

A one phase interior point method (IPM) for non-convex optimization

Oliver Hinder, Yinyu Ye

ohinder@stanford.edu

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Outline

1 Introduction

- Current non-convex IPMs
- Local infeasibility certificates
- Algorithm outline

2 Theory

- Global convergence
- Local convergence
- How to return L1 minimizer of constraint violation

3 Practice

- Implementation details
- Preliminary empirical results with CUTEst

Problem we wish to solve:

$$\begin{aligned} \min f(x) \\ a(x) \leq 0 \end{aligned}$$

Assume:

- 1 The constraints and objective are C^2
- 2 The set $a(x) \leq \theta$ is bounded for any $\theta \in \mathbb{R}^m$

One phase IPMs for convex optimization

Either declare problem infeasible (primal or dual) or optimal in one phase:

- 1 Infeasible start IPM (Mehortra, 1992)
- 2 Homogenous algorithm (Anderson and Ye, 1998, 1999)

State of the art for convex optimization

Why one phase non-convex IPM do not exist (yet)

- 1 Some infeasible start algorithms have been tested, but no theoretical guarantees (i.e. LOQO)
- 2 Ideally our algorithm would either find a local optimum or a local minimizer the violation of the constraints
- 3 Infeasible start IPMs may fail this criterion (Wachter and Biegler in 2000).

Solutions in literature to this problem

- ① Two phases (IPOPT)
 - ① Each phase has a different variables
 - ② Awkward convergence assumptions
 - ③ Poor performance on infeasible problems
- ② Compute two directions (KNITRO)
- ③ Penalty (or big-M) method (Curtis, 2012)

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It doesn't need to be this complicated!

Local infeasibility certificates

Farkas certificate, there exists $y \geq 0$ such that x^* is a stationary point of:

$$\min y^T a(x)$$

With $y^T a(x^*) > 0$.

L1 constraint violation, x^* is a stationary point of:

$$\min e^T (a(x))^+$$

With $e^T (a(x^*))^+ > 0$.

One phase barrier problem

Original problem:

$$\begin{aligned} \min f(x) \\ a(x) \leq 0 \end{aligned}$$

Barrier problem:

$$\begin{aligned} \min f(x) - (1 - \eta^k) \mu^k \sum_i \log s_i \\ a(x) + s = (1 - \eta^k)(a(x^k) + s^k) \\ s \geq 0 \end{aligned}$$

Primal-dual direction computation

$$H^k d_x + (A^k)^T d_y = -(\nabla f(x) + y^T \nabla a(x))$$

$$A^k d_x + d_s = -\eta^k (a(x^k) + s^k)$$

$$S^k d_y + Y^k d_s = (1 - \eta^k) \mu^k e - Y^k s^k$$

$$H^k = \nabla_x^2 L(x^k, y^k) + \delta^k I$$

$$A^k = \nabla a(x^k)$$

One phase non-convex algorithm

for $k \leftarrow 1, \dots, \infty$ **do**

Factor newton system using IPOPT strategy to compute δ^k

if $\|\nabla f(x^k) + (y^k)^T \nabla a(x^k)\| < \mu^k$ **then**

Aggressive step:

Compute direction with $\eta^k \in (0, 1]$

Take largest step while maintaining complementary

else

Stabilization step:

Compute direction with $\eta^k = 0$

Backtracking line search on direction using merit function

end if

end for

Iterate update

Backtrack on α and update iterates:

$$x^{k+1} = x^k + \alpha^k d_x^k$$

$$y^{k+1} = y^k + \alpha^k d_y^k$$

$$s^{k+1} = (1 - \alpha^k \eta^k)(a(x^k) + s^k) - a(x^{k+1})$$

$$\mu^{k+1} = (1 - \eta^k \alpha^k) \mu^k$$

Only accept iterates that approximately satisfy complementary:

$$\beta \mu^{k+1} \leq s_i^{k+1} y_i^{k+1} - \mu^{k+1} \leq \frac{1}{\beta} \mu^{k+1}$$

Merit function line search

Backtrack on α until armijo rule is satisfied for the merit function:

$$\psi^k(x) = f(x) - \mu^k \sum_i \log(a_i(x^k) + s_i^k - a_i(x))$$

Termination criterion

Declare problem first-order ϵ -locally optimal if:

$$\left\| \nabla f(x) + y^T \nabla a(x) \right\| + \|a(x) + s\| + \mu < \epsilon_{opt}$$

Declare problem first-order ϵ -locally (farkas) infeasible if:

$$\frac{\mu + \|y^T \nabla a(x)\|}{y^T (a(x) + s)} < \epsilon_{inf}$$

Global convergence

If δ^k and η^k are judiciously chosen then it possible to show the following result:

Theorem

For any $\epsilon > 0$, after a finite number of steps the algorithm has found either a first-order ϵ -local optimum or a first-order ϵ -local (farkas) infeasibility certificate.

Local convergence

Assume:

- 1 Converging towards a KKT solution that satisfies strict complementary.
- 2 The sufficient conditions for local optimality are satisfied at the KKT solution.

If η^k is judiciously chosen then it is possible to show:

Theorem

If (1-2) hold, the algorithm converges Q-quadratically to a local optimality certificate.

How to return L1 minimizer of constraint violation

We wish to solve:

$$\begin{aligned} \min f(x) \\ a(x) \leq 0 \end{aligned}$$

This problem is trivially equivalent to:

$$\begin{aligned} \min f(x) \\ a(x) \leq z \\ e^T z \leq 0 \\ z \geq 0 \end{aligned}$$

The local Farkas infeasibility certificate to this problem is an certificate of L1 minimization of the constraint violation.

Implementation details

- 1 Use fraction to boundary rule i.e. $y^{k+1} \geq 0.05y^k$ and $s^{k+1} \geq 0.05s^k$
- 2 Choose η^k on aggressive steps using predictor-corrector technique
- 3 Search on negative eigenvector to guarantee second-order necessary conditions are satisfied

Scaled KKT merit function

Scaled KKT residuals is an alternate merit function:

$$\frac{\|\nabla f(x) + y^T \nabla a(x)\|}{\|y\|}$$

Accept step that makes progress on either:

- 1 Classic merit function
- 2 Scaled KKT residuals

Using a filter to prevent cycling

Empirical results

Table: Results on 30 selected CUTEst problems, number of variables between 100 and 1000

	One phase	IPOPT (2004)	KNITRO (2004)
Median	71 (59)	41	53
SEGFAULTS	4	0	0
MAXIT	3	3	3

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The End

References



Itai Ashlagi, Mark Braverman, and Avinatan Hassidim (2014)
Stability in Large Matching Markets with Complementarities
Operations Research 62:4, 713-732.

Conjectures

Conjecture

Assume (x^*, s^*) is a limit point of the algorithm and there exists finite dual multipliers at (x^*, s^*) then for some subsequence of the dual multipliers $y^{\pi(k)} \rightarrow y^*$.

Conjecture

After a polynomial number of steps a simple variant of the algorithm has found either a first-order ϵ -local optimum or a first-order ϵ -local infeasibility certificate.