GOAL: define the notion of a
quasitriangular structure
(universal R-matrix) on a Hopf algebra H.

DEF A quasitriangular structure on H is
$$R \in H \otimes H$$
such that

1. R is invertible in $H \otimes H$
2. $\forall x \in H \qquad x_{(2)} \otimes x_{(1)} = R \cdot (x_{(1)} \otimes x_{(2)}) \cdot R^{-1}$
3. $$\left.\begin{aligned} (\Delta \otimes id) R &= R_{13} R_{23} \\ (id \otimes \Delta) R &= R_{13} R_{12} \end{aligned}\right\} \text{ in } H \otimes H \otimes H$$

where $\quad R_{12} = R \otimes 1_H \in H^{\otimes 3}$

$R_{23} = 1_H \otimes R \in H^{\otimes 3}$

$R_{13} = (\tau \otimes id) R_{23}$

Observation Every cocommutative Hopf algebra has at least one quasitriangular structure, namely the trivial QT structure $1 \otimes 1$. $= 1_{H \otimes H}$.

Indeed, $x_{(1)} \otimes x_{(2)} = x_{(2)} \otimes x_{(1)} \quad \forall x \in H$

PROP Let R be a quasitriangular structure on H. Then $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$.

(R satisfies the Quantum Yang-Baxter equation without the spectral parameter)

Proof. Recall:

2. $\forall x \in H \qquad x_{(2)} \otimes x_{(1)} = R \cdot (x_{(1)} \otimes x_{(2)}) \cdot R^{-1}$

3. $\left.\begin{aligned} (\Delta \otimes id) R &= R_{13} R_{23} \\ (id \otimes \Delta) R &= R_{13} R_{12} \end{aligned}\right\}$ in $H \otimes H \otimes H$

So
$$\underbrace{R_{12} R_{13} R_{23}} = R_{12} (\Delta \otimes id) R$$
$$= \underbrace{R_{12} (\Delta \otimes id) R \cdot R_{12}^{-1}} R_{12}$$
$$\underset{(2)}{=} (\tau \otimes id)\big((\Delta \otimes id) R\big) R_{12}$$
$$= (\tau \otimes id)(R_{13} R_{23}) R_{12}$$
$$= R_{23} R_{13} R_{12} \qquad \checkmark$$

Example Quasitriangular structures on $H = (\mathbb{C}G)^*$.

Let G be a finite group with identity e.
The dual space $(\mathbb{C}G)^*$ of $\mathbb{C}G$ has basis $\{\delta_g \mid g \in G\}$ where $\delta_g \in \mathrm{Lin}(\mathbb{C}G, \mathbb{C})$ is defined on $h \in G$ by

$$\delta_g(h) = \begin{cases} 1, & h = g \\ 0, & h \neq g \end{cases}$$

Recall:
- commutative product on $(\mathbb{C}G)^*$: $\quad \delta_g \delta_h = \begin{cases} \delta_g, & g = h \\ 0, & g \neq h \end{cases}$
- $1_{(\mathbb{C}G)^*} = \sum_{g \in G} \delta_g$
- coproduct $\quad \Delta \delta_g = \sum_{h,k \in G,\, hk = g} \delta_h \otimes \delta_k$
- counit $\quad \varepsilon(\delta_g) = \delta_g(e)$

What are quasitriang. structures $R \in H^{\otimes 2} = (\mathbb{C}G)^* \otimes (\mathbb{C}G)^*$?

Expand R in the basis of $H^{\otimes 2}$: $R = \sum_{g,h \in G} \alpha_{g,h} \delta_g \otimes \delta_h$

Axiom 1. R is invertible.

$$R^{-1} = \sum_{g,h \in G} \beta_{g,h} \delta_g \otimes \delta_h ; \quad RR^{-1} = \sum_{g,h \in G} \overbrace{\alpha_{g,h} \beta_{g,h}}^{=1} \delta_g \otimes \delta_h$$

$$1_{H \otimes H} = 1_H \otimes 1_H = \Big(\sum_{g \in G} \delta_g\Big) \otimes \Big(\sum_{h \in G} \delta_h\Big) = \sum_{g,h \in G} \delta_g \otimes \delta_h$$

Axiom 1 $\Longleftrightarrow \alpha_{g,h} \neq 0 \ \forall g,h \Longleftrightarrow \alpha_{g,h} \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

Axiom 2. $R(x_{(1)} \otimes x_{(2)}) R^{-1} = x_{(2)} \otimes x_{(1)}, \quad \forall x \in (\mathbb{C}G)^*$

$H \otimes H$ is commutative (as H is) so LHS $= x_{(1)} \otimes x_{(2)}$

So Axiom 2 $\Longleftrightarrow \Delta = \Delta^{op}$ on $(\mathbb{C}G)^* \Longleftrightarrow \mathbb{C}G$ is commutative

$\Longleftrightarrow G$ is abelian.

