Suggested exercises Sections 2 and 3

Section 2: Modules

Exercise 2.21. Let R be a commutative ring and let M be an R-module. We say that M is *faithfully flat* if it satisfies the following property: A sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ in $R-$mod is exact if and only if the sequence $0 \longrightarrow M \otimes_R A \xrightarrow{f} M \otimes_R B \xrightarrow{g} M \otimes_R C \longrightarrow 0$ is exact in $R-$mod.

i. Prove that an R-module M is faithfully flat if and only if M is flat and if $M \otimes_R N = 0$ implies $N = 0$ for any R-module N.

ii. As $\mathbb{Z}$-module, is $\mathbb{Q}$ faithfully flat, flat or neither?

iii. Let $\varphi : R \to S$ be a ring homomorphism, and let M be an S-module. Prove that $\operatorname{res}_\varphi M$ is flat as R-module if and only if the localisation $\operatorname{res}_\varphi(M_P)$ is flat over R_{P^c} for all $P \in \operatorname{Spec}(R)$, where $P^c = \varphi^{-1}(P)$ is the contraction of P.

Solution.

i. By definition, if M is faithfully flat, then M is flat and if there is an R-module N such that $M \otimes_R N = 0$, then $N = 0$. Conversely, suppose that $M \otimes_R A \xrightarrow{1\otimes f} M \otimes_R B \xrightarrow{1\otimes g} M \otimes_R C$ is an exact sequence in $R-$mod. Since M is flat, $M \otimes_R -$ maps kernels to kernels, and images to images. Therefore, $\operatorname{im}(1 \otimes (gf)) = \operatorname{im}((1 \otimes g)(1 \otimes f)) = 0$. By assumption, $\operatorname{im}(gf) = 0$, i.e. the composition gf is zero. Therefore, $A \xrightarrow{f} B \xrightarrow{g} C$ is a complex. Now, $M \otimes_R \ker(g)/\operatorname{im}(f) = \ker(1 \otimes g)/\operatorname{im}(1 \otimes f) = 0$, which implies that $\ker(g)/\operatorname{im}(f) = 0$ by assumption.

ii. $\mathbb{Q}$ is not faithfully flat since $\mathbb{Q} \otimes_\mathbb{Z} \mathbb{Z}/2 = 0$, but $\mathbb{Q}$ is flat since $\mathbb{Q} = \mathbb{Z}_{(0)}$ is the localisation of $\mathbb{Z}$ at (0), see Proposition 2.35.

iii. Localisation and restriction preserve exact sequences.

Section 3: Integral dependence

Exercise 3.1. i. Let $f \in \mathbb{Z}[x]$ and let $\frac{a}{b} \in \mathbb{Q}$, in reduced form, such that $f(\frac{a}{b}) = 0$. Prove that b divides the leading coefficient of f and that a divides the constant term

ii. Deduce from it that $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

Solution.

i. By assumption, let $f = c_0x^n + \cdots + c_{n-1}x + c_n \in \mathbb{Z}[x]$ such that $f(\frac{a}{b}) = 0$, i.e. $0 = c_0\frac{a^n}{b^n} + \cdots + c_{n-1}\frac{a}{b} + c_n$. Multiply by b^n both sides, and observe that we get two equations of the form

$$c_0a^n = bg \quad \text{and} \quad c_nb^n = ah, \quad \text{for some } g, h \in \mathbb{Z}[a, b],$$

from which we deduce that b divides c_0 and a divides c_n as required.

ii. By the first part, if $\frac{a}{b} \in \mathbb{Q}$ is integral over $\mathbb{Z}$, then b divides the leading coefficient of any polynomial $f \in \mathbb{Z}[x]$ that has $\frac{a}{b}$ as root. So $b = \pm 1$, i.e. $\frac{a}{b} \in \mathbb{Z}$. The result generalises to any UFD.

Exercise 3.2. Let $z = e^{\frac{2\pi i}{3}} \in \mathbb{C}$ and let $R = \mathbb{Z}[z]$. Prove that R is integrally closed in $\mathbb{Q}[z]$.

