§ 3

Coalgebras. Coactions

In the previous section, we adopted an approach to defining algebraic objects as vector spaces, V, equipped with several linear maps between tensor powers of V which satisfy certain "laws" (equations involving compositions of the maps). In particular, we recast the traditional definition of an associative unital algebra in this language.

We will now construct the definition of a coalgebra, using the "reversing the arrows" recipe. Namely, laws defining an algebra are written as commutative diagrams. Reversing the direction of all the arrows gives us the laws which define a coalgebra.

"Reversing the arrows" is not just an abstract trick to construct new definitions. If $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ are linear maps between vector spaces, then $X^* \xleftarrow{\alpha^*} Y^* \xleftarrow{\beta^*} Z^*$ is the "dual" diagram (contragredient maps between the dual spaces) where $\alpha^*\beta^* = (\beta\alpha)^*$, see Exercise E1.4. This creates a reasonable expectation that the algebraic structure defined by "reversing the arrows" will be defined on the dual space V^*.

This expectation is fulfilled when V is a *finite-dimensional* vector space. Below, we show that if A is a finite-dimensional algebra, then A^* is a coalgebra. The restriction arises because $(A \otimes A)^*$ is the same as $A^* \otimes A^*$ only if $\dim A < \infty$. On the other hand, if C is a coalgebra then C^* is always an algebra, as we will see.

3.1 Definition A **coalgebra** is a non-zero vector space C over $\mathbb{C}$ equipped with linear maps $\Delta\colon C \to C \otimes C$ (coproduct map) and $\epsilon\colon C \to \mathbb{C}$ (counit map) which satisfy the **coassociative law,**

$$(\mathrm{id}_C \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_C)\Delta \qquad \text{as linear maps from } C \text{ to } C \otimes C \otimes C,$$

and the **counit law,**

$$(\epsilon \otimes \mathrm{id}_C)\Delta = \mathrm{id}_C = (\mathrm{id}_C \otimes \epsilon)\Delta \qquad \text{as linear maps from } C \text{ to } \mathbb{C} \otimes C \cong C \cong C \otimes \mathbb{C}$$

Here is the diagrammatic representation of the coassociative law and the counit law. The following diagrams must commute:

$$\begin{tikzcd}
& C \otimes C \otimes C & \\
C \otimes C \arrow[ur, "\Delta\otimes\mathrm{id}_C"] & & C \otimes C \arrow[ul, "\mathrm{id}_C\otimes\Delta"'] \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& C & \\
\mathbb{C} \otimes C \arrow[ur, "\cong"] & & C \otimes \mathbb{C} \arrow[ul, "\cong"'] \\
C \otimes C \arrow[u, "\epsilon\otimes\mathrm{id}_C"] & & C \otimes C \arrow[u, "\mathrm{id}_C\otimes\epsilon"'] \\
& C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] \arrow[uuu, equal, "\mathrm{id}_C"] &
\end{tikzcd}$$

These diagrams are obtained from the diagrams for the associative law and the identity law in an algebra, by reversing the direction of the arrows.

Subcoalgebras, coideals, factorcoalgebras

3.2 Definition (subcoalgebra, coideal, quotient) A **subcoalgebra** of a coalgebra C is a subspace $D \subseteq C$ such that $\Delta D \subseteq D \otimes D$ and $\epsilon(D) \neq \{0\}$. Here $D \otimes D$ is viewed as a subspace of $C \otimes C$.

A **coideal** of C is a subspace $K \subseteq C$ such that $\Delta K \subseteq K \otimes C + C \otimes K$ and $\epsilon(K) = \{0\}$.

