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Upside and Downside Capture Ratios:

How to Make Them Come Out the Way You Want

By

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Abstract

Upside and downside capture ratios are used to assess the quality of investment managers and investment strategies. We propose a theoretical model which predicts that the upside capture ratio is an increasing function of the measurement interval length and that the downside capture ratio is a decreasing function of the measurement interval length. The model also predicts that all measurement intervals' capture ratios depend strongly on betas, not just alphas, and that short measurement intervals' capture ratios are dominated by betas, hence are unreliable for assessing alphas. Consequently, capture ratios are problematic for assessing managers' skill, but offer investment managers a wonderful opportunity to mislead clients.

Introduction

Investors often approach the task of selecting investment managers, or assessing the effectiveness of their investment strategies, by analyzing backtests or live performance using various performance metrics. Two popular performance metrics are "upside capture" and "downside capture." For example, Clarke and Arnott (1987) discuss various tradeoffs of portfolio insurance by examining the impact of the insured portfolio's upside and downside capture ratios. Lee, Phoon, and Wong (2006) are concerned with how the inclusion of Asian hedge funds in various portfolios impact upside and downside capture ratios. Finally, Xiong, Sullivan, and Wang (2012) propose a model of portfolio selection that adjusts an investors' portfolio allocation in accordance with changing market liquidity premium environments and market conditions, and use upside and downside capture ratios to test the efficacy of their model.

The upside capture ratio is defined as the ratio of the portfolio's average return to the benchmark's average return for measurement intervals when the benchmark's return is positive. The downside capture ratio is defined as the ratio of the portfolio's average return to the benchmark's average return for measurement intervals when the benchmark's return is negative. These capture ratios are widely used as measures of a portfolio's attractiveness. For example, a portfolio with an upside capture ratio that is larger than its downside capture ratio is often thought to have a positive alpha, while a portfolio with a lower upside capture ratio than its downside capture ratio is often thought to have a negative alpha. [1] While this may be true in some contexts, it not nearly as reliable an indicator of portfolio attractiveness as many investors believe. For example, this standard of interpretation implies that a fair priced stock+put combination has a positive alpha and that a fair priced stock-call combination has a negative alpha. Since neither payoff has anything to do with alpha, or talent, substantial investment management fees would be paid for nothing. This paper shows that, even in the absence of non-linear payoffs, the typical uses of capture ratios are problematic for assessing a portfolio's attractiveness.

A "measurement interval" can be a day, a week, a month, a quarter, or longer and the computations are typically performed using non-overlapping calendar measurement intervals. As a result, the measurement interval length will impact the overall results since, for example, while a quarterly market return is either "up" or "down," not all months in that particular calendar quarter are "up" months or "down" months. Worse, there is a tendency for the short measurement interval capture ratios often used by

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[1] This kind of comparison is problematic. Consider an upside capture ratio of 2 and a downside capture ratio of 0.5. Suppose that the upside capture ratio is due to an average portfolio return of 1% and an average benchmark return of 0.5%, and that the downside capture ratio is due to an average portfolio return of -25% and an average benchmark return of -50%. Are these capture ratios comparable? Suppose, instead, that the same upside capture ratio stems from an average portfolio return of 20% and an average benchmark return of 10%, and that the same downside capture ratio stems from an average portfolio return of -1% and an average benchmark return of -2%? Are these two identical pairs of capture ratios comparable?

investors to be dominated by beta, hence they are unreliable indicators of alpha. At a minimum, using the same measurement interval length is essential to ensure a fair comparison among managers. It is also important to understand the tradeoffs between shorter measurement intervals (e.g. monthly or quarterly) and longer measurement intervals (e.g. annual or longer).

To the best of our knowledge there is no study exploring the characteristics of upside and downside capture ratios, even in the simplest case where a portfolio is viewed as having a constant positive alpha and a constant beta. This is particularly true concerning how capture ratios vary with the measurement interval length, beta, and alpha. If capture ratios are problematic when alphas and betas are constant, they are likely to be unreliable, generally. For this simple case, we find that the length of the interval over which returns are measured (e.g., monthly or quarterly, versus annually or longer) can impact the results materially, as can beta and alpha. The relationship is especially strong for low volatility and low beta portfolios, which have been shown to outperform cap weighted indices over time with lower risk levels,[2] but should hold for many investment strategies. We explore some of the properties of upside and downside capture ratios and develop a model for the case of constant alpha and beta which explains their dependence on the measurement interval length, beta, and alpha for investment strategies with positive alphas.[3] The model also predicts that all measurement intervals' capture ratios depend strongly on betas, not just alphas, and that short measurement intervals' capture ratios are dominated by betas. Consequently, capture ratios are problematic for assessing managers' skill, but offer investment managers a wonderful opportunity to mislead clients.

Low volatility portfolios provide good intuition about why and how the measurement interval length should affect upside and downside capture ratios. Low volatility strategies typically have a low beta; hence on any given day the strategy is more likely to underperform in an up market and outperform in a down market. However, over the course of longer periods, a low volatility strategy with a positive alpha may outperform even in an up market, if the alpha more than offsets the beta source return's shortfall relative to the benchmark.

A portfolio's and benchmark's short measurement interval returns are dominated by randomness. Thus, a portfolio's short measurement interval capture ratios are dominated by its beta. A portfolio's and benchmark's long measurement interval returns are dominated by trend. Thus, a portfolio's long measurement interval capture ratios reflect both its beta and alpha. We show that long measurement interval and short measurement interval capture ratios can vary materially due to their varying dependence on beta and alpha.

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[2] See, for example, Haugen and Baker (1991).

[3] We focus on the case of positive alphas because practitioners presume that they exist. The analysis can be extended to strategies with negative alphas, and to some degree to time varying or stochastic alphas.

To illustrate the impact of the measurement interval length on upside and downside capture ratios, we examined the historical performance of the MSCI World Minimum Volatility Index versus the MSCI World Index from 1989 to 2012 and at the performance of the S&P 500 Low Volatility Index relative to the S&P 500 Index from 1991 to 2012. We computed upside and downside capture ratios while conditioning on positive or negative monthly, quarterly, semi-annual, annual, 2-year, 3-year, and 4-year benchmark returns. We used the following conventional formula for upside capture:

$$UpCapture = \frac{\left(\prod_{i=1}^{n_p}\left(1+R_i\right)\right)^{\frac{1}{y}}-1}{\left(\prod_{i=1}^{n_p}\left(1+B_i\right)\right)^{\frac{1}{y}}-1} \tag{1}$$

Where:

$n_p \equiv$ Number of positive benchmark returns.

