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CSED261: Discrete Mathematics for Computer Science
Homework 2: Logic and Proof

Question 1. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.

1. "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."

2. "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."

3. "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."

4. "Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."

5. "All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."

6. "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."

Solutions

1. let p is "I play hockey", q is "I am sore", r is "I use the whirlpool".

Step	Reason
1. $q \rightarrow r$	Premise
2. $\neg r$	Premise
3. $\neg q$	Modus Tollens using (1) and (2).
4. $p \rightarrow q$	Premise
5. $\neg p$	Modus Tollens using (3) and (4) **Conclusion**.

   So, Conclusion is $\neg p$, I didn't play hockey.

2. let $p(x)$ is "I work on x", $q(x)$ is "It is sunny on x", $r(x)$ is "It is partly sunny on x".

Step	Reason
1. $\forall x p(x) \rightarrow (q(x) \vee r(x))$	Premise
2. $\neg(q(\text{Tuesday}) \vee r(\text{Tuesday}))$	Premise
3. $p(\text{Tuesday}) \rightarrow (q(\text{Tuesday}) \vee r(\text{Tuesday}))$	Universal instantiation using (1).
4. $\neg p(\text{Tuesday})$	Modus Tollens using (2) and (3). **Conclusion 1**

   So, Conclusion is "I didn't work on Tuesday".

3. let $p(x)$ is "x is insect" and $q(x)$ is "x has six legs". $r(x, y)$ is "x eats y".

Step	Reason
1. $\forall x p(x) \rightarrow q(x)$	Premise
2. $p(\text{dragonfly})$	Premise
3. $p(\text{dragonfly}) \rightarrow q(\text{dragonfly})$	Universal instantiation using (1).
4. $q(\text{dragonfly})$	Modus Ponens using (2) and (3). **Conclusion 1**
5. $\neg q(\text{spider})$	Premise
6. $p(\text{spider}) \rightarrow q(\text{spider})$	Universal instantiation using (1).
7. $\neg p(\text{spider})$	Modus Tollens using (5) and (6). **Conclusion 2**

   So. Conclusions are "dragonflies have six legs" and "spiders are not insects".

4. let $p(x)$ is "x is student" and $q(x)$ is "x has an internet account".

Step	Reason
1. $\forall x p(x) \to q(x)$	Premise
2. $\neg q(\text{Homer})$	Premise
3. $p(\text{Homer}) \to q(\text{Homer})$	Universal instantiation using (1).
4. $\neg p(\text{Homer})$	Modus Tollens using (2) and (3). **Conclusion**

So, Conclusion is "Homer is not a student".

5. let $p(x)$ is "x is healthy to eat", $q(x)$ is "x tastes good", and $r(x)$ is "you eat x".

Step	Reason
1. $\forall x p(x) \to \neg q(x)$	Premise
2. $p(\text{tofu})$	Premise
3. $p(\text{tofu}) \to \neg q(\text{tofu})$	Universal instantiation using (1).
4. $\neg q(\text{tofu})$	Modus Ponens using (2) and (3). **Conclusion**

So, Conclusion is "tofu does not taste good". There are not any other conclusions computed from the premises.

6. let p is "I am dreaming", q is "I am hallucinating", and r is "I see elephants running down the road".

Step	Reason
1. $p \vee q$	Premise
2. $\neg p$	Premise
3. q	Disjunctive syllogism using (1) and (2).
4. $q \to r$	Premise
5. r	Modus Ponens using (3) and (4). **Conclusion**

So, Conclusion is "I see elephants running down the road".

Question 2. Determine whether these are valid arguments.

1. If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

2. If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

Solutions

1. Argument : If a^2 is positive, where a is a real number, then a is a positive real number.

   This argument is false. I'll show that this argument is false by giving a counter example.