For a coideal K of C we define the **quotient coalgebra** (or, the same, **factorcoalgebra**) C/K to be the quotient space C/K with coproduct and counit defined as follows. Let $q\colon C \twoheadrightarrow C/K$ be the quotient map. Note that both $K \otimes C$ and $C \otimes K$, hence also their sum $K \otimes C + C \otimes K$, are in the kernel of the map $q \otimes q\colon C \otimes C \to C/K \otimes C/K$. This means that the map $C \xrightarrow{\Delta} C \otimes C \xrightarrow{q \otimes q} C/K \otimes C/K$ vanishes on K, and so there is the well-defined map

$$\bar{\Delta}\colon C/K \to C/K \otimes C/K,$$

taken to be the coproduct on C/K. The counit $\bar{\epsilon}\colon C/K \to \mathbb{C}$ is also well-defined, induced by the counit $\epsilon\colon C \to \mathbb{C}$ of C whose kernel must contain K by definition of a coideal.

The dual space of a coalgebra is an algebra

For the next result we recall several facts about the dual space $C^* = \mathrm{Lin}(C, \mathbb{C})$ of a vector space C. Some of them are discussed in E1.3 and E1.5.

- If D is a subspace of C then $D^\perp := \{\xi \in C^* : \xi(D) = \{0\}\}$ is a subspace of C^*.
- The restriction map $C^* \to D^*$, given by $\phi \mapsto \phi|_D$, is a linear map with kernel $D^\perp$.
- The restriction map is surjective because every linear functional $D \to \mathbb{C}$ can be extended to a linear functional $C \to \mathbb{C}$ (by completing a basis of D to a basis of C).
- The restriction map therefore induces a linear isomorphism $C/D^\perp \xrightarrow{\cong} D^*$.
- $C^* \otimes C^*$ is canonically a subspace of $(C \otimes C)^*$, by viewing a tensor $\phi \otimes \psi \in C^* \otimes C^*$ (where $\phi, \psi \in C^*$) as the linear map

$$\phi \otimes \psi\colon C \otimes C \to \mathbb{C}, \qquad \langle x \otimes y, \phi \otimes \psi \rangle := \langle x, \phi \rangle \langle y, \psi \rangle \quad \forall x, y \in C.$$

3.3 Theorem *Let (C, Δ, ϵ) be a coalgebra. Put $A = C^*$, $m = \Delta^*|_{C^* \otimes C^*}$, $\eta = \epsilon^*$. Then (A, m, η) is an algebra. Furthermore,*

- *$D \subseteq C$ is a subcoalgebra iff $I = D^\perp$ is a proper ideal of A; in this case, $D^* \cong A/I$ as algebras;*
- *$K \subseteq C$ is a coideal iff $B = K^\perp$ is a subalgebra of A; in this case, $(C/K)^* \cong B$ as algebras.*

Proof. To show that $(C^*, \Delta^*, \epsilon^*)$ is an algebra, one needs to check that the maps Δ^* and ϵ^* obey the associative law and the identity law. But this follows immediately by taking the dual spaces and contragredient maps in the commutative diagrams which represent the two laws: the diagrams still commute.

A full proof of duality between subcoalgebras and ideals, respectively, coideals and subalgebras, is technical if C is infinite-dimensional. The reader is referred to Radford 2012, Proposition 2.3.7. □

What about the dual of an algebra?

If A is an algebra, will A^* be a coalgebra? Not necessarily if A is infinite-dimensional: the product map $m\colon A \otimes A \to A$ of A has contragredient map $m^*\colon A^* \to (A \otimes A)^*$. Yet for A^* to be a coalgebra, we need a coproduct which maps A^* to $A^* \otimes A^*$. If $\dim A^* = \infty$ then it is not guaranteed that the image of m^* lies in the proper subspace $A^* \otimes A^*$ of $(A \otimes A)^*$, and so the coproduct on A^* is, in general, not defined.

However, if $\dim A < \infty$, there is no such obstacle because the canonical embedding of $A^* \otimes A^*$ in $(A \otimes A)^*$ is a linear isomorphism. We therefore have the following result, proved by the method of reversing the arrows in the diagrams describing the associative and identity laws:

3.4 Theorem *If (A, m, η) is a **finite-dimensional** algebra, then (A^*, m^*, η^*) is a coalgebra.* $\square$

Sweedler notation for the coproduct

The product $m(a \otimes b)$ of a, b in an algebra A is written more simply as ab. What about the coproduct in a coalgebra C? Elements of $C \otimes C$ have the form $\sum_{i=1}^{N} y_i \otimes z_i$ where $y_i, z_i \in C$; that is, every tensor from $C \otimes C$ can be written as a sum of finitely many pure tensors. Hence one may write $\Delta x = \sum_{i=1}^{N} x_{(1)_i} \otimes x_{(2)_i}$ for some $x_{(1)_i}, x_{(2)_i} \in C$. This decomposition into pure tensors is not unique, and N is not unique.

