Answer **<u>THREE</u>** of the four questions. If more than THREE questions are attempted, then credit will be given for the best THREE answers.

1. As usual, $\mathbb{C}\langle x\rangle$ denotes the free algebra with one generator x. We equip $H = \mathbb{C}\langle x\rangle$ with the unique Hopf algebra structure where x is a primitive element. An element of H of the form $c1_H$, with $c \in \mathbb{C}$ (where the coefficient of any monomial x^n with $n > 0$ is zero) will be called a *constant*.

(a) If $f \in H$ is a constant, explain, referring to the Hopf algebra axioms, why $\Delta f = f \otimes 1_H$. Let now $h \in H$ be such that $\Delta h = h \otimes 1_H$; show that h is a constant.

Answer. If f is a constant then $f = c \cdot 1_H$ for some $c \in \mathbb{C}$. By definition of a Hopf algebra, the coproduct $\Delta\colon H \to H \otimes H$ is a homomorphism of associative unital algebras; in particular it takes the identity element 1_H of H to the identity element $1_H \otimes 1_H$ of $H \otimes H$. Moreover, linearity of Δ means that $\Delta f = c\Delta 1_H = c(1_H \otimes 1_H) = (c1_H) \otimes 1_H = f \otimes 1_H$.

Assume that $\Delta h = h \otimes 1_H$. One way to show that h is constant is to recall the counit law: $h = \epsilon(h_{(1)})h_{(2)}$ for all $h \in \mathbb{C}\langle x\rangle$, where $\epsilon\colon \mathbb{C}\langle x\rangle \to \mathbb{C}$ is the counit. Substituting $h_{(1)} \otimes h_{(2)} = h \otimes 1_H$, we obtain $h = \epsilon(h)1_H$. Since $\epsilon(h) \in \mathbb{C}$, this shows h is constant.

Write $\mathbb{C}[y]$ for the commutative algebra of polynomials in the variable y. Define the linear map $D\colon \mathbb{C}[y] \to \mathbb{C}[y]$ by $Dg = g'$ for all $g \in \mathbb{C}[y]$ (i.e., D is the differentiation operator on $\mathbb{C}[y]$). Given $f = a_0 + a_1x + \cdots + a_nx^n \in H$, let $f(D)$ be the element $a_0 + a_1D + \cdots + a_nD^n$ of $\operatorname{End}\mathbb{C}[y]$. You are given that the bilinear map $\triangleright\colon H \otimes \mathbb{C}[y] \to \mathbb{C}[y]$ defined for $f \in H$, $g \in \mathbb{C}[y]$ by $f \triangleright g = f(D)(g)$, is an action of the algebra H on the vector space $\mathbb{C}[y]$, and you do not have to prove this.

(b) Prove that the action $\triangleright$ makes the algebra $\mathbb{C}[y]$ an H-module algebra.

Answer. Since we are given that $\triangleright$ is an action, we only need to check the axioms

$$f \triangleright (gh) = (f_{(1)} \triangleright g)(f_{(2)} \triangleright h), \tag{1}$$
$$f \triangleright 1_{\mathbb{C}[y]} = \epsilon(f) \tag{2}$$

for all $f \in \mathbb{C}\langle x\rangle$. Since both sides of each axiom are linear in f, it is enough to check them in the case where $f = x^n$ is a monomial, $n \geq 0$. Since $\Delta x^n = (x \otimes 1 + 1 \otimes x)^n = \sum_{i=0}^n \binom{n}{i} x^i \otimes x^{n-i}$, axiom (1) for $f = x^n$ reads

$$D^n(gh) = \sum_{i=0}^n \binom{n}{i} (D^ig)(D^{n-i}h),$$

which is true by a well-known higher-order Leibniz rule [*easy to prove by induction*]. Since $\epsilon(x^n) = 0$ if $n > 0$, $\epsilon(1) = 1$, axiom (2) reads

$$D^n 1 = 0 \text{ if } n > 0,\ D^0 1 = 1$$

which is clearly true and finishes the proof.

(c) Given $u \in \mathbb{C}[y]$, define $I_u = \{f \in H : f \triangleright u = 0\}$. Show that I_u is an ideal of the algebra H.

Answer. Since $\triangleright$ is bilinear, the condition $f \triangleright u = 0$ is linear in f, so I_u is a subspace of H.

