Problem Sheet 3. Solutions

Question 1

$$O(1,1) = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) \mid X^T E_{1,1} X = E_{1,1} \right\}$$

where $E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The matrix equation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

is equivalent to the following system of 3 scalar equations

$$\begin{cases} a^2 - c^2 = 1 \\ d^2 - b^2 = 1 \\ ab - cd = 0 \end{cases}$$

Solving this system we obtain the following 4 series of solutions:

$$X = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \begin{pmatrix} -\cosh t & \sinh t \\ -\sinh t & \cosh t \end{pmatrix}$$

$$\begin{pmatrix} \cosh t & -\sinh t \\ \sinh t & -\cosh t \end{pmatrix}, \begin{pmatrix} -\cosh t & -\sinh t \\ -\sinh t & -\cosh t \end{pmatrix}$$

These series represent 4 connected components of ~~G~~ $O(1,1)$.

The identity component is $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, and the map

$$t \xrightarrow{\varphi} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \varphi(t)$$

is an isomorphism between $(\mathbb{R}, +)$ and this identity component.

This map is smooth, bijective and satisfies

$$\varphi(t_1 + t_2) = \varphi(t_1)\varphi(t_2) \quad \text{(straightforward verification)}.$$

Question 2

$$O(p,q) = \{ X \in GL(p+q, \mathbb{R}) \mid X^T E_{p,q} X = E_{p,q} \}$$

$$E_{p,q} = \begin{pmatrix} \overbrace{1 \; \ddots \; 1}^{p} & \\ & \overbrace{-1 \; -1 \; \ddots \; -1}^{q} \end{pmatrix}$$

If we write X as $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$, then $X^T E_{p,q} X = E_{p,q}$

can be rewritten as:

$$\begin{cases} X_{11}^T X_{11} - X_{21}^T X_{21} = I_p = \underbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}_{p} \Big\} p \\ X_{22}^T X_{22} - X_{12}^T X_{12} = I_q = \underbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}_{q} \Big\} q \\ X_{11}^T X_{12} - X_{21}^T X_{22} = 0 \end{cases}$$

Consider the first equation: $X_{11}^T X_{11} = I_p + X_{21}^T X_{21}$

Notice that $X_{11}^T X_{11}$ is symmetric. To prove that $\det X_{11} \neq 0$ we use the following facts from Linear Algebra.

> A symmetric $p \times p$ matrix B is called positive definite if for each $v \in \mathbb{R}^p$, $v \neq 0$, we have $v^T B v > 0$. If B is positive definite, then $\det B > 0$

We are going to check that $X_{11}^T X_{11}$ is positive definite.

Indeed, let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix} \neq 0$ be an arbitrary (non zero) vector. Then

$$v^T (X_{11}^T X_{11}) v = v^T (I_p + X_{21}^T X_{21}) v = v^T I_p v + v^T X_{21}^T X_{21} v =$$

$$= v^T v + \underbrace{(X_{21} v)^T}_{u^T} \underbrace{(X_{21} v)}_{u} = \langle v, v \rangle + \langle u, u \rangle > 0$$

as $v \neq 0 \Rightarrow \langle v, v \rangle > 0$ and $\langle u, u \rangle \geq 0$

Thus $X_{11}^T X_{11}$ is positive definite and therefore $\det(X_{11}^T X_{11}) > 0$

On the other hand $\det(X_{11}^T X_{11}) = \det X_{11}^T \cdot \det X_{11} = (\det X_{11})^2$, i.e.

$\det X_{11} \neq 0$.

Similarly, we can prove that $\det X_{22} \neq 0$.

Thus, $O(p,q)$ can be partitioned into 4 pieces:

$G_0 = \{ \det X_{11} > 0, \det X_{22} > 0 \}$ ← identity components.

$G_1 = \{ \ldots > 0, \ldots < 0 \}$

$G_2 = \{ \ldots < 0, \ldots > 0 \}$

$G_3 = \{ \ldots < 0, \ldots < 0 \}$

These subsets are obviously open and disjoint. Moreover, they are not empty because:
$$\begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix} \in G_0, \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \in G_1$$
$$\begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \in G_2, \begin{pmatrix} -I_p & 0 \\ 0 & -I_q \end{pmatrix} \in G_3$$

Remark. In fact, each of G_i is connected, i.e. $O(p,q)$ has exactly 4 connected components.

Question 3 The condition $x_{11} x_{22} \ldots x_{nn} \neq 0$ implies that $x_{ii} \neq 0$ (i.e. either > 0 or < 0) for all $i = 1, \ldots, n$.

Each connected component of T is determined by the signs of x_{ii}. Thus, for each i we can choose the sign of x_{ii} in two different way, so the total number of possibilities is 2^n. Therefore T has 2^n connected components.

Q. 5 See Solution for Problem Sheet (revision 1).

Question 4. See Lecture Notes (Lecture 1)