2024-2025 MAGIC 009 exam solutions

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1 Question 1

1. Let $f : X \to Y$ be a morphism in $\mathbb{A}$. We need to show that $X \mapsto \theta_{FX}$ varies naturally in f. Observe that by naturality of θ on Ff, the following diagram commutes in $\mathbb{B}$.

$$\begin{tikzcd} PFX \arrow[r, "PFf"] \arrow[d, "\theta_{FX}"'] & PFY \arrow[d, "\theta_{FY}"] \\ QFX \arrow[r, "QFf"'] & QFY \end{tikzcd}$$

   But this is the desired naturality square for $\theta.F$.

2. Let $f : X \to Y$ be a morphism in $\mathbb{A}$. Naturality of $P.\phi$ follows from the following calculation.

$$\begin{aligned} P\phi_Y \circ PFf &= P(\phi_Y \circ Ff) && \text{functoriality of } P \\ &= P(Gf \circ \phi_X) && \text{naturality of } \phi \text{ on } f \\ &= PGf \circ P\phi_X && \text{functoriality of } P \end{aligned}$$

3. Let $X \in \mathbb{A}$ and consider the following diagram in $\mathbb{C}$.

$$\begin{tikzcd} PFX \arrow[r, "P\phi_X"] \arrow[d, "\theta_{FX}"'] & PGX \arrow[d, "\theta_{GX}"] \\ QFX \arrow[r, "Q\phi_X"'] & QGX \end{tikzcd}$$

   Observe that this diagram commutes by naturality of θ in the morphism $\phi_X : FX \to GX$. The proof is complete by also observing that the clockwise traversal of this diagram is the component of $(\theta.G) \circ (P.\phi)$ on X, and the anticlockwise traversal is the component of $(Q.\phi) \circ (\theta.F)$ on X.

4. We need to check that the assignments described in the question give a well-defined functor $[\mathbb{B}, \mathbb{C}] \times [\mathbb{A}, \mathbb{B}] \to [\mathbb{A}, \mathbb{C}]$. For the unit law, observe that when both ϕ and θ are identities, the pair (θ, ϕ) is assigned to a natural transformation whose components are given by the composite of two identity morphisms. This uses the fact that identity natural transformations have identity components and the unit laws for P and Q. The unit law follows since the composite of two identity morphisms is again an identity morphism. For the composition law, consider a composable pair in $[\mathbb{B}, \mathbb{C}] \times [\mathbb{A}, \mathbb{B}]$ as displayed below.

$$(P,F) \xrightarrow{(\theta,\phi)} (Q,G) \xrightarrow{(\chi,\psi)} (R,H)$$

   The composition law follows from the following calculation.

$$
\begin{aligned}
((\chi \circ \theta) * (\psi \circ \phi))_X &:= (\chi \circ \theta)_{HX} \circ P(\psi \circ \phi)_X && \text{definition of } * \\
&:= (\chi_{HX} \circ \theta_{HX}) \circ P(\psi \circ \phi)_X && \text{definition of } \chi \circ \theta \text{ and } \psi \circ \phi \\
&= (\chi_{HX} \circ \theta_{HX}) \circ P\psi_X \circ P\phi_X && \text{functoriality of } P \\
&= (\chi_{HX} \circ (\theta_{HX} \circ P\psi_X)) \circ P\phi_X && \text{associativity} \\
&= (\chi_{HX} \circ (Q\psi_X \circ \theta_{GX})) \circ P\phi_X && \text{naturality of } \theta \text{ on } \psi_X \\
&= (\chi_{HX} \circ Q\psi_X) \circ (\theta_{GX} \circ P\phi_X) && \text{associativity} \\
&:= (\chi * \psi)_X \circ (\theta * \phi)_X && \text{definition of } \chi * \psi \text{ and } \theta * \phi \\
&:= ((\chi * \psi) \circ (\theta * \phi))_X && \text{definition of } \alpha \circ \beta \text{ where } \alpha = \chi * \psi \text{ and } \beta = \theta * \phi
\end{aligned}
$$

2 Question 2

1. There are many examples but the simplest one (and the one briefly mentioned in lectures) is $\mathbf{1} \to \mathbb{I}$ where $\mathbb{I}$ is the free-living isomorphism and $\mathbf{1}$ is the category with one object and one morphism.

