Exercise Sheet 9

MAGIC009 - Category Theory

December 3rd, 2023

1. Let $\mathbb{C}$ be a small category and $c \in \mathbb{C}$. Recall that we have a contravariant functor

$$\mathbb{C}(-, c) \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}.$$

Prove that, for any presheaf $X \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, any natural transformation

$$\phi \colon \mathbb{C}(-, c) \to X$$

is completely determined by the value of its component ϕ_c on the identity map $1_c \in \mathbb{C}(c, c)$. *Hint.* Use the naturality of ϕ and the fact that we are working with **sets** of maps.

2. Let $\mathbb{C}$ be a category. For any category $\mathbb{B}$ and diagram $X \in \mathbb{C}^{\mathbb{B}}$, there is a functor

$$\mathrm{Cone}_X(-) \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$$

mapping an object $C \in \mathbb{C}$ to the set of natural transformations from ΔC to X. Unfold explicitly what it means for this functor to be representable.

3. Fix two sets B and C.

   (i) Check that the function sending a set A to the set of functions $f \colon A \times B \to C$ extends to a functor
   $$\mathrm{Hom}(- \times B, C) \colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$$

   (ii) Prove that this functor is represented by the set C^B of functions from B to C.

4. Define the category of pointed sets $\mathbf{Set}_*$ as follows:

   - Objects are pointed sets, i.e. pairs (X, x) consisting of a set X and an element $x \in X$, called the *basepoint*,
   - Maps $f \colon (X, x) \to (Y, y)$ are basepoint-preserving functions, i.e. functions $f \colon X \to Y$ such that $f(x) = y$.

   (i) Prove that the function mapping a pointed set (X, x) to its underlying set X extends to a functor
   $$U \colon \mathbf{Set}_* \to \mathbf{Set}$$

   (ii) Prove that this functor U is representable.

5. Let $F \colon \mathbb{C} \to \mathbb{D}$ be a functor. For $A \in \mathbb{D}$, define a functor $Q_A \colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ such that the following conditions are equivalent:

   (a) Q_A is representable,

   (b) The comma category $F \downarrow A$ (as defined in Exercise 4 of Exercise Sheet 4) has a terminal object.