Preface

These are the lecture notes for MAGIC109 Introduction to Hopf Algebras and Quantum Groups given in Semester 2 of the academic year 2024/25. The notes are being developed as the course progresses. You are welcome to contact me at yuri.bazlov@manchester.ac.uk with any corrections, ideas and suggestions.

Yuri Bazlov

Introduction

In this Introduction, we try to convey an idea behind a Hopf algebra as an algebraic structure. We do not give formal definitions here — they will come later.

The Introduction is not covered in lectures. Readers may skip it and go straight to §1 where we begin to develop the theory rigorously.

0.1 Example (a family of transformations of $\mathbb{R}[x]$) Let $\mathbb{R}[x]$ be the vector space of all polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$. Let us consider the following linear transformations of $\mathbb{R}[x]$: for each real number a there is the transformation

$$T_a\colon \mathbb{R}[x] \to \mathbb{R}[x], \qquad T_a f \text{ is the polynomial defined by } (T_a f)(x) = f(x-a);$$

also,

$$R\colon \mathbb{R}[x] \to \mathbb{R}[x], \qquad Rf \text{ is the polynomial defined by } (Rf)(x) = f(-x).$$

The transformations $\{T_a\}_{a\in\mathbb{R}}$ and R are not independent. Indeed, applying T_a then R one obtains

$$(R(T_a f))(x) = (T_a f)(-x) = f(-x-a) = f(-(x+a)) = (Rf)(x+a) = (T_{-a}(Rf))(x).$$

As is typical in abstract algebra, the above calculation is interpreted as a *relation* between operators T_a, $a \in \mathbb{R}$ and R, and is written without mentioning the argument f:

$$RT_a = T_{-a}R. \tag{0.2}$$

One can similarly show that further relations hold:

$$T_{a+b} = T_a T_b, \qquad T_a T_{-a} = 1, \qquad R^2 = 1 \tag{0.3}$$

where 1 denotes the identity transformation.

0.4 Observation (the Euclidean group $E(1)$) At this point, we note that there is a well-known abstract algebraic structure "containing" the symbols T_a ($a \in \mathbb{R}$) and R which obey the relations (0.2)–(0.3). This structure is called a group.

A group can be defined by specifying a set of *generators* (symbols) and a set of *relations* (a relation is an equation which says that a product of generators is equal to another product of generators).

The group generated by the symbols T_a ($a \in \mathbb{R}$) and R subject to the relations (0.2)–(0.3) is known in geometric group theory as the *Euclidean group* $E(1)$, the group of isometries of the real line $\mathbb{R}^1$ viewed as a one-dimensional Euclidean space. Isometry roughly means that the real line is moved as a rigid object, without dilation. The transformation T_a on polynomials corresponds to the *translation* $\mathbb{R}^1$ by a. The transformation R on polynomials corresponds to the reflection of $\mathbb{R}^1$ about the origin.

0.5 Example (a new transformation D) Now consider a new linear transformation of $\mathbb{R}[x]$ defined by

$$D\colon \mathbb{R}[x] \to \mathbb{R}[x], \qquad Df \text{ is the derivative } f' \text{ of } f.$$

Surely D cannot be an element of a group, at least because D is not an invertible transformation: one has $D^{n+1}f = 0$ where n is the degree of the polynomial f. The following relations involving D hold:

$$T_a D = D T_a, \qquad RD = -DR. \tag{0.6}$$

Note the minus sign in the second relation — this is something we did not have in the group relations (0.2)–(0.3).

Is there an algebraic structure which is a natural "home" for T_a, R and D? Yes: one needs to consider the group $E(1)$ as a Lie group. The operator D comes from the Lie algebra of that Lie group.

0.7 Remark (a glimpse of Hopf algebras) Let us emphasise the important difference between the transformations T_a, R which come from the group structure of $E(1)$, and the transformation D which comes from the Lie algebra of $E(1)$. When these transformations are applied to a product of polynomials, one has

$$T_a(fg) = (T_a f)(T_a g), \qquad R(fg) = (Rf)(Rg), \tag{0.8}$$

i.e., T_a and R act as *automorphisms* of the ring $\mathbb{R}[x]$. However, D obeys the product rule

$$D(fg) = (Df)g + f(Dg). \tag{0.9}$$

Transformations which obey the product rule are called *derivations*.

