MAGIC 009 Exam

Category Theory

To be released on 13th January 2025

Instructions

- This paper contains four questions. You should submit answers to **all** questions.
- The maximum number of marks is 100.
- All questions carry equal weight (25 marks each).
- The pass mark is set at 50 marks.
- The deadline for submitting your answers is **11:00am** Friday 24 January 2025
- You should submit your work by uploading **clearly legible** examination scripts to the course filespace.
- The filename of the submitted work should show your name and 'MAGIC-009'.
- Each student participating in MAGIC assessment is asked to declare that they understand that MAGIC assessment will be subject to the academic misconduct rules of their own institution.

Questions

1. Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ be categories. Let $F, G, H : \mathbb{A} \to \mathbb{B}$ and $P, Q : \mathbb{B} \to \mathbb{C}$ be functors. Let $\phi : F \Rightarrow G$, $\psi : G \Rightarrow H$ and $\theta : P \to Q$ be natural transformations. You may assume the following facts from Lecture 2.

   - The composite of functors F and P is well-defined as a functor $PF : \mathbb{A} \to \mathbb{C}$, and similarly for all other pairs of composable functors.
   - The natural transformations ϕ and ψ has a well-defined composite natural transformation $\psi \circ \phi : F \to H$ whose component on $X \in \mathbb{A}$ is given by $\psi_X \circ \phi_X$.

   (a) Prove that there is a natural transformation $\theta.F : PF \Rightarrow QF$ whose component on $X \in \mathbb{A}$ is given by $\theta_{FX} : PFX \to QFX$.

   (b) Prove that there is a natural transformation $P.\phi : PF \Rightarrow PG$ whose component on $X \in \mathbb{A}$ is given by $P(\phi_X) : PFX \to PGX$.

   (c) Prove that the natural transformations $(\theta.G) \circ (\phi.P)$ and $(Q.\phi) \circ (\theta.F)$ from the functor $PF : \mathbb{A} \to \mathbb{C}$ to the functor $QG : \mathbb{A} \to \mathbb{C}$ are equal.

   (d) Let $[\mathbb{A}, \mathbb{B}]$ and $[\mathbb{B}, \mathbb{C}]$ be the respective functor categories. You may assume that these are well-defined categories as described in Lecture 2. Consider the following assignments of data in the category $[\mathbb{B}, \mathbb{C}] \times [\mathbb{A}, \mathbb{B}]$ to data in the category $[\mathbb{A}, \mathbb{C}]$

- On objects, a pair of functors (P, F) is sent to their composite PF,
- On morphisms, a pair of natural transformations of the form $(\theta, \phi) : (P, F) \to (Q, G)$ is sent to $\theta * \phi$, where $\theta * \phi$ denotes either of the two natural transformations that were proved to be equal in part (c),

Prove that these assignments extend to a well-defined functor $[\mathbb{B}, \mathbb{C}] \times [\mathbb{A}, \mathbb{B}] \to [\mathbb{A}, \mathbb{C}]$.

2. (a) Give an example of an equivalence of categories $F : \mathbb{A} \to \mathbb{B}$ for which the function $\mathbf{ob}(F) : \mathbf{ob}(\mathbb{A}) \to \mathbf{ob}(\mathbb{B})$ is injective but not surjective.

   (b) A morphism $e : A \to A$ in a category $\mathbb{C}$ is called *idempotent* if $e \circ e = e$. A *splitting* of an idempotent e consists of a pair of morphisms (r, s) of the form $r : A \to B$ and $s : B \to A$ satisfying $rs = 1_B$ and $sr = e$. Let $e : A \to A$ be an idempotent with splitting (r, s) and consider the parallel pair in $\mathbb{C}$ displayed below.

$$\begin{tikzcd} A \arrow[r, shift left, "1_A"] \arrow[r, shift right, "e"'] & A \end{tikzcd}$$

      i. Prove that $s : B \to A$ is the equaliser of the parallel pair displayed above.
      ii. Hence prove that $r : A \to B$ is the coequaliser of the same parallel pair displayed above.
      iii. Prove that any functor $F : \mathbb{C} \to \mathbb{D}$ preserves the equaliser of part (i) and the coequaliser of part (ii).

3. Let $\mathbb{A}$ and $\mathbb{B}$ be categories and suppose $\mathbb{B}$ has binary products with projections denoted $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$.

   (a) Let $F, G : \mathbb{A} \to \mathbb{B}$ be functors. Given a morphism $f : X \to Y$ in $\mathbb{A}$, let $Ff \times Gf$ denote the morphism induced by the universal property of $FY \times GY$ given the pair of morphisms $(Ff \circ \pi_{FX}, Gf \circ \pi_{GX})$. Prove that there is a functor $F \times G : \mathbb{A} \to \mathbb{B}$ defined on objects via $(F \times G)(X) = FX \times GX$ and on morphisms via $(F \times G)(f : X \to Y) = Ff \times Gf$.

   (b) Prove that the functor category $[\mathbb{A}, \mathbb{B}]$ has binary products.

   (c) Suppose that $\mathbb{A}$ has n objects for some $n \geq 2$, but that all the morphisms in $\mathbb{A}$ are identities. What does it mean for the diagonal $\Delta : \mathbb{B} \to [\mathbb{A}, \mathbb{B}]$ to have a right adjoint?

   (d) Suppose that $\mathbb{B}$ has binary products and $\mathbb{A}$ has $n \geq 2$ objects and only identity morphisms. Prove by induction on n that the diagonal $\Delta : \mathbb{B} \to [\mathbb{A}, \mathbb{B}]$ does indeed have a right adjoint.

4. (a) Let $\mathbb{B}$ be a locally small category.

      i. For an object $X \in \mathbb{B}$, prove that the representable $\mathbb{B}(X, -) : \mathbb{B} \to \mathbf{Set}$ preserves any binary products that exist in $\mathbb{B}$.
      ii. Prove that the terminal object in $[\mathbb{B}^{\mathrm{op}}, \mathbf{Set}]$ is the functor which sends every object to the singleton set and which sends every morphism to the identity function on that set.

   (b) Let $(G, ., e)$ be a group and let $\overline{G}$ be the corresponding category with one object, with morphisms given by the elements of G, with identity e and with composition law given by $h \circ g = h.g$. You may assume that this is well-defined as a category. Denote the unique object in $\overline{G}$ by $*$.

      i. Define explicitly the behaviour of the representable presheaf $\overline{G}(-, *) : \overline{G}^{\mathrm{op}} \to \mathbf{Set}$, and of representable natural transformations $\overline{G}(-, g) : \overline{G}(-, *) \to \overline{G}(-, *)$ for $g \in G$. You do not need to prove that these are well-defined as functors or as natural transformations.

ii. Describe explicitly what it means for the Yoneda embedding $Y_{\overline{G}} : \overline{G} \to [\overline{G}^{\mathrm{op}}, \mathbf{Set}]$ to be well-defined as a functor.

iii. Describe explicitly what it means for the Yoneda embedding $Y_{\overline{G}} : \overline{G} \to [\overline{G}^{\mathrm{op}}, \mathbf{Set}]$ to be fully faithful.