MAGIC assessment cover sheet

Course:	**MAGIC073**
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$\frac{57}{100}$

You've lost many mark because
(1) your solutions have not been scanned properly (you should have checked your file before submitting)
(2) You've "mis-used" the notation.

Commutative Algebra Q1

Ibraheem Sajid

(i) ~~[illegible]~~

FALSE: $I = 2\mathbb{Z} \times \mathbb{Z} \triangleleft \mathbb{Z}\times\mathbb{Z}$ is prime since $(a,b)(x,y) \in I$

$\Rightarrow (ax, by) \in I \Rightarrow ax \in 2\mathbb{Z} \Rightarrow a \in 2\mathbb{Z}$ or $x \in 2\mathbb{Z}$

$\Rightarrow (a,b) \in I$ or $(x,y) \in I$.

But $\mathbb{Z} \notin \operatorname{Spec}(\mathbb{Z})$ so this is not of the claimed form. #

✓ $\frac{2}{2}$

(ii) FALSE: $2\mathbb{Z} \triangleleft \mathbb{Z}$. $\mathbb{Z}/(2\mathbb{Z})(2\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ has an element of order 4

but $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has no elements of order 4.

(i.e. ~~[illegible]~~ these differ as abelian groups so can't be isomorphic rings) ✓ #

✓ $\frac{2}{2}$

(iii) TRUE: ① $\mathbb{Z}[x,y]$ is Noetherian: ~~since~~ $\mathbb{Z}$ is a PID hence Noetherian (Cor 2.47) and $\mathbb{Z}[x,y]$ is a multivariate polynomial ring (Cor 2.49) over a Noetherian ring hence Noetherian //

✓

② R Noetherian, $I \triangleleft R \Rightarrow R/I$ Noetherian:

Ideals of R/I correspond to ideals of R containing I (Cor 1.18), so ascending chains in R/I correspond to ascending chains in R starting with I. The latter all terminate since R is Noetherian so the former must also all terminate. $\therefore R/I$ Noetherian. //

✓

① + ② $\Rightarrow \mathbb{Z}[x,y]/(x^2 - y^3)$ is Noetherian.

✓ $\frac{2}{2}$

(iv) FALSE: If we take the localisation of $\mathbb{Z}$ at $\mathbb{Z} \setminus p\mathbb{Z}$, note $\mathbb{Z}_{(p)} = \{ \frac{a}{s} \in \mathbb{Q} \mid p \nmid s \} \subseteq \mathbb{Q}$ is an inclusion of rings.

By Cor 1.40, $\mathbb{Z}_{(p)}$ is a local ring with maximal ideal $p\mathbb{Z}_{(p)}$. Note, $\frac{p}{1} \in p\mathbb{Z}_{(p)}$ so $p\mathbb{Z}_{(p)} \neq 0$.
$\frac{p}{1} \neq 0$

Also, $\mathbb{Q}$ is local (it is a field) with maximal ideal (0). Thus we have $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$ with $\mathrm{Rad}(\mathbb{Z}_{(p)}) = p\mathbb{Z}_{(p)} \nsubseteq \mathrm{Rad}(\mathbb{Q}) = (0)$.
#

2/2

(v) TRUE: Let F be freely generated by $\{x_i\}_{i \in I}$, where I is finite.
A generic element of $F \otimes M$ is $\sum_J r_j y_j \otimes m_j = x$ for some finite indexing set J and $r_j \in R$, $y_j \in F$, $m_j \in M$ for all $j \in J$. Writing each y_j as a linear combination of $x_i, i \in I$ we can ~~rewrite~~ assume $x = \sum_I r_i x_i \otimes m_i$.

Now, $f_*(x) = 0 \iff \sum_I r_i x_i \otimes f(m_i) = 0$

We have homomorphisms $F \to R$ given by $g_i(x_j) = \delta_{ij}$ (since F is free on $\{x_i\}$)
Each g_i extends to $g_{i*} : F \otimes N \to R \otimes N \cong N$
So $f_*(x) = 0 \Rightarrow g_{i*} \circ f_*(x) = 0 \quad \forall i \in I$

$\Rightarrow r_i \otimes f(m_i) = 0 \quad \forall i \in I$
$\Rightarrow 1 \otimes r_i f(m_i) = 0 \quad \forall i \in I$
$\Rightarrow \forall i \in I$ either $r_i = 0$ or $m_i = 0$
$\Rightarrow x = 0$. //

2/2

TRUE:

(vi) Let M be a fractional ideal of a PID R, and F be the field of fractions of R.

