§ 4

Hopf algebras: definition, standard examples. Module algebras

A vector space with two structures — algebra and coalgebra — is **not** a Hopf algebra. The algebra and coalgebra structures need to be compatible, as we explain in this chapter, to form a bialgebra. Then, just like a group is a semigroup with an inverse operation, a Hopf algebra is defined to be a bialgebra with an antipode map. Hopf algebras can act on tensor products of modules and on dual spaces to modules; in this chapter we will see how they act on algebras, which will motivate the idea of a quantum group.

1 Bialgebra: definition. Algebra of a monoid is a bialgebra. Module algebra

4.1 Definition (bialgebra) A **bialgebra** is a vector space H is equipped with linear maps

- $m\colon H \otimes H \to H$, the multiplication map;
- $\eta\colon \mathbb{C} \to H$, the unit map;
- $\Delta\colon H \to H \otimes H$, the coproduct map;
- $\epsilon\colon H \to \mathbb{C}$, the counit map,

such that (H, m, η) is an algebra, (H, Δ, ϵ) is a coalgebra, and the algebra and coalgebra structures are compatible in the following way:

- Δ is an algebra homomorphism from H to $H \otimes H$, where the algebra structure on $H \otimes H$ is the tensor product of algebras;
- $\epsilon\colon H \to \mathbb{C}$ is an algebra homomorphism.

4.2 Remark (notation for multiplication and coproduct) When working with a bialgebra, we combine juxtaposition notation for multiplication, $m(a \otimes b) = ab$ for $a, b \in H$, with Sweedler notation for coproduct, $\Delta a = a_{(1)} \otimes a_{(2)}$ for $a \in H$.

4.3 Remark (algebra-coalgebra symmetry of the compatibility condition) It may seem that the compatibility condition in the definition of a bialgebra singles out Δ and ϵ, forcing them to be homomorphisms with respect to the algebra structure. However, this is just a result of choices made in earlier chapters. We claim that the compatibility condition can be equivalently written as

- m is a coalgebra homomorphism from $H \otimes H$ to H;
- $\eta\colon \mathbb{C} \to H$ is a coalgebra homomorphism,

had we formally defined coalgebra homomorphism and tensor product of coalgebras. The symmetry is illustrated by drawing a diagram in the language of graphical tensor calculus, see E4.4.

4.4 Example (algebra of a monoid is a bialgebra) The algebra $\mathbb{C}\mathcal{B}$ of a monoid $\mathcal{B}$ is a bialgebra, where the coalgebra structure is introduced by declaring elements of $\mathcal{B}$ grouplike. This means that $\Delta g = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in \mathcal{B}$, extended by linearity to $\mathbb{C}\mathcal{B}$. From Example 3.6 we know that this makes $\mathbb{C}\mathcal{B}$ a coalgebra.

To show that $\mathbb{C}\mathcal{B}$ is a bialgebra, we need to check that Δ and ϵ are algebra homomorphisms.

We have $\Delta(ab) = (\Delta a)(\Delta b)$ when $a, b \in \mathcal{B}$ as both sides are $ab \otimes ab$, and by bilinearity of both sides in a, b the same is true for arbitrary $a, b \in \mathbb{C}\mathcal{B}$. We also have $\Delta 1_{\mathbb{C}\mathcal{B}} = \Delta e = e \otimes e = 1_{\mathbb{C}\mathcal{B}\otimes\mathbb{C}\mathcal{B}}$ because the identity element e lies in $\mathcal{B}$ and so is grouplike. Thus, Δ is an algebra homomorphism; ϵ is left as an exercise.

4.5 Definition (covariant action, module algebra) Let H be a bialgebra and A be an algebra. An action $\triangleright$ of H on the (underlying vector space of) A is **covariant** if it is compatible with the multiplication and identity element of A in the following way:

1. $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$ for all $h \in H$, $a, b \in A$;
2. $h \triangleright 1_A = \epsilon(h)1_A$ for all $h \in H$.

An algebra A equipped with a covariant action of H is an **H-module algebra.**

In this course we focus on a class of bialgebras with extra structure which we now define.