Sweedler notation is an optimisation of this way of writing Δx: one simply writes

$$\Delta x = \sum x_{(1)} \otimes x_{(2)}$$

without mentioning N and subsripting the summands by i. Also popular is even more optimised, compact notation:

$$\Delta x = x_{(1)} \otimes x_{(2)}$$

where the symbol $\sum$ is omitted, but summation is understood.

It is important to remember that $x_{(1)} \otimes x_{(2)}$ is not necessarily a pure tensor. However, **bilinear** maps on $C \otimes C$ can be applied to $x_{(1)}$ and $x_{(2)}$ in this compact notation.

For example, assume that $\Delta x = a \otimes b + c \otimes d$ with $a, b, c, d \in C$. Let us write down the counit law explicitly:

$$(\epsilon \otimes \mathrm{id})\Delta x = \epsilon(a)b + \epsilon(c)d \qquad \text{must be equal to } x.$$

However, in compact notation we write the counit law as follows:

$$\epsilon(x_{(1)})x_{(2)} \qquad \text{must be equal to } x.$$

Further examples will be seen in the course.

Grouplikes, primitive elements. A simple example of a coalgebra

3.5 Definition **(grouplike)** Let C be a coalgebra. An element $g \in C$ is a **grouplike** (or a **grouplike element**) if

$$\Delta g = g \otimes g, \qquad \epsilon(g) = 1.$$

3.6 Example **(coalgebra with a basis of grouplikes)** Take any set $\Gamma \neq \emptyset$. Put $C = \mathbb{C}\Gamma$ and define $\Delta\colon C \to C \otimes C$ by $\Delta g = g \otimes g$ for $g \in \Gamma$. Thus, Δ is defined by extension from the basis Γ of C: the coproduct, say, of $g_1 + g_2$ with $g_1, g_2 \in \Gamma$ is $\Delta(g_1 + g_2) = g_1 \otimes g_1 + g_2 \otimes g_2$, by linearity of Δ.

Similarly, ϵ is defined by extending, from the basis Γ, the function $\epsilon(g) = 1$ for $g \in \Gamma$.

How to check the coassociative law and the counit law? The coassociative law is the equality of two linear maps, $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$. If $g \in \Gamma$, then both sides of this equation evaluate to $g \otimes g \otimes g$. Since the linear maps $(\mathrm{id} \otimes \Delta)\Delta$ and $(\Delta \otimes \mathrm{id})\Delta$ agree on a basis Γ of $\mathbb{C}\Gamma$, they are equal as linear maps. The coassociative law is proved.

Similarly, the counit law is the equality of three linear maps, $(\epsilon \otimes \mathrm{id})\Delta$, id, and $(\mathrm{id} \otimes \epsilon)\Delta$. Again, they agree on $g \in \Gamma$: both sides evaluate to g. Hence the maps are equal.

In the next definition, we must assume that the coalgebra C has a distinguished grouplike element, denoted $\mathbf{1}$. This element is referrd to as **unit**. However, nothing more is assumed of $\mathbf{1}$, other that it is grouplike.

3.7 Definition (primitive element, $P(C)$) Let C be a coalgebra with unit $\mathbf{1}$. We call $x \in C$ **primitive** if $\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x$.

The set of primitive elements of C is denoted $P(C)$. Since the condition defining a primitive is linear, $P(C)$ is a subspace of the vector space C.

3.8 Lemma *If C is a coalgebra with unit, $P(C)$ is a coideal of C.*

Proof. If $x \in P(C)$ then $\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x$ is in $P(C) \otimes C + C \otimes P(C)$, verifying the first axiom of a coideal.

