Model answers to Week 08 review worksheet — exercises for §8

E8.1 **(the $\hbar$-adic algebra $U_\hbar = U_\hbar(\mathfrak{sl}_2)$)** The algebra $U_\hbar := U_\hbar(\mathfrak{sl}_2)$ is defined, as an $\hbar$-adic algebra over the ring $\mathbb{C}[[\hbar]]$ of formal power series in $\hbar$, by the presentation

$$U_\hbar = \mathbb{C}\langle E,\, H,\, F \mid HE - EH = 2E,\, HF - FH = -2F,\, EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\rangle$$

where q denotes the formal power series $e^\hbar \in \mathbb{C}[[\hbar]]$, K denotes $e^{\hbar H}$ and K^{-1} denotes $e^{-\hbar H}$.

(a) Which facts are used to deduce the following relations? Remind yourself of a proof of those facts, or try to prove them.

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2 EK, \quad KF = q^{-2}FK$$

(b) Expand $\frac{K-K^{-1}}{q-q^{-1}}$, as a formal power series in $\hbar$, to $O(\hbar^7)$ (that is, ignoring all terms which contain $\hbar^7$ or higher power of $\hbar$).

Answer to E8.1. (a) That $KK^{-1} = 1$, equivalently $e^{\hbar H}e^{-\hbar H} = 1$, follows from the exponential law: if a, b commute then $e^{\hbar a}e^{\hbar b} = e^{\hbar(a+b)}$, see the answer to E7.2. Clearly H and $-H$ commute so $e^{\hbar H}e^{-\hbar H} = e^{\hbar(H-H)} = e^0 = 1$.

Now $KE = q^2EK$ is equivalent to $KEK^{-1} = q^2E$. To show this, we need the following

> **Lemma on** ad **and** Ad**.** Let A be an associative algebra, $a \in A$. Define the operator $\mathrm{ad}_a \in \mathrm{End}(A)$ by $\mathrm{ad}_a(x) = ax - xa$. If $b \in A$ is invertible, define $\mathrm{Ad}_b \in \mathrm{End}(A)$ by $\mathrm{Ad}_b(x) = bxb^{-1}$, for all $x \in A$.
>
> Assume that A is an $\hbar$-adic algebra. Then, in $\mathrm{End}(A)$,
>
> $$\exp(\hbar\,\mathrm{ad}_a) = \mathrm{Ad}_{\exp(\hbar a)}\,.$$
>
> **Proof of the lemma.** Given $a \in A$, we consider two more elements of $\mathrm{End}(A)$: λ_a, the operator of left multiplication by a, $\lambda_a(x) = ax$, and ρ_a, the right multiplication by a, $\rho_a(x) = xa$ for all $x \in A$. Note that $\hbar\,\mathrm{ad}_a = \lambda_{\hbar a} + \rho_{-\hbar a}$. Crucially, the operators $\lambda_{\hbar a}$ and $\rho_{-\hbar a}$ commute in $\mathrm{End}(A)$, so by the exponential law
>
> $$\exp(\hbar\,\mathrm{ad}\,a) = \exp(\lambda_{\hbar a} + \rho_{-\hbar a}) = \exp(\lambda_{\hbar a})\exp(\rho_{-\hbar a}).$$
>
> Clearly $\exp(\lambda_{\hbar a}) = \lambda_{\exp(\hbar a)}$ and $\exp(\rho_{-\hbar a}) = \rho_{\exp(-\hbar a)}$, and the product of these two operators sends $x \in A$ to $\exp(\hbar a)x\exp(-\hbar a)$, which is $\mathrm{Ad}_{\exp\hbar a}\,x$. The Lemma is proved.

From the Lemma,

$$KEK^{-1} = \mathrm{Ad}_{\exp\hbar H}\,E = \exp(\hbar\,\mathrm{ad}_H)E.$$

As E is an eigenvector for $\hbar\,\mathrm{ad}_H$ with eigenvalue $2\hbar$, E is also an eigenvector for $\exp(\hbar\,\mathrm{ad}_H)$ with eigenvalue $e^{2\hbar} = q^2$. Thus, $KEK^{-1} = q^2K$.

