Model answers to Week 04 review worksheet — exercises for §4

Part A. Exercises for interactive discussion

E4.1 (grouplike elements of a Hopf algebra form a group) (a) If H is a Hopf algebra, show that the set $G(H)$ of grouplike elements of H is a group where the operation is multiplication in H.

(b) Let $\mathbb{C}\Gamma$ be a group algebra of a group Γ, viewed as a Hopf algebra. What is the group $G(\mathbb{C}\Gamma)$?

Answer to E4.1. (a) Recall that the notion of *grouplike* is defined in terms of the coproduct and counit only and makes sense for any coalgebra. However, in this exercise we have to calculate products of grouplikes; thus, we need both coproduct and product on H.

Let H be a Hopf algebra with coproduct Δ and counit ϵ. We will show that $G(H) = \{g \in H : g \text{ is grouplike}\}$ is closed with respect to the product in H and forms a group.

Closure: we need to show that the product of grouplikes is a grouplike.

Recall that by bialgebra axioms, Δ and ϵ of H are multiplicative, meaning that $\Delta(gh) = (\Delta g)(\Delta h)$ as elements of the algebra $H \otimes H$, and $\epsilon(gh) = \epsilon(g)\epsilon(h)$ in $\mathbb{C}$. Let now $g, h \in G(H)$. Then by definition of grouplike,

$$\Delta g = g \otimes g, \qquad \Delta h = h \otimes h, \qquad \epsilon(g) = \epsilon(h) = 1$$

and so, multiplying in $H \otimes H$ and in $\mathbb{C}$, respectively, we have

$$\Delta(gh) = (\Delta g)(\Delta h) = (g \otimes g)(h \otimes h) = gh \otimes gh, \qquad \epsilon(gh) = \epsilon(g)\epsilon(h) = 1 \cdot 1 = 1.$$

Hence gh verifies the definition of grouplike, that is, $gh \in G(H)$.

Associativity of multiplication in $G(H)$: since the product in $G(H)$ is induced from H, it is associative.

The identity element: Let 1_H be the identity element of H. We show that $1_H \in G(H)$.

Since $\Delta\colon H \to H \otimes H$ is a homomorphism of associative algebras with identity, and since $1_H \otimes 1_H$ is the identity element of $H \otimes H$ by definition of the algebra structure on $H \otimes H$, we have $\Delta 1_H = 1_H \otimes 1_H$. Also since $\epsilon\colon H \to \mathbb{C}$ is also a homomorphism, $\epsilon(1_H) = 1$. Hence 1_H is grouplike, and plays the role of multiplicative identity of $G(H)$.

The inverse: let S be the antipode of H. We show that $S(g) = g^{-1}$ for grouplike g.

By definition of grouplike and the antipode law,

$$1_H = \epsilon(g)1_H = S(g_{(1)})g_{(2)} = S(g)g, \qquad 1_H = \epsilon(g)1_H = g_{(1)}S(g_{(2)}) = gS(g).$$

Hence $S(g)$ is the inverse of g for all $g \in G(H)$.

(b) By definition of the coproduct and counit on $\mathbb{C}\Gamma$, every element of Γ is grouplike in $\mathbb{C}\Gamma$. Thus, $\Gamma \subseteq G(\mathbb{C}\Gamma)$.

Now recall that by E3.2, $G(\mathbb{C}\Gamma)$ is a linearly independent set. A basis Γ of $\mathbb{C}\Gamma$ cannot be contained in a strictly larger linearly independent set. We conclude that $G(\mathbb{C}\Gamma) = \Gamma$.

E4.2 (primitive elements) An element x of a Hopf algebra H is called **primitive** if $\Delta x = x \otimes 1 + 1 \otimes x$. (Here $1 = 1_H$.) What is true? (*select one or more*):

A) 0 is primitive
B) 1 is primitive
C) if x, y are primitive, then $x + y$ is primitive
D) if x, y are primitive, then xy is primitive
E) an element of H cannot be grouplike <u>and</u> primitive

Answer to E4.2. A),C) are true: $P(H)$ is a vector space. B) is false because $\Delta 1 = 1 \otimes 1 \neq 2(1 \otimes 1)$.

