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CSED261: Discrete Mathematics for Computer Science
Homework 8: Counting & Discrete Probability

Question 1. Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion–exclusion to find the number of positive integers not exceeding n that are relatively prime to n.

Solution

The positive integer n is composed of p and q which are prime numbers. So, We can say that the relatively prime to n are the numbers that are not divisible by p and q. There are $\frac{n}{p}$ numbers that are divisible by p, and $\frac{n}{q}$ numbers that are divisible by q. But, There are duplicated numbers that are divisible by both p and q. So, we need to subtract the number of numbers that are divisible by p and q (principle of inclusion-exclusion). There are $\frac{n}{pq}$ numbers. Finally, the number of positive integers not exceeding n that are relatively prime to n is $n - \frac{n}{p} - \frac{n}{q} + \frac{n}{pq} = n(1 - \frac{1}{p})(1 - \frac{1}{q})$.

Question 2. Let $n_1, n_2, ..., n_t$ be positive integers. Show that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some $i, i = 1, 2, ..., t$, the ith box contains at least n_i objects.

Solution

We will use the pigeonhole principle with contradiction. Suppose that for all $i, i = 1, 2, \ldots, t$, the ith box contains less than n_i objects. Then, the number of total objects should be less than $\sum_{i=1}^{t} n_i = n_1 + n_2 + \cdots + n_t - t$. However, we have $n_1 + n_2 + \cdots + n_t - t + 1$ objects, which is a contradiction. Therefore, if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some $i, i = 1, 2, ..., t$, the ith box contains at least n_i objects by contrardicion.

Question 3. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

Solution

There are the pairs of number $(\text{the number of men}, \text{the number of woman}) = (0,6), (1,5), (2,4)$ for the committee which have more women then men. We will calculate the probability of each case and sum them up. In composing the committee, the order of the members does not matter.

(Case 1) (man, women) = (0, 6)
There are $\binom{10}{0} = 1$ ways to choose man, and $\binom{15}{6} = 5005$ ways to choose women. So there are 5005 ways in this case.

(Case 2) (man, women) = (1, 5)
There are $\binom{10}{1} = 10$ ways to choose man, and $\binom{15}{5} = 3003$ ways to choose women. So there are 30030 ways in this case.

(Case 3) (men, women) = (2, 4)
There are $\binom{10}{2} = 45$ ways to choose man, and $\binom{15}{4} = 1365$ ways to choose women. So there are 61425 ways in this case.

Therefore, the total number of ways to form a committee with six members which must have more women then men is $5005 + 30030 + 61425 = 96460$ ways.

In non-calculated form, the number of ways is $\binom{10}{0}\binom{15}{6} + \binom{10}{1}\binom{15}{5} + \binom{10}{2}\binom{15}{4} = 96460$ ways.

Question 4. Give a combinatorial proof that if n is a positive integer then $\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$. [**Hint:** Show that both sides count the ways to select a subset of a set of n elements together with two not necessarily distinct elements from this subset. Furthermore, express the righthand side as $n(n-1)2^{n-2} + n2^{n-1}$.]

Solution

Given expression is,

$$\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2} = n(n-1)2^{n-2} + n2^{n-1}.$$

We can think about the situation that there are n candidates for committee members and we want to make a committee of any legal size with a leader and two manager, which are not necessarily distinct (i.e., the leader can be one of the managers).
On the one hand, we can divide the situation into two cases.

1. Select a leader first and determine the others will be or not be the committee members. (2 ways for each member) Then, the number of ways is, $n \cdot 2^{n-1}$.

2. Select two managers first and determine the others will be or not be the committee members. Then, the number of ways is, $n(n-1) \cdot 2^{n-2}$.

Because the leader can be one of the managers, the total number of ways is $n(n+1)2^{n-2}$ (sum).
On the other hand, by the same fixed size k, we can select k committee members and choose a leader or two manager from the committee members.
Then, the number of ways is,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} + \sum_{k=0}^{n} k(k-1)\binom{n}{k} = \sum_{k=0}^{n} k^2\binom{n}{k}.$$

Since we count the same situation in two different ways, the two expressions are equal. So, we can get the result,

$$\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}.$$

Question 5. How many different strings can be made from the letters in *MISSISSIPPI*, using all the letters?

