Counting Finite Magmas

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Abstract

Given a non-negative integer n, we establish a formula for the number of finite magmas on a set with cardinality n up to isomorphism. We then generalize the method to operations with arbitrary finite arity, which yields a corrected version of Harrison's formula. Moreover, we present the cycle index as a helpful tool for practical computations and, based on that, we give a suitable code in Sage with a few generated examples.

1 Introduction

While algebraic structures such as fields or Abelian groups are quite easy to classify in the finite case, a classification becomes more challenging if one drops certain conditions, such as the existence of inverse elements, the commutativity, or the associativity. On the other hand, by dropping all conditions, it is sometimes possible to compute the number of isomorphy classes depending on the cardinality of the underlying set. Since the most common algebraic structures have binary operations, we will first focus on those with one such operation.

Definition 1. A magma is a set M equipped with a map $*: M \times M \to M$, also called (internal binary) operation on M, which we formally write together as a pair (M, *).

Throughout the paper, let n be a non-negative integer and $[n] := \{1, ..., n\}$. Moreover, we consider ([n], *) as a prototype for a finite magma on a set with cardinality n. Note that for n = 0, there is the empty map as the only possible operation. If one wants to write * explicitly, one can do this via an operation table, also called Cayley table:

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A natural question that arises is how many different operations exist on [n]. The answer is easy: For each of the n^2 entries of the Cayley table, there are n different possibilities to assign a value, and thus, there are n^{n^2} different operations in total. We set $0^0 := 1$ in order to be consistent.

Although binary operations, i.e., Cayley tables, can be formally different, it might be possible to obtain one such table by renaming the elements of some other, and vice versa.

Example 2. Consider the Cayley tables

which are, of course, formally different, but by interchanging the elements 1 and 2 in the first table, we get

$$\begin{array}{c|cccc}
*' & 2 & 1 \\
\hline
2 & 2 & 2 \\
1 & 2 & 1
\end{array}$$

which, after rearranging, is obviously the same as the second table. In the same way we also obtain the first table from the second. This means that the two initial tables are structurally the same.

Instead of asking how many different operations exist on [n] in total, we can ask how many *structurally* different operations exist on [n]. To put it another way:

Problem 3. How many structurally different magmas exist on a set with cardinality n?

Theorem 4, which goes back to Harrison [1, Theorem 4], provides an answer; the corresponding sequence is $\underline{A001329}$. Harrison also gives generalizations for finite algebraic structures with an operation of arbitrary finite arity k, which we will call k-magmas. However, his corresponding formula [1, Theorem 5] contains a typing error concerning a factorial symbol as well as a more serious error concerning an exponent. For that reason, we will deduce a corrected version in Section 3; see Theorem 10. This means that sequence $\underline{A001331}$ based on Harrison's formula is incorrect, while sequence $\underline{A091510}$ describing the same (namely 3-magmas) is correct.

Theorem 4 (Harrison [1]). There are

$$\sum_{j \in J_n} \frac{1}{\prod_{i=1}^n i^{j_i} j_i!} \prod_{r,s=1}^n \left(\sum_{d \mid lcm(r,s)} dj_d \right)^{\gcd(r,s)j_r j_s}$$

pairwise non-isomorphic magmas on a set with cardinality n, where J_n denotes the set of all sequences $j = (j_i)_{i=1}^{\infty}$ of non-negative integers such that $\sum_{i=1}^{\infty} ij_i = n$.

2 Proof of Theorem 4

We first formalize what we did in Example 2.

Definition 5. Given two magmas (M, *) and (N, *), a magma homomorphism is a map $\phi \colon M \to N$ such that $\phi(a*a') = \phi(a) \star \phi(a')$ for all $a, a' \in M$. A magma isomorphism is a bijective magma homomorphism. If there exists a magma isomorphism $M \to N$, we say that M and N are isomorphic, which we write as $M \cong N$, or to be more precise, as $(M, *) \cong (N, \star)$; if we additionally have $(M, *) = (N, \star)$, an isomorphism is also called automorphism.

