

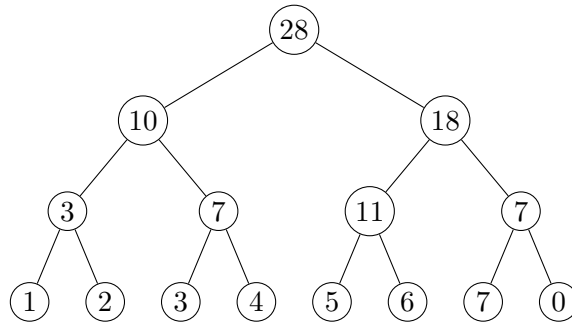
CSC265: Assignment 1

Caroline (Ruo yi) Lin, written solution
Junjie Cheng, proofreading and revision

October 4th, 2016

1. (a) Describe in detail a data structure implementing the Partial-Sums ADT. Draw an example with $n = 8$, and values a_1, \dots, a_8 of your choosing, but not $a_1 = \dots = a_8 = 0$.

Solution: The correct data structure is a full binary tree of height k, B_k . There are $2^k = n$ leaves in the lowest row of this tree. The i th leaf holds value a_i . The values of the parent nodes are defined recursively as the sum of their two children. An example is below with $a_1, \dots, a_n = 1, 2, 3, 4, 5, 6, 7, 0$.



- (b) Describe in clear and precise English how to implement the three operations INIT, ADD(i, t), and SUM(i). Give pseudocode for ADD(i, t), and SUM(i). For each operation argue why it computes the correct output, and show that it runs in the required worst-case running time.
- INIT must run in time $O(n)$
 - ADD(i, t) must run in time $O(\log(n))$
 - SUM(i) must run in time $O(\log(n))$

Solution:

INIT: Since $n = 2^k$, the binary tree required by the Partial-Sums ADT will always be full. Then the tree can be easily implemented as an array of size $2^{k+1} - 1$, with initial values all set to 0. We assume that the first index of the array is 1.

There is no need to construct pointers, since for a complete binary tree there are simple closed-form expressions for parent and child nodes. The functions are reproduced from CLRS Section 6.1.

Then INIT runs in time $O(2^{k+1} - 1) = O(2(2^k)) = O(2n) = O(n)$.

PARENT(i)

1 **return** $\lfloor i/2 \rfloor$

LEFT-CHILD(i)

1 **return** $2i$

RIGHT-CHILD(i)

1 **return** $2i + 1$

ADD(i, t): Add is simple to implement. The i th leaf of the tree is given by index $2^k + i$. We add t to this leaf, and then add t to each of i 's parents, grandparents, great-grandparents ... until we reach the root.

ADD(i, t)

1 node_index = $2^k + (i - 1)$

2 **while** node_index $\neq 0$

3 node_index.value = node_index.value + t

4 node_index = PARENT(node_index)

The resultant tree has an updated leaf and satisfies the sum property specified in Q1(a). This is because only the parents of the leaf specified by i will have their values affected.

ADD(i, t) runs in $O(\log(n))$. One iteration of the while loop is executed for each level of the tree, so the algorithm is proportional to the tree height. This in turn is proportional to the log of the number of nodes in the tree, which is proportional to the number of leaves.

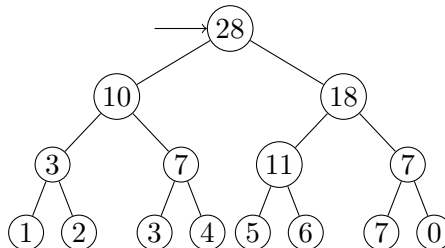
We may reach the same conclusion by noticing that node_index is proportional to n and that it is halved on each iteration when PARENT is called. It takes $\lfloor \log(n) \rfloor + 1$ iterations of the loop for node_index to reach zero.

SUM(i): Notice that for any node N at depth h of the tree, there are 2^{n-h} descendants of N which are leaves. By induction, the value of N is equal to the sum of its leaf descendants.

This is key to our algorithm, because we can decompose the sum $\sum_{j=1}^i a_j$ into a sum of groups of powers of two. This binary representation of i is unique.

For example, if $i = 7$ and $n = 2^3 = 8$, we may write $\sum_{j=1}^i a_j$ as $\sum_{j=1}^{2^2} a_j + \sum_{j=2^2+1}^{2^2+2^1} a_j + \sum_{j=2^2+2^1+1}^{2^2+2^1+2^0} a_j$. This first sum is a group of 4, the second a group of 2, and the third a group of 1. Now since each node at level h of our tree corresponds to a group of 2^{k-h} leaf terms, we may choose nodes from levels 1, 2, and 3 respectively to represent these terms.

