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Traffic Current Fluctuation and the Burgers Equation

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The car concentration on an expressway must obey the Burgers equation if the concentration is linearly related to the drift speed. The power spectral density of the random Burgers flow was numerically evaluated based on this model and compared with the observed spectrum of the car flow. They are in approximate agreement.

§1. Introduction

In a previous paper[1)] we reported observation of traffic current fluctuation on an expressway and derived from it a power spectral density as a function of frequency. The spectrum consists of two parts: the shot noise part for high frequencies and the $1/f$ fluctuation part for low frequencies. This spectrum looks very much like the spectrum of electric current noise on a dc current passing a vacuum tube as was first noticed by Johnson more than 50 years ago.[2)]

The low frequency behavior of the spectrum is reflecting the nature of interactions among the cars traveling on an expressway. We postulated that the speed of the cars is reduced as the car concentration is increased. A linear relationship was assumed between these quantities as the simplest mathematical expression for the interactions; as a result the Burgers nonlinear differential equation was reached for the car concentration.

The present paper presents some numerical evaluation of power spectra of a steady random flow which obeys the Burgers equation and they are compared with the observed power spectrum of the car current. We have an approximate agreement between theory and observation. The present result is an evidence that a random particle flow in general will manifest the $1/f$-like spectrum in its fluctuation if the flow rate is related to the concentration.

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§2. Particle Flow and the Burgers Equation

Let us consider a one-dimensional particle flow imbedded in another medium, which has drift velocity $v(x, t)$, concentration $n(x, t)$, random velocity v_r, and current $J(x, t)$. We will briefly repeat the process in which the Burgers equation is reached for n.

The continuity equation for the flow is given by

$$\frac{\partial n}{\partial t}+\frac{\partial J}{\partial x}=0. \tag{1}$$

The flow rate J consists of the drift current and the diffusion current:

$$J=nv-D\frac{\partial n}{\partial x}, \tag{2}$$

where D is the diffusion coefficient given by

$$D=\tau v_r^2, \tag{3}$$

τ being the mean collision time of the particles.

Here we postulate that the drift velocity is influenced by the particle concentration such that their relationship is formulated as

$$v=v_0(1-n/n_s). \tag{4}$$

If the drift velocity increases with the concnetration, a constant n_s should be regarded as taking a negative value. From eqs. (1), (2) and (4), one obtains

$$\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial x}\right)n-2\left(\frac{n}{n_s}\right)n\frac{\partial n}{\partial x}-D\frac{\partial^2 n}{\partial x^2}=0. \tag{5}$$

Let this equation be rewritten in a moving frame of reference x' as defined by

$$x'=-x+v_0 t, \tag{6}$$

to obtain

$$\frac{\partial n}{\partial t}+an\frac{\partial n}{\partial x'}=D\frac{\partial^2 n}{\partial x'^2}, \tag{7}$$

where

$$a=2v_0/n_s. \tag{8}$$

When the drift velocity is an increasing function of the particle concentration, the same formula as eq. (7) will be achieved if eq. (6) is replaced by

$$x'=x+v_0t. \tag{9}$$

Equation (7) may be brought to a dimensionless form for the convenience of numerical treatments:

$$\frac{\partial n}{\partial t}+n\frac{\partial n}{\partial x'}=\frac{1}{R}\frac{\partial^2 n}{\partial x'^2}, \tag{10}$$

where the particle concentration has been normalized to $n_s/2$, the time to a constant t_0 (for instant, the mean time between the successive transits of particles), and the length to x_0 which is equal to v_0t_0. Here R is a dimensionless constant defined by

$$R=(v_0/v_r)^2t_0/\tau, \tag{11}$$

which is equivalent to the Reynolds number in fluid dynamics. After numerical treatments the original particle concentration can be restored, if necessary, in terms of the normalization parameters. The Burgers equation is a one-dimensional version of the Navier-Stokes equation where a pressure term is ignored.

