MAGIC assessment cover sheet

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Magic 109 Ibraheem Sajid

Q1. (a) $\varepsilon(S(x)) = \varepsilon(x_{(1)})\varepsilon(S(x_{(2)}))$ Given

$= \varepsilon(x_{(1)}S(x_{(2)}))$ Since ε is an algebra morphism ✓

$= \varepsilon(\varepsilon(x)1_H)$ By the antipode law ✓

$= \varepsilon(x)\varepsilon(1_H)$ As ε is linear and $\varepsilon(x)$ is scalar ✓

$= \varepsilon(x)$ ■ Since $\varepsilon(1_H) = 1_{\mathbb{C}}$ (ε algebra morphism) ✓

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(b) ~~(a)~~ (i) $y := g - g^2$

$S(y) = S(g) - S(g^2)$ ✓ S is linear

$= g^2 - g = -y$ ✓ $S(g) := g^{-1} = g^2$, $S(g^2) = (g^2)^{-1} = g$

$\Delta y = \Delta g - \Delta g^2$ Δ is linear

$= g\otimes g - g^2 \otimes g^2$ g, g^2 are grouplike

But note $1\otimes y + y \otimes 1 = 1\otimes g - 1\otimes g^2 + g\otimes 1 - g^2\otimes 1 \neq \Delta y$

$H\otimes H$ has basis $\{x_1\otimes x_2 \mid x_1, x_2 \in \{\underset{=1}{e}, g, g^2\}\}$ ✓

(ii) ~~[illegible]~~ $y := x^3$

$S(y) = S(x)^3 = (-x)^3 = -x^3 = -y$

S antimorphism, $S(x) := -x$ ✓

$\Delta y = (\Delta x)^3 = (1\otimes x + x\otimes 1)^3$

$= 1\otimes x^3 + 3x\otimes x^2 + 3x^2\otimes x + x^3\otimes 1$

Δ algebra morphism, x primitive

But note $1\otimes y + y\otimes 1 = 1\otimes x^3 + x^3\otimes 1 \neq \Delta y$

$x\otimes x^2$, $x^2\otimes x$ are basis elements of $H\otimes H$ so non-zero ✓

(iii) $y := K - K^{-1}$

$S(y) = S(K) - S(K^{-1})$ S linear

$= K^{-1} - K = -y$ $S(K^{\pm 1}) := K^{\mp 1}$

$\Delta y = \Delta K - \Delta K^{-1}$ Δ linear

$= K \otimes K - K^{-1} \otimes K^{-1}$ $K^{\pm 1}$ grouplike ✓

But note $1 \otimes y + y \otimes 1 = 1 \otimes K - 1 \otimes K^{-1} + K \otimes 1 - K^{-1} \otimes 1$ 6/6

$\neq \Delta y$

$H \otimes H$ has basis $\{ x_1 \otimes x_2 \mid x_1, x_2 \in \{E, F, K^{\pm 1}\}\}$

Not quite, but these elements are linearly independent

(c) Let $a, b \in B$. We have ~~...~~ $a - b \in A$ and we

(Jack Hett) compute $h \triangleright (a-b) = h \triangleright a - h \triangleright b$ $\triangleright$ linear

$= \varepsilon(h) a - \varepsilon(h) b$ ✓

$= \varepsilon(h)(a-b)$

$\therefore a - b \in B$ so B is an additive subgroup of A.

$\triangleright$ linear

Also for $\lambda \in \mathbb{C}$, $h \triangleright \lambda a = \lambda h \triangleright a = \lambda \varepsilon(h) a = \varepsilon(h)(\lambda a)$

$\therefore \lambda a \in B$ so B is a vector subspace of A.