$\exists \frac{a}{b} \in F^\times$ st. $\frac{a}{b} M \subseteq R$. Thus $\frac{a}{b}M$ is a ideal of R, ✓

So $\frac{a}{b}M = (x)$, some $x \in R$. Then $\frac{a}{bx} \in F$ ✓ (assume $M \neq 0$ so $x \neq 0$ as $\frac{a}{b} \in F^\times$) ✓

Let M' be the R-submodule of F generated by $\frac{a}{bx}$. M' is a fractional ideal since $\frac{bx}{a} \in F^\times$ ($a \neq 0$ since $\frac{a}{b} \in F^\times$)

and $\frac{bx}{a} M' = R \subseteq R$

Now, ~~[illegible]~~ $MM' = \{ M \{ \frac{ar}{bx} \mid r \in R \}$

$= \left\{ \frac{arm}{bx} \mid r \in R,\ m \in M \right\}$

but $m \in M \Rightarrow \frac{a}{b} m \in (x) \Rightarrow \frac{a}{bx} \in R$

$\Rightarrow \frac{ar}{bx} \in R$ OK

So $MM' \subseteq R$.

~~[illegible]~~ Since $\frac{a}{b} M = (x)$

$\exists\, m \in M$ st. $\frac{a}{b} m = x$

then $\frac{a}{bx} \cdot x = 1$, so $R \subseteq MM'$

i.e. $MM' = R$ ✓

//

2/2

(vii) FALSE: Let $I = (2, 1+\sqrt{5}i)$

Suppose $I = (x)$. Then $x | 2$ ~~and $x | 1+\sqrt{5}i$~~.

Have $x = a + b\sqrt{5}i$, $a, b \in \mathbb{Z}$

and $(a+b\sqrt{5}i)(c+d\sqrt{5}i) = 2$, $c, d \in \mathbb{Z}$

$\therefore (ac - 5bd) + (ad+bc)\sqrt{5}i = 2$

$\therefore ac - 5bd = 2$, $\underline{ad+bc = 0}$ (✓)

So $(a-b\sqrt{5}i)(c-d\sqrt{5}i) = 2$ as well and then

$(a^2+5b^2)(c^2+5d^2) = 4$ ok.

$\therefore b = d = 0$ (else we blow up part 4).

could be negative { Have $a = 1$, $c = 2$ $(I = (1))$

or $a = 2$, $c = 1$ $(I = (2))$

$f, g \in \mathbb{Z}$

But $2 | 1+\sqrt{5}i \Rightarrow 2(f+g\sqrt{5}i) = 1+\sqrt{5}i$

$\Rightarrow 2f = 1, 2g = 1$ #

$\therefore (2) \subsetneq (2, 1+\sqrt{5}i)$

Suppose $A \cdot 2 + B(1+\sqrt{5}i) = 1$, $A, B \in \mathbb{Z}$

Then $B = 0$ ($\sqrt{5}i$ coefficient)

and $2A = 1$ #

OK

But too long! $\frac{2}{2}$

(viii) TRUE: We know $\mathbb{Z}$ is a UFD. Then, by Lem 1.15, $\mathbb{Z}[x,y] = \mathbb{Z}[x][y]$ is also a UFD. So by Prop 3.5, $\mathbb{Z}[x,y]$ is integrally closed //

✓ $\frac{2}{2}$

Q1 $\frac{16}{16}$

No! you can't: if the
... is prime ...

I will write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$ in this question. ring $\mathbb{Z}_n$ denotes the n-adic integers

Commutative Algebra Q2

Ibraheem Sajid

(i) (a) A generic element of $\frac{R}{I} = \frac{\mathbb{Z}_7[x]}{(x^2+2)}$ is $a + bx + (x^2+2)$. $a, b \in \mathbb{Z}_7$.

$(a+bx)(c+dx) + (x^2+2) = 1 + (x^2+2)$

$\iff ac + (bc+ad)x + bdx^2 + (x^2+2) = 1 + (x^2+2)$

$\iff ac + (bc+ad)x - 2bd + (x^2+2) = 1 + (x^2+2)$

$\iff ac - 2bd \equiv 1, \quad bc + ad \equiv 0 \pmod 7$

7-adic integers.