Solution

The word *MISSISSIPPI* has 11 letters with 4 I's, 4 S's, 2 P's, and 1 M. This case is multinomial coefficient. Ths answer is,

$$\frac{11!}{4!\,4!\,2!\,1!} = 34650.$$

If all letters are distinct, the answer would be 11!. However, since there are repeating letters, so we have to divide the effect of repeating letters. For example, I has 4 repeating letters, so there are 4! ways to arrange them. We have to do the same for S and P as well, then we get the answer.

Question 6. Two events E_1 and E_2 are called independent if $p(E_1 \cap E_2) = p(E_1)p(E_2)$. For each of the following pairs of events, which are subsets of the set of all possible outcomes when a coin is tossed three times, determine whether or not they are independent.

- E_1: tails comes up with the coin is tossed the first time; E_2: heads comes up when the coin is tossed the second time.

Solution

The probability of E_1 is $p(E_1) = \frac{1}{2}$ and the probability of E_2 is $p(E_2) = \frac{1}{2}$. The probability of $E_1 \cap E_2$ is $p(E_1 \cap E_2) = \frac{1}{4}$. Since $p(E_1 \cap E_2) = p(E_1)p(E_2)$, E_1 and E_2 are independent events.

Question 7. Show that if E and F are independent events, then $\overline{E}$ and $\overline{F}$ are also independent events.

Solution

Because E and F are independent events, we have:

$$P(E \cap F) = P(E) \cdot P(F)$$

To check if $\overline{E}$ and $\overline{F}$ are independent events, we need to check if $P(\overline{E} \cap \overline{F}) = P(\overline{E}) \cdot P(\overline{F})$ holds.
Since the definition of complement,

$$P(\overline{E}) = 1 - P(E),\ P(\overline{F}) = 1 - P(F).$$

Then,

$$\begin{aligned} P(\overline{E} \cap \overline{F}) &= 1 - P(\overline{\overline{E} \cap \overline{F}}) = 1 - P(E \cup F) \\ &= 1 - (P(E) + P(F) - P(E \cap F)) \\ &= 1 - P(E) - P(F) + P(E) \cdot P(F) \\ &= (1 - P(E)) \cdot (1 - P(F)) = P(\overline{E}) \cdot P(\overline{F}) \end{aligned}$$

Therefore, $\overline{E}$ and $\overline{F}$ are independent events.

Question 8. Suppose that a Bayesian spam filter is trained on a set of 500 spam messages and 200 messages that are not spam. The word "exciting" appears in 40 spam messages and in 25 messages that are not spam. Would an incoming message be rejected as spam if it contains the word "exciting" and the threshold for rejecting spam is 0.9?

Solution

Let S be the event that message is spam, and E be the event that message contains the word "exciting". The number of spam messages is 500, and the number of messages that are not spam is 200. The number of spam messages that contain the word "exciting" is 40, and the number of messages that are not spam that contain the word "exciting" is 25.
Then, We know $P(E|S) = \frac{40}{500} = 0.08$ and $P(E|\overline{S}) = \frac{25}{200} = 0.125$. Also, $P(S) = 5/7$ and $P(\overline{S}) = 2/7$
The probability that the message is spam when it contains the word "exciting" is:

$$P(S|E) = \frac{P(E|S)P(S)}{P(E)} = \frac{P(E|S)P(S)}{P(E|S)P(S) + P(E|\overline{S})P(\overline{S})} = \frac{0.08 \times 5/7}{0.08 \times 5/7 + 0.125 \times 2/7} \approx 0.615$$

Then, $P(S|E) \approx 0.615$. So, the message will not be rejected as spam because $0.615 < 0.9$.

Question 9. Prove the law of total expectations.

Solution

The law of total expectation asserts that the expected value of a random variable is equal to the expected value of its conditional expectation,

$$E[X] = E[E[X|Y]].$$

$$\begin{aligned}
E[X] &= \sum_x x \cdot P(X = x) \\
E[E[X|Y]] &= \sum_x \sum_y x \cdot P(X = x|Y = y) \cdot P(Y = y) \\
&= \sum_x \sum_y x \cdot P(X = x, Y = y) \because \text{law of conditional probability} \\
&= \sum_x x \cdot \sum_y P(X = x, Y = y) \\
&= \sum_x x \cdot P(X = x) \\
&= E[X]
\end{aligned}$$