Model answers to Week 03 review worksheet — exercises for §3

Part A. Exercises for interactive discussion

E3.1 (Sweedler notation) Let C be a coalgebra with coproduct Δ and counit ϵ. Let $x \in C$. Review the Sweedler notation, $\Delta x = \sum x_{(1)} \otimes x_{(2)}$, for the coproduct; we will write it without the $\sum$ symbol.

Which of the following are necessarily the same as x?

(Exclude the options where the Sweedler notation is used incorrectly or which are ill-defined.)

$A = \epsilon(x_{(1)})x_{(2)}$ $\qquad$ $D = \epsilon(x_{(1)})\epsilon(x_{(2)})$

$B = x_{(1)}\epsilon(x_{(2)})$ $\qquad$ $E = \frac{1}{2}(x_{(1)} + x_{(2)})$

$C = x_{(2)}$ $\qquad$ $F = \epsilon(x_{(1)})\epsilon({x_{(2)}}_{(1)}){x_{(2)}}_{(2)}$

Answer to E3.1. A, B, F are equal to x. D is well-defined but is equal to $\epsilon(x)$.

$A = x$ by the counit law.

B might be better written as $\epsilon(x_{(2)})x_{(1)}$ scalars precede elements of C, but is still equal to x by the counit law (and is still commonly written in the form $x_{(1)}\epsilon(x_{(2)})$ which refers to the canonical isomorphism $C \otimes \mathbb{C} \cong C$).

C is not well-defined: an expression cannot involve the second leg of Δx without the first leg. For example, if $\Delta x = x \otimes x$, then Δx can also be written as $2x \otimes \frac{1}{2}x$, so what is $x_{(2)}$: x or $\frac{1}{2}x$?

D is clearly a scalar (an element of the field $\mathbb{C}$) so it cannot be equal to x. D is well-defined because it is an expression bilinear in $x_{(1)}$ and $x_{(2)}$. We can simplify D as follows: $\epsilon(x_{(1)})\epsilon(x_{(2)}) = \epsilon\left(x_{(1)}\epsilon(x_{(2)})\right)$ because ϵ is linear; but $x_{(1)}\epsilon(x_{(2)}) = x$ by the counit law, hence $D = \epsilon(x)$.

E is not well-defined since it is not bilinear in $x_{(1)}$ and $x_{(2)}$: try substituting $x \otimes x = 2x \otimes \frac{1}{2}x$ for $x_{(1)} \otimes x_{(2)}$.

F is well-defined and can be simplified: $\epsilon(x_{(1)})\epsilon({x_{(2)}}_{(1)}){x_{(2)}}_{(2)} = \epsilon(x_{(1)})x_{(2)} = x$ where we apply the counit law twice.

E3.2 (grouplike elements) Let (C, Δ, ϵ) be a coalgebra. Review the definition of a grouplike element of C.

(a) Prove that any $g \in C$ such that $g \neq 0$ and $\Delta g = g \otimes g$, is grouplike.

(b) Let $G(C)$ denote the set of all grouplike element of C. Prove that $G(C)$ is a linearly independent set. (*Hint:* use the algebra-coalgebra duality and an exercise from last week!)

(c) What are the grouplikes in A^* where A is a finite-dimensional algebra?

(d) What are the possible 1-dimensional coalgebras?

Answer to E3.2. (a) Applying the counit to the left leg of Δg gives $\epsilon(g)g$, but by the counit law this must equal g. So, $(\epsilon(g) - 1)g = 0$. Since g is, by assumption, a non-zero vector in C, we must have $\epsilon(g) = 1$. Thus, g verifies the definition of grouplike.

(b) Recall that the space C^* is an algebra with product map $m_{C^*} = \Delta^*$ and unit map $\eta_{C^*} = \epsilon^*$. This means that the product $\phi\psi$ of $\phi, \psi \in C^*$ and the unit 1_{C^*} are defined by their evaluation against arbitrary $x \in C$, as follows:

$$\langle x, \phi\psi\rangle \stackrel{\text{def}}{=} \langle x_{(1)}, \phi\rangle\langle x_{(2)}, \psi\rangle, \qquad \langle x, 1_{C^*}\rangle = \epsilon(x).$$

We now turn this around and consider C as a space of linear functionals on C^*, thus embedding C injectively in $\mathrm{Lin}(C^*, \mathbb{C})$:

$$x \in C \quad \text{is viewed as} \quad x\colon C^* \to \mathbb{C}, \quad x(\phi) = \langle x, \phi\rangle.$$

Then by definition of grouplike, "g is grouplike" means that, for all $\phi, \psi \in C^*$,

$$(*) \qquad g(\phi\psi) = g(\phi)g(\psi), \qquad g(1_{C^*}) = 1.$$

Thus, $g \in C$ is grouplike iff the linear functional $g\colon C^* \to \mathbb{C}$ is an algebra homomorphism, i.e., a **character** of C^*.

