Two Weeks

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO HOPF ALGEBRAS AND QUANTUM GROUPS

Exam Released:	—	**(BST)**
End of Submission Window:	—	**(BST)**

Answer **THREE** of the **FOUR** questions. If more than **THREE** questions are attempted, then credit will be given for the best **THREE** answers.

The maximum possible mark for this paper is 60. The pass mark is 30 out of 60.

This is a take-home open book exam. Your solutions should be written on paper, preferably using blue or black ink (not pencil), on a tablet using a stylus, or typeset. You may use without proof any of the formulae given in the course, unless you are asked to prove them.

Your answers must be your own work. Identical answers will be flagged for plagiarism.

Answer **THREE** of the four questions. If more than THREE questions are attempted, then credit will be given for the best THREE answers.

1. As usual, $\mathbb{C}\langle x\rangle$ denotes the free algebra with one generator x. We equip $H = \mathbb{C}\langle x\rangle$ with the unique Hopf algebra structure where x is a primitive element. An element of H of the form $c1_H$, with $c \in \mathbb{C}$ (where the coefficient of any monomial x^n with $n > 0$ is zero) will be called a *constant*.

(a) If $f \in H$ is a constant, explain, referring to the Hopf algebra axioms, why $\Delta f = f \otimes 1_H$. Let now $h \in H$ be such that $\Delta h = h \otimes 1_H$; show that h is a constant.

Write $\mathbb{C}[y]$ for the commutative algebra of polynomials in the variable y. Define the linear map $D\colon \mathbb{C}[y] \to \mathbb{C}[y]$ by $Dg = g'$ for all $g \in \mathbb{C}[y]$ (i.e., D is the differentiation operator on $\mathbb{C}[y]$). Given $f = a_0 + a_1x + \cdots + a_nx^n \in H$, let $f(D)$ be the element $a_0 + a_1D + \cdots + a_nD^n$ of $\operatorname{End}\mathbb{C}[y]$. You are given that the bilinear map $\triangleright\colon H \otimes \mathbb{C}[y] \to \mathbb{C}[y]$ defined for $f \in H$, $g \in \mathbb{C}[y]$ by $f \triangleright g = f(D)(g)$, is an action of the algebra H on the vector space $\mathbb{C}[y]$, and you do not have to prove this.

(b) Prove that the action $\triangleright$ makes the algebra $\mathbb{C}[y]$ an H-module algebra.

(c) Given $u \in \mathbb{C}[y]$, define $I_u = \{f \in H : f \triangleright u = 0\}$. Show that I_u is an ideal of the algebra H.

(d) Determine all polynomials $u \in \mathbb{C}[y]$ such that I_u is a coideal of H. *Hint*: if I_u is a coideal, then $\mathbb{C}\langle x\rangle \to \mathbb{C}\langle x\rangle/I_u$ must be a coalgebra morphism.

[20 marks]

2. In this question, Γ denotes a finite cyclic group of order n with identity element e. Let $\mathbb{C}\Gamma$ be the group algebra of Γ, viewed as a Hopf algebra in the standard way, and let $(\mathbb{C}\Gamma)^*$ be the Hopf algebra dual to $\mathbb{C}\Gamma$. Choose a generator γ of Γ so that $e, \gamma, \ldots, \gamma^{n-1}$ is a basis of $\mathbb{C}\Gamma$, and let $\delta_0, \ldots, \delta_{n-1}$ be the dual basis of $(\mathbb{C}\Gamma)^*$.

(a) Formally prove that $\mathbb{C}\langle x \mid x^n = 1\rangle$ is a presentation of the algebra $\mathbb{C}\Gamma$. You need to define an algebra homomorphism from the free tensor algebra $\mathbb{C}\langle x\rangle$ onto $\mathbb{C}\Gamma$ and prove that the kernel of this homomorphism is the ideal generated by $x^n - 1$ in $\mathbb{C}\langle x\rangle$. You can identify $\mathbb{C}\langle x\rangle$ with the polynomial algebra $\mathbb{C}[x]$.

(b) Let ω be any nth root of unity in $\mathbb{C}$. Show that $\sum_{i=0}^{n-1} \omega^i\delta_i$ is a grouplike element of $(\mathbb{C}\Gamma)^*$. Give a brief but convincing explanation why all grouplike elements of $(\mathbb{C}\Gamma)^*$ are of this form.

(c) Deduce from the result of (b) that the group $G((\mathbb{C}\Gamma)^*)$ of grouplike elements in $(\mathbb{C}\Gamma)^*$ is cyclic.

