SIAM J. APPL. MATH.
Vol. 21, No. 2, September 1971

STOCHASTIC DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO RANDOM HARMONIC OSCILLATORS AND WAVE PROPAGATION IN RANDOM MEDIA*

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Abstract. The two-time method is used to obtain an expansion, valid for ε small and t large, of the vector solution $u(t, \varepsilon)$ of an abstract ordinary differential equation involving ε. The same method is used to get expansions of functions of u. The results are shown to apply to the solutions of stochastic equations. They are used to find the first two moments and the transition probability of the displacement of a harmonic oscillator with spring constant a random function of t. The result contains the condition for mean square stability due to Stratonovich. The results are also applied to one-dimensional wave propagation through a layer with refractive index a random function of position. They are used to find the mean square amplitude reflection and transmission coefficients, which are just the mean power reflection and transmission coefficients. A graph of the mean square transmission coefficient as a function of layer thickness is presented. The results are also compared with those obtainable by other methods.

1. Introduction. Suppose an ordinary differential equation for a vector $u(t, \varepsilon)$ is solvable for $\varepsilon = 0$. Then we can solve it for ε small by a perturbation method. The usual perturbation expansion contains secular terms which grow with t, so it is valid only for a bounded range of t. To obtain an expansion valid for a much larger range of t, we shall use the two-time method, which is described in the next section (see Cole [1]). We shall also use this method to obtain expansions of functions of u. Our purpose is to treat stochastic differential equations, in which the coefficient $A(t, \varepsilon)$ is a stochastic process. We assume that $A(t, 0)$ is deterministic so that for ε small the random part of $A(t, \varepsilon)$ is small. Then we shall get the expansion of the solution u and of functions of it by the two-time method. From these expansions we shall calculate various statistics of u.

As an application of this procedure, we shall consider the linear second order ordinary differential equation governing the displacement $y(t)$ of a damped harmonic oscillator. We shall assume that the spring "constant" differs from a constant by ε times a random function of t. Then we shall determine the mean displacement $\langle y(t) \rangle$, the mean square displacement $\langle y^2(t) \rangle$, the two-time correlation $\langle y(t)y(t + s) \rangle$ and the distribution of the displacement for ε small and $t \leqq O(\varepsilon^{-2})$. One particular consequence of the result for $\langle y^2(t) \rangle$ is the mean square stability criterion first found by Stratonovich [2] and also considered by Samuels and Eringen [3]. In the special case in which the spring constant has an exponential correlation function, we have obtained the same results by assuming that the spring constant is Markovian and solving an appropriate Fokker–Planck equation [4].

Another application concerns one-dimensional wave propagation through a layer within which the refractive index differs from a constant by ε times a random

* Received by the editors July 9, 1970, and in revised form January 11, 1971.

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function of position. In this case we calculate $\langle |T|^2 \rangle$, the mean square amplitude transmission coefficient of the layer. This also yields $\langle |R|^2 \rangle = 1 - \langle |T|^2 \rangle$, the mean square amplitude reflection coefficient of the layer. A graph of $\langle |T|^2 \rangle$ as a function of the layer thickness is presented which shows how $\langle |T|^2 \rangle$ decreases from unity to zero as the thickness increases from zero to infinity.

For the case in which the refractive index is a Gaussian Markov process, specifically the Orstein–Ohlenbeck process, and also for the random telegraph process, $\langle |T|^2 \rangle$ has been found by Papanicolau, Morrison and Keller [5], who solved the relevant backward Kolmogoroff equation for ε small. Their results agree with the present one specialized to those cases. Exact results for moments valid for all ε, when the refractive index is the random telegraph process, have been obtained by McKenna and Morrison [6], [7], [8]. The use of a Markovian refractive index in wave propagation problems was introduced by Frisch [9], who also gives a good survey of other work on waves in random media. Recently, Kupiec, Felsen, Rosenbaum, Keller and Chow [10] determined the mean reflection and transmission coefficients of a layer of random medium using a method given by Keller [11], Bourret [12] and others.

The present method can be justified, for a large class of problems, in a manner that makes it clear why it works so well. These results will be published in the near future.

2. The two-time method. Let us consider the initial value problem

$$du/dt = \tilde{A}(t, \varepsilon)u, \qquad u(0, \varepsilon) = f(\varepsilon). \tag{2.1}$$

For each $t \geqq 0$ and $\varepsilon \geqq 0$, $u(t, \varepsilon)$ and $f(\varepsilon)$ are vectors in a linear space S and $\tilde{A}$ is an operator on S. We assume that we can solve (2.1) for u when $\varepsilon = 0$, and we wish to solve it for ε positive but small. To do so we shall use the two-time perturbation method.

The solution of (2.1) when $\varepsilon = 0$ is expressible in terms of the operator $Y(t)$ which is the solution of the following initial value problem:

$$dY/dt = \tilde{A}(t, 0)Y, \qquad Y(0) = I. \tag{2.2}$$

It is convenient to introduce a new unknown vector $w(t, \varepsilon)$ related to u by

$$u(t, \varepsilon) = Y(t)w(t, \varepsilon). \tag{2.3}$$

Then it follows from (2.1)–(2.3) that w satisfies the equations

$$dw/dt = A(t, \varepsilon)w, \qquad w(0, \varepsilon) = f(\varepsilon). \tag{2.4}$$

Here A is defined by

$$A(t, \varepsilon) = Y^{-1}(t)[\tilde{A}(t, \varepsilon) - \tilde{A}(t, 0)]Y(t). \tag{2.5}$$

We note that $A(t, 0) = 0$. If $\tilde{A}(t, 0) = 0$, then $Y(t) = I$, $A = \tilde{A}$ and $w = u$.

We begin by introducing a second time variable τ defined by

$$\tau = \varepsilon^r t. \tag{2.6}$$

Here r is a positive integer to be determined. Then we write w as a function of both t and τ:

$$w(t, \varepsilon) = v(t, \tau, \varepsilon). \tag{2.7}$$

Next we use (2.7) in (2.4), use the fact that $w_t = v_t + \varepsilon^r v_\tau$, and obtain

$$v_t + \varepsilon^r v_\tau = A(t, \varepsilon)v, \qquad v(0, 0, \varepsilon) = f(\varepsilon). \tag{2.8}$$

We now assume that f and $\tilde{A}$ have the asymptotic expansions

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} f_n \varepsilon^n, \qquad \tilde{A}(t, \varepsilon) \sim \sum_{n=0}^{\infty} \tilde{A}_n(t)\varepsilon^n. \tag{2.9}$$

Then A has the expansion

$$A(t, \varepsilon) \sim \sum_{n=1}^{\infty} A_n(t)\varepsilon^n, \qquad A_n(t) = Y^{-1}(t)\tilde{A}_n(t)Y(t). \tag{2.10}$$

To solve (2.8) we seek for v an expansion of the form

$$v(t, \tau, \varepsilon) \sim \sum_{n=0}^{\infty} v_n(t, \tau)\varepsilon^n. \tag{2.11}$$

Then we insert (2.9)–(2.11) into (2.8) and equate coefficients of ε^n to obtain

$$v_{n,t} = \sum_{j=0}^{n-1} A_{n-j}(t)v_j(t, \tau) - v_{n-r,\tau}, \qquad v_n(0, 0) = f_n, \quad n = 0, 1, \cdots. \tag{2.12}$$

Here and below quantities with negative subscripts are defined to be zero. The equations (2.12) can be solved successively for the v_n, starting with $n = 0$. The solution of (2.12) is

$$v_n(t, \tau) = \alpha_n(\tau) + \int_0^t \left[\sum_{j=0}^{n-1} A_{n-j}(\sigma)v_j(\sigma, \tau) - v_{n-r,\tau}(\sigma, \tau)\right] d\sigma. \tag{2.13}$$

Here $\alpha_n(\tau)$ is a vector function of τ which is undetermined, except that it must satisfy the initial condition

$$\alpha_n(0) = f_n, \qquad n = 0, 1, \cdots. \tag{2.14}$$

We shall choose the integer r and the functions $\alpha_n(\tau)$ to prevent v_n from growing too rapidly as $t \to \infty$. This is how the two-time method eliminates secular terms from the perturbation expansion, and thus yields approximations which are valid for large ranges of t. To do so we first use (2.13) to write v_0 and v_1 explicitly:

$$v_0(t, \tau) = \alpha_0(\tau), \tag{2.15}$$

$$v_1(t, \tau) = \alpha_1(\tau) + \int_0^t A_1(\sigma)\, d\sigma \alpha_0(\tau) - t\alpha_{0\tau}(\tau)\delta_{r1}. \tag{2.16}$$