Remark. A bijective magma homomorphism $\phi: M \to N$ already implies that its inverse map $\phi^{-1}: N \to M$ is also a homomorphism and thus, by being bijective as well, an isomorphism of magmas. To see this, let $b, b' \in N$, and let $a, a' \in M$ be the respective preimages under ϕ . Then we have

$$\phi^{-1}(b \star b') = \phi^{-1}(\phi(a) \star \phi(a')) = \phi^{-1}(\phi(a * a')) = a * a' = \phi^{-1}(b) * \phi^{-1}(b').$$

Thus, there exists a magma isomorphism $M \to N$ if and only if there exists a magma isomorphism $N \to M$, which justifies the symmetric notation $M \cong N$.

When are two finite magmas ([n], *) and ([m], *) isomorphic? By definition, this is the case if and only if there is a bijective map $\sigma: [n] \to [m]$ such that

$$\sigma(a * a') = \sigma(a) \star \sigma(a') \qquad \text{for all } a, a' \in [n], \tag{1}$$

which we can, after defining $b := \sigma(a)$ and $b' := \sigma(a')$, equivalently write as

$$b \star b' = \sigma(\sigma^{-1}(b) \star \sigma^{-1}(b')) \qquad \text{for all } b, b' \in [m].$$

In this case, we necessarily have m = n, and $\sigma \in S_n$ is a permutation of [n]. The reason for rewriting (1) into (2) is that, given a magma ([n], *) and a permutation $\sigma \in S_n$, we can define a new operation on [n] by

$$b *_{\sigma} b' := \sigma(\sigma^{-1}(b) * \sigma^{-1}(b')) \qquad \text{for all } b, b' \in [n]. \tag{3}$$

Then the corresponding Cayley table looks as follows:

It is immediately clear through the Cayley table that σ is an isomorphism $([n], *) \to ([n], *_{\sigma})$. Conversely, if $([n], *) \cong ([n], *)$, then we see, by combining (2) and (3), that there exists a permutation $\sigma \in S_n$ such that $\star = *_{\sigma}$. This means that, given a finite magma, we obtain an isomorphic magma by applying a permutation to all entries of the corresponding Cayley table, as seen in Example 2 with the cyclic permutation $\sigma = (1\ 2)$; Bergeron et al. [2] call such a procedure transport of structures. On the other hand, this is the only way to obtain an isomorphic magma from a given finite magma on the same underlying set.

Let Mag_n denote the set of all magmas on [n]. Hence, we can write the subset of Mag_n consisting of magmas that are isomorphic to ([n], *) as

$$\{([n], \star) : ([n], \star) \cong ([n], \star)\} = \{([n], \star_{\sigma}) : \sigma \in S_n\}. \tag{4}$$

We call it isomorphy class of ([n], *) because isomorphic objects form an equivalence class (i.e., isomorphy is an equivalence relation); a likewise suitable term is isomorphy type. Equivalence classes also arise in group actions, and we can see that the right-hand side of (4) looks like the orbit of ([n], *), provided that S_n acts on Mag_n by σ .([n], *) := ([n], * σ). This is indeed the case, which becomes clear via id_{S_n} . = $\mathrm{id}_{\mathrm{Mag}_n}$ and

$$a *_{\sigma \circ \tau} a' = \sigma \circ \tau((\sigma \circ \tau)^{-1}(a) * (\sigma \circ \tau)^{-1}(a'))$$

$$= \sigma(\tau(\tau^{-1}(\sigma^{-1}(a)) * \tau^{-1}(\sigma^{-1}(a'))))$$

$$= \sigma(\sigma^{-1}(a) *_{\tau} \sigma^{-1}(a'))$$

$$= a (*_{\tau})_{\sigma} a' \quad \text{for all } a, a' \in [n] \text{ and } \sigma, \tau \in S_n,$$

and thus, we get $(\sigma \circ \tau)$. $= \sigma$. $\circ \tau$. for all $\sigma, \tau \in S_n$. Our approach so far shows that each orbit of the action of S_n just detected is nothing else than an isomorphy class of magmas on [n]; Bergeron et al. [2] call such an action functoriality of the transports of structures. As we are interested in computing the number of structurally different (i.e., non-isomorphic) magmas, we can thus try to find a way to compute the number of the respective orbits. The following lemma helps us.

