Onedimensional Screening without Single-Crossing: Numerical Guided Approach.

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Abstract

In this paper we solve stylized problems in onedimensional screening without the single-crossing condition. With the aid of numerical optimization tools, we guess which are the relevant binding incentive compatibility constraints. Then we use this information to build a candidate solution that matches the numerical pattern.

Keywords: unidimensional screening, non single-crossing

Palavras chaves: screening unidimensional, no single-crossing.

JEL Codes: D42, D82.

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Revision *1.18*

1. Introduction

When the SMC is violated the local IC constraints are no longer sufficient for implementability and additional (global) IC constraints have to be taken into account.
With the aid of numerical optimization tools, we guess which are the relevant binding incentive compatibility constraints.

2. Model

We use the Principal-Agent framework to analyse the monopolistic screening problem. In this model, each agent has a quase-linear preference,

$$V(q, t, \theta) = v(q, \theta) - t,$$

were t represents the monetary transfer. The type of consumer $\theta \in \Theta$ is a random variable with a density p positive and continuous. The firm is a profit-maximizing monopolist which can produce any quality $q \in Q \subset \mathbb{R}_+$ incurring in a cost $C(q)$. Q represents the quality spectrum. The monopolist revenue is given by

$$\Pi(q, t) = t - C(q).$$

Using the *Revelation Principle* [2] the mopolist's problem can be stated as choosing the allocation rule $(q, t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ that solves

$$\underset{\{q(\cdot),t(\cdot)\}}{max} \int_{\Theta} \Pi(q(\theta), t(\theta))p(\theta)d\theta, \tag{1}$$

subject to the *Individual-Racionality* constraints

$$v(q(\theta), \theta) - t(\theta) \geq 0 \ \forall \theta \in \Theta,$$

and the *Incentive Compatibility* constraints

$$\theta \in \underset{\theta' \in \Theta}{argmax}\{v(q(\theta'), \theta) - t(\theta')\} \ \forall \theta \in \Theta.$$

Remark 1. *The 'Taxation Principle'* [3] *states that any allocation (q, t) satisfying the Incentive Compatibility constraints can be implemented by a nonlinear tariff $T : Q = q(\Theta) \to \mathbb{R}$ where*

$$T(q(\theta)) = t(\theta), \ \forall \theta \in \Theta.$$

[2] The Revelation Principle has been enunciated in Gibbard [3]

[3] The principle can be found in Guesnerie [4], Hammond [5] and Rochet [10].

One of the greatest difficulties related to the monopolist's problem is how to deal with the IC constraints. In general, the binding IC constraints may be determined only endogenously which makes it a rather difficult task.

Definition 1. *The single-crossing or Spence-Mirrlees condition (SMC) is the constant sign of the cross partial derivative with respect to decision and type*

$$v_{q\theta} > 0 \ \ on\ Q{\times}\Theta \tag{CS_+}$$

or

$$v_{q\theta} < 0 \ \ on\ Q{\times}\Theta \tag{CS_-}$$

We assume that the allocation rule (q, t) is bounded and incentive compatible. The informational rent $V : \Theta \to \mathbb{R}_+$ is given by

$$V(\theta) = v(q(\theta), \theta) - t(\theta), \tag{2}$$

and $T : q(\Theta) \to \mathbb{R}$ is the tariff resulting from the *Revelation Principle*.

Lemma 1. *The tariff T and the informational rent V are Lipschitz continuous.*

The Lemma 1 guarantees that T and V are a.e differentiable.

Lemma 2.