Axiom 3. $(\Delta \otimes \mathrm{id})R = R_{13}R_{23}$, $(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}$.

$$\mathrm{LHS} = \sum_{g,h \in G} \alpha_{g,h} \sum_{k,\ell : k\ell = g} \delta_k \otimes \delta_\ell \otimes \delta_h = \sum_{h,k,\ell \in G} \alpha_{k\ell,h} \delta_k \otimes \delta_\ell \otimes \delta_h$$

$$\mathrm{RHS} = \sum_{g,h,m,n \in G} \alpha_{g,h} \alpha_{m,n} \delta_g \otimes \delta_m \otimes \delta_h \delta_n = \sum_{g,h,m} \alpha_{g,h} \alpha_{m,h} \delta_g \otimes \delta_m \otimes \delta_h$$

$$= \sum_{k,\ell,h} \alpha_{k,h} \alpha_{\ell,h} \delta_k \otimes \delta_\ell \otimes \delta_h$$

So $\forall k,\ell,h$ $\alpha_{k\ell,h} = \alpha_{k,h}\alpha_{\ell,h}$.

Also $\alpha_{h,k\ell} = \alpha_{h,k}\alpha_{h,\ell}$

So $\alpha_{g,h}$ is bimultiplicative, $\alpha_{\cdot,\cdot} : G \times G \to \mathbb{C}^\times$ is a bicharacter of the abelian group G.

$\{$bicharacters of $G\} \overset{1:1}{\longleftrightarrow} \{$qt structures on $(\mathbb{C}G)^*\}$

A universal R-matrix for $\mathcal{U}_\hbar$:

$$R_\hbar = \exp\left(\frac{\hbar}{2} H\otimes H\right) \exp_{q^{-2}}\left((q-q^{-1}) E\otimes F\right)$$

Here $\exp\left(\frac{\hbar}{2} H\otimes H\right) = \sum_{i=0}^{\infty} \frac{\hbar^i}{2^i\, i!} H^i \otimes H^i$ [⚠ this is NOT $K^{1/2}\otimes K^{1/2}$]

$$\in \mathcal{U}_\hbar \hat{\otimes} \mathcal{U}_\hbar$$

Notation: the q-exponential series.

DEF

$$[n;q^{-2}] = \overbrace{1 + q^{-2} + q^{-4} + \dots + (q^{-2})^{n-1}}^{= n + \hbar(\dots)} = \frac{1-q^{-2n}}{1-q^{-2}}$$

$$[n;q^{-2}]! = [1;q^{-2}][2;q^{-2}]\cdots[n;q^{-2}]$$

$$\exp_{q^{-2}}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n;q^{-2}]!}$$

note that $q - q^{-1} = 2\hbar + \hbar^2(\dots)$

An action of $\mathcal{U}_\hbar$ on a 2-dimensional space

$V = \text{span}_{\mathbb{C}}\{x, y\}$ We know that $U(sl_2)$ acts on V via

$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$E \triangleright x = 0$ $E \triangleright y = x$ $F \triangleright x = y$ $F \triangleright y = 0$ $H \triangleright x = x$ $H \triangleright y = -y$

PROP There exists a $\mathbb{C}[[\hbar]]$-linear action of $\mathcal{U}_\hbar$ on $V[[\hbar]]$ where E, F and H act as above.

$$\overset{\shortparallel}{\left\{ \sum_{n=0}^{\infty} \hbar^n v_n \mid v_n = \lambda_n x + \mu_n y,\ \lambda_n, \mu_n \in \mathbb{C} \right\}}$$

Proof Any action of the generators E, H, F can be uniquely extended to an action of the free $\hbar$-adic algebra $T_\hbar = \mathbb{C}[[\hbar]]\langle E, H, F\rangle$. We need to show that the $\hbar$-adic ideal $I_\hbar$ generated by

$$HE - EH - 2E, \quad HF - FH + 2F, \quad EF - FE - \frac{K - K^{-1}}{q - q^{-1}}$$

acts by 0. (Recall $K = e^{\hbar H}$, $q = e^{\hbar}$)