Now, $h \triangleright 1_A = \varepsilon(h) 1_A$ Definition of covariant action ✓

$\therefore 1_A \in B$

Lastly $h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b)$ ✓ Def^n of cov. action

$= \varepsilon(h_{(1)}) a \, \varepsilon(h_{(2)}) b$

$= \varepsilon(h_{(1)} \varepsilon(h_{(2)}))\, ab$ ε linear, $\varepsilon(h_{(2)})$ scalar

$= \varepsilon(h) ab$ counit law ✓

$\therefore ab \in B$

Thus, B is a subalgebra

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(d) $I = ATA$. Let $x \in I$, i.e. ~~$x = ata$~~

$x = a_1 t_1 b_1 + \cdots + a_n t_n b_n$

for $n \geq 0$, $a_i, b_i \in A$, $t_i \in T$ $\forall i$.

By linearity of $\triangleright$, it suffices to show that

$h \triangleright atb \in I$

for $a, b \in A$, $t \in T$.

Indeed, $h \triangleright atb = (h_{(1)} \triangleright a)(h_{(2)} \triangleright tb)$

$= (h_{(1)} \triangleright a)(h_{(2)_{(1)}} \triangleright t)(h_{(2)_{(2)}} \triangleright b)$

$= (h_{(1)} \triangleright a)(\varepsilon(h_{(2)_{(1)}}) t)(h_{(2)_{(2)}} \triangleright b)$

$= \varepsilon(h_{(2)_{(1)}})(h_{(1)} \triangleright a) t (h_{(2)_{(2)}} \triangleright b) \in I$

since $\varepsilon(h_{(2)_{(1)}})(h_{(1)} \triangleright a) \in A$ and $h_{(2)_{(2)}} \triangleright b \in A$

(i.e. each term in the sum is in A)

$\therefore$ $h \triangleright x \in I$ $\forall x \in I$, so $h \triangleright I \subset I$.

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(Q1: 20/20)

Q2.(a) Note ~~[illegible]~~ $ab = c$, $bc = a$, $ac = b$ in this group K.

~~[illegible]~~ Let $q_\gamma = \frac{1}{4}(e + \gamma_1 a + \gamma_2 b + \gamma_3 c)$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$

Then for $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$,

$$q_\alpha q_\beta = \frac{1}{16}\Big[(1 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)e + (\alpha_1 + \beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)a$$

$$+ (\alpha_2 + \beta_2 + \alpha_1\beta_3 + \alpha_3\beta_1)b$$

$$+ (\alpha_3 + \beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1)c\Big]$$

Suppose α satisfies $\left(\begin{array}{l}\alpha_1\alpha_2 = \alpha_3, \quad \alpha_1\alpha_3 = \alpha_2, \quad \alpha_2\alpha_3 = \alpha_1 \\ \text{and } \alpha_i \in \{1, -1\} \ \forall i.\end{array}\right)$ $(*)$

Then $q_\alpha^2 = \frac{1}{16}\left[(1 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2)e + 4\alpha_1 a + 4\alpha_2 b + 4\alpha_3 c\right]$

$$= \frac{1}{16}\left[4e + 4\alpha_1 a + 4\alpha_2 b + 4\alpha_3 c\right] = q_\alpha$$ ✓

But note ~~[illegible]~~ for all i, $p_i = q_\alpha$ for some α satisfying $(*)$

$\therefore p_i^2 = p_i \ \forall i.$ ✓

Now suppose α, β satisfy (*) and further

$$\left(\begin{array}{c}\text{exactly one of } (\alpha_1=\beta_1), (\alpha_2=\beta_2), \\ (\alpha_3=\beta_3) \text{ holds}\end{array}\right) \quad (\dagger)$$

Then $\{\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3\} = \{1, -1, -1\}$ as multisets.

$$\therefore \quad 1 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$$

Next we consider the multiset $M = \{\alpha_i, \beta_i, \alpha_j\beta_k, \alpha_k\beta_j\}$ where $i, j, k \in \{1,2,3\}$ are distinct.

