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CSED261: Discrete Mathematics for Computer Science
Homework 6: Relations

Question 1. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

1. $x + y = 0$.
2. $x = \pm y$.
3. $x - y$ is a rational number.

Solutions

1. $x + y = 0$
   - **Reflexive:** $\forall x[x \in A \to (x, x) \in R]$.
     Let's assume that $x = 1$, then $1 + 1 = 2 \neq 0$. Thus, it is not reflexive.
   - **symmetric:** $\forall x \forall y[(x, y) \in R \to (y, x) \in R]$.
     Let's assume that $(x, y) \in R$, then $x + y = 0$, also $y + x = 0$. Thus, it is symmetric.
   - **Antisymmetric:** $\forall x \forall y[(x, y) \in R \wedge (y, x) \in R \to x = y]$.
     Let's assume that $(x, y) \in R, (y, x) \in R$. then, $x + y = 0, y + x = 0, y = -x$. Thus, it is not antisymmetric.
   - **Transitive:** $\forall x \forall y \forall z[(x, y) \in R \wedge (y, z) \in R \to (x, z) \in R]$.
     Let's assume that $(x, y) \in R, (y, z) \in R$. then, $x + y = 0, y + z = 0$. However, $x - z = 0$, not $x + z$. So, it is not transitive.

   Relation 1 has: symmetry.

2. $x = \pm y$
   (a) **Reflexive:** $\forall x[x \in A \to (x, x) \in R]$.
   For all x, $x = \pm x \to x = x$. Thus, it is reflexive.
   (b) **symmetric:** $\forall x \forall y[(x, y) \in R \to (y, x) \in R]$.
   Let's assume that $(x, y) \in R$, then $x = \pm y$, also $y = \pm x$. Thus, it is symmetric.
   (c) **Antisymmetric:** $\forall x \forall y[(x, y) \in R \wedge (y, x) \in R \to x = y]$.
   Let's assume that $(x, y) \in R, (y, x) \in R$. then, $x = \pm y, y = \pm x$. x and y can be 1 and -1 (not equal). So, it is not antisymmetric.
   (d) **Transitive:** $\forall x \forall y \forall z[(x, y) \in R \wedge (y, z) \in R \to (x, z) \in R]$.
   Let's assume that $(x, y) \in R, (y, z) \in R$. then, $x = \pm y, y = \pm z$. This means, $z = \pm(\pm x) = \pm x$. Thus, it is transitive.

   Relation 2 has: reflexivity, symmetry, transitivity.

Problem 1.3 is on the next page

3. $x - y$ is a rational number

   (a) **Reflexive:** $\forall x[x \in A \to (x, x) \in R]$.
   $\forall x \in \mathbb{R}, x - x = 0$ is rational number, so it is reflexive.

   (b) **symmetric:** $\forall x \forall y[(x, y) \in R \to (y, x) \in R]$.
   Let's assume that $(x, y) \in R$, then $x - y$ is rational number, then also $-(x - y) = y - x$ is rational number. Thus, it is symmetric.

   (c) **Antisymmetric:** $\forall x \forall y[(x, y) \in R \land (y, x) \in R \to x = y]$.
   When x is 5 and y is 3, $x - y = 2$ is rational number, $y - x = -2$ is also rational number but x is not equal to y. Thus, it is not antisymmetric.

   (d) **Transitive:** $\forall x \forall y \forall z[(x, y) \in R \land (y, z) \in R \to (x, z) \in R]$.
   Let's assume that $(x, y) \in R, (y, z) \in R$. then, $x - y$ is rational number, $y - z$ is rational number. Thus, $x - z = (x - y) + (y - z)$ is rational number(sum of two rational numbers). So, it is transitive.

   Relation 3 has: reflexivity, symmetry, transitivity.

Question 2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).

1. $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
2. $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
3. $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
4. $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$

Solutions

1.
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2.
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

3.
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

4.
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 3. Suppose that the relation R is reflexive. Show that R* is reflexive.

Solutions

R^* is the transitive closure of R, so $R^* = \cup_{i=1}^{\infty} R^i$ when $R^1 = R$, $R^n = R^{n-1} \circ R$. To prove this problem, we should show two things:

1. **The composition of two reflexive relations is reflexive.**
   Let R_1, R_2 are reflexive relations, and $R = R_1 \circ R_2$. Then for all $x \in A$, xRx holds because xR_1x and xR_2x hold. Therefore, R is reflexive.

