Suggested exercises Sections 2 and 3

Section 2: Modules

Exercise 2.21. Let R be a commutative ring and let M be an R-module. We say that M is *faithfully flat* if it satisfies the following property: A sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ in $R-$mod is exact if and only if the sequence $0 \longrightarrow M \otimes_R A \xrightarrow{f} M \otimes_R B \xrightarrow{g} M \otimes_R C \longrightarrow 0$ is exact in $R-$mod.

i. Prove that an R-module M is faithfully flat if and only if M is flat and if $M \otimes_R N = 0$ implies $n = 0$ for any R-module N.

ii. As $\mathbb{Z}$-module, is $\mathbb{Q}$ faithfully flat, flat or neither?

iii. Let $\varphi : R \to S$ be a ring homomorphism, and let M be an S-module.

iv. Let $\varphi : R \to S$ be a ring homomorphism, and let M be an S-module. Prove that $\mathrm{res}_\varphi M$ is flat as R-module if and only if the localisation $\mathrm{res}_\varphi(M_P)$ is flat over R_{P^c} for all $P \in \mathrm{Spec}(R)$, where $P^c = \varphi^{-1}(P)$ is the contraction of P.

Section 3: Integral dependence

Exercise 3.1. i. Let $f \in \mathbb{Z}[x]$ and let $\frac{a}{b} \in \mathbb{Q}$, in reduced form, such that $f(\frac{a}{b}) = 0$. Prove that b divides the leading coefficient of f and that a divides the constant term

ii. Deduce from it that $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

Exercise 3.2. Let $z = e^{\frac{2\pi \mathrm{i}}{3}} \in \mathbb{C}$ and let $R = \mathbb{Z}[z]$. Prove that R is integrally closed in $\mathbb{Q}[z]$.

Exercise 3.3. i. Prove that $\sqrt{2} + \sqrt{3} \in \mathbb{R}$ is integral over $\mathbb{Z}$.

ii. Find its *minimal polynomial* in $\mathbb{Q}[x]$, that is, the unique monic irreducible polynomial $f \in \mathbb{Q}[x]$ such that $f(\sqrt{2} + \sqrt{3}) = 0$.

Exercise 3.8. Let k be a field and let $R = k[x, y]$. Calculate I^{-1} where $I = (x, y)$, and prove that $(I^{-1})^{-1} \neq I$.

Exercise 3.9. Let R be a Dedekind domain with field of fractions K, and let M_1, M_2 be fractional ideals of R. Prove the following.

i. Every nonzero ideal of R is fractional.

ii. The sum $M_1 + M_2$ and the product

$$M_1 M_2 = \{\sum_{i=1}^{n} \frac{a_i}{b_i}\frac{c_i}{d_i} \mid \frac{a_i}{b_i} \in M_1,\ \frac{c_i}{d_i} \in M_2,\ n \in \mathbb{N}\} \quad \text{are fractional ideals of } R.$$

iii. M_1^{-1} is a fractional ideal of R.

iv. If $M_1 M_2 = R$, then $M_2 = M_1^{-1}$.

v. The set of invertible fractional ideals of R forms an abelian multiplicative group with multiplicative identity R.

Exercise 3.10. Let R be a Dedekind domain and let U be a multiplicative subset of R. Prove that R_U is Dedekind too.

Exercise 3.11. Let $\mathbb{Z}[t^2] = R \subseteq S = \mathbb{Z}[t, \sqrt{3}]$. Apply the going-up theorem to find chains of prime ideals in S lifting the following chains in R. In each case describe the inclusions $R/P_1 \hookrightarrow S/Q_1$ and $R/P_2 \hookrightarrow S/Q_2$.

(i) $0 \subseteq (13) \subseteq (t^2 - 1, 13)$.

(ii) $0 \subseteq (t^2 - 1) \subseteq (t^2 - 1, 13)$.

(iii) $0 \subseteq (t^2 + 1) \subseteq (t^2 + 1, 13)$.

(iv) $0 \subseteq (t^2) \subseteq (t^2, 13)$.