Model answers to Week 01 review worksheet — exercises for §1

Part A. Exercises for interactive discussion

E1.1 Let V be a vector space over $\mathbb{C}$. Which vector spaces from the following list must be isomorphic to V?

(A) V^* **(B)** $\mathrm{Lin}(\mathbb{C}, V)$ **(C)** V/V **(D)** $V/\{0\}$ **(E)** $V \otimes \mathbb{C}$ **(F)** $\mathrm{Lin}(V^*, \mathbb{C})$

Let $L\colon V \to W$ be a linear map. Which formula describes the correct way to apply L^* to $\psi \in W^*$?

(G) $L^*(\psi) = L \circ \psi$ **(H)** $L^*(\psi) = \psi \circ L$

Answer to E1.1. (A) False: if V is infinite-dimensional, V^* has a strictly larger dimension than V, so cannot be isomorphic to V. See an example in Exercise E1.2 below.

(B) True. Consider the basis $\{1\}$ of $\mathbb{C}$. By Prop. 1.7 (Linear Extension), linear maps $\mathbb{C} \to V$ are in one-to-one correspondence with functions from the set $\{1\}$ to V; such a function is nothing but a choice of an element of V (the image of 1).

(C) False. The space V/V is a zero vector space (the space which consists of one element, 0).

(D) True. The quotient map $V \twoheadrightarrow V/\{0\}$ is surjective (all quotient maps are) and its kernel is $\{0\}$ so it is injective, hence bijective.

(E) True. See Corollary 1.33.

(F) False. $\mathrm{Lin}(V^*, \mathbb{C})$ is the space V^{**}. As in (A), if V is infinite-dimensional, this has dimension strictly larger than that of V^* and hence larger than the dimension of V.

(G) False, (H) True. By Def. 1.16, $L^*\psi$ is a linear functional on V, and the value of $L^*\psi$ at $v \in V$ is $\psi(Lv)$. (*Note: Lv is the shorthand notation for $L(v)$; we will often omit brackets after linear maps.*)

But the latter is $(\psi \circ L)(v)$, by definition of composition of functions. Hence $L^*\psi$ is equal to $\psi \circ L$.

E1.2 (multiplicative characters are linearly independent) Let $\mathbb{R}[x]$ be the vector space of all polynomials in one variable x over the field $\mathbb{R}$ of real numbers. Given a point a of the real line, define the functional $e_a\colon \mathbb{R}[x] \to \mathbb{R}$ by $e_a(f(x)) = f(a)$.

(a) Show that the subset $\{e_a\}_{a\in\mathbb{R}}$ of $\mathbb{R}[x]^*$ is linearly independent. (*Hence the space $\mathbb{R}[x]^*$ is of uncountably infinite dimension. Easy to conclude: no $\mathbb{R}$-vector space V has $\dim V^* = \aleph_0$.*)

(b) Give an example of a linear functional $\xi \in \mathbb{R}[x]^*$ which does not belong to $\mathrm{span}\{e_a\}_{a\in\mathbb{R}}$. (*Hence $\{e_a\}_{a\in\mathbb{R}}$ is not a spanning set for $\mathbb{R}[x]^*$.*)

(c) More generally, let M be a set with multiplication $(x, y) \mapsto xy$ (not necessary associative) and L be a field. A function $\sigma\colon M \to L$, not identically 0, is called a **multiplicative character** if $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in M$. Prove that multiplicative characters are linearly independent over L; that is, if $\lambda_1, \dots, \lambda_n \in L$ and distinct $\sigma_1, \dots, \sigma_n$ are such that $\lambda_1\sigma_1(x) + \dots + \lambda_n\sigma_n(x) = 0$ for all $x \in M$, then $\lambda_1 = \dots = \lambda_n = 0$.