Lemma 6 (Cauchy–Frobenius–Burnside [3; 4, Theorem 14.19]). Given a finite group G acting on a finite set X, the number of orbits is

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where $X^g := \{x \in X : g.x = x\}$ denotes the fixed-point set of $g \in G$.

Proof. Let $x \in X$, and let $G.x := \{g.x : g \in G\}$ and $G_x := \{g \in G : g.x = x\}$ denote the orbit and the stabilizer of x, respectively. By the orbit-stabilizer theorem [4, Theorem 14.11], we have $|G| = |G.x||G_x|$, and thus, we get

$$|\{G.x : x \in X\}| = \sum_{x \in X} \frac{1}{|G.x|}$$

$$= \frac{1}{|G|} \sum_{x \in X} |G_x| = \frac{1}{|G|} |\{(g, x) \in G \times X : g.x = x\}| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Transferred to our situation, where $G = S_n$ and $X = \mathrm{Mag}_n$, our goal is to compute $|\mathrm{Mag}_n^{\sigma}|$ for a given permutation $\sigma \in S_n$. First we see that

$$([n], *) \in \operatorname{Mag}_{n}^{\sigma} \iff *_{\sigma} = *$$

$$\iff \forall b, b' \in [n] \colon b *_{\sigma} b' = b * b'$$

$$\iff \forall b, b' \in [n] \colon \sigma(\sigma^{-1}(b) * \sigma^{-1}(b')) = b * b'$$

$$\iff \forall a, a' \in [n] \colon \sigma(a * a') = \sigma(a) * \sigma(a')$$

$$\iff \sigma \text{ is an automorphism of } ([n], *).$$

Let $\sigma \in S_n$ and $([n], *) \in \operatorname{Mag}_n^{\sigma}$. We arrange the rows and columns of the Cayley table of * according to the cycle decomposition of σ , which means that we obtain a grid of rectangular sections.

Example 7. Let n = 5 and $\sigma = (134)(25)$. Then a Cayley table arranged according to σ has the shape

Now consider a section of the Cayley table of * of size $r \times s$:

Because σ is an automorphism, we have

$$\sigma^k(x_{p,q}) = \sigma^k(\sigma^p(a) * \sigma^q(b)) = \sigma^{k+p}(a) * \sigma^{k+q}(b) = x_{(k+p) \bmod r, (k+q) \bmod s}$$

for all $k \in \mathbb{Z}$ and $(p,q) \in \{0,\ldots,r-1\} \times \{0,\ldots,s-1\}$, which implies

$$\sigma^k(x_{p,q}) = x_{p,q} \iff x_{(k+p) \bmod r, (k+q) \bmod s} = x_{p,q} \iff r \mid k \wedge s \mid k \iff \operatorname{lcm}(r,s) \mid k.$$

Thus, each entry $x_{p,q}$ already determines $\operatorname{lcm}(r,s)$ entries, which we can imagine as a diagonal chain directed downwards to the right starting and ending at $x_{p,q}$ (while jumping from one edge to the opposite possibly several times). Because we have $rs = \gcd(r,s) \operatorname{lcm}(r,s)$ entries, we obtain $\gcd(r,s)$ such chains within a rectangular section of size $r \times s$, i.e., we have $\gcd(r,s)$ independent choices of entries for a value assignment.