$B_i \equiv$ i-th positive benchmark return.

$R_i \equiv$ The manager's return for the same measurement interval as the i-th positive benchmark return.

$y \equiv$ The number of years counting measurement intervals with positive benchmark returns only.

The downside capture is computed in similarly using only the non-positive returns for the benchmark.

The empirical results we obtained are presented in Table 1.

Table 1 – Upside and downside capture for MSCI World Minimum Volatility Index and S&P 500 Low Volatility Index using conditioning on various benchmark return horizons.

	Monthly	Quarterly	Semi-Annual	Annual	2-Year	3-year	4-Year
MSCI World Minimum Volatility Index upside capture (vs. MSCI World Index - 1989 to 2012)	69.9%	74.0%	80.8%	83.7%	88.4%	101.0%	106.1%
MSCI World Minimum Volatility Index downside capture (vs. MSCI World Index - 1989 to 2012)	64.2%	56.2%	50.6%	51.6%	42.4%	15.3%	-1.4%
S&P 500 Low Volatility Index upside capture (vs. S&P 500 Index - 1991 to 2012)*	66.1%	70.7%	73.8%	76.5%	89.6%	75.9%	84.2%
S&P 500 Low Volatility Index downside capture (vs. S&P 500 Index -1991 to 2012)*	52.6%	37.7%	11.1%	5.9%	36.2%	-20.0%	-58.2%

* 3-year measurement interval is for the period from 1991 to 2011; 4-year measurement interval is for the period from 1991 to 2010.

The salient features include:

- The upside capture ratios tend to increase (improve) materially as the measurement interval length increases. Even differences between quarterly and monthly measurement intervals (most commonly used in practice) can impact the strategies' ranking. For example, the S&P 500 Low Volatility Index has a lower upside capture ratio than the MSCI World Minimum Volatility Index when a monthly measurement interval is used for both. However, if we compare the S&P 500 Low Volatility Index's upside capture ratio computed using a quarterly measurement interval to the MSCI World Minimum Volatility Index's upside capture ratio calculated using a monthly measurement interval, then the former ranks higher than the latter.

- The downside capture ratios tend to decrease (improve) materially as the measurement interval length increases and can even turn strongly negative, implying that the portfolios tend to outperform even when their benchmarks underperform. Here too, the measurement interval can make a difference when ranking strategies.

The results in Table 1 suggest that, consistent with the intuition discussed earlier, upside capture ratios can increase materially and downside capture ratios can decrease materially with increases in the measurement interval length over even surprisingly short measurement intervals, for talented managers. Our theoretical model also predicts this dependence of the upside and downside capture ratios on the measurement interval length.

It is important to understand the reason for the dependence of capture ratios on measurement interval lengths, betas, and alphas to ensure properly interpreting them. For example, consider a talented manager with a positive alpha and a beta of 1, and relatively long measurement intervals for computing the capture ratios (say five years). The benchmark and the portfolio are likely to be up in most measurement intervals and the portfolio is likely to be ahead of the benchmark most of the time, due to its alpha. The manager's upside capture ratio will typically be greater than 1. In the measurement intervals that the benchmark is down, the portfolio is still likely to outperform the benchmark, due to its alpha, resulting in a downside capture ratio that is less than 1. As a more extreme and unusual case, consider the same talented manager and very long measurement intervals for computing the capture ratios (say, twenty years). Ignoring the practicality that there are few, if any, products with a live track record that spans multiple twenty-year intervals, the benchmark and the portfolio are almost surely going to be up in virtually all twenty-year non-overlapping measurement intervals and the portfolio is almost surely going to be ahead of the benchmark in those measurement intervals, due to its alpha. The manager's upside capture ratio will almost surely be greater than 1 and his downside capture ratio will rarely be observed. In the unlikely event that the benchmark is down, the portfolio is still likely to be up, due to its alpha, resulting in a negative downside capture ratio.

In contrast, since alphas are roughly proportional to time and standard deviations are roughly proportional to the square root of time, short measurement intervals' capture ratios reflect mostly noise, hence tend to depend on betas, not alphas. Consequently, short measurement interval capture ratios are almost useless as a guide to alphas.

Capture ratios depend on the measurement interval length, in addition to a portfolio's alpha and beta. A manager with the same alpha and beta as another manager will tend to have less attractive capture ratios than the other manager if his measurement interval length is shorter. A manager with a higher alpha and slightly lower beta than another manager can have less attractive capture ratios than the other manager even if the same measurement interval length is used for both managers.

Comparing capture ratios across managers is problematic.

Consider the following method for computing the upside and downside capture ratios, using continuous returns.[4]

- Choose a measurement interval length for determining if the benchmark's return is positive or negative.

[4] Equation (1) is how upside (downside) capture ratios are often computed in practice, i.e., by dividing the annualized geometric return of the portfolio over the intervals when the benchmark is up (down) by the annualized geometric return of the benchmark over the intervals where the benchmark is up (down). Our use of continuous returns simplifies the math involved. While upside capture and downside capture ratios using continuous returns will be slightly different from when annualized geometric returns are used, the conclusions will be the same.

- Classify the measurement intervals in the analysis as either positive benchmark return or negative benchmark return.

- Compute the average portfolio return for the positive benchmark return measurement intervals.

- Compute the average benchmark return for the positive benchmark return measurement intervals.

- Compute the upside capture ratio by dividing the average portfolio return for the positive benchmark return measurement intervals by the average benchmark return for these measurement intervals.

- Compute the average portfolio return for the negative benchmark return measurement intervals.

- Compute the average benchmark return for the negative benchmark return measurement intervals.

- Compute the downside capture ratio by dividing the average portfolio return for the negative benchmark return measurement intervals by the average benchmark return for the negative benchmark return measurement intervals.

The result of applying these computations to some real, low tracking error, positive alpha portfolios is that the upside capture ratios increase and the downside capture ratios decrease as the measurement interval length increases.

This paper shows that a market model relating a portfolio's return to a benchmark's return replicates this behavior for portfolios with positive alphas. The model's features that give rise to the dependence of the computed upside capture ratio and downside capture ratio on the measurement interval length for determining if the benchmark's return is positive or negative are common to other reasonable models of portfolio return in relation to benchmark return. This suggests that real portfolios' upside capture ratios and downside capture ratios are not unique and that investors should be aware of the measurement interval lengths used when comparing upside capture ratios and downside capture ratios across managers.

The paper's model also shows that upside and downside capture ratios depend importantly on betas, not just alphas. The model predicts that short measurement interval upside capture ratios and downside capture ratios are dominated by beta and that long measurement interval upside capture ratios depend strongly on beta. Consequently, capture ratios are problematic for assessing managers' alphas.

