Extremal Dependence

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Motivation

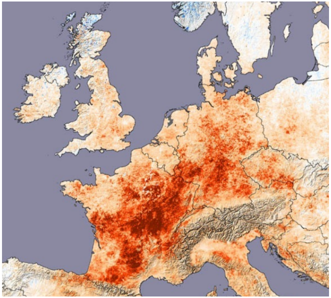
> "Human influence has likely increased the chance of compound extreme events since the 1950s."
>
> — IPCC 6th Assessment Report on Climate Change 2021

Compound extremes

Definition

Compound extremes, referred to as *simultaneous, concurrent, or coincident* extremes, may lead to larger *impacts* to human society and the environment than individual extremes alone.

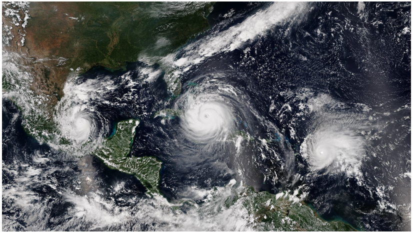
Example of compound extremes



2003 Heat wave in the South of France

⇒ **Impact**: the larger the number of cities suffered from the heatwaves the higher the death toll is likely to be.

Example of compound extremes



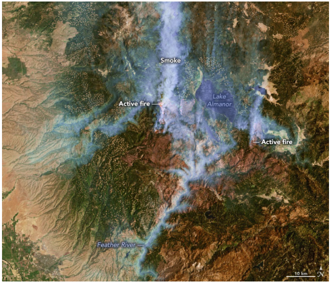
Cyclones Irma, Jose and Katia.

$\Rightarrow$ **Impact**: the larger the number of infrastructures suffers from severe winds, the higher the potential of damages (and induced insurance losses).

Example of compound extremes

$\Rightarrow$ **Impact**: directional alignment of wind speed and waves heights are likely to cause platform destruction / sink tankers yielding high risk of pollution ("black tides").

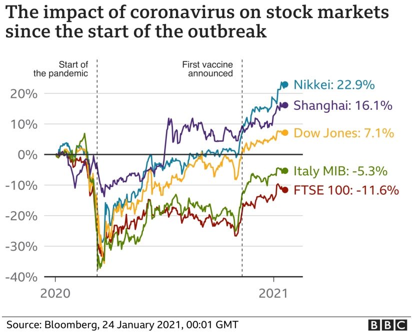
Example of compound extremes



Wildfire "Dixie" in July 2021 in California.

$\Rightarrow$ **Impact**: combination of severe drought and high temperatures increase wildfire risk causing forests and properties destructions.

Example of compound extremes



Global Financial crisis.

$\Rightarrow$ **Impact**: increased poverty and unemployment rates.

Notion of Dependence

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Statistical dependence

Dependence refers to any statistical association between two variables. Most of the time, it is replaced by the notion correlation, the degree to which a pair of variables are *linearly* related.

Notion of Spatial Dependence

Tobler's First Law of Geography

Everything is related to everything else, but near things are more related than distant things.

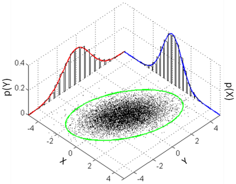
Covariance

- Classical definition of pairwise dependence:

$$\mathrm{Cov}(X, Y) = \mathrm{E}\{(X - \mu_X)(Y - \mu_Y)\} = \mathrm{E}(XY) - \mu_X \mu_Y,$$

where X and Y are two random variables with finite mean μ_X and μ_Y and satisfying

$$\mathrm{E}(X^2) < \infty, \quad \mathrm{E}(Y^2) < \infty.$$



- Covariance is very popular as it simple and fully specifies multivariate normal / Gaussian distributions.

Limits of the covariance measure

- Covariance quantifies association for bulk of the distribution.
- For random variables with sufficiently heavy tails ($\xi > 1/2$) covariance does not exist!
- Outside of the family of Gaussian distribution, pairwise summaries does not fully characterize dependence.

