Commutative Algebra Q1

Ibraheem Sajid

(i) ~~[illegible]~~

FALSE: $I = 2\mathbb{Z}\times\mathbb{Z} \triangleleft \mathbb{Z}\times\mathbb{Z}$ is prime since

$(a,b)(x,y) \in I$

$\Rightarrow (ax, by) \in I \Rightarrow ax \in 2\mathbb{Z} \Rightarrow a \in 2\mathbb{Z}$ or $x \in 2\mathbb{Z}$

$\Rightarrow (a,b) \in I$ or $(x,y) \in I$.

But $\mathbb{Z} \notin \mathrm{Spec}(\mathbb{Z})$ so this is not of the claimed form. #

(ii) FALSE: $2\mathbb{Z} \triangleleft \mathbb{Z}$. $\mathbb{Z}/(2\mathbb{Z})(2\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ has an element of order 4

but $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ has no elements of order 4.

(~~simply~~ i.e. these differ as abelian groups so can't be isomorphic rings) #

(iii) TRUE: ① $\mathbb{Z}[x,y]$ is Noetherian: $\mathbb{Z}$ is a PID hence Noetherian (Cor 2.47) and $\mathbb{Z}[x,y]$ is a multivariate polynomial ring (Cor 2.49) over a Noetherian ring hence Noetherian //

② R Noetherian, $I \triangleleft R \Rightarrow R/I$ Noetherian:

Ideals of R/I correspond to ideals of R containing I (Cor 1.18), so ascending chains in R/I correspond to ascending chains in R starting with I. The latter all terminate since R is Noetherian so the former must also all terminate. $\therefore R/I$ Noetherian. //

① + ② $\Rightarrow$ $\mathbb{Z}[x,y]/(x^2-y^3)$ is Noetherian.

(iv) FALSE: If we take the localisation of $\mathbb{Z}$ at $\mathbb{Z} \setminus p\mathbb{Z}$, note $\mathbb{Z}_{(p)} = \{ \frac{a}{s} \in \mathbb{Q} \mid p \nmid s \} \subseteq \mathbb{Q}$ is an inclusion of rings.

By Cor 1.40, $\mathbb{Z}_{(p)}$ is a local ring with maximal ideal $p\mathbb{Z}_{(p)}$. Note, $\underset{\substack{\neq \\ 0}}{\frac{p}{1}} \in p\mathbb{Z}_{(p)}$ so $p\mathbb{Z}_{(p)} \neq 0$.

(it is a field)

Also, $\mathbb{Q}$ is local with maximal ideal (0). Thus we have $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$ with $\mathrm{Rad}(\mathbb{Z}_{(p)}) = p\mathbb{Z}_{(p)} \not\subseteq \mathrm{Rad}(\mathbb{Q}) = (0)$.

#

(v) TRUE: Let F be freely generated by $\{x_i\}_{i \in I}$, where I is finite.

A generic element of $F \otimes M$ is $\sum_J r_j y_j \otimes m_j = x$ for some finite indexing set J and $r_j \in R$, $y_j \in F$, $m_j \in M$ for all $j \in J$. Writing each y_j as a linear combination of $x_i, i \in I$ we can assume $x = \sum_I r_i x_i \otimes m_i$.

Now, $f_*(x) = 0 \iff \sum_I r_i x_i \otimes f(m_i) = 0$

We have homomorphisms $F \to R$ given by $g_i(x_j) = \delta_{ij}$ (since F is free on $\{x_i\}$)

Each g_i extends to $g_{i*} : F \otimes N \to R \otimes N \cong N$

So $f_*(x) = 0 \Rightarrow g_{i*} \circ f_*(x) = 0 \quad \forall i \in I$

$\Rightarrow r_i \otimes f(m_i) = 0 \quad \forall i \in I$

$\Rightarrow 1 \otimes r_i f(m_i) = 0 \quad \forall i \in I$

$\Rightarrow \forall i \in I$ either $r_i = 0$ or $m_i = 0$

$\Rightarrow x = 0$. //

TRUE:

(vi) Let M be a fractional ideal of a PID R, and F be the field of fractions of R.

