Week 04 review worksheet — exercises for §4

Part A. Exercises for interactive discussion

E4.1 (grouplike elements of a Hopf algebra form a group) (a) If H is a Hopf algebra, show that the set $G(H)$ of grouplike elements of H is a group where the operation is multiplication in H.

(b) Let $\mathbb{C}\Gamma$ be a group algebra of a group Γ, viewed as a Hopf algebra. What is the group $G(\mathbb{C}\Gamma)$?

E4.2 (primitive elements) An element x of a Hopf algebra H is called **primitive** if $\Delta x = x \otimes 1 + 1 \otimes x$. (Here $1 = 1_H$.) What is true? (*select one or more*):

A) 0 is primitive
B) 1 is primitive
C) if x, y are primitive, then $x + y$ is primitive
D) if x, y are primitive, then xy is primitive
E) an element of H cannot be grouplike <u>and</u> primitive

E4.3 (a) Let $P(H)$ be the set of primitive elements of H. Prove: if $x, y \in P(H)$ then $xy - yx \in P(H)$.

(b) We know from E4.1 that the set $G(H)$ of grouplikes in H is a group under multiplication. What kind of algebraic structure does (a) impose on the space $P(H)$ of primitives in H?

(c) Show: $\epsilon(x) = 0$ and $S(x) = -x$ if x is primitive.

E4.4 Write the Hopf algebra axioms as commutative diagrams.

Moreover, use the **graphical tensor calculus** to produce the diagrams (*to be discussed in the interactive session*).

E4.5 (tensor product of modules; the dual module) Our goal is to show that the class of modules over a Hopf algebra H is closed under tensor products and duals.

(a) Given an algebra A and A-modules V and W, define an $A \otimes A$-module structure on $V \otimes W$.

(b) Let H be a bialgebra. Use the coproduct $\Delta\colon H \to H \otimes H$ and (a) to make $V \otimes W$ an H-module whenever V and W are.

(c) If V is an A-module, show that $\triangleleft\colon V^* \otimes A \to V^*$ where, for $\phi \in V^*$, $\phi \triangleleft a$ is the linear functional on V defined by $\langle \phi \triangleleft a, v\rangle = \langle \phi, a \triangleright v\rangle$, is a *right action* of A on V^*. (Write down the definition of a right action.)

(d) If $\triangleleft$ is a right action of a Hopf algebra H, show that $\triangleright$ defined by the rule "$h\triangleright\, = \,\triangleleft Sh$" where $S\colon H \to H$ is the antipode, is a (left) action. Conclude from (c) that if V is an H-module then so is V^*.

Part B. Extra exercises

Attempt these exercises and compare your answers with the model solutions, published after the session.

E4.6 (uniqueness of antipode) Let H be a bialgebra. Show that an antipode $S\colon H \to H$, if exists, is unique. *Hint:* let S, S' be two antipodes; compute $S(a_{(1)})a_{(2)}S'(a_{(3)})$ in two ways.

E4.7 Let H be a Hopf algebra which acts via $\triangleright$ on an H-module algebra A. Prove: $x \in P(H)$ is primitive, then $x \triangleright (ab) = (x \triangleright a)b + a(x \triangleright b)$ for all $a, b \in A$ (*the Leibniz law*).

E4.8 Recall that the dual space to a coalgebra is an algebra, and the dual space to a finite-dimensional algebra is a coalgebra. Extend this to show: if H is a finite-dimensional Hopf algebra, then H^* is also a Hopf algebra.

E4.9 (the Hopf algebras $\mathbb{C}\Gamma$ and $(\mathbb{C}\Gamma)^*$) Let $\Gamma = \{e, g, g^2\}$ be the cyclic group of order 3. Use earlier results to show that the Hopf algebras $\mathbb{C}\Gamma$ and $(\mathbb{C}\Gamma)^*$ are isomorphic. (You need to write down a correct definition of an *isomorphism between Hopf algebras*.) What do you think happens for other finite groups?