We see that (2.16) contains an integral and for $r = 1$, a term proportional to t. Suppose the integral approaches tA_1 for t large. Then A_1 is the time average of A_1. For any n, $\bar{A}_n$ is defined by

$$\bar{A}_n = \lim_{t \to \infty} t^{-1} \int_0^t A_n(\sigma)\, d\sigma. \tag{2.17}$$

If $\bar{A}_1 \neq 0$, we choose $r = 1$ and let α_0 satisfy the equation

$$\alpha_{0\tau}(\tau) = \bar{A}_1 \alpha_0(\tau). \tag{2.18}$$

This equation and the initial condition (2.14) for $n = 0$ determine $\alpha_0(\tau)$. Then (2.16) becomes

$$v_1(t, \tau) = \alpha_1(\tau) + \left[\int_0^t A_1(\sigma)\,d\sigma - t\bar{A}_1\right]\alpha_0(\tau). \tag{2.19}$$

The terms linear in t cancel from the coefficient of $\alpha_0(\tau)$ in (2.19) and thus the most rapidly growing term has been removed from v_1. The subsequent $\alpha_n(\tau)$ can be found in the same way.

If $\bar{A}_1 = 0$, we do not set $r = 1$ and do not require that α_0 satisfy (2.18). Instead we use (2.13) to write v_2 explicitly:

$$\begin{aligned} v_2(t, \tau) &= \alpha_2(\tau) + \int_0^t A_2(\sigma)\,d\sigma\alpha_0(\tau) + \int_0^t A_1(\sigma)\,d\sigma\alpha_1(\tau) \\ &\quad + \int_0^t\int_0^\sigma A_1(\sigma)A_1(s)\,ds\,d\sigma\alpha_0(\tau) - t\alpha_{0\tau}(\tau)\delta_{r2}. \end{aligned} \tag{2.20}$$

Then we define $\overline{A_1A_1}$ by

$$\overline{A_1A_1} = \lim_{t\to\infty} t^{-1}\int_0^t\int_0^\sigma A_1(\sigma)A_1(s)\,ds\,d\sigma. \tag{2.21}$$

If $\bar{A}_2 + \overline{A_1A_1} \neq 0$, we choose $r = 2$ and make α_0 satisfy the equation

$$\alpha_{0\tau} = (\bar{A}_2 + \overline{A_1A_1})\alpha_0. \tag{2.22}$$

This equation and the initial condition (2.14) for $n = 0$ determine $\alpha_0(\tau)$, and thus the term linear in t is eliminated from v_2. The other α_n can be determined similarly.

If both $\bar{A}_1 = 0$ and $\bar{A}_2 + \overline{A_1A_1} = 0$, we do not set $r = 2$ nor require that α_0 satisfy (2.22). Instead we choose r and α_0 to eliminate from v_3 the terms linear in t, etc. This completes the presentation of the method for obtaining the expansion of $u(t, \varepsilon)$.

When $\bar{A}_1 \neq 0$ or $\bar{A}_2 + \overline{A_1A_1} \neq 0$, (2.3), (2.7), (2.15) and (2.18) or (2.22) yield for the leading term in u the results

$$u(t, \varepsilon) = Y(t)e^{\varepsilon t\bar{A}_1}f_0 + O(\varepsilon), \qquad \bar{A}_1 \neq 0, \tag{2.23}$$

$$u(t, \varepsilon) = Y(t)e^{\varepsilon^2 t(\bar{A}_2 + \overline{A_1A_1})}f_0 + O(\varepsilon), \quad \bar{A}_1 = 0, \quad \bar{A}_2 + \overline{A_1A_1} \neq 0. \tag{2.24}$$

3. Expansions of functions of u. Once the expansion of $u(t, \varepsilon)$ has been found, the expansion of any function $q[u(t, \varepsilon)]$ can be calculated from it. However, a second way to get an expansion of $q(u)$ is to derive a differential equation for $q(u)$ and then solve this equation by the two-time method. The expansion obtained in this second way may not be identical with that obtained in the first way, but the two expansions are necessarily asymptotically equal for ε small. The second way, which we shall now develop, is preferable in the application of the two-time method to stochastic equations, as we shall show in the next section.

We first consider the function $u(t) \otimes u(t)$, the tensor or outer product of $u(t)$ with itself. This is a vector in the product space $S \times S$. If A and B are two operators on S and if u and v are in S, then the operator $A \otimes B$ on $S \times S$ is defined by $(A \otimes B)(u \otimes v) = (Au) \otimes (Bv)$. Thus from (2.3) we have

$$u \otimes u = (Y \otimes Y)(w \otimes w). \tag{3.1}$$

If I denotes the identity operator on S, we define the operator $[A]$ on $S \times S$ by

$$[A] = A \otimes I + I \otimes A. \tag{3.2}$$

Now by differentiating $w \otimes w$ and using (2.4), we find that $w \otimes w$ satisfies the differential equation

$$\frac{dw \otimes w}{dt} = [A(t, \varepsilon)]w \otimes w, \qquad w(0, \varepsilon) \otimes w(0, \varepsilon) = f(\varepsilon) \otimes f(\varepsilon). \tag{3.3}$$

Since (3.3) is of the same form as (2.4), the expansion of $w \otimes w$ is given by (2.23) or (2.24) with A replaced by $[A]$, Y replaced by I and f replaced by $f \otimes f$. Then (3.1) yields

$$u(t, \varepsilon) \otimes u(t, \varepsilon) = Y(t) \otimes Y(t)e^{\varepsilon t[\bar{A}_1]}f_0 \otimes f_0 + O(\varepsilon), \qquad \bar{A}_1 \neq 0, \tag{3.4}$$

$$\begin{aligned} u(t, \varepsilon) \otimes u(t, \varepsilon) = Y(t) \otimes Y(t)&e^{\varepsilon^2 t([\bar{A}_2] + \overline{[A_1][A_1]})}f_0 \otimes f_0 + O(\varepsilon), \\ &\bar{A}_1 = 0, \quad [\bar{A}_2] + \overline{[A_1][A_1]} \neq 0. \end{aligned} \tag{3.5}$$

The leading term in (3.4) is just the outer product with itself of the leading term in the expression (2.23) for u. However, the leading terms in (3.5) and (2.24) are not related in this way.

We next consider $u(t) \otimes u(t + s)$ which can be written in terms of $w(t) \otimes w(t)$ as follows:

$$\begin{aligned} u(t) \otimes u(t + s) &= Y(t)w(t) \otimes Y(t + s)w(t + s) \\ &= [Y(t) \otimes Y(t + s)U(t + s, t)]w(t) \otimes w(t). \end{aligned} \tag{3.6}$$

The operator U introduced here is the solution of the problem

$$dU(t + s, t)/ds = A(t + s, \varepsilon)U(t + s, t), \qquad U(t, t) = I. \tag{3.7}$$

Since (3.7) is of the form (2.4), the expansion of U is given by (2.23) or (2.24) with f_0 and Y replaced by I, and t replaced by s. Now we assume that

$$\lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \int_{t_0}^{s} A_1(s)A_1(\sigma)\, d\sigma\, ds = \overline{A_1 A_1},$$

independently of t_0. We have also seen above that the expansion of $w(t) \otimes w(t)$ is given by (3.4) or (3.5) with Y replaced by I. Upon combining these expansions in (3.6) we obtain

$$u(t) \otimes u(t + s) = [Y(t) \otimes Y(t + s)e^{\varepsilon s \bar{A}_1}]e^{\varepsilon t[\bar{A}_1]}f_0 \otimes f_0 + O(\varepsilon), \qquad \bar{A}_1 \neq 0, \tag{3.8}$$

$$\begin{aligned} u(t) \otimes u(t + s) = [Y(t) \otimes Y(t + s)&e^{\varepsilon^2 s(\bar{A}_2 + \overline{A_1 A_1})}]e^{\varepsilon^2 t([\bar{A}_2] + \overline{[A_1][A_1]})} \\ \cdot f_0 \otimes f_0 + O(\varepsilon), &\qquad \bar{A}_1 = 0, \quad [\bar{A}_2] + \overline{[A_1][A_1]} \neq 0. \end{aligned} \tag{3.9}$$