It turns out that to correctly describe these transformations by means of an abstract algebraic structure, we need to record how they act on products. We would like to do that without writing the arguments f, g. In the course, we will learn how this is expressed via the *coproduct*, Δ, of each generator. Here is how equations (0.8)–(0.9) are written in terms of Δ:

$$\Delta T_a = T_a \otimes T_a, \qquad \Delta R = R \otimes R, \qquad \Delta D = D \otimes 1 + 1 \otimes D \tag{0.10}$$

where $\otimes$ is the tensor product symbol; the exact meaning of (0.10) will be explained in the course.

Informally, a Hopf algebra can be defined by specifying a set of generators and

- rules for multiplication of generators (example: relations (0.2)–(0.6));
- rules for coproduct, Δ (example: (0.10));
- rules for the antipode, S. The antipode of a transformation shows how the transformation acts on functions with their argument inverted; thus, in the above setup we will have

$$S(T_a) = RT_aR = T_{-a}, \qquad S(R) = RRR = R, \qquad S(D) = RDR = -D.$$

Note that in the theory of Hopf algebras, which we will develop in the course, elements which have the "automorphism" property, like T_a and R in (0.10), are called *grouplike*. We will axiomatise this property and insist that, whenever the Hopf algebra acts by linear transformations on some algebra A, a grouplike element acts as an automorphism of A.

Elements like D in (0.10), which have the "derivation" property, will be called *primitive* and required to act as derivations of A.

In general, the coproduct of an element of a Hopf algebra may be more complicated than the examples in (0.10).

0.11 "Definition" (of a quantum group) There is no universally accepted definition of a "quantum group". In many texts, quantum groups are synonymous with Hopf algebras.

The rationale for an abstract algebraic structure more general than a Lie group is that Hopf algebras can act on rings which, unlike the polynomial ring $\mathbb{R}[x]$, are not commutative. Such actions are not natural for groups, Lie algebras and Lie groups.

The need for formalisation of noncommutative structures arises, for example, from quantum physics. An algebraic manifestation of Heisenberg's uncertainty principle is that observables are no longer functions on the classical phase space, but are non-commuting entities.

Just as the above set of transformations of the polynomial ring $\mathbb{R}[x]$ arises from symmetries of the Euclidean space $\mathbb{R}^1$ which form a group $E(1)$, we will employ a figure of speech and say that "symmetries of a noncommutative space" form a "quantum group". In reality, we will be talking about algebraic transformations of a noncommutative ring which will form a Hopf algebra.

So, in this course a quantum group will mean a Hopf algebra viewed together with its action on some noncommutative algebra.

0.12 Example (preview of a quantum group) The algebra $A = \mathbb{R}[x, y]$ of polynomials in two variables is the algebra of (polynomial) functions on the plane $\mathbb{R}^2$. There is a Hopf algebra $U(\mathfrak{sl}_2)$ which is generated by all linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ with trace zero. Naturally, $U(\mathfrak{sl}_2)$ acts on the algebra A. The algebra A is, of course, commutative.

Define a new algebra A_q, generated by x, y subject to the relation $xy = qyx$, where q is a non-zero number. This new algebra is noncommutative (unless $q = 1$) and is called the "quantum plane". Of course, a noncommutative algebra cannot literally be the algebra of functions on anything.

An important example of a quantum group which we will study in this course is the Drinfeld-Jimbo quantum group $U_q(\mathfrak{sl}_2)$, also called "quantised $U(\mathfrak{sl}_2)$". This will be a Hopf algebra which will act by transformations on the noncommutative algebra A_q.

§ 1

Vector spaces. Tensor products

In this section, we provide some background necessary to introduce Hopf algebras. We assume the knowledge of undergraduate linear algebra but take care to avoid statements that fail without assuming that the vector space is finite-dimenional. Most of the material in this section can be found in Knapp 2016, Chapter II, Sections 1, 3, 4, 5, 6, 9. Since this section is preliminary, we do not include the proofs of most results, citing the literature instead.

We will generally use the field $\mathbb{C}$ of complex numbers as the ground field, though some examples will use reals, $\mathbb{R}$, or rational numbers, $\mathbb{Q}$.

Vector spaces, bases and their properties

The following two definitions are standard.