D) is, in general, false, but giving a counterexample requires at least a Hopf algebra H with $P(H) \neq \{0\}$. We can use the free algebra, $H = \mathbb{C}\langle X\rangle$, where $X \neq \emptyset$. Let $x \in X$, then x is primitive by definition of coproduct on

$\mathbb{C}\langle X\rangle$. Yet we have

$$\Delta(x^2) = (\Delta x)^2 = (x \otimes 1 + 1 \otimes x)^2 = x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2,$$

where the middle term is nonzero (because $x \otimes x$ is an element of a standard basis of $\mathbb{C}\langle X\rangle^{\otimes 2}$), showing that x^2 is not primitive.

E) is false: by Lemma 3.8, $\epsilon(x) = 0$ for all primitive x. Yet $\epsilon(g) = 1$ for all grouplike g, by definition of grouplike.

E4.3 (a) Let $P(H)$ be the set of primitive elements of H. Prove: if $x, y \in P(H)$ then $xy - yx \in P(H)$.

(b) We know from E4.1 that the set $G(H)$ of grouplikes in H is a group under multiplication. What kind of algebraic structure does (a) impose on the space $P(H)$ of primitives in H?

(c) Show: $\epsilon(x) = 0$ and $S(x) = -x$ if x is primitive.

Answer to E4.3. (a) By definition of "primitive", $x, y \in P(H)$ means

$$\Delta x = x \otimes 1 + 1 \otimes x, \qquad \Delta y = y \otimes 1 + 1 \otimes y.$$

By multiplicativity of Δ and by definition of multiplication in $H \otimes H$, it follows that

$$\Delta(xy) = (\Delta x)(\Delta y) = xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy,$$
$$\Delta(yx) = (\Delta y)(\Delta x) = yx \otimes 1 + y \otimes x + x \otimes y + 1 \otimes yx.$$

Subtracting, we can see that the intermediate terms cancel:

$$\Delta(xy) - \Delta(yx) = xy \otimes 1 - yx \otimes 1 + 1 \otimes xy - 1 \otimes yx,$$

which by linearity rewtites as

$$\Delta(xy - yx) = (xy - yx) \otimes 1 + 1 \otimes (xy - yx).$$

Hence by definition of "primitive", $xy - yx \in P(H)$.

(b) On any associative algebra A one can introduce the bilinear operation $[-,-]_{\text{comm}} \colon A \otimes A \to A$ by $[a,b]_{\text{comm}} = ab - ba$ for $a, b \in A$. This operation is called **the commutator bracket.** It is a particular case of a **Lie bracket** because it satisfies the antisymmetry axiom $[a,a] = 0$ and the Jacobi identity $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ for all a, b, c. A vector space equipped with a Lie bracket is called a **Lie algebra.**

By part (a), the subspace $P(H)$ of a Hopf algebra H is a Lie algebra with respect to the commutator bracket.

Note, however, that $P(H)$ can be $\{0\}$. In fact, $P(H)$ is the zero space for any finite-dimensional Hopf algebra H over a field of characteristic 0. This will be shown in the answer to E5.4.

(c) For $\epsilon(x) = 0$, see Lemma 3.8. By antipode law, $S(x)1 + S(1)x = \epsilon(x) = 0$, but $S(1) = 1$ as S is an antihomomorphism, so $S(x) + x = 0$ forcing $S(x) = -x$.

E4.4 Write the Hopf algebra axioms as commutative diagrams.

Moreover, use the **graphical tensor calculus** to produce the diagrams (*to be discussed in the interactive session*).

Answer to E4.4. The algebra axioms (associative law, unit law) were written as commutative diagrams in Remark 2.4, and coalgebra axioms, after Definition 3.1.In this exercise, we are interested in Hopf algebra axioms which link the algebra and coalgebra structure.

The following two diagrams represent the axiom that the coproduct and the counit are multiplicative. We especially note that multiplicativity of the coproduct involves the multiplication map on the tensor product algebra $H \otimes H$, whose definition (2.18) forces the use of the flip map $\tau \colon H \otimes H \to H \otimes H$ in the first diagram in Figure 4.1.

$$
\begin{tikzcd}
H \otimes H \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "m"'] & H \otimes H \otimes H \otimes H \arrow[d, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \\
H \arrow[d, "\Delta"'] & H \otimes H \otimes H \otimes H \arrow[dl, "m\otimes m"] \\
H \otimes H &
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H \arrow[d, "m"'] \arrow[dr, "\epsilon\otimes\epsilon"] & \\
H \arrow[d, "\epsilon"'] & \mathbb{C} \otimes \mathbb{C} \arrow[dl, "\cong"] \\
\mathbb{C} &
\end{tikzcd}
$$

Figure 4.1: *the commutative diagrams representing the axiom "Δ and ϵ are algebra homomorphisms"*

The commutative diagram for the antipode law is given in Definition 4.6.