2. **The union of two reflexive relations is reflexive.**
   Let R_1, R_2 are reflexive relations, and $R = R_1 \cup R_2$. Then for all $x \in A$, xRx holds because xR_1x or xR_2x holds. Therefore, R is reflexive.

Because R^* is constructed as the union and composition of reflexive relations R, R^* is reflexive.

Question 4. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

Solutions

No, I found a counterexample. It is relation R,

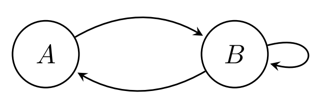
$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

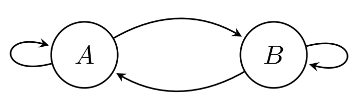
$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(This is not matrix multiplication, but the composition of relations.)
The relation $R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is reflexive because all diagonal elements are 1. Therefore, the relation R^2 is not necessarily irreflexive even though the relation R is irreflexive.

Additional)
R is,



R^2 is,



R does not have a self-loop of node A, but R^2 has a self-loop of every node. Therefore, R^2 is not irreflexive.

Question 5. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

1. $\{(a, b)|$ a and b are the same age$\}$
2. $\{(a, b)|$ a and b have the same parents$\}$
3. $\{(a, b)|$ a and b share a common parent$\}$
4. $\{(a, b)|$ a and b have met$\}$
5. $\{(a, b)|$ a and b speak a common language$\}$

Solutions

We have to check the three properties of an equivalence relation: reflexivity, symmetry, and transitivity.

1. $\{(a, b)|$ a and b are the same age$\}$
   reflexivity: a is the same age as a because a is a.
   symmetry: a and b are the same age if and only if b and a are the same age.
   transitivity: If a is the same age as b and b is the same age as c, then a is the same age as c.
   So, this relation is an equivalence relation.

2. $\{(a, b)|$ a and b have the same parents$\}$
   reflexivity: a has the same parents as a because a is a.
   symmetry: a and b have the same parents if and only if b and a have the same parents.
   transitivity: If a has same parents as b and b has the same parents as c, then a has the same parents as c.
   So, this relation is an equivalence relation.

3. $\{(a, b)|$ a and b share a common parent$\}$
   reflexivity: a shares a common parent with a because a is a.
   symmetry: a and b share a common parent if and only if b and a share a common parent.
   transitivity: If a shares a common parent with b and b shares a common parent with c, but a may not share a common parent with c. (only one common parent is enough to satisfy the condition)
   So, this relation is not an equivalence relation.

4. $\{(a, b)|$ a and b have met$\}$
   reflexivity: a has met a because a is a. However, we cannot ensure that a can meet itself, it is awkward.
   symmetry: a has met b if and only if b has met a.
   transitivity: If a has met b and b has met c, but a may not have met c.
   So, this relation is not an equivalence relation.

5. $\{(a, b)|$ a and b speak a common language$\}$
   reflexivity: a speaks a common language with a because a is a.
   symmetry: a speaks a common language with b if and only if b speaks a common language with a.
   transitivity: If a speaks a common language with b and b speaks a common language with c, However, if b speaks English and Korean, a speaks English and c speaks Korean, then a and c don't speak a common language.
   So, this relation is not an equivalence relation.

So, the equivalence relations are 1 and 2. relations 3, 4, and 5 are not equivalence relations because they do not satisfy the transitivity property.

Question 6. Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

Solutions

We have to check the three properties of an equivalence relation: reflexivity, symmetry, and transitivity.
Reflexivity: $((a,b),(a,b))$ is in R because $ab = ba$, So R is reflexive.
Symmetry: If $((a,b),(c,d))$ is in R, then $ad = bc$. So, $((c,d),(a,b))$ is in R because $cb = da$. So, R is symmetric.
Transitivity: If $((a,b),(c,d))$ and $((c,d),(e,f))$ are in R, then $ad = bc$ and $cf = de$. So, $((a,b),(e,f))$ is in R because $af = \frac{bc}{d} \cdot \frac{de}{c} = be$. So, R is transitive.

So, R is an equivalence relation.