# § 2

# Algebras. Actions

Although the term "algebra" can refer to a wide class of algebraic structures, including Lie algebras and more exotic objects such as Jordan algebras, this chapter considers *associative unital* algebras. The goal is to formalise the definition of an algebra in the language of tensor products and commutative diagrams. This is necessary in order to introduce the dual notion of coalgebra in the next chapter.

*Note:* "associative algebra with identity" is the same as "associative unital algebra".

**2.1 Definition** An **associative unital algebra** (or simply an **algebra**) is a vector space $A$ over $\mathbb{C}$ equipped with linear maps $m\colon A\otimes A\to A$ (multiplication map) and $\eta\colon \mathbb{C}\to A$ (unit map) which satisfy the associative law,

(2.2) $$m(m\otimes \mathrm{id}_A) = m(\mathrm{id}_A\otimes m) \qquad \text{as linear maps from } A\otimes A\otimes A \text{ to } A,$$

and the identity law,

(2.3) $$m(\eta\otimes \mathrm{id}_A) = \mathrm{id}_A = m(\mathrm{id}_A\otimes \eta) \qquad \text{as linear maps from } \mathbb{C}\otimes A\cong A\cong A\otimes\mathbb{C} \text{ to } A,$$

**2.4 Remark** Traditionally, $m(a\otimes b)$ is written as $ab$ for $a,b\in A$. The requirement that $m$ is a linear map defined on $A\otimes A$ is because the product $ab$ must be bilinear in $a,b$. The unit map $\eta$ is the unique linear map such that $\eta(1)=1_A$, the identity element of $A$. Recasting the choice of an element, $1_A$, of $A$ as a linear map from $\mathbb{C}$ to $A$ allows us to consider a dual picture in the next chapter. The associative law and the identity law mean that the following diagrams commute:

$$\begin{tikzcd}
 & A\otimes A\otimes A \arrow[dl, "m\otimes\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A\otimes m"] & \\
A\otimes A \arrow[dr, "m"] & & A\otimes A \arrow[dl, "m"'] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \arrow[dl, "\cong"'] \arrow[dr, "\cong"] \arrow[dd, equal, "\mathrm{id}_A"] & \\
\mathbb{C}\otimes A \arrow[d, "\eta\otimes\mathrm{id}_A"'] & & A\otimes\mathbb{C} \arrow[d, "\mathrm{id}_A\otimes\eta"] \\
A\otimes A \arrow[r, "m"] & A & A\otimes A \arrow[l, "m"']
\end{tikzcd}$$

**2.5 Example** The ground field $\mathbb{C}$ is itself an algebra, because it has associative product and identity. In fact, the multiplication map $m_{\mathbb{C}}\colon \mathbb{C}\otimes\mathbb{C}\to\mathbb{C}$ is the isomorphism which identifies the vector spaces $\mathbb{C}\otimes\mathbb{C}$ and $\mathbb{C}$, see Corollary 1.33, and the unit map $\eta\colon\mathbb{C}\to\mathbb{C}$ is the identity map $\mathrm{id}_{\mathbb{C}}$ because $\eta(1)=1$ and $\{1\}$ is a basis of $\mathbb{C}$.

**2.6 Example** For every vector space $V$, the space $\mathrm{End}(V)$ of all linear endomorphisms of $V$ is an algebra, where the multiplication is the composition $f\circ g$ of $f,g\in\mathrm{End}(V)$, and the identity element is the identity endomorphism $\mathrm{id}_V$ of $V$.

If $\dim V = n < \infty$, $\mathrm{End}(V)$ is an $n^2$-dimensional algebra. A choice of a basis in $V$ gives rise to an isomorphism between $\mathrm{End}(V)$ and the algebra of $n \times n$ matrices.

However, if $V$ is infinite-dimensional, there is no description for a basis of $\mathrm{End}(V)$ — this algebra is "too large".

## The algebra of a monoid

One way to construct a unital associative algebra is to define an associative multiplication with identity on a set, and use that set as a basis for the algebra. This is done in the following

**2.7 Definition (algebra of a monoid)** A **monoid** is a set $\mathcal{B}$ equipped with an associative product $\cdot\colon \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ which has an idenity element $e \in \mathcal{B}$.

The **algebra of the monoid** $\mathcal{B}$ is the space $\mathbb{C}\mathcal{B}$ with multiplication $m\colon \mathbb{C}\mathcal{B} \otimes \mathbb{C}\mathcal{B} \to \mathbb{C}\mathcal{B}$, defined on the basis $\{x \otimes y\}_{x,y\in\mathcal{B}}$ of $\mathbb{C}\mathcal{B} \otimes \mathbb{C}\mathcal{B}$ by $m(x \otimes y) = x \cdot y \in \mathcal{B} \subset \mathbb{C}\mathcal{B}$. The identity element of $\mathbb{C}\mathcal{B}$ is $1_{\mathbb{C}\mathcal{B}} = e$.