We finally consider a general function $q[u(t, \varepsilon)]$, where $u(t, \varepsilon)$ is a solution of

$$du/dt = B(t, \varepsilon)u, \qquad u(t_0, \varepsilon) = u^0. \tag{3.10}$$

By using $U(t, t_0, \varepsilon)$ defined in (3.7) with B instead of A, we can write $u(t, \varepsilon) = U(t, t_0, \varepsilon)u^0$. Now we denote $q(u)$ by

$$Q(t, t_0, u^0, \varepsilon) = q[u(t, \varepsilon)] = q[U(t, t_0, \varepsilon)u^0]. \tag{3.11}$$

To obtain an equation for Q we first differentiate (3.11) with respect to t_0 to get

$$Q_{t_0} = q_u U_{t_0}(t, t_0, \varepsilon)u^0. \tag{3.12}$$

From the definition of U it follows that $U(t_0, t)U(t, t_0) = I$. Differentiation of this identity with respect to t_0 and the use of (3.7) yields

$$\begin{aligned} U_{t_0}(t, t_0) &= -U^{-1}(t_0, t)U_{t_0}(t_0, t)U(t, t_0) = -U^{-1}(t_0, t)B(t_0)U(t_0, t)U(t, t_0) \\ &= -U^{-1}(t_0, t)B(t_0) = -U(t, t_0)B(t_0). \end{aligned} \tag{3.13}$$

By using (3.13) in (3.12) we get

$$Q_{t_0} = -q_u U(t, t_0)B(t_0)u^0. \tag{3.14}$$

Next we differentiate (3.11) with respect to u^0 and obtain

$$Q_{u^0} = q_u U(t, t_0). \tag{3.15}$$

Upon using (3.15) in (3.14) we find the following equation for Q:

$$Q_{t_0} = -Q_{u^0}B(t_0, \varepsilon)u^0, \qquad Q(t, t, u^0) = q(u^0). \tag{3.16}$$

The "final" condition in (3.16) follows by setting $t_0 = t$ in (3.11). We may call (3.16) a "backward" equation for $Q(t, t_0, u^0, \varepsilon)$ because it involves differentiation with respect to the variable t_0 at which the initial condition is imposed on $u(t)$, rather than with respect to t itself. We note that (3.16) is an initial value problem (or final value problem) for a functional differential equation. It is just a partial differential equation if u^0 is a finite-dimensional vector and q is a scalar. The problem (3.16) is of the form (2.1) so the results of § 2 are applicable for the determination of the expansion of Q.

To apply those results we introduce a fixed basis in S and denote by u_i^0, $i = 1, \cdots$, the components of u^0 in this basis, and by $B_{ij}(t_0, \varepsilon)$, the matrix elements of the operator $B(t_0, \varepsilon)$. Then (3.16) becomes the partial differential equation

$$Q_{t_0} = -\sum_{i,j} u_j^0 B_{ij}(t_0, \varepsilon)\frac{\partial}{\partial u_i^0}Q, \qquad Q(t, t, u^0, \varepsilon) = q(u^0). \tag{3.17}$$

Since $t_0 < t$, this equation is of the form (2.1) with $-t_0$ instead of t and with $\tilde{A} = \sum_{i,j} u_j^0 B_{ij}\partial/\partial u_i^0$. If $B \sim \sum_{n=0}^{\infty} B^{(n)}(t_0)\varepsilon^n$, then $\tilde{A}$ is of the form (2.9) with $\tilde{A}_n = \sum_{i,j} u_j^0 B_{ij}^{(n)}\partial/\partial u_i^0$. For simplicity we shall assume that $B^{(0)} = 0$ so that $A_0 = 0$ and then (2.2) yields $Y = I$. It follows that $A_n = \tilde{A}_n$ and then (2.17) and (2.21) yield

$$\bar{A}_n = \sum_{i,j} \overline{B_{ij}^{(n)}} u_j^0 \frac{\partial}{\partial u_i^0}, \tag{3.18}$$

$$\overline{A_1 A_1} = \sum_{i,j,k,p} \overline{B_{ij}^{(1)} B_{kp}^{(1)}} u_j^0 \frac{\partial}{\partial u_i^0} u_p^0 \frac{\partial}{\partial u_k^0}. \tag{3.19}$$

In (3.17) the initial condition is given at $t_0 = t$, so we define τ by

$$\tau = \varepsilon^r(t - t_0). \tag{3.20}$$

If $\bar{A}_1 \neq 0$, we set $r = 1$ and use (3.18) and (2.15) in (2.18). Thus we obtain for $v_0(\tau, u_0)$, which is the leading term in the expansion of Q, the equations

$$\frac{\partial}{\partial \tau} v_0(\tau, u^0) = \sum_{i,j} \overline{B_{ij}^{(1)}} u_j^0 \frac{\partial}{\partial u_i^0} v_0, \qquad v_0(0, u^0) = q(u^0). \tag{3.21}$$

If $\bar{A}_1 = 0$ but $\bar{A}_2 + \overline{A_1 A_1} \neq 0$, we set $r = 2$ and use (3.18), (3.19) and (2.15) in (2.22). Then $v_0(\tau, u_0)$, the leading term in Q, satisfies the equations

$$\begin{gathered} \frac{\partial}{\partial \tau} v_0(\tau, u^0) = \sum_{i,j} \overline{B_{ij}^{(2)}} u_j^0 \frac{\partial}{\partial u_i^0} v_0 + \sum_{i,j,k,p} \overline{B_{ij}^{(1)} B_{kp}^{(1)}} u_j^0 \frac{\partial}{\partial u_i^0} u_p^0 \frac{\partial}{\partial u_k^0} v_0, \\ v_0(0, u^0) = q(u^0). \end{gathered} \tag{3.22}$$

We note from (3.21) that when $\bar{A}_1 \neq 0$, v_0 satisfies a first order partial differential equation, while when $\bar{A}_1 = 0$, v_0 satisfies the second order parabolic equation (3.22).

To obtain "forward" equations analogous to (3.21) and (3.22) we consider the density function $G(t, z, u^0, \varepsilon)$ defined by

$$G(t, z, u^0, \varepsilon) = \delta[z - u(t, \varepsilon)] = \delta[z - U(t, 0, \varepsilon)u^0], \qquad t \geqq 0. \tag{3.23}$$

Here $u(t, \varepsilon)$ is the solution of (3.10), z is an independent vector and $U(t, 0, \varepsilon)$ is the solution of (3.7) with $t_0 = 0$ and B in place of A. It is convenient to rewrite G by changing variables in the delta function to obtain

$$G(t, z, u^0, \varepsilon) = [\det U(t, 0, \varepsilon)]^{-1} \delta[U^{-1}(t, 0, \varepsilon)z - u^0]. \tag{3.24}$$

We now differentiate (3.24) with respect to t and use the facts that $(U^{-1})_t = -U^{-1}B$ and $(\log \det U)_t = \operatorname{tr} B(t, \varepsilon)$, where tr denotes trace. Thus we get

$$G_t = -\frac{(\det U)_t}{(\det U)^2}\delta + \frac{1}{\det U}\delta'(U^{-1})_t z = -G \operatorname{tr} B - \frac{1}{\det U}\delta' U^{-1} B z. \tag{3.25}$$

By combining (3.24) and (3.25) we find that G satisfies the equation

$$G_t + G \operatorname{tr} B + G_z B z = 0, \qquad G(0, z, u^0) = \delta(z - u^0). \tag{3.26}$$

This is the Liouville equation corresponding to (3.10). It can also be written in divergence form as

$$G_t + \frac{\partial}{\partial z} \cdot (GBz) = 0, \qquad G(0, z, u^0) = \delta(z - u^0). \tag{3.27}$$

Since (3.26) is of the form (2.1) we can use the method of § 2 to solve it. The operator $\tilde{A}$ is defined by $\tilde{A}G = -G \operatorname{tr} B - G_z B z$. When B has an expansion in ε, $\tilde{A}$ has the corresponding expansion (2.9). Let us assume that $B^{(0)} = 0$ and then $\tilde{A}_0 = 0$ so $A = \tilde{A}$ and $Y = I$. Then the leading term in the expansion of G is $v_0 = \alpha_0$, which is the solution of (2.18) if $\bar{A}_1 \neq 0$ or of (2.22) if $\bar{A}_1 = 0$. In terms of a basis in S, the operator A_n is represented by $-\sum_i B_{ii}^{(n)} - \sum_{ij} B_{ij}^{(n)} z_j \partial/\partial z_i$.

