Invariants of knots and links

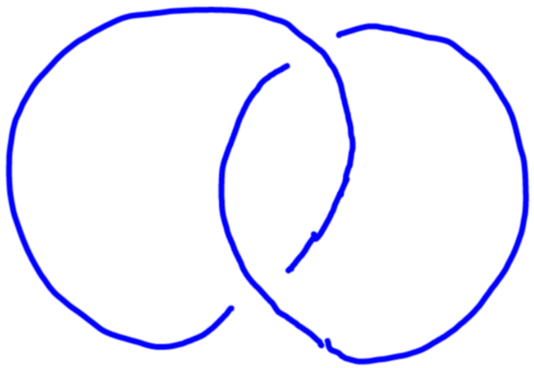
A knot is a smooth injective map $S^1 \to \mathbb{R}^3$. Two knots are isotopic (equivalent) if one is obtained from the other by a self-diffeomorphism of $\mathbb{R}^3$.

Link: $S^1 \sqcup \dots \sqcup S^1 \to \mathbb{R}^3$

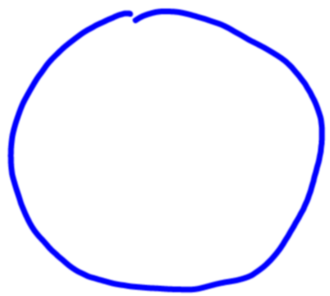
Knot invariant: $f : \{\text{knots}\} \longrightarrow X$

f depends only on the equivalence class

trefoil knot

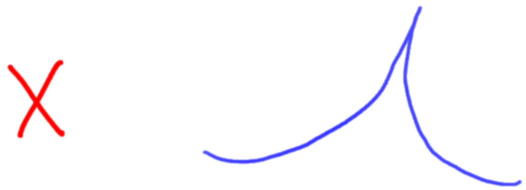


Hopf link



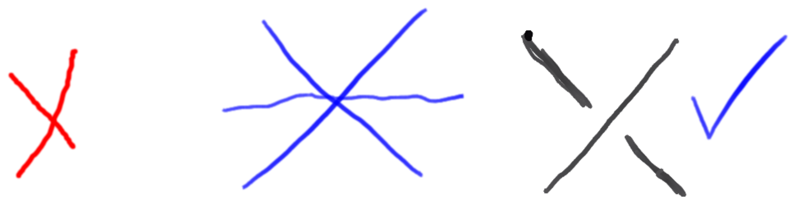
unknot

Instead of 3D knots, one considers f defined on 2D projections of knots, i.e. **knot (or link) diagrams** (see examples above). These are (unions of) smooth closed plane curves with finitely many self-intersections (crossings) where



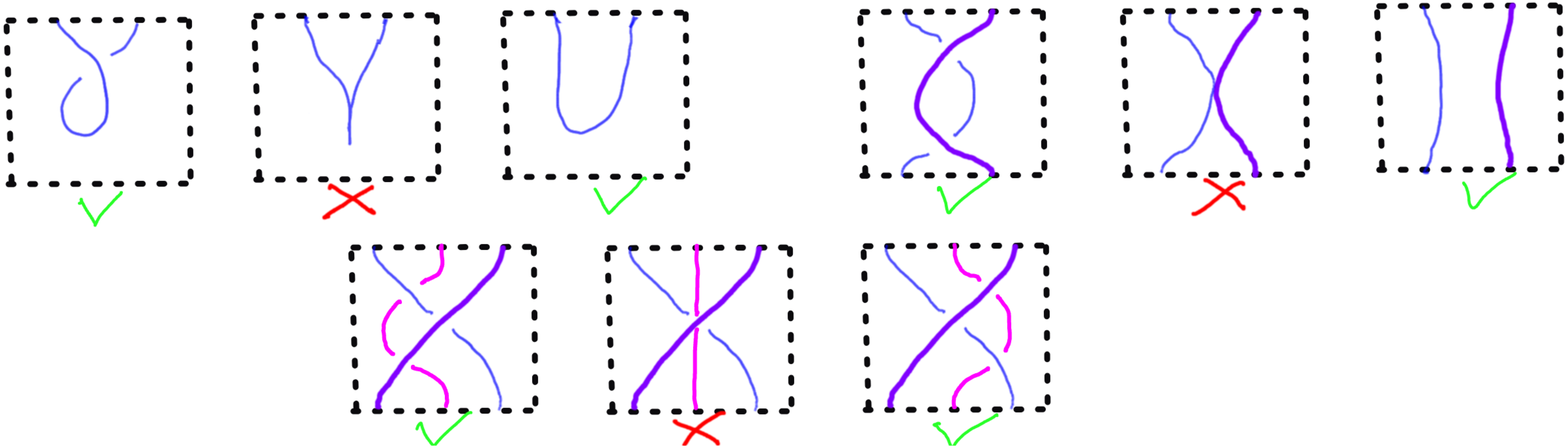
- no "cusps" are allowed;

