# Exercise Sheet 5

MAGIC009 - Category Theory

November 8th, 2024

**Note.** This set of exercises aims at understanding better what we called 'naturality conditions' in one of the characterisation of adjunctions in Lecture 4. As you will see in Exercise 4, these conditions express the naturality condition for a particular natural transformation. Along the way, you will also encounter the opposite of a category, which will feature in some of the next lectures.

1. Let $F, G \colon \mathbb{C} \to \mathbb{D}$ be functors. Prove that a natural transformation $\phi \colon F \Rightarrow G$ is a natural isomorphism, *i.e.* an isomorphism in the functor category $[\mathbb{C}, \mathbb{D}]$, if and only if for every $X \in \mathbb{C}$, the map $\phi_X \colon FX \to GX$ has an inverse $\phi_X^{-1} \colon GX \to FX$. *Hint:* the essence of the exercise is to show that the maps $\phi_X^{-1}$ form a natural transformation.

2. Let $\mathbb{C}$, $\mathbb{D}$ be categories. We can define a new category $\mathbb{C} \times \mathbb{D}$, called the *product* of $\mathbb{C}$ and $\mathbb{D}$, as follows:

   - the objects of $\mathbb{C} \times \mathbb{D}$ are pairs $(X, Y)$, where $X \in \mathbb{C}$ and $Y \in \mathbb{D}$,
   - the morphisms $(f, g) \colon (X, Y) \to (X', Y')$ in $\mathbb{C} \times \mathbb{D}$, are pairs consisting of a morphism $f \colon X \to X'$ in $\mathbb{C}$ and a morphism $g \colon Y \to Y'$ in $\mathbb{C}$.

   (a) Complete the definition of $\mathbb{C} \times \mathbb{D}$ by defining its composition and identity morphisms and proving the axioms for a category.

   (b) Let $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ be categories. Assume that there are
   - a functor $F^X \colon \mathbb{D} \to \mathbb{E}$, for every $X \in \mathbb{C}$,
   - a functor $F_Y \colon \mathbb{C} \to \mathbb{E}$, for every $Y \in \mathbb{D}$,

   such that
   - $F^X(Y) = F_Y(X)$, for all $X \in \mathbb{C}$ and $Y \in \mathbb{D}$,
   - the diagram

   $$\begin{tikzcd} F^X(Y) \arrow[r, "F_Y(f)"] \arrow[d, "F^X(g)"'] & F^{X'}(Y) \arrow[d, "F^{X'}(g)"] \\ F^X(Y') \arrow[r, "F_{Y'}(f)"'] & F^{X'}(Y') \end{tikzcd}$$

   commutes for all $f \colon X \to X'$ in $\mathbb{C}$ and $g \colon Y \to Y'$ in $\mathbb{D}$.

   Show that there is a functor

   $$F \colon \mathbb{C} \times \mathbb{D} \to \mathbb{E}$$

   such that $F(X, Y) = F^X(Y) = F_Y(X)$.

   (c) Let $F, G \colon \mathbb{C} \times \mathbb{D} \to \mathbb{E}$ be functors. Let us assume we have a family of maps

   $$\phi_{X,Y} \colon F(X, Y) \to G(X, Y)$$

   in $\mathbb{E}$, where $X \in \mathbb{C}$ and $Y \in \mathbb{D}$. Prove that $\phi$ is a natural transformation from $F$ to $G$ if and only if the following two conditions hold:
   - the diagram

   $$\begin{tikzcd} F(X, Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "F(f,1_Y)"'] & G(X, Y) \arrow[d, "G(f,1_Y)"] \\ F(X', Y) \arrow[r, "\phi_{X',Y}"'] & G(X', Y) \end{tikzcd}$$

   commutes for every $f \colon X \to X'$ in $\mathbb{C}$,

- the diagram

$$\begin{tikzcd} F(X,Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "F(f,1_Y)"'] & G(X,Y) \arrow[d, "G(f,1_Y)"] \\ F(X',Y) \arrow[r, "\phi_{X',Y}"'] & G(X',Y) \end{tikzcd}$$

commutes for every $g\colon Y \to Y'$.

3. For a category $\mathbb{C}$, we define a new category $\mathbb{C}^{\mathrm{op}}$, called the *opposite* of $\mathbb{C}$, as follows:

   - the objects of $\mathbb{C}^{\mathrm{op}}$ are the objects of $\mathbb{C}$,
   - for $X, Y \in \mathbb{C}$,

   $$\mathbb{C}^{op}(X,Y) =_{\mathrm{def}} \mathbb{C}(Y,X)\,.$$

   Thus, a morphism $f\colon Y \to X$ in $\mathbb{C}^{\mathrm{op}}$ is a morphism $f\colon X \to Y$ in $\mathbb{C}$. *Note that $f\colon Y \to X$ in $\mathbb{C}^{\mathrm{op}}$ is not an inverse to $f\colon X \to Y$ in $\mathbb{C}$, which may not even exist.* We do not distinguish notationally between maps in $\mathbb{C}$ and maps in $\mathbb{C}^{\mathrm{op}}$, but we keep track in which category we consider them to be living in, as in the previous sentence.

   (a) Complete the definition of the category $\mathbb{C}^{\mathrm{op}}$, defining its composition and identity morphisms and proving the axioms for a category.

   (b) Let $\mathbb{C}$ be locally small. Prove that the function mapping a pair of objects $X, Y$ of $\mathbb{C}$ to the set $\mathbb{C}(X,Y)$ extends to a functor

   $$\mathbb{C}(-,-)\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}\,.$$

   *Hint:* use Exercise 2b.

4. Let $\mathbb{C}$ and $\mathbb{D}$ be locally small categories. Let $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{C}$ be functors.

   (a) Prove that the function sending $X \in \mathbb{C}$ and $A \in \mathbb{D}$ to the set $\mathbb{D}(FX, A)$ extends to a functor

   $$\mathbb{D}(F-,-)\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathbf{Set}\,.$$

   *Hint:* use Exercise 2b.

   (b) Prove that the function sending $X \in \mathbb{C}$ and $A \in \mathbb{D}$ to the set $\mathbb{C}(X, GA)$ extends to a functor

   $$\mathbb{C}(-,G-)\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathbf{Set}\,.$$

   *Hint:* use Exercise 2b.

   (c) Check that the naturality condition in item (ii) of the theorem with equivalent characterisations of adjunctions (Lecture 4) expresses that the bijections

   $$\varphi_{X,A}\colon \mathbb{C}(X,GA) \to \mathbb{D}(FX,A)\,,$$

   for $X \in \mathbb{C}$ and $A \in \mathbb{D}$, form a natural transformation

   $$\varphi\colon \mathbb{C}(-,G-) \Rightarrow \mathbb{D}(F-,-)\,.$$

   *Hint:* use Exercise 2c.

   (d) Prove that $\varphi\colon \mathbb{C}(-,G-) \Rightarrow \mathbb{D}(F-,-)$ is a natural isomorphism. *Hint:* use Exercise 1.

   (e) State the naturality condition for the inverse of the natural transformation $\varphi$ and restate it in a simpler equivalent form using Exercise 2c.