CSC265: Assignment 1

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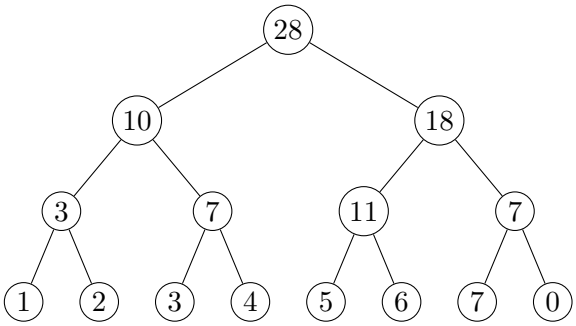
1. (a) Describe in detail a data structure implementing the Partial-Sums ADT. Draw an example with $n = 8$, and values $a_1, \ldots, a_8$ of your choosing, but not $a_1 = \ldots = a_8 = 0$.

   Solution: The correct data structure is a full binary tree of height k, B_k. There are $2^k = n$ leaves in the lowest row of this tree. The ith leaf holds value a_i. The values of the parent nodes are defined recursively as the sum of their two children. An example is below with $a_1, \ldots, a_n = 1, 2, 3, 4, 5, 6, 7, 0$.



   (b) Describe in clear and precise English how to implement the three operations Init, Add(i, t), and Sum(i). Give pseudocode for Add(i, t), and Sum(i). For each operation argue why it computes the correct output, and show that it runs in the required worst-case running time.

   - Init must run in time $O(n)$
   - Add(i, t) must run in time $O(\log(n))$
   - Sum(i) must run in time $O(\log(n))$

   Solution:

   Init: Since $n = 2^k$, the binary tree required by the Partial-Sums ADT will always be full. Then the tree can be easily implemented as an array of size $2^{k+1} - 1$, with initial values all set to 0. We assume that the first index of the array is 1.

   There is no need to construct pointers, since for a complete binary tree there are simple closed-form expressions for parent and child nodes. The functions are reproduced from CLRS Section 6.1.

   Then Init runs in time $O(2^{k+1} - 1) = O(2(2^k)) = O(2n) = O(n)$.

PARENT(i)

1 **return** $\lfloor i/2 \rfloor$

LEFT-CHILD(i)

1 **return** $2i$

RIGHT-CHILD(i)

1 **return** $2i + 1$

ADD(i, t): Add is simple to implement. The ith leaf of the tree is given by index $2^k + i$. We add t to this leaf, and then add t to each of i's parents, grandparents, great-grandparents ... until we reach the root.

ADD(i, t)

1 node_index $= 2^k + i$
2 **while** node_index $\neq 0$
3 node_index.value $=$ node_index.value $+ t$
4 node_index $=$ PARENT(node_index)

The resultant tree has an updated leaf and satisfies the sum property specified in Q1(a). This is because only the parents of the leaf specified by i will have their values affected.

ADD(i, t) runs in O($\log(n)$). One iteration of the while loop is executed for each level of the tree, so the algorithm is proportional to the tree height. This in turn is proportional to the log of the number of nodes in the tree, which is proportional to the number of leaves.

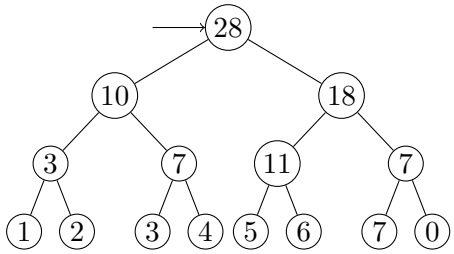
We may reach the same conclusion by noticing that node_index is proportional to n and that it is halved on each iteration when PARENT is called.

SUM(i): Notice that for any node N at depth h of the tree, there are 2^{n-h} descendants of N which are leaves. By induction, the value of N is equal to the sum of its leaf descendants.

This is key to our algorithm, because we can decompose the sum $\sum_{j=1}^{i} a_j$ into a sum of groups of powers of two. This binary representation of i is unique.

