# MATH 122: Calculus I

## Introduction to Limits

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# Introduction

In the last lecture, we examined the behavior of the function

$$\boxed{f(x) = \frac{1}{x}}$$

at several points.

$f(x)$ was *unbounded* at 0.

$f(x)$ grew smaller and smaller when $x$ became large (either positive or negative.)

Let us give an intuitive idea of a limit and consider some examples.

## Definition

We say that the real number $L$ **is the limit of a function $f(x)$ as $x$ approaches $a$** if as $x$ gets close to $a$, $f(x)$ gets close to $L$.

We write $\boxed{\lim_{x \to a} f(x) = L.}$

## Note

- If $\lim_{x \to a} f(x) = L$, then $L$ is unique, so a function can have at most one limit at each point.
- $f(x)$ does not have to be defined at $x = a$ for the limit to exist.

## Example

Let us consider the function

$$f(x) = \frac{x^2 - 1}{x + 1}, \qquad x \neq -1.$$

Notice that $f(x)$ is not defined at $x = -1$. Let us investigate the limit of $f(x)$ as $x$ approaches $-1$. We summarize the results in the following table:

| $x$ | $-1.03$ | $-1.02$ | $-1.01$ | $-1$ | $-0.99$ | $-0.98$ |
|---|---|---|---|---|---|---|
| $f$ | $-2.03$ | $-2.02$ | $-2.01$ | $X$ | $-1.99$ | $-1.98$ |

$f(x)$ gets close to $-2$ as $x$ approaches $-1$. This leads us to guess that

$$\lim_{x \to -1} f(x) = -2.$$

We can also express $f(x)$ as follows:

$$
\begin{aligned}
f(x) &= \frac{x^2-1}{x-1}, \quad x \neq -1 \\
&= \frac{(x+1)(x-1)}{x+1}, \quad x \neq -1 \\
&= x-1, \qquad x \neq -1
\end{aligned}
$$

So, as long as $x \neq -1$, we can replace $\dfrac{x^2-1}{x-1}$ by $x-1$ and still evaluate the limit.

We have that $x-1 \to -1-1=-2$ as $x \to -1$, which is the same as the previous solution (limits are unique.)

## Example

Let us consider the polynomial function $g(x) = x^2 + 2x + 3$. What is $\lim_{x \to 1} g(x)$?

Notice that $g(x)$ is defined at $x = 1$ (since it is a polynomial). We can investigate the limit as follows:

| $x$ | 0.98 | 0.99 | 1 | 1.01 | 1.02 |
|---|---|---|---|---|---|
| $g(x)$ | 5.9204 | 5.9601 | 6 | 6.0401 | 6.0804 |

We have that $\lim_{x \to 1} g(x) = 6$ (note that this equals $g(1)$.)

## Example

Consider $g(x) = \dfrac{\sin x}{x}, \quad x \neq 0.$

We cannot evaluate the limit by factoring. Let us consider the following values:

| $x$ | $g(x)$ |
|---|---|
| $-1$ | 0.841471 |
| $-0.5$ | 0.958851 |
| $-0.1$ | 0.998334 |
| $-0.01$ | 0.999983 |
| 0 | $X$ |
| 0.01 | 0.999983 |
| 0.1 | 0.998334 |
| 0.5 | 0.958851 |
| 1 | 0.841471 |

We see that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1.$

## Example

Find the following limits:

1. $\lim\limits_{x \to -2} (x^2 + 2x - 1)$
2. $\lim\limits_{t \to -1} (t^2 - 1)$
3. $\lim\limits_{t \to -1} (t^2 - x^2)$
4. $\lim\limits_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$
5. $\lim\limits_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$

## Example

6. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$
7. $\lim_{x \to 0} \frac{\sin x}{2x}$.

## Example

6. $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$
7. $\lim_{x \to 0} \dfrac{\sin x}{2x}$.

