Exercise Sheet 1

MAGIC009 - Category Theory

October 11th, 2024

1. Check in detail that the following are indeed categories.
   (a) **Rel**, the category of sets and relations. For this, recall that a relation from a set X to a set Y is a subset $R \subseteq X \times Y$. We define the composite of $R \subseteq X \times Y$ and $S \subset Y \times Z$ as the relation $S \circ R \subset X \times Z$ given by
   $$S \circ R = \{(x, z) \in X \times Z \mid (\exists y \in Y)\ (x, y) \in R \wedge (y, z) \in S\} .$$
   The identity relation on a set X is the relation $1_X = \{(x, y) \in X \times X \mid x = y\}$.
   (b) **Grp**, the category of groups and group homomorphisms.
   (c) **Top**, the category of topological spaces and continuous functions.
2. Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be categories. There is a category, written $\mathbb{C}_1 \times \mathbb{C}_2$ and called the product of $\mathbb{C}_1$ and $\mathbb{C}_2$, such that
   - $\mathrm{Ob}(\mathbb{C}_1 \times \mathbb{C}_2) = \mathrm{Ob}(\mathbb{C}_1) \times \mathrm{Ob}(\mathbb{C})_2$
   - For $(X_1, X_2)\,, (Y_1, Y_2) \in \mathrm{Ob}(\mathbb{C}_1 \times \mathbb{C}_2)$,
   $$\mathbb{C}_1 \times \mathbb{C}_2\big((X_1, X_2), (Y_1, Y_2)\big) = \mathbb{C}_1(X_1, Y_1) \times \mathbb{C}_2(X_2, Y_2) .$$

   Complete this definition in the evident way and show that $\mathbb{C}_1 \times \mathbb{C}_2$ is a category.
3. Let $\mathbb{C}$ be a category. Recall from Lecture 1 that an an **inverse** of a map $f \colon X \to Y$ is a map $g \colon Y \to X$ such that
   $$g \circ f = 1_X \,, \quad f \circ g = 1_Y \,.$$
   and that we say that f is an **isomorphism** if it admits an inverse.
   (a) Prove that an inverse of f, if it exists, is unique.
   (b) Prove that the composite of two isomorphisms is an isomorphism and that identities are isomorphisms.
4. Let $(M, \cdot, 1)$ be a monoid and $\underline{M}$ the associated category.
   (a) Describe what is an isomorphism in the category $\underline{M}$ in terms of the monoid M.
   (b) Assume that every map in $\underline{M}$ is an isomorphism. What can be said about the monoid M?
5. Let $(P, \leq)$ be a poset and consider the category $\underline{P}$ associated to it.
   (a) Describe what is a terminal object in $\underline{P}$ in terms of the poset P.
   (b) Find a finite poset $(P, \leq)$ such that $\underline{P}$ does not have a terminal object.
6. Check that:
   (a) $\{*\}$ is a terminal object in **Set**,
   (b) $\varnothing$ is a terminal object in **Rel**.
7. The distributive law for a ring $(R, +, 0, \cdot, 1)$ says that
   $$a \cdot (b + c) = ab + ac$$
   for all $a, b, c \in R$. Express this equation as a commutative diagram in **Set**.