For example, if $i = 7$ and $n = 2^3 = 8$, we may write $\sum_{j=1}^{i} a_j$ as $\sum_{j=1}^{2^2} a_j + \sum_{j=2^2+1}^{2^2+2^1} a_j + \sum_{j=2^2+2^1+1}^{2^2+2^1+2^0} a_j$. This first sum is a group of 4, the second a group of 2, and the third a group of 1. Now since each node at level h of our tree corresponds to a group of 2^{k-h} leaf terms, we may choose nodes from levels 1, 2, and 3 respectively to represent these terms.

The resulting pseudocode is difficult to parse, so we also illustrate the procedure graphically below.



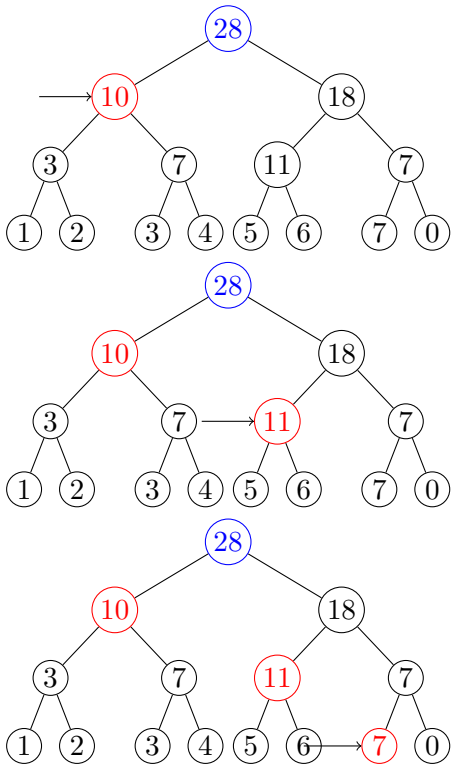


Figure 1: Procedure SUM(i) on the tree in Q1(a) with $i = 7$. The procedure begins at the root of top of the tree. The arrow represents the node being considered. If the node is red, it is added to the running sum x. If it is blue, it has been visited but has not been added.

SUM(i)

```
 1  d = 1
 2  b = i
 3  c = n
 4  x = 0
 5  while b > 0
 6        if (⌊b/c⌋ = 1)
 7              b = b − c
 8              x = x + value(d)
 9              d = 2d + 2
10        else
11              d = 2d
12        c = ⌊c/2⌋
13  return x
```

Note value(d) is a function which returns the value of the node with index d.

Proof of Correctness.
We will begin by proving a lemma which essentially states that the value held by a node is equal to the sum of the values of its leaf descendants.

Lemma 1. *If the index of a node d in an array-implementation of the Sum-ADT binary tree of height k is $2^h + l$, then* value(d) *is equal to*

$$\sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-h}} a_j.$$

Proof. We fix the size of the tree, k, and proceed by induction up the levels of the tree.

Base case: $h = k$. Fix l. Then d is a leaf and

$$\sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-h}} a_j = \sum_{j=l+1}^{l+1} a_j = a_{l+1} = \text{value}(d)$$

So the base case holds. Note $l = 0$ for the first node at depth h in the tree.

Inductive Case: Suppose that the lemma holds for all nodes at depth $h + 1$ of the tree. We show that it holds for nodes at depth h.

Fix l. By construction, value(d) is equal to the sum of the values of its left and right children. The index of the left child is $2(2^h + l) = 2^{h+1} + 2l$ and the index of the right child is $2(2^h + l) + 1 = 2^{h+1} + 2l + 1$. We apply the induction hypothesis on the children of d to obtain

$$\begin{aligned}
\text{value}(d) &= \sum_{j=(2^{k-(h+1)})(2l)+1}^{(2^{k-(h+1)})(2l)+2^{k-(h+1)}} a_j + \sum_{j=(2^{k-(h+1)})(2l+1)+1}^{(2^{k-(h+1)})(2l+1)+2^{k-(h+1)}} a_j \\
&= \sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-(h+1)}} a_j + \sum_{j=(2^{k-h})l+2^{k-h}+1}^{(2^{k-h})l+(2^{k-(h+1)})+2^{k-(h+1)}} a_j \\
&= \sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-(h+1)}} a_j + \sum_{j=(2^{k-h})l+2^{k-h}+1}^{(2^{k-h})l+2^{k-h}} a_j \\
&= \sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-h}} a_j
\end{aligned}$$

So the lemma holds. Whew! □

We will proceed to show termination, then partial correctness. The lemma will not be needed until the proof of partial correctness.