Since H is commutative, it is enough to show that $h \in H$, $f \in I_u$ implies $hf \in I_u$. Indeed, $(hf) \triangleright u = h \triangleright (f \triangleright u) = h \triangleright 0 = 0$, so $hf \in I_u$.

(d) Determine all polynomials $u \in \mathbb{C}[y]$ such that I_u is a coideal of H. *Hint*: if I_u is a coideal, then $\mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle/I_u$ must be a coalgebra morphism.

Answer. Suppose that I_u is a coideal of $\mathbb{C}\langle x\rangle$.

Note that, as coalgebras, $\mathbb{C}\langle x\rangle = \mathbb{C}[x]$ because all monomials in one variable x are obviously standard (with respect to the only total order that exists on the one-element set $X = \{x\}$). So, using the Hint, we have a coalgebra morphism

$$\pi\colon \mathbb{C}[x] \to \mathbb{C}[x]/I_u.$$

In the course, we proved the Heyneman-Radford theorem for the polynomial coalgebra, which says that π is injective if, and only if, the restriction $\pi|_{\mathbb{C}x}$ is injective.

Yet π cannot be injective: we have $\ker \pi = I_u$ which is not zero because I_u contains x^N for all N such that $D^N u = 0$, that is for $N > \deg u$.

So if I_u is a coideal, $\pi|_{\mathbb{C}x}$ cannot be injective, and so we must have $\pi(cx) = 0$ for some $c \neq 0$, equivalently $\pi(x) = 0$, equivalently $x \triangleright u = 0$, equivalently $u' = 0$ so that u is a constant.

Note that $u = 0$ must be excluded because $I_0 = H$ is not a coideal: the definition of a coideal requires $\epsilon(I) = \{0\}$. Thus, u can be any non-zero constant in $\mathbb{C}[y]$, so that $I_u = \ker \epsilon$, the subspace of $\mathbb{C}\langle x\rangle$ spanned by $x, x^2, x^3 \ldots$

[20 marks]

2. In this question, Γ denotes a finite cyclic group of order n with identity element e. Let $\mathbb{C}\Gamma$ be the group algebra of Γ, viewed as a Hopf algebra in the standard way, and let $(\mathbb{C}\Gamma)^*$ be the Hopf algebra dual to $\mathbb{C}\Gamma$. Choose a generator γ of Γ so that $e, \gamma, \ldots, \gamma^{n-1}$ is a basis of $\mathbb{C}\Gamma$, and let $\delta_0, \ldots, \delta_{n-1}$ be the dual basis of $(\mathbb{C}\Gamma)^*$.

(a) Formally prove that $\mathbb{C}\langle x \mid x^n = 1\rangle$ is a presentation of the algebra $\mathbb{C}\Gamma$. You need to define an algebra homomorphism from the free tensor algebra $\mathbb{C}\langle x\rangle$ onto $\mathbb{C}\Gamma$ and prove that the kernel of this homomorphism is the ideal generated by $x^n - 1$ in $\mathbb{C}\langle x\rangle$. You can identify $\mathbb{C}\langle x\rangle$ with the polynomial algebra $\mathbb{C}[x]$.

Answer. Consider the map $\{x\} \to \mathbb{C}\Gamma$, $x \mapsto \gamma$. By the universal mapping property of the free tensor algebra, this extends to an algebra homomorphism $\pi\colon \mathbb{C}[x] = \mathbb{C}\langle x\rangle \to \mathbb{C}\Gamma$. We have $\pi(x^i) = \pi(x)^i = \gamma^i$ so π is surjective (the image contains a spanning set).

If $f(x) \in \mathbb{C}[x]$, divide f by $x^n - 1$ with remainder and write $f(x) = (x^n - 1)g(x) + r(x)$ with $\deg r < n$. Note that $\pi(x^n - 1) = \gamma^n - e = 0$. So $r(x) = r_0 + r_1 x + \cdots + r_{n-1}x^{n-1}$ and $\pi(f) = 0 + \pi(r) = r_0 e + r_1\gamma + \cdots + r_{n-1}\gamma^{n-1}$. Since $e, \gamma, \ldots, \gamma^{n-1}$ are a basis, $\pi(f) = 0$ iff $r_0 = r_1 = \cdots = r_{n-1} = 0$ iff $r = 0$, so $f \in \ker \pi$ iff $f \in (x^n - 1)\mathbb{C}[x]$, as required.