The proof of $KF = q^{-2}FK$ is the same, using $\hbar\,\mathrm{ad}_H\,F = -2\hbar F$.

(b) We need to expand $\dfrac{e^{\hbar H} - e^{-\hbar H}}{e^\hbar - e^{-\hbar}}$. Note that, both in the numerator and in the denominator, terms with even powers of $\hbar$ cancel:

$$\frac{e^{\hbar H} - e^{-\hbar H}}{e^\hbar - e^{-\hbar}} = \frac{2\hbar H + 2(\hbar^3/3!)H^3 + 2(\hbar^5/5!)H^5 + 2(\hbar^7/7!)H^7 + O(\hbar^9)}{2\hbar + 2(\hbar^3/3!) + 2(\hbar^5/5!) + 2(\hbar^7/7!) + O(\hbar^9)}$$

We had to expand both numerator and denominator to $O(\hbar^8)$ because we need to cancel the factor of $2\hbar$ from both. In fact, there are no terms of order $\hbar^8$ so we got an expansion to $O(\hbar^9)$. After cancellation, we get

$$\frac{H + (\hbar^2/3!)H^3 + (\hbar^4/5!)H^5 + (\hbar^6/7!)H^7 + O(\hbar^8)}{1 + \hbar^2/3! + \hbar^4/5! + \hbar^6/7! + O(\hbar^8)}.$$

We need to invert the formal power series in the denominator. Note that, for any formal power series $g(\hbar) \in \mathbb{C}[[\hbar]]$, the inverse of $1+\hbar g(\hbar)$ in $\mathbb{C}[[\hbar]]$ is always well defined and is calculated according to the geometric series sum formula:

$$(1+\hbar g(\hbar))^{-1} = 1 - \hbar g(\hbar) + \hbar^2 g(\hbar)^2 - \hbar^3 g(\hbar)^3 + \cdots = \sum_{k=0}^{\infty} (-\hbar g(\hbar))^k.$$

Putting $\hbar g(\hbar) = \frac{\hbar^2}{3!} + \frac{\hbar^4}{5!} + \frac{\hbar^6}{7!} + O(\hbar^8)$ gives us

$$\begin{aligned}
\Big(1+\frac{\hbar^2}{3!}+\frac{\hbar^4}{5!}+\frac{\hbar^6}{7!}+O(\hbar^8)\Big)^{-1} &= 1 - \Big(\frac{\hbar^2}{6}+\frac{\hbar^4}{120}+\frac{\hbar^6}{5040}+O(\hbar^8)\Big) \\
&\quad + \Big(\frac{\hbar^4}{6^2} + 2\frac{\hbar^6}{6\times 120} + O(\hbar^8)\Big) \\
&\quad - \Big(\frac{\hbar^6}{6^3} + O(\hbar^8)\Big) + O(\hbar^8) \\
&= 1 - \frac{\hbar^2}{6} + \Big(-\frac{\hbar^4}{120}+\frac{\hbar^4}{36}\Big) + \Big(-\frac{\hbar^6}{5040}+\frac{\hbar^6}{360}-\frac{\hbar^6}{216}\Big) + O(\hbar^8) \\
&= 1 - \frac{\hbar^2}{6} + \frac{7\hbar^4}{360} - \frac{31\hbar^6}{15120} + O(\hbar^8).
\end{aligned}$$

Thus, $\dfrac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}$ expands as

$$\begin{aligned}
&\Big(H+\frac{\hbar^2}{6}H^3+\frac{\hbar^4}{120}H^5+\frac{\hbar^6}{5040}H^7\Big)\Big(1-\frac{\hbar^2}{6}+\frac{7\hbar^4}{360}-\frac{31\hbar^6}{15120}\Big)+O(\hbar^8) \\
&\quad = H + \frac{\hbar^2}{6}(-H+H^3) + \frac{\hbar^4}{360}(7H-10H^3+3H^5) + \frac{\hbar^6}{15120}(-31H+49H^3-21H^5+3H^7) + O(\hbar^8).
\end{aligned}$$

Under the specialisation $\hbar = 0$, the whole expression collapses to just H which is the classical commutator bracket between E and F, as expected.