(i) *If V is differentiable at $\theta \in int(\Theta)$ and $q \in q(\theta)$, then*

$$V'(\theta) = v_\theta(q, \theta). \tag{3}$$

(ii) *If T is differentiable at $q \in q(\theta) \cap int(q(\Theta))$, then*

$$T'(\theta) = v_q(q, \theta). \tag{4}$$

Lemma 2 (ii) is the first order condition of the $\theta-$customer maximization problem

$$\theta \in \max_{q\in Q}\{v(q, \theta) - T(q)\}.$$

Remark 2. *With the SMC a dcision is incentive compatible if and only if is monotonic. However, without the SMC we many have a nonmonotonic incentive compatible decision.*

3. The Monopolist's Problem

We assume the following conditions:

(i) $v(q, \theta) \in C^3$,
(ii) $v_{qq} < 0$ and $v_\theta > 0$
(iii) $v_{q^2\theta} > 0$ and $v_{q\theta^2} > 0$,

(iv) $C(0)=0$, $C'(q)>0$ and $C''(q)<0$

Suppose that the alocation rule (q,t) is incentive compatible and define the informational rent by $V(\theta)=v(q(\theta),\theta)-t(\theta)$. Let us now deduce the monopolist's maximization problem, using the same derivation as Mussa and Rosen. From the definition of the informational rent V we can write the monetary transfer as $t(\theta)=v(q(\theta),\theta)-V(\theta)$ and then substitute it in equation (1). The result is the following problem

$$\max_{\{q(\cdot)\}} \int_{\Theta} \{v(q(\theta),\theta)-C(q(\theta))-V(\theta)\}d\theta. \tag{5}$$

Using the Lemma2 and integration by parts, we get $V(\theta)=\int_{\underline{\theta}}^{\theta} v_\theta(q(\theta),s)ds$ [4] and we can rewrite the monopolist's problem as

$$\max_{\{q(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} f(q(\theta),\theta)p(\theta)d\theta, \tag{Π_R}$$

were $f(q,\theta)=v(q(\theta),\theta)-C(q(\theta))+\dfrac{P(\theta)-1}{p(\theta)}v_\theta(q(\theta),\theta)$.

This problem is called the relaxed version of the mopolist's maximization problem. The Euler's equation gives the necessary condition for an extremum of problem (Π_R).

$$f_q(q,\theta)=0 \tag{6}$$

Let us denote the solution of equation (6) by $Q_1(\theta)$. If $Q_1(\theta)$ satisfies the constraints then it is the solution of the monopolist's problem.

Remark 3. *In many situations the solution of problem (Π_R) is far from being incentive compatible. We are going to derive the monopolist's optimization problem in cases were the globals (IC) can be taken into account.*

Unlike the case with SMC, a decision function satisfying the first- and second-order conditions of the customer's maximization problem may not be implementable.

Araujo and Moreira [1] propose the following generalization of the (SMC):

[4] By assumption (ii), V is increasing and we can set $V(\underline{\theta})=0$, eliminating the IR constraints.

(AM1) $\quad v_{q\theta}(\xi,\theta)=0$ defines a decreasing function $Q_0:\Theta\to\mathbb{R}_+$ such that

$$\forall\,\theta\in\Theta\ v_{q\theta}(\xi,\theta)\geq 0\Leftrightarrow \xi\geq Q_0(\theta). \tag{7}$$

So the curve Q_0 divides the (θ,q) plane in two regions: CS_+, where $v_{q\theta}>0$ and CS_-, where $v_{q\theta}<0$.

In this case, without the Single-Crossing Condition, globals incentive compatibility constrains can be binding.

Example, in the horizontal case ilustrated in Figure 1.

In this case, we will impose the global incentive compatibility constrains relating θ_d and θ_2 customers. The same procedure used in [2] [5] impose:

$$v(q(\theta_2),\theta_2)-t(\theta_2)\geq v(q(\theta_d),\theta_2)-t(\theta_d). \tag{8}$$

Lemma 3. *Consider an allocation rule $(q,t):\Theta\to\mathbb{R}_+\times\mathbb{R}$. If (q,t) is incentive compatible then*

$$\int_{\hat\theta}^{\theta}\int_{q(\hat\theta)}^{q(\tilde\theta)} v_{q\theta}(\tilde q,\tilde\theta)d\tilde q d\tilde\theta\geq 0,\ \ \forall\,\theta,\hat\theta\in\Theta \tag{9}$$