Graphical tensor calculus is a language for representing linear maps between tensor products of vector spaces by means of two-dimensional diagrams. We will use graphical tensor calculus of the same style as in Majid 2002.

In graphical tensor calculus, a linear map f between a tensor product $V_1 \otimes V_2 \otimes \ldots \otimes V_m$ of m vector spaces and a tensor product $W_1 \otimes W_2 \otimes \ldots \otimes W_n$ of n vector spaces is represented by a box with m inputs and n outputs. Diagrams are read **top to bottom,** so

- the inputs are drawn as strands which come into the box from the top, and outputs are strands which come out from the bottom;
- composition of maps is represented by connecting outputs of the first map to inputs of the second map;
- $f \otimes g$ is represented by placing the diagram for g to the right of the diagram for f.

For visual clarity, some maps are drawn as specific shapes rather than a generic box. Importantly, the identity map $\mathrm{id}_V\colon V \to V$ is just a vertical strand, and the flip map τ is a crossing between the two strands. Thus, the diagram in Figure 4.2 features a map $f\colon V_1 \otimes V_2 \otimes V_4 \to W_1 \otimes X$ represented by a box with three inputs and two outputs, and a map $g\colon X \otimes V_3 \to W_2$ represented by a box with two inputs and one output. (Here X is some vector space which does not feature among the inputs nor outputs of the whole diagram.) The map represented by the diagram is $(\mathrm{id} \otimes g)(f \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \tau)$ from $V_1 \otimes V_2 \otimes V_3 \otimes V_4$ to $W_1 \otimes W_2$.

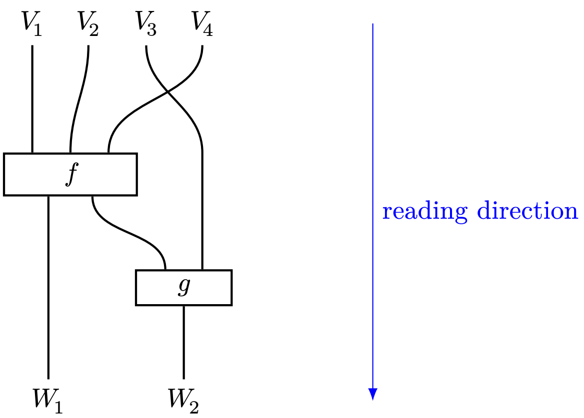

Figure 4.2: *graphical tensor calculus representation of* $(\mathrm{id} \otimes g)(f \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \tau)$

Let us now draw graphical tensor calculus diagrams which will represent the Hopf algebra axioms. We will use special shapes for the Hopf algebra structure maps, shown in Figure 4.3. These maps are between tensor powers of H. Note that the shape for the unit $\eta\colon \mathbb{C} \to H$ has **no inputs:** this is because the domain of η is the ground field, identified with $H^{\otimes 0}$. Likewise, the shape for ϵ has **no outputs.**

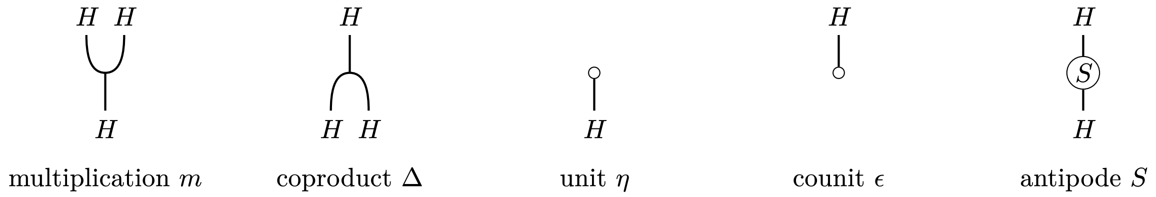

Figure 4.3: *shapes used for the structure maps of a Hopf algebra in graphical tensor calculus*

Axioms that require equality of two or more linear maps are represented in graphical tensor calculus by putting the = sign between diagrams. Thus, the **algebra axioms** (associative law and unit law) are given in Figure 4.4. Turning these diagrams upside down gives the **coalgebra axioms** (coassociative law and counit law), also shown in Figure 4.4.

