MAGIC assessment cover sheet

Course:	**MAGIC009**
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Category Theory Q1 Ibraheem Sajid

$$\begin{tikzcd}
A \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=50, "H"'{name=H}] \arrow[from=F, to=G, Rightarrow, "\varphi"'] \arrow[from=G, to=H, Rightarrow, "\psi"'] & B \arrow[r, bend left=50, "P"{name=P}] \arrow[r, bend right=50, "Q"'{name=Q}] \arrow[from=P, to=Q, Rightarrow, "\theta"'] & C
\end{tikzcd}$$

(a) The naturality condition for $\theta.F : PF \to QF$ is; given any $f : X \to X'$ in A, we have

$$\begin{tikzcd}
PFX \arrow[r, "PFf"] \arrow[d, "(\theta.F)_X"'] \arrow[dr, phantom, "\circlearrowright ?"] & PFX' \arrow[d, "(\theta.F)_{X'}"] \\
QFX \arrow[r, "QFf"'] & QFX'
\end{tikzcd}$$

commutes.

Indeed, note that $Ff : FX \to FX'$ is a morphism in B, so by naturality of θ, we have

$$\begin{tikzcd}
P(FX) \arrow[r, "P(Ff)"] \arrow[d, "\theta_{FX}"'] \arrow[dr, phantom, "\circlearrowright"] & P(FX') \arrow[d, "\theta_{FX'}"] \\
Q(FX) \arrow[r, "Q(Ff)"'] & Q(FX')
\end{tikzcd}$$

which gives the required condition, since $P(FX) = PFX$, ~~and~~ $\theta_{FX} = (\theta.F)_X$ and $P(Ff) = PFf$ by definition (and similar for Q, X'). //

(b) The naturality condition for $P.\varphi : PF \to PG$ is : given any $f : X \to X'$, we have

$$\begin{tikzcd} PFX \arrow[r, "PFf"] \arrow[d, "(P.\varphi)_X"'] \arrow[dr, phantom, "?"] & PFX' \arrow[d, "(P.\varphi)_{X'}"] \\ PGX \arrow[r, "PGf"'] & PGX' \end{tikzcd}$$

commutes.

Indeed, note that naturality of φ gives

$$\varphi_{X'} \circ Ff = Gf \circ \varphi_X$$

Then, functoriality of P gives

$$P(\varphi_{X'} \circ Ff) = P(Gf \circ \varphi_X)$$

$$\Rightarrow \quad P(\varphi_{X'}) \circ P(Ff) = P(Gf) \circ P(\varphi_X)$$

$$\Rightarrow \quad (P.\varphi)_{X'} \circ PFf = PGf \circ (P.\varphi)_X \quad \text{by definition.}$$

which is the required condition //

(c) We want to show the following diagram commutes in the functor category $[\mathbb{A}, \mathbb{C}]$:

$$\begin{tikzcd} PF \arrow[r, "P.\varphi"] \arrow[d, "\theta.F"'] \arrow[dr, phantom, "\circlearrowright ?"] & PG \arrow[d, "\theta.G"] \\ QF \arrow[r, "Q.\varphi"'] & QG \end{tikzcd}$$

Note this is typoed as $\varphi.P$ in the question

Since the morphisms in $[\mathbb{A}, \mathbb{C}]$ (i.e. natural transformations) are given by families of maps, this is equivalent to showing for every $X \in \mathbb{A}$ that the following diagram ~~commutes~~ in $\mathbb{C}$:

$$\begin{tikzcd} PFX \arrow[r, "(P.\varphi)_X"] \arrow[d, "(\theta.F)_X"'] \arrow[dr, phantom, "\circlearrowright ?"] & PGX \arrow[d, "(\theta.G)_X"] \\ QFX \arrow[r, "(Q.\varphi)_X"'] & QGX \end{tikzcd}$$

Indeed, since $\varphi_X : FX \to GX$ in $\mathbb{B}$, we have by naturality of $\theta : P \Rightarrow Q$ that the following diagram commutes:

$$\begin{tikzcd} P(FX) \arrow[r, "P(\varphi_X)"] \arrow[d, "\theta_{FX}"'] \arrow[dr, phantom, "\circlearrowright"] & P(GX) \arrow[d, "\theta_{GX}"] \\ Q(FX) \arrow[r, "Q(\varphi_X)"'] & Q(GX) \end{tikzcd}$$

which gives the required condition ~~state~~ by definition of $\theta.F$, $P.\varphi$ and similar. //

~~Denote this mapping by γ~~

~~Required to check~~

(d) Denote this mapping by γ (abuse notation and use γ for objects and morphisms as usual).

