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**CSED261: Discrete Mathematics for Computer Science**
**Homework 6: Graphs**

**Question 1.** Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

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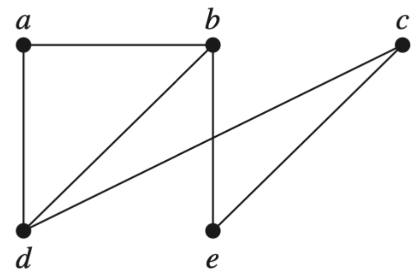
### Solutions

Let $G$ be a simple graph. Let $n$ be the number of vertices in $G$. Let $d_1, d_2, \ldots, d_n$ be the degrees of the vertices of $G$. Since $G$ is simple, the degrees of the vertices are non-negative integers. Since there are $n$ vertices and $n - 1$ possible degrees, by the Pigeonhole Principle, there must be two vertices with the same degree.

**Question 2.** Represent the below graph with an adjacency matrix.

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**Solutions**

The adjacency matrix for the graph is as follows:

| | a | b | c | d | e |
|---|---|---|---|---|---|
| a | 0 | 1 | 0 | 1 | 0 |
| b | 1 | 0 | 0 | 1 | 1 |
| c | 0 | 0 | 0 | 1 | 1 |
| d | 1 | 1 | 1 | 0 | 0 |
| e | 0 | 1 | 1 | 0 | 0 |

,

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

**Question 3.** What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

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**Solutions**

The sum of the entries in a row of the adjacency matrix for an undirected graph is the number of edges connected to the node(vertex), in other words, the degree of the node. For a directed graph, the sum of them is the out-degree of the node. In addition, the sum of the entries in a column of the adjacency matrix is also the degree of the node for an undirected graph, and the in-degree of the node for a directed graph.

**Question 4.** Suppose that $G$ and $H$ are isomorphic simple graphs. Show that their complementary graphs $\overline{G}$ and $\overline{H}$ are also isomorphic.

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**Solutions**

Let $f : V(G) \to V(H)$ be an isomorphism between garph $G$ and $H$. This means, for any two vertices $a, b \in V(G)$, $a$ and $b$ are adjacent in $G$, then $f(a)$ and $f(b)$ are adjacent in $H$.

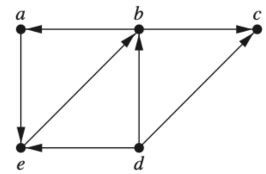
Now, let's consider an isomorphism between complementary graphs $\overline{G}$ and $\overline{H}$. We define a function $g : V(\overline{G}) \to V(\overline{H})$ which $f = g$. Because $G$ and $H$ are isomorphic, we know that $f$ is a bijection, also $g$ is. Then, let's prove that $g$ is an isomorphism between $\overline{G}$ and $\overline{H}$.

- Assume that $u, v \in V(\overline{G})$ and $u$ and $v$ are adjacent in $\overline{G}$, then $u$ and $v$ are not adjacent in $G$ because of property of complementary. Since $f$ is an isomorphism between $G$ and $H$, $f(u)$ and $f(v)$ are not adjacent in $H$. This implies that $g(u)$ and $g(v)$ are adjacent in $\overline{H}$.
- Assume that $u, v \in V(\overline{G})$ and $u$ and $v$ are not adjacent in $\overline{G}$. Then, $u$ and $v$ are adjacent in $G$, because of complementary. Then, $f(u)$ and $f(v)$ are adjacent in $H$ because $f$ is an isomorphism. This implies that $g(u)$ and $g(v)$ are not adjacent in $\overline{H}$.

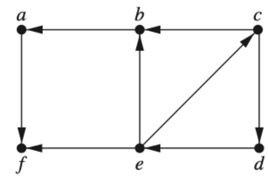
So, the function $g$ between $\overline{G}$ and $\overline{H}$ maintains isomorphism from the function $f$ between $G$ and $H$. Therefore, $\overline{G}$ and $\overline{H}$ are isomorphic.

**Question 5.** Find the strongly connected components of each of these graphs.

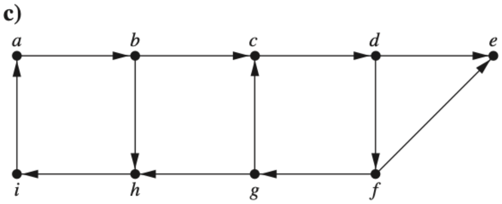
**a)**




**b)**




**c)**

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**Solutions**

**a).** (a, b, e), (d), (c)

**b).** (a), (f), (b), (c, d, e)

**c).** (a, b, c, d, f, g, h, i), (e)

**Question 6.** Show that every connected graph with $n$ vertices has at least $n-1$ edges.

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### Solutions

Let's prove this statement using induction. When the number of vertices $n = 1$, the graph has no edges, so the statement is true. So, we have to prove that if the statement is true for $n = k$, then it is also true for $n = k + 1$.

Let $G$ be a connected graph with $k+1$ vertices. Let's remove an arbitrary vertex $v$ from $G$. Then, the graph $G - v$ has $k$ vertices and at least $k-1$ edges. When we add the vertex $v$ back to the graph, we have to add at least one edge to connect $v$ to the rest of the graph. So, the graph $G$ has at least $k - 1 + 1 = k$ edges. Therefore, the statement is true for $n = k + 1$.

By induction, the statement is true for all $n \in \mathbb{N}$.

**Question 7.** Show that a directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

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**Solutions**

Prove the following two statements:

($\leftarrow$) A directed multigraph having no isolated vertices has an Euler circuit, then the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

- Euler circuit visits every nodes, so the graph is weakly connected.
- The start and end node is equal, so the in-degree and out-degree of start(end) node is equal now. During the Euler circuit, we have to enter and exit the node, so the in-degree and out-degree of each node(contain start and end node) are equal.

($\rightarrow$) A directed multigraph is weakly connected and the in-degree and out-degree of each vertex are equal, then the graph has an Euler circuit.

We already know the theorem that a connected multigraph with a least two vertices has an Euler circuit if and only if each of its vertices has even degree (during class). A graph is weakly connected, so we can know that the every nodes are connected if we ignore the direction. Also, we know that the in-degree and out-degree of each vertex are equal, so every nodes have even degree. Therefore, the graph has an Euler circuit by theorem.