2. First observe that by the following calculation, the equaliser diagram commutes.

$$
\begin{aligned}
es &= srs \\
&= s.1_B \\
&= s \\
&= 1_A.s
\end{aligned}
$$

Let $f : X \to A$ satisfy $ef = f$. We need to show that there is a unique morphism $u : X \to B$ satisfying $su = f$. For existence, take $u := rf$ and observe that $su = f$ as per the following calculation.

$$
\begin{aligned}
su &= srf \\
&= ef \\
&= f
\end{aligned}
$$

For uniqueness, suppose $u' : X \to B$ also satisfies $su' = f$. Then $u' = u$ by the calculation displayed below.

$$
\begin{aligned}
u' &= 1_B.u' \\
&= rsu' \\
&= rf \\
&= u
\end{aligned}
$$

Hence s is indeed the equaliser of 1_A and e.

3. Apply part (2) to $\mathbb{C}^{\mathrm{op}}$, in which the idempotent e has splitting (s, r).

4. By parts (2) and (3) it suffices to show that if e is an idempotent with splitting (r, s) then Fe is an idempotent with splitting (Fr, Fs). Idempotence follows from the following calculation.

$$
\begin{aligned}
Fe \circ Fe &= F(e \circ e) && \text{composition law for } F \\
&= Fe && \text{idempotence of } e
\end{aligned}
$$

But also $Fs \circ Fr = F(s \circ r) = Fe$, and finally

$$\begin{aligned}
Fr \circ Fs &= F(r \circ s) && \text{composition law for } F \\
&= F1_A && (r,s) \text{ is a splitting of } e \\
&= 1_{FA} && \text{unit law for } F
\end{aligned}$$

This completes the proof.

3 Question 3

1. Functoriality follows from the uniqueness aspect of the universal property of products of the form $FX \times GX$ in $\mathbb{B}$. This is immediate for the unit law, while for the composition law it may be observed via the following commutative diagram, in which the composite of the dotted arrows is $(F\times G)(g) \circ (F \times G)(f)$.

$$\begin{tikzcd}
FX \arrow[d, "Ff"'] \arrow[dd, bend right=60, "F(gf)"'] & FX \times GX \arrow[l, "\pi_{FX}"'] \arrow[r, "\pi_{GX}"] \arrow[d, dashed, "Ff \times Gf"] & GX \arrow[d, "Gf"] \arrow[dd, bend left=60, "G(gf)"] \\
FY \arrow[d, "Fg"'] & FY \times GY \arrow[l, "\pi_{FY}"'] \arrow[r, "\pi_{GY}"] \arrow[d, dashed, "Fg \times Gg"] & GY \arrow[d, "Gg"] \\
FZ & FZ \times GZ \arrow[l, "\pi_{FZ}"] \arrow[r, "\pi_{GZ}"'] & GZ
\end{tikzcd}$$

2. The functor $F \times G$ of part (1) defines a product in $[\mathbb{A}, \mathbb{B}]$. Naturality of the projection maps is clear from the definition of $F \times G$ on morphisms. We need to verify the universal property. Let $P : \mathbb{A} \to \mathbb{B}$ be a functor and let $\phi : P \to F$ and $\psi : P \to G$ be natural transformations. By the universal property of $FX \times GX$ for each $X \in \mathbb{A}$, there is a unique morphism $u_X : PX \to FX \times GX$ in $\mathbb{B}$. It therefore suffices to check that the assignment $X \mapsto u_X$ is natural in X. This is verified by the universal property of $FX \times GX$, as per the following calculations. Note that we suppress mention of the associativity axiom for $\mathbb{B}$.

