Suggested exercises Section 1 Solutions

Exercise 1.2. Let R be the ring of continuous real-valued functions on $[0, 1]$. Show that $\{f \in R \mid f(0) = 0\}$ is a maximal ideal of R.

Solution. By definition, $R = \{f : [0, 1] \to \mathbb{R} \mid f \text{ continuous}\}$. Let $I = \{f \in R \mid f(0) = 0\}$. We can assume (or else prove is as a health check) that R is a commutative ring for pointwise addition and multiplication of functions. The zero constant function and the difference of two functions in I is an element of I. So I is an additive subgroup of R. Let $f \in I$ and let $g \in R$. By definition, $(gf)(0) = g(0)f(0) = 0$. So I is an ideal.

Let $\varphi : R \to \mathbb{R}$ be the evaluation at zero, i.e. $\varphi(f) = f(0)$. So φ is a ring homomorphism. For all $a \in \mathbb{R}$, the constant function $f_a : [0, 1] \to \mathbb{R}$, $f_a(x) = a$ for all $x \in [0, 1]$ is an element of R. So φ is surjective. We have $\ker(\varphi) = I$, and therefore $R/I \cong \mathbb{R}$, which proves that I is maximal.

Exercise 1.4. Let R be a ring, and let $e \in R$. Assume that e is a nontrivial idempotent.

i. Prove that e is a zero divisor, and that $1 - 2e \in R^\times$.

ii. Prove that eR is a ring with multiplicative identity e. Hence, find a ring S and a ring isomorphism $R \cong eR \times S$.

iii. Let S be a ring. Describe $\mathrm{Nil}(R \times S)$. Deduce that if R and S are reduced rings, then $R \times S$ is reduced too.

iv. Prove that an ID is reduced, but there are reduced rings which are not IDs, and find an example of such a reduced commutative ring.

Solution.

i. By assumption, $e, 1 - e \neq 0$, and we have $e(1 - e) = 0$. So e is a zero divisor. For the second part, we calculate $(1 - 2e)^2 = 1$.

ii. For any $ea \in eR$, we have $e(ea) = ea = eae$, which shows that $e = 1_{eR}$. Similarly, let $S = (1-e)R$. Then S is a commutative ring with multiplicative identity $1 - e$. Therefore, $(e, 1 - e) = 1_{eR\times S}$. Define $\varphi : R \to eR \times S$ by $\varphi(a) = (ea, (1 - e)a)$. It is routine to check that φ is a ring homomorphism. Define $\psi : eR \times S \to R$ by $\psi(ea, (1 - e)b) = ea + (1 - e)b$. It is routine to check, using that $e(1 - e) = 0$ and that $e, 1 - e$ are idempotents, that φ is a ring homomorphism. We then check that $\varphi\psi$ and $\psi\varphi$ are the identity maps.

iii. $\mathrm{Nil}(R \times S) = \{(a, b) \in R \times S \mid \exists\, n \in \mathbb{N} \text{ with } (a, b)^n = 0\}$. Since $(a, b)^n = (a^n, b^n)$, we have $(a, b)^n = 0$ if and only if a, b are both nilpotent (we can pick n large enough for such an equality to hold). So, $\mathrm{Nil}(R \times S) = \mathrm{Nil}(R) \times \mathrm{Nil}(S)$. Hence, if R and S are reduced (i.e. they have no nonzero nilpotent elements), then $R \times S$ is reduced too.

iv. An ID is reduced since it has no zero divisors. There are reduced rings which are not IDs, for instance the cartesian product $\mathbb{Z} \times \mathbb{Z}$.

Exercise 1.5. Find the maximal ideals in the local rings $\mathbb{Z}_{(p)}$, $k[x]/(x^2)$ and $k[[x]]$, where p is a prime and k is a field.

Solution. $\mathbb{Z}_{(p)}$ has maximal ideal $p\mathbb{Z}_{(p)}$, $k[x]/(x^2)$ has maximal ideal $xk[x]/(x^2)$ and $k[[x]]$ has maximal ideal $xk[[x]]$. The quotients are $\mathbb{F}_p$, k and k, respectively.

Exercise 1.11. Let R be a commutative ring and let U be a multiplicative subset of R. Let I be an ideal of R such that I is maximal in the poset

$$\{J \text{ ideal of } R \mid J \cap U = \emptyset\},$$

where the order relation is given by the inclusion of ideals. Prove that I is prime.