1.1 Definition **(vector space, linear map, isomorphism of vector spaces)** Let $\Bbbk$ be a field. A **vector space** over $\Bbbk$ is an abelian group $(V, +)$ equipped with an operation $(\lambda \in \Bbbk, v \in V) \mapsto \lambda v \in V$ called scalar multiplication which satisfies

$$(\lambda\mu)v = \lambda(\mu v), \qquad 1v = v, \qquad \lambda(u+v) = \lambda u + \lambda v, \qquad (\lambda+\mu)v = \lambda v + \mu v,$$

for all $\lambda, \mu \in \Bbbk$, $u, v \in V$.

Attention:

- In most, though not all, situations the ground field $\Bbbk$ can be assumed to be the field $\mathbb{C}$ of complex numbers. **We will routinely write $\mathbb{C}$ instead of $\Bbbk$ in lectures.**
- Some examples will use $\mathbb{R}$ or $\mathbb{Q}$ as the ground field.
- In this course, we do not work over fields of positive characteristic.

A **linear map** between vector spaces V, W is a function $L \colon V \to W$ such that $L(u + \lambda v) = Lu + \lambda(Lv)$ for all $\lambda \in \mathbb{C}$, $u, v \in V$.

An **isomorphism of vector spaces** is an invertible linear map.

1.2 Definition **(linearly independent set, span, linear combination, basis)** Let A be a subset of a vector space V. The set A is **linearly independent** if for all $n \geq 1$ and for all distinct elements $a_1, \ldots, a_n$ of A the implication

$$\lambda_1 a_1 + \cdots + \lambda_n a_n = 0 \text{ in } V \implies \lambda_1 = \cdots = \lambda_n = 0 \text{ in } \mathbb{C}$$

holds. The **span** of the set A is the subset

$$\operatorname{span}(A) = \{\lambda_1 a_1 + \cdots + \lambda_n a_n : n \geq 0,\ \lambda_1, \ldots, \lambda_n \in \mathbb{C},\ a_1, \ldots, a_n \in A\}$$

of V. A vector of the form $\lambda_1 a_1 + \dots + \lambda_n a_n$ is a **linear combination** of $a_1, \dots, a_n$, and is taken to be 0 if $n = 0$. Finally, a **basis** of V is a linearly independent subset $\mathcal{B}$ of V such that $\operatorname{span}(\mathcal{B}) = V$.

1.3 Remark (finiteness) It is important to remember that linear independence is only tested on *finite* subsets of A, and the span is formed by *finite* linear combinations. An infinite sum of elements of a vector space V is defined *only* if all but finitely many summands are zero.

If some kind of topology is introduced on V, one can calculate some genuinely infinite sums; we will not do it in this course.

The following result is true if one assumes the Axiom of Choice. We do assume the Axiom.

1.4 Theorem (existence and cardinality of bases) *Every vector space V has a basis. Every two bases of V have the same cardinality.*

Proof. Not proved here; for a proof based on Zorn's Lemma, see Knapp 2016, Theorem 2.42. ☐

1.5 Definition (dimension) The cardinality of a basis of V is called the **dimension** of V and is denoted $\dim V$. Thus, $\dim V$ is either a non-negative integer or an infinite cardinal.

1.6 Example (every set X is a basis of a vector space, $\mathbb{C}X$) Let X be an arbitrary set. We construct a vector space $\mathbb{C}X$, called the space of **formal linear combinations** of elements X, as follows. As a set, $\mathbb{C}X$ is formed by all formal expressions of the form $\alpha_1 x_1 + \dots + \alpha_n x_n$ where $n \geq 0$, $x_1, \dots, x_n$ are distinct elements of X and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. The expression where $n = 0$ is taken to be the zero vector in $\mathbb{C}X$. Addition in $\mathbb{C}X$ is by adding two expressions together and gathering like terms. Multiplication by a scalar is termwise. One can easily see that this defines on $\mathbb{C}X$ the structure of a vector space. Identify each element $x \in X$ with the expression $1x \in \mathbb{C}X$ so that X becomes a subset of $\mathbb{C}X$. One can check, using the definitions above, that X is a basis of $\mathbb{C}X$.

The following **key fact** shows why bases are useful: a linear map is uniquely defined by its values on a basis, and these values can be arbitrary. We will often construct linear maps using this approach.