E8.2 Recall that $U_\hbar$ acts on the 2-dimensional space V with basis $\{x, y\}$ (more precisely, on $V[[\hbar]]$) via

$$E \triangleright x = 0, \quad E \triangleright y = x, \quad F \triangleright x = y, \quad F \triangleright y = 0, \quad H \triangleright x = x, \quad H \triangleright y = -y,$$

and that this action extends to a covariant action of $U_\hbar$ on the quantum plane $\mathbb{C}_\hbar[x,y] = \mathbb{C}[[\hbar]]\langle x, y \mid yx = e^\hbar xy\rangle$. Calculate the action of E, H and F on a standard monomial $x^a y^b$ in $\mathbb{C}_\hbar[x,y]$, expressing $E \triangleright (x^a y^b)$ etc in terms of standard monomials in $\mathbb{C}_\hbar[x,y]$. You will need (*why?*) the coproduct on $U_\hbar$, given by

$$\Delta H = 1 \otimes H + H \otimes 1, \quad \Delta E = E \otimes K + 1 \otimes E, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F.$$

Answer to E8.2. [partial answer] For brevity, write the algebra $\mathbb{C}_\hbar[x,y] = \mathbb{C}[[\hbar]]\langle x, y \mid yx = qxy\rangle$ (the quantum plane) as A_q.

The coproduct $\Delta h = h_{(1)} \otimes h_{(2)}$ on $U_\hbar$ is needed because it is part of the definition of a covariant action: $h \triangleright (uv) = (h_{(1)} \triangleright u)(h_{(2)} \triangleright v)$ for all $u, v \in A_q$. Likewise, the counit is needed to ensure $h \triangleright 1_{A_q} = \epsilon(h) 1_{A_q}$.

The formula for $H \triangleright (x^a y^b)$: recall that H is primitive in $U_\hbar$, $\Delta H = H \otimes 1 + 1 \otimes H$. By E4.7, this means that the action of H obeys the Leibniz law,

$$H \triangleright (uv) = (H \triangleright u)v + u(H \triangleright v), \quad \forall u, v \in A_q.$$

It follows that

$$H \triangleright (x^{a+1}) = H \triangleright (x^a x) = (H \triangleright x^a)x + x^a x = (H \triangleright x^a)x + x^{a+1}$$

for all $a \geq 0$, as $H \triangleright x = x$. So $H \triangleright (x^2) = (H \triangleright x)x + x^2 = xx + x^2 = 2x^2$, $H \triangleright (x^3) = (2x^2)x + x^3 = 3x^3$, and so on. We can see that

$$H \triangleright (x^a) = a x^a, \quad \forall a \geq 0,$$

which is proved by induction using $a = 0$, $x^0 = 1$, $H \triangleright 1 = \epsilon(H)1 = 0$ and $a = 1$, $H \triangleright x = x$ as base cases, and the previous displayed formula as the inductive step.

Using $H \triangleright y = -y$ and considering the $-H$ which is also primitive, one proves that $H \triangleright (y^b) = -by^b$ for all $b \geq 0$. Hence, by the Leibniz law,

$$H \triangleright (x^a y^b) = (H \triangleright x^a) y^b + x^a (H \triangleright y^b) = a x^a y^b - b x^a y^b = (a-b) x^a y^b,$$

for all $a, b \geq 0$. That is, $\{x^a y^b\}_{a,b \geq 0}$ is an ($\hbar$-adic) H-eigenbasis of A_q.