Solution. Let $a, b \in R \setminus I$. We need to show that $ab \notin I$. Let $X = \{J \text{ ideal of } R \mid J \cap U = \emptyset\}$. Since $(a)+I, (b)+I$ contain I properly, the assumption implies that $(a)+I \cap X \neq \emptyset$ and $(b)+I \cap X \neq \emptyset$. That is, there exist $c, d \in I,\ r, s \in R$ and $u, v \in U$ such that $u = ra + c$ and $v = sb + d$, which gives (using commutativity of R)

$$uv = rsab + rda + scb + cd \in U \quad \text{since } U \text{ is a multiplicative subset or } R.$$

Since $c, d \in I$, we have $rda + scb + cd \in I$. Since $uv \notin I$, we cannot have $rsab \in I$. Hence $ab \notin I$ as we wanted to show.

Exercise 1.12. Let R be a ring and let I be an ideal of R. Prove that the following statements are equivalent:

i. I is prime.

ii. If J, K are ideals of R such that $JK \subseteq I$, then at least one of J or K must be contained in I.

iii. There do not exist ideals J, K of R with $J \not\subseteq I$ and $K \not\subseteq I$, and such that $JK \subseteq I$.

Solution. $\boxed{\text{i}\Rightarrow\text{ii}}$ Suppose i, and let J, K be ideals of R such that the product ideal JK is contained in I. Suppose that $J \not\subseteq I$. We prove that $K \subseteq I$. Pick $a \in J$ such that $a \notin I$. Now, for all $b \in K$, we have $ab \in JK \subseteq I$, and since I is prime, we obtain that $b \in I$ for all $b \in K$. Therefore ii holds.

$\boxed{\text{ii}\Rightarrow\text{iii}}$ Suppose ii. By contrapositive, whenever J, K are ideals of R which are not contained in I, then JK is not contained in I. Thus iii holds.

$\boxed{\text{iii}\Rightarrow\text{i}}$ Finally, suppose that iii holds, and let $a, b \in R$ such that $ab \in I$. Suppose that $a \notin I$. We need to show that $b \in I$. Consider the ideals

$$J = aR + I = \{ax + u \ :\ x \in R,\ u \in I\} \quad \text{and} \quad K = bR + I = \{by + v \ :\ y \in R,\ v \in I\}.$$

Our assumption implies that $I \subsetneq J$. Now, using the commutativity of R,

$$JK = \{ \sum_{\text{finite sum}} (ax+u)(by+v) = \sum_{\text{finite sum}} abxy + \underbrace{axv + uby + uv}_{\in I} \ :\ x, y \in R,\ u, v \in I\},$$

where the finite sum is on finitely many elements $x, y \in R,\ u, v \in I$.

Since $ab \in I$ by assumption, we conclude that $JK \subseteq I$. By hypothesis iii, and since we assume that $J \not\subseteq I$, we must have $K \subseteq I$. That is $bR + I \subseteq I$, which shows that $b \in I$, and i holds.

Exercise 1.13. Describe $\mathrm{Spec}(\mathbb{Z}[x])$ and $\mathrm{MaxSpec}(\mathbb{Z}[x])$. Same question with $\mathbb{R}$ and with $\mathbb{C}$ instead of $\mathbb{Z}$. (Hint: Theorem 1.13 may be useful.)

Solution. An ideal I of $R[x]$ is prime if and only if $R[x]/I$ is an ID and maximal if and only if the quotient ring is a field. Since $R[x]$ is a PID if and only if R is a field, the question is much easire to answer for $R = \mathbb{R}$ and $\mathbb{C}$. In both cases, $\mathrm{Spec}(R[x]) = \mathrm{MaxSpec}(R[x]) \sqcup \{0\}$, where $\mathrm{MaxSpec}(R[x]) = \{fR[x] \mid \ f \in R[x] \text{ is irreducible}\}$. (Recall that in a PID, prime and irreducible elements coincide.) For $\mathbb{Z}[x]$, the principal prime ideals have the form $p\mathbb{Z}[x]$ or $f\mathbb{Z}[x]$, where $p \in \mathbb{Z}$ is prime or zero (regarded as constant polynomials) and $f \in \mathbb{Z}[x]$ irreducible. The quotient rings are isomorphic to $\mathbb{F}_p[x]$ and $\mathbb{Z}$, respectively. Hence, we obtain maximal ideals of the form $p\mathbb{Z}[x] + f\mathbb{Z}[x]$, for $p \in \mathbb{Z}$ prime, regarded as constant polynomial, and $f \in \mathbb{Z}[x]$ irreducible.

