The quantum group $\mathcal{U}_\hbar(sl_2)$

We have the universal enveloping algebra

$$\mathcal{U}(sl_2) = \langle E, H, F \mid HE - EH = 2E,\ \ HF - FH = -2F,\ \ EF - FE = H \rangle$$

$$\Delta H = 1 \otimes H + H \otimes 1, \quad \Delta E = 1 \otimes E + E \otimes 1, \quad \Delta F = 1 \otimes F + F \otimes 1$$

$\mathbb{C}$-basis of $\mathcal{U}(sl_2)$: $\{ E^m H^n F^p : m, n, p \in \mathbb{Z}_{\geq 0} \}$

PBW Thm

We construct: $\hbar$-adic Hopf algebra $\mathcal{U}_\hbar := \mathcal{U}_\hbar(sl_2)$ over the ring $\mathbb{C}[[\hbar]]$ of formal power series in $\hbar$.

$\hbar$ is a formal parameter ("Planck constant") which appears in new $EF - FE$, new ΔE, new ΔF

$$\mathcal{U}_\hbar \big|_{\hbar=0} = \mathcal{U}_\hbar / \hbar \cdot \mathcal{U}_\hbar \cong \mathcal{U}(sl_2)$$

The (commutative) ring $\mathbb{C}[[\hbar]]$ of formal power series in $\hbar$:

$\{ c_0 + c_1\hbar + c_2\hbar^2 + \dots \mid c_i \in \mathbb{C} \ \forall i \}$ e.g. $(1-\hbar)(1+\hbar+\hbar^2+\hbar^3+\dots) = 1$

so $(1-\hbar)^{-1} = \sum_{n \geq 0} \hbar^n$

<u>Def</u> Multiplication in $\mathbb{C}[[\hbar]]$:

$$(a_0 + a_1\hbar + a_2\hbar^2 + \dots)(b_0 + b_1\hbar + b_2\hbar^2 + \dots) = \sum_{n=0}^{\infty} \Big(\sum_{i+j=n} a_i b_j \Big) \hbar^n$$

$\hbar$-adic vector space M: a $\mathbb{C}[[\hbar]]$-module which is complete: any series $\sum_{i=0}^{\infty} m_i$, such that for any $n \geq 0$, $\exists$ only finitely many i: $m_i \notin \hbar^n M$, has a sum in M.

$\hbar$-adic algebra A: an $\hbar$-adic vector space with multiplication which is bilinear with respect to convergent infinite sums.

$\hbar$-adic Hopf algebra: is an $\hbar$-adic algebra A with

$\Delta : A \to A \hat{\otimes} A$ ← completed tensor square

$\varepsilon : A \to \mathbb{C}[[\hbar]]$

$S : A \to A$

linear with respect to taking (convergent) infinite sums

Example. If $a \in A$ then $e^{\hbar a} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a^n$ is always well-defined.

Every Hopf algebra A (over $\mathbb{C}$) gives rise to a trivial $\hbar$-adic Hopf algebra $A[[\hbar]]$:

- elements $a_0 + \hbar a_1 + \hbar^2 a_2 + \cdots$, $a_0, a_1, a_2, \ldots \in A$
- operations extended from A (bi)linearly with respect to convergent infinite sums.

$\hbar$-adic deformation of a Hopf algebra $(B, m, 1, \Delta, \varepsilon, S)$:

$B_\hbar := B[[\hbar]] = \{\sum_{i=0}^{\infty} \hbar^i b_i \mid b_i \in B\}$ underlying vector space

$m_\hbar : B \otimes B \to B_\hbar$ $\quad m_\hbar = m + \hbar m_1 + \hbar^2 m_2 + \ldots\ldots$

$m_i : B \otimes B \to B$

$1_\hbar = 1$

$\Delta_\hbar : B \to B_\hbar \hat{\otimes} B_\hbar$ $\quad \Delta_\hbar = \Delta + \hbar \Delta^1 + \hbar^2 \Delta^2 + \ldots\ldots$

$\varepsilon : B \to \mathbb{C}$

extended $\mathbb{C}[[\hbar]]$-linearly

DEF $\mathcal{U}_\hbar = \mathcal{U}_\hbar(sl_2)$ is an $\hbar$-adic Hopf algebra generated by E, H, F subject to $HE - EH = 2E$, $HF - FH = -2F$, $EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}$

where K is defined as $K = e^{\hbar H} = 1 + \hbar H + \dfrac{\hbar^2}{2} H^2 + \dots = \sum_{i=0}^{\infty} \dfrac{\hbar^i}{i!} H^i$

and $q = e^{\hbar} \in \mathbb{C}[[\hbar]]$. One further has $\varepsilon(E) = \varepsilon(H) = \varepsilon(F) = 0$,

$\Delta_\hbar H = H \otimes 1 + 1 \otimes H$, $\quad \Delta_\hbar E = 1 \otimes E + E \otimes \underline{K}$, $\quad \Delta_\hbar F = \underline{K^{-1}} \otimes F + F \otimes 1$.

