Problem Sheet 2. (Lie groups and Lie algebras) Solutions

Questions 4 and 5

Question 4

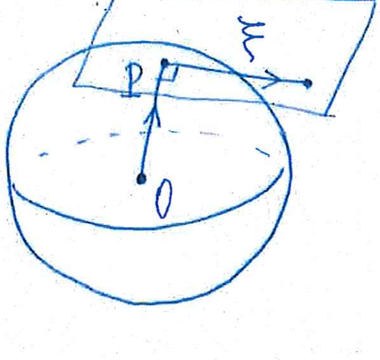
(a) $[\xi, \eta] = 0$

(b) $[\xi, \eta] = 0$

(c) $$[\xi, \eta] = (\sin y + \sin z - (z+y)\cos x)\,\partial/\partial x + (\sin x + \sin z - (x+z)\cos y)\,\partial/\partial y + (\sin x + \sin y - (x+y)\cos z)\,\partial/\partial z$$

Question 5

To verify that ξ and η are tangent to the unit sphere $S^3 = \{x^2+y^2+u^2+v^2\} = 1$, we use the following geometric fact:



A vector $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ is tangent to S^3 at the point $P \in S^3$ iff ξ is perpendicular to the radius vector $\overline{OP}$, in other words:

$$\xi_1 x + \xi_2 y + \xi_3 u + \xi_4 v = 0$$

where (x, y, u, v) are the coordinates of P. (here we consider the centre of the sphere to be the origin of the coordinate system).

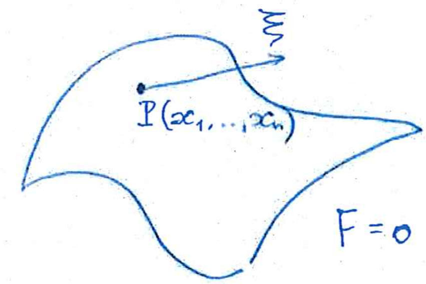
After this remark, the verification is straightforward.

$\xi = (y, -x, u, -v)$, $P = (x, y, \cancel{z}\, u, v) \Rightarrow \langle \xi, \overline{OP} \rangle = 0$

$\eta = (u, v, -x, -y)$, $P = (x, y, u, v) \Rightarrow \langle \eta, \overline{OP} \rangle = 0$

Hence, ξ and η are tangent to S^3.

<u>General remark.</u> If $M \subset \mathbb{R}^n$ is given by an equation $F(x_1, x_2, \dots, x_n) = 0$. and ξ is a vector at point $P \in M$,



then ξ is tangent to M iff the following condition holds:

$$\frac{\partial F}{\partial x_1} \cdot \xi_1 + \frac{\partial F}{\partial x_2} \xi_2 + \dots + \frac{\partial F}{\partial x_n} \xi_n = 0$$

or shortly,

$$\langle \operatorname{grad} F, \xi \rangle = 0.$$

Now, the straightforward computation gives:

$[\xi, \eta] = 0$ (By definition, the zero vector field is tangent to S^3 as well as to any other submanifold of $\mathbb{R}^4$).

$\zeta = (u, -v, -x, y)$ is tangent to S^3 too and the straightforward computation shows that

$$[\xi, \zeta] = 2v\, \partial/\partial x + 2u\, \partial/\partial y - 2y\, \partial/\partial u - 2x\, \partial/\partial v = (2v, 2u, -2y, -2x)$$

It is easy to see that $[\xi, \zeta]$ is tangent to S^3.

This example is a particular case of the following general statement:

Let $M \subset \mathbb{R}^n$ be a submanifold and ξ and η be vector fields on $\mathbb{R}^n$ which are tangent to M. Then $[\xi, \eta]$ is tangent to M too.