13 days

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO HOPF ALGEBRAS AND QUANTUM GROUPS

Exam Released:	**22 Apr 2025 00:00 (BST)**
End of Submission Window:	**05 May 2025 11:00 (BST)**

Answer **THREE** of the **FOUR** questions. If more than **THREE** questions are attempted, then credit will be given for the best **THREE** answers.

The maximum possible mark for this paper is 60. The pass mark is 30 out of 60.

This is a take-home, open-book exam. Your solutions should be written on paper, preferably using blue or black ink (not pencil), on a tablet using a stylus, or typeset. You may use without proof any of the formulae given in the course, unless you are asked to prove them.

Your answers must be your own work. Identical answers will be flagged for plagiarism.

Answer **<u>THREE</u>** of the four questions. If more than THREE questions are attempted, then credit will be given for the best THREE answers.

1. In this question, H denotes a Hopf algebra over the field $\mathbb{C}$ of complex numbers with counit $\epsilon\colon H \to \mathbb{C}$ and antipode $S\colon H \to H$.

(a) Let $x \in H$. Consider the following calculation:

$$\begin{aligned} \epsilon(S(x)) &= \epsilon(S(\epsilon(x_{(1)})x_{(2)})) && \text{by the counit law} \\ &= \epsilon(x_{(1)})\epsilon(S(x_{(2)})) && \text{as } \epsilon \circ S \text{ is a linear map, and } \epsilon(x_{(1)}) \text{ is a scalar.} \end{aligned}$$

Continue the calculation to show that $\epsilon(S(x)) = \epsilon(x)$. You should give as much detail as possible and explain each step in your calculation, similarly to how the first two steps are explained above.

(b) For each example of H given in (i)–(iii), construct an element $y \in H$ such that $y + S(y) = 0$ but y is not primitive. You need to explain why y satisfies the required conditions:

(i) $H = \mathbb{C}\Gamma$, the group algebra of the cyclic group $\Gamma = \{e, g, g^2\}$ of order 3;

(ii) $H = \mathbb{C}\langle x\rangle$, the free algebra with one generator x where x is primitive;

(iii) $H = U_q(\mathfrak{sl}_2)$, the Drinfeld-Jimbo quantum group generated over $\mathbb{C}$ by E, F, K, K^{-1} subject to the relations and coproduct formulae given in the course.

In the rest of the question, $\triangleright$ denotes an action of H on an associative unital algebra A which makes A an H-module algebra. Let the subset B of the algebra A be given by

$$B = \{a \in A : h \triangleright a = \epsilon(h)a \text{ for all } h \in H\}.$$

(c) Prove that B is a subalgebra of A.

(d) Let $T \subset B$ and let I be the ideal of A generated by T. Show that $h \triangleright I \subseteq I$ for all $h \in H$.

[20 marks]

2. In this question, K is the abelian group $\{e, a, b, c\}$ with identity element $e = a^2 = b^2 = c^2$, and $\mathbb{C}K$ is the group algebra of K viewed as a Hopf algebra in the standard way. Denote $p_0 = \frac{1}{4}(e + a + b + c)$, $p_1 = \frac{1}{4}(e + a - b - c)$, $p_2 = \frac{1}{4}(e - a + b - c)$ and $p_3 = \frac{1}{4}(e - a - b + c)$. View $(\mathbb{C}K)^*$ as the dual Hopf algebra of $\mathbb{C}K$. Let $\{\delta_e, \delta_a, \delta_b, \delta_c\}$ be the basis of $(\mathbb{C}K)^*$ dual to the basis $\{e, a, b, c\}$ of $\mathbb{C}K$.

(a) Give a calculation to show that $p_i^2 = p_i$ and $p_ip_j = 0$ if $i \neq j$, for all $i, j = 0, 1, 2, 3$.

(b) Find all grouplike elements of $(\mathbb{C}K)^*$, expressing them as linear combinations of δ_e, δ_a, δ_b and δ_c. For each $\xi \in G((\mathbb{C}K)^*)$ that you find, calculate $\xi(p_0)$, $\xi(p_1)$, $\xi(p_2)$ and $\xi(p_3)$.

