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CSED261: Discrete Mathematics for Computer Science
Homework 7: Induction and Recursion

Question 1. Prove that

$$\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^{j} = \frac{2^{n+1} + (-1)^{n}}{3 \cdot 2^{n}}$$

whenever n is a nonnegative integer.

Solutions

Proof. We will prove by induction. If n is 0, then the both sides are equal to 1. Next, we assume that the statement is true when $n = k$, then we have to prove that the statement is true when $n = k + 1$.

$$\begin{aligned}
\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^{j} &= \sum_{j=0}^{k} \left(-\frac{1}{2}\right)^{j} + \left(-\frac{1}{2}\right)^{k+1} \\
&= \frac{2^{k+1} + (-1)^{k}}{3 \cdot 2^{k}} + (-1)^{k+1} \frac{1}{2 \times 2^{k}} \\
&= \frac{2 \cdot 2^{k+1} + 2(-1)^{k} + 3(-1)^{k+1}}{6 \cdot 2^{k}} = \frac{2^{k+2} + (-1)^{k+1}}{3 \cdot 2^{k+1}}
\end{aligned}$$

Finally, by the mathmetical induction, the statement is true for all nonnegative integers n. □

Question 2. Use mathematical induction to show that $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ whenever $p_1, p_2, \ldots, p_n$ are propositions.

Solutions

Proof. We will use induction. If n is 1, $\neg(p_1)$ is equivalent to $\neg p_1$. If n is 2, the statement $\neg(p_1 \vee p_2) = \neg p_1 \wedge \neg p_2$ is true by De Morgan's law. Next, we assume that the statement is true when $n = k$, then we have to prove that the statement is true when $n = k + 1$.

$$\begin{aligned}
\neg(p_1 \vee p_2 \vee \cdots \vee p_{k+1}) &= \neg((p_1 \vee p_2 \vee \cdots \vee p_k) \vee p_{k+1}) \\
&= \neg(p_1 \vee p_2 \vee \cdots \vee p_k) \wedge \neg p_{k+1} && \text{by De Morgan's law}(n = 2) \\
&= \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_k \wedge \neg p_{k+1} && \text{by induction hypothesis}
\end{aligned}$$

Finally, by induction, the statement is true for all $n \in \mathbb{N}$. □

Question 3. Let b be a fixed integer and j a fixed positive integer. Show that if $P(b), P(b+1), \ldots, P(b+j)$ are true and $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for every integer $k \geq b+j$, then $P(n)$ is true for all integers n with $n \geq b$.

Solutions

Proof. Let Q be the statement $P(b) \wedge P(b+1) \wedge \cdots \wedge P(b+j)$. Then, let $R(k)$ be the statement $Q \wedge P(b+j+1) \wedge \cdots \wedge P(b+j+k)$ for $k \geq b+j$. We will prove that $R(k)$ is true for all $k \geq b+j$ by induction on k.
By assumption, $P(b), P(b+1), \ldots, P(b+j)$ are true. Thus, Q is true.
Base case: $R(0) = Q$ is true.
Inductive step: Suppose that $R(k)$ is true for some $k \geq 1$. Then, $Q \wedge P(b+j+1) \wedge \cdots \wedge P(b+j+k)$ is true. By assumption, $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for every integer $k \geq b+j$. Thus, $P(b) \wedge P(b+1) \wedge \cdots \wedge P(b+j+k) \to P(b+j+k+1)$ is true. Therefore, $Q \wedge P(b+j+1) \wedge \cdots \wedge P(b+j+k) \to P(b+j+k+1)$ is true. Hence, $R(k+1)$ is true.
Finally, by induction, $R(n)$ is true for all $n \geq b+j$. Therefore, $P(n)$ is true for all integers n with $n \geq b$ by assumption and definition of $R(k)$. $\square$

Question 4. Prove that $\sum_{j=1}^{n} j(j+1)(j+2)\cdots(j+k-1) = n(n+1)\ (n+2)\cdots(n+k)/(k+1)$ for all positive integers k and n. [Hint: Use a technique from Exercise 33]

> **33.** Show that we can prove that $P(n, k)$ is true for all pairs of positive integers n and k if we show
>
> **a)** $P(1, 1)$ is true and $P(n, k) \to [P(n+1, k) \wedge P(n, k+1)]$ is true for all positive integers n and k.
>
> **b)** $P(1, k)$ is true for all positive integers k, and $P(n, k) \to P(n+1, k)$ is true for all positive integers n and k.
>
> **c)** $P(n, 1)$ is true for all positive integers n, and $P(n, k) \to P(n, k+1)$ is true for all positive integers n and k.

Solutions

Proof. Let $P(n,k)$ be the statement $\sum_{j=1}^{n} j(j+1)(j+2)\cdots(j+k-1) = n(n+1)\ (n+2)\cdots(n+k)/(k+1)$ for all positive integers k and n. We will prove that $P(n,k)$ is true for all pairs of positive integers n and k by induction on n and k followinng the step b of Exercise 33.

