MAGIC008: Lie Groups and Lie algebras

Problem Sheet 1: Basics of Differential Geometry and Topology

1. Using the Implicit Function Theorem, prove that $SL(n, \mathbb{R})$ carries the structure of a smooth manifold. What is its dimension?

2. Using the Implicit Function Theorem, prove that the subset M in $\mathbb{R}^4(x, y, u, v)$ given by
$$\begin{cases} x^2 + y^2 + u^2 + v^2 = 1 \\ x^2 + y^2 - u^2 - v^2 = 0 \end{cases}$$
is a smooth manifold. Prove that M is diffeomorphic to the 2-torus T^2.

3. Describe a smooth atlas for the 2-sphere S^2 which consists of two charts. Describe the corresponding transition functions.

4. Let θ and ϕ be standard spherical coordinates on the 2-sphere $S^2 = \{x^2+y^2+z^2 = 1\}$. Recall, that θ, ϕ are defined by
$$\begin{aligned} x &= \cos\phi\cos\theta \\ y &= \sin\phi\cos\theta \\ z &= \sin\theta \end{aligned}$$
and can be considered as local coordinates on the whole S^2 except for the north and south poles. Let $P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) \in S^2$ and $\xi = (1, -1, 1)$. Check that ξ (as a vector from $\mathbb{R}^3$) is tangent to S^2 at P. Find the components of ξ in the spherical coordinates (ϕ, θ).

5. Consider $GL(3, \mathbb{R})$ as a smooth manifold and $F = \det : GL(3, \mathbb{R}) \to R$ as a smooth function on it. Compute the directional derivative of $F(X) = \det X$ at point $X_0 = Id$ along the vector
$$\xi = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$
(Here we naturally identify the tangent space to $GL(3, \mathbb{R})$ at the point $X_0 = Id$ with the space of ALL 3×3 matrices.)

   Prove the following general formula $\xi(\det) = \operatorname{tr}\xi$, where ξ is an arbitrary $n \times n$ matrix viewed as a tangent vector to $GL(n, \mathbb{R})$ at the identity.

6. Prove that the special orthogonal group $SO(3)$ is diffeomorphic to the subset in $\mathbb{R}^6$ given by the following 3 equations: $x^2+y^2+z^2 = 1$, $u^2+v^2+w^2 = 1$, $xu+yv+zw = 0$.

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