1.7 Proposition (linear extension from basis) *Let V and W be vector spaces, and let $\mathcal{B}$ be a basis of V. Then to each **function** $\ell \colon \mathcal{B} \to W$ corresponds one and only one **linear map** $L \colon V \to W$ whose restriction to $\mathcal{B}$ coincides with ℓ; that is, $L|_{\mathcal{B}} = \ell$.*

Proof. Not proved here; see Knapp 2016, Proposition 2.13. ☐

Subspaces

1.8 Definition (subspace) A **subspace** of a vector space V is an additive subgroup $U \subseteq V$ closed under multiplication by scalars.

Note that a subspace of V is a vector space in its own right.

1.9 Example Examples of subspaces:

- If $A \subseteq V$ is a subset of V, then $\operatorname{span}(A)$ is a subspace of V (**subspace spanned by** A).
- If $L \colon V \to W$ is a linear map, then the **kernel** $\ker L = \{v \in V : Lv = 0\}$ of L is a subspace of V, and the **image** $L(V)$ of L is a subspace of W.

Spaces of linear maps

1.10 Definition (spaces of linear maps) Let V, W be vector spaces. Since the sum of two linear maps from V to W, and a linear map multiplied by a scalar, are again linear maps, the set

$$\mathrm{Lin}(V,W) = \{L\colon V \to W \mid L \text{ is a linear map}\}$$

is a vector space, called **the space of linear maps** from V to W. The vector space

$$\mathrm{End}(V) = \mathrm{Lin}(V,V)$$

is the **endomorphism space** of V.

1.11 Remark (bases of $\mathrm{Lin}(V,W)$ and $\mathrm{End}(V)$) If V is a vector space with a known **infinite** basis, there is no known way to construct a basis for $\mathrm{Lin}(V,W)$ and $\mathrm{End}(V)$. We know that these bases exist, but we do not appear to be able to describe them, even in the simplest case $W = \mathbb{C}$; see below.

The dual space

1.12 Definition (dual space, linear functional) The **dual space** V^* of a vector space V is defined as $V^* = \mathrm{Lin}(V,\mathbb{C})$. Elements of V^* are called **linear functionals** on V.

1.13 Notation If $v \in V$ and $\xi \in V^*$, the value of the linear functional ξ at v is traditionally written as $\xi(v)$, but can also be denoted by $\langle v, \xi\rangle$. Note that $\xi(v) \in \mathbb{C}$.

1.14 Definition If $\mathcal{B}$ is a basis of V, the **dual system** $\mathcal{B}' \subset V^*$ is the set $\{\delta_b : b \in \mathcal{B}\}$. The linear functional $\delta_b\colon V \to \mathbb{C}$ is defined by

$$\delta_b(c) = \begin{cases} 1, & c = b, \\ 0, & c \neq b \end{cases} \qquad \text{for } c \in \mathcal{B},$$

extending linearly from $\mathcal{B}$ to V. (Recall: linear extensions from a basis exist and are unique by Proposition 1.7.)

1.15 Proposition (the dual system and the dual basis) *For a basis $\mathcal{B}$ of V, the dual system $\mathcal{B}'$ is linearly independent in V^*. If $\dim V < \infty$ then $\mathcal{B}'$ is a basis of V^* (the **dual basis** of V^* relative to $\mathcal{B}$).*

Proof. If $\lambda_1\delta_{b_1} + \dots + \lambda_n\delta_{b_n} = 0$ in V^* where $b_1, \dots, b_n$ are distinct elements of the basis $\mathcal{B}$, evaluate the left-hand side at b_k to obtain $\lambda_k \cdot 1 = 0$. Since k can be any of $1, \dots, n$, we have $\lambda_1 = \dots = \lambda_n = 0$, proving linear independence of the set $\mathcal{B}' = \{\delta_b : b \in \mathcal{B}\}$.