In order to compute the number of possible value assignments, we first write the cycle type of σ as the sequence $j(\sigma) := (j_i(\sigma))_{i=1}^{\infty}$ meaning that σ , written as a composition of pairwise disjoint cycles, consists of $j_i(\sigma)$ cycles of length i, also called i-cycles. Clearly, we have $j_i(\sigma) = 0$ for all i > n and $\sum_{i=1}^n i j_i(\sigma) = n$, which together is equivalent to $\sum_{i=1}^{\infty} i j_i(\sigma) = n$. Now, because of $\sigma^{\text{lcm}(r,s)}(x_{p,q}) = x_{p,q}$, the value of $x_{p,q}$ must lie in a cycle of σ whose length is a divisor of lcm(r,s). Hence, we have $\sum_{d|\text{lcm}(r,s)} dj_d(\sigma)$ possible value assignments for each diagonal chain and thus

$$\left(\sum_{d|\text{lcm}(r,s)} dj_d(\sigma)\right)^{\gcd(r,s)}$$

possible value assignments for a rectangular section of size $r \times s$. Taking all rectangular sections (according to the cycle type of σ) into account, we obtain

$$|\mathrm{Mag}_n^{\sigma}| = \prod_{r,s=1}^n \left(\sum_{d|\mathrm{lcm}(r,s)} dj_d(\sigma) \right)^{\gcd(r,s)j_r(\sigma)j_s(\sigma)}.$$

Note that this expression does not depend on the explicit form of σ , but only on its cycle type. Together with Lemma 6, we finally obtain the number of isomorphy classes of finite magmas on a set with cardinality n:

$$\frac{1}{n!} \sum_{\sigma \in S_n} \prod_{r,s=1}^n \left(\sum_{d \mid \text{lcm}(r,s)} dj_d(\sigma) \right)^{\gcd(r,s)j_r(\sigma)j_s(\sigma)}. \tag{6}$$

Due to the fact that there exist

$$\frac{n!}{1^{j_1} \cdots n^{j_n} j_1! \cdots j_n!} = \frac{n!}{\prod_{i=1}^n i^{j_i} j_i!}$$

permutations in S_n whose cycle type is $(j_i)_{i=1}^{\infty}$, we can rewrite (6) into

$$\sum_{i \in J_n} \frac{1}{\prod_{i=1}^n i^{j_i} j_i!} \prod_{r,s=1}^n \left(\sum_{d \mid lem(r,s)} dj_d \right)^{\gcd(r,s)j_r j_s}, \tag{7}$$

where J_n denotes the set of all sequences $j = (j_i)_{i=1}^{\infty}$ of non-negative integers such that $\sum_{i=1}^{\infty} i j_i = n$. Thus, we have proved Theorem 4.

3 Generalizations

Let us first recall how we started in Section 2. Given a magma ([n],*) and a permutation $\sigma \in S_n$, we can write the operation $*_{\sigma}$ given by (3) also as

$$*_{\sigma} = \sigma \circ * \circ (\sigma^{-1} \times \sigma^{-1}),$$

where $f \times f$ denotes the componentwise application of a given map f to a pair; clearly, we have $(\sigma \times \sigma)^{-1} = \sigma^{-1} \times \sigma^{-1}$. This means that the diagram

$$\begin{array}{ccc} [n] \times [n] & \stackrel{*}{\longrightarrow} & [n] \\ \\ \sigma \times \sigma \downarrow & & \downarrow \sigma \\ \\ [n] \times [n] & \stackrel{*}{\longrightarrow} & [n] \end{array}$$

commutes, and thus, the permutation σ is an isomorphism $([n], *) \to ([n], *_{\sigma})$. We generalize this phenomenon to operations with higher arity. For that, we introduce analogous notions for the respective algebraic structures.

Definition 8. Let k be a non-negative integer. A k-magma is a set M equipped with a map $*: M^k \to M$, also called (internal k-ary) operation on M, which we formally write together as a pair (M, *).

This means that 2-magmas are magmas in the usual sense. Note that a 0-magma is nothing else than a set together with a constant (from that set), and thus, the set must necessarily be non-empty. In general, for a finite set of n elements, there are n^{n^k} formally different operations in total, which we can imagine explicitly visualized as k-dimensional hypercubes with side length n and thus n^k entries. We get the following analogous definition.

