MAGIC assessment cover sheet

Course:	**MAGIC008**
Session:	**2024-2025**
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$19+17+20 = 56/60$

PASS

MAGIC 008 Ibraheem Sajid

Q1. (a) Let X be a arbitrary 3×3 matrix, and write

$$X = \begin{pmatrix} Y & \begin{matrix} d \\ e \end{matrix} \\ \begin{matrix} a & b \end{matrix} & c \end{pmatrix}$$

where Y is an arbitrary 2×2 matrix and $a,b,c,d,e \in \mathbb{R}$.

We compute,

$$XA = \begin{pmatrix} Y & \begin{matrix} -d \\ -e \end{matrix} \\ \begin{matrix} a & b \end{matrix} & -c \end{pmatrix} \quad \text{and} \quad AX = \begin{pmatrix} Y & \begin{matrix} d \\ e \end{matrix} \\ \begin{matrix} -a & -b \end{matrix} & -c \end{pmatrix}$$

So $(XA = AX) \iff (a=b=d=e=0)$

Now suppose $XA = AX$ and note $\det X = c \det Y$

So $X \in GL(3,\mathbb{R}) \iff (c \neq 0,\ \det Y \neq 0)$

Thus we have $G_A = \left\{ X\ 3\times 3 \text{ matrix} \;\middle|\; X = \begin{pmatrix} Y & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & c \end{pmatrix},\ Y \in GL(2,\mathbb{R}),\ c \neq 0 \right\}$

$\dim G_A = 2^2 + 1 = 5$ as it locally (near I_3) looks like $GL(2,\mathbb{R}) \times \mathbb{R}$, and $\dim GL(n,\mathbb{R}) = n^2$

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~~(b)~~

~~We have a well-defined map $G_A \to \mathbb{Z}_2 \times \mathbb{Z}_2$~~
~~$X \mapsto (\operatorname{sgn} \det Y, \operatorname{sgn} c)$~~

Why is G_A a Lie group?

(b) We have the map $\varphi : G_A \longrightarrow \mathbb{R}^2$

$$X = \begin{pmatrix} Y & 0 \\ 0 & c \end{pmatrix} \longmapsto (\det Y, c)$$

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This is smooth (given by polynomials of coefficients of X) and has image in the disconnected space

Why each component is connected? Why?

$$\mathbb{R}^2 \setminus (\{(\lambda, 0) \mid \lambda \in \mathbb{R}\} \cup \{(0,\lambda) \mid \lambda \in \mathbb{R}\})$$

which has 4 connected components.

$\therefore$ The preimage of each component is a union of components in G_A, so G_A is not connected.

OK

We show directly that the identity component is all of $\varphi^{-1}(\{(\lambda,\mu) \mid \lambda > 0, \mu > 0\})$.

$$\parallel$$

$$G_A^{+,+} := \left\{ X = \begin{pmatrix} Y & \\ & c \end{pmatrix} \;\middle|\; \det Y > 0,\ c > 0 \right\}$$

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Indeed, let $X = \begin{pmatrix} Y & \\ & c \end{pmatrix}$ with $\det Y > 0$, $c > 0$.

We know that $GL(2,\mathbb{R})$ has two components given by determinant >0 and <0, so there is a path $Y(t)$ from Y to I_2.

Then $X(t) = \begin{pmatrix} Y(t) & \\ & c(1-t)+1 \end{pmatrix}$ is a path from X to I_3.

and we've shown that $G_A^{+,+}$ is connected.

Note we can get the other 3 preimages under φ by left-multiplying the identity component by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

so they are also connected, and G_A has 4 components.

(c) We know a matrix $X \in G_A$ looks like $\begin{pmatrix} Y & \\ & c \end{pmatrix}$

where $Y \in GL(2,\mathbb{R})$, $c \in \mathbb{R} \setminus \{0\}$

so the map $f : G_A \longrightarrow GL(2,\mathbb{R}) \times \mathbb{R}^*$

$X \longmapsto (Y, c)$

is well-defined.