It remains to show that $\epsilon(x) = 0$. By the counit law, $\epsilon(x)\mathbf{1} + x\epsilon(\mathbf{1}) = x$. Applying ϵ to both sides, we have $2\epsilon(x)\epsilon(\mathbf{1}) = \epsilon(x)$. By definition of grouplike, $\epsilon(\mathbf{1}) = 1$ so $2\epsilon(x) = \epsilon(x)$ implying $\epsilon(x) = 0$, as required. □

Coactions. Comodules. The Fundamental Theorem of Coalgebras

Just as the definition of a coalgebra is obtained by arrow reversal from the definition of an algebra, one can reverse the arrows in the definition of an action and arrive at the following.

3.9 Definition (coaction, comodule) An **coaction** of a coalgebra $C = (C, \Delta, \epsilon)$ on a vector space V is a linear map

$$\delta \colon V \to C \otimes V,$$

such that

- $(\Delta \otimes \mathrm{id}_V)\delta = (\mathrm{id}_C \otimes \delta)\delta$ as maps from V to $C \otimes C \otimes V$;
- $(\epsilon \otimes \mathrm{id}_V)\delta = \mathrm{id}_V$ as maps from V to V.

We say that V is an C-**comodule** if V is a vector space with a coaction of C.

The rest of this chapter is optional material and is not discussed in the video lecture.

As expected, the algebra-coalgebra duality results, Theorem 3.3 and Theorem 3.4, extend to duality between coactions of coalgebras and actions of algebras. The proof of the following theorem is an exercise in verification of axioms:

3.10 Theorem *A coaction $\delta \colon V \to C \otimes V$ of a coalgebra C on a space V gives rise to the action $\triangleright \colon C^* \otimes V \to V$ of the algebra C^* on V, where $\triangleright$ is the composite map*

$$C^* \otimes V \xrightarrow{\mathrm{id}_{C^*} \otimes \delta} C^* \otimes C \otimes V \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_V} \mathbb{C} \otimes V \cong V.$$

Here $\mathrm{ev} \colon C^* \otimes C \to \mathbb{C}$, $\phi \otimes c \mapsto \phi(c)$ is the evaluation map.

Likewise, if A is a finite-dimensional algebra acting on V, then the coalgebra A^* coacts on V via

$$V \to A^* \otimes V, \qquad v \mapsto \sum_{i=1}^{n} \alpha^i \otimes (a_i \triangleright v),$$

where $\alpha^1, \dots, \alpha^n$ is the basis dual to a basis $a_1, \dots, a_n$ of A. □

The above may suggest that actions and coactions are interchangeable via duality. This is indeed so for finite-dimensional algebras and coalgebras. Yet the following results for comodules and coalgebras, meaningful if the coalgebra C has infinite dimension, have **no analogue for algebras.**

3.11 Proposition *If $\delta \colon V \to C \otimes V$ is a coaction, then every $u \in V$ is contained in a subspace $V_u \subseteq V$ such that $\delta(V_u) \subseteq C \otimes V_u$ and $\dim V_u < \infty$. In other words, every vector in a comodule is contained in a finite-dimensional subcomodule.*

Proof. Write $\delta(u) = \sum_{i=1}^{N} c_i \otimes u_i$ where $u_1, \dots, u_N \in V$ and $c_1, \dots, c_N \in C$ are linearly independent elements of C (any element of $C \otimes V$ can be written in this form). Let V_u be the span of $u_1, \dots, u_N$, a finite-dimensional subspace of V. Then $u \in V_u$ because $u = (\epsilon \otimes \mathrm{id}_V)\delta(u) = \sum_{i=1}^{N} \epsilon(c_i)u_i$. It remains to show that $\delta(V_u) \subseteq C \otimes V_u$, that is, $\delta(u_i) \in C \otimes V_u$ for all i; it is enough to show this for $i = 1$.

