Homology of Configuration Spaces of Surfaces as Mapping Class Group Representations

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Declaration

I declare that this thesis is the result of my own work, and includes no work done in collaboration, except for Sections 3.1-3.4 and the entirety of Chapter 4, which were done and written jointly with Andrea Bianchi.

It is not substantially the same as any work which has already been submitted before for any degree or other qualification.

Andreas Stavrou
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Abstract

In this thesis, we study the homology of configuration spaces of surfaces viewed as representations of the mapping class group of the surface, distinguishing between various flavours: ordered and unordered configurations, of closed surfaces and surfaces with boundary, and with different homology coefficients.

In Chapter 2, we prove a version of the scanning isomorphism that is "untwisted" and equivariant with the mapping class group action. We further prove that scanning remembers a product arising from superposing configurations. We apply this equivariant scanning to compute the rational cohomology of unordered configurations of surfaces with boundary.

In Chapter 3, we adapt certain cellular decompositions of compactified configuration spaces to obtain the kernel of the mapping class group action on the homology of unordered configurations of both kinds of surfaces and with any coeffiecients.

Finally, in Chapter 4, we geometrically construct mapping classes deep in the Johnson filtration that act non-trivially on the homology of ordered configurations, in support of a conjecture by Bianchi, Miller and Wilson.

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And to the innumerable other people who made my last eight years in Cambridge this incredible journey...

Contents

Chapter 1

Introduction

A note on coefficients. Omitted coefficients of (co)homology are assumed to be $\mathbb{Z}$ in this Introduction, Chapter 3 and Chapter 4. In Chapter 2, omitted coefficients are assumed to be $\mathbb{Q}$.

1.1 Configuration spaces and Scanning

We shall be interested in the following spaces.

Definition 1. The *ordered configuration space* of n points in the topological space X is the space

$$F_n(X) = \{(x_1, ..., x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\},$$

topologised as a subspace of X^n. The *(unordered) configuration space* is the quotient

$$C_n(X) = F_n(X)/\mathfrak{S}_n,$$

by the permutation action of the symmetric group $\mathfrak{S}_n$.

In the 70's, Segal introduced *scanning* as a method to study iterated loopspaces of spheres via configuration spaces of discs [47]. Soon after, McDuff upgraded scanning to all manifolds, replacing loopspaces with certain section spaces [35]. Specifically, let M be a smooth compact manifold, possibly with boundary. Let τ^+M be the fibrewise one-point compactification of its tangent bundle, and define $\Gamma_n(M)$ to be the space of sections of τ^+M over M, which are supported in the interior of M, and have degree

n. The scanning map is an explicitly constructed map $s_n : C_n(M) \to \Gamma_n(M)$, and McDuff's result is as follows.

Theorem 2 (McDuff [35]). *For every $n \geq 0$, s_n induces an injection on integral homology, and an isomorphism in a range of degrees increasing with n.*

One might hope to use the scanning map to compute the entire homology of configuration spaces, but the induced map $(s_n)_*$ is not an isomorphism in all degrees, even in the case of discs[1]. For this purpose, there is a more appropriate version of scanning following from the work of Bödigheimer, Cohen and Taylor in the '80s [6–8]. To state the theorem, let $\Gamma_\partial(M, S^{2m})$ be the space of sections, with fixed boundary behaviour, of the bundle $\tau^+ M \wedge_f S^{2m}$, which is the fibrewise smash of $\tau^+ M$ with a sphere S^{2m} of positive even dimension. Furthermore, for a graded vector space $V^* = \oplus_i V^i$ and integer n, denote by $V[n]^*$ the shift of V with $(V[n])^i = V^{i-n}$.

Theorem 3 (Bödigheimer, Cohen and Taylor). *If M is orientable, there is an isomorphism of bigraded vector spaces*

$$\bigoplus_{n \geq 0} H^*(C_n(M); \mathbb{Q})[2mn] \cong H^*(\Gamma_\partial(M, S^{2m}); \mathbb{Q}).$$

Finally, a more modern and computationally powerful method to compute $H^*(C_n(M); \mathbb{Q})$ is by Knudsen.

Theorem 4 (Knudsen [29]). *There is an isomorphism of bigraded spaces*

$$\bigoplus_{n \geq 0} H_*(C_n(M); \mathbb{Q}) \cong H^{\mathcal{L}}(H_c^{-*}(M; \mathcal{L}(\mathbb{Q}^w[d-1])))$$

where $d = \dim M$, $H^{\mathcal{L}}$ is Lie-algebra homology, $H_c^{-*}(M; \mathcal{L}(\mathbb{Q}^w[d-1]))$ is the compactly supported cohomology of M with coefficients in the free graded Lie algebra generated by the orientation sheaf $\mathbb{Q}^w$ of M in degree $d-1$.

A direct consequence is that $H^*(C_n(M); \mathbb{Q})$ only depends on the graded abelian group $H_*(M; \mathbb{Q})$ if d is odd, or on the cup-product $H_c^{-*}(M; \mathbb{Q}^w)^{\otimes 2} \to H_c^{-*}(M; \mathbb{Q})$ if d is even.

My motivation is to study configuration spaces of surfaces with the mapping class group action as discussed in Section 1.2, but I will first discuss results for general

[1] see, e.g. Randal-Williams [45, Proof of Proposition 3.3].

manifolds. I specify that the *diffeomorphism group of* M, $\mathrm{Diff}^+_\partial(M)$, is the topological group of orientation preserving diffeomorphisms of M that pointwise fix the boundary ∂M, and the *mapping class group*, $\mathrm{MCG}(M)$, is the group of its connected components, that is, diffeomorphisms up to isotopy. From the natural continuous actions of $\mathrm{Diff}^+_\partial(M)$ on $F_n(M)$ and $C_n(M)$ descend to actions of $\mathrm{MCG}(M)$ on the homology $H_*(F_n(M))$ and $H_*(C_n(M))$. In both Theorems 3 and 4, there are also $\mathrm{MCG}(M)$-actions on the right hand side. As will be observed in Section 1.2.2, the isomorphism of Theorem 4 is not, in general, $\mathrm{MCG}(M)$-equivariant. I will focus on Theorem 3.

1.1.1 Equivariant untwisted rational scanning

In Chapter 1, I use work of Manthorpe–Tillmann [33] to prove that the isomorphism of Theorem 3 is indeed $\mathrm{MCG}(M)$-equivariant. Furthermore, inspired by Bendersky–Miller [1], I replace $\Gamma_\partial(M, S^{2m})$ by the "untwisted" mapping space $\mathrm{map}_\partial(M, S^{d+2m}_{\mathbb{Q}})$ (the sphere can be freely replaced by its rationalisation while working over $\mathbb{Q}$-coefficients). This has the following two advantages: (i) mapping spaces are easier to study from the point of view of rational homotopy theory, and (ii) the $\mathrm{Diff}^+_\partial(M)$-action on $\mathrm{map}_\partial(M, S^{d+2m})$ is simpler: it only acts by precomposition, whereas on $\Gamma_\partial(M, S^{2m})$, it also acts by the derivative $D\phi$, for $\phi \in \mathrm{Diff}^+_\partial(M)$.

Finally, the direct sum decomposition on the left hand side in Theorem 3 corresponds to a weightspace decomposition of $H^*(\mathrm{map}_\partial(M, S^{d+2m}))$ arising from the action of the group $\mathbb{Q}^\times \cong \pi_0(\mathrm{Homeo}(S^{d+2m}_{\mathbb{Q}}))^\times$. Any vector space V defined functorially in terms of $S^{d+2m}_{\mathbb{Q}}$ is naturally acted upon by $\mathbb{Q}^\times$, and for $k \in \mathbb{Z}$, its k-weightspace $V^{(k)}$ is the subspace of $v \in V$ on which each $q \in \mathbb{Q}^\times$ acts by multiplication by q^k. I prove that the space $H^*(\mathrm{map}_\partial(M, S^{d+2m}))$ decomposes as a direct sum of its weightspaces.

Theorem A (Theorem 18)**.** *Let M be a compact, connected, oriented manifold of dimension d, and $m \geq 1$ an arbitrary integer. There is a* $\mathrm{MCG}(M)$*-equivariant isomorphism of bigraded vector spaces*

$$\bigoplus_{i,n\geq 0} H^i(C_n(M);\mathbb{Q})[2mn] \cong H^*(\mathrm{map}_\partial(M, S^{d+2m}_{\mathbb{Q}});\mathbb{Q}),$$

where the second bidegree on the right hand side is the weight of the $\mathbb{Q}^\times$*-action.*

1.1.2 The superposition product

The cup-product of the mapping space corresponds to a meaningful topologically defined product on configuration spaces. There is a partially defined map $C_n(M) \times C_m(M) \dashrightarrow C_{n+m}(M)$ by superposing two configurations, or taking their unions, when these are disjoint. In Section 2.6, I provide two ways[2] to upgrade this map to the *superposition product*, sup, a well defined graded commutative product on $\bigoplus_{n\geq 0} H^*(C_n(M))$.

Theorem B (Theorem 76)**.** *The isomorphism from Theorem A is an isomorphism of algebras*

$$\left(\bigoplus_{n\geq 0} H^*(C_n(M);\mathbb{Q})[2mn], \mathrm{sup}\right) \cong \left(H^*\left(\mathrm{map}_\partial(M, S^{d+2m});\mathbb{Q}\right), \smile\right).$$

1.1.3 Coloured scanning

One may interpolate between ordered and unordered configuration spaces by considering configuration with colours, so that only points of the same colour are indistinguishable. Specifically for a number of colours $k \geq 1$, and numbers $c_1, ..., c_k \geq 1$, $C_{c_1,c_2,...,c_k}(M)$ is the *space of configurations of k colours with c_i points of colour i*, for $i = 1, ..., k$. For $k = 1$, we recover $C_{c_1}(M)$, and for $c_1 = ... = c_k = 1$, we recover $C_{1,...,1}(M) = F_k(M)$.

I prove a version of equivariant scanning for coloured configurations. The sphere $S^{d+2m}_{\mathbb{Q}}$ in the target is replaced by the wedge of k spheres, one for each colour. Correspondingly, there is an action of the product $(\mathbb{Q}^\times)^k = \pi_0(\mathrm{Homeo}(\vee_{i=1}^k S^{2m_i}_{\mathbb{Q}}))^\times$ which defines multi-weightspaces $V^{(c_1,...,c_k)}$ on a representation V, the i^{th} index corresponding to the weight, as defined above, from the action of the i^{th} $\mathbb{Q}^\times$-factor.

Theorem C (Theorem 97)**.** *For any integers $m_1, ..., m_k \geq 1$, there is a* MCG(M)-*equivariant isomorphism*

$$H^j(C_{c_1,...,c_k}(M);\mathbb{Q}) \cong H^{j+2\sum_i c_i m_i}\left(\mathrm{map}_\partial\left(M, \bigvee_{i=1}^k S^{d+2m_i}\right);\mathbb{Q}\right)^{(c_1,...,c_k)}.$$

[2] One of these ways already appears in Moriyama [37] and Bianchi–Miller–Wilson [3].

In particular, there is a $\mathrm{MCG}(M)$*-equivariant isomorphism*

$$H^j(F_k(M);\mathbb{Q}) \cong \widetilde{H}^{j+2km}\left(\mathrm{map}_\partial\left(M, \bigvee_{i=1}^{n} S^{d+2m}\right);\mathbb{Q}\right)^{(1,1,\ldots,1)}.$$

1.2 Surfaces I: application of scanning

We now focus exclusively on surfaces. For $g \geq 1$, $\Sigma_{g,1}$ is the compact, connected, orientable surface of genus g, with one boundary component; its closed counterpart is Σ_g. Their respective mapping class groups, $\Gamma_{g,1}$ and Γ_g, are important mathematical objects, and we will study their representations arising from configuration spaces.

The inclusion $\Sigma_{g,1} \hookrightarrow \Sigma_g$ induces a group homomorphism $\Gamma_{g,1} \twoheadrightarrow \Gamma_g$ which is surjective, so all Γ_g-representations are also $\Gamma_{g,1}$-representations (but not vice versa). The pattern will often be that we first prove a statement for $\Sigma_{g,1}$ and then transfer it to the harder case Σ_g.

The author has used Farb–Margalit [15] as the main reference for properties of $\Gamma_{g,1}$ and Γ_g.

1.2.1 On $\Gamma_{g,1}$- and Γ_g-representations

The *standard symplectic representation* of $\Gamma_{g,1}$ and Γ_g is $H := H_1(\Sigma_{g,1}) \cong H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$. The two mapping class groups act on H preserving the homology intersection form $\lambda : H \otimes H \to \mathbb{Z}$, giving surjections to the *symplectic group* $\mathrm{Sp}_{2g}(\mathbb{Z})$. The kernels of these actions are the *Torelli groups* $\mathcal{T}_g$ and $\mathcal{T}_{g,1}$, respectively.

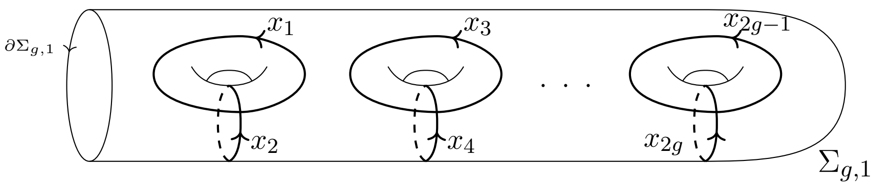


Figure 1.1 Under the basis $x_1,\ldots,x_{2g}$ of $H = H_1(\Sigma_{g,1};\mathbb{Z})$, the intersection form takes the values $\lambda(x_{2i-1},x_{2i}) = -\lambda(x_{2i},x_{2i-1}) = 1$, for $i = 1,\ldots,g$ and vanishes on all other pairings. A basis of H on which λ evaluates like so is called a *standard symplectic basis*.

Representations of $\Gamma_{g,1}$ and Γ_g pulled back from $\mathrm{Sp}_{2g}(\mathbb{Z})$ (or, equivalently, with trivial Torelli action), will be called *symplectic*. In addition, we will call symplectic representations coming from the algebraic group Sp_{2g} *algebraic*. All algebraic representations are subquotients[3] of the tensor powers of $H^{\otimes n}$, for $n \geq 0$. Rationally, they are semi-simple (that is, they split as direct sums of irreducible representations) and the irreducibles are classified. In this thesis, all symplectic representations shall be algebraic.

That the intersection form λ is preserved by $\Gamma_{g,1}$ and Γ_g has the consequences: (i) H is $\Gamma_{g,1}$- and Γ_g-equivariantly self-dual, and (ii) there is a $\Gamma_{g,1}$- and Γ_g-invariant element $\omega \in \Lambda^2 H$ dual to λ; using the basis of Figure 1.1, $\omega = \sum_{i=1}^{g} x_{2i-1} \wedge x_{2i}$.

1.2.2 Configuration spaces of surfaces as representations

In [2], Bianchi computed $\bigoplus_{n\geq 0} H_*(C_n(M);\mathbb{Z}/2)$ as a $\Gamma_{g,1}$-representation, proving, in particular, that it is symplectic. The paper, however, ends with this observation.

Theorem 5 (Bianchi [2, Theorem 7.1]). *For $g,n \geq 2$, $\mathcal{T}_{g,1}$ acts non-trivially on $H_2(C_n(\Sigma_{g,1});\mathbb{Q})$.*

One consequence is that Knudsen's Theorem 4 is not mapping class group equivariant: the action on its right hand side factors through the action on $H^*(\Sigma_{g,1})$ on which $\mathcal{T}_{g,1}$ acts trivially. For us, this means dealing with $\Gamma_{g,1}$-representations that are not algebraic. A generalisation of this notion comes from a filtration $0 = F_n \subset F_{n-1} \subset ... \subset F_1 \subset F_0 = V$ of V by $\Gamma_{g,1}$-subrepresentations; if the filtration quotients F_i/F_{i+1}, for $i = 0,...,n-1$, are algebraic, we say V is *gr-algebraic (of length n)* (see e.g. Hain [20]) and denote by $\mathrm{gr}^F V = \oplus_i F_i/F_{i+1}$ its *associated graded* representation. The definition naturally extends to Γ_g.

Around the same time as Bianchi, Pagaria computed explicitly an associated graded for the gr-algebraic representation $\oplus_{n\geq 0} H_*(C_n(\Sigma_g);\mathbb{Q})$ [38, Theorem 4.11]. The formula is unnecessarily complicated for this exposition, so I only present two corollaries.

Corollary 6 (Pagaria [38, Remark 3.1]). *For $g \geq 0$, in cohomological degrees $* = 0,1,2$, and $* = 3$ if $g = 2$, $H^*(C_n(\Sigma_g);\mathbb{Q})$ is algebraic.*

Therefore, the smallest possibly non-algebraic case is $H_3(C_3(\Sigma_g);\mathbb{Q})$.

[3] I will use this as shorthand for "subrepresentations of quotients".

Corollary 7 (Pagaria [38, Theorem 4.11]). *There is a short-exact sequence of Γ_g-representations*

$$0 \longrightarrow \mathbb{Q} \oplus \mathrm{Sym}^2 H \longrightarrow H_3(C_3(\Sigma_g);\mathbb{Q}) \longrightarrow \Lambda^3 H/\omega H \longrightarrow 0. \qquad (1.2.1)$$

1.2.3 Resolving an extension via submanifolds

In a project that resulted to [48], I completely determined $H_3(C_3(\Sigma_g);\mathbb{Q})$ and, in particular, proved it is not symplectic. I have decided to omit [48] from this thesis: (i) for the sake of space, (ii) because its results can be deduced from Chapter 2 (see, e.g. Section 3.5.6), and (iii) the proof of non-symplecticity was independently done by Looijenga [31]. However, I will discuss here some points from [48] as I believe they illuminate much of what follows.

Let us hide the particularities of the problem by setting $V_1 = \mathbb{Q}$, $V_2 = \mathrm{Sym}^2 H$, $V = V_1 \oplus V_2$ and $U = \Lambda^3 H/\omega H$, remembering that V_1, V_2, and U are irreducible algebraic representations. By picking a linear splitting of (1.2.1), we identify $H_3(C_3(\Sigma_g);\mathbb{Q}) = U \oplus V$ as vector spaces, so the Γ_g-action on $H_3(C_3(\Sigma_g);\mathbb{Q})$ results in an "upper triangular" action on $U \oplus V$ of the form

$$\phi(u \oplus v) = \phi_U(u) \oplus (\phi_V(v) \oplus \xi_\phi(u)), \text{ for } \phi \in \Gamma_g, u \in U, v \in V, \qquad (1.2.2)$$

where ϕ_U and ϕ_V are the symplectic actions of ϕ on U and V, respectively, and $\xi : \Gamma_g \to \mathrm{Hom}(U,V)$ is an extra contribution witnessing that our splitting may not be Γ_g-equivariant. This ξ is, in general, not a group-homomorphism, but what is known as a *crossed homomorphism*[4]. The additive group of crossed homomorphisms up to *coboundaries* (which account for our arbitrary choice of splitting) is precisely the group cohomology $H^1(\Gamma_g;\mathrm{Hom}(U,V))$. Alternatively, extensions of the form (1.2.1) are classified (up to isomomorphism of extensions) by the $\mathbb{Q}$-vector space $\mathrm{Ext}^1_{\mathbb{Q}[\Gamma_g]}(U,V)$, which is naturally isomorphic to $H^1(\Gamma_g;\mathrm{Hom}(U,V))$.

Proposition 8 (Stavrou [48]). *For $g \geq 3$, there is a direct sum decomposition of vector spaces*

$$\mathrm{Ext}^1_{\mathbb{Q}[\Gamma_g]}(U,V) \cong \mathrm{Ext}^1_{\mathbb{Q}[\Gamma_g]}(U,V_1) \oplus \mathrm{Ext}^1_{\mathbb{Q}[\Gamma_g]}(U,V_2),$$

[4] The definition from the literature of ξ being a crossed homomorphism is precisely equivalent to equation (1.2.2) defining a Γ_g-action on $U \oplus V$.

and

$$\dim \mathrm{Ext}^1_{\mathbb{Q}[\Gamma_g]}(U, V_1) = \dim \mathrm{Ext}^1_{\mathbb{Q}[\Gamma_g]}(U, V_2) = 1.$$

In particular, up to isomorphism of representations, there are precisely four Γ_g-representations E that fit into a sequence of the form (1.2.1)*, and they are distinguished by the four cases of whether each of E/V_1 and E/V_2 is symplectic or not.*

To detect these non-symplecticities, we restrict to acting by Torelli elements: as U is symplectic, for any $x \in H_3(C_3(\Sigma_g);\mathbb{Q})$ the difference $\phi(x) - x$ lies in $V_1 \oplus V_2$. So, in [48], I prove that $H_3(C_3(\Sigma_g);\mathbb{Q})$ is the maximally non-trivial case where E/V_1 and E/V_2 are both non-symplectic, by exhibiting, for $i = 1,2$, a Torelli element $\phi_i \in \mathcal{T}_g$ and $x_i \in H_3(C_3(\Sigma_g);\mathbb{Q})$ such that the difference $\phi_i(x_i) - x_i$ projects non-trivially to V_i. The elements x_i are constructed as submanifold classes and the projections to V_i are given by geometric submanifold intersection (cf. Figure 1.2).

Theorem 9 (Looijenga, Stavrou [31, 48])**.** *For $n, g \geq 3$, $\mathcal{T}_g$ acts non-trivially on $H_3(C_n(\Sigma_g);\mathbb{Q})$.*

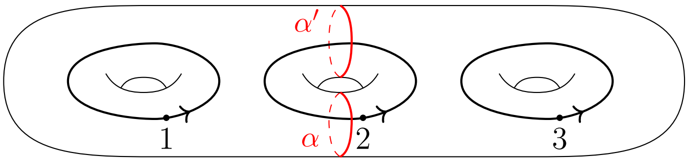


Figure 1.2 A construction from [48]. (A) The Torelli element ϕ_2 is the bounding pair twist on α and α'. (B) The class x_2 is the submanifold class of $M_2 \subset C_3(\Sigma_g)$ given by the torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1 \subset F_3(\Sigma_g)$ with points 1, 2, 3 moving on the drawn circles and then projected to $C_3(\Sigma_g)$ by forgetting the labels.

I want to remark that a key ingredient to the proof of this Proposition 8 is the *Johnson homomorphism*, defined originally by Johnson [23]. It takes the form of a group homomorphism $\tau_{g,1} : \mathcal{T}_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H)$ that (i) has image a certain $\Lambda^3 H \subset \mathrm{Hom}(H, \Lambda^2 H)$, and (ii) modulo 2-torsion, it coincides with the abelianisation of $\mathcal{T}_{g,1}$. Its closed counterpart which is more conveniently stated as $\tau_g : \mathcal{T}_g \to \Lambda^3 H/H$ is (i) also the modulo 2-torsion abelianisation of $\mathcal{T}_g$, and (ii) the target is rationally irreducible. The Johnson homomorphism shall play a key role in this story.

1.2.4 Statement of the result for $C_n(\Sigma_{g,1})$

My work in [48] gave hope that it is feasible to compute exactly the non-algebraic representations $H_*(C_n(\Sigma_g))$. However, Ext-group considerations become too cumbersome for the longer gr-algebraic representations predicted by Pagaria's theorem for $*, n \geq 3$. This had been the initial motivation for proving the equivariant scanning Theorem A: it will now be used to compute $\bigoplus_{n\geq 0} H^*(C_n(\Sigma_{g,1});\mathbb{Q})$ $\Gamma_{g,1}$-equivariantly.

First, let us see the non-equivariant computation produced by Bodigheimer and Cohen as an application of Theorem 3. We will denote by $\mathbb{Q}[S]$ and $\Lambda[S]$, respectively, the free polynomial and exterior algebra on a set S of generators.

Theorem 10 (Bödigheimer–Cohen [7])**.** *There is an isomorphism of bigraded vector spaces*

$$\bigoplus_{n\geq 0} H^*(C_n(\Sigma_{g,1});\mathbb{Q}) \cong \mathbb{Q}[y_1,...,y_{2g},w] \otimes H^*(\Lambda[x_1,...,x_{2g},v],d) \tag{1.2.3}$$

where the differential d takes the value $d(v) = 2\omega$ and vanishes on the x_i; the bidegrees on the left are $(*,n)$, and on the right are $|x_i| = (1,1)$, $|y_i| = (2,2)$, $|w| = (0,1)$ and $|v| = (1,2)$.

Similar computations have been done for all compact surfaces by Drummond-Cole and Knudsen [14] as an application of Theorem 4.

The product of the algebra on the right hand side of (1.2.3) comes from the cup-product of $\Gamma_\partial(\Sigma_{g,1}, S^{2m})$, so, in light of Theorem B, the action on the right hand side that makes the isomorphism (1.2.3) $\Gamma_{g,1}$-equivariant will be compatible with this product. A naive attempt to define an action on this algebra is: (i) define an action on the algebra $\mathbb{Q}[y_1,...,y_{2g},w] \otimes \Lambda[x_1,...,x_{2g},v]$ by declaring that $\Gamma_{g,1}$ acts trivially on v and w and treats the $x_1,...,x_{2g}$ and $y_1,...,y_{2g}$ as two copies of the standard symplectic basis of H from Figure 1.1, (ii) extend via the algebra structure, and (iii) take homology with respect to the differential d. This naive action is symplectic, and so it cannot be correct or it would violate Bianchi's Theorem 5, but the correct action only differs from the naive one by a crossed homomorphism $\xi : \Gamma_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H)$, which I shall now construct.

Let F_{2g} denote the free group on the $2g$ letters α_1, ..., α_{2g}, identified with $\pi_1(\Sigma_{g,1})$ so that α_i is a based loop with homology class $x_i \in H := H_1(\Sigma_{g,1};\mathbb{Z})$. The *standard Magnus expansion* [32] is the function $T : \mathrm{F}_{2g} \to \hat{T}H$, whose target is the free non-commutative algebra of formal sums of tensors in H, sending $\alpha_i \mapsto 1 + x_i$ and $\alpha_i^{-1} \mapsto 1 - x_i + x_i^{\otimes 2} - \ldots$, and extended multiplicatively. Now, compose T with the quotient morphism to the exterior algebra $\hat{T}H \to \Lambda^* H$, and then with the projection to the $\Lambda^2 H$ summand of

Λ^*H. I call this composition $c_2 : \mathrm{F}_{2g} \to \Lambda^2 H$ the *content*[5] of a word, and define the function $\xi : \Gamma_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H)$ by $\xi_\phi : x_i \mapsto c_2(\phi(\alpha_i))$ for $i = 1, ..., 2g$, where $\phi \in \Gamma_{g,1}$. The *Johnson action* on the y_i generators in the right hand side of (1.2.3) is given by

$$\phi^{\text{Johnson}}(y_i) := \phi^{\text{naive}}(y_i) + \xi_\phi(x_i),$$

for every $\phi \in \Gamma_{g,1}$, where the $\Lambda^2 H$ in the target of ξ is viewed as $\Lambda^2\langle x_1, ..., x_{2g}\rangle$.

Theorem D (Theorem 66)**.** *There is a $\Gamma_{g,1}$-equivariant isomomorphism of bigraded algebras*

$$\bigoplus_{n\geq 0} H^*(C_n(\Sigma_{g,1});\mathbb{Q}) \cong \mathbb{Q}[y_1, ..., y_{2g}, w] \otimes H^*(\Lambda[x_1, ..., x_{2g}, v], d)$$

with differential and bidegrees as given in Theorem 10, and the $\Gamma_{g,1}$-action on the right hand side is trivial on v and w, symplectic on the x_i, and the Johnson on the y_i.

1.2.5 Discussion

The function ξ, which is a crossed homomorphism as a consequence of the theorem, is intimately related to other objects from the literature. It differs only by a coboundary and a factor of 2 from the *Johnson–Morita crossed homomorphism* $\widetilde{k}$ defined by Morita [36], and its restriction on $\mathcal{T}_{g,1}$ coincides with twice the Johnson homomorphism $\tau_{g,1}$. Furthermore, the Magnus expansion has been used similarly to above to produce "higher" crossed homomorphisms leading to higher and mod-p Johnson homomorphisms, see Kawazumi, Kitano, and Perron [28, 25, 39].

The fact that ξ is enough to express the entire complexity of the $\Gamma_{g,1}$-action on $\bigoplus_{n\geq 0} H^*(C_n(\Sigma_{g,1});\mathbb{Q})$ may be surprising when, in light of Section 1.2.3, ξ is equivalent to a single representation J fitting in the short exact sequence of $\Gamma_{g,1}$-representations

$$0 \longrightarrow \Lambda^2 H \longrightarrow J \longrightarrow H \longrightarrow 0.$$

Crucially, this was made possible using the multiplicative structure of the superposition or cup product. In the land of representations, this fact translates to: $H^*(C_n(\Sigma_{g,1});\mathbb{Q})$

[5]The content was initially conceived as a function out of F_{2g} with multiplicative-like properties. The fundamental theorem of Fox calculus, see e.g. [17], means that any multiplicative function from F_{2g} to an algebra factors through T.

is a subquotient $\Gamma_{g,1}$-representation of tensor powers of H and J. Ultimately, I believe there is a connection between the longer gr-algebraic representations and the cohomology of $\Gamma_{g,1}$ with coefficients in tensor powers of H. In fact, this cohomology is generated by $\widetilde{k}$, viewed as a class in $H^1(\Gamma_{g,1}; H^{\otimes 3})$, see Morita–Kawazumi [26, 27], via operations which seem parallel to taking tensor products of representations.

Finally, the statement of Theorem D might be disappointing, if one required it as a direct sum of named $\Gamma_{g,1}$-representations. Along those lines is the work of Powell and Vespa on extensions between functors out of the free group category [42]. In the language of Chapter 2, they compute the $\mathrm{Aut}(F_n)$-representations $H^*(\mathrm{map}_*(\vee_n S^1, S^{2m}); \mathbb{Q})$ stably, i.e. in the limit $n \to \infty$. The language of algebraic and gr-algebraic representations still applies there by replacing GL_n with Sp_{2g}, and their answers, which do not involve the product, are given as direct sums of gr-algebraic representations constructed through universal properties. Doing this for $\bigoplus_{n\geq 0} H^*(C_n(\Sigma_{g,1}); \mathbb{Q})$ would require first naming these longer gr-algebraics for $\Gamma_{g,1}$. At the end of the day, one may generate an algebra either by giving many linear generators, or fewer algebraic.

1.3 Surfaces II: polysimplicial decompositions

Theorem D is a proof of concept for the equivariant scanning of Theorem A; its reach, however, goes only as far as $\mathbb{Q}$-coefficients. Furthermore, my attempts to apply Theorem A to closed surfaces remained inconclusive. Chapter 3 builds on these shortcomings and, in collaboration with Andrea Bianchi, I compute the actions of the mapping class groups $\Gamma_{g,1}$ and Γ_g on the homologies of $C_n(\Sigma_{g,1})$ and $C_n(\Sigma_g)$ with various coefficients.

We use a method of Bianchi [2][6], and later used by Bianchi–Miller–Wilson [3]. The idea is to work with the compactification $C_n(\Sigma_{g,1})^\infty$ which collapses all configurations with a point on $\partial\Sigma_{g,1}$ to a new point ∞, which is also declared as the limit point for two configuration points colliding. We endow the new space with a *polysimplicial decomposition*, that is a cellular-decomposition whose cells are products of simplices, see Figure 1.3. This allows for a very explicit computation of the cellular differential. The passage from the homology of the compactification to the homology of $C_n(\Sigma_{g,1})$ is via Poincaré duality. All these steps are compatible with the action of $\Gamma_{g,1}$.

[6] Also, partly, of Moriyama [37].

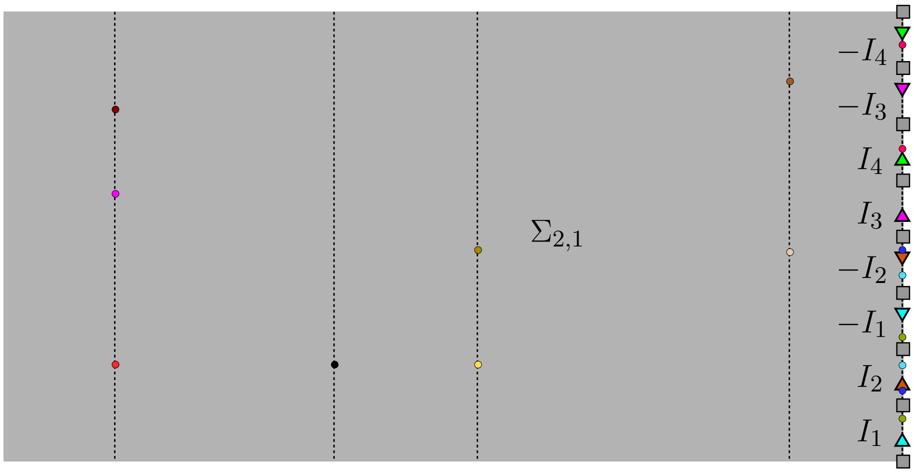


Figure 1.3 An element of the polysimplex corresponding to a datum $\mathfrak{t} = (4,(3,1,2,2),(1,2,0,1))$ in the decomposition of $C_{12}(\Sigma_{2,1})^\infty$. (A) The grey rectangle is our model for $\Sigma_{2,1}$. The right side is partitioned in segments identified according to the names, signs and arrows, yielding the 4 generating loops $\gamma_i = [I_i]$ for $i = 1,...,4$ of $\pi_1(\Sigma_{2,1})$; the other three sides form the boundary $\partial\Sigma_{g,1}$. (B) The tuple $\mathfrak{t} = (4,(3,1,2,2),(1,2,0,1))$ defines the subset of configurations whose points in the interior of $\Sigma_{2,1}$ are organised in 4 vertical bars with 3, 1, 2 and 2 points respectively from left to right; and with precisely 1, 2, 0 and 1 points on the segments I_1, I_2, I_3 and I_4, respectively. (C) The corresponding polysimplex is $\Delta^4 \times (\Delta^3 \times \Delta^1 \times \Delta^2 \times \Delta^2) \times (\Delta^1 \times \Delta^2 \times \Delta^0 \times \Delta^1)$ where: (i) the first simplex Δ^4 parametrises the x-coordinates of the 4 vertical bars, (ii) each of the middle simplices parametrises the vertical positions of points on each bar, and (iii) the final simplices parametrise the points on I_i.

1.3.1 Statement of result for $\Sigma_{g,1}$

We obtain the kernels of the action of $\Gamma_{g,1}$ on the homology of $C_n(\Sigma_{g,1})$ with various coefficients. To state the theorem, we recall the *Johnson kernels* $\mathcal{K}_{g,1}$ and $\mathcal{K}_g$, which are the kernels of the Johnson homomorphisms $\tau_{g,1}$ and τ_g, respectively. By Johnson [23], they are the subgroups of $\Gamma_{g,1}$ and Γ_g, respectively, generated by separating Dehn twists.

We also define their analogues modulo a prime p. The *mod-p Torelli groups* $\mathcal{T}_{g,1}(p)$ and $\mathcal{T}_g(p)$ are the kernels of the $\Gamma_{g,1}$ and Γ_g actions, respectively on $H \otimes \mathbb{Z}/p = H_1(\Sigma_{g,1};\mathbb{Z}/p) \cong H_1(\Sigma_g;\mathbb{Z}/p)$. For p an odd prime, using the crossed homomorphism ξ from Section 1.2.4, we define the *mod-p Johnson homomorphisms* $\tau^p_{g,1} : \mathcal{T}_{g,1}(p) \to \Lambda^3 H \otimes \mathbb{Z}/p$ and, similarly, $\tau^p_g : \mathcal{T}_g(p) \to \Lambda^3 H/\omega H \otimes \mathbb{Z}/p$, as in Perron's work [39]. Their kernels shall be called the *mod-p Johnson kernels* $\mathcal{K}_{g,1}(p)$ and $\mathcal{K}_g(p)$. From [39], all of $\mathcal{T}_{g,1}(p)$, $\mathcal{T}_g(p)$, $\mathcal{K}_{g,1}(p)$ and $\mathcal{K}_g(p)$ are generated by their standard counterparts and the p^{th} powers of all Dehn twists.

Theorem E (Bianchi–Stavrou, Theorems 155 and 158)**.** *Let $n,g \geq 2$. The kernel of the $\Gamma_{g,1}$-action on $H_*(C_n(\Sigma_{g,1});R)$ with $R = \mathbb{Z}$ or $\mathbb{Q}$ is the Johnson kernel $\mathcal{K}_{g,1}$. For p*

an odd prime, the kernel of the $\Gamma_{g,1}$*-action on* $H_*(C_n(\Sigma_{g,1});\mathbb{Z}/p)$ *is the mod-p Johnson kernel* $\mathcal{K}_{g,1}(p)$*.*

We remark that the Torelli group $\mathcal{T}_{g,1}$ acts trivially in the case $g=1$, and, for all $g \geq 1$, with $\mathbb{Z}/2$-coefficients (this is by Bianchi [2]).

1.3.2 Statement of result for Σ_g

Adapting the methods for $\Sigma_{g,1}$ to Σ_g, I prove the following results which are the closed surface analogues of, respectively, Theorem E, Theorem D and Bianchi's $\mathbb{Z}/2$-Theorem from [2].

Theorem F (Theorem 180)**.** *Let* $n,g \geq 3$*. The kernel of the* Γ_g*-action on* $H_*(C_n(\Sigma_g);R)$ *with* $R=\mathbb{Z}$ *or* $\mathbb{Q}$ *is the Johnson kernel* $\mathcal{K}_g$*. For p an odd prime, the kernel of the* Γ_g*-action on* $H_*(C_n(\Sigma_g);\mathbb{Z}/p)$ *is the mod-p Johnson kernel* $\mathcal{K}_g(p)$*.*

Theorem G (Theorem 183)**.** *There is a* Γ_g*-equivariant isomorphism of bigraded algebras*

$$\bigoplus_{n\geq 0} H^*(C_n(\Sigma_g);\mathbb{Q}) \cong H^*(\Lambda[v,x_1,...,x_{2g}]\otimes\mathbb{Q}[w,y_1,...,y_{2g},u]/(u^2),d)$$

where, on the left, the bidegrees are $(*,n)$ *and the product is the superposition product, and on the right,*

- *the bidegrees are* $|w|=(0,1)$*,* $|v|=(1,2)$*,* $|x_i|=(1,1)$*,* $|y_i|=(2,2)$ *and* $|u|=(2,1)$*,*
- *the differential maps* $d(v)=2\omega-2uw$ *and* $d(y_i)=2ux_i$*, and vanishes on the other generators,*
- *with the terminology of Section 1.2.4, the* Γ_g*-action is trivial on* u,v,w*, symplectic on the* x_i *and the Johnson on the* y_i*.*

A version of the above Theorem, but without the Γ_g-action, appeared in Drummond-Cole and Knudsen [14].

Theorem H (Theorem 186)**.** *There is a* Γ_g*-equivariant isomorphism of bigraded algebras*

$$\bigoplus_{n\geq 0} H^*(C_n(\Sigma_g);\mathbb{Z}/2) \cong \Big(\bigoplus_{n\geq 0} H^*(C_n(D^2);\mathbb{Z}/2)\Big)\otimes\mathbb{Z}/2[x_i,y_{i,m},u]/(x_i^2,y_{i,m}^2,u^2),$$

where the indices take the values $i = 1,...,2g, m \geq 1$*, the bidegrees are* $|x_i| = (1,1)$, $|y_{i,m}| = (2m,2m)$ $|u| = (2,1)$*, and the* Γ_g*-action is trivial on* u *and symplectic on the* x_i *and* $y_{i,m}$ *for* $m \geq 1$*. In particular, if* $n \geq 1$*, the kernel of the* Γ_g*-action on* $H_*(C_n(\Sigma_g);\mathbb{Z}/2)$ *is the modulo 2 Torelli group.*

1.4 Ordered configurations: Foggy Roller Coasters[7]

There is a filtration of the mapping class group $\Gamma_{g,1}$, called the *Johnson filtration*,

$$\Gamma_{g,1} = J(0) \supseteq J(1) \supseteq J(2) \supseteq J(3) \supseteq \dots,$$

extending the Torelli group $\mathcal{T}_{g,1} = J(1)$ and the Johnson kernel $\mathcal{K}_{g,1} = J(2)$. To define it, recall the lower central series of a group G, with $G^{(0)} = G$ and $G^{(i+1)} = [G^{(i)}, G]$ for $i \geq 0$. The subgroup $J(i) \subseteq \Gamma_{g,1}$ is the kernel

$$J(i) = \ker\left(\Gamma_{g,1} \curvearrowright \pi_1(\Sigma_{g,1})/\pi_1(\Sigma_{g,1})^{(i)}\right) \tag{1.4.1}$$

of the natural $\Gamma_{g,1}$-action on the nilpotent quotient $\pi_1(\Sigma_{g,1})/\pi_1(\Sigma_{g,1})^{(i)}$.

In this sense, the homology of unordered configurations $H^*(C_n(\Sigma_{g,1}))$ sees only up to the second Johnson subgroup $J(2) = \mathcal{K}_{g,1}$. The ordered configurations $F_n(\Sigma_{g,1})$, however, will allow us to see deeper.

In [37], Moriyama studied certain quotients of compactifications of $F_n(\Sigma_{g,1})$ (whose unordered variants appear in Chapter 3) giving rise to a sequence of representations Mor_n, for $n \geq 1$. By a comparison with the Magnus expansion, Moriyama proved that, for every $n \geq 1$, the kernel of Mor_n is $J(n)$.

Bianchi, Miller and Wilson combined this with the polysimplicial decompositions of Section 1.3 in the ordered context. They proved that the homology of ordered configurations $H_*(F_n(\Sigma_{g,1});\mathbb{Z})$ is a subquotient of tensor products and sums of representations involving only Mor_k for $k \leq n$. It immediately follows that:

Theorem 11 (Bianchi–Miller–Wilson [3])**.** *For* $n \geq 1$*,* $J(n)$ *acts trivially on the homology* $H_*(F_n(\Sigma_{g,1});\mathbb{Z})$*.*

They make the following conjecture.

[7] The title evokes the private name we gave to the main construction seen in Figure 1.4, which was conceived in Copenhagen, not far from Tivoli.

Conjecture 12 (Bianchi–Miller–Wilson [3]). *For $n \geq 1$, the kernel of the $\Gamma_{g,1}$ action on $H_*(F_n(\Sigma_{g,1});\mathbb{Z})$ is generated by $J(n)$ and the boundary Dehn twist $T_{\partial\Sigma_{g,1}}$.*

This conjecture guided my other collaboration with Andrea Bianchi, leading to the preprint [4] which forms Chapter 4.

1.4.1 Statement of results

Our aim is to bound the kernel of the $\Gamma_{g,1}$ from the other direction, by constructing mapping classes $\phi \in \Gamma_{g,1}$ deep in the Johnson filtration, that is $\phi \in J(k)$ with large k, acting non-trivially on $H_*(F_n(\Sigma_{g,1}))$. In this sense, we prove the best possible result:

Theorem I (Theorem 188). *For $g \geq 2$ and $n \geq 1$, the Johnson subgroup $J(n-1)$ acts non-trivially on $H_n(F_n(\Sigma_{g,1}))$.*

There is a corresponding theorem for closed surfaces.

Theorem J (Theorem 229). *For $g \geq 3$ and $n \geq 2$, the closed Johnson subgroup $J_g(n-1) \subset \Gamma_g$ acts non-trivially on $H_{n+1}(F_{n+1}(\Sigma_g))$.*

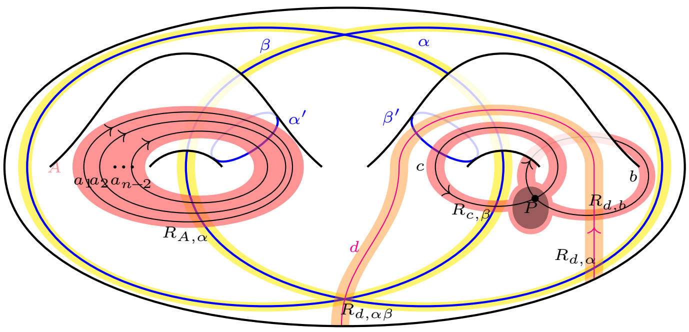


Figure 1.4 A *foggy roller coaster.* (A) The submanifold $M \subset (\Sigma_{g,1})^n$ is the n-torus supported in the pink region, with i^{th} configuration point orbiting on the parallel curves α_i if $i = 1, \ldots, n-2$, the $(n-1)^{th}$ point on curve b and n^{th} point on curve c. The fact that b and c intersect at P means that M does not really lie in $F_n(\Sigma_{g,1})$, but we fix it through a surgery. (B) The mapping class $\phi \in J(n-1)$ is the commutator of the Dehn twist along α repeated $(n-2)$ times and the Dehn twist along β. (C) The open proper submanifold $N \subset F_n(\Sigma_{g,1})$ corresponds to points 1, ..., n travelling freely along d, always remaining in strictly increasing order. (D) The required intersection $[N]^\vee(\phi_*[M] - [M])$ takes place in the yellow neighbourhood of α and β. Ignoring what happens out of it can be thought of as *putting fog.*

The argument is similar to the constructive part of my work in [48] discussed briefly in Section 1.2.3. We construct (i) a class $[M]$ in the homology $H_n(F_n(\Sigma_{g,1}))$ arising from a closed submanifold $M \subset F_n(\Sigma_{g,1})$, (ii) a mapping class $\phi \in J(n-1)$ arising as a commutator of Dehn twists, and (iii) a class $[N]^\vee \in H^n(F_n(\Sigma_{g,1}))$ arising from a proper submanifold $N \subset F_n(\Sigma_{g,1})$, cf. Figure 1.4. The class $[N]^\vee$ witnesses that $\phi_*[M] \neq [M]$ by evaluating $[N]^\vee(\phi_*[M] - [M]) \neq 0$, which is computed by evaluating a new kind of content, as in Section 1.2.4.