(i) (c) Suppose x^2+2 was reducible in $\mathbb{Z}_7[x]$. Then $x^2+2 = (x+a)(x+b)$ for $a, b \in \mathbb{Z}_7$, so it would have a root in $\mathbb{Z}_7$. But

Notation -2

x	x^2+2
0	2
1	3
2	6
3	4
4	4
5	6
6	3

$\therefore$ x^2+2 is irreducible. By lemma 1.15, (x^2+2) is maximal, noting that $\mathbb{Z}_7[x]$ is a PID since $\mathbb{Z}_7$ is a field.

$\therefore$ R/I is a field by Thm 1.10.

OK but $\frac{0}{2}$ because of notation

(b) $\bar{x}^3 = \overline{-2x}$. OK $\overline{-2x(c+dx)} = \bar{1}$ $(c, d \in \mathbb{Z}_7)$

$\iff \overline{-2c} = \bar{0}$, $\overline{4d} = \bar{1}$

So let $c = 0$, $d = 2$ (note $4 \times 2 = 8 = 1$)

Note indeed: $(\bar{x}^3)(\overline{2x}) = \overline{2x^4}$

$= \overline{2(x^2+2-2)^2}$

$= \pi(2(-2)^2 + 4(x^2+2)(-2) + 2(x^2+2)^2)$

$= \pi(2(-2)^2) = \bar{8} = 1$. //

$\frac{4}{4}$

(c) Note $\mathrm{Frac}(R) = \cancel{\mathbb{Z}_7(x)}$ (rational ~~fractions~~ ~~of~~ polynomials in $\mathbb{Z}_7$)

$\quad \| $

$\quad F$

$I^{-1} = \left\{ \frac{a}{b} \in F \;\middle|\; \frac{a}{b} I \subseteq R \right\}$

Let $\frac{p}{q} \in F$, ie. $p, q \in R$. wlog p, q are coprime (since R is UFD).

Then $\frac{p}{q} \cdot \langle x^2+2 \rangle \subseteq R$

$\iff \frac{p}{q}(x^2+2) \in R$

$\iff p(x^2+2) = q \cdot s$, $s \in R$.

Since x^2+2 is prime and divides qs, it must divide q or s. Since p is coprime to q, it must divide s. OK.

Thus, $x^2+2 \mid q \Rightarrow p = s$ and $x^2+2 \mid s \Rightarrow q = 1$ and $s = p(x^2+2)$

)

So $I^{-1} = R \cup \left\{ \frac{a}{b} \;\middle|\; a \in R,\ (x^2+2) \mid b \right\}$

$= R\langle 1, \frac{1}{x^2+2} \rangle$ as an R-module ok.

$= R\langle \frac{1}{x^2+2} \rangle$ since $x^2+2 \cdot \frac{1}{x^2+2} = 1$. $\frac{2}{2}$

$\therefore$ I^{-1} is invertible with inverse I. ✓

$$\left(I I^{-1} = R\langle \frac{1}{x^2+2} \cdot (x^2+2) \rangle = R \right).$$

(ii). First, note that prime ideals of $R \times S$ for rings R, S are either of the form $R \times Q$ or $P \times S$ ~~for~~ ~~prime ideals of R, S~~ $P \in \mathrm{Spec}(R)$, $Q \in \mathrm{Spec}(S)$. ✓

Indeed suppose $Q \in \mathrm{Spec}(S)$ ~~prime~~. Then $R \times S(R \times Q) = R \times Q$

so $R \times Q$ is an ideal. If $(ab, cd) \in R \times Q$, then c or $d \in Q$ since $cd \in Q$. $\therefore$ (a,c) or $(b,d) \in R \times Q$ //

$P \times S$ is similar.

Conversely suppose $I \triangleleft R \times S$ is an ideal. Define $P = \{ r \in R \mid \exists s \in S,\ (r,s) \in I \}$

Note $P \triangleleft R$ since $\exists s,\ (r,s) \in I \Rightarrow (a,1)(r,s) \in I$ $\forall a \in R \Rightarrow ar \in P$ $\forall a \in R$.

If I is prime then P is prime ~~by a similar~~ since we have $ab \in P \Rightarrow \exists s \in S$ st. $(ab, s) \in I \Rightarrow (a,s)$ or $(b,1) \in I$

$\Rightarrow$ a or b $\in P$. ~~Now if ...~~

~~we have ...~~

Since $(0,1)(1,0) = (0,0) \in I$, we have $(0,1)$ or $(1,0)$ $\in I$.