To check formally that $m$ on $\mathbb{C}\mathcal{B}$ is associative, we need to show that $m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)$ as maps from $(\mathbb{C}\mathcal{B})^{\otimes 3}$ to $\mathbb{C}\mathcal{B}$. We note that these two linear maps agree on the basis $\{x \otimes y \otimes z\}_{x,y,z\in\mathcal{B}}$ of $(\mathbb{C}\mathcal{B})^{\otimes 3}$ by associativity of $\cdot$; indeed, when evaluated on $x \otimes y \otimes z$, both return $x \cdot y \cdot z \in \mathcal{B}$. Hence they agree everywhere on $(\mathbb{C}\mathcal{B})^{\otimes 3}$.

The following two classes of algebras are particular cases of the algebra of a monoid construction. These algebras are highly important in the course, and we will turn each of them into a Hopf algebra very soon.

## The group algebra $\mathbb{C}G$

**2.8 Example (the group algebra of a group $G$)** Let $G$ be a group. Then in particular $G$ is a monoid (in fact, $G$ is a monoid with extra structure, namely the inverses). Hence the **group algebra of $G$**, $\mathbb{C}G$, given by Definition 2.7, is an associative unital algebra.

## The free algebra $\mathbb{C}\langle X\rangle$ on a set $X$. The Universal Mapping Property

If $X$ is an arbitrary set, let $\mathrm{Mon}(X)$ denote the set of all **noncommutative monomials** in $X$: elements of $\mathrm{Mon}(X)$ are strings of finite length in the alphabet $X$. The operation of **concatenation**, $a \dots b \cdot y \dots z = a \dots by \dots z$, is associative and equips $\mathrm{Mon}(X)$ with the structure of a monoid, with the empty string $\emptyset$ as the identity element. This is the **free monoid on $X$**, sometimes denoted $\langle X\rangle$.

**2.9 Definition (free algebra)** The algebra $\mathbb{C}\langle X\rangle$ of the free monoid is called the **free algebra** on the set $X$.

Note that if $X$ has more than one element, $\mathbb{C}\langle X\rangle$ is not commutative: indeed, if $a, b \in X$, $a \neq b$, then $ab, ba \in \mathrm{Mon}(X)$ are two distinct elements of the basis of $\mathbb{C}\langle X\rangle$ and so $ab \neq ba$ in $\mathbb{C}\langle X\rangle$.

**2.10 Remark (the free tensor algebra $T(V)$)** (*not discussed in the video lecture*) In the literature, the free algebra $\mathbb{C}\langle X\rangle$ often appears in a different, but equivalent, form: the **free tensor algebra** $T(V)$. Let $V$ be a vector space, and for each $n \geq 0$, let $V^{\otimes n}$ denote the **$n$-fold tensor product** $\underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}}$. By convention, $V^{\otimes 0} = \mathbb{C}$. To define

$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \dots = \bigoplus_{n=0}^{\infty} V^{\otimes n},$$

we need the operation $\oplus$ of external direct sum of vector spaces: essentially, $U \oplus W$ is the space whose basis is a disjoint union of the bases for $U$ and for $W$. The multiplication map $m$ is defined on $V^{\otimes m} \otimes V^{\otimes n}$ by

$$m\colon V^{\otimes m} \otimes V^{\otimes n} \xrightarrow{\cong} V^{\otimes m+n},$$

the canonical isomorphism given by rebracketing as in Remark 1.38, and extended to the whole of $T(V)$ by bilinearity.

The moment we choose a basis $X$ of $V$, the space $V^{\otimes n}$ becomes equipped with the basis $\{x_1 \otimes \cdots \otimes x_n : x_1, \ldots, x_n \in X\}$. We thus obtain a basis for $T(V)$ identified with $\mathrm{Mon}(X)$, and an isomorphism $T(V) \cong \mathbb{C}\langle X\rangle$. The reason to use $T(V)$ instead of $\mathbb{C}\langle X\rangle$ is to avoid fixing an explicit basis for the algebra, yet in our course we do not need to avoid that.