Definition 9. Given two k-magmas (M, *) and (N, *), a k-magma homomorphism is a map $\phi: M \to N$ such that

$$\phi(*(a_1,\ldots,a_k)) = \star(\phi(a_1),\ldots,\phi(a_k)) \quad \text{for all } a_1,\ldots,a_k \in M.$$
 (8)

The remaining notions of Definition 5 are defined analogously.

Let $f^{\times k} := f \times \cdots \times f$ denote the componentwise application of a given map f to a k-tuple. Then we can write the homomorphism condition (8) also as

$$\phi \circ * = \star \circ \phi^{\times k}.$$

For composable maps f and g, we clearly have $(f \circ g)^{\times k} = f^{\times k} \circ g^{\times k}$. Also, if f is bijective, we clearly have $(f^{\times k})^{-1} = (f^{-1})^{\times k}$. Now, given a non-negative integer k, a k-magma ([n], *), and a permutation $\sigma \in S_n$, we analogously define

$$*_{\sigma} := \sigma \circ * \circ (\sigma^{-1})^{\times k}.$$

This means that the diagram

$$\begin{array}{ccc}
[n]^k & \xrightarrow{*} & [n] \\
\sigma^{\times k} \downarrow & & \downarrow \sigma \\
[n]^k & \xrightarrow{*} & [n]
\end{array}$$

commutes, and thus, the permutation σ is an isomorphism $([n], *) \to ([n], *_{\sigma})$.

Let k-Mag_n denote the set of all k-magmas on [n]. The next analogous step is to see again that $\sigma.([n],*) := ([n],*_{\sigma})$ defines a group action of S_n on k-Mag_n. Indeed, we have $\mathrm{id}_{S_n}.=\mathrm{id}_{k\text{-Mag}_n}$ and

$$*_{\sigma \circ \tau} = \sigma \circ \tau \circ * \circ ((\sigma \circ \tau)^{-1})^{\times k}
= \sigma \circ \tau \circ * \circ (\tau^{-1})^{\times k} \circ (\sigma^{-1})^{\times k}
= \sigma \circ *_{\tau} \circ (\sigma^{-1})^{\times k}
= (*_{\tau})_{\sigma} \quad \text{for all } \sigma, \tau \in S_n,$$

and thus, we get $(\sigma \circ \tau)$. = σ . $\circ \tau$. for all σ , $\tau \in S_n$. This means that we can apply Lemma 6, and for that, we compute the cardinality of the fixed-point set k-Mag $_n^{\sigma}$ for a given permutation $\sigma \in S_n$. We get an analogous equivalence:

$$([n], *) \in k\text{-Mag}_n^{\sigma} \iff *_{\sigma} = *$$

$$\iff \sigma \circ * \circ (\sigma^{-1})^{\times k} = *$$

$$\iff \sigma \circ * = * \circ \sigma^{\times k}$$

$$\iff \sigma \text{ is an automorphism of } ([n], *).$$

Let $\sigma \in S_n$ and $([n], *) \in k\text{-Mag}_n^{\sigma}$. We continue analogously as in Section 2 by arranging and dividing the respective k-dimensional operation hypercube of * into hyperrectangular sections, i.e., k-dimensional boxes, according to the cycle type of σ . Considering such a box of size $r_1 \times \cdots \times r_k$, we analogously obtain due to the automorphism property of σ that each entry already determines $lcm(r_1, \ldots, r_k)$ entries, and thus, we have

$$\frac{r_1\cdots r_k}{\mathrm{lcm}(r_1,\ldots,r_k)}$$

independent choices of entries for a value assignment. In order to be consistent, we set $lcm(r_1, \ldots, r_k) := 1$ and $lcm(r_1, \ldots, r_k) := r_1$ for the cases k = 0 and k = 1, respectively. Note that for k > 2, we have

$$\frac{r_1\cdots r_k}{\mathrm{lcm}(r_1,\ldots,r_k)}\neq \gcd(r_1,\ldots,r_k)$$

in general, as we can see by the example $(r_1, \ldots, r_k) = (2, \ldots, 2)$, and therefore, we cannot take the greatest common divisor as Harrison mistakenly did [1, Theorem 5]. The remaining analogous steps are to put the boxes together and eventually apply Lemma 6. First we get