We claim that these are the only prime and the only maximal ideals of $\mathbb{Z}[x]$. Let $P \in \mathrm{Spec}(\mathbb{Z}[x])$. The inclusion $\mathbb{Z} \to \mathbb{Z}[x]$ implies that the contraction P^c is a prime ideal of $\mathbb{Z}$, ie. $P^c = (p)$ with $p = 0$ or p prime.

Suppose that $P^c = (0)$. Consider $U = \mathbb{Z}\backslash\{0\}$ as a multiplicative subset of $\mathbb{Z}[x]$. Then, the localisation $\mathbb{Z}[x]_U \cong \mathbb{Q}[x]$, and the canonical map $\theta : \mathbb{Z}[x] \hookrightarrow \mathbb{Q}[x]$ maps $\theta(P) \in \mathrm{Spec}(\mathbb{Q}[x])$, by Theorem 1.37. Since $\mathbb{Q}[x]$ is a PID, there is $f \in \mathbb{Q}[x]$ irreducible such that $\theta(P) = f\mathbb{Q}[x]$. Say $f = \frac{g}{d}$, with $g \in \mathbb{Z}[x]$ and $d \in \mathbb{Z} \setminus \{0\}$. But by definition of $\theta(P)$, we can choose such g in P. Hence $df \in P$. Since P is prime and $d \notin P$, we must have $f \in P$. Therefore $f\mathbb{Z}[x] \subseteq P$. Conversely, if $h \in P$, wlog h primitive, then $\theta(h) = \frac{fg}{d} \in f\mathbb{Q}[x]$, for some $g \in \mathbb{Z}[x]$ and $d \in \mathbb{Z} \setminus \{0\}$ such that d does not divide the content $c(g)$ of g. Since $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$, it follows from Gauss's lemma that the factorisation $h = \frac{fg}{d} \in \mathbb{Q}[x]$ can be realised in $\mathbb{Z}[x]$. Thus $\frac{c(g)}{d} f \in \mathbb{Z}[x]$. Writin f in reduced form, i.e. such that if α is the least common denominator of the coefficients of f, then $\alpha f \in \mathbb{Z}[x]$ is primitive, we see that $h \in f\mathbb{Z}[x]$.

Suppose that $P^c = (p)$ with p prime. The reduction mod p of the coefficients induces a surjective ring homomorphism $\pi : \mathbb{Z}[x] \to \mathbb{F}_p[x]$. Hence $\pi(P) \in \mathrm{Spec}(\mathbb{F}_p[x])$ for all $P \in \mathrm{Spec}(\mathbb{Z}[x])$. Suppose taht $\pi(P) \neq (0)$. Since $\mathbb{F}_p[x]$ is a PID, $\pi(P) = f\mathbb{F}_p[x]$, with $f \in \mathbb{F}_p[x]$ irreducible. Thus $P = (p) + (\dot{f})$ for some $\dot{f} \in \pi^{-1}(f)$ irreducible.

Exercise 1.14. Let $f : R \to S$ be a ring homomorphism.

i. Prove that the preimage $f^{-1}(J)$ is an ideal of R for every ideal J of S. If the image $f(I)$ an ideal of S for an ideal I of R?

ii. If I is a prime ideal of S, is $f^{-1}(I)$ a prime ideal of R? Same question for maximal ideals.

Solution.

i. The first part is a routine checking of the axioms. The answer to the question is NO: for instance, if $f : \mathbb{Z} \to \mathbb{Q}$ is the inclusion, then the image of any nonzero ideal of $\mathbb{Z}$ is not an ideal of $\mathbb{Q}$.

ii. If $I \in \mathrm{Spec}(S)$, then $f^{-1}(I) \in \mathrm{Spec}(R)$, but this is not true for maximal ideals. Indeed, if $a, b \in R$ such that $ab \in f^{-1}(I)$, then $f(a)f(b) \in I$, and since I is prime conclusion follows.

   This is no longer true for maximal ideals. The same example as above proves the claim.

Exercise 1.15. Let R be a commutative ring, let I be an ideal in R and let X be a nonempty subset of R. Define

$$(I \ : \ X) = \{a \in R \ : \ aX \subseteq I\}, \quad \text{where} \quad aX = \{ax \ : \ x \in X\}.$$

i. Prove that $(I \ : \ X)$ is an ideal of R.

ii. Let J be the ideal in R generated by the subset X of R. Prove that $(I \ : \ X) = (I \ : \ J)$, for any ideal I.

iii. Let I, J be two ideals in R. Prove the following.