Figure 4.4: *the associative law, unit law, coassociative law and counit law*

The remaining axioms, which involve both m and Δ, are shown in Figure 4.5.

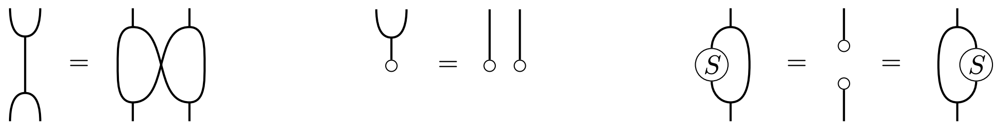

Figure 4.5: *multiplicativity of Δ and ϵ; the antipode law*

E4.5 (tensor product of modules; the dual module) Our goal is to show that the class of modules over a Hopf algebra H is closed under tensor products and duals.

(a) Given an algebra A and A-modules V and W, define an $A \otimes A$-module structure on $V \otimes W$.

(b) Let H be a bialgebra. Use the coproduct $\Delta \colon H \to H \otimes H$ and (a) to make $V \otimes W$ an H-module whenever V and W are.

(c) If V is an A-module, show that $\triangleleft \colon V^* \otimes A \to V^*$ where, for $\phi \in V^*$, $\phi \triangleleft a$ is the linear functional on V defined by $\langle \phi \triangleleft a, v \rangle = \langle \phi, a \triangleright v \rangle$, is a *right action* of A on V^*. (Write down the definition of a right action.)

(d) If $\triangleleft$ is a right action of a Hopf algebra H, show that $\triangleright$ defined by the rule "$h\triangleright = \triangleleft Sh$" where $S \colon H \to H$ is the antipode, is a (left) action. Conclude from (c) that if V is an H-module then so is V^*.

Answer to E4.5. Postponed to week 05, see E5.1.

Part B. Extra exercises

E4.6 (uniqueness of antipode) Let H be a bialgebra. Show that an antipode $S \colon H \to H$, if exists, is unique. *Hint:* let S, S' be two antipodes; compute $S(a_{(1)})a_{(2)}S'(a_{(3)})$ in two ways.

Answer to E4.6. We have $\left(S(a_1)a_{(2)}\right) S'(a_{(3)}) = \epsilon(a_{(1)})S'(a_{(2)}) = S'(a)$ by the antipode law for S. In the same way $S(a_1)\left(a_{(2)}S'(a_{(3)})\right) = S(a)$. By associativity, the two expressions must be equal, so $S'(a) = S(a)$. Here $a \in H$ is arbitrary so $S' = S$.

E4.7 Let H be a Hopf algebra which acts via $\triangleright$ on an H-module algebra A. Prove: if $x \in P(H)$ is primitive, then $x \triangleright (ab) = (x \triangleright a)b + a(x \triangleright b)$ for all $a, b \in A$ (*the Leibniz law*).

Answer to E4.7. By definition of a covariant action (or, the same, H-module algebra)

$$x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b).$$

Since $x_{(1)} \otimes x_{(2)} = x \otimes 1 + 1 \otimes x$, where $1 = 1_H$, the above expression is really a sum of two expressions,

$$(x \triangleright a)(1 \triangleright b) + (1 \triangleright a)(x \triangleright b) = (x \triangleright a)b + a(x \triangleright b),$$

for $1 \triangleright a = a$ and $1 \triangleright b = b$ by definition of an action.

E4.8 Recall that the dual space to a coalgebra is an algebra, and the dual space to a finite-dimensional algebra is a coalgebra. Extend this to show: if H is a finite-dimensional Hopf algebra, then H^* is also a Hopf algebra.

