Exercise Sheet 6

MAGIC009 - Category Theory

November 22nd, 2024

1. Prove explicitly that the following categories have an initial object, binary coproducts, coequalisers and pushouts:

    (a) **Set**,the category of sets and functions,

    (b) **Grp**, the category of groups and group homomorphisms,

    (c) **Top**, the category of topological spaces and continuous functions.

2. A *zero object* in a category $\mathbb{C}$ is an object that is both initial and terminal. The *kernel* of a morphism $f : X \to Y$ in a category $\mathbb{C}$ with a zero object 0 is the pullback of $f : X \to Y$ along the unique morphism $! : 0 \to Y$. Meanwhile, the *cokernel* of f is the kernel of f in $\mathbb{C}^{\mathrm{op}}$.

    (a) What is the zero object in that category of groups?

    (b) Describe explicitly the kernel and cokernel of a group homomorphism. Does this match with the group theoretic notion of kernel and cokernel?

3. Let X be a set and let $u : U \to X$ and $v : V \to X$ be inclusions of subsets $U \subseteq X$ and $V \subseteq X$.

    (a) Show that the intersection $U \cap V$ is the pullback of u and v.

    (b) Let $p : U \cap V \to U$ and $q : U \cap V \to V$ denote the pullback projections. Show that the union $U \cup V$ is the pushout of p and q.