MAD 103 Lie groups and Lie algebras

Problem Sheet 1 Solutions

No 1.

$$SL(n,\mathbb{R}) = \{ n\times n \text{ matrices } A \text{ s.t. } \det A = 1\}$$

let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{pmatrix}$

Then $\det A = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}$ (expansion w.r.t. the first column)

where $A_{1k} = (-1)^{k-1} \cdot \det \begin{pmatrix} \text{matrix obtained by removing} \\ \text{the } 1^{st} \text{ column and } k\text{th row} \end{pmatrix}$

Thus $SL(n,\mathbb{R})$ is defined as a subset in the n^2-dimensional vector space of all $n\times n$ matrices satisfying one polynomial equation $\det(A) = F(a_{11}, a_{12}, \dots, a_{nn}) = 1$.

We have $dF = \left(\frac{\partial F}{\partial a_{11}}, \frac{\partial F}{\partial a_{21}}, \dots, \frac{\partial F}{\partial a_{n1}}, \dots\right) = (A_{11}, A_{12}, \dots, A_{1n}, \dots)$

if $dF = 0$, then $A_{11} = A_{12} = \dots = A_{1n} = 0$ implying that $\det A = 0$.

Therefore $dF \neq 0$ on $SL(n,\mathbb{R})$ and by the Implicit function theorem, $SL(n,\mathbb{R})$ is a smooth manifold of dimension $n^2 - 1$.

(number of variables — number of equations)

No 2.

$$\begin{aligned} f_1(x,y,u,v) &= x^2+y^2+u^2+v^2=1 \\ f_2(x,y,u,v) &= x^2+y^2-u^2-v^2=0 \end{aligned} \qquad (*)$$

We compute the Jacobi matrix for these two functions

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2u & 2v \\ 2x & 2y & -2u & -2v \end{pmatrix}$$

To compute the rank of J, we simplify this matrix by using elementary transformations

$$\begin{pmatrix} 2x & 2y & 2u & 2v \\ 2x & 2y & -2u & -2v \end{pmatrix} \xrightarrow[r_2-r_1]{} \begin{pmatrix} 2x & 2y & 2u & 2v \\ 0 & 0 & -4u & -4v \end{pmatrix} \xrightarrow[r_1+\frac{1}{2}r_2]{} \begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & -4u & -4v \end{pmatrix}$$

$$\xrightarrow[r_2 \rightsquigarrow -\frac{1}{4}r_2]{r_1 \rightsquigarrow \frac{1}{2}r_1} \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & u & v \end{pmatrix}$$

It is easy to see that rank $J=2$ unless $x=y=0$ or $u=v=0$.

But the assumption $x=y=0$ leads to contradiction $\begin{matrix} 0+0+u^2+v^2=1 \\ 0+0-u^2-v^2=0 \end{matrix}$

Similarly for $u=v=0$.

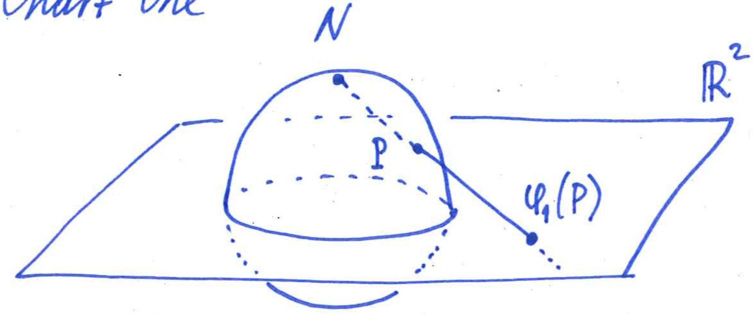
Therefore, rank $J=2$ at each point $(x,y,u,v)\in M$ and from the Implicit Function Theorem we conclude that M is a smooth manifold of dimension $4-2=2$.

To see that M is diffeomorphic to the torus T^2, we rewrite the system $(*)$ in the form
$$\begin{cases} \frac{1}{2}(f_1+f_2) = x^2+y^2=1/2 \\ \frac{1}{2}(f_1-f_2) = u^2+v^2=1/2 \end{cases}$$

Each equation defines a circle S^1 and since these equations are uncoupled, they define the product of two circles $S^1\times S^1$ which is the torus T^2 (by definition).

No3. Let us define two charts for S^2 via stereographic projection from the north pole, $N=(0,0,1)$, and from the south pole, $S=(0,0,-1)$.

