MAGIC008: Lie Groups and Lie algebras

Problem Sheet 2: Vector fields and flows. Lie bracket of vector fields

1. Let $\xi = (x, y)$. Describe the flow of ξ on $\mathbb{R}^2$.

2. Let $\xi = (ax + b, -ay + c)$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Describe the flow of ξ on $\mathbb{R}^2$.

3. Consider the map $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Phi^t(x, y) = \left(\frac{x}{ty + 1} \ , \ \frac{y}{ty + 1} \right).$$

   Show that Φ^t is a flow (one-parameter group of diffeomorphisms), i.e., $\Phi^t \circ \Phi^s = \Phi^{t+s}$. Find the vector field ξ that generates Φ^t.

4. Compute the Lie bracket $[\xi, \eta]$ for the following vector fields:

   (a) $\xi = (x, y)$, $\eta = (-y, x)$ in $\mathbb{R}^2$;

   (b) $\xi = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$, $\eta = (-y, x)$ in $\mathbb{R}^2$;

   (c) $\xi = (\sin x, \sin y, \sin z)$, $\eta = (z + y, x + z, x + y)$ in $\mathbb{R}^3$.

5. Consider two vector fields in $\mathbb{R}^4(x, y, u, v)$ given by $\xi = (y, -x, v, -u)$ and $\eta = (u, v, -x, -y)$. Show that ξ and η are tangent to the 3-sphere $x^2 + y^2 + u^2 + v^2 = 1$. Compute the Lie bracket $[\xi, \eta]$. Is $[\xi, \eta]$ tangent to this sphere? Do the same for ξ and ζ, where $\zeta = (u, -v, -x, y)$.

6. Consider two linear vector fields in $\mathbb{R}^n$

$$\xi(x) = Ax \quad \text{and} \quad \eta(x) = Bx, \quad x \in \mathbb{R}^n$$

   where A and B are $n \times n$ matrices. Show that the Lie bracket $\zeta = [\xi, \eta]$ is a linear vector field, i.e., can be written as $\zeta(x) = Cx$ and find the matrix C.

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