$$\begin{aligned}
\pi_{FY} \circ u_Y \circ Pf &= \phi_Y \circ Pf && \text{definition of } u_Y \\
&= Ff \circ \phi_X && \text{naturality of } \phi \text{ on } f \\
&= Ff \circ \pi_X \circ u_X && \text{definition of } u_X \\
&= \pi_{FY} \circ (Ff \times Gf) \circ u_X && \text{definition of } Ff \times Gf
\end{aligned}$$

$$\begin{aligned}
\pi_{GY} \circ u_Y \circ Pf &= \psi_Y \circ Pf && \text{definition of } u_Y \\
&= Gf \circ \psi_X && \text{naturality of } \psi \text{ on } f \\
&= Gf \circ \pi_X \circ u_X && \text{definition of } u_X \\
&= \pi_{GY} \circ (Ff \times Gf) \circ u_X && \text{definition of } Ff \times Gf
\end{aligned}$$

3. This means that the category $\mathbb{B}$ has n-ary products. Specifically, for any family of objects $\{X_1, ..., X_n\}$ in $\mathbb{B}$, there is an object $X_1 \times ... \times X_n \in \mathbb{B}$ and there are morphisms $\pi_i : X_1 \times ... \times X_n \to X_i$ for each $i \in \{1, ..., n\}$. Moreover, these data satisfy the universal property that given any other object $P \in \mathbb{B}$ and morphisms $p_i : P \to X_i$ for $i \in \{1, ..., n\}$ there is a unique morphism $u : P \to X_1 \times ... \times X_n$ satisfying $\pi_i \circ u = p_i$ for all $i \in \{1, ..., n\}$.

4. The proof is by induction on n for $n \geq 2$, with base case $n = 2$ being true by the assumption that $\mathbb{B}$ has binary products. For the induction hypothesis, suppose that $\mathbb{B}$ has n-ary products $X_1 \times ... \times X_n$ for some $n \geq 2$ and all families of objects $X_1, ..., X_n$. Let X_{n+1} be an object in $\mathbb{B}$. Since $\mathbb{B}$ has binary products, the product $(X_1 \times ... \times X_n) \times X_{n+1}$ also exists in $\mathbb{B}$. Let $\pi_{1,...,n} : (X_1 \times ... \times X_n) \times X_{n+1} \to X_1 \times ... \times X_n$ and $\pi_{n+1} : (X_1 \times ... \times X_n) \times X_{n+1} \to X_{n+1}$ denote the projection maps for the binary product.

   We claim that $(X_1 \times ... \times X_n) \times X_{n+1}$ also forms an $(n + 1)$-ary product with projection maps $\pi_i \circ \pi_{1,...,n} : (X_1 \times ... \times X_n) \times X_{n+1} \to X_i$ for $i \in \{1, ..., n\}$ and $\pi_{n+1} : (X_1 \times ... \times X_n) \times X_{n+1} \to X_{n+1}$. We now check the universal property of an $(n + 1)$-ary product. Let P be an object in $\mathbb{B}$ and for $i \in \{1, ..., n + 1\}$ let $p_i : P \to X_i$ be a morphism in $\mathbb{B}$. Since by the induction hypothesis $X_1 \times ... \times X_n$ is an n-ary product in $\mathbb{B}$, the maps $p_1, ..., p_n$ induce a unique map $q : P \to X_1 \times ... \times X_n$ satisfying $\pi_i \circ q = p_i$ for all $i \in \{1, ..., n\}$. But since $(X_1 \times ... \times X_n) \times X_{n+1}$ is a binary product, the pair of maps $(q : P \to X_1 \times ... \times X_n, p_{n+1} : P \to X_{n+1})$ induce a unique map $u : P \to (X_1 \times ... \times X_n) \times X_{n+1}$ satisfying $\pi_{n+1} \circ u = p_{n+1}$ and $\pi_{1,...,n} \circ u = q$. Thus by the definition of q the family of maps $p_1, ..., p_{n+1}$ indeed induce a unique map $u : P \to (X_1 \times ... \times X_n) \times X_{n+1}$ which compose with each of the i-th projection maps to give p_i. This completes the proof.

