Exercise Sheet 4

MAGIC009 - Category Theory

November 1st, 2024

1. Consider the forgetful functor $U\colon \mathbf{Mon} \to \mathbf{Set}$ and its left adjoint $F\colon \mathbf{Set} \to \mathbf{Mon}$ given by the functor mapping a set X to the free monoid on X.
   (i) Show that F is indeed a functor.
   (ii) Find the counit $\varepsilon\colon FG \to 1_{\mathbf{Mon}}$ of the adjunction.
   (iii) Define a family of bijections
   $$\mathbf{Set}(X, UA) \cong \mathbf{Mon}(FX, A)$$
   where $X \in \mathbf{Set}$ and $A \in \mathbf{Mon}$, as in item (ii) of the characterisation theorem for adjunctions, and explain explicitly what the naturality of this family means.

2. Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets and consider the associated categories $\underline{P}$ and $\underline{Q}$. Describe explictly what is an adjunction between $\underline{P}$ and $\underline{Q}$. *Hint:* Write f and g for the left and right adjoint, respectively, and use the characterisation of adjoints in item (ii) of the characterisation theorem for adjunctions.

3. Let $f\colon X \to Y$ be a function between sets and let $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ be the inverse image function, defined by
   $$f^*(V) = \{x \in X \mid f(x) \in V\}\,.$$
   (i) Check that f^* is order-preserving and hence can be regarded as a functor.
   (ii) Show that the function $\exists_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by
   $$\exists_f(U) = \{y \in Y \mid (\exists x \in U) f(x) = y\}$$
   is a left adjoint to f^*. *Hint:* Use the solution to Exercise 1.
   (iii) Show that the function $\forall_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by
   $$\forall_f(U) = \{y \in Y \mid (\forall x \in U) f(x) = y\}$$
   is a right adjoint to f^*. *Hint:* Use the solution to Exercise 1.

4. Let $F\colon \mathbb{D} \to \mathbb{C}$ be a functor, $A \in \mathbb{D}$. We define the **comma category** $F \downarrow A$ as the category with
   - objects: pairs (X, f), where $X \in \mathbb{C}$ and $f\colon FX \to A$ in $\mathbb{D}$.
   - maps $u\colon (X, f) \to (Y, g)$ are maps $u\colon X \to Y$ in $\mathbb{C}$ such that $F(u) \circ g = f$.

   (i) Prove in detail that $F \downarrow A$ is indeed a category.
   (ii) Describe what is a terminal object in $F \downarrow A$.
   (iii) Observe the connection between solution to part (ii) and the characterisation of right adjoint to F in the Remark on page 6 of the slides for Lecture 4.

5. Let F and G be functors with common codomain,
   $$\mathbb{B} \xrightarrow{F} \mathbb{C} \xleftarrow{G} \mathbb{D}\,.$$
   (i) Define the comma category $F \downarrow G$ by filling the dots below:
   - The objects are triples $(B, D, \ldots)$ where $B \in \mathbb{B}$, $D \in \mathbb{D}$ and ....

- The maps are pairs ...

(ii) Define 'projection' functors $P\colon F \downarrow G \to \mathbb{B}$ and $Q\colon F \downarrow G \to \mathbb{D}$.

(iii) Consider the special case where $\mathbb{D} = \mathbb{B}$, so that we have

$$\mathbb{B} \xrightarrow{F} \mathbb{C} \xleftarrow{G} \mathbb{B}$$

i.e. two functors with common domain and codomain. Unfold explicitly what is a functor $\alpha\colon \mathbb{B} \to F \downarrow G$ such that $P \circ \alpha = Q \circ \alpha = 1_{\mathbb{B}}$.