Hopf algebra: definition. Group algebras as Hopf algebras

4.6 Definition (Hopf algebra) A **Hopf algebra** is a bialgebra H equipped with a linear map $S\colon H \to H$, called the **antipode map,** which satisfies the antipode law: for all $h \in H$,

$$S(h_{(1)})h_{(2)} = \epsilon(h)1_H = h_{(1)}S(h_{(2)}).$$

The antipode law is expressed by saying that the following diagram commutes:

$$\begin{tikzcd}
& H \arrow[dl, "\Delta"'] \arrow[dd, "\epsilon"] \arrow[dr, "\Delta"] & \\
H\otimes H \arrow[dd, "S\otimes\mathrm{id}_H"'] & & H\otimes H \arrow[dd, "\mathrm{id}_H\otimes S"] \\
& \mathbb{C} \arrow[dd, "\eta"] & \\
H\otimes H \arrow[dr, "m"'] & & H\otimes H \arrow[dl, "m"] \\
& H &
\end{tikzcd}$$

4.7 Remark (antipode, if exists, is unique) Not every bialgebra can be equipped with an antipode map, but if it can, there is only one way to do that: see E4.6.

The most standard example of a Hopf algebra is as follows.

4.8 Example (a group algebra is a Hopf algebra) Let G be a group. The group algebra $\mathbb{C}G$, already a bialgebra by Example 4.4, becomes a Hopf algebra by setting $S(g) = g^{-1}$ for all $g \in G$. Here g^{-1} denotes the inverse in the group G.

It is enough to verify the antipode law $S(h_{(1)})h_{(2)} = \epsilon(h)1_H = h_{(1)}S(h_{(2)})$ only for $h = g \in G$, because all parts are linear in h, and G is a basis of $\mathbb{C}G$. We have $g^{-1}g = e = gg^{-1}$ which is true by definition of g^{-1}.

4.9 Remark (a covariant action of a Hopf algebra on an algebra) A covariant action of a Hopf algebra H on an algebra A is the same as Definition 4.5 — one ignores the antipode and considers H as a bialgebra.

4.10 Example (covariant actions of a group algebra on an algebra) Let $H = \mathbb{C}G$. What are covariant actions of $\mathbb{C}G$ on an algebra A over $\mathbb{C}$?

The action of a grouplike element $g \in G$ on A must satisfy

$$g \triangleright (ab) = (g \triangleright a)(g \triangleright b), \quad g \triangleright 1_A = 1_A.$$

This means that the map $a \mapsto g \triangleright a$ is an algebra homomorphism from A to A. Moreover, this homomorphism has inverse $a \mapsto g^{-1} \triangleright a$. Invertible homomorphisms $A \to A$ are called automorphisms of A. Thus, A is a $\mathbb{C}G$-module algebra, if and only if the group G **acts on A by algebra automorphisms.**

Antimultiplicativity of the antipode

The definition of a Hopf algebra requires Δ and ϵ to be algebra homomorphisms, but what about the antipode S? The example of a group algebra of a non-abelian group G convinces us that S is not, in general, a homomorphism, because $S(gh) = (gh)^{-1} = h^{-1}g^{-1}$ if $g, h \in G$, and this is not the same as $g^{-1}h^{-1}$. We extend this observation to arbitrary Hopf algebras:

4.11 Proposition *The antipode $S\colon H \to H$ is an **algebra antihomomorphism,** meaning that*

$$S(ab) = S(b)S(a) \quad \textit{for all } a, b \in H, \quad S(1_H) = 1_H.$$

Proof. The coassociative law for the coproduct implies that, in Sweedler notation,

$$a_{(1)_{(1)}} \otimes a_{(1)_{(2)}} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)_{(1)}} \otimes a_{(2)_{(2)}}, \quad b_{(1)_{(1)}} \otimes b_{(1)_{(2)}} \otimes b_{(2)} = b_{(1)} \otimes b_{(2)_{(1)}} \otimes b_{(2)_{(2)}}.$$