Answer to E1.2. (a) We present a proof based on an interpolating polynomial. *Another approach is to note that each $e_a\colon \mathbb{R}[x] \to \mathbb{R}$ is a multiplicative character, and so (a) follows from (c).*

Assume that for some distinct $a_1, \dots, a_n \in \mathbb{R}$, the linear combination $\lambda_1 e_{a_1} + \dots + \lambda_n e_{a_n}$ is the zero functional. Evaluating it on a polynomial f, we get $\lambda_1 f(a_1) + \lambda_2 f(a_2) + \dots + \lambda_n f(a_n)$, and so this expession must be zero for all $f \in \mathbb{R}[x]$.

Yet there is a polynomial $f(x)$ with the property that $f(a_i) = \lambda_i$ for $i = 1, \dots, n$, given for example by Lagrange's or Newton's interpolation formula. Using this f, we conclude that $\lambda_1^2 + \dots + \lambda_n^2 = 0$ and so $\lambda_i = 0$ for all i. We have proved linear independence, according to Definition 1.2.

Exercise: *modify the proof given above to make it work for any field F in place of $\mathbb{R}$. In general, $\lambda_i \in F$ and $\lambda_1^2 + \cdots + \lambda_n^2 = 0$ does not imply that $\lambda_i = 0$ for all i.*

(b) **Method 1 (using the Euclidean topology on $\mathbb{R}$).** The span of $\{e_a\}$ consists of finite linear combinations, but *some* infinite linear combinations of $e_0, e_1, e_2, \dots$ are well-defined linear functionals on $\mathbb{R}[x]$. For example, for any polynomial f the series $\sum_{n=0}^{\infty} \frac{f(n)}{2^n}$ is absolutely convergent, by the Ratio Test as $\lim\limits_{n\to\infty} \frac{f(n+1)/2^{n+1}}{f(n)/2^n} = \frac{1}{2}$. This defines a linear functional ξ which may be written

$$\xi = \sum_{n=0}^{\infty} \frac{e_n}{2^n}.$$

To show that ξ is not expressible as $\lambda_1 e_{a_1} + \cdots + \lambda_n e_{a_n}$, evaluate this finite linear combination at the polynomial $f(x) = \prod_{i=1}^{n}(x - a_i)^2$ and get 0, yet $\xi(f) > 0$.

Another example: $\xi(f) = \int_0^1 f(x)\,dx$. Exercise: *show that this ξ is not in the span of the e_a.*

Method 2 (purely algebraic, works for any field F). Define the linear functional $\xi\colon F[x] \to F$ by

$$\xi(f) = f'(0).$$

Explicitly, if $f \in a_0 + a_1 x + \cdots + a_n x^n \in F[x]$ then $\xi(f) = a_1 \in F$. To show that a finite linear combination $\lambda_1 e_{a_1} + \cdots + \lambda_n e_{a_n}$, where $a_1, \dots, a_n$ are distinct elements of F, cannot equal ξ, we construct a polynomial $f \in F[x]$ such that $f(a_1) = \cdots = f(a_n) = 0$ yet $f'(0) \neq 0$.

If 0 occurs among $a_1, \dots, a_n$, put $f(x) = \prod_{i=1}^{n}(x - a_i)$, otherwise put $f(x) = x\prod_{i=1}^{n}(x - a_i)$. Clearly, $f(a_i) = 0$ for all i. Also, $f(x) = xg(x)$ with $g(0) \neq 0$. It remains to note that $f'(0) = 1g(0) + 0g'(0) = g(0)$ which is not zero, as required.

Remark. In fact, for $\mathbb{R}[x]$, Method 1 and Method 2 are **not really different:** namely, the functional $\xi(f) = f'(0)$ is expressible as an *infinite* linear combination of the evaluation functionals $e_0, e_1, e_2, \dots$. This can be seen from the fact that there exist coefficients c_k such that, for all $f \in \mathbb{R}[x]$,

$$f'(x) = \sum_{k=0}^{\infty} \frac{c_k}{k!} (\Delta^k f)(x),$$

where Δ is the difference operator, $(\Delta f)(x) = f(x+1) - f(x)$. Substituting $x = 0$, we obtain an expression for $f'(0)$ in terms of the values of f at $0, 1, 2, \dots$. See Theorem 2 in this paper by G.-C. Rota, D. Kahaner and A. Odlyzko (1973).