$\exists \frac{a}{b} \in F^\times$ st. $\frac{a}{b}M \subseteq R$. Thus $\frac{a}{b}M$ is an ideal of R,

so $\frac{a}{b}M = (x)$, some $x \in R$. Then $\frac{a}{bx} \in F$ (assume $M \neq 0$ so $x \neq 0$ as $\frac{a}{b} \in F^\times$).

Let M' be the R-submodule of F generated by $\frac{a}{bx}$. M' is a fractional ideal since $\frac{bx}{a} \in F^\times$ ($a \neq 0$ since $\frac{a}{b} \in F^\times$)

and $\frac{bx}{a}M' = R \subseteq R$

Now, $MM' = M\left\{\frac{ar}{bx} \,\middle|\, r \in R\right\}$

$= \left\{\frac{arm}{bx} \,\middle|\, r \in R, m \in M\right\}$

but $m \in M \Rightarrow \frac{a}{b}m \in (x) \Rightarrow \frac{a}{bx} \in R$

$\Rightarrow \frac{ar}{bx} \in R$

so $MM' \subseteq R$.

Since $\frac{a}{b}M = (x)$

$\exists\, m \in M$ st. $\frac{a}{b}m = x$

then $\frac{a}{bx} \cdot x = 1$, so $R \subseteq MM'$

i.e. $MM' = R$

//

(vii) <u>FALSE:</u> Let $I = (2, 1+\sqrt{5}i)$

Suppose $I = (x)$. Then $x \mid 2$ ~~and $x \mid 1+\sqrt{5}i$~~.

Have $x = a + b\sqrt{5}i$, $a, b \in \mathbb{Z}$

and $(a+b\sqrt{5}i)(c+d\sqrt{5}i) = 2$, $c, d \in \mathbb{Z}$

$\therefore (ac - 5bd) + (ad+bc)\sqrt{5}i = 2$

$\therefore ac - 5bd = 2$, $\underline{ad+bc = 0}$

So $(a-b\sqrt{5}i)(c-d\sqrt{5}i) = 2$ as well and then

$(a^2+5b^2)(c^2+5d^2) = 4$

$\therefore b = d = 0$ (else we blow up part 4).

Have $a = 1, c = 2$ $\quad (I = (1))$

or $a = 2, c = 1$ $\quad (I = (2))$

$f, g \in \mathbb{Z}$

But $2 \mid 1+\sqrt{5}i \implies 2(f+g\sqrt{5}i) = 1+\sqrt{5}i$

$\implies 2f = 1, 2g = 1$ #

$\therefore (2) \subsetneq (2, 1+\sqrt{5}i)$

Suppose $A \cdot 2 + B(1+\sqrt{5}i) = 1$, $A, B \in \mathbb{Z}$

Then $B = 0$ ($\sqrt{5}i$ coefficient)

and $2A = 1$ #

(viii) <u>TRUE:</u> We know $\mathbb{Z}$ is a UFD. Then, by Lem 1.15, $\mathbb{Z}[x,y] = \mathbb{Z}[x][y]$ is also a UFD. So by Prop 3.5, $\mathbb{Z}[x,y]$ is integrally closed. //

I will write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$ in this question.

Commutative Algebra Q2 — Ibraheem Sajid

~~(i) (a) A generic element of $\frac{R}{I} = \frac{\mathbb{Z}_7[x]}{(x^2+2)}$ is $a+bx+(x^2+2)$. $a,b \in \mathbb{Z}_7$.~~

~~$(a+bx)(c+dx)+(x^2+2) = 1+(x^2+2)$~~

~~$\iff ac+(bc+ad)x+bdx^2+(x^2+2) = 1+(x^2+2)$~~

~~$\iff ac+(bc+ad)x-2bd+(x^2+2) = 1+(x^2+2)$~~

$\iff ac-2bd \equiv 1,\ bc+ad \equiv 0 \pmod 7$

(i) (c) Suppose x^2+2 was reducible in $\mathbb{Z}_7[x]$. Then $x^2+2 = (x+a)(x+b)$ for $a,b \in \mathbb{Z}_7$, so it would have a root in $\mathbb{Z}_7$. But

x	x^2+2
0	2
1	3
2	6
3	4
4	4
5	6
6	3

$\therefore$ x^2+2 is irreducible. By lemma 1.15, (x^2+2) is maximal, noting that $\mathbb{Z}_7[x]$ is a PID since $\mathbb{Z}_7$ is a field.