If V is finite-dimensional, $\dim V = n$, write $\mathcal{B} = \{b_1, \dots, b_n\}$ and let $\xi \in V^*$ be an arbitrary linear functional. Consider $\xi' = \xi(b_1)\delta_{b_1} + \dots + \xi(b_n)\delta_{b_n}$. Substitution shows that $\xi'(b_k) = \xi(b_k)$ for all k, so ξ' coincides with ξ on the basis $\mathcal{B}$ and so $\xi = \xi'$ by Proposition 1.7. Since ξ' is manifestly in $\mathrm{span}(\mathcal{B}')$ and ξ was arbitrary, this proves that $\mathrm{span}(\mathcal{B}') = V^*$. □

1.16 Definition (contragredient linear map) Let $L\colon V \to W$ be a linear map. The **contragredient** (or **adjoint**) of L is the map $L^*\colon W^* \to V^*$ defined by

$$\psi \in W^* \mapsto L^*\psi \in V^*, \qquad \langle v, L^*\psi\rangle = \langle Lv, \psi\rangle.$$

1.17 Remark Note that L^* goes in “reverse direction” relative to L. It is easy to see that the expression $\langle Lv, \psi\rangle$ is linear in v, which means that $L^*\psi$ is a wel-defined linear functional on V; and that $\langle Lv, \psi\rangle$ is linear in ψ, which means that L^* is a linear map.

Warning: if V is infinite-dimensional, the space V^* will have dimension which is strictly larger than $\dim V$. This means that $(V^*)^*$ is not isomorphic to V, and $(L^*)^*$ is not the same as L. However, in the finite-dimensional case $(V^*)^*$ can be naturally identified with V.

The quotient space

1.18 Definition (coset, quotient space, quotient map) Let U be a subspace of V. A **coset** of U is a set of the form $v + U = \{v + u : u \in U\}$ for a fixed vector $v \in V$. The space U defines an equivalence relation on V by $v_1 \sim v_2$ iff $v_1 + U = v_2 + U$, same as $v_1 - v_2 \in U$.

The set V/U of all cosets of U has well-defined operations

$$(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U, \qquad \lambda(v + U) = \lambda v + U$$

for $v_1, v_2, v \in V$, $\lambda \in \mathbb{C}$. These operations make V/U into a vector space. This vector space is the **quotient space** of V by U. The linear map

$$q\colon V \twoheadrightarrow V/U, \qquad q(v) = v + U$$

is the **quotient map** of V onto V/U. (*The quotient map is surjective; we indicate surjective maps by double-headed arrows,* $\twoheadrightarrow$.)

1.19 Remark (the kernel of the quotient map) Note that the kernel of the quotient map $q\colon V \twoheadrightarrow V/U$ is the subspace U. This is because the trivial coset $0 + U$ plays the role of the zero vector in the quotient space V/U, and $q(v) = v + U$ equals $0 + U$ if and only if $v - 0 = v \in U$.

1.20 Proposition (the universal mapping property of the quotient space) *Let U be a subspace of V and let $q\colon V \twoheadrightarrow V/U$ be the quotient map. To each linear map $L\colon V \to W$ such that $U \subseteq \ker L$, there corresponds one and only one linear map $\overline{L}\colon V/U \to W$ such that $L = \overline{L}q$.*

Proof. Not given here; see Knapp 2016, Proposition 2.25. $\square$

1.21 Remark A concise statement equivalent to Proposition 1.20 is: a linear map L *factors* through the quotient by the kernel of L. (To "factor" here means to be written as a composition of maps.)

The universal mapping property of the quotient space is extremely important, and we will use its versions for algebras and Hopf algebras (which are vector spaces wth extra structure).

1.22 Remark It is convenient to illustrate the universal mapping property of the quotient space by a commutative diagram. In such diagrams, linear maps are shown as arrows between nodes representing vector spaces:

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "q"] & W \\ V/\ker L \arrow[ur, dashed, "\exists!"', "\overline{L}"] & \end{tikzcd}$$

The universal mapping property says that, for any arrow L, one has a unique arrow $\overline{L}$ such that *the diagram commutes.*

"Commutes" means that if two nodes of the diagram are connected by more than one path, the composition of arrows along each of the paths is the same. In the above diagram, this exactly means that q followed by $\overline{L}$ is the same as L, i.e., $\overline{L}q = L$.

Tensor product of vector spaces

Tensor product of vector spaces is thought to be a difficult concept in algebra, but understanding it is crucial for the course; it is impossible to progress towards Hopf algebras without it. Tensors and tensor products arise in a number of areas of pure and applied mathematics, as well as in physics.