(b) Let ω be any nth root of unity in $\mathbb{C}$. Show that $\sum_{i=0}^{n-1} \omega^i \delta_i$ is a grouplike element of $(\mathbb{C}\Gamma)^*$. Give a brief but convincing explanation why all grouplike elements of $(\mathbb{C}\Gamma)^*$ are of this form.

Answer. Clearly $z_\omega := \sum_{i=0}^{n-1} \omega^i \delta_i$ is not zero (as $\{\delta_i\}$ is a basis) so to show z_ω is grouplike in $(\mathbb{C}\Gamma)^*$, it is enough to check that $\Delta z_\omega = z_\omega \otimes z_\omega$. By evaluating both sides against a basis of $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$, this is equivalent to

$$(\Delta z_\omega)(\gamma^i \otimes \gamma^j) = (z_\omega \otimes z_\omega)(\gamma^i \otimes \gamma^j) \qquad \text{for all } i, j = 0, \ldots, n-1,$$

which by definition of the coproduct in $(\mathbb{C}\Gamma)^*$ rewrites as $z_\omega(\gamma^i\gamma^j) = z_\omega(\gamma^i)z_\omega(\gamma^j)$. Both sides are equal to ω^{i+j}, proving that z_ω is grouplike.

We found n distinct grouplikes, labelled by the nth roots of unity in $\mathbb{C}$. Since grouplikes are linearly independent by a result from the course, there cannot be more grouplikes in a Hopf algebra of dimension n. Therefore, we found all the grouplikes in $(\mathbb{C}\Gamma)^*$.

(c) Deduce from the result of (b) that the group $G((\mathbb{C}\Gamma)^*)$ of grouplike elements in $(\mathbb{C}\Gamma)^*$ is cyclic.

Answer. By (b), we have $G((\mathbb{C}\Gamma)^*) = \{z_\omega : \omega^n = 1\}$.

We show that $z_\omega z_\eta = z_{\omega\eta}$. Evaluate both sides against γ^i: $(z_\omega z_\eta)(\gamma^i) = (z_\omega \otimes z_\eta)(\Delta\gamma^i) = (z_\omega \otimes z_\eta)(\gamma^i \otimes \gamma^i) = z_\omega(\gamma^i)z_\eta(\gamma^i) = \omega^i\eta^i$, which is equal to $z_{\omega\eta}(\gamma^i)$.

It follows that the group $G((\mathbb{C}\Gamma)^*)$ is isomorphic to the group $\{\omega \in \mathbb{C} : \omega^n = 1\}$, which is cyclic and is generated by $e^{2\pi\sqrt{-1}/n}$.

(d) Show: if ω is a primitive nth root of unity, then $R(\omega) = \frac{1}{n}\sum\limits_{a,b=0}^{n-1} \omega^{ab}\gamma^a \otimes \gamma^b$ is a universal R-matrix on $\mathbb{C}\Gamma$. You may assume that $R(\omega)R(\omega^{-1}) = 1 \otimes 1$.

Answer. Write $R = R(\omega)$. We will verify the axioms by direct calculation. To make the calculation more streamlined, we rewrite R as $\sum\limits_{a=0}^{n-1}\Big(\gamma^a \otimes \frac{1}{n}\sum\limits_{b=0}^{n-1}(\omega^a)^b\gamma^b\Big)$. We recognise the elements $\frac{1}{n}\sum\limits_{b=0}^{n-1}(\omega^a)^b\gamma^b$ of $\mathbb{C}\Gamma_n$ as the **orthogonal idempotents** p_a, $a = 0, 1, \dots, n-1$. Indeed, one has

$$p_a^2 = p_a, \qquad p_a p_c = 0 \quad \text{if } a \neq c.$$

To verify this property of the p_a, calculate

$$p_a p_c = \frac{1}{n^2}\sum_{b,d=0}^{n-1}(\omega^a)^b(\omega^c)^d\gamma^{b+d} = \frac{1}{n^2}\sum_{k=0}^{n-1}\Big(\sum_{b=0}^{n-1}(\omega^a)^b(\omega^c)^{k-b}\Big)\gamma^k = \frac{1}{n^2}\sum_{k=0}^{n-1}\omega^{ck}\Big(\sum_{b=0}^{n-1}(\omega^{a-c})^b\Big)\gamma^k.$$

Now, if $a = c$ then $\omega^{a-c} = 1$ and $\sum\limits_{b=0}^{n-1}(\omega^{a-c})^b = n$. We get $p_c p_c = p_c$.