$\therefore$ R/I is a field by Thm 1.10.

(b) $\bar{x}^3 = \overline{-2x}$. $\overline{-2x(c+dx)} = \bar{1}$ $(c,d \in \mathbb{Z}_7)$

$\iff \overline{-2c} = \bar{0},\ \overline{4d} = \bar{1}$

So let $c=0$, $d=2$ (note $4\times 2 = 8 = 1$)

Note indeed: $(\bar{x}^3)(\overline{2x}) = \overline{2x^4}$

$= \overline{2(x^2+2-2)^2}$

$= \pi\left(2(-2)^2 + 4(x^2+2)(-2) + 2(x^2+2)^2\right)$

$= \pi(2(-2)^2) = \bar{8} = 1.$ //

(c) Note $\mathrm{Frac}(R) = \underset{F}{\underset{\shortparallel}{\mathbb{Z}_7(x)}}$ (rational ~~functions~~ of polynomials in $\mathbb{Z}_7$)

$I^{-1} = \left\{ \frac{a}{b} \in F \;\middle|\; \frac{a}{b} I \subseteq R \right\}$

Let $\frac{p}{q} \in F$, ie. $p,q \in R$. wlog p,q are coprime (since R is UFD).

Then $\frac{p}{q} \cdot \langle x^2+2 \rangle \subseteq R$

$\iff \frac{p}{q}(x^2+2) \in R$

$\iff p(x^2+2) = q \cdot s$, $s \in R$.

Since x^2+2 is prime and divides qs, it must divide q or s. Since p is coprime to q, it must divide s.

Thus, $x^2+2 \mid q \Rightarrow p = s$ and $x^2+2 \mid s \Rightarrow q=1$ and $s = p(x^2+2)$

So $I^{-1} = R \cup \left\{ \frac{a}{b} \;\middle|\; a \in R,\ (x^2+2) \mid b \right\}$

$= R\langle 1, \frac{1}{x^2+2} \rangle$ as an R-module

$= R\langle \frac{1}{x^2+2} \rangle$ since $x^2+2 \cdot \frac{1}{x^2+2} = 1$.

$\therefore$ I^{-1} is invertible with inverse I.

$$\left(II^{-1} = R\langle \frac{1}{x^2+2} \cdot (x^2+2) \rangle = R \right).$$

(ii). First, note that prime ideals of $R \times S$ for rings R, S are either of the form $R \times Q$ or $P \times S$ $P \in \operatorname{Spec}(R)$, $Q \in \operatorname{Spec}(S)$.

Indeed suppose $Q \in \operatorname{Spec}(S)$. Then $R\times S\,(R \times Q) = R \times Q$

So $R \times Q$ is an ideal. If $(ab, cd) \in R \times Q$, then c or $d \in Q$ since $cd \in Q$. $\therefore$ (a,c) or $(b,d) \in R \times Q$ //

$P \times S$ is similar.

Conversely suppose $I \triangleleft R \times S$ is an ideal.

Define $P = \{ r \in R \mid \exists s \in S,\ (r,s) \in I \}$

Note $P \triangleleft R$ since $\exists s,\ (r,s) \in I \Rightarrow (a,1)(r,s) \in I$ $\forall a \in R \Rightarrow ar \in P$ $\forall a \in R$.

If I is prime then P is prime since we have $ab \in P \Rightarrow \exists s \in S$ st. $(ab, s) \in I \Rightarrow (a,s)$ or $(b,1) \in I$

$\Rightarrow a$ or $b \in P$. ~~Now if $(1,0),(0,1) \notin I$,~~
~~we have $(1,0)(0,1)$~~

Since $(0,1)(1,0) = (0,0) \in I$, we have $(0,1)$ or $(1,0)$ $\in I$.

If ~~$(1,0)$~~ $(0,1) \in I$, we've shown $I = P \times S$, otherwise define a Q similarly for $I = R \times Q$.