Solution. Note that $\mathbb{Q}[z]$ is the field of fractions of R, since $z^{-1} = e^{\frac{4\pi}{3}} = z^2 \in R$, and $\mathbb{Q}[z]$ is a field. So the result follows since $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

Exercise 3.3. i. Prove that $\sqrt{2} + \sqrt{3} \in \mathbb{R}$ is integral over $\mathbb{Z}$.

ii. Find its *minimal polynomial* in $\mathbb{Q}[x]$, that is, the unique monic irreducible polynomial $f \in \mathbb{Q}[x]$ such that $f(\sqrt{2} + \sqrt{3}) = 0$.

Solution.

i. We calculate $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$ and $(\sqrt{2} + \sqrt{3})^4 = 49 + 20\sqrt{6}$, showing that $(\sqrt{2} + \sqrt{3})$ is a root of $x^4 - 10x^2 + 1 \in \mathbb{Z}[x]$.

ii. The minimal polynomial is $f = x^4 - 10x^2 + 1$. Indeed, f is monic irreducible in $\mathbb{Q}[x]$, and $f(\sqrt{2} + \sqrt{3}) = 0$.

Exercise 3.8. Let k be a field and let $R = k[x, y]$. Calculate I^{-1} where $I = (x, y)$, and prove that $(I^{-1})^{-1} \neq I$.

Solution. Note that R is not a Dedekind domain (since nonzero prime ideals need not be maximal). By definition, $I^{-1} = \{\frac{f}{g} \in k(x, y) \mid \frac{f}{g} I \subseteq k[x, y]\}$ (wlog, in any such fraction, we can assume that f, g are coprime, and with little work, we may assume that $g = 1$ too). Let $\frac{f}{g} \in I^{-1}$. Since $y \in I$, we have $\frac{f}{g} y \in R$, which forces $g \mid y$ in $k[x, y]$, for all denominators $g \in I^{-1}$, and similarly, we see that $g \mid x$ in $k[x, y]$ for all such denominators g. Therefore, g must be a nonzero constant, i.e. $I^{-1} = k[x, y]$, and we have $k[x, y]^{-1} = k[x, y] \neq I$.

Exercise 3.9. Let R be a Dedekind domain with field of fractions K, and let M_1, M_2 be fractional ideals of R. Prove the following.

i. Every nonzero ideal of R is fractional.

ii. The sum $M_1 + M_2$ and the product

$$M_1 M_2 = \{\sum_{i=1}^{n} \frac{a_i}{b_i} \frac{c_i}{d_i} \mid \frac{a_i}{b_i} \in M_1,\ \frac{c_i}{d_i} \in M_2,\ n \in \mathbb{N}\} \quad \text{are fractional ideals of } R.$$

iii. M_1^{-1} is a fractional ideal of R.

iv. If $M_1 M_2 = R$, then $M_2 = M_1^{-1}$.

v. The set of invertible fractional ideals of R forms an abelian multiplicative group with multiplicative identity R.

Solution.

i. If I is an ideal of R, then I is a finitely generated R-submodule of K.

ii. The sum and product of finitely generated R-submodules of K are also finitely generated R-submodules of K.

iii. We know that $M_1^{-1} = \{\frac{a}{b} \in K \mid \frac{a}{b} M_1 \subseteq R\}$ is an R-submodule of K. Moreover, $aM_1^{-1} \subseteq R$ for any $a \in M_1$.

iv. If $M_1M_2 = R$, then $M_1^{-1} = M_1^{-1}R = M_1^{-1}M_1M_2 = M_2$.

v. The set of fractional ideals of R forms a multiplicative group with multiplicative identity R since we've seen that it is closed under multiplication and taking inverses, and moreover, multiplication is associative.

Exercise 3.10. Let R be a Dedekind domain and let U be a multiplicative subset of R. Prove that R_U is Dedekind too.

Solution. If R is a Noetherian ID, then R_U is a Noetherian ID. Moreover, $\mathrm{Spec}(R_U) = \{P_U \mid P \in \mathrm{Spec}(R),\ P \cap U = \emptyset\}$, and the expansion of ideals is an order preserving correspondence, by Theorem 1.36. Therefore, $\mathrm{Spec}(R_U) = \mathrm{MaxSpec}(R_U) \cup \{(0)\}$. Proposition 3.14 shows that R_U is integrally closed too.