§3. Power Spectrum of the Burgers Flow

The Burgers equation would take the same form as the KdV equation for solitons if the second derivative of n with respect to x' on the rhs of eq. (7) is replaced by the third derivative. The nonlinear term in eq. (7) makes the waveform more and more triangular as time goes on, while the term on the rhs of eq. (7) acts to diffuse the sharpened waveform. As a result of the balance between these two tendencies the waveform eventually takes a stable shape but the amplitude decays. When R is large enough the diffusion is slow, and after some time the waveform is almost independent of the initial form especially for large wavelength components. Tatsumi and Mizushima[3)] calculated the power spectrum of a Burgers flow by giving an initial disturbance and found that it is proportional to k^{-q} (k is the wave number) where q is initially equal to four and approaches to unity as time goes on; for small wavelengths it is proportional to $\exp(-k)$.

For the sake of numerical evaluation of the spectrum, eq. (7) is approximated by a set of difference equations:

$$(n_{i,j+1}-n_{i,j})/k+n_{i,j}(n_{i+1,j}-n_{i-1,j})/2h$$
$$=(n_{i+1,j}-2n_{i,j}+n_{i-1,j})/Rh^2. \tag{12}$$

3.1 *Transient flow*

We first discuss a transient Burgers flow, in which Gaussian waveform is initially given. Evolution of the waveform is shown in Fig. 1 for $R=20$ (this value has been chosen to simulate the traffic current problem and it suggests we have no turbulence). The waveform becomes sharpest and then tends to diffuse. Figure 2 shows waveforms at normalized time $t=16$ starting with a Gaussian disturbance for various values of the Reynolds number.

The power spectral densities with respect to the wave number for $R=20$ and for various times are plotted in Fig. 3; as time evolves a high wave number tail is relatively enhanced.

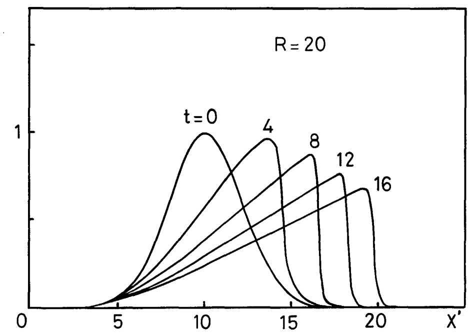

Fig. 1. Evolution of waveform of a Burgers flow starting with a Gaussian disturbance.

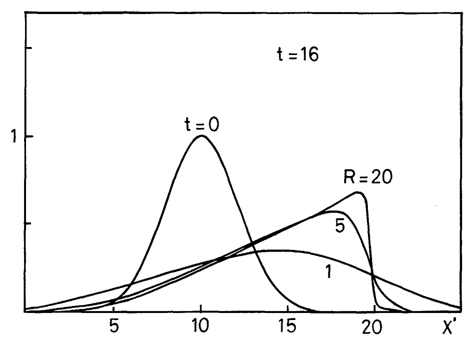

Fig. 2. Waveforms at normalized time $t=16$ for different Reynolds number R.

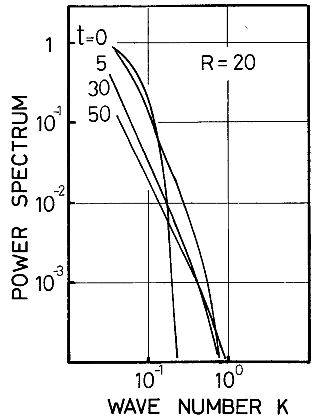

Fig. 3. Power spectral densities of a transient Burgers flow starting with a Gaussian disturbance; K is the normalized wave number.

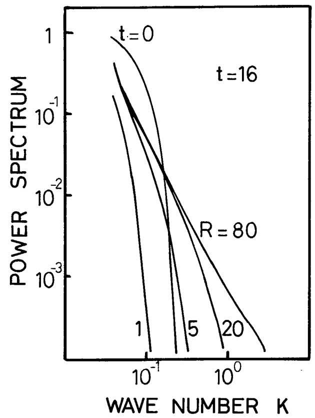

Fig. 4. Power spectral densities of a transient Burgers flow at normalized time 16 for various Reynolds numbers; the initial wave form is Gaussian.

Figure 4 shows power spectra for various values of the Reynolds number at normalized time 16. The spectrum for $R=80$ is proportional to $K^{-1.5}$.