Next note prime ideals of $\mathbb{Z}$ containing $n\mathbb{Z}$ are in ~~bijection~~ correspondence with prime ideals of $\mathbb{Z}_n$.

$\therefore$ $\mathrm{Spec}(\mathbb{Z}_{10}) = \{$ ~~(0)~~ $(2), (5)\}$

and $\mathrm{Spec}(\mathbb{Z}_{21}) = \{$ ~~(0)~~ $(3), (7)\}$.

so $\mathrm{Spec}(\mathbb{Z}_{10} \times \mathbb{Z}_{21})$ is the ~~8~~ 4 elements as described previously
(eg. $(2) \times \mathbb{Z}_{21}$ and $\mathbb{Z}_{10} \times (3)$).

By Thm 4.12, clopen sets are precisely U_e for $e \in R$ idempotent.

Lets enumerate:

x	0	1	2	3	4	5	6	7	8	9
x^2 mod 10	0	1	4	9	6	5	6	9	4	1
x^2 mod 21	0	1	4	9	16	4	15	7	1	18

x	10	11	12	13	14	15	16	17	18	19	20
x^2 mod 21	16	16	18	1	7	15	4	16	9	4	1

$= 16$ unique subsets of $\cancel{\ }$ $\mathrm{Spec}(\mathbb{Z}_{10}\times\mathbb{Z}_{21})$

They idempotents are the 16 combinations
$$\{0,1,5,6\}\times\{0,1,7,15\}$$

In fact, since $|\mathrm{Spec}(\mathbb{Z}_{10}\times\mathbb{Z}_{21})| = 4$,

~~therefore~~ $|\mathcal{P}(\mathrm{Spec}(\mathbb{Z}_{10}\times\mathbb{Z}_{21}))| = 16$

and thus every set is clopen! So each singleton is a connected component.

$$\left(\mathcal{C} = \{\text{set of Zariski closed sets of } \mathrm{Spec}(\mathbb{Z}_{10}\times\mathbb{Z}_{21})\} \subseteq \mathcal{P}\,\mathrm{Spec}(\mathbb{Z}_{10}\times\mathbb{Z}_{21}) \right)$$

so we must have equality

= contradiction

Commutative Algebra Q3 — Ibraheem Sajid

(i) First, (0) is prime $((0) \subsetneq R)$ so R is an integral domain.
Now suppose $\exists r \notin R^{\times}$.

If $(r^2) = (r)$, then $ar^2 = r$ for some $a \in R$,
$\Rightarrow ar^2 - r = 0 \Rightarrow (ar-1)r = 0$
$\Rightarrow ar - 1 = 0 \Rightarrow a \cdot r = 1$
i.e. $r \in R^{\times}$ #

$\therefore (r^2) \subsetneq (r)$. But then $r \cdot r \in (r^2)$
but $r \notin (r^2)$, so (r^2) is not prime. # as it is a proper ideal.

$\therefore r \in R^{\times}$.

~~2(ii) Since $\mathbb{Z}[\sqrt{3}]$ is a PID, prime ideals are of the form (f) for $f \in \mathbb{Z}[\sqrt{3}]$ irreducible~~ (then 4+5)

~~Rett~~

~~let $f = a + b\sqrt{3}$, $a, b \in \mathbb{Z}$.~~

~~...for some $c, d \in \mathbb{Z}$~~

~~and $ad + bc = 0$~~

3 12 27

$|1 - 2\sqrt{3}| = -1$

$a^2 - 3b^2$ prime

$25 - 11 = 14$ $\quad 36 - 11 = 25$

$16 - 11 = 4$ $\quad 49 - 11 = 38$

$9 - 3$ $\quad 11 + 1 = 3 \times 2^2$ $\quad 9 - 11 = -2$ $\quad 64 - 11 = 53$

$4 - 11 = -7$

$16 - 3 = 13$

$(4 + \sqrt{3})(4 - \sqrt{3}) = 13$

$\therefore 4 + \sqrt{3}$ prime

$(1 + 2\sqrt{3})(1 - 2\sqrt{3}) = 1 - 4 \times 3$

11 12

(ii) Note $\mathbb{Z}[\sqrt{3}]$ is integral over $\mathbb{Z}$ $((\sqrt{3})^2-3=0)$ and integrality is preserved under multiplication, addition and subtraction.

