January 2015

Lie groups and Lie algebras
14MAD103

①

Solutions.

No 1. (a) First we check that G_A is a subgroup of $GL(3,\mathbb{R})$.
Let $X_1, X_2 \in G_A$, i.e. $X_1 A = A X_1$ and $X_2 A = A X_2$.
Then $(X_1 X_2) A = X_1 (X_2 A) = X_1 (A X_2) = (X_1 A) X_2 = A(X_1 X_2)$,
i.e. $X_1 X_2 \in G_A$ and G_A is closed under multiplication.
Similarly, if $X \in G_A$, i.e. $AX = XA$, then
$X^{-1}(AX)X^{-1} = X^{-1}(XA)X^{-1}$, $X^{-1}A = AX^{-1}$, i.e. $X^{-1} \in G_A$.
Thus, G_A is a subgroup of $GL(3,\mathbb{R})$. On the other hand,
$XA = AX$ can be considered as a system of linear equations, i.e
G_A is an algebraic linear group and, therefore, a Lie group.

To describe G_A explicitly, we solve the equation

$$X \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot X$$

The ~~se~~ result is:

$$G_A = \left\{ X = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}, \ \det A = (ad-cb)\cdot f \neq 0 \right\}$$

This group has 5 independent parameters a,b,c,d,f.

Hence, $\dim G_A = 5$

[4]

(standard problem)

(2)

(b) G_A is disconnected and consists of 4 connected components

$G_0 = \{ X \in G_A \mid$ ~~ab~~ $ad-bc>0,\ f>0\}$

$G_1 = \{ X \in G_A \mid ad-bc<0,\ f>0\}$

$G_2 = \{ X \in G_A \mid ad-bc>0,\ f<0\}$

$G_3 = \{ X \in G_A \mid ad-bc<0,\ f<0\}$

$G_A = G_0 \sqcup G_1 \sqcup G_2 \sqcup G_3$ (disjoint union)

Each of G_i $(i=0,1,2,3)$ is open and non-empty.

Moreover, each of G_i is connected, indeed topologically G_0 can be considered as the Cartesian product of $GL_+(2,\mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ ad-bc>0\}$ and $\mathbb{R}^+ = \{ f \in \mathbb{R},\ f>0$.

Since $GL_+(2,\mathbb{R})$ and $\mathbb{R}^+$ are both connected, so is $GL_+(2,\mathbb{R}) \times \mathbb{R}^+ \simeq G_0$. For $i=1,2,3$, the proof is similar.

[4]

(standard problem)

(c) Consider $GL(2,\mathbb{R}) \times \mathbb{R}^*$ as the set of pairs (Y, f) where $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$, $f \in \mathbb{R}^*_{\neq 0}$.

Let $F : GL(2,\mathbb{R}) \times \mathbb{R}^* \longrightarrow G_A$ be defined by

$$F(Y,f) = \begin{pmatrix} \boxed{Y} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\ \ 0 & f \end{pmatrix}.$$

Then F is a bijection and

$$F\big((Y_1,f_1) * (Y_2,f_2)\big) = F(Y_1Y_2,\ f_1f_2) =$$

$$= \begin{pmatrix} \boxed{Y_1Y_2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\ \ 0 & f_1f_2 \end{pmatrix} = \begin{pmatrix} \boxed{Y_1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\ \ 0 & f_1 \end{pmatrix} \begin{pmatrix} \boxed{Y_2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\ \ 0 & f_2 \end{pmatrix} = F(Y_1,f_1) \cdot F(Y_2,f_2)$$

Thus, F is an isomorphism

[4] (standard problem)

(d) Let $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $X = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} \in G_A$

Then $Xv = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ c \\ f \end{pmatrix}$

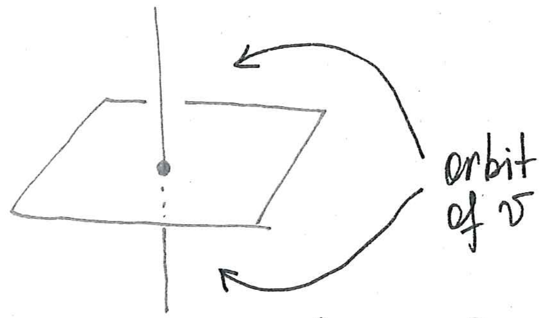
Since $\det X \neq 0$, then $f \neq 0$ and a, c do not vanish simultaneously.