$\Rightarrow$ We need alternative measures of dependence tailored for extremes events.

Recap on univariate EVT: Problem definition

Using observations $X_1, \ldots, X_n \sim F$, for some set $A \subset \mathbb{R}$, we want to estimate

$$\Pr(X \in A).$$

- Use:
  - empirical estimate if A is in the body of the distribution;
  - parametric model if A contains no, or limited, data.
- If A is far from data, bias due to the fit being dominated by the body of the data.
- To compensate for the lack of data in the tails, we need a asymptotically motivated parametric model.

Recap on univariate EVT: Block maxima

For $X_1, \ldots, X_n \sim F$, the distribution of block-maxima

$$M_n = \max(X_1, \ldots, X_n),$$

converges as $n \to \infty$ to

$$\Pr\left(\frac{M_n - b_n}{a_n} < x\right) = \{F(a_n x + b_n)\}^n \to G(x),$$

where G is non-degenerate.

Recap on univariate EVT: Block maxima

Then, the distribution of G is

$$G(x) = GEV(\mu, \sigma, \xi)(x) = \exp\left[-\left\{1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right\}_+^{-1/\xi}\right\},$$

with $x_+ = \max(x, 0)$ and

- Location parameter μ;
- Scale parameter $\sigma < 0$;
- Shape parameter / Tail index ξ.

For large n, we then assume that $M_n \sim GEV(\mu, \sigma, \xi)$.

Recap on univariate EVT: Peaks-over-threshold

From

$$\{F(a_n x + b_n)\}^n \to G(x),$$

for large n, logarithm and Taylor expansion yields

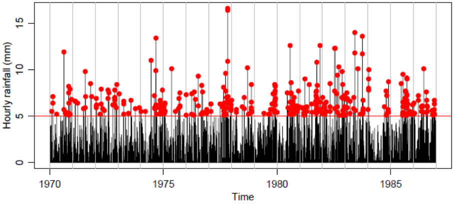
$$n\{1 - F(a_n x + b_n)\} \to -\log G(x).$$

For $x > 0$, as $\to \infty$,

$$\begin{aligned}
\Pr(X > a_n x + b_n | X > b_n) &= \frac{1 - F(a_n x + b_n)}{1 - F(b_n)} \\
&= \frac{n\{1 - F(a_n x + b_n)\}}{n\{1 - F(a_n u + b_n)\}} \\
&\to \frac{\log G(x)}{\log G(u)} \\
&\to \left\{1 + \xi \frac{x}{\sigma_u}\right\}_+^{-1/\xi},
\end{aligned}$$

where $\sigma_u = \sigma + \xi(u - \mu)$.

Recap on univariate EVT: Peaks-over-threshold



For large n, the sequence b_n tends to the upper end point of F and motivate the approximation for large threshold $u > 0$

$$\Pr(X > x + u | X > u) \approx GPD(\sigma_u, \xi) = \left(1 + \xi \frac{x}{\sigma_u}\right)_+,$$

where $\sigma_u = \sigma + \xi(u - \mu)$.

Multivariate / Bivariate Definition of Extremes

- We now suppose that $X_1, \ldots, X_n$ are i.i.d. random vectors with multivariate distribution function F (density f) and we note

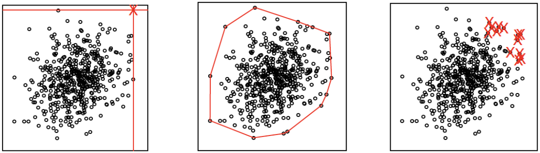
$$X_i = (X_{i,1}, \ldots, X_{i,D}) \in \mathbb{R}^D.$$

- There is no natural "ordering" or unique definition of extremes in a multivariate setting.

When / How can we write $X_1 \geq X_2$ or $X_1 \leq X_2$?

Multivariate / Bivariate Definition of Extremes

- Barnett (1976) proposes several possible orderings:
  - Componentwise maxima:
  $$M_n = (M_{X_1,n}, \ldots, M_{X_D,n});$$
  - Convex hull: $f(X) = c$;
  - Structure variable: extreme of univariate variable $r(X)$ with r chosen depending of the problem, e.g., $r(X) = \sum_{i=1}^{D} X_i$.



