Problem Sheet 5
Solutions

Question 1 Natural action of the upper triangular group T.

Describe orbits and stabiliser subgroups for

(a) $v = e_1 = (1,0,0)$
(b) $v = e_2 = (0,1,0)$
(c) $v = e_3 = (0,0,1)$

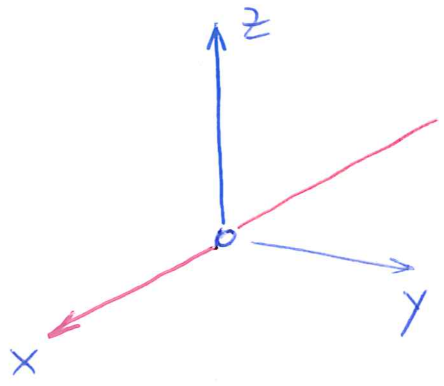
(a) By definition, the orbit of $v = e_1$ is

$$\mathcal{O}(e_1) = \left\{ Xe_1 ,\ X \in T \right\} = \left\{ \underbrace{\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}}_{X} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} t_{11} \\ 0 \\ 0 \end{pmatrix},\ \begin{array}{l} t_{11} \in \mathbb{R} \\ t_{11} \neq 0 \end{array} \right\}$$

because $\det X \neq 0$

$$\boxed{\dim \mathcal{O}(e_1) = 1}$$

Geometrically, $\mathcal{O}(e_1)$ is the vetrical axis Ox in $\mathbb{R}^3$ with the origin $(0,0,0)$ removed.



By definition, the stabiliser of $v = e_1$ is the subgroup of T defined by:

$$St(e_1) = \left\{ X \in T \text{ such that } Xe_1 = e_1 \right\}$$

$$= \left\{ X \in T \text{ s.t. } \begin{pmatrix} t_{11} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix},\ t_{22} \cdot t_{33} \neq 0 \right\}$$

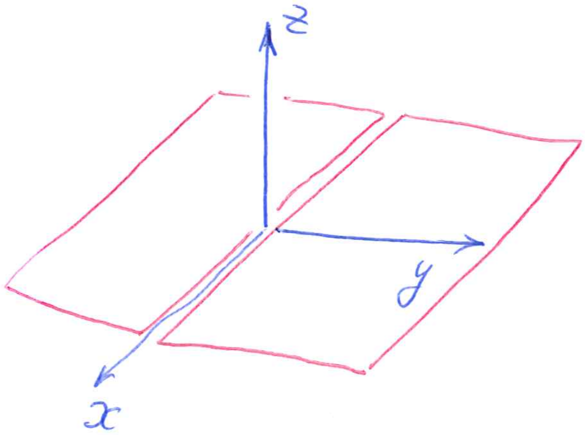
$$\boxed{\dim St(e_1) = 5}$$

(b) Similarly:

$$O(e_2) = \left\{ \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t_{12} \\ t_{22} \\ 0 \end{pmatrix}, \begin{matrix} t_{12}, t_{22} \in \mathbb{R} \\ t_{22} \neq 0 \end{matrix} \right\} =$$

$$= \left\{ \begin{pmatrix} t_{12} \\ t_{22} \\ 0 \end{pmatrix}, t_{12}, t_{22} \in \mathbb{R},\ t_{22} \neq 0 \right\}$$

Geometrically, this orbit is the coordinate plane O_{xy} without O_x-axis.



$$St(e_2) = \left\{ X \in T \text{ such that } \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ \begin{pmatrix} t_{11} & 0 & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}, t_{11} \cdot t_{33} \neq 0 \right\} \subset T$$

$$\boxed{\dim O(e_2) = 2, \quad \dim St(e_2) = 4}$$

(c) $$O(e_3) = \left\{ \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t_{13} \\ t_{23} \\ t_{33} \end{pmatrix}, \begin{matrix} t_{33} \neq 0 \\ t_{13}, t_{23}, t_{33} \in \mathbb{R} \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} t_{13} \\ t_{23} \\ t_{33} \end{pmatrix}, t_{33} \neq 0,\ t_{13}, t_{23}, t_{33} \in \mathbb{R} \right\}.$$ This is the whole space $\mathbb{R}^3$ without the O_{xy} plane.

$$St(e_3) = \{ X \in T \text{ such that } Xe_3 = e_3 \} =$$

$$= \left\{ X \in T \text{ s.t. } \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} t_{11} & t_{12} & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}, t_{11} t_{22} \neq 0 \right\}$$

$$\boxed{\dim O(e_3) = 3, \quad \dim St(e_3) = 3}$$

We finally notice that the whole space $\mathbb{R}^3$ is the disjoint union of $\mathcal{O}(e_1), \mathcal{O}(e_2), \mathcal{O}(e_3)$ and the origin $\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$, which is the orbit of itself, i.e. $\mathcal{O}(v)$, $v=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

Hence, $\mathbb{R}^3$ is partitioned into FOUR orbits.