(a) $I \subseteq (I : J)$.

(b) $J(I : J) \subseteq I$.

(c) if $I = I_1 \cap I_2$, then $(I : J) = (I_1 : J) \cap (I_2 : J)$.

(d) if $J = J_1 + J_2$, then $(I : J) = (I : J_1) \cap (I : J_2)$.

Solution.

i. Routine. $0 \in (I : X)$ since $0x = 0 \in I$ for all $x \in X$. So $(I : X) \neq \emptyset$. Let $a, b \in (I : X)$ and $c \in R$. We have $(a-b)x = ax - bx$ for all $x \in X$. Since $ax, bx \in I$, and I is an ideal, then $ax - bx \in I$ for all $x \in X$, showing that $a - b \in (I : X)$. Finally, $(ca)x = c(ax)$ for all $x \in X$. Since $ax \in I$, and I is an ideal, then $c(ax) \in I$ for all $x \in X$, showing that $ca \in (I : X)$. Therefore $(I : X)$ is an ideal of R.

ii. If $a \in (I : X)$, then $ax \in I$ for all $x \in X$, and so, for any $b = \sum_{x \in X} \lambda_x x \in J$, we have $ab = \lambda_x ax \in I$ because I is an ideal. So $(I : X) \subseteq (I : J)$.

iii. (a) $I \subseteq (I : X)$ because $ax \in I$ for all $a \in I$ and all $x \in R$.

(b) $J(I : J)$ is the set of all the finite sums $\sum_i b_i a_i$ of elements $b_i \in J$ and $a_i \in (I : J) = \{a \in R : ab \in I \;\forall\; b \in J\}$. So each summand $b_i a_i \in I$, and therefore $\sum_i b_i a_i \in I$ for all $\sum_i b_i a_i \in J(I : J)$. So $J(I : J) \subseteq I$.

(c) Suppose that $I = I_1 \cap I_2$. If $a \in (I_1 : J) \cap (I_2 : J)$, then $ab \in I_i$ for $i = 1, 2$, and so $ab \in I$ for all $b \in J$. Therefore $a \in (I : J)$, and we conclude that $(I_1 : J) \cap (I_2 : J) \subseteq (I : J)$. Conversely, if $a \in (I : J)$, then for any $b \in J$, we have $ab \in I \subseteq I_i$, which shows that $a \in (I_i : J)$, for $i = 1, 2$. So $(I_1 : J) \cap (I_2 : J) \supseteq (I : J)$.

(d) Suppose that $J = J_1 + J_2$. Let $a \in (I : J)$. Since $aJ_i \subseteq a(J_1 + J_2) \subseteq I$ for $i = 1, 2$, we conclude that $(I : J) \subseteq (I : J_1) \cap (I : J_2)$.

Conversely, any $a \in (I : J_1) \cap (I : J_2)$ is such that $a(b_1 + b_2) = ab_1 + ab_2 \in I$ for all $b_i \in J_i$, for $i = 1, 2$. Therefore $(I : J) \supseteq (I : J_1) \cap (I : J_2)$.

Exercise 1.16. Use the Chinese remainder theorem with $R = \mathbb{Q}[x]$, $I = (x^3 - 8)R$ and $J = (x^2 + 1)R$, and find a polynomial $f \in R$ such that $f \equiv x \mod I$ and $f \equiv (x + 1) \mod J$.

Solution. We want to solve the system of modular equations

$$\begin{cases} f & \equiv x \pmod{x^3 - 8}, \\ f & \equiv x + 1 \pmod{x^2 + 1}. \end{cases}$$

We start by using the Euclidean algorithm to find $a, b \in \mathbb{Q}[x]$ such that $1 = a(x^3 - 8) + b(x^2 + 1)$. Since $\deg x^3 + 8 > \deg x^2 + 1$, we divide $x^3 - 8$ by $x^2 + 1$, and we obtain

$$x^3 - 8 = (x^2 + 1)x + (-x - 8).$$

Then dividing $x^2 + 1$ by $-x - 8$ gives

$$x^2 + 1 = (-x - 8)(-x + 8) + 65.$$

Since 65 is invertible, the algorithm stops and we write

$$1 = \frac{1}{65}\big((x^2+1) - (-x-8)(-x+8)\big)$$
$$1 = \frac{1}{65}\big((x^2+1) + (x-8)\big((x^3-8) - (x^2+1)x\big)\big)$$
$$1 = \frac{1}{65}\big((x-8)(x^3-8) + (-x^2+8x+1)(x^2+1)\big)$$

