Model answers to Week 05 review worksheet — exercises for §5

E5.1 (tensor product of modules; the dual module — carried over from the week 04 worksheet) Our goal is to show that the class of modules over a Hopf algebra H is closed under tensor products and duals.

(a) Given an algebra A and A-modules V and W, define an $A\otimes A$-module structure on $V\otimes W$.

(b) Let H be a bialgebra. Use the coproduct $\Delta\colon H\to H\otimes H$ and (a) to make $V\otimes W$ an H-module whenever V and W are.

(c) If V is an A-module, show that $\triangleleft\colon V^*\otimes A\to V^*$ where, for $\phi\in V^*$, $\phi\triangleleft a$ is the linear functional on V defined by $\langle\phi\triangleleft a, v\rangle=\langle\phi, a\triangleright v\rangle$, is a *right action* of A on V^*. (Write down the definition of a right action.)

(d) If $\triangleleft$ is a right action of a Hopf algebra H, show that $\triangleright$ defined by the rule "$h\triangleright = \triangleleft Sh$" where $S\colon H\to H$ is the antipode, is a (left) action. Conclude from (c) that if V is an H-module then so is V^*.

Answer to E5.1. (a) The algebra $A\otimes A$ acts on the space $V\otimes W$ via

$$a,b\in A,\ v\in V,\ w\in W \qquad\longmapsto\qquad (a\otimes b)\triangleright(v\otimes w)=(a\triangleright v)\otimes(b\triangleright w).$$

(As usual, we write an action of a pure tensor on a pure tensor, implying that it is extended bilinearly.) It is easy to check that this is an action, using the definition of multiplication and unit on $A\otimes A$.

(b) To act by $h\in H$ on $V\otimes W$, we first map h to $H\otimes H$ using Δ, then act by Δh on $V\otimes W$. This results in the following H-action on $V\otimes W$:

$$h\triangleright(v\otimes w)=(h_{(1)}\triangleright v)\otimes(h_{(2)}\triangleright w).$$

One checks that this is an action using the fact that $\Delta\colon H\to H\otimes H$ is a homomorphism of algebras.

(c) A right action of an algebra A on a space V is a linear map $\triangleleft : V\otimes A\to V$, $v\otimes a\mapsto v\triangleleft a$, which obeys

$$v\triangleleft(ab)=(v\triangleleft a)\triangleleft b,\qquad v\triangleleft 1_A=v,\qquad \forall v\in V,\ a,b\in A.$$

The definition of $\triangleleft\colon V^*\otimes A\to V^*$ can be rewritten as follows. If $a\in A$, write $a\triangleright$ for the linear map $v\mapsto a\triangleright v$ from V to V. The axioms of left action say $(ab)\triangleright=(a\triangleright)\circ(b\triangleright)$ and $1_A\triangleright=\mathrm{id}_V$.

Likewise, write $\triangleleft a$ for the linear map $\phi\mapsto\phi\triangleleft a$ from V^* to V^*. Then the definition of $\triangleleft$ means

$$\triangleleft a=(a\triangleright)^*,$$

that is, $\triangleleft a$ is the adjoint (contragredient) of the map $a\triangleright$, see Definition 1.16.

We now use E1.4 which says that $(ML)^*=L^*M^*$ for linear maps L and M. Therefore,

$$\triangleleft(ab)=((ab)\triangleright)^*=((a\triangleright)\circ(b\triangleright))^*=(b\triangleright)^*\circ(a\triangleright)^*.$$

But this is exactly the statement $\phi\triangleleft(ab)=(\phi\triangleleft a)\triangleleft b$ that we needed to verify: the effect of ab acting on ϕ is the same as of a acting on ϕ first, followed by b.

Finally, $\triangleleft 1_A=\mathrm{id}_{V^*}$ follows from $(\mathrm{id}_V)^*=\mathrm{id}_{V^*}$.

(d) That $h\triangleright$ defined as $\triangleleft Sh$ is a left action follows easily from the fact that $S\colon H\to H$ is an antihomomorphism.

E5.2 (primitive elements in $\mathbb{C}\langle X\rangle$) The free algebra $\mathbb{C}\langle X\rangle$ is a Hopf algebra where all $x\in X$ are primitive.

(a) Let $x\in X$, $n\ge 2$. Show that x^n is not primitive in $\mathbb{C}\langle X\rangle$. ($x^n$ is the monomial $xx\ldots x$ of length n.)

(b) Suppose $|X|>1$. Show that $\mathbb{C}\langle X\rangle$ has primitive elements of every positive degree. Here we refer to a linear combination of monomials of length d as a (homogeneous) element of degree d.