If $a \neq c$ then $\omega^{a-c} \neq 1$ because ω is a primitive nth root of unity. Then $\sum_{b=0}^{n-1}(\omega^{a-c})^b = 0$ (use the formula for the sum of geometric progression with ratio ω^{a-c}) which proves $p_a p_c = 0$.

We now return to $R = \sum_{a=0}^{n-1} g^a \otimes p_a$. We are given that $R(\omega)R(\omega^{-1}) = 1 \otimes 1$. Therefore, R is invertible with $R^{-1} = R(\omega^{-1})$. Axiom 1. of quasitriangular structure is thus verified.

Axiom 2. of quasitriangular structure is trivial in this case: indeed, $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$ is a commutative algebra so $R(x_{(1)} \otimes x_{(2)})R^{-1} = RR^{-1}(x_{(1)} \otimes x_{(2)}) = x_{(1)} \otimes x_{(2)}$; and also $x_{(2)} \otimes x_{(1)} = x_{(1)} \otimes x_{(2)}$ because $\mathbb{C}\Gamma$ is cocommutative.

To verify axiom 3., we use the orthogonal idempotent property of the p_a to calculate

$$(\Delta \otimes \mathrm{id})R = \sum_{a=0}^{n-1}\gamma^a \otimes \gamma^a \otimes p_a = \sum_{a,b=0}^{n-1}\gamma^a \otimes \gamma^b \otimes p_a p_b = R_{13}R_{23}.$$

The identity $(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}$ is verified similarly, by writing $R = \sum\limits_{a=0}^{n} p_a \otimes \gamma^a$.

[20 marks]

3. Let the Lie algebra $\mathfrak{sl}_2$ be spanned over $\mathbb{C}$ by X, H, Y with the Lie bracket given by $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. Let $U = U(\mathfrak{sl}_2)$ be the universal enveloping algebra of $\mathfrak{sl}_2$; consider $\mathfrak{sl}_2$ as a subspace of U. Let $\mathfrak{b}$ be the subspace of $\mathfrak{sl}_2$ spanned by X and H; clearly, $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{sl}_2$.

(a) Explain why the inclusion $\mathfrak{b} \subset \mathfrak{sl}_2$ gives rise to a homomorphism $\psi\colon U(\mathfrak{b}) \to U$ of associative unital algebras. Your answer should refer to the universal mapping property of the universal enveloping algebra. (*Note that ψ, if exists, is uniquely determined by the condition $\psi(z) = z$ for all $z \in \mathfrak{b}$.*)

Answer. By the universal mapping property proved in the course, any Lie map $\mathfrak{b} \xrightarrow{f} A$ where A is an associative algebra, uniquely extends to a morphism $U(\mathfrak{b}) \xrightarrow{\widehat{f}} A$ of associative algebras. A Lie map is such that $f([x, y]_{\mathfrak{b}}) = f(x)f(y) - f(y)f(x)$ in A. The embedding $\mathfrak{b} \subset \mathfrak{sl}_2 \subset U(\mathfrak{sl}_2)$ is a Lie map, so the above applies.

(b) Prove that the map $\psi\colon U(\mathfrak{b}) \to U$ from part (a) is injective.

Answer. By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{b})$ has basis $\{X^m H^n : m, n \in \mathbb{Z}_{\geq 0}\}$. Since ϕ is an algebra homomorphism and $\psi(X) = X$, $\psi(H) = H$, one has $\psi(X^m H^n) = X^m H^n \in U(\mathfrak{sl}_2)$. But the standard monomials $X^m H^n$ are linearly independent in $U(\mathfrak{sl}_2)$, again by the PBW theorem. Thus, ψ carries a basis to a linearly independent set, hence is injective.

(c) Give an example of a coideal K of U such that $U = \psi(U(\mathfrak{b})) \oplus K$; justify your example. Is your choice of K a Hopf ideal? Give brief reasons.