1) If $\alpha_i = \beta_i$ then $M = \{\alpha_i, \beta_i, \alpha_i\alpha_k\beta_k, \alpha_k\beta_i\beta_k\}$

$$= \{\alpha_i, \beta_i, -\alpha_i, -\beta_i\} = \{1,1,-1,-1\}$$

2) If $\alpha_j = \beta_j$ then $M = \{\alpha_j\alpha_k, \beta_j\beta_k, \alpha_j\beta_k, \alpha_k\beta_j\}$

$$= \{\alpha_j\alpha_k, \alpha_j(-\alpha_k), \alpha_j(-\alpha_k), \alpha_j\alpha_k\}$$

$$= \{1,1,-1,-1\}$$

3) If $\alpha_k = \beta_k$ this is similar to (2) and $M = \{1,1,-1,-1\}$.
(symmetry)

In any case, $M = \{1,1,-1,-1\}$ so

$$\alpha_i + \beta_i + \alpha_j\beta_k + \alpha_k\beta_j = 0$$

$\therefore \; q_\alpha q_\beta = 0$ (each term has coefficient equal to one of the two sums above).

Lastly note that for $i \neq j$ we can choose

$$p_i = q_\alpha, \quad p_j = q_\beta \quad \text{for } \alpha, \beta \text{ satisfying } (*) \text{ and } (\dagger).$$

$\therefore$ $p_i p_j = 0$ for $i \neq j$.

✓ 4/4

(b) Note $\mathbb{C}K \otimes \mathbb{C}K$ has basis $\{e,a,b,c\} \otimes \{e,a,b,c\}$. ✓

$((\mathbb{C}K)^*$ has coproduct m^* and for $g \in K$,

$$m^* \delta_g = \delta_g \circ m$$

(~~[illegible]~~ note $(\mathbb{C}K)^* \otimes (\mathbb{C}K)^* \cong (\mathbb{C}K \otimes \mathbb{C}K)^*$ as $\mathbb{C}K$ finite dimensional).

$\therefore$ ~~[illegible]~~ Evaluating $m^* \delta_g$ on the dual basis gives

$$m^* \delta_g = \sum_{h,k \in K,\, hk = g} \delta_h \otimes \delta_k .$$

$\therefore$

$$m^* \delta_e = \delta_e \otimes \delta_e + \delta_a \otimes \delta_a + \delta_b \otimes \delta_b + \delta_c \otimes \delta_c$$
$$m^* \delta_a = \delta_e \otimes \delta_a + \delta_a \otimes \delta_e + \delta_b \otimes \delta_c + \delta_c \otimes \delta_b$$
$$m^* \delta_b = \delta_e \otimes \delta_b + \delta_b \otimes \delta_e + \delta_a \otimes \delta_c + \delta_c \otimes \delta_a$$
$$m^* \delta_c = \delta_e \otimes \delta_c + \delta_c \otimes \delta_e + \delta_a \otimes \delta_b + \delta_b \otimes \delta_a$$

✓

Using linearity of m^* and that $\{\delta_e, \delta_a, \delta_b, \delta_c\}$ is a basis of $(\mathbb{C}K)^*$, we can find all grouplikes.

Let $\varphi = \lambda_1 \delta_e + \lambda_2 \delta_a + \lambda_3 \delta_b + \lambda_4 \delta_c \quad , \lambda_i \in \mathbb{C}.$

! For this question we write δ_{gh} as shorthand for $\delta_g \otimes \delta_h$

We want $m^* \varphi = \varphi \otimes \varphi$

i.e. $\lambda_1(\delta_{ee} + \delta_{aa} + \delta_{bb} + \delta_{cc}) + \lambda_2(\delta_{ea} + \delta_{ae} + \delta_{bc} + \delta_{cb})$
$+ \lambda_3(\delta_{eb} + \delta_{be} + \delta_{ac} + \delta_{ca}) + \lambda_4(\delta_{ec} + \delta_{ce} + \delta_{ab} + \delta_{ba})$

$=$

$\lambda_1^2 \delta_{ee} + \lambda_2^2 \delta_{aa} + \lambda_3^2 \delta_{bb} + \lambda_4^2 \delta_{cc} + \lambda_1\lambda_2(\delta_{ea} + \delta_{ae})$
$+ \lambda_1\lambda_3(\delta_{eb} + \delta_{be}) + \lambda_1\lambda_4(\delta_{ec} + \delta_{ce}) + \lambda_2\lambda_3(\delta_{ab} + \delta_{ba})$
$+ \lambda_2\lambda_4(\delta_{ac} + \delta_{ca}) + \lambda_3\lambda_4(\delta_{bc} + \delta_{cb})$.