Proposition 1. *Write i as $\sum_{j=1}^{k} \phi_j 2^j$. Then the while loop in* Sum(i) *terminates, and does so after the $(k-h)$th iteration, where $h = \{\min(j) | \phi_j = 1\}$.*

Proof. Denote by b_p, c_p the values of the variables b, c after the pth iteration of the while loop. We quickly prove two loop invariants:

$$b_p = \sum_{j=1}^{k-p} \phi_j 2^j$$
$$c_p = 2^{k-p}$$

c_p: $c_0 = n = 2^k$, so the base case holds. Inductively, we have $c_p = c_{p-1}/2 = 2^{k-(p-1)}/2 = 2^{n-p}$.

b_p: Before the start of the first iteration, $b = b_0 = i$, so the loop invariant holds. Assume that the loop invariant holds on the $p-1$th iteration. Then $b_{p-1} = \sum_{j=1}^{k-(p-1)} \phi_j 2^j$. Notice that $c_{p-1} = 2^{k-(p-1)}$. So $\lfloor b/c \rfloor = 1$ iff $\phi_{k-(p-1)} = 1$. Then $b_p = \sum_{j=1}^{k-p} \phi_j 2^j$ in either branch of the if/else block, so the loop invariant holds.

In particular, $b_{k-h+1} = \sum_{j=1}^{k-(k-h+1)} \phi_j 2^j = \sum_{j=1}^{h-1} \phi_j 2^j = 0$, by construction of h. So the loop quits after the $(k-h)$th iteration of the while loop, and the algorithm exits.

□

Corollary 1. *The worst case running time of* Sum(i) *is* O($\log(n)$).

Proof. Choose $i = n - 1 = 2^k - 1 = \sum_{j=1}^{k-1} 2^j$. Then $h = 1$ and the while loop quits after $k-1$ iterations. But $k \in \mathrm{O}(\log(n))$. □

Proposition 2. Sum(i) *is partially correct; i.e. it returns* $\sum_{j=1}^{i} a_j$.

Proof. Again, we begin by proving some loop invariants. We have

$$\text{value}(d_p) = \sum_{j=\psi_{p-1}+1}^{2^{k-h}+\psi_{p-1}} a_j$$
$$x_p = \sum_{j=1}^{\psi_{p-1}} a_j$$
$$\text{where } \psi_p = \sum_{\alpha=k-p}^{k} \phi_\alpha 2^\alpha.$$

value(d_p). Base case: before the start of the loop, value(d_0) = value(2^0) = $\sum_{j=1}^{2^k} a_j$ by the lemma. We have $\psi_{-1} = 0$, so the theorem holds.

Inductive case: Suppose the loop invariant holds for d_{p-1}.

Now $d_{p-1} = 2^{k-(p-1)} + l$ for some l, where $1 \le l \le 2^{k-(p-1)}$. By the induction hypothesis, we can equate the lower bounds on the summations in the lemma and loop invariant. This allows us to solve for l:

$$
\begin{aligned}
2^{k-(p-1)}l + 1 &= \psi_{p-2} + 1 \\
2^{k-(p-1)}l + 1 &= \sum_{j=k-(p-2)}^{k} a_j + 1 \\
l &= \frac{\sum_{j=k-(p-2)}^{k} a_j}{2^{k-(p-1)}} \\
l &= \sum_{j=1)}^{p-1} a_j
\end{aligned}
$$

Then there are two cases.
Case 1. $\lfloor b_{p-1}/c_{p-1} \rfloor = 1$. Notice $\phi_{k-(p-1)} = 1$ and so $\psi_{p-1} = 2^{k-(p-1)} + \psi_{p-2}$.