Its smooth as the composition of elementary functions (identity and lifting maps etc).

We also have the smooth inverse

$$\begin{pmatrix} Y & \\ & c \end{pmatrix} \longleftarrow\!\shortmid (Y, c)$$

so we just need to check that ~~...~~ $f(id) = id$ and $f(X_1 X_2) = f(X_1) f(X_2)$

Indeed $id_{G_A} = I_3$ and $f(I_3) = (I_2, 1)$

Which is the identity of $GL(2,\mathbb{R}) \times \mathbb{R}^*$ as the product of identities.

Next let $X_1, X_2 \in G_A$ with $X_1 = \begin{pmatrix} Y_1 & \\ & c_1 \end{pmatrix}$, $X_2 = \begin{pmatrix} Y_2 & \\ & c_2 \end{pmatrix}$

$$X_1 X_2 = \begin{pmatrix} Y_1 Y_2 & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\ 0 & c_1 c_2 \end{pmatrix} \quad \text{so} \quad f(X_1X_2) = (Y_1Y_2, c_1c_2)$$

$$= (Y_1, c_1) \cdot (Y_2, c_2)$$

$$= f(X_1) \cdot f(X_2)$$

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(d) Let $X \in G_A$ and $X = \begin{pmatrix} Y & \\ & c \end{pmatrix}$ then let $Y = \begin{pmatrix} y_1 & y_3 \\ y_2 & y_4 \end{pmatrix}$

We see $Xv = \begin{pmatrix} y_1 \\ y_2 \\ c \end{pmatrix}$ where $c \neq 0$.

Note (y_1, y_2) can be any pair in $\mathbb{R}^2 \setminus \{0\}$ since we can find a $Y \in GL(2,\mathbb{R})$ which works.

$$\therefore \mathcal{O}(v) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; (a,b) \neq (0,0),\ c \neq 0 \right\}$$

OK

ie the 3-dimensional space with a hyperplane removed, along with a perpendicular line through its centre also removed.

We still have $\dim \mathcal{O}(v) = 3$ though.

Suppose $Xv = v$. Then $\begin{pmatrix} y_1 \\ y_2 \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

So $y_1 = 1$, $y_2 = 0$, $c = 1$

and $X = \begin{pmatrix} 1 & y_3 & \\ 0 & y_4 & \\ & & 1 \end{pmatrix}$ We can't have $y_4 = 0$ for determinant reasons, but y_3 is ~~f~~ arbitrary

$\therefore$ $St(v) = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{R},\ b \neq 0 \right\}$

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So $\dim St(v) = 2$.

This matches $\dim \mathcal{O}(v) + \dim St(v) = \dim G_A$

(e) We consider points in $\mathbb{R}^3 \setminus \mathcal{O}(v)$.

Clearly $\mathcal{O}(0) = 0$

Also $\mathcal{O}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \left\{ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \in \mathbb{R}^3 \middle| c \neq 0 \right\}$

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and $\mathcal{O}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \in \mathbb{R}^3 \middle| (a,b) \neq (0,0) \right\}$

by inspecting the previous work.

$\therefore$ this action has 4 orbits as described.

19/20 for Q1

Q2. (a) First we show G is a subgroup of $GL(4,\mathbb{R})$.

- $I_4 \in G$, taking $A = I_2$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- let $X = \begin{pmatrix} A & B \\ & A \end{pmatrix} \in G$

Then note $\begin{pmatrix} A^{-1} & -A^{-1}BA^{-1} \\ & A^{-1} \end{pmatrix}$ is both

left and right inverse to X, so is X^{-1}. Also, $A^{-1} \in O(2)$ so $X^{-1} \in G$.

