Exercise Sheet 2

MAGIC009 - Category Theory

October 18th, 2024

1. Prove that, for functors $F\colon \mathbb{C} \to \mathbb{D}$ and $G\colon \mathbb{D} \to \mathbb{E}$, their composite $GF\colon \mathbb{C} \to \mathbb{E}$, as defined on page 6 of the notes for Lecture 2, is indeed a functor.

2. Prove that every functor preserves isomorphisms, in the sense that if $F\colon \mathbb{C} \to \mathbb{D}$ is a functor and if $f\colon X \to Y$ is an isomorphism in $\mathbb{C}$, then $F(f)\colon FX \to FY$ is an isomorphism in $\mathbb{D}$.

3. Is it true or false that functors preserve terminal objects, in the sense that if $F\colon \mathbb{C} \to \mathbb{D}$ is a functor and T is a terminal object in $\mathbb{C}$, then $F(T)$ is a terminal object in $\mathbb{D}$?

4. Let **CRing** be the category of commutative rings and **Mon** the category of monoids. Fix $n \geq 1$.
   (a) Check that the function mapping a commutative ring R to the monoid $M_n(R)$ of $(n \times n)$-matrices with coefficients in R extends to a functor $M_n\colon \mathbf{CRing} \to \mathbf{Mon}$.
   (b) Check that the function mapping a commutative ring R to its underlying monoid $U(R)$ extends to a functor $U\colon \mathbf{CRing} \to \mathbf{Mon}$.
   (c) Check that the functions
   $$\phi_R\colon M_n(R) \to U(R)$$
   where R is a commutative ring, defined by $\phi_R(A) = \det(A)$, are monoid homomorphisms and that they form a natural transformation $\phi\colon M_n \Rightarrow U$. See also page 10 of the notes for Lecture 2.

5. (a) Let $F\colon \mathbb{C} \to \mathbb{D}$, $F'\colon \mathbb{C} \to \mathbb{D}$ be functors and $\alpha\colon F \Rightarrow F'$ a natural transformation between them. For a functor $G\colon \mathbb{D} \to \mathbb{E}$, define a family of maps $GFX \to GF'X$ in $\mathbb{E}$, for $X \in \mathbb{C}$, and prove that it is a natural transformation from GF to GF', which we will denote $G\alpha\colon GF \Rightarrow GF'$.
   (b) Let $G\colon \mathbb{D} \to \mathbb{E}$, $G'\colon \mathbb{D} \to \mathbb{E}$ be functors and $\beta\colon G \Rightarrow G'$ a natural transformation between them. For a functor $F\colon \mathbb{C} \to \mathbb{D}$, define a family of maps $GFX \to G'FX$ in $\mathbb{E}$, for $X \in \mathbb{C}$, and prove that it is a natural transformation from GF to $G'F$, which we will denote $\beta F\colon GF \Rightarrow G'F$.
   (c) Let $F\colon \mathbb{C} \to \mathbb{D}$, $F'\colon \mathbb{C} \to \mathbb{D}$, $G\colon \mathbb{D} \to \mathbb{E}$, $G'\colon \mathbb{D} \to \mathbb{E}$ be functors, $\alpha\colon F \Rightarrow F'$ and $\beta\colon G \Rightarrow G'$ be natural transformations. By part (a), we have natural transformations
   $$G\alpha\colon GF \Rightarrow GF', \qquad G'\alpha\colon G'F \Rightarrow G'F'.$$
   By part (b), we also have natural transformations
   $$\beta F\colon GF \Rightarrow G'F, \qquad \beta F'\colon GF' \Rightarrow G'F'.$$
   Prove that the following diagram in the functor category $[\mathbb{C}, \mathbb{E}]$ commutes:
   $$\begin{tikzcd} GF \arrow[r, "G\alpha"] \arrow[d, "\beta F"'] & GF' \arrow[d, "\beta F'"] \\ G'F \arrow[r, "G'\alpha"'] & G'F'. \end{tikzcd}$$

6. Let G be a group and consider the associated category $\underline{G}$. Describe explicitly the functor category
   $$\left[\, \underline{G}, \mathbf{Vect}_k \,\right],$$
   where $\mathbf{Vect}_k$ is the category of finite-dimensional vector spaces and linear maps over a fixed field k.