(c) This fact is **Dedekind's independence of characters** and is used in Galois theory; the proof ought to be more widely known.

Assume for contradiction that there are distinct multiplicative characters $\sigma_1, \dots, \sigma_n \colon M \to L$ such that

$$\lambda_1\sigma_1(x) + \lambda_2\sigma_2(x) + \cdots + \lambda_n\sigma_n(x) = 0 \quad \forall x \in M$$

for some $\lambda_1, \dots, \lambda_n \in L$, not all zero. Suppose n is the least possible for this to occur. Then clearly $\lambda_i \neq 0$ for all i, and $n > 1$ since a single multiplicative character is, by definition, not identically 0. If the equation holds for all $x \in M$ then it holds for $x = yz$, $y, z \in M$, so by multiplicativity we get

$$\lambda_1\sigma_1(y)\sigma_1(z) + \lambda_2\sigma_2(y)\sigma_2(z) + \cdots + \lambda_n\sigma_n(y)\sigma_n(z) = 0 \quad \forall y, z \in M.$$

On the other hand, the first equation at $x = z$, premultiplied by the scalar $\sigma_n(y) \in L$, gives

$$\lambda_1\sigma_n(y)\sigma_1(z) + \lambda_2\sigma_n(y)\sigma_2(z) + \cdots + \lambda_n\sigma_n(y)\sigma_n(z) = 0 \quad \forall y, z \in M.$$

Subtracting, we obtain $\mu_1\sigma_1(z) + \cdots + \mu_{n-1}\sigma_{n-1}(z) = 0$ for all $z \in M$, where $\mu_i = \lambda_i(\sigma_i(y) - \sigma_n(y))$; we note that the nth term cancels.

Fixing $y \in M$ such that, say, $\sigma_1(y) \neq \sigma_n(y)$ (possible, because the functions $\sigma_1, \dots, \sigma_n$ are distinct), we ensure that not all the μ_i are zero. We thus obtain $n-1$ linearly dependent multiplicative characters, which contradicts minimality of n.

E1.3 (dual space of tensor product) Let V, W be vector spaces. The goal of this exercise is to show that $V^* \otimes W^*$ is always a subspace of, but may not be equal to, $(V \otimes W)^*$.

To view any element of $V^* \otimes W^*$ as a linear function on $V \otimes W$, first observe that a pure tensor $\phi \otimes \psi$, where $\phi \in V^*$ and $\psi \in W^*$, can be evaluated on $v \otimes w \in V \otimes W$ according to the formula

$$(\phi \otimes \psi)(v \otimes w) = \phi(v)\psi(w).$$

This formula is bilinear in ϕ, ψ and so extends to the whole of $V^* \otimes W^*$. We have embedded $V^* \otimes W^*$ inside $(V \otimes W)^*$ (it is not difficult to show that this embedding is injective).

- Let $V = \mathbb{R}[x]$, $W = \mathbb{R}[y]$. Exhibit an element of $(V \otimes W)^* = \mathbb{R}[x,y]^*$ which is not in $V^* \otimes W^*$. *Thus, $V^* \otimes W^*$ is a proper subspace of $(V \otimes W)^*$ if V, W are infinite dimensional.*

Answer to E1.3. Here is an idea based on infinite sums (using convergence in the Euclidean topology on $\mathbb{R}$), similar to the example in E1.2(b). Let $F(x,y) \in \mathbb{R}[x,y]$. Put

$$\Xi(F) = \sum_{n=0}^{\infty} \frac{F(n,n)}{2^n}.$$

One shows in the same way as in the answer to E1.2(b) that this series converges for all polynomials F and so Ξ is a well-defined element of $\mathbb{R}[x,y]^*$.