$$|k\operatorname{-Mag}_{n}^{\sigma}| = \prod_{(r_{1},\dots,r_{k})\in[n]^{k}} \left(\sum_{d|\operatorname{lcm}(r_{1},\dots,r_{k})} dj_{d}(\sigma)\right)^{\frac{r_{1}\cdots r_{k}}{\operatorname{lcm}(r_{1},\dots,r_{k})}j_{r_{1}}(\sigma)\cdots j_{r_{k}}(\sigma)}.$$

Now, by applying the lemma and using the same notation as in (7), we obtain the following result.

Theorem 10. There are

$$\sum_{j \in J_n} \frac{1}{\prod_{i=1}^n i^{j_i} j_i!} \prod_{(r_1, \dots, r_k) \in [n]^k} \left(\sum_{d \mid \text{lcm}(r_1, \dots, r_k)} dj_d \right)^{\frac{r_1 \cdots r_k}{\text{lcm}(r_1, \dots, r_k)} j_{r_1} \cdots j_{r_k}}$$
(9)

pairwise non-isomorphic k-magmas on a set with cardinality n, where J_n denotes the set of all sequences $j = (j_i)_{i=1}^{\infty}$ of non-negative integers such that $\sum_{i=1}^{\infty} ij_i = n$.

4 Using the cycle index

Riedel [5] detected a helpful tool for practical computations of (7), namely the cycle index, which is a formal polynomial whose monomials represent the cycle types that arise in a given permutation group, i.e., in a subgroup of the group S_X of all permutations of a given finite set X. If a group G acts faithfully on X, then we can think of G as a permutation group, namely as the subgroup $G := \{g : g \in G\}$ embedded in S_X . It is important to mention that the cycle types arising in this way depend not only on G but in fact on its action on X. Analogously to the previous sections, we write $j(g) = (j_i(g))_{i=1}^{\infty}$ for the cycle type of the permutation $g \in S_X$ where $g \in G$. Also, like in Theorems 4 and 10, let J_n denote the set of all sequences $j = (j_i)_{i=1}^{\infty}$ of non-negative integers such that $\sum_{i=1}^{\infty} ij_i = n$.

Definition 11. Given a finite group G acting on a finite set X, the cycle index of G with respect to its action on X is defined as the polynomial

$$\frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{|X|} t_i^{j_i(g.)} \in \mathbb{Q}[t_1, \dots, t_{|X|}].$$

This means that the monomial $\prod_{i=1}^{|X|} t_i^{j_i(g,i)}$ represents the cycle type $(j_i(g,i))_{i=1}^{\infty}$, and its coefficient indicates the corresponding relative frequency among all elements in G.. Transferred to our case, where $G = S_n$ plays the key role, we first obtain

$$Z_n := \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^n t_i^{j_i(\sigma)} = \sum_{j \in J_n} \frac{1}{\prod_{i=1}^n i^{j_i} j_i!} \prod_{i=1}^n t_i^{j_i} \in \mathbb{Q}[t_1, \dots, t_n]$$
 (10)

as the cycle index of S_n with respect to its natural action on [n], which is given by $\sigma a = \sigma(a)$.

Example 12. We have $Z_3 = \frac{1}{6}(t_1^3 + 3t_1t_2 + 2t_3)$ because S_3 consists of one identity permutation ([1³2⁰3⁰]), three transpositions ([1¹2¹3⁰]), and two 3-cycles ([1⁰2⁰3¹]).

A nice property and computational advantage of Z_n is that it satisfies a recursion.

