Lie Groups and Lie Algebras
(MAGIC 008)

2025

Answer **THREE** questions.

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1. (a) Prove that the set G_A of all invertible 3×3-matrices that commute with the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, i.e., $G_A = \{X \in GL(3, \mathbb{R}) \mid XA = AX\}$, is a Lie group. Describe G_A explicitly and find its dimension. [4]

   (b) Is G_A connected? How many connected components does this Lie group consist of? Justify your answer. [4]

   (c) Prove that G_A is isomorphic to the direct product $GL(2, \mathbb{R}) \times \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is considered as a group under multiplication. [4]

   (d) Consider the natural action of G_A on $\mathbb{R}^3$ and describe the orbit $\mathcal{O}(v)$ and stationary subgroup $\mathrm{St}(v)$ of the vector $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. What are the dimensions of $\mathcal{O}(v)$ and $\mathrm{St}(v)$? [4]

   (e) How many distinct orbits does this action have? Describe all of them. [4]

2. Consider the subset $G \subset GL(4, \mathbb{R})$ consisting of block-diagonal matrices of the form

$$X = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix},$$

   where A is an orthogonal 2×2 matrix and B is an arbitrary 2×2 matrix.

   (a) Prove that G is an algebraic linear group. What is the dimension of G? Is G compact? [5]

   (b) Describe the Lie algebra $\mathfrak{g}$ of G. [5]

   (c) State the definitions of a solvable Lie algebra and a nilpotent Lie algebra. [5]

   (d) Is $\mathfrak{g}$ solvable? Is $\mathfrak{g}$ nilpotent? Justify your answer. [5]

3. Consider 3 vector fields in $\mathbb{R}^2(x,y)$:

$$\xi_1 = (xy, 1+y^2), \quad \xi_2 = (y,0), \quad \xi_3 = (1,0).$$

(a) Compute the Lie brackets $[\xi_i, \xi_j]$, $i,j = 1,2,3$, and prove that the 3-dimensional space $\mathfrak{g}$ spanned by these vector fields is a Lie algebra. [4]

(b) Find the structure constants of this Lie algebra $\mathfrak{g}$ in the basis ξ_1, ξ_2, ξ_3 and describe the adjoint representation of $\mathfrak{g}$. [4]

(c) Give the definition of an ideal of a Lie algebra. [3]

(d) Let $\mathfrak{h}_1 = \mathrm{span}(\xi_2) \subset \mathfrak{g}$ and $\mathfrak{h}_2 = \mathrm{span}(\xi_2, \xi_3) \subset \mathfrak{g}$.
Is $\mathfrak{h}_1$ an ideal of $\mathfrak{g}$? Is $\mathfrak{h}_2$ an ideal of $\mathfrak{g}$? [4]

(e) Prove that $\mathfrak{g}$ is isomorphic to the Lie algebra of the matrix Lie group

$$G = \left\{ A = \begin{pmatrix} \cos t & -\sin t & a \\ \sin t & \cos t & b \\ 0 & 0 & 1 \end{pmatrix}, \quad t, a, b \in \mathbb{R} \right\} \subset GL(3,\mathbb{R}).$$

[5]

4. Consider the matrix Lie group

$$G = \left\{ X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{R} \right\} \subset GL(3,\mathbb{R}),$$

and its three-dimensional representation on the space V of skew-symmetric 3×3 matrices $\Psi : G \to GL(V)$:

$$\Psi_X(B) = XBX^\top, \quad X \in G, B \in V.$$

(a) Let $B = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \in V$. Prove that $f(B) = \gamma$ is an invariant of Ψ. [4]

(b) Let $\mathfrak{g} \subset \mathrm{gl}(3,\mathbb{R})$ be the Lie algebra of G. For the representation Ψ, describe the induced representation ψ of $\mathfrak{g}$. [4]

(c) Give the definition of an irreducible representation. Is ψ irreducible? [4]

(d) Consider the natural representation ϕ of $\mathfrak{g}$ on $\mathbb{R}^3$:

$$\phi : \mathfrak{g} \to \mathrm{gl}(3,\mathbb{R}), \quad \phi_A = A, \ A \in \mathfrak{g}.$$

Is ϕ irreducible? [4]

(e) Prove that ϕ and ψ are not isomorphic, i.e. there is no invertible linear map $P : \mathbb{R}^3 \to V$ such that $P \circ \phi_A = \psi_A \circ P$ for all $A \in \mathfrak{g}$. [4]