Answer to E5.2. (a) It is useful to recall

The Binomial Theorem. Suppose that a, b are elements of an associative algebra A. If $ab = ba$ then, for all $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Here $\binom{n}{k}$ is the non-negative integer defined by $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$ and equal to the number of k-element subsets of an n-element set. (If A is an algebra over a field $\Bbbk$, an integer N is interpreted as $1_\Bbbk + 1_\Bbbk + \cdots + 1_\Bbbk$ with N terms.)

The Binomial Theorem is proved by a standard induction argument. We note that it is really an "if and only if" statement: if the equation holds for all n then ab must be equal to ba. In fact, already the equation for $n = 2$ implies $ab = ba$ as $(a+b)^2$ is always equal to $a^2 + ab + ba + b^2$ rather than $a^2 + 2ab + b^2$.

If x is a primitive element of some bialgebra H then $\Delta x = x \otimes 1 + 1 \otimes x$. Putting $a = x \otimes 1$, $b = 1 \otimes x$, we note that $ab = ba$ in $H \otimes H$: both equal $x \otimes x$. Hence we are in the situation of the Binomial Theorem, and

$$\Delta(x^n) = (\Delta x)^n = \sum_{k=0}^{n} \binom{n}{k} (x \otimes 1)^{n-k} (1 \otimes x)^k = \sum_{k=0}^{n} \binom{n}{k} (x^{n-k} \otimes 1)(1 \otimes x^k) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k.$$

If $H = \mathbb{C}\langle X\rangle$ and $x \in X$, the coefficient $\binom{n}{1}$ of $x^{n-1} \otimes x$ is not zero. Since the monomials $x^r \otimes x^s$ are linearly independent in $\mathbb{C}\langle X\rangle^{\otimes 2}$ as they form part of the standard basis of $\mathbb{C}\langle X\rangle^{\otimes 2}$, we conclude that $\Delta(x^n) \neq x^n \otimes 1 + 1 \otimes x^n$ and so x^n is not primitive.

Remark. If $\mathbb{C}$ is replaced by a field $\Bbbk$ of positive characteristic, the above argument fails as binomial coefficients may become zero in $\Bbbk$. For example, any prime p divides $\binom{p}{k}$ for $1 \le k \le p-1$, so if $p = \operatorname{char} \Bbbk$ then $\Delta(x^p) = x^p \otimes 1 + 1 \otimes x^p$ as all intermediate coefficients vanish, so x^p is primitive.

(b) Suppose x, y are two distinct elements of X. Define $p_1 = y$. Then p_1 is primitive of degree 1 in $\mathbb{C}\langle X\rangle$.

For each $n \in \mathbb{N}$, define $p_{n+1} = xp_n - p_n x$. Then $p_{n+1} \in P(\mathbb{C}\langle X\rangle)$ by part (a). So if we show that $p_{n+1} \neq 0$, then we found a primitive of degree $n+1$.

To show that $p_n \neq 0$ for all n, we show that the coefficient of the monomial $x^{n-1}y$ in p_n is 1. Recall that noncommutative monomials in x, y are linearly independent in $\mathbb{C}\langle X\rangle$, by definition of $\mathbb{C}\langle X\rangle$.

We do this by induction in n. The base case is $n = 1$, $p_1 = y = x^0 y$ for which the claim is true.

For the inductive step $n \to n+1$, assume that $x^{n-1}y$ occurs in p_n with coefficient 1. Note that a monomial in p_{n+1} can only arise from either pre-multiplication or post-multiplication by x of a monomial in p_n. The only way the monomial $x^n y$, which does not end in x, arises in $p_{n+1} = xp_n - p_n x$ is as $x \cdot x^{n-1}y$. It follows that $x^n y$ occurs in p_{n+1} with coefficient 1, as claimed.

E5.3 (The universal mapping property of $U(\mathfrak{g})$) Review the definition of **Lie bracket**, $[\cdot,\cdot]$, **Lie algebra**, $\mathfrak{g}$, and the **universal enveloping algebra** $U(\mathfrak{g})$ of $\mathfrak{g}$.

Let X be a basis of $\mathfrak{g}$ and let $f\colon \mathfrak{g} \to A$ be a **Lie map** from $\mathfrak{g}$ to some associative algebra A, so f is linear and

$$f([x,y]) = f(x)f(y) - f(y)f(x) \quad \forall x, y \in \mathfrak{g}.$$

That is, f takes the Lie bracket on $\mathfrak{g}$ to the commutator bracket on A.