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Structure variables: notations

- Using observation $X_1, \ldots, X_n$, we want to estimate

$$\Pr(X \in A)$$

  for some set $A \subset \mathbb{R}^D$.

- For some function $r : \mathbb{R}^D \to \mathbb{R}$ and threshold $u > 0$, we analyze

$$\Pr(X \in A) = \Pr\{r(X) > u\}.$$

- Coles and Tawn (1994) showed that this can be tackeld as a univariate problem.

Examples of Structure Variables

1. Force on offshore structure :

$$r(X) = a_1 X_1^2 + a_2 X_2^2,$$

   where X_1 and X_2 are the wave and wind forces.

2. Water volume over a region :

$$r(X) = \sum_{i=1}^{D} a_i X_i,$$

   where X_i is the rainfall measured at station i and a_i the estimated region covered by the station.

3. Portfolio risk :

$$r(X) = \sum_{i=1}^{D} a_i X_i,$$

   where X_i is return loss of asset i. The aim is here to find a_i such that the risk of losses is minimized.

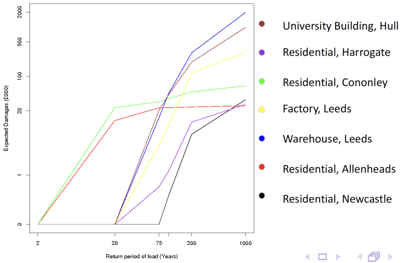
Examples of Structure Variables

4. Financial loss from spatial river flows:

$$r(X) = \sum_{i=1}^{D} r_i(X_i),$$

where X_i is the river flow at gauging station i and r_i is the (highly non-linear) property loss function.

Individual property damage data



Univariate approach for Structure Variables

- Observations $r(X_1), \ldots, r(X_n)$ are univariate, so we apply GEV/GPD approximations.
- Stenghts:
  - very simple and flexible;
  - no need to model dependence.
- Limits:
  - cannot handle missing data;
  - the estimated model is strongly influenced by the choice of r;
  - We have no information on the structure of the underlying event.

Multivariate Approach for Structure Variables

- Using observation $X_1, \ldots, X_n$, we estimate their joint distribution F

$$\Pr(X \in A) = \int_A dF(x)$$

for some set $A \subset \mathbb{R}^D$.

- For some function $r : \mathbb{R}^D \to\to$ and threshold $u > 0$, this is equivalent to

$$\Pr\{r(X) > u\} = \int_{\mathbb{R}^D} 1\{r(x) > u\} dF(x).$$

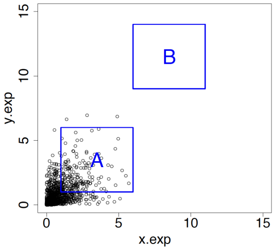
- The corresponding probability might not be available in closed form (explicit formula) but can be evaluated numerically.

Multivariate Extremes: Problem Definition

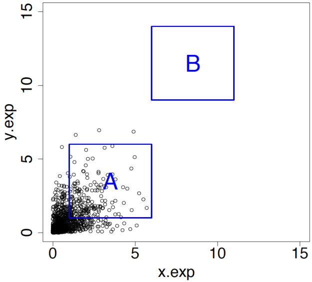
- Aim: estimate $\Pr(X \in B)$ when there is little to no data in B.

- Tool: we use asymptotically motivated models to extrapolate into the multivariate tail.

- The methodology relies on linking $\Pr(X \in B)$ to $\Pr(X \in A)$ by modelling the tail decay.



Multivariate Extremes: Difficulties



- No natural direction of extrapolation.
- Each "direction" can be considered with different underlying hypothesis.
- Multiple models are available with different hypothesis.
- Diagnostics are required to asses the quality of the proposed models.

