MAGIC008: Lie Groups and Lie algebras

Problem Sheet 3: Lie groups: connectedness and the identity component

1. Let $O(1,1) = \{X \in GL(2,\mathbb{R}) \mid X^\top E_{1,1} X = E_{1,1}\}$, where $E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Describe the matrices $X \in O(1,1)$ explicitly and show that $O(1,1)$ consists of 4 connected components each of which is diffeomorphic to $\mathbb{R}$ (moreover, the identity component is isomorphic to $\mathbb{R}$ as a Lie group).

2. Let $O(p,q) = \{X \in GL(p{+}q,\mathbb{R}) \mid X^\top E_{p,q} X = E_{p,q}\}$, where $E_{p,q} = \text{diag}(\underbrace{1,\dots,1}_{p}, \underbrace{-1,\dots,-1}_{q})$.

   Consider
   $$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in O(p,q)$$
   where $X_{11}, X_{12}, X_{21}, X_{22}$ are sub-matrices of dimension $p \times p$, $p \times q$, $q \times p$, $q \times q$ respectively. Show that $\det X_{11} \neq 0$ and $\det X_{22} \neq 0$. Derive from this that $O(p,q)$ contains at least 4 connected components.

3. Consider the group of invertible upper triangular matrices:
   $$T = \left\{ X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ 0 & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{nn} \end{pmatrix} \quad : \quad \det X = x_{11}x_{22} \cdot \ldots \cdot x_{nn} \neq 0 \right\}$$
   How many connected components are there in T? Describe the identity component.

4. Prove that $SL(2,\mathbb{R})$ and $SO(2,\mathbb{R})$ are connected. Describe the topology of these two groups.

5. (*) Prove that $SL(n;\mathbb{R})$ and $SO(n;\mathbb{R})$ are connected.

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