Notice that

$$0\leq\int_{\theta_d}^{\theta_2}\int_{q(\theta_d)}^{q(\tilde\theta)} v_{q\theta}(\tilde q,\tilde\theta)d\tilde q d\tilde\theta=\int_{\theta_d}^{\theta_2}\int_{q_1}^{q_2} v_{q\theta}(\tilde q,\tilde\theta)d\tilde q d\tilde\theta+\int_{\theta_d}^{\theta_2}\int_{q_2}^{q(\tilde\theta)} v_{q\theta}(\tilde q,\tilde\theta)d\tilde q d\tilde\theta \tag{10}$$

we can rewrite the integrals on the right hand side and determinate the isoperimetric condition (ISO).

$$0\leq\int_{\theta_d}^{\theta_2} v_\theta(q(s),s)ds-\int_{\theta_d}^{\theta_2} v_\theta(q(\theta_d),s)ds\ \ (ISO) \tag{11}$$

In the interval $[\theta_d,\theta_2]$, we will optimally choose $q(\theta)$ such that the condition (ISO) is fulfilled. So we have the following isoperimetric problem

$$\max_{\{q(\cdot)\}}\int_{\theta_d}^{\theta_2} f(q(\theta),\theta)d\theta \tag{12}$$
$$s.t\ \ (ISO)$$

The following theorem can be founded in [2].

[5] i.e that the θ_2-type does not envy θ_d-type

Theorem 1. *The solution of (12) is characterized by the following condition*

$$f_q(q,\theta) + \lambda v_{q\theta}(q,\theta) = 0 \tag{13}$$

were λ is chosen to satisfy the condition (ISO) with equality.

4. Numerical Optimization

The monopolist's problem (Π) can be approximated by their discretization and be solved numerically as a monopolist's problem with a finite number of types. We use the language AMPL and the solver Knitro for to obtain information about the solution to the problem with incentive compatibility constraints, more precisely, information about the global I.C that must be binding. This information together with the Theorem 1 allow us to find the optimal decision in several cases.

5. Examples

5.1. Q_0 Horizontal

- **Q_1 increasing**

  Given a parameter $a \in [0, \frac{\sqrt{2}}{2}]$, suppose that the types's θ are uniformly distributed in $[0, 1]$ and their preferences are given by

  $$v(q,\theta,a) = \theta(\frac{q^2}{2} - aq) + \theta + q$$

  and the costs

  $$C(q,\theta,a) = -1 + 2\theta + q^2\theta + q(1 + a - \theta - 2a\theta).$$

  Using the Euler Equation (6), the relaxed solution to the monopolist's problem is $Q_1(\theta) = \theta$ and the curve Q_0 separating the regions CS_+ and CS_- is given by a constant $Q_0(\theta) = a$,

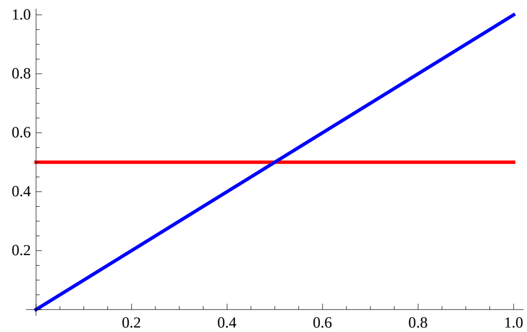


Figure 1.

The numerical solution found by Knitro for 100 types is given by Figure 2.

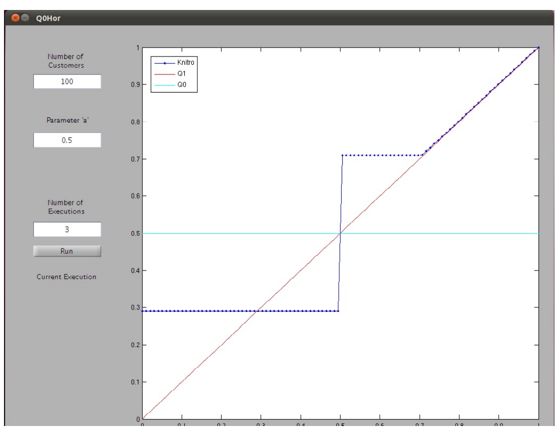


Figure 2.