Answer. Recall that by the PBW theorem, $\{X^m H^n Y^p : m, n, p \in \mathbb{Z}_{\geq 0}\}$ is a basis of U. So if

$$K = UY = \operatorname{span}\{X^m H^n Y^p : p > 0\}$$

then $U = \operatorname{span}\{X^m H^n\} \oplus K = \psi(U(\mathfrak{b})) \oplus K$. To show that UY is a coideal, observe that if $p > 0$, $\Delta(X^m H^n Y^p) = (\Delta(X^m H^n Y^{p-1}))(Y \otimes 1 + 1 \otimes Y) \in UY \otimes U + U \otimes UY$, and that $\epsilon(UY) = 0$ because $\epsilon(Y) = 0$. The coideal UY is not an ideal, because $Y \in UY$ but $XY - YX = H \notin UY$.

(d) (i) Show that there exists an action $\triangleright$ of the algebra U on the vector space U, which is defined on the generators $z = X, H, Y$ of U by $z \triangleright A = zA - Az$ for $A \in U$. (*Note that the action, if exists, is unique because it is defined on generators. You only need to prove existence.*)

Answer. Let $d_z\colon U \to U$ be defined by $d_z(A) = zA - Az$. The linear map $\mathfrak{sl}_2 \to \operatorname{End} U$, $z \mapsto d_z$ is a Lie morphism because

$$\begin{aligned}(d_z d_t - d_t d_z)(A) &= z(tA - At) - (tA - At)z - t(zA - Az) + (zAt - Az)t \\ &= (zt - tz)A - A(zt - tz) = d_{[z,t]}(A).\end{aligned}$$

Hence $z \mapsto d_z$ extends to a map $U \to \operatorname{End} U$ of associative algebras, which is the action of U on the vector space U.

(ii) Let $U^0 = \{A \in U : u \triangleright A = \epsilon(u)A$ for all $u \in U\}$ where ϵ is the counit of U. Find $\lambda \in \mathbb{C}$ such that $XY + YX + \lambda H^2 \in U^0$.

Answer. We need to ensure that $\Omega_\lambda := XY + YX + \lambda H^2 = 2XY - H + \lambda H^2$ commutes with X, Y and H in U. It is easy to see that $[H, \Omega_\lambda] = 0$ for all λ. Now,

$$[Y, H] = 2Y, \qquad [Y, XY] = [Y, X]Y - X[Y, Y] = -HY - 0 = -HY,$$

$$[Y,H^2] = H[Y,H] + [Y,H]H = 2HY + 2YH = 2HY + 2HY - 2[H,Y] = 4HY + 4Y,$$

so $[Y,\Omega_\lambda] = -2HY - 2Y + \lambda(4HY + 4Y)$ and we must put $\lambda = 1/2$. One checks that $[X,\Omega_{1/2}] = 0$ so $\lambda = 1/2$ is the answer.

[20 marks]

4. Let the Hopf algebra $U_q = U_q(\mathfrak{sl}_2)$ be generated over $\mathbb{C}$ by E, F, K, K^{-1} subject to relations $KK^{-1} = 1$, $K^{-1}K = 1$, $KE = q^2EK$, $KF = q^{-2}FK$, $EF - FE = (q-q^{-1})^{-1}(K - K^{-1})$, where the coalgebra structure is given by $\Delta E = 1 \otimes E + E \otimes K$, $\Delta F = K^{-1} \otimes F + F \otimes 1$, $\epsilon(E) = \epsilon(F) = 0$ and K being grouplike. Let $S\colon U_q \to U_q$ be the antipode. For the purposes of this question, we treat q as a complex number which is neither 0 nor a root of unity. Let $\mathcal{B} = \{E^mK^nF^p : m,p \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}$ be the PBW-type basis of U_q.

(a) Express $S(EKF^2)$ as a linear combination of elements of $\mathcal{B}$.