Chart One



$$\varphi_1 : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$$

Let us find the relation between the coordinates of P and $\varphi_1(P) = (x, y, 0)$

P belongs to the straight line through $\underset{(1,0,0)}{\underset{\shortparallel}{N}}$ with the generating vector

$\overrightarrow{\varphi_1(P)N} = (x, y, -1)$

The parametric equation of this line is $(\lambda x, \lambda y, 1-\lambda)$, $\lambda \in \mathbb{R}$

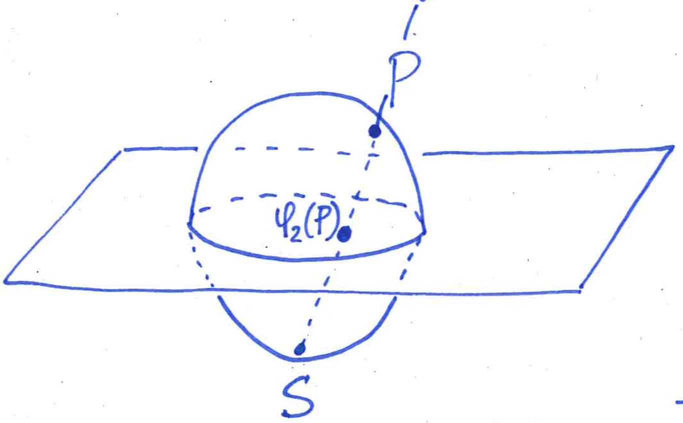
Since P belongs to the unit sphere we have

$$(\lambda x)^2 + (\lambda y)^2 + (1-\lambda)^2 = 1$$

$$\lambda^2(x^2+y^2+1) - 2\lambda = 0$$

$$\lambda = \frac{2}{x^2+y^2+1} \quad \text{so that} \quad P = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

Chart Two is defined in a similar way: $\varphi_2 : S^2 \setminus \{S\} \longrightarrow \mathbb{R}^2$



The relation between P and $\varphi_2(P)$ is as follows:

if $\varphi_2(P) = (u, v, 0)$, then

$$P = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\right)$$

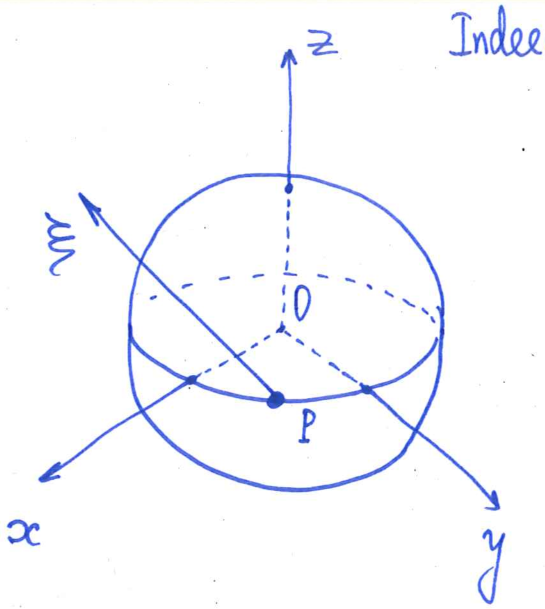
One can check that the relation between (x,y) and (u,v) (i.e. the transition functions) are

$$\begin{cases} u = \dfrac{x}{x^2+y^2} \\ v = \dfrac{y}{x^2+y^2} \end{cases}$$

Remark this formula can be rewritten in the complex form $u+iv = \overline{\left(\frac{1}{x+iy}\right)}$

$w = \bar{z}^{-1}$

No 4. $P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$, $\vec{\xi} = (1, -1, 1)$



Indeed, ξ is tangent to the sphere as $\vec{\xi} \perp \vec{OP}$

To compute the components of $\vec{\xi}$ in coordinates φ, θ we use the standard parametrisation of S^2 $\cancel{\xi}$:

$$\vec{r}(\varphi, \theta) = (\cos\varphi \cos\theta, \sin\varphi \cos\theta, \sin\theta)$$

By definition, the components of $\vec{\xi}$ w.r.t. φ and θ are defined from the relation

$$\vec{\xi} = \xi_1 \cdot \vec{r}_\varphi + \xi_2 \vec{r}_\theta, \text{ where}$$

$$\vec{r}_\varphi = \frac{\partial \vec{r}}{\partial \varphi}, \quad \vec{r}_\theta = \frac{\partial \vec{r}}{\partial \theta}$$