To state the key property of the free algebra, we need the definition of a homomorphism. As with most classes of algebraic objects, a homomorphism, a map which "respects" the operations on a given object. For associative unital algebras, homomorphisms must respect the vector space structure, the product and the identity element. Formally:

**2.11 Definition** **(homomorphism, isomorphism)** Let $A, B$ be algebras. A **homomorphism** from $A$ to $B$ is a linear map $f\colon A \to B$ such that

$$f(xy) = f(x)f(y) \text{ for all } x, y \in A, \qquad f(1_A) = 1_B.$$

An **isomorphism** is a bijective homomorphism.

**2.12 Proposition** **(the universal mapping property of a free algebra)** *Let $X$ be a set. Denote the inclusion map of $X$ in the free algebra $\mathbb{C}\langle X\rangle$ by $i\colon X \hookrightarrow \mathbb{C}\langle X\rangle$. For any algebra $A$ and any map $f\colon X \to A$ of sets, there exists a unique algebra homomorphism $F\colon \mathbb{C}\langle X\rangle \to A$ such that $Fi = f$, i.e., the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"] & A \\ \mathbb{C}\langle X\rangle \arrow[ur, dashed, "\exists!", "F"'] & \end{tikzcd}$$

*Proof.* Not given, similar to Knapp 2016, Proposition 6.22. □

## Ideals and quotients. The First Isomorphism Theorem for algebras

**2.13 Definition** A **subalgebra** of an associative unital algebra $A$ is a subspace $B \subseteq A$ such that

$$m(B \otimes B) \subseteq B, \qquad \eta(\mathbb{C}) \subseteq B.$$

Equivalently, for all $b, b' \in B$ one has $bb' \in B$, and $1_A \in B$.

**2.14 Definition** **(ideal, ideal generated by a set, quotient)** Let $A$ be an associative unital algebra. An **ideal** of $A$ is a subspace $I \subseteq A$ such that

$$m(I \otimes A + A \otimes I) \subseteq I.$$

Equivalently, for all $a \in A$ and $x \in I$ one has $ax \in I$ and $xa \in I$.

Let now $T$ be any subset of an algebra $A$. The **ideal generated by** $T$ is the ideal of $A$ given by

$$ATA = \{a_1t_1b_1 + a_2t_2b_2 + \cdots + a_nt_nb_n : n \geq 0,\ a_i, b_i \in A,\ t_i \in T \text{ for all } i\}.$$

For an ideal $I$ of $A$ which is proper (i.e., $I \neq A$), we define the **quotient algebra** (or, the same, **factor algebra**) $A/I$ to be the quotient vector space $A/I$ with multiplication and identity given on cosets by

$$(a + I)(b + I) = ab + I, \qquad 1_{A/I} = 1_A + I.$$

A standard argument shows that the multiplication on $A/I$ is well defined, i.e., does not depend on the choice of a representative within a coset.

The following result is a version for associative unital algebras of the theorem which is well-known for groups.

**2.15 Theorem (the First Isomorphism Theorem)** *If $f\colon A \to B$ is an algebra homomorphism, then the **kernel** $\ker f = \{a \in A : f(a) = 0\}$ is a proper ideal of $A$, the **image** $f(A)$ of $f$ is a subalgebra of $B$, and the map*

$$\overline{f}\colon A/\ker f \to f(A)$$

*is an isomorphism of algebras.*

*Proof.* Note that $\overline{f}$ is the map on the quotient space $A/\ker f$ given by Proposition 1.20. We omit the rest of this standard proof, which can be found for example in Dummit and Foote 2004, chapter 7, theorem 7. □

## Presentations

The First Isomorphism Theorem leads to a powerful way to construct algebras, discussed next.

**2.16 Definition (presentation)** Let $X$ be a set and $\mathcal{R}$ be a subset of the free tensor algebra $\mathbb{C}\langle X\rangle$. Denote by $I_{\mathcal{R}}$ the ideal of $\mathbb{C}\langle X\rangle$ generated by $\mathcal{R}$. We define the following quotient algebra:

$$\mathbb{C}\langle X \mid \mathcal{R}\rangle = \mathbb{C}\langle X\rangle/I_{\mathcal{R}}.$$

An isomorphism $\mathbb{C}\langle X \mid \mathcal{R}\rangle \xrightarrow{\cong} A$ is called a **presentation** of an algebra $A$, where $X$ is referred to as **generators** and $\mathcal{R}$ is referred to as **relations**.