The mathematical details are contained in the Appendix.

I. An Illustrative Model of Portfolio Behavior in Relation to Benchmark Behavior.

All returns in the model are presumed to be continuous.[5] Expected returns are presumed to be proportional to time. Disturbance terms are presumed to be independently normal, with means of zero and standard deviations that are proportional to the square root of time.

$$R_{\pi T} = \alpha_\pi T + \beta_\pi R_{\mu T} + \varepsilon_{\pi T} \tag{2}$$

$$R_{\mu T} = \alpha_\mu T + \varepsilon_{\mu T} \tag{3}$$

$R_{\mu T} \equiv$ The benchmark's return over a measurement interval of length T.

$\alpha_\mu \equiv$ The benchmark's expected return rate over a measurement interval of length 1. It is assumed that $\alpha_\mu > 0$, so that the benchmark's long term return is positive.[6]

$\varepsilon_{\mu T} \equiv$ The benchmark's disturbance term over a measurement interval of length T.

$R_{\pi T} \equiv$ The portfolio's return over a measurement interval of length T.

$\alpha_\pi \equiv$ The portfolio's expected relative return rate over a measurement interval of length 1.

$\varepsilon_{\pi T} \equiv$ The portfolio's residual disturbance term over a measurement interval of length T.

$\beta_\pi \equiv$ The portfolio's beta, measured with respect to the benchmark.

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5 Continuous returns are the logarithm of the ratio of final value to initial value. The use of continuous returns has the advantage that compounding is replaced with addition and annualizing is replaced by averaging.

6 If this were not the case, an investor would be better off using cash as a benchmark.

The portfolio's return is annualized by dividing by the number of years in the measurement interval, T. Hence, the portfolio's return and annualized return for a measurement interval of length T can be written as:

$$R_{\pi T} = \alpha_\pi T + \beta_\pi \left(\alpha_\mu T + \varepsilon_{\mu T}\right) + \varepsilon_{\pi T} = \left(\alpha_\pi + \beta_\pi \alpha_\mu\right) T + \left(\beta_\pi \varepsilon_{\mu T} + \varepsilon_{\pi T}\right) \tag{4}$$

$$\frac{R_{\pi T}}{T} = \left(\alpha_\pi + \beta_\pi \alpha_\mu\right) + \left(\frac{1}{T}\right)\left(\beta_\pi \varepsilon_{\mu T} + \varepsilon_{\pi T}\right) \tag{5}$$

The benchmark's annualized return for a measurement interval of length T can be written as:

$$\frac{R_{\mu T}}{T} = \alpha_\mu + \left(\frac{1}{T}\right)\varepsilon_{\mu T} \tag{6}$$

II. The Capture Ratios.

The upside capture ratio includes only measurement intervals when the benchmark return is positive and the downside capture ratio includes only measurement intervals when the benchmark return is negative. Denote the number of measurement intervals with positive benchmark returns by n_U and the number of measurement intervals with negative benchmark returns by n_D. Define $\sum\left(A \mid B\right)$ as the sum of the values of A for measurement intervals that satisfy the condition B. Then the upside and downside capture ratios are:[7]

$$C_U = \frac{\frac{1}{n_U}\sum\left(\frac{R_{\pi T}}{T} \mid R_{\mu T} > 0\right)}{\frac{1}{n_U}\sum\left(\frac{R_{\mu T}}{T} \mid R_{\mu T} > 0\right)} \tag{7}$$

$$C_D = \frac{\frac{1}{n_D}\sum\left(\frac{R_{\pi T}}{T} \mid R_{\mu T} < 0\right)}{\frac{1}{n_D}\sum\left(\frac{R_{\mu T}}{T} \mid R_{\mu T} < 0\right)}$$

[7] Each measurement interval is T years long and there is a cumulative return for each T year interval. Since continuous returns are used, these cumulative returns are annualized by dividing by T. Averaging over all such intervals gives the average annualized return.

(8)

If there are many measurement intervals of length T, the average capture ratios are approximated by their probability limits. Roughly, these probability limits can be viewed as the values that realized capture ratios fluctuate around. Thus, they provide guidance about the true capture ratios underlying a realized capture ratio. If capture ratios' probability limits indicate that capture ratios are problematic, realized capture ratios are likely to be more so.

The capture ratios' probability limits are as follows.[8]

$$P\lim C_U = \frac{E\left(\frac{R_{\pi T}}{T} \mid R_{\mu T} > 0\right)}{E\left(\frac{R_{\mu T}}{T} \mid R_{\mu T} > 0\right)} \tag{9}$$

$$P\lim C_D = \frac{E\left(\frac{R_{\pi T}}{T} \mid R_{\mu T} < 0\right)}{E\left(\frac{R_{\mu T}}{T} \mid R_{\mu T} < 0\right)} \tag{10}$$

Denote the standard deviation of the benchmark's disturbance term by σ_{ε_μ}. The appendix shows that, for the model above, the probability limits of the upside capture ratio and the downside capture ratio are:

[8] Probability limits are obtained by replacing each quantity's value with its expected value. This is justified by the Law of Large Numbers. The capture ratios' probability limits are not the capture ratios' expected values, since, for a ratio, the numerator's expected value divided by the denominator's expected value is not generally equal to the expected value of the ratio. Our preference for using probability limits for the capture ratios is because they are what the capture ratios approach as the sample size approaches infinity.

$$P\lim C_U = \frac{\left(\alpha_\pi + \beta_\pi \alpha_\mu\right) + \beta_\pi \left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}}{\alpha_\mu + \left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}} \tag{11}$$

$$P\lim C_D = \frac{\left(\alpha_\pi + \beta_\pi \alpha_\mu\right) - \beta_\pi \left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}}{\alpha_\mu - \left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}} \tag{12}$$

In Equations (11) and (12), $n(x)$ denotes the standard normal density, with mean 0 and standard deviation 1, and $N(x)$ denotes the standard cumulative normal distribution.

Equations (11) and (12) show that upside and downside capture ratios depend strongly on the measurement interval length, beta, and alpha.

III. The Behavior of the Upside Capture Ratio with respect to Measurement Interval Length, Beta, and Alpha.

Refer to the upside capture ratio formula in Equation (11).

As the measurement interval length approaches 0.