The numerical approach suggests that the optimal decision is a discontinuous function that consists of three parts: two flat pieces with constant decision, q_1 and q_2, and part of the curve relaxed $Q_1(\theta)$.

Thus, the optimal decision in this case, has the form

$$q(\theta, a) = q_1 \mathbb{I}_{[0,\theta_d]}(\theta, a) + q_2 \mathbb{I}_{[\theta_d \leq \theta \leq \theta_2]}(\theta, a) + \theta \mathbb{I}_{[\theta_2 \leq \theta \leq 1]}(\theta, a)$$

were $\theta_2 = q_2$.

Monopolist's profit can be rewritten as

$$\Pi(q, \theta, a) = \int_0^{\theta_d} f(q_1, \theta) d\theta + \int_{\theta_d}^{q_2} f(q_2, \theta) d\theta + \int_{q_2}^{1} f(\theta, \theta) d\theta \qquad (14)$$

and the global constraint that we consider is that the type θ_2 does not envy θ_d-type, which can be written, from (ISO), as

$$\int_{\theta_d}^{\theta_2} v_\theta(q_2, s, a)ds - \int_{\theta_d}^{\theta_2} v_\theta(q(\theta_d), s, a)ds \geq 0$$

or, in this case

$$\int_{\theta_d}^{\theta_2} v_\theta(q_2, s, a)ds - \int_{\theta_d}^{\theta_2} v_\theta(q_1, s, a)ds = 0 \tag{15}$$

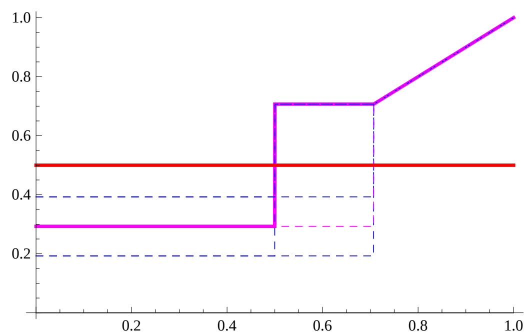


Figure 3.

The equation (15) implies the relationship

$$q_1 = 2a - q_2,$$

moreover, the continuity of $f(q, \theta, a)$ implies the additional condition

$$f(q_1, \theta_d, a) = f(q_2, \theta_d, a)$$

which can be rewritten, using (ISO), as

$$f(2a - q_2, \theta_d, a) - f(q_2, \theta_d, a) = 0 \tag{16}$$

obtaining from (16) the value for θ_d,

$$\theta_d = a.$$

Finally, write the profit considering the global constraint (15) as

$$\Pi(q_2, a) = \int_0^a f(2a - q_2, \theta, a)d\theta + \int_a^{q_2} f(q_2, \theta, a)d\theta + \int_{q_2}^1 f(\theta, \theta, a)d\theta \tag{17}$$

In this (horizontal) case, the monopolist's problem without single-crossing was reduced to maximization problem with a single variable

$$\begin{aligned} &\underset{\{q_2\}}{max}\, \Pi(q_2, a) \\ &s.t \;\; a \leq q_2 \leq 1 \end{aligned} \tag{18}$$

where the constraint $0 \leq \theta_d \leq \theta_2 \leq 1$ was rewritten as $a \leq q_2 \leq 1$.

The expected profit $\Pi(q_2, a) = \frac{1}{6}(1-6a^3+6a^2q_2-q_2^3)$ is a strongly concave function in q_2 for all a, then the KKT optimality conditions are necessary and sufficient.