The resulting pseudocode is difficult to parse, so we also illustrate the procedure graphically below.



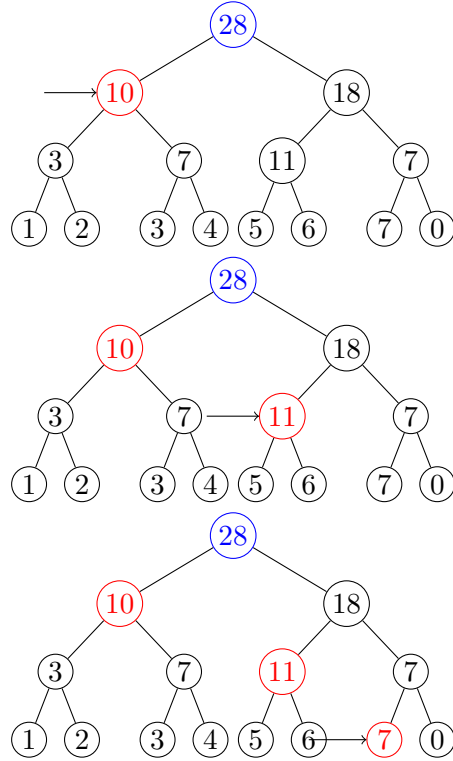


Figure 1: Procedure $\text{SUM}(i)$ on the tree in Q1(a) with $i = 7$. The procedure begins at the root of top of the tree. The arrow represents the node being considered. If the node is red, it is added to the running sum x . If it is blue, it has been visited but has not been added.

$\text{SUM}(i)$

```

1   $d = 1$ 
2   $b = i$ 
3   $c = n$ 
4   $x = 0$ 
5  while  $b > 0$ 
6      if  $(\lfloor b/c \rfloor = 1)$ 
7           $b = b - c$ 
8           $x = x + \text{value}(d)$ 
9           $d = 2d + 2$ 
10     else
11          $d = 2d$ 
12          $c = \lfloor c/2 \rfloor$ 
13 return  $x$ 
```

Note $\text{value}(d)$ is a function which returns the value of the node with index d . Intuitively, b and c are helper variables that decompose i into a sum of powers of two, while d holds the index of the next node to be visited by the algorithm, and x holds the running sum of nodes visited so far.

Proof of Correctness.

We will begin by proving a lemma which essentially states that the value held by a node is equal to the sum of the values of its leaf descendants.

Lemma 1. *If the index of a node d in an array-implementation of the Sum-ADT binary tree of height k is $2^h + l$, then $\text{value}(d)$ is equal to*

$$\sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-h}} a_j.$$

Proof. We fix the size of the tree, k , and proceed by induction up the levels of the tree.

Base case: $h = k$. Fix l . Then d is a leaf and

$$\sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-h}} a_j = \sum_{j=l+1}^{l+1} a_j = a_{l+1} = \text{value}(d)$$

So the base case holds. Note $l = 0$ for the first node at depth h in the tree.

Inductive Case: Suppose that the lemma holds for all nodes at depth $h + 1$ of the tree. We show that it holds for nodes at depth h .

Fix l . By construction, $\text{value}(d)$ is equal to the sum of the values of its left and right children. The index of the left child is $2(2^h + l) = 2^{h+1} + 2l$ and the index of the right child is $2(2^h + l) + 1 = 2^{h+1} + 2l + 1$. We apply the induction hypothesis on the children of d to obtain

$$\begin{aligned} \text{value}(d) &= \sum_{j=(2^{k-(h+1)})(2l)+1}^{(2^{k-(h+1)})(2l)+2^{k-(h+1)}} a_j + \sum_{j=(2^{k-(h+1)})(2l+1)+1}^{(2^{k-(h+1)})(2l+1)+2^{k-(h+1)}} a_j \\ &= \sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-(h+1)}} a_j + \sum_{j=(2^{k-h})l+(2^{k-(h+1)})+2^{k-(h+1)}}^{(2^{k-h})l+(2^{k-(h+1)})+2^{k-(h+1)}} a_j \\ &= \sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-(h+1)}} a_j + \sum_{j=(2^{k-h})l+2^{k-h}+1}^{(2^{k-h})l+2^{k-h}} a_j \\ &= \sum_{j=(2^{k-h})l+1}^{(2^{k-h})l+2^{k-h}} a_j \end{aligned}$$

So the lemma holds. Whew! □

We will proceed to show termination, then partial correctness. The lemma will not be needed until the proof of partial correctness.