As the measurement interval length, T, approaches 0, $\left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right)$ gets very large, $n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)$ approaches 0.3989, and $N\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)$ approaches 0.5. Thus the coefficient of β_π gets very large and the terms $\left(\alpha_\pi + \beta_\pi \alpha_\mu\right)$ in the numerator and α_μ in the denominator of the upside capture ratio can be ignored. This implies that the upside capture ratio's probability limit approaches β_π as the measurement interval length approaches 0.

$$\lim_{T \to 0} P \lim C_U = \beta_\pi \tag{13}$$

Equation (13) shows that upside capture ratios computed using short measurement intervals are dominated by beta. If two portfolios have different betas, the portfolio with the highest beta will have the highest upside capture ratio for short enough measurement interval lengths. Consequently, upside capture ratios computed over short measurement intervals are essentially useless for comparing managers' alphas. A high beta investment manager will look better than a low beta investment manager, no matter their alphas.

This makes sense because, over many short measurement intervals:

- The portfolio's trend (which reflect the portfolio's beta and alpha) and benchmark's trend (which reflects the benchmark's beta), are proportional to the measurement interval's length.
- The portfolio's and benchmark's disturbance terms are proportional to the square root of the measurement interval's length.
- The portfolio's and benchmark's trends get smaller faster than their disturbance terms, as the measurement interval's length approaches 0.
- Only the portfolio's and benchmark's disturbance terms become important.
- The portfolio's own disturbance term, ε_π, averages out over many short measurement intervals, independently of the benchmark's returns.
- The benchmark's disturbance term, ε_μ, does not tend to average out over short intervals with only positive benchmark returns.

- Only the beta source disturbance terms, $\beta_\pi \varepsilon_{\mu T}$,and $\varepsilon_{\mu T}$, are material in both the numerator and denominator of the upside capture ratio.

- The ratio of the portfolio's to the benchmark's average disturbance terms is determined by the portfolio's beta, $\frac{\beta_\pi \bar{\varepsilon}_{\mu T}}{\bar{\varepsilon}_{\mu T}} = \beta_\pi$.

As the measurement interval length approaches infinity.

As the measurement interval length, T , approaches infinity, $\left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right)$ gets very small, $n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)$ approaches 0.0, and $N\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)$ approaches 1. Thus the coefficient of β_π gets very small and the terms $\left(\alpha_\pi + \beta_\pi \alpha_\mu\right)$ in the numerator and α_μ in the denominator of the upside capture ratio are the only ones that matter. This implies that the upside capture ratio's probability limit approaches $\beta_\pi + \frac{\alpha_\pi}{\alpha_\mu}$ as the measurement interval approaches infinity.

$$\lim_{T \to \infty} P \lim C_U = \frac{\alpha_\pi + \beta_\pi \alpha_\mu}{\alpha_\mu} = \beta_\pi + \frac{\alpha_\pi}{\alpha_\mu} \tag{14}$$

Equation (14) shows that upside capture ratios computed using long measurement intervals depend strongly on beta, hence are problematic for comparing managers' alphas. For example, a higher beta can more than offset a lower alpha to produce a higher upside capture ratio and a lower beta can more than offset a higher alpha to produce a lower upside capture ratio. A high beta investment manager will tend to look better than a low beta investment manager.

This makes sense because, over many long measurement intervals:

- The portfolio's and benchmark's trends, determined by the portfolio's beta and alpha, and the benchmark's alpha, are proportional to the measurement interval's length.

- The portfolio's and benchmark's disturbance terms are proportional to the square root of the measurement interval's length.

- The portfolio's and benchmark's trends get larger faster than their disturbance terms as the measurement interval lengthens.

- Only the portfolio's and benchmark's trend terms become important.

- The ratio of the portfolio's to the benchmark's average trend terms determines the portfolio's upside capture ratio.

Over long measurement intervals, the upside capture ratio converges to a value that is related to the portfolio's beta, the portfolio's expected relative return (its alpha), and the benchmark's expected return. The limiting upside capture ratio is likely to be only moderately greater than the portfolio's beta, because the manager's alpha is likely to be much less than the benchmark's long term return. For example, if the manager's alpha is 2% annually, the benchmark's long-term return is 9% annually, and the portfolio's beta is 0.7, then the portfolio's limiting upside capture ratio is only about 92%. In contrast, if the manager's alpha is zero and the portfolio's beta is 1, the portfolio's limiting upside capture ratio is 100%. These examples suggest that upside capture ratios computed over long measurement intervals are problematic for comparing managers' alphas.

IV. The Behavior of the Downside Capture Ratio with respect to Measurement Interval Length.

Refer to the downside capture ratio formula in Equation (12).

As the measurement interval length approaches 0.

For the same reasons as with the upside capture ratio, the downside capture ratio's probability limit approaches β_π as the measurement interval length approaches 0.

$$\lim_{T\to 0} P\lim C_D = \beta_\pi \tag{15}$$

Downside capture ratios computed using short measurement intervals are dominated by beta, hence are essentially useless for comparing managers' alphas.

As the measurement interval length approaches infinity.

As the measurement interval length, T, approaches infinity, $\left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right)$ gets very small, $n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)$ approaches 0.0, and $N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)$ approaches 0. Thus,

$$\left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right)\frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}$$

is undefined at 0, and its limit must be evaluated by applying L'hopital's rule. The result is that the downside capture ratio's probability limit becomes increasingly negative. However, it also becomes increasingly rare to have a negative benchmark return over a measurement interval.

An example.

