§ 5

Hopf ideals and quotients. Universal enveloping algebras. The Poincaré-Birkhoff-Witt theorem

We can obtain new Hopf algebras as quotients of a Hopf algebra modulo a Hopf ideal. An example of this construction is the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, which has a presentation by quadratic-linear relations.

A general, and generally difficult, problem of noncommutative algebra is to determine a basis of an algebra defined by generators and relations. This problem is solved for the algebra $U(\mathfrak{g})$. The solution, known as the Poincaré-Birkhoff-Witt (or PBW) theorem, was found by Poincaré in 1900, and a full proof was given by Birkhoff and Witt in 1937; for the so-called linear Lie algebras, the result can be deduced from the work of Capelli in 1880s.

The PBW theorem and its original proof predate Hopf algebras. Our goal here is to present a proof which uses Hopf algebra methods. We will not prove the most general case of the theorem, but will make an additional assumption, explained below. This assumption is seen to hold for all finite-dimensional Lie algebras, all abelian Lie algebras and Lie algebras formed by primitive elements in any Hopf algebra H. A more general proof which works for arbitrary Lie algebras can be found in the literature, e.g., Procesi 2007, chapter 5, §7.1.

Hopf ideals

5.1 Definition (Hopf ideal, quotient Hopf algebra) A subspace I of a Hopf algebra H is a **Hopf ideal** if

- I is an ideal of the algebra H;
- I is a coideal of the coalgebra H;
- $S(I) \subseteq I$, where S is the antipode of H.

The quotient space H/I modulo a Hopf ideal I has the structure of a Hopf algebra (it is an algebra and a coalgebra by earlier results, and it is easy to see that the map $h + I \mapsto S(h) + I$ from H/I to H/I obeys the antipode law). The Hopf algebra H/I is the **quotient Hopf algebra** of H modulo I.

Hopf algebras obtained as a quotient of a free algebra

Recall that $\mathbb{C}\langle X\rangle$ is a free algebra generated by a set X, which is made a Hopf algebra by declaring each element of X to be primitive, see Theorem 4.15. A presentation $\langle X \mid \mathcal{R}\rangle = \mathbb{C}\langle X\rangle/I_{\mathcal{R}}$ defines a quotient Hopf algebra of the Hopf algebra $\mathbb{C}\langle X\rangle$ if $I_{\mathcal{R}}$ is a Hopf ideal of $\mathbb{C}\langle X\rangle$. Yet not all ideals of $\mathbb{C}\langle X\rangle$ are Hopf ideals.

A condition on $\mathcal{R}$ which is **sufficient** for $\mathcal{R}$ to generate a Hopf ideal is given by the following

5.2 Proposition *If T is a set of primitive elements of a Hopf algebra H, that is, T is a subset of $P(H)$, then the ideal I of H generated by T is a Hopf ideal.*

Proof. We can assume that T is a subspace of $P(H)$, otherwise replace T by the span of T (which still lies in $P(H)$ because $P(H)$ is a vector space, and generates the same ideal). The ideal generated by T can be written as $I = HTH$. It is already an ideal; to check that I is a coideal, write

$$\begin{aligned}
\Delta(I) = \Delta(HTH) &\subseteq \Delta(H)\Delta(T)\Delta(H) && \text{as } \Delta \text{ is multiplicative}\\
&\subseteq (H\otimes H)(T\otimes 1 + 1\otimes T)(H\otimes H) && \text{as } T \text{ consists of primitive elements}\\
&\subseteq HTH\otimes H + H\otimes HTH = I\otimes H + H\otimes I,
\end{aligned}$$

where we freely use $U(V+W)\subseteq UV+UW$ whenever U, V, W are subspaces of some algebra, and $HH = H$. Moreover,

$$\epsilon(I) = \epsilon(HTH) = \epsilon(H)\epsilon(T)\epsilon(H) = \mathbb{C}\cdot\{0\}\cdot\mathbb{C} = \{0\}$$

as ϵ is multiplicative and $\epsilon(T)\subseteq\epsilon(P(H))$ which is $\{0\}$ by E4.3. We have verified the definition of a coideal for I. Finally,

$$S(I) = S(HTH) = S(H)S(T)S(H) \subseteq HTH = I$$

by antimultiplicativity of the antipode S and the fact that $S(x) = -x$ if $x\in P(H)$ (E4.3) and so $S(T) = -T = T$. We conclude that I is a Hopf ideal. □

Lie algebras

We will not require a lot of theory on Lie algebras (which is a topic of a separate course), but we need to recall the basic definition.