I consider this work as an indication for the validity of Conjecture 12, but I expect that a proof of the conjecture will recourse to algebra, as was the fate of [48] compared to Chapters 2 and 3.

Chapter 2

Scanning and application to surfaces

Sections 2.1-2.5 of this Chapter are adapted from the paper [49] of the author appearing in Transactions of the American Mathematical Society.

2.1 Preliminaries

2.1.1 Conventions

In this chapter, M shall be a *smooth, compact, connected* and *oriented* manifold, of dimension d, and possibly with boundary; $\mathrm{Diff}^+_\partial(M)$ its *diffeomorphism group*, and $\mathrm{MCG}(M)$ its *mapping class group*.

Specifically for manifolds, we modify Definition 1 so that $F_n(M)$ and $C_n(M)$ shall stand for $F_n(\mathring{M})$ and $C_n(\mathring{M})$, respectively, that is, for configurations in the interior of M. This affects neither the homotopy type of these spaces (and thus their homology), nor the $\mathrm{Diff}^+_\partial(M)$-action on them, and is better suited for our purposes.

Finally, all (co)homology will be with $\mathbb{Q}$ coefficients unless explicitly stated.

2.1.2 The group $\mathbb{Q}^\times$, sphere actions, and weights

Let $\mathbb{Q}^\times$ denote the group of non-zero rationals with multiplication. For a simply connected space X, a map $r : X \to X_\mathbb{Q}$ is a *rationalisation* if it induces rationalisations on homotopy or, equivalently, homology groups. The rationalisation $X_\mathbb{Q}$ is uniquely defined up to weak equivalence.

Definition 13. A based $\mathbb{Q}^\times$-action on a rational sphere $(S^n_{\mathbb{Q}}, *)$, $n \geq 1$, by homeomorphisms is called a *sphere action* if the induced action on $\pi_n(S^n_{\mathbb{Q}}) = \mathbb{Q}$ is by multiplication by q for every $q \in \mathbb{Q}^\times$. When no ambiguity can arise, we will abusively refer to the induced $\mathbb{Q}^\times$-action on a space constructed functorially out of $S^n_{\mathbb{Q}}$ as a sphere action.

Remark 14. Sphere actions exist (see Section 2.3.7 for an example), but we will not specify one. Furthermore, for any $d, n \geq 1$, specifying a sphere action on $S^n_{\mathbb{Q}}$ induces a sphere action on $S^d \wedge S^n_{\mathbb{Q}}$ which is a rational $(n+d)$-sphere.

Definition 15 (Weightspaces)**.** For a $\mathbb{Q}^\times$-representation V and $k \in \mathbb{Z}$, the *k-weightspace* $V^{(k)}$ is the subspace of V on which each $q \in \mathbb{Q}^\times$ acts by multiplication by q^k. We say that $\mathbb{Q}^\times$ acts on V *purely by weights* if V splits as the direct sum of its weightspaces.

Definition 16 (Weighted algebras)**.** A *weighting* on a (differential) multi-graded commutative algebra $A^{*,\dots,*}$ is an extra grading denoted by $A^{*,\dots,*,(*)}$ that is compatible with the product of A, but *does not* contribute a Koszul sign in the graded commutativity rule (nor in the Leibniz rule). If A has a weighting, we call A *weighted.* Once defined, the weighting index will be usually omitted to alleviate notation.

Remark 17. A multi-graded algebra $A^{*,\dots,*}$ on which $\mathbb{Q}^\times$ acts purely by weights inherits a *weighting* $A^{*,\dots,*,(*)}$. The cohomology of a space functorially constructed out of $S^n_{\mathbb{Q}}$ is naturally weighted from the sphere action.

2.1.3 On algebras

For a group G, a *G-algebra* is an algebra with an action of G by algebra automorphisms.

For a vector space V, denote by $\mathbb{Q}[V]$ (resp. $\Lambda[V]$) the *free polynomial (resp. exterior) algebra generated by V*.

If V is a G-representation, then $\mathbb{Q}[V]$ and $\Lambda[V]$ are naturally G-algebras, and if V is multigraded. then so are $\mathbb{Q}[V]$ and $\Lambda[V]$.

2.2 Configuration spaces in terms of mapping spaces, equivariantly

In this section, we prove our version of *untwisted equivariant scanning* and explain its name. We define the space $\mathrm{map}_\partial(M, X)$ of maps $M \to X$ relative to the boundary, that is maps of pairs $(M,\partial M)\to(X,x_0)$ from a manifold and its (possibly empty) boundary to a based space. For $X=S^{d+2l}_{\mathbb{Q}}$ a rational sphere, the cohomology algebra $H^*(\mathrm{map}_\partial(M,S^{d+2l}_{\mathbb{Q}}))$ inherits two commuting actions: (i) a left action of $\mathrm{MCG}(M)$ via precomposition of maps and (ii) a right action of $\mathbb{Q}^\times$ via postcomposition with the sphere action. Altogether, $H^*(\mathrm{map}_\partial(M,S^{d+2l}_{\mathbb{Q}}))$ is a $\mathrm{MCG}(M)\times\mathbb{Q}^\times$-algebra, and we view it as a weighted $\mathrm{MCG}(M)$-algebra, the weights encoding the $\mathbb{Q}^\times$-action.

Theorem 18. *For $i\geq 0$, $k\geq 0$, and any $l\geq 1$, the sphere action on $H^*(\mathrm{map}_\partial(M,S^{d+2l}_{\mathbb{Q}}))$ is purely by weights, and there is a* $\mathrm{MCG}(M)$*-equivariant isomorphism*

$$H^i(C_k(M))\cong H^{i+2lk}(\mathrm{map}_\partial(M,S^{d+2l}_{\mathbb{Q}}))^{(k)}. \tag{2.2.1}$$

For the proof of this theorem, we interpolate the claimed isomorphism using the auxiliary space $\Gamma_\partial(M;S^{2l}_{\mathbb{Q}})$, and then deal with the two intermediate isomorphisms in two separate steps.

Let us define the space $\Gamma_\partial(M;S^{2l}_{\mathbb{Q}})$. Let τ^+M be the fibrewise one point compactification of the tangent bundle of the manifold M. For a based space (X,x_0), let $\tau^+(M,X):=\tau^+M\wedge_f X$ be the fibrewise smash product of $(\tau^+_xM,0)$ with (X,x_0). This has a canonical section s_0 defined by the smash point. Denote by $\Gamma_\partial(M;X)$ the space of continuous sections of $\tau^+(X,M)$ defined on M that agree with s_0 on ∂M. The diffeomorphism group $\mathrm{Diff}^+_\partial(M)$ acts on $\Gamma_\partial(M;X)$ via bundle maps on τM. For $X=S^{2l}_{\mathbb{Q}}$, there is also an induced $\mathbb{Q}^\times$-action on $\Gamma_\partial(M;S^{2l}_{\mathbb{Q}})$ which commutes with the $\mathrm{Diff}^+_\partial(M)$ action, just like in the case of mapping spaces.

The first step of the argument is proving the $\mathrm{MCG}(M)$-equivariant isomorphism

$$H^i(C_k(M))\cong H^{i+2lk}\Big(\Gamma_\partial(M;S^{2l}_{\mathbb{Q}})\Big)^{(k)}.$$

This is done in Theorem 19 through an equivariant version of an argument due to Bödigheimer–Cohen [7], which uses the Thom space of a bundle over $C_n(M)$, coupled with an equivariant stable splitting result due to Manthorpe–Tillmann [33].

In the second step, we untwist the section space to the corresponding mapping space to obtain the $\mathrm{MCG}(M)$-equivariant isomorphisms

$$H^{i+2lk}\Big(\Gamma_\partial(M;S^{2l}_{\mathbb{Q}})\Big)^{(k)}\cong H^{i+2lk}(\mathrm{map}_\partial(M,S^{d+2l}_{\mathbb{Q}}))^{(k)}. \tag{2.2.2}$$

This isomorphism should come as a surpise: in the case that M is not parallelisable, there is no obvious such map in the level of spaces, and if M is parallelisable, the resulting homeomorphism $\Gamma_\partial(M;S^n_{\mathbb{Q}}) \cong \mathrm{map}_\partial(M;S^{n+d}_{\mathbb{Q}})$ is not $\mathrm{Diff}^+_\partial(M)$-equivariant. To produce this isomorphism in Theorem 21, we use a trick inspired from Bendersky and Miller [1].

Theorem 19. *For $l \geq 1$ and $i,k \geq 0$, the sphere action on $H^*(\Gamma_\partial(M;S^{2l}_{\mathbb{Q}}))$ is purely by weights and there is a* $\mathrm{MCG}(M)$*-equivariant isomorphism*

$$H^i(C_k(M)) \cong H^{i+2lk}\Big(\Gamma_\partial(M;S^{2l}_{\mathbb{Q}})\Big)^{(k)}.$$

Proof. We first observe that $H^0\Big(\Gamma_\partial(M;S^{2l}_{\mathbb{Q}})\Big) \cong \mathbb{Q}$ with the trivial $\mathbb{Q}^\times$-action; it is thus equal to its 0-weightspace and only contributes to $H^*(C_0(M))$ which is $\mathbb{Q}$ in degree $* = 0$. So, in the rest of the proof, we may assume that $k \geq 1$, and replace the right hand side of the asserted equality with the reduced homology $\widetilde{H}^{i+2lk}\Big(\Gamma_\partial(M;S^{2l}_{\mathbb{Q}})\Big)$.

Define the *configuration space of M with labels in the based space* (X,x_0)

$$C(M;X) = \Big(\bigsqcup_{n\geq 1} F_n(M) \times_{\mathfrak{S}_n} X^n\Big)\Big/\sim \tag{2.2.3}$$

where $(z_1,...,z_n;x_1,...,x_n) \sim (z_1,...,z_{n-1};x_1,...,x_{n-1})$ if $x_n = x_0$, and we define its filtration $C_{\leq k}(M;X) \subset C(M;X)$ by configurations with at most k points. Denote by $D_k(M;X) := C_{\leq k}(M;X)/C_{\leq k-1}(M;X)$ the filtration quotients, which are naturally given by the formula

$$D_k(M;X) = (F_k(M)_+ \wedge X^{\wedge k})/\mathfrak{S}_k. \tag{2.2.4}$$

Observe that $D_0(M;X) \simeq *$ for all M and X.

Let G be a group acting by based homeomorphisms on X. A special case of Corollary 4.2 from Manthorpe–Tillman [33] is that, for X connected, there is a $\mathrm{MCG}(M) \times G$-equivariant isomorphism

$$\widetilde{H}_*(\Gamma_\partial(M;X)) \cong \bigoplus_{k\geq 1} \widetilde{H}_*(D_k(M;X)). \tag{2.2.5}$$

We apply this to $X = S^{2l}_{\mathbb{Q}}$ and $G = \mathbb{Q}^\times$ with a sphere action and obtain a $\mathrm{MCG}(M) \times \mathbb{Q}^\times$-equivariant isomorphism

$$\widetilde{H}_*(\Gamma_\partial(M; S^{2l}_{\mathbb{Q}})) \cong \bigoplus_{k \geq 1} \widetilde{H}_*(D_k(M; S^{2l}_{\mathbb{Q}})). \tag{2.2.6}$$

From formula (2.2.4), the rationalisation map $r : S^{2l} \to S^{2l}_{\mathbb{Q}}$ induces a $\mathrm{Diff}^+_\partial(M)$-equivariant $\mathbb{Q}$-homology isomorphism $D(r) : D_k(M; S^{2l}) \to D_k(M; S^{2l}_{\mathbb{Q}})$. For each $n \in \mathbb{Q}^\times \cap \mathbb{Z}$, we construct a based map $f_n : S^{2l} \to S^{2l}$ of degree n by taking the standard degree n map on S^1 and suspending it $2l-1$ times. Then f_n is of degree n, it fixes the S^{2l-2} complementary to S^1 so, in particular, two antipodal points $0, \infty \in S^{2l}$, and $f_n^{-1}(0) = \{0\}$ and $f_n^{-1}(\infty) = \{\infty\}$. The diagram

$$\begin{array}{ccc} D_k(M; S^{2l}) & \xrightarrow{D(r)} & D_k(M; S^{2l}_{\mathbb{Q}}) \\ \downarrow{\scriptstyle D(f_n)} & & \downarrow{\scriptstyle n_D} \\ D_k(M; S^{2l}) & \xrightarrow{D(r)} & D_k(M; S^{2l}_{\mathbb{Q}}) \end{array}, \tag{2.2.7}$$

with n_D the sphere action of $n \in \mathbb{Q}^\times$ and $D(f_n)$ induced from f_n, commutes up to homotopy.

Claim 20. For each $k \geq 0$, there is an $\mathrm{MCG}(M)$-equivariant isomorphism

$$\widetilde{H}^i(D_k(M; S^{2l})) \cong H^{i-2lk}(C_k(M)) \tag{2.2.8}$$

and, for each $n \in \mathbb{Z} \cap \mathbb{Q}^\times$, the induced map $D(f_n)^*$ on $\widetilde{H}^*(D_k(M; S^{2l}))$ is by multiplication by n^k.

From claim 20, it immediately follows that $\widetilde{H}^i(D_k(M; S^{2l}_{\mathbb{Q}})) \cong H^{i-2lk}(C_k(M))$, $\mathrm{MCG}(M)$-equivariantly. Thus the direct sum in the right hand side of isomorphism (2.2.6) is finite, so the isomorphism dualises to

$$\widetilde{H}^i(\Gamma_\partial(M; S^{2l}_{\mathbb{Q}})) \cong \bigoplus_{k \geq 1} H^{i-2lk}(C_k(M)) \tag{2.2.9}$$

for $i \geq 0$, where, from the second part of Claim 20, the $\mathbb{Q}^\times$-action on each $H^{i-2lk}(C_k(M))$ is by weight k. Taking k-weightspaces gives the desired result.

It now remains to prove Claim 20. As in Bödigheimer–Cohen [7], we define the vector bundle

$$\eta_{k,2l} : F_k(M) \times_{\mathfrak{S}_k} (\mathbb{R}^{2l})^k \to C_k(M).$$

By identifying the label-space S^{2l} with the one-point compactification $(\mathbb{R}^{2l})^+$, we obtain a $\mathrm{Diff}^+_\partial(M)$-equivariant homeomorphism

$$h : D_k(M; S^{2l}) \to \mathrm{Th}(\eta_{k,2l}) \tag{2.2.10}$$

to the Thom space of $\eta_{k,2l}$. Denote by E the total space of $\eta_{k,2l}$ and by $E^\#$ the complement of the 0-section.

The bundle $\eta_{k,2l}$ is orientable since, in the definition of $\eta_{k,2l}$, the symmetric group $\mathfrak{S}_k$ permutes copies of the even dimensional $\mathbb{R}^{2l}$. Fixing an orientation on $\eta_{k,2l}$, we obtain a *Thom class* $u_E \in H^{2lk}(E, E^\#)$. The Thom isomorphism is given by

$$\begin{aligned} \tau : H^i(C_k(M)) &\longrightarrow H^{i+2lk}(E, E^\#) \cong \widetilde{H}^{i+2lk}(\mathrm{Th}(\eta_{k,2l})) \\ x &\longmapsto u_E \cup \eta^*_{k,2l}(x). \end{aligned}$$

Now, each $\phi \in \mathrm{Diff}^+_\partial(M)$ induces an orientation preserving automorphism of $\eta_{k,2l}$ which preserves the unique Thom class u_E. Thus τ is $\mathrm{MCG}(M)$-equivariant and we conclude that

$$h^* \circ \tau : H^i(C_k(M)) \to \widetilde{H}^{i+2lk}(D_k(M; S^{2l})) \tag{2.2.11}$$

is a $\mathrm{MCG}(M)$-equivariant isomorphism.

Finally, we show that that $g_n = h \circ D(f_n) \circ h^{-1}$ acts on $H^*(\mathrm{Th}(\eta_{k,2l}))$ by multiplication by n^k. Observe that by the definition of the map $D(f_n)$, the map g_n restricts on $E = \mathrm{Th}(\eta_{k,2l}) - \{\infty\}$ to a (non-linear) $(\mathbb{R}^{2l})^k$-bundle map that covers the identity on $C_k(M)$, fixes the 0-section and, for each $s \in C_k(M)$, maps $E^\#_s \cong ((\mathbb{R}^{2l})^k)^\#$ into itself. Furthermore, the map g_n acts by degree n^k on the smash product of k spheres $(S^{2l})^{\wedge k}$ that appears as $((\mathbb{R}^{2l})^k)^+ \cong E^+_s$ in $\mathrm{Th}(\eta_{k,2l})$. Therefore, g_n acts on each pair $(E_s, E^\#_s)$ by degree n^k. So $g^*_n(u_E) = n^k \cdot u_E$ and, since g_n covers the identity on $C_k(M)$, then $g^*_n(u_E \cup \eta^*_{k,2l}(x)) = n^k \cdot u_E \cup \eta^*_{k,2l}(x)$. We conclude that g_n acts on $\widetilde{H}^*(\mathrm{Th}(\eta_{k,2l}))$ by multiplication by n^k and, thus, so does $D(f_n)$ on $\widetilde{H}^*(D_k(M; S^{2l}))$. This concludes the proof of Claim 20 and of the theorem. □

Theorem 21. *For $n \geq 2$, there is a* $\mathrm{MCG}(M) \times \mathbb{Q}^{\times}$*-algebra isomorphism*

$$H^*(\Gamma_\partial(M; S^n_{\mathbb{Q}})) \cong H^*(\mathrm{map}_\partial(M, S^{n+d}_{\mathbb{Q}})). \tag{2.2.12}$$

Proof. We fix (i) an integer $0 \leq k \leq n-1$ so that $d+k$ is odd, (ii) a model for $S^k_{\mathbb{Q}}$, (iii) a model for $S^{n-k}_{\mathbb{Q}}$ with a sphere action, and (iv) $S^d \wedge S^k_{\mathbb{Q}}$ as our model for $S^{d+k}_{\mathbb{Q}}$. [1]

Let $E^+ \wedge_f S^n_{\mathbb{Q}}$ be the fibrewise one-point-compactification of an oriented vector bundle $E^d \to B$ fibrewise-smashed with $S^n_{\mathbb{Q}}$. By decomposing $S^n_{\mathbb{Q}} = S^k_{\mathbb{Q}} \wedge S^{n-k}_{\mathbb{Q}}$, we observe that

$$E^+ \wedge_f S^n_{\mathbb{Q}} = E^+ \wedge_f S^k_{\mathbb{Q}} \wedge_f S^{n-k}_{\mathbb{Q}}$$

is, in particular, an oriented $S^d \wedge S^n_{\mathbb{Q}}$-fibration-with-a-section fibrewise-smashed with $S^{n-k}_{\mathbb{Q}}$. The universal oriented $S^{d+k}_{\mathbb{Q}}$-fibration-with-a-section is

$$\pi_S : E_{d+k} \longrightarrow BhAut^1_*(S^{d+k}_{\mathbb{Q}}), \tag{2.2.13}$$

whose base is the classifying space of the monoid $hAut^1_*(S^{d+k}_{\mathbb{Q}})$ consisting of based degree-1 homotopy self-equivalences of $S^{d+k}_{\mathbb{Q}}$. Therefore $E^+ \wedge_f S^n_{\mathbb{Q}}$ is pulled back from $E_{d+k} \wedge_f S^{n-k}_{\mathbb{Q}}$ via a classifying map $B \to BhAut^1_*(S^{d+k}_{\mathbb{Q}})$.

We apply this to the universal oriented rank-d vector bundle $\pi_{SO} : \gamma_d \to BSO(d)$ and fix a classifying map $g : BSO(d) \to BhAut^1_*(S^{d+k}_{\mathbb{Q}})$ for $\gamma_d^+ \wedge_f S^n_{\mathbb{Q}}$. We further fix a classifying map $\tau_M : M \to BSO(d)$ for the tangent bundle of M. In all, we obtain the following diagram with two pullback squares

$$\begin{tikzcd}
\tau^+ M \wedge_f S^n_{\mathbb{Q}} \arrow[r] \arrow[d, "\pi_M"] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_d^+ \wedge_f S^n_{\mathbb{Q}} \arrow[r] \arrow[d, "\pi_{SO(d)}"] \arrow[dr, phantom, "\lrcorner", very near start] & E_{d+k} \wedge_f S^{n-k} \arrow[d, "\pi_S"] \\
M \arrow[r, "\tau_M"] & BSO(d) \arrow[r, "g"] & BhAut^1_*(S^{d+k}_{\mathbb{Q}}).
\end{tikzcd} \tag{2.2.14}$$

For a fibre bundle $\pi : E \to B$ with section s and a map $f : M \to B$, let us denote by $L_\partial(f, \pi)$ the space of lifts l of f along π agreeing with s on ∂M, i.e.

$$L_\partial(f,\pi) := \left\{ \begin{tikzcd} & E \arrow[d, "\pi"'] \\ (M, \partial M) \arrow[ur, dashed, "l"] \arrow[r, "f"] & B \arrow[u, bend right, "s"'] \end{tikzcd} : l|_{\partial M} = s \circ f \right\} \tag{2.2.15}$$

[1]Both sides of isomorphism (2.2.12) are independent of the choice of model for the respective rational sphere.

By definition, $\Gamma_\partial(M;S^n_\mathbb{Q}) = L_\partial(\mathrm{id}_M,\pi_M)$. Furthermore, any homotopy $g \simeq_H f$ produces a homotopy equivalence

$$L_H : \; L_\partial(g,\pi) \xrightarrow{\;\sim\;} L_\partial(f,\pi) \;, \tag{2.2.16}$$

and homotopic homotopies H,H' produce homotopic maps $L_H \simeq L_{H'}$. Finally, from the universal property of pullbacks in the diagram (2.2.14), we can canonically identify

$$\Gamma_\partial(M;S^n_\mathbb{Q}) = L_\partial(\mathrm{id}_M,\pi_M) \cong L_\partial(g\circ\tau_M,\pi_S). \tag{2.2.17}$$

Now, for general $N \geq 1$,

$$hAut^1_*(S^N_\mathbb{Q}) \simeq \Omega^N_1 S^N_\mathbb{Q} \simeq \begin{cases} * \text{ , if } N \text{ is odd,} \\ K(\mathbb{Q},N-1) \text{ , if } N \text{ is even,} \end{cases} \tag{2.2.18}$$

so, in particular, $BhAut^1_*(S^{d+k}_\mathbb{Q})$ is contractible, since $d+k$ was chosen to be odd. Therefore, $g\circ\tau_M$ is nullhomotopic, say via a null-homotopy H, so $L_\partial(g\circ\tau_M,\pi_S) \simeq_{L_H} L_\partial(c_*,\pi_S)$ where c_* is the constant map. Lifts of the constant map are just maps to the fibre, i.e. $L_\partial(c_*,\pi_S) = \mathrm{map}_\partial(M, S^{d+k}_\mathbb{Q}\wedge S^{n-k}_\mathbb{Q})$. In all, we obtain a homotopy equivalence

$$\gamma_H : \; \Gamma_\partial(M;S^n_\mathbb{Q}) \xrightarrow{\;\sim\;} \mathrm{map}_\partial(M, S^{d+k}_\mathbb{Q}\wedge S^{n-k}_\mathbb{Q}) \tag{2.2.19}$$

and the induced map γ^*_H is the desired isomorphism. Since all the constructions above leave the final $S^{n-k}_\mathbb{Q}$ coordinate untouched, the equivalence γ_H commutes with the sphere action on $S^{n-k}_\mathbb{Q}$, and so γ^*_H is $\mathbb{Q}^\times$-equivariant. It remains to check that it is also $\mathrm{Diff}^+(M,\partial M)$-equivariant.

For $\psi \in \mathrm{Diff}^+(M,\partial M)$, write $d\psi : \tau M \to \tau M$ for the derivative bundle map. Then ψ acts on $s \in \Gamma_\partial(M;S^n_\mathbb{Q})$, by $\psi * s = d\psi \circ s \circ \psi^{-1}$ and on $f \in \mathrm{map}_\partial(M;S^{d+n}_\mathbb{Q})$ by $\psi * f = f\circ\psi^{-1}$. We obtain the commuting diagram

$$\begin{tikzcd}
\Gamma_\partial(M; S^n_{\mathbb{Q}}) \arrow[r, "\cong"'] \arrow[d, "\psi_* -"] \arrow[rr, bend left, "\gamma_{H\circ\psi}"] & L_\partial(g\circ\tau_M\circ\psi, \pi_{Aut^1_*}) \arrow[r, "L_{H\circ\psi}", "\simeq"'] \arrow[d, "-\circ\psi^{-1}"] & \mathrm{map}_\partial(M; S^{d+k}_{\mathbb{Q}} \wedge S^{n-k}_{\mathbb{Q}}) \arrow[d, "\psi_* -"] \\
\Gamma_\partial(M; S^n_{\mathbb{Q}}) \arrow[r, "\cong"] \arrow[rr, bend right, "\gamma_H"'] & L_\partial(g\circ\tau_M, \pi_{Aut^1_*}) \arrow[r, "L_H", "\simeq"'] & \mathrm{map}_\partial(M; S^{d+k}_{\mathbb{Q}} \wedge S^{n-k}_{\mathbb{Q}}).
\end{tikzcd}$$

But, again, $BhAut^1_*(S^{d+k}_{\mathbb{Q}})$ is contractible, so the two homotopies H and $H\circ\psi$ are homotopic and thus give homotopic maps $\gamma_H \simeq \gamma_{H\circ\psi}$. So the cohomology maps γ^*_H and $\gamma^*_{H\circ\psi}$ are equal and thus γ^*_H commutes with the action of ψ on the cohomology of the two spaces. □

2.2.1 Two simplifications

We may state Theorem 18 avoiding the rational sphere, albeit still with $\mathbb{Q}$-coefficients on homology. Take the submonoid $\mathbb{Z}\cap\mathbb{Q}^\times \cong \pi_0(\mathrm{map}_*(S^N, S^N))$ which acts by the *integral sphere action* on vector spaces constructed functorially out of S^N. Weightspaces are defined as for $\mathbb{Q}^\times$.

Corollary 22. *For any $l \geq 1$, the integral sphere action on $H^*(\mathrm{map}_\partial(M, S^{d+2l}))$ is purely by weights, and there is a* MCG(M)*-equivariant isomorphism*

$$H^i(C_k(M)) \cong H^{i+2lk}(\mathrm{map}_\partial(M, S^{d+2l}))^{(k)}.$$

Proof. Immediate from Theorem 18 using the $\mathrm{Diff}^+_\partial(M)$-equivariant rationalisation $S^{d+2l} \to S^{2l}_{\mathbb{Q}}$ and observing that $\mathbb{Z}\cap\mathbb{Q}^\times$-weightspaces recover the original $\mathbb{Q}^\times$-weightspaces. □

Finally, we can get rid of the weights, if we choose large enough l.

Corollary 23. *For $i, k \geq 1$ and $2l \geq max(2, i, kd)$, there is a* MCG(M)*-equivariant isomorphism*

$$H^i(C_k(M)) \cong \widetilde{H}^{i+2lk}(\mathrm{map}_\partial(M; S^{d+2l}_{\mathbb{Q}})).$$

Proof. The isomorphism (2.2.9) states that

$$\widetilde{H}^{i'}(\Gamma_\partial(M; S^{2l}_{\mathbb{Q}})) \cong \bigoplus_{k' \geq 1} H^{i'-2lk'}(C_k(M)).$$

By setting $i' = i + 2lk$, with i,k,l satisfying the given conditions, the summand $H^{i'-2lk'}(C_k(M)) = H^{i+2l(k-k')}(C_k(M))$ in the right hand side vanishes for $k' \neq k$, since $\widetilde{H}^*(C_k(M))$ vanishes in degrees $* \leq 0$ and $* > kd$, because $C_k(M)$ is a kd-dimensional manifold. We conclude that $\widetilde{H}^{i+2lk}(\Gamma_\partial(M;S^{2l}_{\mathbb{Q}}))^{(k)} = \widetilde{H}^{i+2lk}(\Gamma_\partial(M;S^{2l}_{\mathbb{Q}}))$. □

2.3 Based maps to spheres

In the remainder of this chapter, we apply theorem 18 to the case where $M = \Sigma_{g,1}$ and thus $d = 2$. In this case, the theorem requires us to compute

$$H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S^{2+2l}_{\mathbb{Q}})), \tag{2.3.1}$$

for $l \geq 1$, as a $\Gamma_{g,1} \times \mathbb{Q}^\times$-algebra. We henceforth set $m = l+1$, to simplify notation. As a first step, we relax the mapping space to that of merely basepoint-preserving maps and compute the algebra

$$H^*(\mathrm{map}_*(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}})), \tag{2.3.2}$$

for $m \geq 2$ as a $\Gamma_{g,1} \times \mathbb{Q}^\times$-algebra (Theorem 38). This will be used in section 2.4 to compute (2.3.1). We start with the following definition.

Definition 24. Let $\mathbb{Z}^{*n}$ be the free product of n copies of the group $\mathbb{Z}$. Define the *category of marked*[2] *free groups* $\mathcal{F}$ whose

- objects are the non-negative integers $n \geq 0$ and
- morphisms from n to n' are the group homomorphism $\phi : \mathbb{Z}^{*n} \to \mathbb{Z}^{*n'}$.

Definition 25. For a category $\mathcal{C}$, an *$\mathcal{F}$-object in $\mathcal{C}$* is a functor $\mathcal{F} \to \mathcal{C}$.

By exploiting the deformation retraction of $\Sigma_{g,1}$ to its 1-skeleton $\vee_{2g} S^1 \simeq B\mathbb{Z}^{*2g}$, we will deduce the computation of (2.3.2) from the $\mathcal{F}$-object

$$HMB : n \mapsto H^*(\mathrm{map}_*(B\mathbb{Z}^{*n}, S^{2m}_{\mathbb{Q}})) \tag{2.3.3}$$

in the category of graded-commutative $\mathbb{Q}^\times$-algebras (Theorem 37). We start by constructing enough $\mathcal{F}$-objects to compute HMB.

[2] The generators of the copies of $\mathbb{Z}$ give $\mathbb{Z}^{*n}$ a preferred basis. That is why we call these free groups *marked*.

2.3.1 The content of free words

The rational abelianisation

$$H(n) = H_1(B\mathbb{Z}^{*n}) \cong \mathbb{Q}^n,$$

is naturally an $\mathcal{F}$-object. For each n, define the vector space

$$J(n) = H(n) \oplus \Lambda^2 H(n)$$

and the truncated exterior algebra

$$\Theta(n) = \mathbb{Q} \oplus H(n) \oplus \Lambda^2 H(n).$$

We equip $\Theta(n)$ with a (graded-commutative) $\mathbb{Q}$-algebra structure as the quotient of the exterior algebra $\Lambda^* H(n)$ by wedges of length greater than 2. Both J and Θ can be extended to $\mathcal{F}$-objects using the object H, but this is not what we will do here.

Instead, first observe that $1 + J(n)$ is a subgroup of the multiplicative group of the algebra $\Theta(n)$, with inverses given by

$$(1+z)^{-1} = (1-z) \in \Theta(n)$$

for $z \in J(n)$. We define an exterior version of the standard Magnus expansion (see Kawazumi [25]), namely a group homomorphism

$$\theta : \mathbb{Z}^{*n} \to 1 + J(n) \subset \Theta(n),$$

by mapping the standard basis $\alpha_1, ..., \alpha_n \in \mathbb{Z}^{*n}$ to

$$\theta(\alpha_i) = 1 + [\alpha_i]$$

where $[-] : \mathbb{Z}^{*n} \to \mathbb{Z}^n \subset \mathbb{Q}^n$ is the abelianisation map. The direct sum decomposition $\Theta(n) = \mathbb{Q} \oplus H(n) \oplus \Lambda^2 H(n)$, decomposes θ into $\theta = \theta_0 + \theta_1 + \theta_2$ and it is easy to see that $\theta_0 \equiv 1$ and $\theta_1 = [-]$. We call θ_2 the *content of a word* which is function $c := \theta_2 : \mathbb{Z}^{*n} \to \Lambda^2 H$.

Denote by $l(w)$ the *length* of a word $w \in \mathbb{Z}^{*n}$ with respect to the basis α_i and $\mathcal{A} = \{\alpha_1^{\pm}, ..., \alpha_n^{\pm}\}$ the set of length-one words of $\mathbb{Z}^{*n}$.

Proposition 26. *The content has the following properties:*

1. $c(ab) = c(a) + c(b) + [a] \wedge [b]$ *for all* $a, b \in \mathbb{Z}^{*n}$*;*

2. $c(a) = 0$ *if* $a = 1$ *or* $a \in \mathcal{A}$*;*

3. $c(a^{-1}) = -c(a)$ *for all* $a \in \mathbb{Z}^{*n}$*;*

4. *for each word* $w = w_1 w_2 ... w_k \in \mathbb{Z}^{*n}$ *with* $k \geq 0$ *and* $w_i \in \mathcal{A}$ *for all* i
$$c(w) = \sum_{1 \leq i < j \leq k} [w_i] \wedge [w_j]; \tag{2.3.4}$$

5. $c([a,b]) = 2[a] \wedge [b]$ *where* $[a,b] := aba^{-1}b^{-1}$ *is the commutator;*

6. $c([a,[b,c]]) = 0$ *for all* $a, b \in \mathbb{Z}^{*n}$ *i.e.* c *vanishes on double commutators.*

Proof. Property (1) immediately follows from

$$\theta(ab) = \theta(a) \wedge \theta(b) = (1 + [a] + c(a)) \wedge (1 + [b] + c(b)) = 1 + [a] + [b] + c(a) + [a] \wedge [b] + c(b).$$

Since $\theta(a^{-1}) = 1 - [a]$ for $a \in \mathcal{A}$, (2) and (3) follow, and (4) is a recursive application of the previous three. For property (5), we write $[a,b] = aba^{-1}b^{-1}$, apply property (4) three times to obtain

$$\begin{aligned} c([a,b]) &= c(a) + c(b) + c(a^{-1}) + c(b) \\ &+ [a] \wedge [b] + [a] \wedge [b^{-1}] + [a] \wedge [a^{-1}] + [b] \wedge [a^{-1}] + [b] \wedge [b^{-1}] + [a^{-1}] \wedge [b^{-1}] \end{aligned}$$

and cancel using property (2). Finally, from (5) we get $c([a,[b,c]) = 2[a] \wedge [[b,c]] = 0$ since commutators vanish in the abelianisation and deduce (6). □

Example 27. For $n = 2g$ and $\zeta = [\alpha_1, \alpha_2]...[\alpha_{2g-1}, \alpha_{2g}]$,

$$c(\zeta) = 2\Big([\alpha_1] \wedge [\alpha_2] + ... + [\alpha_{2g-1}] \wedge [\alpha_{2g}]\Big). \tag{2.3.5}$$

The word ζ represents the boundary loop of $\Sigma_{g,1}$ in a standard basis of $\pi_1(\Sigma_{g,1})$. This computation will be used in the proof Lemma 53.

2.3.2 Remarks on the Johnson representation

For a group G, denote by $G^{(i)}$ its lower central series, namely $G^{(0)} = G$ and $G^{(i)} = [G, G^{(i-1)}]$ for $i \geq 1$. We thus obtain a short exact sequence of groups

$$1 \to G^{(1)}/G^{(2)} \to G/G^{(2)} \to G/G^{(1)} = G_{ab} \to 1. \tag{2.3.6}$$

In the case $G = \mathbb{Z}^{*n}$, this sequence has a natural action of $\mathrm{Aut}(\mathbb{Z}^{*n})$. The outermost groups are $G_{ab} = \mathbb{Z}^n$ and $G^{(1)}/G^{(2)} \cong \Lambda^2\mathbb{Z}^n$ through the map

$$\begin{aligned} L^2 : (\mathbb{Z}^{*n})^{(1)} = [\mathbb{Z}^{*n}, \mathbb{Z}^{*n}] &\to \Lambda^2\mathbb{Z}^n \\ [a,b] &\mapsto [a] \wedge [b] \end{aligned}$$

which has kernel precisely $(\mathbb{Z}^{*n})^{(1)}/(\mathbb{Z}^{*n})^{(2)}$ (from [12, 30] the Lie ring of $\mathbb{Z}^{*n}$ is the free Lie ring on $\mathbb{Z}^n$ in degree 1). In this respect the content $c : \mathbb{Z}^{*n} \to \Lambda^2\mathbb{Z}^n$ is a non-canonical (i.e. basis dependent) extension of the map $2L^2$ to the whole of $\mathbb{Z}^{*n}$.

2.3.3 The data $J(\phi)$

With the aim of making J into an $\mathcal{F}$-object, we define candidate data $J(\phi) : J(n) \to J(n')$ for each $\phi : n \to n'$, which we will later show define a functor.

We start with Θ. The extra index $\alpha_{i,(n)} = \alpha_i$ is used here to remember it is an element of $\mathbb{Z}^{*n}$. For each $\phi : n \to n'$, define $\Theta(\phi) : \Theta(n) \to \Theta(n')$ by setting

$$\Theta(\phi) : [\alpha_{i,(n)}] \mapsto \theta(\phi(\alpha_{i,(n)})) - 1 \in J(n'). \tag{2.3.7}$$

Proposition 28. $\Theta(\phi)$ *extends to an algebra morphism.*

Proof. We must check that the properties $z \wedge z = 0$ and $z_1 \wedge z_2 \wedge z_3 = 0$ for $z, z_1, z_2, z_3 \in H(n)$ are preserved by $\Theta(\phi)$. For the first, we have

$$(\theta(w) - 1)^{\wedge 2} = ([w] + c(w))^{\wedge 2} = [w] \wedge [w] (\mathrm{mod}\ \Lambda^{\geq 3}H(m)) = 0$$

for all $w \in \mathbb{Z}^{*m}$, so $\Theta(\phi)([\alpha_{i,(n)}])^{\wedge 2} = (\theta(\phi(\alpha_{i,(n)})) - 1)^{\wedge 2} = 0$. For the second,

$$\Theta(\phi)([\alpha_{i,(n)}]) \wedge \Theta(\phi)([\alpha_{j,(n)}]) \wedge \Theta(\phi)([\alpha_{k,(n)}]) \in J(m)^{\wedge 3} = (0) \subset \Theta(m).$$

□

Manifestly, $\Theta(\phi)(J(n)) \subset J(n')$ for all $\phi : n \to n'$, so we may declare

$$J(\phi) := \Theta(\phi)|_{J(n)}.$$

Furthermore, since $J(\phi)(\Lambda^2 H(n)) \subset \Lambda^2 H(m)$, we get the short exact sequences of vector spaces

$$0 \longrightarrow \Lambda^2 H(n) \xrightarrow{i(n)} J(n) \longrightarrow H(n) \longrightarrow 0 \tag{2.3.8}$$

where $i(n)$ is the inclusion. The sequence (2.3.8) is not obviously split (in fact, it cannot be: see Proposition 33) and we measure its failure to be split by the quantity ξ. This is defined for each $\phi : n \to n' \in \mathcal{F}$, to be the linear map $\xi(\phi) : H(n) \to \Lambda^2 H(m)$,

$$\xi(\phi)([\alpha_{i,(n)}]) = c(\phi(\alpha_{i,(n)})). \tag{2.3.9}$$

That the data $\{\Theta(n), \Theta(\phi)\}$ and $\{J(n), J(\phi)\}$ form an $\mathcal{F}$-object is shown in Corollary 32 as a consequence of Theorem 29. The logic here is not circular: Theorem 29 only constructs a natural isomorphism from a functorial construction on $\{J(n), J(\phi)\}$ onto the *a priori* topologically constructed $\mathcal{F}$-object $n \to H^*(\mathrm{map}_*(B\mathbb{Z}^{*n}, S^{2m}_{\mathbb{Q}}))$.

2.3.4 Maps from $B\mathbb{Z}^{*n}$ to spheres

Fix $m \geq 1$ and set $\mu = 2m - 1$.

Theorem 29. *The graded-commutative $\mathbb{Q}^\times$-algebra (with the sphere action) $HMB(n) = H^*(\mathrm{map}_*(B\mathbb{Z}^{*n}, S^{2m}_{\mathbb{Q}}))$ is isomorphic to the algebra*

$$A(n) := \mathbb{Q}\Big[J(n)[\mu,(1)]\Big] \otimes \Lambda\Big[H(n)[\mu,(2)]\Big]\Big/ I$$

where I is the ideal

$$I = \Big(z - i(z) : z \in \Lambda^2 H(n)\Big)$$

where $i : \Lambda^2 H(n) \to J(n)$ is the inclusion, and the weighting is to insist that $\mathbb{Q}^\times$ acts on $H(2)$ and $J(n)$ by weights 1 and 2, respectively.

Furthermore, under these isomorphisms, for every $\mathcal{F}$-morphism $\phi : n \to n'$ the map $HMB(\phi)$ agrees with the map $A(\phi)$ extended from $J(\phi)$ and $H(\phi)$ via the algebra structure.

Remark 30. Observe that as graded-commutative algebras

$$\mathbb{Q}\Big[J(n)\Big] \otimes \Lambda\Big[H(n)\Big]\Big/ I \cong \mathbb{Q}[H(n)] \otimes \Lambda[H(n)] \cong \mathbb{Q}[y_1, ..., y_n] \otimes \Lambda[x_1, ..., x_n]$$

where the x_i and y_i are the standard bases of $H(n)$ in the the exterior and polynomial algebra respectively.

Proof of Theorem 29. We will replace $B\mathbb{Z}^{*n}$ by $\vee_n S^1$ based at the wedge point, so that $\alpha_i \in \pi_1(\vee_n S^1)$ is the generator of the ith loop. By abuse of notation ϕ will denote a map $\phi : (\vee_n S^1, *) \to (\vee_m S^1, *)$ that induced ϕ on π_1.

We recall a few facts about the Hopf algebra $H^*(\Omega S^{2m}_{\mathbb{Q}}) \cong H^*(\Omega S^{2m})$ (see e.g. sections 3.C and 4.J of Hatcher [21]). First, as graded commutative algebras,

$$H^*(\Omega S^{2m}) \cong \Lambda[x] \otimes \mathbb{Q}[y],$$

with degrees $|x| = \mu, |y| = 2\mu$. We choose the generators x and y as follows. The Pontrjagin product of ΩS^{2m} makes $H_*(\Omega S^{2m}; \mathbb{Z})$ into the polynomial ring $\mathbb{Z}[t]$ with $|t| = \mu$, where t is the Hurewicz image of the generator of $\pi_{2m-1}(\Omega S^{2m})$ adjoint to the identity map of S^{2m}. We define x, y so that they evaluate to 1 on t, t^2 respectively. The coproduct of $H^*(\Omega S^{2m})$ is induced by the loop-concatenation map $C : \Omega S^{2m} \times \Omega S^{2m} \to \Omega S^{2m}$ and given by $C^*(x) = x \otimes 1 + 1 \otimes x$ and $C^*(y) = y \otimes 1 + x \otimes x + 1 \otimes y$, and the involution is induced by the the loop-inversion map $I : \Omega S^{2m} \to \Omega S^{2m}$ and given by $I^*(x) = -x$ and $I^*(y) = -y$.