Let $F\colon \mathbb{C}\langle X\rangle \to A$ be the unique algebra homomorphism such that $F|_X = f$, given by the universal mapping property of the free algebra, Proposition 2.12. Prove: F factors through $U(\mathfrak{g})$, i.e., is the composite map

$$F\colon \mathbb{C}\langle X\rangle \twoheadrightarrow \mathbb{C}\langle X\rangle / I(\mathfrak{g}) = U(\mathfrak{g}) \xrightarrow{\overline{F}} A$$

for some (unique) algebra homomorphism $\overline{F}$.

Answer to E5.3. Briefly, by properties of the quotient space and quotient algebra, the claim is equivalent to saying that $I(\mathfrak{g}) \subseteq \ker F$. To check this, we recall that the kernel of any algebra homomorphism is a two-sided ideal. So if $xy - yx - [x,y] \in \ker F$ for all $x, y \in X$ (the chosen basis of $\mathfrak{g}$), then the ideal $I(\mathfrak{g})$, which is generated by the elements $xy - yx - [x,y]$ of $\mathbb{C}\langle X\rangle$, lies in $\ker F$.

We are thus left to verify that $F(xy - yx - [x,y]) = 0$ for all $x, y \in X$. Since F is an algebra homomorphism, we have $F(xy - yx - [x,y]) = F(x)F(y) - F(y)F(x) - F([x,y])$. Recall $F|_X = f$, and F and f are both linear, so F agrees with f on $\operatorname{span} X = \mathfrak{g}$. Hence

$$F(x)F(y) - F(y)F(x) - F([x,y]) = f(x)f(y) - f(y)f(x) - f([x,y]) = 0$$

by the assumption that f is a Lie map, finishing the proof.

E5.4 (A Milnor-Moore theorem) Let H be a Hopf algebra over $\mathbb{C}$. View the subspace $P(H)$ of H as a Lie algebra with the commutator bracket $[x,y]_{\mathrm{comm}} = xy - yx$, then the embedding $P(H) \hookrightarrow H$ is a Lie map which by the Universal Mapping Property, E5.3, extends to an algebra homomorphism

$$U(P(H)) \to H.$$

Prove that this homomorphism is **injective.** (Hint: use the Heyneman-Radford theorem for the polynomial coalgebra.) Conclude that if $P(H) \neq 0$ then H must be infinite-dimensional.

Answer to E5.4. Denote $\mathfrak{g} = P(H)$ and choose some ordered basis X of $\mathfrak{g}$. In the lecture, we proved the PBW theorem which says that the through map

$$\mathbb{C}[X] \hookrightarrow \mathbb{C}\langle X\rangle \twoheadrightarrow U(\mathfrak{g})$$

is an isomorphism of coalgebras. We now observe that the map

$$U(\mathfrak{g}) \xrightarrow{\phi} H,$$

which we want to show to be injective, is a Hopf algebra morphism, and in particular a coalgebra morphism. Indeed, "coalgebra morphism" means that the equations

$$\Delta \circ \phi = (\phi \otimes \phi) \circ \Delta, \quad \epsilon \circ \phi = \epsilon$$

are satisfied. In our case, both sides of each equations are algebra homomorphisms. Since they agree on primitive generators of $U(\mathfrak{g})$ (elements of $\mathfrak{g}$), they agree everywhere on $U(\mathfrak{g})$.

The isomorphism $\mathbb{C}[X] \cong U(\mathfrak{g})$ of coalgebras identifies the space $\mathbb{C}X$ with $\mathfrak{g}$. In the lecture we proved the Heyneman-Radford theorem for the polynomial coalgebra, which says that a coalgebra morphism is injective on $\mathbb{C}[X]$ if it is injective on $\mathbb{C}X$. The morphism $U(\mathfrak{g}) \to H$ is injective on $\mathfrak{g}$ by definition, hence is injective on $U(\mathfrak{g})$ by Heyneman-Radford.

It remains to note that $\mathbb{C}[X]$ is infinite-dimensional when X is not empty: indeed, if $x \in X$, the standard monomials $1, x, x^2, \ldots$ are linearly independent. Hence $\dim U(P(H)) = \infty$ if $P(H) \neq \{0\}$. (This result only holds in characteristic zero.)

E5.5 (expand in standard monomials, calculate the antipode in $U(\mathfrak{sl}_2)$) Recall the presentation

$$U(\mathfrak{sl}_2) = \langle X,\, H,\, Y \mid HX - XH = 2X,\ HY - YH = -2Y,\ XY - YX = H\rangle\,.$$

The Hopf algebra structure of $U(\mathfrak{sl}_2)$ is fully determined by saying that the generators X, H, Y are primitive.

We order the generators so that $X \prec H \prec Y$, so that the standard monomials are $X^m H^n Y^p$ with $m, n, p \geq 0$.

(a) Express YHX as a linear combination of standard monomials.

(b) Think of a way to justify the claim that an arbitrary monomial in X, H, Y can be expressed, in $U(\mathfrak{sl}_2)$, as a linear combination of standard monomials.