Firstly, for $(P,F) \in [\mathbb{B},\mathbb{C}] \times [\mathbb{A},\mathbb{B}]$, we need to check that $1_{(P,F)} \overset{\gamma}{\longmapsto} 1_{PF}$.

Note $1_{(P,F)} = (1_P, 1_F)$ (standard result for product categories).

So $\gamma 1_{(P,F)} = 1_P * 1_F$

$= (1_P . F) \circ (P . 1_F)$

On objects we have $(\gamma 1_{(P,F)})_X = (1_P . F)_X \circ (P . 1_F)_X$

$= 1_{PFX} \circ P(1_{FX})$

(functoriality of P) $= 1_{PFX} \circ 1_{PFX} = 1_{PFX}$

$\therefore \gamma 1_{(P,F)} = 1_{PF}$ //

~~Next, for $(\theta,\varphi), (\rho,\psi) \in ([\mathbb{B},\mathbb{C}] \times [\mathbb{A},\mathbb{B}])((P,F),(Q,G))$~~

~~we need to show that $\gamma(\theta,\varphi) \circ \gamma(\rho,\psi) = \gamma((\theta,\varphi)\circ(\rho,\psi))$~~

~~i.e. $(\theta * \varphi) \circ (\rho * \psi) = (\theta \circ \rho) * (\varphi \circ \psi)$~~

Next, we need to check that for

$$(\theta,\varphi) : (P,F) \to (Q,G)$$
and $$(\rho,\psi) : (Q,G) \to (R,H) \quad \text{in } [\mathbb{B},\mathbb{C}]\times[\mathbb{A},\mathbb{B}]$$

we have

$$\gamma((\rho,\psi)\circ(\theta,\varphi)) = \gamma(\rho,\psi)\circ\gamma(\theta,\varphi)$$

ie $$(\rho\circ\theta) * (\psi\circ\varphi) = (\rho*\psi)\circ(\theta*\varphi)$$

$$\begin{tikzcd}
A \arrow[r, bend left=50, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=50, "H"'{name=H}] \arrow[from=F, to=G, phantom, "\varphi\Downarrow"] \arrow[from=G, to=H, phantom, "\psi\Downarrow"] & B \arrow[r, bend left=50, "P"{name=P}] \arrow[r, "Q"{name=Q}] \arrow[r, bend right=50, "R"'{name=R}] \arrow[from=P, to=Q, phantom, "\theta\Downarrow"] \arrow[from=Q, to=R, phantom, "\rho\Downarrow"] & C
\end{tikzcd}$$

Note $\rho*\psi = (\rho.H)\circ(Q.\psi) = (R.\psi)\circ(\rho.G)$

and $\theta*\varphi = (\theta.G)\circ(P.\varphi) = (Q.\varphi)\circ(\theta.F)$

So $(\rho*\psi)\circ(\theta*\varphi) = (\rho.H)\circ(Q.\psi)\circ(Q.\varphi)\circ(\theta.F)$

On objects, we get

$$((\rho*\psi)\circ(\theta*\varphi))_X = \rho_{HX}\circ Q(\psi_X)\circ Q(\varphi_X)\circ\theta_{FX}$$
$$= \rho_{HX}\circ Q(\psi\circ\varphi)_X\circ\theta_{FX}$$

Since $\rho : Q \Rightarrow R$ is natural, and $(\psi\circ\varphi)_X : FX \to HX$,

we have $\rho_{HX}\circ Q(\psi\circ\varphi)_X = R(\psi\circ\varphi)_X\circ\rho_{FX}$

$$\therefore ((\rho*\psi)\circ(\theta*\varphi))_X = R(\psi\circ\varphi)_X\circ\rho_{FX}\circ\theta_{FX} = R(\psi\circ\varphi)_X\circ(\rho\circ\theta)_{FX}$$

$$= \big(R\cdot(\psi\circ\varphi)\big)_X \circ \big((\rho\circ\theta)\cdot F\big)_X$$

$$= \Big(\big(R\cdot(\psi\circ\varphi)\big)\circ\big((\rho\circ\theta)\cdot F\big)\Big)_X$$

But $(\rho\circ\theta) * (\psi\circ\varphi) = \big(R\cdot(\psi\circ\varphi)\big)\circ\big((\rho\circ\theta)\cdot F\big)$

So we are done. //.