$HE - EH - 2E$, $HF - FH + 2F$ act by 0 (from sl_2)

$$K \triangleright x = e^{\hbar H} \triangleright x = e^{\hbar \cdot 1} x = qx; \quad K \triangleright y = q^{-1} y$$

$$\left(EF - FE - \frac{K - K^{-1}}{q - q^{-1}}\right) \triangleright x = E \triangleright y - 0 - \frac{q - q^{-1}}{q - q^{-1}} x = 0$$

$\blacksquare$

Corollary $\mathcal{U}_q(sl_2)$ acts on V via

$E \triangleright x = 0$, $E \triangleright y = x$, $F \triangleright x = y$, $F \triangleright y = 0$, $K^{\pm 1} \triangleright x = q^{\pm 1} x$, $K^{\pm 1} \triangleright y = q^{\mp 1} y$

The quantum plane $\mathbb{C}_\hbar[x,y]$ is a $\mathcal{U}_\hbar$-module algebra

Def The quantum plane $\mathbb{C}_\hbar[x,y]$ is the noncommutative $\hbar$-adic algebra $\mathbb{C}[[\hbar]]\langle x, y \mid yx = e^{\hbar} xy \rangle$.

Exercise $\mathbb{C}_\hbar[x,y]$ has $\hbar$-adic basis $\{x^a y^b \mid a, b \in \mathbb{Z}_{\geq 0}\}$.

THM $U_\hbar$ acts on $\mathbb{C}_\hbar[x,y]$ covariantly, extending the above action on $V[[\hbar]]$.

Proof $\mathbb{C}_\hbar[x,y] = \mathbb{C}[[\hbar]]\langle x,y\rangle / I_{yx-qxy}$

It is trivial to extend the action of $U_\hbar$ from V to the free algebra $\mathbb{C}[[\hbar]]\langle x,y\rangle$ covariantly using the coproduct on $U_\hbar$. We need to show that

$$E \triangleright I \subseteq I, \quad H \triangleright I \subseteq I, \quad F \triangleright I \subseteq I$$

where $I = I_{yx-qxy}$. This will define the $U_\hbar$-action on $\mathbb{C}[[\hbar]]\langle x,y\rangle / I$, automatically covariant.

$$E \triangleright (yx - qxy) = (1 \triangleright y)(\underbrace{E \triangleright x}_{}^{=0}) - q\,\underbrace{(1 \triangleright x)(E \triangleright y)}_{-qxx = -qx^2}$$

$\Delta E = 1 \otimes E + E \otimes K$

$$+ \underbrace{(E \triangleright y)(K \triangleright x)}_{+x(qx)} - q\,(\underbrace{E \triangleright x}_{=0})(K \triangleright y) = 0$$

Similarly for $F \triangleright (yx - qxy). = 0$

$$H \triangleright (yx - qxy) = y(H \triangleright x) + (H \triangleright y)x - qx(H \triangleright y) - q(H \triangleright x)y$$

$\Delta H = 1 \otimes H + H \otimes 1$

$$= + yx - yx + qxy - qxy = 0$$

So: $\mathcal{U}_\hbar \triangleright I_{yx - qxy} = I_{yx - qxy}$ $\blacksquare$

Idea of proof of the "PBW theorem" for $\mathcal{U}_\hbar$.

THM In the algebra $\mathcal{U}_\hbar$, standard monomials

$$E^m H^n F^p, \quad m, n, p \in \mathbb{Z}_{\geq 0},$$

form an $\hbar$-adic basis.

Idea of proof (of linear independence)

If $\sum_{k=0}^{\infty} \hbar^k u_k = 0$ in $\mathcal{U}_\hbar$, where u_k is a finite linear combination of standard monomials as above, let $k_0 = \min\{k : u_k \neq 0\}$. We can construct a monomial $x^a y^b$ in $\mathbb{C}_\hbar[x,y]$ such that $u_{k_0} \triangleright x^a y^b \notin \hbar\ \mathbb{C}_\hbar[x,y]$, leading to a contradiction. $\square$

Corollary In $U_q(sl_2)$, the monomials of the form $E^m K^n F^p$, $m, p \in \mathbb{Z}_{\geqslant 0}$, $n \in \mathbb{Z}$, form a $\mathbb{C}$-basis.