Informally, the idea behind the tensor product is as follows. As mentioned in the Introduction, we need to "linearise" everything; maps between algebraic structures need to be linear maps. However, an important class of maps are not linear. For example, multiplication defined on polynomials takes two arguments f, g and outputs fg; viewed as a map from the direct product $\mathbb{R}[x] \times \mathbb{R}[x]$ to the vector space $\mathbb{R}[x]$, $(f, g) \mapsto fg$ is not a linear map.

The situation is rectified by considering the tensor product $\mathbb{R}[x] \otimes \mathbb{R}[x]$ instead of the direct product. Since the multiplication map is *bilinear*, we can use the *universal mapping property* of the tensor product (see below) to convert it to a *linear* map $m\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x]$.

We will now formalise this idea.

Tensor product: definition and construction

We define the tensor product as a space which has a certain universal mapping property.

1.23 Definition Let E, F, U be vector spaces. A map $b\colon E \times F \to U$ is **bilinear** if

$$\begin{aligned} b(e_1 + e_2, f) &= b(e_1, f) + b(e_2, f) \\ b(e, f_1 + f_2) &= b(e, f_1) + b(e, f_2) \\ b(\lambda e, f) &= b(e, \lambda f) = \lambda b(e, f) \end{aligned}$$

for all $e, e_1, e_2 \in E$, $f, f_1, f_2 \in F$, $\lambda \in \mathbb{C}$.

1.24 Remark (linear in each argument) The axioms of a bilinear map can be equivalently written as follows:

$$\begin{aligned} &\forall f \in F, \qquad b(\cdot, f)\colon E \to U \text{ is a linear map;} \\ &\forall e \in E, \qquad b(e, \cdot)\colon F \to U \text{ is a linear map.} \end{aligned}$$

Here $b(\cdot, f)$ is shorthand notation for the map $e \in E \mapsto b(e, f) \in U$ for fixed f; that is, the dot, $\cdot$, is a placeholder for the argument of the map. One says that *the map $b(\cdot, \cdot)$ is bilinear iff it is linear in each argument.*

1.25 Example (examples of bilinear maps) 1. For each vector space V over $\mathbb{C}$, there is the bilinear map $\mathrm{ev}_V\colon V^* \times V \to \mathbb{C}$, $(\xi, v) \mapsto \xi(v)$, called the evaluation map.

2. Let $M_{p\times q}(\mathbb{C})$ denote the vector space of all matrices with complex entries, p rows and q columns. The matrix multiplication map $M_{p\times q} \times M_{q\times r} \to M_{p\times r}$, $(A, B) \mapsto AB$, is bilinear.

3. (*non-example*) The map $\mathbb{R}[x] \times \mathbb{R}$, $(f, a) \mapsto f(a)$ is *not* bilinear: the expression $f(a)$ is not linear in a unless $f = 0$ or f is a polynomial of degree 1 with zero constant term.

1.26 Definition (tensor product of two vector spaces) A **tensor product** of vector spaces E, F is a vector space V equipped with a bilinear map $i\colon E \times F \to V$ which has the following universal mapping property: whenever $b\colon E \times F \to U$ is a bilinear map, there exists a unique *linear* map $B\colon V \to U$ such that $b = Bi$, equivalently the following diagram commutes:

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[d, "i"'] & U \\ V \arrow[ur, dashed, "\exists!", "B"'] & \end{tikzcd}$$

1.27 Theorem *For any pair E, F of vector spaces, a tensor product $E \times F \xrightarrow{i} V$ of E and F exists. It is unique up to a canonical isomorphism in the following sense: if $E \times F \xrightarrow{i'} V'$ is another tensor product, the linear map $I\colon V' \to V$ such that $Ii' = i$ is an isomorphism.*

Proof. Omitted here: see Knapp 2016, Theorem 6.10. ☐

1.28 Notation Due to uniqueness up to a canonical isomorphism, one speaks of *the* tensor product of the vector spaces E, F and writes $E \otimes F$. The tensor product comes with the bilinear map from $E \times F$ to $E \otimes F$, denoted instead of i by

$$\otimes\colon E \times F \to E \otimes F, \qquad (e, f) \mapsto e \otimes f.$$

The universal mapping property of the tensor product says that *whenever $b\colon E \times F \to W$ is bilinear, b can be uniquely written as a linear map*

$$E \otimes F \xrightarrow{b} W, \qquad e \otimes f \mapsto b(e, f)$$

(the linear map on $E \otimes F$ is typically also denoted b, by convenient abuse of notation).