We tensor-multiply these two equations and change the order of tensor factors in the same way on each side:

$$a_{(1)_{(1)}} \otimes b_{(1)_{(1)}} \otimes a_{(1)_{(2)}} \otimes b_{(1)_{(2)}} \otimes b_{(2)} \otimes a_{(2)} = a_{(1)} \otimes b_{(1)} \otimes a_{(2)_{(1)}} \otimes b_{(2)_{(1)}} \otimes b_{(2)_{(2)}} \otimes a_{(2)_{(2)}}.$$

We apply the map $m \otimes m \otimes \mathrm{id}_H \otimes \mathrm{id}_H$ to both sides — here $m\colon H \otimes H \to H$ is the multiplication map of H:

$$a_{(1)_{(1)}} b_{(1)_{(1)}} \otimes a_{(1)_{(2)}} b_{(1)_{(2)}} \otimes b_{(2)} \otimes a_{(2)} = a_{(1)} b_{(1)} \otimes a_{(2)_{(1)}} b_{(2)_{(1)}} \otimes b_{(2)_{(2)}} \otimes a_{(2)_{(2)}}.$$

Since Δ is a homomorphism, we can rewrite the left-hand side:

$$(a_{(1)} b_{(1)})_{(1)} \otimes (a_{(1)} b_{(1)})_{(2)} \otimes b_{(2)} \otimes a_{(2)} = a_{(1)} b_{(1)} \otimes a_{(2)_{(1)}} b_{(2)_{(1)}} \otimes b_{(2)_{(2)}} \otimes a_{(2)_{(2)}}.$$

We apply the map $S \otimes \mathrm{id} \otimes S \otimes S$ to both sides, then multiply all the tensor legs together:

$$S\left((a_{(1)} b_{(1)})_{(1)}\right) \cdot (a_{(1)} b_{(1)})_{(2)} \cdot S(b_{(2)}) \cdot S(a_{(2)}) = S(a_{(1)} b_{(1)}) \cdot a_{(2)_{(1)}} b_{(2)_{(1)}} \cdot S(b_{(2)_{(2)}}) \cdot S(a_{(2)_{(2)}}).$$

We use the antipode law in the form of $S(x_{(1)})x_{(2)} = \epsilon(x)$ on the left, and $x_{(1)}S(x_{(2)}) = \epsilon(x)$ on the right:

$$\epsilon(a_{(1)} b_{(1)}) S(b_{(2)}) \cdot S(a_{(2)}) = S(a_{(1)} b_{(1)}) \cdot a_{(2)_{(1)}} \epsilon(b_{(2)}) \cdot S(a_{(2)_{(2)}}).$$

We use $\epsilon(xy) = \epsilon(x)\epsilon(y)$, multilinearity of multiplication and linearity of S to move the scalars around:

$$S(\epsilon(b_{(1)}) b_{(2)}) \cdot S(\epsilon(a_{(1)}) a_{(2)}) = S(a_{(1)} b_{(1)} \epsilon(b_{(2)})) \cdot a_{(2)_{(1)}} \cdot S(a_{(2)_{(2)}}).$$

Using the counit law on both ides and the antipode law on the right, we arrive at

$$S(b) \cdot S(a) = S(a_{(1)} b) \epsilon(a_{(2)}),$$

and one more application of the counit law takes us to the required $S(b)S(a) = S(ab)$. Finally, $\Delta 1_H = 1_H \otimes 1_H$ implies (antipode law, ϵ is a homomorphism) $S(1_H)1_H = \epsilon(1_H)1_H = 1_H$ whence $S(1_H) = 1_H$. $\square$

4.12 Remark We do not say that S is an antiautomorphism of H because S does not have to be bijective.

Grouplike and primitive elements

Recall that **grouplike elements of a Hopf algebra** H are defined exactly as in any coalgebra, that is, g is grouplike iff $\Delta g = g \otimes g$ and $\epsilon(g) = 1$. Denote by $G(H)$ the set of all grouplikes in H. We record here the basic properties of grouplikes — proofs are in the exercises to this chapter:

- $G(H)$ is a linearly independent subset of H.
- $G(H)$ is a group with respect to multiplication in H, where the identity element is 1_H and the inverse is given by $g^{-1} = S(g)$ for all $g \in G(H)$.