Answer to E4.8. For a vector space H, the space $H^* \otimes H^*$ is canonically a subspace of $(H \otimes H)^*$ via

$$\phi, \psi \in H^*, \qquad \phi \otimes \psi \colon H \otimes H \to \mathbb{C}, \qquad \langle x \otimes y, \phi \otimes \psi \rangle := \langle x, \phi \rangle \langle y, \psi \rangle \quad \forall x, y \in H.$$

If $\dim H < \infty$ then $\dim H^* \otimes H^* = \dim(H \otimes H)^* = (\dim H)^2 < \infty$ and so $H^* \otimes H^* = (H \otimes H)^*$.

Let $(H, m, \eta, \Delta, \epsilon, S)$ be a finite-dimensional Hopf algebra. We claim that $(H^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ is also a Hopf algebra.

We already know that $(H^*, \Delta^*, \epsilon^*)$ is an algebra and that (H^*, m^*, η^*) is a coalgebra. Namely, the coassociative law and the counit law for (H, Δ, ϵ) imply the associative law and the identity law for $(H^*, \Delta^*, \epsilon^*)$, as explained in the pre-recorded lectures. Similarly, the associative law and the identity law for (H, m, η) imply the coassociative law and the counit law for (H, Δ^*, η^*).

Let us check the remaining axioms of a Hopf algebra which prescribe compatibility between the coalgebra and algebra structure, and the antipode law.

The coproduct $m^* \colon H^* \to H^* \otimes H^*$ is an algebra homomorphism. Let us write the axiom which says that $\Delta \colon H \to H \otimes H$ is multiplicative:

$$\Delta(xy) = x_{(1)} y_{(1)} \otimes x_{(2)} y_{(2)}.$$

Thus, applying m and then Δ must produce the same effect as

- applying Δ to x and to y, which produces $x_{(1)} \otimes x_{(2)} \otimes y_{(1)} \otimes y_{(2)}$;
- swapping $x_{(2)}$ with $y_{(1)}$ which produces $x_{(1)} \otimes y_{(1)} \otimes x_{(2)} \otimes y_{(2)}$;
- multiplying the first two legs as well as the last two legs, which produces $x_{(1)} y_{(1)} \otimes x_{(2)} y_{(2)}$.

In other words, the axiom which says that Δ is multiplicative is written

$$\Delta m = (m \otimes m)(\mathrm{id} \otimes \tau_{H,H} \otimes \mathrm{id})(\Delta \otimes \Delta),$$

where $\tau_{H,H} \colon H \otimes H \to H \otimes H$ is the flip map, $\tau_{H,H}(a \otimes b) = b \otimes a$.

Both sides are linear maps $H \otimes H \to H \otimes H$. Let us take the adjoint map of both sides, producing two equal linear maps $H^* \otimes H^* \to H^* \otimes H^*$. By E1.4, we need to reverse the order of factors in the composition:

$$m^* \Delta^* = (\Delta^* \otimes \Delta^*)(\mathrm{id} \otimes \tau_{H,H}^* \otimes \mathrm{id})^*(m^* \otimes m^*).$$

We note that $(\alpha \otimes \beta)^* = \alpha^* \otimes \beta^*$ for linear maps α, β between finite-dimensional spaces, and that $\tau_{H,H}^* = \tau_{H^*,H^*}$ (*exercise*). We can now see that the coproduct m^* on H^* satisfies the multiplicativity axiom, with respect to the product Δ^*.

There is the second axiom which is required for m^* to be an algebra homomorphism is that it must map the identity element to the identity element. In the algebra H, this axiom is $\Delta 1_H = 1_H \otimes 1_H$, or, in terms of linear maps, $\Delta\eta = \eta \otimes \eta$. The version for m^* must, of course, involve m^* and the unit map ϵ^* of the algebra H^*. This gives us a hint that we should dualise the axiom involving m and ϵ: namely, the multiplicativity of ϵ axiom,

$$\epsilon(xy) = \epsilon(x)\epsilon(y) \quad \text{or, in terms of linear maps,} \quad \epsilon m = \epsilon \otimes \epsilon.$$

Taking the adjoints, we obtain

$$m^*\epsilon^* = \epsilon^* \otimes \epsilon^*,$$

which says exactly that m^* takes the identity element of H^* to the identity element of $H^* \otimes H^*$, as required.

