Problem Sheet 6. Solutions

1. $$\mathrm{Ad}_X A = XAX^{-1}$$
$$\Psi_X B = XBX^T$$
$$\Phi_X\left(p(x,y)\right) = p(\alpha x + \gamma y, \beta x + \delta y)$$
$$X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{R})$$

(a) Prove that the above formulae indeed define linear representations.

We need to verify that
$$\mathrm{Ad}_{X_1} \circ \mathrm{Ad}_{X_2} = \mathrm{Ad}_{X_1X_2}$$
$$\Psi_{X_1} \circ \Psi_{X_2} = \Psi_{X_1X_2}$$
$$\Phi_{X_1} \circ \Phi_{X_2} = \Phi_{X_1X_2}$$
for any $X_1, X_2 \in SL(2,\mathbb{R})$.

$$\mathrm{Ad}_{X_1} \circ \mathrm{Ad}_{X_2}(A) = \mathrm{Ad}_{X_1}(\mathrm{Ad}_{X_2}(A)) = \mathrm{Ad}_{X_1}(X_2AX_2^{-1}) =$$
$$= X_1X_2AX_2^{-1}X_1^{-1} = (X_1X_2)A(X_1X_2)^{-1} = \mathrm{Ad}_{X_1X_2}(A), \text{ i.e.}$$
$$\mathrm{Ad}_{X_1} \circ \mathrm{Ad}_{X_2} = \mathrm{Ad}_{X_1X_2} \text{ as needed.}$$

Similarly,
$$\Psi_{X_1} \circ \Psi_{X_2}(B) = \Psi_{X_1}(\Psi_{X_2}(B)) = \Psi_{X_1}(X_2BX_2^T) =$$
$$= X_1X_2BX_2^TX_1^T = (X_1X_2)B(X_1X_2)^T = \Psi_{X_1X_2}(B), \text{ i.e.}$$
$$\Psi_{X_1} \circ \Psi_{X_2} = \Psi_{X_1X_2}, \text{ as needed.}$$

For Φ, the verification is a bit more difficult.

Let $X_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$, $X_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$

Then $X_1 X_2 = \begin{pmatrix} \alpha_1\alpha_2 + \beta_1\gamma_2 & \alpha_1\beta_2 + \beta_1\delta_2 \\ \gamma_1\alpha_2 + \delta_1\gamma_2 & \gamma_1\beta_2 + \delta_1\delta_2 \end{pmatrix}$

We have:

$$\Phi_{X_1} \circ \Phi_{X_2} (p(x,y)) = \Phi_{X_1}(\Phi_{X_2}(p(x,y))) =$$

$$= \Phi_{X_1}\left(p\left(\alpha_2 x + \gamma_2 y,\ \beta_2 x + \delta_2 y\right)\right) =$$

$$= p\left(\alpha_2(\alpha_1 x + \gamma_1 y) + \gamma_2(\beta_1 x + \delta_1 y),\ \beta_2(\alpha_1 x + \gamma_1 y) + \delta_2(\beta_1 x + \delta_1 y)\right)$$

$$= p\left((\alpha_2\alpha_1 + \gamma_2\beta_1)x + (\alpha_2\gamma_1 + \gamma_2\delta_1)y,\ (\beta_2\alpha_1 + \delta_2\beta_1)x + (\beta_2\gamma_1 + \delta_2\delta_1)y\right)$$

On the other hand,

$$\Phi_{X_1X_2}(p(x,y)) = p\left((\alpha_1\alpha_2 + \beta_1\gamma_2)x + (\gamma_1\alpha_2 + \delta_1\gamma_2)y,\ (\alpha_1\beta_2 + \beta_1\delta_2)x + (\gamma_1\beta_2 + \delta_1\delta_2)y\right)$$

The right hand sides are the same, and therefore

$\Phi_{X_1} \circ \Phi_{X_2} = \Phi_{X_1X_2}$, as needed.