5.3 Definition (Lie algebra) A **Lie algebra** is a vector space $\mathfrak{g}$ over $\mathbb{C}$ equipped with a bilinear map $[\ ,\ ]\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ (called **Lie bracket**) such that for all $x, y, z\in\mathfrak{g}$,

- $[x,y] = -[y,x]$ (*antisymmetry*);
- $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ (*Jacobi identity*).

5.4 Example (an abelian Lie algebra) On an arbitrary vector space V, introduce the **zero Lie bracket** $[x,y]=0$ for all $x,y\in V$. This gives an **abelian Lie algebra** on V.

5.5 Example (the commutator bracket on an associative algebra) Let A be an associative algebra. The Lie bracket on the space A, defined by

$$[a,b]_{\text{comm}} = ab - ba,$$

is the **commutator bracket** on A. (Exercise: show that associativity of the product on A implies the Jacobi identity for the commutator bracket.)

The commutator bracket construction allows us to define new Lie algebras, not only as associative algebras A equipped with $[\ ,\ ]_{\text{comm}}$ but also subspaces of A which are closed under $[\ ,\ ]_{\text{comm}}$ (they do not have to be associative subalgebras of A). The most important example of this is as follows.

5.6 Example (the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$) Consider the vector space

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ a + d = 0 \right\}$$

of traceless complex 2×2 matrices. The Lie bracket $[A, B] = AB - BA$ turns $\mathfrak{sl}_2(\mathbb{C})$ into a 3-dimensional Lie algebra.

Explanation why $\mathfrak{sl}_2(\mathbb{C})$ is a Lie algebra. Recall the well-known fact from linear algebra: for matrices A, B such that the products AB and BA are defined, one has $\operatorname{tr} AB = \operatorname{tr} BA$ where $\operatorname{tr} A$ denotes the trace (i.e., the sum of entries on the main diagonal) of the matrix A. In particular, for square matrices A, B of the same size one has $\operatorname{tr}(AB - BA) = 0$ because the operation of taking the trace is linear. This shows that the subspace $\mathfrak{sl}_2(\mathbb{C})$ of the associative algebra $M_{2\times 2}(\mathbb{C})$ of all 2×2 matrices is closed under the commutator bracket.

The universal enveloping algebra of a Lie algebra

For every Lie algebra $\mathfrak{g}$, we construct an associative unital algebra $U(\mathfrak{g})$ which contains $\mathfrak{g}$ and has the desirable property of being a Hopf algebra.

5.7 Definition Let $\mathfrak{g}$ be a Lie algebra with a basis X over $\mathbb{C}$ and Lie bracket $[\ ,\ ]$. Define the algebra $U(\mathfrak{g})$ by the presentation

$$U(\mathfrak{g}) = \langle X \mid xy - yx = [x, y],\ \forall x, y \in X \rangle.$$

The algebra $U(\mathfrak{g})$ is the **universal enveloping algebra** of the Lie algebra $\mathfrak{g}$.

Note that the set of defining relations for $U(\mathfrak{g})$ consists of quadratic-linear elements of the free algebra $\mathbb{C}\langle X\rangle$: in $xy - yx - [x, y]$, the monomials xy and yx are quadratic, and the Lie bracket $[x, y]$ is an element of $\mathfrak{g}$ and so can be written as a linear combination of elements of X which are degree 1 monomials.

The following result allows us to introduce on $U(\mathfrak{g})$ a structure of a Hopf algebra.

