Week 02 review worksheet — exercises for §2

The general form of a presentation of an associative unital algebra is $\mathbb{C}\langle X \mid \mathcal{R}\rangle$ where X is a set and $\mathcal{R}$ is a subset of the free tensor algebra $\mathbb{C}\langle X\rangle$. There are several notational conventions:

- inside $\langle \quad \rangle$, sets can be written without $\{ \}$;
- relations (elements of $\mathcal{R}$) can be written in the form "$P = Q$" where $P, Q \in \mathbb{C}\langle X\rangle$; this is interpreted to mean that $P - Q$ is an element of $\mathcal{R}$.

Part A. Exercises for interactive discussion

Attempt the part A exercises and be prepared to discuss them in the interactive session.

E2.1 (a group algebra of a finite cyclic group) Let $\Gamma = \{e, g, g^2\}$ denote the cyclic group of orger 3, where $g^3 = e$. Let $\mathbb{C}\Gamma$ be the group algebra of Γ. Which statements about $\mathbb{C}\Gamma$, given below, are true? Explain your answers.

(A) The subspace of $\mathbb{C}\Gamma$ spanned by e is a subalgebra of $\mathbb{C}\Gamma$.
(B) $\mathbb{C}\langle x \mid x^3 = 1\rangle$ is a presentation of $\mathbb{C}\Gamma$.
(C) The map $\epsilon\colon \mathbb{C}\Gamma \to \mathbb{C}$ given by $\epsilon(\alpha e + \beta g + \gamma g^2) = \alpha + \beta + \gamma$ is a homomorphism of algebras.
(D) The subspace $Z = \{\alpha e + \beta g + \gamma g^2 : \alpha + \beta + \gamma = 0\}$ is a subalgebra of $\mathbb{C}\Gamma$.
(E) The subspace $Z = \{\alpha e + \beta g + \gamma g^2 : \alpha + \beta + \gamma = 0\}$ is an ideal of $\mathbb{C}\Gamma$;
(F) If $x, y \in \mathbb{C}\Gamma$, $x \neq 0$, $y \neq 0$, then $xy \neq 0$.

E2.2 (algebra characters are lin. independent) If A is an algebra over $\mathbb{C}$, let $\mathrm{Alg}(A, \mathbb{C})$ be the subset of A^* formed by **algebra homomorphisms** from A to $\mathbb{C}$. Show: $\mathrm{Alg}(A, \mathbb{C})$ is a linearly independent set in A^*.

E2.3 (multiplicative characters in $(\mathbb{C}\Gamma)^*$) Let $\mathbb{C}\Gamma$ be the group algebra of $\Gamma = \{e, g, g^2\}$ from E2.1.

(a) Calculate $\mathrm{Alg}(\mathbb{C}\Gamma, \mathbb{C})$ and show that this set is a basis of $(\mathbb{C}\Gamma)^*$.

(b) Will the result obtained in (a) still hold if:

- the group Γ is replaced by another finite cyclic group?
- the group Γ is replaced by another finite abelian group?
- the group Γ is replaced by a finite non-abelian group?
- the field $\mathbb{C}$ is replaced by a smaller field of characteristic 0, say, $\mathbb{R}$ or $\mathbb{Q}$?

E2.4 (a presentation for the polynomial algebra) The algebra $\mathbb{C}[x, y]$ of polynomials in two variables has, by definition, a basis of **standard monomials:** monomials of the form $x^m y^n$ where $m, n \geq 0$, i.e., where all instances of x precede all instances of y. Note that the monoid $\mathrm{StMon}(x, y)$ of standard monomials is **not** a submonoid of $\mathrm{Mon}(x, y)$: it has different multiplication, $x^m y^n \cdot x^p y^q = x^{m+p} y^{n+q}$. The algebra $\mathbb{C}[x, y]$ can be viewed as the algebra of the monoid $\mathrm{StMon}(x, y)$.

Suggest a presentation for the algebra $\mathbb{C}[x, y]$. Prove that what you suggest is indeed a presentation.

Part B. Extra exercises

Attempt these exercises and compare your answers with the model solutions, published after the session.

E2.5 (actions are homomorphisms to $\mathrm{End}(V)$) Let A be an algebra and V be a vector space over the field $\mathbb{C}$. Prove that there is a 1-to-1 correspondence between actions $\triangleright\colon A \otimes V \to V$ of A on V and algebra homomorphisms $\rho\colon A \to \mathrm{End}(V)$, where an action $\triangleright$ corresponds to the homomorphism

$$\rho_\triangleright : A \to \mathrm{End}(V), \qquad \rho_\triangleright(a) \text{ is the element of } \mathrm{End}(V) \text{ defined by } (\rho_\triangleright(a))(v) = a \triangleright v.$$