The formula for $E \triangleright (x^a y^b)$: we use $\Delta E = 1 \otimes E + E \otimes K$. Recall that $K \triangleright x = e^{\hbar H} \triangleright x$; as x is an eigenvector for H with eigenvalue 1, we have $K \triangleright x = e^{\hbar 1} x = qx$. Similarly, $K \triangleright y = q^{-1} y$.

First, we note that, for all $u \in A_q$, $E \triangleright (xu) = x(E \triangleright u) + (E \triangleright x)(K \triangleright u) = x(E \triangleright u)$ since $E \triangleright x = 0$. In particular, $E \triangleright (x^a) = 0$ for all $a \geq 0$, and it follows that

$$E \triangleright (x^a y^b) = x^a (E \triangleright y^b).$$

To find the formula for $E \triangleright y^b$, we consider small b first. We have $E \triangleright y = x$ and

$$E \triangleright (y^2) = y(E \triangleright y) + (E \triangleright y)(K \triangleright y) = yx + x(q^{-1} y) = (q + q^{-1}) xy$$

where we have used the defining relation $yx = qxy$ of A_q.

Exercise left for next week: calculate $E \triangleright (y^3)$ and $E \triangleright (y^4)$, then write down the general formula for $E \triangleright (y^b)$ and prove it by induction. Hint: $E \triangleright (y^b)$ will be proportional to xy^{b-1}.

Some of the exercises which appeared on the draft version of the current worksheet have been moved to the Week 09 review worksheet.

Part B. Extra exercises

E8.3 (the antipode of $U_q(\mathfrak{sl}_2)$) Deduce or review the formulas for $S(E)$ and $S(F)$ in $U_q(\mathfrak{sl}_2)$. Calculate $S(EF)$ and $S(EKF)$, expressing the answer as a linear combination of monomials from the PBW-type basis of $U_q(\mathfrak{sl}_2)$. Show that $S^2 \neq \mathrm{id}$ on $U_q(\mathfrak{sl}_2)$.

Answer to E8.3. Let us remind ourselves how the only possible formula for $S(E)$ is found. We have $\Delta E = 1 \otimes E + E \otimes K$. Then by the antipode law, $0 = \epsilon(E) = 1S(E) + ES(K)$. Hence $S(E) = -ES(K) = -EK^{-1}$, remembering that K is grouplike so $S(K) = K^{-1}$. Similarly, $S(F) = -KF$.

We use the antihomomorphism property of the antipode S then the relations $KF = q^{-2} FK$, $KE = q^2 EK$ from E8.1 to compute

$$S(EF) = S(F)S(E) = (-KF)(-EK^{-1}) = q^{-2} FKEK^{-1} = FEKK^{-1} = FE.$$

Yet in a PBW monomial, according to our convention, E must precede F. So we use the relation $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$ to conclude:

$$S(EF) = EF - \frac{K - K^{-1}}{q - q^{-1}}.$$

Similarly, using $K^{-1} E = q^{-2} EK^{-1}$,

$$S(EKF) = q^{-2} S(EFK) = q^{-2} S(K) S(EF) = q^{-2} K^{-1} (EF - \tfrac{K - K^{-1}}{q - q^{-1}}) = q^{-4} EK^{-1} F - \tfrac{q^{-2}}{q - q^{-1}} (1 - K^{-2}).$$

To show $S^2 \neq \mathrm{id}$, we can for example calculate

$$S^2(E) = S(-EK^{-1}) = -S(K^{-1}) S(E) = -K(-EK^{-1}) = KEK^{-1} = q^2 E.$$

Since $E \neq 0$ in $U_q(\mathfrak{sl}_2)$ (E is a vector from the PBW basis) and $q^2 \neq 1$ (q^2 is the formal series $e^{2\hbar}$), we have $q^2 E \neq E$.

Extra exercise: show that $S^2(EF) = EF$.