5.8 Proposition *In the above notation, let $I(\mathfrak{g})$ denote the ideal of $\mathbb{C}\langle X\rangle$ generated by the elements of the form $xy - yx - [x, y]$ where $x, y \in X$. Then $I(\mathfrak{g})$ is a Hopf ideal of $\mathbb{C}\langle X\rangle$, so that $U(\mathfrak{g}) = \mathbb{C}\langle X\rangle / I(\mathfrak{g})$ is a Hopf algebra.*

Proof. By definition of the coproduct on $\mathbb{C}\langle X\rangle$ (see Theorem 4.15), $x, y \in X$ are primitive elements of $\mathbb{C}\langle X\rangle$. Recall the important property of primitive elements given in E4.3(a): if x, y are primitive then $xy - yx$ is primitive. Note also that the set $P(\mathbb{C}\langle X\rangle)$ of primitive elements is a vector space, so $[x, y]$ is primitive as a linear combination of elements of X, and $xy - yx - [x, y]$ is primitive.

Thus, $I(\mathfrak{g})$ is generated by primitive elements and so by Proposition 5.2 is a Hopf ideal, as claimed. □

5.9 Remark Our definition of $U(\mathfrak{g})$ involves choosing a basis X of $\mathfrak{g}$. Yet the end result, the Hopf algebra $U(\mathfrak{g})$, does not depend on this choice. This can be shown as an exercise (which we omit), or the definition can be given in terms of the tensor algebra $T(\mathfrak{g})$ discussed in Remark 2.10. In practice it is convenient to work with explicit generators and relatons in $U(\mathfrak{g})$ as demonstrated in the next section.

Example: the Hopf algebra $U(\mathfrak{sl}_2)$

We will write $\mathfrak{sl}_2$ for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The general construction of a universal enveloping algebra gives us the Hopf algebra $U(\mathfrak{sl}_2)$. Let us study $U(\mathfrak{sl}_2)$ in more detail; later on, we will construct a quantum deformation of this Hopf algebra.

It is convenient to work with the basis $\{X, H, Y\}$ of the Lie algebra $\mathfrak{sl}_2$ where

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A direct calculation shows that the Lie bracket between the basis elements is given by

$$[H, X] = 2X, \quad [H, Y] = 2Y, \quad [X, Y] = H.$$

It is enough to know these values of the Lie bracket to compute, using bilinearity and antisymmtery, the Lie bracket between two arbitrary elements of $\mathfrak{sl}_2$. We obtain the standard presentation of $U(\mathfrak{sl}_2)$:

5.10 Proposition *As an algebra, $U(\mathfrak{sl}_2)$ is written as*

$$U(\mathfrak{sl}_2) = \langle X, H, Y \mid HX - XH = 2X,\ HY - YH = -2Y,\ XY - YX = H\rangle.$$

The Hopf algebra structure on $U(\mathfrak{sl}_2)$ is defined by declaring the generators X, H and Y primitive. □

Standard monomials. The polynomial coalgebra

Recall that $\mathrm{Mon}(X)$ denotes the set of all noncommutative monomials in X, where X is an arbitrary set. The definition of $\mathrm{Mon}(X)$ does not involve any additional structure on the set X.

We will now assue that X is equipped with a **total order** $\preccurlyeq$. Recall that a total order is a reflexive, antisymmetric and transitive binary relation. If x, y are in relation, we write $x \preccurlyeq y$ and say, informally, that "x is less than or equal to y". We write $x \prec y$ to mean "$x \preccurlyeq y$ and $x \neq y$". Given a pair of elements $x, y \in X$, precisely one out of three statements is true: $x \prec y$, $x = y$ or $y \prec x$.

5.11 Definition (standard monomials) A monomial $ab \dots z \in \mathrm{Mon}(X)$ is **standard** with respect to the order $\preccurlyeq$ if it consists of symbols arranged in a non-strictly increasing order: $a \preccurlyeq b \preccurlyeq \dots \preccurlyeq z$.

