Week 05 review worksheet — exercises for §5

E5.1 (tensor product of modules; the dual module — carried over from the week 04 worksheet) Our goal is to show that the class of modules over a Hopf algebra H is closed under tensor products and duals.

(a) Given an algebra A and A-modules V and W, define an $A \otimes A$-module structure on $V \otimes W$.

(b) Let H be a bialgebra. Use the coproduct $\Delta\colon H \to H \otimes H$ and (a) to make $V \otimes W$ an H-module whenever V and W are.

(c) If V is an A-module, show that $\triangleleft\colon V^* \otimes A \to V^*$ where, for $\phi \in V^*$, $\phi \triangleleft a$ is the linear functional on V defined by $\langle \phi \triangleleft a, v\rangle = \langle \phi, a \triangleright v\rangle$, is a *right action* of A on V^*. (Write down the definition of a right action.)

(d) If $\triangleleft$ is a right action of a Hopf algebra H, show that $\triangleright$ defined by the rule "$h\triangleright = \triangleleft Sh$" where $S\colon H \to H$ is the antipode, is a (left) action. Conclude from (c) that if V is an H-module then so is V^*.

E5.2 (primitive elements in $\mathbb{C}\langle X\rangle$) The free algebra $\mathbb{C}\langle X\rangle$ is a Hopf algebra where all $x \in X$ are primitive.

(a) Let $x \in X$, $n \geq 2$. Show that x^n is not primitive in $\mathbb{C}\langle X\rangle$. ($x^n$ is the monomial $xx\ldots x$ of length n.)

(b) Suppose $|X| > 1$. Show that $\mathbb{C}\langle X\rangle$ has primitive elements of every positive degree. Here we refer to a linear combination of monomials of length d as a (homogeneous) element of degree d.

E5.3 (The universal mapping property of $U(\mathfrak{g})$) Review the definition of **Lie bracket**, $[\cdot,\cdot]$, **Lie algebra**, $\mathfrak{g}$, and the **universal enveloping algebra** $U(\mathfrak{g})$ of $\mathfrak{g}$.

Let X be a basis of $\mathfrak{g}$ and let $f\colon \mathfrak{g} \to A$ be a **Lie map** from $\mathfrak{g}$ to some associative algebra A, so f is linear and

$$f([x,y]) = f(x)f(y) - f(y)f(x) \quad \forall x, y \in \mathfrak{g}.$$

That is, f takes the Lie bracket on $\mathfrak{g}$ to the commutator bracket on A.

Let $F\colon \mathbb{C}\langle X\rangle \to A$ be the unique algebra homomorphism such that $F|_X = f$, given by the universal mapping property of the free algebra, Proposition 2.12. Prove: F factors through $U(\mathfrak{g})$, i.e., is the composite map

$$F\colon \mathbb{C}\langle X\rangle \twoheadrightarrow \mathbb{C}\langle X\rangle/I(\mathfrak{g}) = U(\mathfrak{g}) \xrightarrow{\overline{F}} A$$

for some (unique) algebra homomorphism $\overline{F}$.

E5.4 (A Milnor-Moore theorem) Let H be a Hopf algebra over $\mathbb{C}$. View the subspace $P(H)$ of H as a Lie algebra with the commutator bracket $[x,y]_{\text{comm}} = xy - yx$, then the embedding $P(H) \hookrightarrow H$ is a Lie map which by the Universal Mapping Property, E5.3, extends to an algebra homomorphism

$$U(P(H)) \to H.$$

Prove that this homomorphism is **injective.** (Hint: use the Heyneman-Radford theorem for the polynomial coalgebra.) Conclude that if $P(H) \neq 0$ then H must be infinite-dimensional.

E5.5 (expand in standard monomials, calculate the antipode in $U(\mathfrak{sl}_2)$) Recall the presentation

$$U(\mathfrak{sl}_2) = \langle X,\, H,\, Y \mid HX - XH = 2X,\ HY - YH = -2Y,\ XY - YX = H\rangle\,.$$

The Hopf algebra structure of $U(\mathfrak{sl}_2)$ is fully determined by saying that the generators X, H, Y are primitive.

We order the generators so that $X \prec H \prec Y$, so that the standard monomials are $X^m H^n Y^p$ with $m, n, p \geq 0$.

(a) Express YHX as a linear combination of standard monomials.

(b) Think of a way to justify the claim that an arbitrary monomial in X, H, Y can be expressed, in $U(\mathfrak{sl}_2)$, as a linear combination of standard monomials.

(c) What is the antipode of XY: (A) XY; (B) $-XY$; (C) $XY - H$; (D) $XY + H$?

Part B. Extra exercises

Attempt these exercises and compare your answers with the model solutions, published after the session.

E5.6 (tensor product exercise) The following fact is used in the proof of the PBW theorem: if X, Y are vector spaces and $f\colon X \to Y$ is an injective linear map, then $f \otimes f\colon X \otimes X \to Y \otimes Y$ is injective. Prove it.