Base Case:

$P(1,k) \iff \sum_{j=1}^{1} j(j+1)(j+2)\cdots(j+k-1) = 1(1+1)(1+2)\cdots(1+k-1) = 1 = 1(1+1)(1+2)\cdots(1+k)/(k+1)$ is true for all positive integer k.

$$
\begin{aligned}
P(n+1,k) \iff & \sum_{j=1}^{n+1} j(j+1)(j+2)\cdots(j+k-1) \\
= & \sum_{j=1}^{n} j(j+1)(j+2)\cdots(j+k-1) + (n+1)(n+2)\cdots(n+k) \\
= & \frac{n(n+1)\cdots(n+k)}{k+1} + (n+1)(n+2)\cdots(n+k) && \because P(n,k) \text{ is true} \\
= & \frac{n(n+1)\cdots(n+k) + (n+1)(n+2)\cdots(n+k)(k+1)}{k+1} \\
= & \frac{(n+1)(n+2)\cdots(n+k)(n+k+1)}{k+1} && \text{So, } P(n+1,k) \text{ is true}
\end{aligned}
$$

Therefore, by induction, $P(n,k)$ is true for all positive integers n and k. $\square$

Question 5. Give a recursive definition of the functions max and min so that max $(a_1, a_2, \ldots, a_n)$ and min $(a_1, a_2, \ldots, a_n)$ are the maximum and minimum of the n numbers $a_1, a_2, \ldots, a_n$, respectively.

Solutions

Definition 1. ***Max of*** $a_1, a_2, \ldots, a_n$

$\max(a_1) = a_1$

$\max(a_1, a_2) = \begin{cases} a_1 & \textit{if } a_1 \geq a_2 \\ a_2 & \textit{if } a_1 < a_2 \end{cases}$

$\max(a_1, a_2, \ldots, a_n) = \max(a_1, \max(a_2, \ldots, a_n))$

Definition 2. ***Min of*** $a_1, a_2, \ldots, a_n$

$\min(a_1) = a_1$

$\min(a_1, a_2) = \begin{cases} a_1 & \textit{if } a_1 \leq a_2 \\ a_2 & \textit{if } a_1 > a_2 \end{cases}$

$\min(a_1, a_2, \ldots, a_n) = \min(a_1, \min(a_2, \ldots, a_n))$

Question 6. Give a recursive definition of the set of bit strings that are palindromes.

Solutions

Definition 3. .
Base case: *The empty string* ϵ_0 *is a palindrome. Also, the string* 0 *and* 1 *are palindromes.*
Recursive case: *If* w *is a palindrome, then* $0w0$ *and* $1w1$ *are palindromes.*

Question 7. Prove that Algorithm 1 for computing n ! when n is a non-negative integer is correct.

ALGORITHM 1 A Recursive Algorithm for Computing n!.

procedure *factorial*(n: nonnegative integer)
if $n = 0$ **then return** 1
else return $n \cdot factorial(n - 1)$
{output is n!}

Solutions

Proof. We will prove that the algorithm 1 is correct with the mathmetical induction.
Basis. For $n = 0$, the algorithm return 1. This is correct since $0! = 1$.
Induction. we assume that the algorithm is correct for $n = k$, then we will prove that the algorithm is correct for $n = k+1$. By the assumption, we have that the algorithm returns $k!$ for $n = k$. Then, when $n = k+1$, the algorithm returns $(k+1)k!$ and $(k+1)k! = (k+1)!$ by definition of factorial. Therefore, the algorithm is correct for $n = k+1$. Finally, by the mathemtical induction, the algorithm 1(computing n!) is correct for all non-negative integer n. $\square$

Question 8. Use a merge sort to sort $4, 3, 2, 5, 1, 8, 7, 6$ into increasing order. Show all the steps used by the algorithm.

Solutions

Algorithm 1 MergeSort (recursive)

```
Input: An array A of n elements
if n = 1 then
    Return A
else if n > 1 then
    mid ← ⌊n/2⌋
    L ← MergeSort(A[1 : mid])
    R ← MergeSort(A[mid + 1 : n])
    return Merge(L, R)
```

Algorithm 2 Merge

```
Input: Two sorted arrays left and right
result ← []
i ← 0
j ← 0
while i < length of left and j < length of right do
    if left[i] ≤ right[j] then
        append left[i] to result
        i ← i + 1
    else
        append right[j] to result
        j ← j + 1
while i < length of left do
    append left[i] to result
    i ← i + 1
while j < length of right do
    append right[j] to result
    j ← j + 1
return result
```

Step	Array
Initial	$(4, 3, 2, 5, 1, 8, 7, 6)$
divide-1	$(4, 3, 2, 5)$ $(1, 8, 7, 6)$
divide-2	$(4, 3)$ $(2, 5)$ $(1, 8)$ $(7, 6)$
divide-3	(4) (3) (2) (5) (1) (8) (7) (6)
merge-1	$(3, 4)$ $(2, 5)$ $(1, 8)$ $(6, 7)$
merge-2	$(2, 3, 4, 5)$ $(1, 6, 7, 8)$
merge-3	$(1, 2, 3, 4, 5, 6, 7, 8)$

Done!