So φ is grouplike ($\neq 0$) $\iff$
$$\begin{cases} \lambda_1 = \lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2 & (1) \\ \lambda_2 = \lambda_1\lambda_2 = \lambda_3\lambda_4 & (2) \\ \lambda_3 = \lambda_1\lambda_3 = \lambda_2\lambda_4 & (3) \\ \lambda_4 = \lambda_1\lambda_4 = \lambda_2\lambda_3 & (4) \end{cases}$$

Using E3.2(a) here implicitly ~~[illegible]~~

~~The~~ (1) gives $\lambda_1 = 0$ or 1. But $\lambda_1 = 0 \Rightarrow \lambda_i = 0 \ \forall i$
$\Rightarrow \varphi = 0$ is not grouplike.

$\therefore \lambda_1 = 1$ and $\lambda_2, \lambda_3, \lambda_4 \in \{1, -1\}$ from (1).

(2), (3), (4) imply we must have $\{\lambda_2, \lambda_3, \lambda_4\} = \{1, -1, -1\}$ as multisets.

In fact these implications are all reversible, so or $\lambda_2 = \lambda_3 = \lambda_4 = 1$

So φ grouplike $\iff$ $\varphi = \delta_e + \delta_a + \delta_b + \delta_c$

① holds

Since $\varphi \neq 0$, ~~①~~ $\iff \lambda_1 = 1$ and $\lambda_2, \lambda_3, \lambda_4 \in \{1, -1\}$.

Then ①-④ holds $\iff \lambda_1 = 1$ and either $\lambda_2 = \lambda_3 = \lambda_4 = 1$ or $\{\lambda_2, \lambda_3, \lambda_4\} = \{1, -1, -1\}$ as multisets.

$$\therefore \varphi \text{ is grouplike} \iff \varphi = \begin{cases} \delta_e + \delta_a + \delta_b + \delta_c =: \xi_1 \\ \delta_e + \delta_a - \delta_b - \delta_c =: \xi_2 \\ \delta_e - \delta_a + \delta_b - \delta_c =: \xi_3 \\ \delta_e - \delta_a - \delta_b + \delta_c =: \xi_4 \end{cases}$$

✓

We have $\xi_i(p_j) = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$

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Since either all signs are equal (squaring to give 1) or two pairs of signs match and two differ (cancelling out).

(c) ~~$R^2 = (e \otimes (p_0+p_1))^2 + (b \otimes (p_0+p_1))^2$~~

$+ (e \otimes (p_0+p_1))(b \otimes (p_0+p_1)) + (b \otimes (p_0+p_1))(e \otimes (p_0+p_1))$

$= \cancel{e \otimes (p_0+p_1)} + e \otimes (p_0+p_1) + \cancel{b \otimes (p_0+p_1)} + b \otimes (p_0+p_1)$

~~(since $p_0^2 = p_0$, $p_0 p_1 = 0$, $p_1^2 = p_1$).~~ $= p_1 p_0$

~~[illegible]~~

(c) $R^2 = (e\otimes(p_0+p_1))^2 + (b\otimes(p_2+p_3))^2$
$+ (e\otimes(p_0+p_1))(b\otimes(p_2+p_3)) + (b\otimes(p_2+p_3))(e\otimes(p_0+p_1))$

$= e\otimes(p_0+p_1) + e\otimes(p_2+p_3) + b\otimes 0 + b\otimes 0$ 4/4

$= e\otimes(p_0+p_1+p_2+p_3) = e\otimes e$

$\therefore \lambda = 1.$ ✓

$H := \mathbb{C}K$

(d) Axiom 1: R is invertible.
This holds with $R^{-1} = R$ since $R\cdot R = e\otimes e = 1_{H\otimes H}$ ✓

Axiom 2: $x_{(2)}\otimes x_{(1)} = R\cdot x_{(1)}\otimes x_{(2)}\cdot R^{-1} \quad \forall x\in H$
Since multiplication in H and hence $H\otimes H$ is commutative, this becomes $x_{(2)}\otimes x_{(1)} = x_{(1)}\otimes x_{(2)}$. ✓