Copulas to model dependence structure

- Let $X \sim F$, the copula C is given by

$$F(x) = C\{F_1(x_1), \ldots, F_D(x_D)\},$$

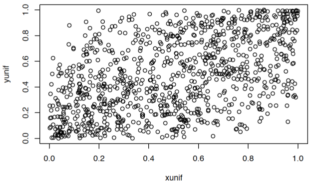
where F_i is the marginal distribution of compoent X_i of X.

- The copula C is unique and determines the dependent structure of X;

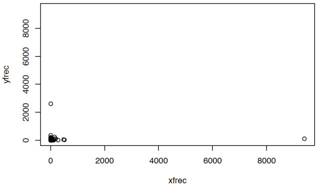
- C defines a multivariate distribution with uniform marginals.

$\Rightarrow$ The Sklar theorem, ensuring the existence and uniqueness of the copula, allows to separate marginal (univariate) distributions and dependence modeling.

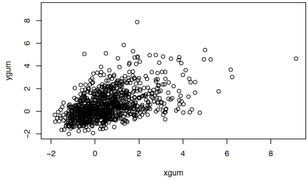
Does the Marginal Choice Matter?



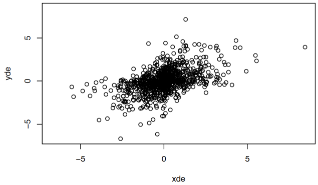
Uniform - $F_i(x) = x$



Fréchet - $F_i(x) = x^{-1}$



Gumbel - $F_i(x) = \exp\{-\exp(-x)\}$



Laplace - $F_i(x) = \exp -|x|$

Copulas with different marginals

- Copulas can have any marginal, the choice is arbitrary:

$$\begin{aligned} F(x) &= C_U\{F_1(x_1), \ldots, F_D(x_D)\}, \text{ Uniform} \\ &= C_F\{-1/\log F_1(x_1), \ldots, -1/\log F_D(x_D)\} \text{ Fréchet} \\ &= C_G[-\log\{-\log F_1(x_1)\}, \ldots, -\log\{-\log F_D(x_D)\}] \text{ Gumbel.} \end{aligned}$$

- There is thus an equivalence between C_U, C_F and C_G, e.g.,

$$C_U(u_1, \ldots, u_D) = C_F(-1/\log u_1, \ldots, -1/\log u_D).$$

- Changing of marginals can thus simply be made by repeated use of probability integral transform.

- The choice of marginal is in general made for convenience.

Bivariate PoT: Marginal standardization

- We focus on bivariate peaks-over-threshold for simplicity, but multivariate can be generalized very similarly.
- Suppose we have observations $X_1, \ldots, X_n \subset \mathbb{R}^2$, that we first normalize to a common marginal distribution to model dependence (Sklar theorem).
- We use standard Pareto margins, i.e.,

$$(Y_1, Y_2) = \left(\frac{1}{1 - F_{X_1}(X_1)}, \frac{1}{1 - F_{X_2}(X_2)} \right),$$

and then

$$F_{Y_1}(y) = F_{Y_2}(y) = 1 - y^{-1}.$$

Bivariate PoT: Marginal standardization

- We need to estimate F_{X_1} and F_{X_2}.
- For inference purposes only, we can simply use

$$F_X(x) = \tilde{F}(x),$$

where $\tilde{F}$ is the empirical distribution function.
- An alternative that can be used also for simulation and extrapolation is, for some high threshold u,

$$F_X(x) = \begin{cases} \tilde{F}(x), & x \leq u, \\ 1 - \{1 - \tilde{F}(u)\} \left(1 + \xi \frac{x-u}{\sigma_u}\right)^{-1/\xi}, & x > u, \end{cases}$$

where σ_u and ξ are the parameters of a GP distribution estimated on exceedances above threshold u.

Bivariate PoT: Regular variation

- We now assume that the vector X is regularly varying, i.e., that for any $A \subset (0,\infty)^2$
$$t\Pr\{(Y_1, Y_2) \in tA\} \to \Lambda(A) > 0, \quad t \to \infty, \tag{1}$$
where Λ is a non-degenerate measure on $(0,\infty)^2$.