Example 4.10 extends to show that

- if A is an H-module algebra, then every grouplike $g \in H$ acts on A by an algebra automorphism of A.

Another important class of Hopf algebra elements are primitives.

4.13 Definition (primitive element) If H is a Hopf algebra, $x \in H$ is **primitive** if $\Delta x = x \otimes 1_H + 1_H \otimes x$. The set of all primitive elements of H is denoted $P(H)$.

We record here the basic properties of primitive elements of a Hopf algebra H — again, proofs are in the exercises to this chapter:

- $0 \in P(H)$, and $P(H)$ is a subspace of the vector space H;
- if x is primitive, then $\epsilon(x) = 0$ and $S(x) = -x$;
- if $x, y \in P(H)$ then $xy - yx \in P(H)$.

The Hopf algebra $\mathbb{C}\langle X\rangle$ generated by primitive elements

Let X be a set. We define a Hopf algebra structure on the free algebra $\mathbb{C}\langle X\rangle$. This construction will play an important role in the next chapter, when we will obtain new Hopf algebras by constructing a presentation, that is, as quotients of $\mathbb{C}\langle X\rangle$.

We would like all elements of $X \subset \mathbb{C}\langle X\rangle$ to be **primitive**. This defines Δ, ϵ and S on X. These maps can be extended to the whole $\mathbb{C}\langle X\rangle$ by the universal mapping property; but do the extensions satisfy the axioms in the definition of Hopf algebra? Yes, as the following result implies.

4.14 Lemma (extending Δ, ϵ, S from generators) *Let H be an algebra, and $X \subset H$ be a generating set for H (that is, H is spanned by products of finitely many elements of X). Suppose that $\Delta\colon H \to H \otimes H$ and $\epsilon\colon H \to \mathbb{C}$ are algebra homomorphisms, and $S\colon H \to H$ is an algebra antihomomorphism, such that the equations*

$$(\Delta \otimes \mathrm{id})\Delta x = (\mathrm{id} \otimes \Delta)\Delta x, \quad (\epsilon \otimes \mathrm{id})\Delta x = x = (\mathrm{id} \otimes \epsilon)\Delta x, \quad m(S \otimes \mathrm{id})\Delta x = \epsilon(x)1_H = m(\mathrm{id} \otimes S)\Delta x$$

hold for all $x \in X$. Then these equations hold for all $x \in H$.

Proof. It is enough to show that if the given equations hold for $x = a$ and for $x = b$, then they hold for $x = ab$. If true, then by induction the equations will hold when x is any finite product of elements of X. They also hold when $x = 1$ by the (anti)homomorphism assumption, and all parts of the equations are linear in x. Hence the equations will be true for any linear combinations of products of element of X — that is, for the whole of H.

In the equation $(\Delta \otimes \mathrm{id})\Delta x = (\mathrm{id} \otimes \Delta)\Delta x$, the maps on both sides are algebra homomorphisms $H \to H^{\otimes 3}$, which means that they are multiplicative in x, as required. The same goes for $(\epsilon \otimes \mathrm{id})\Delta x = x = (\mathrm{id} \otimes \epsilon)\Delta x$, where all the maps are homomorphisms $H \to H$.

Finally, $m(S \otimes \mathrm{id})\Delta(ab) = S((ab)_{(1)})(ab)_{(2)}$ which equals $S(a_{(1)}b_{(1)})a_{(2)}b_{(2)}$ since Δ is a homomorphism. By the antihomomorphism property of S, this rewrites as $S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}$. By the assumption that the equation

$m(\mathrm{id} \otimes S)\Delta(x) = \epsilon(x)$ holds for $x = a$, this becomes $\epsilon(a)S(b_{(1)})b_{(2)}$, and by the assumption that the equation holds for $x = b$, we get $\epsilon(a)\epsilon(b)$. Finally, by the homomorphism property of ϵ, this is $\epsilon(ab)$, so that the equation $m(S \otimes \mathrm{id})\Delta(ab) = \epsilon(ab)$ holds. The equation $m(\mathrm{id} \otimes S)\Delta(ab) = \epsilon(ab)$ is dealt with in the same way. □