Solving the problem (18) as a *KKT system* we obtain the optimum values for q_1 and q_2

$$q_1(a) = 2a - \sqrt{2}a$$
$$q_2(a) = \sqrt{2}a.$$

So, the optimal decision is given by

$$q(\theta, a) = (2a - \sqrt{2}a)\mathbb{I}_{[0\leq\theta\leq a]} + \sqrt{2}a\mathbb{I}_{[a\leq\theta\leq\sqrt{2}a]} + \theta\mathbb{I}_{[\sqrt{2}a\leq\theta\leq 1]}$$

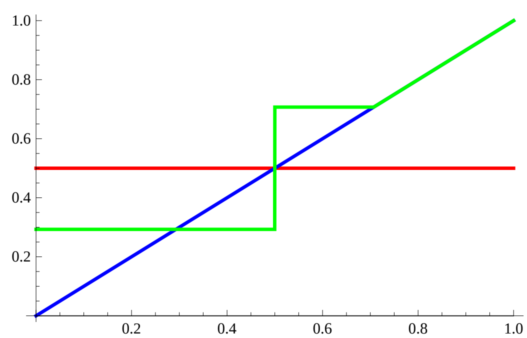
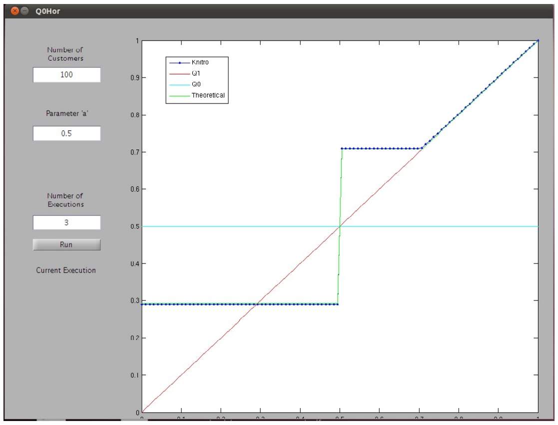


Figure 4.

Figure 5.

and the monopolist's expected profit from $q(\theta, a)$ is

$$\Pi(q_2(a), a) = \frac{1}{6}(1 - 6a^3 + 4\sqrt{2}a^3)$$



Given a parameter $a \in [\frac{3}{10}, \frac{1}{2}]$, consider the types's θ uniformly distributed in $[0, 1]$ with preferences given by

$$v(q, \theta, a) = \theta(-\frac{q^2}{2} + aq) + \theta + q$$

if costs are

$$C(q, \theta, a) = -1 - q^2(-1 + \theta) + 2\theta + q(1 - \theta + a(-1 + 2\theta)),$$

then, we get the same curves Q_0 and Q_1, but the region under the curve Q_0 is CS_+. So, we expect a decision function flat for the higher types.

The numerical solution found by Knitro for 100 types is given by Figure 6.

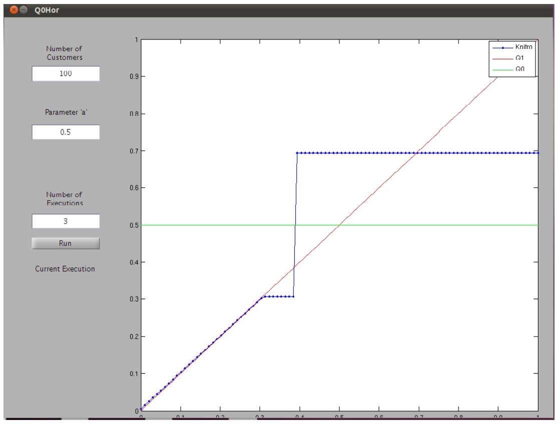
Figure 6.

The numerical approach suggests that the optimal decision is a discontinuous function that consists of three parts: a first part, for the lower types, were the optimal decision is the relaxed curve $Q_1(\theta)$ and other two flat pieces with constant decision, q_1 and q_2, for the higher types.