$\therefore$ By ~~Pro~~ Thm 3.11, $\mathrm{Spec}(\mathbb{Z}/11) \longleftrightarrow \mathrm{Spec}(\mathbb{Z}[\sqrt{3}])$ are bijective, so there is just one prime ideal I with $I \cap \mathbb{Z} = (11)$.

Note $1+2\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ has norm $1^2-4\cdot 3 = -11$, which is prime so ~~PR~~ $1+2\sqrt{3}$ is irreducible and $(1+2\sqrt{3})$ is prime by Lem 1.15.

Note $11 = -(1-2\sqrt{3})(1+2\sqrt{3})$ so $11 \in (1+2\sqrt{3})$

and $(11) \subseteq (1+2\sqrt{3}) \cap \mathbb{Z}$.

But then $(11) = (1+2\sqrt{3}) \cap \mathbb{Z}$ since $(1+2\sqrt{3}) \cap \mathbb{Z}$ is prime.

$\therefore I = (1+2\sqrt{3})$ is the only possibility.

Use that $\mathbb{Z}[x,y]$ is UFD below to cancel common factors.

(iii) Suppose $I = (f)$. Then $f = rxy + sy^2$

$\Rightarrow y \mid f$

Let $f = yg$. Have $xy = hyg$ and $y^2 = kyg$

$\therefore x = hg \Rightarrow g = 1$ or $g = x$.

~~But~~ But

Since $y \notin R$, have $g = x$. But $y^2 \notin (xy)$

so contradiction; R is non PID.

$I^2 = (xy, y^2)(xy, y^2) = (x^2y^2, xy^3, y^4)$

Since $y^2x^2 \in I^2$. If I^2 primary, we have

$y^2 \in I^2$ or $x^{2n} \in I^2$ $n \geq 0$

But $y^2 \notin I^2$ since $I^2 \cap (y^2)$

has $\pi(I^2 \cap (y^2)) = (0)$ where $\pi : R \to R/(x)$
evaluation at $x=0$ map.

but $\pi(y^2) \neq 0$.

$\therefore$ I^2 not primary.

~~(ii) Suppose $f = \frac{p}{q}$, $p, q \in R\setminus\{0\}$ w.l.o.g. $\gcd(p,q) = 1$~~ $\in K$

~~f integral over $R \iff \exists\, a_0, \ldots, a_{n-1} \in R$~~

~~s.t. $a_0 + \frac{p}{q}a_1 + \cdots + \frac{p^{n-1}}{q^{n-1}}a_{n-1} + \frac{p^n}{q^n} = 0$.~~

~~$\iff a_0q^n + a_1pq^{n-1} + \cdots + a_{n-1}p^{n-1}q + p^n = 0$~~

~~Hence $p \mid a_0$~~ $\mathbb{Z}[\sqrt{5}] \subseteq \mathbb{Q}[\sqrt{5}] \subseteq \mathbb{Q}(\sqrt{5})$.

$a + b\sqrt{5}$

 Ibraheem Sajid

(ii) The homomorphism $f: R \to R/I \times R/J$

$$r \mapsto (r+I, r+J)$$

has $\ker f = \{r \mid r \in I, r \in J\}$

$= I \cap J = (0)$

$\therefore f$ is injective.

Suppose $K_1 \subseteq K_2 \subseteq \dots$ is an ascending chain of ideals in R. Since $R \to R/I$ and $R \to R/J$ are surjective, $f(K_i) = f(K_i) \cdot R/I \times R/J$ is an ideal (direct products of two ideals is an ideal).

$\therefore K_1/I \times K_1/J \subseteq K_2/I \times K_2/J \subseteq \dots$ is an ascending chain of ideals in $R/I \times R/J$.

Note $f^{-1}(K_i/I \times K_i/J) = K_i$ since f injective.