Extra check for completeness (well-definedness):

for $(\theta,\varphi):(P,F)\longrightarrow(Q,G)$, do we have

$$\gamma(\theta,\varphi):\gamma(P,F)\longrightarrow\gamma(Q,G) \quad ?$$

Well, $\gamma(\theta,\varphi) = \theta*\varphi : PF \longrightarrow QG$

and $\gamma(P,F) = PF$, $\gamma(Q,G) = QG$,

So yes. ∎

This does need to be checked but it is known at this stage.

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Category Theory Q2 Ilorcheeu Sayid

(a) Let $\mathbb{C}$ be given by $ob(\mathbb{C}) = \{*\}$ and $\mathbb{C}(*,*) = \{1_*\}$ $\qquad \bullet *$

and $\mathbb{D}$ by $ob(\mathbb{D}) = \{X, Y\}$
$\mathbb{C}(X,X) = \{1_X\}$, $\mathbb{C}(Y,Y) = \{1_Y\}$,
$\mathbb{C}(X,Y) = \{f\}$, $\mathbb{C}(Y,X) = \{g\}$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

Note we must have
$fg = 1_Y$ and $gf = 1_X$. ✓

Now the functor $F: \mathbb{C} \to \mathbb{D}$ sending $*$ to X and 1_* to 1_X (it is trivially so)

Since there are no non-trivial compositions in $\mathbb{C}$.

is • essentially surjective ✓ ($X \cong_{1_X} F*$ and $Y \cong_g F*$)

• faithful ✓ ($F_{*,*}: \{1_*\} \to \{1_X\}$ is bijective).
and full ✓

$\therefore$ F is an equivalence of categories by the theorem from lecture 3.

But note $F_0: \{*\} \to \{X, Y\}$ is injective but not surjective.

Alternatively, define $G: \mathbb{D} \to \mathbb{C}$ by
$$\begin{Bmatrix} X \mapsto * \\ Y \mapsto * \\ 1_X \mapsto 1_* \\ 1_Y \mapsto 1_* \\ f \mapsto 1_* \\ g \mapsto 1_* \end{Bmatrix}$$

G is again trivially a functor since all maps go to 1_*.

Then $FG: \mathbb{D} \to \mathbb{D}$
$X, Y \mapsto X$
$1_X, 1_Y, f, g \mapsto 1_X$

Have $\varepsilon: FG \to 1_{\mathbb{D}}$ by $\varepsilon_X = 1_X$, $\varepsilon_Y = g = f^{-1}$

and we can check the only interesting naturality condition: for $g: Y \to X$ we get

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\varepsilon_Y"'] \arrow[dr, phantom, "\circlearrowright ?"] & X \arrow[d, "\varepsilon_X"] \\ FGY \arrow[r, "FGg"'] & FGX \end{tikzcd}$$

which is asking if $\varepsilon_X \circ g = FGg \circ \varepsilon_Y$

ie. $1_X \circ g = 1_X \circ g$ ✓ which holds.

$\therefore$ F is an equivalence with inverse G

$(1_{\mathcal{C}} \xrightarrow{\eta} GF$ is trivial as is the rest of $FG \xrightarrow{\varepsilon} 1_{\mathcal{D}})$.

//

(b) (i) Suppose $s': X \to A$ makes the following diagram commute: $X \xrightarrow{s'} A \overset{e}{\underset{1_A}{\rightrightarrows}} A$.

ie $1_A \circ s' = e \circ s'$, then $s' = e \circ s' = srs'$.

Note $1_A \circ s = s = s \circ 1_B = s \circ rs = sr \circ s = e \circ s$

Thus, the following diagram commutes:

8/10 You need to first show e.s = 1_A.s

$$\begin{tikzcd} X \arrow[d, "rs'"'] \arrow[dr, "s'"] & & \\ B \arrow[r, "s"'] & A \arrow[r, shift left, "1_A"] \arrow[r, shift right, "e"'] & A \end{tikzcd}$$

Suppose now that $s' = s \circ u$. Then $r \circ s' = r \circ s \circ u$

$= rs \circ u = 1_B \circ u$

$= u$

So u is uniquely determined.

We have shown that $s: A \to B$ satisfies the universal property of $Eq(e, 1_A)$. //

(ii) If we consider the same setup in C^{op}, we have an idempotent morphism $e: A \to A$ such that $e \circ_{op} e = e$ as before. But now the splitting

$$\begin{matrix} rs = 1_B \\ sr = e \end{matrix} \quad \text{for} \quad \begin{matrix} r: A \to B \\ s: B \to A \end{matrix} \quad \text{in } \mathcal{C}$$

becomes

$$\begin{matrix} sr = 1_B \\ rs = e \end{matrix} \quad \text{for} \quad \begin{matrix} s: B \to A \\ r: A \to B \end{matrix} \quad \text{in } \mathcal{C}^{op}$$

Thus, the same proof as in (i) shows that r is the equaliser of 1_A and e in $\mathcal{C}^{op}$.