This is true for all basis elements of H (as they are grouplike), and it extends linearly to all of H. ✓

Axiom 3: $(\Delta\otimes id)R = R_{13}R_{23}, \quad (id\otimes\Delta)R = R_{13}R_{12}$

First, $(\Delta\otimes id)R = e\otimes e\otimes(p_0+p_1) + b\otimes b\otimes(p_2+p_3)$
and $R_{13}R_{23} = \left(e\otimes e\otimes(p_0+p_1) + b\otimes e\otimes(p_2+p_3)\right)\cdot$
$\left(e\otimes e\otimes(p_0+p_1) + e\otimes b(p_2+p_3)\right)$

$= e\otimes e\otimes(p_0+p_1) + e\otimes b\otimes 0 + b\otimes e\otimes 0 + b\otimes b\otimes(p_2+p_3)$
$= (\Delta\otimes id)R$ ✓

~~Next, $(id\otimes\Delta)R = $ ...~~

Next, we have $p_0+p_1 = \frac{1}{2}(e+a)$

$$\therefore \Delta(p_0+p_1) = \frac{1}{2}(e\otimes e + a\otimes a)$$

and $p_2+p_3 = \frac{1}{2}(e-a)$ $\therefore \Delta(p_2+p_3) = \frac{1}{2}(e\otimes e - a\otimes a)$.

$$\therefore (id\otimes\Delta)R = \frac{1}{2}e\otimes e\otimes e + \frac{1}{2}e\otimes a\otimes a + \frac{1}{2}b\otimes e\otimes e - \frac{1}{2}b\otimes a\otimes a$$ ✓

and $R_{13}R_{12} = \big(e\otimes e\otimes(p_0+p_1) + b\otimes e\otimes(p_2+p_3)\big)\cdot$
$\big(e\otimes(p_0+p_1)\otimes e + b\otimes(p_2+p_3)\otimes e\big)$

$$= e\otimes(p_0+p_1)\otimes(p_0+p_1) + b\otimes(p_2+p_3)\otimes(p_0+p_1) + b\otimes(p_0+p_1)\otimes(p_2+p_3) + e\otimes(p_2+p_3)\otimes(p_2+p_3)$$

$$= \left[\frac{1}{4}e\otimes e\otimes e + \frac{1}{4}e\otimes e\otimes a + \frac{1}{4}e\otimes a\otimes e + \frac{1}{4}e\otimes a\otimes a\right]$$
$$+ \left[\frac{1}{4}b\otimes e\otimes e - \frac{1}{4}b\otimes a\otimes e + \frac{1}{4}b\otimes e\otimes a - \frac{1}{4}b\otimes a\otimes a\right]$$
$$+ \left[\frac{1}{4}b\otimes e\otimes e + \frac{1}{4}b\otimes a\otimes e - \frac{1}{4}b\otimes e\otimes a - \frac{1}{4}b\otimes a\otimes a\right]$$
$$+ \left[\frac{1}{4}e\otimes e\otimes e - \frac{1}{4}e\otimes e\otimes a - \frac{1}{4}e\otimes a\otimes e + \frac{1}{4}e\otimes a\otimes a\right]$$

$$= \frac{1}{2}e\otimes e\otimes e + \frac{1}{2}e\otimes a\otimes a + \frac{1}{2}b\otimes e\otimes e - \frac{1}{2}b\otimes a\otimes a = (id\otimes\Delta)R.$$ ✓

great

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(Q2: 20/20)

Q3 (a) ~~Let $M \in$ ... be a non-standard monomial. Then~~

Let $M \in \underline{NonSt}$ be a non-standard monomial.

For $t_1, t_2 \in T$, we have $t_1 M t_2 \in J$

Since every monomial in $t_1 M t_2$ is non-standard (as M is a submonomial of each) ✓

By linearity, this means $t_1 j t_2 \in J \quad \forall j \in J$ too.

J is ~~already~~ a vector subspace, so it is clear now that it's an ✓ ideal.