Thus (2.18) and (2.22) become the following equations for $v_0(\tau, z, u^0)$, in which the summation convention is used:

$$\frac{\partial v_0}{\partial \tau} + \overline{B_{ii}^{(1)}} v_0 + \overline{B_{ij}^{(1)}} z_j \frac{\partial v_0}{\partial z_i} = 0, \qquad v_0(0, z, u^0) = \delta(z - u^0), \qquad \bar{B}^{(1)} \neq 0, \tag{3.28}$$

$$\begin{aligned} &\frac{\partial v_0}{\partial \tau} + (\bar{B}_{ii}^{(2)} - \overline{B_{ii}^{(1)} B_{jj}^{(1)}}) v_0 + (\bar{B}_{ij}^{(2)} - \overline{2B_{kk}^{(1)} B_{ij}^{(1)}} - \overline{B_{kj}^{(1)} B_{ik}^{(1)}}) z_j \frac{\partial v_0}{\partial z_i} \\ &\qquad - \overline{B_{ij}^{(1)} B_{km}^{(1)}} z_j z_m \frac{\partial^2 v_0}{\partial z_i \partial z_k} = 0, \\ &v_0(0, z, u^0) = \delta(z - u^0), \qquad \bar{B}^{(1)} = 0. \end{aligned} \tag{3.29}$$

By using the results of this section and those of § 2, we can obtain two different expressions for the leading term in the expansion of $u(t) \otimes u(t)$. One is given by (3.4) or (3.5), obtained by solving the equation satisfied by $u \otimes u$. The other is the tensor product with itself of the leading term in the expansion of u. When $\bar{A}_1 \neq 0$, these two expressions, one given by (3.4) and the other formed from (2.23), are the same. However, when $\bar{A}_1 = 0$ the corresponding expressions obtained from (3.5) and (2.24) are not the same, since in general $\overline{[A_1 A_1]} \neq [\bar{A}_1][\bar{A}_1]$. Therefore the two expressions differ by $O(\varepsilon^2 t)$. As each expression has an error $O(\varepsilon)$, it is necessary that $O(\varepsilon^2 t) \leqq O(\varepsilon)$ when both expressions are valid. This shows that both can be valid only for $t \leqq O(\varepsilon^{-1})$, although one of them may be valid for a longer time. In fact, when (3.5) is used for a stochastic equation and its expectation is taken, the result is valid for $t \leqq O(\varepsilon^{-2})$ in the examples of § 5 and § 6. However, this is not so for the expression for $u \otimes u$ formed from (2.24). Similar remarks apply to the other functions of u considered in this section. We have shown that expectations of the expressions obtained in this section have a greater range of validity than the corresponding expressions formed from (2.24). (G. Papanicolau, to be published.)

4. Stochastic equations. We now suppose that $\tilde{A}(t, \varepsilon)$ is a random operator and f is a random vector so that (2.1) is a stochastic equation and its solution $u(t, \varepsilon)$ is a random vector. By this we mean that $\tilde{A}$, f and u also depend upon a variable ω which ranges over a probability space Ω, but we shall not write ω explicitly. The expectation or mean of any function $f(\omega)$ will be denoted by $\langle f \rangle$. We assume that for almost all ω, $\tilde{A}$ and f have the expansion (2.9) with A_0 and f_0 not random. Then $Y(t)$ is not random but $\tilde{A}_n$, f_n and A_n are random for $n \geqq 1$. The calculation in § 2 of the leading term in the expansion of u applies for almost all realizations of the random operator $\tilde{A}$ and therefore the results hold for almost all solutions u. The same is true of the calculations in § 3 of the leading terms in the expansions of $u(t) \otimes u(t)$, $u(t) \otimes u(s)$, $g(u)$ and $\delta(z - u)$. Therefore, by taking expectations of those leading terms we obtain the leading terms in the expansions of the first and second moments $\langle u(t) \rangle$ and $\langle u(t) \otimes u(t) \rangle$, of the two-time correlation $\langle u(t) \otimes u(s) \rangle$, of the expectation $\langle g(u) \rangle$ and of the probability density $\langle \delta(z - u) \rangle$.

The error in each of the original expansions is $O(\varepsilon)$, so the error in the mean value $\langle u \rangle$ and in the expectations of each function of u is $\langle O(\varepsilon) \rangle$. That part of the error $O(\varepsilon)$ linear in ε is the sum of a part linear in A_1 and a part linear in f_1. An important special case is that in which the mean values of these quantities are zero; i.e., $\langle A_1 \rangle = 0$, $\langle f_1 \rangle = 0$. Then that part of the mean error $\langle O(\varepsilon) \rangle$ linear in ε is also zero, so $\langle O(\varepsilon) \rangle = O(\varepsilon^2)$. Thus in this case the error in each expectation is $O(\varepsilon^2)$. As a consequence the range of t in which the resulting expressions for the expectations are valid may be larger than that for the corresponding unaveraged expressions. When this occurs it is because secular terms linear in ε average out to zero.

We shall now make an assumption about $A_1(t)$ to facilitate the calculation of the expectations referred to above. We first observe that the only random quantity in the expression (2.23) for u is $\overline{A}_1$, while in (2.24) it is $\overline{A}_2 + \overline{A_1A_1}$. Therefore we assume that $A_1(t)$ and $A_2(t)$ belong to the class of stochastic operators for which the limits $\overline{A}_1$, $\overline{A}_2$ and $\overline{A_1A_1}$ exist and are the same for almost all realizations of $A_1(t)$ and $A_2(t)$. Then the leading term in u, given by (2.23) or (2.24), is almost surely independent of the realization. As a consequence, the leading term in $\langle u(t, \varepsilon) \rangle$, the mean value of u, is also given by the right side of (2.23) or (2.24). Furthermore, $\overline{A}_1$, $\overline{A}_2$ and $\overline{A_1A_1}$ are also equal to their mean values. Therefore we have from (2.17) and (2.21), upon taking mean values,

$$\overline{A}_n = \lim_{t\to\infty} t^{-1} \int_0^t \langle A_n(\sigma) \rangle \, d\sigma, \qquad n = 1, 2, \tag{4.1}$$

$$\overline{A_1A_1} = \lim_{t\to\infty} t^{-1} \int_0^t \int_0^\sigma \langle A_1(\sigma)A_1(s) \rangle \, ds \, d\sigma. \tag{4.2}$$

Sufficient conditions for (4.1) and (4.2) to hold can be given when S is a finite-dimensional space, in which case the operators can be represented by finite matrices. Then the almost sure stability theorem for random functions given by Loève [13, p. 488] yields the desired conditions. These conditions are adequate for the examples in § 5 and § 6.

It is of interest to compare the present results for $\langle u(t) \rangle$ with those given by the method of Keller [11] and Bourret [12]. For simplicity we shall make the comparison for the case in which $\tilde{A}(t, 0) = \tilde{A}_0(t) = 0$, so that $Y(t) = I$ and $u = w$. Then that method applied to (2.4) yields for $\langle u(t) \rangle$ the following integro-differential equation:

$$\begin{aligned}\left[\frac{\partial}{\partial t} - \varepsilon\langle A_1(t) \rangle - \varepsilon^2 \langle A_2(t) \rangle\right]\langle u(t) \rangle - \varepsilon^2 \int_0^t [\langle A_1(t)A_1(s) \rangle - \langle A_1(t) \rangle \langle A_1(s) \rangle] \\ \cdot \langle u(s) \rangle \, ds + O(\varepsilon^3) = 0, \qquad \langle u(0) \rangle = \langle f(\varepsilon) \rangle.\end{aligned} \tag{4.3}$$

On the other hand, let us suppose that (4.1) and (4.2) hold. Then upon setting $Y(t) = I$ and differentiating (2.23) and (2.24) with respect to t, we find that $\langle u \rangle$ given by the present method satisfies one of the following differential equations:

$$\left[\frac{\partial}{\partial t} - \varepsilon \overline{A}_1\right]\langle u(t) \rangle + O(\varepsilon) = 0, \qquad \langle u(0) \rangle = f_0 + O(\varepsilon), \qquad \overline{A}_1 \neq 0, \tag{4.4}$$

$$\begin{aligned}\left[\frac{\partial}{\partial t} - \varepsilon^2(\overline{A}_2 + \overline{A_1A_1})\right]\langle u(t) \rangle + O(\varepsilon) = 0, \\ \langle u(0) \rangle = f_0 + O(\varepsilon), \qquad \overline{A}_1 = 0.\end{aligned} \tag{4.5}$$