There is another way to come to the same conclusion. You may notice that the definition of Φ can be rewritten as

$\Phi_X\, p(\bar{v}) = p(X^T \bar{v})$, where $\bar{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\Phi_{X_1} \circ \Phi_{X_2}(p(\bar{v})) = \Phi_{X_1}(p(X_2^T \bar{v})) = p(X_2^T(X_1^T \bar{v})) =$$

$$= p((X_1X_2)^T \bar{v}) = \Phi_{X_1X_2}(p(\bar{v})).$$

These two proof are basically equivalent.

(b) What is the dimension of each representation?

By definition, the dimension of a representation is the dimension of the space V on which a given group acts.

Ad acts on the Lie algebra $sl(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$

$\dim sl(2,\mathbb{R}) = \underline{\underline{3}}$

Ψ acts on the space of 2×2 symmetric matrices $\left\{ B = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \right\}$, its dimension is $\underline{\underline{3}}$

Φ acts on the space of homogeneous polynomials of degree k, i.e.

$$p(x,y) = a_0 x^k + a_1 x^{k-1} y + a_2 x^{k-2} y^2 + \dots + a_{k-1} x y^{k-1} + a_k y^k$$

The dimension of this space is $\underline{\underline{k+1}}$ (because, each polynomial is defined by $k+1$ coefficients $a_0, a_1, \dots, a_k$)

(c) Describe the induced representations ad, ψ and ϕ of the Lie algebra $sl(2,\mathbb{R})$.

In general, to describe the induced representation $\psi = d\Psi$ of the Lie algebra $\mathfrak{g}$ for a given representation Ψ of the corresponding Lie group G, we use the following formula:

$$\psi_B(v) = \frac{d}{dt}\Big|_{t=0} \Psi_{\exp tB}(v)$$

To apply this formula one should remember several "rules" related to matrix exponents:

$$\frac{d}{dt}\Big|_{t=0} \exp tB = B \text{, or more general } \frac{d}{dt}\exp tB = B\cdot\exp tB = \exp tB \cdot B$$

$$\exp tB\Big|_{t=0} = Id$$ (or simply $\exp 0 = Id$ where 0 denotes the zero matrix)

$$\exp(-B) = (\exp B)^{-1}$$

$$\exp(B^T) = (\exp B)^T$$

$$\det(\exp B) = e^{\operatorname{tr} B}$$

Sometimes, it is convenient to use the definition of $\exp tB$:

$$\exp tB = Id + tB + \frac{t^2B^2}{2!} + \frac{t^3B^3}{3!} + \dots$$

Thus:

$$ad_B A = \frac{d}{dt}\Big|_{t=0} Ad_{\exp tB} A = \frac{d}{dt}\Big|_{t=0}\left(\exp tB \cdot A \cdot (\exp tB)^{-1}\right) =$$

$$= \frac{d}{dt}\Big|_{t=0}\left(\exp tB \cdot A \cdot \exp(-tB)\right) = \left[\frac{d\exp tB}{dt} A \exp(-tB) + \exp tB \cdot A \cdot \frac{d\exp(-tB)}{dt}\right]_{t=0}$$

$$= BA\cdot Id + Id\, A\,(-B) = BA - AB = [B, A].$$

Similarly: (C denotes a 2×2 symmetric matrix, $B \in sl(2,\mathbb{R})$)

$$\psi_B C = \frac{d}{dt}\Big|_{t=0} \Psi_{\exp tB} C = \frac{d}{dt}\Big|_{t=0}\left(\exp tB \cdot C\,(\exp tB)^T\right) =$$

$$= \frac{d}{dt}\Big|_{t=0}\left(\exp tB \cdot A \cdot \exp(tB^T)\right) = \left[\frac{d\exp tB}{dt} A \cdot \exp(tB^T) + \exp tB \cdot A \cdot \frac{d\exp tB^T}{dt}\right]_{t=0} =$$

$$= B\cdot A\cdot Id + Id\cdot A\cdot B^T = BA + AB^T.$$

To make this compution for ϕ, we take $B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in sl(2,\mathbb{R})$ and

$$\exp tB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}^2 + \dots =$$

$$= \begin{pmatrix} 1+t\alpha & t\beta \\ t\gamma & 1-t\alpha \end{pmatrix} + \dots \text{ (terms of higher order in } t\text{)}.$$