The subset of $\mathrm{Mon}(X)$ formed by standard monomials is denoted $\mathrm{StMon}(X)$, or more fully $\mathrm{StMon}(X, \preccurlyeq)$.

5.12 Lemma (the polynomial coalgebra) *Assume that a total order $\preccurlyeq$ on the set X is chosen. The subspace*

$$\mathbb{C}[X] = \operatorname{span} \mathrm{StMon}(X) \subseteq \mathbb{C}\langle X\rangle$$

is a subcoalgebra of the free algebra $\mathbb{C}\langle X\rangle$.

We will refer to the coalgebra $\mathbb{C}[X]$ as the **polynomial coalgebra** on X (with respect to $\preccurlyeq$). There is the more familiar associative commutative algebra structure on $\mathbb{C}[X]$, which is not a subalgebra of $\mathbb{C}\langle X\rangle$ and which we are not using at the moment. In fact, the commutative algebra and the coalgebra structures on $\mathbb{C}[X]$ together give a Hopf algebra. We will identify this Hopf algebra with a universal enveloping algebra of an abelian Lie algebra.

Proof of Lemma 5.12. A formula for the coproduct of a monomial $M \in \mathrm{Mon}(X) \subset \mathbb{C}\langle X\rangle$ was given in Remark 4.16, and we recall it here:

$$\Delta M = \sum \{N \otimes (M \setminus N) : N \text{ is a submonomial of } M\}.$$

Note that both N and $M \setminus N$ in this formula are obtained from M by deleting some symbols without changing the order of the symbols. It follows that if M is a standard monomial, then both N and $M \setminus N$ are also standard, proving that

$$\Delta \mathbb{C}[X] \subseteq \mathbb{C}[X] \otimes \mathbb{C}[X].$$

We further note that the empry monomial 1 is (trivially) standard, so $1 \in \mathbb{C}[X]$; and $\epsilon(1) = 1$ so that

$$\epsilon(\mathbb{C}[X]) \neq \{0\}.$$

We have verified Definition 3.2 of coalgebra for $\mathbb{C}[X]$. □

The following useful result is a particular case of the **Heyneman-Radford theorem** for coalgebras. We will not prove the Theorem, which applies to arbitrary coalgebras, in full; interested students may find the general proof in Montgomery 1993, Theorem 5.3.1. Our proof is specific to the polynomial coalgebra.

5.13 Proposition (the Heyneman-Radford theorem for the polynomial coalgebra) *Let D be a coalgebra and $f\colon \mathbb{C}[X] \to D$ be a coalgebra morphism. If the restriction $f|_{\mathbb{C}X}$ of f is injective, then f is injective.*

For clarity, we define a **coalgebra morphism** as a linear map $C \xrightarrow{f} D$ between two coalgebras such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"] & D \arrow[d, "\Delta_D"] \\ C\otimes C \arrow[r, "f\otimes f"] & D\otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\ & \mathbb{C} & \end{tikzcd} \tag{5.14}$$

where Δ_C, Δ_D, ϵ_C, ϵ_D denote the coproducts and the counits of the respective coalgebras.

The subspace $\mathbb{C}X$ of the coalgebra $\mathbb{C}[X]$ is spanned by elements of X (i.e., monomials of degree 1) and can be referred to as the space of homogeneous linear polynomials.

Proof of Proposition 5.13. Let $C = \mathbb{C}[X]$ and for $d = 0, 1, 2, \ldots$ denote

$$C_d = \text{span of standard monomials in } X \text{ of degree } \le d.$$

In particular, writing $\mathbf{1}$ for the empty monomial in $\mathbb{C}[X]$, we have

$$C_0 = \mathbb{C}\cdot\mathbf{1}, \qquad C_1 = C_0 + \mathbb{C}X.$$

Assume that $f_{\mathbb{C}X}$ is injective. We will prove by induction that the restriction f_{C_d} is injective for all $d \ge 1$. Since

$$C = \bigcup_{d=1}^{\infty} C_d,$$

this will show that f is injective on the whole of $C = \mathbb{C}[X]$.