These three equations become equations for approximations to $\langle u \rangle$ when the error terms $O(\varepsilon^3)$ and $O(\varepsilon)$ are omitted. Since (4.4) and (4.5) have greater errors, the resulting equations yield poorer approximations to $\langle u \rangle$ than does that which results from (4.3), but the equations are simpler to solve. Therefore we may view the equations resulting from (4.4) and (4.5) as approximations to that which results from (4.3). The approximation (4.4) consists in replacing $\langle A_1(t) \rangle$ by $\overline{A}_1$ and dropping terms of order ε^2. The approximation (4.5), which applies when $\langle A_1(t) \rangle = \overline{A}_1 = 0$, consists in replacing $\langle A_2(t) \rangle$ by $\overline{A}_2$ and $\int_0^t \langle A_1(t)A_1(s) \rangle \langle u(s) \rangle \, ds$ by $\overline{A_1 A_1} \langle u(t) \rangle$. Approximations of this type have been considered by Filatov [14] and Vahabov [15] who have given conditions under which they are valid.

5. Random harmonic oscillator. We shall now apply the preceding considerations to the analysis of a damped harmonic oscillator with its spring constant a random function of time. The displacement $y(t)$ of the oscillator is assumed to be unity at $t = 0$, and its initial velocity is assumed to be zero. Then $y(t)$ is determined by the following initial value problem:

$$\frac{d^2y}{dt^2} + 2\beta\varepsilon^2\frac{dy}{dt} + k^2[1 + \varepsilon x(t)]y = 0, \qquad y(0) = 1, \qquad \frac{dy}{dt}(0) = 0. \tag{5.1}$$

Here β, k and ε are given nonnegative constants while $x(t)$ is a random function with mean value zero and correlation function R:

$$\langle x(t) \rangle = 0, \qquad \langle x(t)x(t') \rangle = R(|t - t'|). \tag{5.2}$$

Therefore $y(t)$ is also a random function. We wish to determine various statistics of $y(t)$ such as its mean, its mean square, its two-point correlation and its distribution.

First we shall write (5.1) as a first order system by introducing the two-component vector $u(t)$ defined by

$$u = \frac{1}{2}\begin{pmatrix} y + \dfrac{1}{ik}\dfrac{dy}{dt} \\ y - \dfrac{1}{ik}\dfrac{dy}{dt} \end{pmatrix}. \tag{5.3}$$

Then (5.1) becomes

$$\frac{du}{dt} = [\tilde{A}_0 + \varepsilon\tilde{A}_1 + \varepsilon^2\tilde{A}_2]u, \qquad u(0) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}. \tag{5.4}$$

The matrices $\tilde{A}_0$, $\tilde{A}_1$ and $\tilde{A}_2$ are defined by

$$\tilde{A}_0 = ik\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{A}_1 = \frac{ikx(t)}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \tilde{A}_2 = \beta\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5.5}$$

Thus (5.4) is of the form (2.1) with f and $\tilde{A}$ of the form (2.9) in which $\tilde{A}_n = 0$ for $n \geqq 3$ and $f_n = 0$ for $n \geqq 1$.

To apply the results of § 2, we first note from (5.5) that $\tilde{A}(t, 0) = \tilde{A}_0$ is diagonal, so the solution $Y(t)$ of (2.2) is

$$Y(t) = e^{\tilde{A}_0 t} = \begin{pmatrix} e^{ikt} & 0 \\ 0 & e^{-ikt} \end{pmatrix}. \tag{5.6}$$

Now we assume that (4.1) and (4.2) hold. Then by using (5.2), (5.5) and (5.6) in (4.1) and (4.2) we obtain

$$\overline{A}_1 = 0, \tag{5.7}$$

$$\overline{A}_2 = \beta \lim_{t\to\infty} t^{-1} \int_0^t \begin{pmatrix} e^{-ikt'} & 0 \\ 0 & e^{ikt'} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{ikt'} & 0 \\ 0 & e^{-ikt'} \end{pmatrix} dt' = -\beta I, \tag{5.8}$$

$$\begin{aligned} \overline{A_1 A_1} &= -\frac{k^2}{4} \lim_{t\to\infty} t^{-1} \int_0^t \int_0^\sigma \begin{pmatrix} e^{-ik\sigma} & 0 \\ 0 & e^{ik\sigma} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{ik(\sigma-s)} & 0 \\ 0 & e^{-ik(\sigma-s)} \end{pmatrix} \\ &\qquad \cdot \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{iks} & 0 \\ 0 & e^{-iks} \end{pmatrix} R(|\sigma - s|)\, ds\, d\sigma \\ &= \frac{k^2}{4} \begin{pmatrix} S^*(2k) - S(0) & 0 \\ 0 & S(2k) - S(0) \end{pmatrix}. \end{aligned} \tag{5.9}$$

Here $S^*(\omega)$ is the complex conjugate of $S(\omega)$ which is defined by

$$S(\omega) = \int_0^\infty R(\sigma) e^{i\omega\sigma}\, d\sigma. \tag{5.10}$$

In view of (5.7), we set $r = 2$ and use (2.24) for u. Both $\overline{A}_2$ and $\overline{A_1 A_1}$, given by (5.8) and (5.9), are diagonal. From the initial condition in (5.4) we see that $f_0 = (1/2, 1/2)$. Then (2.24) yields, with probability one,

$$u(t, \varepsilon) = \begin{pmatrix} \exp\left(ikt + \varepsilon^2 t\left[-\beta + \frac{k^2}{4} S^*(2k) - \frac{k^2}{4} S(0)\right]\right) & 0 \\ 0 & \exp\left(-ikt + \varepsilon^2\left[-\beta + \frac{k^2}{4} S(2k) - \frac{k^2}{4} S(0)\right]\right) \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + O(\varepsilon). \tag{5.11}$$

By using (5.11) in (5.3) we obtain for y the following result with probability one:

$$\begin{aligned} y(t, \varepsilon) &= \exp\left(\left\{\frac{k^2}{4}[\operatorname{Re} S(2k) - S(0)] - \beta\right\}\varepsilon^2 t\right) \\ &\qquad \cdot \cos\left\{\left[k - \frac{\varepsilon^2 k^2}{4} \operatorname{Im} S(2k)\right] t\right\} + O(\varepsilon). \end{aligned} \tag{5.12}$$

Now $\langle y(t, \varepsilon)\rangle$, the mean value of y, is also given by the right side of (5.12). However, that part of the error linear in ε is also linear in $x(t)$, and by (5.2), its mean is zero. Thus $\langle O(\varepsilon)\rangle = O(\varepsilon^2)$ and we obtain, from (5.12),

$$\begin{aligned} \langle y(t, \varepsilon)\rangle &= \exp\left(\left\{\frac{k^2}{4}[\operatorname{Re} S(2k) - S(0)] - \beta\right\}\varepsilon^2 t\right) \\ &\qquad \cdot \cos\left\{\left[k - \frac{\varepsilon^2 k^2}{4} \operatorname{Im} S(2k)\right] t\right\} + O(\varepsilon^2). \end{aligned} \tag{5.13}$$

We shall now set $r = 2$ and use (3.5) to obtain the expansion of $u \otimes u$, from which we will get the expansion of $y^2(t, \varepsilon)$ and then of $\langle y^2(t, \varepsilon)\rangle$. First we note from their definitions and from (5.7) and (5.8) that

$$\overline{[A_1]} = 0, \qquad \overline{[A_2]} = -\beta[I], \tag{5.14}$$

$$\overline{[A_1][A_1]} = \frac{k^2}{2}\begin{bmatrix} S^*(2k) - 2S(0) & 0 & 0 & 0 \\ 0 & \operatorname{Re} S(2k) & \operatorname{Re} S(2k) & 0 \\ 0 & \operatorname{Re} S(2k) & \operatorname{Re} S(2k) & 0 \\ 0 & 0 & 0 & S(2k) - 2S(0) \end{bmatrix}. \tag{5.15}$$