The energy density of the flow over a length L, E, is defined by

$$E=\frac{1}{L}\int_0^L n^2\,\mathrm{d}x, \tag{13}$$

Variation of the energy density with time is plotted in Fig. 5 where the initial amplitude on each lattice point has been determined by a random number. The energy goes down as $t^{-2/3}$ as was already found.[4)] This tendency is not sensitive to the initial waveform.

3.2 *Steady random flow*

We here apply random forces to the Burgers

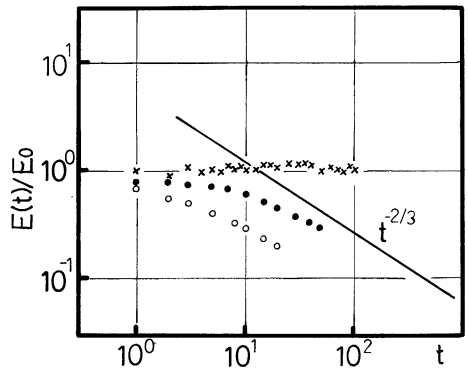

Fig. 5. Decay of the wave energy. Dots and open circles refer to initial disturbances which are Gaussian and random, respectively; crosses refer to a stationary random flow discussed in the text.

equation to maintain a steady flow:

$$\frac{\partial n}{\partial t}+n\frac{\partial n}{\partial x}=\frac{1}{R}\frac{\partial^2 n}{\partial x^2}+f. \tag{14}$$

The force f is random in space as well as in time. The ensemble average of eq. (14) yields

$$\frac{\partial\langle n\rangle}{\partial t}+\frac{1}{2}\frac{\partial\langle n^2\rangle}{\partial x}=\frac{1}{R}\frac{\partial^2\langle n\rangle}{\partial x^2}+\langle f\rangle. \tag{15}$$

If $\langle n\rangle$ is constant and $\langle f\rangle=0$ in space and time, we must have

$$\frac{1}{2}\frac{\partial\langle n^2\rangle}{\partial x}=0. \tag{16}$$

The time dependence of $\langle n^2\rangle$ is derived as follows: by multiplying eq. (14) by n and by taking the ensemble average, one has

$$\frac{1}{2}\frac{\partial\langle n^2\rangle}{\partial t}+\frac{1}{3}\frac{\partial\langle n^3\rangle}{\partial x}=\frac{1}{R}\left\{\frac{1}{2}\frac{\partial^2\langle n^2\rangle}{\partial x^2}-\left\langle\left(\frac{\partial n}{\partial x}\right)^2\right\rangle\right\}+\langle nf\rangle. \tag{17}$$

If $\langle n^2\rangle$ is constant in space, this equation becomes

$$\frac{1}{2}\frac{\partial\langle n^2\rangle}{\partial t}=-\frac{1}{R}\left\langle\left(\frac{\partial n}{\partial x}\right)^2\right\rangle+\langle nf\rangle. \tag{18}$$

The energy decay rate without an external force is, therefore, determined by the gradient of n. To maintain a steady flow it is required that $\partial\langle n^2\rangle/\partial t=0$; the external random force to maintain a steady random flow should be related to the gradient of n. We carried out the calculation as follows: at a regular time interval Δt, $\langle n\rangle$ and $\langle n^2\rangle$ were evaluated and a Gaussian random number was used to determine the

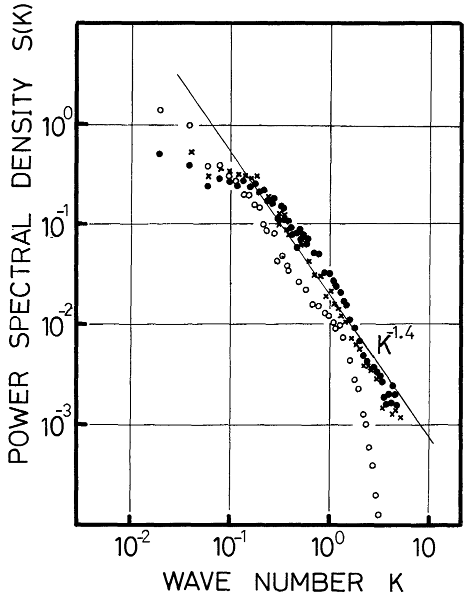

Fig. 6. Power spectral densities of a steady-state Burgers flow with a random force. Dots, crosses and open circles refer to Reynolds numbers 40, 20 and 4, respectively.

