Suggested exercises Section 1: Commutative algebra - the essentials

Exercise 1.2. Let R be the ring of continuous real-valued functions on $[0, 1]$. Show that $\{f \in R \mid f(0) = 0\}$ is a maximal ideal of R.

Exercise 1.4. Let R be a ring, and let $e \in R$. Assume that e is a nontrivial idempotent.

i. Prove that e is a zero divisor, and that $1 - 2e \in R^\times$.

ii. Prove that eR is a ring with multiplicative identity e. Hence, find a ring S and a ring isomorphism $R \cong eR \times S$.

iii. Let S be a ring. Describe $\mathrm{Nil}(R \times S)$. Deduce that if R and S are reduced rings, then $R \times S$ is reduced too.

iv. Prove that an ID is reduced, but there are reduced rings which are not IDs, and find an example of such a reduced commutative ring.

Exercise 1.5. Find the maximal ideals in the local rings $\mathbb{Z}_{(p)}$, $k[x]/(x^2)$ and $k[[x]]$, where p is a prime and k is a field.

Exercise 1.11. Let R be a commutative ring and let U be a multiplicative subset of R. Let I be an ideal of R such that I is maximal in the poset

$$\{J \text{ ideal of } R \mid J \cap U = \emptyset\},$$

where the order relation is given by the inclusion of ideals. Prove that I is prime.

Exercise 1.12. Let R be a ring and let I be an ideal of R. Prove that the following statements are equivalent:

i. I is prime.

ii. If J, K are ideals of R such that $JK \subseteq I$, then at least one of J or K must be contained in I.

iii. There do not exist ideals J, K of R with $J \not\subseteq I$ and $K \not\subseteq I$, and such that $JK \subseteq I$.

Exercise 1.13. Describe $\mathrm{Spec}(\mathbb{Z}[x])$ and $\mathrm{MaxSpec}(\mathbb{Z}[x])$. Same question with $\mathbb{R}$ and with $\mathbb{C}$ instead of $\mathbb{Z}$. (Hint: Theorem 1.14 may be useful.)

Exercise 1.14. Let $f : R \to S$ be a ring homomorphism.

i. Prove that the preimage $f^{-1}(J)$ is an ideal of R for every ideal J of S. If the image $f(I)$ an ideal of S for an ideal I of R?

ii. If I is a prime ideal of S, is $f^{-1}(I)$ a prime ideal of R? Same question for maximal ideals.

Exercise 1.15. Let R be a commutative ring, let I be an ideal in R and let X be a nonempty subset of R. Define

$$(I : X) = \{a \in R : aX \subseteq I\}, \quad \text{where} \quad aX = \{ax : x \in X\}.$$

i. Prove that $(I : X)$ is an ideal of R.

ii. Let J be the ideal in R generated by the subset X of R. Prove that $(I : X) = (I : J)$, for any ideal I.

iii. Let I, J be two ideals in R. Prove the following.

(a) $I \subseteq (I : J)$.
(b) $J(I : J) \subseteq I$.
(c) if $I = I_1 \cap I_2$, then $(I : J) = (I_1 : J) \cap (I_2 : J)$.
(d) if $J = J_1 + J_2$, then $(I : J) = (I : J_1) \cap (I : J_2)$.

Exercise 1.16. Use the Chinese remainder theorem with $R = \mathbb{Q}[x],\ I = (x^3 - 8)R$ and $J = (x^2 + 1)R$, and find a polynomial $f \in R$ such that $f \equiv x \mod I$ and $f \equiv (x+1) \mod J$.

Exercise 1.17. Let $f = 2x^3 + 3x^2 + 5x + a \in \mathbb{Z}/7[x]$.

i. Find all $a \in \mathbb{Z}/7$ such that f is irreducible.

ii. Let $a = 1$.

(a) Prove that the principal ideal $I = f\mathbb{Z}/7[x]$ is maximal. Let $\pi : \mathbb{Z}/7[x] \longrightarrow (\mathbb{Z}/7[x])/I$ be the projection map.
(b) Prove that $\pi(g) \neq 0$, for $g = 3x^2 + 4x + 5 \in \mathbb{Z}/7[x]$.
(c) Find $\big(\pi(g)\big)^{-1}$ in $(\mathbb{Z}/7[x])/I$.

Exercise 1.18. Let R be a commutative ring.

i. Prove that $\mathrm{Nil}(R)$ and $\mathrm{Rad}(R)$ are ideals of R and that $\mathrm{Nil}(R) \subseteq \mathrm{Rad}(R)$.

ii. Prove that if $I \in \mathrm{Spec}(R)$, then $\sqrt{I} = I$.

iii. Find a commutative ring R and a radical ideal $I = \sqrt{I}$ such that $I \notin \mathrm{Spec}(R)$. (Hint: consider $\mathbb{Z}$ and an ideal $n\mathbb{Z}$ with n a product of distinct primes.)

iv. Prove that $\mathrm{Nil}(R/\,\mathrm{Nil}(R)) = \{0\}$.

Exercise 1.19. Let p be a prime number. Prove that the saturated ideals of $\mathbb{Z}$ with respect to $\mathbb{Z} \setminus (p)$ are those generated by the powers of p, i.e. of the form $(p^n) = p^n\mathbb{Z}$ for some $n \in \mathbb{N}$. (Note that there is a unique prime ideal of $\mathbb{Z}$ which does not meet $\mathbb{Z} \setminus (p)$, and that $\mathbb{Z}_{(p)}$ has a unique nonzero prime - hence maximal - ideal.)

Exercise 1.20. Let k be a field, let $R = k[x, y]$ and let $\lambda \in k$. Consider the ideal $I = (x - \lambda y)$ of R.

i. Prove that the quotient ring R/I is isomorphic to $k[y]$.

ii. Deduce from the above that the ideal $I = (x - \lambda y)$ is prime.