- We recall that for the univariate case, we had assumed that $X_1 \in MDA(G)$ which translated into
$$n\{1 - F(a_n x_1 + b_n)\} \to -\log G(x_1), \tag{2}$$
but because of the Pareto normalization, $\xi = 1$ and we can choose $b_n \equiv 0$ and $a_n = n$, thus obtain
$$n\Pr\{X_1 > nx_1\} = n\Pr\{X_1 \in n(x_1, \infty)\} \to -\log G(x_1),$$
as $n \to \infty$

$\Rightarrow$ Equation (1) is the direct generalization of univariate EVT (2).

Return on structure variables

- We focus on a specific case of structure variables, i.e., variables for which the function r satisfies an homogeneity property:

$$r(tx) = tr(x), \quad t > 0, \quad x \in (0, \infty)^2.$$

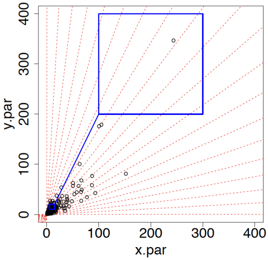
- For example, this includes:
  - $r(x) = x_1$;
  - $r(x) = \max(x_1, x_2)$;
  - $r(x) = (x_1 + x_2)/2$.

Pseudo-polar decomposition and illustration

- Regular variation implies that for some $u > 1$

$$\begin{aligned}\Pr\{Y_1 > ut, Y_2 > tu \mid r(Y) > t\} &= \frac{\Pr\{Y_1 > tu, Y_2 > tu\}}{\Pr\{r(Y) > u\}}, \\ &\to \frac{1}{u} \frac{\Lambda\{y \in (1,\infty)^2\}}{\Lambda\{y \in (0,\infty)^2 : r(y) > 1\}},\end{aligned}$$

as $t \to \infty$.



$\Rightarrow$ Regular variation implies a specific "direction" for extrapolation.

Measure of extremal dependence: the χ coefficient

- We now focus on $r(y_1, y_2) = y_1$, regular variation implies

$$\Pr\{Y_1 > u | Y_2 > u\} = \frac{u\Pr\{Y_1 > u, Y_2 > u\}}{u\Pr\{Y_2 > u\}}$$

$$\chi(u) \to \frac{\Lambda\{y \in (1,\infty) \times (1,\infty)\}}{\Lambda\{y \in (0,\infty) \times (1,\infty)\}}$$

$$\to \chi > 0.$$

- The limit χ is used as a pairwise measure of (asymptotic) extremal dependence.

- This measure is very popular for its simplicity and interpretability.

- χ is linked to the so-called extremal coefficient θ through

$$\chi = 2 - \theta.$$

Asymptotic Independence

- What happens if Y_1 and Y_2 are independent?
- We have simply $F(y_1, y_2) = F(y_1)F(y_2)$, and thus

$$\begin{aligned}\Pr\{Y_1 > u | Y_2 > u\} &= \frac{\Pr\{Y_1 > u, Y_2 > u\}}{\Pr\{Y_2 > u\}} \\ &= \frac{\{1 - F(u)\}\{1 - F(u)\}}{\{1 - F(u)\}}, \\ &= 1 - F(u), \\ \chi &\to 0,\end{aligned}$$

  as $u \to \infty$.
- More generally *asymptotic independence* is defined when $\chi = 0$.
- ⚠ Asymptotic independence does not mean that Y_1 and Y_2 are exactly independent.

Asymptotic independence

- If $\chi = 0$, then regular variation cannot hold and

$$t\Pr(Y_1 > ty_1, Y_2 > ty_2) \to 0.$$

- Once again if Y_1 and Y_2 were exactly independent:

$$\begin{aligned}
\Pr(Y_1 > ty, Y_2 > ty) &= \frac{1}{(ty)^2} \\
&= \frac{1}{t^2}\frac{1}{y^2} \\
&= \frac{1}{t^2}\Pr(Y_1 > y, Y_2 > y) \\
&\neq \frac{1}{t}\Pr(Y_1 > y, Y_2 > y).
\end{aligned}$$

- In case of perfect independence, we observe a faster decay rate than regular variation.