4.15 Theorem (the Hopf algebra structure on $\mathbb{C}\langle X\rangle$) *Let X be a set. There exists a unique Hopf algebra structure on the free algebra $\mathbb{C}\langle X\rangle$ in which every element $x \in X$ is primitive.*

Proof. The set-theoretical map $X \to \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$ which sends $x \in X$ to $x \otimes 1 + 1 \otimes x$ uniquely extends, by the Universal Mapping Property (Proposition 2.12), to an algebra homomorphism $\Delta\colon \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle$.

The set-theoretical map $X \to \mathbb{C}$ sending $x \in X$ to 0 likewise extends to a unique homomorphism $\epsilon\colon \mathbb{C}\langle X\rangle \to \mathbb{C}$.

The set-theoretical map $X \to \mathbb{C}\langle X\rangle$ sending x to $-x$ extends to an antihomomorphism $S\colon \mathbb{C}\langle X\rangle \to \mathbb{C}\langle X\rangle$. Strictly speaking, the Universal Mapping Property was proved for homomorphisms not antihomomorphisms, but here we can define S explicitly on the basis of $\mathbb{C}\langle X\rangle$ by $S(x_1x_2\ldots x_n) = (-x_n)(-x_{n-1})\ldots(-x_1)$ and extend linearly; the antihomomorphism property holds on the monomial basis hence holds everywhere.

The required equations from Lemma 4.14 hold for $x \in X$: indeed, we calculate

$$(\Delta \otimes \mathrm{id})\Delta x = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x = (\mathrm{id} \otimes \Delta)\Delta x,$$

and the other two equations are also easily verified. By Lemma 4.14, we have a Hopf algebra. □

4.16 Remark (Δ, ϵ and S on monomials) Recall that the free algebra $\mathbb{C}\langle X\rangle$ has a basis $\mathrm{Mon}(X)$ of noncommutative monomials in X. It is useful to write down the coproduct, counit and antipode in this basis. If $x_1, \ldots, x_n \in X$, we have

$$\Delta(x_1x_2\ldots x_n) = (x_1 \otimes 1 + 1 \otimes x_1)(x_2 \otimes 1 + 1 \otimes x_2)\ldots(x_n \otimes 1 + 1 \otimes x_n).$$

The right-hand side expands to a sum of 2^n terms. We can describe this sum as follows. To each subset $\{i_1 < i_2 < \cdots < i_k\}$ of $\{1, 2, \ldots, n\}$ there corresponds the **submonomial** $N = x_{i_1}x_{i_2}\ldots x_{i_k}$ of $M = x_1x_2\ldots x_n$. Let us also use the shorthand notation $M \setminus N$ for the complementary submonomial of M which corresponds to the set $\{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$; that is, $M \setminus N$ is obtained by deleting from M the letters which form N. We then have

$$\Delta M = \sum\{N \otimes (M \setminus N) : N \text{ is a submonomial of } M\}.$$

The sum on the right has 2^n terms, where the empty submonomial is understood to be 1.

We also have

$$\epsilon(M) = \begin{cases} 1, & M = 1, \\ 0, & M \neq 1, \end{cases} \qquad S(x_1x_2\ldots x_n) = (-1)^n x_nx_{n-1}\ldots x_1.$$

Example: Let $x, y \in X$. $\Delta(x^2y) = \Delta(xxy) = 1\otimes xxy + x\otimes xy + x\otimes xy + y\otimes xx + xx\otimes y + xy\otimes x + xy\otimes x + xxy\otimes 1$. Gathering terms, we obtain $1 \otimes x^2y + x^2y \otimes 1 + 2(x \otimes xy + xy \otimes x) + 2(y \otimes x^2 + x^2 \otimes y)$.

Furthermore, $\epsilon(x^2y) = 0$ and $S(x^2y) = -yx^2$.