Recall from E2.2 that characters of an algebra are linearly independent. It follows that grouplikes in C are linearly independent.

(c) Similarly to (b), we can see that the grouplikes in the coalgebra A^* are exactly the algebra characters of A. That is, $G(A^*) = \mathrm{Alg}(A, \mathbb{C})$.

(d) Let C be a 1-dimensional coalgebra with a single-element basis $\{v\}$. Then by Proposition 1.30, $C \otimes C$ has basis $\{v \otimes v\}$, and $\Delta v = \lambda v \otimes v$ for some λ in the ground field. Note that $\lambda \neq 0$, because Δ must always be injective as, by the counit law, $(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id}$. Hence the element λv also forms a basis of C. Note that $\Delta(\lambda v) = \lambda \Delta v = \lambda(\lambda v \otimes v) = \lambda v \otimes \lambda v$ and so, by part (a), λv is grouplike.

To conclude: **every one-dimensional coalgebra is spanned by a grouplike element.**

E3.3 Let G be a finite monoid (e.g., a finite group), so that $\mathbb{C}G$ is a finite-dimensional algebra. The coalgebra $(\mathbb{C}G)^*$ has a basis $\{\delta_g\}_{g\in G}$ dual to the basis $\{g\}_{g\in G}$ of $\mathbb{C}G$. Give formulae for $\Delta\delta_g$ and $\epsilon(\delta_g)$.

Answer to E3.3. We woud like to expand $\Delta\delta_g$ in terms of the basis $\{\delta_h \otimes \delta_k \mid h, k \in G\}$, given by Proposition 1.30. Since this basis of $(\mathbb{C}G)^* \otimes (\mathbb{C}G)^*$ is dual to the basis $\{h \otimes k\}$ of $\mathbb{C}G \otimes \mathbb{C}G$, the coefficient of $\delta_h \otimes \delta_k$ must be

$$\langle \Delta\delta_g, h \otimes k\rangle.$$

Yet by definition of Δ on $(\mathbb{C}G)^*$ we have $\Delta = m^*$, where $m\colon \mathbb{C}G \otimes \mathbb{C}G \to \mathbb{C}G$ is the product map. This means that

$$\langle \Delta\delta_g, h \otimes k\rangle = \langle \delta_g, m(h \otimes k)\rangle = \langle \delta_g, hk\rangle = \begin{cases} 1, & hk = g, \\ 0, & hk \neq g. \end{cases}$$

Hence we arrive at the expansion

$$\Delta\delta_g = \sum_{h,k\in G,\ hk=g} \delta_h \otimes \delta_k.$$

To calculate $\epsilon(\delta_g)$, we recall that, by definition, the counit ϵ on $(\mathbb{C}G)^*$ is η^* where $\eta\colon \mathbb{C} \to \mathbb{C}G$ is the unit map. We have $\eta(1) = e$ where e is the identity element of G. Thus,

$$\epsilon(\delta_g) = \langle \delta_g, e\rangle = \begin{cases} 1, & g = e, \\ 0, & g \neq e. \end{cases}$$

E3.4 (a) Assume that a coalgebra C has basis $\{\chi_0, \chi_1, \chi_2\}$ of grouplikes. Let $A = C^*$ be the dual algebra. Describe the multiplication on the dual basis $\{e_0, e_1, e_2\}$ of A.

(b) In the case when $A = \mathbb{C}\Gamma$ is the group algebra of the cyclic group $\Gamma = \{e, g, g^2\}$, and $C = (\mathbb{C}\Gamma)^*$, take χ_k to be the character of Γ which sends g to ω^k with $\omega = e^{2\pi i/3} \in \mathbb{C}$. Calculate the basis $\{e_0, e_1, e_2\}$ of $\mathbb{C}\Gamma$. Check directly that the multiplication on this basis is as you expect from (a).

Answer to E3.4. (a) Since $\{\chi_0, \chi_1, \chi_2\}$ is the dual basis to the basis $\{e_0, e_1, e_2\}$ of A, every element $a \in A$ can be expanded as $\chi_0(a)e_0 + \chi_1(a)e_1 + \chi_2(a)e_2$.

First, let us calculate $e_0^2 \in C^*$. For each $i = 0, 1, 2$, we have

$$\chi_i(e_0^2) = \chi_i(e_0)^2,$$

because by E3.2 χ_i is an algebra character of A. We therefore have $\chi_0(e_0) = 1$ and $\chi_i(e_0) = 0$ for $i \neq 0$ (by dual basis), giving

$$e_0^2 = e_0.$$

In the same way, $e_1^2 = e_1$ and $e_2^2 = e_2$, so that e_0, e_1, e_2 are **idempotents** in A. Recall that an idempotent is an element of an algebra which squares to itself.