If ~~...~~ $(0,1) \in I$, we've shown $I = P \times S$, otherwise define a Q similarly for $I = R \times Q$. OK

Next, note prime ideals of $\mathbb{Z}$ containing $n\mathbb{Z}$ are in ~~bijection~~ correspondence with prime ideals of $\mathbb{Z}_n$?could be read $\mathbb{Z}_n$...

$\therefore$ $\text{Spec}(\mathbb{Z}_{10}) = \{ \cancel{(0)}, (2), (5) \}$ Notation

and $\text{Spec}(\mathbb{Z}_{21}) = \{ \cancel{(0)}, (3), (7) \}$.

so $\text{Spec}(\mathbb{Z}_{10} \times \mathbb{Z}_{21})$ is the 4 elements as described previously

(eg. $(2) \times \mathbb{Z}_{21}$ and $\mathbb{Z}_{10} \times (3)$).

By Thm 4.12, clopen sets are precisely U_e for each idempotent.

Lets enumerate:

x	0	1	2	3	4	5	6	7	8	9
x^2 mod 10	0	1	4	9	6	5	6	9	4	1
x^2 mod 21	0	1	4	9	16	4	15	7	1	18

OK

x	10	11	12	13	14	15	16	17	18	19	20
x^2 mod 21	16	16	18	1	7	15	4	16	9	4	1

$= 16$ unique subsets of ~~$\mathbb{Z}$~~ $\mathrm{Spec}(\mathbb{Z}_{10} \times \mathbb{Z}_{21})$

Then idempotents are the 16 combinations ok.

$\{0, 1, 5, 6\} \times \{0, 1, 7, 15\}$

Proof?

In fact, since $|\mathrm{Spec}(\mathbb{Z}_{10} \times \mathbb{Z}_{21})| = 4$,

Power set?

~~then~~ $|\mathcal{P}(\mathrm{Spec}(\mathbb{Z}_{10} \times \mathbb{Z}_{21}))| = 16$

and thus every set is clopen! So each singleton is a connected component. ok.

$\left(\mathcal{C} = \{\text{set of Zariski closed sets of } \mathrm{Spec}(\mathbb{Z}_{10} \times \mathbb{Z}_{21})\} \subseteq \mathcal{P}\,\mathrm{Spec}(\mathbb{Z}_{10} \times \mathbb{Z}_{21}) \right)$

so we must have equality

$\frac{8}{12}$

Q2

$\frac{14}{20}$

$\#$ = contradiction

Commutative Algebra Q3 — Ibraheem Sajid

(1) First, (0) is prime $((0) \subsetneq R)$ so R is an integral domain.
Now suppose $\exists\, r \notin R^\times$.

If $(r^2) = (r)$, then $ar^2 = r$ for some $a \in R$,
$\Rightarrow ar^2 - r = 0 \Rightarrow (ar-1)r = 0$
$\Rightarrow ar - 1 = 0 \Rightarrow a \cdot r = 1$
i.e. $r \in R^\times$ $\#$

contradiction symbol ?

$\therefore (r^2) \subsetneq (r)$. But then $r \cdot r \in (r^2)$
but $r \notin (r^2)$, so (r^2) is
not prime. $\#$ as it is a proper ideal.

$\therefore r \in R^\times$.

OK

$\frac{2}{2}$

~~2(i) Since $\mathbb{Z}[\sqrt{3}]$ is a PID, prime ideals are of the form (f) for $f \in \mathbb{Z}[\sqrt{3}]$ irreducible~~
(then 4+5)

~~But Let $f = a + b\sqrt{3}$, $a, b \in \mathbb{Z}$.~~

~~Hence $f \ldots \therefore \ldots$ for some $c, d \in \mathbb{Z}$~~
~~$\therefore \ldots$~~

~~and $ad + bc = 0$~~

3 12 27

$|1-2\sqrt{3}| = -1$

$a^2 - 3b^2$ prime

$25 - 11 = 14$ $\quad 36 - 11 = 25$
$16 - 11 = 4$ $\quad 49 - 11 = 38$

$9 - 3$ $\quad 11+1 = 3 \times 2^2$ $\quad 9 - 11 = -2$ $\quad 64 - 11 = 53$

$4 - 11 = -7$

$16 - 3 = 13$

$(4+\sqrt{3})(4-\sqrt{3}) = 13$

$\therefore 4 + \sqrt{3}$ prime

$(1+2\sqrt{3})(1-2\sqrt{3}) = 1 - 4 \times 3$

Not sure of what you're doing here... $\quad$ 11 12 2

(ii) Note $\mathbb{Z}[\sqrt{3}]$ is integral over $\mathbb{Z}$ ($(\sqrt{3})^2 - 3 = 0$) ✓

? and integrality is preserved under multiplication, addition and subtraction.