1.29 Remark (pure tensors) One needs to be aware of what the formula $e \otimes f \mapsto b(e, f)$ is meant to express. Elements of $E \otimes F$ of the form $e \otimes f$, called **pure tensors**, do not exhaust all of $E \otimes F$, so a linear map needs to be extended from pure tensors to the whole of $E \otimes F$ by linearity. Every element of $E \otimes F$ is a sum of finitely many pure tensors, by Proposition 1.30 below.

Properties of tensor products

The next Proposition assumes that E and F have countable bases, although this is for notational convenience only; the result is true without this assumption.

1.30 Proposition (basis of the tensor product) *Let $\{e_i\}_{i\geq 1}$ be a basis of the vector space E and $\{f_j\}_{j\geq 1}$ be a basis of the vector space F. Then*

$$\{e_i \otimes f_j\}_{i\geq 1, j\geq 1}$$

is a basis of $E \otimes F$.

Idea of proof (omitted from the lectures). An *ad hoc* proof of this is as follows. Assume for contradiction that $\{e_i \otimes f_j\}_{i\geq 1, j\geq 1}$ is linearly dependent; up to rearranging and renumbering, this means that

$$e_1 \otimes f_1 = \sum_{(i,j)\neq(1,1)} \lambda_{ij} e_i \otimes f_j \qquad \text{(the sum must be finite).} \tag{1.31}$$

Let $\{\epsilon_i\}_{i\geq 1}$ be the dual system of $\{e_i\}$ in E^*, so that $\epsilon_i(e_k) = \delta_{ik}$, see Proposition 1.15. Likewise, let $\{\phi_j\}_{j\geq 1}$ be the dual system of $\{f_j\}$ in F^*. The map $b\colon E \times F \to \mathbb{C}$ defined by $b(e, f) = \epsilon_1(e)\phi_1(f)$ is easily seen to be bilinear and so defines a linear map $B\colon E \otimes F \to \mathbb{C}$. One has $B(e_1 \otimes f_1) = b(e_1, f_1) = \epsilon_1(e_1)\phi_1(f_1) = 1 \cdot 1 = 1$ and $B(e_i \otimes f_j) = 0$ if $(i, j) \neq (1, 1)$. Applying B to both sides of (1.31) gives $1 = 0$, a contradiction.

Assume for contradiction that the linearly independent set $\{e_i \otimes f_j\}_{i\geq 1, j\geq 1}$ does not span $E \otimes F$, then by results from linear algebra there exists a basis of $E \otimes F$ of the form $\{e_i \otimes f_j\}_{i\geq 1, j\geq 1} \sqcup \mathcal{S}$ where $\mathcal{S}$ is a non-empty set. If $b\colon E \times F \to U$ is a bilinear map and $B\colon E \otimes F \to U$ is such that the composition $E \times F \to E \otimes F \xrightarrow{B} U$ is b, then B can be redefined on $\mathcal{S}$ without changing it on $\{e_i \otimes f_j\}_{i\geq 1, j\geq 1}$, and the composition $E \times F \to E \otimes F \xrightarrow{B} U$ will not change. This contradicts the uniqueness of B which is part of Definition 1.26 of the tensor product. Thus $\{e_i \otimes f_j\}_{i\geq 1, j\geq 1}$ is a spanning set, hence a basis.

For a different and more conceptual approach, see Knapp 2016, Proposition 6.14. $\square$

1.32 Corollary (dimension of the tensor product)

$$\dim E \otimes F = (\dim E)(\dim F). \quad \square$$

Recall that the ground field, $\mathbb{C}$, is a vector space of dimension 1 over itself. A convenient choice of basis of $\mathbb{C}$ is $\{1\}$.