- crossings must be transversal



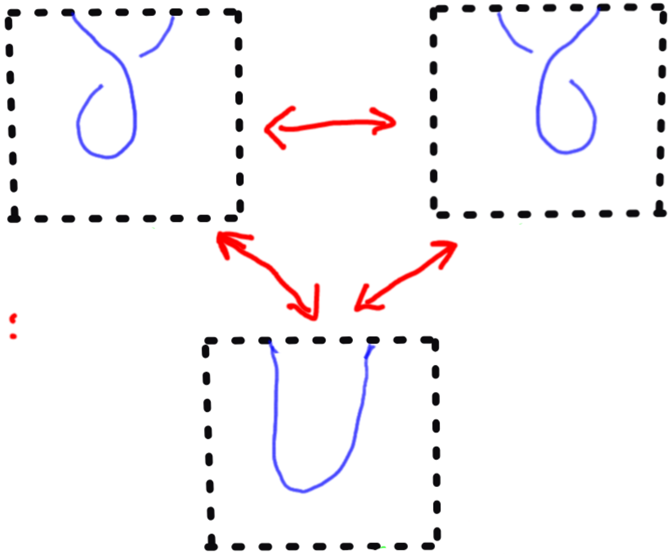
- only 2 strands can cross at a given point and each crossing is labelled to show which strand is "above".

When $\mathbb{R}^3$ is continuously deformed, a knot $K: S^1 \to \mathbb{R}^3$ is deformed to K' isotopic to K. Yet the plane diagrams D of K and D' of K' may not be isotopic in $\mathbb{R}^2$: this happens if, during deformation, the diagram passes throug a "prohibited" configuration, such as



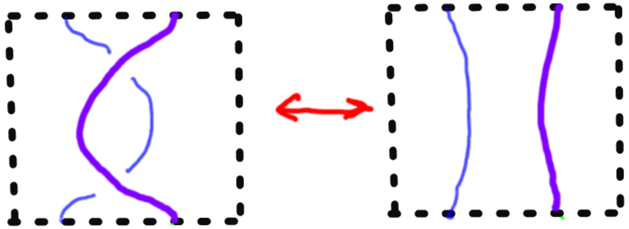
Reidemeister moves

R1:

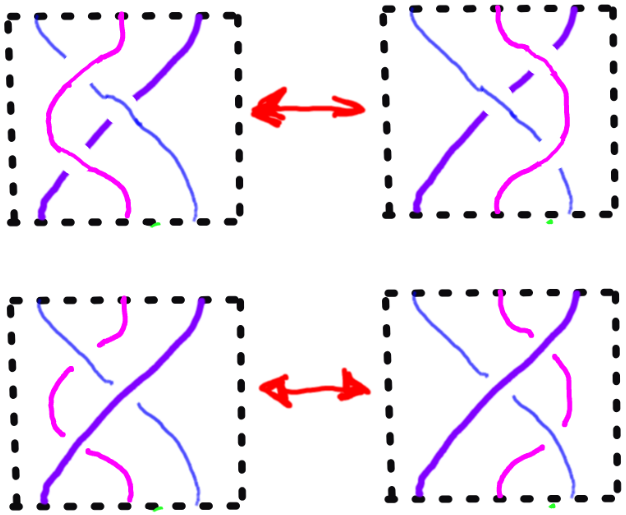


R0: an isotopy of $\mathbb{R}^2$

R2:



R3:



THM If D, D' are diagrams of isotopic knots K, K' then $D \underset{\text{Reidemeister}}{\sim} D'$.

Corollary A function $f : \left\{ \begin{array}{c} \text{knot diagrams} \\ \text{in } \mathbb{R}^2 \end{array} \right\} \to X$ gives a knot invariant if, and only if,

$$D \underset{\text{Reidemeister moves}}{\sim} D' \implies f(D) = f(D').$$

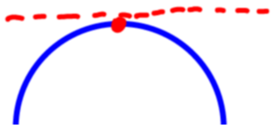
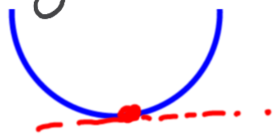
Sliced tangle diagrams

We fix a "vertical direction" (= "time arrow") in $\mathbb{R}^2$ which we will always draw downwards.

time arrow

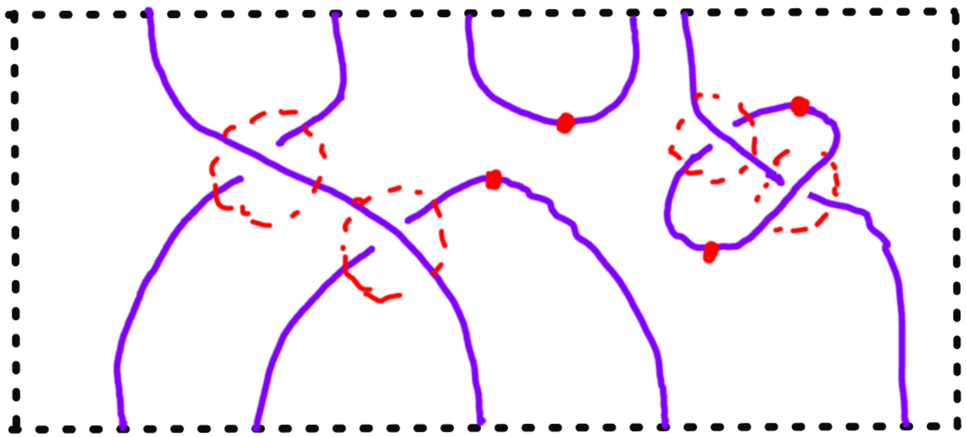
We consider only diagrams satisfying (1) & (2) below:

(1) there are finitely many critical points,

either  or 

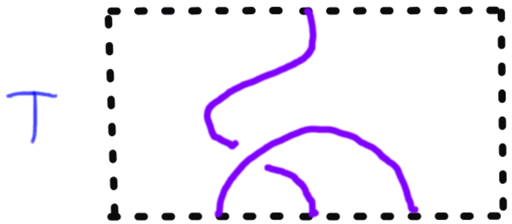
(2) all crossings are at 45° to the time arrow

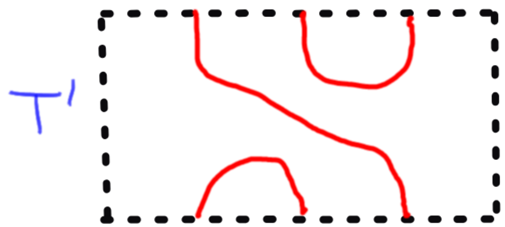
A tangle diagram is an intersection of a "regular" link diagram with a box (horizontal/vertical rectangle) where strands intersect only top & bottom of the box and always vertically.

