Problem Sheet 7
Solutions

Question 1

Let $\mathfrak{g}$ be a Lie algebra

Denote $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$ and so on

Recall that $\mathfrak{g}$ is solvable if for some $k \in \mathbb{N}$ we have $\mathfrak{g}^{(k)} = \{0\}$.

Similarly, denote $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}]$, $\mathfrak{g}_3 = [\mathfrak{g}_2, \mathfrak{g}]$ and so on.

$\mathfrak{g}$ is nilpotent if for some $k \in \mathbb{N}$, we have $\mathfrak{g}_k = \{0\}$.

(a) $\mathfrak{g}$ is defined by $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$

Hence $\mathfrak{g}^{(1)} = \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] = \operatorname{Span}([e_1, e_2], [e_2, e_3], [e_3, e_1]) =$

$= \operatorname{Span}(e_3, e_1, e_2) = \mathfrak{g}$

Therefore $\mathfrak{g}_k = \mathfrak{g}^{(k)} = \mathfrak{g}$ for any $k \in \mathbb{N}$ $\Rightarrow$

$\mathfrak{g}$ is neither solvable, nor nilpotent

(b) $\mathfrak{g}$ is defined by $[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_2, e_3] = e_1$

Hence $\mathfrak{g}^{(1)} = \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] = \operatorname{Span}([e_1, e_2], [e_2, e_3], [e_3, e_1]) =$

$= \operatorname{Span}(2e_2, e_1, 2e_3) = \mathfrak{g}$,

implying that $\mathfrak{g}$ is neither solvable nor nilpotent (as in (a)).

(c) $\mathfrak{g}$ is defined by $[e_1, e_2] = e_3$

$$\mathfrak{g}^{(1)} = \mathfrak{g}_1 = \operatorname{Span}\left([e_1, e_2], [e_2, e_3], [e_3, e_1]\right) =$$
$$= \operatorname{Span}(e_3)$$

$$\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}] = \operatorname{Span}\left([e_3, \xi], \xi \in \mathfrak{g}\right) =$$
$$= \operatorname{Span}(0) = \{0\}$$

Hence $\mathfrak{g}$ is nilpotent and therefore solvable (Recall that every nilpotent Lie algebra is solvable but not vice versa)

(d) $\mathfrak{g}$ is defined by $[e_1, e_3] = e_3$, $[e_2, e_3] = -e_3$

$$\mathfrak{g}^{(1)} = \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] = \operatorname{Span}\left([e_1, e_2], [e_2, e_3], [e_3, e_1]\right) =$$
$$= \operatorname{Span}(0, -e_3, -e_3) = \operatorname{Span}(e_3)$$

$$\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \operatorname{Span}([e_3, e_3]) = \{0\} \Longrightarrow$$

$\mathfrak{g}$ is solvable

However

$$\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}] = \operatorname{Span}\left([e_3, e_1], [e_3, e_2], [e_3, e_3]\right) =$$
$$= \operatorname{Span}(-e_3, e_3, 0) = \underset{\mathfrak{g}_1}{\underset{\parallel}{\operatorname{Span}(e_3)}}$$

That is, $\mathfrak{g}_2 = \mathfrak{g}_1$ and therefore $\mathfrak{g}_k = \mathfrak{g}_1$ for any $k \in \mathbb{N}$, implying that $\mathfrak{g}$ is not nilpotent.

Question 2

$$e(n) = \left\{ \left(\begin{array}{c|c} A & \bar{x} \\ \hline 0 & 0 \end{array} \right), \ A \in so(n), \ \bar{x} \in \mathbb{R}^n \right\}$$

(a) Take two matrices from $e(n)$ and compute the matrix commutator (Lie bracket)

$$\left[\begin{pmatrix} A_1 & x_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A_2 & x_2 \\ 0 & 0 \end{pmatrix} \right] =$$

$$= \begin{pmatrix} A_1 & x_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_2 & x_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A_2 & x_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1 & x_1 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} A_1A_2 & A_1x_2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A_2A_1 & A_2x_1 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} [A_1, A_2] & A_1x_2 - A_2x_1 \\ 0 & 0 \end{pmatrix} \in e(n)$$ since $[A_1, A_2] \in so(n)$ (indeed $so(n)$ is a Lie algebra closed under matrix commutator)

$$\dim e(n) = \dim so(n) + \dim \mathbb{R}^n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}.$$