No! that's not what the thm states!!!

$\therefore$ By ~~Thm~~ Thm 3.11, $\mathrm{Spec}(\mathbb{Z}) \longleftrightarrow \mathrm{Spec}(\mathbb{Z}[\sqrt{3}])$ are bijective, so there is just one prime ideal I with $I \cap \mathbb{Z} = (11)$.

Note $1+2\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ has norm $1^2 - 4\cdot 3 = -11$, which is prime so ~~ISR~~ $1+2\sqrt{3}$ is irreducible and $(1+2\sqrt{3})$ is prime by Lem 1.15.

Note $11 = -(1-2\sqrt{3})(1+2\sqrt{3})$ so $11 \in (1+2\sqrt{3})$

and $(11) \subseteq (1+2\sqrt{3}) \cap \mathbb{Z}$.

But then $(11) = (1+2\sqrt{3}) \cap \mathbb{Z}$ since $(1+2\sqrt{3}) \cap \mathbb{Z}$ is prime.

$\therefore I = (1+2\sqrt{3})$ is the only possibility.

$\frac{0}{3}$

illegible.

Use that $\mathbb{Z}[x,y]$ is UFD, helarte to cancel common factors.

(iii) Suppose $I = (f)$. Then $f = rxy + sy^2$

~~[illegible]~~ $\Rightarrow y \mid f$ in what ring? It can't be R since $y \notin R$.

in R?

$\therefore$ Let $f = yg$. Have $xy = hyg$ and $y^2 = kyg$

$\therefore x = hg \Rightarrow g = 1$ or $g = x$. $\therefore$ ~~[illegible]~~

~~[illegible]~~ But

Since $y \notin R$, have $g = x$. But $y^2 \notin (xy)$

So contradiction; R is non PID.

$I^2 = (xy, y^2)(xy, y^2) = (x^2y^2, xy^3, y^4)$ ✓

Since $y^2x^2 \in I^2$. If I^2 primary, we have

for all? for some?

$\therefore y^2 \in I^2$ or $x^{2n} \in I^2$ $n \geq 0$

But $y^2 \notin I^2$ since ~~[illegible]~~ $I^2 \cap (y^2)$?

hence $\pi(I^2 \cap (y^2)) = (0)$ where $\pi : R \to R/(x)$ evaluation at $x=0$ map.

but $\pi(y^2) \neq 0$.

The argument is incomprehensible as written.

$\frac{0}{5}$

$\therefore I^2$ not primary.

~~(iv) Suppose $f = \frac{p}{q}$, $p, q \in R\setminus\{0\}$ $\in K$ wlog: $\gcd(p,q)=1$~~

~~f integral over $R \iff \exists\, a_0, \dots, a_{n-1} \in R$~~

~~st. $a_0 + \frac{p}{q} a_1 + \dots + \frac{p^{n-1}}{q^{n-1}} a_{n-1} + \frac{p^n}{q^n} = 0$.~~

~~$\iff a_0 q^n + a_1 p q^{n-1} + \dots + a_{n-1} p^{n-1} q + p^n = 0$~~

~~Hence $p \mid a_0$~~ $\mathbb{Z}[\sqrt{5}] \subseteq \mathbb{Q}[\sqrt{5}] \subseteq \mathbb{Q}(\sqrt{5})$.

$a + b\sqrt{5}$

Q3

iv $\frac{0}{10}$

$\frac{2}{20}$

Commutative Algebra Q4 Ibraheem Sajid

R-module?

(i) The homomorphism $f: R \to R/I \times R/J$
$r \mapsto (r+I, r+J)$
has $\ker f = \{r \mid r \in I, r \in J\}$
$= I \cap J = (0)$

$\therefore f$ is injective. ✓

Suppose $K_1 \subseteq K_2 \subseteq \dots$ is an ascending chain of ideals in R. Since $R \to R/I$ and $R \to R/J$ are surjective, $f(K_i) = f(K_i) \cdot R/I \times R/J$ is an ideal (direct products of two ideals is an ideal). ok.