Then

$$\phi_B(p(x,y)) = \frac{d}{dt}\Big|_{t=0} \Phi_{\exp tB}(p(x,y)) = \frac{d}{dt}\Big|_{t=0} p\big((1+\alpha t)x + \gamma t y + \dots,\ \beta t x + (1-\alpha t)y + \dots\big) =$$

$$= \frac{d}{dt}\Big|_{t=0} p\big(x + t(\alpha x + \gamma y) + \dots,\ y + t(\beta x - \alpha y) + \dots\big) = \text{we differentiate by } \text{~~use~~ using the chain rule} =$$

$$= \frac{\partial p}{\partial x}\cdot(\alpha x + \gamma y) + \frac{\partial p}{\partial y}\cdot(\beta x - \alpha y)$$

where $B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$

We can also rewrite this formula as

$$\phi_B(p(x,y)) = \frac{\partial p}{\partial x}\cdot v_1 + \frac{\partial p}{\partial y}\cdot v_2 \quad \text{where } \bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = B^T\begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\phi_B(p(x,y)) = \bar{v}(p(x,y))$$

where $v(\cdot)$ denotes the directional derivative w.r.t. the vector field $\bar{v}$ and $\bar{v} = B^T\begin{pmatrix} x \\ y \end{pmatrix}$.

To describe a representation in matrix form, we need to chose a certain basis in the space of representation and to find the matrix of ϕ_B w.r.t. this basis.

For example, in the case of the adjoint representation ad, the space V is the Lie algebra $sl(2,\mathbb{R})$ itself and we can choose a basis in the following way:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then for $B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ we have

$$ad_B e_1 = [B, e_1] = Be_1 - e_1 B = \begin{pmatrix} 0 & -2\beta \\ 2\gamma & 0 \end{pmatrix} = 0 \cdot e_1 + (-2\beta) e_2 + 2\gamma \cdot e_3$$

Thus, the first column of the matrix of ad_B is $\begin{pmatrix} 0 \\ -2\beta \\ 2\gamma \end{pmatrix}$.

The second and third columns can be found in the same way:

$$ad_B e_2 = [B, e_2] = Be_2 - e_2 B = \begin{pmatrix} -\gamma & 2\alpha \\ 0 & \gamma \end{pmatrix} = -\gamma \cdot e_1 + 2\alpha \cdot e_2 + 0 \cdot e_3$$

$$ad_B e_3 = [B, e_3] = Be_3 - e_3 B = \begin{pmatrix} \beta & 0 \\ -2\alpha & -\beta \end{pmatrix} = \beta e_1 + 0 \cdot e_2 + (-2\alpha) e_3$$

Thus,
$$ad_B \simeq \begin{pmatrix} 0 & -\gamma & \beta \\ -2\beta & 2\alpha & 0 \\ 2\gamma & 0 & -2\alpha \end{pmatrix}$$

The adjoint representation of $sl(2,\mathbb{R})$ in matrix form is:

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \xmapsto{ad} ad_B = \begin{pmatrix} 0 & -\gamma & \beta \\ -2\beta & 2\alpha & 0 \\ 2\gamma & 0 & -2\alpha \end{pmatrix}$$

For the representation ψ, we choose the following basis in the space of symmetric 2×2 matrices