Also notice that

$$\dim \mathcal{O}(v) + \dim St(v) = \dim T$$

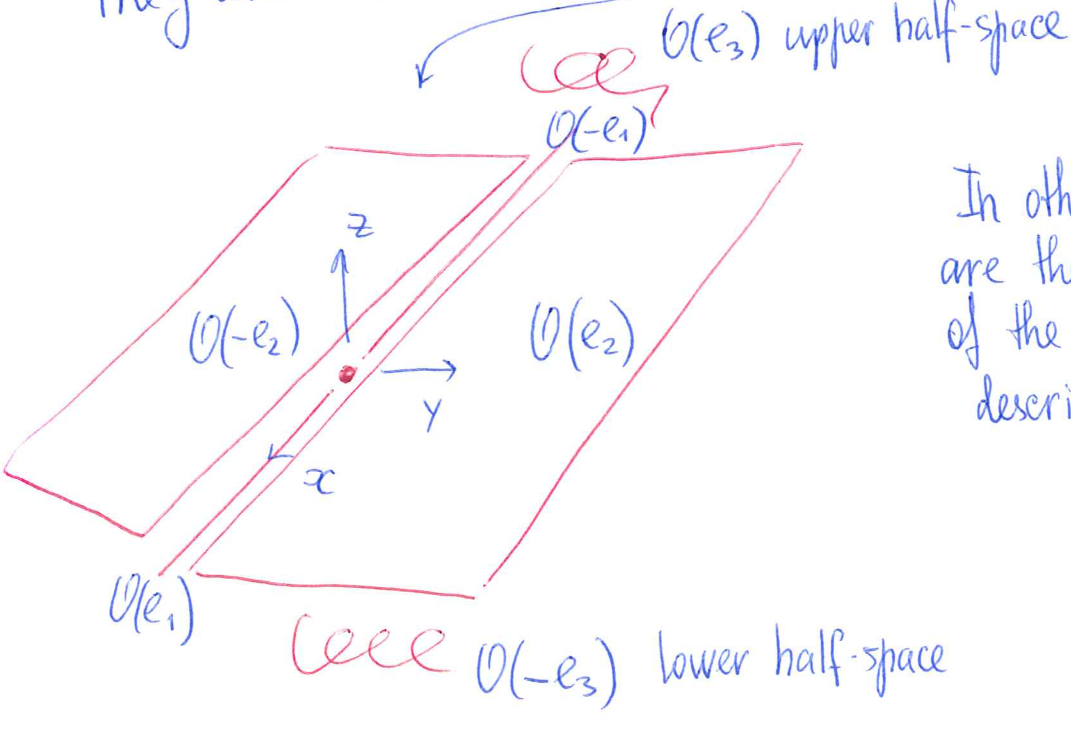
for any v. This is an important general property of actions of Lie groups.

Finally, if we replace T with T_0 (identity component of T), then the natural action will have 7 orbits, namely

$$\mathcal{O}(e_1), \mathcal{O}(-e_1), \mathcal{O}(e_2), \mathcal{O}(-e_2), \mathcal{O}(e_3), \mathcal{O}(-e_3), \mathcal{O}(0)$$

zero vector

They are shown here



In other words, these orbits are the connected components of the orbits of T described in (a), (b), (c).

Question 2

(a) The stabilizer subgroup of $v = e_3 = (0,0,1)$ consists of the matrices $X \in G_B$ satisfying $Xe_3 = e_3$, i.e.

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \begin{matrix}0\\0\end{matrix} \\ \hline b & c & d \end{array}\right)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\d\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \implies d = 1,$$

Thus $St(e_3) = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \begin{matrix}0\\0\end{matrix} \\ \hline b & c & 1 \end{array}\right), \; A \in O(2) \right\}$

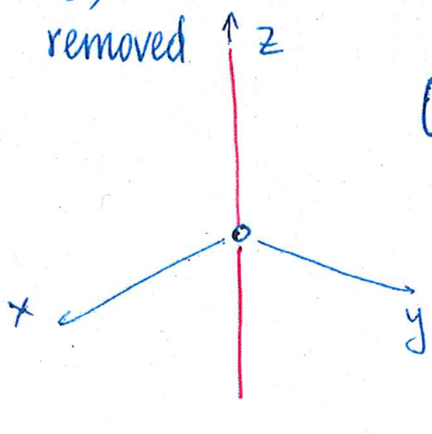
$\dim St(e_3) = 3$ because we have 3 independent parameters, namely b, c and φ (φ is related to the submatrix $A \in O(2)$ which, as we know, depends on one parameter).