It follows from this that the orbit of v is

$\mathcal{O}(v) = \mathbb{R}^3 \setminus \left(\{z=0\} \cup \{x=y=0\} \right)$, i.e.

$\mathcal{O}(v)$ is the space $\mathbb{R}^3$ from which the plane $\{z=0\}$ and the vertical axis (z-axis) are removed.

$\dim \mathcal{O}(v) = 3$.



$St(v) = \{ X \in G_A \mid Xv = v \} = \left\{ X = \begin{pmatrix} 1 & b & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, d \neq 0 \right\}$

This group contains 2 parameters and therefore
$\dim St(v) = 2$

[4]
(standard question)

(e) Take $u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}$, $f \neq 0$

Thus, the orbit of u is the vertical axis without the origin

Take $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Then $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ d \\ 0 \end{pmatrix}$ where b and d do not vanish simultaneously.

Thus, the orbit of w is the horizontal plane $\{z=0\}$ without the origin.

Finally, the orbit of the zero vector is the zero vector itself (one-point orbit)

It is easy to see that the union of these 4 orbits $\mathcal{O}(v), \mathcal{O}(u), \mathcal{O}(w), \mathcal{O}(\bar{0})$ is $\mathbb{R}^3$. Therefore, this action has 4 distinct orbits. [4]

(unseen).

No 2.

(a) The fact that G is a subgroup of $GL(4,\mathbb{R})$ can be checked in the same way as in 1(a), i.e.

G is closed under multiplication:

$$\begin{pmatrix} A_1 & B_1 \\ 0 & A_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 & A_1B_2 + B_1A_1 \\ 0 & A_1A_2 \end{pmatrix} \in G \quad \text{as } A_1A_2 \text{ is orthogonal}$$

and

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BA^{-1} \\ 0 & A^{-1} \end{pmatrix} \in G \quad \text{as } A^{-1} \text{ is orthogonal.}$$

G is algebraic because the elements $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of G can be characterised as follows: $\begin{cases} A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ A = D \\ C = 0 \end{cases}$

Thus, G is an algebraic linear group.

$\dim G = 5$, as $A = \begin{pmatrix} \cos\varphi & \mp\sin\varphi \\ \sin\varphi & \pm\cos\varphi \end{pmatrix}$ contains one paramete φ,

and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ contains 4 parameters b_1, b_2, b_3, b_4.

G is not compact as the entries of B are unbounded.

[5]

(standard question)

(b) The Lie algebra $\mathfrak{g}$ of G can be described as follows:

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a & b_1 & b_2 \\ -a & 0 & b_3 & b_4 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix}, \quad a, b_1, b_2, b_3, b_4 \in \mathbb{R} \right\}$$

[5]

(unseen)

(c) Def. 1 Let $\mathfrak{g}$ be a Lie algebra and

$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}]$, ... $\mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}]$, ...

$\mathfrak{g}$ is called ~~solvable~~ nilpotent if there exists $n \in \mathbb{N}$ such that

$$\mathfrak{g}_n = \{0\}.$$

Def 2. Let $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$,

$\mathfrak{g}^{(3)} = [\mathfrak{g}^{(2)}, \mathfrak{g}^{(2)}]$, ..., $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$, ...

$\mathfrak{g}$ is called solvable if there exists $n \in \mathbb{N}$ such that

$$\mathfrak{g}^{(n)} = \{0\}.$$

[5]

(bookwork)

(d) Let us compute $\mathfrak{g}_1 = \mathfrak{g}^{(1)}$

$$\left[\begin{pmatrix} A_1 & B_1 \\ 0 & A_1 \end{pmatrix}, \begin{pmatrix} A_2 & B_2 \\ 0 & A_2 \end{pmatrix} \right] = \begin{pmatrix} 0 & [A_1, B_2] + [B_1, A_2] \\ 0 & 0 \end{pmatrix}$$

where $A_1 = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & a_2 \\ -a_2 & 0 \end{pmatrix}$

The matrix $[A_1, B_2] + [B_1, A_2]$ has the form $\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$.

Thus, $\mathfrak{g}_1 = \mathfrak{g}^{(1)}$ is 2-dimensional commutative subalgebra.

In particular, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \{0\}$, i.e. $\mathfrak{g}$ is solvable.

(unseen)

However,

$$\left[\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}, \left(\begin{array}{c|cc} 0 & \alpha & \beta \\ & \beta & -\alpha \\ \hline 0 & & 0 \end{array}\right) \right]$$

is again a matrix of the form $\left(\begin{array}{c|cc} 0 & \alpha' & \beta' \\ & -\beta' & \alpha' \\ \hline 0 & & 0 \end{array}\right)$ with $\alpha', \beta' \in \mathbb{R}$ $\cancel{\begin{pmatrix} \alpha' & \beta' \\ \beta' & -\alpha' \end{pmatrix}}$, i.e.

$\mathfrak{g}_2 = \mathfrak{g}_1$, and therefore $\mathfrak{g}_k = \mathfrak{g}_{k-1} = \ldots = \mathfrak{g}_1 \neq \{0\}$

So, $\mathfrak{g}$ is not nilpotent. [5]

No 3.

(a) $\xi_1 = (xy, 1+y^2)$, $\xi_2 = (y, 0)$, $\xi_3 = (1, 0)$.

$[\xi_1, \xi_2] = \cancel{\xi}\, [xy\, \partial/\partial x + (1+y^2)\, \partial/\partial y,\ y\, \partial/\partial x] =$

$= (1+y^2)\, \partial/\partial x - y^2\, \partial/\partial x = \partial/\partial x = \xi_3$

$[\xi_1, \xi_3] = [xy\, \partial/\partial x + (1+y^2)\, \partial/\partial y,\ \partial/\partial x] = -y\, \partial/\partial x = -\xi_2$

$[\xi_2, \xi_3] = [y\, \partial/\partial x,\ \partial/\partial x] = 0$

Thus, $\mathfrak{g} = \mathrm{span}(\xi_1, \xi_2, \xi_3)$ is closed under the Lie bracket

$\Rightarrow \mathfrak{g}$ is a Lie algebra. [4]

(standard problem).

(b) $[\xi_1, \xi_2] = \xi_3 \Rightarrow C^1_{12} = C^2_{12} = C^1_{21} = C^2_{21} = 0$
and $C^3_{12} = -C^3_{21} = 1$

$[\xi_1, \xi_3] = -\xi_2 \Rightarrow C^1_{13} = C^3_{13} = C^1_{31} = C^3_{31} = 0$
and $C^2_{13} = -C^2_{31} = -1$

$[\xi_2, \xi_3] = 0 \Rightarrow C^k_{23} = C^k_{32} = 0$.

Let $\xi = a\xi_1 + b\xi_2 + c\xi_3$, $a, b, c \in \mathbb{R}$

Then $\mathrm{ad}_\xi \xi_1 = -b\xi_3 + c\xi_2$

$\mathrm{ad}_\xi \xi_2 = a\xi_3$

$\mathrm{ad}_\xi \xi_3 = -a\xi_2$

Hence,

$$\xi \longmapsto \mathrm{ad}_\xi = \begin{pmatrix} 0 & 0 & 0 \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

[4]

(standard problem)

(c) Definition

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra. $\mathfrak{h}$ is called an ideal of $\mathfrak{g}$, if for all $h \in \mathfrak{h}$ and $a \in \mathfrak{g}$ we have $[h, a] \in \mathfrak{h}$. [3]

(bookwork)

(d) Consider $\mathfrak{h}_1 = \text{span}(\xi_2)$.

We have $[\xi_1, \xi_2] = \xi_3 \notin \mathfrak{h}_1$. [2]

Thus, $\mathfrak{h}_1$ is not an ideal.

Consider $\mathfrak{h}_2 = \text{span}(\xi_2, \xi_3)$

We have for $\xi = \alpha\xi_1 + \beta\xi_2 + \gamma\xi_3$: [2]

$$[\xi, \xi_2] = +\alpha\xi_3 \in \mathfrak{h}_2$$

(standard problem).

$$[\xi, \xi_3] = -\alpha\xi_2 \in \mathfrak{h}_2$$

Thus, $\mathfrak{h}_2$ is an ideal of $\mathfrak{g}$.

(e) The Lie algebra $\tilde{\mathfrak{g}}$ of G has the form

$$\tilde{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & \alpha & a \\ -\alpha & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \alpha, a, b \in \mathbb{R} \right\},$$ Take $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ [5]

Then $[E_1, E_2] = E_3$,

(standard problem)

$[E_1, E_3] = -E_2$ and $[E_2, E_3] = 0$.

These commutation relations coincide with those for the Lie algebra $\mathfrak{g}$.

Hence, $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are isomorphic.

<u>No 4</u> (a) We need to check that

$$f(XBX^T) = f(B) \text{ for all } B \in V,\ X \in G.$$

$$XBX^T = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} = \begin{pmatrix} 0 & \alpha + ac\gamma - b\gamma + c\beta & \beta + a\gamma \\ -\alpha - ac\gamma - \beta c + b\gamma & 0 & \gamma \\ -\beta - a\gamma & -\gamma & 0 \end{pmatrix}$$

The element γ of the matrix B remains unchanged, i.e.

$f(B) = \gamma$ is an invariant of the action Ψ. [4]
(unseen)

(b) Let $\mathfrak{g} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix},\ a, b, c \in \mathbb{R} \right\}$ be the Lie algebra of G.

Then for $A \in \mathfrak{g}$: $\psi_A(B) = \frac{d}{dt}\Big|_{t=0} \Psi_{\exp tA}(B) =$

$$= \frac{d}{dt}\Big|_{t=0} \exp tA \cdot B \cdot (\exp tA)^T = AB + BA^T$$

[4]
(standard problem)

(c) <u>Definition</u> Let $\varphi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$ on a vector space V. (bookwork)
φ is called irreducible, if there is no non-trivial invariant subspace $L \subset V$ (i.e. such that $\varphi_A(v) \in L$ for all $v \in L, A \in \mathfrak{g}$)

Let $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$, $B = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \in V$

Then $\psi_A(B) = AB + BA^T = \begin{pmatrix} 0 & -b\gamma + c\beta & a\gamma \\ -c\beta + b\gamma & 0 & 0 \\ -a\gamma & 0 & 0 \end{pmatrix}$

If we consider $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ as a basis of V

then the matrix of ψ_A is $\begin{pmatrix} 0 & c & -b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$.

This shows that $\psi_A(E_1) = 0$ for all $A \in \mathfrak{g}$, i.e.

$L = \mathrm{span}(E_1)$ is a one-dimensional invariant subspace.

Hence, ψ is ~~reducib~~ <u>not</u> irreducible. [4]

(standard problem)

(d) Let $\phi : \mathfrak{g} \longrightarrow gl(3, \mathbb{R})$ be the natural representation of $\mathfrak{g}$.

$$\phi_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}}_{= A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ay + bz \\ cz \\ 0 \end{pmatrix}$$

This formula shows that $L = \mathrm{span}(e_1) = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, x \in \mathbb{R} \right\}$

is a one-dimensional invariant subspace.

Conclusion: ϕ_A is <u>not</u> irreducible. [4]

(standard problem)

(e) The relation $P \circ \phi_A = \psi_A \circ P$ can be understood as the following matrix equation

$$P \cdot \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c & -b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \cdot P \quad \text{for all } a, b, c \in \mathbb{R} \text{ identically}$$

The general solution to this equation is

$P = \begin{pmatrix} 0 & \lambda & \mu \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix}$. This P is degenerate. Thus, there is no invertible P s.t. $P \circ \phi_A = \psi_A \circ P$, i.e.

ϕ and ψ are <u>not</u> isomorphic. [4]

(unseen).