Therefore, $\dfrac{1}{65}(-x^2+8x+1)(x^2+1) \equiv 1 \pmod{x^3-8}$ and $\dfrac{1}{65}(x-8)(x^3-8) \equiv 1 \pmod{x^2+1}$, and we put

$$x = \frac{1}{65}x(-x^2+8x+1)(x^2+1) + \frac{1}{65}(x+1)(x-8)(x^3-8) = \frac{1}{65}(x^4-8x^3+57x+64).$$

(The full solution set is $\{\frac{1}{65}(x^4-8x^3+57x+64) + (x^3-8)(x^2+1)f \mid f \in \mathbb{Q}[x].\}$)

Exercise 1.17. Let $f = 2x^3+3x^2+5x+a \in \mathbb{Z}/7[x]$.

i. Find all $a \in \mathbb{Z}/7$ such that f is irreducible.

ii. Let $a = 1$.

   (a) Prove that the principal ideal $I = f\mathbb{Z}/7[x]$ is maximal. Let $\pi : \mathbb{Z}/7[x] \longrightarrow (\mathbb{Z}/7[x])/I$ be the projection map.

   (b) Prove that $\pi(g) \neq 0$, for $g = 3x^2+4x+5 \in \mathbb{Z}/7[x]$.

   (c) Find $\big(\pi(g)\big)^{-1}$ in $(\mathbb{Z}/7[x])/I$.

Solution.

i. $\mathbb{Z}/7$ is a field, and so f is irreducible if and only if f has no root in $\mathbb{Z}/7$. To find the possible roots, we evaluate f at each element of $\mathbb{Z}/7$:

$$f(0) = f(4) = f(5) = a,\ f(1) = f(2) = f(6) = 3+a \quad \text{and} \quad f(3) = 5+a.$$

So f is irreducible if and only if $a \neq 0, 4, 2$, respectively, i.e. if and only if $a \in \{1,3,5,6\}$.

ii. Let $a = 1$.

   (a) $\mathbb{Z}/7[x]$ is a PID because $\mathbb{Z}/7$ is a field. So $I = f\mathbb{Z}/7[x]$ is maximal if and only if f is irreducible, which we have just proved for $a = 1$.
   Let $\pi : \mathbb{Z}/7[x] \longrightarrow (\mathbb{Z}/7[x])/I$ be the canonical projection map.

   (b) $\pi(g) \neq 0$, for $g = 3x^2+4x+5 \in \mathbb{Z}/7[x]$, if and only if $f \nmid g$. This holds because $\deg g < \deg f$.

   (c) f is irreducible (hence prime in the PID $\mathbb{Z}/7[x]$), so that $\gcd(f,g) = 1$ can be written as a $\mathbb{Z}/7[x]$-linear combination of f and g. We calculate, using the Euclidean algorithm twice:

$$f = g(3x+4) + (2x+2), \quad \text{and then} \quad g = (2x+2)(5x+4) + 4 \in (\mathbb{Z}/7[x])^{\times}.$$

   So the algorithm stops and we can first write: (recall that $-1 = 6 \in \mathbb{Z}/7$)

$$4 = g + (2x+2)\big(6(5x+4)\big)$$
$$4 = g + \big(f + g\big(6(3x+4)\big)\big)\big(6(5x+4)\big)$$
$$4 = g + \big(f + g(4x+3)\big)(2x+3)$$
$$4 = g\big(1 + (4x+3)(2x+3)\big) + f(2x+3)$$
$$4 = g(x^2+4x+3) + f(2x+3) \quad \text{and so}$$
$$1 = g(2x^2+x+6) + f(4x+6).$$

It follows that $1 = \pi(1) = \pi\big(g(2x^2+x+6) + f(4x+6)\big) = \pi(g)\pi(2x^2+x+6) + 0$, giving

$$\big(\pi(g)\big)^{-1} = \pi(2x^2+x+6) \quad \text{in } (\mathbb{Z}/7[x])/I.$$

Exercise 1.18. Let R be a commutative ring.

i. Prove that $\mathrm{Nil}(R)$ and $\mathrm{Rad}(R)$ are ideals of R and that $\mathrm{Nil}(R) \subseteq \mathrm{Rad}(R)$.

ii. Prove that if $I \in \mathrm{Spec}(R)$, then $\sqrt{I} = I$.

iii. Find a commutative ring R and a radical ideal $I = \sqrt{I}$ such that $I \notin \mathrm{Spec}(R)$. (Hint: consider $\mathbb{Z}$ and an ideal $n\mathbb{Z}$ with n a product of distinct primes.)

iv. Prove that $\mathrm{Nil}(R/\mathrm{Nil}(R)) = \{0\}$.