Now, acting on ΩS^{2m} by a based map of S^{2m} of degree n multiplies t by n and thus t^2 by n^2. By dualising this, we conclude that the sphere action on $H^*(\Omega S^{2m}_{\mathbb{Q}}) = \Lambda[x] \otimes \mathbb{Q}[y]$ is by weight 1 on x and weight 2 on y. We identify $\mathrm{map}_*(S^1, S^{2m}_{\mathbb{Q}}) = \Omega S^{2m}_{\mathbb{Q}}$ and hence

$$\mathrm{map}_*(\vee_n S^1, S^{2m}_{\mathbb{Q}}) = \Big(\mathrm{map}_*(S^1, S^{2m}_{\mathbb{Q}})\Big)^{\times n} = (\Omega S^{2m}_{\mathbb{Q}})^{\times n}, \tag{2.3.10}$$

so by the Künneth theorem we obtain the $\mathbb{Q}^\times$-equivariant identification

$$H^*(\mathrm{map}_*(\vee_n S^1, S^{2m}_{\mathbb{Q}})) = \Lambda[x_1, ..., x_n] \otimes \mathbb{Q}[y_1, ..., y_n]$$

with degrees $|x_i| = \mu$, $|y_i| = 2\mu$ and weights $W(x_i) = 1$, $W(y_i) = 2$. The x_i, y_i are the pullbacks of x, y via the projection p_i to the i-th factor in (2.3.10).

It remains to prove that the action of $\phi \in \mathrm{map}_*(\vee_n S^1, \vee_{n'} S^1)$ is given by $J(\phi)$ on the x_i and y_i. Using the decomposition (2.3.10), it suffices to check only for $n = 1$. In this case, the map $\phi : S^1 \to \vee_n S^1$ induces $\Omega\phi : \left(\Omega S^{2m}_{\mathbb{Q}}\right)^{\times n} \to \Omega S^{2m}_{\mathbb{Q}}$ which is homotopic to a sequence of compositions of C and I, as prescribed by the word $w = \phi(\alpha) \in \mathbb{Z}^{*n}$. More specifically, if $w = \alpha_{i_1}^{\varepsilon_1} \alpha_{i_2}^{\varepsilon_2} ... a_{i_l}^{\varepsilon_l}$ with $\varepsilon_i \in \{\pm\}$, then

$$\Omega\phi \simeq C \circ \left(\iota_{i_1}^{\varepsilon_1} \times \left(C \circ \left(\iota_{i_2}^{\varepsilon_2} \times \left(... \times \iota_{i_l}^{\varepsilon_l}\right)...\right)\right)\right) \tag{2.3.11}$$

where ι_i is the inclusion of the i-th factor in (2.3.10), $\iota_i^+ = \iota_i$ and $\iota_i^- = \iota_i \circ I$. Now, write $[-]_x$, $[-]_y$ for versions of the abelianisation so that $[\alpha_i]_x = x_i$ and $[\alpha_i]_y = y_i$ and assume that $\xi(\phi)$ is expressed in terms of the x_i. An induction on the length l of w concludes that the right hand side composition C_ϕ in (2.3.11) maps $C_\phi^*(x) = [w]_x$ and $C_\phi^*(y) = [w]_y + \xi(\phi)([w])$, that is, by $J(\phi)$, as desired. The inductive step uses the Properties (1) and (2) from Proposition 26 and the observation that $(\iota_i^\varepsilon)^*$ maps $x \mapsto \varepsilon x_i$ and $y \mapsto \varepsilon y_i$, so $\left(C \circ (f \times \iota_i^\varepsilon)\right)^*$ maps $x \mapsto f^*(x) + \varepsilon x_i$ and $y \mapsto f^*(y) + \varepsilon y_i + f^*(x) \wedge \varepsilon x_i$. $\square$

Remark 31. For a general space X, the $\mathcal{F}$-object $n \mapsto H^*(\mathrm{map}_*(\vee_n S^1, X))$ can be determined using the Hopf-structure of $H^*(\Omega X)$, see Conant–Kassabov [10].

Corollary 32. *J and Θ are $\mathcal{F}$-objects and* (2.3.8) *is an exact sequence of $\mathcal{F}$-objects in vector spaces.*

Proof. The isomorphisms $J(n) \cong H^2(\mathrm{map}_*(B\mathbb{Z}^{*n}, S^2_{\mathbb{Q}}))$, for $n \geq 0$, identify $J(\phi)$ with the restriction of $HMB(\phi)$ on H^2 which is *a priori* functorial for composable morphisms. Thus J is an $\mathcal{F}$-object, and the same follows for Θ because J captures its non-trivial part. The short exact sequence is thus functorial in $\mathcal{F}$. $\square$

2.3.5 The free Johnson representation and the Johnson action on algebras

We henceforth fix $n \geq 1$ and recall some representation theory of $\mathrm{Aut}(\mathbb{Z}^{*n})$. The action of $\mathrm{Aut}(\mathbb{Z}^{*n})$ on the abelianisation $\mathbb{Z}^n = (\mathbb{Z}^{*n})_{ab}$ induces the short exact sequence of groups

$$1 \longrightarrow IA_n \longrightarrow \mathrm{Aut}(\mathbb{Z}^{*n}) \longrightarrow GL_n(\mathbb{Z}) \longrightarrow 1, \tag{2.3.12}$$

where the *free Torelli group* IA_n is the kernel of this action. Any $\mathrm{Aut}(\mathbb{Z}^{*n})$-representation that factors through $GL_n(\mathbb{Z})$, or equivalently with trivial IA_n-action, is called *linear*. The *standard linear representation* $H(n)$ is linear, and so is its dual $H^\vee$ and all the subquotients of their tensor powers.

Proposition 33 (The free Johnson representation)**.** *For $\phi \in \mathrm{Aut}(\mathbb{Z}^{*n})$, the linear maps $J(\phi)$ make $J(n)$ into an $\mathrm{Aut}(\mathbb{Z}^{*n})$-representation which fits into a short exact sequence*

$$0 \longrightarrow \Lambda^2 H(n) \xrightarrow{\ i\ } J(n) \longrightarrow H(n) \longrightarrow 0. \tag{2.3.13}$$

of $\mathrm{Aut}(\mathbb{Z}^{*n})$-representations. Furthermore, provided $n \geq 2$, the representation J does not factor through $GL_n(\mathbb{Z})$, i.e. it is not linear, nor through $\mathrm{Out}(\mathbb{Z}^{*n})$.

Proof. The first sentence follows immediately from Corollary 32 and the short exact sequence (2.3.8).

That it does not factor through $GL_n(\mathbb{Z})$ nor $\mathrm{Out}(\mathbb{Z}^{*n})$ can be shown by exhibiting an element $\phi \in \mathrm{Inn}(\mathbb{Z}^{*n}) \subset \mathrm{Aut}(\mathbb{Z}^{*n})$ which acts non-trivially in J, as the subgroup $\mathrm{Inn}(\mathbb{Z}^{*n})$ lies in the kernel of both maps from $\mathrm{Aut}(\mathbb{Z}^{*n})$ to $\mathrm{Out}(\mathbb{Z}^{*n})$ and $\mathrm{GL}_n(\mathbb{Z})$. We choose

$$\phi(a_i) = a_1 a_i a_1^{-1}$$

for $i = 1, ..., n$. Then we have $\phi \cdot y_i = y_i + 2x_1 x_i \neq y_i$ for $i \geq 2$. □

Corollary 34. *The function $\xi : \mathrm{Aut}(\mathbb{Z}^{*n}) \to \hom(H, \Lambda^2 H)$ defined in (2.3.9) is a crossed homomorphism.*

Proof. Immediate from Proposition 33 and the correspondence of extensions and crossed-homomorphisms. □

Remark 35. The naming of J is to refer to Johnson. It is a free group version of the $\Gamma_{g,1}$-representation appearing in Morita [36] and our ξ corresponds to the Johnson-Morita crossed homomorphism $\tilde{k}$ appearing there.

It will often be the case that the vector space $V = H(n) \oplus \Lambda^2 H(n)$ will appear in an algebra generated by two copies of $H(n)$ and we will want to insist that as the $\mathrm{Aut}(\mathbb{Z}^{*n})$-representation V is isomorphic to $J(n)$. We formalise this as follows. Recall the $\mathrm{Aut}(\mathbb{Z}^{*n})$-algebra

$$\mathbb{Q}[y_1, ..., y_n] \otimes \Lambda[x_1, ..., x_n] \cong \mathbb{Q}[J(n)] \otimes \Lambda[H(n)]/I$$

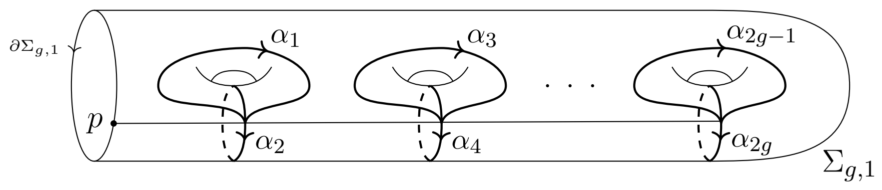


Figure 2.1 A standard symplectic basis of $\pi_1(\Sigma_{g,1}, p)$.

from the statement of Theorem 29.

Definition 36. Let T be an $\mathrm{Aut}(\mathbb{Z}^{*n})$-algebra with a compatible $\Lambda[x_1,...,x_n]$-module structure. The $\mathrm{Aut}(\mathbb{Z}^{*n})$-algebra structure on $\mathbb{Q}[y_1,...,y_n]\otimes T$ naturally inherited from the identification

$$\mathbb{Q}[y_1,...,y_n]\otimes T := (\mathbb{Q}[y_1,...,y_n]\otimes\Lambda[x_1,...,x_n])\otimes_{\Lambda[x_1,...,x_{2g}]} T$$

will be called *the Johnson action on the* y_i.

The following is a rephrasing of Theorem 29 in light of this new definition.

Corollary 37. *There is an isomorphism of weighted graded* $\mathrm{Aut}(\mathbb{Z}^{*n})$*-algebras*

$$H^*(\mathrm{map}_*(\vee_n S^1, S^{2m}_{\mathbb{Q}})) \cong \mathbb{Q}[y_1,...,y_n]\otimes\Lambda[x_1,...,x_n], \tag{2.3.14}$$

with degree/weight on the right given by $|x_i| = (\mu,(1))$ *and* $|y_i| = (2\mu,(2))$*, and* $\mathrm{Aut}(\mathbb{Z}^{*n})$*-action the standard linear on the* x_i *and the Johnson action on the* y_i*.*

2.3.6 $\Gamma_{g,1}$ and maps from surfaces

We fix a genus $g\geq 1$ and turn our attention to the surface $\Sigma_{g,1}$. We fix a basepoint on the boundary $p\in\partial\Sigma_{g,1}$ and a 1-skeleton $\vee_{2g}S^1$ shown in Figure 2.1, together with a based deformation retraction $\pi : (\Sigma_{g,1}, p) \to (\vee_{2g}S^1, *)$. This provides an identification $\pi_1(\Sigma_{g,1}, p) = \mathbb{Z}^{*2g}$ so that the basis $\alpha_1,...,\alpha_{2g}$ is a *standard symplectic basis*, satisfying that

$$[\partial\Sigma_{g,1}] = [\alpha_1,\alpha_2]...[\alpha_{2g-1},\alpha_{2g}] =: \zeta \in \pi_1(\Sigma_{g,1})$$

and, on first homology, the only non-trivial intersection products among basis vectors are $[\alpha_{2i-1}]\cdot[\alpha_{2i}] = 1 = -[\alpha_{2i}]\cdot[\alpha_{2i-1}]$.

The action of $\Gamma_{g,1}$ on $\pi_1(\Sigma_{g,1}) = \mathbb{Z}^{*2g}$ defines a homomorphism $\rho : \Gamma_{g,1} \to \mathrm{Aut}(\mathbb{Z}^{*2g})$, that is injective with image

$$\operatorname{im}\rho = \{\phi \in \mathrm{Aut}(\mathbb{Z}^{*2g}) : \phi(\zeta) = \zeta\}, \tag{2.3.15}$$

so that $\Gamma_{g,1}$ can be viewed as a subgroup of $\mathrm{Aut}(\mathbb{Z}^{*n})$. Consequently, we define the *symplectic crossed homomorphism* ξ, the *symplectic Johnson representation* J and $\Gamma_{g,1}$-*algebras with Johnson action on the* y_i by restricting Definition (2.3.9), Proposition 33 and Definition 36 to $\Gamma_{g,1}$ via ρ.

Theorem 38. *The sphere action on* $H^*(\mathrm{map}_*(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}))$ *is purely by weights and there is an isomorphism of weighted graded* $\Gamma_{g,1}$*-algebras*

$$H^*(\mathrm{map}_*(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}})) \cong \mathbb{Q}[y_1, ..., y_{2g}] \otimes \Lambda[x_1, ..., x_{2g}] \tag{2.3.16}$$

with degree/weight on the right given by $|x_i| = (\mu, (1))$ *and* $|y_i| = (2\mu, (2))$*, and* $\Gamma_{g,1}$*-action the standard symplectic on the* x_i *and the Johnson action on the* y_i*.*

Proof. Immediate from Corollary 37 for $n = 2g$, using the deformation retraction i and the embedding of groups $\rho : \Gamma_{g,1} \to \mathrm{Aut}(\mathbb{Z}^{*2g})$. □

2.3.7 Eilenberg-MacLane spaces and the Postnikov fibration of $S^{2m}_{\mathbb{Q}}$

Definition 39. For $m, r \geq 1$, a based $\mathbb{Q}^\times$-action on an Eilenberg MacLane space $(K(\mathbb{Q}, m), *)$ is called a *degree r action* if, for every $q \in \mathbb{Q}^\times$, the induced action on $\pi_m(K(\mathbb{Q}, m)) = \mathbb{Q}$ is multiplication by q^r. We write $K(\mathbb{Q}(r), m)$ to remember such an action.

Example 40. An odd sphere $S^{2m+1}_{\mathbb{Q}}$ with a sphere action is a $K(\mathbb{Q}(1), 2m+1)$.

We now pick a topological model for the Postnikov fibration of $S^{2m}_{\mathbb{Q}}$, for $m \geq 1$, with a commuting sphere action. We use Chapter 17 of Felix–Halperin–Thomas [16] to sketch such a construction. A Sullivan model for S^{2m} is $(\Lambda^*(u[2m], v[4m-1]); d(v) = u^2)$. It fits in the short exact sequence of augmented Sullivan algebras

$$\begin{tikzcd} (\Lambda^*(v[4m-1]),0) \arrow[dr] & (\Lambda^*(u[2m],v[4m-1]),d(v)=u^2) \arrow[l] \arrow[d] & (\Lambda^*(u[2m]),0) \arrow[l] \arrow[dl] \\ & (\mathbb{Q},0) & \end{tikzcd} \tag{2.3.17}$$

and the group $\mathbb{Q}^\times$ acts on the diagram by algebra automorphisms via $q \cdot u = qu$ and $q \cdot v = q^2 v$. Applying the *spacial realization* functor from *dgcas* to *topological spaces* translates the diagram (2.3.17) to a fibration

$$(K(\mathbb{Q}(2), 4m-1), *) \longrightarrow (S^{2m}_{\mathbb{Q}}, *) \xrightarrow{u} (K(\mathbb{Q}(1), 2m), *) \tag{2.3.18}$$

which is a equipped with a commuting $\mathbb{Q}^\times$-action that is the sphere action in the middle and the degree actions on the base and fibre.

We compute the $\Gamma_{g,1} \times \mathbb{Q}^\times$-algebra $H^*(\mathrm{map}_*(\Sigma_{g,1}, K(\mathbb{Q}(r), m)))$ for $m, r \geq 1$, and how the functor $H^*(\mathrm{map}_*(\Sigma_{g,1}, -))$ applies to the Postnikov fibration (2.3.18).

Proposition 41. *For all $m \geq 2$, the action of $\mathbb{Q}^\times$ on $H^*(\mathrm{map}_*(\Sigma_{g,1}, K(\mathbb{Q}(r), m)))$ is purely by weights and there are isomorphisms of weighted graded commutative $\Gamma_{g,1}$-algebras*

$$H^*\Big(\mathrm{map}_*(\Sigma_{g,1}, K(\mathbb{Q}(r), m))\Big) \cong \begin{cases} \mathbb{Q}[e_1, ..., e_{2g}], & \text{if } m \text{ is odd,} \\ \Lambda[e_1, ..., e_{2g}], & \text{if } m \text{ is even,} \end{cases} \tag{2.3.19}$$

with degrees/weights $|e_i| = (m-1, (r))$ and the standard symplectic action on the e_i.

Furthermore, under these isomorphisms and the isomorphism (2.3.16)*, the functor $H^*(\mathrm{map}_*(\Sigma_{g,1}, -))$ applied to fibration* (2.3.18)

$$K(\mathbb{Q}(1), 2m) \longleftarrow S^{2m}_{\mathbb{Q}} \longleftarrow K(\mathbb{Q}(2), 4m-1))$$

is given by

$$\Lambda[x_1, ..., x_{2g}] \to \mathbb{Q}[y_1, ..., y_{2g}] \otimes \Lambda[x_1, ..., x_{2g}] \to \mathbb{Q}[y_1, ..., y_{2g}], \tag{2.3.20}$$

where the first map is the inclusion and the second is the projection.

Proof. For the first part, the argument is identical to what we have done for spheres. If m is odd, identify $H^*(\Omega K(\mathbb{Z}, m)) = \mathbb{Q}[x[m-1, (1)]]$ with Hopf-algebra structure defined by $x \mapsto x \otimes 1 + 1 \otimes x$. An identical argument to Proposition 29, shows that $H^*(\mathrm{map}_*(\vee_n S^1, K(\mathbb{Q}, m))) = \mathbb{Q}[e_1, ..., e_{2g}]$ with the linear action on the e_i. Transferring to surfaces the action is symplectic. Changing $K(\mathbb{Q}, m)$ to $K(\mathbb{Q}(r), m)$ only changes the weights by a factor of r. If m is even, replace $\mathbb{Q}[-]$ with $\Lambda[-]$ throughout.

Now, the spherical fibration (2.3.18) has rational cohomology maps

$$H^*(\Omega K(\mathbb{Q}(1), 2m)) \to H^*(\Omega S^{2m}_{\mathbb{Q}}) \to H^*(\Omega K(\mathbb{Q}(2), 4m-1))$$

given by

$$\Lambda[x[2m-1,(1)]] \to \Lambda[x[2m-1,(1)]] \otimes \mathbb{Q}[y[4m-2,(2)]] \to \mathbb{Q}[y[4m-2,(2)]].$$

Tensoring this n times gives (2.3.20). □

2.3.8 Appendix: Free maps to spheres

We compute the weighted $\mathrm{Aut}(\mathbb{Z}^{*n}) \times \mathbb{Q}^\times$-algebra structure of $H^*(\mathrm{map}(\vee_n S^1, S^{2m}_{\mathbb{Q}}))$ for free (i.e. unbased) maps. This result is not needed for the remainder of this chapter except briefly in Section 2.5.2. We fix $n, m \geq 1$ and recall the graded weighted $\mathrm{Aut}(\mathbb{Z}^{*n})$-algebra

$$A^*_n := \mathbb{Q}[y_1, ..., y_n] \otimes \Lambda[x_1, ..., x_n]$$

with the Johnson action on the y_i, defined in Theorem 29 with degrees/weights $|x_i| = (\mu, (1))$ and $|y_i| = (2\mu, (2))$.

Definition 42. The *Koszul differential* $d_K : A^*_n \to A^*_n$ is given by $d_K(x_i) = 0$ and $d_K(y_i) = 2x_i$ for $i = 1, ..., n$, and extended as a derivation of degree $(-\mu, (0))$ via the Leibniz rule.

Proposition 43. *The derivation d_K is an* $\mathrm{Aut}(\mathbb{Z}^{*n})$*-equivariant differential with homology $H_*(A_n, d_K) \cong \mathbb{Q}$ generated by the class of the unit $1 \in K$.*

Proof. It is easy to check equivariance and that it is a differential. The homology follows from writing the generators as y_i and $d_K(y_i)$, and observing that this is the homology of polynomial differential forms on an n-simplex. □

The kernel $\ker d_K$ is a weighted graded $\mathrm{Aut}(\mathbb{Z}^{*n})$-subalgebra of A^*_n and the cokernel $\mathrm{coker} d_K$ inherits a weighted graded module structure over $\ker d_K$ from the multiplication in A^*_n. Therefore we can speak of the *square-zero extension*

$$\ker d_K \ltimes \mathrm{coker} d_K[2m, (1)] \tag{2.3.21}$$

where $\mathrm{coker} d_K[\mu+1,(1)]$ is $\mathrm{coker} d_K$ shifted up by $(\mu+1,(1))$ degrees. The square-zero extension is naturally a weighted $\mathrm{Aut}(\mathbb{Z}^{*n})$-algebra.

Theorem 44. *There is an isomorphism of weighted* $\mathrm{Aut}(\mathbb{Z}^{*n})$*-algebras*

$$H^*(\mathrm{map}(\vee_n S^1, S^{2m}_{\mathbb{Q}})) \cong \ker d_K \ltimes \mathrm{coker} d_K[\mu+1,(1)]. \tag{2.3.22}$$

Proof. The evaluation fibration

$$\mathrm{map}_*(\vee_n S^1, S^{2m}_{\mathbb{Q}}) \longrightarrow \mathrm{map}(\vee_n S^1, S^{2m}_{\mathbb{Q}}) \xrightarrow{\mathrm{ev}} S^{2m}_{\mathbb{Q}}, \tag{2.3.23}$$

has associated Serre spectral sequence

$$E_2^{p,q} = H^p(S^{2m}_{\mathbb{Q}}) \otimes H^q(\mathrm{map}_*(\vee_n S^1, S^{2m}_{\mathbb{Q}})) \implies H^{p+q}(\mathrm{map}(\vee_n S^1, S^{2m}_{\mathbb{Q}})). \tag{2.3.24}$$

Its only two non-trivial columns are $p=0$ and $p=2m=\mu+1$, and so the only possibly non-trivial differentials are

$$d_{2m} : \mathbb{Q}\{1\} \otimes S^q(H[\mu] \oplus H[2\mu]) \to \mathbb{Q}\{u\} \otimes S^{q-\mu}(H[\mu] \oplus H[2\mu])$$

where $u \in H^{2m}(S^{2m}_{\mathbb{Q}})$ is the orientation class. Here $S[V] = \Lambda[V^{\mathrm{odd}}] \otimes \mathbb{Q}[V^{\mathrm{even}}]$ is the free *graded* commutative algebra on the graded vector space V.

We claim that d_{2m} is given by $d_{2m}(1 \otimes z) = u \otimes d_K(z)$. Since both $d_{2m}(1 \otimes -)$ and $u \otimes d_K(-)$ are derivations, it suffices to check their equality on the generators $z = x_i, y_i$ for $i = 1,...,n$. By using the retractions of $\vee_n S^1$ to its wedge circles, we need only check for $n=1$. But in that case the fibration is precisely the free loopspace fibration of $S^{2m}_{\mathbb{Q}}$. This is studied in Seeliger [46] for S^{2m} and indeed has $d_{2m}(1 \otimes x) = 0$ and $d_{2m}(u \otimes y) = 2u \otimes x$. It immediately follows that the bigraded $\mathrm{Aut}(\mathbb{Z}^{*n})$-algebra $E_\infty^{*,*}$ is given by

$$1 \otimes \ker d_K \ltimes u \otimes \mathrm{coker} d_K$$

where $(\ker d_K)^i$ and $(\mathrm{coker} d_K)^i$ are in degrees $(i,0)$, and $|u| = (0,2m)$.

Finally, the $\mathbb{Q}^\times$-action on the E_2-page is by weight k on $H^{k\mu}(\mathrm{map}_*(\vee_n S^1, S^{2m}_{\mathbb{Q}}))$ and by weight 1 on u. So it is by weight k on $E_\infty^{0,k\mu}$ and weight $k+1$ on $E_\infty^{2m,k\mu}$, and these are the only non-trivial entries of E_∞. If $m>1$, there is at most one non-trivial entry on every diagonal $p+q=d$, so $H^d(\mathrm{map}(\vee_n S^1, S^{2m}_{\mathbb{Q}}))$ is equal to that entry $\mathrm{Aut}(\mathbb{Z}^{*n}) \times \mathbb{Q}^{\times}$-equivariantly. The cup-product, the $\mathrm{Aut}(\mathbb{Z}^{*n})$-action and the weights on $H^*(\mathrm{map}(\vee_n S^1, S^{2m}_{\mathbb{Q}}))$ are therefore as stated.

In the case $m = 1$, we have $\mu = 1$ and $H^{k+2}(\mathrm{map}(\vee_n S^1, S^{2m}_{\mathbb{Q}}))$ is an extension of $E_\infty^{0,k+2}$ by $E_\infty^{2,k}$ which a priori could be non-trivial. However, the two have different weights, namely $k+2$ and $k+1$ and the weightspaces thus provide an $\mathrm{Aut}(\mathbb{Z}^{*n}) \times \mathbb{Q}^{\times}$-equivariant splitting that is compatible with the cup-product. We conclude again the desired weighted $\mathrm{Aut}(\mathbb{Z}^{*n})$-algebra structure. $\square$

Remark 45 (Relations with higher Hochschild homology). The $\mathrm{Aut}(\mathbb{Z}^{*n})$-representations in Corollary 37 and Theorem 44 are related to the ones studied by Turchin–Willwacher, Powell–Vespa and Gadish–Hainaut [50, 42, 19] via the identification of higher Hochschild homology with homology of mapping spaces as established by Pirashvili [41].

2.4 Maps with fixed boundary behaviour

In this section, we compute the $\Gamma_{g,1} \times \mathbb{Q}^{\times}$-algebra $H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}))$, for $g \geq 0$ and $m \geq 2$, which are fixed throughout the section. As before $\mu = 2m-1$.

2.4.1 The Serre spectral sequence

We first give a summary of some properties of the Serre spectral sequence that we will use in sequel (our main reference is Hatcher [22]).

For a fibration $f : E \to B$ with fibre F, denote by $E_r^{p,q}(f)$ the entries of its Serre spectral sequence, with p called the *column*, q the *row* and r the *page*. Provided B is simply connected, the spectral sequence will be of the form

$$E_2^{p,q}(f) = H^p(B) \otimes H^q(F) \implies H^{p+q}(E). \tag{2.4.1}$$

More specifically, the fibration $f : E \to B$ induces a *Serre filtration* $F_*^*(f)$ of $H^*(E)$

$$0 = F_k^k(f) \subset \ldots \subset F_{i+1}^k(f) \subset F_i^k(f) \subset \ldots \subset F_0^k(f) = H^k(E),$$

and canonical isomorphisms

$$\mathrm{gr}^f(H^*(E))^{p,q} := F_q^{p+q}(f)/F_{q+1}^{p+q} \cong E_\infty^{p,q}(f).$$

2.4.2 The algebra R

The end result of this section is

Theorem 46. *The sphere action on* $H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}))$ *is purely by weights and there is a weighted* $\Gamma_{g,1}$*-equivariant isomorphism*

$$H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S^{2m+2}_{\mathbb{Q}})) \cong H^*(\mathbb{Q}[y_1,...,y_{2g},w] \otimes \Lambda[x_1,...,x_{2g},v], d_R), \tag{2.4.2}$$

where on the right hand side

- *the degrees/weights are* $|x_i| = (\mu,(1))$, $|y_i| = (2\mu,(2))$, $|w| = (\mu-1,(1))$ *and* $|v| = (2\mu-1,(2))$,
- *the differential* d_R *is given by*

$$d_R(v) = 2\omega := 2(x_1x_2 + ... + x_{2g-1}x_{2g}),$$

  vanishing on the x_i, y_i *and* w, *and extended using the Leibniz rule, and*
- *the* $\Gamma_{g,1}$ *action is trivial on* w *and* v, *symplectic on the* x_i *and the Johnson on the* y_i.

Definition 47. Define

$$(R^{*,(*)}, d_R) = (\mathbb{Q}[y_1,...,y_{2g},w] \otimes \Lambda[x_1,...,x_{2g},v], d_R)$$

for the weighted graded differential $\Gamma_{g,1}$-algebra appearing in Theorem 46.

Remark 48. Up to regrading and without the $\Gamma_{g,1}$-action, this is the same as the algebra appearing in Theorem A of Bödigheimer–Cohen [7].

Definition 49. For a bigraded algebra $B^{*,*}$, its *total algebra* $\mathrm{Tot}(B)^*$ is the graded algebra $\mathrm{Tot}(B)^k = \oplus_{p+q=k} B^{p,q}$, with product inherited from $B^{*,*}$.

Remark 50. If $B^{*,*}$ is weighted, then $\mathrm{Tot}(B)^*$ is also weighted. If $(B^{*,*}, d_B)$ is a differential algebra, then $\mathrm{Tot}(B)^*$ inherits a differential d_B and $H^*(\mathrm{Tot}(B), d_B) = \mathrm{Tot}(H(B, d_B))^*$.

Definition 51. An element $b \in B^{*,*}$ of a bigraded algebra is a *column* element (resp. *row* element) if it is of pure degree $|b| = (p,q)$ and $p = 0$ (resp. $q = 0$).

The generating set $\{x_1,...,x_{2g},y_1,...,y_{2g},v,w\}$ of $R^{*,(*)}$ has four obvious types of elements which we call x, y, v and w.

Definition 52. Let S,T be two disjoint subsets of $\{x,y,v,w\}$. The algebra $R(S;T)$ is the *weighted bigraded differential* $\Gamma_{g,1}$*-algebra* so that

- as algebra is freely (in the graded sense) generated by all generators of R of types $S\cup T$ so that the type S generators are row and the type T are column, and each of x, y, w, v have total degree μ, 2μ, $\mu-1$, $2\mu-1$ respectively;
- the *differential* d_R on $R(S;T)$ is defined with the formula from 46 if both $v,x \in S\cup T$ and is trivial otherwise;
- $\Gamma_{g,1}$ acts trivially on v,w and via the symplectic action on the x_i and Johnson on the y_i if both $x,y\in S$ or both $x,y\in T$, and via the symplectic on both the x_i and y_i, otherwise.

Given only one subset $S\subset\{x,y,v,w\}$, $R(S) := \mathrm{Tot}(R(S;\emptyset))$. In practice, we drop the set-braces so, for example, $R(x,y,v,w) = R^{*,(*)}$.

2.4.3 Strategy of proof

A fibration f with total space $\mathrm{map}_\partial(\Sigma_{g,1},S^{2m+2}_{\mathbb{Q}})$ that commutes with the $\mathrm{Diff}^+_\partial(\Sigma_{g,1})$ and the sphere actions, gives a spectral sequence whose pages $E^{*,*}_r(f)$ are weighted bigraded $\Gamma_{g,1}$-algebras with weight preserving $\Gamma_{g,1}$-equivariant differentials. Running the sequence computes only the weighted $\Gamma_{g,1}$-algebra $\mathrm{gr}^f(\mathrm{map}_\partial(\Sigma_{g,1},S^{2m+2}_{\mathbb{Q}}))\cong E^{*,*}_\infty(f)$ and not $H^*(\mathrm{map}_\partial(\Sigma_{g,1},S^{2m+2}_{\mathbb{Q}}))$ per se. Unfortunately, when going from associated graded to the original homology we might not fully recover the algebra structure and the $\Gamma_{g,1}$-action. We will salvage this using *bottom entries* (see 2.4.6) i.e. situations where $F^k_{i+1}=0$ so that $F^k_i(f) = (\mathrm{gr}^f)^{k-i,i} = E^{k-i,i}_\infty$ is indeed a $\Gamma_{g,1}$-subrepresentation of $H^{k-1}(\mathrm{map}_\partial(\Sigma_{g,1},S^{2m+2}_{\mathbb{Q}}))$. With these ideas, the logic of the argument can be summarised into three steps.

1. Show that without the $\Gamma_{g,1}$ action, $H^*(R(x,y,v,w),d_R) = R(y,w)\otimes H^*(R(x,v),d_R)$ and the right hand side is generated by the $y_1,...,y_{2g},w$ and the subalgebra $H^*(R(x,v),d_R)$.

2. Produce enough fibrations whose spectral sequences will give a $\Gamma_{g,1}$-equivariant algebra inclusion $H^*(R(x,v),d_R) \to H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S_{\mathbb{Q}}^{2m+2}))$ as a bottom entry and a (non-equivariant) inclusion $\mathbb{Q}\{y_1,...,y_{2g},w\} \to H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S_{\mathbb{Q}}^{2m+2}))$ so that the extended algebra morphism

$$R(y,w) \otimes H^*(R(x,v),d_R) \to H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S_{\mathbb{Q}}^{2m+2}))$$

is an isomorphism.

3. Check that under this isomorphism, the $\Gamma_{g,1}$-action on w is trivial and on the y_i is Johnson.

2.4.4 The topological input

We define fibrations whose Serre spectral sequences will be of the form $(R(S;T),d_R)$. For reference, we present them collectively in the commutative diagram (2.4.3), with the corresponding cohomology algebras of the spaces presented in diagram (2.4.4).

$$\begin{tikzcd}
\Omega^2 S_{\mathbb{Q}}^{2m} \arrow[r, "j"] \arrow[d, equal] & F_{\chi\circ r} \arrow[r, "\bar{r}"] \arrow[d, hook, "i_{\chi\circ r}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{map}_*\left(\Sigma'_{g,1}, K(\mathbb{Q}(2),4m-1)\right) \arrow[d, hook, "i_\chi"] \\
\Omega^2 S_{\mathbb{Q}}^{2m} \arrow[r] & \mathrm{map}_\partial\left(\Sigma_{g,1}, S_{\mathbb{Q}}^{2m}\right) \arrow[r, "r"] \arrow[d, "\chi\circ r"] & \mathrm{map}_*\left(\Sigma'_{g,1}, S_{\mathbb{Q}}^{2m}\right) \arrow[d, "\chi"] \\
 & \mathrm{map}_*\left(\Sigma'_{g,1}, K(\mathbb{Q}(1),2m)\right) \arrow[r, equal] & \mathrm{map}_*\left(\Sigma'_{g,1}, K(\mathbb{Q}(1),2m)\right).
\end{tikzcd} \tag{2.4.3}$$

$$\begin{tikzcd}
R(v,w) \arrow[d, equal] & R(y,w,v) \arrow[l, "j^*"'] & R(y) \arrow[l, "\bar{r}^*"'] \\
R(v,w) & H^*(R(x,y,v,w),d_R) \arrow[l] \arrow[u, "i_{\chi\circ r}"] & R(x,y) \arrow[l, "r^*"'] \arrow[u, "i_\chi^*"] \\
 & R(x) \arrow[u, "(\chi\circ r)^*"] \arrow[r, equal] & R(x). \arrow[u, "\chi^*"]
\end{tikzcd} \tag{2.4.4}$$

Middle row: fibration r

We fix a smooth self-embedding $\rho : \Sigma_{g,1} \to \Sigma_{g,1}$ isotopic to the identity and supported in a collar neighbourhood of the boundary $\partial\Sigma_{g,1}$ that fixes the basepoint $p \in \partial\Sigma_{g,1}$ and pushes the rest of the boundary slightly inwards. Write $\Sigma'_{g,1} := \operatorname{im}\rho \subset \Sigma_{g,1}$. The map r is defined as the restriction $\operatorname{map}_\partial(\Sigma_{g,1}, S^{2m}) \to \operatorname{map}_*(\Sigma'_{g,1}, S^{2m})$ and it is a fibration because ρ is a cofibration. The complement $\Sigma_{g,1} - \mathring{\Sigma}'_{g,1}$ is a closed disc D^2, so the fibre of r is $\operatorname{map}_\partial(D^2, S^{2m}_{\mathbb{Q}}) = \Omega^2 S^{2m}_{\mathbb{Q}}$. We have the fibration sequence

$$\Omega^2 S^{2m}_{\mathbb{Q}} \longrightarrow \operatorname{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}) \xrightarrow{\ r\ } \operatorname{map}_*(\Sigma'_{g,1}, S^{2m}_{\mathbb{Q}}). \tag{2.4.5}$$

appearing in the middle row of diagram (2.4.3).

Right column: fibration χ

We recall the Postnikov fibration (2.3.18) of $S^{2m}_{\mathbb{Q}}$ and apply on it the fibration preserving functor $\operatorname{map}_*(\Sigma'_{g,1}, -)$ to obtain the fibration sequence

$$\operatorname{map}_*(\Sigma'_{g,1}, K(\mathbb{Q}(2), 4m-1)) \xrightarrow{i_\chi} \operatorname{map}_\partial(\Sigma'_{g,1}, S^{2m}_{\mathbb{Q}}) \xrightarrow{\chi} \operatorname{map}_*(\Sigma'_{g,1}, K(\mathbb{Q}(1), 2m)). \tag{2.4.6}$$

appearing in the right column of diagram (2.4.3).

Middle column: fibration $\chi \circ r$

Composing two fibrations produces another fibration, thus we obtain the fibration sequence

$$F_{\chi\circ r} \xrightarrow{i_{\chi\circ r}} \operatorname{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}) \xrightarrow{\chi\circ r} \operatorname{map}_*(\Sigma'_{g,1}, K(\mathbb{Q}(1), 2m)). \tag{2.4.7}$$

appearing in the middle column of diagram (2.4.3). By virtue of both maps χ and r being fibrations, the fibre $F_{\chi\circ r}$ is the topological pullback in the top-right square of diagram (2.4.3)

$$\begin{tikzcd} F_{\chi\circ r} \arrow[r, "\bar{r}"] \arrow[d, hook, "i_{\chi\circ r}"] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{map}_*\left(\Sigma'_{g,1}, K(\mathbb{Q}(2), 4m-1)\right) \arrow[d, hook, "i_\chi"] \\ \operatorname{map}_\partial\left(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}\right) \arrow[r, "r"] & \operatorname{map}_*\left(\Sigma'_{g,1}, S^{2m}_{\mathbb{Q}}\right) \end{tikzcd} \tag{2.4.8}$$

Consequently, the map $\bar{r}$ is the pullback of the fibration r under i_χ, so it is a fibration as well, and we write $j : \Omega^2 S^{2m}_{\mathbb{Q}} \to F_{\chi\circ r}$ for the inclusion of its fibre.

$\Gamma_{g,1}$ and sphere action

The map r and all other maps of diagram (2.4.3) are $\mathrm{Diff}^+_\partial(\Sigma'_{g,1})$-equivariant. Since the inclusion $\rho : \Sigma_{g,1} \to \Sigma_{g,1}$ is an isotopy equivalence, the mapping class group $\Gamma_{g,1}$ will henceforth be viewed as $\mathrm{MCG}(\Sigma'_{g,1})$ instead of $\mathrm{MCG}(\Sigma_{g,1})$ without affecting the validity of our computations. As for the sphere action, all maps are obviously natural.

2.4.5 The spectral sequences of r and $\chi \circ r$

Lemma 53. *The Serre spectral sequence of the fibration r has only non-trivial differential d_{4m-2} and there is an isomorphism*

$$(E_2^{*,*}(r), d_{4m-2}) \cong (R(x,y;v,w), d_R). \tag{2.4.9}$$

of weighted differential $\Gamma_{g,1}$-algebras.

Proof. The fibration r is the pullback of the path-fibration of $\Omega S^{2m}_{\mathbb{Q}}$ along the restriction map $b : \mathrm{map}_*(\Sigma'_{g,1}, S^{2m}) \to \mathrm{map}_*(\partial\Sigma'_{g,1}, S^{2m}) = \Omega S^{2m}$. Observe $\Omega^2 S^{2m}_{\mathbb{Q}} = \mathrm{map}_\partial(D^2, S^{2m}_{\mathbb{Q}})$.

$$\begin{tikzcd}
\Omega^2 S^{2m}_{\mathbb{Q}} \arrow[r] \arrow[d, equal] & \mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}) \arrow[r, "r"] \arrow[d, "\bar{b}"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}_*(\Sigma'_{g,1}, S^{2m}_{\mathbb{Q}}) \arrow[d, "b"] \\
\Omega^2 S^{2m}_{\mathbb{Q}} \arrow[r] & \mathrm{map}_*(D^2, S^{2m}_{\mathbb{Q}}) \arrow[r, "r_{\partial D^2}"] & \Omega S^{2m}_{\mathbb{Q}}
\end{tikzcd} \tag{2.4.10}$$

The total space $\mathrm{map}_*(D^2, S^{2m}_{\mathbb{Q}})$ of fibration $r_{\partial D^2}$ is contractible (as D^2 is contractible) and so the spectral sequence of $r_{\partial D^2}$ converges to the cohomology of a point. As the base has $H^*(\Omega S^{2m}_{\mathbb{Q}}) \cong \Lambda\Big[x[\mu,(1)]\Big] \otimes \mathbb{Q}\Big[y[2\mu,(2)]\Big]$, a reverse engineering argument shows that the only possible E_2 page of such a spectral sequence is

$$E_2^{*,*}(r_{\partial D^2}) = \mathbb{Q}\Big[w[0,\mu-1,(1)]y[2\mu,0,(2)]\Big] \otimes \Lambda\Big[x[\mu,0,(1)], v[0,2\mu-1,(2)]\Big] \tag{2.4.11}$$

with only non-trivial differentials $d_\mu(w) = x$ and $d_{2\mu}(v) = y$. Therefore the fibre of $r_{\partial D^2}$, and hence of r, has weighted cohomology algebra $H^*(\Omega^2 S^{2m}_{\mathbb{Q}}) \cong \mathbb{Q}\Big[w[0,\mu-1,(1)]\Big] \otimes \Lambda\Big[v[0,2\mu-1,(2)]\Big]$ and the $\Gamma_{g,1}$-action is trivial, as $\mathrm{Diff}^+_\partial(\Sigma'_{g,1})$ acts trivially on the fibre.

As $r_{\partial D^2}$ and r share the same fibre, we combine the calculation of $H^*(\Omega S^{2m}_{\mathbb{Q}})$ with the weighted $\Gamma_{g,1}$-algebra structure of $H^*(\mathrm{map}_*(\Sigma'_{g,1}, S^{2m}_{\mathbb{Q}}))$ from Theorem 38, to obtain the desired isomorphism $E_2^{*,*}(r) \cong R(x,y;v,w)$. We use the map b to compute the differentials.

Up to homotopy b is the map $\mathrm{map}_*(\vee_{2g} S^1, S^{2m}) \to \mathrm{map}_*(S^1, S^{2m})$ corresponding to the homomorphism $\mathbb{Z} \to \mathbb{Z}^{*2g}$ defined by $1 \mapsto \zeta$ where

$$\zeta = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}...\alpha_{2g-1}\alpha_{2g}\alpha_{2g-1}^{-1}\alpha_{2g}^{-1} \in \mathbb{Z}^{*2g}. \tag{2.4.12}$$

From Proposition 29 and Example 27, we get that $b^* : \mathbb{Q}[y] \otimes \Lambda[x] \to \mathbb{Q}[y_1,...,y_{2g}] \otimes \Lambda[x_1,...,x_{2g}]$ maps $b^*(x) = 0$ and $b^*(y) = 2\omega$. Therefore, by comparing with the differentials of (2.4.11), we obtain that $d_\mu(w) = 0$ and $d_{2\mu}(v) = 2\omega$ in $E_*^{*,*}(r)$. □

We compare fibrations r and $\chi \circ r$ via the map of fibrations (id_M, χ) in the diagram

$$\begin{tikzcd}
\Omega^2 S^{2m}_{\mathbb{Q}} \arrow[r, hook, "i"] \arrow[d, "j"] & \mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}) \arrow[r, "r"] \arrow[d, equal] & \mathrm{map}_*(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}) \arrow[d, "\chi"] \\
F_{\chi\circ r} \arrow[r, "i_{\chi\circ r}"] & \mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}) \arrow[r, "\chi\circ r"] & \mathrm{map}_*(\Sigma'_{g,1}, K(\mathbb{Q}(1), 2m)).
\end{tikzcd} \tag{2.4.13}$$

The following lemma shows that taking cohomology in the above diagram we obtain

$$\begin{tikzcd}
R(v,w) & H^*(R(x,y,v,w), d_R) \arrow[l, "i^*"'] & R(x,y) \arrow[l, "r^*"'] \\
R(y,w,v) \arrow[u, "j^*"] & H^*(R(x,y,v,w), d_R) \arrow[l, "i^*_{\chi\circ r}"'] \arrow[u, equal] & R(x). \arrow[l, "(\chi\circ r)^*"'] \arrow[u, "\chi^*"']
\end{tikzcd} \tag{2.4.14}$$

Lemma 54. *The Serre spectral sequence of $\chi \circ r$ has only non-trivial differential d_{4m-2} and there is an isomorphism*

$$(E_2^{*,*}, d_{4m-2}) \cong (R(x;y,v,w), d_R). \tag{2.4.15}$$

of weighted differential $\Gamma_{g,1}$-algebras, where the action on the y on the right is symplectic.