Dualising the universal property back into terms of $\mathcal{C}$ shows that r satisfies the universal property of the coequaliser of 1_A and e in $\mathcal{C}$.

(iii) In $\mathcal{D}$ we have $Fe \circ Fe = F(e \circ e) = Fe$. [NOTE: $Fe: FA \to FA$]
Moreover, since $rs = 1_B$ and $sr = e$ in $\mathcal{C}$, we get
$F(rs) = F(1_B)$, i.e. $F(r)F(s) = 1_{FB}$ and
$F(sr) = F(e)$, i.e. $F(s)F(r) = F(e)$.

So $F(e)$ is an idempotent with splitting $(F(r), F(s))$.
By (i) and (ii), then, we have

$$Eq(F(e), 1_{FA}) = F(s) = FEq(e, 1_A)$$
$$\text{and } Coeq(Fe, 1_{FA}) = Fr = FCoeq(e, 1_A)$$

which exactly says that F preserves the equaliser and coequaliser of e and 1_A.

23/25 for Q2

Category Theory Q3

Ibraheem Sajid

(a) $\mathbb{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{B}$, $\mathbb{B}$ has binary products.

(in particular the)
We have, binary products $(FX \times GX, \pi_{FX}, \pi_{GX})$ and $(FY \times GY, \pi_{FY}, \pi_{GY})$ in $\mathbb{B}$

Given $f : X \to Y$ in $\mathbb{A}$, the universal property for $FY \times GY$ for the maps $Ff \circ \pi_{FX} : FX \times GX \to FY$ and $Gf \circ \pi_{GX} : FX \times GX \to GY$ defines a unique map u ($: FX \times GX \to FY \times GY$) making the following diagram commute:

$$\begin{tikzcd}
 & FX \times GX \arrow[dl, "Ff \circ \pi_{FX}"'] \arrow[d, "u"] \arrow[dr, "Gf \circ \pi_{GX}"] & \\
FY & FY \times GY \arrow[l, "\pi_{FY}"] \arrow[r, "\pi_{GY}"'] & GY
\end{tikzcd}$$

We've defined $Ff \times Gf$ as this u. Let's check functoriality.

Firstly, $(F \times G)(f : X \to Y) = Ff \times Gf : FX \times GX \to FY \times GY$

$$FX \times GX = (F \times G)X, \quad FY \times GY = (F \times G)Y$$

So we have well-definedness. //

(identity) $(F \times G)(1_X) = F1_X \times G1_X$ is u in the following commutative diagram

$$\begin{tikzcd}
 & FX \times GX \arrow[dl, "\pi_{FX}"'] \arrow[d, "u"] \arrow[dr, "\pi_{GX}"] & \\
FX & FX \times GX \arrow[l, "\pi_{FX}"] \arrow[r, "\pi_{GX}"'] & GX
\end{tikzcd}$$

(using that F, G preserve identity)

But $u = 1_{FX\times GX}$ satisfies the diagram, so by uniqueness of u (in UP.) we have

$$(F\times G)(1_X) = 1_{(F\times G)X} \quad //$$

(composition) Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{A}$.

~~[illegible]~~ We will "stack" the universal property diagram in the definitions of $(F\times G)f$ and $(F\times G)g$ as follows:

$$\begin{tikzcd}
 & FX\times GX \arrow[dl, "Ff\circ\pi_{FX}"'] \arrow[d, "Ff\times Gf"] \arrow[dr, "Gf\circ\pi_{GX}"] & \\
FY \arrow[d, "Fg"'] & FY\times GY \arrow[l, "\pi_{FY}"] \arrow[r, "\pi_{GY}"'] \arrow[dl, "Fg\circ\pi_{FY}"] \arrow[d, "Fg\times Gg"] \arrow[dr, "Gg\circ\pi_{GY}"] & GY \arrow[d, "Gg"] \\
FZ & FZ\times GZ \arrow[l, "\pi_{FZ}"] \arrow[r, "\pi_{GZ}"'] & GZ
\end{tikzcd}$$

(ignoring pink)

This christmas tree diagram is ~~[illegible]~~ commutative ~~[illegible]~~ in each of the four corners.

Adding in two new maps in pink, the two new triangles commute by definition of the maps.