Answer. Recall that $S(K^{\pm 1}) = K^{\mp 1}$ as K is grouplike, $S(E) = -EK^{-1}$, $S(F) = -KF$. The first step is to use antimultiplicativity of S: $S(EKF^2) = S(F)^2 \cdot S(K)S(E) = -KFKF \cdot K^{-1}EK^{-1}$, then use $KF = q^{-2}FK$ and $EK^{-1} = q^2K^{-1}E$ to write this as $-q^{-6}F^2K^2 \cdot q^2K^{-2}E$, so that

$$S(EKF^2) = -q^{-4}F^2E.$$

We have

$$F^2E = EF^2 - [E,F^2] = EF^2 - [E,F]F - F[E,F] = EF^2 - \{F,[E,F]\},$$

where we write $[a,b] := ab - ba$, $\{a,b\} := ab + ba$. Since

$$[E,F] = \tfrac{K-K^{-1}}{q-q^{-1}}, \qquad \{F,K\} = (q^2+1)KF, \qquad \{F,K^{-1}\} = (1+q^{-2})K^{-1}F,$$

we have

$$F^2E = EF^2 - \frac{(q^2+1)KF - (1+q^{-2})K^{-1}F}{q-q^{-1}},$$

so the final answer is $-q^{-4}EF^2 + \frac{q^{-2}+q^{-4}}{q-q^{-1}}KF - \frac{q^{-4}+q^{-6}}{q-q^{-1}}K^{-1}F$.

(b) Let $\rho\colon U_q \to M_{n\times n}(\mathbb{C})$ be a homomorphism of associative unital algebras. Explain why $\widetilde{\rho}\colon U_q \to M_{n\times n}(\mathbb{C})$ defined by $\widetilde{\rho}(h) = \rho(S(h))^T$ is also a homomorphism of associative unital algebras. (Here ^{T} denotes matrix transposition.) Prove that if $n = 1$, then necessarily $\rho = \widetilde{\rho}$.

Answer. An acceptable conceptual explanation is: interpret the matrix algebra $M_{n\times n}(\mathbb{C})$ as the endomorphism algebra $\mathrm{End}(V)$ where $V = \mathbb{C}^n$, and consider the action $\triangleright$ of U_q on the space V, defined by ρ. As seen in the course, since U_q is a Hopf algebra, it acts on the dual space V^* via $(h \blacktriangleright \xi)(v) = \xi(Sh \triangleright v)$ for $\xi \in V^*$, $v \in V$. Thus, the map $h\,\blacktriangleright\colon V^* \to V^*$ is the contragredient of $Sh\triangleright\colon V \to V$, and so its matrix the transpose of $\rho(Sh)$.

However, the fact that $\widetilde{\rho}$ is a homomorphism can be checked directly: it is linear since S and $(\ \ )^T$ are linear, and is multiplicative, because both S and $(\ \ )^T$ are antimultiplicative.

Assume now that $n = 1$. Since the matrix algebra $M_{1\times 1}(\mathbb{C}) \cong \mathbb{C}$ is commutative, $\rho(K)\rho(E) = q^2\rho(E)\rho(K)$ translates into $(1-q^2)\rho(K)\rho(E) = 0$. Note that $1 - q^2 \neq 0$ as q is not a root of unity, and $\rho(K)\rho(K^{-1}) = 1$ so $\rho(K) \neq 0$. It follows that $\rho(E) = 0$, and $\widetilde{\rho}(E) = \rho(-EK^{-1}) = -\rho(K^{-1})0 = 0$.

In the same way $\widetilde{\rho}(F) = \rho(F) = 0$. Finally, $\rho(K) - \rho(K^{-1}) = (q - q^{-1})(\rho(E)\rho(F) - \rho(F)\rho(E)) = 0$, so one has $\rho(K) = \rho(K^{-1}) = \rho(S(K))^T = \widetilde{\rho}(K) = \widetilde{\rho}(K^{-1})$.

We conclude that the homomorphisms ρ and $\widetilde{\rho}$ agree on generators of U_q, hence are equal.

(c) You are given that $\sigma\colon U_q \to M_{3\times 3}(\mathbb{C})$ is such that $\sigma(E) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q^3+q & 0 \end{pmatrix}$, $\sigma(K) = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix}$ and σ is a homomorphism of associative unital algebras.

Show that $\sigma(F)$ is uniquely determined by these conditions. Find $\sigma(F)$.