In writing (5.15) we have put the components of $u \otimes u$ in the order 11, 12, 21, 22. We next use (5.14) and (5.15) in (3.5) and evaluate the exponential function by utilizing the eigenvalues and eigenvectors of $\overline{[A_1]} + \overline{[A_1][A_1]}$. Then we find from (5.3) that $y = u_1 + u_2$ so $y^2 = u_1^2 + u_1u_2 + u_2u_1 + u_2^2$, which is the sum of all the elements of $u \otimes u$. By summing all the elements of (3.5) we obtain, with probability one,

$$\begin{aligned} y^2(t, \varepsilon) &= \frac{1}{2}\exp\left(\left\{\frac{k^2}{2}[\operatorname{Re} S(2k) - 2S(0)] - 2\beta\right\}\varepsilon^2 t\right) \\ &\quad \cdot \cos\left\{\left[2k - \frac{\varepsilon^2k^2}{2}\operatorname{Im} S(2k)\right]t\right\} \\ &\quad + \tfrac{1}{2}\exp(\{k^2 \operatorname{Re} S(2k) - 2\beta\}\varepsilon^2 t) + O(\varepsilon). \end{aligned} \tag{5.16}$$

Thus for the reasons given above,

$$\begin{aligned} \langle y^2(t, \varepsilon)\rangle &= \frac{1}{2}\exp\left(\left\{\frac{k^2}{2}[\operatorname{Re} S(2k) - 2S(0)] - 2\beta\right\}\varepsilon^2 t\right) \\ &\quad \cdot \cos\left\{\left[2k - \frac{\varepsilon^2k^2}{2}\operatorname{Im} S(2k)\right]t\right\} \\ &\quad + \tfrac{1}{2}\exp(\{k^2 \operatorname{Re} S(2k) - 2\beta\}\varepsilon^2 t) + O(\varepsilon^2). \end{aligned} \tag{5.17}$$

From (5.17) we see that for $\langle y^2(t, \varepsilon)\rangle$ to be bounded, and therefore for the oscillator to be stable in the mean square sense, the damping coefficient β must be large enough, i.e., it must satisfy

$$\beta > \frac{k^2}{2}\operatorname{Re} S(2k). \tag{5.18}$$

This result was first obtained by Stratonovich [2] in a different way.

Next we want to find the correlation function $\langle y(t)y(t + s)\rangle$. We note that since $y = u_1 + u_2$, it follows that $y(t)y(t + s)$ is the sum of the four elements of $u(t) \otimes u(t + s)$. To calculate $u(t) \otimes u(t + s)$ we shall use (3.9). We have already calculated all the matrices which occur in it, and therefore we need merely substitute them into (3.9) and then sum the four elements of the resulting matrix. Upon doing this we obtain

$$
\begin{aligned}
y(t)y(t+s) = \frac{1}{2}\Bigg[& \exp\left(\left\{\frac{k^2}{2}[\operatorname{Re} S(2k) - 2S(0)] - 2\beta\right\}\varepsilon^2 t\right) \\
& \cdot \cos\left[\left\{2k - \frac{\varepsilon^2 k^2}{2}\operatorname{Im} S(2k)\right\} t + \left\{k - \frac{\varepsilon^2 k^2}{4}\operatorname{Im} S(2k)\right\} s\right] \\
& + \exp(\{k^2 \operatorname{Re} S(2k) - 2\beta\}\varepsilon^2 t) \\
& \cdot \cos\left[\left\{k^2 - \frac{\varepsilon^2 k^2}{4}\operatorname{Im} S(2k)\right\} s\right]\Bigg] \\
& \cdot \exp\left(\left\{\frac{k^2}{4}[\operatorname{Re} S(2k) - S(0)] - \beta\right\}\varepsilon^2 s\right) + O(\varepsilon).
\end{aligned} \tag{5.19}
$$

Since (5.19) holds almost surely, it also yields $\langle y(t)y(t+s)\rangle$, but then the error is $O(\varepsilon^2)$. This result for the correlation of y has also been obtained by J. A. Morrison [8].

Let us now specialize the preceding results to the case of the exponential correlation function:

$$R(s) = e^{-|s|/\gamma}, \qquad \gamma > 0. \tag{5.20}$$

From (5.10), $S(\omega) = (\gamma^{-1} + i\omega)^{-1}$ so (5.13) and (5.17) become

$$\langle y(t,\varepsilon)\rangle = e^{-(2k^2\gamma^2\sigma + \varepsilon^2\beta)t}\cos(1-\gamma\sigma)kt + O(\varepsilon^2), \tag{5.21}$$

$$\langle y^2(t,\varepsilon)\rangle = \tfrac{1}{2}e^{-(8k^2\gamma^2\sigma+\sigma+2\varepsilon^2\beta)t}\cos(1-\gamma\sigma)2kt + \tfrac{1}{2}e^{2(\sigma-\varepsilon^2\beta)t} + O(\varepsilon^2). \tag{5.22}$$

Here σ is defined by

$$\sigma = (\varepsilon^2 k^2 \gamma)/(2 + 8k^2\gamma^2). \tag{5.23}$$

The results (5.21) and (5.22) agree exactly with the results of an independent analysis of (5.1) based upon the assumption that $x(t)$ is the Ornstein–Uhlenbeck process, which is Markovian with the correlation function (5.20). In that case, the multivariate process $[y(t), y'(t), x(t)]$ is also Markovian and its distribution function satisfies a Fokker–Planck equation. This equation is derived and solved by G. Papanicolau [4], and the mean values of y and y^2 computed from the solution are just (5.21) and (5.22).

Let us now consider the process $w(t) = Y^{-1}(t)u(t)$, where u satisfies (5.4) and Y is given by (5.6). Then w satisfies (2.4) with A given by (2.5) and $w(0) = (\frac{1}{2}, \frac{1}{2})$. We wish to calculate any function $g[w(t)]$ of the process $w(t)$. To do so we consider the function Q defined by (3.11). The leading term in the expansion of Q is $v_0(\tau, z, w^0)$, which is the solution of (3.22) if $\overline{A} = 0$. Since (4.1) and (4.2) hold, the leading term in $\langle Q\rangle$ is also v_0.

The matrices $B^{(n)}$ in (3.23) are just the coefficients A_n in the expansion of A given by (2.5) in which the $\tilde{A}_n$ are defined by (5.5) and Y by (5.6). By combining these equations we find $B^{(1)}$ and $B^{(2)}$. Then we use them in (4.1) and (4.2) to get $\overline{B^{(1)}}$, $\overline{B^{(2)}}$ and $\overline{B^{(1)}_{ij}B^{(1)}_{kp}}$. We obtain $\overline{B^{(1)}} = 0$, $\overline{B^{(2)}} = -\beta I$, $\overline{B^{(1)}_{11}B^{(1)}_{11}} = \overline{B^{(1)}_{22}B^{(1)}_{22}}$

$= -\overline{B_{11}^{(1)}B_{22}^{(1)}} = -\overline{B_{22}^{(1)}B_{11}^{(1)}} = -a, \overline{B_{12}^{(1)}B_{21}^{(1)}} = b + ic, \overline{B_{21}^{(1)}B_{12}^{(1)}} = b - ic$ and all other $\overline{B_{ij}^{(1)}B_{kp}^{(1)}} = 0$. Here

$$a = k^2 S(0)/4, \qquad b = k^2 \operatorname{Re} S(2k)/4, \qquad c = k^2 \operatorname{Im} S(2k)/4. \tag{5.24}$$

Then we substitute these results into (3.22) to obtain the following equation for v_0:

$$\begin{aligned}\frac{\partial v_0}{\partial \tau} &= [-a(w_1^0)^2\partial_1^2 + 2(a+b)w_1^0 w_2^0 \partial_1\partial_2 - a(w_2^0)^2\partial_2^2 \\ &\quad + (-\beta - a + b - ic)w_1^0\partial_1 + (-\beta - a + b + ic)w_2^0\partial_2]v_0, \\ &v_0(0, z, w^0) = q(w^0).\end{aligned} \tag{5.25}$$

Here $\partial_i = \partial/\partial w_i^0, i = 1, 2$.