Thus, the optimal decision in this case, has the form

$$q(\theta, a) = \theta \mathbb{I}_{[0 \leq \theta \leq \theta_1]}(\theta, a) + q_1 \mathbb{I}_{[\theta_1, \theta_d]}(\theta, a) + q_2 \mathbb{I}_{[\theta_d \leq \theta \leq \theta_1]}(\theta, a)$$

were $\theta_1 = q_1$.



Monopolist's profit can be rewritten as

$$\Pi(q,\theta,a) = \int_0^{q_1} f(\theta,\theta)d\theta + \int_{q_1}^{\theta_d} f(q_1,\theta)d\theta + \int_{\theta_d}^{1} f(q_2,\theta)d\theta \tag{19}$$

From the global constraint(ISO) applied to the types θ_1 and θ_d and continuity of f, we get the following relationships

$$q_1 = 2a - q_2$$

and

$$\theta_d = a.$$

The monopolist's profit can be expressed as function of one variable

$$\Pi(q_2,a) = \int_0^{2a-q_2} f(\theta,\theta,a)d\theta + \int_{2a-q_2}^{a} f(2a-q_2,\theta,a)d\theta + \int_a^1 f(q_2,\theta,a)d\theta$$

or

$$\Pi(q_2,a) = \frac{1}{6}(2a^3 - 6a^2q_2 + 6aq_2^2 - q_2(-3+3q_2+q_2^2)),$$

the expected profit is a strongly concave function in q_2 for all $a \in [0,1/2]$, then the KKT optimality conditions are necessary and sufficient.

Solving the problem as a *KKT system* we obtain the optimum values for q_1 and q_2

$$q_1(a) = 1 - \sqrt{2}\sqrt{(-1+a)^2}$$

$$q_2(a) = -1 + \sqrt{2}\sqrt{(-1+a)^2} + 2a.$$

So, the optimal decision is given by

$$q(\theta,a) = \theta\mathbb{I}_{[0\leq\theta\leq q_1(a)]} + q_1(a)\mathbb{I}_{[q_1(a)\leq\theta\leq a} + q_2(a)\mathbb{I}_{[a\leq\theta\leq 1]}$$

and the monopolist's expected profit from $q(\theta,a)$ is

$$\Pi(a) = \frac{1}{6}(-5+4\sqrt{2}\sqrt{(-1+a)^2}+2(9-4\sqrt{2}\sqrt{(-1+a)^2})a+2(-9+2\sqrt{2}\sqrt{(-1+a)^2})a^2+6a^3)$$

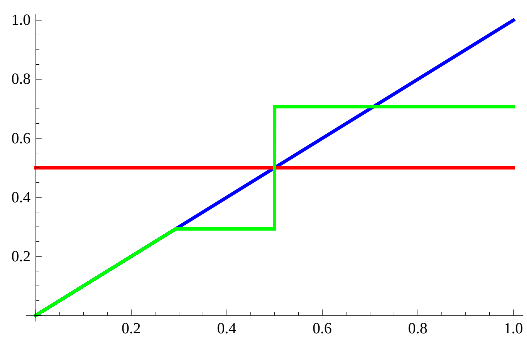


Figure 7.

- Q_1 **decreasing**

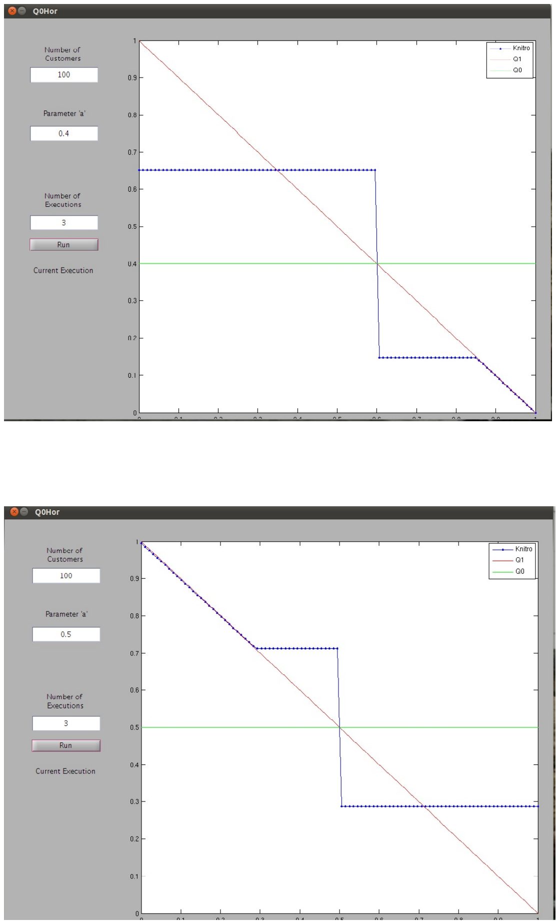
With the following specifications for the distribution of types, preferences and costs we get the same curve $Q_0 = a$ and the relaxed solution $Q_1 = 1-\theta$.