Note $(X+J)(Y+J) = XY + J$
and $(Y+J)(X+J) = YX + J$

$\therefore (X+J)(Y+J) - (Y+J)(X+J) = XY - YX + J = XY + J \neq 0$ ✓

since $XY \notin J$

$(XY$ is standard$)$. ✓

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(b) ~~...~~

We have ~~...~~ $\mathcal{E}(J) = \{0\}$ does not fail. ✓
Neither do the axioms for an ideal fail by (a).

$\mathcal{E}(X) = \mathcal{E}(H) = \mathcal{E}(Y) = 0$ and $\mathcal{E}$ is an algebra morphism
Note $1 \notin \underline{NonSt}$.

However $\Delta(YX) = (\Delta Y)(\Delta X)$

$$= (1\otimes Y + Y\otimes 1)(1\otimes X + X\otimes 1)$$

$$= 1\otimes YX + Y\otimes X + X\otimes Y + YX\otimes 1$$

I would like a more detailed explanation here, using bases and linear independence. E.g.: Even though Y⊗X is not in T⊗J+J⊗T, and X⊗Y is not, why is their sum not an element of that space?

But note eg. $Y\otimes X \notin T\otimes J + J\otimes T$

(neither X or Y are non-standard)

So $\Delta(YX) \notin T\otimes J + J\otimes T$

and so the axiom $\left(\Delta(J) \subseteq T\otimes J + J\otimes T \text{ fails}\right)$.

~~Re~~ [This also means it is not a coideal]

Moreover, $S(YX) = XY \notin J$ ✓

$\therefore \left(S(J) \subseteq J\right)$ fails too.

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(b) ~~XYHH~~ ~~XYXHH+2X~~

~~$X^2YH^2 = 2X^2(HX+2X)...$~~

~~$XYXH = X(XY-H)H = X^2YH - XH^2$~~

~~and $XH^2 = (HX-2X)H = HXH - 2XH$~~

and $HXH =$

(c) Have $XYXH = X(XY-H)H = X^2YH - XH^2$

$= X^2YH - (HX-2X)H = X^2YH - HXH + 2XH$

$= X^2YH - H(HX-2X) + 2XH$

$= X^2YH - H^2X + 2HX + 2XH = X^2YH - H^2X + 2HX + 2(HX-2X)$

$= X^2YH - H^2X + 4HX - 4X$

$\therefore X^2YH - XYXH - H^2X + 4HX = 4X$ i.e. $\mu = 4$. ✓ 4/4

(d) The monomials $X^2YH, XYXH, H^2X, HX$ are all non-standard, so are in $\pi(J)$. $\therefore 4X \in \pi(J)$ ✓ so $X \in \pi(J)$

(using that $\pi(J)$ is an ideal so closed under addition and multiplication).

In T, $\ker \varepsilon = \text{span}\{X^pH^qY^r \mid p,q,r \in \mathbb{Z}_{>0}\}$

and this same formula descends to U.

We show that $X, Y, H \in \pi(J)$ and then it must be the whole kernel since

$\pi(\text{span}\{X^pH^qY^r \mid p,q,r \in \mathbb{Z}_{>0}\}) =$

$U\{X,H,T\}U$ i.e. the ideal generated by X,H,T.

$(1 \notin U\{X,H,T\}U$

$I :=$

(d) cont. ~~Consider the ideal $T\{X,H,Y\}T$ of T generated by $\{X,H,Y\}$. We claim $I = \mathrm{Span}_{\mathbb{C}} St_+$, where $St_+ := \{X^pH^qY^r \mid p,q,r \in \mathbb{Z}_{\geq 0}\}$.~~

~~Clearly $\mathrm{Span}_{\mathbb{C}} St_+ \subseteq I$. For the converse, suppose~~

(d) cont. ~~Consider the ideal $I := T\{X,H,Y\}T$ of T generated by $\{X,H,Y\}$. We claim $I = \mathrm{Span}_{\mathbb{C}} \underline{Mon}_+$, where $\underline{Mon}_+$ is all monomials of positive degree, ie. $\underline{Mon}_+ = \underline{Mon} \setminus \{1\}$.~~

