Suggested exercises Section 2: Modules

Exercise 2.1. Let R be a commutative ring, let M be an R-module and let I be an ideal of R. Suppose that $IM = \{\sum_{\text{finite}} a_i x_i \mid a_i \in I,\ x_i \in M\} = 0$. Define a structure of R/I-module on M.

Exercise 2.2. Recall that a $\mathbb{Z}$-module is the same as an abelian group. Determine the group structure of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$ as an abelian group, for all integers $m, n \geq 2$. More generally, let R be a commutative ring and I, J two ideals of R. Describe $\mathrm{Hom}_R(R/I, R/J)$ as an R-module.

Exercise 2.3. Let R be a commutative ring and let M, N be two R-modules. Verify that $\mathrm{Hom}_R(M, N)$ is an R-module for the R-action $(af)(x) = af(x)$ for all $a \in R, f \in \mathrm{Hom}_R(M, N)$ and $x \in M$. How can you generalise the construction to arbitrary rings and left modules?

Exercise 2.4. Let M be an R-module.

i. Prove that the torsion subgroup $M_{\mathbb{Z}-tor}$ of M, formed by the elements of finite order, is an R-module.

ii. Suppose that R is a commutative ring.

  (a) Prove that the (R-)torsion submodule $M_{tor} = \{x \in M \mid \exists\, a \in R,\ a \neq 0,$ such that $ax = 0\}$ is an R-submodule of M.

  (b) Prove that $\big(M/M_{tor}\big)_{tor} = \{0\}$.

iii. Find an example of (non-commutative) ring R and R-module M for which M_{tor} is not an R-submodule of M.

Exercise 2.5. Let R be a commutative ring. The *annihilator* of an R-module M is $\mathrm{Ann}(M) = \{a \in R \mid ax = 0,\ \forall\, x \in M\}$.

i. Prove that M is faithful if and only if $\mathrm{Ann}(M) = \{0\}$.

ii. Prove that $\mathrm{Ann}(M)$ is a two-sided ideal of R.

iii. Prove that M is a faithful module as a module for the quotient ring $R/\mathrm{Ann}(M)$.

Exercise 2.6. Let R be a commutative ring.

- Let M be an R-module. Prove that $\mathrm{Hom}_R(R, M)$ is an R-module isomorphic to M.
- Prove that $\mathrm{End}_R M$ is a unital ring, generally not commutative.

Exercise 2.8. Let R be a commutative ring. Prove that, given any R-homomorphism $\varphi \in \mathrm{Hom}_R(M, N)$, we obtain an exact sequence

$$0 \longrightarrow \ker(\varphi) \xrightarrow{incl} M \xrightarrow{\varphi} N \xrightarrow{\pi} \mathrm{coker}(\varphi) \longrightarrow 0\ .$$

Exercise 2.9. Let R be a ring. Prove that a short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of R-modules splits if and only if C is isomorphic to $A \oplus B$.

Exercise 2.10 (Five Lemma)**.** Let R be a ring. Suppose that we have a commutative diagram of R-modules and R-homomorphisms, of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & D \arrow[r, "i"] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "i'"] & E' \end{tikzcd} ,$$

with exact rows. *Commutative* means that all the 'paths' between two modules are equal, e.g. $f'a = bf$.

i. Suppose that b, d are surjective and e injective. Prove that c is surjective.

ii. Suppose that b, d are injective and a surjective. Prove that c is injective.

iii. Suppose that a, b, d, e are isomorphisms. Prove that c is an isomorphism.

Exercise 2.14. Adapt the proof of Hilbert's basis theorem to show that $R[[x]]$ is Noetherian if R is Noetherian.

Exercise 2.15. Let R be a ring, let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of R-modules and let M be an R-module.

i. Prove that the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f_*} \mathrm{Hom}_R(M, B) \xrightarrow{g_*} \mathrm{Hom}_R(M, C) \quad \text{is exact,}$$

where $f_*(\varphi) = f\varphi : M \to A \to B$ and similarly for g_*.

ii. Prove that the sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, M) \xrightarrow{g^*} \mathrm{Hom}_R(B, M) \xrightarrow{f^*} \mathrm{Hom}_R(A, M) \quad \text{is exact,}$$

where $g^*(\varphi) = \varphi f : B \to C \to M$ and similarly for f^*.

Exercise 2.18. Let R be a commutative ring. An R-module M is *divisible* if $aM = M$ for all $a \in R$. That is, the multiplication by a map $M \to M$ is a surjective R-homomorphism.
i. Prove that $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module (i.e. abelian group).
iv. Prove that an injective R-module is divisible.

Exercise 2.20. Let R be a PID and let M be a nonzero finitely generated torsionfree R-module. Prove that M is free. (Hint: proceed by induction on the number of generators of M.)