Furthermore, under this isomorphism and the one from Lemma 53, the map of fibrations (id_M, χ) induces the map of spectral sequences $E_2^{*,*}(\chi \circ r) \to E_2^{*,*}(r)$ given by the map

$$\phi : R(x,y;v,w) \to R(x;y,v,w)$$

that maps the w, v, x_i to themselves and vanishes on the y_i.

Proof. We compute $H^*(F_{\chi\circ r})$ by running the Serre spectral sequence of $\bar{r}$ from diagram (2.4.3). Importing the cohomology of the fibre $\Omega^2 S^{2m}_{\mathbb{Q}}$ from Lemma 53 and of the base $\mathrm{map}_*(\Sigma'_{g,1}, K(\mathbb{Q}(2), 4m-1))$ from Proposition 41, we deduce that

$$E_2^{*,*}(\bar{r}) = R(y; w, v). \tag{2.4.16}$$

Using the map of fibrations $(i_{\chi\circ r}, i_\chi)$ to compare with r and Lemma 53, we conclude that all differentials in $E_*^{*,*}(\bar{r})$ are trivial, so $E_\infty^{*,*}(\bar{r}) = E_2^{*,*}(\bar{r})$ is a free graded commutative algebra, with the generators y_i, w, v separated by weight and total degree. Therefore

$$H^*(F_{\chi\circ r}) = R(y, w, v) \tag{2.4.17}$$

with the symplectic $\Gamma_{g,1}$-action on the y_i and the trivial on v, w.

From (2.4.17) and Proposition 41, the Serre spectral sequence of $\chi\circ r$ has $E_2^{*,*}(\chi\circ r) \cong R(x; y, v, w)$ as weighted $\Gamma_{g,1}$-algebras.

Now, the induced map on $E_2^{*,*}(\chi\circ r) \to E_2^{*,*}(r)$ from the map of fibrations (id_M, χ) in (2.4.13) is precisely $\chi^*\otimes j^*$. We compute it under the identifications $E_2^{*,*}(r) = R(x, y; v, w)$ and $E_2^{*,*}(\chi\circ r) = R(x; y, v, w)$. Using the last part of Proposition 41, the map $\chi^* : H^*(\mathrm{map}_*\left(\Sigma'_{g,1}, K(\mathbb{Q}, 2m)\right)) \to H^*(\mathrm{map}_*\left(\Sigma'_{g,1}, S^{2m}_{\mathbb{Q}}\right))$ is the inclusion of algebras

$$\chi^* : \Lambda[x_1, ..., x_{2g}] \to \mathbb{Q}[y_1, ..., y_{2g}] \otimes \Lambda[x_1, ..., x_{2g}]. \tag{2.4.18}$$

The map j is the inclusion of the fibre of $\bar{r}$, so, from the spectral sequence of $\bar{r}$, the map $j^* : H^*(F_{\chi\circ r}) \to H^*(\Omega^2 S^{2m}_{\mathbb{Q}})$ is the algebra surjection

$$\chi^* : \mathbb{Q}[y_1, ..., y_{2g}, w] \otimes \Lambda[v] \to \mathbb{Q}[w] \otimes \Lambda[v] \tag{2.4.19}$$

vanishing on the y_i. Therefore $\chi^*\otimes j^*$ is precisely given by ϕ from the statement of the lemma.

Finally, since ϕ commutes with the differentials, we have $d_{4m-2}(v) = 2\omega$ in $E_*^{*,*}(\chi\circ r)$. There can be no other non-trivial differentials, because $E_2^{*,*}(r) \cong R(x, y; v, w)$ and $E_2^{*,*}(\chi\circ r) \cong R(x; y, w, v)$ have equal total algebras $R(x, y, v, w)$ and they converge to the same algebra $H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}))$; any more differentials and the dimension of

$E_\infty^{*,*}(\chi \circ r)$ in some total degree would be too small. Thus, under $E_2^{*,*} \cong R(x;y,w,v)$, the differential is $d_{4m-2} = d_R$ as desired. $\square$

2.4.6 Bi-graded algebras and filtered graded algebras

We will here make the notion of *bottom* more precise and prove the *lifting lemma* 60.

Definition 55. A *filtered algebra* (A^*, F_*^*) is a graded algebra A^* with a filtration

$$0 = F_{k+1}^k \subset F_k^k \subset ... \subset F_{i+1}^k \subset F_i^k \subset ... \subset F_0^k = A^k,$$

that is multiplicative in the sense that $F_i^k \cdot F_{i'}^{k'} \subset F_{i+i'}^{k+k'}$. The *associated graded* algebra $\mathrm{gr}^F(A)^{*,*}$ is given by the filtration quotients

$$\mathrm{gr}^F(A)^{p,q} = F_q^{p+q}/F_{q+1}^{p+q}$$

and inherits a product from A^*. A morphism $\phi : (A^*, F_*^*) \to (B^*, G_*^*)$ of graded algebras is *filtered* if $\phi(F_i^k) \subset G_i^k$ for all $0 \le i \le k$. In that case, ϕ has *associated graded* morphism

$$\mathrm{gr}(\phi) : \mathrm{gr}^F(A) \to \mathrm{gr}^G(B)$$

which is also a morphism of bigraded algebras.

Definition 56. For a bigraded algebra $B^{*,*}$, the *tautological filtration* $F_*^*(\mathrm{Tot})$ of $\mathrm{Tot}(B)^*$ is

$$F_i \,\mathrm{Tot}(B)^k = \bigoplus_{p+q=k, q \le i-k} B^{p,q}.$$

Remark 57. Naturally, $\mathrm{gr}^F(\mathrm{Tot}(B))^{*,*} \cong B^{*,*}$, but in general $\mathrm{Tot}(\mathrm{gr}^F(A))^* \ncong A^*$.

For a subset $S \subset \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ define $B^S = \oplus_{(p,q)\in S} B^{p,q}$ as a bigraded vector subspace of $B^{*,*}$ and denote its inclusion by $i_S : B^S \hookrightarrow B^{*,*}$. If S is additively closed and contains $(0,0)$, then $B^S \subset B^{*,*}$ is in fact a subalgebra.

Definition 58. For $k \ge 0$, we say that $(p,q) \in \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ is *a bottom entry in total degree* k of $B^{*,*}$ if $p+q=k$ and, for $p'+q'=k$ and $q' < q$, $B^{p',q'} = 0$.

Definition 59. For a filtered algebra (A^*, F_*^*), a set of indices $S \subset \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ is *bottom and closed* if it satisfies

1. $(0,0) \in S$,
2. each $(p,q) \in S$ is a bottom entry in its total degree for $\mathrm{gr}^F(A)^{*,*}$ and
3. for each $(p,q),(p',q') \in S$, then $(p+p',q+q')$ is also a bottom entry of $\mathrm{gr}^F(A)^{*,*}$ and furthermore, if $\mathrm{gr}^F(A)^{p+p',q+q'} \neq 0$, then $(p+p',q+q') \in S$.

Lemma 60 (Lifting Lemma)**.** *Let (A^*, F_*^*) be a filtered graded algebra with a bottom and closed set of indices $S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Then, $\mathrm{gr}^F(A)^S \subset \mathrm{gr}^F(A)^{*,*}$ is a bigraded subalgebra, and this inclusion of algebras, i_S, has a canonical lift*

$$\phi_S : \Big(\mathrm{Tot}(\mathrm{gr}^F(A)^S), F_*^*(\mathrm{Tot})\Big) \to (A^*, F_*^*), \tag{2.4.20}$$

in the sense that ϕ_S is a filtered monomorphism of algebras whose associated graded

$$\mathrm{gr}(\phi_S) : \mathrm{gr}^{F(\mathrm{Tot})}\Big(\mathrm{Tot}(\mathrm{gr}^F(A)^S)\Big) = \mathrm{gr}^F(A)^S \to \mathrm{gr}^F(A)^{*,*} \tag{2.4.21}$$

is the inclusion i_S.

Proof. We first construct the linear map ϕ_S and prove it is a lift. If (p,q) is a bottom entry then $F_i^{p+q} = 0$ for all $i > q$ and so $\mathrm{gr}^F(A)^{p,q}$ is canonically isomorphic to F_q^{p+q}. Composing this isomorphism with the inclusion $F_q^{p+q} \subset A^{p+q}$, provides a canonical linear monomorphism $\phi_{p,q} : \mathrm{gr}^F(A)^{p,q} \to A^{p,q}$, and the monomorphisms $\phi_{p,q}$ piece together to form the graded linear monomorphism $\phi_S : \mathrm{Tot}(\mathrm{gr}^F(A)^S) \to A^*$. But now, as S contains only bottom entries, it has at most one non-trivial entry in each diagonal, and so it is tautological that ϕ_S is filtered with respect to $F_*^*(\mathrm{Tot})$ and F_*^* and has $\mathrm{gr}(\phi_S) = i_S$

That $\mathrm{gr}^F(A)^S$ is a subalgebra of $\mathrm{gr}^F(A)^{*,*}$ follows from Definition 59 of *bottom and closed*: from property (1), $\mathrm{gr}^F(A)^S$ contains the unit; from (3), it is multiplicatively closed. It remains to prove that ϕ_S is multiplicative.

For $i = 1,2$, pick $(p_i,q_i) \in S$ and $z_i \in F_{q_i}^{p_i+q_i}$ and write

$$[z_i] \in \mathrm{gr}^F(A)^{p_i,q_i} = F_{q_i}^{p_i+q_i} / F_{q_i+1}^{p_i+q_i} = F_{q_i}^{p_i+q_i}.$$

The last equality holds because all $(p_i,q_i) \in S$ are bottom. Observe that $\phi_S([z_i]) = \phi_{p_i,q_i}([z_i]) = z_i$ and denote by $\cdot_A$ the product in A and $\cdot_{\mathrm{gr}}$ the product in $\mathrm{gr}(A)$. Then by definition,

$$[z_1] \cdot_{\mathrm{gr}} [z_2] = [z_1 \cdot_A z_2] \in \mathrm{gr}(A)^{p_1+p_2,q_1+q_2}.$$

By property (3) of Definition 59, (p_1+p_2, q_1+q_2) is also a bottom entry, so we have two cases:

1. $\mathrm{gr}(A)^{p_1+p_2,q_1+q_2} \neq 0$ and so $(p_1+p_2, q_1+q_2) \in S$, therefore

$$\phi_S([z_1] \cdot_{\mathrm{gr}} [z_2]) = \phi_S([z_1 \cdot_A z_2]) = z_1 \cdot_A z_2 = \phi_S(z_1) \cdot \phi_S(z_2),$$

   or

2. $\mathrm{gr}(A)^{p_1+p_2,q_1+q_2} = 0$, so $F^{p_1+p_2+q_1+q_2}_{q_1+q_2} = 0$ and $z_1 \cdot_A z_2 = 0$, and therefore $\phi_S([z_1] \cdot_{\mathrm{gr}} [z_2]) = 0 = \phi_S(z_1) \cdot \phi_S(z_2)$.

In either case, ϕ_S is multiplicative. □

2.4.7 The exterior algebra $R(x; v)$

The differential d_R vanishes on $R(y; w)$, so

$$H^{*,*}(R(x, y; v, w), d_R) = R(y; w) \otimes H^{*,*}(R(x; v), d_E). \tag{2.4.22}$$

Even if the inclusion of the exterior part $(R(x; v), d_R) \subset (R(x, y; v, w), d_R)$ is $\Gamma_{g,1}$-equivariant, the equality (2.4.22) is not: on the right hand side the action on the y_i is symplectic whereas on the left it is Johnson. Let us denote by $d_E := d_R|_{R(x;v)}$ the differential on $R(x; v)$ and study $H^{*,*}(R(x; v), d_E)$.

The $\Gamma_{g,1}$-algebra $\Lambda^* = \Lambda^{*,0} := \Lambda[x_1, ..., x_{2g}]$, with $|x_i| = (\mu, 0)$ has a $\Gamma_{g,1}$-equivariant endomorphism

$$\Phi : z \mapsto z \wedge \omega,\ z \in \Lambda^*.$$

Denote by $V^* := \Lambda^*/(\omega)$ the cokernel $\Gamma_{g,1}$-algebra and $K^* := \ker(\Phi)$, a graded $\Gamma_{g,1}$-representation. Both V^* and K^* are naturally Λ^*-modules.

Lemma 61 (The structure of $H^{*,*}(R(x; v), d_E)$)**.** *The following properties hold:*

1. *the algebra $R(x; v)$ is the free Λ^*-module,*

$$R^{*,*}(x; v) = \Lambda^{*,0} \otimes \mathbb{Q}\{1, v\}; \tag{2.4.23}$$

2. *the differential d_E is given by $d_E(z) = 0$ and $d_E(zv) = z \wedge \omega = 2\Phi(z)$, for $z \in \Lambda^*$, and is a Λ^*-module map;*

3. *the homology $H^{*,*}(R(x;v), d_E)$ is the weighted bigraded Λ^*-module*

$$V^{*,0} \otimes \mathbb{Q}\{1\} \oplus K^{*,0} \otimes \mathbb{Q}\{v\} \tag{2.4.24}$$

4. *and it is non-zero precisely in the set of bidegrees*

$$S_E := \{\Big(\mu k, 0\Big), k = 0, 1, ..., g\} \cup \{\Big(\mu k, 2\mu - 1\Big), k = g, g+1, ..., 2g\}. \tag{2.4.25}$$

Proof. All $\Gamma_{g,1}$-algebras and representations here factor through the surjection $\Gamma_{g,1} \to \mathrm{Sp}_{2g}(\mathbb{Z})$, and so this is a question of $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation theory. Properties (1) and (2) are immediate and (3) follows from (2). Now, forgetting any gradings, the $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant map $\Phi : \Lambda^i(H) \to \Lambda^{i+2}(H)$ given by wedging with ω is injective for $i \le g-1$ and surjective for $i \ge g-1$ with $\Lambda^i(H)/\operatorname{im}(\Phi)$ a non-zero $\mathrm{Sp}_{2g}(\mathbb{Z})$ irreducible for $i = 0, ..., g$ (see Fulton–Harris [18]). In particular $\Lambda^i(H)/\operatorname{im}(\Phi) = 0$ for $i > g$, and by duality $\Lambda^i(H)/\operatorname{im}(\Phi) \cong \ker^{2g-i}(\Phi)$ for all $i \ge 0$. Remembering the bigradings on the x_i, we get that V^* is non-zero precisely in bidegrees $(\mu k, 0), k = 0, 1, ..., g$ and K^* in bidegrees $(\mu k, 0), k = g, g+1, ..., 2g$. Using $|v| = (0, 2\mu - 1)$ and (3), part (4) follows. □

Lemma 62. *The set of indices S_E is bottom and closed for the bigraded algebra $E_\infty^{*,*}(\chi \circ r)$.*

Proof. For this argument we write $E_\infty^{*,*} := E_\infty^{*,*}(\chi \circ r)$. From Lemma 54, we have

$$\mathrm{gr}^{\chi \circ r}(H^*) = E_\infty^{*,*} = R(\emptyset; y, w) \otimes H^{*,*}(R(x;v), d_E)$$

and all the polynomial generators are column. This means that $E_\infty^{*,*}$ is the direct sum of copies of $H^{*,*}(R(x;v), d_E)$ shifted upwards in different rows. This immediately shows S_E satisfies the first two conditions of Definition 59 for $E_\infty^{*,*}$. It also shows that $E_\infty^{p,q}$ vanishes for columns $p > 2g\mu$ so any entry (p,q) with $p \ge 2g\mu$ is automatically bottom. We check the third condition of Definition 59.

1. For $(i_1, 0), (i_2, 0) \in \{\Big(\mu k, 0\Big), k = 0, 1, ..., g\}$, the entry $(i_1 + i_2, 0)$ is bottom by being in the bottom row, and either $i_1 + i_2 \le g\mu$ so $(i_1 + i_2, 0) \in S_E$ or otherwise $E_\infty^{i_1+i_2,0} = 0$.

2. For $(i_1,0) \in \{\big(\mu k,0\big), k = 0,1,...,g\}$ and $(i_2,2\mu-1) \in \{\big(\mu k,2\mu-1\big), k = g,g+1,...,2g\}$, the entry $(i_1+i_2,2\mu-1)$ is either in S_E if $i_1+i_2 \leq 2g\mu$, or $i_1+i_2 \leq 2g\mu$ and so $(i_1+i_2,2\mu-1)$ is bottom and $E_\infty^{i_1+i_2,2\mu-1} = 0$.

3. For $(i_1,2\mu-1),(i_2,2\mu-1) \in \{\big(\mu k,2\mu-1\big), k = g,g+1,...,2g\}$, $i_1+i_2 \geq 2g\mu$ so $(i_1+i_2,4\mu-2)$ is bottom. If $i_1+i_2 > 2g\mu$, then $E_\infty^{i_1+i_2,4\mu-2} = 0$. Otherwise $i_1 = i_2 = g\mu$, we also check that $E_\infty^{2g\mu,4\mu-2} = 0$. Every $E_\infty^{2g\mu,q}$ is a vertical translation of $E_\infty^{2g\mu,2\mu-1}$ by a product of w and y_i and any such monomial has bidegree $(0,l(\mu-1)+2k\mu)$, $l,k \in \mathbb{Z}_{\geq 0}$. The equation $2\mu-1+l(\mu-1)+2k\mu = 4\mu-2$ has no solutions for $l,k \in \mathbb{Z}_{\geq 0}$.

□

2.4.8 Definition and proof of the isomorphism

From now on, we use the shorthand

$$H^* := H^*(\mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}})).$$

Both fibrations r and $\chi\circ r$ have total space $\mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}})$ so they define different Serre filtrations $F^*_*(r)$ and $F^*_*(\chi\circ r)$ on H^*. The map of fibrations (id_M,χ) makes the identity map id_M of $\mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}})$ into a filtered map

$$\mathrm{id}_{H^*} : (H^*, F^*_*(\chi\circ r)) \to (H^*, F^*_*(r)). \tag{2.4.26}$$

Proposition 63. *There is a weighted $\Gamma_{g,1}$-equivariant filtered algebra monomorphism*

$$\sigma_E : \Big(\mathrm{Tot}(H^*(R(x;v),d_E))^*, F^*_*(\mathrm{Tot})\Big) \to \Big(H^*, F^*_*(r)\Big)$$

whose associated graded is the inclusion

$$H^{*,*}(R(x;v),d_E) \to H^{*,*}(R(x,y;v,w),d_R) \cong E^{*,*}_\infty(r),$$

where the last isomorphism is from Lemma 53.

Proof. Observe that $H^{*,*}(R(x;v),d_E) = E_\infty^{S_E}(\chi\circ r)$. From Lemma 62, S_E is bottom and closed, so we apply the Lifting Lemma 60 to deduce that: 1. $H^{*,*}(R(x;v),d_E)$ is a subalgebra of $E_\infty^{*,*}(\chi\circ r)$ and 2. the subalgebra-inclusion $H^{*,*}(R(x;v),d_E)\hookrightarrow E_\infty^{*,*}(\chi\circ r)$ lifts naturally to a map

$$\sigma_H : (H^*(\mathrm{Tot}(R(x;v),d_E)), F_*^*(\mathrm{Tot})) \to (H^*, F_*^*(\chi\circ r)),$$

with $\mathrm{Tot}(R(x;v),d_E) = (R(x,v),d_E)$. Finally, we compose with the filtered identity map $\mathrm{id}_{H^*} : (H^*, F_*^*(\chi\circ r)) \to (H^*, F_*^*(r))$, and take associated graded. The second part of Lemma 54 gives the associated graded of this map. □

Proposition 64. *There is a weighted filtered algebra monomorphism*

$$\sigma_P : \Big(\mathrm{Tot}(R(y;w))^*, F_*^*(\mathrm{Tot})\Big) \to \Big(H^*, F_*^*(r)\Big)$$

whose associated graded map is the inclusion

$$R(y;w) \to E_\infty^{*,*}(r) = H^{*,*}(R(x;v),d_E)\otimes R(y;w).$$

Remark 65. The morphism σ_P is *not* $\Gamma_{g,1}$-equivariant.

Proof. The degrees of classes $y_1,...,y_{2g},w$ in $E_\infty^{*,*}(r)$ are bottom entries, and so there are canonical lifts of these classes into $y_1,...,y_{2g},w\in H^*$. Extend this to an algebra map $\sigma_P : \mathbb{Q}[y_1,...,y_{2g},w]\to H^*$ which is naturally filtered with respect to $F_*^*(\mathrm{Tot})$ and $F_*^*(r)$, respectively. The associated graded map is therefore the claimed inclusion, which is a monomorphism. Therefore σ_P itself is a monomorphism. □

Finally, we define the algebra morphism

$$\sigma := \sigma_P\otimes\sigma_E : R(y,w)\otimes H^*(R(x,v),d_E) = H^*(R(x,y,v,w),d_R)\to H^*. \tag{2.4.27}$$

We prove Theorem 46 by showing that it is an isomorphism of $\Gamma_{g,1}$-algebras if we insist that the action on the y_i is Johnson in $R(y,w)\otimes H^*(R(x,v),d_E)$.

Proof of Theorem 46. Since both σ_P and σ_E are filtered maps, the tensor product

$$\sigma = \sigma_P\otimes\sigma_E : \Big(\mathrm{Tot}(H(R(x,y;v,w),d_R))^*, F_*^*(\mathrm{Tot})\Big) \to \Big(H^*, F_*^*(r)\Big)$$

is also a filtered map and the associated graded $\mathrm{gr}(\sigma)$ is the tensor product

$$\mathrm{gr}(\sigma_P)\otimes\mathrm{gr}(\sigma_E): H^{*,*}(R,d_R)\to E_\infty^{*,*}(r).$$

Combining the explicit expressions of $\mathrm{gr}(\sigma_P)$ from Proposition 63 and $\mathrm{gr}(\sigma_E)$ from Proposition 64, and under the isomorphism $E_\infty^{*,*}(r)\cong H^{*,*}(R,d_R)$ from Lemma 53, the map $\mathrm{gr}(\sigma)$ is the identity. In particular, σ is an isomorphism. It is weighted because both σ_E and σ_P are.

It remains to prove that σ is $\Gamma_{g,1}$-equivariant. We do so by exhibiting a generating set of $H^*(R,d_R)$ and checking that σ is $\Gamma_{g,1}$-equivariant on the degrees that contain these generators. Obviously, $H^*(R,d_R)$ is generated by the subalgebra $H(R(x,v),d_E)=H^*(\Lambda[x_1,...,x_{2g},v],d(v)=2\omega)$ and the elements w and $y_1,...,y_{2g}$. We check these three cases separately.

1. The restriction of σ on $H(R(x,v),d_E)\subset\mathrm{Tot}(H(R,d_R))^*$ is σ_E which is $\Gamma_{g,1}$-equivariant from Proposition 63.

2. The element $w\in H^{\mu-1}(R,d_R)$ has the trivial $\Gamma_{g,1}$ action. Furthermore, $H^{\mu-1}(R,d_R)$ is generated by w as a vector space because all other algebra generators of R^* have higher degrees, namely $\mu, 2\mu-1$ and 2μ. The Serre spectral sequence of r from Lemma 53 provides a $\Gamma_{g,1}$-equivariant isomorphism $H^{\mu-1}\to H^{\mu-1}(\Omega^2S^{2m}_{\mathbb{Q}})\cong\mathbb{Q}$, with the trivial action on $\mathbb{Q}$. Thus $\sigma:\mathbb{Q}\langle w\rangle = H^{\mu-1}(R,d_R)\to H^{\mu-1}=\mathbb{Q}$ is (trivially) $\Gamma_{g,1}$-equivariant.

3. The y_i have degree/weight $(2\mu,(2))$ in $H^*(R,d_R)$. By comparing degrees and weights of the x_i,y_i,v and w, we see that the weight 2 part $H^{2\mu,(2)}(R,d_R)$ is generated by the double wedges x_ix_j and the y_i, in particular it is $\Gamma_{g,1}$-isomorphic to $H^{2\mu,0}(R,d_R)=H^{2\mu,0,(2)}(R,d_R)$. The associated graded map $\mathrm{gr}(\sigma)^{2\mu,0}: H^{2\mu,0}(R,d_R)\to E_\infty^{2\mu,0}$ is the $\Gamma_{g,1}$-equivariant isomorphism from Lemma 53. Finally, the entry $E_\infty^{2\mu,0}$ is bottom and therefore there is a natural, in particular $\Gamma_{g,1}$-equivariant, inclusion $E_\infty^{2\mu,0}\subset H^{2\mu}$. Combining these, the restriction of σ on degree/weight $(2\mu,(2))$ is given by the $\Gamma_{g,1}$-equivariant composition

$$\sigma: H^{2\mu,(2)}(R,d_R)= \ H^{2\mu,0}(R,d_R)\xrightarrow[\cong]{\mathrm{gr}(\sigma)^{2\mu,0}} E_\infty^{2\mu,0}\subset H^{2\mu}\ .$$

Thus the action on the y_i is Johnson.

□

2.5 Cohomology of configuration spaces of surfaces as $\Gamma_{g,1}$-representation

Theorem 66. *For $g,i,n \geq 0$, the $\Gamma_{g,1}$-representation $H^i(C_n(\Sigma_{g,1}))$ is isomorphic to the bidegree (i,n) part of the weighted graded commutative $\Gamma_{g,1}$-algebra*

$$\mathbb{Q}[y_1,...,y_{2g},w] \otimes H^{*,*}(\Lambda[x_1,...,x_{2g},v],d(v) = 2\omega)$$

with

- *bidegrees $|x_i| = (1,1)$, $|y_i| = (2,2)$, $|w| = (0,1)$ and $|v| = (1,2)$,*
- *differential d given on v by*

  $$d(v) = 2\omega = 2(x_1x_2 + ... + x_{2g-1}x_{2g})$$

  and vanishing on the x_i, and extended using the Leibniz rule, and
- *the $\Gamma_{g,1}$-action trivial on w and v, symplectic on the x_i and the Johnson on the y_i.*

Proof. From Theorem 18, we have the isomorphism

$$H^i(C_n(\Sigma_{g,1})) \cong H^{i+2ln}(\mathrm{map}_\partial(\Sigma_{g,1},S^{2(l+1)}_{\mathbb{Q}}))^{(n)} \tag{2.5.1}$$

for any $l \geq 1$. Fix $m = l+1$, recall $\mu = 2m-1$ and apply Theorem 46 to the right hand side to obtain the isomorphism

$$H^{i+(\mu-1)n}(\mathrm{map}_\partial(\Sigma_{g,1},S^{2m}_{\mathbb{Q}}))^{(n)} \cong H^{i+(\mu-1)n,(n)}\Big(R(x,y,v,w),d_R\Big).$$

Apart from the degrees, this is the same as the required $\Gamma_{g,1}$-algebra and the isomorphism is $\Gamma_{g,1}$-equivariant. As for the degrees, let us write $R_m := R(x,y,v,w)$ to remember that the degrees depended on m, but not the underlying weighted $\Gamma_{g,1}$-algebra. The degree shifts give the isomorphism $H^{i+(\mu-1)n,(n)}\Big(R_m,d_R\Big) \cong H^{i,(n)}\Big(R_1,d_R\Big)$ which by, treating the weighting as a usual grading, is the desired bigraded $\Gamma_{g,1}$-algebra. □

Remark 67. By writing $A^{i,(\leq n)} = \oplus_{j\leq n} A^{i,(j)}$, we may omit the generator w and rephrase the theorem as the isomorphism of $\Gamma_{g,1}$-representations

$$H^{\bullet}(C_n(\Sigma_{g,1})) \cong \Big(\mathbb{Q}[y_1,...,y_{2g}] \otimes H^{*,*}(\Lambda[x_1,...,x_{2g},v], d(v) = 2\omega)\Big)^{\bullet,(\leq n)} \tag{2.5.2}$$

Finally, we present an alternative "coordinate-free" statement of the theorem.

Theorem 68. *For $i,n \geq 0$, there are isomorphisms of $\Gamma_{g,1}$-representations*

$$H^i(C_n(\Sigma_{g,1})) \cong H^{i,n}\Big(\Big(\mathbb{Q}\Big[J[2,2] \oplus \mathbb{Q}[0,1]\Big] \otimes \Lambda\Big[H[1,1] \oplus \mathbb{Q}[1,2]\Big]\Big)\Big/ I, d\Big)$$

where

- *I is the ideal*
$$I = \Big(z - i(z) : z \in \Lambda^2\Big[H[1,1]\Big]\Big),$$
with $i : \Lambda^2\Big[H[1,1]\Big] \to J[2,2]$ the $\Gamma_{g,1}$-equivariant inclusion defined in (2.3.13), *and*

- *differential d given by*
$$d(1[1,2]) = 2\omega \in \Lambda^2\Big[H[1,1]\Big],$$
vanishing on the other generating vector spaces and extended by the Leibniz rule.

2.5.1 The action of the Johnson filtration

We determine the action of the Johnson filtration

$$... \subset J(i+i) \subset J(i) \subset ... \subset J(0) = \Gamma_{g,1}$$

from Section 1.4, on $H^*(C_n(\Sigma_{g,1}))$.

Proposition 69. *The action of $J(2)$ on the representation J is trivial.*

Proof. Since $J(2) \subset J(1) = \mathcal{T}_{g,1}$, $J(2)$ acts trivially on each of $H, \Lambda^2 H$, therefore by the definition of J, it suffices to check that for all $\phi \in J(2)$, $\xi(\phi)(y)$ vanishes for $y \in H$ and furthermore it suffices to check this on the basis y_i, $i = 1,...,2g$.

From the definition of $J(2)$, a mapping class $\phi \in J(2)$ maps every $\alpha \in \pi_1(\Sigma_{g,1})$ to $\phi(\alpha) = \alpha\beta$ for some $\beta \in \pi_1(\Sigma_{g,1})^{(2)}$. Then by definition 2.3.9, $\xi(\phi, y_i) = c(\phi(\alpha_i)) = c(\alpha_i\beta)$ for some double commutator $\beta \in (\mathbb{Z}^{*2g})^{(2)}$. From proposition 26, we get $c(\alpha_i h) = c(\alpha_i) + c(h) + [\alpha_i] \wedge [h]$, $c(\alpha_i) = 0$, $c(h) = 0$ and $[h] = 0$ because h is a double commutator, so $\xi(\phi)(y_i) = 0$. $\square$

Corollary 70. *The action of $J(2)$ on $H^*(C_n(\Sigma_{g,1}))$ is trivial for all $n \geq 0$ and $g \geq 0$.*

Proof. The vector space $H^i(C_n(\Sigma_{g,1});\mathbb{Q})$ is a $\Gamma_{g,1}$-subrepresentation of $H^*(R, d_R)$ which is a subquotient of the free tensor power on the direct sum of $\Gamma_{g,1}$-representation $J \oplus H \oplus \mathbb{Q} \oplus \mathbb{Q}$ with appropriate gradings. All have trivial $J(2)$-action. $\square$

Remark 71. This is in contrast with $J(2)$ acting non-trivially on $H^*(F_n(\Sigma_{g,1}))$ for *ordered* configuration spaces, see Bianchi–Miller–Wilson [3].

On the other hand, the $\Gamma_{g,1}$-representation $H^2(C_2(\Sigma_{g,1}))$ was already known by Bianchi to be non-symplectic [2]. We compute this representation explicitly and conclude this result independently.

Definition 72. The *reduced Johnson representation* is the quotient $\widetilde{J} := J/\langle\omega\rangle$, fitting in an extension of $\Gamma_{g,1}$-representations

$$0 \longrightarrow \Lambda^2 H/\langle\omega\rangle \xrightarrow{i} \widetilde{J} \longrightarrow H \longrightarrow 0. \tag{2.5.3}$$

Here $\omega \in \Lambda^2 H$ is the invariant vector

$$x_1 \wedge x_2 + \ldots + x_{2g-1} \wedge x_{2g}.$$

Proposition 73. *If $g \geq 2$, the reduced Johnson representation $\widetilde{J}$ is not symplectic and the extension* (2.5.3) *is not $\Gamma_{g,1}$-split. In particular, J is also not symplectic.*

Proof. If J were symplectic, then so would be its quotient $\widetilde{J}$. On the other hand, if the sequence were $\Gamma_{g,1}$-split, then $\widetilde{J}$ would be a direct sum of symplectic representations, thus symplectic. So it suffices to prove that $\widetilde{J}$ is not symplectic. For this, it is enough to show that there are $\phi \in \mathcal{T}_{g,1}$ and $y \in H$ so that $\xi(\phi, y) \notin \mathbb{Q}\langle\omega\rangle$, and this is equivalent to

$$\operatorname{im}\left(\xi|_{\mathcal{T}_{g,1}} : \mathcal{T}_{g,1} \to \hom(H, \Lambda^2 H)\right) \not\subset \hom(H, \mathbb{Q}\langle\omega\rangle) \subset \hom(H, \Lambda^2 H). \tag{2.5.4}$$

We will show (2.5.4) by comparing the crossed homomorphism ξ with the Johnson homomorphism $\tau : \mathcal{T}_{g,1} \to \hom(H_\mathbb{Z}, \Lambda^2 H_\mathbb{Z})$ from [23].

Let $\alpha_1, ..., \alpha_{2g} \in \pi := \pi_1(\Sigma_{g,1})$ and $x_1, ..., x_{2g} \in H_1(\Sigma_{g,1}; \mathbb{Z})$ be the standard bases and define homomorphism $j : [\pi, \pi] \to \Lambda^2 H_\mathbb{Z}$ given by $j([a,b]) = [a] \wedge [b]$. Then Johnson defined τ so that for $\phi \in \mathcal{T}_{g,1}$,

$$\tau(\phi)(x_i) = j(\phi(\alpha_i)\alpha_i^{-1}). \tag{2.5.5}$$

Now by definition of $\mathcal{T}_{g,1}$, for each $i = 1, ..., n$ there is a conjugator $c = [a,b] \in [\pi, \pi]$ so that $\phi(\alpha_i) = \alpha_i c$. Then

$$\tau(\phi)(x_i) = j(\alpha_i c \alpha_i^{-1}) = j([\alpha_i, c]c) = j([\alpha_i, c]) + j(c) = [a] \wedge [b] \tag{2.5.6}$$

since $[c] = 0$, being a commutator. On the other hand,

$$\xi(\phi)(x_i) = c(\phi(\alpha_i)) = c(c\alpha_i) = c(c) + c(\alpha_i) + [c] \wedge [\alpha_i] = c(c) = 2[a] \wedge [b], \tag{2.5.7}$$

by Proposition 26. Extending linearly we conclude $\xi|_{\mathcal{T}_{g,1}} = 2\tau$ and therefore ξ is, up to a scalar factor and a coboundary, equal to the Morita crossed homomorphism $\widetilde{k}$ from [36].

Finally, from Johnson [23], the image of τ is $\Lambda^3 H \subset H_\mathbb{Z} \otimes \Lambda^2 H_\mathbb{Z} \cong \hom(H_\mathbb{Z}, \Lambda^2 H_\mathbb{Z})$. Thus $\operatorname{im}(\xi|_{\mathcal{T}_{g,1}}) = \Lambda^3 H \subset H \otimes \Lambda^2 H$, which is not contained in $H \otimes \omega$, provided that $g \geq 2$, so $\dim(\Lambda^3 H) = \binom{2g}{3} > 1$. □

Remark 74. Compare Proposition 73 with Lemma 153 from Chapter 3, which studies the same representation over $\mathbb{Z}$.

Proposition 75. *As $\Gamma_{g,1}$ representations, $H^2(C_2(\Sigma_{g,1})) \cong \widetilde{J}$. In particular, $H^2(C_2(\Sigma_{g,1}))$ is not symplectic.*

Proof. Immediate from Theorem 66 and Proposition 73. □

2.5.2 A comment on closed surfaces

Let Σ_g be a closed oriented genus g surface and Γ_g its mapping class group. Looijenga proved in [31] that the Γ_g representation $H^3(C_3(\Sigma_g))$ is non-symplectic for

$g \geq 3$. We briefly sketch how the same representation arises from results of this chapter but defer more detailed elaboration of this argument to future work.

The inclusion $\mathrm{Diff}^+_\partial(\Sigma_{g,1}) \subset \mathrm{Diff}^+(\Sigma_g)$ gives rise to surjection $\Gamma_{g,1} \to \Gamma_g$, and so we can speak of $\Gamma_{g,1}$ representations instead. By applying Theorem 18, $H^3(C_3(\Sigma_g))$ is isomorphic to the $\Gamma_{g,1}$ representation $\widetilde{H}^{3\mu}(\mathrm{map}(\Sigma_g, S^{2m}_{\mathbb{Q}}))^{(3)}$. We can compute this by analysing the Serre spectral sequence of the evaluation fibration

$$\mathrm{map}_\partial(\Sigma_{g,1}, S^{2m}_{\mathbb{Q}}) \cong \mathrm{map}_*(\Sigma_g, S^{2m}_{\mathbb{Q}}) \to \mathrm{map}(\Sigma_g, S^{2m}_{\mathbb{Q}}) \to S^{2m}_{\mathbb{Q}}, \tag{2.5.8}$$

and comparing it with the Serre spectral sequence from Section 2.3.8 by deformation retracting $\vee_{2g} S^1$ to $\Sigma_{g,1}$. After computing differentials, we get a surjection from $\widetilde{H}^{3\mu}(\mathrm{map}(\Sigma_g, S^{2m}_{\mathbb{Q}}))^{(3)}$ to

$$E_\infty^{0,3\mu,(3)} = \mathbb{Q}\{x_i \wedge x_j \wedge x_k, x_i y_j : i,j,k = 1,\ldots,2g\}/\langle \omega \wedge x_i, x_i y_j = x_j y_i \rangle$$

with the symplectic action on the x_i and the Johnson action on the y_i. Thus we obtain a non-symplectic $\Gamma_{g,1}$-extension of symplectic representations

$$0 \to \Lambda^3 H/(\omega \wedge H) \to E_\infty^{0,3\mu,(3)} \to \mathrm{Sym}^2 H \to 0.$$

Since $H^3(C_3(\Sigma_g))$ surjects onto $E_\infty^{0,3\mu,(3)}$, it is also itself non-symplectic.

In Section 3.5.6, we will prove that the kernel of the Γ_g-representation $E_\infty^{0,3\mu,(3)}$ is precisely the image of $J(2)$ in Γ_g.

2.6 Addendum I – Superposition product

We construct a binary operation, sup, on $\bigoplus_{n\geq 1} H^*(C_n(M))[2ln]$ arising from *superposing* (or taking unions of) configurations, and prove that under the scanning isomorphism of Theorem 19 it corresponds to the cup-product, $\smile$, of the section space.

Theorem 76. *The isomorphism of Theorem 19*

$$s : \left(H^*\left(\Gamma_\partial(M, S^{2l}); \mathbb{Q}\right), \smile\right) \longrightarrow \left(\bigoplus_{n\geq 0} H^*(C_n(M);\mathbb{Q})[2ln], \mathrm{sup}\right) \tag{2.6.1}$$

is an isomorphism of unital algebras.

Remark. This superposition product was used by Bianchi–Miller–Wilson in [3] for the same reason as the author used the cup-product in Sections 2.3 and 2.4 of this Chapter, that is, to bound the kernel of the mapping class group action on $H^*(C_n(M))$ for M a surface. In light of this theorem, it was no coincidence.

Structure of this addendum

In section 2.6.1, we define and discuss cohomology maps induced by multivalued maps of spaces, as opposed to the usual treatment in literature of considering induced *homology* maps via the Dold–Thom isomorphism [13]. In section 2.6.2, we revisit the definition of the scanning map σ that induces the isomorphism s, following Bödigheimer–Cohen and Manthorpe–Tillmann [7, 33]. The next two sections define the superposition product and prove the theorem. We conclude with an alternative but equivalent definition of sup in section 2.6.5.

For the sake of generality, we will now work with $\mathbb{Z}$-coefficients: it is easy to see that all results we shall prove over $\mathbb{Z}$ automatically hold over $\mathbb{Q}$. We will only revert to $\mathbb{Q}$-coefficients in the proof of Theorem 76.

2.6.1 Symmetric products, multivalued functions, cohomology and products

Definition 77. Given a based space (X,e), the *symmetric product* $\mathrm{SP}(X)$ is the free unital abelian monoid on X with identity e, topologised as the quotient

$$\mathrm{SP}(X) := \Big(\bigsqcup_{n\geq 1} X^n/\mathfrak{S}_n\Big)\Big/\sim$$

where $\sim$ identifies $(x_1,...,x_n,e)\sim(x_1,...,x_n)$. If X is an unbased space, then $\mathrm{SP}^+ = \mathrm{SP}(X\sqcup *)$, with $*$ the basepoint.

Elements of $\mathrm{SP}(X)$ are formal sums $\sum_{i\in I} x_i$ with $x_i \in X$ and I a finite indexing set.

Definition 78. A *multivalued function* of based spaces is a based map $(A,*) \to (\mathrm{SP}(B),e)$. A multivalued function of unbased spaces is a map $A \to \mathrm{SP}^+(B)$.

Dold–Thom on cohomology

Usually, from a multivalued function $X \to \mathrm{SP}(Y)$, one defines a homology map $H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ via the Dold–Thom process [13]. Here, we demonstrate how to produce a cohomology map $H^*(Y;\mathbb{Z}) \to H^*(X;\mathbb{Z})$ instead[3].

Fix a map $f : X \to \mathrm{SP}(Y)$, and identify $\widetilde{H}^n(-) = [-,K(\mathbb{Z},n)]_*$. A model of $K(\mathbb{Z},n)$ is $\mathrm{SP}(S^n)$ which is an abelian monoid. Now, the functor $\mathrm{SP}(-)$ satisfies the universal property that any based map from a based space (A,a) to an abelian topological monoid (M,e) factors uniquely through the natural inclusion $(A,a) \hookrightarrow (\mathrm{SP}(A),a)$ via a homomorphism $\mathrm{SP}(A) \to M$. Therefore, a map $\alpha : Y \to \mathrm{SP}(S^n)$ representing a cohomology class $[\alpha] \in \widetilde{H}^n(Y)$, factors uniquely through

$$Y \xhookrightarrow{\iota_Y} \mathrm{SP}(Y) \xrightarrow{\widetilde{\alpha}} \mathrm{SP}(S^n).$$

where ι_Y is the inclusion of Y into $\mathrm{SP}(Y)$ and $\widetilde{\alpha}$ is a homomorphism. We define

$$f^* : \widetilde{H}^*(Y;\mathbb{Z}) \to \widetilde{H}^*(X;\mathbb{Z})$$

as mapping $[\alpha] \mapsto [\widetilde{\alpha}\circ f] \in [X,K(\mathbb{Z},n)]_*$.

[3] This is a trick suggested by Oscar Randal-Williams.

Proposition 79 (Naturality). *Let $h:(Y_1,y_1)\to(Y_2,y_2)$ be a based map so that the diagram of based maps*

$$\begin{array}{ccc} (X_1,x_1) & \xrightarrow{f_1} & (\mathrm{SP}(Y_1),y_1) \\ \Big\downarrow g & & \Big\downarrow \mathrm{SP}(h) \\ (X_2,x_2) & \xrightarrow{f_2} & (\mathrm{SP}(Y_2),y_2) \end{array} \tag{2.6.2}$$

commutes. Then $f_1^*\circ h^*=g^*\circ f_2^*$.

Proof. Using the definitions above, a representative $\alpha:(Y_2,y_2)\to(\mathrm{SP}(S^n),e)$ of a class in $\widetilde{H}^*(Y_2;\mathbb{Z})$ is sent under $f_1^*\circ h^*$ to $\widetilde{\alpha\circ h}\circ f_1$ and by $g^*\circ f_2^*$ to $\widetilde{\alpha}\circ f_2\circ g$, which is equal to $\widetilde{\alpha}\circ\mathrm{SP}(h)\circ f_1$ by the commutativity of diagram (2.6.2). It suffices to check that $\widetilde{\alpha\circ h}=\widetilde{\alpha}\circ\mathrm{SP}(h)$. The two maps both restrict to $\alpha\circ h$ on Y_1, which is a generating set for $\mathrm{SP}(Y_1)$. As both $\widetilde{\alpha\circ h}$ and $\widetilde{\alpha}\circ\mathrm{SP}(h)$ are homomorphisms, their equality follows. □

It is furthermore easy to check that if f is really a single-valued map $f:X\to Y$, then this definition of f^* agrees with the standard definition.