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then for $B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ we have

$$\psi_B e_1 = Be_1 + e_1B^T = \begin{pmatrix} 2\alpha & \gamma \\ \gamma & 0 \end{pmatrix} = 2\alpha \cdot e_1 + \gamma e_2 + 0 \cdot e_3$$

$$\psi_B e_2 = Be_2 + e_2B^T = \begin{pmatrix} 2\beta & 0 \\ 0 & 2\gamma \end{pmatrix} = 2\beta e_1 + 0 \cdot e_2 + 2\gamma \cdot e_3$$

$$\psi_B e_3 = Be_3 + e_3B^T = \begin{pmatrix} 0 & \beta \\ \beta & -2\alpha \end{pmatrix} = 0 \cdot e_1 + \beta \cdot e_2 - 2\alpha \cdot e_3$$

Thus, the matrix form of the representation ψ is:

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \xmapsto{\psi} \psi_B = \begin{pmatrix} 2\alpha & 2\beta & 0 \\ \gamma & 0 & \beta \\ 0 & 2\gamma & -2\alpha \end{pmatrix}$$

For the representation ϕ, we choose the following basis in the space of ~~symmetric~~ homogeneous polynomials in x and y of degree 2: $e_1 = x^2$, $e_2 = xy$, $e_3 = y^2$.

Then for $B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ we have (we use the formula obtained above).

$$\varphi_B(x^2) = \frac{\partial}{\partial x}(x^2) \cdot (\alpha x + \gamma y) = 2x(\alpha x + \gamma y) =$$
$$= 2\alpha x^2 + 2\gamma \cdot xy =$$
$$= 2\alpha e_1 + 2\gamma \cdot e_2 + 0 \cdot e_3$$

$$\varphi_B(xy) = \frac{\partial}{\partial x}(xy) \cdot (\alpha x + \gamma y) + \frac{\partial}{\partial y}(xy) \cdot (\beta x - \alpha y) =$$
$$= y(\alpha x + \gamma y) + x(\beta x - \alpha y) = \beta x^2 + \gamma y^2 =$$
$$= \beta e_1 + 0 \cdot e_2 + \gamma \cdot e_3$$

$$\varphi_B(y^2) = \frac{\partial}{\partial y}(y^2) \cdot (\beta x - \alpha y) = 2y(\beta x - \alpha y) = 2\beta xy - 2\alpha y^2 =$$
$$= 0 \cdot e_1 + 2\beta e_2 - 2\alpha \cdot e_3$$

Thus, the matrix form of the representation ϕ is:

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \xmapsto{\ \phi\ } \phi_B = \begin{pmatrix} 2\alpha & \beta & 0 \\ 2\gamma & 0 & 2\beta \\ 0 & \gamma & -2\alpha \end{pmatrix}$$

It is not hard to notice that ψ_B and ϕ_B are very similar in matrix form. In fact, if we ~~#~~ take $e_2 = 2xy$ (instead of $e_2 = xy$), we'll see that $\psi_B = \phi_B$, i.e. these two representations are equivalent.

No 2.

Explicit description of ad_ξ in matrix form

- $\mathfrak{g}_1$ $\quad [e_1,e_2]=e_3, \ [e_2,e_3]=e_1, \ [e_3,e_1]=e_2$

$\xi = ae_1+be_2+ce_3$

$ad_\xi e_1 = [ae_1+be_2+ce_3, e_1] = -be_3+ce_2$

$ad_\xi e_2 = [ae_1+be_2+ce_3, e_2] = ae_3-ce_1$

$ad_\xi e_3 = [ae_1+be_2+ce_3, e_3] = -ae_2+be_1$

Hence, $$ad_\xi = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

- $\mathfrak{g}_2$ $\quad [e_1,e_2]=2e_2, \ [e_1,e_3]=-2e_3, \ [e_2,e_3]=e_1$

$\xi = ae_1+be_2+ce_3$

$ad_\xi e_1 = [ae_1+be_2+ce_3, e_1] = -2be_2+2ce_3$

$ad_\xi e_2 = [ae_1+be_2+ce_3, e_2] = 2ae_2-ce_1$

$ad_\xi e_3 = [ae_1+be_2+ce_3, e_3] = -2ae_3+be_1$

Hence, $$ad_\xi = \begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}$$

- $\mathfrak{g}_3$ $\quad [e_1, e_2] = e_3$

$$\xi = ae_1 + be_2 + ce_3$$

$$ad_\xi e_1 = [ae_1 + be_2 + ce_3, e_{\cancel{3}1}] = -be_3$$

$$ad_\xi e_2 = [ae_1 + be_2 + ce_3, e_2] = ae_3$$

$$ad_\xi e_3 = [ae_1 + be_2 + ce_3, e_3] = 0$$

Hence, $$ad_\xi = \underline{\underline{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b & a & 0 \end{pmatrix}}}$$