Answer. $\sigma(F) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$\Rightarrow \sigma(F)\sigma(K) - q^2\sigma(K)\sigma(F) = \begin{pmatrix} (q^{-2}-1)a & 0 & (q^2-1)c \\ (-q^2+q^{-2})d & (-q^2+1)e & 0 \\ (-q^4+q^{-2})g & (-q^4+1)h & (-q^4+q^2)i \end{pmatrix}$, this must be zero so

$\sigma(F) = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix}$ as the polynomials in q which appear above are not 0 due to q not being a root of unity. Calculate $\sigma(E)\sigma(F) - \sigma(F)\sigma(E) = \mathrm{diag}(-b,\ b - (q^3+q)f,\ (q^3+q)f)$, this must be equal to $(\sigma(K) - \sigma(K)^{-1})/(q - q^{-1}) = \mathrm{diag}(-(q+q^{-1}),\ 0,\ q+q^{-1})$. It follows that

$$\sigma(F) = \begin{pmatrix} 0 & q+q^{-1} & 0 \\ 0 & 0 & q^{-2} \\ 0 & 0 & 0 \end{pmatrix}.$$

(d) Let $q = e^{\hbar}$ and let $\mathcal{R} = \exp(\frac{\hbar}{2} H \otimes H)\exp_{q^{-2}}((q - q^{-1})E \otimes F)$ be the universal R-matrix for $U_q(\mathfrak{sl}_2)$ constructed in the course; in particular, H is a primitive element such that $K = \exp(\hbar H)$. Show, by explicit calculation, that $(\epsilon \otimes \mathrm{id})\mathcal{R} = 1_{U_q}$, briefly explaining the assumptions made in your calculation.

Answer. We assume that the usual properties of ϵ, e.g., linearity and multiplicativity, extend to infinite sums and their products. (This is because $U_q(\mathfrak{sl}_2)$ is a $\mathbb{C}$-subalgebra of the $\hbar$-adic $\mathbb{C}[[\hbar]]$-algebra $U_\hbar(\mathfrak{sl}_2)$.) Recall that $\exp_{q^{-2}}((q - q^{-1})E \otimes F)$ is defined as

$$\sum_{n=0}^{\infty} \frac{1}{[n;\, q^{-2}]!}((q - q^{-1})E \otimes F)^n = 1_{U_q} \otimes 1_{U_q} + \frac{1}{[n;\, q^{-2}]!}(q - q^{-1})E \otimes F + \dots,$$

and observe that the terms where $n \geq 1$ contain E^n in the left leg. Since $\epsilon(E^n) = \epsilon(E)^n = 0^n = 0$, the operator $\epsilon \otimes \mathrm{id}$ gives zero when applied to each of the $n \geq 1$ terms; only the $n = 0$ term survives. Thus,

$$(\epsilon \otimes \mathrm{id})\left(\exp_{q^{-2}}((q - q^{-1})E \otimes F)\right) = (\epsilon \otimes \mathrm{id})(1_{U_q} \otimes 1_{U_q}) = 1_{\mathbb{C}} \otimes 1_{U_q} = 1_{U_q}.$$

We are left to calculate $(\epsilon \otimes \mathrm{id})\left(\exp(\frac{\hbar}{2} H \otimes H)\right)$. Similarly to the above,

$$\exp(\frac{\hbar}{2} H \otimes H) \sum_{n=0}^{\infty} \frac{(\hbar/2)^n}{n!} H^n \otimes H^n = 1_{U_q} \otimes 1_{U_q} + \frac{\hbar}{2} H \otimes H + \dots,$$

where the $n > 0$ terms contain a positive power of H in the left leg. Since H is primitive, $\epsilon(H) = 0$ and $\epsilon(H^n) = \epsilon(H)^n = 0$. Therefore,

$$(\epsilon \otimes \mathrm{id}) \left(\exp(\frac{\hbar}{2} H \otimes H) \right) = (\epsilon \otimes \mathrm{id})(1_{U_q} \otimes 1_{U_q}) = 1_{\mathbb{C}} \otimes 1_{U_q} = 1_{U_q}.$$

We conclude:

$$(\epsilon \otimes \mathrm{id})\mathcal{R} = (\epsilon \otimes \mathrm{id}) \left(\exp(\frac{\hbar}{2} H \otimes H) \right) \cdot (\epsilon \otimes \mathrm{id}) \big(\exp_{q^{-2}}((q - q^{-1}) E \otimes F) \big) = 1_{U_q} \cdot 1_{U_q} = 1_{U_q}.$$

[20 marks]

END OF EXAMINATION PAPER