Let us show that Ξ is not in $\mathbb{R}[x]^* \otimes \mathbb{R}[y]^*$, that is, Ξ is not representable as $\phi_1 \otimes \psi_1 + \dots + \phi_N \otimes \psi_N$ where $\phi_i \in \mathbb{R}[x]^*$ and $\psi_i \in \mathbb{R}[y]^*$. Find a polynomial $f \neq 0$ such that $\phi_i(f(x)) = 0$ for all i and $\psi_j(f(y)) = 0$ for all j. Such polynomials exist, because the linear map $\mathbb{R}[x] \to \mathbb{R}^{2N}$ given by $(\phi_1, \dots, \psi_N)$ must have a non-trivial kernel as an infinite-dimensional space $\mathbb{R}[x]$ cannot be injectively mapped by a linear map into $\mathbb{R}^{2N}$. Now put

$$F(x,y) = f(x)f(y)$$

and observe that $\phi_1 \otimes \psi_1 + \dots + \phi_N \otimes \psi_N$ evaluates at F to 0. Yet $\Xi(F) = \sum_{n=0}^{\infty} \frac{f(n)^2}{2^n}$ is strictly positive as f cannot vanish at all $n \in \mathbb{N}$.

Exercise: *let L be an arbitrary field. Try to construct an element of $(L[x] \otimes_L L[y])^* = L[x,y]^*$ which does not lie in $L[x]^* \otimes_L L[y]^*$.*

Part B. Extra exercises

E1.4 (the contragredient of a composition) Let $U \xrightarrow{L} V \xrightarrow{M} W$ be linear maps and let $W^* \xrightarrow{M^*} V^* \xrightarrow{L^*} U^*$ be the corresponding contragredient maps. Prove that $(ML)^* = L^*M^*$ (both sides are linear maps $W^* \to U^*$).

Answer to E1.4. By E1.1(H), for any $\psi \in W^*$ one has $(ML)^*\psi = \psi \circ (ML) = \psi \circ (M \circ L)$ as linear functionals on V. *Note that ML is shorthand notation for $M \circ L$; we will often omit the symbol $\circ$ for composition when composing linear maps.*

Since the composition of functions is associative, the above functional is $(\psi \circ M) \circ L$. By E1.1(H) again, this is $(M^*\psi) \circ L$, and by E1.1(H) yet again, this is $L^*(M^*\psi) = (L^* \circ M^*)\psi$. So $(ML)^*\psi = (L^* \circ M^*)\psi$ for all $\psi \in W^*$, meaning that $(ML)^* = L^* \circ M^*$, as claimed.

E1.5 (duality exchanges subspaces and quotients) Let U be a subspace of V. Show that the dual space U^* is canonically isomorphic to the quotient space $V^*/U^\perp$ where $U^\perp$ is defined as $\{\xi \in V^* : \xi(U) = \{0\}\}$.

(*Hint*: you can use the "first isomorphism theorem for vector spaces", $L(V) \cong V/\ker L$ for any linear map $L \colon V \to W$; you should know how to deduce this "theorem" from the material in §1.)

Answer to E1.5. First, let us establish that $L(V) \cong V/\ker L$. View L as a map from V to $L(V)$; the map is then surjective with kernel $\ker L$. Let $q\colon V \twoheadrightarrow V/\ker L$ be the quotient map.

Proposition 1.20 gives us the linear map $\overline{L}\colon V/\ker L \to L(V)$ such that $L = \overline{L}q$. Since the composition $\overline{L}q$ is surjective, $\overline{L}$ is surjective.

If a coset $v + \ker L$ is in the kernel of $\overline{L}$, then $\overline{L}(v + \ker L) = 0$, but $v + \ker L = q(v)$ by definition of q; so $\overline{L}q(v) = L(v) = 0$, $v \in \ker L$ and so $v + \ker L$ is the zero element of $V/\ker L$. This shows that the kernel of $\overline{L}$ consists only of the zero element, i.e., $\overline{L}$ is injective.

Being surjective and injective, $\overline{L}$ is the required isomorphism between $V/\ker L$ and $L(V)$.