- $\mathfrak{g}_4$ $\quad [e_1, e_3] = e_3 \;,\; [e_2, e_3] = -e_3$

$$\xi = ae_1 + be_2 + ce_3$$

$$ad_\xi e_1 = [ae_1 + be_2 + ce_3, e_1] = -ce_3$$

$$ad_\xi e_2 = [ae_1 + be_2 + ce_3, e_2] = ce_3$$

$$ad_\xi e_3 = [ae_1 + be_2 + ce_3, e_3] = (a-b)e_3$$

Hence, $$ad_\xi = \underline{\underline{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -c & c & a-b \end{pmatrix}}}$$

(b) Prove that $\mathfrak{g}_1 \simeq so(3)$, $\mathfrak{g}_2 \simeq \cancel{so}\; sl(2,\mathbb{R})$.

$so(3)$ is the Lie algebra of 3×3 skew-symmetric matrices.

If we put

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

(as a basis in $so(3)$), then

it is easy to check that

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2$$

These are just the same commutation relations as for $\mathfrak{g}_1$.

So $\quad so(3) \simeq \mathfrak{g}_1$.

Similarly, for $sl(2,\mathbb{R})$ we choose the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$[E_1, E_2] = 2E_2, \quad [E_1, E_3] - 2E_3, \quad [E_2, E_3] = E_1$$

These are just the same commutation relations as for $\mathfrak{g}_2$.

So $\quad sl(2,\mathbb{R}) \simeq \mathfrak{g}_2$.

(c) Is the adjoint representation for $\mathfrak{g}_i$ faithful?

$\mathfrak{g}_1$: yes because $ad_\xi \neq 0$ for all $\xi \neq 0$

$\mathfrak{g}_2$: yes because $ad_\xi \neq 0$ for all $\xi \neq 0$

$\mathfrak{g}_3$: no because $ad_\xi = 0$ for $\xi = e_3$

$\mathfrak{g}_4$: no because $ad_\xi = 0$ for $\xi = ae_1 + ae_2$

(d) What is the center of $\mathfrak{g}_i$?

Recall that the center of $\mathfrak{g}$ is the subalgebra that consists of all elements $\xi \in \mathfrak{g}$ such that $ad_\xi \eta = [\xi, \eta] = 0$ for any $\eta \in \mathfrak{g}$. In other words, $ad_\xi = 0$. One can also say that the center of $\mathfrak{g}$ is the kernel of the adjoint representation.

$\mathfrak{g}_1$: the center is trivial, i.e. $\{0\}$

$\mathfrak{g}_2$: the center is trivial, i.e. $\{0\}$

$\mathfrak{g}_3$: the center is one-dimensional and generated by e_3, i.e. center $= \langle e_3 \rangle$ (or, $= \text{span}\{e_3\}$)

$\mathfrak{g}_4$: the center is one-dimensional and generated by $\xi = e_1 + e_2$, i.e. center $= \langle e_1+e_2 \rangle$ (or, $= \text{span}\{e_1+e_2\}$)

(e) Find a faithful representation for $\mathfrak{g}_3$, and for $\mathfrak{g}_4$.