It is convenient to introduce the new variables r and θ, related to w_1^0 and w_2^0 by

$$w_1^0 = e^{r - i\theta}, \qquad w_2^0 = e^{r + i\theta}. \tag{5.26}$$

From (5.3) and the definition of w in terms of u, it follows that $w_1 = w_2^*$. Therefore we assume that r and θ are real with $-\infty < r < \infty$ and $0 \leqq \theta \leqq 2\pi$. We also define V by

$$V(\tau, r, \theta) = v_0(\tau, e^{r-i\theta}, e^{r+i\theta}). \tag{5.27}$$

Then (5.27) becomes

$$\frac{\partial V}{\partial \tau} = \frac{b}{2}V_{rr} + (b - \beta)V_r + \left(a + \frac{b}{2}\right)V_{\theta\theta} + cV_\theta, \qquad V(0, r, \theta) = q(e^{r-i\theta}, e^{r+i\theta}). \tag{5.28}$$

By separation of variables we find that the fundamental solution of (5.28) is

$$\begin{aligned}G(\tau, r', \theta', r, \theta) &= (2\pi b\tau)^{-1/2} \exp\left\{-\frac{1}{2b\tau}[r - r' - (\beta - b)\tau]^2\right\} \\ &\quad \cdot \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{im(\theta' - \theta) - \left[m^2\left(a + \frac{b}{2}\right) + imc\right]\tau\right\}.\end{aligned} \tag{5.29}$$

Thus v_0, the leading term in the expansion of Q, is given by

$$\begin{aligned}v_0(\tau, e^{r-i\theta}, e^{r+i\theta}) &= V(\tau, r, \theta) \\ &= \int_{-\infty}^{\infty}\int_0^{2\pi} G(\tau, r', \theta', r, \theta) q(e^{r'-i\theta'}, e^{r'+i\theta'})\, d\theta'\, dr'.\end{aligned} \tag{5.30}$$

Since (4.1) and (4.2) hold, the leading term in the expansion of $\langle Q \rangle$ is also v_0,

$$\langle Q(t, t_0, e^{r-i\theta}, e^{r+i\theta}, \varepsilon)\rangle = v_0[\varepsilon^2(t - t_0), e^{r-i\theta}, e^{r+i\theta}] + O(\varepsilon^2). \tag{5.31}$$

From (5.31) the preceding results of this section can be obtained independently. The fundamental solution G given by (5.29) is just the leading term in the transition probability density of the solution, expressed in terms of r and θ. In the special case mentioned above in which $x(t)$ is the Ornstein–Uhlenbeck process, (5.29) agrees with the result obtained by solving the Fokker-Planck equation governing that case [4].

6. Wave propagation in random media. We shall now apply the preceding considerations to one-dimensional wave propagation through a layer within which the refractive index is a random function of position. We assume that the wave is time harmonic with complex amplitude $y(x)$. Here x denotes position along a line in the direction of propagation. The random medium extends from $x = 0$ to $x = L$, and we assume that in it $y(x)$ satisfies the equation

$$d^2y/dx^2 + k^2[1 + \varepsilon z(x)]y = 0, \qquad 0 \leqq x \leqq L. \tag{6.1}$$

The constant k is the propagation constant of the field, $z(x)$ is a given random function and ε is a constant. Thus the refractive index $[1 + \varepsilon z(x)]^{1/2}$ is a random function.

Outside the interval $0 < x < L$, we assume that (6.1) holds with $z(x) \equiv 0$. Therefore we can write $y(x)$ for $x \leqq 0$ as

$$y(x) = e^{ikx} + re^{-ikx}, \qquad x \leqq 0. \tag{6.2}$$

We interpret e^{ikx} in (6.2) as an incident wave of unit amplitude traveling toward the layer and re^{-ikx} as a reflected wave. Then the complex constant r is the amplitude reflection coefficient. For $x \geqq L$ we assume that

$$y(x) = Te^{ik(x-L)}, \qquad x \geqq L. \tag{6.3}$$

The complex constant T is the amplitude transmission coefficient. The problem we consider is to find $y(x)$ satisfying (6.1) and constants r and T such that

$$y, y' \quad \text{are continuous at} \quad x = 0 \quad \text{and} \quad x = L. \tag{6.4}$$

This problem is a two-point boundary value problem for (6.1).

We can write (6.1) as a first order equation for the vector (y, y'). However, it is more convenient to use the fundamental matrix $Y(x)$, whose columns are two linearly independent solution vectors. This matrix satisfies the equations

$$\frac{dY}{dx} = \begin{pmatrix} 0 & 1 \\ -k^2[1 + \varepsilon z(x)] & 0 \end{pmatrix} Y, \qquad Y(0) = I. \tag{6.5}$$

From (6.2) we have $(y, y') = (1 + r, ik - ikr)$ at $x = 0$, and from (6.3), $(y, y') = (T, ikT)$ at $x = L$. Now for any x, $(y, y') = Y(x)[y(0), y'(0)]$ and thus we find that the preceding values are related by

$$\begin{pmatrix} T \\ ikT \end{pmatrix} = Y(L)\begin{pmatrix} 1 + r \\ ik - ikr \end{pmatrix}. \tag{6.6}$$

We may view (6.6) as a pair of equations for r and T. Upon solving for T and computing $|T|^2$ we obtain

$$|T|^2 = 4[k^{-2}Y_{21}^2(L) + k^2Y_{12}^2(L) + Y_{11}^2(L) + Y_{22}^2(L) + 2]^{-1}. \tag{6.7}$$

Furthermore, from (6.6) or from the relation $\det Y(x) = 1$, we obtain the energy conservation relation

$$|r|^2 = 1 - |T|^2. \tag{6.8}$$

The relations (6.7) and (6.8) show that $|r|^2$ and $|T|^2$ can be expressed in terms of the four elements of the matrix $Y(L)$.

The equation (6.5) for $Y(x)$ can be simplified by introducing the new matrix $u(x)$ defined by

$$u(x) = \begin{pmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{pmatrix} C Y(x). \tag{6.9}$$

Here C is given by

$$C = \frac{1}{2}\begin{pmatrix} 1 & 1/ik \\ 1 & -1/ik \end{pmatrix}. \tag{6.10}$$

From (6.5) and (6.9) it follows that $u(x)$ satisfies the equations

$$du/dx = \varepsilon B^{(1)}(x)u, \qquad u(0) = C. \tag{6.11}$$

The matrix $B^{(1)}$ is found to be

$$B^{(1)}(x) = \frac{ikz(x)}{2}\begin{pmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{pmatrix}. \tag{6.12}$$

From (6.7) and (6.9) we find that $|T|^2 = q[u(L)]$, where $q[u]$ is defined by

$$q[u] = \frac{2}{1 + 2(u_{11}u_{21} + k^2 u_{12}u_{22})}. \tag{6.13}$$

To find $|T|^2$ we must determine the function $q[u(x)]$ at $x = L$, where $u(x)$ is the solution of (6.11). We note that (6.11) is of the form (3.10) with $t = x$, $B = \varepsilon B^{(1)}(x)$, $t_0 = x_0 = 0$ and $u^0 = C$. Therefore $q[u(L)] = Q(L, 0, C, \varepsilon)$, where Q is defined by (3.11). Then $v_0(\tau, u^0)$, the leading term in the expansion of Q, satisfies (3.21) if $\overline{B^{(1)}} \neq 0$ or (3.22) if $\overline{B^{(1)}} = 0$. We now assume that the random function $z(x)$ is such that (4.1) and (4.2) hold for $B^{(1)}$, and that

$$\langle z(x) \rangle = 0, \qquad \langle z(x)z(x') \rangle = R(|x - x'|). \tag{6.14}$$

Then from (4.1) it follows that $\overline{B^{(1)}} = 0$ so v_0 satisfies (3.22) with $\tau = \varepsilon^2(x - x_0)$. We must now calculate the sixteen quantities $\overline{B_{ij}^{(1)}B_{kp}^{(1)}}$ by using (6.12) and (6.14) in (4.2). However, the matrix $B^{(1)}(x)$ is exactly the same as $B^{(1)}(t)$ in § 5 with $z(x)$ in place of $x(t)$. Therefore the results given just above (5.24) apply in the present case as well. Furthermore, $B^{(2)} \equiv 0$.