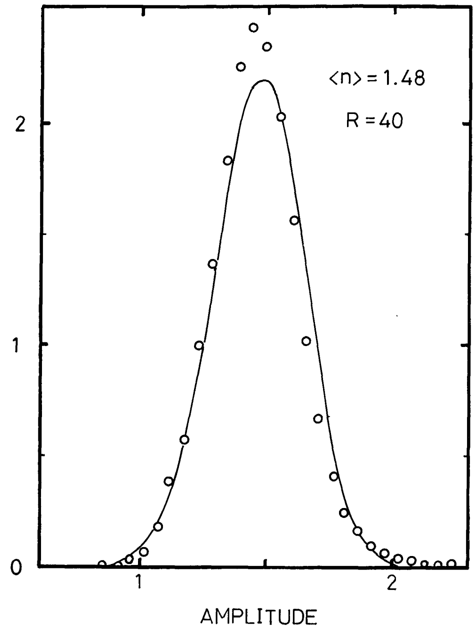

Fig. 7. Normalized amplitude distribution of a steady-state Burgers flow disturbed by a random force. The mean particle concentration is equal to 1.5 and $R=40$.

force in such a way that the variance of the amplitudes became equal to $0.01=\{\langle n^2\rangle-\langle n\rangle^2\}$; in other words the variance of the steady fluctuations was set to 0.01 and furthermore $\langle n\rangle=1.5$ to give a moderate nonlinearity to the results. The power spectra of the steady flow with respect to the normalized wave number K averaged over $\Delta t=0.05$, 0.25 and 1 for some values of R are plotted in Fig. 6, from which one finds

$$S(K)=2\times 10^{-2}/K^{1.4}. \tag{19}$$

The steady flow spectrum does not largely differ from the transient power spectrum.

The amplitude distribution of the steady fluctuations is shown in Fig. 7 where the calculated values are compared with a Gaussian distribution with the same variance. The power spectrum of a steady Burgers flow with respect to the normalized frequency F is plotted in Fig. 8, from which one obtains

$$P(F)=3\times 10^{-2}/F^2. \tag{20}$$

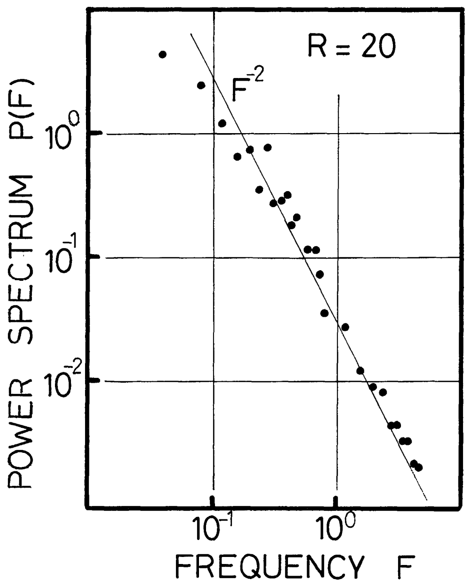

Fig. 8. Power spectral density of a steady Burgers flow for $R=20$; F is the normalized frequency.

§4. Fluctuations of a Traffic Flow

The concentration fluctuations of the particle flow is found to obey the Burgers nonlinear differential equation in a moving frame of reference provided the concentration is linearly related to the drift velocity. The spatial pattern of the particle concentration in the moving frame of reference manifests itself as a temporal variation in a stationary frame of reference. The power spectral density of the particle flow with respect to angular frequency ω, $\langle|J_\omega|^2\rangle$, observed by a stationary observer is given by

$$\langle |J_\omega|^2\rangle = \langle |n_\omega|^2\rangle\langle v\rangle^2 + \langle n\rangle^2\langle |v_\omega|^2\rangle + 2\langle n\rangle\langle v\rangle \ \mathrm{Re}\ \langle n_\omega v_\omega^*\rangle, \qquad (21)$$

where quantities with subscript ω should be understood as the Fourier transforms of the original quantities, the asterisk denotes the complex conjugate, and Re means the real part. The second and third terms in eq. (21) are much smaller than the first term,[1] and one approximately obtains

$$\langle |J_\omega|^2\rangle \simeq \langle v\rangle^2 \langle |n_k(k=\omega/\langle v\rangle)|^2\rangle \partial k/\partial\omega = \langle v\rangle\langle |n_k(k=\omega/\langle v\rangle)|^2\rangle. \qquad (22)$$