In the rest of the question, R denotes the element $e \otimes (p_0 + p_1) + b \otimes (p_2 + p_3)$ of $\mathbb{C}K \otimes \mathbb{C}K$.

(c) Calculate R^2 and show that $R^2 = \lambda e \otimes e$, where $\lambda \neq 0$ is a scalar to be determined.

(d) Prove that R is a quasitriangular structure on $\mathbb{C}K$.

[20 marks]

3. View the free algebra $T = \mathbb{C}\langle X, H, Y\rangle$ as a Hopf algebra in the standard way, by making X, H, Y primitive. Let Mon be the basis of T which consists of all (noncommutative) monomials in X, H, Y, and let the set $St \subset Mon$ of standard monomials be $\{X^p H^q Y^r : p, q, r \in \mathbb{Z}_{\geq 0}\}$. Let the set $NonSt$ of non-standard monomials be given by $NonSt = Mon \setminus St$. Let J be the subspace of T spanned by $NonSt$.

(a) Prove that J is an ideal of T. By considering the elements $X + J$ and $Y + J$ of the quotient algebra T/J, or otherwise, show that T/J is not commutative.

(b) Show that J is not a Hopf ideal of T, indicating all axioms from the definition of a Hopf ideal which fail.

Let $U = \mathbb{C}\langle X, H, Y \mid HX - XH = 2X,\ HY - YH = -2Y,\ XY - YX = H\rangle$ be the universal enveloping algebra of $\mathfrak{sl}_2$, and let $\pi\colon T \twoheadrightarrow U$ be the quotient map. You are given that $\pi(J)$ is an ideal of U.

(c) Show that in U, $X^2YH - XYXH - H^2X + 4HX = \mu X$, where $\mu \neq 0$ is a scalar to be determined.

(d) Deduce from (c) that the ideal $\pi(J)$ of U contains X and is equal to the kernel of the counit of U.

[20 marks]

4. Assume that some associative unital algebra H contains elements E, F, K, K^{-1} which satisfy

$$KK^{-1} = K^{-1}K = 1,\ KE = q^2EK,\ KF = q^{-2}FK,\ EF - FE = \tfrac{K-K^{-1}}{q-q^{-1}}$$

where $q \in \mathbb{C} \setminus \{0, \pm 1\}$. Let $Z \in H$ be given by $Z = (q - q^{-1})^2 EF + q^{-1}K + qK^{-1}$.

(a) Prove that $EZ = ZE$ in H.

Assume further that H is a Hopf algebra where the coproduct Δ and counit ϵ satisfy $\Delta E = 1 \otimes E + E \otimes K$, $\Delta F = K^{-1} \otimes F + F \otimes 1$, $\epsilon(E) = \epsilon(F) = 0$, and where K, K^{-1} are grouplike.

(b) Prove that the antipode of F must be equal to $-KF$.

You are now given an action $\triangleright$ of H on the quantum plane $A_q = \mathbb{C}\langle x, y \mid yx = qxy\rangle$, which makes A_q an H-module algebra and satisfies $E \triangleright x = F \triangleright y = 0$, $E \triangleright y = x$, $F \triangleright x = y$, $K \triangleright x = qx$, $K \triangleright y = q^{-1}y$.

(c) Show that $Z \triangleright (y^2) = (q^m + q^{-m})y^2$, where m is a positive integer to be determined. Also calculate $Z \triangleright (x^2)$.

Finally, assume that $H = U_q(\mathfrak{sl}_2)$, the Hopf algebra defined by generators and relations as set out above. Let $q = e^{\hbar}$ and let $\mathcal{R} = \exp(\frac{\hbar}{2} H \otimes H) \exp_{q^{-2}}((q - q^{-1})E \otimes F)$ be the universal R-matrix for $U_q(\mathfrak{sl}_2)$ constructed in the course; in particular, H is a primitive element such that $H \triangleright x = x$, $H \triangleright y = -y$ and $K = \exp(\hbar H)$.

(d) Show that $\mathcal{R} \triangleright (y \otimes y) = f(\hbar) y \otimes y$, where $f(\hbar) \in \mathbb{C}[[\hbar]]$ is a formal power series in $\hbar$, to be determined. Briefly explain the notation and assumptions used in your calculation.

[20 marks]

END OF EXAMINATION PAPER