We have $\vec{r}_\varphi = (-\sin\varphi \cos\theta, \cos\varphi \sin\theta, 0)$

$\vec{r}_\theta = (-\cos\varphi \sin\theta, -\sin\varphi \sin\theta, \cos\theta)$

At the point $P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$ we have $\varphi = \pi/4$, $\theta = 0$, hence

$\vec{r}_\varphi = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$, $\vec{r}_\theta = (0, 0, 1)$

Therefore, $\vec{\xi} = -\sqrt{2} \cdot r_\varphi + 1 \cdot r_\theta$. Finally, the components of ξ are $(-\sqrt{2}, 1)$.

No 5.

By definition, the directional derivative of a function $F(x)$ along a tangent vector ξ can be computed as follows:

$$\xi(F)(x) = \frac{d}{dt}\Big|_{t=0} F(x+t\xi).$$

~~For~~ In our case: $X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\xi = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$$\xi(\det)(X_0) = \frac{d}{dt}\Big|_{t=0} \det(X_0 + t\xi) =$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{vmatrix} 1+2t & 0 & 0 \\ 0 & 1+3t & 0 \\ 0 & 0 & 1+4t \end{vmatrix} = \frac{d}{dt}\Big|_{t=0} (1+2t)(1+3t)(1+4t) = $$
$$= 9$$

For an arbitrary ξ (considered as a tangent vector at $X_0 = Id$), we have

$$\xi(\det)(Id) = \frac{d}{dt}\Big|_{t=0} \begin{vmatrix} 1+t\xi_{11} & t\xi_{12} & t\xi_{13} \\ t\xi_{21} & 1+t\xi_{22} & t\xi_{23} \\ t\xi_{31} & t\xi_{32} & 1+t\xi_{33} \end{vmatrix} =$$

$$= \frac{d}{dt}\Big|_{t=0} \left(1 + t(\xi_{11}+\xi_{22}+\xi_{33}) + \underset{\text{in } t}{\text{higher order terms}}\right) =$$

$$= \xi_{11}+\xi_{22}+\xi_{33} = \operatorname{tr}(\xi)$$ as required.

No. 6

$$SO(3) = \{ 3\times 3 \text{ matrices } A \text{ s.t. } A^T A = I,\ \det A = 1 \}$$

The condition $A^T A = I$ means that the columns of A form an orthonormal basis in $\mathbb{R}^3$.

Indeed, if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $A^T A = I$ canbe rewritten as 6 equations of the form

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = 1, \quad a_{12}^2 + a_{22}^2 + a_{32}^2 = 1, \quad a_{13}^2 + a_{23}^2 + a_{33}^2 = 1$$

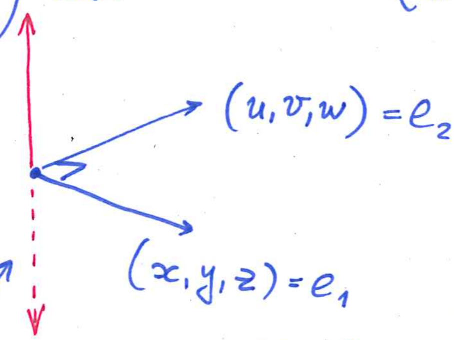
$$a_{11} a_{12} + a_{21} a_{22} + a_{31} a_{32} = 0$$

$$a_{11} a_{13} + a_{21} a_{23} + a_{31} a_{33} = 0$$

$$a_{12} a_{13} + a_{22} a_{23} + a_{32} a_{33} = 0.$$

Denote the first column of A as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the second $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$,

i.e. $A = \begin{pmatrix} x & u & * \\ y & v & * \\ z & w & * \end{pmatrix}$



Then for the third column of A we have 2 possibilities illustrated here

If $\det A = 1$, then the choice is unique: the third column-vector is $e_1 \times e_2$ (vector product of e_1 and e_2).

Conclusion: an element $A \in SO(3)$ is uniquely determined by the first and second columns $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ which have to satisfy

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ xu + yv + zw = 0 \\ u^2 + v^2 + w^2 = 1 \end{cases}$$

as required. This gives a bijection between $SO(3)$ and the subset defined by this system.

This bijection is a diffeomorphism.