To do this, we need first to "understand" what $\mathfrak{g}_3$ and $\mathfrak{g}_4$ are. These are three-dimensional Lie algebras. It is not hard to recognize them. We know two natural Lie algebras of dimension 3:

$$\left\{\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}\right\} \quad \text{strictly upper triangular } 3\times 3 \; = \mathfrak{n}_3$$

and

$$\left\{\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\right\} \quad \text{upper triangular } 2\times 2 \; = \mathfrak{t}_2$$

It is not hard to notice that $\mathfrak{n}_3 \simeq \mathfrak{g}_3$ and $\mathfrak{t}_2 \simeq \mathfrak{g}_4$

Indeed, if we put

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(basis in $\mathfrak{n}_3$)

then $[E_1, E_2] = E_3$

$[E_2, E_3] = 0$

$[E_1, E_3] = 0$

these are exactly the same commutation relations as for $\mathfrak{g}_3$.

Similarly, if we put

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(basis in $\mathfrak{t}_2$)

then

$[E_1, E_2] = 0$

$[E_1, E_3] = E_3$

$[E_2, E_3] = -E_3$

These are exactly the same commutation relations as for $\mathfrak{g}_4$.

But the fact that $\mathfrak{n}_3 \simeq \mathfrak{g}_3$ and $\mathfrak{t}_2 \simeq \mathfrak{g}_4$ is just another way of saying that

$\mathfrak{g}_3$ has the following faithful representation: $\xi = ae_1 + be_2 + ce_3 \longmapsto \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$

and similarly

$\mathfrak{g}_4$ has the following faithful representation: $\xi = ae_1 + be_2 + ce_3 \longmapsto \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$

(f) Prove that these Lie algebras are not isomorphic to each other.

$\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semisimple

whereas

$\mathfrak{g}_3$ is nilpotent

and

$\mathfrak{g}_4$ is solvable (but not nilpotent).

Thus $\mathfrak{g}_1 \not\cong \mathfrak{g}_3 \not\cong \mathfrak{g}_4 \not\cong \mathfrak{g}_1 \quad \mathfrak{g}_2 \not\cong \mathfrak{g}_3$, $\mathfrak{g}_2 \not\cong \mathfrak{g}_4$

~~Th~~ To show that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are not isomorphic we may notice that $\mathfrak{g}_1 \simeq so(3)$ has no two-dimensional subalgebras, whereas $\mathfrak{g}_2 \simeq sl(2,\mathbb{R})$ has for example

$$\mathfrak{b} = \operatorname{span}\{e_1, e_2\} =$$

$$= \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right\}.$$

No 3.

(a) Verify that Φ_i is indeed a representation.

We simply need to verify that (see Ex. 1)

$(\Phi_i)_{X_1} \circ (\Phi_i)_{X_2} = (\Phi_i)_{X_1 X_2}$. It's almost obvious:

- $(\Phi_1)_{X_1} \circ (\Phi_1)_{X_2} A = (\Phi_1)_{X_1} (X_2 A) = X_1 X_2 A = (X_1 X_2) A =$
$$= (\Phi_1)_{X_1 X_2} A$$
- $(\Phi_2)_{X_1} \circ (\Phi_2)_{X_2} A = (\Phi_1)_{X_1} (X_2^{T^{-1}} A) = X_1^{T^{-1}} X_2^{T^{-1}} A =$
$$= (X_2^T X_1^T)^{-1} A = (X_1 X_2)^{T^{-1}} A = (\Phi_2)_{X_1 X_2} A$$
- $(\Phi_3)_{X_1} \circ (\Phi_3)_{X_2} A = (\Phi_3)_{X_1} (X_2 A X_2^T) = X_1 X_2 A X_2^T X_1^T =$
$$= (X_1 X_2) A (X_1 X_2)^T = (\Phi_3)_{X_1 X_2} A$$
- $(\Phi_4)_{X_1} \circ (\Phi_4)_{X_2} A = (\Phi_4)_{X_1} (X_2^{T^{-1}} A X_2^T) = X_1^{T^{-1}} X_2^{T^{-1}} A X_2^T X_1^T =$
$$= (X_1 X_2)^{T^{-1}} A (X_1 X_2)^T = (\Phi_4)_{X_1 X_2} A$$

(b) Describe the corresponding induced representation $\phi_i = d\Phi_i$ of the Lie algebra $\mathfrak{g}$.