Thus, the whole diagram (including pink) is commutative, ~~so~~ using functoriality of F and G we can simplify this diagram into:

$$\begin{tikzcd}
 & FX \times GX \arrow[dl, "F(g\circ f)\circ\pi_{FX}"'] \arrow[d, "(Fg\times Gg)\circ(Ff\circ Gf)"] \arrow[dr, bend left, "G(g\circ f)\circ \pi_{GX}"] & \\
FZ & FZ \times GZ \arrow[l, "\pi_{FZ}"] \arrow[r, "\pi_{GZ}"'] & GZ
\end{tikzcd}$$

This is the universal property for $(F\times G)(g\circ f)$, so by uniqueness, we have

$$(F\times G)(g\circ f) = (F\times G)g \circ (F\times G)f \quad //$$

(b) A product of $F, G \in [\mathbb{A}, \mathbb{B}]$ is some $P \in [\mathbb{A}, \mathbb{B}]$ with maps

$$F \overset{\varphi}{\Longleftarrow} P \overset{\gamma}{\Longrightarrow} G$$

such that given $Q \in [\mathbb{A}, \mathbb{B}]$ with maps

$$F \overset{\psi}{\Longleftarrow} Q \overset{\delta}{\Longrightarrow} G$$

there is a unique $\omega : Q \Rightarrow P$ making the following diagram commute:

$$\begin{tikzcd}
 & Q \arrow[dl, Rightarrow, "\psi"'] \arrow[d, Rightarrow, "\omega"] \arrow[dr, Rightarrow, "\delta"] & \\
F & P \arrow[l, Rightarrow, "\varphi"] \arrow[r, Rightarrow, "\gamma"'] & G
\end{tikzcd}$$

Since the maps are all natural transformations, they are given by families of maps indexed by $\overset{ob}{\mathbb{A}}$.

Define $P = F\times G$ and for $X \in \mathbb{A}$,

$\varphi_X = \pi_{FX}$, $\gamma_X = \pi_{GX}$.

φ, γ are natural by definition of $F\times G$ on morphisms, ie given $f : X \to Y$ we have

(by definition)

$$\begin{tikzcd} & FX\times GX \arrow[dl, "Ff\circ\pi_{FX}"'] \arrow[d, "Ff\times Gf"] \\ FY & FY\times GY \arrow[l, "\pi_{FY}"] \end{tikzcd}$$

commutes

This is exactly the naturality square for φ:

$$\begin{tikzcd} FX \arrow[d, "Ff"'] & FX\times GX \arrow[l, "\varphi_X"'] \arrow[d, "(F\times G)f"] \\ FY & FY\times GY \arrow[l, "\varphi_Y"] \end{tikzcd}$$

The argument is similar for γ.

Now, for $X \in \mathbb{A}$, the universal property of F and G's product becomes

$$\begin{tikzcd} & QX \arrow[dl, "\psi_X"'] \arrow[d, "\omega_X"] \arrow[dr, "\delta_X"] & \\ FX & FX\times GX \arrow[l, "\pi_{FX}"] \arrow[r, "\pi_{GX}"'] & GX \end{tikzcd}$$

where ω_X is to be determined to show that $F\times G$ is the correct candidate. Indeed, the universal property for $FX\times GX$ in $\mathbb{B}$ gives us the map ω_X uniquely, and it remains to check ω is natural.

For $f: X \to Y$ in $\mathcal{A}$, we have the following diagram:

$$\begin{tikzcd}
 & QY \arrow[dl, "\psi_Y"'] \arrow[d, "\omega_Y"] \arrow[dr, "\delta_Y"] & \\
FY & FY \times GY \arrow[l, "\pi_{FY}"] \arrow[r, "\pi_{GY}"'] & GY \\
 & QX \arrow[uu, "Qf"] \arrow[dl, "\psi_X"'] \arrow[d, "\omega_X"] \arrow[dr, "\delta_X"] & \\
FX \arrow[uu, "Ff"] & FX \times GX \arrow[l, "\pi_{FX}"] \arrow[r, "\pi_{GX}"'] \arrow[uu, "Ff \times Gf"] & GX \arrow[uu, "Gf"']
\end{tikzcd}$$

$\pi_{FX} = \varphi_X$

By naturality, all the outer faces of this prism commute. Only the central square involving ω is not known to commute, and we must check this for ω's naturality.