Hence, $\dim O(e_3) = \dim G_B - \dim St(e_3) = 4 - 3 = 1$

Another explanation:

$O(e_3)$ consists of all vectors of the form Xe_3 where X runs over G_B, i.e. $O(e_3) = \left\{ \begin{pmatrix}0\\0\\d\end{pmatrix}, \; d \neq 0 \right\}$

In other words $O(e_3)$ is the vertical axes O_z with one point (origin) removed



Obviously, $\dim O(e_3) = 1$

(b) Stabilizer of $v = (v_1, v_2, 0) \neq \bar{0}$

We first consider a particular case: $v = (1,0,0) = e_1$

$$Xe_1 = e_1$$

$$\begin{pmatrix} \cos\varphi & \mp\sin\varphi & 0 \\ \sin\varphi & \pm\cos\varphi & 0 \\ b & c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\cos\varphi = 1$
$\sin\varphi = 0$
$b = 0$

Thus $$St(e_1) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & c & d \end{pmatrix}, \ d \neq 0 \right\}$$

$\dim St(e_1) = 2$ (because we have 2 parameters, c and d)

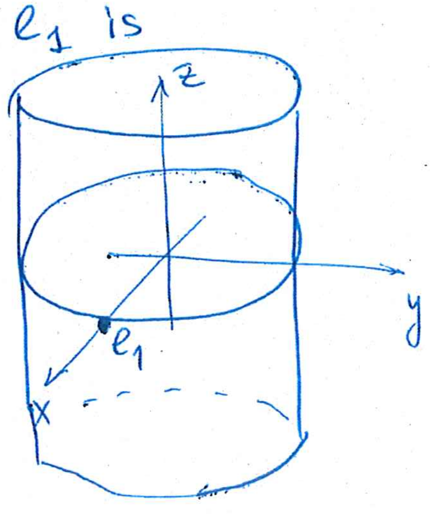
Hence $\dim \mathcal{O}(e_1) = \dim G_B - \dim St(e_1) = 4 - 2 = 2$

The orbit $\mathcal{O}(e_1)$ is easy to describe:

$$\mathcal{O}(e_1) = \left\{ \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ b \end{pmatrix}, \ \text{where } \varphi \text{ and } b \text{ are arbitrary parameters} \right\}$$

thus the first two components x, y obey $x^2 + y^2 = 1$, whereas the third one z is arbitrary.

This simply means that the orbit of e_1 is the cylinder of radius one (around Oz)



Clearly, $\dim \mathcal{O}(e_1) = 2$

For $v=(v_1,v_2,0)$ arbitrary $\neq 0$, the computation is slightly more complicated:

$Xv=v$ We consider two cases separately

$$X=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ b & c & d\end{pmatrix} \text{ and } X=\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ \sin\varphi & -\cos\varphi & 0\\ b & c & d\end{pmatrix}$$

Case 1 gives

$$\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}v_1\\ v_2\end{pmatrix} \quad\text{and}\quad bv_1+cv_2=0$$

the first equation (remember that the unknowns are φ, b, c, d, but not v_1, v_2!)

gives $\cos\varphi=1$, $\sin\varphi=0$;

the second has the following general solution $b=\lambda v_2$, $c=-\lambda v_1$, $\lambda\in\mathbb{R}$

Case 2 gives

$$\begin{pmatrix}\cos\varphi & \sin\varphi\\ \sin\varphi & -\cos\varphi\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}v_1\\ v_2\end{pmatrix} \quad\text{and}\quad bv_1+cv_2=0$$

the first equation can be more or less easily solved, and we get

$$\cos\varphi=\frac{v_1^2-v_2^2}{v_1^2+v_2^2},\quad \sin\varphi=\frac{2v_1v_2}{v_1^2+v_2^2}$$

the second equation has the same solution as before: $b=\lambda v_2$, $c=-\lambda v_1$

Thus the stabilizer subgroup $St(v)$ is as follows

$$St(v)=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \lambda v_2 & -\lambda v_1 & d\end{pmatrix}, \begin{pmatrix}\frac{v_1^2-v_2^2}{v_1^2+v_2^2} & \frac{2v_1v_2}{v_1^2+v_2^2} & 0\\ \frac{2v_1v_2}{v_1^2+v_2^2} & -\frac{v_1^2-v_2^2}{v_1^2+v_2^2} & 0\\ \lambda v_2 & -\lambda v_1 & d\end{pmatrix};\ d\neq 0\right\}$$

$\dim St(v) = 2$ because we have 2 independent parameters λ and d

(Notice, that v_1 and v_2 are fixed!)