We use the same formula as in Ex. 1:

$$(\phi_i)_B A = \frac{d}{dt}\Big|_{\substack{\cancel{\exp tB} \\ t=0}} (\Phi_i)_{\exp tB} A$$

- $(\Phi_1)_B A = \frac{d}{dt}\Big|_{t=0} \exp tB \cdot A = BA$

- $(\Phi_2)_B A = \frac{d}{dt}\Big|_{t=0} (\exp tB)^{T^{-1}} A = \frac{d}{dt}\Big|_{t=0} \exp(-tB^T) A = -B^T A$

- $(\Phi_3)_B A = \frac{d}{dt}\Big|_{t=0} \left(\exp tB \cdot A \cdot (\exp tB)^T\right) = BA + AB^T$

- $(\Phi_4)_B A = \frac{d}{dt}\Big|_{t=0} \left((\exp tB)^{T^{-1}} A (\exp tB)^T\right) = \frac{d}{dt}\Big|_{t=0} \left(\exp(-tB^T) A \exp(tB^T)\right) =$

$$= -B^T A + AB^T = [A, B^T]$$

(c) Show that Φ_1, Φ_2, Φ_3 are reducible.

We need to find non-trivial invariant subspaces for Φ_1, Φ_2, Φ_3.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

It is easy to see that the matrices $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ (and, similarly, ~~mat~~ matrices $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$) form an invariant subspace. Indeed, for any $X \in G$ and $A = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$, we have

$$XA = X \cdot \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \leftarrow \text{this matrix belongs to the same subspace.}$$

Therefore, Φ_1 is reducible.

The same subspace is invariant for Φ_2, so Φ_2 is reducible too.

For Φ_3, the situation is quite different, but still we can find an invariant subspace.
For example, we can take the subspace of symmetric matrices $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$. To verify that this subspace is invariant under Φ_3, we only need to check the XAX^T is symmetric (as soon as A is so!).

It's easy:

$$\left(XAX^T\right)^T = X^{T^T} A^T X^T = XA^TX^T = XAX^T \text{ (since } A \text{ is symmetric).}$$

Hence XAX^T is symmetric, and the space of symmetric matrices is invariant.
Thus, Φ_3 is reducible.

(d) For $G = SL(2, \mathbb{R})$, describe the invariants of Φ_1.

We need to find functions $f(A)$ (where A is a 2×2 matrix) such that $f(A) = f(XA)$ for any $X \in SL(2,\mathbb{R})$.
One of such functions is evident: $f(A) = \det A$.
Indeed $f(XA) = \det(XA) = \det X \cdot \det A = 1 \cdot \det A = \det A = f(A)$.
OK.

Now let us ask ourselves the following question: how many independent invariants exist for Φ_1?

We know that the number of independent invariants cannot exceed $\dim V - \dim \mathcal{O}$, where V is the space of representation

(in our case, V is the space of 2×2 matrices, so $\dim V = 4$)

and $\dim \mathcal{O}$ is the maximal dimension of orbits ~~for~~

Let us try to compute $\dim \mathcal{O}$.

Take, for example, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and find

$$\dim \mathcal{O}(A) = \dim G - \dim St(A)$$

$G = SL(2,\mathbb{R})$, so $\dim G = 3$.

$$St(A) = \{X \in SL(2,\mathbb{R}) \text{ s.t. } XA = A\}$$

if A is non-degenerate, then $X = Id$

So $St(A) = \{Id\}$, i.e. consists of the identity matrix only.

Thus ~~$St(A)$~~ $\dim St(A) = 0$

Hence, $\dim \mathcal{O}(A) = 3 - 0 = 3$, and therefore the number of independent invariants cannot exceed

$$\dim V - \dim \mathcal{O} = 4 - 3 = 1.$$

But we have already found one invariant $f(A) = \det A$.

Conclusion: the only independent invariant of this representation is $f(A) = \det A$.