Loughborough University

Lie Groups and Lie Algebras
(MAGIC008 TEST)

2011 TEST

Answer **3** questions.

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1. (a) State the definition of a Lie group. [4]

   (b) Prove that the set G_A of all invertible 3×3-matrices that commute with the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, i.e.,

   $$G_A = \{X \in GL(3, \mathbb{R}) \mid XA = AX\},$$

   is a Lie group. Describe G_A explicitly and find its dimension. [4]

   (c) Is G_A connected? How many connected components does this Lie group consist of? Justify your answer. [4]

   (d) Let G be a Lie group and $G_0 \subset G$ be its identity component. Prove that G_0 is a subgroup of G. [4]

   (e) Describe the identity component $(G_A)_0$ of the Lie group G_A and prove that $(G_A)_0$ is diffeomorphic to $\mathbb{R}^5$. [4]

2. Consider 3 vector fields in $\mathbb{R}^2(x, y)$:

   $$\xi_1 = (-y, x), \quad \xi_2 = (1, 0), \quad \xi_3 = (0, 1).$$

   (a) Describe explicitly the action of the one-parameter group of diffeomorphisms Φ^t_ξ generated by the vector field $\xi = a\xi_1 + b\xi_2 + c\xi_3$, where $a, b, c \in \mathbb{R}^3$ are some constants. [4]

   (b) Compute the Lie brackets $[\xi_i, \xi_j]$, $i, j = 1, 2, 3$, and prove that the 3-dimensional space spanned by these vector fields is a Lie algebra. [4]

   (c) Describe the structure constants of this Lie algebra $\mathfrak{g}$ in the basis ξ_1, ξ_2, ξ_3. [4]

   (d) Describe explicitly the adjoint representation of this Lie algebra. Is this representation faithful? What is the center of $\mathfrak{g}$? [4]

(e) Prove that $\mathfrak{g}$ is isomorphic to the matrix Lie algebra

$$\left\{ \begin{pmatrix} 0 & -x & y \\ x & 0 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\} \subset gl(3, \mathbb{R}).$$

[4]

3. (a) Give the definition of the special linear group $SL(n, \mathbb{R})$.
Explain why $SL(n, \mathbb{R})$ is a Lie group. [4]

(b) Is $SL(n, \mathbb{R})$ compact? Justify your answer. [4]

(c) State the definition of an (abstract) Lie algebra. Describe the Lie algebra $sl(n, \mathbb{R})$ of the Lie group $SL(n, \mathbb{R})$ and compute its dimension. Justify your answer. [4]

(d) Let $\mathfrak{g}$ be a Lie algebra. State the definition of an ideal of $\mathfrak{g}$.
Prove that $sl(n, \mathbb{R})$ is an ideal of $gl(n, \mathbb{R})$. [4]

(e) Is $so(n)$ an ideal of $sl(n, \mathbb{R})$? Justify your answer. [4]

4. (a) Let $\Phi : G \to GL(V)$ be a linear representation of a Lie group G on a vector space V. State the definition of an invariant subspace $L \subset V$.
State the definition of an irreducible representation. [4]

(b) Let V be the space of $n \times n$-matrices. Consider the map

$$\Psi : SL(n, \mathbb{R}) \to GL(V), \quad X \mapsto \Psi_X$$

where $\Psi_X B = XBX^\top, \quad X \in SL(n, \mathbb{R}), \; B \in V$. Verify that Ψ is a linear representation. Is this representation irreducible? [4]

(c) Describe explicitly the induced representation $\psi = d\Psi : sl(n, \mathbb{R}) \to gl(V)$ for the Lie algebra $sl(n, \mathbb{R})$. [4]

(d) Let $n = 3$ and

$$B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in V, \qquad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} \in V, \quad b_i > 0.$$

Prove that the stationary subgroups $\mathrm{St}(B_0) = \{X \in SL(3, \mathbb{R}) \mid \Psi_X(B_0) = B_0\}$ and $\mathrm{St}(B) = \{X \in SL(3, \mathbb{R}) \mid \Psi_X(B) = B\}$ are isomorphic. [4]

(e) Let $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in V$. Describe the stationary subgroup $\mathrm{St}(C)$ and find the dimension of the orbit $\mathcal{O}(C) = \{B \in V \mid B = XCX^\top, \; X \in SL(3, \mathbb{R})\}$. [4]