- let $X = \begin{pmatrix} A & B \\ & A \end{pmatrix} \in G$, $Y = \begin{pmatrix} C & D \\ & C \end{pmatrix} \in G$

Then $XY = \begin{pmatrix} AC & AD+BC \\ & AC \end{pmatrix}$ and since $O(2)$ is a group, $AC \in O(2)$ and $XY \in G$.

$\therefore$ G is a subgroup.

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G is algebraic as the defining equations

- bottom-left block $= 0$
- top-left and bottom-right blocks are equal
- top-left block is orthogonal

are all polynomial in the matrix components.

$A^TA = I$ is a system of polynomial equations

A has 1 free variable (parameter) and B has 4, so $\dim G = 5$.

G is not compact as it is not bounded.

Eg consider the subset $\left\{ \begin{pmatrix} 1 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \in G \;\middle|\; \lambda \in \mathbb{R} \right\}$

(b) The Lie algebra $\mathfrak{g}$ is generated by vectors

$$F := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \quad E_{1,1} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix} \quad E_{1,2} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

$$E_{2,1} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}, \quad E_{2,2} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}$$

(since the vector generating $\mathfrak{so}(2)$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$).

We perform 8 calculations and then put these together into the bracket structure.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

We can put this together to see that

$FE_{1,1} = -E_{2,1}$	$E_{1,1}F = E_{1,2}$
$FE_{2,1} = E_{1,1}$	$E_{2,1}F = E_{2,2}$
$FE_{1,2} = -E_{2,2}$	$E_{1,2}F = -E_{1,1}$
$FE_{2,2} = E_{1,2}$	$E_{2,2}F = -E_{2,1}$

Also note $E_{i,j}E_{k,l} = 0$ for all matrices of this type

~~∴ we get the Lie bracket structure as in the table below~~

	F	$E_{1,1}$	$E_{2,1}$	$E_{1,2}$	$E_{2,2}$
F	0				
$E_{1,1}$		0			
$E_{2,1}$			0		
$E_{1,2}$				0	
$E_{2,2}$					0

And so the Lie bracket structure is as follows:

$$[F, E_{1,1}] = -E_{2,1} - E_{1,2}$$
$$[F, E_{2,1}] = E_{1,1} - E_{2,2}$$
$$[F, E_{1,2}] = -E_{2,2} + E_{1,1} = -[F, E_{2,1}]$$
$$[F, E_{2,2}] = E_{1,2} + E_{2,1} = -[F, E_{1,1}]$$

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With all other brackets of basis elements trivial.

(c) Let $\mathfrak{g}$ be a Lie algebra and define:

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$$
$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$$

Then recursively define for $n > 0$

$$\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$$
$$\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$$

We note that there are descending chains

$$\mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots$$
$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots$$

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$\mathfrak{g}$ is nilpotent if for some $k > 0$ we have $\mathfrak{g}_k = \{0\}$.

$\mathfrak{g}$ is solvable if for some $k > 0$ we have $\mathfrak{g}^{(k)} = \{0\}$.

(d) $\mathfrak{g}$ is solvable, since we have $\mathfrak{g}^{(2)} = \{0\}$.

Note all elements of $\mathfrak{g}^{(1)}$ are linear combinations of $E_{i,j}$ elements. But these have trivial bracket ~~with~~ with eachother, so by bilinearity of the bracket, all brackets $[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$ are trivial.

For nilpotency:

Note $[F, [F, E_{1,1}]] = [F, -E_{2,1} - E_{1,2}]$

$$= -[F, E_{2,1}] - [F, E_{1,2}]$$

$$= -[F, E_{2,1}] + [F, E_{2,1}] = 0$$

So also $[F, [F, E_{2,2}]] = 0$

And $[F, [F, E_{2,1}]] = [F, E_{1,1} - E_{2,2}]$

$$= [F, E_{1,1}] - [F, E_{2,2}]$$

$$= [F, E_{1,1}] + [F, E_{1,1}] = 2[F, E_{1,1}]$$

$$= -2E_{2,1} - 2E_{1,2}$$

So ~~also~~ $[F, [F, E_{1,2}]] = 2E_{2,1} + 2E_{1,2}$

~~Also~~ Also $[E_{i,j}, \mathfrak{g}_1] = \{0\}$ for similar reasons as before.