$\therefore$ $K_1/I \times K_1/J \subseteq K_2/I \times K_2/J \subseteq \dots$ is an ascending chain of ideals in $R/I \times R/J$.

for this you need $K_i \supseteq I+J$

Note $f^{-1}(K_i/I \times K_i/J) = K_i$ since f injective.

Now, the chain $K_1/I \subseteq K_2/I \subseteq \dots$ in R/I terminates since R/I noetherian.
Similarly ~~the chain~~ $K_1/J \subseteq K_2/J \subseteq \dots$ in R/J terminates.

ie. $\exists\, n > 0$ st. $K_i/I = K_n/I \quad \forall i \geq n$
$\exists\, m > 0$ st $K_j/J = K_m/J \quad \forall j \geq m$

Let $N = \max(n, m)$

$\therefore$ $K_i/I \times K_i/J = K_N/I \times K_N/J \quad \forall i \geq N$, so the chain in $R/I \times R/J$ terminates. Thus, its preimage

$K_1 \subseteq K_2 \subseteq \dots$ in R terminates //.

$\frac{3}{5}$

ARGHH! ?
you should have checked your file!

(a)
(ii) By Cor 1.18, ~~...~~ maximal ideals of R
are in correspondence with maximal i
$\operatorname{im}\varphi = S$, via the maps ~~...~~

$$\left\{\begin{array}{c}\text{maximal ideals}\\ \text{of } R \text{ containing}\\ \ker\varphi\end{array}\right\} \quad \begin{array}{c} I \longmapsto \varphi(\\ \varphi^{-1}(J) \longleftarrow J \end{array}$$

Then, for all maximal ideals $J \triangleleft$
a maximal ideal $\varphi^{-1}(J) \triangleleft R$ st.

$$\varphi(\operatorname{Jac}(R)) = \varphi\Big(\bigcap_{I \in \operatorname{MaxSpec}(R)} I\Big) \subseteq \bigcap_{I \in \operatorname{MaxSpec}(R)} \varphi(I)$$

$$\subseteq \bigcap_{J \in \operatorname{MaxSpec}(S)} \varphi(\varphi^{-1}(J)) = \operatorname{Jac}(S).$$

?

$\frac{0}{5}$

Thm 1.10

(b) Let $R = k[x]$ for some infinite field k and note $(x-a)$ is maximal for all $a \in k$ $\boxed{R/(x-a) \cong k \text{ a field}}$ ok

$\therefore (0) \subseteq Jac(R) \subseteq \bigcap_{a \in k} (x-a) = (0)$

You must use the notation of this course! -2

infinite intersection of coprime ideals is infinite product $= 0$. needed?

Let $S = k[x]/(x^2)$ and φ the quotient map. From Ex 1.5, S is local with maximal ideal (x).

$\therefore Jac(S) = (x)$

So $\varphi(Jac(R)) = \varphi((0)) = (0) \subsetneq (x) = Jac(S)$.

$\frac{2}{4}$

(iii) Let $0 \neq m \in M$. $0 \subsetneq \langle m \rangle \leq M$ (submodule). Since M is simple, we must have $\langle m \rangle = M$.

Thus the homomorphism $\varphi : R \to M$, $r \mapsto r \cdot m$ is a surjection

and $\ker\varphi = \{r \mid rx = 0 \ \forall x \in M\} = Ann(M) = I$ ok. "$Ann(m)$".

$\therefore$ By isomorphism theorem (2.10), $M \cong R/I$

~~Now suppose $I \subseteq J$ is a inclusion of ideals of~~ R

~~By the correspondence theorem~~

We will show that there is a correspondence (as R-modules)

$$\left\{ \begin{array}{c} \text{Submodules of } R \\ \text{containing } I \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Submodules of} \\ R/I \end{array} \right\}$$ ✓

Since R-submodules of R are precisely
if we show that R-submodules ~~are ide~~
of R/I are ideals of R/I, th
we done by the correspondence theorem
(

But in fact R and R/I actions on R/I a
equal so R-modules are the same as
(when looking at submodules of R/I)

Thus, an ideal $J \supsetneq I$ corresponds to a
$N \supsetneq \{0\}$. ~~and~~ Since M is simple, $N=M$ an

ie. I is maximal.

//

0/8

Again missing a part of the page... I can't give any mark.