Measuring asymptotic independence

- Ledford and Tawn (1996) propose to study

$$\begin{aligned}\Pr(Y_1 > ty, Y_2 > ty) &\sim \frac{1}{t^{1/\eta}} \Pr(Y_1 > y, Y_2 > y) \\ &\sim \frac{1}{t^{1/\eta}} \mathcal{L}(y),\end{aligned}$$

  where $0 < \eta \neq 1$ and $\mathcal{L}$ is a "slowly varying function".

- The coefficient η measure the strength of "subasymptotic dependence", i.e., the speed at which the vector converge to independence.

- For $\eta = 1$, we retrieve regular variation and asmyptotic independence, so the larger is η the slower is the convergence.

Alternative measure of asymptotic independence

- We define the coefficient

$$\bar{\chi} = 2\eta - 1, \quad \bar{\chi} \in (-1, 1].$$

- $\bar{\chi} = 1$ asymptotic dependence with $\chi = \lim_{y\to\infty} \mathcal{L}(y) > 0$;
- $\bar{\chi} < 1$ asymptotic independence and $\chi = 0$;
- We distinguish three regimes of asymptotic independence:
  - $0 < \bar{\chi} < 1$ positive extremal association;
  - $\bar{\chi} = 0$ near extremal independence;
  - $-1 < \bar{\chi} < 0$ negative extremal association;

Example: Gaussian Copula

- Suppose that the dependence of Y is driven by a Gaussian copula with correlation ρ,

$$C_U(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \phi(y, \rho) dy,$$

  where
  - ϕ is the density function of a bivariate normal distribution with mean 0 and correlation $\rho \in (-1; 1)$,
  - Φ is the cumulative distribution function of a univariate Gaussian distribution.
- In this case, we have

$$\chi = 0, \quad \bar{\chi} = \rho < 1.$$

- The Gaussian copula motivated the parametrization of $\bar{\chi}$.

Measure of extremal dependence: Summary

Asymptotic dependence	Asymptotic independence
$\chi > 0$	$\chi = 0$
$\bar{\chi} = 1$	$-1 < \bar{\chi} < 1$

- Recall that χ and $\bar{\chi}$ are *asymptotic* quantities.
- In practice, we need to estimate these measures from a finite number of observations / finite threshold, i.e., for large enough threshold

$$\Pr(Y_1 > u | Y_2 > u) = \chi(u) \approx \chi$$

and for $t > 1$,

$$\begin{aligned}\Pr(Y_1 > tu, Y_2 > tu | Y_1 > u, Y_2 > u) &= \Pr(\min(Y_1, Y_2) > tu | \min(Y_1, Y_2) > u) \\ &\approx \left\{\frac{u}{t}\right\}^{1/\eta(u)}\end{aligned}$$

Estimating χ and $\bar{\chi}$

- Suppose we have observations $X_1, \ldots, X_n$ that have been normalized to $Y_1, \ldots, Y_n$ with unit Pareto margins.

- Empirical estimator for χ for $u \gg 0$:

$$\hat{\chi}(u) = \frac{\sum_{i=1}^{n} 1\{Y_{1,i} > u, Y_{2,i} > u\}}{\sum_{i=1}^{n} 1\{Y_{2,i} > u\}}$$

- Estimator for $\widehat{\bar{\chi}} = 2\hat{\eta} - 1$:
  - compute pairwise minimas $Z_i = \min(Y_{1,i}, Y_{2,i})$;
  - Estimate the tail index $\hat{\eta}$ of the Pareto distribution using observations Z_i exceeding threshold u.

Link between χ and $\bar{\chi}$ and copulas

- The is an explicit link between the bivariate copula C_U and both extremal coefficients (Coles et al., 1999).