Note $\omega_Y \circ Qf$ and $Ff \times Gf \circ \omega_X$ both work as u in the following diagram (universal property of $FY \times GY$)

$$\begin{tikzcd}
 & QX \arrow[dl, "\psi_Y \circ Qf"'] \arrow[d, "u"] \arrow[dr, "\delta_Y \circ Qf"] & \\
FY & FY \times GY \arrow[l, "\pi_{FY}"] \arrow[r, "\pi_{GY}"'] & GY
\end{tikzcd}$$

Indeed, $u = \omega_Y \circ Qf$ works due to the definition of ω_Y (and composing with Qf).

$u = Ff \times Gf \circ w_X$ works since

$$\begin{aligned}\pi_{FY} \circ Ff \times Gf \circ w_X &= Ff \circ \pi_{FX} \circ w_X && \text{(nat. of } \varphi) \\ &= Ff \circ \psi_X && \text{(def of } w) \\ &= \psi_Y \circ Qf && \text{(nat. of } \psi).\end{aligned}$$

$\therefore$ By uniqueness, we have

$$Ff \times Gf \circ w_X = w_Y \circ Qf \text{, ie. } w : Q \Rightarrow F \times G \text{ is natural.}$$

$\quad\; \shortparallel$

$(F \times G)f \circ w_X$ 9/9

//

(c) ~~The~~ Denote the n objects of $\mathbb{A}$ by $1, 2, \dots, n$.

The universal property definition of adjoint functors says that the functor $\Delta : \mathbb{B} \to [\mathbb{A}, \mathbb{B}]$

$b \mapsto (a \mapsto b)$

(I won't prove this is a functor as we assumed it in lectures.)

has a right adjoint R if $[\mathbb{A}, \mathbb{B}] \ni F \mapsto RF \in \mathbb{B}$ is a mapping (assignment of objects) and for all $F \in [\mathbb{A}, \mathbb{B}]$, $\exists$ $\varepsilon_F \in [\mathbb{A}, \mathbb{B}](\Delta R F, F)$

Such that for all $f : \Delta X \to F$ $(X \in \mathbb{B})$ there exists a unique $f^{\#} : X \to RF$ ~~such~~ satisfying $f = \varepsilon_F \circ \Delta f^{\#}$.

UNPACK ↓

Since $\mathbb{A}$ has n objects and no (interesting) morphisms, we see that functors $F : \mathbb{A} \to \mathbb{B}$ simply pick out n objects of $\mathbb{B}$, $\{F1, F2, \dots, Fn\}$.

(functoriality is automatic for any n objects)

Since ε_F is a morphism in $[A, B]$, it is actually a collection of morphisms $\Delta RF_i \longrightarrow F_i$ for $i \in A$ in B.

ε_F is natural in i, but since there are no non-identity morphisms we can forget about this.

Similarly, given some $f : \Delta X \longrightarrow F$, we actually specify a collection $(f_i : \Delta X_i \longrightarrow F_i)_{i \in A}$

Note that $\Delta X_i = X \in B$ by definition, so these f_i are in $B(X, F_i)$.

[Also $\Delta RF_i = RF$ so $(\varepsilon_F)_i : RF \longrightarrow F_i$
and $(\Delta f^\#)_i = f^\#$ by def^n of Δ]

with morphisms

Rewriting the universal property ~~of the~~ of this functor category unpacked and using the restriction on A gives:

$R \vdash \Delta$ means R maps a collection of n objects of B $(F_1, F_2, \dots, F_n)$ to a single object RF of B, and we have morphisms $(\varepsilon_F)_i : RF \longrightarrow F_i$ for each such collection, satisfying; $(i = 1, \dots, n)$

If $X \in B$ with $f_i : X \longrightarrow F_i$ for $1 \leq i \leq n$, then $f_i = (\varepsilon_F)_i \circ f^\#$ for some unique $f^\# : X \longrightarrow RF$.

If we write X_i for $F_i \in B$, $X_1 \times \cdots \times X_n$ for $RF \in B$, and π_i for $(\varepsilon_F)_i$, we see that this gives the universal property of the n-ary product in B.

($f^\#$ is usually denoted u).

So $R \vdash \Delta \iff B$ has n-ary products.

//

has
it ~~is~~ the same assumption

(d) The base case $n=2$ is direct from (c) ✓ since $R \vdash \Delta$ precisely means $\mathbb{B}$ has binary products and it does. //

Suppose $\mathbb{B}$ also has $(n-1)$-ary products ✓ (and binary products from the underlying assumption)

Given $X_1, \dots, X_n \in \mathbb{B}$, we denote the $(n-1)$-ary product of $X_1, \dots, X_{n-1}$ by $(X_1 \times \cdots \times X_{n-1}, \pi_1, \dots, \pi_{n-1})$ and claim that the binary product

$$\left((X_1 \times \cdots \times X_{n-1}) \times X_n, f_0, f_n\right)$$

✓ gives rise to the n-ary product as

$$\left((X_1 \times \cdots \times X_{n-1}) \times X_n, \pi_1 \circ f_0, \dots, \pi_{n-1} \circ f_0, f_n\right).$$

Indeed, suppose $Y \in \mathbb{B}$ with $p_i : Y \to X_i$ for all $1 \le i \le n$.