4 Question 4

1. Let $Y, Z \in \mathbb{B}$ be objects and suppose that the product $Y \times Z$ exists in $\mathbb{B}$. Denote the product projections $\pi_Y : Y \times Z \to Y$ and $\pi_Z : Y \times Z \to Z$. We need to show that there is a bijective function $\mathbb{B}(X, Y \times Z) \to \mathbb{B}(X, Y) \times \mathbb{B}(X, Z)$. We claim that the function ϕ defined via $f \mapsto (\pi_Y \circ f, \pi_Z \circ f)$ is a bijection, with inverse ψ given by the function which sends a pair of morphisms $g : X \to Y$, $h : X \to Z$ to the morphism $< g, h >: X \to Y \times Z$ induced by the universal property of the product.

   The fact that $\psi \circ \phi$ is the identity is verified by the calculations displayed below.

$$\begin{aligned} \psi(\phi(f)) &= \psi(\pi_Y \circ f, \pi_Z \circ f) && \text{definition of } \phi \\ &= f && \text{uniqueness aspect of the universal property of the product} \end{aligned}$$

   The fact that $\phi \circ \psi$ is the identity is verified by the calculations displayed below.

$$\begin{aligned} \phi(\psi(g,h)) &= (\pi_Y \circ \psi(g,h), \pi_Z \circ \psi(g,h)) && \text{definition of } \phi \\ &= (g,h) && \text{definition of } \psi \end{aligned}$$

2. Let $F : \mathbb{B}^{\text{op}} \to \mathbf{Set}$ be a functor. Then for each $X \in \mathbb{B}$ there is a unique function $!_X : FX \to \mathbf{1}$. It suffices to show that the assignment $X \mapsto !_X$ gives rise to a natural transformation $! : F \to \Delta(\mathbf{1})$. But naturality is an equation for pairs of maps into $\mathbf{1}$ and hence follows from the fact that $\mathbf{1}$ is the terminal object in $\mathbf{Set}$.

3. The representable presheaf $\overline{G}(-, *) : \overline{G}^{\text{op}} \to \mathbf{Set}$ on $\overline{G}$ assigns the unique object of $\overline{G}$ to the underlying set G of the group $(G, ., e)$, and it assigns each element $h \in G$, viewed as a morphism in $\overline{G}$, to the function $h.(-) : G \to G$ given by multiplicationon the left with h. The representable natural transformation $\overline{G}(-, g) : \overline{G}(-, *) \to \overline{G}(-, *)$ has a single component corresponding to the unique object $* \in \overline{G}$. This component is a function $(-).g : G \to G$ which acts by multiplication with g on the right.

4. The Yoneda embedding $Y_{\overline{G}} : \overline{G} \to [\overline{G}^{\text{op}}, \mathbf{Set}]$ assigns $*$ to the presheaf $\overline{G}(-, *)$, and assigns $g : * \to *$ to the representable natural transformation $\overline{G}(-, g) : \overline{G}(-, *) \to \overline{G}(-, *)$. There are two requirements for $Y_{\overline{G}}$ to be well-defined as a functor. The first is that $\overline{G}(-, e) : \overline{G}(-, *) \to \overline{G}(-, *)$ should be the identity natural transformation. This is to say that multiplication by e on the right acts as the identity on G, which is true by the right unit law for G. The other requirement is that $\overline{G}(-, h) \circ \overline{G}(-, g) = \overline{G}(-, h \circ g)$. This means that for an element $f \in G$, $h.(g.f) = (h.g).f$. This is true by the associativity law for G.

5. The Yoneda embedding $Y_{\overline{G}}$ being fully faithful is precisely to say that elements of G are in bijection with natural transformations of the form $\overline{\phi} : \overline{G}(-,*) \to \overline{G}(-,*)$. But these natural transformations are precisely functions $\phi : G \to G$ which satisfy $h.\phi(k) = \phi(h.k)$. Fully faithfulness of the Yoneda embedding says that the assignation of an element $g \in G$ to the function $G \to G$ which multiplies by g on the right is a bijection.