(b) $h_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\} \subset e(n)$, $h_2 = \left\{ \begin{pmatrix} 0 & \bar{x} \\ 0 & 0 \end{pmatrix} \right\} \subset e(n)$

h_1 is a subalgebra as it is a vector subspace closed under matrix commutator (indeed, $\left[\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} [A_1, A_2] & 0 \\ 0 & 0 \end{pmatrix} \in h_1$)

h_2 is a subalgebra of $e(n)$ as it is a vector subspace closed under matrix commutator, moreover, this subalgebra is commutative, i.e. we have $\left[\begin{pmatrix} 0 & \bar{x}_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bar{x}_2 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(c) To verify if $\mathfrak{h}_1$ is an ideal of $\mathfrak{e}(n)$ we compute $[X,Y]$ for $X=\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{h}_1$ and $Y=\begin{pmatrix} B & \bar{y} \\ 0 & 0 \end{pmatrix} \in \mathfrak{e}(n)$

We have $[X,Y] = \left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & \bar{y} \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} [A,B] & A\bar{y} \\ 0 & 0 \end{pmatrix} \notin \mathfrak{h}_1$

since $A\bar{y} \neq 0$

Thus, $\mathfrak{h}_1$ is not an ideal.

Similarly, to verify if $\mathfrak{h}_2$ is an ideal of $\mathfrak{e}(n)$ we compute $[X,Y]$ for $X=\begin{pmatrix} 0 & \bar{x} \\ 0 & 0 \end{pmatrix} \in \mathfrak{h}_2$ and $Y=\begin{pmatrix} B & \bar{y} \\ 0 & 0 \end{pmatrix} \in \mathfrak{e}(n)$

We have $[X,Y] = \left[\begin{pmatrix} 0 & \bar{x} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & \bar{y} \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -B\bar{x} \\ 0 & 0 \end{pmatrix} \in \mathfrak{h}_1$

Thus, $\mathfrak{h}_2$ is an ideal.

(d) Consider the cases $n=1$, $n=2$, $n=3,4,\dots$

For $n=1$ $\quad \mathfrak{e}(1) = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, x \in \mathbb{R} \right\}$ is commutative $\Rightarrow$ nilpotent and solvable

For $n=2$ $\quad \mathfrak{e}(2) = \left\{ \begin{pmatrix} 0 & a & x_1 \\ -a & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}, a, x_1, x_2 \in \mathbb{R} \right\}$

Take the following basis of $\mathfrak{e}(2)$

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We have $[E_1,E_2] = -E_3$, $[E_1,E_3] = E_2$, $[E_2,E_3]=0$

Hence (see Q1) $[\mathfrak{e}(2),\mathfrak{e}(2)] = \operatorname{Span}(E_2,E_3)$

$\mathfrak{e}(2)^{(1)} = \mathfrak{e}(2)_1 = \operatorname{Span}(E_2,E_3)$

Next,

$$e(2)^{(2)} = [e(2)^{(1)}, e(2)^{(1)}] = [\mathrm{Span}(E_2,E_3), \mathrm{Span}(E_2,E_3)]$$

$$= \mathrm{Span}([E_2,E_3]) = \{0\} \Rightarrow e(2) \text{ is solvable}$$

However

$$e(2)_2 = [e(2)_2, e(2)] = [\mathrm{Span}(E_2,E_3), e(2)] =$$

$$= \mathrm{Span}\Big([E_2,\xi],[E_3,\xi]\Big)_{\xi \in e(2)} = \mathrm{Span}(E_2,E_3) = e(2)_1$$

implying that $e(2)_k = e(2)_1 = \mathrm{Span}(E_2,E_3)$ for all $k \in \mathbb{N}$ $\Rightarrow$ $e(2)$ is not nilpotent.

Finally $e(n)$ starting from $n=3$ is neither solvable nor nilpotent.

Indeed, straightforward computation gives:

$$\left[\begin{pmatrix} A & \bar{x} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & \bar{y} \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} [A,B] & A\bar{y}-B\bar{x} \\ 0 & 0 \end{pmatrix}$$

where $[A,B]$ can be an arbitrary skew-symm. matrix from $so(n)$ and $A\bar{y}-B\bar{x}$ can be arbitrary vector from $\mathbb{R}^n$. In other words,

$$\underbrace{[e(n), e(n)]}_{= e(n)^{(1)} = e(n)_1} = e(n)$$

and therefore

$$e(n)^{(k)} = e(n)_k = e(n) \neq \{0\}$$

for all $k \in \mathbb{N}$

Question 3.

Solution is similar to Q2 (b, c)

All of these subspaces are subalgebras of $\mathfrak{g}$.

$\left\{\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\right\}$, $\left\{\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix}\right\}$ are ideal

$\left\{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right\}$, $\left\{\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}\right\}$, $\left\{\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}\right\}$ are not ideals.