Lie groups and Lie algebras
Problem Sheet 4.
Solutions

No 1. $G_B = \{X \in GL(3,\mathbb{R}) \mid X^T B T = B\}$ where $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

We first describe the matrices $X \in GL(3,\mathbb{R})$ explicitly.

Let $X = \left(\begin{array}{c|c} A & \begin{matrix} e \\ f \end{matrix} \\ \hline \begin{matrix} b & c \end{matrix} & d \end{array}\right)$, then $X^T B X = B$ gives the following relations:

$$\left(\begin{array}{c|c} A^T & \begin{matrix} b \\ c \end{matrix} \\ \hline \begin{matrix} e & f \end{matrix} & d \end{array}\right) \left(\begin{array}{c|c} \begin{matrix} 1 & \\ & 1 \end{matrix} & \\ \hline & 0 \end{array}\right) \left(\begin{array}{c|c} A & \begin{matrix} e \\ f \end{matrix} \\ \hline \begin{matrix} b & c \end{matrix} & d \end{array}\right) = \left(\begin{array}{c|c} \begin{matrix} 1 & \\ & 1 \end{matrix} & \\ \hline & 0 \end{array}\right) \Rightarrow$$

$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e. $A \in O(2)$

$A^T \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which implies $\begin{pmatrix} e \\ f \end{pmatrix} = 0$ as $\det A \neq 0$

and no relations for b, c, d.

— Thus, $X = \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} b & c \end{matrix} & d \end{array}\right)$ where $A \in O(2)$, $d \neq 0$
b, c are arbitrary real numbers.

$$A = \begin{pmatrix} \cos\varphi & \mp\sin\varphi \\ \sin\varphi & \pm\cos\varphi \end{pmatrix}$$

— $\dim G_B = 4$ as G_B is defined by means of 4 parameters: φ, b, c, d.

The Lie algebra $\mathfrak{g}_B$ can be described in the following way

Consider a smooth curve

$$X(t) = \begin{pmatrix} \cos\varphi(t) & -\sin\varphi(t) & 0 \\ \sin\varphi(t) & \cos\varphi(t) & 0 \\ b(t) & c(t) & d(t) \end{pmatrix} \quad \text{such that } X(0) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

i.e. $\varphi(0) = 0$
$b(0) = 0$
$a(0) = 0$
$d(0) = 1$.

We know that $\mathfrak{g}_B$ consists of the matrices of the form $A = \frac{dX(t)}{dt}\Big|_{t=0}$. This "velocity vector" is easy to compute:

$$A = \begin{pmatrix} 0 & -\varphi' & 0 \\ \varphi' & 0 & 0 \\ b' & c' & d' \end{pmatrix}$$

where φ', b', c', d' are, in general, arbitrary real numbers.

– Thus, $\mathfrak{g}_B = \left\{ \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & \delta \end{pmatrix} \right\} \subset gl(3, \mathbb{R})$

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$

– G_B consists of 4 connected components:

the identity component $G_0 = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline b & c & d \end{array} \right), \ \det A = 1, \ d > 0 \right\}$

and 3 similar components which are defined by the signs of $\det A$ and d

$G_1 = \{ \det A = 1, d < 0 \}$, $G_2 = \{ \det A = -1, d > 0 \}$, $G_3 = \{ \det A = -1, d < 0 \}$.

– Each of these components is diffeomorphic to $S^1 \times \mathbb{R}^3$.

For the identity component G_0, the diffeomorphism $\phi : S^1 \times \mathbb{R}^3 \longrightarrow G_0$ is as follows:

if we define a point $P \in S^1$ by an angle $\varphi \in \mathbb{R} \bmod 2\pi$, then

$$\Phi(\underset{S^1}{\underset{\cap}{\varphi}}, \underbrace{b, c, t}_{\underset{\mathbb{R}^3}{\cap}}) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ b & c & e^t \end{pmatrix} \in G_0 .$$

No 2. To describe $C(L)$ for $L = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$, λ_i are all distinct.

we need to solve the matrix equation $LX = XL$.
It's easy to see that X is a diagonal matrix $\begin{pmatrix} x_{11} & & & 0 \\ & x_{22} & & \\ & & \ddots & \\ 0 & & & x_{nn} \end{pmatrix}$ with arbitrary diagonal elements $x_{ii} \in \mathbb{R}$

Thus, $$C(L) = \left\{ X = \begin{pmatrix} x_{11} & & \\ & \ddots & \\ & & x_{nn} \end{pmatrix}, \quad \det X \neq 0 \right\}$$

In particular, $\dim C(L) = n$.

Now assume that some of diagonal elements λ_i of L coincide:

$$L = \begin{pmatrix} \left.\begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_1\end{matrix}\right\} k_1 \text{ times} & & \\ & \left.\begin{matrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_2\end{matrix}\right\} k_2 \text{ times} & \\ & & \ddots \end{pmatrix}$$

Then the solution of $XL = LX$ is

$$X = \begin{pmatrix} \overbrace{\boxed{X_{11}}}^{k_1} & & \\ & \overset{k_2}{\boxed{X_{22}}} & \\ & & \ddots \end{pmatrix}$$

, i.e. X is a block diagonal matrix with arbitrary diagonal blocks X_{ii} (of size $k_i \times k_i$).

In this case $\dim C(L) = k_1^2 + k_2^2 + \ldots > n$.

Now, let $L = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ Jordan block.

The equation $LX = XL$ is equivalent to $NX = XN$ where $N = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$.

The general solution is

$$X = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ & a_1 & a_2 & \ddots & \vdots \\ & & \ddots & \ddots & a_3 \\ & 0 & & \ddots & a_2 \\ & & & & a_1 \end{pmatrix}$$

with arbitrary entries $a_1, a_2, \ldots, a_n$.

In particular, $\dim C(L) = n$ in this case.

Q. 3

Show that $Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$.

We have

$$Sp(2,\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; A^T J A = J \right\},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Let us solve the matrix equation $A^T J A = J$

$$\underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\parallel} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} c & d \\ -a & -b \end{pmatrix} = \begin{pmatrix} ac-ca & ad-cb \\ bc-da & bd-db \end{pmatrix} = \begin{pmatrix} 0 & ad-bc \\ -(ad-bc) & 0 \end{pmatrix}$$

Hence, $A \in Sp(2,\mathbb{R})$ if and only if $\underset{\det A}{\underset{\parallel}{ad-bc}} = 1$,

which coincides with the condition that $A \in SL(2,\mathbb{R})$.

Finally we conclude $Sp(2,\mathbb{R}) = SL(2,\mathbb{R})$, as stated.