Remark 80. A common way to obtain a multivalued function is by "inverting" a finite-sheeted covering map $p:\widetilde{X}\to X$, by mapping $p^{-1}:X\to\mathrm{SP}^+(\widetilde{X})$,

$$\widetilde{x}\mapsto\sum_{x\in p^{-1}(\widetilde{x})}x.$$

Similarly, if $p:(\widetilde{X},e)\to(\mathrm{SP}(X),e)$ is a covering space away from the basepoint, we get the corresponding based multivalued function $p^{-1}:(X,e)\to(\mathrm{SP}(\widetilde{X}),e)$. The cohomology maps $(p^{-1})^*$ are equal to the transfer maps $p_!$ on unreduced or reduced cohomology.

Dold–Thom and products of maps

Given two based multivalued functions $f:A\to\mathrm{SP}(B)$ and $g:C\to\mathrm{SP}(D)$, there is a multivalued function $f\triangle g:A\wedge C\to\mathrm{SP}(B\wedge D)$ given by

$$(f\triangle g)(a,c)=\sum_{(i,j)\in I\times J}(a_i,c_j)$$

where $f(a)=\sum_I a_i$ and $g(c)=\sum c_i$. More precisely, there is the tautological continuous map

$$\iota_{X,Y}:\mathrm{SP}(X)\wedge\mathrm{SP}(Y)\to\mathrm{SP}(X\wedge Y), \tag{2.6.3}$$

and $f \triangle g$ is the composition of the continuous maps $f \wedge g$ with $\iota_{X,Y}$.

In the interpretation of cohomology as $[-, K(\mathbb{Z}, n)]_*$, the cross product $\times$ is given by the composition

$$\begin{tikzcd}
{[X, K(\mathbb{Z}, m)]_* \times [Y, K(\mathbb{Z}, n)]_*} \arrow[d, "- \wedge -"] \\
{[X \wedge Y, K(\mathbb{Z}, m) \wedge K(\mathbb{Z}, n)]_*} \arrow[d, "\iota_{m,n} \circ -"] \\
{[X \wedge Y, K(\mathbb{Z}, m+n)]_*,}
\end{tikzcd}$$

where $\iota_{m,n} : K(\mathbb{Z}, m) \wedge K(\mathbb{Z}, n) \to K(\mathbb{Z}, m+n)$ is the map corresponding to a generator of $H^{m+n}(K(\mathbb{Z}, m) \wedge K(\mathbb{Z}, n); \mathbb{Z}) \cong \mathbb{Z}$. Again, identifying $K(\mathbb{Z}, n)$ with $\mathrm{SP}(S^n)$, the map $\iota_{m,n}$ can be taken to be ι_{S^m, S^n} in the sense of (2.6.3). We thus obtain a diagram

$$\begin{tikzcd}
\widetilde{H}^*(C; \mathbb{Z}) \otimes \widetilde{H}^*(D; \mathbb{Z}) \arrow[r, "f^* \otimes g^*"] \arrow[d, "\times"] & \widetilde{H}^*(A; \mathbb{Z}) \otimes \widetilde{H}^*(B; \mathbb{Z}) \arrow[d, "\times"] \\
\widetilde{H}^*(C \wedge D; \mathbb{Z}) \arrow[r, "(f \triangle g)^*"] & \widetilde{H}^*(A \wedge B; \mathbb{Z}).
\end{tikzcd} \tag{2.6.4}$$

Proposition 81. *The diagram* (2.6.4) *commutes.*

Proof. We evaluate the two directions of the diagram on a representative $\alpha \otimes \beta$ for a class in $\widetilde{H}^*(C; \mathbb{Z}) \otimes \widetilde{H}^*(D; \mathbb{Z})$. Going right-then-down, we have

$$\begin{aligned}
\alpha \otimes \beta &\mapsto (\tilde{\alpha} \circ f) \otimes (\tilde{\beta} \circ g) \\
&\mapsto \iota_{n,m} \circ \Big((\tilde{\alpha} \circ f) \wedge (\tilde{\beta} \circ g) \Big) = \iota_{n,m} \circ (\tilde{\alpha} \wedge \tilde{\beta}) \circ (f \wedge g),
\end{aligned}$$

whereas going down-then-right,

$$\begin{aligned}
\alpha \otimes \beta &\mapsto \iota_{n,m} \circ (\alpha \wedge \beta) \\
&\mapsto \widetilde{\iota_{n,m} \circ (\alpha \wedge \beta)} \circ (f \triangle g).
\end{aligned}$$

These are equal because the diagram

$$\begin{tikzcd}
 & C \wedge D \arrow[r, "\alpha\wedge\beta"] \arrow[d, "\iota_C\wedge\iota_D"] & \mathrm{SP}(S^n)\wedge\mathrm{SP}(S^m) \arrow[r, "\iota_{n,m}"] & \mathrm{SP}(S^n\wedge S^m) \\
 & \mathrm{SP}(C)\wedge\mathrm{SP}(D) \arrow[ur, "\widetilde{\alpha}\wedge\widetilde{\beta}"] \arrow[d, "\iota_{C,D}"] & & \\
A\wedge B \arrow[ur, "f\wedge g"] \arrow[r, "f\triangle g"] & \mathrm{SP}(C\wedge D) \arrow[uurr, "\widetilde{\iota_{n,m}\circ(\alpha\wedge\beta)}"'] & &
\end{tikzcd}$$

commutes: every map with a tilde is defined as the unique homomorphism extending the un-tilded map through the inclusion into the free abelian monoid on the domain. $\square$

Let $\Sigma X = S^1\wedge X$ denote the *reduced suspension* of a based space X. A based multivalued function $f:(X,x)\to(\mathrm{SP}(Y),y)$ induces a suspension map $\Sigma f = \mathrm{id}_{S^1}\triangle f : (\Sigma X,\Sigma x)\to(\mathrm{SP}(\Sigma Y),\Sigma y)$.

Corollary 82. *The suspended map Σf commutes with the suspension isomorphism, i.e. if $s_Z:\widetilde{H}^*(Z;\mathbb{Z})\to\widetilde{H}^{*+1}(\Sigma Z;\mathbb{Z})$ is the suspension of space Z, then $s_X\circ(\Sigma f)^* = f^*\circ s_Y$.*

Proof. Apply Proposition 81 with $C=A=S^1$ and $f=\mathrm{id}_{S^1}$. The cross-product maps are the suspension isomorphisms. $\square$

Lemma 83. *Let $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ be based cofibration sequences for $i=1,2$, and suppose the following diagram of based maps commutes*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "a"] & B_1 \arrow[r, "g_1"] \arrow[d, "b"] & C_1 \arrow[d, "c"] \\
\mathrm{SP}(A_2) \arrow[r, "\mathrm{SP}(f_2)"] & \mathrm{SP}(B_2) \arrow[r, "\mathrm{SP}(g_2)"] & \mathrm{SP}(C_2).
\end{tikzcd} \tag{2.6.5}$$

Then the induced cohomology maps a^*,b^*,c^* commute with the long exact sequences of the two cofibration sequences, that is

$$\begin{tikzcd}
\dots \arrow[r, "\partial_2^*"] & \widetilde{H}^*(C_2) \arrow[r, "g_2^*"] \arrow[d, "c^*"] & \widetilde{H}^*(B_2) \arrow[r, "f_2^*"] \arrow[d, "b^*"] & \widetilde{H}^*(A_2) \arrow[r, "\partial_2^*"] \arrow[d, "a^*"] & \widetilde{H}^{*+1}(C_2) \arrow[r, "g_2^*"] \arrow[d, "c^*"] & \dots \\
\dots \arrow[r, "\partial_1^*"] & \widetilde{H}^*(C_1) \arrow[r, "g_1^*"] & \widetilde{H}^*(B_1) \arrow[r, "f_1^*"] & \widetilde{H}^*(A_1) \arrow[r, "\partial_1^*"] & \widetilde{H}^{*+1}(C_1) \arrow[r, "g_1^*"] & \dots
\end{tikzcd} \tag{2.6.6}$$

commutes. (Here the $\mathbb{Z}$-coefficients are kept implicit).

Proof. The left two squares commute by the naturality proved in Proposition 79, so we focus on the rightmost square. We recall the construction of the connecting homomorphism ∂_i^* using the reduced mapping cone $\widetilde{C}A_i$ of A_i. Since f_i are cofibrations, we have

$C_i = B_i/A_i$ and we have the sequence of maps $B_i/A_i \xleftarrow[\iota]{\sim} \widetilde{C}A_i \cup_{A_i} B_i \xrightarrow{k} \Sigma A_i$ where ι collapses the cone and is a homotopy equivalence, whereas k collapses the subspace B. The connecting homomorphism ∂_i^* is thus the composition $(\iota^*)^{-1} \circ k^* \circ s_{A_i}$ of maps that commute with the induced cohomology maps either by the naturality of the construction in spaces and Proposition 79 or by Corollary 82. $\square$

2.6.2 Scanning revisited

We recall the set-up for the space-level model of the scanning σ from Manthorpe–Tillmann [33]. Let (M, M_0) be a manifold submanifold pair, and (X, x_0) a based space. The labelled configuration space $C(M, M_0; X)$ is defined as

$$C(M, M_0; X) = \Big\{ \{(m_i, x_i)\}_{i \in I} : I \text{ finite set }, m_i \in M, x_i \in X, m_i \neq m_j \text{ if } i \neq j \Big\} / \sim,$$

where $\sim$ is the relation generated by $\xi_I \sim \xi_I \cup \{(m, x)\}$ if $m \in M_0$ or $x = x_0$. We will denote a generic element of $C(M, M_0; X)$ by ξ_I, and given a subset $J \subset I$, we will write $\xi_J := \{(m_i, x_i)\}_{i \in J}$.

The space $C := C(M, M_0; X)$ has a filtration $C_1 \subset ... \subset C_k \subset C_{k+1} \subset ... \subset C$ of labelled configurations of at most k points. The filtration quotients $D_k = C_k / C_{k-1}$ are naturally based at ∞, corresponding to when $(m, x) \in \xi_I$ with $m \in M_0$ or $x = x_0$, and we take the wedge sum $V = \bigvee_{k \geq 1} D_k$. The scanning map is

$$\begin{aligned} \sigma : C &\to \mathrm{SP}(V) \\ \xi_I &\mapsto \sum_{J \subseteq I} \xi_J. \end{aligned}$$

We define further maps

- $\Delta : C \to C \wedge C$, the diagonal map $\xi \mapsto (\xi, \xi)$;
- $\Phi : V \to \mathrm{SP}(V \wedge V)$ mapping $\xi_I \mapsto \sum_{I = A \sqcup B} (\xi_A, \xi_B)$;
- $\Psi : V \to \mathrm{SP}(V \wedge V)$ mapping $\xi_I \mapsto \sum_{I = A \cup B} (\xi_A, \xi_B)$.

By abuse of notation, Φ and Ψ will also denote their natural extensions of the form $\mathrm{SP}(V) \to \mathrm{SP}(V \wedge V)$. We thus obtain a square of maps

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[d, "\Delta"] & \mathrm{SP}(V) \arrow[d, shift right, "\Psi"'] \arrow[d, shift left, "\Phi"] \\ C \wedge C \arrow[r, "\sigma\triangle\sigma"] & \mathrm{SP}(V \wedge V), \end{tikzcd} \tag{2.6.7}$$

which should be thought of as two diagrams, one with Φ and one with Ψ.

Proposition 84. *The cohomology map $\sigma^* : \widetilde{H}^*(V;\mathbb{Z}) \to \widetilde{H}^*(C;\mathbb{Z})$ is an isomorphism.*

Proof. This is analogous to the proof of Theorem 4.1 from Manthorpe–Tillmann [33]. Write $V_k = \vee_{i=1}^k D_i$ and observe that σ restricts to a multivalued function of cofibrations from $C_k \to C_{k+1} \to C_{k+1}/C_k$ to $V_k \to V_{k+1} \to V_{k+1}/V_k$, in both of which the third space is D_{k+1}, of the form

$$\begin{tikzcd} C_k \arrow[r, hook] \arrow[d, "\sigma_k"] & C_{k+1} \arrow[r, two heads] \arrow[d, "\sigma_{k+1}"] & D_{k+1} \arrow[d, hook, "\iota_{D_{k+1}}"] \\ \mathrm{SP}(V_k) \arrow[r, hook] & \mathrm{SP}(V_{k+1}) \arrow[r, two heads] & \mathrm{SP}(D_{k+1}) \end{tikzcd} \tag{2.6.8}$$

The map σ_1^* is the identity map. We induct using the map of long exact sequences

$$\begin{tikzcd} \dots \arrow[r, "\partial^*"] & \widetilde{H}^*(D_k;\mathbb{Z}) \arrow[r] \arrow[d, equal] & \widetilde{H}^*(V_{k+1};\mathbb{Z}) \arrow[r] \arrow[d, "\sigma_{k+1}^*"] & \widetilde{H}^*(V_k;\mathbb{Z}) \arrow[r, "\partial^*"] \arrow[d, "\sigma_k^*"] & \dots \\ \dots \arrow[r, "\partial^*"] & \widetilde{H}^*(D_k;\mathbb{Z}) \arrow[r] & \widetilde{H}^*(C_{k+1};\mathbb{Z}) \arrow[r] & \widetilde{H}^*(C_k;\mathbb{Z}) \arrow[r, "\partial^*"] & \dots \end{tikzcd} \tag{2.6.9}$$

from Lemma 83, and the 5-lemma, to conclude that σ_k^* is an isomorphism for all $k \geq 1$. As C and V are the colimits of the C_k and V_k, σ^* is an isomorphism by naturality of cohomology maps. $\square$

Commutativity of the diagram 2.6.7

Proposition 85. *The diagram 2.6.7 with Ψ commutes, that is $\Psi \circ \sigma = (\sigma\triangle\sigma) \circ \Delta$.*

Proof. By direct computations

$$(\Psi \circ \sigma)(\xi_I) = \Psi\Big(\sum_{J \subseteq I} \xi_J\Big) = \sum_{A,B,J \subseteq I : A \cup B = J} (\xi_A, \xi_B)$$

and

$$(\sigma\triangle\sigma)\circ\Delta(\xi_I) = (\sigma\triangle\sigma)(\xi_I,\xi_I) = \sum_{A,B\subseteq I}(\xi_A,\xi_B).$$

The two summations agree: in the first summation J is fully determined by A and B and can be ignored. $\square$

Proposition 86. *If $M_0 = \emptyset$ and $X = S^{2d}$ is an even sphere, the maps Φ and Ψ are based homotopic.*

The proof of Proposition 86, relies on the technical Lemma 87 which requires defining coloured configurations. Fix a *number of colours* $k \geq 1$, and a sequence of finite sets $\mathcal{I} = \{I_i : i \in [k]\}$ called *type.* The *coloured configuration space on M of type $\mathcal{I}$* is the space

$$C_{\mathcal{I}}(M) := F_{\sqcup_{i\in[k]} I_i}(M)/\prod_{i\in[k]}\mathrm{Sym}(I_i),$$

where $F_S(M)$ is the ordered configuration on M of points labeled by S and the product of symmetric groups acts $\prod_{i\in[k]}\mathrm{Sym}(I_i)$ acts by permuting the corresponding co-ordinates. We may make a similar definition

$$D_{\mathcal{I}}(M,X) = F_{\sqcup_{i\in[k]} I_i}(M)\times_{\prod_{i\in[k]}\mathrm{Sym}(I_i)} X^{\sqcup_{i\in[k]} I_i}/\sim,$$

where $\sim$ collapses all configurations with some $x_i = x_0$ to one point called ∞.

Lemma 87. *Let $d > 0$, and A,B,C be three disjoint sets. If A is non-empty, then the map*

$$\begin{aligned}\mu : D_{A,B,C}(M,S^{2d}) &\to D_{A\cup B}(M,S^{2d})\wedge D_{A\cup C}(M,S^{2d}),\\ (\xi_A,\xi_B,\xi_C) &\mapsto (\xi_A\cup\xi_B,\xi_A\cup\xi_C)\end{aligned}$$

is based-null-homotopic.

Proof. For brevity, we write $D_{\mathcal{I}} = D_{\mathcal{I}}(M,S^{2d})$. As observed by Bödigheimer–Cohen [7], $D_{\mathcal{I}}$ given any type $\mathcal{I}$, is homeomorphic to the Thom space of the vector bundle

$$\eta_{\mathcal{I}} : E_{\mathcal{I}} := F_{\mathcal{I}}\times_{\prod_{i\in[k]}\mathrm{Sym}(I_i)}\mathbb{R}^{2d}\otimes\left(\mathbb{R}\langle\sqcup_{i\in[k]} I_i\rangle\right)\longrightarrow C_{\mathcal{I}}.$$

Under these homeomorphisms, the map μ is identified with the Thomification, $\mathrm{Th}(\widetilde{f})$, of the bundle morphism

$$\begin{tikzcd}
E_{A,B,C} \arrow[r, "\widetilde{f}"] \arrow[d, "\eta_{A,B,C}"] & E_{A\sqcup B} \times E_{A\sqcup C} \arrow[d, "\eta_{A\sqcup B}\times\eta_{A\sqcup C}"] \\
C_{A,B,C} \arrow[r, "f"] & C_{A\sqcup B} \times C_{A\sqcup C},
\end{tikzcd}$$

which on $F_{\mathcal{I}} \times \mathbb{R}^{2d} \otimes \mathbb{R}\langle \sqcup_{i\in[k]} I_i\rangle$ is defined as the product of the map

$$(x_A, x_B, x_C) \to (x_A \cup x_B, x_A \cup x_C),$$

the identity on $\mathbb{R}^{2d}$ and the map

$$\begin{aligned}\mathbb{R}\langle A\sqcup B\sqcup C\rangle &\to \mathbb{R}\langle A\sqcup B\rangle \oplus \mathbb{R}\langle A\sqcup C\rangle \\ (a\oplus b\oplus c) &\mapsto \big((a\oplus b)\oplus(a\oplus c)\big),\end{aligned}$$

and descends after quotienting by the diagonal action. Observe that $\widetilde{f}$ is an injective linear map on the fibres, which is a necessary condition for a morphism of vector bundles to give a map of Thom spaces.

We factor $(f, \widetilde{f})$ as the composition of two bundle maps both injective on the fibres. First, we include $\eta_{A,B,C}$ into its stabilisation $\eta_{A,B,C} \oplus \mathbb{R}$. Now, observe that for any finite set I, the action of $\mathrm{Sym}(I)$ on $\mathbb{R}\langle I\rangle$ has the invariant vector $v_I := \sum_{i\in I} i$. Using these for $I = A\cup B, A\cup C$, and picking an arbitrary non-trivial vector $v \in \mathbb{R}^{2d}$, we obtain the nowhere vanishing section s of $\eta_{A\cup B} \times \eta_{A\cup C}$ given by $p \mapsto (p, v\otimes(v_{A\cup B} \oplus -v_{A\cup C}))$. Furthermore, on every fibre, s does not lie in the image of $\widetilde{f}$: this is because the $\mathbb{R}\langle A\rangle$ components in $\mathbb{R}\langle A\sqcup B\rangle$ and $\mathbb{R}\langle A\sqcup C\rangle$ agree for all vectors in the image of $\widetilde{f}$, whereas s has them with a difference of a minus sign. We obtain the diagram of bundle maps

$$\begin{tikzcd}
E_{A,B,C} \arrow[r, hook] \arrow[d] & E_{A,B,C} \oplus \mathbb{R} \arrow[r, "\widetilde{f}\oplus s"] \arrow[d] & E_{A\cup B} \times E_{A\cup C} \arrow[d] \\
C_{A,B,C} \arrow[r, equal] & C_{A,B,C} \arrow[r, "f"] & C_{A\cup B} \times C_{A\cup C} \arrow[u, bend left, "s"]
\end{tikzcd}$$

and both squares are injective on fibres. We conclude that the map μ factors through

$$\mathrm{Th}(E_{A,B,C}) \to \mathrm{Th}(E_{A,B,C} \oplus \mathbb{R}) \cong \mathrm{Th}(E_{A,B,C}) \wedge S^1 \to \mathrm{Th}(E_{A\sqcup B}) \wedge \mathrm{Th}(E_{A\sqcup C}).$$

The canonical inclusion of $\mathrm{Th}(E_{A,B,C})$ in its reduced suspension is based-null-homotopic, and thus so is μ. $\square$

Proof of Proposition 86. We recall that $V = \vee_{k\geq 1} D_k$ is a wedge of spaces and so the $V\wedge V$ is similarly the wedge $\vee_{l,m\geq 1} D_l \wedge D_m$. Furthermore SP of a wedge is homeomorphic and isomorphic as a monoid to the direct product of the wedge spaces. As a result, the maps $\Phi, \Psi : V \to \mathrm{SP}(V\wedge V)$ decompose to their $k-(l,m)$ parts

$$D_k \to \mathrm{SP}(D_l \wedge D_m).$$

Now, the $k-(l,m)$-part of Φ is given explicitly by

$$\Phi(\xi_I) = \sum_{\substack{A,B:|A|=l,\\ |B|=m,\\ A\sqcup B=I}} (\xi_A, \xi_B),$$

where $|I| = k$, which vanishes if $k \neq l+m$. The $k-(l,m)$-part of Ψ has the similar formula on ξ_I with $|I| = k$,

$$\Psi(\xi_I) = \sum_{\substack{A,B:|A|=l,\\ |B|=m,\\ A\cup B=I}} (\xi_A, \xi_B).$$

This too clearly vanishes if $l+m < k$ and, when $k = l+m$, it is precisely equal to the $k-(l,m)$ part of Φ. So the maps Φ and Ψ are precisely equal on their $k-(l,m)$ parts for $k \leq l+m$.

For $k > l+m$, we observe that $\Psi^{k-(l,m)} : D_k \to \mathrm{SP}(D_l\wedge D_m)$ factors through the composition:

$$D_k \to \mathrm{SP}(D_{m+l-k,k-m,k-l}) \to \mathrm{SP}(D_l \wedge D_m) \tag{2.6.10}$$

by first partitioning the set I of size k into three disjoint sets C, A', B' of sizes $m+l-k, k-m, k-l$, respectively, and then mapping $\xi_{C,A',B'}$ to $\xi_{C\cup A'} \wedge \xi_{C\cup B'}$. This is the map μ from Lemma 87 which is based-null homotopic. Thus $\Psi^{k-(l,m)}$ is based-homotopic to the trivial map which is equal to $\Phi^{k-(l,m)}$ and we are done. $\square$

Corollary 88. *The diagram*

$$\begin{tikzcd}
\widetilde{H}^*(V;\mathbb{Z}) \arrow[r, "\sigma^*"] & \widetilde{H}^*(C;\mathbb{Z}) \\
\widetilde{H}^*(V\wedge V;\mathbb{Z}) \arrow[u, "\Delta^*"] \arrow[r, "(\sigma\triangle\sigma)^*"] & \widetilde{H}^*(C\wedge C;\mathbb{Z}). \arrow[u, "\Phi^*"']
\end{tikzcd}$$

commutes.

2.6.3 The superposition product and its relation with Φ

Let $C_{n,m}(M)$ be the coloured configuration space of n blue points and m red points on M, all distinct. The space $C_{n,m}(M)$ is both an open subset of the product $C_n \times C_m$, and a degree $\binom{n+m}{n}$ covering map of C_{n+m}, giving the maps

$$C_n \times C_m \xleftarrow{\ i\ } C_{n,m} \xrightarrow{\ p\ } C_{n+m}$$

Using the cohomology transfer map $p_!$, we obtain the *superposition product*

$$\sup_{n,m} = p_! \circ i^* : H^*(C_n \times C_m;\mathbb{Z}) \to H^*(C_{n+m};\mathbb{Z}).$$

We will suppress the index and merely write sup.

We recall that D_n is a Thom space over C_n and $D_n \wedge D_m$ is also a Thom space over $C_n \times C_m$. We recall that the map Φ restricted to a multivalued function $\Phi^{(n+m)-(n,m)}$: $D_{n+m} \to \mathrm{SP}(D_n \wedge D_m)$. Again we suppress the index of Φ, and using section 2.6.1, we obtain a cohomology map Φ^*.

Proposition 89. *The diagram*

$$\begin{tikzcd} H^*(C_n \times C_m;\mathbb{Z}) \arrow[r, "\sup"] \arrow[d, "Th", "\cong"'] & H^*(C_{n+m};\mathbb{Z}) \arrow[d, "Th", "\cong"'] \\ \widetilde{H}^*(D_n \wedge D_m;\mathbb{Z}) \arrow[r, "\Phi^*"] & \widetilde{H}^*(D_{n+m};\mathbb{Z}), \end{tikzcd}$$

commutes. Here the degree shifts of the Thom isomorphisms are kept implicit.

Proof. We use the coloured and labelled configuration space $D_{n,m}$ which is the Thom space of an oriented vector bundle over $C_{n,m}$. Defining the maps $\widetilde{i}$ and $\widetilde{p}$ in the obvious way, we get two pullback diagrams of bundles

$$\begin{tikzcd} D_n \wedge D_m - \{\infty\} \arrow[d] & D_{n,m} - \{\infty\} \arrow[l, hook', "\widetilde{i}"] \arrow[r, "\widetilde{p}"] \arrow[d] & D_{n+m} - \{\infty\} \arrow[d] \\ C_n \times C_m & C_{n,m} \arrow[l, hook', "i"] \arrow[r, "p"] & C_{n+m}. \end{tikzcd}$$

Therefore, the Thom isomorphisms commute with the compactified maps $\widetilde{i}^*_\infty$ and $\widetilde{p}^*_\infty$. Moreover, from Lemma 90, as p is a covering space, there are transfer maps $p_!$ and $(\widetilde{p}_\infty)_! : \widetilde{H}^*(D_{n,m}) \to \widetilde{H}^*(D_{n+m})$, that also commute with the Thom isomorphisms.

Now $p_! \circ i^*$ is by definition the superposition product. But also $(\widetilde{p}_\infty)_! \circ \widetilde{i}^*_\infty$ equals Φ^*, because $\Phi^{(n+m)-(n,m)}$ factors is the composition $D_{n+m} \to \mathrm{SP}(D_{n,m}) \to \mathrm{SP}(D_n \wedge D_m)$, where the first map is the "inverse" of the covering map $\widetilde{p}_\infty$ and the second map is $\mathrm{SP}(\widetilde{i}_\infty)$; in cohomology these give the equality $(\widetilde{p}_\infty)_! \circ \widetilde{i}_\infty^* = \Phi^*$. As a result, the desired diagram commutes. $\square$

Lemma 90. *Let $\pi : E \to B$ be an oriented rank d vector bundle, and $p : \widetilde{B} \to B$ a degree n covering space. Then there is a transfer map $\bar{p}_! : \widetilde{H}^*(\mathrm{Th}(p^*E);\mathbb{Z}) \to \widetilde{H}^*(\mathrm{Th}(E);\mathbb{Z})$ given that comes from the multivalued functions $\mathrm{Th}(E) \to \mathrm{SP}(\mathrm{Th}(p^*E))$ inverse to the covering map $\bar{p} : p^*E \to E$, and it commutes with the transfer map corresponding to p, i.e. the diagram*

$$\begin{tikzcd}
\widetilde{H}^{*+d}(\mathrm{Th}(p^*E);\mathbb{Z}) \arrow[r, "\bar{p}_!"] & \widetilde{H}^{*+d}(\mathrm{Th}(E);\mathbb{Z}) \\
H^*(\widetilde{B};\mathbb{Z}) \arrow[u, "\mathrm{Th}", "\cong"'] \arrow[r, "p_!"] & H^*(B;\mathbb{Z}) \arrow[u, "\mathrm{Th}"]
\end{tikzcd}$$

commutes.

Proof. For both bundles $F = E, p^*E$, we replace the pair $(\mathrm{Th}(F), \infty)$ with the pair $(\mathbb{D}(F), \mathbb{S}(F))$ which is isomorphic by a radial homeomorphism, so that the map $\bar{p} : (\mathbb{D}(p^*E), \mathbb{S}(p^*E)) \to (\mathbb{D}(E), \mathbb{S}(E))$ is a degree n covering space of pairs. Under the excision isomorphisms $\bar{p}_! : H^*(\mathbb{D}(p^*E), \mathbb{S}(p^*E)) \to H^*(\mathbb{D}(E), \mathbb{S}(E))$ is equal to the transfer map defined in the classical way, namely on singular cochains, it is the dual the chain map:

$$\bar{p}^! : C_*(\mathbb{D}(E), \mathbb{S}(E)) \to C_*(\mathbb{D}(p^*E), \mathbb{S}(p^*E))$$
$$\sigma \mapsto \sum_n \widetilde{\sigma}$$

where the summation is taken over all n preimages of the simplex σ over $\bar{p}$. The transfer map $p_!$ has a similar definition. We pick a cochain $u_E \in C^d(\mathbb{D}(E), \mathbb{S}(E))$ representing the Thom class of $E \to B$, so that p^*u_E represents the Thom class of $p^*E \to \widetilde{B}$. We check that the diagram

$$\begin{tikzcd}
C^{i+d}(\mathbb{D}(E), \mathbb{S}(E)) \arrow[r, "\bar{p}_!"] & C^{i+d}(\mathbb{D}(E), \mathbb{S}(E)) \\
C^i(\widetilde{B}) \arrow[u, "\widetilde{\pi}^*(-)\smile \bar{p}^* u_E"] \arrow[r, "p_!"] & C^i(B) \arrow[u, "\pi^*(-)\smile u_E"']
\end{tikzcd}$$

commutes. In other words we check that for any class α,

$$\pi^*(p_!(\alpha)) \smile u_E = \bar{p}_!(\widetilde{\pi}^*(\alpha) \smile \bar{p}^* u_E).$$

But it is easy to see from the definition of the cup product that $\bar{p}_!(\tilde{\pi}^*(\alpha) \smile \bar{p}^* u_E) = \bar{p}_!(\tilde{\pi}^*(\alpha)) \smile u_E$, and then from evaluating on a simplex, that $\bar{p}_!(\tilde{\pi}^*(\alpha)) = \pi^*(p_!(\alpha))$. □

2.6.4 Conclusion

Proposition 91. *For any $l \geq 1$, the multivalued function σ induces a multiplicative isomorphism*

$$\sigma^* : \widetilde{H}^*(C(M; S^{2l}); \mathbb{Z}) \to \bigoplus_{n \geq 1} \widetilde{H}^*(C_n(M); \mathbb{Z})[2ln]$$

where mutliplication on the codomain is given by the superposition product.

Proof. To simplify the notation, we assume $\mathbb{Z}$-coefficients for homology throughout.

For positive l, and a fixed i, the abelian group $H^{i-2ln}(C_n(M)) \cong \widetilde{H}^i(D_n)$ is non-zero only for finitely many n, which justifies the identification

$$\widetilde{H}^i(\vee_{n\geq 1} D_n) \cong \bigoplus_{n \geq 1} \widetilde{H}^i(D_n),$$

with a direct sum, as opposed to a direct product. The superposition product is thus the direct sum of all the maps $\sup_{m,n}$ for all m and n. That it is an isomorphism is proved in Proposition 84.

We have the following diagram (where C_n is shorthand for $C_n(M)$ for economy of space)

$$\begin{tikzcd}
\widetilde{H}^*(C) \otimes \widetilde{H}^*(C) \arrow[r, "\sigma^* \otimes \sigma^*"] \arrow[d, "\times"] & \widetilde{H}^*(\vee_{n\geq 1} D_n) \otimes \widetilde{H}^*(\vee_{n\geq 1} D_n) \arrow[r, "\mathrm{Th} \otimes \mathrm{Th}"] \arrow[d, "\times"] & \bigoplus_{n,m\geq 1} H^*(C_n) \otimes H^*(C_n) \arrow[d, "\times"] \\
\widetilde{H}^*(C \wedge C) \arrow[r, "(\sigma \triangle \sigma)^*"] \arrow[d, "\Delta^*"] & \widetilde{H}^*(\vee_{n\geq 1} D_n \wedge \vee_{n\geq 1} D_n) \arrow[r, "\mathrm{Th}"] \arrow[d, "\Phi^*"] & \bigoplus_{n,m\geq 1} H^*(C_n \times C_m) \arrow[d, "\sup"] \\
\widetilde{H}^*(C) \arrow[r, "\sigma^*"] & \widetilde{H}^*(\vee_{n\geq 1} D_n) \arrow[r, "\mathrm{Th}"] & \bigoplus_{n\geq 1} H^*(C_n)
\end{tikzcd}$$

in which every square commutes: the top-left from Proposition 81, the bottom-left from Corollary 88, the top-right because the Thom isomorphism commutes with the cross-product, and the bottom-right by Proposition 89. Looking only at the corners of the big square, the leftmost vertical composition is the cup product, and the righmost column is the superposition product, whereas the top-most and bottom maps are the scanning maps. □

Proof of Theorem 76. Observe that everything we have proved in this addendum also holds for $\mathbb{Q}$-coefficients of homology; in particular, Proposition 91 holds rationally. Combining it with Proposition 3.6 and Corollary 4.2 from Manthorpe–Tillmann [33], gives the isomorphism of non-unital algebras

$$s: \left(\widetilde{H}^*\left(\Gamma_\partial(M, S^{2l}); \mathbb{Q}\right), \smile\right) \longrightarrow \left(\bigoplus_{n\geq 1} H^*(C_n(M); \mathbb{Q})[2ln], \sup\right).$$

Both domain and codomain naturally extend to unital algebras: replace $\widetilde{H}^*$ by H^* on the left hand side, and $\bigoplus_{n\geq 1}$ by $\bigoplus_{n\geq 0}$ on the right hand side (superposition with the empty configuration is the identity map). The isomorphism exteds. $\square$

2.6.5 An alternative definition of the superposition product

We recall an alternative superposition product appearing in Bianchi–Miller–Wilson [3] and Moriyama [37]. Let $C_n(M)^\infty$ be the one point compactification of $C_n(M)$, where the point ∞ corresponds to configurations where two points collide or one point goes to the boundary. It is straightforward to define a superposition map

$$\begin{aligned} \mu_{m,n}: C_n(M)^\infty \wedge C_m(M)^\infty &\to C_{n+m}(M)^\infty \\ (s,t) &\mapsto \begin{cases} \infty \text{ if } s \cap t \neq \emptyset \text{ or } s = \infty \text{ or } t = \infty, \\ s \cup t \text{ otherwise.} \end{cases} \end{aligned}$$

From now on assume that M is orientable. We observe that $C_n(M)^\infty \wedge C_m(M)^\infty = (C_n(M) \times C_n(M))^\infty$. The Poincare–Lefschetz duality isomorphisms PL and the homology map μ_*, we define $\widetilde{\sup}$ so that the diagram

$$\begin{tikzcd} H^*(C_n(M) \times C_m(M); \mathbb{Z}) \arrow[r, dashed, "\widetilde{\sup}_{n,m}"] \arrow[d, "\mathrm{PL}"', "\cong"] & H^*(C_{n+m}(M); \mathbb{Z}) \arrow[d, "\mathrm{PL}"', "\cong"] \\ \widetilde{H}_{d(n+m)-*}(C_n(M)^\infty \wedge C_m(M)^\infty; \mathbb{Z}) \arrow[r, "\mu_*"] & \widetilde{H}_{d(n+m)-*}(C_{n+m}(M)^\infty; \mathbb{Z}) \end{tikzcd} \tag{2.6.11}$$

commutes.

Proposition 92. *The map* $\widetilde{\sup}_{n,m}$ *coincides with the superposition product* $\sup_{n,m}$ *from Section 2.6.3.*

Proof. An open inclusion gives an opposite map after compactifying, factoring μ as

$$C_n(m)^\infty \wedge C_m(m)^\infty \xhookrightarrow{i^\infty} C_{n,m}(M)^\infty \xrightarrow{p^\infty} C_{n+m}(M)^\infty.$$

Therefore, we factor

$$\widetilde{\sup} = (\mathrm{PL}^{-1} \circ p_*^\infty \circ \mathrm{PL}) \circ (\mathrm{PL}^{-1} \circ i_*^\infty \circ \mathrm{PL}),$$

using the PL isomoprhism for $C_{n,m}(M)$.

By standard properties of Poincaré Lefschetz duality $(\mathrm{PL}^{-1} \circ i_*^\infty \circ \mathrm{PL})$ is equal to the cohomology map i^* and the Gysin map $(\mathrm{PL}^{-1} \circ p_*^\infty \circ \mathrm{PL})$ is equal to the transfer map $p_!$. As a result $\widetilde{\sup} = p_! \circ i^* = \sup$. □

Remark 93. The proposition holds also for non-orientable manifolds, but care must taken in using the orientation sheaf for Poincaré-Lefschetz duality.

2.7 Addendum II – Coloured scanning

The results of this section are for homology with $\mathbb{Q}$-coefficients.

For $k \in \mathbb{Z}_{\geq 0}$, write $[k] = \{1, ..., k\}$. We will denote by $\mathbf{v}$ elements of $[k]^n$. The *counting function* $c : \coprod_{n \geq 1} [k]^n \to (\mathbb{Z}_{\geq 0})^k$ is

$$c(\mathbf{v})_i = |\{j \in [n] : v_j = i\}|.$$

For $\mathbf{c} \in (\mathbb{Z}_{\geq 0})^k$ write $|\mathbf{c}| = \sum_i c_i$ the *size* of $\mathbf{c}$.

Remark 94. The symmetric group $\mathfrak{S}_n$ acts on $[k]^n$ by permutations. Its orbits are the preimage sets $c^{-1}(\mathbf{c})$.

Definition 95. For $\mathbf{c} \in (\mathbb{Z}_{\geq 0})^k$ of size n, the *configuration of n points on M with coulour-pattern* $\mathbf{c}$ is

$$C_{n,\mathbf{c}}(M) := F_n(M)/\mathfrak{S}_{c_1} \times ... \times \mathfrak{S}_{c_k}$$

where $\mathfrak{S}_{c_1} \times ... \times \mathfrak{S}_{c_k}$ acts as the obvious subset of $\mathfrak{S}_n$. The *total configuration of n points on X with k colours* is the disjoint union

$$C_{n,[n]}(M) := \coprod_{\substack{\mathbf{c} \in (\mathbb{Z}_{\geq 0})^k \\ |\mathbf{c}|=n}} C_{n,\mathbf{c}}$$

which, by Remark 94 coincides with the quotient $(F_n(M) \times [k]^n)/\mathfrak{S}_n$, with $\mathfrak{S}_n$ acting with the diagonally.

Definition 96. Suppose V is $\mathbb{Q}$-vector space and a representation of the group$(\mathbb{Q}^*)^k$. For $\mathbf{c} \in \mathbb{Z}^n$ define the $\mathbf{c}$-*weightspace*

$$V^{(\mathbf{c})} = \{v \in V : (q_1, ..., q_n) * v = q_1^{c_1} ... q_k^{c_k} v \, \forall \mathbf{q} \in (\mathbb{Q}^*)^k\}.$$

If for some $\mathbf{c} \in \mathbb{Z}^n$, $V = V^{(\mathbf{c})}$ then we say that $(\mathbb{Q}^*)^k$ *acts on V by weight* $\mathbf{c}$.

We let the group $(\mathbb{Q}^*)^k$ also act on a wedge of rational spheres $\vee_{i=1}^k S_{\mathbb{Q}}^{m_i}$ by degree homeomorphisms. Then any vector space defined functorially in terms $\vee_{i=1}^k S_{\mathbb{Q}}^{m_i}$ is naturally a $(\mathbb{Q}^*)^k$-representation.

Theorem 97 (Coloured scanning)**.** *For colour-pattern* $\mathbf{c}$ *of size* $n \geq 1$ *and integers* $d_i \geq 1$ *for* $i = 1, ..., k$*, there is a* MCG(M)*-equivariant isomorphism*

$$H^j(C_{n,\mathbf{c}}(M);\mathbb{Q}) \cong \widetilde{H}^{j+2\sum_i c_i d_i}(\mathrm{map}_\partial(M, \vee_{i=1}^k S^{d+2d_i});\mathbb{Q})^{(\mathbf{c})}.$$

2.7.1 The proof

We check that the steps of the proof of Theorem 18 go through.

Lemma 98. *For every* $\mathbf{c} \in (\mathbb{Z}_{\geq 0})^n$ *with* $|\mathbf{c}| = k \geq 1$ *there is a* MCG(M)*-equivariant isomorphism*

$$\widetilde{H}^j\Big(\Big(F_n(M)_+ \wedge \bigvee_{\substack{\boldsymbol{v}\in[k]^n \\ c(\boldsymbol{v})=\mathbf{c}}} (\wedge_{i=1}^k S^{2d_{v_i}})\Big)/\mathfrak{S}_n;\mathbb{Q}\Big) \cong H^{j-\sum_i d_{v_i}}\Big(C_{n,\mathbf{c}}(M);\mathbb{Q}\Big).$$

Furthermore $(\mathbb{Q}^*)^k$ *acts naturally on the LHS by weight* $\mathbf{c}$*.*

Proof. The space on the left hand side is the Thom space of the orientable vector bundle

$$\Big(F_n(M) \times \bigsqcup_{\substack{\mathbf{v}\in[k]^n \\ c(\mathbf{v})=\mathbf{c}}} (\oplus_{i=1}^k \mathbb{R}^{2d_{v_i}})\Big)/\mathfrak{S}_n \to C_{n,\mathbf{c}}(M)$$

of rank $2\sum_i d_{v_i}$. The Thom isomorphism is MCG(M) equivariant. We conclude identically to the proof of Claim 20. $\square$

Recall the *configuration space of* M *with labels in the based space* (X, x_0)

$$C(M;X) = \Big(\bigsqcup_{n\geq 1} F_n(M) \times_{\mathfrak{S}_n} X^n\Big)\Big/\sim$$

where $(z_1, ..., z_n; x_1, ..., x_n) \sim (z_1, ..., z_{n-1}; x_1, ..., x_{n-1})$ if $x_n = x_0$. The subset $C_{\leq n}(M;X) \subset C(M;X)$ contains configurations with at most n configuration points. Denote by $D_n(M;X) := C_{\leq n}(M;X)/C_{\leq n-1}(M;X)$ the filtration quotients. These can be better understood via the formula

$$D_n(M;X) = (F_n(M)_+ \wedge X^{\wedge n})/\mathfrak{S}_n. \tag{2.7.1}$$

Proposition 99. *For colour-pattern* $\mathbf{c}$ *of size* $n \geq 1$ *and integers* $d_i \geq 1$ *for* $i = 1, \ldots, k$*, there is a* $\mathrm{MCG}(M)$*-equivariant isomorphism*

$$H^j\Big(C_{n,\mathbf{c}}(M);\mathbb{Q}\Big) \cong \widetilde{H}^{j+2\sum_i c_i d_i}\Big(D_n(M;\vee_{i=1}^k S^{2d_i});\mathbb{Q}\Big)^{(\mathbf{c})}.$$

Proof. The key observation is the decomposition

$$\begin{aligned} D_b(M;\vee_{i=1}^k S^{2d_i}) &= \Big(F_n(M)_+ \wedge (\vee_{i=1}^k S^{2d_i})^{\wedge k}\Big)/\mathfrak{S}_n \\ &= \vee_{\mathbf{v}\in[k]^n}(F_n(M)_+ \wedge S^{2|\mathbf{v}|})/\mathfrak{S}_n \\ &= \vee_{|\mathbf{c}|=n}(\vee_{\mathbf{v}:c(\mathbf{v})=\mathbf{c}}(F_n(M)_+ \wedge S^{2|\mathbf{v}|})/\mathfrak{S}_n) \end{aligned}$$

which combined with Lemma 98 gives the desired statement. □

Proposition 100 (Untwisting)**.** *Let a group* G *act on a based space* (X, x_0) *by based homeomorphisms. Let* $d = \dim M$*, and fix* $i \geq 0$*, if* d *is odd, or* $i \geq 1$*, if* d *is even. Then there is a* $\mathrm{MCG}(M) \times G$*-equivariant isomorphism of algebras*

$$H^*\Big(\Gamma_\partial(M;S^i_\mathbb{Q} \wedge X);\mathbb{Q}\Big) \cong H^*\Big(\mathrm{map}_\partial(M;S^{d+i}_\mathbb{Q} \wedge X);\mathbb{Q}\Big). \tag{2.7.2}$$

Proof. The proof is identical to the proof of Theorem 21. Fix $k \in \{0,1\}$ congruent to $d+1$ modulo 2. Observe that $k \leq i$. Then $\Gamma_\partial(M;S^i_\mathbb{Q} \wedge X)$ is sections of the bundle $\tau^+M \wedge_f S^i_\mathbb{Q} \wedge X \to M$ having fibres $S^{d+i}_\mathbb{Q} \wedge X$. This can be viewed as an oriented[4] fibration with a section of fibre $S^{d+k}_\mathbb{Q}$ fibrewise smashed with $S^{i-k}_\mathbb{Q} \wedge X$. The classifying space $BhAut^1_*(S^{d+k}_\mathbb{Q})$ is contractible as $d+k$ is odd. Thus we can untwist $\mathrm{MCG}(M)$-equivariantly up to homotopy and the untwisting map is automatically G-equivariant as well because we have not affected the X coordinate. □

Proof of Theorem 97. We have the $\mathrm{MCG}(M)$-equivariant isomorphisms

$$\begin{aligned} H^j(C_{n,\mathbf{c}}(M);\mathbb{Q}) &\cong \widetilde{H}^{j+2\sum_i c_i d_i}\Big(D_n(M;\vee_{i=1}^k S^{2d_i});\mathbb{Q}\Big)^{(\mathbf{c})} \\ &\cong \widetilde{H}^{j+2\sum_i c_i d_i}\Big(\Gamma_\partial(M;\vee_{i=1}^k S^{2d_i});\mathbb{Q}\Big)^{(\mathbf{c})} \\ &\cong \widetilde{H}^{j+2\sum_i c_i d_i}\Big(\mathrm{map}_\partial(M,\vee_{i=1}^k S^{d+2d_i});\mathbb{Q}\Big)^{(\mathbf{c})}, \end{aligned}$$

[4] i.e. the classifying group acts by degree 1.

where the first isomorphism comes from Proposition 99, and the last isomorphism comes from Proposition 100 applied to $X = \vee_i S_{\mathbb{Q}}^{2d_i-1}$. The second isomorphism uses the special case of Corollary 4.2 from Manthorpe–Tillmann [33] that there is a $\mathrm{MCG}(M) \times G$-equivariant isomorphism

$$\widetilde{H}_*(\Gamma_\partial(M;X);\mathbb{Q}) \cong \bigoplus_{n\geq 1} \widetilde{H}_*(D_n(M;X);\mathbb{Q}).$$

Weights $\mathbf{c}$ of size n pick exactly the $\widetilde{H}_*(D_n(M;X);\mathbb{Q})$ summand. Dualising, gives the result in cohomology. $\square$

Corollary 101 (Ordered configurations)**.** *For $n \geq 0$ and $m \geq 1$, there is a* $\mathrm{MCG}(M)$-*equivariant isomorphism*

$$H^j(F_n(M);\mathbb{Q}) \cong \widetilde{H}^{j+2nm}(\mathrm{map}_\partial(M, \vee_{i=1}^k S^{d+2m});\mathbb{Q})^{(1,1,\ldots,1)}. \tag{2.7.3}$$

Chapter 3

Polysimplicial decompositions

In this chapter, we construct chain complexes that compute $H^*(C_n(\Sigma_{g,1}))$ and $H^*(C_n(\Sigma_g))$, with various coefficients, as mapping class groups representations. We will study the kernels of these representations.