Now we can write out (3.22). In doing so we must view u^0 as a four-component vector with components in the order 11, 21, 12, 22. Then $(B)^1$ must be treated as the following compound matrix:

$$\begin{pmatrix} B^{(1)} & 0 \\ 0 & B^{(1)} \end{pmatrix}. \tag{6.15}$$

We find that (3.22) can be written as follows, in which $\partial_{ij} = \partial/\partial u_{ij}^0$:

$$\begin{aligned} \frac{\partial v_0}{\partial \tau} &= [-a(u_{11}^0\partial_{11} + u_{12}^0\partial_{12} - u_{21}^0\partial_{21} - u_{22}^0\partial_{22})^2 \\ &\quad + (b + ic)(u_{12}^0\partial_{11} + u_{22}^0\partial_{12})(u_{11}^0\partial_{21} + u_{12}^0\partial_{22}) \\ &\quad + (b - ic)(u_{11}^0\partial_{21} + u_{12}^0\partial_{22})(u_{12}^0\partial_{11} + u_{22}^0\partial_{12})]v_0, \\ v_0(0, u^0) &= q[u^0]. \end{aligned} \tag{6.16}$$

The function $q[u^0]$ depends only upon the combination

$$\rho = 2(u_{11}^0 u_{21}^0 + k^2 u_{12}^0 u_{22}^0). \tag{6.17}$$

Therefore we shall seek a solution $v_0(\tau, \rho)$ of (6.16) which depends upon τ and ρ. From (6.16) and (6.17) we find

$$\frac{\partial v_0}{\partial \tau} = 2b \frac{\partial}{\partial \rho}\left[(\rho^2 - 1)\frac{\partial v_0}{\partial \rho}\right], \qquad \rho \geqq 1, \quad v_0(0, \rho) = \frac{2}{1 + \rho}. \tag{6.18}$$

The range of ρ indicated in (6.18) results from the fact that $|T|^2 = q[u(L)]$ must be real and less than or equal to unity. We also note that $b > 0$ because it is the real part of the Fourier transform of a correlation function. We shall require v_0 to be finite at $\rho = 1$, the singular point of (6.18), and to decay sufficiently rapidly at $\rho = \infty$.

Separation of variables in (6.18) yields the product solution

$$e^{-2b(s^2 + 1/4)\tau} P_{-1/2 + is}(\rho).$$

Here $P_{-1/2+is}$ is the conical Legendre function which satisfies the equation

$$\frac{d}{d\rho}[(\rho^2 - 1)\frac{d}{d\rho}P_{-1/2+is}] + (s^2 + \tfrac{1}{4})P_{-1/2+is}(\rho) = 0, \qquad \rho \geqq 1. \tag{6.19}$$

Therefore we can solve (6.18) by using the Mehler transform formulas [16]

$$\tilde{v}(s) = \int_1^\infty v(\rho) P_{-1/2+is}(\rho)\, d\rho, \qquad s \geqq 0, \tag{6.20}$$

$$v(\rho) = \int_0^\infty s \tanh \pi s P_{-1/2+is}(\rho) \tilde{v}(s)\, ds, \qquad \rho \geqq 1. \tag{6.21}$$

The solution is

$$\begin{aligned} v_0(\tau, \rho) = 2e^{-b\tau/4} &\int_0^\infty e^{-2b\tau s^2} s \tanh \pi s P_{-1/2+is}(\rho) \\ &\cdot \left[\int_1^\infty (1 + \xi)^{-1} P_{-1/2+is}(\xi)\, d\xi\right] ds. \end{aligned} \tag{6.22}$$

This is the leading term in the expansion of Q.

To find $|T|$ we must set $x = L$ and $x_0 = 0$ in τ, so that τ becomes $\varepsilon^2 L$. We must also set $u^0 = C$, and then (6.17) yields $\rho = 1$. Then we have, with probability one,

$$\begin{aligned} |T|^2 = v_0(\varepsilon^2 L, 1) + O(\varepsilon) = 2e^{-\varepsilon^2 bL/4} &\int_0^\infty e^{-2\varepsilon^2 bLs^2} \\ \cdot s \tanh \pi s &\left[\int_1^\infty (1 + \xi)^{-1} P_{-1/2+is}(\xi)\, d\xi\right] ds + O(\varepsilon). \end{aligned} \tag{6.23}$$

The integral in the brackets in (6.23) can be evaluated by making use of the integral representation [16]

$$P_{-1/2+is}(\cosh \alpha) = \frac{2^{1/2}}{\pi} \cosh \pi s \int_0^\infty (\cosh t + \cosh \alpha)^{-1/2} \cos st\, dt, \qquad \alpha \geqq 0. \tag{6.24}$$

Then (6.23) becomes, with probability one,

$$|T|^2 = 2e^{-\varepsilon^2 bL/2}\int_0^\infty e^{-2\varepsilon^2 bLs^2}\frac{\pi s\sinh \pi s}{\cosh^2 \pi s}\,ds + O(\varepsilon). \tag{6.25}$$

Then the mean square transmission coefficient $\langle |T|^2\rangle$, which is just the mean power transmission coefficient, is also given by the right side of (6.25) but the error is then $O(\varepsilon^2)$.

From (6.25) we see that the leading term in $|T|^2$ and in $\langle |T|^2\rangle$ is a function of the dimensionless thickness $L' = 2\varepsilon^2 bL$. From (5.24) and (5.10) we find

$$L' = 2\varepsilon^2 bL = \tfrac{1}{2}\varepsilon^2 k^2 L\int_0^\infty \cos 2k\sigma R(\sigma)\,d\sigma. \tag{6.26}$$

Evaluation of (6.25) for $L' = 0$ and asymptotically for L' large yields the results

$$|T|^2 = 1 + O(\varepsilon) \quad \text{at} \quad L' = 0, \tag{6.27}$$

$$|T|^2 \sim \frac{\pi^{5/2}e^{-L'/4}}{2(L')^{3/2}} + O(\varepsilon), \qquad L' \gg 1. \tag{6.28}$$

A graph of the leading term in $|T|^2$ and in $\langle |T|^2\rangle$, given by the first term on the right side of (6.25), is shown in Fig. 1, obtained by numerical evaluation of the integral. It shows that $|T|^2$ and $\langle |T|^2\rangle$ decrease monotonically as L' increases, from unity at $L' = 0$ to zero at $L' = \infty$, as one would expect.

Acknowledgment. We wish to thank John Morrison for the observation following (6.17) and for bringing to our attention reference [14].

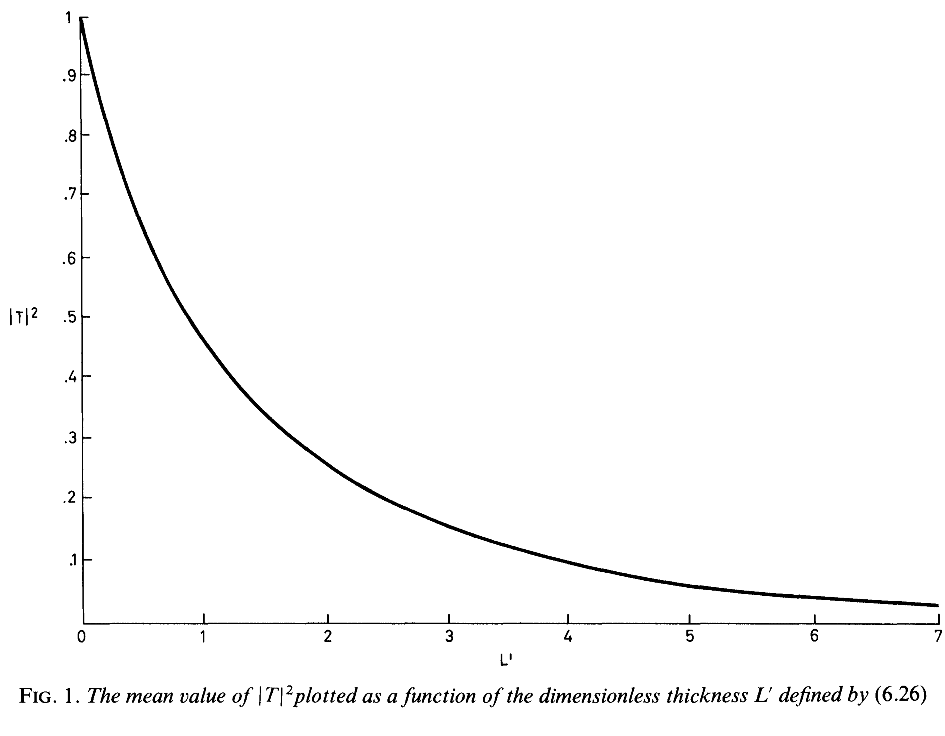

FIG. 1. *The mean value of* $|T|^2$ *plotted as a function of the dimensionless thickness* L' *defined by* (6.26)

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