- For $u \in (0;1)$,

$$\chi(u) = \frac{1 - 2u + C_U(u,u)}{1-u} = 2 - \frac{\log C_U(u,u)}{\log u}$$

and

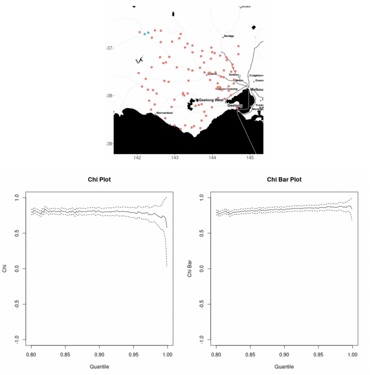
$$\bar{\chi}(u) = \frac{2\log(1-u)}{\bar{C}_U(u,u)} - 1,$$

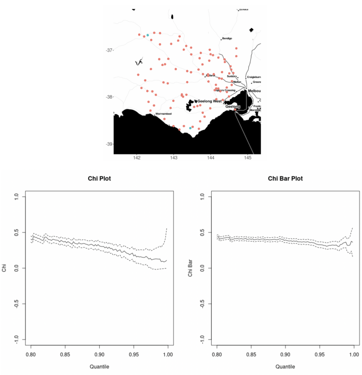
where $\bar{C}_U$ is the inverted copular, i.e., $C(u) = \bar{C}_U(1-u, 1-u)$.

- Both quantities can thus be estimated by replacing C_U and $\bar{C}_U$ by their empirical estimates.

Estimating χ and $\bar{\chi}$ with R

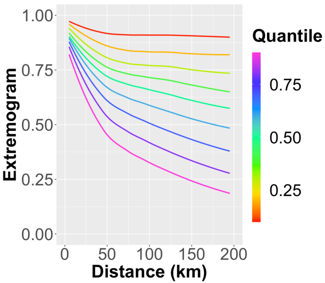
- The function `chiplot` in the package `evd` will estimate both coefficient as function of the threshold u.
- Example of rainfall in the state of Victoria, AU

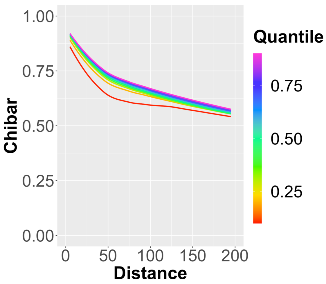




Spatial dependence

- In a spatial setting, we can estimate χ and $\bar{\chi}$ as function of the distance.





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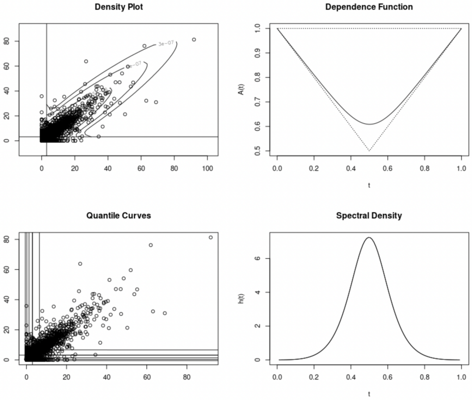
Parametric Copula Models

- The function `fbvpot` (resp. `fbvevd` for block maxima) in the package `evd` enable automatic fit of a large number of parametric models.
- Example of an asymmetric logistic model fit for rainfall measurements from station 6 and 16 in the state of Victoria, AU:



Parametric Copula Models

Model	AIC	χ
Logistic	34830.66	0.784
Bi-logistic	34832.67	0.784
Asymmetric logistic	34690.47	0.783
Negative logistic	34879.74	0.785
Negative bi-logistic	34881.73	0.785
Asymmetric negative logistic	36498.8	0.5
Coles-Tawn	35091.2	0.760
Husler-Reiss	35187.61	0.762
Asymmetric mixed	36401.8	0.5

Model fit summary for all models available in `fbvpot`.

Disclaimer

This presentation has been developed as teaching material for the course "Extreme Value Statistics" and does not reflect the position of the Swiss Confederation.