We will use the definitions $\Gamma_{g,1} := \pi_0\mathrm{Homeo}(\Sigma_{g,1}, \partial\Sigma_{g,1})$ and $\Gamma_g := \pi_0\mathrm{Homeo}^+(\Sigma_g)$, which, for surfaces, coincide with the definition via diffeomorphism groups we have used so far.

3.1 Preliminaries

3.1.1 Partial compactifications of configuration spaces

Given a pair of spaces (X, X'), for $n \geq 0$ we define two subspaces of X^n:

$$\Delta_n(X) := \{(x_1, ..., x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\},$$
$$A_n(X, X') := \{(x_1, ..., x_n) \in X^n : x_i \in X' \text{ for some } i\}.$$

Any continuous map $f : X \to Y$ induces a map $f^n \colon X^n \to Y^n$ that restricts to a map $\Delta_n(f) : \Delta_n(X) \to \Delta_n(Y)$. If f is moreover a map of pairs $(X, X') \to (Y, Y')$, then f^n also restricts to a map $A_n(f) : A_n(X, X') \to A_n(Y, Y')$. Finally, the permutation action of $\mathfrak{S}_n$ on X^n preserves the subsets $\Delta_n(X)$ and $A_n(X, X')$.

Definition 102. For $n \geq 1$ and a pair (X, X'), we define the space

$$C_n(X, X')^\infty := \Big(X^n/(\Delta_n(X) \cup A_n(X, X'))\Big)/\mathfrak{S}_n.$$

We denote by ∞ the image of $(\Delta_n(X) \cup A_n(X,X'))_+$ in the quotient, which serves as basepoint for $C_n(X,X')^\infty$. For $n = 0$, we define $C_0(X,X')^\infty$ to be the discrete space $\{\infty, \emptyset\}$ based at ∞. We write the wedge of pointed spaces $C_\bullet(X,X')^\infty := \bigvee_{n\geq 0} C_n(X,X')^\infty$.

In the case of a pointed space (X,x_0), we also write $\widetilde{C}_n(X)^\infty := C_n(X,\{x_0\})^\infty$, and $\widetilde{C}_\bullet(X)^\infty = C_\bullet(X,\{x_0\})^\infty$.

The image of $(x_1,...,x_n)$ in $C_n(X,X')^\infty$ is denoted by $[x_1,...,x_n]$; we often denote by s a generic point in $C_n(X,X')^\infty$. Note that a map of pairs $f : (X,X') \to (Y,Y')$ induces a map $C_n(f)^\infty : C_n(X,X')^\infty \to C_n(Y,Y')^\infty$.

3.1.2 Connection to $C_n(M)$ via Poincaré duality

If M is a compact manifold with boundary, we will often use the shorthand $C_n(M)^\infty := C_n(M,\partial M)^\infty$, which coincides with the one-point compactification of $C_n(\mathring{M})$, where $\mathring{M} \subseteq M$ denotes the interior of M. The group $\mathrm{Homeo}(M,\partial M)$ of orientation-preserving homeomorphisms of M fixing ∂M pointwise acts on both $C_n(\mathring{M})$ and $C_n(M)^\infty$ by homeomorphisms.

Proposition 103. *Let R be a commutative ring, and M be an oriented manifold of even dimension $2d$. For every $n,i \geq 0$, there is a* $\mathrm{Homeo}(M,\partial M)$*-equivariant isomorphism $H^i(C_n(M);R) \cong \widetilde{H}_{2dn-i}(C_n(M)^\infty;R)$.*

Proof. First, as in Chapter 2, the inclusion $C_n(\mathring{M}) \hookrightarrow C_n(M)$ is a homotopy equivalence, hence $H^i(C_n(M);R) \cong H^i(C_n(\mathring{M});R)$. Second, $C_n(\mathring{M})$ is an open manifold of dimension $2dn$, and the chosen orientation on M induces an orientation on $C_n(\mathring{M})$ (here we use that $2d$ is even). The one-point compactification of $C_n(\mathring{M})$ is $C_n(M)^\infty$; Poincaré duality gives an isomorphism $H^i(C_n(\mathring{M});R) \cong H^{BM}_{2dn-i}(C_n(\mathring{M});R)$ where H^{BM}_* is Borel-Moore homology. For an open manifold X, $H^{BM}_*(X;R) \cong \widetilde{H}_*(X^+;R)$, the homology of the one-point compactification. All the above isomorphisms are natural with the action of orientation-preserving homeomorphisms of M. $\square$

We will use the above proposition as a principle throughout the chapter.

3.1.3 Superposition of configurations

For $m, n \geq 0$, we define the maps

$$\begin{aligned} \mu_{m,n} &: C_m(X,X')^\infty \times C_n(X,X')^\infty \to C_{m+n}(X,X')^\infty \\ (s_1,s_2) &= ([x_1,...,x_m],[y_1,...,y_n]) \mapsto s_1 \cup s_2 = [x_1,...,x_m,y_1,...,y_m], \end{aligned}$$

where, if $m=0$ or $n=0$, we mean $\mu_{n,0}(s,\emptyset)=\mu_{0,n}(\emptyset,s)=s$ and $\mu_{n,0}(s,\infty)=\mu_{0,n}(\infty,s)=\infty$. Then $\mu := \bigvee_{m,n\geq 0}\mu_{m,n}$ is a binary operation on $C_\bullet(X,X')^\infty$.

Proposition 104. *The pair $\left(C_\bullet(X,X')^\infty,\mu\right)$ is an abelian topological monoid with unit $\emptyset$ and absorbing element ∞.*

Proof. The associative monoid $X^\bullet := \coprod_{n\geq 0} X^n$, with operation given by concatenation, admits an ideal given by the subspace

$$\Delta_\bullet(X)\cup A_\bullet(X,X') := \coprod_{n\geq 0}\Delta_n(X)\cup A_n(X,X');$$

the abelianisation of the quotient monoid $X^\bullet/\Delta_\bullet(X)\cup A_\bullet(X,X')$ is precisely the abelian topological monoid $C_\bullet(X,X')^\infty$. □

3.1.4 On polysimplicial decompositions

Definition 105. For $n \geq 0$, we fix the following model for the *n-simplex*

$$\Delta^n := \{(t_1,\dots,t_n)\in\mathbb{R}^n : 0\leq t_1\leq t_2\leq\cdots\leq t_n\leq 1\}. \tag{3.1.1}$$

There are $n+1$ *face maps* $f_i : \Delta^{n-1}\to\Delta^n$, for $i=0,...,n$, each mapping

$$f_i : (s_1,...,s_{n-1}) \longmapsto (t_1,...,t_n)$$

with $t_j = s_j$, if $j\leq i$, and $t_j = s_{j-1}$, if $j\geq i+1$. Here we have set $s_0\equiv 0$ and $s_n\equiv 1$. The *boundary* $\partial\Delta^n$ is the union $\bigcup_{i=0}^n f_i(\Delta^{n-1})$, and is homeomorphic to the $(n-1)$-sphere. The *interior* of Δ^n is the complement $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$.

Definition 106. For $k\geq 0$, and $\underline{n}\in(\mathbb{Z}_{\geq 0})^k$, the $\underline{n}$-*polysimplex* is the product $\Delta^{\underline{n}} := \Delta^{n_1}\times...\times\Delta^{n_k}$. Its *boundary* is $\partial\Delta^{\underline{n}} = \bigcup_{j=1}^k \Delta^{n_1}\times...\times\partial\Delta^{n_j}\times...\Delta^{n_k}$, and its *interior* is $\mathring{\Delta}^{\underline{n}} = \Delta^{\underline{n}} - \partial\Delta^{\underline{n}}$. The *dimension* of $\Delta^{\underline{n}}$, also referred to as the *dimension of* $\underline{n}$, is $d(\underline{n}) = n_1 + ... + n_k$, so that $\Delta^{\underline{n}} \cong D^{d(\underline{n})}$.

Definition 107. A *polysimplicial decomposition* of a pair of spaces (X,Y) is a relative CW-decomposition of (X,Y) together with a chosen homeomorphism of every cell with a polysimplex.

A polysimplicial decomposition comes with the following data:

- the graded set $S_\bullet = \bigsqcup_{d\geq 0} S_d$ of *polysimplices*; for $\mathfrak{s} \in S_\bullet$, write $d(\mathfrak{s}) = d$ if $\mathfrak{s} \in S_d$;
- the function $\underline{n} : S_\bullet \to \bigsqcup_{k\geq 0}(\mathbb{Z}_{\geq 1})^k$ satisfying $d(\underline{n}(\mathfrak{s})) = d(\mathfrak{s})$ and recording the type of the polysimplex;
- the collection of embeddings $\Phi^{\mathfrak{s}} : \Delta^{\mathfrak{s}} := \Delta^{\underline{n}(\mathfrak{s})} \longrightarrow X$, for each $\mathfrak{s} \in S_\bullet$.

The relative cellular chain complex associated to a polysimplicial decomposition of (X,Y) is

$$\mathrm{Ch}_*^{\mathrm{cell}}(X,Y) = \mathbb{Z}\langle e_{\mathfrak{s}} : \mathfrak{s} \in S_\bullet\rangle,$$

which computes the the relative homology $H_*(X,Y)$. We will now provide a formula for the cellular differential d^{cell}.

Convention 108. We make the following choices of orientation: (i) $\Delta^{\underline{n}}$ inherits its orientation as a subspace of $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$ with the standard orientation, (ii) the boundary $\partial\Delta^{\underline{n}}$ has the orientation that makes the face inclusion $f_0 \times \mathrm{id} : \Delta^{n_1-1} \times \Delta^{n_2} \times ... \times \Delta^{n_k} \to \partial\Delta^{\underline{n}}$ orientation preserving. These only affect d^{cell} by a sign, and do not affect the computed homology.

Any map of pairs $f : (\Delta^{\underline{n}}, \partial\Delta^{\underline{n}}) \to (X^{d(\underline{n})}, X^{d(\underline{n})-1})$ defines a canonical element $[f] \in \mathrm{Ch}_{d(\underline{n})}(X,Y)$. In particular, $[\Phi^{\mathfrak{s}}] = e_{\mathfrak{s}} \in \mathrm{Ch}_{d(\mathfrak{s})}(X,Y)$.

Proposition 109. *The cellular differential* d^{cell} *on* $\mathrm{Ch}_*^{\mathrm{cell}}(X,Y)$ *is given by*

$$d^{\mathrm{cell}}(e_{\mathfrak{s}}) = \sum_{j=1}^{k}(-1)^{n_1+...+n_{j-1}}\sum_{i=0}^{n_j}(-1)^i[\Phi^{\mathfrak{s}} \circ (\mathrm{id}_{\Delta^{n_1}\times...\times\Delta^{n_{j-1}}} \times f_i \times \mathrm{id}_{\Delta^{n_{j+1}}\times...\times\Delta^{n_k}})].$$

Proof. The boundary $\partial\Delta^{\underline{n}}$ is itself a polysimplicial complex: its polysimplices are products of faces of its factors. The orientation cycle $[\partial\Delta^{\underline{n}}] \in \mathrm{Ch}_{d(\underline{n})-1}(\partial\Delta^{\underline{n}})$ is a signed sum of the top-dimensional poly-simplices, and we identify these with their embeddings

$$f_{j,i} := \mathrm{id}_{\Delta^{n_1}\times\ldots\times\Delta^{n_{j-1}}} \times f_i \times \mathrm{id}_{\Delta^{n_{j+1}}\times\ldots\times\Delta^{n_k}} : \\ (\Delta^{n_1}\times\ldots\times\Delta^{n_{j-1}})\times\Delta^{n_j-1}\times(\Delta^{n_{j+1}}\times\ldots\times\Delta^{n_k}) \longrightarrow \partial\Delta^{\underline{\mathrm{n}}},$$

for $j=1,\ldots,k$ and $i=0,\ldots,n_j$. The coefficient of $[f_{j,i}]$ in the summation of $[\partial\Delta^{\underline{\mathrm{n}}}]$ is the degree of the embedding $f_{j,i}$. The face inclusion f_i has degree $(-1)^i$: it is standard that the degrees of the f_i alternate and the degree of f_0 is 1 by convention. Furthermore, the orientation of the boundary of a product $M\times N$ follows the rule $\partial(M\times N)=\partial(M)\times N+(-1)^{\dim(M)}M\times\partial N$. We deduce that the degree of $f_{j,i}$ is $(-1)^{n_1+\ldots+n_{j-1}}(-1)^i$. We conclude that

$$\begin{aligned} d^{\mathrm{cell}}(e_{\mathfrak{s}}) = (\Phi^{\mathfrak{s}}|_{\partial\Delta^{\mathfrak{s}}})_*[\partial\Delta^{\mathfrak{s}}] &= \sum_{j,i}(-1)^{n_1+\ldots+n_{j-1}}(-1)^i(\Phi^{\mathfrak{s}}|_{\partial\Delta^{\mathfrak{s}}})_*[f_{j,i}] \\ &= \sum_{j,i}(-1)^{n_1+\ldots+n_{j-1}}(-1)^i[\Phi^{\mathfrak{s}}\circ f_{j,i}]. \end{aligned}$$

□

3.2 Configurations on Bouquets of Circles

We introduce a model for bouquets of circles.

Definition 110. For $k\geq 1$, construct the bouquet of k circles as the quotient

$$V_k := \coprod_{i=1}^{k}[0,1]\times\{i\}/\sim$$

where $\sim$ collapses the endpoints of all intervals to a single point $*$. By the loops $\gamma_i:([0,1],\{0,1\})\to(V_k,*)$ given by $\gamma_i(t)=[t,i]$, we fix once and for all a standard basis $\gamma_1,\ldots,\gamma_k\in\pi_1(V_k)$, exhibiting an isomorphism $\pi_1(V_k)\cong\mathbb{Z}^{*k}$.

3.2.1 A polysimplicial decomposition for $\widetilde{C}_\bullet(V_k)^\infty$

For any vector $\mathfrak{v}=(v_1,\ldots,v_k)\in(\mathbb{Z}_{\geq 1})^{\times k}$, with $k\geq 1$, define the polysimplex $\Delta^{\mathfrak{v}} := \Delta^{v_1}\times\ldots\times\Delta^{v_k}$ of dimension $d(\mathfrak{v}):=v_1+\ldots+v_k$, and the map $\Phi^{\mathfrak{v}}:\Delta^{\mathfrak{v}}\to\widetilde{C}_{d(\mathfrak{v})}(V_k)^\infty$, sending

$$\Big((t_1^{(1)},\dots,t_{v_1}^{(1)}),(t_1^{(2)},\dots,t_{v_2}^{(2)}),\dots,(t_1^{(k)},\dots,t_{v_k}^{(k)})\Big)$$
$$\mapsto \Big[(t_1^{(1)},1),\dots,(t_{v_1}^{(1)},1),(t_1^{(2)},2),\dots,(t_{v_2}^{(2)},2),\dots,(t_1^{(k)},k),\dots,(t_{v_k}^{(k)},k)\Big]. \tag{3.2.1}$$

Observe that the image of $\Phi^{\mathfrak{v}}$ is the subspace in $\widetilde{C}_\bullet(V_k)^\infty$ of configurations with precisely v_i points on the i^{th} circle of V_k, for $1 \le i \le k$.

Proposition 111. *For $k \ge 1$, the set of vectors $\mathfrak{v} = (v_1,\dots,v_k) \in (\mathbb{Z}_{\ge 1})^k$, with dimensions $d(\mathfrak{v})$, and the collection of maps $\Phi^{\mathfrak{v}}$ define a polysimplicial decomposition for the based space $(\widetilde{C}_\bullet(V_k)^\infty,\infty)$. The boundaries of all polysimplices are constantly mapped to ∞. In particular, for $n \ge 0$, $\widetilde{C}_n(V_k)^\infty$ is homeomorphic to the wedge of $\binom{n+k-1}{k-1}$ spheres of dimension n. Furthermore, the cellular differential on $\widetilde{\mathrm{Ch}}_*\Big(\widetilde{C}_\bullet(V_k)^\infty\Big) = \mathbb{Z}\langle e_{\mathfrak{v}} : \mathfrak{v} \in (\mathbb{Z}_{\ge 0})^k\rangle$ vanishes, so that*

$$\widetilde{\mathrm{Ch}}_*\Big(\widetilde{C}_\bullet(V_k)^\infty\Big) \cong \bigoplus_{n\ge 0} \widetilde{\mathrm{Ch}}_*\Big(\widetilde{C}_n(V_k)^\infty\Big) \cong \bigoplus_{n\ge 0} \widetilde{H}_*(\widetilde{C}_n(V_k)^\infty).$$

Proof. The map $\Phi^{\mathfrak{v}}$ restricts to a topological embedding of the interior of $\Delta^{\mathfrak{v}}$, with image the subspace $\prod_{i=1}^k C_{\mathfrak{c}_i}((0,1)\times\{i\}) \subset \widetilde{C}_{d(\mathfrak{v})}(V_k)^\infty$. Conversely, for all $n \ge 0$, the configuration $s \in \widetilde{C}_n(V_k)^\infty \setminus \{\infty\}$ lies in the image of $\Phi^{\mathfrak{v}}$ for exactly one vector $\mathfrak{v}$, namely the vector $\mathfrak{v}(s)$ whose i^{th} coordinate counts the number of points of s lying on $(0,1)\times\{i\} \subset V_k$. This proves that the collection of $\mathfrak{v}$ and $\Phi^{\mathfrak{v}}$ forms a polysimplicial decomposition. Each map $\Phi^{\mathfrak{v}}$ maps $\partial\Delta^{\mathfrak{v}}$ constantly to the point ∞ so the differential vanishes by the formula of Proposition 109. The rest follows. $\square$

We henceforth denote

$$\mathrm{UMor}_{\bullet,*}(k) := \widetilde{\mathrm{Ch}}_*\Big(\widetilde{C}_\bullet(V_k)^\infty\Big),$$

viewed as a *weighted* chain complex[1]: in particular $\mathrm{UMor}_*(k)$ is a bigraded abelian group, with the first grading being the homological degree $*$, and the second being the size of the configuration n, generically denoted by the symbol $\bullet$. Unless we want to emphasise $\bullet$, we will keep it implicit in the notation.

[1]The notation "UMor" comes from the fact that $\mathrm{UMor}_n(k)$ is a version of a construction of Moriyama [37] but in the context of unordered configuration spaces.

For $1 \leq i \leq k$, let $\mathfrak{e}_i = (0,\ldots,0,1,0,\ldots,0) \in (\mathbb{Z}_{\geq 0})^{\times k}$ be the unit vector in the i^{th} direction, and write $e(i,n) := e_{n\mathfrak{e}_i}$ for the cell corresponding to the vector $n\mathfrak{e}_i$, parametrising configurations of n points, all lying on $(0,1)\times\{i\}$.

Definition 112. For $m,n \geq 0$ the *signed shuffle* coefficient is

$$\mathbf{ss}(m,n) = \begin{cases} 0 & \text{if both } m,n \text{ are odd,} \\ \binom{\lfloor \frac{m}{2}\rfloor + \lfloor \frac{n}{2}\rfloor}{\lfloor \frac{m}{2}\rfloor} = \binom{\lfloor \frac{m+n}{2}\rfloor}{\lfloor \frac{m}{2}\rfloor} & \text{otherwise.} \end{cases} \tag{3.2.2}$$

Proposition 113. *The product μ endows $\mathrm{UMor}_*(k)$ with a ring structure, under which*

- $e_{\mathfrak{v}} \cdot e_{\mathfrak{v}'} = (-1)^{|\mathfrak{v}||\mathfrak{v}'|} e_{\mathfrak{v}'} \cdot e_{\mathfrak{v}}$,
- $e_{\mathfrak{v}} \cdot e_{\mathfrak{v}'} = \lambda(\mathfrak{v},\mathfrak{v}') e_{\mathfrak{v}+\mathfrak{v}'}$*, for some* $\lambda(\mathfrak{v},\mathfrak{v}') \in \mathbb{Z}$,
- $\lambda(\mathfrak{v},\mathfrak{v}') = \pm 1$*, if* $\mathfrak{v}$ *and* $\mathfrak{v}'$ *have disjoint support, and*
- $e(i,m)\cdot e(i,n) = \mathbf{ss}(m,n) e(i,m+n)$.

Furthermore, by setting $x_i := e(i,1)$ and $y_i := e(i,2)$, we have the isomorphism of bigraded rings

$$\mathrm{UMor}_*(k) \cong \Lambda_{\mathbb{Z}}[x_1,\ldots,x_k] \otimes \Gamma_{\mathbb{Z}}[y_1,\ldots,y_k]$$

where $\Lambda_{\mathbb{Z}}$ and $\Gamma_{\mathbb{Z}}$ denote, respectively, the free exterior algebra[2] *and the free divided power algebra. In particular, a homomorphism of rings with source $\mathrm{UMor}_*(k)$ is characterised by its values on the x_i and y_i, for $i = 1,\ldots,k$, if the target ring is torsion-free as an abelian group.*

Proof. We check that for $m,n \geq 0$ the map $\mu_{m,n}$ from Subsection 3.1.3 is cellular. The cells of $\widetilde{C}_{n+m}(V_k)^\infty$ are in dimension 0 and $(m+n)$, and $\widetilde{C}_m(V_k)^\infty \times \widetilde{C}_n(V_k)^\infty$ has product cells of dimension $0,m,n$ and $(m+n)$. The union of all 0-cells, m-cells and n-cells in the product $\widetilde{C}_m(V_k)^\infty \times \widetilde{C}_n(V_k)^\infty$ consists of all pairs (s_1,s_2) in which at least one between s_1 and s_2 is ∞; for such pairs we have $\mu_{m,n}(s_1,s_2) = \infty$, which is a 0-cell. So $\mu_{m,n}$ preserves skeleta, i.e. it is cellular. Thus μ is cellular and induces a map μ_* that makes $(\mathrm{UMor}_*(k),\mu_*)$ into a bigraded ring.

By the observation before Proposition 111, all points in the image of $e_{\mathfrak{v}} \times e_{\mathfrak{v}'}$ along $\mu_{m,n}$ are contained in the union $\{\infty\} \cup e_{\mathfrak{v}+\mathfrak{v}'}$, thus $e_{\mathfrak{v}} \cdot e_{\mathfrak{v}'} = \lambda(\mathfrak{v},\mathfrak{v}') e_{\mathfrak{v}+\mathfrak{v}'}$ for some

[2] i.e. the quotient of the free tensor algebra on V by the ideal $(v \otimes v : v \in V)$.

$\lambda(\mathfrak{v},\mathfrak{v}')\in\mathbb{Z}$. Furthermore, if $\mathfrak{v}$ and $\mathfrak{v}'$ have disjoint supports, then up to a permutation of the k coordinates in $(\mathbb{Z}_{\geq 0})^{\times k}$ the polysimplex $\Delta^{\mathfrak{v}+\mathfrak{v}'}$ can be identified with $\Delta^{\mathfrak{v}}\times\Delta^{\mathfrak{v}'}$, and $\Phi^{\mathfrak{v}+\mathfrak{v}'}$ coincides with $\mu\circ(\Phi^{\mathfrak{v}}\times\Phi^{\mathfrak{v}'})$. Thus $e_{\mathfrak{v}}\cdot e_{\mathfrak{v}'}=\pm e_{\mathfrak{v}+\mathfrak{v}'}$; the sign depends on whether or not the identification $\Delta^{\mathfrak{v}+\mathfrak{v}'}\cong\Delta^{\mathfrak{v}}\times\Delta^{\mathfrak{v}'}$ is orientation-preserving. We deduce that $\mathrm{UMor}_*(k)$ is generated as a ring by the elements $e(i,n)$ for $i=1,\dots,k$ and $n\geq 1$.

A particular case of the previous discussion is when $\mathfrak{v}=m\mathfrak{e}_i$ and $\mathfrak{v}'=n\mathfrak{e}_j$ for some $1\leq i,j\leq k$ with $i\neq j$: then the sign $\epsilon\in\{\pm 1\}$ such that $e(i,m)\cdot e(j,n)=\epsilon\, e(j,n)\cdot e(i,m)$ is equal to the degree of the map $\Delta^m\times\Delta^n\to\Delta^n\times\Delta^m$ swapping the two coordinates. Thus $\epsilon=(-1)^{mn}$, that is $\epsilon=-1$ if both m,n are odd, and $\epsilon=1$ otherwise.

We next study the value of the coefficient $\lambda(m\mathfrak{e}_i,n\mathfrak{e}_i)$ for a fixed $i\in\{1,\dots,k\}$, and show that $\lambda(m\mathfrak{e}_i,n\mathfrak{e}_i)$ is equal to $\mathbf{ss}(m,n)$. For $l\geq 0$, write Φ^l for the map

$$\Phi^{l\mathfrak{e}_i}/\partial\Delta^l\colon \Delta^l/\partial\Delta^l \overset{\cong}{\to} \widetilde{C}_l(S^1\times\{i\})^\infty\subset\widetilde{C}_l(V_k)^\infty,$$

induced by $\Phi^{l\mathfrak{e}_i}$ on the quotient: this map sends the interior of Δ^l homeomorphically onto $C_l((0,1)\times\{i\})\subset\widetilde{C}_l(V_k)^\infty$.

The composition $(\Phi^{n+m})^{-1}\circ\mu_{m,n}\circ(\Phi^m\times\Phi^n)$ is equal to the map

$$\begin{aligned}\underline{\mathrm{sort}}_{m,n}:(\Delta^m\times\Delta^n,\partial(\Delta^m\times\Delta^n))&\to(\Delta^{m+n},\partial\Delta^{m+n})\\ ((s_1,\dots,s_m),(t_1,\dots,t_n))&\mapsto(p_1,\dots,p_{m+n})\end{aligned}$$

where $(p_1,\dots,p_{m+n})$ is the weakly increasing sorting of the sequence of real numbers $s_1,\dots,s_m,t_1,\dots,t_n$, repeated with multiplicity. Thus $\lambda(m\mathfrak{e}_i,n\mathfrak{e}_i)$ is the degree of the map $\underline{\mathrm{sort}}_{m,n}$. We compute this degree as the sum of local degrees at the preimages of a generic point $p=(p_1,\dots,p_{m+n})\in\mathring{\Delta}^{n+m}$, i.e. with $0<p_1<\dots<p_{m+n}<1$. The preimages of p along $\underline{\mathrm{sort}}_{m,n}$ are in natural bijection wtih the permutations $\sigma\in\mathfrak{S}_{m+n}$ that satisfy

$$\sigma(1)<\dots<\sigma(m)\text{ and }\sigma(m+1)<\dots<\sigma(m+n).$$

Such permutations are also called *shuffles* of type (m,n). Near the preimage p_σ corresponding to a given σ, $\underline{\mathrm{sort}}_{m,n}$ is equal to the linear automorphism of $\mathbb{R}^{m+n}$ permuting coordinates as σ, giving a local degree $\mathrm{sgn}(\sigma)$ of $\underline{\mathrm{sort}}_{m,n}$ at p_σ. We conclude that

$$\lambda(m\mathfrak{e}_i,n\mathfrak{e}_i)=\sum_\sigma\mathrm{sgn}(\sigma)$$

where the sum is taken over all shuffles of type (m,n). The fact that the latter sum equals $\mathbf{ss}(m,n)$ is standard, and we sketch an argument for convenience of the reader.

For $m,n \geq 0$ let $\alpha_{m,n}, \beta_{m+n} \in \mathfrak{S}_{m+n}$ be the permutations defined as follows:

- $\alpha_{m,n}(i) = m+1-i$ for $1 \leq i \leq m$ and $\alpha_{m,n}(i) = 2m+n+1-i$ for $m+1 \leq i \leq m+n$;
- $\beta_{m+n}(i) = m+n+1-i$.

Observe that $\mathrm{sgn}(\alpha_{m,n}) \cdot \mathrm{sgn}(\beta_{m+n}) = -1$ whenever both m and n are odd, and it is 1 otherwise.

If both m,n are odd, then σ is a shuffle of type (m,n) if and only if $\beta_{m+n} \circ \sigma \circ \alpha_{m,n}$ is a shuffle of type (m,n), and the two permutations have opposite signs. It follows that $\lambda(m\mathfrak{e}_i, n\mathfrak{e}_i) = 0 = \mathbf{ss}(m,n)$ in this case.

If either m,n is even, then σ is a shuffle of type (m,n) if and only if $\sigma \circ \beta_{m+n} \circ \alpha_{m,n}$ is a shuffle of type (n,m), and the two permutations have the same signs. It follows that $\lambda(m\mathfrak{e}_i, n\mathfrak{e}_i) = \lambda(n\mathfrak{e}_i, m\mathfrak{e}_i)$.

We further observe that if σ is a shuffle of type (m,n), then either $\sigma(m+n) = m+n$ or $\sigma(m) = m+n$:

- in the first case, the permutation $\sigma|_{\{1,\dots,m+n-1\}} \in \mathfrak{S}_{m+n-1}$ has the same sign as σ, and moreover it is a shuffle of type $(m,n-1)$;
- in the second case, the permutation $\sigma \circ \beta_{m+n} \circ \alpha_{m,n}$ sends $m+n \mapsto m+n$, it has the same sign as σ (we assume m or n even) and it is a shuffle of type (n,m), so its restriction to $\{1,\dots,m+n\}$ is a shuffle of type $(n,m-1)$.

Thus if m or n is even we have $\lambda(m\mathfrak{e}_i, n\mathfrak{e}_i) = \lambda(m\mathfrak{e}_i + (n-1)\mathfrak{e}_i) + \lambda((m-1)\mathfrak{e}_i, n\mathfrak{e}_i)$. Similarly, for m or n odd, one has $\mathbf{ss}(m,n) = \mathbf{ss}(m,n-1) + \mathbf{ss}(m-1,n)$. Finally, one checks that $\mathbf{ss}(0,0) = \lambda(0,0) = 1$: for the last equality, observe that $\mu_{0,0}$ sends the pair of zero-cells $(\emptyset,\emptyset)$ to the zero-cell $\emptyset \in \widetilde{C}_\bullet(V_k)^\infty$.

By a double induction argument in $m,n \geq 0$ we obtain for all $m,n \geq 0$ the desired equality $\lambda(m\mathfrak{e}_i, n\mathfrak{e}_i) = \mathbf{ss}(m,n)$.

We conclude that the $\mathrm{UMor}_*(k)$ is generated as a ring by the elements $e(i,n)$ for $i = 1,\dots,k$ and $n \geq 1$ (the elements $e(i,0)$ are equal to each other and give the unit of the ring); since $\mathbf{ss}(2m,1) = 1$ for all $m \geq 0$, we can in fact factor $e(i,2m+1) = e(i,2m)e(2i,1)$, so that the elements $e(i,1)$ and $e(i,2m)$ also generated $\mathrm{UMor}_*(k)$.

The above discussion shows that that the elements $e(i,1)$ anticommute, the elements $e(i,2m)$ are central, and $e(i,2m)\cdot e(i,2n) = \binom{m+n}{m} e(i,2(m+n))$. Therefore $\mathrm{UMor}_*(k)$ receives a natural homomorphism of rings with source the tensor product of $\mathbb{Z}$-algebras $\Lambda_{\mathbb{Z}}[x_1,\dots,x_k]\otimes\Gamma_{\mathbb{Z}}[y_1,\dots,y_k]$, sending $x_i\mapsto e(i,1)$ and $y_i\mapsto e(i,2)$, and the above discussion shows that this is in fact a bijection of rings. □

Remark 114. Every even degree element of $\mathrm{UMor}_*(k)$ has well defined *divided powers* $y^{[m]} := \frac{y^m}{m!}\in\mathrm{UMor}_*(k)$, for $m\geq 1$.

3.2.2 Contents

Let $\mathrm{G}_k := \pi_1(V_k) = \langle\gamma_1,\dots,\gamma_k\rangle\cong\mathbb{Z}^{*k}$ as in Definition 110. Write $H_k := (G_k)^{\mathrm{ab}}\cong\mathbb{Z}^k$ and $[-]:\mathrm{G}_k\to H_k$ for the abelianisation map.

Definition 115. The *content* is the ring homomorphism $c:\mathbb{Z}[\mathrm{G}_k]\longrightarrow\Lambda H_k$ defined on the generators as $\gamma_i\mapsto 1+[\gamma_i]$. Write c_i for the component of c in the i^{th} exterior power $\Lambda^i H_k$; note that $c_i\colon\mathbb{Z}[\mathrm{G}_k]\to\Lambda^i H_k$ is a homomorphism of abelian groups.

Remark 116. This is the same construction as in Section 2.3.1.

Lemma 117. *Suppose that $C:\mathrm{G}_k\to\Lambda^2 H_k$ is a function of sets satisfying:*

1. *C vanishes on the generators $\gamma_1,\dots,\gamma_k$, and*
2. *$C(w_1w_2) = C(w_1)+C(w_2)+[w_1]\wedge[w_2]$ for all $w_1,w_2\in G_k$.*

Then C is the restriction of c_2 to $G_k\subset\mathbb{Z}[G_k]$.

Proof. The extension of C to the function $\bar{C} := 1+[-]+C:\mathrm{G}_k\to\Lambda^*H_k/\Lambda^{\geq 3}H_k$ is multiplicative by property (2). As a result, $\bar{C}$ is defined by its values on the generators of G_k, on which, by property (1), it takes the value $\bar{c}(\gamma_i) = 1+[\gamma_i]$. Therefore $\bar{C}$ agrees with c modulo $\Lambda^{\geq 3}H_k$ and therefore $C = c_2$. □

3.2.3 Induced maps on UMor_*

A based map $f:(V_k,*)\to(V_l,*)$ induces a map $\mathrm{UMor}_*(f):\mathrm{UMor}_*(k)\to\mathrm{UMor}_*(l)$ which depends on f only up to based homotopy. So, by abuse of notation, for a homomorphism $\phi:G_k\to G_l$, we write $\mathrm{UMor}_*(\phi) := \mathrm{UMor}_*(f)$, where $f\colon V_k\to V_l$ is any based map inducing ϕ on π_1. Furthermore, $\mathrm{UMor}_*(-)$ is *functorial*, that is for composable homomorphisms $\phi : G_k \to G_l$ and $\psi : G_l \to G_m$, $\mathrm{UMor}_*(\psi \circ \phi) = \mathrm{UMor}_*(\psi) \circ \mathrm{UMor}_*(\phi)$. Note that $\mathrm{UMor}_*(\phi)$ is a *ring homomorphism*, since the superposition product μ, making $\widetilde{C}_\bullet(V_k)^\infty$ into an abelian topological monoid, is natural with respect to pointed maps $f \colon V_k \to V_l$.

The next theorem computes the natural $\mathrm{Aut}(G_k)$-action on $\mathrm{UMor}_*(k)$.

Notation 118. We denote by $(-)_x \colon H_k \to \bigoplus_{i=1}^k \mathbb{Z}x_i$ the isomorphism of abelian groups sending $[\gamma_i] \mapsto x_i$. Similarly $(-)_y \colon H_k \to \bigoplus_{i=1}^k \mathbb{Z}y_i$ is the isomorphism sending $[\gamma_i] \mapsto y_i$.

Theorem 119. *For a group homomorphism $\phi \colon G_k \to G_l$, the induced ring homomorphism $\mathrm{UMor}_*(\phi) : \mathrm{UMor}_*(k) \to \mathrm{UMor}_*(l)$ is given on the ring generators from Proposition 113 by*

$$\begin{aligned} x_i &\mapsto [\phi(\gamma_i)]_x \\ y_i &\mapsto [\phi(\gamma_i)]_y + [c_2(\phi(\gamma_i))]_x \,. \end{aligned} \tag{3.2.3}$$

Proof. We observe that there is a natural inclusion $V_k \cong \widetilde{C}_1(V_k)^\infty \hookrightarrow \widetilde{C}_\bullet(V_k)^\infty$, inducing on first homology the map $(-)_x \colon H_k \to \mathrm{UMor}_*(k)$ sending $[\gamma_i] \mapsto x_i$. This justifies the first part of the assertion (3.2.3). We prove the second part in four steps, dealing with different types of maps ϕ.

Step 1: Simple maps Given a function $g : \{1,\dots,k\} \to \{0\} \sqcup \{1,\dots,l\}$, consider the group homomorphism $\phi_g : G_k \to G_l$ sending $\gamma_i \mapsto \gamma_{g(i)}$ if $g(i) \geq 1$ and sending $\gamma_i \mapsto 1$ if $g(i) = 0$. We can realise ϕ_g by the based map of bouquets $f_g : V_k \to V_l$ sending $* \mapsto *$ and sending

$$f_g : (t,i) \in (0,1) \times \{i\} \subset V_k \mapsto \begin{cases} (t, g(i)) & \text{if } g(i) \geq 1, \\ * & \text{otherwise.} \end{cases}$$

We observe that $\widetilde{C}_n(f_g)^\infty$ maps the cell $e(i,n)$ of $\widetilde{C}_n(V_k)^\infty$ to the cell $e(g(i),n)$ of $\widetilde{C}_n(V_l)^\infty$ via an oriented homeomorphism, and collapses it to the 0-cell ∞ otherwise. In particular, $\mathrm{UMor}_*(f_g)$ sends $y_i \mapsto y_{g(i)}$ if $g(i) \geq 1$, and it sends $y_i \mapsto 0$ otherwise. This agrees with the assertion (3.2.3), since $c_2(\gamma_i) = 0$.

Step 2: Pinch map The pinch map $P : V_1 \to V_2$, given by

$$P\colon (t,1) \mapsto \begin{cases} (2t,1) & \text{if } 0 \le t \le \frac{1}{2} \\ (2t-1,2) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

realises the group homomorphism $p : G_1 \to G_2$ sending $\gamma_1 \mapsto \gamma_1\gamma_2$. According to the expression (3.2.3), we must show that $\mathrm{UMor}_*(P)$ sends $y_1 \mapsto y_1 + y_2 + x_1 \cdot x_2$. We compute $\mathrm{UMor}_*(P)(y_1)$. The composition of the characteristic map $\Phi^{(2)}$ relative to the cell $e(1,2) \subset \widetilde{C}_2(V_1)^\infty$ with the map $\widetilde{C}_2(P)^\infty$ is given explicitly as the map $\widetilde{C}_2(P)^\infty \circ \Phi^{(2)} : \Delta^2 \to \widetilde{C}_2(V_2)^\infty$ sending

$$(t_1,t_2) \mapsto \begin{cases} [(2t_1,1),(2t_2,1)] & \text{if } t_1,t_2 \le \frac{1}{2}, \\ [(2t_1,1),(2t_2-1,2)] & \text{if } t_1 \le \frac{1}{2} \le t_2, \\ [(2t_1-1,2),(2t_2-1,2)] & \text{if } \frac{1}{2} \le t_1,t_2. \end{cases}$$

The three cases partition the domain Δ^2 into three regions $A^{(2,0)}$, $A^{(1,1)}$, $A^{(0,2)}$ which are parametrised by Δ^2, $\Delta^1 \times \Delta^1$, Δ^2, respectively, via the oriented affine transformations $a_{(2,0)} : (t_1,t_2) \mapsto (2t_1, 2t_2)$, $a_{(1,1)} : (t_1,t_2) \mapsto (2t_1, 2t_2-1)$, $a_{(0,2)} : (t_1,t_2) \mapsto (2t_1, 2t_2-1)$, as in Figure 3.1. For each of $\mathfrak{v} = (2,0),(1,1),(0,2)$, we have an equality of maps $\Phi^{\mathfrak{v}} \circ a_{\mathfrak{v}}^{-1} = \widetilde{C}_2(P)^\infty \circ \Phi^{(2)} : A^{\mathfrak{v}} \to \widetilde{C}_2(V_2)^\infty$. We conclude that $\mathrm{UMor}_*(P) : y_1 = e_{(2)} \mapsto e_{(2,0)} + e_{(0,2)} + e_{(1,1)} = y_1 + y_2 + x_1 \cdot x_2$.

Step 3: Maps from G_1 Let $\psi_w : G_1 \to G_l$ be the group homomorphism sending $\gamma_1 \mapsto w \in G_l$. For words $w,u \in G_l$, the homomorphism ψ_{wu} factors through

$$G_1 \xrightarrow{\quad P \quad} G_2 \cong G_1 * G_1 \xrightarrow{\psi_w * \psi_u} G_l * G_l \xrightarrow{\quad \phi_g \quad} G_l \tag{3.2.4}$$

where $g : \{1,\ldots,l,1',\ldots,l'\} \to \{0,1,\ldots,l\}$ sends $i,i' \mapsto i$.

We can now apply UMor_* to the composition (3.2.4), and evaluate at y_1 leveraging Steps 1 and 2, to obtain

$$\mathrm{UMor}_*(\psi_{wu})(y_1) = \mathrm{UMor}_*(\psi_w)(y_1) + \mathrm{UMor}_*(\psi_u)(y_1) + [w]_x \cdot [u]_x. \tag{3.2.5}$$

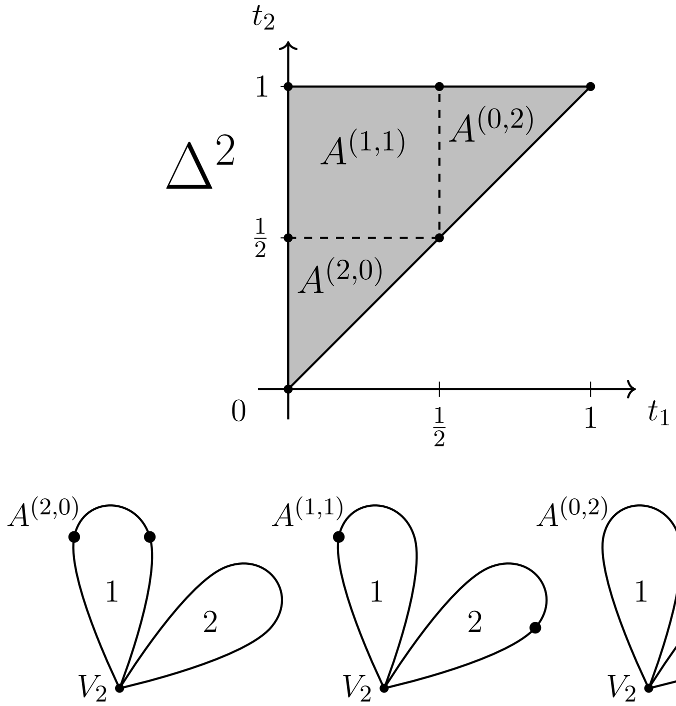


Figure 3.1 The 2-simplex Δ^2 (above) maps under $\widetilde{C}_2(P)^\infty \circ \Phi^{(2)}$ to configurations of 2 points in V_2. Region $A^{(2,0)}$ corresponds (below) to configurations of two points on the first loop; region $A^{(1,1)}$ to configurations of one point on each loop; region $A^{(0,2)}$ to configurations of two points on the second loop.