Now, the chain $K_1/I \subseteq K_2/I \subseteq \dots$ in R/I terminates since R/I noetherian.
Similarly $K_1/J \subseteq K_2/J \subseteq \dots$ in R/J terminates.

ie. $\exists\, n > 0$ st. $K_i/I = K_n/I \ \forall i \geq n$

$\exists\, m > 0$ st $K_j/J = K_m/J \ \forall j \geq m$

Let $N = \max(n, m)$

$\therefore K_i/I \times K_i/J = K_N/I \times K_N/J \ \forall i \geq N$, so the chain in $R/I \times R/J$ terminates. Thus, its preimage

$K_1 \subseteq K_2 \subseteq \dots$ in R terminates //.

(a)

(ii) By Cor 1.18, maximal ideals of R containing $\ker\varphi$ are in correspondence with maximal ideals of $\operatorname{im}\varphi = S$, via the maps ~~...~~

$$\left\{\begin{matrix}\text{maximal ideals}\\ \text{of } R \text{ containing}\\ \ker\varphi\end{matrix}\right\} \quad \begin{matrix} I \longmapsto \varphi(I) \\ \varphi^{-1}(J) \longleftarrow J \end{matrix} \quad \left\{\begin{matrix}\text{maximal}\\ \text{ideals of } S\end{matrix}\right\}$$

Then, for all maximal ideals $J \triangleleft S$, there is a maximal ideal $\varphi^{-1}(J) \triangleleft R$ st. $\varphi\circ\varphi^{-1}(J) = J$.

$$\varphi(\operatorname{Jac}(R)) = \varphi\Big(\bigcap_{I\in \operatorname{MaxSpec}(R)} I\Big) \subseteq \bigcap_{I\in\operatorname{MaxSpec}(R)} \varphi(I)$$

$$\subseteq \bigcap_{J\in\operatorname{MaxSpec}(S)} \varphi(\varphi^{-1}(J)) = \operatorname{Jac}(S). \quad //$$

Thm 1.10

(b) Let $R = k[x]$ for some infinite field k and note $(x-a)$ is maximal for all $a \in k$ $\boxed{R/(x-a) \cong k \text{ a field}}$

$\therefore$ $(0) \subseteq \text{Jac}(R) \subseteq \bigcap_{a \in k} (x-a) = (0)$

infinite intersection of coprime ideals is infinite product $= 0$.

Let $S = k[x]/(x^2)$ and φ the quotient map. From Ex 1.5, S is local with maximal ideal (x).

$\therefore \text{Jac}(S) = (x)$

So ~~Res~~ $\varphi(\text{Jac}(R)) = \varphi((0)) = (0) \subsetneq (x) = \text{Jac}(S)$.

(iii) Let $0 \neq m \in M$. $0 \subsetneq \langle m \rangle \leq M$ (submodule). Since M is simple, we must have $\langle m \rangle = M$.

Thus the homomorphism $\varphi : R \to M$, $r \mapsto r \cdot m$ is a surjection

and $\ker\varphi = \{ r \mid rx = 0 \ \forall x \in M \} = \text{Ann}(M) = I$

$\therefore$ By isomorphism theorem (2.10), $M \cong R/I$

~~Now suppose $I \subseteq J$ is an inclusion of ideals of R~~

~~By the correspondence theorem~~

We will show that there is a correspondence (as R-modules)

$$\left\{ \begin{matrix} \text{Submodules of } R \\ \text{containing } I \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Submodules of} \\ R/I \end{matrix} \right\}$$

Since R-submodules of R are precisely ideals of R, if we show that R-submodules ~~are ideals~~ of R/I are ideals of R/I, then we are done by the correspondence theorem for rings (Cor 1.18)

But in fact R and R/I actions on R/I are equal so R-modules are the same as R/I-modules (when looking at submodules of R/I)

R/I-modules $=$ ideals of R/I

//

Thus, an ideal $J \supsetneq I$ corresponds to a submodule $N \supsetneq \{0\}$. ~~and~~ Since M is simple, $N = M$ and $J = R$.

ie. I is maximal.

//

~~1.~~ Commutative Algebra Q5 Ibraheem Sajjad

(i)(a) $\left(\sqrt{\mathrm{Ann}(M/IM)} \subseteq \sqrt{\mathrm{Ann}(M)+I} \right)$

Suppose $a \in \sqrt{\mathrm{Ann}(M/IM)}$, ie. $\exists n>0$ st. $a^n m \in IM$ $\forall m \in M$. By Prop 2.21, the map $\varphi : M \to M$, $m \mapsto a^n m$

satisfies some equation $\varphi^N + x_1\varphi^{N-1} + \dots + x_{N-1}\varphi + x_N = 0$ where all $x_i \in I$.