Lemma 13 (Pletsch [6]). We have $Z_0 = 1$ and

$$Z_n = \frac{1}{n} \sum_{i=1}^n t_i Z_{n-i} \qquad \text{for all } n > 0.$$

Proof. If n = 0, then we have $|S_n| = |J_n| = 1$, so either sum in (10) runs over one element yielding the empty product 1. If n > 0, then we first note that, like Z_n , the polynomial $n!Z_n$ is also related to S_n (with respect to the natural action), but with the difference that its coefficients indicate absolute frequencies instead of relative ones. Now, if we choose an element $m \in [n]$, then there are $\frac{(n-1)!}{(n-i)!}$ i-cycles in S_n containing m. Each such cycle that appears in a permutation in S_n contributes with a factor t_i to the corresponding monomial of Z_n , while the remaining contribution comes from a cycle type of a permutation in S_{n-i} , i.e., a monomial of Z_{n-i} . Thus, the polynomial

$$\frac{(n-1)!}{(n-i)!}t_i(n-i)!Z_{n-i} = (n-1)!t_iZ_{n-i},$$

with absolute frequencies as coefficients, is related to the subset of S_n consisting of permutations that have an *i*-cycle containing m. By summing this expression over all i from 1 to n, we cover all permutations

in S_n eventually because, given any such permutation, the element m must lie in a cycle whose length is between 1 and n. Thus, we obtain

$$\sum_{i=1}^{n} (n-1)! t_i Z_{n-i} = n! Z_n,$$

and dividing by n! on both sides finishes the proof.

Let us assume that we have computed a specific Z_n , possibly via the recursion given in the previous lemma. How can we use it to compute (7)? First we can map each monomial $\prod_{i=1}^n t_i^{j_i}$ of Z_n to $\prod_{r,s=1}^n t_{\mathrm{lcm}(r,s)}^{\mathrm{gcd}(r,s)j_rj_s}$ and obtain

$$Z_n^{[2]} := \sum_{j \in J_n} \frac{1}{\prod_{i=1}^n i^{j_i} j_i!} \prod_{r,s=1}^n t_{\text{lcm}(r,s)}^{\gcd(r,s) j_r j_s} \in \mathbb{Q}[t_1, \dots, t_{n^2}].$$

In a similar way as in Section 2 starting from (5), we can see that $Z_n^{[2]}$ is the cycle index of S_n with respect to its action on $[n]^2$ given by $\sigma(a, a') = (\sigma(a), \sigma(a'))$, which looks technically the same as the automorphism property $\sigma(a * a') = \sigma(a) * \sigma(a')$ there. The main difference is that in the case of the action on $[n]^2$, one does not have to deal with value assignments, and hence, a chain of length $\operatorname{lcm}(r, s)$ leads to $t_{\operatorname{lcm}(r,s)}$.

Remark. For $n \leq 4$ and $n \geq 4$, the highest index of an indeterminate appearing in $\mathbb{Z}_n^{[2]}$ is bounded by n and $\left\lfloor \frac{n^2}{4} \right\rfloor$, respectively.

Example 14. We have $Z_3^{[2]} = \frac{1}{6}(t_1^9 + 3t_1t_2^4 + 2t_3^3)$ because the identity permutation, a transposition, and a 3-cycle lead to a permutation fixing nine pairs, a permutation fixing one pair and having four transpositions of pairs, and a permutation consisting of three 3-cycles of pairs, respectively.

Finally, by plugging $\sum_{d|\text{lcm}(r,s)} dj_d$ into each $t_{\text{lcm}(r,s)}$ of $Z_n^{[2]}$, we obtain (7). Of course, we can generalize the method just described by defining

$$Z_n^{[k]} := \sum_{j \in J_n} \frac{1}{\prod_{i=1}^n i^{j_i} j_i!} \prod_{\substack{(r_1, \dots, r_k) \in [n]^k}} t_{\text{lcm}(r_1, \dots, r_k)}^{\frac{r_1 \cdots r_k}{\text{lcm}(r_1, \dots, r_k)} j_{r_1} \cdots j_{r_k}} \in \mathbb{Q}[t_1, \dots, t_{n^k}]$$

