Complex zeros of the 2d Ising model on dynamical random lattices [*]

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We study the zeros in the complex plane of the partition function for the Ising model coupled to 2d quantum gravity for complex magnetic field and for complex temperature. We compute the zeros by using the exact solution coming from a two matrix model and by Monte Carlo simulations of Ising spins on dynamical triangulations. We present evidence that the zeros form simple one-dimensional patterns in the complex plane, and that the critical behaviour of the system is governed by the scaling of the distribution of singularities near the critical point.

1. THE MODEL

We study the Ising model on dynamical square and triangular lattices with the spins located on the N faces and on the N_v vertices, respectively. The partition function for a fixed lattice G_N is given by

$$Z_{\text{flat}}(G_N, \beta, H) = \sum_{\{\sigma\}} e^{\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j + H \sum_i \sigma_i} \quad (1)$$

and the partition function $Z(N, \beta, H)$ for the model coupled to quantum gravity is obtained by summing $Z_{\text{flat}}(G_N, \beta, H)$ over all lattices G_N with spherical topology. The system undergoes a third order phase transition [5]. Assuming scaling, the critical exponents (that are simply related to the flat space exponents [4]) can be computed directly from the equivalence of $Z(N, \beta, H)$ to the free energy of a hermitean two–matrix model [5]. In the usual notation, $\beta = 1/2$, $\gamma = 2$, $\delta = 5$, $\nu d_H = 3$ ($d_H \approx 4$ is the Hausdorff dimension of the system).

If we fix β (H), $Z_{\text{flat}}(G_N, \beta, H)$ is a polynomial in $y \equiv e^{-2H}$ ($c \equiv e^{-2\beta}$). In the thermodynamic limit its zeros (called Lee–Yang [1] or Fisher [2] zeros respectively) form dense sets on lines, which are Stokes lines for Z. The Lee–Yang zeros lie on the unit circle [1] for fixed G_N. One then defines a density of zeros $\rho_{YL}(\beta, \theta)$ ($H = i\theta$) ($\rho_F(\beta, H)$ for Fisher zeros). In the high temperature phase the Lee–Yang zeros form a gap such that $\rho_{YL}(\beta, \theta) = 0$ for $|\theta| < \theta_0$. As $\beta \to \beta_c$, the gap closes and the

zeros pinch the real axis at $y = 1$, signalling the onset of the phase transition.

2. EXACT CALCULATION

The exact solution of the Ising model on a dynamical lattice was given in [5]. The grand canonical partition function $Z(g, \beta, H)$ is given by $Z(g, \beta, H) \equiv \sum_{N=1}^{\infty} k^N Z(N, \beta, H)$, where $k \equiv -4gc/(1 - c^2)^2$. Using the exact solution we determined $Z(N, \beta, H)$ for $N \le 14$. Then its roots for either fixed β or fixed H were determined.

Our results indicate that the Lee–Yang zeros lie on the unit circle (cf. [6]), *also after summing over all* G_N. Fig. 1 indicates a vanishing gap in $\rho_{YL}(\beta_c, \theta)$ as $N \to \infty$. We also observed the vanishing of the gap for a fixed lattice size as $\beta \to \beta_c$ from the hot phase.
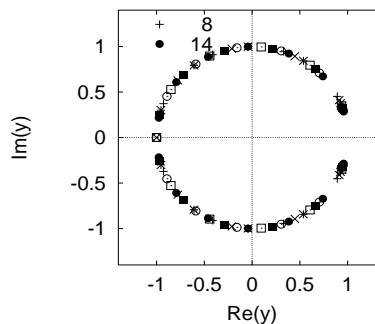

Figure 1. Lee–Yang zeros for dynamical lattices of varying size $N = 8$, 9, ..., 14 at β_c.