By the UP of $X_1 \times \cdots \times X_{n-1}$ we have a unique ✓ $u_0 : P \to X_1 \times \cdots \times X_{n-1}$ such that $\pi_i \circ u_0 = p_i$ for all $1 \le i \le n-1$.

Now by the UP of $(X_1 \times \cdots \times X_{n-1}) \times X_n$ we have a ✓ unique $u : P \to (X_1 \times \cdots \times X_{n-1}) \times X_n$ such that $f_0 \circ u = u_0$ and $f_n \circ u = p_n$

But then $(\pi_i \circ f_0) \circ u = \pi_i \circ u_0 = p_i$ for all $1 \le i \le n-1$ and $f_n \circ u = p_n$, ✓ which is the UP of $X_1 \times \cdots \times X_n$.

So (n-ary) products exist for all $n \ge 2$ by induction //

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Category Theory Q4 Ibraheem Sajid

$X \in \mathbb{B}$ fixed

(a) (i) Suppose $A, B \in \mathbb{B}$ and $(A \times B, \pi_A, \pi_B)$ is their product.

We want to show that $\mathbb{B}(X,A)$ and $\mathbb{B}(X,B)$ has a product isomorphic to $\mathbb{B}(X, A\times B)$ with projection maps ~~$\mathbb{B}(X,A\times B)$~~

$$\mathbb{B}(X,\pi_A) : \mathbb{B}(X, A\times B) \longrightarrow \mathbb{B}(X,A)$$
$$f \longmapsto \pi_A \circ f$$
$$\mathbb{B}(X,\pi_B) : \mathbb{B}(X, A\times B) \longrightarrow \mathbb{B}(X,B)$$
$$f \longmapsto \pi_B \circ f$$

It suffices to show that this satisfies the UP of the product since that determines the product up to unique isomorphism.

So, suppose that $Y \in \underline{\mathrm{Set}}$ with $g_A : Y \to \mathbb{B}(X,A)$
$g_B : Y \to \mathbb{B}(X,B)$

Then $\forall y \in Y$, $g_A(y) : X \to A$ in $\mathbb{B}$
and $g_B(y) : X \to B$ in $\mathbb{B}$

So $\forall y \in Y$ we have a unique $u_y : X \to A\times B$ such that $\pi_A \circ u_y = g_A(y)$ and $\pi_B \circ u_y = g_B(y)$.

Define $u : Y \to \mathbb{B}(X, A\times B)$
$y \longmapsto u_y$. ~~By the above, we have the commutativity part of UP satisfied, so it remains to show u is unique~~.

Note $\mathbb{B}(X,\pi_A) \circ u\,(y)$
$= \mathbb{B}(X,\pi_A)(u(y))$
$= \pi_A \circ u(y) = \pi_A \circ u_y = g_A(y) \quad \forall y \in Y$

So $\mathbb{B}(X,\pi_A) \circ u = g_A$ and similar for $\mathbb{B}(X,\pi_B) \circ u = g_B$.

What remains of the UP is uniqueness of u.

Suppose
$$\begin{tikzcd} & Y \arrow[dl, "g_A"'] \arrow[d, "a"] \arrow[dr, "g_B"] & \\ \mathbb{B}(X,A) & \mathbb{B}(X,A\times B) \arrow[l, "\pi_A\circ -"] \arrow[r, "\pi_B\circ -"'] & \mathbb{B}(X,B) \end{tikzcd}$$

Then $\forall y\in Y$ $g_A(y) = \pi_A\circ a(y)$, so $a(y) = u_y$ by uniqueness of u_y

But ~~then~~ then $a(y)=u_y$ $\forall y\in Y$, so $a=u$. //

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(ii) Suppose T is terminal in $[\mathbb{B}^{op}, \underline{Set}]$. Then for all $F\in[\mathbb{B}^{op}, \underline{Set}]$ $\exists$ unique $u: F \Rightarrow T$.

~~[illegible]~~ We show that $T: B \longmapsto *$
$(f: B\to B') \longmapsto 1_*$

satisfies this UP.