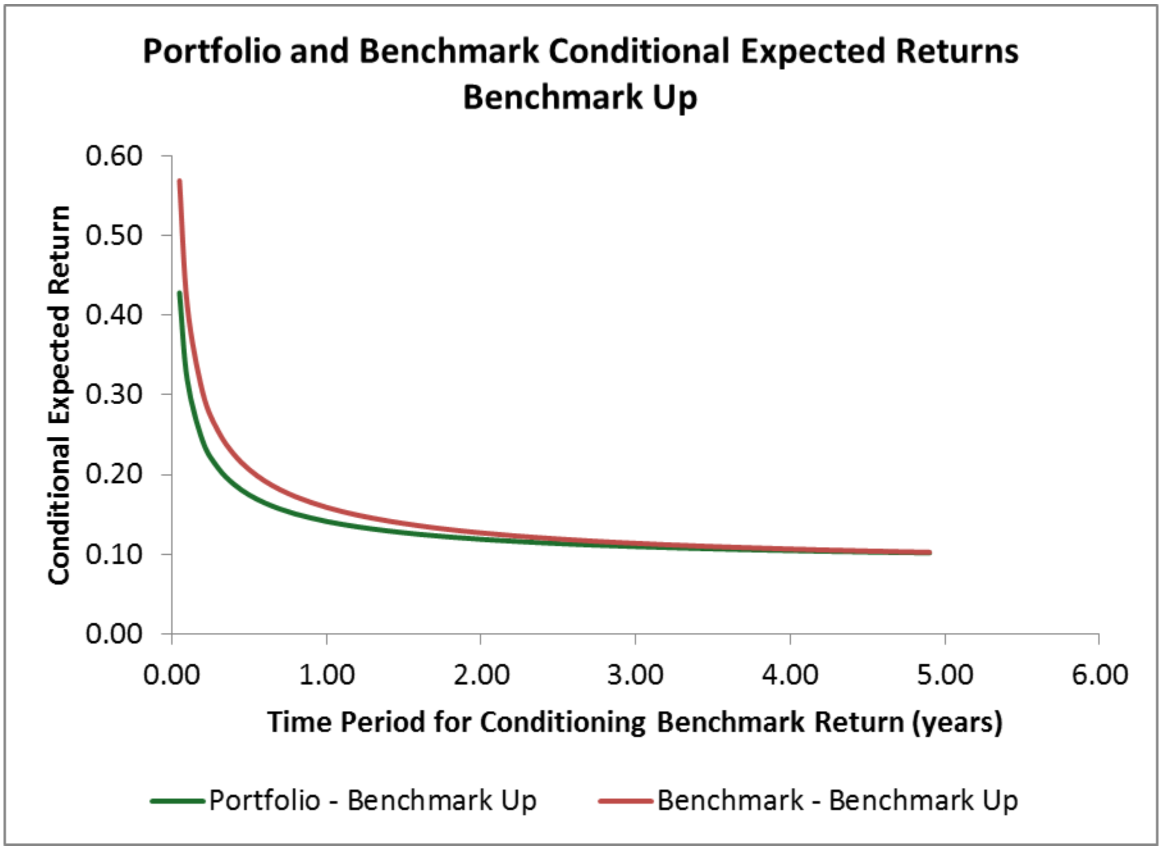
The following example illustrates how a reasonable portfolio's and benchmark's conditional expected returns and upside capture and downside capture probability limits behave with respect to the measurement interval's length.

The benchmark is presumed to have a trend return of 9% annually, consistent with the long term average for stocks in the CRSP universe. The portfolio's beta is assumed to be 0.70, consistent with the betas of low volatility portfolios. The portfolio is presumed to have a relative return of 3% annually, hence a trend return of 9.3% annually. The benchmark's standard deviation of return is assumed to be 15% annually.

$$\alpha_\mu = 0.09$$
$$\sigma_{\varepsilon_\mu} = 0.15$$
$$\alpha_\pi = 0.03$$
$$\beta_\pi = 0.70$$

Figure 1 presents the portfolio's and benchmark's expected annualized returns, conditional on a positive benchmark return, for measurement intervals of from close to 0 to 5 years. Both the portfolio's and benchmark's expected annualized returns are always positive and decrease as the measurement interval's length increases.

Figure 1



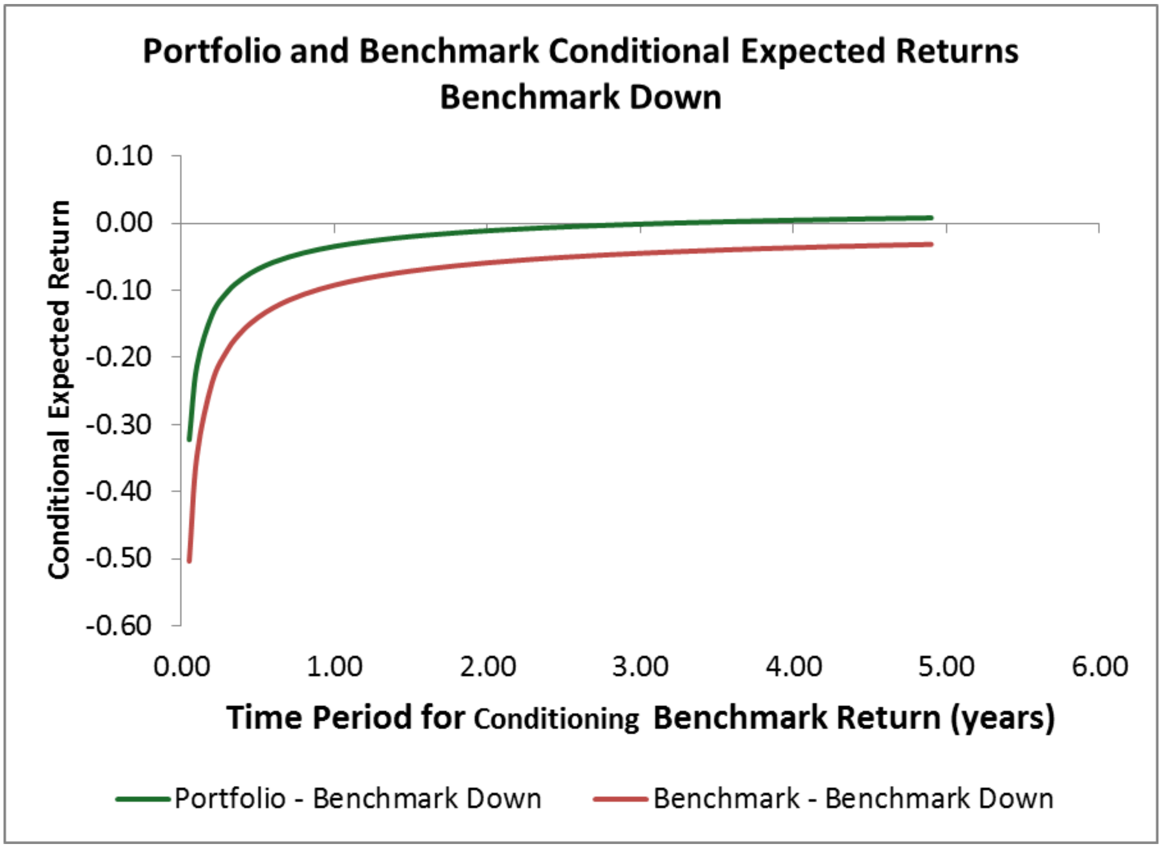
Over short measurement intervals, the portfolio's and benchmark's expected annualized returns are positive and large. This reflects the dominance of randomness over trend over short measurement intervals and annualizing short interval returns. Since the portfolio's beta is materially less than 1 and there is little time for its alpha to contribute much, the portfolio's expected annual return is less than the benchmark's over short measurement interval lengths.