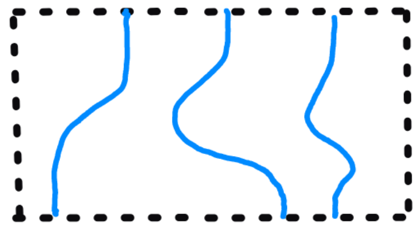


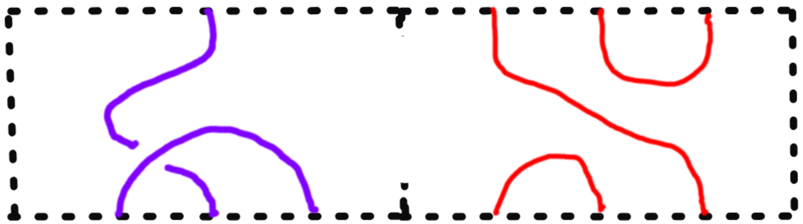
Operations on tangle diagrams:

a trivial diagram
no crit. points
no crossings

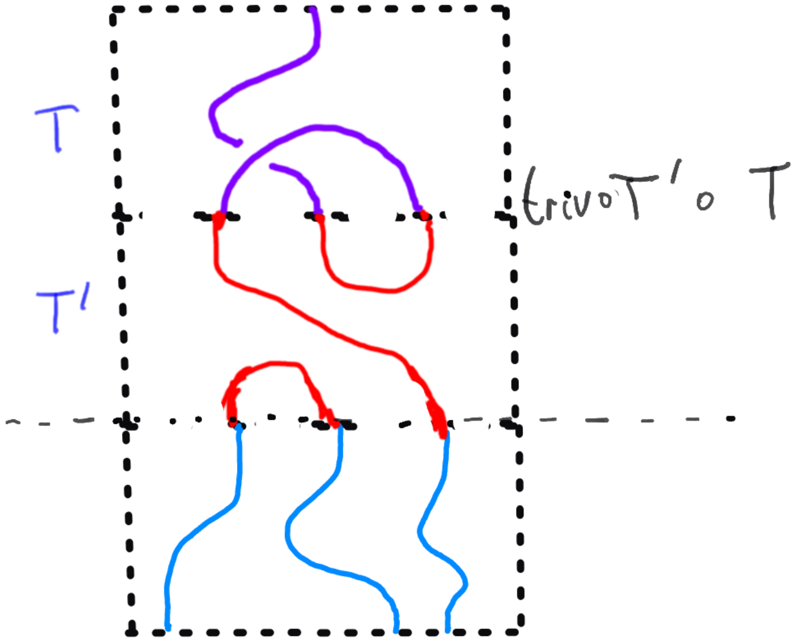






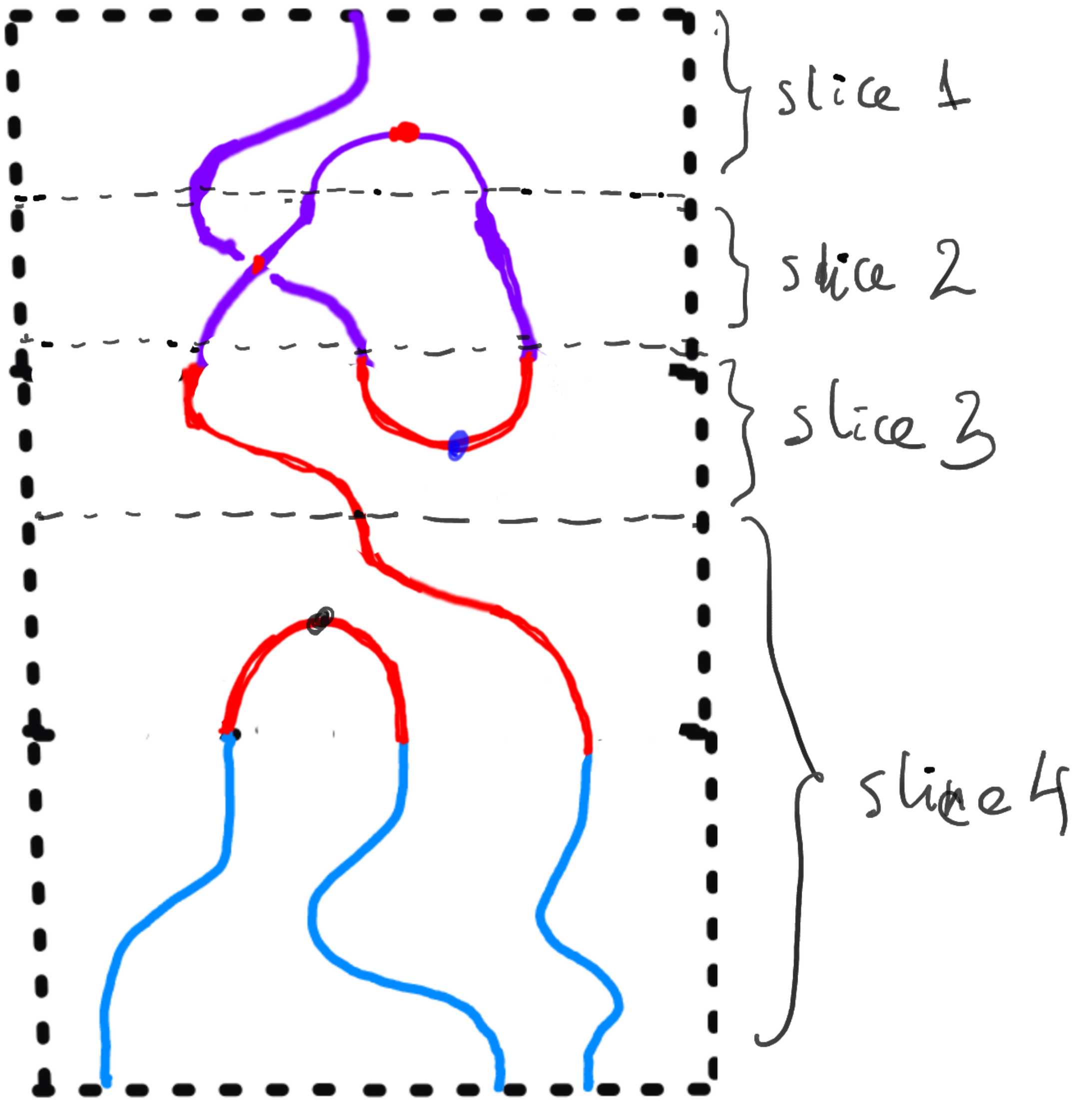


$T \otimes T'$



Up to a regular isotopy preserving all crossings and all critical points,

every regular tangle diagram is equivalent to a sliced tangle diagram: a composition of boxes each of which contains at most one crossing or critical point.


slice 1
slice 2
slice 3
slice 4

We would like to construct $f : \left\{ \begin{matrix} \text{sliced} \\ \text{link diagrams} \end{matrix} \right\} \to X$ which would give a knot and link invariant.

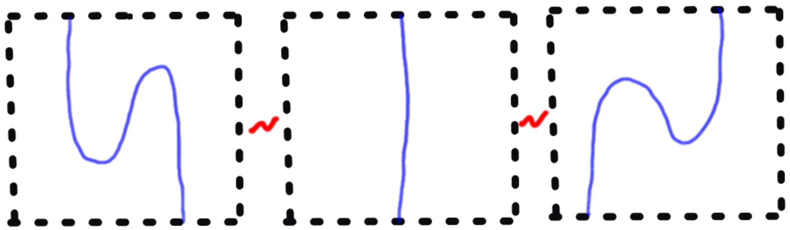
What kind of equivalence of sliced diagrams should preserve the value of f to guarantee it's an invariant?