By decomposing $\mathrm{UMor}_*(\psi_w)(y_2) \in \mathrm{UMor}_2(l)$ as the sum of components $Y(w) \in \bigoplus_{i=1}^l \mathbb{Z}y_i$ and $C(w) \in \Lambda^2\left(\bigoplus_{i=1}^l \mathbb{Z}x_i\right)$, equality (3.2.5) gives

$$\begin{aligned} Y(wu) &= Y(w) + Y(u); \\ C(wu) &= C(w) + C(u) + [w]_x \cdot [u]_x. \end{aligned} \tag{3.2.6}$$

Thus $Y: G_l \to \bigoplus_{i=1}^l \mathbb{Z}y_i$ is a group homomorphism, and it coincides with $[-]_y$ on the generators $\gamma_1, \ldots, \gamma_l$ by Step 1, and so $Y(-) = [-]_y$. Finally, C vanishes on words of length 1 by Step 1 and satisfies (3.2.6), so $C = c_2$ by Lemma 117.

Step 4: Any map $\phi: G_k \to G_l$ If $\phi\colon G_k \to G_l$ is any group homomorphism, by Step 1 we have that the generator $y_i \in \mathrm{UMor}_2(k)$ is equal to the image of $y_1 \in \mathrm{UMor}_2(1)$ along the map $\phi_{g_i}\colon G_1 \to G_k$, where $g_i : 1 \in \{1\} \mapsto i \in \{0, ..., k\}$; we can therefore compute $\mathrm{UMor}_*(\phi)(y_i) = \mathrm{UMor}_*(\phi \circ \phi_{g_i})(y_1)$, and the latter expression is equal to what was predicted by assertion (3.2.3) by Step 3. $\square$

3.3 Cellular chains for configurations of surfaces

Let $g \geq 0$ be fixed. Henceforth, we unambiguously denote $\pi := G_{2g}$, $H := H_{2g}$, and $\mathrm{UMor}_* := \mathrm{UMor}_*(2g)$.

3.3.1 A model for the surface $\Sigma_{g,1}$

In the rectangle $\mathbf{R} = [0,2] \times [0,1]$, we decompose the side $\{2\} \times [0,1]$ into $4g$ consecutive closed intervals of equal length called $J_1, ..., J_{4g}$ ordered and oriented with increasing second co-ordinate. Denote $P_0 \subset \{2\} \times [0,1]$ the set of their endpoints.

Definition 120. The space $\mathcal{M}$ is the quotient of $\mathbf{R}$ by identifying J_{4i+1} (resp. J_{4i+2}) with J_{4i+3} (resp. J_{4i+4}), via their unique orientation-*reversing* isometry, for $i = 0, ..., g-1$. (See Figure 3.2).

The quotient $\mathcal{M}$ is homeomorphic to $\Sigma_{g,1}$ and has a cell decomposition consisting of:

- one 0-cell p_0, the image of P_0;
- $2g+1$ 1-cells attached on p_0:
  - the images I_{2k+i} of J_{4k+i}, for $k = 0, ..., g-1$ and $i = 1, 2$;
  - the image I of $\partial\mathbf{R} - \{2\} \times (0,1)$, forming the boundary $\partial\mathcal{M}$;
- one 2-cell, the interior of $\mathbf{R}$.

Definition 121. The embedding $\iota : (V_{2g}, *) \hookrightarrow (\mathcal{M}, p_0)$ identifies the i-th circle of V_{2g} with the oriented interval I_i, for $i = 1, ..., 2g$.

3.3.2 Polysimplicial decomposition for $C_n(\mathcal{M})^\infty$

Recall from Section 3.1.2 the shorthand $C_n(\mathcal{M})^\infty := C_n(\mathcal{M}, \partial\mathcal{M})^\infty$.

Definition 122. A *polysimplicial datum*, denoted generically $\mathfrak{t} = (b, \underline{\mathrm{P}}, \mathfrak{v})$, is a choice of

- an integer $b \geq 0$;
- a sequence $\underline{\mathrm{P}} = (P_1, ..., P_b)$ with positive integer values $P_i \in \mathbb{Z}_{\geq 1}$;

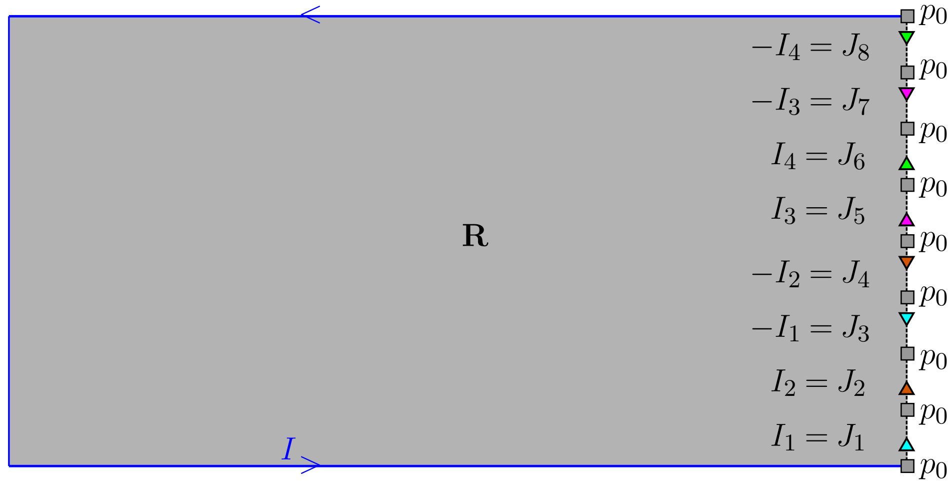


Figure 3.2 A cell decomposition of $\mathcal{M}$.

- a sequence $\mathfrak{v} = (v_1, ..., v_{2g})$ with non-negative integer values $v_i \in \mathbb{Z}_{\geq 0}$,

We call $n(\mathfrak{t}) := P_1 + ... + P_b + v_1 + ... + v_{2g}$ the *size*, $b(\mathfrak{t}) := b$ the *bar-length*, and $d(\mathfrak{t}) = n(\mathfrak{t}) + b(\mathfrak{t})$ the *dimension* of $\mathfrak{t}$. Denote by $\mathcal{T}$ the set of all polysimplicial data.

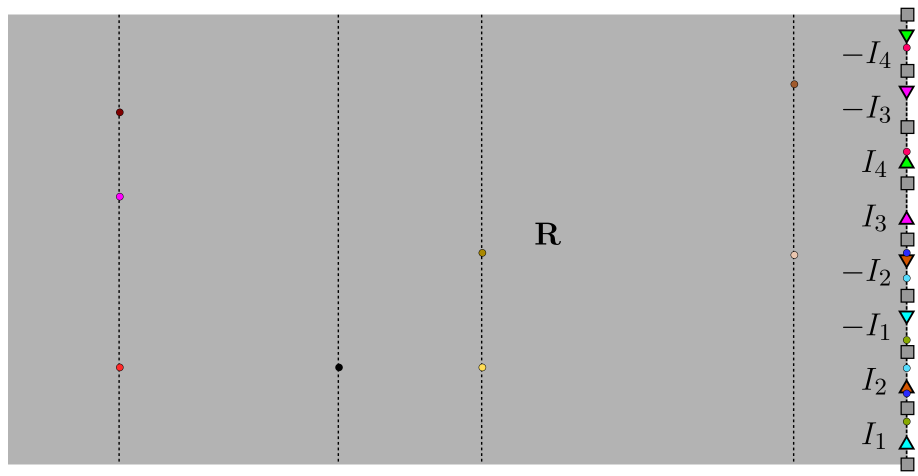


Figure 3.3 A configuration lying in the interior of the cell $e_{\mathfrak{t}} \subset C_{12}(\mathcal{M})$, where $\mathfrak{t} = (4, (3,1,2,2), (1,2,0,1))$. On the right side, every point is drawn on both copies of the interval I_i it lies.

For each $\mathfrak{t} \in \mathcal{T}$, we define the polysimplex

$$\Delta^{\mathfrak{t}} := \Delta^b \times \prod_{i=1,...,b} \Delta^{P_b} \times \prod_{i=1,...,2g} \Delta^{v_i}$$

and the polysimplicial embedding $\Phi^{\mathfrak{t}} : \Delta^{\mathfrak{t}} \to C_{n(\mathfrak{t})}(\mathcal{M})^{\infty}$ so that

$$\Phi^{\mathfrak{t}}\Big((x_1,...,x_b),\big((s_1^{(1)},...,s_{P_1}^{(1)}),...,(s_1^{(b)},...,s_{P_b}^{(b)})\big),\big((t_1^{(1)},...,t_{v_1}^{(1)}),...,(t_1^{(2g)},...,t_{v_{2g}}^{(2g)})\big)\Big)$$

is the configuration

$$\Big[(2x_i, s_j^{(i)}) \text{ for } i=1,...,b, j=1,...,P_i;\ \iota((t_j^{(i)},i)) \text{ for } i=1,...,2g, j=1,...,v_i\Big]. \quad (3.3.1)$$

In other words, $\Phi^{\mathfrak{t}}$ parametrises the family of configurations $C_{n(\mathfrak{t})}(\mathcal{M})^{\infty}$ with

1. exactly v_i points in the interior of the interval I_i;
2. points in the interior of $\mathbf{R}$ are grouped in b vertical bars with the i^{th} bar from the left having precisely b_i points.

Proposition 123. *The based space $(C_{\bullet}(\mathcal{M})^{\infty},\infty)$ admits a polysimplicial decomposition with set of polysimplices $\mathcal{T}$ and the maps $\Phi^{\mathfrak{t}}$ as polysimplicial embeddings.*

Proof. Each map $\Phi^{\mathfrak{t}}$ restricts to a homeomorphism $\mathring{\Delta}^{\mathfrak{t}} \to e_{\mathfrak{t}}$, and maps the boundary $\partial\Delta^{\mathfrak{t}}$ to ∞ or into the images of $\Phi^{\mathfrak{t}'}$ for $d(\mathfrak{t}') < d(\mathfrak{t})$. Furthermore, each $s \in C_n(\mathcal{M})^{\infty}$ lies in a unique $e_{\mathfrak{t}}$, and $e_{\mathfrak{t}}$ is homeomorphic to an open $d(\mathfrak{t})$-disc. $\square$

3.3.3 The cellular chain ring

The reduced cellular chain complex $\widetilde{\mathrm{Ch}}_*(C_{\bullet}(\mathcal{M})^{\infty}) = \mathbb{Z}\langle e_{\mathfrak{t}}\rangle$ is bigraded: the first degree $*$ is homological degree, and the second degree $\bullet$ is the size of the configuration, so that $|e_{\mathfrak{t}}| = (d(\mathfrak{t}), n(\mathfrak{t}))$ and the differential d has bidegree $(-1,0)$.

Proposition 124. *The superposition product μ is cellular making $\widetilde{\mathrm{Ch}}_*(C_{\bullet}(\mathcal{M})^{\infty})$ into a differential bigraded ring. Under this multiplication, $e_{\mathfrak{t}} \cdot e_{\mathfrak{t}'} = (-1)^{d(\mathfrak{t})d(\mathfrak{t}')} e_{\mathfrak{t}'} \cdot e_{\mathfrak{t}}$, and $e_{(b,\underline{P},\mathfrak{v})} = e_{(b,\underline{P},\underline{0})} \cdot e_{(0,0,\mathfrak{v})}$.*

Proof. For a configuration $s \in C_{\bullet}(\mathcal{M})^{\infty}$, let $n(s)$, $d(s)$ and $b(s)$ be $n(\mathfrak{t})$, $d(\mathfrak{t})$ and $b(\mathfrak{t})$, respectively, where $e_{\mathfrak{t}}$ is the unique cell containing s in its interior. To show that μ is cellular with the product cellular structure on $C_{\bullet}(\mathcal{M})^{\infty} \times C_{\bullet}(\mathcal{M})^{\infty}$ it suffices that $d(\mu(s,s')) \le d(s) + d(s')$ for all $s,s' \in C_{\bullet}(\mathcal{M})^{\infty}$. If $\mu(s,s') = \infty$, this is obvious as $d(\mu(s,s')) = 0$. Otherwise, $b(\mu(s,s')) \le b(s) + b(s')$ and $n(\mu(s,s')) = n(s) + n(s')$, and by using $d(s) = b(s) + n(s)$ we are also done.

Furthermore, the product μ is strictly commutative and so $e_{\mathfrak{t}} \cdot e_{\mathfrak{t}'} = (-1)^{d(\mathfrak{t})d(\mathfrak{t}')} e_{\mathfrak{t}'} \cdot e_{\mathfrak{t}}$ follows immediately. From the decomposition $\Delta^{(b,\underline{P},\mathfrak{v})} = \Delta^{(b,\underline{P},\underline{0})} \times \Delta^{(0,\underline{0},\mathfrak{v})}$, we have the equality of maps $\Phi^{(b,\underline{P},\mathfrak{v})}(-,-) = \mu(\Phi^{(b,\underline{P},\underline{0})}(-), \Phi^{(0,\underline{0},\mathfrak{v})}(-))$, proving $e_{(b,\underline{P},\mathfrak{v})} = e_{(b,\underline{P},\underline{0})} \cdot e_{(0,0,\mathfrak{v})}$. □

Proposition 125. *The map $C_\bullet(\iota)^\infty : \widetilde{C}_\bullet(V_{2g})^\infty \to C_\bullet(\mathcal{M})^\infty$, induced from the embedding ι from Definition 121, is cellular and commutes with μ. The map $\widetilde{\mathrm{Ch}}_*(C_\bullet(\iota)^\infty) :$ $\mathrm{UMor}_*(2g) \to \mathbb{Z}\langle e_{\mathfrak{t}}\rangle$ is the inclusion of differential bigraded algebras given by $e_{\mathfrak{v}} \to e_{(0,\underline{0},\mathfrak{v})}$.*

Proof. Clear by construction. □

Let $\widetilde{\mathrm{Ch}}^B_*(\mathcal{M})$ be the subgroup of $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty)$ linearly generated by the cells of the form $e_{(b,\underline{P},\underline{0})}$. Combining the last two propositions, we obtain the tensor product decomposition

$$\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty) = \widetilde{\mathrm{Ch}}^B_*(\mathcal{M}) \otimes \mathrm{UMor}_* \tag{3.3.2}$$

where both factors are subrings with the product μ and, furthermore, UMor_* is a *differential* subring with the trivial differential.

Remark 126. The quotient complex $\widetilde{\mathrm{Ch}}^B_*(\mathcal{M})$ obtained from $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty)$ after quotienting by the ideal generated by $\mathrm{UMor}_{\geq 1}$ is independent of g. Setting $g = 0$ and combining with Proposition 103, we see that it computes the homology of configurations on the disk. So, we may view $\widetilde{\mathrm{Ch}}^B_*(\mathcal{M})$ as a bar construction for the homology of $C_\bullet(D^2)$ (*bar* literally corresponding to the bars in our polysimplices).

3.3.4 The differential

Since $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty)$ is a differential bigraded ring, its differential d is uniquely determined via the graded Leibniz rule by its behaviour on a set of ring generators, such as the elements $e_{(b,\underline{P},\underline{0})}$ and $e_{(0,\underline{0},\mathfrak{v})}$ for varying $(b,\underline{P})$ and $\mathfrak{v}$; this uses Proposition 124. On generators of the second kind the differential d vanishes, so we will focus on computing $d(e_{(b,\underline{P},\underline{0})})$.

Definition 127. Let ω denote the element

$$\omega := x_1 \wedge x_2 + x_3 \wedge x_4 + \cdots + x_{2g-1} \wedge x_{2g} \in \Lambda^2[x_1, \ldots, x_{2g}] \subset \mathrm{UMor}_2,$$

where we use the notation from Proposition 113. For $k \geq 1$ define $\Omega_{2k-1} := 0$ and define $\Omega_{2k} \in \Lambda^{2k}[x_1, \dots, x_{2g}] \subset \mathrm{UMor}_{2k}$ as the element

$$\Omega_{2k} := \frac{(2\omega)^k}{k!} = 2^k \sum_{1 \leq i_1 < \cdots < i_k \leq g} x_{2i_1 - 1} \wedge x_{2i_1} \wedge \cdots \wedge x_{2i_k - 1} \wedge x_{2i_k}.$$

Definition 128. We denote by $\zeta_g \in \pi_1(V_{2g}) = \mathbb{Z}^{*2g}$ the element

$$\zeta_g = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} \dots \gamma_{2g-1} \gamma_{2g} \gamma_{2g-1}^{-1} \gamma_{2g}^{-1} \in \pi_1(V_{2g}) = \mathbb{Z}^{*2g}.$$

Lemma 129. *Recall Definition 115. Then* $c_2(\zeta_g) = 2\omega \in \Lambda^2 H_{2g}$.

Proof. This is Example 27. □

Proposition 130. *The differential d restricted to $\widetilde{\mathrm{Ch}}_*^B(\mathcal{M})$ is the sum of two components $d_B + d_M$ given on a generator $e_{(b,\underline{P},\underline{0})}$ by*

$$d_B(e_{(b,\underline{P},\underline{0})}) = \sum_{i=1}^{b-1} (-1)^i \mathbf{ss}(P_i, P_{i+1})\, e_{(b-1,(P_1,\dots,P_{i-1},P_i+P_{i+1},P_{i+1},\dots P_b),\underline{0})};$$
$$d_M(e_{(b,\underline{P},\underline{0})}) = (-1)^b e_{(b-1,(P_1,\dots,P_{b-1})),\underline{0},0)} \cdot \Omega_{P_b}.$$

Proof. We apply the formula for the differential from Proposition 109. The multi-faces coming from a face of Δ^{P_i} in the polysimplex $\Delta^{(b,\underline{P},\underline{0})} = \Delta^b \times \Delta^{P_1} \times \dots \times \Delta^{P_b}$ do not contribute to $d(e_{\mathfrak{t}})$: they correspond to either two configuration points of a bar colliding, or a point of the bar going to $\partial\mathcal{M}$; both give rise to the ∞-configuration.

We now check the contributions from the faces of Δ^b. Firstly, the 0^{th}-face also does not contribute: it corresponds to the left-most bar being pushed to $\partial\mathcal{M}$ giving rise again to the ∞-configuration.

Secondly, for $k = 1, \dots, b$, the map $\Phi^{\mathfrak{t}} \circ (f_k \times \mathrm{id}_{\Delta^{P_1} \times \dots \times \Delta^{P_b}})$ from Proposition 109 factors as the composition

$$\begin{array}{c} \Delta^{b-1} \times \Delta^{P_1} \times \dots \times \Delta^{P_k} \times \Delta^{P_{k+1}} \times \dots \times \Delta^{P_b} \\ \Big\downarrow \mathrm{id} \times \underline{\mathrm{sort}}_{P_k, P_{k+1}} \times \mathrm{id} \\ \Delta^{b-1} \times \Delta^{P_1} \times \dots \times \Delta^{P_k + P_{k+1}} \times \dots \times \Delta^{P_b} \xrightarrow{\Phi^{(b-1,\underline{P}^k,0)}} C_n(\mathcal{M})^\infty \end{array}$$

where $\underline{\mathrm{P}}^k = (P_1, ..., P_{k-1}, P_k + P_{k+1}, P_{k+2}, ..., P_b)$ and $\underline{\mathrm{sort}}_{P_k,P_{k+1}}$ is the map from the proof of Theorem 119, which has degree $\mathbf{ss}(P_k, P_{k+1})$. Thus

$$[\Phi^{\mathfrak{t}} \circ (f_k \times \mathrm{id}_{\Delta^{P_1} \times ... \times \Delta^{P_b}})] = \mathbf{ss}(P_k, P_{k+1}) e_{(b-1,\underline{\mathrm{P}}^k,0)} \in \widetilde{\mathrm{Ch}}_{d(\mathfrak{t})-1}(C_\bullet(\mathcal{M})^\infty).$$

Finally, the case $k = b$ corresponds to the right-most bar hitting $\iota(V_{2g})$. The bar, positively oriented, defines the loop

$$\zeta = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma^{-2} ... \gamma_{2g-1} \gamma_{2g} \gamma_{2g-1}^{-1} \gamma 2g^{-1} \in \pi_1(V_{2g}) = \mathbb{Z}^{*2g}.$$

Let $f_\zeta : V_1 \to V_{2g}$ be the based map realising $1 \mapsto \zeta$ with constant speed. Then, $\Phi^{\mathfrak{t}} \circ (f_k \times \mathrm{id}_{\Delta^{P_1} \times ... \times \Delta^{P_b}})$ is the composition $\mu(\Phi^{(b-1,(P_1,...,P_{b-1}),\underline{0})}, \widetilde{C}_{P_b}(\iota)^\infty \circ C_{P_b}(f_\zeta)^\infty)$, and, as a result,

$$[\Phi^{\mathfrak{t}} \circ (f_k \times \mathrm{id}_{\Delta^{P_1} \times ... \times \Delta^{P_b}})] = e_{(b-1,(P_1,...,P_{b-1}),\underline{0})} \cdot \mathrm{UMor}(f_\zeta)(e(1, P_b)).$$

By Theorem 119, if $P_b = 2k+1$, then

$$\mathrm{UMor}(f_\zeta)(e(1, P_b)) = \mathrm{UMor}(f_\zeta)(x \frac{y^k}{k!}) = [\zeta]_x \frac{([c_2(\zeta)]_x)^k}{k!} = 0,$$

and if $P_b = 2k$, then $\mathrm{UMor}(f_\zeta)(e(1, P_b)) = \mathrm{UMor}(f_\zeta)(\frac{y^k}{k!}) = \frac{(2\omega)^k}{k!}$. In both cases this equals Ω_{P_b}. □

3.3.5 Cellular approximation of homeomorphisms

The polysimplicial decomposition from Section 3.3.2 is not at all $\Gamma_{g,1}$-equivariant. Nevertheless, it can still be used to study the $\Gamma_{g,1}$-action on homology, by the following two propositions. The trick is due to Bianchi–Miller–Wilson [3].

We fix the disc $D = [0,1] \times [0,1] \subset \mathcal{M}$ and identify the mapping class group $\Gamma_{g,1}$ with $\pi_0(\mathrm{Homeo}(\mathcal{M}, D \cup \partial\mathcal{M}))$. We define the self-map of the pair $(\mathcal{M}, \partial M)$ given by

$$\tau : (x,y) \in \mathcal{M} \mapsto \begin{cases} (2x, y) \text{ if } x \leq \frac{1}{2}, \\ (2, y) \text{ if } x \geq \frac{1}{2}. \end{cases}$$

Proposition 131. *For any $f \in \mathrm{Homeo}(\mathcal{M}, D \cup \partial\mathcal{M})$, $C_n(\tau \circ f)^\infty$ is cellular.*

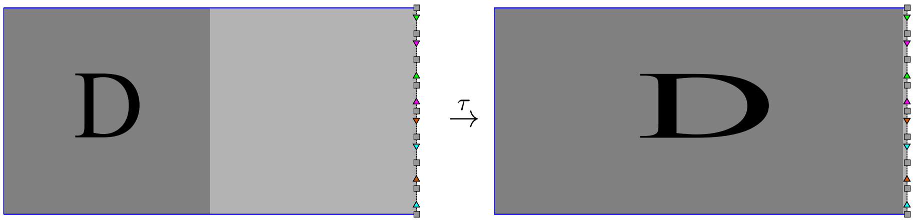


Figure 3.4 The map τ expands D and contracts horizontally the rest of $\mathcal{M}$.

Proof. In the notation of the proof of 124, it suffices to show $d(C_n(\tau \circ f)^\infty(s)) \leq d(s)$, for all $s \in C_n(\mathcal{M})^\infty$. If $C_n(\tau \circ f)^\infty(s) = \infty$, this holds immediately. Otherwise, $n(C_n(\tau \circ f)^\infty(s)) = n(s)$, and as $\tau \circ f$ sends bars to bars or into $\iota(V_{2g})$, then $b(C_n(\tau \circ f)^\infty(s)) \leq b(s)$. Using $d(-) = n(-) + b(-)$, the conclusion follows. $\square$

The map τ is homotopic to $\mathrm{id}_{\mathcal{M}}$ via self-maps of the pair $(\mathcal{M}, \partial\mathcal{M})$ (e.g. via the family $(x,y,t) \in \mathcal{M} \times [1,2] \mapsto (tx, y)$ if $x \leq \frac{1}{t}, (2,y)$ if $x \geq \frac{1}{t}$). Therefore for any self maps f, g of $(\mathcal{M}, \partial\mathcal{M})$, we have homotopies $\tau \circ f \simeq f$ and $\tau \circ f \circ \tau \circ g \simeq \tau \circ f \circ g$ via maps of pairs $(\mathcal{M}, \partial\mathcal{M})$. Therefore $C_n(\tau \circ f)^\infty$ is homotopic to $C_n(f)^\infty$, and we model the natural action of $\Gamma_{g,1}$ on $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty)$ by insisting that $f \in \mathrm{Homeo}(\mathcal{M}, D \cup \mathcal{M})$ acts on $C_n(\mathcal{M})^\infty$ via $C_n(\tau \circ f)^\infty$.

Proposition 132. *Under the decomposition* (3.3.2) *the subrings* $\widetilde{\mathrm{Ch}}_*^B(\mathcal{M})$ *and* UMor_* *of* $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty)$ *are* $\Gamma_{g,1}$*-subrepresentations, the* $\Gamma_{g,1}$*-action on* $\widetilde{\mathrm{Ch}}_*^B(\mathcal{M})$ *is trivial, and* $f \in \mathrm{Homeo}(\mathcal{M}, D \cup \mathcal{M})$ *acts on* UMor_* *via* $C_\bullet(\tau \circ f)_* = \mathrm{UMor}(\tau \circ f) = \mathrm{UMor}(\phi)$*, where* $\phi \in \mathrm{Aut}(\pi)$ *is the map on* $\pi_1(\Sigma_{g,1}) \cong \pi_1(V_{2g})$ *induced by* f*.*

Proof. The maps $C_\bullet(\tau \circ f)^\infty$ with μ commute so the multiplicative decomposition is $\Gamma_{g,1}$-equivariant, proving the first assertion. Now for any $f \in \mathrm{Homeo}^+(\mathcal{M}, D \cup \partial\mathcal{M})$, the maps $\tau \circ f$ and τ agree in the interior of $\mathcal{M}$ and $C_\bullet(\tau)^\infty$ is homotopic to $C_\bullet(\mathrm{id})^\infty$. As the cell $e_{(b,\underline{P},\underline{0})}$ is supported in $\mathring{\mathcal{M}}$, $(C_\bullet(\tau \circ f)^\infty)_*(e_{(b,\underline{P},\underline{0})}) = (C_\bullet(\tau)^\infty)_*(e_{(b,\underline{P},\underline{0})}) = e_{(b,\underline{P},\underline{0})}$. Thus $(C_n(\tau \circ f)^\infty)_*$ is the identity on $\widetilde{\mathrm{Ch}}_*^B(\mathcal{M})$. Finally, the action on UMor_* follows from Theorem 119. $\square$

3.4 The $\Gamma_{g,1}$-action on the homology of configurations

Henceforth, p shall denote a prime or the number 0, so that $\mathbb{Z}/p$ shall be the field of p elements or the ring of integers, respectively. We also fix $g \geq 0$, and recall that

$\pi = \mathbb{Z}^{*2g} \cong \pi_1(\Sigma_{g,1})$ is the free group on $2g$ generators, and $H = \pi^{\mathrm{ab}} \cong H_1(\Sigma_{g,1})$ its abelianisation.

3.4.1 Recollections on Torelli groups and Johnson homomorphisms

Definition 133. The *Torelli group modulo p*, denoted $\mathcal{T}_{g,1}(p)$, is the kernel of the $\Gamma_{g,1}$-action on $H_1(\Sigma_{g,1};\mathbb{Z}/p)$.

The group $\mathcal{T}_{g,1} := \mathcal{T}_{g,1}(0)$ is the standard Torelli group.

Theorem 134 (Cooper [11], Perron [39])**.** *For a prime p, the group $\mathcal{T}_{g,1}(p)$ is generated by $\mathcal{T}_{g,1}$ and p^{th} powers of Dehn twists.*

Remark 135. We may replace "all Dehn twists" in the Theorem by "all non-separating Dehn twists" as $\mathcal{T}_{g,1}$ already contains all separating Dehn twists.

In [23], Johnson proved that the quotient $[\pi,\pi]/[\pi,[\pi,\pi]]$ of lower central subgroups of G is a free abelian group, isomorphic to $\Lambda^2 H$ as $\Gamma_{g,1}$-representation. The explicit isomorphism $j\colon [\pi,\pi]/[\pi,[\pi,\pi]] \cong \Lambda^2 H$ is given by sending the class of $w_1 w_2 w_1^{-1} w_2^{-1}$ in $[\pi,\pi]/[\pi,[\pi,\pi]]$ to $[w_1] \wedge [w_2] \in \Lambda^2 H$, for $w_1, w_2 \in G$.

Definition 136 (Johnson [23])**.** The *Johnson homomorphism* is the group homomorphism $\tau_{g,1} : \mathcal{T}_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H)$ sending $\phi \in \mathcal{T}_{g,1}$ to the homomorphism of abelian groups $[w] \mapsto j([\phi(w)w^{-1})]_{[\pi,\pi]/[\pi,[\pi,\pi]]})$.

Johnson showed that $\tau_{g,1}$ is a well defined group homomorphism, and computed its image using the identification $\mathrm{Hom}(H,\Lambda^2 H) \cong H^\vee \otimes \Lambda^2 H \cong H \otimes \Lambda^2 H$, where the self duality $H \cong H^\vee$ comes from the symplectic form on $H = H_1(\Sigma_{g,1})$.

Theorem 137 (Johnson [23])**.** *The image of $\tau_{g,1}$ is the subgroup $\Lambda^3 H \subset H \otimes \Lambda^2 H$, where the inclusion is given by sending the triple wedge $a \wedge b \wedge c \in \Lambda^3 H$ to the sum $a \otimes b \wedge c + b \otimes c \wedge a + c \otimes a \wedge b \in H \otimes \Lambda^2 H$.*

We observe that $H \otimes \Lambda^2 H$ is a free abelian group of rank $2g\binom{2g}{2}$, and $\Lambda^3 H$ is a split free abelian subgroup of rank $\binom{2g}{3}$.

The kernel of $\tau_{g,1}$ is the so-called *Johnson kernel* $\mathcal{K}_{g,1} \subseteq \mathcal{T}_{g,1}$.

Theorem 138 (Johnson [23])**.** *If $g \geq 0$, the group $\mathcal{K}_{g,1}$ is generated by all Dehn twists around separating simple closed curves in $\Sigma_{g,1}$.*

3.4.2 The maps ξ^p

Notation 139. Recall Proposition 113 and Theorem 119; we identify the abelian group UMor_2 with

$$\mathrm{UMor}_2 = \bigoplus_{i=1}^{2g} \mathbb{Z}y_i \oplus \Lambda^2 \left[\bigoplus_{i=1}^{2g} \mathbb{Z}x_i\right] \cong H \oplus \Lambda^2 H.$$

We observe that UMor_2 has a naïve $\Gamma_{g,1}$-action as the direct sum of the representations H and $\Lambda^2 H$. In light of Proposition 132, it also inherits an action via $\mathrm{UMor}_2(\tau \circ -)$.

Notation 140. For $\phi \in \Gamma_{g,1}$ denote the naïve action by $\phi *_H -$ and the action via $\mathrm{UMor}_*(\tau \circ -)$ by $\phi * -$.

For both actions of $\Gamma_{g,1}$ on UMor_2 we have a short exact sequence of $\Gamma_{g,1}$-representations

$$0 \longrightarrow \Lambda^2 H \longrightarrow \mathrm{UMor}_2 \longrightarrow H \longrightarrow 0,$$

and for the naïve action there is a $\Gamma_{g,1}$-equivariant splitting $H \to \mathrm{UMor}_2$.

Definition 141. We define a function of sets $\xi^p : \Gamma_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H) \otimes \mathbb{Z}/p$ by sending $\phi \in \Gamma_{g,1}$ to the homomorphism of abelian groups sending the basis element $[\gamma_i] \in H$ to the element $c_2(\phi(\gamma_i)) \in \Lambda^2 H \otimes \mathbb{Z}/p$. We let ξ_I^p be the restriction of ξ^p on $\mathcal{T}_{g,1}(p)$. For shorthand, we write $\xi = \xi^0$ and $\xi_I = \xi_I^0$.

The function ξ enjoys a special relationship with the Johnson homorphism.

Proposition 142. $\xi_I = 2\tau_{g,1}$.

Proof. See proof of Proposition 73. □

The rest of this section has the dual purpose of relating ξ with the representation UMor_2, and using ξ^p to define modulo p Johnson homomorphisms and kernels.

Proposition 143. *Let $a \in H \subset \mathrm{UMor}_2$ and $b \in \Lambda^2 H \subset \mathrm{UMor}_2$, then for $\phi \in \Gamma_{g,1}$, we have $\phi * (a \oplus b) = \phi *_H a \oplus (\phi *_H b + \xi(\phi)(a))$.*

Proof. Applying Theorem 119 proves the equality on the basis of UMor_2 given by the elements y_i and $x_j \wedge x_k$, for $1 \leq i,j,k \leq 2g$. □

Proposition 144. *For $\phi, \psi \in \Gamma_{g,1}$ and $a \in H \subset \mathrm{UMor}_2$, we have*

$$\xi^p(\phi\psi)(a) = \phi *_H \xi^p(\psi)(a) + \xi^p(\phi)(\psi *_H a).$$

Proof. Using Proposition 143, we get a chain of equalities

$$\begin{aligned}\phi * (\psi * (a \oplus 0)) &= \phi * (\psi *_H a \oplus \xi(\psi)(a)) \\ &= \phi *_H (\psi *_H a) \oplus (\phi *_H \xi(\psi)(a) + \xi(\phi)(\psi *_H a)).\end{aligned}$$

As $- * -$ and $- *_H -$ are actions, we also have $\phi * (\psi * (a \oplus 0)) = (\phi\psi) * (a \oplus 0) = (\phi\psi) *_H a \oplus \xi(\phi\psi)(a) = \psi *_H (\phi *_H a) \oplus \xi(\phi\psi)(a)$. Reducing modulo p if necessary, the result follows. $\square$

Corollary 145. *The map ξ^p_I is a group homomorphism.*

Proof. By definition of $\mathcal{T}_{g,1}(p)$, for any $\phi \in \mathcal{T}_{g,1}(p)$, the map $\phi *_H -$ is the identity map modulo p. Then, for $\phi, \psi \in \mathcal{T}_{g,1}(p)$, Proposition 144 yields $\xi^p_I(\phi\psi)(a) \equiv \xi^p_I(\phi)(a) + \xi^p_I(\phi)(a) \pmod p$ for all $a \in H$. Thus ξ^p_I is a group homomorphism. $\square$

Remark 146. It can be shown that ξ^p_I is equal to $2\tau^Z_1(p)$, where $\tau^Z_1(p)\colon \mathcal{T}_{g,1}(p) \to \mathrm{Hom}(H, \Lambda^2 H) \otimes \mathbb{Z}/p$ is the Johnson–Zassenhaus homomorphism from [11, 51]. However the definition of $\tau^Z_1(p)$ is a bit complicated, hence we prefer to avoid comparing ξ^p_I to $\tau^Z_1(p)$ and directly prove the needed properties of ξ^p_I.

The normal subgroups $\mathcal{T}_{g,1}(p) \subset \Gamma_{g,1}$ admit actions of $\Gamma_{g,1}$ by conjugation, and $\mathrm{Hom}(H, \Lambda^2 H)$ has a $\Gamma_{g,1}$-action defined by $f \mapsto (\phi *_H -) \circ f \circ (\phi^{-1} *_H -)$ for $\phi \in \Gamma_{g,1}$ and $f \in \mathrm{Hom}(H, \Lambda^2 H)$.

Lemma 147. *Under the above actions, ξ^p_I is $\Gamma_{g,1}$-equivariant.*

Proof. Applying Proposition 144 twice, we get, for $a \in H \subset \mathrm{UMor}_2$ and $\phi, \psi \in \mathcal{T}_{g,1}(p)$, a chain of equivalences

$$\begin{aligned}&\xi^p_I(\phi\psi\phi^{-1})(a) \equiv (\phi\psi) *_H \xi^p_I(\phi^{-1})(a) + \xi^p_I(\phi\psi)(\phi^{-1} *_H a) \pmod p \\ &\equiv \phi *_H \psi *_H \xi^p_I(\phi^{-1})(a) + \Big(\phi *_H \xi^p_I(\psi)(\phi^{-1} *_H a) + \xi^p_I(\phi)(\psi *_H \phi^{-1} *_H a)\Big).\end{aligned}$$

But, as $\psi \in \mathcal{T}_{g,1}(p)$, the sum of the first and third summand vanishes:

$$\phi *_H \xi^p_I(\phi^{-1})(a) + \xi^p_I(\phi)(\phi^{-1} *_H a) \equiv \xi^p_I(\phi\phi^{-1})(a) \equiv \xi^p_I(\mathrm{id})(a) \equiv 0.$$

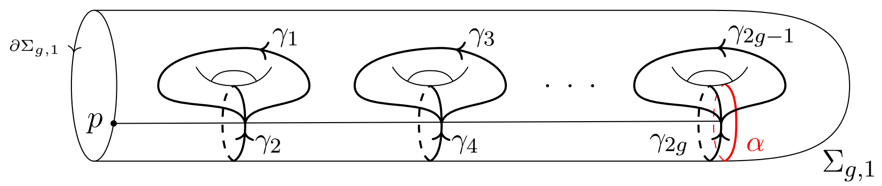


Figure 3.5 A standard generators of $\pi = \pi_1(\Sigma_{g,1}, p)$ and the curve α.

Again, we used by Proposition 144. The conclusion follows. □

Remark 148. The definition of the maps of sets ξ^p really depends on a choice of generators for the free group π; Lemma 147 however proves that the restrictions ξ_I^p are independent of such choice.

Recall from Theorem 138 that all separating Dehn twists lie in $\mathcal{T}_{g,1} \subseteq \mathcal{T}_{g,1}(p)$, and from Theorem 134 that all p^{th} powers of Dehn twists lie in $\mathcal{T}_{g,1}(p)$.

Lemma 149. *All p^{th} powers of non-separating Dehn twists are in the kernel of ξ_I^p. All separating Dehn twists are in the kernel of ξ_I.*

Proof. All non-separating Dehn twists in $\Gamma_{g,1}$ are conjugate to each other, so it suffices to show that the p^{th} power of a specific Dehn twist is in the kernel of ξ^p. Let a be a simple closed curve freely homotopic to γ_{2g}, as in Figure 3.5, and let $D_a \in \Gamma_{g,1}$ be the associated Dehn twist. Then $D_a(\gamma_{2g}) = \gamma_{2g-1}\gamma_{2g}$, whereas $D_a(\gamma_i) = \gamma_i$ for all $1 \le i \le 2g-1$. It follows that D_a^p sends $\gamma_{2g} \mapsto \gamma_{2g-1}^p\gamma_{2g}$ and fixes the other generators γ_i; therefore the group homomorphism $\xi(D_a^p)\colon H \to \Lambda^2 H$ sends $[\gamma_{2g}] \mapsto c_2(D_a^p(\gamma_{2g})) = \binom{p}{2}[\gamma_{2g-1}] \wedge [\gamma_{2g-1}] + p[\gamma_{2g-1}] \wedge [\gamma_{2g}] = p[\gamma_{2g-1}] \wedge [\gamma_{2g}]$ and sends $\gamma_i \mapsto c_2(D_a^p(\gamma_i)) = c_2(\gamma_i) = 0$ for $i \ne 2g$. Modulo p, we obtain the zero group homomorphism, i.e. $\xi^p(D_a^p) = 0$.

To prove that separating Dehn twists lie in the kernel of ξ_I, we use Proposition 142 together with the fact that by Theorem 138 the kernel of $\tau_{g,1}$ is precisely the subgroup of $\Gamma_{g,1}$ generated by all separating Dehn twists.

□

Proposition 150. *Let p be an odd prime. Then the image of ξ_I^p is isomorphic to $\Lambda^3 H \otimes \mathbb{Z}/p \subset H \otimes \Lambda^2 H \otimes \mathbb{Z}/p$, and the kernel of ξ_I^p is generated by the kernel of ξ_I and p^{th} powers of Dehn twists.*

Proof. By Lemma 149 all p^{th} powers of Dehn twists are in the kernel of ξ_I^p, and by Theorem 134 they generate $\mathcal{T}_{g,1}(p)$ together with $\mathcal{T}_{g,1}$; it follows that

$$\mathrm{Im}(\xi_I^p) = \mathrm{Im}(\xi_I^p|_{\mathcal{T}_{g,1}}) = \mathrm{Im}(\xi_I \otimes \mathbb{Z}/p).$$

By Proposition 142 and the assumption that p is odd, we have that the maps $\xi_I \otimes \mathbb{Z}/p, \tau_{g,1} \otimes \mathbb{Z}/p \colon \mathcal{T}_{g,1} \to \mathrm{Hom}(H, \Lambda^2 H) \otimes \mathbb{Z}/p$ have the same image and the same kernel.

Recall from Theorem 137 that $\mathrm{Im}(\tau_{g,1}) \subset \mathrm{Hom}(H, \Lambda^2 H) \cong H \otimes \Lambda^2 H$ is the split subgroup $\Lambda^3 H$; thus we can identify $\mathrm{Im}(\xi_I \otimes \mathbb{Z}/p) = \mathrm{Im}(\tau_{g,1} \otimes \mathbb{Z}/p) \subset H \otimes \Lambda^2 H \otimes \mathbb{Z}/p$ with $\mathrm{Im}(\tau_{g,1}) \otimes \mathbb{Z}/p \cong \Lambda^3 H \otimes \mathbb{Z}/p$; this proves the first claim.

For the second claim, we first observe that $\ker(\xi_I^p)$ is generated by p^{th} powers of Dehn twists together with $\ker(\xi_I \otimes \mathbb{Z}/p) = \ker(\tau_{g,1} \otimes \mathbb{Z}/p) \subset \mathcal{T}_{g,1}$. From the first isomorphism theorem, there is a short sequence of groups

$$\ker(\tau_{g,1}) \to \ker(\tau_{g,1} \otimes \mathbb{Z}/p) \to p\mathrm{Hom}(H, \Lambda^2 H) \cap \mathrm{Im}(\tau_{g,1});$$

the third term can be identified with $p\Lambda^3 H = p\mathrm{Im}(\tau_{g,1})$, using again that the inclusion $\Lambda^3 H \subset H \otimes \Lambda^2 H$ is split. In particular $\ker(\tau_{g,1} \otimes \mathbb{Z}/p)$ can be generated by $\ker(\tau_{g,1})$ and lifts of generators of the group $p\Lambda^3 H$ in $\mathcal{T}_{g,1}$; for the latter we may take the p^{th} powers of a generating set of $\mathcal{T}_{g,1}$. We now observe that $\mathcal{T}_{g,1}$ is generated by separating twists and *bounding pairs* $D_a D_{a'}^{-1}$, i.e. products of a Dehn twist D_a with the inverse of a Dehn twist $D_{a'}$, where $a, a' \subset \Sigma_{g,1}$ are disjoint simple closed curves cobounding a subsurface of $\Sigma_{g,1}$. We conclude that the following elements generate $\ker(\xi_I^p)$:

1. all p^{th} powers of all Dehn twists;
2. all separating Dehn twists;
3. all p^{th} powers of bounding pairs.

We may drop the third kind of generators from the generating set: for a bounding pair $D_a D_{a'}^{-1}$, we have that the Dehn twists D_a and $D_{a'}$ commute, hence $(D_a D_{a'}^{-1})^p = D_a^p D_{a'}^{-p}$, which is already an element in the subgroup of $\Gamma_{g,1}$ generated by p^{th} powers of Dehn twists. $\square$

Lemma 149 and Proposition 150 justify the following notation.