Since $c_1, \dots, c_N$ are linearly independent, there exists a linear functional $\xi \in C^*$ such that $\xi(c_1) = 1$ and $\xi(c_2) = \dots = \xi(c_N) = 0$. Therefore,

$$\begin{aligned}
\delta(u_1) = \sum_{i=1}^{N} \xi(c_i)\delta(u_i) &= (\xi \otimes \mathrm{id}_C \otimes \mathrm{id}_V) \sum_{i=1}^{N} c_i \otimes \delta(u_i) \\
&= (\xi \otimes \mathrm{id}_C \otimes \mathrm{id}_V)(\mathrm{id}_C \otimes \delta) \sum_{i=1}^{N} c_i \otimes u_i \\
&= (\xi \otimes \mathrm{id}_C \otimes \mathrm{id}_V)(\mathrm{id}_C \otimes \delta)\delta(v) \\
&= (\xi \otimes \mathrm{id}_C \otimes \mathrm{id}_V)(\Delta \otimes \mathrm{id}_V)\delta(v) \\
&= (\xi \otimes \mathrm{id}_C \otimes \mathrm{id}_V) \sum_{i=1}^{N} \Delta c_i \otimes u_i,
\end{aligned}$$

which is clearly in $C \otimes V_u$. We used the coaction axiom $(\mathrm{id}_C \otimes \delta)\delta = (\Delta \otimes \mathrm{id}_V)\delta$. □

The next result is proved in a similar way.

3.12 Theorem (Fundamental Theorem on Coalgebras) *Every element of a coalgebra C lies in a finite-dimensional subcoalgebra of C.*

Proof. For each $u \in C$, write $(\Delta \otimes \mathrm{id})\Delta u \in C^{\otimes 3}$ as $\sum_{i=1}^{M} \sum_{j=1}^{N} a_i \otimes b_{ij} \otimes c_j$ where $a_1, \dots, a_M$ are linearly independent and $c_1, \dots, c_N$ are linearly independent. Put D_u to be the span of $b_{11}, \dots, b_{MN}$. Note that D contains u: indeed, by the counit law $\sum_{i,j} \epsilon(a_i)b_{ij} \otimes c_j = \Delta u$ and so $\sum_{i,j} \epsilon(a_i)b_{ij}\epsilon(c_j) = u$.

We claim that D is a subcoalgebra of C. Without the loss of generality, we can prove this by showing that $\Delta b_{11} \in D \otimes D$. Since $D \otimes D = D \otimes C \cap C \otimes D$ (by linear algebra), it is enough to show that $\Delta b_{11} \in D \otimes C$ and $\Delta b_{11} \in C \otimes D$. The two arguments are completely similar, and we will only show that $\Delta b_{11} \in D \otimes C$, leaving the other one to the reader.

By linear independence of the a_i and of the c_j, there exist $\xi, \eta \in C^*$ such that $\xi(a_1) = 1$, $\xi(a_2) = \dots = \xi(a_M) = 0$, $\eta(c_1) = 1$ and $\eta(c_2) = \dots = \eta(c_N) = 0$. We have

$$\begin{aligned}
\Delta b_{11} = (\xi \otimes \Delta \otimes \eta) \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \otimes b_{ij} \otimes c_j &= (\xi \otimes \Delta \otimes \eta)(\Delta \otimes \mathrm{id})\Delta u \\
&= (\xi \otimes \mathrm{id} \otimes \mathrm{id} \otimes \eta)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta u.
\end{aligned}$$

By coassociativity,

$$(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\mathrm{id} \otimes \Delta)\Delta,$$

and so we can continue the above calculation by

$$(\xi \otimes \mathrm{id} \otimes \mathrm{id} \otimes \eta)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta) \sum_{i,j} a_i \otimes b_{ij} \otimes c_j = \sum_{i,j} \xi(a_i)b_{ij} \otimes (\mathrm{id} \otimes \eta)\Delta c_j,$$

which is in $D \otimes C$, as claimed. □

References

Radford, David E. (2012). *Hopf algebras.* Vol. 49. Series on Knots and Everything. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, pp. xxii+559. ISBN: 978-981-4335-99-7; 981-4335-99-1.