We would like to construct $f : \left\{ \begin{array}{l} \text{sliced} \\ \text{link diagrams} \end{array} \right\} \to X$ which would give a knot and link invariant.

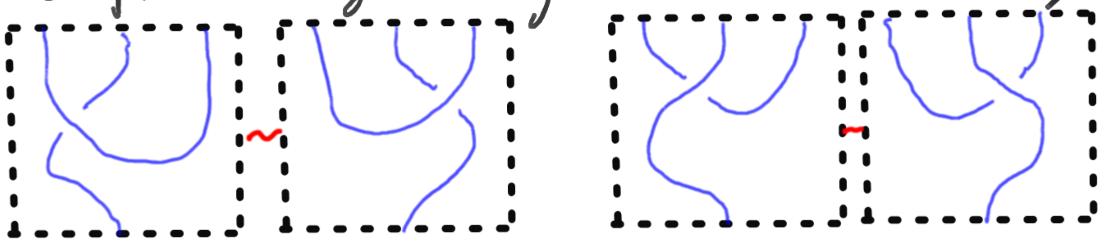
What kind of equivalence of sliced diagrams should preserve the value of f to guarantee it's an invariant?

THM (Turaev, Freyd-Yetter, 1988-9) f is an isotopy invariant iff f is unchanged under the Turaev moves:

- Reidemeister moves R1-R3 (for tangle diagrams - in boxes)



T1



T2

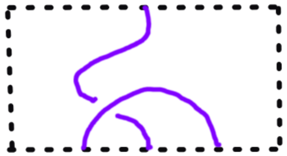
– as well as

$$\begin{array}{|c|}\hline T \\ \hline \text{triv.} \\ \hline\end{array} \sim \begin{array}{|c|}\hline \text{triv}' \\ \hline T \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline T & \text{triv}' \\ \hline \text{triv} & T' \\ \hline\end{array} \sim \begin{array}{|c|c|}\hline \text{triv}'' & T' \\ \hline T & \text{triv}''' \\ \hline\end{array}$$

The Operator Bracket

The next construction assigns, to each sliced tangle diagram T, a value $\langle T \rangle$ in the set of linear operators $V^{\otimes k} \to V^{\otimes \ell}$ where V is a fixed vector space,

k = # of inputs of T
ℓ = # of outputs of T
V = vector space which is fixed throughout

T  $k=1$, $\ell=3$

The bracket $\langle T \rangle$ will be defined for all sliced tangle diagrams T and will automatically satisfy

$$\langle T \otimes T' \rangle = \langle T \rangle \otimes \langle T' \rangle \qquad \langle T' \circ T \rangle = \langle T' \rangle \circ \langle T \rangle$$

$$\langle \text{trivial with } k \text{ strands} \rangle = \mathrm{id}_{V^{\otimes k} \to V^{\otimes k}}$$

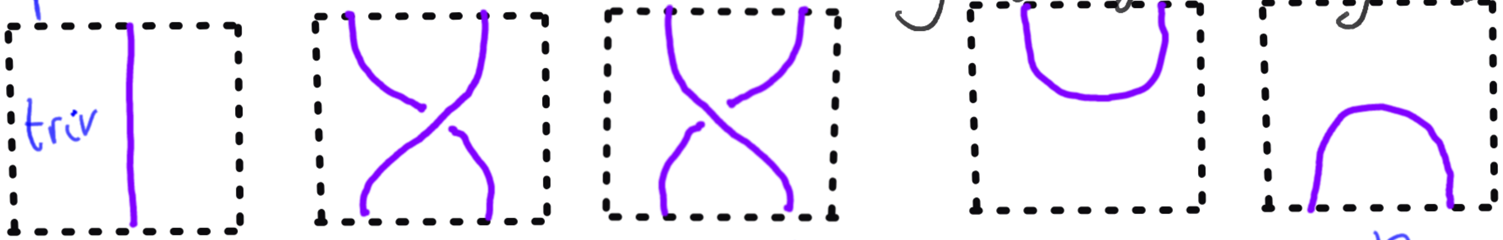
To define an operator bracket, we must choose V (a vector space) and three linear maps:

$R : V \otimes V \to V \otimes V$ called "R-matrix"

$u : V \otimes V \to \mathbb{C}$ cup map

$n : \mathbb{C} \to V \otimes V$ cap map

We assign operator value to elementary tangle diagrams as follows:



$id = id_V$ $\qquad \Psi$ $\qquad \Psi^{-1}$ $\qquad u$ $\qquad n$

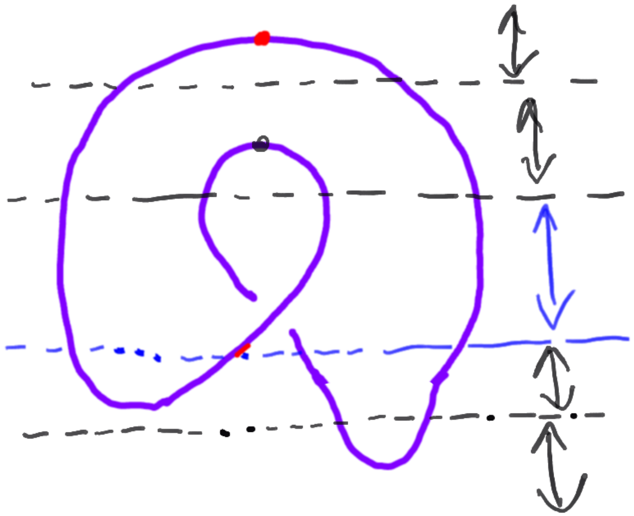
$u : V^{\otimes 2} \to V^{\otimes 0} = \mathbb{C}$

$$\Psi = \tau \circ R : V \otimes V \longrightarrow V \otimes V$$

A slice is: $\langle T_1 \otimes T_2 \otimes \cdots \otimes T_m \rangle \overset{\text{def}}{=} \langle T_1 \rangle \otimes \cdots \otimes \langle T_m \rangle$

define $\langle T' \circ T \rangle = \langle T' \rangle \circ \langle T \rangle$,

Example Suppose V, $R : V^{\otimes 2} \to V^{\otimes 2}$, $u : V^{\otimes 2} \to \mathbb{C}$, $n : \mathbb{C} \to V^{\otimes 2}$ are chosen. Calculate $\langle T \rangle$ for the knot diagram given below.

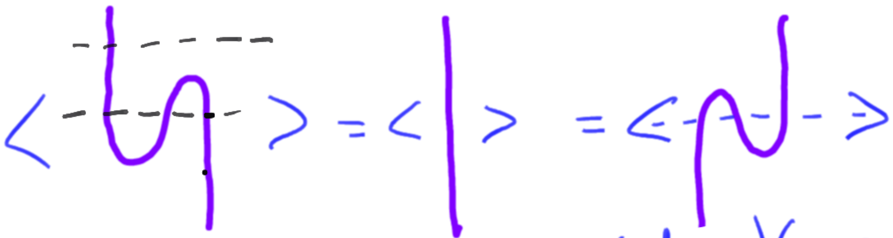


$$\mathbb{C} \xrightarrow{n} V \otimes V \xrightarrow{id \otimes n \otimes id} V \otimes V \otimes V \otimes V \xrightarrow{id \otimes \Psi \otimes id} V \otimes V \otimes V \otimes V \xrightarrow{u \otimes id \otimes id} V \otimes V \xrightarrow{u} \mathbb{C}$$

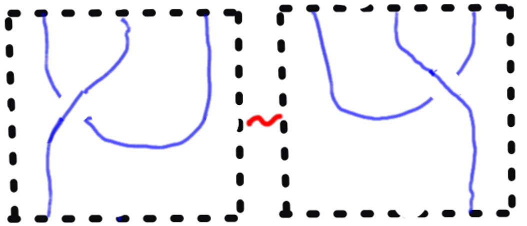
$$\{\text{linear maps } \mathbb{C} \to \mathbb{C}\} \cong \mathbb{C}$$

Can we make sure that the operator bracket $\langle\ \rangle = \langle\ \rangle_{R,u,n}$ is an isotopy invariant?