   Counter example : $\exists x \neg P(x) \equiv \neg \forall x P(x)$

   Let $P(x)$ be "x^2 is positive, where x is real number, then a is a positive real number."

   let $c = -1$, then $c^2 = 1$ which is positive. But c is not a positive real number.

   So, $\neg P(c)$, therefore $\exists x \neg P(x)$ by EG.

   Eventually, $\neg \forall x P(x)$, so the argument is false.

2. Argument : Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

   Direct Proof.

   Let $P(x)$ be "$x^2 \neq 0$, where x is a real number, then $x \neq 0$."

   $\forall x P(x)$, so $P(a)$ is true by UI.

   So the argument is valid.

Question 3. Prove that if n is an integer and $3n + 2$ is even, then n is even using

1. a proof by contraposition.

2. a proof by contradiction.

Solutions

Conjecture: If n is an integer and $3n + 2$ is even, then n is even.

1. Let's Assume that n is odd. Then $n = 2k + 1$ for some integer k. Then

$$3n + 2 = 3(2k + 1) + 2 = 6k + 3 + 2 = 6k + 5 = 2(3k + 2) + 1$$

   So, $3n + 2$ is odd. Since we have shown $\neg q \to \neg p$, $p \to q$ must hold as well. Conjecture is true.

2. Let Conjecture be $p \to q$. Let's assume that $p \wedge \neg q$ is true. Then $3n + 2$ is even and n is odd. Then $n = 2k + 1$ for some integer k. Then

$$3n + 2 = 3(2k + 1) + 2 = 6k + 3 + 2 = 6k + 5 = 2(3k + 2) + 1$$

   So, $3n + 2$ is odd. This is a contradiction. So, $p \to q$ must hold. Conjecture is true.

Question 4. Show that at least three of any 25 days chosen must fall in the same month of the year.

Solutions

$r \equiv p \to q$ is the original statement, where p is "pick 25 days" and q is "at least three days fall in the same month".

Then, $\neg r \equiv \neg(p \to q) \equiv p \wedge \neg q.$, where $\neg q$ means "at most 2 fall on the same month of the year", which is equivalent to "at most 24 days are picked", $\neg p$.

Eventually, $\neg r \equiv p \wedge \neg q \equiv p \wedge \neg p$: Contradiction.

So, the original statement is true.

Question 5. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n + \epsilon$, n is an integer, and $0 \leq \epsilon < 1$

Solutions

Existence Proof

Let x be a real number. Then, $n = \lfloor x \rfloor$ and $\epsilon = x - \lfloor x \rfloor$. Then $\exists x P(x)$ is true by EG.

Uniqueness Proof

Let's assume that there are two pairs of numbers (n_1, ϵ_1) and (n_2, ϵ_2) such that $x = n_1 + \epsilon_1 = n_2 + \epsilon_2$, n_1, n_2 are integers, and $0 \leq \epsilon_1, \epsilon_2 < 1$.

Then, $n_1 - n_2 = \epsilon_2 - \epsilon_1$. Since $n_1 - n_2$ is an integer, $\epsilon_2 - \epsilon_1$ must be an integer. But $0 \leq \epsilon_1, \epsilon_2 < 1$, so $-1 < \epsilon_2 - \epsilon_1 < 1$. Therefore, $\epsilon_2 - \epsilon_1 = 0$. Then, $n_1 - n_2 = 0$, so $n_1 = n_2$ and $\epsilon_1 = \epsilon_2$. So, the pair of numbers is unique.

Question 6. Prove that between every rational number and every irrational number there is an irrational number.

Solutions

Let r be rational number and i be an irrational number. Then,

$$\text{Arithmetic Mean} = \frac{i + r}{2}$$

is irrational. Let's prove artihmetic mean is between r and i. When $r < i$, then $r < \dfrac{i + r}{2} < i$. When $r > i$, then $i < \dfrac{i + r}{2} < r$. Also, addition and division are the closed under irrational number.

So, there is an irrational number between every rational number and every irrational number.