Exercise 3.11. Let $\mathbb{Z}[t^2] = R \subseteq S = \mathbb{Z}[t, \sqrt{3}]$. Apply the going-up theorem to find chains of prime ideals in S lifting the following chains in R. In each case describe the inclusions $R/P_1 \hookrightarrow S/Q_1$ and $R/P_2 \hookrightarrow S/Q_2$.

(i) $0 \subseteq (13) \subseteq (t^2 - 1, 13)$.

(ii) $0 \subseteq (t^2 - 1) \subseteq (t^2 - 1, 13)$.

(iii) $0 \subseteq (t^2 + 1) \subseteq (t^2 + 1, 13)$.

(iv) $0 \subseteq (t^2) \subseteq (t^2, 13)$.

Solution. In all cases we have $P_0 = (0)$ and $Q_0 = (0)$. We want to find prime ideals $Q_1 \subset Q_2$ in S such that $Q_1 \cap R = P_1$ and $Q_2 = P_2$.

(i) We can choose $Q_1 = (4 - \sqrt{3})$ and $Q_2 = (t - 1, 4 - \sqrt{3})$. We have $S/Q_1 = \mathbb{Z}[t, \sqrt{3}]/(4 - \sqrt{3}) \cong \mathbb{Z}[t, s]/(s^2 - 3, 4 - s) = \mathbb{Z}[t, s]/(s - 4, 13) \cong \mathbb{F}_{13}[t]$ is an ID, and so Q_1 is prime. The inclusion $R/P_1 \hookrightarrow S/Q_1$ is the inclusion $\mathbb{F}_{13}[t^2] \hookrightarrow \mathbb{F}_{13}[t]$. Similarly, we have $S/Q_2 = \mathbb{Z}[t, \sqrt{3}]/(t - 1, 4 - \sqrt{3}) = \mathbb{F}_{13}[t]/(t - 1) \cong \mathbb{F}_{13}$, and so Q_2 is a prime ideal of S. The inclusion $R/P_2 \hookrightarrow S/Q_2$ is the identity map on $\mathbb{F}_{13} = \mathbb{Z}/(13)$.

(ii) We can choose $Q_1 = (t - 1)$ and $Q_2 = (t - 1, 4 - \sqrt{3})$. We have $R/P_1 \cong \mathbb{Z} \hookrightarrow S/Q_1 \cong \mathbb{Z}[\sqrt{3}]$. Similarly, $S/Q_2 \cong \mathbb{Z}[\sqrt{3}]/(4 - \sqrt{3}) \cong \mathbb{Z}[s]/(s^2 - 3, 4 - s) = \mathbb{Z}[s]/(s - 4, 13) \cong \mathbb{F}_{13}$. In particular, the map $R/P_2 \hookrightarrow S/Q_2$ is the identity on $\mathbb{F}_{13}$.

(iii) We have $Q_1 = (t^2 + 1)$, since $\mathbb{Z}[t, \sqrt{3}]/(t^2 + 1) \cong \mathbb{Z}[\mathrm{i}, \sqrt{3}]$ is an ID. But $(t^2 + 1, 4 - \sqrt{3})$ is not a prime ideal of S since $S/(t^2+1, 4-\sqrt{13}) \cong \mathbb{Z}[s, t]/(t^2+1, s^2-3, 4-s) = \mathbb{Z}[s, t]/(t^2+1, 13, s-4) \cong \mathbb{F}_{13}[t]/(t^2+1)$ is not an ID because $t^2+1 = (t+5)(t-5) \in \mathbb{F}_{13}[t]$. Hence let $Q_2 = (t-5, 4-\sqrt{13}) \subsetneq (t^2 + 1, 4 - \sqrt{3})$. Note that $S/Q_2 = \mathbb{Z}[s, t]/(s^2 - 3, t - 5, s - 4) = \mathbb{Z}[s, t]/(13, s - 4, t - 5) \cong \mathbb{F}_{13}$, and so the map $R/P_2 \hookrightarrow S/Q_2$ is the identity.

(iv) Let $Q_1 = (t)$ and $Q_2 = (t, 4 - \sqrt{3})$. Note that $R/P_1 \hookrightarrow S/Q_1$ is the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}[\sqrt{3}]$. Similarly, $R/P_2 \hookrightarrow S/Q_2$ is the identity on $\mathbb{Z}/(13)$.