Also the Fisher zeros form $1d$ patterns. For $H = 0$ and $N \to \infty$ they approach the (physical

and non-physical) critical points $c = \pm 1/4$. There are also Fisher zeros on the imaginary axis, that flow to $c = \pm i\infty$. Fig. 2 shows the Fisher zeros for $H = 0$ and $H \neq 0$ at fixed N. Mapping the zeros onto the $\tilde{c}$-plane, where the tilde refers to the *dual* spin model (recall the usual duality relation $\tilde{c} \equiv \mathrm{e}^{-2\tilde{\beta}} = \tanh(\beta)$), we observed the existence of an *antiferromagnetic* phase transition in the dual model in agreement with ref. [8]. This AF transition is absent in the original model where the spins are located on the faces of the lattice.
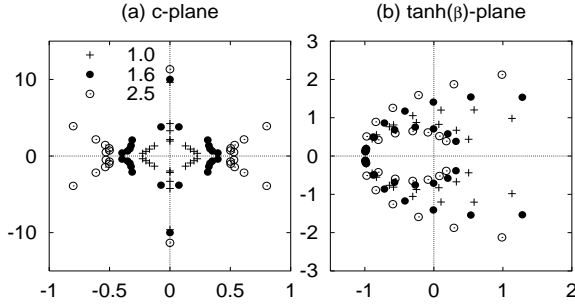

(a) c-plane   (b) tanh(β)-plane

Figure 2(a). Fisher zeros in the complex c plane for fixed $N = 14$ and varying y. Only for $y = 1$ ($H = 0$) the zeros pinch the real axis. **(b)** The trajectories in the $\tilde{c}$-plane. Note the FM and AFM phase transitions at $\tilde{c} = 3/5,\ 5/3$ for $y = 1$. The approach to these points is observed as N increases (not shown).

A priori, it is not evident that the partition function zeros will exhibit scaling in the presence of quantum gravity. However, the evidence in the range of central charge $0 < c < 1$ [7] for a diverging correlation length at the critical point makes such a scenario more plausible.

From scaling arguments [3] one deduces that the jth Lee–Yang zero should scale as $H_j \sim N^{-\beta\delta/(\nu d_H)}$. Fig. 3 shows the first three Lee–Yang zeros vs. N. Fig. 3 gives $\beta\delta/(\nu d_H) \approx 0.871 \pm 0.002,\ 0.935 \pm 0.002$ and 0.951 ± 0.002 for $j = 1, 2, 3$ respectively. The value expected from the scaling assumption is $5/6 \approx 0.8333...$ for all j.

It is expected from renormalization group arguments [3] that the trajectories of the motion of Fisher zeros in the K plane ($K \equiv u - u_c$ and $u \equiv \mathrm{e}^{-4\beta}$) with varying real H, will form an angle $\psi = \pi/(2\beta\delta)$ with the real K axis as $N \to \infty$. For our case $\psi = 36°$. The approach to this value (dashed line) can be observed in Fig. 4.

Next we tested the scaling relation [3] $H_j^2(N/j)^{2\delta/(\delta+1)} = F\left(K(N/j)^{1/(\nu d_H)}\right)$ where F is a universal scaling function. Taking K real, we plotted $H_j(N/j)^{5/6}$ vs. $(\beta - \beta_c)(N/j)^{1/3}$ (not shown) for all zeros of order $j > 1$ and all lattice sizes. The points fell on one universal curve representing the function F for large j in the hot phase. By varying the value of the exponent $5/6$ we observed a broadening of the curve and obtained a (somewhat subjective) determination $\delta/(\delta+1) = \beta\delta/(\nu d_H) \approx 0.85 \pm 0.05$, in excellent agreement with the KPZ value.

From the definition of F we expect for the jth Fisher zero and large N: $\ln|K_j| \sim \ln|\beta_j - \beta_c| = 1/(\nu d_H)\ln(j/N) + C$ Fig. 5 shows $|K_1|$ and $|\beta_1 - \beta_c|$ vs. N on a log–log plot. K_j did not yield a straight line at these lattice sizes, but $|\beta_j - \beta_c|$ scales and gives $1/(\nu d_H) \approx 0.327 \pm 0.001$, 0.377 ± 0.002, for $j = 1, 2$ respectively. The value expected from scaling is $1/3$.
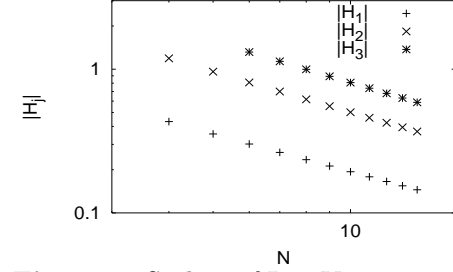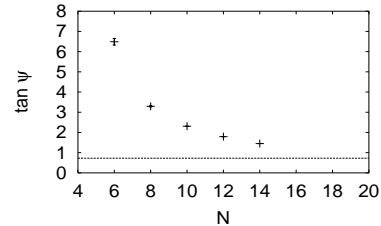

Figure 3. Scaling of Lee–Yang zeros H_j.