We now apply this to describe U^*. Consider the *restriction map*

$$R\colon V^* \to U^*, \qquad R(\xi) = \xi|_U$$

which takes a linear functional $\xi\colon V \to \mathbb{C}$ as an argument and outputs the restriction of ξ to the subspace U of V. Clearly, R is a linear map.

To show that R is surjective, take an arbitrary $\eta \in U^*$, take a basis $\mathcal{B}$ of U, extend it to a basis $\hat{\mathcal{B}} \supseteq \mathcal{B}$ of V and define $\hat{\eta}\colon V \to \mathbb{C}$ by putting

$$\hat{\eta}(b) = \begin{cases} \eta(b), & b \in \mathcal{B}, \\ 0, & b \in \hat{\mathcal{B}} \setminus \mathcal{B} \end{cases}$$

The linear functional $\hat{\eta}\colon V \to \mathbb{C}$ is extended linearly from the basis $\hat{\mathcal{B}}$ onto V (Proposition 1.7) and has the property that $R(\hat{\eta}) = \eta$. Surjectivity of R is proved.

Then by the above, $V^*/\ker R \cong R(V^*) = U^*$. It remains to observe that

$$\ker R = \{\xi \in V^* : \xi|_U = 0\} = \{\xi \in V^* : \xi(U) = \{0\}\} = U^\perp.$$

E1.6 (bilinear maps) Review the definition of the *tensor product* $E \otimes F$ of vector spaces E and F. Let $E = F = \mathbb{R}[x]$, a vector space over $\mathbb{R}$; in each of the following, determine whether the given formula is a well-defined bilinear map on $\mathbb{R}[x] \times \mathbb{R}[x]$ hence a linear map on $\mathbb{R}[x] \otimes \mathbb{R}[x]$:

$A\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x] \otimes \mathbb{R}[x], \quad A(f \otimes g) = g \otimes f.$
$B\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x], \quad B(f \otimes g) = f.$
$C\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x], \quad C(f \otimes g) = f + g.$
$D\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x], \quad D(f \otimes g) = fg.$
$E\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x] \otimes \mathbb{R}[x] \otimes \mathbb{R}[x], \quad E(f \otimes g) = f \otimes 1 \otimes g.$
$F\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x] \otimes \mathbb{R}[x] \otimes \mathbb{R}[x], \quad F(f \otimes g) = f \otimes f \otimes g.$

Answer to E1.6. ***A* is bilinear:** the expression $g \otimes f$ is linear in each argument, by definition of $\otimes$. The map $E \otimes F \to F \otimes E$, $e \otimes f \mapsto f \otimes e$ is known as the *flip map.*

***B* is not bilinear:** e.g., $B(2x \otimes x)$ is not equal to $B(x \otimes 2x)$ as would be expected for a bilinear map. In fact, this formula does not give a well-defined map on the tensor product $\mathbb{R}[x] \otimes \mathbb{R}[x]$: the elements $2x \otimes x$ and $x \otimes 2x$ of $\mathbb{R}[x] \otimes \mathbb{R}[x]$ are equal (these are just two ways to write the same pure tensor in $\mathbb{R}[x] \otimes \mathbb{R}[x]$), so any map from $\mathbb{R}[x] \otimes \mathbb{R}[x]$ to anywhere must return the same value on $2x \otimes x$ and $x \otimes 2x$.

***C* is not bilinear:** the pure tensors $x \otimes x$ and $2x \otimes \frac{1}{2}x$ are equal in $\mathbb{R}[x] \otimes \mathbb{R}[x]$ but this map would output different values on these.

***D* is bilinear:** indeed, the product map on polynomials satisfies $(f_1 + \lambda f_2)g = (f_1 g) + \lambda(f_2 g)$ so it is linear in f. In the same way, it is linear in g. This is the product map on the *algebra* $\mathbb{R}[x]$ — see next chapter.

***E* is bilinear:** linearity in f follows from bilinearity of the tensor product in $f \otimes (1 \otimes g)$, similarly for g.

***F* is not bilinear:** outputs different values on equal pure tensors $2x \otimes x$ and $x \otimes 2x$.