$\dim O(v) = \dim G_B - \dim \cancel{O(v)}\, St(v) = 4-2=2.$

It is not hard to see that the orbit of $v=(v_1, v_2, 0)$ is the cylinder of radius $\sqrt{v_1^2+v_2^2}$.

(c) $v = \bar{0}$

$X\bar{0} = \bar{0}$ ~~B~~ true for all $X \in G_B$

Thus $St(\bar{0}) = G_B$, $\dim St(\bar{0}) = 4$

$\dim O(\bar{0}) = \dim G_B - \dim St(\bar{0}) = 4-4=0$

This simply mean that $O(\bar{0})$ is a one-point orbit which consists of $\bar{0}$ itself only.

(d) v arbitrary vector (v_1, v_2, v_3), we assume $v_3 \neq 0$

$Xv = v$

$$X = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & 0 \\ & & 0 \\ \hline b & c & d \end{array}\right) \Rightarrow \begin{cases} A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ bv_1 + cv_2 + dv_3 = v_3 \end{cases}$$

We have solved the first equation $A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in (b)

The general solution of $bv_1 + cv_2 + dv_3 = v_3$ is

$$b = \lambda, \quad c = \mu, \quad d = \frac{v_3 - \lambda v_1 - \mu v_2}{v_3}$$

the same as in (b)

$$St(v) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & \mu & \frac{v_3 - \lambda v_1 - \mu v_2}{v_3} \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ \lambda & \mu & \frac{v_3 - \lambda v_1 - \mu v_2}{v_3} \end{pmatrix} \right\}$$

$\dim St(v) = 2$

because we have 2 parameters: λ and μ

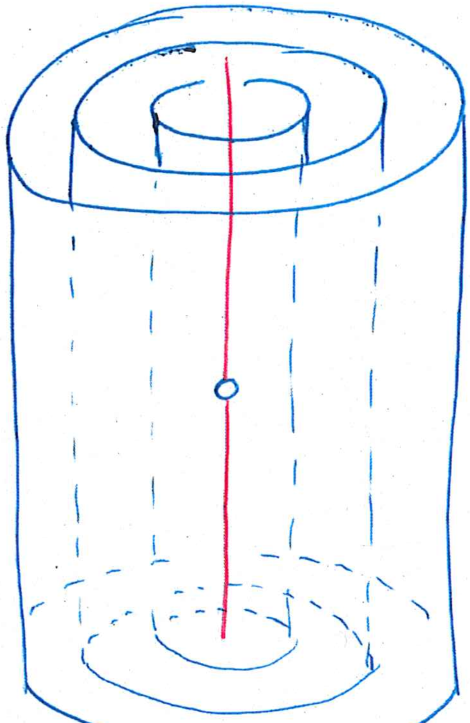
We finally need to describe all the orbits.

There are infinitely many orbits of 3 kinds:

① two-dimensional orbits which are vertical cylinders of an arbitrary radius $r \in (0, +\infty)$

② one-dimensional (disconnected) orbit which is the vertical axes O_z with the origin removed

③ one zero-dimensional orbit $= \{\bar{0}\}$ which consists of a single point, namely the origin $\bar{0} = (0,0,0)$.



Since the maximal dimension of orbits is 2 and the dimension of the space is 3, we should expect $3-2=1$, one non-trivial invariant $f(x,y,z)$.

It is easy to see that

$$f = x^2 + y^2$$

is constant on orbits.

So this function is an invariant of the action.

Question 3

(a) Orbits and invariants of the natural action.

Take an arbitrary $v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{R}^{n+1}$ and try to describe its orbit

$$\mathcal{O}(v) = \{ Xv \,,\ X \in G \} \quad , \quad X = \left(\begin{array}{c|c} A & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \\ \hline 0 & 1 \end{array} \right)$$

$$Xv = \left(\begin{array}{c|c} A & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \\ \hline 0 \dots 0 & 1 \end{array} \right) \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} v_{n+1} \\ \\ v_{n+1} \end{pmatrix}$$

if $v_{n+1} \neq 0$, then the first n components of Xv may take any values. Indeed, if we take an arbitrary $(u_1, \dots, u_n) \in \mathbb{R}^n$ then we can always find A and $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that

$$A \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot v_{n+1} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}.$$

For example, we take $A = I_n$ and

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{v_{n+1}} \cdot \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{pmatrix}$$

The last component of Xv is v_{n+1} and therefore is constant for the given $v = (v_1, v_2, \dots, v_{n+1})$.