1.33 Corollary (tensor product with the ground field) *For any vector space E, there is a canonical isomorphism $E \otimes \mathbb{C} \cong E$ given by $e \otimes 1 \mapsto e$ for all $e \in E$. Similarly there is a canonical isomorphism $\mathbb{C} \otimes E \cong E$.* $\square$

1.34 Example (tensor product of two polynomial algebras) We denote by $\mathbb{C}[x]$ the vector space with basis $x^0 = 1, x^1, x^2, \dots$, similarly for $\mathbb{C}[y]$. Also, $\mathbb{C}[x, y]$ is the vector space with basis $\{x^i y^j : i, j \geq 0\}$.

We claim that the tensor product of $\mathbb{C}[x]$ and $\mathbb{C}[y]$ is isomorphic to the space $\mathbb{C}[x, y]$ via the following map:

$$\begin{aligned} \mathbb{C}[x] \otimes \mathbb{C}[y] &\to \mathbb{C}[x, y] \\ f(x) \otimes g(y) &\mapsto f(x)g(y). \end{aligned}$$

Indeed, the expression $f(x)g(y)$ is bilinear in $f(x)$ and $g(y)$ hence gives a well-defined linear map from $\mathbb{C}[x]\otimes\mathbb{C}[y]$ to $\mathbb{C}[x, y]$. The pure tensors $x^i \otimes y^j$, $i, j \geq 0$, which by Proposition 1.30 form a basis of $\mathbb{C}[x] \otimes \mathbb{C}[y]$, are carried by this map to the $x^i y^j$ which form a basis of $\mathbb{C}[x, y]$ by definition of $\mathbb{C}[x, y]$. A linear map which carries a basis to a basis is an isomorphism. Thus $\mathbb{C}[x] \otimes \mathbb{C}[y] \cong \mathbb{C}[x, y]$.

1.35 Definition (tensor product of linear maps) Let $E \xrightarrow{L} V$ and $F \xrightarrow{M} W$ be linear maps. The linear map

$$L \otimes M\colon E \otimes F \to V \otimes W$$

is defined by

$$(L \otimes M)(e \otimes f) = L(e) \otimes M(f)$$

for all $e \in E$, $f \in F$.

1.36 Remark In light of Remark 1.29, one should remember that not all elements of $E \otimes F$ are simple tensors of the form $e \otimes f$. What we mean here is that the expression $L(e) \otimes M(f)$ is bilinear in e, f. By definition of the tensor product, this guarantees that $L \otimes M$ is well-defined on pure tensors and uniquely extends to a linear map on $E \otimes F$.

1.37 Proposition (associativity of tensor products) *For vector spaces E, F, G there is a canonical isomorphism*

$$(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$$

given by $(e \otimes f) \otimes g \mapsto e \otimes (f \otimes g)$ for all $e \in E$, $f \in F$, $g \in G$.

Sketch of proof. One checks that the map $(e \otimes f) \otimes g \mapsto e \otimes (f \otimes g)$ is linear in $e \otimes f$ (which means that it is linear in e and linear in f) and linear in g, hence well-defined. If $\{e_i\}$, $\{f_j\}$, $\{g_k\}$ are respective bases of E, F, G, by Proposition 1.30 $\{(e_i \otimes f_j) \otimes g_k\}_{i,j,k}$ is a basis of $(E \otimes F) \otimes G$, and it is carried by the map to $\{e_i \otimes (f_j \otimes g_k)\}_{i,j,k}$ which is a basis of $E \otimes (F \otimes G)$. A linear map which carries a basis to a basis is an isomorpism. $\square$

1.38 Remark (triple and n-fold tensor products) The above result allows us to consider *triple tensor product*

$$E \otimes F \otimes G$$

which can be identified with $(E \otimes F) \otimes G$ and with $E \otimes (F \otimes G)$. One can give an abstract definition of the triple tensor product via a universal mapping property for *trilinear maps* $E \times F \times G \to V$. This point of view is adopted in Knapp 2016, Chapter VI, Section 7.

Generalising, one defines an n-fold tensor product

$$E_1 \otimes E_2 \otimes \ldots \otimes E_n.$$

This space is canonically isomorphic to

$$\underbrace{((\ldots(}_{n-1} E_1 \otimes E_2) \otimes \ldots \otimes E_{n-1}) \otimes E_n$$

and to any tensor products obtained by rebracketing.

References

Knapp, Anthony W. (2016). *Basic Algebra: Digital Second Edition.* East Setauket, NY. ISBN: 9781429799980. URL: https://projecteuclid.org/euclid.bia/1531879291.