Solution.

i. By definition $\mathrm{Nil}(R) = \displaystyle\bigcap_{P \in \mathrm{Spec}(R)} P \subseteq \bigcap_{P \in \mathrm{MaxSpec}(R)} P = \mathrm{Rad}(R)$.

ii. Let $I \in \mathrm{Spec}(R)$ and let $a \in R$. By induction on $n \in \mathbb{N}$, we have $a^n \in I$ if and only if $a \in I$. The equality follows.

iii. For example the ideal $6\mathbb{Z}$ of $\mathbb{Z}$ is radical but not prime.

iv. We know that the prime ideals of $R/\mathrm{Nil}(R)$ are in 1-1 correspondence with the prime ideals of R containing $\mathrm{Nil}(R)$. So their intersection is equal to $\mathrm{Nil}(R)$, i.e. we have $\mathrm{Nil}(R/\mathrm{Nil}(R)) = \{0\}$.

Exercise 1.19. Let p be a prime number. Prove that the saturated ideals of $\mathbb{Z}$ with respect to $\mathbb{Z} \setminus (p)$ are those generated by the powers of p, i.e. of the form $(p^n) = p^n\mathbb{Z}$ for some $n \in \mathbb{N}$. (Note that there is a unique prime ideal of $\mathbb{Z}$ which does not meet $\mathbb{Z} \setminus (p)$, and that $\mathbb{Z}_{(p)}$ has a unique nonzero prime - hence maximal - ideal.)

Solution. By definition, $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \mid \gcd(p,b) = 1\}$ is the localisation of $\mathbb{Z}$ with respect to $\mathbb{Z} - (p) = \{a \in \mathbb{Z} \mid \gcd(a,p) = 1\}$. Let $\theta : \mathbb{Z} \to \mathbb{Z}_{(p)}$ the inclusion $\theta(a) = \frac{a}{1}$. The proper ideals of $\mathbb{Z}_{(p)}$ are precisely the ideals of the form $\frac{p^n}{1}\mathbb{Z}_{(p)} = \theta(p^n\mathbb{Z}) = \theta(p^n d\mathbb{Z})$, for some $n \in \mathbb{N}$ and $d \in \mathbb{Z}$ with $\gcd(p,d) = 1$. Indeed, every integer coprime to p is mapped by θ to a unit of $\mathbb{Z}_{(p)}$. Now, we note that $\theta^{-1}(\frac{p^n}{1}\mathbb{Z}_{(p)}) = p^n\mathbb{Z}$, which proves the assertion.

Exercise 1.20. Let k be a field, let $R = k[x,y]$ and let $\lambda \in k$. Consider the ideal $I = (x - \lambda y)$ of R.

i. Prove that the quotient ring R/I is isomorphic to $k[y]$.

ii. Deduce from the above that the ideal $I = (x - \lambda y)$ is prime.

Solution.

i. Let $\varepsilon : R \to k[y]$ be the evaluation at $x = \lambda y$, i.e. $\varepsilon(f) = f(\lambda y, y) \in k[y]$ for all $f \in R$. Observe that ε is a surjective ring homomorphism, and $I \subseteq \ker(\varepsilon)$. It remains to check that $I = \ker(\varepsilon)$, since the first isomorphism theorem gives the required isomorphism.

   Let $f \in \ker(\varepsilon)$, and write $f = (x - \lambda y)g + h$, for some $g \in R$ and some $h \in k[y]$. Note that we can do so by grouping all the monomials containing x together. So we calculate

$$0 = \varepsilon(f) = f(\lambda y, y) = 0g(\lambda y, y) + h(y) \quad \text{implying that} \quad h = 0.$$

   That is, $f = (x - \lambda y)g \in I$, for all $f \in \ker(\varepsilon)$.

ii. The isomorphism $R/I \cong k[y]$, with $k[x]$ an ID shows that I is prime. (Note that I is not maximal since $k[y]$ is not a field.)