So we have $(a^{Nn} + x_1 a^{(N-1)n} + \dots + x_N)m = 0$ $\forall m \in M$. ie. $a^{Nn} + x_1 a^{(N-1)n} + \dots + x_N \in \mathrm{Ann}(M)$

Also $x_1 a^{(N-1)n} + \dots + x_N \in I$, so $a^{Nn} \in \mathrm{Ann}(M) + I$

$\therefore a \in \sqrt{\mathrm{Ann}(M)+I}$ //

$\left(\sqrt{\mathrm{Ann}(M)+I} \subseteq \sqrt{\mathrm{Ann}(M/IM)} \right)$

Suppose $a \in \sqrt{\mathrm{Ann}(M)+I}$. Then $\exists n>0$ st $a^n = b+c$ where $b \in \mathrm{Ann}(M)$, $c \in I$.

$\forall m+IM \in M/IM$, $a^n m + IM = bm + cm + IM = cm + IM = 0 + IM$ (since $cm \in IM$).

$\therefore a \in \sqrt{\mathrm{Ann}(M/IM)}$

(b) Since $M+N$ is finitely generated, so is the quotient $\frac{M+N}{N}$ (just take the image of the generators).

Let $\varphi : M+N \to \frac{M}{M\cap N}$ (note $M\cap N$ is a submodule)

$m+n \mapsto m + M\cap N$ $\left(\begin{smallmatrix} m\in M \\ n \in N \end{smallmatrix} \right)$

φ is surjective since $\forall m + M\cap N \in M/M\cap N$, $m \in M \subseteq M+N$; $\varphi(m) = m + M\cap N$

Suppose $m+n \xrightarrow{\varphi} 0$. Then $m \in M\cap N$

So $\ker\varphi = N + M \cap N = N$.

Note φ is well-defined since if $m+n = m'+n'$ $\left(\begin{array}{l} m,m' \in M \\ n,n' \in N \end{array}\right)$

then $m - m' = n' - n$

i.e. $m - m' \in M \cap N$

So $\varphi(m+n - (m'+n')) = m - m' + M \cap N = 0$

$\therefore \varphi(m+n) = \varphi(m'+n')$.

So $\dfrac{M}{M \cap N}$ is finitely generated, say by

$x_1 + M\cap N, \dots, x_n + M \cap N$ ~~$x_1, \dots, x_n$~~.

Also $M \cap N$ is f.g., say by $y_1, \dots, y_m$.

Let $m \in M$. Then $m + M\cap N = \sum_{i=1}^{n} r_i x_i + M \cap N$

so $(m - \sum_1^n r_i x_i) + M \cap N = 0$

$\therefore m - \sum_1^n r_i x_i \in M \cap N$

i.e. $m - \sum_{i=1}^n r_i x_i = \sum_{i=1}^{p} s_i y_i$ i.e. $m = \sum_{i=1}^n r_i x_i + \sum_{i=1}^p s_i y_i$.

i.e. $\{x_i\}_{i=1}^n \cup \{y_i\}_{i=1}^p$ generate M.

By symmetry, N is also f.g.

(ii) Assuming $\pi(M)$ means $M/\mathrm{Rad}(R)M$.

Define $\bar{\pi} : M \to M/\mathrm{Rad}(R)M$, the quotient map.

Let N be the submodule of M generated by X.

We have $\bar{\pi}(N) = M/\mathrm{Rad}(R)M$, so $\bar{\pi}^{-1} \circ \bar{\pi}(N) = M$

But also $\bar{\pi}^{-1} \circ \bar{\pi}(N) = N + \mathrm{Rad}(R)M$, so $M = N + \mathrm{Rad}(R)M$.

By Nakayama's lemma corollary (Cor 2.26), $M = N$.

