Exercises for §8

E8.1 (the $\hbar$-adic algebra $U_\hbar = U_\hbar(\mathfrak{sl}_2)$) The algebra $U_\hbar := U_\hbar(\mathfrak{sl}_2)$ is defined, as an $\hbar$-adic algebra over the ring $\mathbb{C}[[\hbar]]$ of formal power series in $\hbar$, by the presentation

$$U_\hbar = \mathbb{C}\langle E,\, H,\, F \mid HE - EH = 2E,\, HF - FH = -2F,\, EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\rangle$$

where q denotes the formal power series $e^\hbar \in \mathbb{C}[[\hbar]]$, K denotes $e^{\hbar H}$ and K^{-1} denotes $e^{-\hbar H}$.

(a) Which facts are used to deduce the following relations? Remind yourself of a proof of those facts, or try to prove them.

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2 EK, \quad KF = q^{-2} FK$$

(b) Expand $\frac{K-K^{-1}}{q-q^{-1}}$, as a formal power series in $\hbar$, to $O(\hbar^7)$ (that is, ignoring all terms which contain $\hbar^7$ or higher power of $\hbar$).

E8.2 Recall that $U_\hbar$ acts on the 2-dimensional space V with basis $\{x, y\}$ (more precisely, on $V[[\hbar]]$) via

$$E \triangleright x = 0, \quad E \triangleright y = x, \quad F \triangleright x = y, \quad F \triangleright y = 0, \quad H \triangleright x = x, \quad H \triangleright y = -y,$$

and that this action extends to a covariant action of $U_\hbar$ on the quantum plane $\mathbb{C}_\hbar[x, y] = \mathbb{C}[[\hbar]]\langle x, y \mid yx = e^\hbar xy\rangle$. Calculate the action of E, H and F on a standard monomial $x^a y^b$ in $\mathbb{C}_\hbar[x, y]$, expressing $E \triangleright (x^a y^b)$ etc in terms of standard monomials in $\mathbb{C}_\hbar[x, y]$. You will need (*why?*) the coproduct on $U_\hbar$, given by

$$\Delta H = 1 \otimes H + H \otimes 1, \quad \Delta E = E \otimes K + 1 \otimes E, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F.$$

Some of the exercises which appeared on the draft version of the current worksheet have been moved to the Week 09 review worksheet.

Part B. Extra exercises

E8.3 (the antipode of $U_q(\mathfrak{sl}_2)$) Deduce or review the formulas for $S(E)$ and $S(F)$ in $U_q(\mathfrak{sl}_2)$. Calculate $S(EF)$ and $S(EKF)$, expressing the answer as a linear combination of monomials from the PBW-type basis of $U_q(\mathfrak{sl}_2)$. Show that $S^2 \neq \mathrm{id}$ on $U_q(\mathfrak{sl}_2)$.

E8.4 (the PBW-type theorem for $U_q(\mathfrak{sl}_2)$) Review the formulae which describe the action of the generators $E, K^{\pm 1}, F$ of the $\mathbb{C}$-algebra U_q on the basis $\{x^m y^n \mid m, n \in \mathbb{Z}_{\geq 0}\}$ of the quantum plane $A_q := \mathbb{C}_\hbar[x, y]$. Use this action to show that the PBW-like monomials $E^m K^n F^p$, where $m, p \geq 0$, $n \in \mathbb{Z}$, are linearly independent in U_q.

Hint: assume that there is a non-trivial linear combination L of the PBW-like monomials is zero in U_q. Take the least p such that a monomial of the form $E^m K^n F^p$ appears in L with non-zero coefficient. Act by L on $x^p y^b$; note that all monomials $E^{m'} K^{n'} F^{p'}$ with $p' > p$ act on $x^p y^b$ by zero. You are left to choose b so as to get a contradiction.