t} + 7e^t \cos t + 11e^t \sin t.$$

SECTION 7.5

1. $y'' - 2y' + 5y = 0,\ y(0) = 2,\ y'(0) = 4$

Taking the Laplace transform of both sides of the equation gives

$$(s^2 Y(s) - s(2) - 4) - 2(sY(s) - 2) + 5Y(s) = 0$$

$$(s^2 - 2s + 5)Y(s) - 2s = 0$$

$$Y(s) = \frac{2s}{s^2 - 2s + 5}$$

$$Y(s) = \frac{2s}{(s-1)^2 + 2^2}$$

$$Y(s) = \frac{2(s-1) - 2}{(s-1)^2 + 2^2}$$

$$Y(s) = 2\Big(\frac{(s-1)}{(s-1)^2 + 2^2}\Big) + \frac{2}{(s-1)^2 + 2^2}.$$