Figure 4. $\tan\psi$ versus N. The dashed line is $\tan 36°$.
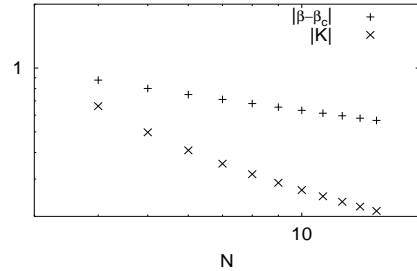

Figure 5. Scaling of the first Fisher zero.

3. MULTIHISTOGRAMMING

For systems with more than a few spins, the study of complex zeros must rely on numerical methods. We used multihistogramming [9] to determine the partition function accurately throughout a continuous range of couplings. The lattice sizes that we simulated ranged from 32 to 256 vertices (60 – 508 triangles). We refer to [10] for details. Lee–Yang and Fisher zeros were determined from the minima of $Z(\beta, H)/Z(\beta)$ and $Z(\beta, 0)/Z(\text{Re}\beta)$, respectively, for real β and imaginary H and for complex β and $H = 0$. Tables with the observed zeros can be found in [10]. We observed Lee–Yang zeros as follows: for $N_v = 64$, only the first zero could be seen, for $N_v = 96$, the first two zeros, for $N_v = 128$, three zeros, for $N_v = 256$, five zeros. For the Fisher zeros, only the first zero was visible for all lattice sizes $N_v = 32$, 64, 96, 128, 256.
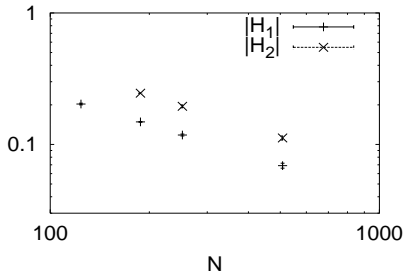

Figure 6. The first and second Lee–Yang zero observed using multihistogramming.
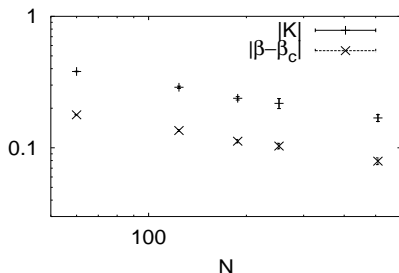

Figure 7. The first Fisher zero observed using multihistogramming.

Fits to $H_j \sim N^{-\beta\delta/(\nu d_H)}$ give for the first and second Lee–Yang zero (Fig. 6) $\beta\delta/(\nu d_H) = 0.773 \pm 0.013$, 0.788 ± 0.033 ($j = 1, 2$). Using N_v instead of N for volume, the corresponding values are 0.787 ± 0.013, 0.800 ± 0.034 ($j = 1, 2$), indicating finite size effects. The results are in quite good agreement with the expected value 5/6. From

$\nu d_H = 3$ [5] we obtain $\beta\delta = 2.36 \pm 0.04$, 2.40 ± 0.10 ($j = 1, 2$ respectively); the exact value is 2.5.

The scaling of the observed Fisher zeros is shown in Fig. 7. The extracted values of $1/(\nu d_H)$ are: for K 0.392 ± 0.016; for $|\beta - \beta_c|$ 0.386 ± 0.014 (expected value: 1/3 for both as $N \to \infty$). They increase slightly if N_v is used instead of N. Including only the three largest lattices we obtain (using N for volume): $1/(\nu d_H) = 0.350 \pm 0.067$.

4. CONCLUSION

We conclude that the critical behaviour of the system is described by the scaling of the distribution of the complex zeros of the partition function. Our result for Lee–Yang zeros presents us with the challenge of proving a corresponding Lee–Yang theorem for the case when a fluctuating metric contributes an additional quantum degree of freedom.

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