We can also discuss limits of functions defined *piecewise*:

Consider the function $f(x)$ defined as follows:

$$f(x) = \begin{cases} x & x < 1 \\ 2x + 1 & x \geq 1 \end{cases}$$

Does $f(x)$ have a limit at $x = 1$?

| $x$ | 0.9 | 0.99 | 1 | 1.01 | 1.1 |
|---|---|---|---|---|---|
| $f(x)$ | 0.9 | 0.99 | ? | 3.02 | 3.1 |

The limit of $f(x)$ *does not exist* at $x = 1$. This is so even though $f(x)$ is defined at $x = 1$.

Can we say something about each "piece" of the function, and what happens as we approach 1 from only one side?

# One-sided Limits

## Definition

Let $f(x)$ be a function. We say that

- $\lim_{x \to a^-} f(x) = L$ if $f(x)$ approaches $L$ as $x$ approaches $a$ from the left *[i.e. $x$ is near to $a$ but less than $a$]*
- $\lim_{x \to a^+} f(x) = L$ if $f(x)$ approaches $L$ as $x$ approaches $a$ from the right *[i.e. $x$ is near to $a$ but greater than $a$]*

From the previous example, the limit of $f(x)$ as $x \to 1$ does not exist, but

$$\lim_{x \to 1^-} f(x) = 1, \qquad \lim_{x \to 1^+} f(x) = 3.$$

## Fact

If $\lim_{x \to a} f(x) = L$, then both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and equal $L$:

$$\boxed{\left\{ \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x) \right\} \Leftrightarrow \lim_{x \to a} f(x) = L.}$$

## Example

Consider $g(x) = \sin\left(\frac{1}{x}\right)$. Can we find $\lim_{x \to 0} g(x)$?

## Example

Let $g(x) = \begin{cases} 1 - x, & x < 1 \\ x - 1, & 1 \leq x < 2 \\ 5 - x^2, & x \geq 2 \end{cases}$

Find $\lim_{x \to 1} g(x)$, $\lim_{x \to 2} g(x)$, and $\lim_{x \to 2^+} g(x)$.

# Theorems on Limits

**Theorem**

*Suppose $f(x)$ and $g(x)$ are functions with limits at $x = a$. Let $k$ be a constant and $n \in \mathbb{Z}^+$. Then the following are true:*

1. $\lim\limits_{x \to a} k = k$
2. $\lim\limits_{x \to a} x = a$
3. $\lim\limits_{x \to a} [k \cdot f(x)] = k \cdot \lim\limits_{x \to a} f(x)$
4. $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$

# Theorems on Limits

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*Suppose $f(x)$ and $g(x)$ are functions with limits at $x = a$. Let $k$ be a constant and $n \in \mathbb{Z}^+$. Then the following are true:*

5. $\lim\limits_{x \to a}[f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$
6. $\lim\limits_{x \to a}[f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
7. $\lim\limits_{x \to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, *provided* $\lim\limits_{x \to a} g(x) \neq 0$

# Theorems on Limits

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*Suppose $f(x)$ and $g(x)$ are functions with limits at $x = a$. Let $k$ be a constant and $n \in \mathbb{Z}^+$. Then the following are true:*

8. $\lim\limits_{x \to a} [f(x)]^n = [\lim\limits_{x \to a} f(x)]^n$
9. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$, *provided* $\lim\limits_{x \to a} f(x) \geq 0$ *when $n$ is even.*

**Theorem**

*Suppose $f(x)$ is a polynomial or rational function. If $f(x)$ is defined at $x = a$, then*

$$\lim_{x \to a} f(x) = f(a).$$

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**Theorem**

*If $f(x) = g(x)$ for all $x$ in an open interval containing $x = a$ except possibly at $a$, and if $\lim_{x \to a} g(x)$ exists, then so does $\lim_{x \to a} f(x)$, and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

# Some Examples

Let us consider the following:

1. $$\lim_{x\to 4}(3x^2 - 2x) = \lim_{x\to 4}(3x^2) - \lim_{x\to 4}(2x)$$
$$= 3\lim_{x\to 4}(x^2) - 2\lim_{x\to 4}(x)$$
$$= 3(\lim_{x\to 4} x)^2 - 2(\lim_{x\to 4} x)$$
$$= 3 \cdot 4^2 - 2 \cdot 4 = 40$$

2. We can similarly evaluate $\displaystyle\lim_{x\to 3} \frac{\sqrt{x^2+16}}{x}$.

How about $\lim_{x \to 2} \frac{x^2 + 3x - 10}{x^2 + x - 6}$?

