Week 01 review worksheet — exercises for §1

Part A. Exercises for interactive discussion

Attempt the part A exercises and be prepared to discuss them in the interactive session.

E1.1 Let V be a vector space over $\mathbb{C}$. Which vector spaces from the following list must be isomorphic to V?

(A) V^* **(B)** $\mathrm{Lin}(\mathbb{C}, V)$ **(C)** V/V **(D)** $V/\{0\}$ **(E)** $V \otimes \mathbb{C}$ **(F)** $\mathrm{Lin}(V^*, \mathbb{C})$

Let $L\colon V \to W$ be a linear map. Which formula describes the correct way to apply L^* to $\psi \in W^*$?

(G) $L^*(\psi) = L \circ \psi$ **(H)** $L^*(\psi) = \psi \circ L$

E1.2 (multiplicative characters are linearly independent) Let $\mathbb{R}[x]$ be the vector space of all polynomials in one variable x over the field $\mathbb{R}$ of real numbers. Given a point a of the real line, define the functional $e_a\colon \mathbb{R}[x] \to \mathbb{R}$ by $e_a(f(x)) = f(a)$.

(a) Show that the subset $\{e_a\}_{a\in\mathbb{R}}$ of $\mathbb{R}[x]^*$ is linearly independent. (*Hence the space* $\mathbb{R}[x]^*$ *is of uncountably infinite dimension. Easy to conclude: no* $\mathbb{R}$*-vector space* V *has* $\dim V^* = \aleph_0$*.*)

(b) Give an example of a linear functional $\xi \in \mathbb{R}[x]^*$ which does not belong to $\mathrm{span}\{e_a\}_{a\in\mathbb{R}}$. (*Hence* $\{e_a\}_{a\in\mathbb{R}}$ *is not a spanning set for* $\mathbb{R}[x]^*$*.*)

(c) More generally, let M be a set with multiplication $(x, y) \mapsto xy$ (not necessary associative) and L be a field. A function $\sigma\colon M \to L$, not identically 0, is called a **multiplicative character** if $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in M$. Prove that multiplicative characters are linearly independent over L; that is, if $\lambda_1, \dots, \lambda_n \in L$ and distinct $\sigma_1, \dots, \sigma_n$ are such that $\lambda_1\sigma_1(x) + \dots + \lambda_n\sigma_n(x) = 0$ for all $x \in M$, then $\lambda_1 = \dots = \lambda_n = 0$.

E1.3 (dual space of tensor product) Let V, W be vector spaces. The goal of this exercise is to show that $V^* \otimes W^*$ is always a subspace of, but may not be equal to, $(V \otimes W)^*$.

To view any element of $V^* \otimes W^*$ as a linear function on $V \otimes W$, first observe that a pure tensor $\phi \otimes \psi$, where $\phi \in V^*$ and $\psi \in W^*$, can be evaluated on $v \otimes w \in V \otimes W$ according to the formula

$$(\phi \otimes \psi)(v \otimes w) = \phi(v)\psi(w).$$

This formula is bilinear in ϕ, ψ and so extends to the whole of $V^* \otimes W^*$. We have embedded $V^* \otimes W^*$ inside $(V \otimes W)^*$ (it is not difficult to show that this embedding is injective).

- Let $V = \mathbb{R}[x]$, $W = \mathbb{R}[y]$. Exhibit an element of $(V \otimes W)^* = \mathbb{R}[x, y]^*$ which is not in $V^* \otimes W^*$. *Thus,* $V^* \otimes W^*$ *is a proper subspace of* $(V \otimes W)^*$ *if V, W are infinite dimensional.*

Part B. Extra exercises

Attempt these exercises and compare your answers with the model solutions, published after the session.

E1.4 (the contragredient of a composition) Let $U \xrightarrow{L} V \xrightarrow{M} W$ be linear maps and let $W^* \xrightarrow{M^*} V^* \xrightarrow{L^*} U^*$ be the corresponding contragredient maps. Prove that $(ML)^* = L^*M^*$ (both sides are linear maps $W^* \to U^*$).

E1.5 (duality exchanges subspaces and quotients) Let U be a subspace of V. Show that the dual space U^* is canonically isomorphic to the quotient space $V^*/U^\perp$ where $U^\perp$ is defined as $\{\xi \in V^* : \xi(U) = \{0\}\}$.

(*Hint*: you can use the "first isomorphism theorem for vector spaces", $L(V) \cong V/\ker L$ for any linear map $L\colon V \to W$; you should know how to deduce this "theorem" from the material in §1.)

E1.6 (bilinear maps) Review the definition of the *tensor product* $E \otimes F$ of vector spaces E and F. Let $E = F = \mathbb{R}[x]$, a vector space over $\mathbb{R}$; in each of the following, determine whether the given formula is a well-defined bilinear map on $\mathbb{R}[x] \times \mathbb{R}[x]$ hence a linear map on $\mathbb{R}[x] \otimes \mathbb{R}[x]$:

$A\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x] \otimes \mathbb{R}[x], \quad A(f \otimes g) = g \otimes f.$
$B\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x], \quad B(f \otimes g) = f.$
$C\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x], \quad C(f \otimes g) = f + g.$
$D\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x], \quad D(f \otimes g) = fg.$
$E\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x] \otimes \mathbb{R}[x] \otimes \mathbb{R}[x], \quad E(f \otimes g) = f \otimes 1 \otimes g.$
$F\colon \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x] \otimes \mathbb{R}[x] \otimes \mathbb{R}[x], \quad F(f \otimes g) = f \otimes f \otimes g.$