Q4 $\frac{5}{22}$

~~##~~ Commutative Algebra Q5 — Ibraheem Sajjad

(i)(a) $\left(\sqrt{\mathrm{Ann}(M/IM)} \subseteq \sqrt{\mathrm{Ann}(M)+I}\right)$

Suppose $a \in \sqrt{\mathrm{Ann}(M/IM)}$, ie. $\exists n>0$ st. $a^n m \in IM$ $\forall m \in M$. By Prop 2.21, the map $\varphi: M \to M$, $m \mapsto a^n m$

satisfies some equation $\varphi^N + x_1\varphi^{N-1} + \dots + x_{N-1}\varphi + x_N = 0$ where all $x_i \in I$.

So we have $(a^{Nn} + x_1 a^{(N-1)n} + \dots + x_N)m = 0$ $\forall m \in M$. ie. $a^{Nn} + x_1 a^{(N-1)n} + \dots + x_N \in \mathrm{Ann}(M)$

Also $x_1 a^{(N-1)n} + \dots + x_N \in I$, so $a^{Nn} \in \mathrm{Ann}(M)+I$
$\therefore a \in \sqrt{\mathrm{Ann}(M)+I}$ //

$\left(\sqrt{\mathrm{Ann}(M)+I} \subseteq \sqrt{\mathrm{Ann}(M/IM)}\right)$

Suppose $a \in \sqrt{\mathrm{Ann}(M)+I}$. Then $\exists n>0$ st $a^n = b+c$ where $b \in \mathrm{Ann}(M)$, $c \in I$.

$\forall m+IM \in M/IM$, $a^n m + IM = bm + cm + IM = cm + IM = 0 + IM$ (since $cm \in IM$).

$\therefore a \in \sqrt{\mathrm{Ann}(M/IM)}$

5/5

(b) Since $M+N$ is finitely generated, so is the quotient $\frac{M+N}{N}$ (just take the image of the generators).

Let $\varphi: M+N \to \frac{M}{M\cap N}$ (note $M \cap N$ is a submodule)
$m+n \mapsto m + M\cap N$ ($m \in M$, $n \in N$)

is this function well defined and an R-mod. homomorphism?

φ is surjective since $\forall m + M\cap N \in M/M\cap N$, $m \in M \subseteq M+N$; $\varphi(m) = m + M\cap N$

Suppose $m+n \xrightarrow{\varphi} 0$. Then $m \in M\cap N$

−2

R-mod. hom?

So $\ker\varphi = N + M\cap N = N$.

Note φ is well-defined since if $m+n = m'+n'$ $(m,m'\in M$, $n,n'\in N)$

then $m-m' = n'-n$

i.e. $m-m' \in M\cap N$

So $\varphi(m+n-(m'+n')) = m-m'+M\cap N = 0$

$\therefore \varphi(m+n) = \varphi(m'+n')$.

So $\dfrac{M}{M\cap N}$ is finitely generated, say by

$x_1+M\cap N, \dots, x_n+M\cap N$.

Also $M\cap N$ is f.g., say by $y_1, \dots, y_m$.

Let $m\in M$. Then $m+M\cap N = \sum_{i=1}^{n} r_i x_i + M\cap N$

so $(m-\sum_1^n r_i x_i) + M\cap N = 0$

$\therefore m - \sum_1^n r_i x_i \in M\cap N$

i.e. $m-\sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{p} s_i y_i$ i.e. $m = \sum_{i=1}^{n} r_i x_i + \sum_{i=1}^{p} s_i y_i$.

i.e. $\{x_i\}_{i=1}^{n} \cup \{y_i\}_{i=1}^{p}$ generate M.

$\frac{2}{4}$

By symmetry, N is also f.g.

(ii) Assuming $\pi(M)$ means $M/\mathrm{Rad}(R)M$. yes, it does!

Define $\bar{\pi} : M \to M/\mathrm{Rad}(R)M$, the quotient map. ✓

Let N be the submodule of M generated by X.