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Notice that substituting $x = 2$ into the expression does not work, since we obtain $\dfrac{2^2 + 3 \cdot 2 - 10}{2^2 + 2 - 6}$, which gives $\dfrac{0}{0}$, which is meaningless. This is called an *indeterminate form*.

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Other indeterminate forms are

$$\frac{\infty}{\infty},\ 0 \cdot \infty,\ 0^0,\ \infty^0,\ 1^\infty \text{ and } \infty - \infty.$$

We convert the function $\dfrac{x^2+3x-10}{x^2+x-6}$ into the equivalent form $\dfrac{(x-2)(x+5)}{(x-2)(x+3)}$.

We get

$$\lim_{x\to 2}\frac{x^2+3x-10}{x^2+x-6}=\lim_{x\to 2}\frac{(x-2)(x+5)}{(x-2)(x+3)}=\lim_{x\to 2}\frac{x+5}{x+3}=\frac{7}{5}$$

Using the previous method, we can evaluate

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$$\begin{aligned}
\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1} &= \lim_{x \to 1} \frac{(\sqrt{x})^2 - 1^2}{\sqrt{x}-1} \\
&= \lim_{x \to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} \\
&= \lim_{x \to 1} (\sqrt{x}+1) = 2
\end{aligned}$$

# The Squeeze Principle

**Theorem (The Squeeze Principle)**

*Suppose* $g(x) \leq f(x) \leq h(x)$ *for all* $x$ *in some open interval containing* $a$*, except possibly at* $a$ *itself. Suppose also that* $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$*. Then* $\lim_{x \to a} f(x) = L$*.*

# The Squeeze Principle

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## Example

Suppose $f(x)$ is a function such that $1 - x^2/4 \leq f(x) \leq 1 + x^2/4$. We can use the squeeze principle[a] to find the limit of $f(x)$ as $x \to 0$.

[a] also called the sandwich principle

## Example

Using the squeeze principle, evaluate the following limits:

1. $\lim\limits_{\theta \to 0} (\sin \theta)$
2. $\lim\limits_{\theta \to 0} (\cos \theta)$.

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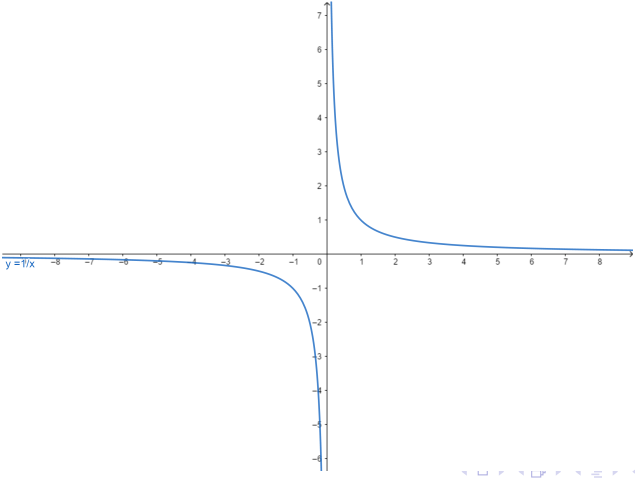
Show that given a function $f(x)$, $\lim_{x \to a} |f(x)| = 0 \Rightarrow \lim_{x \to a} f(x) = 0$.

# Limits at Infinity; Infinite Limits

In the first lecture, we considered the reciprocal function $y = 1/x$.



We observed that as $x$ *grew without bound*, $1/x$ became smaller and smaller.

