Suggested exercises in Sections 3 and 4

Section 3: Integral dependence

Exercise 3.4. Let R be a subring of a commutative ring S and suppose that S is integral over R. Is the contraction map $c : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ injective? surjective? Prove your claims.

Section 4: Prime and maximal ideal spectra

Exercise 4.1. Find $V(1176) \subseteq \mathrm{Spec}(\mathbb{Z})$.

Exercise 4.2. Let $R = \mathbb{Q}[x]$ and let $f = x^3 - 3x^2 + 2x$.

i. Find $V((f))$.

ii. Let $I = (x^2 + 1)$ and set $\overline{R} = R/I$. Find $V((\overline{f})) \subseteq \mathrm{Spec}(\overline{R})$.

Exercise 4.3. Let $R = \mathbb{Z} \times \mathbb{Z}/42$. Find all the idempotents of R.

Exercise 4.5. Let $R = k[x]$ where k is a field. Prove that there exist proper open subsets U, U' of $\mathrm{Spec}(R)$ such that $\mathrm{Spec}(R) = U \cup U'$.

Exercise 4.6. Let $f : R \to S$ be a ring homomorphism with R, S commutative. Suppose that f is surjective. Prove that $\mathrm{im}(f^*) = V(\ker(f))$, where f^* is the induced function $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, defined by $f^*(P) = f^{-1}(P)$ for $P \in \mathrm{Spec}(S)$. (The map f^* defines a homeomorphism $\mathrm{Spec}(S) \to V(\ker(f))$.

Exercise 4.7. Let R be a commutative ring and let $P \in \mathrm{Spec}(R)$. Consider the ideal $I = \big(\{e = e^2 \in P\}\big)$ generated by the idempotents of R lying in P.

i. Prove that the only idempotents of R/I are $0, 1$.

ii. Prove that the prime ideals containing I form the connected component of $\mathrm{Spec}(R)$ containing P.

Exercise 4.8. Let R be a commutative ring and let I be a minimal ideal of R. That is, $I \neq 0$ and the only ideals of R contained in I are 0 and I. Suppose that $I^2 \neq 0$. Prove that there exists an idempotent $e \in R$ such that $I = Re$.