

Exercise Sheet 3

MAGIC009 - Category Theory

October 25th, 2024

1. Find an example of a functor that is both full and faithful, one that is full but not faithful, one that is faithful but not full, and one that is neither.
2. Find an example of a functor $F: \mathbb{C} \rightarrow \mathbb{D}$ such that F is faithful but there exist distinct maps f_1 and f_2 in \mathbb{C} such that $F(f_1) = F(f_2)$.
3. Let $F: \mathbb{C} \rightarrow \mathbb{D}$ be an equivalence of categories. Show that if \mathbb{C} has a terminal object, then so does \mathbb{D} .
4. Prove that every functor $F: \mathbb{C} \rightarrow \mathbb{D}$ admits a factorisation of the form

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{D} \\ & \searrow L \quad \nearrow R & \\ & \mathbb{X} & \end{array}$$

where L is essentially surjective and R is full and faithful.

Let \mathbb{C} be a category. A *subcategory* \mathbb{D} of \mathbb{C} consists of a subclass $\text{Ob}(\mathbb{D}) \subseteq \text{Ob}(\mathbb{C})$ and a family of subclasses $\mathbb{D}(X, Y) \subseteq \mathbb{C}(X, Y)$, for $X, Y \in \text{Ob}(\mathbb{D})$, which contains the identity maps and is closed under composition, so that \mathbb{D} is itself a category.

5. Give an example of a subcategory of the category **Grp** of groups.
6. Let P be a partially ordered set and consider the associated category \underline{P} . What are the subcategories of \underline{P} ?
7. Let \mathbb{C} be a category and \mathbb{D} a subcategory of \mathbb{C} .
 - (a) Show that there is an inclusion functor $F: \mathbb{D} \rightarrow \mathbb{C}$.
 - (b) Is $F: \mathbb{D} \rightarrow \mathbb{C}$ always full?
 - (c) Is $F: \mathbb{D} \rightarrow \mathbb{C}$ always faithful?
8. Let $F: \mathbb{C} \rightarrow \mathbb{D}$ be a full and faithful functor. Prove that \mathbb{C} is equivalent to a subcategory of \mathbb{D} .