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Using the language of limits, we can express this behavior as follows:

$$\boxed{\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0.}$$

Similarly,

$$\boxed{\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0.}$$

## Definition

Let $f(x)$ be a function. We say that $\lim_{x\to\infty} f(x) = L$ (or $\lim_{x\to+\infty} f(x) = L$) if $f(x)$ gets closer and closer to $L$ as $x$ *increases without bound*.
Similarly, $\lim_{x\to-\infty} f(x) = L$ if $f(x)$ gets closer and closer to $L$ as $x$ *decreases without bound*.

Note that all the limit laws in the main theorem hold when the number $a$ is replaced by $\pm\infty$. In particular,

$$\lim_{x\to\pm\infty} k = k; \qquad \lim_{x\to\pm\infty} \frac{k}{x^n} = 0,\ n \geq 1.$$

## Example (Rational functions)

To determine the limit of a rational function $R(x)$ as $x \to \pm\infty$, divide both numerator and denominator by the highest power of $x$ in the denominator, and apply the previous results.

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$$\lim_{x\to\infty} \frac{1-x^2}{1+x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{x^2}{x^2}}{\frac{1}{x^2} + \frac{x^2}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} + 1} = \frac{0-1}{0+1} = -1$$

Sometimes we may use the reciprocal property.
Given $f(x) = \sin\left(\frac{1}{x}\right)$, we can evaluate

$\lim_{x \to \infty} \sin\left(\frac{1}{x}\right)$ by defining $t = 1/x$.

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Given $f(x) = \sin\left(\frac{1}{x}\right)$, we can evaluate

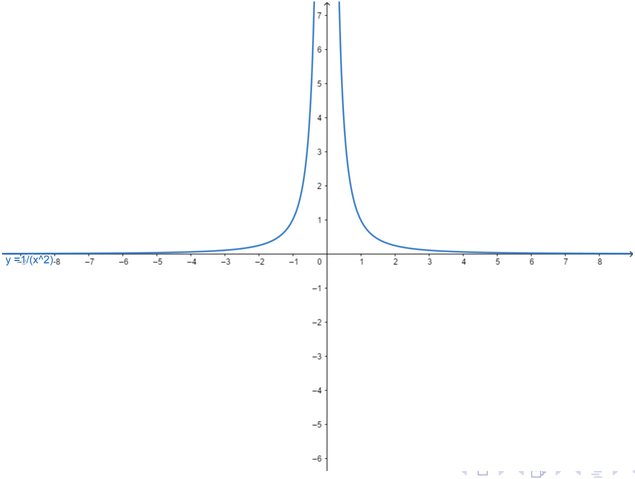
$\lim\limits_{x\to\infty} \sin\left(\frac{1}{x}\right)$ by defining $t = 1/x$.

Therefore, as $x \to \infty$, $t \to 0$. Hence

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0} \sin(t) = 0.$$

# Infinite Limits

Let us consider the graph of the function $f(x) = 1/x^2$.



As $x \to 0$, $f(x)$ increases without bound. We have $\lim_{x \to 0} f(x) = +\infty$. This means the function $f(x)$ is *unbounded* as we move toward 0. This gives us an *infinite limit*.

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An infinite limit occurs when one of the following is true:

- $\lim_{x\to a^-} f(x) = +\infty$
- $\lim_{x\to a^-} f(x) = -\infty$
- $\lim_{x\to a^+} f(x) = +\infty$
- $\lim_{x\to a^+} f(x) = -\infty$

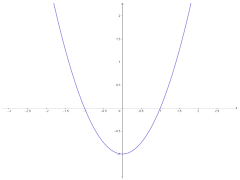
## Example

Evaluate the following limits, or state why they do not exist:

1. $\lim_{x\to 2} \dfrac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$
2. $\lim_{x\to 0} \dfrac{(2 + x)^3 - 8}{x}$
3. $\lim_{t\to 3+} \ln(t - 3)$
4. $\lim_{x\to 0} \dfrac{\tan(2x)}{\tan(\pi x)}$

# Continuity of Functions

Intuitively, we think of a function as *continuous* if the function can be sketched in one movement, with all points joined together. Consider the following examples:



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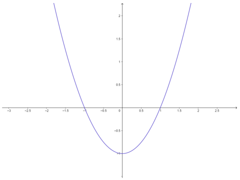

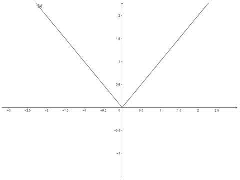

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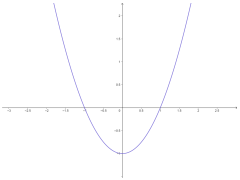

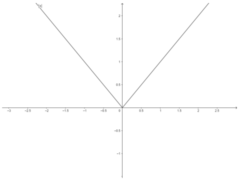

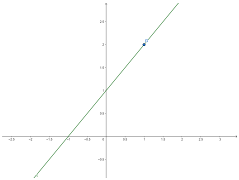

## Definition

The function $f(x)$ is continuous at $x = a$ if

1. $f$ is defined at $a$
2. $\lim_{x \to a} f(x)$ exists and is finite
3. $\lim_{x \to a} = f(x)$.

Note that if the third condition holds, then the other two must be true as well.

We say that the function $f(x)$ is continuous if it is continuous at every point in the domain under discussion.

# Removable and Nonremovable Discontinuities

Suppose $f(x)$ is continuous except at $x = a$. The discontinuity is removable if we can define $f(x)$ at $x = a$ so that $f$ is continuous there.

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If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal, then $f(x)$ has a jump discontinuity at $x = a$.

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If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal, then $f(x)$ has a jump discontinuity at $x = a$.

If either of the one-sided limits do not exist, then $f$ has an infinite discontinuity.

# Continuity of Familiar Functions

## Theorem

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- *Polynomial functions are continuous everywhere*
- *Rational functions are continuous except where the denominator is* $0$
- *The absolute value function is continuous everywhere*
- *If* $n$ *is odd, the radical function* $x^{1/n}$ *is continuous everywhere; if* $n$ *is even,* $x^{1/n}$ *is continuous for* $x > 0$.

## Theorem

*Suppose* $f(x)$ *and* $g(x)$ *are continuous at* $x = a$. *Then* $kf$, $f + g$, $f - g$, $f \cdot g$, $\left(\frac{f}{g}\right)$ *where* $g \neq 0$, $f^n$ *and* $f^{1/n}$ *where* $f(a) > 0$ *if* $n$ *is even are all continuous.*

### Theorem

*Suppose $f(x)$ and $g(x)$ are continuous at $x = a$. Then $kf$, $f + g$, $f - g$, $f \cdot g$, $\left(\frac{f}{g}\right)$ where $g \neq 0$, $f^n$ and $f^{1/n}$ where $f(a) > 0$ if $n$ is even are all continuous.*

### Theorem

*If $\lim_{x \to a}(g(x)) = L$ and $f$ is continuous at $L$, then*

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(L).$$

## Example

Determine whether the following functions are continuous on $\mathbb{R}$:

1. $f(x) = \dfrac{21 - 7x}{x - 3}$
2. $g(t) = \begin{cases} t^3 - 27 & t \neq 3 \\ t - 3 & t = 3 \end{cases}$
3. $h(u) = \begin{cases} -3x + 7 & u \leq 3 \\ -2 & u > 3 \end{cases}$
4. $f(x) = \begin{cases} x & x < 0 \\ x^2 & 0 \leq x \leq 1 \\ x & x > 1 \end{cases}$

## Example

Find $a$, $b$ such that the function

$$f(x) = \begin{cases} x+1 & x < 1 \\ ax+b & 1 \leq x < 2 \\ 3x & x \geq 2 \end{cases}$$

is continuous everywhere.

# The Intermediate Value Theorem

**Theorem**

*Suppose $f(x)$ is a function defined on an interval $[a, b]$, and suppose $W$ is a number between $f(a)$ and $f(b)$. If $f$ is continuous on $[a, b]$ then there is at least one number $c$ between $a$ and $b$ such that $f(c) = W$.*

We can use the IVT to establish that a function has a root in an interval.