Proposition 1. *Write i as $\sum_{j=1}^k \phi_j 2^j$. Then the while loop in $\text{SUM}(i)$ terminates, and does so after the $(k - h)$ th iteration, where $h = \{\min(j) | \phi_j = 1\}$.*

Proof. Denote by b_p, c_p the values of the variables b, c after the p th iteration of the while loop. We quickly prove two loop invariants:

$$\begin{aligned} b_p &= \sum_{j=1}^{k-p} \phi_j 2^j \\ c_p &= 2^{k-p} \end{aligned}$$

c_p : $c_0 = n = 2^k$, so the base case holds. Inductively, we have $c_p = c_{p-1}/2 = 2^{k-(p-1)}/2 = 2^{k-p}$.

b_p : Before the start of the first iteration, $b = b_0 = i$, so the loop invariant holds. Assume that the loop invariant holds on the $p-1$ th iteration. Then $b_{p-1} = \sum_{j=1}^{k-(p-1)} \phi_j 2^j$. Notice that $c_{p-1} = 2^{k-(p-1)}$. So $\lfloor b/c \rfloor = 1$ iff $\phi_{k-(p-1)} = 1$. Then $b_p = \sum_{j=1}^{k-p} \phi_j 2^j$ in either branch of the if/else block, so the loop invariant holds.

In particular, $b_{k-h+1} = \sum_{j=1}^{k-(k-h+1)} \phi_j 2^j = \sum_{j=1}^{h-1} \phi_j 2^j = 0$, by construction of h . So the loop quits after the $(k-h)$ th iteration of the while loop, and the algorithm exits. \square

Corollary 1. *The worst case running time of SUM(i) is $O(\log(n))$.*

Proof. Choose $i = n - 1 = 2^k - 1 = \sum_{j=1}^{k-1} 2^j$. Then $h = 1$ and the while loop quits after $k-1$ iterations. Since $k \in O(\log(n))$, $k-1 \in O(\log(n))$ and SUM(i) runs in $O(\log(n))$ time. \square

Proposition 2. SUM(i) is partially correct; i.e. it returns $\sum_{j=1}^i a_j$.

Proof. Again, we begin by proving some loop invariants. We have

$$\begin{aligned} \text{value}(d_p) &= \sum_{j=\psi_{p-1}+1}^{2^{k-h}+\psi_{p-1}} a_j \\ x_p &= \sum_{j=1}^{\psi_{p-1}} a_j \end{aligned}$$

where

$$\psi_p = \sum_{\alpha=k-p}^k \phi_\alpha 2^\alpha.$$

value(d_p). Base case: before the start of the loop, $\text{value}(d_0) = \text{value}(2^0) = \sum_{j=1}^{2^0} a_j$ by the lemma. We have $\psi_{-1} = 0$, so the theorem holds.

Inductive case: Suppose the loop invariant holds for d_{p-1} .

Now $d_{p-1} = 2^{k-(p-1)} + l$ for some l , where $1 \leq l \leq 2^{k-(p-1)}$. By the induction hypothesis, we can equate the lower bounds on the summations in the lemma and loop invariant. This allows us to solve for l :

$$\begin{aligned}
2^{k-(p-1)}l + 1 &= \psi_{p-2} + 1 \\
2^{k-(p-1)}l + 1 &= \sum_{j=k-(p-2)}^k a_j + 1 \\
l &= \frac{\sum_{j=k-(p-2)}^k a_j}{2^{k-(p-1)}} \\
l &= \sum_{j=1}^{p-1} a_j
\end{aligned}$$

Then there are two cases.

Case 1. $\lfloor b_{p-1}/c_{p-1} \rfloor = 1$. Notice $\phi_{k-(p-1)} = 1$ and so $\psi_{p-1} = 2^{k-(p-1)} + \psi_{p-2}$.