~~Clearly $\mathrm{Span}_{\mathbb{C}} \underline{Mon}_+ \subseteq I$.~~

(d) cont. Consider the ideal $I := T\{X,H,Y\}T$ of T generated by $\{X,H,Y\}$. Note that $I = \mathrm{Span}_{\mathbb{C}} \underline{Mon}_+$, where $\underline{Mon}_+$ is all monomials of positive degree, ie $\underline{Mon}_+ = \underline{Mon} \setminus \{1\}$. ✓

I don't think it's true, considering ideals of the form < X-a, H-b, Y-c >,

Note that I is (the only) maximal ideal of T.

Now, $\langle HX - XY = 2X,\ HY - YH = -2Y,\ XY - YX = H \rangle \subseteq I$.

but this isn't important.

$\therefore$ $\pi(I)$ is (the only) maximal ideal of $\mathcal{U}$, by the correspondence theorem for ideals of quotients.

Note $\pi(I) = \mathcal{U}\{X,H,Y\}\mathcal{U}$, generated by monomials of positive degree in $\mathcal{U}$, is precisely the kernel of the counit map.

Furthermore, $J \subseteq I$ so $\pi(J) \subseteq \pi(I)$. ~~Note~~ ~~See that~~

True

Since counit of quotient agrees with the counit of parent space.

Finally, we show this inclusion is equality.

Indeed, we've shown (at the start of (d)) that $X \in \pi(J)$. ~~[illegible]~~

$\therefore \quad H = XY - YX \in \pi(J)$ ✓ (using multiplication and sum stays in ideal).

$\therefore \quad Y = \frac{1}{2}(YH - HY) \in \pi(J)$.

So $\pi(J) \supseteq \mathcal{U}\{X, H, Y\}\mathcal{U} = \pi(I)$

✓

$\therefore \pi(J) = \pi(I) =$ kernel of counit. ■

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(Q3 : 19/20)

Q4(a). First note that in H, $K^{-1}(KE)K^{-1} = K^{-1}(q^2EK)K^{-1}$

$\therefore \quad EK^{-1} = q^2K^{-1}E$ ✓

Now, $EZ = E\left[(q-q^{-1})^2 EF + q^{-1}K + qK^{-1}\right]$

$= (q-q^{-1})^2 E^2F + q^{-1}EK + qEK^{-1}$

$= (q-q^{-1})^2 E\left(FE + \dfrac{K-K^{-1}}{q-q^{-1}}\right) + q^{-1}EK + qEK^{-1}$ ✓

$= (q-q^{-1})EFE + E(q-q^{-1})(K-K^{-1}) + q^{-1}EK + qEK^{-1}$

$= (q-q^{-1})EFE + qEK + q^{-1}EK^{-1}$

$= (q-q^{-1})EFE + q\cdot q^{-2}KE + q^{-1}\cdot q^2K^{-1}E$

$= \left[(q-q^{-1})EF + q^{-1}K + qK^{-1}\right]E = ZE$ ✓ ∎

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(b) The antipode law says $S(F_{(1)})F_{(2)} = \varepsilon(F)1_H$

$\therefore \quad S(K^{-1})F + S(F)1 = 0$

$\therefore \quad S(F) = -S(K^{-1})F$ ✓

Since K^{-1} is grouplike, we must have $S(K^{-1}) = (K^{-1})^{-1}$
$= K$

$\left(S(K_{(1)})K_{(2)} = \varepsilon(K)1 \Rightarrow S(K)K = 1 \right)$.