Write d_p as $2^{k-p} + l'$. Now $d_p = 2d_{p-1} + 2$ (by line 9 of the pseudocode) so that $l' = 2l + 2$. Using the lemma, we have

$$
\text{value}(d_p) = \sum_{j=(2^{k-p})l'+1}^{(2^{k-p})l'+2^{k-p}} a_j
$$

The lower bound resolves to

$$
\begin{aligned}
(2^{k-p})l' + 1 & \\
&= (2^{k-p})(2l + 2) + 1 \\
&= (2^{k-p})(2\sum_{j=1)}^{p-1} a_j + 2) + 1 \\
&= \sum_{j=k-(p-2)}^{k} a_j + 2^{k-(p-1)} + 1 \\
&= \psi_{p-2} + 2^{k-(p-1)} + 1 \\
&= \psi_{p-1} + 1.
\end{aligned}
$$

Similarly, the upper bound resolves to $2^{k-p} + \psi_{p-1}$.

Case 2. $\lfloor b_{p-1}/c_{p-1} \rfloor = 0$. Then $\phi_{k-(p-1)} = 0$ so that $\psi_{p-1} = \psi_{p-2}$. Also, we have $d_p = 2d_{p-1}$ so that $l' = 2l$. Following the same sequence of calculations as in case 1, we see that the loop invariant holds. It remains only to prove the loop invariant for x.

x_p. Base case: Since $\psi_{-1} = 0$ we have $\sum_{j=1}^{0} a_j = 0 = x_0$, as desired.
Assume that the loop invariant holds before the $(p-1)$th iteration. We show it holds before the pth iteration. There are two cases.

Case 1. If $\lfloor b_{p-1}/c_{p-1} \rfloor = 1$, then

$$\begin{aligned} x_p &= x_{p-1} + \text{value}(d_{p-1}) \\ &= \sum_{j=1}^{\psi_{p-2}} a_j + \sum_{j=\psi_{p-2}+1}^{2^{k-h}+\psi_{p-2}} a_j \\ &= \sum_{j=1}^{2^{k-h}+\psi_{p-2}} a_j \\ &= \sum_{j=1}^{\psi_{p-1}} a_j. \end{aligned}$$

The second step uses the inductive hypothesis, and the final step uses the fact that $\phi_{k-(p-1)} = 1$ in this case.

Case 2. If $\lfloor b_{p-1}/c_{p-1} \rfloor = 0$, then $\phi_{k-(p-1)} = 1$ so that $\psi_{p-1} = \psi_{p-2}$. Then $x_p = x_{p-1} = \sum_{j=1}^{\psi_{p-2}} a_j = \sum_{j=1}^{\psi_{p-1}} a_j$, so the loop invariant holds.
Now we can consider the value of x when the loop terminates. We showed this occurs before the $(k-h+1)$th iteration. Then $\psi_{k-h} = \sum_{\alpha=k-(k-h)}^{k} \phi_\alpha = \sum_{\alpha=h}^{k} \phi_\alpha = \sum_{\alpha=1}^{k} \phi_\alpha 2^\alpha = i$ (the last step follows from construction of h). But this means $x_{k-h+1} = \sum_{j=1}^{\psi_{(k-h+1)-1}} a_j = \sum_{j=1}^{i} a_j$. So SUM($i$) is partially correct.

□

The total correctness of SUM(i) follows.

Theorem 1. SUM(i) *is correct.*

(c) For each $n = 2^k$, and each setting of $a_1, \ldots, a_n$, give an input i, $1 \le i \le n$, such that your implementation of SUM(i) runs in time $\Omega(\log(n))$. Explain clearly and precisely why the running time is $\Omega(\log(n))$ on this input.

Solution: Choose $i = 2^k - 1$. Refer to Corollary **??** for a detailed explanation why the run time is $\Omega(\log(n))$ on this input.
Notice that our implementation would (oddly enough) also be $\Omega(\log(n))$ for $i = 1$. To be precise, it is $\Omega(\log(n))$ on all odd inputs.

2. (a) Give an algorithm for UNION(H_1, H_2) running in worst-case time complexity $O(r_1.d + r_2.d)$ where H_1 and H_2 are R-HEAP structures, r_1 is the root of H_1, r_2 is the root of H_2, and H_1 and H_2 are given to the algorithm by pointers to the roots r_1 and r_2, respectively.
UNION(H_1, H_2) must return a pointer to the root of an R-HEAP that contains the union of the elements of H_1 and H_2. Describe your algorithm in clear in precise English, and give pseudocode. Show that the running time of the algorithm is $O(r_1.d + r_2.d)$.