Write d_p as $2^{k-p} + l'$. Now $d_p = 2d_{p-1} + 2$ (by line 9 of the pseudocode) so that $l' = 2l + 2$. Using the lemma, we have

$$\text{value}(d_p) = \sum_{j=(2^{k-p})l'+1}^{(2^{k-p})l'+2^{k-p}} a_j$$

The lower bound resolves to

$$\begin{aligned}
(2^{k-p})l' + 1 &= (2^{k-p})(2l + 2) + 1 \\
&= (2^{k-p})(2 \sum_{j=1}^{p-1} a_j + 2) + 1 \\
&= \sum_{j=k-(p-2)}^k a_j + 2^{k-(p-1)} + 1 \\
&= \psi_{p-2} + 2^{k-(p-1)} + 1 \\
&= \psi_{p-1} + 1.
\end{aligned}$$

Similarly, the upper bound resolves to $2^{k-p} + \psi_{p-1}$.

Case 2. $\lfloor b_{p-1}/c_{p-1} \rfloor = 0$. Then $\phi_{k-(p-1)} = 0$ so that $\psi_{p-1} = \psi_{p-2}$. Also, we have $d_p = 2d_{p-1}$ so that $l' = 2l$. Following the same sequence of calculations as in case 1, we see that the loop invariant holds. It remains only to prove the loop invariant for x .

x_p . Base case: Since $\psi_{-1} = 0$ we have $\sum_{j=1}^0 a_j = 0 = x_0$, as desired.

Assume that the loop invariant holds before the $(p-1)$ th iteration. We show it holds before the p th iteration. There are two cases.

Case 1. If $\lfloor b_{p-1}/c_{p-1} \rfloor = 1$, then

$$\begin{aligned}
x_p &= x_{p-1} + \text{value}(d_{p-1}) \\
&= \sum_{j=1}^{\psi_{p-2}} a_j + \sum_{j=\psi_{p-2}+1}^{2^{k-h}+\psi_{p-2}} a_j \\
&= \sum_{j=1}^{2^{k-h}+\psi_{p-2}} a_j \\
&= \sum_{j=1}^{\psi_{p-1}} a_j.
\end{aligned}$$

The second step uses the inductive hypothesis, and the final step uses the fact that $\phi_{k-(p-1)} = 1$ in this case.

Case 2. If $\lfloor b_{p-1}/c_{p-1} \rfloor = 0$, then $\phi_{k-(p-1)} = 1$ so that $\psi_{p-1} = \psi_{p-2}$. Then $x_p = x_{p-1} = \sum_{j=1}^{\psi_{p-2}} a_j = \sum_{j=1}^{\psi_{p-1}} a_j$, so the loop invariant holds.

Now we can consider the value of x when the loop terminates. We showed this occurs before the $(k - h + 1)$ th iteration. Then $\psi_{k-h} = \sum_{\alpha=k-(k-h)}^k \phi_{\alpha} 2^{\alpha} = \sum_{\alpha=h}^k \phi_{\alpha} 2^{\alpha} = \sum_{\alpha=1}^k \phi_{\alpha} 2^{\alpha} = i$ (the last step follows from construction of h). But this means $x_{k-h+1} = \sum_{j=1}^{\psi_{(k-h+1)-1}} a_j = \sum_{j=1}^i a_j$. So $\text{SUM}(i)$ is partially correct. □

The total correctness of $\text{SUM}(i)$ follows:

Theorem 1. $\text{SUM}(i)$ is correct.

- (c) For each $n = 2^k$, and each setting of a_1, \dots, a_n , give an input i , $1 \leq i \leq n$, such that your implementation of $\text{SUM}(i)$ runs in time $\Omega(\log(n))$. Explain clearly and precisely why the running time is $\Omega(\log(n))$ on this input.

Solution: Choose $i = 2^k - 1$. Refer to Corollary 1 for a detailed explanation why the run time is $\Omega(\log(n))$ on this input.

Notice that our implementation would (oddly enough) also be $\Omega(\log(n))$ for $i = 1$. To be precise, it is $\Omega(\log(n))$ on all odd inputs.

2. (a) Give an algorithm for $\text{UNION}(H_1, H_2)$ running in worst-case time complexity $O(r_1.d + r_2.d)$ where H_1 and H_2 are R-HEAP structures, r_1 is the root of H_1 , r_2 is the root of H_2 , and H_1 and H_2 are given to the algorithm by pointers to the roots r_1 and r_2 , respectively. $\text{UNION}(H_1, H_2)$ must return a pointer to the root of an R-HEAP that contains the union of the elements of H_1 and H_2 . Describe your algorithm in clear in precise English, and give pseudocode. Show that the running time of the algorithm is $O(r_1.d + r_2.d)$.