E8.4 (the PBW-type theorem for $U_q(\mathfrak{sl}_2)$) Review the formulae which describe the action of the generators $E, K^{\pm 1}, F$ of the $\mathbb{C}$-algebra U_q on the basis $\{x^m y^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$ of the quantum plane $A_q := \mathbb{C}_\hbar[x, y]$. Use this action to show that the PBW-like monomials $E^m K^n F^p$, where $m, p \geq 0$, $n \in \mathbb{Z}$, are linearly independent in U_q.

Hint: assume that there is a non-trivial linear combination L of the PBW-like monomials is zero in U_q. Take the least p such that a monomial of the form $E^m K^n F^p$ appears in L with non-zero coefficient. Act by L on $x^p y^b$; note that all monomials $E^{m'} K^{n'} F^{p'}$ with $p' > p$ act on $x^p y^b$ by zero. You are left to choose b so as to get a contradiction.

Answer to E8.4. Suppose that a finite linear combination L of PBW-like monomials $E^m K^n F^{p'}$ is equal to zero in U_q, and so acts on the quantum plane A_q by 0. Gathering the terms with the same m and p', we can write $L = \sum_{m,p'} E^m f_{m,p'}(K) F^{p'}$ where $f_m(K)$ is a Laurent polynomial, i.e., a $\mathbb{C}$-linear combination of powers of K and K^{-1}.

Assume for contradiction that not all $f_{m,p'}$ are 0, and fix the **least** $p \in \mathbb{Z}_{\geq 0}$ with the property that $f_{m,p} \neq 0$ for some $m \in \mathbb{Z}_{\geq 0}$.

We want to act by L on a monomial $x^p y^b$ which is an element of the monomial basis of the quantum plane A_q, As shown in the course,

$$(*) \qquad E \triangleright (x^a y^b) = [b]_q x^{a+1} y^{b-1}, \quad F \triangleright (x^a y^b) = [a]_q x^{a-1} y^{b+1}, \quad K \triangleright (x^a y^b) = q^{a-b} x^a y^b,$$

and so, iterating the action of F, we obtain

$$(\dagger) \qquad F^p \triangleright (x^p y^b) = [1]_q [2]_q \dots [p]_q y^{b+p} = [p]_q! y^{b+p}, \qquad F^{p'} \triangleright (x^p y^b) = 0 \text{ if } p' > p.$$

Therefore,

$$\begin{aligned}
0 &= \sum_{m,p'} E^m f_{m,p'}(K) F^{p'} \triangleright (x^p y^b) \\
&= \sum_m E^m f_{m,p}(K) F^p \triangleright (x^p y^b) && \text{by } (\dagger),\ F^{p'} \text{ with } p' > p \text{ acts on } x^p y^b \text{ by } 0 \\
&= [p]_q! \sum_m E^m f_{m,p}(K) \triangleright (y^{b+p}) && \text{by } (\dagger) \\
&= [p]_q! \sum_m E^m \triangleright (f_{m,p}(q^{-(b+p)}) y^{b+p}) && \text{by the last formula in } (*) \\
&= [p]_q! \sum_m \lambda_m f_{m,p}(q^{-(b+p)}) y^{b+p-m}
\end{aligned}$$

where $\lambda_m = [b+p]_q [b+p-1]_q \dots [b+p-m+1]_q$, by the first formula in (*).

We choose b such that $b + p$ is greater than all m for which $f_{m,p} \neq 0$, and also that $f_{m,p}(q^{-(b+p)}) \neq 0$ for some m: the latter is possible because q is not a root of unity and so $q^{-(b+p)}$ runs over an infinite set of complex numbers, yet a non-zero Laurent polynomial has finitely many roots. We note that neither $[p]_q!$ nor λ_m are zero: indeed, a q-integer $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$, vanishes only if $q^{2k} = 1$, but q is not a root of unity.

Thus, 0 is written as a linear combination of powers of y where not all coefficients are zero. Yet powers of y are linearly independent in the quantum plane A_q. We have a contradiction, hence our assumption of linear dependency between PBW-like monomials in U_q was false.