MAGIC008: Lie Groups and Lie algebras

Problem Sheet 4: Matrix Lie groups, their Lie algebras and properties

1. Consider the Lie group $G_{\mathcal{B}} = \{X \in GL(3, \mathbb{R}) \mid X^\top B X = B\}$, where $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   In other words, $G_{\mathcal{B}}$ is the group of invertible transformations that preserve the (degenerate) symmetric form given by B.

   - What is $\dim G_{\mathcal{B}}$?
   - Describe all matrices $X \in G_{\mathcal{B}}$.
   - Describe the corresponding Lie algebra $\mathfrak{g}_{\mathcal{B}}$.
   - How many connected components are there in $G_{\mathcal{B}}$?
   - Prove that the identity component $(G_{\mathcal{B}})_0$ is diffeomorphic to $S^1 \times \mathbb{R}^3$.

2. Consider the matrix group $C(L) = \{X \in GL(n, \mathbb{R}) \mid X^{-1} L X = L\}$, where L is a certain $n \times n$ matrix (viewed as an operator but not a bilinear form); $C(L)$ is usually called the centralizer of L.

   - Describe $C(L)$ for $L = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ assuming that all λ_i's are distinct. What happens if some of λ_i's coincide? Describe the corresponding Lie algebra $\mathfrak{c}(L)$.
   - Describe $C(L)$ in the case when L is a Jordan block of maximal dimension:

   $$L = \begin{pmatrix} \lambda & 1 & \ldots & 0 \\ 0 & \lambda & 1 & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \ldots & \lambda \end{pmatrix}$$

     Describe the corresponding Lie algebra $\mathfrak{c}(L)$.
   - Prove that $\dim C(L) \geq n$ for any L.

3. Recall that

   $$SL(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \det A = ad - bc = 1 \right\}$$

   and

   $$Sp(2, \mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; A^\top J A = J \right\}, \qquad \text{where } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

   Show that $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$.

4. (*) Prove that the symplectic group $Sp(2n, \mathbb{R})$ is connected.

5. (*) Let $\mathcal{B}$ be a positive definite symmetric bilinear form on $\mathbb{R}^{2n}$ and $\mathcal{J}$ be a non-degenerate skew symmetric form on the same $\mathbb{R}^{2n}$. Denote their matrices (w.r.t. a certain basis in $\mathbb{R}^n$, the same for both of them) by B and J respectively, and consider the Lie groups that "preserve" $\mathcal{B}$ and $\mathcal{J}$ (see Lecture 8):

$$G_{\mathcal{B}} = \{X \in GL(n,\mathbb{R}) \mid X^\top BX = B\} \text{ and } G_{\mathcal{J}} = \{X \in GL(n,\mathbb{R}) \mid X^\top JX = J\}.$$

We know that $G_{\mathcal{B}} \simeq O(n)$ and $G_{\mathcal{J}} \simeq Sp(n,\mathbb{R})$ (here $\simeq$ means "isomorphic as Lie groups").

The question is to analyze the intersection $G_{\mathcal{B}} \cap G_{\mathcal{J}}$ depending on B and J.

- Prove: $G_{\mathcal{B}+\mathcal{J}} = G_{\mathcal{B}} \cap G_{\mathcal{J}}$.
- Prove that for standard B and J (i.e., $B = E_{2n}$, $J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$), the intersection $G_{\mathcal{B}} \cap G_{\mathcal{J}} = O(2n) \cap Sp(2n,\mathbb{R})$ is isomorphic to the unitary group $U(n)$.
- Describe the intersection $G_{\mathcal{B}} \cap G_{\mathcal{J}}$ for $B = E_{2n}$ and $J = \begin{pmatrix} 0 & D_n \\ -D_n & 0 \end{pmatrix}$ where $D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ with distinct λ_i.
- Describe (up to isomorphism) $G_{\mathcal{B}} \cap G_{\mathcal{J}}$ for B and J arbitrary. (Use the following result: for any skew-symmetric matrix J there exist an orthogonal matrix P such that $P^{-1}JP = P^\top JP = \begin{pmatrix} 0 & D_n \\ -D_n & 0 \end{pmatrix}$ as above but some of λ_i's may coincide.)
- What is the minimal possible dimension for $G_{\mathcal{C}}$, where $\mathcal{C}$ is an arbitrary bilinear form on $\mathbb{R}^{2n}$?

6. (*) Consider the space V of skew symmetric $n \times n$ matrices (over $\mathbb{R}$) endowed with the bilinear operation

$$[A_1, A_2]_C = A_1 C A_2 - A_2 C A_1, \qquad A_1, A_2 \in V,$$

where C is a fixed symmetric matrix. Prove that this operation determines the structure of a Lie algebra on V. Prove that if C is positive definite, then $(V, [\cdot,\cdot]_C)$ is isomorphic to $so(n)$.