$\therefore \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1] = \langle E_{1,2} + E_{2,1} \rangle$

$\mathbb{R}$-span

But $E_{1,2} + E_{2,1} = [F, E_{2,2}]$

$$\therefore [F, E_{1,2} + E_{2,1}] = 0$$

as already shown.

$$\therefore [\mathfrak{g}_1, \mathfrak{g}_2] = \mathfrak{g}_3 = \{0\}.$$

Thus $\mathfrak{g}$ is nilpotent. "Arithmetic errors"

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17/20 for Q2

$$[\xi,\eta]^k = \xi^i \frac{\partial \eta^k}{\partial x^i} - \eta^i \frac{\partial \xi^k}{\partial x^i}$$

Q3 (a)

$\xi_1 = xy\,\partial_x + (1+y^2)\,\partial_y$

$\xi_2 = y\,\partial_x$

$\xi_3 = \partial_x$

$[\xi_1, \xi_2] = (1+y^2)\,\partial_x - y^2\,\partial_x = \partial_x = \xi_3$

$[\xi_1, \xi_3] = 0 - y\partial_x = -\xi_2$

$[\xi_2, \xi_3] = 0 - 0 = 0$

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The bracket is closed on $\langle \xi_1, \xi_2, \xi_3 \rangle_{\mathbb{R}}$

($\mathbb{R}$-span)

So this vector space forms a Lie algebra.

(b) $C^3_{12} = -C^3_{21} = 1$

$C^2_{13} = -C^2_{31} = -1$

All other structure constants are 0.

ad_{ξ_1} has matrix wrt. basis (ξ_1, ξ_2, ξ_3)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\mathrm{ad}\,\xi_2$ has matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

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$\mathrm{ad}\,\xi_3$ has matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Lie algebra with bracket $[\cdot,\cdot]$

(c) A vector subspace $\mathfrak{I} \subseteq \mathfrak{g}$ is an <u>ideal</u> if $[\mathfrak{g}, \mathfrak{I}] \subseteq \mathfrak{I}$.

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(Note this also means $[\mathfrak{I}, \mathfrak{I}] \subseteq \mathfrak{I}$ so $\mathfrak{I}$ is a Lie subalgebra).

(d) $\mathfrak{I}_1 = \mathrm{span}(\xi_2)$ is not an ideal as $[\xi_1, \xi_2] = \xi_3 \notin \mathfrak{I}_1$

$\mathfrak{I}_2 = \mathrm{span}(\xi_2, \xi_3)$ is an ideal since in fact

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$$[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{I}_2$$

so indeed $[\mathfrak{g}, \mathfrak{I}_2] \subseteq \mathfrak{I}_2$.

($\mathfrak{I}_1, \mathfrak{I}_2$ are evidently vector subspaces by definition).

(gothick)

(e) Call the Lie algebra of G by $\mathfrak{h}$

$\mathfrak{h}$ is spaned by $\eta_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\eta_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\eta_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

by considering curves in G and their velocity vectors.

Compute:

$\eta_1\eta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\eta_2\eta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\eta_1\eta_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\eta_3\eta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\eta_2\eta_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\eta_3\eta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

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$\therefore [\eta_1, \eta_2] = -\eta_3$

$[\eta_1, \eta_3] = \eta_2$

$[\eta_2, \eta_3] = 0$

$\therefore$ the map

$\xi_1 \mapsto \eta_1$

$\xi_2 \longmapsto \eta_3$

$\xi_3 \mapsto \eta_2$

is an isomorphism (it is bijective and a homomorphism)

20/20 for Q3