We now consider the traffiic flow problem. The power spectral density of the normalized particle concentration is given by eq. (19) in terms of the normalized wave number K. Restoring the original dimensions yields the power spectral density of the car concentration $\langle |n_k|^2\rangle$ with the aid of eq. (19):

$$\langle |n_k|^2\rangle = \left(\frac{n_s}{2}\right)^2 2\times 10^{-2} \frac{2}{(kx_0)^{1.4}} \frac{\mathrm{d}K}{\mathrm{d}k}. \qquad (23)$$

Therefore, the power spectral density of the car flow fluctuation observed by a stationary observer is given, to a hydrodynamical approximation, by

$$\langle |J_\omega|^2\rangle = v_0\langle |n_k(k=\omega/v_0)|^2\rangle = J_0^2\left(\frac{n_s}{n}\right)^2 \frac{1}{t_0^{0.4}} \frac{10^{-2}}{(2\pi)^{1.4}} \frac{1}{f^{1.4}}, \qquad (24)$$

where

$$J_0 = nv_0. \qquad (25)$$

The mean velocity is equated to v_0 and n is regarded as the mean car concentration. Rewriting eq. (24) in a power spectral density in Hz and adding to it a shot noise term $2J_0$, we finally find

$$S(f) = \frac{10^{-2}}{(2\pi)^{0.4}} J_0^2\left(\frac{n_s}{n}\right)^2 \frac{1}{t_0^{0.4}} \frac{1}{f^{1.4}} + 2J_0. \qquad (26)$$

The best fit of this equation to the observed spectrum would be achieved if one puts the coefficient of $1/f^{1.4}$ to 0.0084. From the observation $J_0 = 0.786$ and we put $t_0 = 2$ sec which with the other parameters yielded $R = 20$; the best fit requires

$$n_s = 1.2\ n. \qquad (27)$$

The theoretical curve thus derived is plotted in Fig. 9 together with the observed data for the power spectrum. They approximately agree but the curve calculated from eq. (26) has a slightly steeper slope for low frequencies than the observed points although they scatter. The present model seems to involve a long-term correlation among the cars which is little larger than the actual case.

§5. Conclusion

When the particle concentration is linearly related to the particle drift velocity in a one-dimensional situation, the concentration fluctuation is subject to the Burgers equation in a moving frame of reference. The power spectral density of a steady Burgers flow is calculated by adding to it a random-force term. The fluctuation is almost random in the frequency

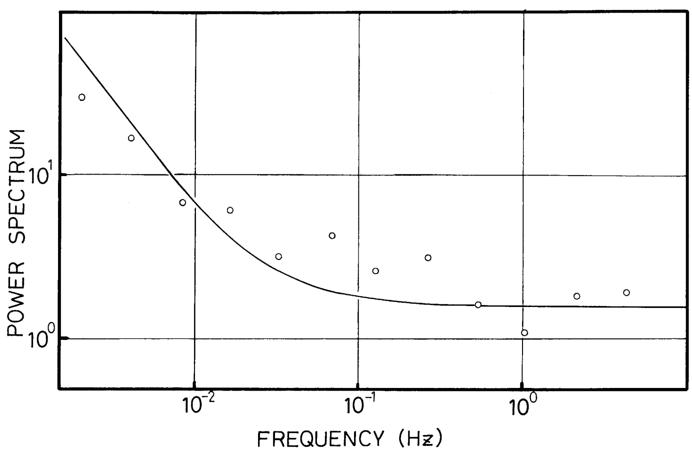

Fig. 9. The observed power spectrum (dots) of the car current and the theoretical spectrum given by eq. (26).

domain but manifests some correlation in the spatial-frequency domain; this tendency is also manifest in the amplitude distribution deviating from a Gaussian form. This theoretical spectrum was applied to the traffic current fluctuation. The best fit was obtained by setting $n_s = 1.2\,n$ where n_s is the saturation concentration at which the drift velocity would vanish and n is the mean car concentration. Agreement between theory and observation is satisfactory.

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