Every associative unital algebra has a presentation. To construct a presentation of an algebra $A$, we can:

1. Choose a subset $X \subset A$ such that $X$ **generates** $A$ — this means that $A$ is spanned by products of the form $x_1x_2 \dots x_n$ where $n \geq 0$ and $x_i \in X$ for all $i = 1, \dots, n$. Every algebra has a generating set: an extreme example is $X = A$ but usually a much smaller generating set exists.
   By the Universal Mapping Property, Proposition 2.12, the inclusion $X \subset A$ extends to an algebra homomorphism $F\colon \mathbb{C}\langle X\rangle \to A$. The monomial $x_1x_2 \dots x_n \in \mathbb{C}\langle X\rangle$ is sent to the product $x_1x_2 \dots x_n \in A$. Since $X$ generates $A$, such products span $A$, so the image of $F$ is the whole algebra $A$.
2. By the First Isomorphism Theorem, Theorem 2.15, $A \cong \mathbb{C}\langle X\rangle/\ker F$ and so we have a presentation

   $$A \cong \mathbb{C}\langle X \mid \mathcal{R}\rangle,$$

   where $\mathcal{R}$ is any generating set of the ideal $\ker F$ of $\mathbb{C}\langle X\rangle$. Every ideal is generated by some set — e.g., the set of all elements of the ideal, but in many cases, a much smaller set which generates the ideal exists.

Key constructions in this course will have algebra structure defined by a presentation.

## Tensor product of algebras

Let $A$ and $B$ be associative algebras. We can construct a new algebra with underlying vector space $A \otimes B$, as follows.

**2.17 Definition (tensor product of algebras)** The **tensor product of algebras** $A$ and $B$ is the space $A \otimes B$ with multiplication satisfying $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for all $a, a' \in A$ and $b, b' \in B$.

Although the formula in the definition only shows how to multiply pure tensors, it is bilinear in $a, b$ (hence linear in $a \otimes b$) and bilinear in $a', b'$ (hence linear in $a \otimes b'$), so it extends to a well-defined bilinear map $(A \otimes B) \otimes (A \otimes B) \to A \otimes B$.

It is easy to check that the product thus defined on $A \otimes B$ is associative: similarly to the argument after Definition 2.7, choose bases for $A$ and $B$ and check associativity on the resulting basis of $(A \otimes B) \otimes (A \otimes B)$.

It is also straightforward to check that $1_A \otimes 1_B$ is the identity element of the algebra $A \otimes B$.

Instead of multiplying elements of $A \otimes B$, we can write the multiplication map as

$$m_{A\otimes B}\colon A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B. \tag{2.18}$$

Here $\tau\colon B \otimes A \to A \otimes B$ is the **flip map,** defined on pure tensors by $\tau(b \otimes a) = a \otimes b$. Likewise, the unit map of $A \otimes B$ is written as

$$\eta_{A\otimes B}\colon \mathbb{C} \cong \mathbb{C} \otimes \mathbb{C} \xrightarrow{\eta_A \otimes \eta_B} A \otimes B, \tag{2.19}$$

where we use the isomorphism $\mathbb{C} \cong \mathbb{C} \otimes \mathbb{C}$ as in Example 2.5 and the tensor product of two unit maps.

Writing the algebra structure of $A \otimes B$ in terms of linear maps is important for *dualisation*, i.e., obtaining a similar structure for coalgebras later.

**2.20 Remark (commuting subalgebras of $A \otimes B$)** The algebra $A$ is isomorphic to the subalgebra $A \otimes 1_B$ of the algebra $A \otimes B$ which is the image of the injective homomorphism $A \to A \otimes B$, $a \mapsto a \otimes 1_B$. Similarly, $B$ is isomorphic to the subalgebra $1_A \otimes B$ of $A \otimes B$. Importantly, $A \otimes 1_B$ and $1_A \otimes B$ are **mutually commuting** subalgebras inside $A \otimes B$.

## Actions and modules

Informally, associative unital algebras are designed to act on vector spaces. Given an algebra $A$ and a vector space $V$, an element $a \in A$ acts on each vector $v \in V$, sending it to some vector $a \triangleright v$. The key principle here that $a \triangleright v$ must be both **linear in** $a$ and **linear in** $v$. In our course, bilinearity is interpreted as linearity on a tensor product:

**2.21 Definition (action, module)** An **action** of an algebra $A$ on a vector space $V$ is a linear map

$$\triangleright\colon A \otimes V \to V, \quad a \otimes v \mapsto a \triangleright v,$$

which is multiplicative and unital in $A$, meaning that for all $a, b \in A$, $v \in V$,

$$a \triangleright (b \triangleright v) = (ab) \triangleright v, \qquad 1_A \triangleright v = v.$$

We say that $V$ is an $A$**-module** if $V$ is a vector space with an action of $A$.

Note that algebraic aspects of Hopf algebras and quantum groups are part of the field of **representation theory,** a major direction in abstract algebra which can be seen as a study of modules over algebras.