To calculate e_0e_1, we again evaluate this element against χ_i for $i=0,1,2$:

$$\chi_i(e_0e_1)=\chi_i(e_0)\chi_i(e_1)=0 \text{ for all } i \quad\Rightarrow\quad e_0e_1=0.$$

In the same way, $e_ie_j=0$ whenever $i\neq j$. We conclude that the dual basis to a basis of grouplikes consists of **pairwise orthogonal idempotents.**

E3.5 (the trigonometric coalgebra) Let C be a two-dimensional space over $\mathbb{R}$ with basis $\{c,s\}$. Define $\Delta\colon C\to C\otimes C$ by

$$\Delta c = c\otimes c - s\otimes s, \quad \Delta s = s\otimes c + c\otimes s.$$

(a) Define a counit $\epsilon\colon C\to\mathbb{R}$ so that (C,Δ,ϵ) becomes a coalgebra.

(b) Does C contain any grouplikes? Does C have proper subcoalgebras?

(c) How does the answer to (b) change if the field $\mathbb{R}$ is replaced by $\mathbb{C}$?

Answer to E3.5. (a) If $\epsilon\colon C\to\mathbb{R}$ is a counit, we must have

$$c=(\epsilon\otimes\mathrm{id})\Delta c = \epsilon(c)c-\epsilon(s)s,$$

and, since $\{c,s\}$ is a basis, $\epsilon(c)=1$ and $\epsilon(s)=0$. This completely determines ϵ, and we should also check that the counit law is satisfied in all other cases, i.e., $(\epsilon\otimes\mathrm{id})\Delta s=s$, $(\mathrm{id}\otimes\epsilon)\Delta c=c$ and $(\mathrm{id}\otimes\epsilon)\Delta s=s$. This is a straightforward calculation.

(b) Assume that $g=\alpha c+\beta s$ is grouplike, then $\epsilon(g)=1$ which says that $\alpha=1$.

Moreover, $\Delta g=g\otimes g$ which is equivalent to

$$(c\otimes c-s\otimes s)+\beta(s\otimes c+c\otimes s)=(c+\beta s)\otimes(c+\beta s).$$

Equating the coefficients of $c\otimes c$, $c\otimes s$, $s\otimes c$ and $s\otimes s$ on both sides, we come to the equation

$$\beta^2=-1.$$

This tells us that over $\mathbb{R}$, the trigonometric coalgebra does not contain grouplikes.

(c) If we extend the scalars to $\mathbb{C}$, then the resulting 2-dimensional coalgebra over $\mathbb{C}$ is spanned by two grouplikes, $e=c+is$ and $\bar{e}=c-is$.

E3.6 Review the definition of an *action* of an associative unital algebra A on a vector space V.

True or false: every algebra can act on a 1-dimensional space?

Answer to E3.6. False. Suppose that an algebra A acts on a 1-dimensional space V with a one-element basis $\{v\}$. This means that every $a\in A$ must act by $a\triangleright v=\lambda(a)v$ with $\lambda(a)\in\mathbb{C}$. By definition of action, λ must be linear in a, $\lambda(ab)=\lambda(a)\lambda(b)$ for all $a,b\in A$, and $\lambda(1_A)=1$. That is, $\lambda\in\mathrm{Alg}(A,\mathbb{C})$ must be an algebra character.

Yet some associative unital algebras have no characters: for example, the algebra $M_{2\times2}(\mathbb{C})$ of complex 2×2-matrices, or the Weyl algebra with presentation $\mathbb{C}\langle x,y\mid yx-xy=1\rangle$ (*exercise or a topic for further discussion*).

Part B. Extra exercises

E3.7 (left regular module and left regular comodule) Let A be an algebra. Prove that the map $\triangleright_{\mathrm{reg}}\colon A\otimes A\to A$ given by $a\triangleright_{\mathrm{reg}} v=av$ (product in A) is an action of the algebra A on the vector space A. (This action of A on A is called the *left regular action.*)

Develop the parallel notion of left regular coaction for coalgebras.

Answer to E3.7. We show that the regular action verifies the definition of an action. First of all, $a \otimes v \mapsto av$ is bilinear in a and v, by definition of product map in the algebra A. Furthermore,

- $a \triangleright_{\text{reg}} (b \triangleright_{\text{reg}} v) = a \triangleright_{\text{reg}} (bv) = a(bv) = (ab)v = (ab) \triangleright_{\text{reg}} v$ by associativity of multiplication in A;
- $1_A \triangleright_{\text{reg}} v = 1_A v = v$ by the identity law in A.

Thus, all axioms of action are satisfied.

The **left regular coaction** of a coalgebra (C, Δ, ϵ) on the vector space C is

$$\delta_{\text{reg}} = \Delta \colon C \to C \otimes C, \quad \delta_{\text{reg}}(x) = x_{(1)} \otimes x_{(2)},$$

where we consider the left leg as being in C the coalgebra, and the right leg as being in C the vector space (the comodule). The axioms of coaction follow from the coassociative law and the counit law for Δ and ϵ.