Conclusion: if $v_{n+1} \neq 0$, then the orbit $\mathcal{O}(v)$ is the hyperplane defined by the equation $u_{n+1} = v_{n+1} = \text{const}$

If $v_{n+1} = 0$, then the orbit consists of the vectors of the form $\begin{pmatrix} A\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ 0 \end{pmatrix}$

Here we have two possibilities

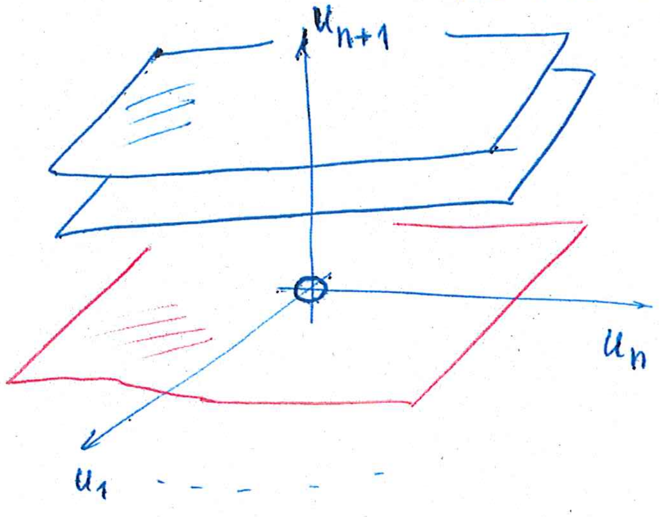
(i) $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \bar{0}$, then $\mathcal{O}(v) = \mathcal{O}(\bar{0}) = \{\bar{0}\}$, i.e. is a one-point orbit

(ii) $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \neq \bar{0}$, then $A\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ can be any non-zero vector,

i.e. $\mathcal{O}(v) = \left\{ \begin{pmatrix} u_1 \\ \vdots \\ u_n \\ 0 \end{pmatrix}, \ (u_1, \dots, u_n) \neq \bar{0} \right\}$

that is the coordinate hyper~~plane~~ $u_{n+1} = 0$ with the origin removed.

We have described all possible orbits (see Fig.)



The maximal dimension of orbits is n, whereas the dimension of the space is $n+1$.

We should expect $(n+1) - n = 1$ one non-trivial invariant.

This is $f = u_{n+1}$. This function is constant on orbits and therefore is an invariant of the natural action

(b) dual action

$$\hat{X} v = (X^T)^{-1} v$$

$$X = \left(\begin{array}{c|c} A & x \\ \hline 0 & 1 \end{array}\right) \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$(X^T)^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -(A^{-1}x)^T & 1 \end{pmatrix} = \left(\begin{array}{c|c} B & 0 \\ \hline y_1 \dots y_n & 1 \end{array}\right)$$

We try to do the same as in (a). Take $v = (v_1, \dots, v_n, v_{n+1})$ and look at its orbit: $\cancel{\mathcal{O}(v) = \{Bv, B}$

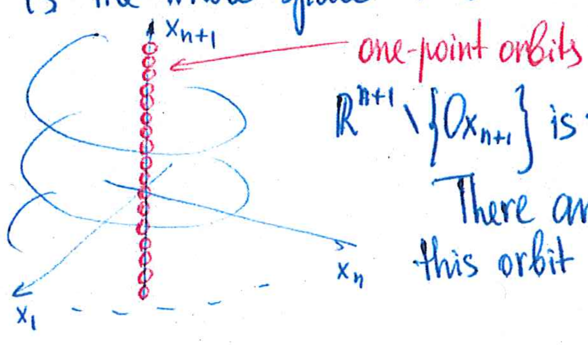
$$\left(\begin{array}{c|c} B & 0 \\ \hline y_1 \dots y_n & 1 \end{array}\right) \begin{pmatrix} v_1 \\ \vdots \\ v_n \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} B \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ y_1 v_1 + y_2 v_2 + \dots + y_n v_n + v_{n+1} \end{pmatrix}$$

if $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \bar{0}$ then the orbit is $\left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_{n+1} \end{pmatrix} \right\}$,

that is, the orbit ~~cos~~ consists of the only point $v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_{n+1} \end{pmatrix}$

if $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \neq \bar{0}$ then $B \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ can be an arbitrary non-zero vector ant

the last component $y_1 v_1 + \dots + y_n v_n + v_{n+1}$ can be an arbitrary real number (if we vary the parameters $y_1, \dots, y_n$). Thus, this orbit is the whole space with the vertical axis Ox_{n+1} removed.