(by assumption)

If $m + \mathrm{Rad}(R)M = \sum \bar{\pi}(x_i) \cdot r_i$ for $x_i \in X$,

then $\bar{\pi}(\sum r_i x_i) = m + \mathrm{Rad}(R)M$

so $M/\mathrm{Rad}(R)M \subseteq \bar{\pi}(N) \subseteq M/\mathrm{Rad}(R)M$

(iii) <u>LEMMA</u>

Suppose

$$\begin{tikzcd} Q \arrow[r, "f"] & M \arrow[d, "\alpha"] \\ N \arrow[r, "d"] & L \end{tikzcd}$$

are maps of finitely generated R-modules for R Noetherian with Q projective and $\mathrm{im}(\alpha \circ f) \subseteq \mathrm{im}(d)$

then there is a free R-module P and maps g, β completing this diagram to

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "\beta"] & Q \arrow[r, "f"] & M \arrow[d, "\alpha"] \\ N \arrow[rr, "d"] & & L \end{tikzcd}$$

such that the square commutes and the sequence is exact (at Q).

i.e. $\mathrm{im}(g) = \ker(f)$

Indeed, note d ~~is~~ is surjective onto its image, and

$\alpha \circ f : Q \to \mathrm{im}(d)$ is well-defined.

By the projective property $\beta : Q \to N$ exists making the square commute

Now, $\ker(f)$ is a submodule of Q, which is a finitely generated module over a Noetherian ring, so is Noetherian (Thm 2.44). Thus, $\ker(f)$ is finitely generated (Thm 2.46), say by $\{x_1, \dots, x_m\}$.

Define $P = R^m$ and $g: P \longrightarrow Q$
$e_i \longmapsto x_i$, where ~~e_i is the~~ $\{e_1, \dots, e_m\}$ is a free set of generators for P.
(so g is well-defined).

By construction, $\operatorname{im}(g) = \ker(f)$, so we are done. //LEMMA

Now we can proceed by induction. ~~on n~~. The base case has

$$\begin{tikzcd} Q_0 \arrow[r] & M \arrow[d] \\ N_0 \arrow[r] & M \end{tikzcd}$$

with Q_0 projective by assumption and because ~~[illegible]~~ $N_0 \to M$ is surjective ~~[illegible]~~, the composition $Q_0 \to M \to M$

has image lying in the image of $N_0 \to M$,

So we obtain

$$\begin{tikzcd} Q_1 \arrow[r] & Q_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ & N_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

by following ~~the~~ our lemma.

and ~~M~~ is n (to be precise, we'd need to define ~~...~~ $Q_{-1} = N_{-1} = M$)

Now, the inductive step. Suppose we have

$$\begin{tikzcd} Q_i \arrow[r] & Q_{i-1} \arrow[r] \arrow[d] & Q_{i-2} \arrow[r] \arrow[d] & \cdots \\ & N_{i-1} \arrow[r] & N_{i-2} \arrow[r] & \cdots \end{tikzcd}$$

as given by ~~...~~ our lemma.

Note since Q_i is free it is projective (Lem 2.24)

Fill in $N_i \to N_{i-1}$ from the assumption in question.
$\mathrm{im}(N_i \to N_{i-1}) = \ker(N_{i-1} \to N_{i-2})$ by exactness.

and $\mathrm{im}(Q_i \to Q_{i-1}) = \ker(Q_{i-1} \to Q_{i-2})$, so

$Q_i \to Q_{i-1} \to Q_{i-2}$ is the zero map

and hence so is
$$\begin{tikzcd} Q_i \arrow[r] & Q_{i-1} \arrow[d] & \\ & N_{i-1} \arrow[r] & N_{i-2} \end{tikzcd}$$

by commutativity. Thus
$$\begin{tikzcd} Q_i \arrow[r] & Q_{i-1} \arrow[d] \\ & N_{i-1} \end{tikzcd}$$

has image in $\ker(N_{i-1} \to N_{i-2}) = \mathrm{im}(N_i \to N_{i-1})$,

Now we meet the conditions of our lemma and we can apply it again.

The induction ends with

$$\begin{tikzcd} Q_{n+1} \arrow[r] & Q_n \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ & N_n \arrow[r] & \cdots \arrow[r] & N_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

//