for all non-negative integers k, which, analogously, is the cycle index of S_n with respect to its action on $[n]^k$ given by $\sigma = \sigma^{\times k}$. Hence, by plugging $\sum_{d \mid \text{lcm}(r_1, \dots, r_k)} dj_d$ into each $t_{\text{lcm}(r_1, \dots, r_k)}$ of $Z_n^{[k]}$, we obtain (9). Note that in the case k = 0, we get $Z_n^{[k]} = t_1$ for all non-negative integers n, but the corresponding group action of S_n is not faithful for n > 1. This means that in order to plug in the respective values, we have to view t_1 still decomposed into summands that correspond to the cycle types of S_n . In the case n = 3, for example, we have $Z_n^{[0]} = t_1 = \frac{1}{6}(t_1 + 3t_1 + 2t_1)$, and now we can plug into each summand the corresponding j_1 (which eventually yields 1).

Finally, we present a suitable code in Sage and generate a few terms of the corresponding sequences for k-magmas in the cases $k \in \{0, ..., 4\}$:

sage: Pol.<t> = InfinitePolynomialRing(QQ)

....: @cached_function

....: def Z(n):

....: if n==0: return Pol.one()

```
return sum(t[k]*Z(n-k)) for k in (1..n)/n
. . . . :
....: def magmas(n,k): # number of isomorphy classes of k-magmas on a set of n elements
           P = Z(n)
. . . . :
           q = 0
. . . . :
            coeffs = P.coefficients()
. . . . :
            count = 0
. . . . :
           for m in P.monomials():
. . . . :
                p = 1
. . . . :
                V = m.variables()
. . . . :
                T = cartesian_product(k*[V])
. . . . :
                for t in T:
. . . . :
                     r = [Pol.varname_key(str(u))[1] for u in t]
. . . . :
                     j = [m.degree(u) for u in t]
. . . . :
                     D = 0
. . . . :
                     lcm_r = lcm(r)
. . . . :
                     for d in divisors(lcm_r):
. . . . :
                          try: D += d*m.degrees()[-d-1]
. . . . :
                          except: break
. . . . :
                     p *= D^(prod(r)/lcm_r*prod(j))
. . . . :
                q += coeffs[count]*p
. . . . :
                count += 1
. . . . :
           return q
. . . . :
sage: [magmas(n,0) for n in (0..10)]
                                      [0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
sage: [magmas(n,1) for n in (0..15)]
                                            # A001372
            [1, 1, 3, 7, 19, 47, 130, 343, 951, 2615, 7318, 20491, 57903, 163898, 466199, 1328993]
sage: [magmas(n,2) for n in (0..6)]
                                              # A001329
             [1, 1, 10, 3330, 178981952, 2483527537094825, 14325590003318891522275680]
sage: [magmas(n,3) for n in (0..4)]
                                             # incorrectly A001331, correctly A091510
                [1, 1, 136, 1270933717887, 14178431955039102651224805804387336192]
sage: [magmas(n,4) for n in (0..3)]
                       [1, 1, 32896, 73904414707172961664120440374585941530]
In the case of a unary operation, we obtain sequence A001372. If we think of the above sequences as
```

rows of a matrix, then fixing the cardinality instead yields the corresponding columns:

```
sage: [magmas(0,k) for k in (0..10)]
                                      [0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
sage: [magmas(1,k) for k in (0..10)]
```

sage: [magmas(2,k) for k in (0..7)] # A191363 for k = 1,...,5

[1, 3, 10, 136, 32896, 2147516416, 9223372039002259456, 170141183460469231740910675752738881536]

sage: [magmas(3,k) for k in (0..4)]

[1, 7, 3330, 1270933717887, 73904414707172961664120440374585941530]

sage: [magmas(4,k) for k in (0..3)]

[1, 19, 178981952, 14178431955039102651224805804387336192]

In the case of cardinality 2, we obtain a sequence whose k-th terms coincide with sequence <u>A191363</u> for $k \in \{1, ..., 5\}$.

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(Concerned with sequences <u>A001329</u>, <u>A001331</u>, <u>A091510</u>, <u>A001372</u>, and <u>A191363</u>.)