$\mathbb{R}^{n+1} \setminus \{Ox_{n+1}\}$ is the only orbit of maximal dimension $n+1$

There are no non-trivial invariants because this orbit fills almost the whole space $\mathbb{R}^{n+1}$.

(c) $\hat{X} u = Au + x$

To prove that this formula defines an action we need to verify that

$$\widehat{X_1 X_2}\, u = \hat{X}_1 (\hat{X}_2 u) \quad \text{for all } u \in \mathbb{R}^n,\ X_1, X_2 \in G$$

Take

$$X_1 = \left(\begin{array}{c|c} A_1 & \bar{x}_1 \\ \hline 0 & 1 \end{array}\right), \quad X_2 = \left(\begin{array}{c|c} A_2 & \bar{x}_2 \\ \hline 0 & 1 \end{array}\right)$$

$$X_1 X_2 = \left(\begin{array}{c|c} A_1 & \bar{x}_1 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} A_2 & \bar{x}_2 \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} A_1 A_2 & A\bar{x}_2 + \bar{x}_1 \\ \hline 0 & 1 \end{array}\right)$$

$$\widehat{X_1 X_2}\, u = A_1 A_2 u + A\bar{x}_2 + \bar{x}_1$$

the same

$$\hat{X}_1(\hat{X}_2 u) = \hat{X}_1 (A_2 u + \bar{x}_2) = A_1 (A_2 u + \bar{x}_2) + \bar{x}_1 =$$
$$= A_1 A_2 u + A_1 \bar{x}_2 + \bar{x}_1$$

Also we need to check that $\hat{I}_{n+1}$ is the identity map.

$$I_{n+1} = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & 1 \end{array}\right), \quad \hat{I}_{n+1} u = I_n u + 0 = u \quad \text{as needed.}$$

This is an action.

Let us consider the orbit of $u = \bar{0}$. $\mathcal{O}(\bar{0}) = \left\{ \begin{array}{l} A \cdot \bar{0} + x = x \\ \text{where } x \in \mathbb{R}^n \text{ is arbitrary} \end{array} \right\}$

in other words $\mathcal{O}(\bar{0}) = \mathbb{R}^n$. There is only one orbit and therefore the action is transitive.

There are no non-trivial invariants, because each invariant must be constant on orbits, so $f|_{\mathbb{R}^n} = \text{const}$, i.e. the invariant is trivial.

the stabilizer subgroup for $u = \bar{0}$ is :

$$A\bar{0} + x = \bar{0}$$

$$A \text{ arbitrary}, \quad x = 0$$

$$St(\bar{0}) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right), A \in GL(n, \mathbb{R}) \right\} \cong GL(n, \mathbb{R})$$

Since the action is transitive, the stabilizer of any other point $u \in \mathbb{R}$ is isomorphic to $St(\bar{0}) = GL(n, \mathbb{R})$ because the stabilizers of any two points from the same orbit are always conjugate and therefore isomorphic to each other.

Question 4.

To compute the dimension of $\mathcal{O}(A)$ we use the standard formula

$\dim \mathcal{O}(A) = \dim GL(n,\mathbb{R}) - \dim St(A)$, where $\dim GL(n,\mathbb{R}) = n^2$

$St(A) = \{ X \in GL(n,\mathbb{R}) \text{ such that } XAX^{-1} = A \}$

This subgroup is called the centrilizer of A

(a) $A = I_n$

$$X I_n X^{-1} = I_n$$

$$I_n = I_n$$

true for any $X \in GL(n,\mathbb{R})$, i.e. $St(I_n) = GL(n,\mathbb{R})$ and

$\dim \mathcal{O}(I_n) = n^2 - n^2 = 0$ which means that the orbit consists of a single matrix, namely I_n itself.

