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CSED261: Discrete Mathematics for Computer Science
Homework 3: Sets and Set Operations & Function and Sequences & Set Cardinality and Matrices

Question 1. Prove or disprove that if A, B and C are nonempty sets and $A \times B = A \times C$, then $B = C$.

Solutions

I'll use proof by contradiction. Let $B = b_0, b_1, \cdots b_n$ and $C = c_0, c_1, \cdots c_m$. Let's assume that $A \times B = A \times C$ however $B \neq C$. Without the generality, Let $a_0 \in A$, and $b_0 \in B$ and $b_0 \notin C$ or vice versa. Then, $(a_0, b_0) \in A \times B$ but $(a_0, b_0) \notin A \times C$. So, $A \times B \neq A \times C$, this is a contradiction.

Question 2. Let A, B, and C be sets. Use the the identity $A - B = A \cap \overline{B}$, which holds for any sets A and B, and the set identities to show that $(A - B) \cap (B - C) \cap (A - C) = \emptyset$.

Solutions

I'll use the notation A^c to denote the complenet of the set instead overline.

Reason	Expression
Given	$(A - B) \cap (B - C) \cap (A - C)$
Def of difference of set	$= (A \cap B^c) \cap (B \cap C^c) \cap (A \cap C^c)$
Associative laws	$= A \cap B^c \cap B \cap C^c \cap A \cap C^c$
Idempotent laws	$= A \cap B \cap B^c \cap C^c$
Complement laws	$= A \cap \phi \cap C^c$
Domination laws	$= \phi$

Question 3. Show that if x is a real number, then $\lceil x \rceil - \lfloor x \rfloor = 1$ if x is not an integer and $\lceil x \rceil - \lfloor x \rfloor = 1$ if x is an integer.

Solutions

The floor function, denoted $f(x) = \lfloor x \rfloor$, is the largest integer less than or equal to x.
The ceiling function, denoted $f(x) = \lceil x \rceil$, is the smallest integer greater than or equal to x.

Let x be a real number. If x is an integer, then $\lceil x \rceil = \lfloor x \rfloor = x$. So, $\lceil x \rceil - \lfloor x \rfloor = 0$.

Else, if x is not integer, x can be presented as $x = n + \alpha$, where n is an integer and $0 < \alpha < 1$. Then, because of the definition of two function, $\lfloor x \rfloor = n$ and $\lceil x \rceil = n + 1$. So, $\lceil x \rceil - \lfloor x \rfloor = 1$.

Question 4. Use the identify $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ and **telescoping** to compute $\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Note. $\sum_{j=1}^{n} a_j - a_{j-1} = a_n - a_0$, where $a_0, a_1, ..., a_n$ is a sequence of real numbers. This type of sum is called telescoping.

Solutions

Using the given identify formula, We can change of the sum as:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

Then, we can rewrite the sum seperately as:

$$\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Only first and last terms are remain, and the sum of others are 0. (This is the main property of telescoping sums)
Finally, we get:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \left(\frac{1}{1} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

Question 5. Show that the set $Z^+ \times Z^+$ is countable.

Solutions

Let's $Z^+ = \{1, 2, 3, \cdots\}$. We can represent the set $Z^+ \times Z^+$ with Cartesian product of two countable sets:

$$Z^+ \times Z^+ = \{(a, b) | a, b \in Z^+\}$$

We can write this set as a grid:

	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$\cdots$
$a = 1$	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$\cdots$
$a = 2$	$(2, 1)$	$(2, 2)$	$(2, 3)$	$(2, 4)$	$\cdots$
$a = 3$	$(3, 1)$	$(3, 2)$	$(3, 3)$	$(3, 4)$	$\cdots$
$a = 4$	$(4, 1)$	$(4, 2)$	$(4, 3)$	$(4, 4)$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Then, we can list all elements of $Z^+ \times Z^+$. The criterion is to move $a + b$ is assending, and if $a + b$ is same, move a is assending. Then, we can list all elements of $Z^+ \times Z^+$ as $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), \cdots$.

Finally, conclusion is that $Z^+ \times Z^+$ is countable.

Question 6. The $n \times n$ matrix $A = [a_{ij}]$ is called a **diagonal matrix** if $a_{ij} = 0$ when $i \neq j$. Show that the product of two $n \times n$ diagonal matrices is again a diagonal matrix. Give a simple rule for determining this product.

Solutions

Let assume that there are two $n \times n$ diagonal matrix A, B.Then,

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix}.$$

We can easily compute the product of two matrices. Let C be $A \times B$, then c_{ij}, the element of i-th row and j-th columes of C, is $\displaystyle\sum_{k=1}^{n} a_{ik} b_{kj}$ because of definition.

For example,

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} \\ &= a_1 b_1 + 0 + \cdots + 0 \end{aligned}$$

Like this example, we can see that $c_{ii} = a_i b_i$ and $c_{ij} = 0$ when $i \neq j$ with easy computation. So, C is also a diagonal matrix.