$\therefore S(F) = -KF$. ✓

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(c) By linearity of $\triangleright$, $Z \triangleright y^2 =$

$$(q-q^{-1})^2 EF \triangleright y^2 + q^{-1} K \triangleright y^2 + q K^{-1} \triangleright y^2$$

Calculate: $K \triangleright y^2 = (K_{(1)} \triangleright y)(K_{(2)} \triangleright y)$

$$= (K \triangleright y)(K \triangleright y) = q^{-2} y^2$$ ✓

and $K^{-1} \triangleright y^2 = (K^{-2} \triangleright y)^2 = q^2 y^2$

(note ~~[illegible]~~

$$K^{-1} \triangleright K \triangleright y = 1 \triangleright y$$

$$\therefore \; K^{-1} \triangleright q^{-1} y = y \quad \text{i.e. } K^{-1} \triangleright y = qy \text{)}.$$

and $F \triangleright y^2 = (K^{-1} \triangleright y)(F \triangleright y) + (F \triangleright y)(1 \triangleright y)$

$$= qy \cdot 0 + 0 \cdot 1 = 0$$ ✓

(so $EF \triangleright y^2 = E \triangleright F \triangleright y^2 = 0$).

$$\therefore \; Z \triangleright y^2 = 0 + q^{-1} \cdot q^{-2} y^2 + q \cdot q^2 y^2$$

$$= (q^{-3} + q^3) y^2$$ ✓ , i.e. $m = 3$.

Similarly, calculate:

$$K \triangleright x^2 = (K \triangleright x)(K \triangleright x) = q^2 x^2$$

$K^{-1} \triangleright x^2 = q^{-2} x^2$ (~~[illegible]~~ as above for $K^{-1} \triangleright y$) ✓

$$E \triangleright x^2 = (1 \triangleright x)(E \triangleright x) + (E \triangleright x)(K \triangleright x)$$

$$= 1 \cdot 0 + 0 \cdot qx = 0$$

(so $FE \triangleright x^2 = 0$).

But $(EF) \triangleright (x^2)$ is not 0.

$$\therefore \; Z \triangleright x^2 = \boxed{0} + q^{-1} \cdot q^2 x^2 + q \cdot q^{-2} x^2 = (q + q^{-1}) x^2$$ ✗

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(d) $\exp_{q^{-2}}((q-q^{-1})E\otimes F) = \cancel{\ldots}$

$$1 + \sum_{n\geq 1} \frac{(q-q^{-1})^n E^n \otimes F^n}{[n;q^{-2}]!}$$

We won't need the value of $[n;q^{-2}]!$, just that it is welldefined and non-zero. ✓

We assume that tensor products act as

$$(X\otimes Y) \triangleright (x\otimes y) = (X\triangleright x)\otimes(Y\triangleright y)$$ ✓

Note $\exp_{q^{-1}}((q-q^{-1})E\otimes F) = 1 + \left(\sum_{n\geq 1} \frac{(q-q^{-1})^n E^{n-1}\otimes F^{n-1}}{[n;q^{-2}]!}\right) E\otimes F.$

and that $(E\otimes F)(y\otimes y) = x\otimes 0 = 0$. [Assuming convergence] ✓

$\therefore \exp_{q^{-1}}((q-q^{-1})E\otimes F)\triangleright(y\otimes y) = 1\triangleright(y\otimes y) = y\otimes y$ ✓

Thus $R\triangleright(y\otimes y) = \exp(\frac{\hbar}{2}H\otimes H)\triangleright(y\otimes y)$

Have $\exp(\frac{\hbar}{2}H\otimes H) = \sum_{i\geq 0} \frac{\hbar^i}{2^i i!} H^i\otimes H^i$

and so

$$\exp(\tfrac{\hbar}{2}H\otimes H)\triangleright(y\otimes y)$$

$$= \sum_{i\geq 0}\frac{\hbar^i}{2^i i!}(H^i\triangleright y)\otimes(H^i\triangleright y) = \sum_{i\geq 0}\frac{\hbar^i}{2^i i!}\, (-1)^i y\otimes(-1)^i y$$ ✓

$$= \sum_{i\geq 0}\frac{\hbar^i}{2^i i!}\, y\otimes y$$ ✓

$$= \left(\sum_{i\geq 0}\frac{\hbar^i}{2^i i!}\right)(y\otimes y). \blacksquare$$

$$= \exp(\tfrac{\hbar}{2})(y\otimes y)$$ ✓

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(Q4: 18/20)

End of submission.

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TOTAL: 20+20+19 = 59 out of 60
98%
Work of outstanding quality, very clearly written.
Thank you.