As the measurement interval length is increased, the portfolio's alpha makes up more and more of the shortfall due to its low beta. The portfolio's five year expected annual return is approximately the same as the benchmark's.

Over long measurement intervals, the expected annualized returns are approximately the portfolio's and benchmark's true trend returns. This reflects the dominance of trend over randomness and the low probability of a negative benchmark return over long measurement intervals.

Figure 2 presents the portfolio's and benchmark's expected annualized returns, conditional on a negative benchmark return, for measurement intervals of from close to 0 to 5 years. Both the portfolio's and benchmark's expected annualized returns increase as the measurement interval's length increases. However, while the benchmark's expected annualized return is always negative, because it is conditioned on a negative return, the portfolio's expected annualized return is positive for sufficiently long measurement intervals.

Figure 2



Over short measurement intervals, the portfolio's and benchmark's expected annualized returns are negative and large. This reflects the dominance of randomness over trend over short measurement intervals and annualizing short interval returns. Since the portfolio's beta is materially less than 1, the portfolio's expected annual return is greater than the benchmark's over short measurement interval lengths.

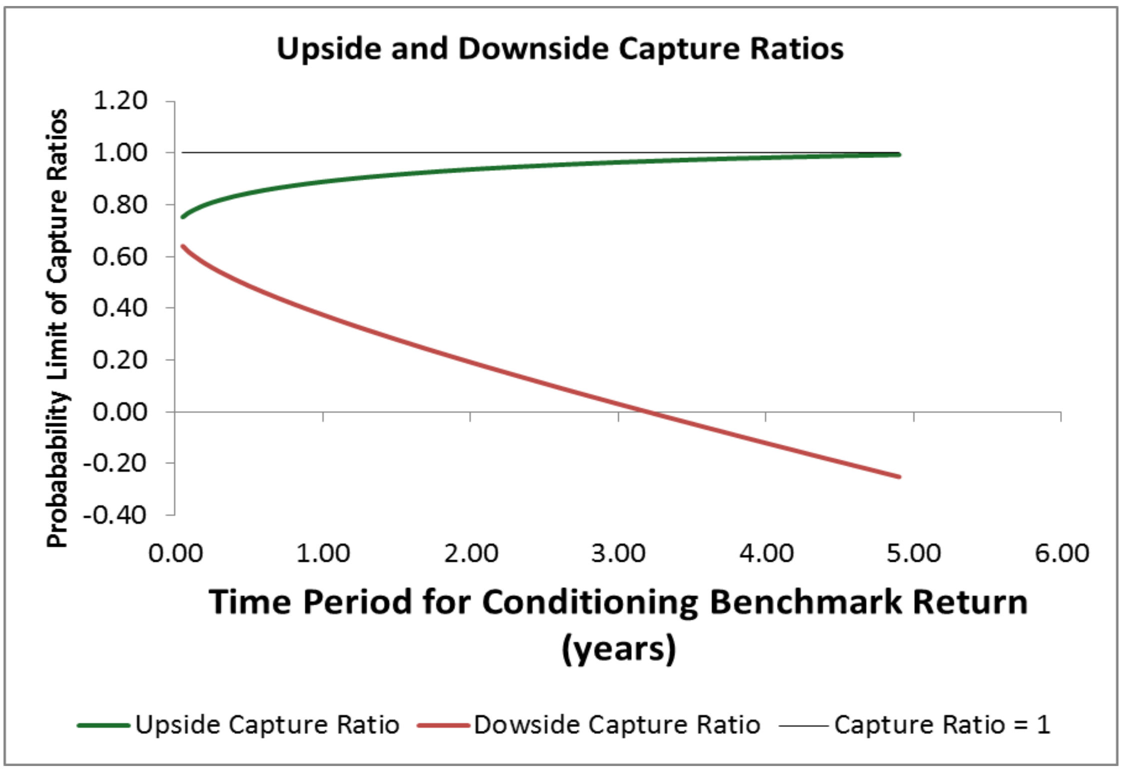
As the measurement interval length is increased, the portfolio's alpha adds more and more to the advantages provided by its low beta. The portfolio's five year expected annual return is positive, despite the benchmark's being negative.

Over long measurement intervals, the benchmark's expected annualized return is negative and small. However, the portfolio's expected annualized return is positive.

Both behaviors reflect the dominance of trend over randomness. For the benchmark, it is increasingly unlikely that it will be down or, if it is down, that it will be materially down. For the portfolio, its superior trend return is likely to offset a small negative benchmark return. For very long measurement intervals, the portfolio's expected annualized return approximates its trend return.

Figure 3 presents the portfolio's upside capture ratio and downside capture ratio for measurement intervals of from close to 0 to 5 years. The upside capture ratio increases as the measurement interval's length increases. The downside capture ratio decreases as the measurement interval's length increases, and becomes negative for sufficiently long measurement intervals.

Figure 3



Both the upside capture ratio and downside capture ratio approximate the portfolio's beta for short measurement interval lengths. Because short measurement interval length capture ratios are dominated by beta, they provide little information about alpha.

As the measurement interval length is increased, the portfolio's alpha makes up more and more of the shortfall due to its low beta, and the upside capture ratio approaches 1.03 (in this example).

Conclusion.

When investors compare the upside and downside capture ratios of different investment managers' portfolios, the measurement interval lengths may differ from one manager to another. This is a problem, because both capture ratios depend on the measurement interval's length. The longer the measurement interval, the more attractive the capture ratios tend to be for a talented manager, other things equal. However, even long measurement interval capture ratios are largely dependent on betas, hence are unreliable indicators of alpha. The short measurement intervals often used by investors tend to be wholly dominated by beta, hence these capture ratios are essentially useless as indicators of alpha.

This paper presents a model whereby the upside capture ratio increases as the length of the measurement interval increases and the downside capture ratio decreases as the length of the measurement interval increases, for a talented manager. Behavior consistent with the model is also observed in actual data. Thus, other things equal, a comparison between the capture ratios of two talented managers using different measurement interval lengths favors the manager with the longest measurement interval.

The paper's model also implies that capture ratios computed using short measurement intervals are dominated by betas, not alphas, and that capture ratios computed using long measurement intervals are strongly dependent on betas, in addition to alphas. For example, a manager with a lower beta and higher alpha than another manager may have a lower upside capture ratio than the other manager, due to his lower beta.

The material dependence of upside and downside capture ratios on the measurement interval length, and portfolios' betas and alphas implies that comparing managers' capture ratios, even using the same measurement interval length, is problematic for assessing managers' skill.

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Appendix (available from the authors)

Upside capture ratio.