$$v(q,\theta,a) = q + \theta + (-aq + \frac{q^2}{2})\theta$$

$$C(q,\theta,a) = -1 + q^2(-1+\theta) + 2\theta + q(2+a-\theta-2a\theta)$$

Using the same technique as in the previous case ((ISO) and continuity of f), the problem of monopoly was reduced to a one-dimensional optimization problem.
In the figures 8 and 9, we show the optimal decision by Knitro for the cases with CS_+ above and with CS_+ below, respectively.

Figure 8.


Q0Hor
Number of Customers
100
Parameter 'a'
0.4
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3
Run
Current Execution
Knitro
Q1
Q0
Q0Hor
Number of Customers
100
Parameter 'a'
0.5
Number of Executions
3
Run
Current Execution
Knitro
Q1
Q0

Figure 9.

5.2. Q_0 Decreasing

Given a parameter $a \in [0, \frac{1}{3}]$, suppose that the types's θ are uniformly distributed in $[0,1]$ and their preferences are given by

$$v(q,\theta,a) = -(1+a)q\theta + \frac{q^2\theta}{2} + \frac{q\theta^2}{2} + q + \theta + 1,$$

and the costs

$$C(q,\theta,a) = 2\theta + q^2\theta + q(1 + a - 2\theta - 2a\theta + \frac{3\theta^2}{2}).$$

Using the Euler Equation (6), the relaxed solution and the curve Q_0 are $Q_1(\theta) = 1 - \theta$ and $Q_0(\theta) = 1 + a - \theta$ respectively.

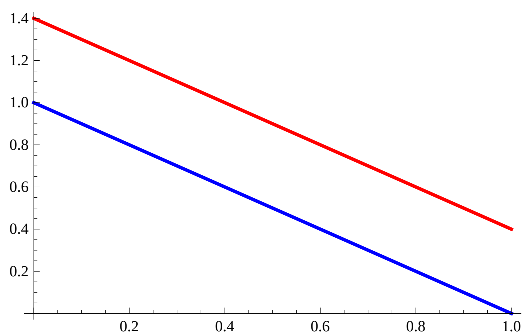
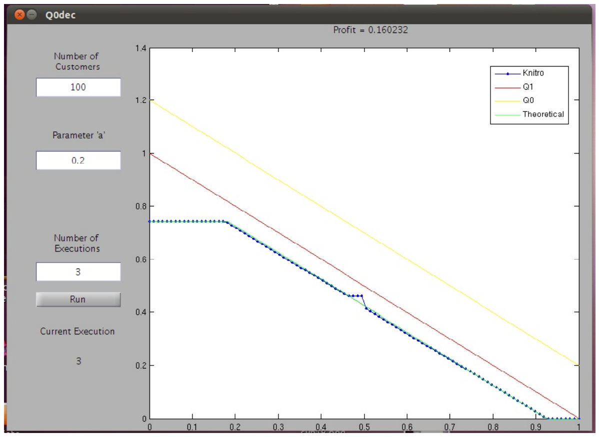


Figure 10.

The numerical solution found by Knitro for 100 types is given by Figure 11.

Figure 11.



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