Exercises in Sections 3 and 4 Solutions

Section 3: Integral dependence

Exercise 3.4. Let R be a subring of a commutative ring S and suppose that S is integral over R. Is the contraction map $c : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ injective? surjective? Prove your claims.

Solution. The contraction map need not be injective or surjective. Take for instance $R = \mathbb{Z}$ and $S = \mathbb{Q}$, or more generally an ID R with $\mathrm{Spec}(R) \neq \{(0)\}$, and a field containing R as subring. Then $\mathrm{Spec}(S) = \{(0)\}$ shows that the contraction $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is not surjective.

Similarly, let $R = k$ be a field and let $S = k[x]$. Then the contraction map is not injective since $\mathrm{Spec}(S) \neq \{(0)\} = \mathrm{Spec}(R)$.

Section 4: Prime and maximal ideal spectra

Exercise 4.1. Find $V(1176) \subseteq \mathrm{Spec}(\mathbb{Z})$.

Solution. We factorise $1176 = 2^3 \cdot 3 \cdot 7^2$. Therefore $V(1176) = \{(2), (3), (7)\}$.

Exercise 4.2. Let $R = \mathbb{Q}[x]$ and let $f = x^3 - 3x^2 + 2x$.

i. Find $V((f))$.

ii. Let $I = (x^2 + 1)$ and set $\overline{R} = R/I$. Find $V((\overline{f})) \subseteq \mathrm{Spec}(\overline{R})$.

Solution.

i. We factorise $f = x(x-1)(x-2)$. Since $\mathbb{Q}[x]$ is a PID and $x, x-1, x-2$ are irreducible (hence prime), $V((f)) = \{(x), (x-1), (x-2)\}$.

ii. Note that $f = (x^2+1)(x-3) + (x+3)$, and so $f + I = (x+3) + I$ in R/I, where $(x+3+I) \in \mathrm{MaxSpec}(R/I)$ since $(R/I)/((x+3+I)/I) \cong \mathbb{Q}$. Therefore $V((f+I)) = \{(x+3+I)\}$.

Exercise 4.3. Let $R = \mathbb{Z} \times \mathbb{Z}/42$. Find all the idempotents of R.

Solution. The idempotents of R are all the elements of the form $(e, f) \in R$ such that $e \in \{0, 1\}$ and $f \in \{0, 1, 7, 15, 21, 22, 28, 36\}$.

Exercise 4.5. Let $R = k[x]$ where k is a field. Prove that there exist proper open subsets U, U' of $\mathrm{Spec}(R)$ such that $\mathrm{Spec}(R) = U \cup U'$.

Solution. Pick $f, g \in R$ two coprime noninvertible polynomials. Note that this is possible, e.g. $x+1$ and, either $x+a$ for $a \neq 1$, if $k \neq \mathbb{F}_2$, or x^2+x+1 if $k = \mathbb{F}_2$. By Bézout's theorem, and since R is a PID, there exist $s, t \in R$ such that $1 = sf + tg$. Then $\emptyset = V(fR + gR) = V(fR \cup gR) = V(fR) \cap V(gR)$. Equivalently, taking complements, $\mathrm{Spec}(R) = U_f \cup U_g$. Since $f, g \notin R^\times$, the open sets U_f, U_g are proper.

Exercise 4.6. Let $f : R \to S$ be a ring homomorphism with R, S commutative. Suppose that f is surjective. Prove that $\mathrm{im}(f^*) = V(\ker(f))$, where f^* is the induced function $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, defined by $f^*(P) = f^{-1}(P)$ for $P \in \mathrm{Spec}(S)$. (The map f^* defines a homeomorphism $\mathrm{Spec}(S) \to V(\ker(f))$.

Solution. By definition,

$$\mathrm{im}(f^*) = \{P \in \mathrm{Spec}(R) \mid \exists\, Q \in \mathrm{Spec}(S) \text{ s.t. } P = f^{-1}(Q)\} = \{P \in \mathrm{Spec}(R) \mid f(P) \in \mathrm{Spec}(S)\},$$

since the surjectivity of f implies that $Q = ff^{-1}(Q) = f(P)$ for all $Q \in \mathrm{Spec}(S)$.

On the other hand, $V(\ker(f)) = \{P \in \mathrm{Spec}(R) \mid \ker(f) \subseteq P\}$, that is, $P \in V(\ker(f))$ if and only if $(0) \subseteq f(P) \in \mathrm{Spec}(S)$, which holds for any prime ideal of S. The result follows.

Exercise 4.7. Let R be a commutative ring and let $P \in \mathrm{Spec}(R)$. Consider the ideal $I = \big(\{e = e^2 \in P\}\big)$ generated by the idempotents of R lying in P.

i. Prove that the only idempotents of R/I are $0, 1$.

ii. Prove that the prime ideals containing I form the connected component of $\mathrm{Spec}(R)$ containing P.

Solution.

i. Let $(e+I)^2 = e+I$ in R/I, for some $e \in R$. Equivalently, $e^2 - e \in I \subseteq P$. Since $e^2 - e = e(e-1)$ and P is prime, we may assume $e \in P$ (else replace e with $1-e$). Write

$$e^2 - e = \sum_{i=1}^{n} a_i e_i, \quad \text{and} \quad f = e\prod_{i=1}^{n}(1-e_i)$$

for some $a_i \in R$ and some idempotents $e_i \in I$ for all i. We calculate

$$f^2 = e^2\prod_{i=1}^{n}(1-e_i)^2 = \big(e + \sum_{i=1}^{n} a_i e_i\big)\prod_{i=1}^{n}(1-e_i) = f + \big(\sum_{i=1}^{n} a_i e_i\big)\prod_{i=1}^{n}(1-e_i) = f$$

since each summand in the right hand term involves an expression of the form $e_i(1-e_i) = 0$. Hence f is an idempotent and $f \in P$ because $e \in P$. It follows that $f \in I$.

Using that $\prod_{i=1}^{n}(1-e_i)$ is an idempotent, we see that $0 = f^2 - f = (e^2-e)\prod_{i=1}^{n}(1-e_i) = (e-1)f$. Moreover, $((f-1)e)^2 = (f-1)e \in P$ implies that $(f-1)e \in I$, and therefore $e = (1-f)e + f \in I$ too.

ii. Any prime ideal containing I contains e for all $e = e^2 \in P$, by definition of I. So the result follows from Theorem 4.10 and the correspondence between the ideals of R containing I and those of R/I.

Exercise 4.8. Let R be a commutative ring and let I be a minimal ideal of R. That is, $I \neq 0$ and the only ideals of R contained in I are 0 and I. Suppose that $I^2 \neq 0$. Prove that there exists an idempotent $e \in R$ such that $I = Re$.

Solution. Since $I^2 \neq 0$, choose $a \in I$ such that $aI \neq 0$. Note that aI is an ideal of R with $aI \subseteq aR \subseteq I$. By minimality of I, we muct have equality, $aI = aR = I$. So there exists $e \in I$ such that $ae = a$. Similarly, $0 \neq eI \subseteq eR \subseteq I$ implies $eR = I$.

Hence the equality $ae^2 = (ae)e = ae$ implies $a(e^2 - e) = 0$. Let $J = \{x \in I \mid ax = 0\}$. Then $e^2 - e \in J$ shows that $J \neq \emptyset$, and note that J is an ideal of R. Moreover, since $ae = a \neq 0$, we have $e \notin J$. Thus J is properly contained in I. By minimality of I, we must have $J = (0)$, and therefore $e^2 - e = 0$, as required.