The counit $\eta^*\colon H^* \to \mathbb{C}$ is an algebra homomorphism. Briefly, taking the adjoints of both sides in the axiom $\Delta\eta = \eta \otimes \eta$ (the coproduct Δ is an algebra homomorphism and so it maps the identity element of H to the identity element of $H \otimes H$), we obtain $\eta^*\Delta^* = \eta * \otimes \eta^*$ which says that the counit η^* of H^* is multiplicative. To see that the counit η^* of H^* takes the identity element of H^* to $1 \in \mathbb{C}$, dualise the axiom $\epsilon\eta = \mathrm{id}_{\mathbb{C}}$ (*exercise: why is this an axiom, what does it say, and how to dualise it?*)

The antipode law for H^*. So far, we proved that H^* is a bialgebra. We now claim that $S^*\colon H^* \to H^*$ is the antipode of H^*. The antipode law for H says

$$m(S \otimes \mathrm{id})\Delta = \eta\epsilon = m(\mathrm{id} \otimes S)\Delta.$$

Taking the adjoints of all three parts of the equation, we obtain the antipode law for H^* (*left as an exercise to the reader*).

E4.9 (the Hopf algebras $\mathbb{C}\Gamma$ and $(\mathbb{C}\Gamma)^*$) Let $\Gamma = \{e, g, g^2\}$ be the cyclic group of order 3. Use earlier results to show that the Hopf algebras $\mathbb{C}\Gamma$ and $(\mathbb{C}\Gamma)^*$ are isomorphic. (You need to write down a correct definition of an *isomorphism between Hopf algebras.*) What do you think happens for other finite groups?

Answer to E4.9. The definition of an isomorphism between Hopf algebras is as follows: $\phi\colon H_1 \to H_2$ is an isomorphism between the Hopf algebras H_1 and H_2 if ϕ is a linear isomorphism and

(a) ϕ is a map of algebras, i.e.,
  - $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in H_1$;
  - $\phi(1_{H_1}) = 1_{H_2}$;

(b) ϕ is a map of coalgebras, i.e.,
  - $\Delta(\phi(x)) = \phi(x_{(1)}) \otimes \phi(x_{(2)})$ for all $x \in H_1$;
  - $\epsilon(\phi(x)) = \epsilon(x)$ for all $x \in H_1$;

(c) ϕ preserves the antipode, i.e., $S(\phi(x)) = \phi(Sx)$ for all $x \in H_1$.

One can show that (c), preservation of the antipode, follows from (a) and (b). Indeed, if (a) and (b) hold then the map $\phi \circ S_{H_1} \circ \phi^{-1}\colon H_2 \to H_2$ obeys the antipode law on H_2, i.e., is an antipode on H_2. Yet H_2 has antipode S_{H_2}. By uniqueness of antipode, E4.6, we must have $\phi \circ S_{H_1} \circ \phi^{-1} = S_{H_2}$, equivalently $\phi \circ S_{H_1} = S_{H_2} \circ \phi$, which is (c).

Isomorphism between $\mathbb{C}\Gamma$ and $(\mathbb{C}\Gamma)^*$. For the group Γ of order 3, as discussed in E3.2 and E3.4, the Hopf algebra $(\mathbb{C}\Gamma)^*$ has a basis $\{\chi_0, \chi_1, \chi_2\}$ of grouplikes. By E4.1, the set $\{\chi_0, \chi_1, \chi_2\}$ is a group under multiplication, and, since it is a basis, we conclude that $(\mathbb{C}\Gamma)^*$ **must be isomorphic to the group algebra of this group.** Every group of order 3 is cyclic, so the Hopf algebras $\mathbb{C}\Gamma$ and $(\mathbb{C}\Gamma)^*$ are isomorphic.

Other finite groups. One can show that if Γ is a finite abelian group, the grouplike elements of $(\mathbb{C}\Gamma)^*$ form a group isomorphic to Γ. If Γ is not abelian, the Hopf algebra $(\mathbb{C}\Gamma)^*$ cannot be isomorphic to $\mathbb{C}\Gamma$ because (1) it has not got enough grouplikes to form a basis, (2) $(\mathbb{C}\Gamma)^*$ is always commutative whereas $\mathbb{C}\Gamma$ is not.

References

Majid, Shahn (2002). *A quantum groups primer.* Vol. 292. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, pp. x+169. ISBN: 0-521-01041-1. DOI: 10.1017/CBO9780511549892.