MAGIC073 – Commutative algebra – EXAM 2024-25

There are 5 questions for a total of 100 marks. You must attempt ALL questions. The exam has been set in a way that it can be completed in three hours by a student who has diligently followed the course. You may use the course material (notes and exercises) and computer algebra. If you do so, you must refer to precise statements in the notes or exercises, or give the details of the algorithms that you have used. Please follow the guidance provided on the MAGIC website regarding the submission of the completed assessment, including deadlines.

Question 1. Are the following true or false? Briefly prove your claims. If you refer to results in the lecture notes, you must indicate which one(s) precisely.

i. Let R, S be commutative rings. Every prime ideal of $R \times S$ is of the form $P \times Q$, where $P \in \mathrm{Spec}(R)$ and $Q \in \mathrm{Spec}(S)$. [2]

ii. Let R be a commutative ring and let I, J be ideals of R. Then $R/IJ \cong R/I \times R/J$ as rings. [2]

iii. The quotient ring $\mathbb{Z}[x,y]/(x^2 - y^3)$ is Noetherian. [2]

iv. Let S be an extension ring of R, with S commutative. Then $\mathrm{Rad}(R) \subseteq \mathrm{Rad}(S)$. [2]

v. Let R be a commutative ring, let F, M, N be finitely generated R-modules. Suppose that F is free and that $f \in \mathrm{Hom}_R(M, N)$ is injective. Then the induced R-homomorphism $f_* \in \mathrm{Hom}_R(F \otimes_R M, F \otimes_R N)$ is injective, where $f_*(x \otimes y) = x \otimes f(y)$ for all $x \otimes y \in F \otimes_R M$. [2]

vi. Every principal fractional ideal in a PID is invertible. [2]

vii. $\mathbb{Z}[i\sqrt{5}]$ is a PID. [2]

viii. $\mathbb{Z}[x,y]$ is integrally closed. [2]

Question 2.

i. Let $R = \mathbb{Z}/7[x]$ and let $I = (x^2 + 2)$. Let $\pi : R \to R/I$ denote the quotient map and write $\pi(f) = \overline{f}$ for all $f \in R$.

  (a) Prove that R/I is a field. [2]

  (b) Find $(\overline{x}^3)^{-1}$ in R/I. [4]

  (c) Calculate the fractional ideal I^{-1} and decide whether I is invertible. Prove your answers. [2]

ii. Describe $\mathrm{Spec}(\mathbb{Z}/10 \times \mathbb{Z}/21)$ by listing its clopen sets in the Zariski topology and give the connected components. Explain your method. [12]

Question 3.

i. Let R be a nontrivial commutative ring in which every proper ideal is prime. Prove that R is a field. [2]

ii. Consider the ring extension $\mathbb{Z} \subseteq \mathbb{Z}[\sqrt{3}]$, and the ideal (11) of $\mathbb{Z}$. Find all the prime ideals I of $\mathbb{Z}[\sqrt{3}]$ lying over (11), i.e. such that $I \cap \mathbb{Z} = (11)$. You may assume without proof that $\mathbb{Z}[\sqrt{3}]$ is a PID, and that if $a + b\sqrt{3}$ is such that $\pm(a^2 - 3b^2)$ is a prime integer, then $a + b\sqrt{3}$ is irreducible in $\mathbb{Z}[\sqrt{3}]$. [3]

iii. Let R be the subring of the polynomial ring $\mathbb{Z}[x,y]$ formed by the polynomials of the form $\sum_{i,j} a_{ij} x^i y^j$ such that $a_{0,1} = 0$; that is, the term of degree 1 in y is divisible by x. Let $I = (xy, y^2)$. Prove that R is not a PID and that I^2 is not primary. [5]

iv. Let $K = \mathbb{Q}(\sqrt{5})$ be the field of fractions of $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Z}\}$. Find the integral closure of R in K. [10]

Question 4. Let R be a commutative ring.

i. Let I, J be ideals of R such that $I \cap J = (0)$. Suppose that R/I and R/J are Noetherian. Prove that R is Noetherian. [5]

ii. Let S be a commutative ring and let $\varphi : R \to S$ be a surjective ring homomorphism.

(a) Prove that $\varphi(\mathrm{Rad}(R)) \subseteq \mathrm{Rad}(S)$. [5]

(b) Find an example with $\varphi(\mathrm{Rad}(R)) \subsetneq \mathrm{Rad}(S)$. [4]

iii. Let M be a simple R-module, that is, the only submodules of M are M and $\{0\}$. Let $I = \{a \in R \mid ax = 0, \ \forall\, x \in M\}$ be the annihilator of M in R. Prove that $I \in \mathrm{MaxSpec}(R)$. [8]

Question 5.

i. Let R be a commutative ring and let M, N be R-modules.

(a) Let I be an ideal of R and suppose that M is finitely generated. Prove that $\sqrt{\mathrm{Ann}(M/IM)} = \sqrt{\mathrm{Ann}(M) + I}$ in R. Recall that $\mathrm{Ann}(V) = \{a \in R \mid ax = 0, \ \forall\, x \in V\}$ for any R-module V. [5]

(b) Suppose that $M \cap N$ and $M + N$ are finitely generated. Prove that M and N are both finitely generated. [4]

ii. Let $\pi : R \to R/\mathrm{Rad}(R)$ be the quotient map. Suppose that M is finitely generated, and that there is a subset X of M such that $\pi(X)$ generates $\pi(M)$. Prove that X generates M. [3]

iii. Let R be a Noetherian ring and let $M, N_0, \ldots, N_n, Q_0$ be finitely generated R-modules with Q_0 projective. Suppose that we have a diagram

$$\begin{array}{ccccccccccc}
 & & & & & & Q_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & & & & & \big\downarrow \mathrm{Id}_M & & \\
N_n & \longrightarrow & \cdots & \longrightarrow & N_1 & \longrightarrow & N_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where the two rows are exact sequences. Prove that we can complete the diagram into a commutative diagram of R-modules and R-homomorphisms

$$\begin{array}{ccccccccccc}
Q_n & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \mathrm{Id}_M & & \\
N_n & \longrightarrow & \cdots & \longrightarrow & N_1 & \longrightarrow & N_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where the two rows are exact sequences. with $Q_1, \ldots, Q_n$ projective R-modules and the two rows exact. [10]