The condition that the benchmark's return in a measurement interval of length T is positive is a condition on the benchmark's disturbance term.

$$\varepsilon_{\mu T} > -\alpha_{\mu} T \tag{16}$$

The probability that the benchmark's return is positive is:

$$P\left(R_{\mu T} > 0\right) = 1 - F_{\varepsilon_{\mu T}}\left(-\alpha_{\mu} T\right) = N\left(\frac{\alpha_{\mu} T}{\sigma_{\varepsilon_{\mu T}}}\right) = N\left(\frac{\alpha_{\mu} T}{\sigma_{\varepsilon_{\mu}}\sqrt{T}}\right) = N\left(\left(\frac{\alpha_{\mu}}{\sigma_{\varepsilon_{\mu}}}\right)\sqrt{T}\right) \tag{17}$$

Equation (17) shows that the probability that the benchmark's return is positive increases and approaches 1 as the length of the measurement interval increases, if $\alpha_{\mu} > 0$.[9]

Noise dominates in the short run.

$$\lim_{T \to 0} P\left(R_{\mu T} > 0\right) = 0.5 \tag{18}$$

Signal dominates in the long run.

$$\lim_{T \to \infty} P\left(R_{\mu T} > 0\right) = 1 \tag{19}$$

If $\alpha_{\mu} = 0$, then there is no signal and:

$$P\left(R_{\mu T} > 0\right) = 0.5 \tag{20}$$

Evaluate the required expectations for the benchmark and the portfolio. For the upside capture ratio, we have:

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9 If $\alpha_{\mu} < 0$, then the probability that the benchmark's return is positive decreases and approaches 0 as the length of the time interval increases. This is unrealistic, since there is no incentive for investors to hold the benchmark.

$$E\left(\frac{R_{\mu T}}{T} \mid R_{\mu T} > 0\right) = E\left(\alpha_\mu + \left(\frac{1}{T}\right)\varepsilon_{\mu T} \mid \varepsilon_{\mu T} > -\alpha_\mu T\right) \tag{21}$$

$$E\left(\frac{R_{\mu T}}{T} \mid R_{\mu T} > 0\right) = \alpha_\mu + \left(\frac{1}{T}\right)E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} > -\alpha_\mu T\right) \tag{22}$$

$$E\left(\frac{R_{\pi T}}{T} \mid R_{\mu T} > 0\right) = E\left(\left(\alpha_\pi + \beta_\pi \alpha_\mu\right) + \left(\frac{1}{T}\right)\left(\beta_\pi \varepsilon_{\mu T} + \varepsilon_{\pi T}\right) \mid \varepsilon_{\mu T} > -\alpha_\mu T\right) \tag{23}$$

Remember that:

$$E\left(\varepsilon_{\pi T} \mid \varepsilon_{\mu T} > -\alpha_\mu T\right) = E\left(\varepsilon_{\pi T}\right) = 0 \tag{24}$$

This essentially means that none of these expressions will depend on the residual disturbance term of the portfolio. The intuition here is that this term has a zero mean and is assumed to be uncorrelated with the benchmark's disturbance term.

$$E\left(\frac{R_{\pi T}}{T} \mid R_{\mu T} > 0\right) = \left(\alpha_\pi + \beta_\pi \alpha_\mu\right) + \beta_\pi \left(\frac{1}{T}\right)E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} > -\alpha_\mu T\right) \tag{25}$$

$$P\lim C_U = \frac{\left(\alpha_\pi + \beta_\pi \alpha_\mu\right) + \beta_\pi \left(\frac{1}{T}\right)E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} > -\alpha_\mu T\right)}{\alpha_\mu + \left(\frac{1}{T}\right)E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} > -\alpha_\mu T\right)} \tag{26}$$

To evaluate $E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} > -\alpha_\mu T\right)$, use the property of a truncated normal distribution, which says that if $X \sim N\left(\mu, \sigma^2\right)$, X lies in the interval $\left(a, \infty\right)$, and $-\infty < a < \infty$, then $E\left(X \mid X > a\right) = \mu + A\left(\frac{a-\mu}{\sigma}\right)\sigma$, where $A\left(x\right) = \frac{n\left(x\right)}{1-N\left(x\right)}$, and $n\left(x\right)$ and $N\left(x\right)$ denote the normal distribution and the cumulative normal distribution, respectively. This leads to the following result.

$$P\lim C_U = \frac{\left(\alpha_\pi + \beta_\pi \alpha_\mu\right) + \beta_\pi \left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}}{\alpha_\mu + \left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}}$$

(27)

$$\lim_{T \to 0} P\lim C_U = \beta_\pi$$

(28)

Assuming that $\alpha_\mu > 0$:

$$\lim_{T \to \infty} P\lim C_U = \frac{\alpha_\pi + \beta_\pi \alpha_\mu}{\alpha_\mu} = \beta_\pi + \frac{\alpha_\pi}{\alpha_\mu}$$

(29)

Downside capture ratio.

The condition that the benchmark's return in a measurement interval of length T is negative is a condition on the benchmark's disturbance term.

$$\varepsilon_{\mu T} < -\alpha_\mu T$$

(30)

The probability that the benchmark's return is negative is:

$$P\left(R_{\mu T} < 0\right) = F_{\varepsilon_{\mu T}}\left(-\alpha_\mu T\right) = N\left(-\frac{\alpha_\mu T}{\sigma_{\varepsilon_{\mu T}}}\right) = N\left(-\frac{\alpha_\mu T}{\sigma_{\varepsilon_\mu}\sqrt{T}}\right) = N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)$$

(31)

Equation (31) shows that the probability that the benchmark's return is negative decreases and approaches 0 as the measurement interval's length increases, if $\alpha_\mu > 0$.

