Model answers to Week 07 review worksheet — exercises for §7

E7.1 (warm-up) Is $S^2 = \mathrm{id}$ on $U(\mathfrak{sl}_2)$? Is $S^2 = \mathrm{id}$ on all Hopf algebras?

Answer to E7.1. If S is the antipode of a Hopf algebra H, then $S^2 \colon H \to H$ is linear as a composition of linear maps, and, using the antihomomorphism property of the antipode (Proposition 4.11), for all $a, b \in H$,

$$S^2(ab) = S(S(ab)) = S(S(b)S(a)) = S(S(a))S(S(b)) = S^2(a)S^2(b), \qquad S^2(1) = S(S(1)) = S(1) = 1.$$

Hence $S^2 \colon H \to H$ is an **algebra homomorphism**.

In the case of a Hopf algebra generated by primitives X, H, Y, we have $S^2(X) = S(-X) = X$, and similarly $S^2(H) = H$ and $S^2(Y) = Y$. Hence S^2 is identity on generators, and, being an algebra homomorphism, S^2 extends to the identity map on the whole of $U(\mathfrak{sl}_2)$.

Remark. If the Hopf algebra H is **commutative** or **cocommutative,** then $S^2 = \mathrm{id}$ on H. Note that $U(\mathfrak{sl}_2)$ is a cocommutative Hopf algebra.

Furthermore, if H is **finite-dimensional,** then $S^n = \mathrm{id}$ for some $n \geq 1$. In particular, the antipode of a finite-dimensional Hopf algebra is invertible. This, and the previous, result can be found in literature on Hopf algebras.

We will see that $S^n \neq \mathrm{id}$ on the Hopf algebra $U_q(\mathfrak{sl}_2)$, which we will consider soon. Yet the antipode of $U_q(\mathfrak{sl}_2)$ is bijective, i.e., invertible. There are inifinite-dimensional Hopf algebras with non-invertible antipode.

E7.2 (exponentiate a primitive element) (a) Let H be a Hopf algebra and let $x \in H$ be primitive. Suppose that we can formally write elements $e^{\hbar x} = \sum_{n=0}^{\infty} \hbar^n \frac{x^n}{n!}$ (where $\hbar$ is a formal parameter) and that we can manipulate infinite series according to the same rules as finite linear combinations.

Show that $e^{\hbar x}$ is grouplike. (In other words, show that, exponentiating a primitive element, we obtain a grouplike element.)

(b) Discuss $\hbar$-adic Hopf algebras where formal exponentials described in (a) actually make sense.

Answer to E7.2. (a) Let us calculate the coproduct, assuming that Δ is linear with respect to infinite sums:

$$\Delta e^{\hbar x} = \sum_{n=0}^{\infty} \hbar^n \frac{\Delta(x^n)}{n!} = \sum_{n=0}^{\infty} \hbar^n \frac{(\Delta x)^n}{n!} = \sum_{n=0}^{\infty} \hbar^n \frac{(x \otimes 1 + 1 \otimes x)^n}{n!} = e^{\hbar(x\otimes 1 + 1 \otimes x)}.$$

Recall:

Theorem (exponential law). If a and b commute, then $e^{\hbar(a+b)} = e^{\hbar a} e^{\hbar b}$.

This follows from **the Binomial Theorem** (see the answer to E5.2). Indeed, the coefficient of $\hbar^n$ in the product of the two series, $e^{\hbar a} = \sum_{i=0}^{\infty} \frac{\hbar^i}{i!} a^i$ and $e^{\hbar b} = \sum_{j=0}^{\infty} \frac{\hbar^j}{j!} b^j$, is

$$\sum_{i+j=n} \frac{a^i}{i!} \frac{b^j}{j!} \underset{i=n-j}{=} \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} a^{n-j} b^j = \frac{1}{n!}(a+b)^n,$$

where the last step is by the Binomial Theorem (true when $ab = ba$). So $e^{\hbar a} e^{\hbar b} = \sum_{n=0}^{\infty} \frac{1}{n!}(\hbar a + \hbar b)^n = e^{\hbar(a+b)}$.

Note that $x \otimes 1$ commutes with $1 \otimes x$, hence $\Delta e^{\hbar x} = e^{\hbar x \otimes 1} e^{1 \otimes \hbar x}$.

Now we note that $(\hbar x \otimes 1)^n = (\hbar x)^n \otimes 1$ for all n. So we consider the series $e^{\hbar x \otimes 1} = 1 \otimes 1 + \frac{\hbar x \otimes 1}{1!} + \frac{(\hbar x \otimes 1)^2}{2!} + \cdots = 1 \otimes 1 + \frac{\hbar x}{1!} \otimes 1 + \frac{(\hbar x)^2}{2!} \otimes 1 + \ldots$, and arrive at $e^{\hbar x \otimes 1} = e^{\hbar x} \otimes 1$. Similarly, $e^{1 \otimes \hbar x} = 1 \otimes e^{\hbar x}$. We conclude that

$$\Delta e^{\hbar x} = (e^{\hbar x} \otimes 1)(1 \otimes e^{\hbar x}) = e^{\hbar x} \otimes e^{\hbar x}.$$

It remains to note that $\epsilon(e^{\hbar x}) = 1$, again by considering the series for $e^{\hbar x}$ and assuming that ϵ is linear for infinite sums. Hence $e^{\hbar x}$ is grouplike.

(b) In $\hbar$-adic Hopf algebras, Δ and ϵ are linear with respect to convergent infinite sums.

We note that by definition of an $\hbar$-adic algebra A, an infinite sum converges if, for each $n \in \mathbb{N}$, the sum contains only finitely many terms which do not lie in $\hbar^n A$.

This in particular implies that for all $a \in A$, $e^{\hbar a}$ is well defined. Indeed, the series $1 + \frac{\hbar}{1!}a + \frac{\hbar^2}{2!}a^2 + ...$ contains at most n terms which do not lie in $\hbar^n A$.