Solution:

Our implementation of UNION(H_1, H_2) works by recursive comparison, and was constructed around the limitation that the procedure must terminate in time $O(r_1.d + r_2.d)$. Informally, our algorithm works as follows:

1. If one heap is empty, return the root of the non-empty heap.
2. If not, compare the values of the two roots r_1, r_2.
3. Redefine r_1 to be the left subtree of the heap with smaller value, and r_2 to be the unaltered second tree. Notice that the R-balance property ensures that the left subtree has a d value 1 less than the d of the original tree.
4. Run the algorithm recursively on r_1 and r_2. Take the result and add it as the left child of the original parent node.
5. Fix the d values of the original heap.

The algorithm terminates because the d value of either one of r_1, r_2 decrements on each recursive call. Eventually, one of the trees being compared will be empty, and the function will return. In the worst case, for instance where $r_1.d$ and $r_2.d$ are decremented in turns, this will take $O(r_1.d + r_2.d)$.

For our formal proof of correctness, we will extend the definition of the d-value of a node to be 0 if the node is empty. It is easy to check that this definition is consistent with the other conditions.

UNION(H_1, H_2)

```
1  // base case
2  if (r1 == NIL)
3       return r2
4  elseif (r2 == NIL)
5       return r1
6  else
7       // find heap with smaller value
8       if (r1.value < r2.value)
9            small = r1
10           large = r2
11      else
12           small = r2
13           large = r1
14      new_child = UNION(small.left, large)
15      new_child.parent = small
16      // Recalculate d values
17      if small.left and small.right
18           small.d = (1 + min(small.left.d, small.right.d)
19      else
20           new_child.d = 1
21      // Finish up while preserving R-balance
22      if small.right == NIL or small.right.d < small.left.d
23           small.left = small.right
24           small.right = new_child
25      else
26           small.left = new_child
27      return small
```

We proceed to prove the correctness of the algorithm.

Theorem 2. UNION(H_1, H_2) *terminates, and its worst-case running time is* $O(r_1.d+r_2.d)$.

Proof. Thankfully, we only have to prove $O(r_1.d+r_2.d)$, and not also $\Omega(r_1.d+r_2.d)$, since the latter would be annoying to construct an example for.

Suppose r_1 and r_2 are min-R-Heaps and let $k = r_1.d + r_2.d$. Without loss of generality, assume $r_1.value \leq r_2.value$. Then on line 14, UNION runs recursively on $r_1.left, r_2$.

Now notice that $r_1.left.d = r_1.d - 1$. If $r_1.left$ is empty, then $r_1.d = 1$ by definition. Otherwise, we know that r_1 has both left and right children and that $r_1.d = \min(r_1.left.d, r_1.right.d) + 1$. Since we also have that $r_1.left.d \leq r_1.right.d$, we know $\min(r_1.left.d, r_1.right.d) = r_1.left.d$, so the equation holds.

We can perform a quick induction to show that the value of k after the pth recursive call to UNION is $k - p$. When $p = k - 1$, $k - p = 1$. Since R-heaps cannot have negative d values, one of the input heaps must be empty, so the algorithm terminates then. Thus the algorithm terminates after at most $k - 1$ recursive calls (it may terminate earlier). There are a constant number of operations performed each recursive call, so UNION runs in $O(r_1.d + r_2.d)$ time. □

Theorem 3. UNION(H_1, H_2) *returns a min-heap which contains all the nodes in* H_1 *and* H_2 *for any R-balanced min-heap inputs* H_1, H_2.

Proof. Base case: One heap is empty. This case is vacuously true – it just returns the non-empty min-heap! Inductive case: Assume, without loss of generality, that $r_1.value \leq r_2.value$, and that UNION$(r_1.subheap, r_2)$ preserves the min-heap property, where $r_1.subheap$ is any subheap of r_1.

Consider UNION(r_1, r_2). At line 14, the procedure runs UNION$(r_1.left, r_2)$. By the induction hypothesis, this returns a R-balanced min-heap new$_c$hild containing all the nodes of $r_1.left$ and r_2. Now $new_child.value \geq r_1.value$, and $r_1.right.value \geq r_1.value$ by hypothesis. Thus regardless of whether lines 23+24 or line 26 is executed, the children of r_1 have values greater than or equal to $r_1.value$. So the min-heap property is preserved.