Noise dominates in the short run.

$$\lim_{T \to 0} P\left(R_{\mu T} < 0\right) = 0.5 \tag{32}$$

Signal dominates in the long run.

$$\lim_{T \to \infty} P\left(R_{\mu T} < 0\right) = 0 \tag{33}$$

If $\alpha_{\mu} = 0$, then there is no signal and:

$$P\left(R_{\mu T} < 0\right) = 0.5 \tag{34}$$

Evaluate the required expectations for the benchmark and the portfolio. For the downside capture ratio, we have:

$$E\left(\frac{R_{\mu T}}{T} \mid R_{\mu T} < 0\right) = E\left(\alpha_{\mu} + \left(\frac{1}{T}\right)\varepsilon_{\mu T} \mid \varepsilon_{\mu T} < -\alpha_{\mu} T\right) \tag{35}$$

$$E\left(\frac{R_{\mu T}}{T} \mid R_{\mu T} < 0\right) = \alpha_{\mu} + \left(\frac{1}{T}\right) E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} < -\alpha_{\mu} T\right) \tag{36}$$

$$E\left(\frac{R_{\pi T}}{T} \mid R_{\mu T} < 0\right) = E\left(\left(\alpha_{\pi} + \beta_{\pi}\alpha_{\mu}\right) + \left(\frac{1}{T}\right)\left(\beta_{\pi}\varepsilon_{\mu T} + \varepsilon_{\pi T}\right) \mid \varepsilon_{\mu T} < -\alpha_{\mu} T\right) \tag{37}$$

$$E\left(\frac{R_{\pi T}}{T} \mid R_{\mu T} < 0\right) = \left(\alpha_{\pi} + \beta_{\pi}\alpha_{\mu}\right) + \beta_{\pi}\left(\frac{1}{T}\right) E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} < -\alpha_{\mu} T\right) \tag{38}$$

$$P\lim C_{D} = \frac{\left(\alpha_{\pi} + \beta_{\pi}\alpha_{\mu}\right) + \beta_{\pi}\left(\frac{1}{T}\right) E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} < -\alpha_{\mu} T\right)}{\alpha_{\mu} + \left(\frac{1}{T}\right) E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} < -\alpha_{\mu} T\right)} \tag{39}$$

To evaluate $E\left(\varepsilon_{\mu T} \mid \varepsilon_{\mu T} < -\alpha_{\mu} T\right)$, use the property of truncated normal distribution, which says that if $X \sim N\left(\mu, \sigma^2\right)$, X lies in the interval $\left(-\infty, b\right)$, and $-\infty < b < \infty$, then $E\left(X \mid X < b\right) = \mu - \left(\frac{n\left(B\right)}{N\left(B\right)}\right)\sigma$, where $B = \frac{b-\mu}{\sigma}$, and $n\left(B\right)$ and $N\left(B\right)$ denote the normal distribution and the cumulative normal distribution, respectively. This leads to the following result.

(40)

$$P\lim C_D = \frac{\left(\alpha_\pi + \beta_\pi \alpha_\mu\right) - \beta_\pi \left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}}{\alpha_\mu - \left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}}$$

(41)

$$\lim_{T \to 0} P\lim C_D = \beta_\pi$$

(42)

As $T \to \infty$, $\left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \to 0$, $n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right) \to 0$, and $N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right) \to 0$. Therefore, the limiting value of

$$\left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right) \frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}$$

(43)

must be evaluated using L'hopital's rule.

Expression (43) can be rewritten as:

$$\alpha_{\mu}\frac{1}{\left(\left(\frac{\alpha_{\mu}}{\sigma_{\varepsilon_{\mu}}}\right)\sqrt{T}\right)}\frac{n\left(\left(\frac{\alpha_{\mu}}{\sigma_{\varepsilon_{\mu}}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_{\mu}}{\sigma_{\varepsilon_{\mu}}}\right)\sqrt{T}\right)}=\alpha_{\mu}\frac{1}{x}\frac{n(x)}{N(-x)} \tag{44}$$

L'hopital's rule then implies the following.

$$\lim_{x\to\infty}\alpha_{\mu}\frac{1}{x}\frac{n(x)}{N(-x)}=\lim_{x\to\infty}\alpha_{\mu}\frac{1}{x}\frac{n'(x)}{N'(-x)} \tag{45}$$

$$N(-x)=1-N(x) \tag{46}$$

$$N'(-x)=-N'(x)=-n(x) \tag{47}$$

$$n'(x)=-xn(x) \tag{48}$$

$$\lim_{x\to\infty}\alpha_{\mu}\frac{1}{x}\frac{n(x)}{N(-x)}=\lim_{x\to\infty}\alpha_{\mu}\frac{x}{x}\frac{n(x)}{n(x)}=\alpha_{\mu} \tag{49}$$

$$\lim_{T\to\infty}\left[\left(\frac{\sigma_{\varepsilon_{\mu}}}{\sqrt{T}}\right)\frac{n\left(\left(\frac{\alpha_{\mu}}{\sigma_{\varepsilon_{\mu}}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_{\mu}}{\sigma_{\varepsilon_{\mu}}}\right)\sqrt{T}\right)}\right]=\alpha_{\mu} \tag{50}$$

Suppose that the quantity

$$\left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right)\frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}$$

has a negative slope as T approaches infinity. Then, for large values of T,

$$\left(\frac{\sigma_{\varepsilon_\mu}}{\sqrt{T}}\right)\frac{n\left(\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}{N\left(-\left(\frac{\alpha_\mu}{\sigma_{\varepsilon_\mu}}\right)\sqrt{T}\right)}=\alpha_\mu+\delta$$

where $\delta >\approx 0$.

Consequently:

$$P\lim C_D \approx \frac{\alpha_\pi+\beta_\pi\alpha_\mu-\beta_\pi\alpha_\mu}{\alpha_\mu-\alpha_\mu-\delta}=-\frac{\alpha_\pi}{\delta}<0 \tag{51}$$

In this case, the downside capture ratio becomes increasingly negative as $T\to\infty$.

The negative slope is established as follows.

$$\frac{d}{dx}\left(\frac{1}{x}\left[\frac{n(x)}{1-N(x)}\right]\right)=-\frac{1}{x^2}\left[\frac{n(x)}{1-N(x)}\right]+\frac{1}{x}\left[\frac{n'(x)}{1-N(x)}\right]+\frac{1}{x}\left[\frac{n(x)}{1-N(x)}\right]^2 \tag{52}$$

$$\frac{d}{dx}\left(\frac{1}{x}\left[\frac{n(x)}{1-N(x)}\right]\right)=\frac{1}{x}\left(\frac{n(x)}{1-N(x)}\right)\left[-\frac{1}{x}+\frac{n'(x)}{n(x)}+\left(\frac{n(x)}{1-N(x)}\right)\right] \tag{53}$$

Equation (49) implies that:

$$\lim_{x\to\infty}\frac{1}{x}\frac{n(x)}{N(-x)}=\lim_{x\to\infty}\frac{1}{x}\frac{n'(x)}{N'(-x)}=1 \tag{54}$$

In addition:

$$\frac{n'(x)}{n(x)} = -x \tag{55}$$

Therefore:

$$\lim_{x\to\infty}\frac{d}{dx}\left(\frac{1}{x}\left[\frac{n(x)}{1-N(x)}\right]\right) = \lim_{x\to\infty}\left(-\frac{1}{x} - x + x\right) = -\frac{1}{x} < 0 \tag{56}$$