(b) $A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$ diagonal

$$XAX^{-1} = A$$

$XA = AX$; if $X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$ then

$$XA = \begin{pmatrix} \lambda_1 x_{11} & \cdots & \lambda_n x_{1n} \\ \vdots & & \vdots \\ \lambda_1 x_{n1} & \cdots & \lambda_n x_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \cdots & \lambda_1 x_{1n} \\ \vdots & & \vdots \\ \lambda_n x_{n1} & \cdots & \lambda_n x_{nn} \end{pmatrix} = AX$$

that is: $\lambda_i x_{ij} = \lambda_j x_{ij}$

$(\lambda_i - \lambda_j) x_{ij} = 0 \implies x_{ij} = 0$ if $i \neq j$

$x_{ii} \in \mathbb{R}$ arbitrary

Hence X is a diagonal matrix, i.e.

$St(A) = \left\{ \begin{pmatrix} x_{11} & & & \\ & x_{22} & & \\ & & \ddots & \\ & & & x_{nn} \end{pmatrix} \in GL(n,\mathbb{R}) \right\}$. Clearly $\dim St(A) = n$

$\dim \mathcal{O}(A) = n^2 - n$

(c) Similar computation

$$A = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = \lambda I_n + \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

We need to solve

$$XAX^{-1} = A$$

$$XA = AX$$

$$X\left(\lambda I_n + \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\right) = \left(\lambda I_n + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\right) X$$

$$X \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} X \quad , \text{ if } X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

then

$$X \cdot \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_{11} & & x_{1,n-1} \\ 0 & x_{21} & & x_{2,n-1} \\ \vdots & \vdots & & \vdots \\ 0 & x_{n1} & & x_{n,n-1} \end{pmatrix} = \begin{pmatrix} x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \\ 0 & & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} X$$

it follows that $x_{11} = x_{22} = \ldots = x_{nn}$

$x_{12} = x_{23} = \ldots = x_{n-1,n}$

$\ldots$

and all the entries of X below the diagonal vanish.

Hence

$$St(A) = \left\{ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\ & x_{11} & x_{12} & & x_{1,n-1} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & x_{12} \\ & & & & x_{11} \end{pmatrix} \right\} \subset GL(n,\mathbb{R})$$

Clearly $\dim St(A) = n$ (because we have n independent parameters $x_{11}, x_{12}, \ldots, x_{1n}$)

Therefore

$$\dim \mathcal{O}(A) = \dim GL(n,\mathbb{R}) - \dim St(A) = n^2 - n$$

(d) $A = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$, $p+q=n$

We solve

$$XAX^{-1} = A$$
$$XA = AX$$

Let $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$

where X_1, X_2, X_3, X_4 are subblocks of size $p\times p$, $p\times q$, $q\times p$ and $q\times q$ respectively

$$XA = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} = \begin{pmatrix} X_1 & -X_2 \\ X_3 & -X_4 \end{pmatrix}$$

$$AX = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ -X_3 & -X_4 \end{pmatrix}$$

Hence

$X_1 = X_1$ no restriction

$X_2 = -X_2 \Rightarrow X_2 = 0$

$X_3 = -X_3 \Rightarrow X_3 = 0$

$-X_4 = -X_4$ no restriction

Therefore $St(A) = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_4 \end{pmatrix} \text{, where } X_1 \text{ and } X_4 \text{ are arbitrary non-degenerate square matrices } p\times p \text{ and } q\times q \text{ respectively} \right\}$

$$\dim St(A) = p^2+q^2$$

$\dim O(A) = n^2 - (p^2+q^2)$, $p+q=n$

Question 5

To prove that $z \longmapsto \varphi_A(z) = \dfrac{az+b}{cz+d}$ defines an action of $SL(2,\mathbb{R})$ on the upper complex plane $\mathcal{H}$, we need to check

$$\varphi_{AB}(z) = \varphi_A(\varphi_B(z))$$

and also that $\varphi_{I_2} = id$, i.e. $\dfrac{1\cdot z+0}{0\cdot z+1} = z$ true

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$. then

$$AB = \begin{pmatrix} a\tilde{a}+b\tilde{c} & a\tilde{b}+b\tilde{d} \\ c\tilde{a}+d\tilde{c} & c\tilde{b}+d\tilde{d} \end{pmatrix}$$

$$\varphi_{AB}(z) = \frac{(a\tilde{a}+b\tilde{c})z + (a\tilde{b}+b\tilde{d})}{(c\tilde{a}+d\tilde{c})z + (c\tilde{b}+d\tilde{d})}$$

the same

$$\varphi_A(\varphi_B(z)) = \varphi_A\left(\frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}\right) = \frac{a\left(\frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}\right)+b}{c\left(\frac{\tilde{a}z+\tilde{b}}{\tilde{c}z+\tilde{d}}\right)+d} =$$

$$= \frac{a(\tilde{a}z+\tilde{b})+b(\tilde{c}z+\tilde{d})}{c(\tilde{a}z+\tilde{b})+d(\tilde{c}z+\tilde{d})} = \frac{(a\tilde{a}+b\tilde{c})z + (a\tilde{b}+b\tilde{d})}{(c\tilde{a}+d\tilde{c})z + (c\tilde{b}+d\tilde{d})}$$