Solution:

Our implementation of $\text{UNION}(H_1, H_2)$ works by recursive comparison, and was constructed around the limitation that the procedure must terminate in time $O(r_1.d + r_2.d)$. Informally, our algorithm works as follows:

1. If one heap is empty, return the root of the non-empty heap.
2. If not, compare the values of the two roots r_1, r_2 .
3. Define the variable *small* to be the node with the smaller value and define the variable *large* to be the node with the larger value. Recursively execute the function with *large* and the left child of *small*. Assign the returned node to be the new left child of the node *small* as well as fix the parent of the child. Notice that the R-balance property ensures that the left subtree has a *d* value 1 less than the *d* of the original tree.
4. Fix the *d* values of the node *small*, and swap the left and right childs when necessary, to maintain the R-balance property.
5. Finally, return the node *small*, which satisfies all postconditions of UNION .

The pseudocode is shown below:

$\text{UNION}(H_1, H_2)$

```

1  // base case
2  if ( $r_1 == \text{NIL}$ )
3      return  $r_2$ 
4  elseif ( $r_2 == \text{NIL}$ )
5      return  $r_1$ 
6  else
7      // find heap with smaller value
8      if ( $r_1.\text{value} < r_2.\text{value}$ )
9          small =  $r_1$ 
10         large =  $r_2$ 
11     else
12         small =  $r_2$ 
13         large =  $r_1$ 
14     new_child =  $\text{UNION}(\text{small.left}, \text{large})$ 
15     new_child.parent = small
16     small.left = new_child
17     // Recalculate d values
18     if small.left and small.right
19         small.d =  $(1 + \min(\text{small.left.d}, \text{small.right.d}))$ 
20     else
21         new_child.d = 1
22     // Swap to preserve R-balance
23     if  $\text{small.right} == \text{NIL}$  or  $\text{small.right.d} < \text{small.left.d}$ 
24         small.left = small.right
25         small.right = new_child
26     return small

```


The algorithm terminates because the d value of either one of r_1, r_2 decrements on each recursive call. Eventually, one of the trees being compared will be empty, and the function will return. In the worst case, for instance where $r_1.d$ and $r_2.d$ are decremented in turns, this will take $O(r_1.d + r_2.d)$.

For our formal proof of correctness, we will extend the definition of the d -value of a node to be 0 if the node is empty. It is easy to check that this definition is consistent with the other conditions.

We proceed to prove the correctness of the algorithm.

Theorem 2. $\text{UNION}(H_1, H_2)$ terminates, and its worst-case running time is $O(r_1.d + r_2.d)$.

Proof. Thankfully, we only have to prove $O(r_1.d + r_2.d)$, and not also $\Omega(r_1.d + r_2.d)$, since the latter would be annoying to construct an example for.

Suppose r_1 and r_2 are min-R-Heaps and let $k = r_1.d + r_2.d$. Without loss of generality, assume $r_1.value \leq r_2.value$. Then on line 14, UNION runs recursively on $r_1.left, r_2$.

Now notice that $r_1.left.d = r_1.d - 1$. If $r_1.left$ is empty, then $r_1.d = 1$ by definition. Otherwise, we know that r_1 has both left and right children and that $r_1.d = \min(r_1.left.d, r_1.right.d) + 1$. Since we also have that $r_1.left.d \leq r_1.right.d$, we know $\min(r_1.left.d, r_1.right.d) = r_1.left.d$, so the equation holds.

We can perform a quick induction to show that the value of k after the p th recursive call to UNION is $k - p$. When $p = k - 1$, $k - p = 1$. Since R-heaps cannot have negative d values, one of the input heaps must be empty, so the algorithm terminates then. Thus the algorithm terminates after at most $k - 1$ recursive calls (it may terminate earlier). There are a constant number of operations performed each recursive call, so UNION runs in $O(r_1.d + r_2.d)$ time. \square

Theorem 3. $\text{UNION}(H_1, H_2)$ returns a min-heap which contains all the nodes in H_1 and H_2 for any R-balanced min-heap inputs H_1, H_2 .

Proof. Base case: One heap is empty. This case is vacuously true – it just returns the non-empty min-heap! Inductive case: Assume, without loss of generality, that $r_1.value \leq r_2.value$, and that $\text{UNION}(r_1.subheap, r_2)$ preserves the min-heap property, where $r_1.subheap$ is any subheap of r_1 .