Note $e^{\hbar a} e^{\hbar b} = e^{\hbar(a+b)}$ $\underline{\text{if } ab = ba}$ $\Rightarrow K^{-1} = e^{-\hbar H}$

Note $\dfrac{K - K^{-1}}{q - q^{-1}}$ is well defined: $K - K^{-1} = e^{\hbar H} - e^{-\hbar H} = \underline{2\hbar} H + \dfrac{\hbar^3}{3} H^3 + \dots$.

$q - q^{-1} = \underline{2\hbar} + \dfrac{\hbar^3}{3} + \dots\dots$ so $\dfrac{K - K^{-1}}{q - q^{-1}} = \left(2 + \dfrac{\hbar^2}{3} + \dots\right)^{-1} \left(2H + \dfrac{\hbar^2}{3} H^3 + \dots\right)$

Why is $\mathcal{U}_\hbar$ an ($\hbar$-adic) Hopf algebra?

$$\mathcal{U}_\hbar = \overbrace{\mathbb{C}[[\hbar]]\langle E,H,F\rangle}^{T_\hbar} / I_\hbar$$

where $I_\hbar$ is the ($\hbar$-adic) ideal generated by the span $J_\hbar$ of

$$\left.\begin{array}{l} HE - EH - 2E, \\ HF - FH + 2F, \\ EF - FE - \frac{K - K^{-1}}{q - q^{-1}} \end{array}\right\} \text{these are not primitive (if } \hbar \neq 0)$$

We need to show that $I_\hbar$ is a Hopf ideal.

We will show: if $\Delta : T_\hbar \to T_\hbar \hat{\otimes} T_\hbar \twoheadrightarrow \mathcal{U}_\hbar \hat{\otimes} \mathcal{U}_\hbar$, homo'm

$\underset{\text{homo'm}}{\varepsilon : T_\hbar \to \mathcal{U}_\hbar}$, $S : T_\hbar \to \mathcal{U}_\hbar$ antihomo'm then

$\Delta(J_\hbar) = 0$, $\varepsilon(J_\hbar) = 0$, $S(J_\hbar) = 0$ $*$

We write $[a,b]$ to denote $ab-ba$. We need to show:

1 $\Delta([H,E]) = 2\Delta E$ 2 $\Delta([H,F]) = -2\Delta F$ 3 $\Delta([E,F]) = \dfrac{\Delta K - \Delta K^{-1}}{q-q^{-1}}$

in $U_\hbar$, i.e. modulo the relations $J_\hbar$

1 $[\Delta H, \Delta E] = [1\otimes H + H\otimes 1,\ 1\otimes E + E\otimes K]$

$$= \underbrace{[1\otimes H, 1\otimes E]}_{= 1\otimes[H,E]} + \underbrace{[H\otimes 1, 1\otimes E]}_{= 0} + \underbrace{[1\otimes H, E\otimes K]}_{= E\otimes[H,K]} + \underbrace{[H\otimes 1, E\otimes K]}_{= [H,E]\otimes K}$$

$K = e^{\hbar H} \Rightarrow [H,K] = 0$

$= 1\otimes 2E + 2E\otimes K = 2\Delta E.$

2 $[\Delta H, \Delta F] = -2\Delta F$ similarly. Use $[H,K^{-1}] = 0$.

③ $[\Delta E, \Delta F] = [1\otimes E, K^{-1}\otimes F] + [E\otimes K, K^{-1}\otimes F] + [1\otimes E, F\otimes 1] + [E\otimes K, F\otimes 1]$

$[1\otimes E, K^{-1}\otimes F] = K^{-1}\otimes [E,F] = \dfrac{\cancel{K^{-1}\otimes K} - K^{-1}\otimes K^{-1}}{q-q^{-1}}$

$[E\otimes K, K^{-1}\otimes F] = EK^{-1}\otimes KF - K^{-1}E\otimes FK \boxed{=0}$

$[1\otimes E, F\otimes 1] = 0$

$[E\otimes K, F\otimes 1] = [E,F]\otimes K = \dfrac{K\otimes K - \cancel{K^{-1}\otimes K}}{q-q^{-1}}$

By Lemma: $KFK^{-1} = e^{\hbar H} F e^{-\hbar H} = e^{ad_{\hbar H}}(F)$

note $[\hbar H, F] = -2\hbar F$, $(ad_{\hbar H})^n F = (-2\hbar)^n F$, $e^{ad_{\hbar H}}(F) = e^{-2\hbar} F =$
$= q^{-2} F$, so $KF = KFK^{-1}\cdot K = q^{-2} FK$. Similarly, $K^{-1}E = q^{-2} EK^{-1}$.

