MAGIC008: Lie Groups and Lie algebras

Problem Sheet 5: Actions of Lie groups: orbits, stabilizers and invariants

1. Consider the natural action of the upper triangular group

$$T = \left\{ \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{pmatrix}, \quad t_{11} \cdot t_{22} \cdot t_{33} \neq 0 \right\}$$

on the 3-dimensional space $\mathbb{R}^3$. Describe the stabilizer subgroup $\mathrm{St}(v)$ and the orbit $\mathcal{O}(v)$. What is the dimension of $\mathcal{O}(v)$?

   (a) $v = e_1 = (1, 0, 0)$
   (b) $v = e_2 = (0, 1, 0)$
   (c) $v = e_3 = (0, 0, 1)$

   How many distinct orbits does this action have? What happens if we replace T by its identity component T_0? (Reminder: the identity component T_0 is determined by the following additional condition: $t_{11} > 0, t_{22} > 0, t_{33} > 0$.)

   Are there any invariants of this action?

2. Consider the Lie group $G_{\mathcal{B}} = \{X \in GL(3, \mathbb{R}) \mid X^\top BX = B\}$, where $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

   (See Problem Sheet 3). Recall that $G_{\mathcal{B}}$ has the following matrix description:

$$G_{\mathcal{B}} = \left\{ \begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} b & c \end{matrix} & d \end{pmatrix} \quad A \in O(2), \ d \neq 0 \right\}$$

   We consider the natural action in $\mathbb{R}^3$. Describe the stabilizer subgroup $\mathrm{St}(v)$ and the orbit $\mathcal{O}(v)$. What is the dimension of $\mathcal{O}(v)$?

   (a) $v = e_3 = (0, 0, 1)$,
   (b) $v = (v_1, v_2, 0) \neq \bar{0}$,
   (c) $v = \bar{0}$,
   (d) v is an arbitrary vector in $\mathbb{R}^3$.

   How many distinct orbits does this action have? Describe all of them. Find the invariants of this action.

3. Consider the following matrix group $G \subset GL(n + 1, \mathbb{R})$:

$$G_{\mathcal{B}} = \left\{ X = \begin{pmatrix} & & & x_1 \\ & A & & \vdots \\ & & & x_n \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad A \in GL(n, \mathbb{R}) \right\}$$

(a) Describe the orbits and invariants of the natural action of G in $\mathbb{R}^{n+1}$.

(b) Describe the orbits and invariants of the dual action which is defined by

$$\hat{X}v = (X^\top)^{-1}v, \quad X \in G,\ v \in \mathbb{R}^{n+1}$$

(c) Consider the action of G in $\mathbb{R}^n$ defined as follows:

$$\hat{X}u = Au + x, \quad \text{where } u \in \mathbb{R}^n,\ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Verify that this formula defines indeed an action, i.e. $\hat{I}_{n+1} = \mathrm{id}$, and $\widehat{XY} = \hat{X} \circ \hat{Y}$. Describe the orbits and invariants of this action.

Is this action transitive? Describe the stabilizer subgroup for $u = 0$. What can you say about the stabilizer subgroup of $u \neq 0$?

4. Consider the adjoint action of $GL(n,\mathbb{R})$ on $gl(n,\mathbb{R})$ (recall that $gl(n,\mathbb{R})$ is simply the space of all $n \times n$ matrices):

$$\mathrm{Ad}_X A = XAX^{-1}, \quad X \in GL(n,\mathbb{R}),\ A \in gl(n,\mathbb{R}).$$

Notice that the orbit $\mathcal{O}(A)$ of this action is just the set of all matrices similar to A. Compute $\dim O(A)$ for

(a) $A = I_n$ identity matrix,

(b) A is a diagonal matrix with distinct diagonal elements,

(c) $A = \begin{pmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & 1 & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \dots & \lambda \end{pmatrix}$,

(d) $A = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$, $p + q = n$.

5. Consider the action Φ of $SL(2,\mathbb{R})$ on the complex upper half-plane $\mathcal{H}$ defined by

$$z \mapsto \Phi_A(z) = \frac{az+b}{cz+d}, \qquad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

Prove that this formula defines indeed an action. Is this action transitive? What is the stabilizer subgroup of $z = i$.

Consider the action of $SL(2,\mathbb{R})$ on the adjoint orbit $\mathcal{O}(B)$ of the matrix $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. What is the stabilizer of B?

Prove that these two actions are, in fact, isomorphic in the sense that there exists a diffeomorphism $f : \mathcal{H} \to \mathcal{O}(B)$ such that $\mathrm{Ad}_A \circ f = f \circ \Phi_A$ for all $A \in SL(n,\mathbb{R})$.

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