Consider $\text{UNION}(r_1, r_2)$. At line 14, the procedure runs $\text{UNION}(r_1.left, r_2)$. By the induction hypothesis, this returns a R-balanced min-heap new_child containing all the nodes of $r_1.left$ and r_2 . Now $new_child.value \geq r_1.value$, and $r_1.right.value \geq r_1.value$ by hypothesis. Thus regardless of whether lines 24+25 or line 27 is executed, the children of r_1 have values greater than or equal to $r_1.value$. So the min-heap property is preserved.

Notice that when r_1 is returned, it has two children: one which was the original right child of r_1 (possibly empty), and one which is the union of the other (possibly empty) child of r_1 , and r_2 . So the algorithm also returns the pointer to the root of a heap which contains all nodes of the two original heaps, as desired. \square

Theorem 4. $\text{UNION}(H_1, H_2)$ also preserves R-balance and the d -value properties.

Proof. This proof is basically an examination of lines 18-21 (d-values) and 23-28 (swap children to preserve R-balance). We'll deal with d-values first. In the base case, one heap is empty, and d-values are preserved vacuously.

In the inductive case, we consider the call $\text{UNION}(r_1, r_2)$ and assume that UNION preserves d-values on r_2 and subtrees of r_1 . Then the only d-value which needs to be recalculated is that of r_1 , denoted by "smallest" in the pseudocode. This is dutifully carried out in lines 18-21.

The proof of R-balance preservation follows in exactly the same way. \square

One might ask, "needn't we recalculate the d-values of r_1 's parents?". But this is unnecessary; when we run $\text{UNION}(r_1, r_2)$, we consider r_1 and r_2 to be the roots of their respective min-heaps; accordingly, they have no parents. If r_1 and r_2 are subheaps, and do in fact have parents, the faulty d-values of parents will be rectified in the *parents'* function call to UNION .

We have finished proving termination, partial correctness, and an upper bound on the time complexity of the procedure. We may thus safely conclude that

Theorem 5. $\text{UNION}(H_1, H_2)$ is totally correct.

- (b) Show that for any n -node R-HEAP H with root node r , we have $r.d = O(\log(n))$.

Solution: Intuitively, we can think of this theorem as saying that it is not possible to create an n -node R-Heap which has a bigger d-value than an n -node complete binary tree, which has height $\log(n)$. We begin with a lemma.

Lemma 2. The R-Heap with d-value $r.d$ containing the fewest number of nodes n is a complete binary tree with $n = 2^d - 1$.

Proof. Base case: $d = 1$. Then a single node will do; obviously this is the smallest possible such heap. Also, $2^1 - 1 = 1$.

Inductive case: Suppose $r.d = d$. Then each of r 's children must have at least d-value $d - 1$ (by properties of d-value). By the inductive hypothesis, then, the children of r are complete binary trees with 2^{d-2} nodes each. Then H has $1 + 2^{d-1} - 1 + 2^{d-1} - 1 = 2^d - 1$. \square

Now this is all we need. For suppose that $r.d \notin O(\log(n))$. Then there is some N , such that for all $n > N$ there exist R-heaps H_n such that $r_n.d > \log(n)$. Without loss of generality we may assume n is a power of two, that \log is the base-two logarithm, and that $n > 4$ so that $d > 2$.

Finally, we assume that H_n is the smallest such R-Heap with d-value d . By the lemma H_n must have $2^d - 1$ nodes. Then

$$\begin{aligned} d &> \log(n) \\ d &\geq \log(n) + 1 \\ \log(2^d) &\geq \log(2^d - 1) + 1 \\ \log\left(\frac{2^d}{2^d - 1}\right) &\geq 1 \end{aligned}$$

But this is a contradiction, since $\log(\frac{2^d}{2^d-1}) < 1$ for $d > 2$. Thus the theorem holds.

Fun fact: Combining parts (a) and (b) gives us an upper bound of $O(\log(n_1 n_2))$ on the run-time of $\text{UNION}(H_1, H - 2)$.

- (c) Recall that the height of a tree T with root r equals the number of edges in the longest path from r to a leaf of T . What is the largest possible height of an R-HEAP with n nodes?

Solution: The largest possible height is $n - 1$.

Note that in general the largest possible height for a binary tree of n nodes is given by $n - 1$: each node in the tree has exactly one child. To construct such a tree which satisfies the R-HEAP properties, make all the children the right-children of the tree, and make all key values equal. More formally, if we want to construct T_n , take a node N with key 1, and set its right child to T_{n-1} . T_1 is the tree with a single node with its key set to 1. Then this tree is an R-HEAP which (trivially) satisfies the min-heap and R-balance properties.