(d) Show: if ω is a primitive nth root of unity, then $R(\omega) = \frac{1}{n}\sum\limits_{a,b=0}^{n-1} \omega^{ab}\gamma^a \otimes \gamma^b$ is a universal R-matrix on $\mathbb{C}\Gamma$. You may assume that $R(\omega)R(\omega^{-1}) = 1 \otimes 1$.

[20 marks]

3. Let the Lie algebra $\mathfrak{sl}_2$ be spanned over $\mathbb{C}$ by X, H, Y with the Lie bracket given by $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$. Let $U = U(\mathfrak{sl}_2)$ be the universal enveloping algebra of $\mathfrak{sl}_2$; consider $\mathfrak{sl}_2$ as a subspace of U. Let $\mathfrak{b}$ be the subspace of $\mathfrak{sl}_2$ spanned by X and H; clearly, $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{sl}_2$.

(a) Explain why the inclusion $\mathfrak{b} \subset \mathfrak{sl}_2$ gives rise to a homomorphism $\psi\colon U(\mathfrak{b}) \to U$ of associative unital algebras. Your answer should refer to the universal mapping property of the universal enveloping algebra. (*Note that ψ, if exists, is uniquely determined by the condition $\psi(z) = z$ for all $z \in \mathfrak{b}$.*)

(b) Prove that the map $\psi\colon U(\mathfrak{b}) \to U$ from part (a) is injective.

(c) Give an example of a coideal K of U such that $U = \psi(U(\mathfrak{b})) \oplus K$; justify your example. Is your choice of K a Hopf ideal? Give brief reasons.

(d) (i) Show that there exists an action $\triangleright$ of the algebra U on the vector space U, which is defined on the generators $z = X, H, Y$ of U by $z \triangleright A = zA - Az$ for $A \in U$. (*Note that the action, if exists, is unique because it is defined on generators. You only need to prove existence.*)

(ii) Let $U^0 = \{A \in U : u \triangleright A = \epsilon(u)A \text{ for all } u \in U\}$ where ϵ is the counit of U. Find $\lambda \in \mathbb{C}$ such that $XY + YX + \lambda H^2 \in U^0$.

[20 marks]

4. Let the Hopf algebra $U_q = U_q(\mathfrak{sl}_2)$ be generated over $\mathbb{C}$ by E, F, K, K^{-1} subject to relations $KK^{-1} = 1$, $K^{-1}K = 1$, $KE = q^2EK$, $KF = q^{-2}FK$, $EF - FE = (q - q^{-1})^{-1}(K - K^{-1})$, where the coalgebra structure is given by $\Delta E = 1 \otimes E + E \otimes K$, $\Delta F = K^{-1} \otimes F + F \otimes 1$, $\epsilon(E) = \epsilon(F) = 0$ and K being grouplike. Let $S\colon U_q \to U_q$ be the antipode. For the purposes of this question, we treat q as a complex number which is neither 0 nor a root of unity. Let $\mathcal{B} = \{E^m K^n F^p : m, p \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}$ be the PBW-type basis of U_q.

(a) Express $S(EKF^2)$ as a linear combination of elements of $\mathcal{B}$.

(b) Let $\rho\colon U_q \to M_{n\times n}(\mathbb{C})$ be a homomorphism of associative unital algebras. Explain why $\widetilde{\rho}\colon U_q \to M_{n\times n}(\mathbb{C})$ defined by $\widetilde{\rho}(h) = \rho(S(h))^T$ is also a homomorphism of associative unital algebras. (Here ^{T} denotes matrix transposition.) Prove that if $n = 1$, then necessarily $\rho = \widetilde{\rho}$.

(c) You are given that $\sigma\colon U_q \to M_{3\times 3}(\mathbb{C})$ is such that $\sigma(E) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q^3 + q & 0 \end{pmatrix}$, $\sigma(K) = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix}$ and σ is a homomorphism of associative unital algebras.

Show that $\sigma(F)$ is uniquely determined by these conditions. Find $\sigma(F)$.

(d) Let $q = e^\hbar$ and let $\mathcal{R} = \exp(\frac{\hbar}{2} H \otimes H) \exp_{q^{-2}}((q - q^{-1}) E \otimes F)$ be the universal R-matrix for $U_q(\mathfrak{sl}_2)$ constructed in the course; in particular, H is a primitive element such that $K = \exp(\hbar H)$. Show, by explicit calculation, that $(\epsilon \otimes \mathrm{id})\mathcal{R} = 1_{U_q}$, briefly explaining the assumptions made in your calculation.

[20 marks]

END OF EXAMINATION PAPER