Lemma Let A be an $\hbar$-adic algebra, $a \in \hbar A$. Define:

$ad_a : A \to A \qquad ad_a(x) = [a, x]$;

$Ad_{e^a} : A \to A \qquad Ad_{e^a}(x) = e^a x e^{-a}$.

Then $$Ad_{e^a} = e^{ad_a} = \sum_{n=0}^{\infty} \frac{1}{n!} (ad_a)^n$$

So, $EK^{-1}\otimes KF - K^{-1}E\otimes FK = EK^{-1}\otimes q^{-2}FK - q^{-2}EK^{-1}\otimes FK = 0$

and
$$[\Delta E, \Delta F] = \frac{K\otimes K - K^{-1}\otimes K^{-1}}{q-q^{-1}}.$$

Exercise x is primitive $\Rightarrow e^{\hbar x}$ is grouplike.

It follows from Exercise that $K = e^{\hbar H}$, $K^{-1} = e^{-\hbar H}$ are grouplike.

Hence
$$[\Delta E, \Delta F] = \frac{\Delta K - \Delta K^{-1}}{q-q^{-1}}$$ as required.

$\varepsilon(J_\hbar)$ is 0 because $\varepsilon(E) = \varepsilon(F) = \varepsilon(H) = 0$ implies

$$\varepsilon(K^{\pm 1}) = \varepsilon(e^{\pm\hbar H}) = e^{\pm\hbar\varepsilon(H)} = 1$$

and so $HE - EH - 2E$, $HF - FH - 2F$, $EF - FE - \frac{K-K^{-1}}{q-q^{-1}} \in \ker\varepsilon$.

What about S? Define S on generators:

$$S(H) = -H \quad \text{because } H \text{ is primitive}; \quad S(K) = K^{-1}, \quad S(K^{-1}) = K$$

$$\Delta E = 1 \otimes E + E \otimes K \quad \overset{\text{antipode law}}{\Longrightarrow} \quad 1\, S(E) + E\, S(K) = \varepsilon(E) = 0$$

$$\Downarrow$$

$$S(E) = -EK^{-1}$$

Similarly $S(F) = -KF$

$$S(HE - EH - 2E) = S(E)S(H) - S(H)S(E) - 2S(E)$$

$$= -EK^{-1}(-H) + H(-EK^{-1}) + 2EK^{-1} = \underbrace{(EH - HE + 2E)}_{=0}K^{-1}$$

$K^{-1}H = HK^{-1}$

Similarly $S(HF - FH + 2F) = 0$.

$$S\left(EF - FE - \frac{K-K^{-1}}{q-q^{-1}}\right) = (-KF)(-EK^{-1}) - (-EK^{-1})(-KF)$$

$$- \frac{K^{-1}-K}{q-q^{-1}} = \underbrace{KFEK^{-1}}_{} - EF + \frac{K-K^{-1}}{q-q^{-1}} = 0.$$

$$KFK^{-1} \cdot KEK^{-1} = q^{-2}Fq^{2}E = FE$$

So $U_\hbar$ is a Hopf algebra.

Is $U_\hbar$ an $\hbar$-adic deformation of $U(sl_2)$?

In other words, is $U_\hbar \cong U(sl_2)[[\hbar]]$ as $\mathbb{C}[[\hbar]]$-modules?

THM Yes: every element of $U_\hbar$ can be uniquely written as $\sum_{n=0}^{\infty} \hbar^n u_n$ where u_n are finite linear combinations of standard monomials $E^m H^n F^p$, $m, n, p \in \mathbb{Z}_{\geq 0}$.

Proof Postponed.

The Hopf algebra $\mathcal{U}_q = \mathcal{U}_q(sl_2)$ (over $\mathbb{C}$)

Consider the $\mathbb{C}$-span of all monomials in E, F, K, K^{-1} inside $\mathcal{U}_h$. We have shown that these generators satisfy $KE = q^2 EK$, $KF = q^{-2} FK$, $KK^{-1} = K^{-1}K = 1$, $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$, $\Delta E = 1 \otimes E + E \otimes K$, $\Delta F = K^{-1} \otimes F + F \otimes 1$, $\Delta K^{\pm 1} = K^{\pm 1} \otimes K^{\pm 1}$, $\varepsilon(E), \varepsilon(F) = 0$, $\varepsilon(K^{\pm 1}) = 1$, formulae for $S(\ )$ as above.

This $\mathbb{C}$-vector space is a Hopf algebra over $\mathbb{C}$ denoted $\mathcal{U}_q(sl_2)$.