Exercise Sheet 6

MAGIC009 - Category Theory

November 15th, 2024

1. Prove explicitly that the following categories have a terminal object, binary products, equalisers and pullbacks:
   (a) **Set**,the category of sets and functions,
   (b) **Grp**, the category of groups and group homomorphisms,
   (c) **Top**, the category of topological spaces and continuous functions.
2. Fix a category $\mathbb{C}$. Let $\mathbb{I}$ be the empty category, with no objects.
   (a) Convince yourself that $\mathbb{C}^{\mathbb{I}}$ is isomorphic to the terminal category $\mathbf{1}$, with a single object and no non-identity maps.
   (b) Describe the diagonal functor $\Delta\colon \mathbb{C} \to \mathbb{C}^{\mathbb{I}}$ in this case.
   (c) Prove that, for this choice of $\mathbb{I}$, the following statements are equivalent:
      i. The functor $\Delta\colon \mathbb{C} \to \mathbb{C}^{\mathbb{I}}$ has a right adjoint.
      ii. $\mathbb{C}$ has a terminal object.
3. Fix a category $\mathbb{C}$. Let I be a set (not necessarily finite) and consider it as a category with no non-identity morphisms.
   (a) Describe explicitly the category $\mathbb{C}^I$.
   (b) Describe the diagonal functor $\Delta\colon \mathbb{C} \to \mathbb{C}^I$ in this case.
   (c) Describe what it means for $\Delta\colon \mathbb{C} \to \mathbb{C}^I$ to have a right adjoint, following the pattern explained in Lecture 6.
   (d) Prove that such a right adjoint exists when $\mathbb{C} = \mathbf{Set}$.
4. Prove the following statements:
   (a) If $\mathbb{C}$ has binary products and equalisers, then it has pullbacks. *Hint.* Consider the description of pullbacks in **Set** obtained in the answer to Exercise 1 (a).
   (b) If $\mathbb{C}$ has pullbacks and a terminal object, then it has binary products and equalisers.