For instance, consider the function $f(x) = x^3 - x$ defined on the interval $[-2, 2]$.

# The Intermediate Value Theorem

For instance, consider the function $f(x) = x^3 - x$ defined on the interval $[-2, 2]$.

$f(-2) = -8 + 2 = -6$ and $f(2) = 8 - 2 = 6$. This means that for any number $W$ between $-6$ and $6$, there exists $x \in [-2, 2]$ such that $f(x) = W$. For instance for $W = 0$, there are three possible choices for $x$: $-1, 0$ and $1$.

## Example

Show that the equation $x - \cos x = 0$ has a solution between $x = 0$ and $x = \pi/2$.

**Solution:**

We use the IVT:

$$\begin{aligned} f(x) &= x - \cos x \\ f(0) &= 0 - \cos 0 = -1 \\ f(\pi/2) &= \pi/2 - \cos(\pi/2) = \pi/2 \end{aligned}$$

We can let $W = 0$, which lies between $-1$ and $\pi/2$. Then by the IVT, there exists $c \in [0, \pi/2]$ such that $f(c) = 0$.

Therefore there exists $c \in [0, \pi/2]$ such that $c - \cos c = 0$.

# Asymptotes

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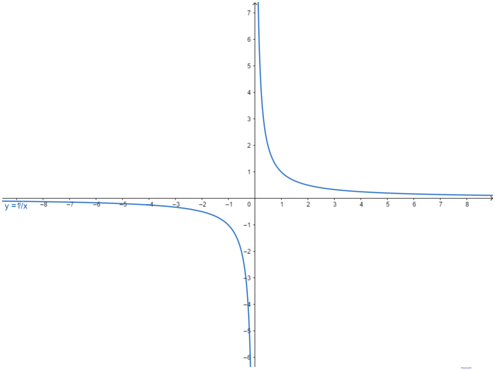
In MATH 122, we will consider three types of asymptotes:

- *Vertical Asymptotes*, which have the equation $x = h$
- *Horizontal Asymptotes*, which have the equation $y = k$
- *Oblique or Slant Asymptotes*, which are lines with equation $y = ax + b$.

# Vertical Asymptotes

The line $x = h$ is a vertical asymptote for the graph of $f(x)$ if and only

$$\lim_{x \to h^-} f(x) = \pm\infty \text{ or } \lim_{x \to h^+} f(x) = \pm\infty$$



# Vertical Asymptotes

In the diagram, the $x$-axis is a vertical asymptote for the graph of the reciprocal function $1/x$.

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**Note**

For a rational function with no common factors in the numerator and denominator, the vertical asymptotes occur where the denominator is 0. In such cases, it is impossible for the graph of the function to cross the vertical asymptote, since this would mean dividing by 0.

# Horizontal Asymptotes

The line $y = k$ is a horizontal asymptote for the graph of $f(x)$ if either $\lim_{x\to\infty} f(x) = k$ or $\lim_{x\to-\infty} f(x) = k$.

# Horizontal Asymptotes

The line $y = k$ is a horizontal asymptote for the graph of $f(x)$ if either $\lim_{x \to \infty} f(x) = k$ or $\lim_{x \to -\infty} f(x) = k$.

In some cases, the graph of $f(x)$ can cross a horizontal asymptote. Horizontal asymptotes model long term (long run) behavior of the function.

# Oblique/Slant Asymptotes

If a rational function $r(x)$ is such that the degree of the numerator is one greater than the degree of the denominator, then the graph of $r(x)$ has an oblique asymptote.

# Oblique/Slant Asymptotes

If a rational function $r(x)$ is such that the degree of the numerator is one greater than the degree of the denominator, then the graph of $r(x)$ has an oblique asymptote.

We can find the asymptote by long division. For instance, consider $r(x) = \dfrac{x^2+1}{x+1}$.

Notice that $r(x) = x - 1 + \dfrac{2}{x+1}$, therefore as $x \to \infty$, the value $\dfrac{2}{x+1}$ disappears, leaving the linear function $y = x - 1$, which is the slant asymptote.