MAGIC008: Lie Groups and Lie algebras

Problem Sheet 7: Lie algebras, ideals, solvable and nilpotent Lie algebras

1. Consider 3-dimensional Lie algebras defined by the following commutator relations:

   - $\mathfrak{g}_1$: $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$;
   - $\mathfrak{g}_2$: $[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_2, e_3] = e_1$;
   - $\mathfrak{g}_3$: $[e_1, e_2] = e_3$;
   - $\mathfrak{g}_4$: $[e_1, e_3] = e_3$, $[e_2, e_3] = -e_3$.

   Which of these Lie algebras are solvable? nilpotent?

2. Consider the set $e(n)$ consisting of the $(n+1) \times (n+1)$ matrices of the form

$$\begin{pmatrix} A & \bar{x} \\ 0 & 0 \end{pmatrix},$$

   where $A \in so(n)$ is a skew-symmetric $n \times n$ matrix and $\bar{x} \in \mathbb{R}^n$ is a column-vector.

   - Show that $e(n)$ is a Lie algebra and compute its dimension.
   - Consider

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\} \subset e(n) \quad \text{and} \quad \mathfrak{h}_2 = \left\{ \begin{pmatrix} 0 & \bar{x} \\ 0 & 0 \end{pmatrix} \right\} \subset e(n).$$

     Is $\mathfrak{h}_i$ a subalgebra $(i = 1, 2)$?
   - Is $\mathfrak{h}_i$ an ideal of $e(n)$ $(i = 1, 2)$?
   - Is $e(n)$ solvable? nilpotent? (The answer depends on n!)

3. Consider the set $\mathfrak{g}$ of $(n+k) \times (n+k)$ matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

   where A is an $n \times n$ matrix, C is a $k \times k$ matrix and B is an $n \times k$ matrix (A, B and C are arbitrary). Check that $\mathfrak{g}$ is a Lie algebra. Which of the following subspaces

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \right\}, \left\{ \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix} \right\},$$

   are subalgebras of $\mathfrak{g}$? ideals of $\mathfrak{g}$?

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