<u>Base case</u>: $d = 1$. Assume that $u \in C_1$ is in the kernel of $f|_{C_1}$, that is $f(u) = 0$. Write $u = \lambda\mathbf{1} + v$ where $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}X$.

Let us compute the counit $\epsilon_C(u)$ of u. Note that v is primitive in C because elements of X are primitive in $\mathbb{C}\langle X\rangle$ hence in the subcoalgebra $C = \mathbb{C}[X]$. Recall that the counit vanishes on primitive elements (see Lemma 3.8), so $\epsilon_C(v) = 0$. Note also that $\epsilon_C(\mathbf{1}) = 1$. We thus have

$$\lambda = \epsilon_C(u) \underset{(5.14)}{=} \epsilon_D(f(u)) = \epsilon_D(0) = 0.$$

Since $\lambda = 0$, we have $u \in \mathbb{C}X$. But $f(u) = 0$ and f is assumed to be injective on $\mathbb{C}X$. Hence $u = 0$.

We have proved that the linear map $f|_{C_1}$ has zero kernel, meaning that it is injective. The base case is proved.

<u>Inductive step $d \to d+1$</u>. Assume that $f|_{C_d}$ is an injective map. We want to show that $f|_{C_{d+1}}$ is also injective. Let us consider the following linear map from $\mathbb{C}[X]$ to $\mathbb{C}[X]\otimes\mathbb{C}[X]$:

$$\tilde{\Delta}\colon C \to C\otimes C, \qquad \tilde{\Delta}a = \Delta a - a\otimes\mathbf{1} - \mathbf{1}\otimes a.$$

By definition of the coproduct on $C = \mathbb{C}[X]$, for all $x \in X$ one has $\Delta x = x\otimes\mathbf{1} - \mathbf{1}\otimes x$ and so $\tilde{\Delta}x = 0$. This means that $\mathbb{C}X \subseteq \ker\tilde{\Delta}$. We claim that, moreover, $\mathbb{C}X = \ker\tilde{\Delta}$.

Indeed, let sort: $\mathrm{Mon}(X) \to \mathrm{StMon}(X)$ denote the function which takes a monomial, writes all of its symbols in the non-strictly increasing order and outputs the resulting standard monomial. Of course, on standard monomials the function sort is identity. We now construct the following linear map:

$$R\colon \mathbb{C}[X]\otimes\mathbb{C}[X] \to \mathbb{C}[X], \qquad R(M\otimes N) = \mathrm{sort}(MN) \text{ for } M, N \in \mathrm{StMon}(X);$$

that is, R takes an element $M \otimes N$ of the standard basis of $\mathbb{C}[X] \otimes \mathbb{C}[X]$, where M, N are standard monomials, concatenates M and N and then sorts the resulting (possibly non-standard) monomial MN. For any standard monomial M we have

$$R(\Delta M) = \sum R(N \otimes (M \setminus N)) = \sum M,$$

where the sum is taken over all submonomials N of M. For each such submonomial, sorting the concatenation $N(M \setminus N)$ in the increasing order obviously returns M. In the last sum, therefore, M is repeated as many times as there are submonomials of M. The number of submonomials of a monomial of degree $\deg M$ is $2^{\deg M}$. Thus,

$$(R \circ \Delta)M = 2^{\deg M} M \quad \Longrightarrow \quad (R \circ \tilde{\Delta})M = (2^{\deg M} - 2)M,$$

because $(R \circ \tilde{\Delta})M = (R \circ \Delta)M - R(M \otimes \mathbf{1}) - R(\mathbf{1} \otimes M)$ where, clearly, $R(M \otimes \mathbf{1}) = R(\mathbf{1} \otimes M) = M$. It follows that the space $\mathbb{C}[X]$ has a basis which consists of eigenvectors of the linear operator $R \circ \tilde{\Delta} \colon \mathbb{C}[X] \to \mathbb{C}[X]$, namely the basis of standard monomials.