(c) What is the antipode of XY?

Answer to E5.5. (a) To rewrite YHX as a linear combination of standard monomials, first use $YH = HY + 2Y$ so that $YHX = HYX + 2YX = (H+2)YX$.

Substitute $YX = XY - H$ to obtain $(H+2)(XY-H) = HXY + 2XY - H^2 - 2H$. The only non-standard monomial here is HXY, which needs to be written as $(XH+2X)Y = XHY + 2XY$. The final answer is therefore $XHY + 4XY - H^2 - 2H$.

(b) We refer to the ordered set of generators of a Lie algebra $\mathfrak{g}$ as an alphabet (thus, for $\mathfrak{sl}_2$ the alphabet will be X, H, Y) and to an element of the alphabet as a symbol.

First, we prove a **Lemma:** if x is a symbol and M is a monomial of degree d in the given alphabet, then $xM - Mx$ is a linear combination of monomials of degree d in $U(\mathfrak{g})$.

To prove this, observe that if $M = a_1 a_2 \dots a_d$ where $a_1, \dots, a_d$ are symbols, then in $U(\mathfrak{g})$ we have

$$\begin{aligned} xM - Mx &= (xa_1 - a_1x)a_2 \dots a_d + a_1(xa_2 - a_2x)a_3 \dots a_d + \dots + a_1 \dots a_{d-1}(xa_d - a_dx) \\ &= [x, a_1]a_2 \dots a_d + a_1[x, a_2]a_3 \dots a_d + \dots + a_1 \dots a_{d-1}[x, a_d] \end{aligned}$$

(intermediate terms in the first line collapse). As $[x, a_i]$ is a linear combination of symbols, the row 2 is a linear combination of monomials of degree d (despite xM and Mx being monomials of degree $d+1$). Lemma is proved.

We can now use **induction** in d to prove that any monomial of degree d is expressible in $U(\mathfrak{g})$ as a linear combination of standard monomials of degree $\le d$.

This is true in the base cases $d = 0$ and $d = 1$, as any monomial of degree ≤ 1 is standard.

Suppose the claim holds for d, and consider a monomial of degree $d+1$. It can be written as xM where x is a symbol and M is a monomial of degree d. By the induction hypothesis, M is equal, in $U(\mathfrak{g})$, to a linear combination of standard monomials; replacing M by such a linear combination, we see that without the loss of generality we may assume that M is standard.

Then there is a place in M where the symbol x can be inserted to obtain a standard monomial. That is, $M = NP$ where N, P are standard monomials such that NxP is standard. (Simply take N to be the submonomial of M formed by all symbols $< x$. We do not exclude the case where either N or P is of length zero.) We have

$$xM = xNP = (xN - Nx)P + NxP.$$

By Lemma, $(xN - Nx)P$ is a linear combination of monomials of degree d, and so is expressible as a linear combination of standard monomials by the induction hypothesis. Moreover, NxP is a standard monomial by construction. We have expressed xM as a linear combination of standard monomials.

By induction, the claim is true for all d.

(c) Since the antipode is antimultiplicative, Proposition 4.11, we have $S(XY) = S(Y)S(X)$. Since $S(x) = -x$ for all primitive x, E4.3, and X, Y are primitive by definition of $U(\mathfrak{sl}_2)$, we have $S(XY) = (-Y)(-X) = YX$.

It is better to express the answer as a linear combination of standard monomials. Since $XY - YX = H$, we have $YX = XY - H$.

Part B. Extra exercises

E5.6 (tensor product exercise) The following fact is used in the proof of the PBW theorem: if X, Y are vector spaces and $f\colon X \to Y$ is an injective linear map, then $f \otimes f\colon X \otimes X \to Y \otimes Y$ is injective. Prove it.

Answer to E5.6. Denote $Y' = f(X)$, so that Y' is a subspace of X. Then $f\colon X \to Y'$ is a bijective linear map, so there exists a linear map $g\colon Y' \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_{Y'}$. Consider the linear map $g \otimes g\colon Y' \otimes Y' \to X \otimes X$. We have

$$(g \otimes g)(f \otimes f) = gf \otimes gf = \mathrm{id}_X \otimes \mathrm{id}_X = \mathrm{id}_{X \otimes X}.$$

In the same way, $(f \otimes f)(g \otimes g) = \mathrm{id}_{Y' \otimes Y'}$. This shows that $f \otimes f\colon X \otimes X \to Y' \otimes Y'$ is invertible. Since $Y' \otimes Y'$ is a subspace of $Y \otimes Y$, $f \otimes f\colon X \otimes X \to Y \otimes Y$ is injective, as claimed.