Exercise Sheet 6

MAGIC009 - Category Theory

November 10th, 2023

1. *Note.* This exercise is essentially a special case of the theorem that right adjoints preserve limits. The purpose of the exercise is to work out the details of the proof in this simple case.

   (i) Let $\mathbb{C}$ and $\mathbb{D}$ be two categories with terminal objects. Prove that a right adjoint functor $G \colon \mathbb{D} \to \mathbb{C}$ preserves the terminal object.

   (ii) Let $\mathbb{C}$ and $\mathbb{D}$ be two categories with binary products. Prove that a right adjoint functor $G \colon \mathbb{D} \to \mathbb{C}$ preserves products.

2. Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets and consider the associated categories $\underline{P}$ and $\underline{Q}$, respectively.

   (i) Unfold what is an adjunction between $\underline{P}$ and $\underline{Q}$ in terms of $(P, \leq)$ and $(Q, \leq)$.

   (ii) Unfold what the statement that right adjoints preserve limits amounts to in this case.

3. Consider the following concepts defined by duality:

   - an initial object in $\mathbb{C}$ is a terminal object in $\mathbb{C}^{\mathsf{op}}$,
   - a coproduct in $\mathbb{C}$ is a product in $\mathbb{C}^{\mathsf{op}}$,
   - a coequalizer in $\mathbb{C}$ is an equalizer in $\mathbb{C}^{\mathsf{op}}$,
   - a pushout in $\mathbb{C}$ is a pullback in $\mathbb{C}^{\mathsf{op}}$.

   Then,

   (i) unfold explicitly what is an initial object, a coproduct, a coequalizer and a pushout purely in terms of $\mathbb{C}$,

   (ii) define explicitly initial object, coproducts, coequalisers and pushouts in the category **Set** of sets and functions.

4. Dualise the following statements (without proving them!)

   - "If a category has binary products and equalisers, then it has pullbacks."
   - "If a category has pullbacks and a terminal object, then it has binary products and equalisers."
   - "A terminal object, if it exists, is unique up to unique isomorphism".
   - "A right adjoint preserves terminal objects".

5. We have seen that the functor $U \colon \mathbf{Top} \to \mathbf{Set}$ mapping a topological space to its set of points has both a left and a right adjoint. Contemplate some consequences of this fact in terms of some limits and colimits in **Top** and **Set**.