MAGIC008: Lie Groups and Lie algebras

Problem Sheet 6: Linear representations of Lie groups and Lie algebras

1. For $SL(2,\mathbb{R})$ consider three linear representations:

   - adjoint representation Ad: $\quad \mathrm{Ad}_X A = XAX^{-1}$;
   - the representation on the space of symmetric 2×2 matrices defined by:
   $$\Psi_X B = XBX^\top;$$
   - the representation of the space V of homogeneous polynomials $p(x,y)$ of two variables x and y of degree k defined by:
   $$\Phi_X(p(x,y)) = p(\alpha x + \gamma y, \beta x + \delta y), \text{ where } X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

   Prove that the above formulae indeed define linear representations.

   What is the dimension of each representation?

   Describe the induced representations ad, ϕ and ψ of the Lie algebra $sl(2,R)$.

   Describe ad, ψ and ϕ for $k=2,3$ in matrix form.

2. Consider 3-dimensional Lie algebras defined by the following commutator relations:

   - $\mathfrak{g}_1$: $\quad [e_1,e_2]=e_3$, $[e_2,e_3]=e_1$, $[e_3,e_1]=e_2$;
   - $\mathfrak{g}_2$: $\quad [e_1,e_2]=2e_2$, $[e_1,e_3]=-2e_3$, $[e_2,e_3]=e_1$;
   - $\mathfrak{g}_3$: $\quad [e_1,e_2]=e_3$;
   - $\mathfrak{g}_4$: $\quad [e_1,e_3]=e_3$, $[e_2,e_3]=-e_3$.

   For each of them, describe explicitly the adjoint representation. Namely, for each element $\xi = ae_1+be_2+ce_3$ find the matrix of the operator ad_ξ.

   Prove that $\mathfrak{g}_1$ is isomorphic to $so(3)$ and $\mathfrak{g}_2$ is isomorphic to $sl(2,\mathbb{R})$.

   Is the adjoint representation for $\mathfrak{g}_i$ faithful?

   What is the center of $\mathfrak{g}_i$?

   Find a faithful representation for $\mathfrak{g}_3$, and for $\mathfrak{g}_4$.

   Prove that these Lie algebras are not isomorphic to each other.

3. Let G be a matrix Lie group, i.e. $G\subset GL(n,\mathbb{R})$. Let V be a vector space of $n\times n$ square matrices. Consider the following linear representations of G on V:

   - Φ_1: $\quad (\Phi_1)_X A = XA$,
   - Φ_2: $\quad (\Phi_2)_X A = (X^\top)^{-1}A$,
   - Φ_3: $\quad (\Phi_3)_X A = XAX^\top$,
   - Φ_4: $\quad (\Phi_4)_X A = (X^\top)^{-1}AX^\top$,

where $X \in G$ and $A \in V$.

Verify that Φ_i is indeed a linear representation and describe the corresponding induced representation $\phi_i = d\Phi_i$ of the corresponding Lie algebra $\mathfrak{g}$.

Show that Φ_1, Φ_2, Φ_3 are reducible, i.e. admit non-trivial invariant subspaces in V.

For $G = SL(n, \mathbb{R})$, describe the invariants of Φ_1.