Notice that when r_1 is returned, it has two children: one which was the original right child of r_1 (possibly empty), and one which is the union of the other (possibly empty) child of r_1, and r_2. So the algorithm also returns the pointer to the root of a heap which contains all nodes of the two original heaps, as desired. □

Theorem 4. UNION(H_1, H_2) *also preserves R-balance and the d-value properties.*

Proof. This proof is basically an examination of lines 17-20 (d-values) and 22-27 (swap children to preserve R-balance). We'll deal with d-values first. In the base case, one heap is empty, and d-values are preserved vacuously.

In the inductive case, we consider the call UNION(r_1, r_2) and assume that UNION preserves d-values on r_2 and subtrees of r_1. Then the only d-value which needs to be recalculated is that of r_1, denoted by "smallest" in the psuedocode. This is dutifully carried out in lines 17-20.

The proof of R-balance preservation follows in exactly the same way. □

One might ask, "needn't we recalculate the d-values of r_1's parents?". But this is unnecessary; when we run UNION(r_1, r_2), we consider r_1 and r_2 to be the roots of their respective

min-heaps; accordingly, they have no parents. If r_1 and r_2 are subheaps, and do in fact have parents, the faulty d-values of parents will be rectified in the *parents'* function call to UNION.

We have finished proving termination, partial correctness, and an upper bound on the time complexity of the procedure. We may thus safely conclude that

Theorem 5. UNION(H_1, H_2) *is totally correct.*

(b) Show that for any n-node R-HEAP H with root node r, we have $r.d = \mathrm{O}(\log(n))$.

Solution: Intuitively, we can think of this theorem as saying that it is not possible to create an n-node R-Heap which has a bigger d-value than an n-node complete binary tree, which has height $\log(n)$. We begin with a lemma.

Lemma 2. *The R-Heap with d-value $r.d$ containing the fewest number of nodes n is a complete binary tree with $n = 2^d - 1$.*

Proof. Base case: d = 1. Then a single node will do; obviously this is the smallest possible such heap. Also, $2^1 - 1 = 1$.

Inductive case: Suppose $r.d = d$. Then each of r's children must have at least d-value $d - 1$ (by properties of d-value). By the inductive hypothesis, then, the children of r are complete binary trees with 2^{d-2} nodes each. Then H has $1+2^{d-1}-1+2^{d-1}-1 = 2^d-1$. □

Now this is all we need. For suppose that $r.d \notin \mathrm{O}(\log(n))$. Then there is some N, such that for all $n > N$ there exist R-heaps H_n such that $r_n.d > \log(n)$. Without loss of generality we may assume n is a power of two, that log is the base-two logarithm, and that $n > 4$ so that $d > 2$.

Finally, we assume that H_n is the smallest such R-Heap with d-value d. By the lemma H_n must have $2^d - 1$ nodes. Then

$$\begin{aligned} d &> \log(n) \\ d &\geq \log(n) + 1 \\ \log(2^d) &\geq \log(2^d - 1) + 1 \\ \log(\frac{2^d}{2^d - 1}) &\geq 1 \end{aligned}$$

But this is a contradiction, since $\log(\frac{2^d}{2^d-1}) < 1$ for $d > 2$. Thus the theorem holds.

Fun fact: Combining parts (a) and (b) gives us an upper bound of $\mathrm{O}(log(n_1 n_2))$ on the run-time of UNION$(H_1, H - 2)$.

(c) Recall that the height of a tree T with root r equals the number of edges in the longest path from r to a leaf of T. What is the largest possible height of an R-HEAP with n nodes?

Solution: The largest possible height is $n - 1$.

Note that in general the largest possible height for a binary tree of n nodes is given by $n-1$: each node in the tree has exactly one child. To construct such a tree which satisfies

the R-Heap properties, make all the children the right-children of the tree, and make all key values equal. More formally, if we want to construct T_n, take a node N with key 1, and set its right child to T_{n-1}. T_1 is the tree with a single node with its key set to 1. Then this tree is an R-Heap which (trivially) satisfies the min-heap and R-balance properties.