Thus, $\varphi_{AB}(z) = \varphi_A(\varphi_B(z))$, it is an action

To find out whether or not this action is transitive, we try to solve $\varphi_A(i) = w$ for any $w = u+iv \in \mathcal{H}$, $v>0$ where $A \in SL(2,\mathbb{R})$ is considered as unknown.

$\dfrac{ai+b}{ci+d} = u+iv$, Obviously, we can find $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ with this property. For example, $A_w = \begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix}$ is OK.

Thus, the action is transitive.

The stabilizer subgroup of $z = i$ is defined by

$$\Phi_A(i) = i \qquad A \in SL(2,\mathbb{R})$$

$$\frac{ai+b}{ci+d} = i$$

$$ai + b = di - c$$

$$a = d$$
$$b = -c$$

Hence $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $\det A = a^2 + b^2 = 1$

We may put $a = \cos\varphi$, $b = \sin\varphi$ (because $a^2 + b^2 = 1$)

Finally, $$\underline{St(i)} = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \right\} = \underline{SO(2)} \subset SL(2,\mathbb{R})$$

The stabilizer subgroup of $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ w.r.t. the adjoint action of $SL(2,\mathbb{R})$ is defined by

$$ABA^{-1} = B$$
$$AB = BA$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$c = -b$$
$$a = d$$
, i.e. $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $\det A = a^2 + b^2 = 1$

So, we get the same result:

$$\underline{St(B)} = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \right\} = \underline{SO(2)} \subset SL(2,\mathbb{R})$$

Thus, $\mathcal{O}(B)$ and $\mathcal{H}$ can both be regarded as the homogeneous space $\widetilde{SL}(2,\mathbb{R})/SO(2)$, because they are both orbits of $SL(2,\mathbb{R})$ with the same stabilizer subgroup $\widetilde{St}(B) = St(i) = SO(2)$.

To construct the desired map $f : \mathcal{H} \longrightarrow \mathcal{O}(B)$ explicitly we first put by definition

$f(i) = B$

If we take any other point $w = u + iv \in \mathcal{H}$, then we know that $w = \varphi_{A_w}(i)$ where $A_w = \begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix}$

~~A~~ We must have:

$$f(w) = f(\varphi_{A_w}(i)) = A_w f(i) A_w^{-1} = A_w B A_w^{-1} =$$

$$= \begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{v}} & -\frac{u}{\sqrt{v}} \\ 0 & \sqrt{v} \end{pmatrix} = \frac{1}{v} \begin{pmatrix} -u & u^2+v^2 \\ -1 & u \end{pmatrix}$$

This gives us the ~~exact~~ explicit formula for f:

for $w = u + iv$ we have $f(w) = \frac{1}{v}\begin{pmatrix} -u & u^2+v^2 \\ -1 & u \end{pmatrix} \in \mathcal{O}(B)$

Also we need to check that $f(\varphi_A z) = A f(z) A^{-1}$ for any $w \in \mathcal{H}$, $A \in \widetilde{SL}(2,\mathbb{R})$

It's easy. First we check this for $z = i$

~~$f(\varphi_A w) = f(\varphi_A \varphi_{A_w}(i)) = f(\ldots$~~ Let $\varphi_A(i) = w = \varphi_{A_w}(i) \Rightarrow A_w^{-1}A \in St(i) = SO(2)$

$$f(\varphi_A(i)) = f(\varphi_{A_w}(i)) = A_w B A_w^{-1} = A_w (A_w^{-1}A) B (A_w^{-1}A)^{-1} A_w^{-1} =$$

$$= A B A_w^{-1} = A f(i) A^{-1}$$

Similarly, for an arbitrary point $z \in \mathcal{H}$ we have $z = \varphi_{A_z}(i)$ and therefore

$$f(\varphi_A(z)) = f(\varphi_A \varphi_{A_z}(i)) = f(\varphi_{AA_z}(i)) =$$

$$= AA_z f(i) A_z^{-1} A^{-1} = A\left(A_z f(i) A_z^{-1}\right) A^{-1} = A\, f(z) A^{-1}$$

as needed.