We need to check that $\langle\ \rangle$ does not change under the Turaev moves, which mean constraints (equations) on R, u and n:

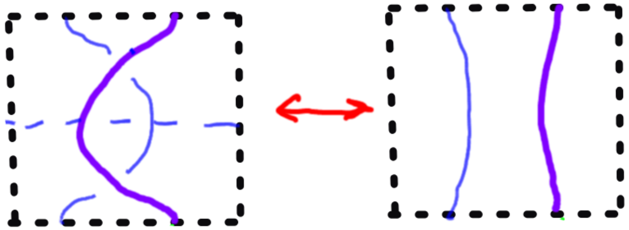


$$\mathrm{id} = (\mathrm{id} \otimes u)(n \otimes \mathrm{id})$$



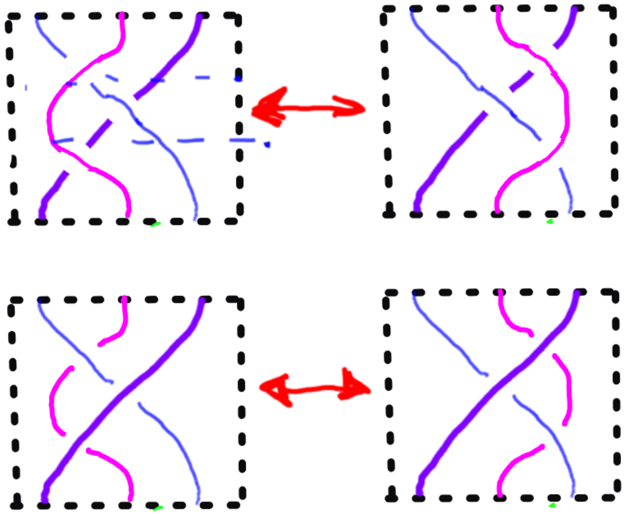
$$(\mathrm{id} \otimes u)(\Psi \otimes \mathrm{id}) = (u \otimes \mathrm{id})(\mathrm{id} \otimes \Psi^{-1})$$

R2:



$$\Psi^{-1} \circ \Psi = \mathrm{id}_{V^{\otimes 2}}$$

R3:



$(\Psi = \tau \circ R)$

constraint on Ψ:

$$\Psi_{12}\Psi_{23}\Psi_{12} = \Psi_{23}\Psi_{12}\Psi_{23}$$

(as maps $V^{\otimes 3} \to V^{\otimes 3}$)

i.e. Braid Equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Q.Y.B.E.

How to construct solutions to the equations given by the Turaev moves R1, R2, R3, T1, T2?

THM Assume that H is a Hopf algebra;

$R \in H \otimes H$ is a quasitriangular structure on H;

V is a finite-dimensional H-module, i.e. vector space with an H-action $\triangleright : H \otimes V \to V$;

$u : V \otimes V \to \mathbb{C}$ is a bilinear pairing which is non-degenerate and is invariant under $\triangleright$, meaning $u(h \triangleright x, y) = u(x, S(h) \triangleright y)$ $\forall x, y \in V$, $h \in H$

Consider R, u, n :

$R : V\otimes V \to V\otimes V,$ $\quad R(x\otimes y) = R \triangleright (x\otimes y)$

$u : V\otimes V \to \mathbb{C}$ $\quad$ as given

$n : \mathbb{C} \to V\otimes V,$ $\quad n(1_{\mathbb{C}}) = \sum_{i=1}^{\dim V} e_i \otimes f_i$ $\quad$ dual bases w.r.t. u

Then the operator bracket $\langle \;\; \rangle$ defined by R, u, n is invariant with respect to: