Week 03 review worksheet — exercises for §3

Part A. Exercises for interactive discussion

E3.1 **(Sweedler notation)** Let C be a coalgebra with coproduct Δ and counit ϵ. Let $x \in C$. Review the Sweedler notation, $\Delta x = \sum x_{(1)} \otimes x_{(2)}$, for the coproduct; we will write it without the $\sum$ symbol.

Which of the following are necessarily the same as x?
(Exclude the options where the Sweedler notation is used incorrectly or which are ill-defined.)

$A = \epsilon(x_{(1)})x_{(2)}$

$B = x_{(1)}\epsilon(x_{(2)})$

$C = x_{(2)}$

$D = \epsilon(x_{(1)})\epsilon(x_{(2)})$

$E = \frac{1}{2}(x_{(1)} + x_{(2)})$

$F = \epsilon(x_{(1)})\epsilon({x_{(2)}}_{(1)}){x_{(2)}}_{(2)}$

E3.2 **(grouplike elements)** Let (C, Δ, ϵ) be a coalgebra. Review the definition of a grouplike element of C.

(a) Prove that any $g \in C$ such that $g \neq 0$ and $\Delta g = g \otimes g$, is grouplike.
(b) Let $G(C)$ denote the set of all grouplike element of C. Prove that $G(C)$ is a linearly independent set. (*Hint:* use the algebra-coalgebra duality and an exercise from last week!)
(c) What are the grouplikes in A^* where A is a finite-dimensional algebra?
(d) What are the possible 1-dimensional coalgebras?

E3.3 Let G be a finite monoid (e.g., a finite group), so that $\mathbb{C}G$ is a finite-dimensional algebra. The coalgebra $(\mathbb{C}G)^*$ has a basis $\{\delta_g\}_{g\in G}$ dual to the basis $\{g\}_{g\in G}$ of $\mathbb{C}G$. Give formulae for $\Delta\delta_g$ and $\epsilon(\delta_g)$.

E3.4 (a) Assume that a coalgebra C has basis $\{\chi_0, \chi_1, \chi_2\}$ of grouplikes. Let $A = C^*$ be the dual algebra. Describe the multiplication on the dual basis $\{e_0, e_1, e_2\}$ of A.

(b) In the case when $A = \mathbb{C}\Gamma$ is the group algebra of the cyclic group $\Gamma = \{e, g, g^2\}$, and $C = (\mathbb{C}\Gamma)^*$, take χ_k to be the character of Γ which sends g to ω^k with $\omega = e^{2\pi i/3} \in \mathbb{C}$. Calculate the basis $\{e_0, e_1, e_2\}$ of $\mathbb{C}\Gamma$. Check directly that the multiplication on this basis is as you expect from (a).

E3.5 **(the trigonometric coalgebra)** Let C be a two-dimensional space over $\mathbb{R}$ with basis $\{c, s\}$. Define $\Delta\colon C \to C \otimes C$ by

$$\Delta c = c \otimes c - s \otimes s, \quad \Delta s = s \otimes c + c \otimes s.$$

(a) Define a counit $\epsilon\colon C \to \mathbb{R}$ so that (C, Δ, ϵ) becomes a coalgebra.

(b) Does C contain any grouplikes? Does C have proper subcoalgebras?

(c) How does the answer to (b) change if the field $\mathbb{R}$ is replaced by $\mathbb{C}$?

E3.6 Review the definition of an *action* of an associative unital algebra A on a vector space V.

True or false: every algebra can act on a 1-dimensional space?

Part B. Extra exercises

Attempt these exercises and compare your answers with the model solutions, published after the session.

E3.7 **(left regular module and left regular comodule)** Let A be an algebra. Prove that the map $\triangleright_{\mathrm{reg}}\colon A \otimes A \to A$ given by $a \triangleright_{\mathrm{reg}} v = av$ (product in A) is an action of the algebra A on the vector space A. (This action of A on A is called the *left regular action.*)

Develop the parallel notion of left regular coaction for coalgebras.