We have $\bar{\pi}(N) = M/\mathrm{Rad}(R)M$, so $\bar{\pi}^{-1} \circ \bar{\pi}(N) = M$ ✓

But also $\bar{\pi}^{-1} \circ \bar{\pi}(N) = N + \mathrm{Rad}(R)M$, so $M = N + \mathrm{Rad}(R)M$. ✓

By Nakayama's lemma corollary (Cor 2.26), $M = N$. ✓

(by assumption)

If $m + \mathrm{Rad}(R)M = \sum \bar{\pi}(x_i) \cdot r_i$ for $x_i \in X$,

then $\bar{\pi}(\sum r_i x_i) = m + \mathrm{Rad}(R)M$

so $M/\mathrm{Rad}(R)M \subseteq \bar{\pi}(N) \subseteq M/\mathrm{Rad}(R)M$ ✓

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(iii) <u>LEMMA</u>

Suppose

$$\begin{tikzcd} Q \arrow[r, "f"] & M \arrow[d, "\alpha"] \\ N \arrow[r, "d"] & L \end{tikzcd}$$

are maps of finitely generated R-modules for R Noetherian with Q projective and $\mathrm{im}(\alpha \circ f) \subseteq \mathrm{im}(d)$

then there is a free R-module P and maps g, β completing this diagram to

$$\begin{tikzcd} P \arrow[r, "g"] & Q \arrow[r, "f"] \arrow[d, "\beta"] & M \arrow[d, "\alpha"] \\ & N \arrow[r, "d"] & L \end{tikzcd}$$

such that the square commutes and the top sequence is exact (at Q). ie. $\mathrm{im}(g) = \ker(f)$ OK.

Indeed, note d is surjective onto its image and $\alpha \circ f : Q \to \mathrm{im}(d)$ is well-defined. by assumption and free modules are projective.

By the projective property $\beta : Q \to N$ exists making the square commute

Now, $\ker(f)$ is a submodule of Q, which is a finitely generated module over a Noetherian ring, so is Noetherian (Thm 2.44). ✓ Thus, $\ker(f)$ is finitely generated (Thm 2.46), say by $\{x_1, \dots, x_m\}$.

Define $P = R^m$ and $g : P \longrightarrow Q$
$e_i \longmapsto x_i$, where ~~e_i is the~~ $\{e_1, \dots, e_m\}$ is a free set of generators for P.
(so g is well-defined).

By construction, $\operatorname{im}(g) = \ker(f)$, so we are done. // LEMMA ✓

Now we can proceed by induction. ~~on n.~~ The base case has

$$\begin{tikzcd} Q_0 \arrow[r] & M \arrow[d] \\ N_0 \arrow[r] & M \end{tikzcd}$$

with Q_0 projective by assumption and because ~~...~~, $N_0 \to M$ is surjective, the composition $Q_0 \to M \to M$ has image lying in the image of $N_0 \to M$,

So we obtain

$$\begin{tikzcd} Q_1 \arrow[r] & Q_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ & N_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

by following our ~~the~~ lemma.

OK.

and ~~[illegible]~~ is n (to be precise, we'd need to define ~~[illegible]~~ $Q_{-1} = N_{-1} = M$) — OK

Now, the inductive step: Suppose we have

$$\begin{tikzcd} Q_i \arrow[r] & Q_{i-1} \arrow[r] \arrow[d, "d"] & Q_{i-2} \arrow[r] \arrow[d, "d"] & \cdots \\ & N_{i-1} \arrow[r] & N_{i-2} \arrow[r] & \cdots \end{tikzcd}$$

as given by ~~[illegible]~~ our lemma.

Note since Q_i is free it is projective (Lem 2.24) ✓

Fill in $N_i \to N_{i-1}$ from the assumption in question.
$\mathrm{im}(N_i \to N_{i-1}) = \ker(N_{i-1} \to N_{i-2})$ by exactness.

and $\mathrm{im}(Q_i \to Q_{i-1}) = \ker(Q_{i-1} \to Q_{i-2})$, so

$Q_i \to Q_{i-1} \to Q_{i-2}$ is the zero map

and hence so is

$$\begin{tikzcd} Q_i \arrow[r] & Q_{i-1} \arrow[d, "d"] \\ & N_{i-1} \arrow[r] & N_{i-2} \end{tikzcd}$$

by commutativity. OK Thus

$$\begin{tikzcd} Q_i \arrow[r] & Q_{i-1} \arrow[d] \\ & N_{i-1} \end{tikzcd}$$

has image in $\ker(N_{i-1} \to N_{i-2})$ ✓ $= \mathrm{im}(N_i \to N_{i-1})$,

Now we meet the conditions of our lemma and we can apply it again. OK.

Q5

The induction ends with

$$\begin{tikzcd} Q_{n+1} \arrow[r] & Q_n \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ & N_n \arrow[r] & \cdots \arrow[r] & N_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

✓ 10/10

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