In this situation, linear algebra tells us that the kernel of $R \circ \tilde{\Delta}$ is the span of eigenvectors with eigenvalue 0 — they are monomials of degree 1:

$$\ker R \circ \tilde{\Delta} = \operatorname{span}\{M \in \operatorname{StMon}(X) : 2^{\deg M} - 2 = 0\} = \mathbb{C}X.$$

Since $\ker R \circ \tilde{\Delta}$ contains $\ker \tilde{\Delta}$ (when you postcompose a linear map with some other linear map, the kernel can only increase), this shows that $\ker \tilde{\Delta} \subseteq \mathbb{C}X$. On the other hand, $\mathbb{C}X \subseteq \ker \tilde{\Delta}$ as noted earlier. We have proved that $\mathbb{C}CX = \ker \tilde{\Delta}$.

We are ready to show that f is injective on C_{d+1}. Let $u \in C_{d+1}$ be such that $f(u) = 0$ in D. Then, obviously,

$$\Delta_D f(u) = 0 \quad \text{and} \quad \Delta_D f(u) - f(u) \otimes f(\mathbf{1}) - f(\mathbf{1}) \otimes f(u) = 0.$$

Since f is a coalgebra morphism, (5.14) tells us that $\Delta_D f(u) - f(u) \otimes f(\mathbf{1}) - f(\mathbf{1}) \otimes f(u) = (f \otimes f)\tilde{\Delta}u$. Therefore, $(f \otimes f)\tilde{\Delta}u = 0$.

Observe that, for all $u \in C_{d+1}$, we have $\tilde{\Delta}u \in C_d \otimes C_d$. Indeed, if $M \in \operatorname{StMon}(X)$, the only terms in $\Delta M = \sum N \otimes (M \setminus N)$ which contain a monomial of degree $\deg M$ in the left leg or in the right leg are $M \otimes \mathbf{1}$ and $\mathbf{1} \otimes M$; but these two terms cancel in $\tilde{\Delta}M$, and so $\tilde{\Delta}M \in C_{\deg M - 1} \otimes C_{\deg M - 1}$.

Thus $\tilde{\Delta}u \in C_d \otimes C_d$. By the inductive hypothesis, f is injective on C_d. By a linear algebra exercise, E5.6, this implies that $f \otimes f$ is injective on $C_d \otimes C_d$. But $(f \otimes f)\tilde{\Delta}u = 0$, so $\tilde{\Delta}u$ must be 0. We proved earlier that $\ker \tilde{\Delta} = \mathbb{C}X$; so $u \in \mathbb{C}X$. Since $f|_{\mathbb{C}X}$ is injective and $f(u) = 0$, we have $u = 0$. This concludes the inductive step. By induction, f is injective on C_d for all d, hence injective on C. □

The Poincaré-Birkhoff-Witt theorem

We are ready to introduce the main result of this chapter.

5.15 Theorem (The PBW theorem) *Let $\mathfrak{g}$ be a Lie algebra. Choose a basis X of $\mathfrak{g}$ and impose a total order on X. Let $U(\mathfrak{g}) = \mathbb{C}\langle X \rangle / I(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then:*

(a) generators from X are linearly independent in $U(\mathfrak{g})$; that is, the linear map $i \colon \mathfrak{g} \to U(\mathfrak{g})$ given by the composition $\mathfrak{g} = \mathbb{C}X \hookrightarrow \mathbb{C}\langle X \rangle \twoheadrightarrow \mathbb{C}\langle X \rangle / I(\mathfrak{g}) = U(\mathfrak{g})$ is injective;

(b) moreover, standard monomials in X form a basis of $U(\mathfrak{g})$; that is, the following composite map is bijective:

$$\mathbb{C}[X] \hookrightarrow \mathbb{C}\langle X \rangle \twoheadrightarrow U(\mathfrak{g}).$$

We will not give a complete proof of both (a) and (b). Rather, we will assume that (a) holds, and deduce (b) from the Heyneman-Radford theorem for $\mathbb{C}[X]$. In fact, (a) is automatic for any Lie algebra which sits inside an associative algebra with the commutator bracket:

5.16 Claim *If a Lie algebra $\mathfrak{g}$ is a subspace of an associative unital algebra A, such that the Lie bracket on $\mathfrak{g}$ is the restriction onto $\mathfrak{g}$ of the commutator bracket $[\ ,\ ]_{\text{comm}}$ of A, then the natural map $i\colon \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

Proof of Claim. If we denote by f the tautological embedding of $\mathfrak{g}$ in A, that is $f(x) = x$ for all $x \in \mathfrak{g}$, then by definition of the Lie bracket on $\mathfrak{g}$ we have

$$f([x, y]) = f(x)f(y) - f(y)f(x).$$

Maps $f\colon \mathfrak{g} \to A$ with this property from $\mathfrak{g}$ to an associative unital algebra A are called **Lie maps**. By construction of $U(\mathfrak{g})$, the natural map $F\colon \mathfrak{g} \to U(\mathfrak{g})$ is a Lie map. Moreover, i has the following **universal mapping property:** for every Lie map $f\colon \mathfrak{g} \to A$ there is a unique homomorphism $F\colon U(\mathfrak{g}) \to A$ such that the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "f"] \arrow[d, "i"] & A \\
U(\mathfrak{g}) \arrow[ur, dashed, "F"'] &
\end{tikzcd}$$

commutes. (For the proof of this Universal Mapping Property, see E5.3.)

Now if f is injective and $f = Fi$, then i must be injective. □

There are many Lie algebras which satisfy the assumption in Claim 5.16. For example, the Lie algebra $\mathfrak{sl}_2$ was constructed as a subspace of the associative algebra $M_{2\times 2}(\mathbb{C})$ with the commutator bracket. Moreover, every finite-dimensional Lie algebra can be embedded, as set out in the Claim, in an associative algebra of the form $M_{N\times N}(\mathbb{C})$: this is Ado's theorem from 1935 (an advanced result from Lie theory not considered in this course.) Finally, every abelian Lie algebra with basis X can be easily seen to be embedded a commutative associative algebra — e.g., consider the presentation $\langle X \mid xy = 0,\ \forall x, y \in X\rangle$. For all such Lie algebras, part (a) of the PBW theorem holds by Claim 5.16.

Proof of the PBW Theorem 5.15 assuming that (a) holds. Let $\mathfrak{g}$ be a Lie algebra such that the natural map $i\colon \mathfrak{g} \to U(\mathfrak{g})$ is injective. Impose a total order on the basis X of $\mathfrak{g}$. The embedding $\mathbb{C}[X] \hookrightarrow \mathbb{C}\langle X\rangle$ is a coalgebra morphism by definition of the coalgebra structure on $\mathbb{C}[X]$. The quotient map $\mathbb{C}\langle X\rangle \twoheadrightarrow \mathbb{C}\langle X\rangle/I(\mathfrak{g}) = U(\mathfrak{g})$ is a morphism of Hopf algebras by definition of the Hopf algebra structure on $U(\mathfrak{g})$, and in particular a morphism of coalgebras.

Hence the composite map $\mathbb{C}[X] \to U(\mathfrak{g})$ is a coalgebra morphism, for it is a composition of two coalgebra morphisms. Note that the restriction of this map onto $\mathbb{C}X = \mathfrak{g}$ is the natural map $i\colon \mathfrak{g} \to U(\mathfrak{g})$ which we assumed to be injective. Hence the map $\mathbb{C}[X] \to U(\mathfrak{g})$ is **injective** by the Heyneman-Radford Theorem for $\mathbb{C}[X]$, Proposition 5.13.

It remains to show that the map $\mathbb{C}[X] \to U(\mathfrak{g})$ is surjective; that is, $U(\mathfrak{g})$ is spanned by standard monoials in X. But this follows from the defining relations in $U(\mathfrak{g})$ without the use of the Hopf algebra structure and is an exercise: a full solution is given in E5.5(b). □

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