Suppose $F: \mathbb{B}^{op} \to \underline{Set}$. Any $u: F\Rightarrow T$ ~~[illegible]~~ ~~[illegible]~~ consists of $u_B: FB \to TB = *$ in $\underline{Set}$. But then u_B is actually uniquely determined as the constant function $FB\to *$ for all B

and so u is uniquely determined. If $f: B\to B'$ in $\mathbb{B}$ we have the naturality condition
$$\begin{tikzcd} FB' \arrow[rr, "Ff"] \arrow[dr] & & FB \arrow[dl] \\ & * & \end{tikzcd}$$

which is trivially satisfied. So u is indeed a (unique) natural transformation $F\Rightarrow T$. //

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(a presheaf, so a functor which acts on objects and morphisms)

Set

(b)(i) Under $\bar{G}(-,*)$, the object $*$ maps to $\bar{G}(*,*) = G$ and morphisms $* \to *$ (i.e. group elements) g map to functions $G \to G$ given by $r_g : h \mapsto hg$.

~~$\bar{G}(-,g)$ (a natural transformation so a~~

$\bar{G}(-,g)$ is a natural transformation with a single component morphism $\bar{G}(-,g)_* : \bar{G}(*,*) \to \bar{G}(*,*)$

$$\underset{\bar{G}(*,g)}{\overset{\shortparallel}{}} \qquad \underset{G}{\overset{\shortparallel}{}} \qquad \underset{G}{\overset{\shortparallel}{}}$$

given by $l_g : h \mapsto gh$.

Naturality just restates associativity $\left(\underset{(kh)g}{\underset{\shortparallel}{r_g l_k(h)}} = \underset{k(hg)}{\underset{\shortparallel}{l_k r_g(h)}} \right)$

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(ii) $Y_{\bar{G}}(g) = \bar{G}(-,g) : \underset{\bar{G}(-,*)}{\underset{\shortparallel}{Y_{\bar{G}}(*)}} \to \underset{\bar{G}(-,*)}{\underset{\shortparallel}{Y_{\bar{G}}(*)}}$

$Y_{\bar{G}}$ being a functor says that 1) $\bar{G}(-,e) = 1_{\bar{G}(-,*)}$

i.e. $\underset{l_e}{\underset{\shortparallel}{\bar{G}(*,e)}} = \underset{1_G : G \to G}{\underset{\shortparallel}{1_{\bar{G}(*,*)}}}$

so the group identity maps to the identity permutation of G as a set.

2) $\underset{l_{gh}}{\underset{\shortparallel}{\bar{G}(-,gh)}} = \underset{l_g}{\underset{\shortparallel}{\bar{G}(-,g)}} \cdot \underset{l_h}{\underset{\shortparallel}{\bar{G}(-,h)}}$ $\quad \left(Y_{\bar{G}}(gh) = Y_{\bar{G}}(g) Y_{\bar{G}}(h) \right)$

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1)+2) $\Rightarrow$ ~~the~~ the map $G \to \mathrm{Perm}(G)$ is a homomorphism.

(iii) $Y_{\bar{G}}$ being faithful says that in fact the map $g \mapsto l_g$ is injective. (since g is a morphism of $\bar{G}$ and faithful means injective on hom-sets).

But now we have the ~~statement~~ proof/statement of Cayley's theorem:

Every group G is isomorphic to a subgroup of $\mathrm{Sym}(G)$

(= there exists an injective homomorphism $G \to \mathrm{Sym}(G)$)

$Y_{\bar{G}}$ is in addition full, so the map $g \mapsto l_g$ is a surjection too (so a bijection). We need to interpret the functor category $[\bar{G}^{op}, \underline{Set}]$ to see what this tells us (more specifically we care about the full subcategory with only the object $\bar{G}(-,*) = Y_{\bar{G}}(*)$)

Suppose $\alpha : \bar{G}(-,*) \Rightarrow \bar{G}(-,*)$.
α consists of only $\alpha_* : G \to G$ in $\underline{Set}$

Such that, for ~~[illegible]~~ $g \in G$, the following naturality condition holds. (since $\bar{G}(g,*) = r_g$)

$$* \quad \begin{tikzcd} G \arrow[r, "\alpha_*"] \arrow[d, "r_g"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "r_g"] \\ G \arrow[r, "\alpha_*"'] & G \end{tikzcd}$$

ie. $\alpha_* \circ r_g = r_g \circ \alpha_*$

or $\forall h \in G$, $\alpha_*(hg) = \alpha_*(h) g$.

So the subgroup of $\mathrm{Sym}(G)$ in question is the one of all right G-equivariant maps. //

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25/25 for Q4

98/100