Notation 151. For an odd prime p, we denote by $\mathcal{K}_{g,1}(p) := \ker(\xi_I^p) \subseteq \mathcal{T}_{g,1}(p)$, i.e. the subgroup generated by separating Dehn twists and p^{th} powers of Dehn twists.

3.4.3 The $\Gamma_{g,1}$-action on integral homology

Proposition 152. *The kernel of the $\Gamma_{g,1}$-action on UMor_* is the Johnson kernel $\mathcal{K}_{g,1}$.*

Proof. Let $\phi \in \Gamma_{g,1}$ act trivially on $\mathrm{UMor}_2 \cong H \oplus \Lambda^2 H$, where we use the splitting from Notation 139. Then ϕ acts trivially on the subrepresentation $\Lambda^2 H \subset \mathrm{UMor}_2$ and on the quotient $\mathrm{UMor}_2/\Lambda^2 H \cong H$, so that $\phi \in \mathcal{T}_{g,1}$. Furthermore, by Proposition 143 $\phi \in \mathcal{T}_{g,1}$ acts trivially on $H \subset \mathrm{UMor}_2$ precisely if $\xi_I(\phi) = 0$. Thus the kernel of the action is $\ker(\xi_I) = \ker(\tau_1)$, using again that $\xi_I = 2\tau_1$.

Conversely, $\mathcal{K}_{g,1} \subseteq \mathcal{T}_{g,1}$ is generated by separating Dehn twists; hence $\mathcal{K}_{g,1}$ acts trivially on UMor_1, and combining Lemma 149 and Proposition 143 we obtain that $\mathcal{K}_{g,1}$ also acts trivially on UMor_2. The action of $\mathcal{K}_{g,1}$ on UMor_* is by ring automorphisms, and the ring UMor_* is generated, at least after tensoring with $\mathbb{Q}$, by UMor_1 and UMor_2. This proves that $\mathcal{K}_{g,1}$ acts trivially on $\mathrm{UMor}_* \otimes \mathbb{Q}$ and, since UMor_* is a free abelian group, the same is true for UMor_* itself. $\square$

The element $\Omega_2 = 2\omega \in \Lambda^2 H \subset \mathrm{UMor}_2$ from Definition 127 is $\Gamma_{g,1}$-invariant, so the quotient $\mathrm{UMor}_2/\mathbb{Z}\Omega_2$ is also a $\Gamma_{g,1}$-representation.

Lemma 153. *The kernel of the $\Gamma_{g,1}$-action on $\mathrm{UMor}_2/\mathbb{Z}\Omega_2$ is $\mathcal{K}_{g,1}$.*

Proof. The fact that $\mathcal{K}_{g,1}$ acts trivially on $\mathrm{UMor}_2/\mathbb{Z}\Omega_2$ is a consequence of Proposition 152. Conversely, if a class $\phi \in \Gamma_{g,1}$ acts trivially on $\mathrm{UMor}_2/\mathbb{Z}\Omega_2$, then it also acts trivially on $\mathrm{UMor}_2/\Lambda^2 H \cong H$, hence ϕ must lie in $\mathcal{T}_{g,1}$. Furthermore, $\phi \in \mathcal{T}_{g,1}$ acts trivially on $\mathrm{UMor}_2/\mathbb{Z}\Omega_2$ if and only if $\xi_I(\phi)(a) \in \mathbb{Z}\Omega_2 \subset \Lambda^2 H$ for all $a \in H$. Since $\xi_I = 2\tau_1$ and $\Omega_2 = 2\omega$, and since multiplication by 2 is injective on $\Lambda^2 H$, the condition on ϕ can be rewritten as $\tau_1(\phi) \in \mathrm{Hom}(H, \mathbb{Z}\omega) \subset \mathrm{Hom}(H, \Lambda^2 H)$. We will prove that

$$\mathrm{Im}(\tau_1) \cap \mathrm{Hom}(H, \mathbb{Z}\Omega_2) = 0 \subset \mathrm{Hom}(H, \Lambda^2 H); \tag{3.4.1}$$

from this the assertion $\ker(\tau_1) = \mathcal{K}_{g,1}$ will follow.

We use the identification $\mathrm{Hom}(H, \Lambda^2 H) \cong H \otimes \Lambda^2 H$ already considered in the discussion before Theorem 137, and embed $\Lambda^2 H \hookrightarrow H \otimes H$ via $a \wedge b \mapsto a \otimes b - b \otimes a$;

we thus obtain an embedding of $H \otimes \Lambda^2 H \hookrightarrow H \otimes H \otimes H$. Under this embedding, $\mathrm{Hom}(H,\mathbb{Z}\omega)$ is the subgroup of $H^{\otimes 3}$ consisting of the elements

$$a \otimes \omega := \sum_{i=1}^{g} a \otimes x_{2i-1} \otimes x_{2i} - a \otimes x_{2i} \otimes x_{2i-1},$$

for varying $a \in H$, and whereas $\mathrm{Im}(\tau_1) \cong \Lambda^3 H$ is the subgroup of $H^{\otimes 3}$ generated by all elements of the form

$$\sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma) a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes a_{\sigma(3)},$$

for varying $a_1, a_2, a_3 \in H$. In particular, each element of $\mathrm{Im}(\tau_1)$ is invariant under the cyclic permutation of tensors $C\colon H^{\otimes 3} \to H^{\otimes 3}$ sending $a_1 \otimes a_2 \otimes a_3 \mapsto a_2 \otimes a_3 \otimes a_1$.

Let now $a \in H$ be such that $a \otimes \omega \in \mathrm{Im}(\tau_1)$, and wirte $a \otimes \omega$ as a linear combination $\sum_{1 \le j,k,l \le 2g} c_{j,k,l} x_j \otimes x_k \otimes x_l$ of the basis elements $x_j \otimes x_k \otimes x_l$ of $H^{\otimes 3}$, with $c_{j,k,l} \in \mathbb{Z}$. Using the definition of $a \otimes \omega$ we get $c_{j,k,l} = 0$ for all j,k,l such that $\{k,l\}$ is not a pair of the form $\{2i+1, 2i+2\}$; using invariance of $a \otimes \omega$ under C, we obtain that $c_{j,k,l} = 0$ for every j,k,l such that at least one of the pairs $\{j,k\}$, $\{k,l\}$ and $\{l,j\}$ is not a pair of the form $\{2i+1, 2i+2\}$; this clearly implies $c_{j,k,l} = 0$ for all j,k,l, so that $a \otimes \omega = 0$. $\square$

Remark 154. Compare Lemma 153 with Proposition 73 which studies the same representation over $\mathbb{Q}$.

Theorem 155. *For $n \ge 2$, the kernel of the $\Gamma_{g,1}$-action on $H^2(C_n(\Sigma_{g,1});\mathbb{Z})$, and overall on $H^*(C_n(\Sigma_{g,1});\mathbb{Z})$, is the second Johnson subgroup $\mathcal{K}_{g,1}$.*

Proof. Propositions 132 and 152 together imply that $\mathcal{K}_{g,1}$ acts trivially on the cellular chains $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty)$, thus also on the homology $\widetilde{H}_*(C_n(\mathcal{M})^\infty;\mathbb{Z})$ for all $n \ge 0$. By Proposition 103 we have an isomorphism of $\Gamma_{g,1}$-representations $\widetilde{H}_*(C_n(\mathcal{M})^\infty;\mathbb{Z}) \cong H^{2n-*}(C_n(\mathcal{M});\mathbb{Z})$, so that $\mathcal{K}_{g,1}$ lies in the kernel of the action on $H^*(C_n(\Sigma_{g,1});\mathbb{Z})$. To prove that the kernel is precisely $\mathcal{K}_{g,1}$, it suffices by Lemma 153 to embed $\mathrm{UMor}(2)/\langle 2\omega \rangle$ as a $\Gamma_{g,1}$-subrepresentation of $H_{2n-2}(C_n(\mathcal{M})^\infty;\mathbb{Z}) \cong H^2(C_n(\Sigma_{g,1});\mathbb{Z})$.

For $l \ge 0$, denote by 1^l and 1^l2, respectively, the polysimpilicial data $(l,(1,\dots,1),\underline{0})$ and $(l,(1,\dots,1,2),\underline{0})$; let $e_{1^l}, e_{1^l2} \in \widetilde{\mathrm{Ch}}^B_*(\mathcal{M})$ be the corresponding generators. By Proposition 130, the differential $d(e_{1^l})$ vanishes for all $l \ge 0$; in particular the differential vanishes on the abelian groups $\mathbb{Z}e_{1^{n-2}} \otimes \mathrm{UMor}_2 \subset \widetilde{\mathrm{Ch}}_{n-2}(C_n(\mathcal{M})^\infty)$ and $\mathbb{Z}e_{1^{n-1}} \otimes$

$\mathrm{UMor}_1 \subset \widetilde{\mathrm{Ch}}_{n-1}(C_n(\mathcal{M})^\infty)$. The first vanishing implies that $\mathbb{Z}e_{1^{n-2}} \otimes \mathrm{UMor}_2$ only consists of cycles. The second vanishing, together with the direct sum decomposition

$$\widetilde{\mathrm{Ch}}_{n-1}(C_n(\mathcal{M})^\infty) \cong \mathbb{Z}e_{1^{n-1}} \otimes \mathrm{UMor}_1 \ \oplus \ \mathbb{Z}e_{1^{n-2}2} \otimes \mathrm{UMor}_0$$

implies that the boundaries in $\widetilde{\mathrm{Ch}}_{n-2}(C_n(\mathcal{M})^\infty)$ are generated by the image of the restriction of the differential on $\mathbb{Z}e_{1^{n-2}2} \otimes \mathrm{UMor}_0$, which is generated by the element $e_{1^{n-2}2} = e_{1^{n-2}2} \otimes 1$, and again by Proposition 130 we have $d(e_{1^{n-2}2}) = e_{1^{n-2}} \otimes \Omega_2$. Thus $e_{1^{n-2}}$ gives an injection of $\mathrm{UMor}_2/\mathbb{Z}\Omega_2$ into $\widetilde{H}_{2n-2}(C_n(\mathcal{M})^\infty;\mathbb{Z})$, which is $\Gamma_{g,1}$-equivariant.

We conclude that the kernel of the action of $\Gamma_{g,1}$ on $\widetilde{H}_{2n-2}(C_n(\mathcal{M})^\infty;\mathbb{Z})$, and overall on $\widetilde{H}_*(C_n(\mathcal{M})^\infty;\mathbb{Z})$, is precisely $\mathcal{K}_{g,1}$, for $n \geq 2$. We can again pass to $H^*(C_n(\Sigma_{g,1});\mathbb{Z})$ thanks to Proposition 103. □

3.4.4 The $\Gamma_{g,1}$-action on $\mathbb{Z}/p$-homology

Let p be an odd prime. The homology $\widetilde{H}_*(C_\bullet(\mathcal{M})^\infty;\mathbb{Z}/p)$ is computed by the complex $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty) \otimes \mathbb{Z}/p$ with the induced differential and $\Gamma_{g,1}$-action. The arguments of this subsection are analogous to the ones of Subsection 3.4.3, so we emphasise only the differences.

Proposition 156. *The kernel of the $\Gamma_{g,1}$-action on $\mathrm{UMor}_* \otimes \mathbb{Z}/p$ is $\mathcal{K}_{g,1}(p)$.*

Proof. The first part of the proof of Proposition 152 can be adapted word by word modulo p to show that $\phi \in \Gamma_{g,1}$ acts trivially on $\mathrm{UMor}_2 \otimes \mathbb{Z}/p$ if and only if $\phi \in \ker(\xi_I^p)$: we use that the short exact sequence of $\Gamma_{g,1}$-representations $\Lambda^2 H \to \mathrm{UMor}_2 \to H$ is split as a sequence of abelian groups, hence after tensoring with $\mathbb{Z}/p$ we obtain a short exact sequence $\Lambda^2 H \otimes \mathbb{Z}/p \to \mathrm{UMor}_2 \otimes \mathbb{Z}/p \to H \otimes \mathbb{Z}/p$. This shows that the kernel of the $\Gamma_{g,1}$-action on $\mathrm{UMor}_* \otimes \mathbb{Z}/p$ is contained in $\mathcal{K}_{g,1}(p)$.

By Proposition 113, the ring UMor_* is generated (as a $\mathbb{Z}$-algebra) by the elements x_i, for $1 \leq i \leq 2g$, and by the divided powers $y_i^{[m]}$, for $1 \leq i \leq 2g$ and $m \geq 1$; it follows that the elements $x_i \otimes 1$ and $y_i^{[m]} \otimes 1$ generate the $\mathbb{Z}/p$-algebra $\mathrm{UMor}_* \otimes \mathbb{Z}/p$. Therefore, to prove that $\mathcal{K}_{g,1}(p)$ acts trivially on $\mathrm{UMor}_* \otimes \mathbb{Z}/p$, it suffices to show that for $\phi \in \mathcal{K}_{g,1}(p) = \ker(\xi_I^p)$ we have $\mathrm{UMor}(\phi)(y_i^{[m]}) \equiv y_i^{[m]} \pmod{p}$, i.e. the difference $\mathrm{UMor}(\phi)(y_i^{[m]}) - y_i^{[m]}$ is a multiple of p in the abelian group UMor_{2m}. We know that

$\mathrm{UMor}(\phi)(y_i) \equiv y_i \pmod p$, i.e. there is $a \in \mathrm{UMor}_2$ such that $\mathrm{UMor}(\phi)(y_i) = y_i + pa$; for $m \geq 1$ we then have

$$\begin{aligned}\mathrm{UMor}(\phi)\left(y_i^{[m]}\right) &= \frac{(y_i+pa)^m}{m!} = \frac{1}{m!}\left(y_i^m + \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} y_i^{m-k} p^k a^k\right) \\ &= \frac{y_i^m}{m!} + \sum_{k=1}^{m} \frac{y_i^{m-k}}{(m-k)!} p^k \frac{a^k}{k!} = y_i^{[m]} + p\sum_{k=1}^{m} y_i^{[m-k]} p^{k-1} a^{[k]} \\ &\equiv y_i^{[m]} \pmod p.\end{aligned}$$

The first equality follows from the fact that $\mathrm{UMor}(\phi)$ is a ring homomorphism and UMor_* is torsion-free. □

Lemma 157. *The kernel of the $\Gamma_{g,1}$-actions on $(\mathrm{UMor}_2 \otimes \mathbb{Z}/p)/\langle\Omega_2\rangle$ is $\mathcal{K}_{g,1}(p)$.*

Proof. The argument is the same as in the proof of Lemma 153. By Proposition 156 we know that $\mathcal{K}_{g,1}(p)$ acts trivially on $\mathrm{UMor}_2 \otimes \mathbb{Z}/p$, hence on its quotient $(\mathrm{UMor}_2 \otimes \mathbb{Z}/p)/\langle\Omega_2\rangle$. Conversely, if $\phi \in \Gamma_{g,1}$ acts trivially on $(\mathrm{UMor}_2 \otimes \mathbb{Z}/p)/\langle\Omega_2\rangle$ then it also acts trivially on $(\mathrm{UMor}_2 \otimes \mathbb{Z}/p)/(\Lambda^2 H \otimes \mathbb{Z}/p) \cong H \otimes \mathbb{Z}/p$, hence $\phi \in \mathcal{T}_{g,1}(p)$; the condition that ϕ acts trivially on $(\mathrm{UMor}_2 \otimes \mathbb{Z}/p)/\langle\Omega_2\rangle$ can then be rephrased as $\xi_I^p(\phi) \in \mathrm{Hom}(H, \mathbb{Z}\Omega_2) \otimes \mathbb{Z}/p = \mathrm{Hom}(H, \mathbb{Z}\omega) \otimes \mathbb{Z}/p \subset \mathrm{Hom}(H, \Lambda^2 H) \otimes \mathbb{Z}/p$, where in the last step we use that p is odd, and that $\mathbb{Z}\omega \subset \Lambda^2 H$ is a split subgroup.

We can now identify $\mathrm{Hom}(H, \Lambda^2 H) \cong H \otimes \Lambda^2 H \otimes \mathbb{Z}/p$ with a subgroup of $H^{\otimes 3} \otimes \mathbb{Z}/p$ as in the proof of Lemma 153; by Proposition 150, the image of ξ_I^p can then be identified with $\Lambda^3 H \otimes \mathbb{Z}/p$, and all we have to prove is that the subgroups $\Lambda^3 H \otimes \mathbb{Z}/p$ and $H \otimes \mathbb{Z}\omega \otimes \mathbb{Z}/p$ of $H^{\otimes 3} \otimes \mathbb{Z}/p$ intersect trivially: this can be done by repeating the final argument used in the proof of Lemma 153. □

Theorem 158. *For $n \geq 2$, the kernel of the $\Gamma_{g,1}$-action on $H^2(C_n(\Sigma_{g,1}); \mathbb{Z}/p)$, and overall on $H^*(C_n(\Sigma_{g,1}); \mathbb{Z}/p)$, is $\mathcal{K}_{g,1}(p)$.*

Proof. By Proposition 156, the group $\mathcal{K}_{g,1}(p)$ acts trivially on the $\Gamma_{g,1}$-equivariant chain complex $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty, \mathbb{Z}/p) \cong \widetilde{\mathrm{Ch}}^B(C_\bullet(\mathcal{M})^\infty) \otimes (\mathrm{UMor}_* \otimes \mathbb{Z}/p)$, and hence on its homology, which by Proposition 103 coincides with $\widetilde{H}^*(C_\bullet(\mathcal{M})^\infty; \mathbb{Z}/p)$.

To prove that the kernel of the $\Gamma_{g,1}$-action on $H^2(C_n(\Sigma_{g,1}); \mathbb{Z}/p)$ is no larger than $\mathcal{K}_{g,1}(p)$, for a fixed $n \geq 2$, it suffices by Lemma 157 to embed a copy of $(\mathrm{UMor}_2 \otimes \mathbb{Z}/p)/\langle\Omega_2\rangle$ inside $H^2(C_n(\Sigma_{g,1}); \mathbb{Z}/p)$, and this can be done by the same argument used in the proof of Theorem 155, tensoring with $\mathbb{Z}/p$. □

3.5 The case of closed surfaces

The quotient $\mathcal{N}$ of $\mathcal{M}$ by collapsing the boundary $I = \partial\mathcal{M}$ to the single point p_0 shall be our model of Σ_g. It has a cellular structure with the 0-cell p_0, the $2g$ 1-cells I_i for $i = 1,\dots,2g$, and the 2-cell $\mathbf{R}$ (see Figure 3.6), so that the projection $p : \mathcal{M} \to \mathcal{N}$ is cellular, mapping each cell except I to the cell of the same name, and collapsing I to p_0.

We will study $C_\bullet(\mathcal{N})^\infty = C_\bullet(\mathcal{N},\emptyset)^\infty$ following the steps we took for $C_\bullet(\mathcal{M})^\infty$. We start by lifting, in a sense, the polysimplicial decompositions for $C_\bullet(V_k)^\infty$ and $C_\bullet(\mathcal{M})^\infty$ to ones for $C_\bullet(V_k,\emptyset)^\infty$ and $C_\bullet(\mathcal{N})^\infty$: each polysimplex will define two new simplices one with and one without a configuration point at the point p_0.

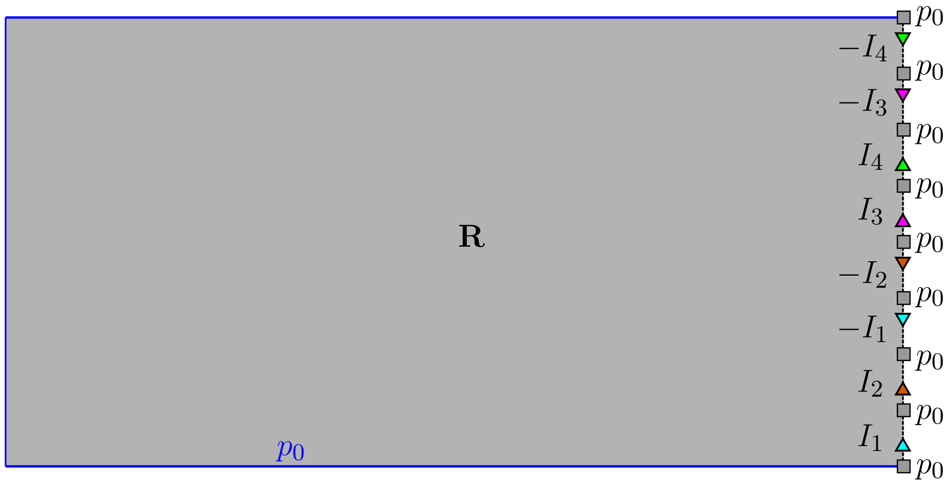


Figure 3.6 A cell decomposition of $\mathcal{N}$. The blue part of the boundary of $\mathbf{R}$ and the grey square points on the right are all the single point p_0.

3.5.1 Polysimplicial decomposition for $C_\bullet(V_k,\emptyset)^\infty$

Given $\mathfrak{v} \in (\mathbb{Z}_{\geq 0})^k$, define the map $\phi^{\mathfrak{v}} : \Delta^{\mathfrak{v}} \to C_\bullet(V_k,\emptyset)^\infty$ by reinterpreting the formula (3.2.1) of $\Phi^{\mathfrak{v}}$, viewed as landing in the new target $C_\bullet(V_k,\emptyset)^\infty$. For a pair $(\mathfrak{v},\varepsilon)$ with $\epsilon \in \{0,1\}$, define the polysimplicial embeddings $\Phi^{(\mathfrak{v},\varepsilon)} : \Delta^{\mathfrak{v}} \longrightarrow C_\bullet(V_k,\emptyset)^\infty$ given by $\Phi^{(\mathfrak{v},0)} = \phi^{\mathfrak{v}}$ and $\Phi^{(\mathfrak{v},1)} = \mu(\phi^{\mathfrak{v}},\{p_0\})$.

Proposition 159. *The pair $(C_\bullet(V_k,\emptyset)^\infty,\infty)$ admits a polysimplicial decomposition with set of polysimplices $(\mathbb{Z}_{\geq 0})^k \times \{0,1\}$ and the maps $\Phi^{(\mathfrak{v},\varepsilon)}$ as polysimplicial embeddings.*

Proof. Analogous to Proposition 111. $\square$

Proposition 160. *The cellular chain ring of $(C_\bullet(V_k,\emptyset)^\infty,\infty)$ is given by*

$$\widetilde{\mathrm{Ch}}_*(C_\bullet(V_k,\emptyset)^\infty) \cong \mathrm{UMor}_*(k)\otimes\mathbb{Z}[u]/(u^2)$$

by identifying $x_i = e_{(\mathfrak{e}_i,0)}$, $y_i = e_{(2\mathfrak{e}_i,0)}$ and $u = e_{(\underline{0},1)}$, and the cellular differential is $d(y_i) = 2x_iu$, $d(x_i) = d(u) = 0$ for $i = 1,...,k$. Furthermore, under the isomorphism, the map $\widetilde{\mathrm{Ch}}_*(C_\bullet(f)^\infty)$ induced from a based map $f : V_k \to V_l$ corresponds to the map $\mathrm{UMor}(f)\otimes\mathrm{id}_{\mathbb{Z}[u]/(u^2)}$.

Proof. Analogous to the discussion of Section 3.2, except for the computation of the differential d. Clearly, d vanishes on u because this lives in $*$ degree 0. It also vanishes on $x_i = e_{(\mathfrak{e}_i,0)}$ which is a 1-cell with both its ends attached to u. Now, the three faces of y_i are the maps $s\mapsto(0,s),(s,s),(s,1)\in\Delta^2$ composed with $\Phi^{(2\mathfrak{e}_i,0)}$, which are $\Phi^{(\mathfrak{e}_i,1)}$, the constant ∞ map and $\Phi^{(\mathfrak{e}_i,1)}$, respectively. The polysimplicial differential formula gives $d(y_i) = 2ux_i$. □

3.5.2 Polysimplicial decomposition for $C_\bullet(\mathcal{N})^\infty$

Recall the set $\mathcal{T}$ from Definition 122. For every $\mathfrak{t}\in\mathcal{T}$, define the map $\phi^{\mathfrak{t}} : \Delta^{\mathfrak{t}} \to C_\bullet(\mathcal{N})^\infty$ by reinterpreting the defining formula (3.3.1) of $\Phi^{\mathfrak{t}}$, viewed as landing in the new target $C_\bullet(\mathcal{N})^\infty$. For a pair $(\mathfrak{t},\varepsilon)$ with $\varepsilon\in\{0,1\}$, define the polysimplicial embeddings $\Phi^{(\mathfrak{t},\varepsilon)} : \Delta^{\mathfrak{t}} \to C_\bullet(\mathcal{N})^\infty$ given by $\Phi^{(\mathfrak{t},0)} = \phi^{\mathfrak{t}}$ and $\Phi^{(\mathfrak{t},1)} = \mu(\phi^{\mathfrak{t}},\{p_0\})$.

Proposition 161. *The pair $(C_\bullet(\mathcal{N})^\infty,\infty)$ admits a polysimplicial decomposition with set of polysimplices $\mathcal{T}\times\{0,1\}$ and the maps $\Phi^{(\mathfrak{t},\varepsilon)}$ as polysimplicial embeddings.*

Proof. Analogous to Proposition 123. □

The reduced cellular chain complex $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty) = \mathbb{Z}\langle e_{(\mathfrak{t},\varepsilon)}\rangle$ is bigraded so that $|e_{(\mathfrak{t},\varepsilon)}| = (d(\bar{\mathfrak{t}}),n(\bar{\mathfrak{t}}))$.

Proposition 162. *The superposition product μ is cellular, making $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty)$ into a differential bigraded ring. Under this multiplication, we have the identities $e_{(\mathfrak{t},\varepsilon)}\cdot e_{(\mathfrak{t}',\varepsilon')} = (-1)^{d(\mathfrak{t})d(\mathfrak{t}')}e_{(\mathfrak{t}',\varepsilon')}\cdot e_{(\mathfrak{t},\varepsilon)}$, $e_{(b,\underline{P},\mathfrak{v},0)} = e_{(b,\underline{P},\underline{0},0)}\cdot e_{(0,0,\mathfrak{v},0)}$ and $e_{(b,\underline{P},\mathfrak{v},1)} = e_{(b,\underline{P},\underline{0},0)}\cdot e_{(0,0,\mathfrak{v},0)}\cdot e_{(0,0,\underline{0},1)}$.*

Proof. For a configuration $s\in C_\bullet(\mathcal{N})^\infty$, let $n(s) := n(\mathfrak{t})+\varepsilon$, $d(s) := d(\mathfrak{t})$, $b(s) := b(\mathfrak{t})$ and $\varepsilon(s) = \varepsilon$ where $e_{(\mathfrak{t},\varepsilon)}$ is the unique polysimplex containing s in its interior. It suffices that for all $s,s' \in C_\bullet(\mathcal{M})^\infty$, we have $d(\mu(s,s')) \leq d(s)+d(s')$ or $\mu(s,s')=\infty$. Indeed, if $\mu(s,s') \neq \infty$, then $b(\mu(s,s')) \leq b(s)+b(s')$, $n(\mu(s,s')) = n(s)+n(s')$ and $\varepsilon(\mu(s,s')) = \varepsilon(s)+\varepsilon(s')$. Using the equality $d(s) = b(s)+n(s)-\varepsilon(s)$, we are done.

The multiplication identities follow as in Proposition 124. ☐

Proposition 163. *Let $j:(V_{2g},*) \hookrightarrow (\mathcal{N},p_0)$ be the composition of the embedding $\iota:(V_{2g},*) \hookrightarrow (\mathcal{M},p_0)$ and the projection $p:\mathcal{M} \to \mathcal{N}$. The map $C_\bullet(j)^\infty : \widetilde{C}_\bullet(V_{2g},\emptyset)^\infty \to C_\bullet(\mathcal{M})^\infty$ is cellular and commutes with μ. The induced map $\widetilde{\mathrm{Ch}}_*(C_\bullet(j)^\infty)$ is the inclusion of differential bigraded algebras sending $e_{(\mathfrak{v},\varepsilon)} \mapsto e_{(0,0,\mathfrak{v},\varepsilon)}$.*

Proof. Clear by construction. ☐

Let $\widetilde{\mathrm{Ch}}_*^B(\mathcal{N})$ be the subgroup of $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty)$ linearly generated by cells of the form $e_{(b,\underline{\mathrm{P}},\underline{0},0)}$. Combining the last two propositions, we obtain the tensor product decomposition

$$\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty) = \widetilde{\mathrm{Ch}}_*^B(\mathcal{N}) \otimes \mathrm{UMor}_* \otimes \mathbb{Z}[u]/(u^2). \tag{3.5.1}$$

where all factors are subrings, and furthermore $\Big(\mathrm{UMor}_* \otimes \mathbb{Z}[u]/(u^2), d(y_i) = 2x_i \otimes u\Big)$ is a sub-differential ring. Under the identificiation $e_{(b,\underline{\mathrm{P}},\underline{0},0)} \mapsto e_{(b,\underline{\mathrm{P}},\underline{0})}$, we have an isomorphism of rings $\widetilde{\mathrm{Ch}}_*^B(\mathcal{N}) \cong \widetilde{\mathrm{Ch}}_*^B(\mathcal{M})$. We will refer by $\widetilde{\mathrm{Ch}}_*^B$ to either of the two rings.

Proposition 164. *The differential d restricted to $\widetilde{\mathrm{Ch}}_*^B(\mathcal{N})$ is the sum of three components $d_B+d_M+d_N$, where d_B and d_M are defined in Proposition 130, and d_N is defined on generators $e_{(b,\underline{\mathrm{P}},\underline{0},0)}$ by*

$$d_N(e_{(b,\underline{\mathrm{P}},\underline{0},0)}) = \sum_{i=1}^{b} (-1)^{b+P_1+\ldots+P_{i-1}} (1+(-1)^{P_i})\, e_{(b,(P_1,\ldots,P_i-1,\ldots,P_b),\underline{0},0)} \cdot u.$$

Proof. We apply the polysimplicial differential formula from Proposition 109 to $e_{(b,\underline{\mathrm{P}},\underline{0},0)}$. The terms without a factor of u do not introduce a configuration point at p_0 and so correspond to the contributions $d_B(e_{(b,\underline{\mathrm{P}},\underline{0},0)})$ and $d_M(e_{(b,\underline{\mathrm{P}},\underline{0},0)})$ identified in Proposition 130. We compute now the terms involving u, whose sum shall constitute the d_N component.

There are two ways to reach the boundary of the cell $e_{(b,\underline{P},\underline{0},0)}$ introducing a single configuration point at p_0: (i) the left-most bar of $e_{(b,\underline{P},\underline{0},0)}$ reaches the left side of $\mathbf{R}$, or (ii) the first or last point on a bar reaches the top or bottom side of $\mathbf{R}$.

Case (i) corresponds to the face $f_0 \times \mathrm{id}_{\Delta^{P_1}\times\ldots\times\Delta^{P_b}}$ of the polysimplex $\Delta^b \times \Delta^{P_1} \times \ldots \times \Delta^{P_b}$. If $P_1 \geq 2$, then all (at least 2) points of the first bar collide at p_0 giving rise to the ∞-configuration, and this contribution vanishes. Otherwise $P_1 = 1$, in which case $\Phi^{(b,\underline{P},\underline{0},0)} \circ (f_0 \times \mathrm{id}_{\Delta^{P_1}\times\ldots\times\Delta^{P_b}})$ factors as

$$\begin{array}{c} \Delta^{b-1} \times \Delta^{P_1} \times \Delta^{P_2} \times \ldots \times \Delta^{P_b} \\ \Big\downarrow \mathrm{proj} \\ \Delta^{b-1} \times \Delta^{P_2} \times \ldots \times \Delta^{P_b} \xrightarrow{\Phi^{(b-1,(P_2,\ldots,P_b),\underline{0},1)}} C_\bullet(\mathcal{N})^\infty, \end{array}$$

where proj is the projection that forgets the Δ^{P_1} co-ordinate. As the projection abruptly drops the dimension, the degree of $\Phi^{(b,\underline{P},\underline{0},0)} \circ (f_0 \times \mathrm{id}_{\Delta^{P_1}\times\ldots\times\Delta^{P_b}})$, and thus its contribution, vanishes.

Case (ii) corresponds to the faces

$$f_{i,k} := \mathrm{id}_{\Delta^{P_1}\times\ldots\times\Delta^{P_{i-1}}} \times f_k \times \mathrm{id}_{\Delta^{P_{i+1}}\times\ldots\times\Delta^{P_b}}$$

with $i = 1,\ldots,b$, and $k = 0$ or $k = P_i$. The composition $\Phi^{(b,\underline{P},\underline{0},0)} \circ f_{i,k}$ coincides with $\Phi^{(b,(P_1,\ldots,P_i-1,\ldots,P_b),\underline{0},1)}$, and gives the summand

$$\begin{aligned} &(-1)^{b+P_1+\ldots+P_{i-1}}(-1)^k e_{(b,(P_1,\ldots,P_i-1,\ldots,P_b),\underline{0},1)} \\ &= (-1)^{b+P_1+\ldots+P_{i-1}}(-1)^k e_{(b,(P_1,\ldots,P_i-1,\ldots,P_b),\underline{0},0)} \cdot u \end{aligned}$$

of $d(e_{(b,\underline{P},\underline{0},0)})$, where the last equality is from Proposition 162. □

3.5.3 Cellular approximation of homeomorphisms

We fix the disc $D = [0,1] \times [0,1] \subset \mathcal{N}$ and identify the mapping class group $\Gamma_{g,1}$ with $\pi_0(\mathrm{Homeo}(\mathcal{N}, D))$. The map τ from Section 3.3.5 descends to a based self-map of $(\mathcal{N}, p_0)$.

Proposition 165. *For any $f \in \mathrm{Homeo}(\mathcal{N}, D)$, $C_n(\tau \circ f)^\infty$ is cellular.*

Proof. In the notation of the proof of Proposition 162, it suffices to show $d(C_n(\tau \circ f)^\infty(s)) \leq d(s)$, for all $s \in C_n(\mathcal{N})^\infty$. If $C_n(\tau \circ f)^\infty(s) = \infty$, this holds immediately. Otherwise, $n(C_n(\tau \circ f)^\infty(s)) = n(s)$ and $\varepsilon(C_n(\tau \circ f)^\infty(s)) \geq \varepsilon(s)$. Furthermore, as $\tau \circ f$ sends bars to bars or into $\iota(V_{2g})$, then $b(C_n(\tau \circ f)^\infty(s)) \leq b(s)$. Using $d(-) = n(-) + b(-) - \varepsilon(-)$, the conclusion follows. □

The map τ is homotopic to $\mathrm{id}_{\mathcal{N}}$ via based self-maps. As before, we model the natural action of $\Gamma_{g,1}$ on $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty)$ by insisting that $f \in \mathrm{Homeo}(\mathcal{N}, D \cup \partial\mathcal{N})$ acts on $C_\bullet(\mathcal{N})^\infty$ via $C_\bullet(\tau \circ f)^\infty$.

Proposition 166. *The subrings $\widetilde{\mathrm{Ch}}_*^B(\mathcal{N})$ and UMor_* of $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty)$ are $\Gamma_{g,1}$-subrepresentations, the $\Gamma_{g,1}$-action on $\widetilde{\mathrm{Ch}}_*^B(\mathcal{N})$ is trivial, and $f \in \mathrm{Homeo}(\mathcal{N}, D)$ acts on UMor_* via $C_\bullet(\tau \circ f)_* = \mathrm{UMor}(\tau \circ f) = \mathrm{UMor}(\phi)$, where $\phi \in \mathrm{Aut}(\pi)$ is the map on $\pi_1(\Sigma_{g,1}) \cong \pi_1(V_{2g})$ induced by f.*

Proof. Analogous to Proposition 132. □

3.5.4 On Torelli groups and Johnson homomorphisms of closed surfaces

We present the closed surface version of the definitions and results from Section 3.4. Henceforth, we let p be an odd prime number or the number 0.

Definition 167. The *Torelli group modulo p*, denoted $\mathcal{T}_g(p)$, is the kernel of the action of Γ_g on $H = H_1(\Sigma_g; \mathbb{Z}/p)$. The standard Torelli group is also denoted $\mathcal{T}_g := \mathcal{T}_g(0)$.

Remark 168. The inclusion $\mathrm{Homeo}^+(\mathcal{N}, D) \hookrightarrow \mathrm{Homeo}^+(\mathcal{N})$ induces a surjection $\eta : \Gamma_{g,1} \to \Gamma_g$, under which $\mathcal{T}_g(p) = \eta(\mathcal{T}_{g,1}(p))$. Furthermore, any Γ_g-representation is also a $\Gamma_{g,1}$-representation. Conversely, a $\Gamma_{g,1}$-representation descends to a Γ_g-representation if and only if $\ker(\eta)$ acts trivially.

In [23], Johnson observed that $\tau_{g,1}(\ker(\eta))$ lies in the subgroup $\omega \wedge H \subset \Lambda^3 H$ of the image of $\tau_{g,1}$, as described in Theorem 137.

Definition 169 (Johnson [23])**.** The *Johnson homomorphism* $\tau_g : \mathcal{T}_g \to \Lambda^3 H / \omega \wedge H$ is descends from $\tau_{g,1}$ via the surjection $\eta : \mathcal{T}_{g,1} \to \mathcal{T}_g$. Its kernel is the *Johnson kernel* $\mathcal{K}_g$.

Theorem 170 (Johnson [23])**.** *The Johnson homomorphism τ_g is surjective, Γ_g-equivariant and, if $g \geq 3$, it constitutes, modulo 2-torsion, the abelianisation of $\mathcal{T}_g$. Furthermore, $\mathcal{K}_g = \eta(\mathcal{K}_{g,1})$ is generated by all separating Dehn twists.*

3.5.5 The function ζ^p

We use η and the crossed-homomorphisms ξ^p from Section 3.4 to define closed modulo p Johnson homomorphisms and kernels.

Definition 171. The function of sets $\zeta^p : \Gamma_g \to (\mathrm{Hom}(H, \Lambda^2 H)/\omega \wedge H) \otimes \mathbb{Z}/p$ is defined as $\zeta(\phi) \mapsto \xi^p(\phi) + \omega \wedge H$, for $\phi \in \Gamma_{g,1}$.

Proposition 172. *The function ζ^p satisfies the following:*

1. *it is well defined;*

2. *it is a crossed homomorphism, that is, for $\phi, \psi \in \Gamma_g$, and $a \in H$,*

$$\zeta^p(\phi\psi)(a) \equiv \phi *_H \zeta^p(\psi)(a) + \zeta^p(\phi)(\psi *_H a);$$

3. *it restricts to a Γ_g-equivariant group homomorphism $\zeta_I^p := \zeta^p|_{\mathcal{T}_g(p)}$ on the Torelli group $\mathcal{T}_g(p)$.*

Proof. For $\phi, \psi \in \Gamma_{g,1}$ with $\eta(\phi) = \eta(\psi)$, then $\phi = \psi f$, for some $f \in \ker(\eta)$ which lies in $\mathcal{T}_{g,1}$. Applying Proposition 144, we have

$$\xi^p(\phi f) \equiv \xi^p(\phi) + \phi * \xi^p(f) \pmod{p},$$

and $\phi * \xi^p(f) \subset \phi * (\omega \wedge H) \subseteq \omega \wedge H$, because ω is a $\Gamma_{g,1}$-invariant element, so $\omega \wedge H$ is a subrepresentation. Therefore, $\zeta^p(\phi)$ and $\zeta^p(\phi f)$ agree in the quotient by $\omega \wedge H$.

Properties 2. and 3. follow directly from the corresponding properties of ξ^p (see Proposition 144, Corollary 145, and Lemma 147). □

Definition 173. For p an odd prime, the *Johnson kernel modulo p* is $\mathcal{K}_g(p) := \ker(\zeta_I^p)$.

Proposition 174. *For p an odd prime, $\mathcal{K}_g(p)$ is generated by $\mathcal{K}_g$ and the p^{th} powers of all Dehn twists.*

Proof. It is clear that $\eta(\mathcal{K}_{g,1}(p)) \subseteq \mathcal{K}_g(p)$, so it suffices to prove the opposite inclusion.

Consider the commutative square

$$\begin{tikzcd}
\mathcal{T}_{g,1}(p) \arrow[r, two heads, "\xi_I^p"] \arrow[d, two heads, "\eta|_{\mathcal{T}_{g,1}(p)}"] & \Lambda^3 H \otimes \mathbb{Z}/p \arrow[d, two heads, "Q"] \\
\mathcal{T}_g(p) \arrow[r, two heads, "\zeta_I^p"] & (\Lambda^3 H/\omega \wedge H) \otimes \mathbb{Z}/p
\end{tikzcd}$$

where Q is the quotient map. As $\eta|_{\mathcal{T}_{g,1}(p)}$ is surjective, the kernel $\ker(\zeta_I^p)$ is precisely the image $\eta(\zeta_I^p \circ \eta|_{\mathcal{T}_{g,1}(p)}) = \eta(\ker(Q \otimes \xi_I^p))$. By the surjectivity of ξ_I^p , then

$$\ker(Q \otimes \xi_I^p) = (\xi_I^p)^{-1}(\ker(Q)) = (\xi_I^p)^{-1}(\omega \wedge H \otimes \mathbb{Z}/p).$$

Johnson [23] provided an explicit generating set of the group $\omega \wedge H$ which lie in the image of $\tau_{g,1}(\ker(\eta))$. Lifts of these generators in $\ker(\eta) \subset \mathcal{T}_{g,1}(p)$, together with $\ker(\xi_I^p)$, generate $(\xi_I^p)^{-1}(\omega \wedge H \otimes \mathbb{Z}/p)$. But $\ker(\eta) \subset \mathcal{K}_{g,1}(p)$, so we may conclude that

$$\mathcal{K}_g(p) = \eta(\ker(Q \otimes \xi_I^p)) \subseteq \eta(\mathcal{K}_{g,1}(p))$$

as required. □

3.5.6 The kernel of the Γ_g-action on the homology of $C_n(\Sigma_g)$

Notation 175. Consider the abelian subgroup of UMor_3

$$\bar{E} := \bigoplus_{1 \leq i \leq 2g} \mathbb{Z} y_i x_i \oplus \bigoplus_{1 \leq i < j \leq 2g} \mathbb{Z}(y_i x_j + y_j x_i) \oplus \Lambda^3 \left[\bigoplus_{i=1}^{2g} \mathbb{Z} x_i \right]$$

and let E be its quotient by the subgroup

$$2\omega \wedge \bigoplus_{i=1}^{2g} \mathbb{Z} x_i \subset \Lambda^3 \left[\bigoplus_{i=1}^{2g} \mathbb{Z} x_i \right].$$

Proposition 176. *Under the action* $\mathrm{UMor}(\tau \circ -)$*, the sequence of inclusions*

$$2\omega \wedge \bigoplus_{i=1}^{2g} \mathbb{Z} x_i \subset \bar{E} \subset \mathrm{UMor}_3$$

are inclusions of $\Gamma_{g,1}$*-representations.*

For p an odd prime or 0*, and* $n \geq 3$*, there is a* $\Gamma_{g,1}$*-equivariant embedding of* $E \otimes \mathbb{Z}/p$ *into a quotient of* $H^3(C_n(\Sigma_g); \mathbb{Z}/p)$*.*

In particular, $E \otimes \mathbb{Z}/p$ *is a* Γ_g*-representation, fitting in a short exact sequence of* Γ_g*-representation*

$$0 \longrightarrow (\Lambda^3 H / \omega \wedge H) \otimes \mathbb{Z}/p \longrightarrow E \otimes \mathbb{Z}/p \longrightarrow \mathrm{Sym}^2 H \otimes \mathbb{Z}/p \longrightarrow 0.$$

Proof. We consider from the polysimplial data $1^{n-3} := (n-3,(1,...,1),\underline{0},0)$, $1^{n-3}2 := (n-2,(1,...,1,2),\underline{0},0)$, the corresponding generators $e_{1^{n-3}} \in \widetilde{\mathrm{Ch}}^B_{2n-6}(\mathcal{N})$, $e_{1^{n-3}2} \in \widetilde{\mathrm{Ch}}^B_{2n-3}(\mathcal{N})$.

The subspace of

$$\mathbb{Z}e_{1^{n-3}} \otimes \mathrm{UMor}_3 \subset \widetilde{\mathrm{Ch}}_{2n-3}(C_n(\mathcal{N})^\infty)$$

lying in the cellular cycles is precisely $\mathbb{Z}e_{1^{n-3}} \otimes \bar{E}$. This is because UMor_3 is generated by monomials of the form $x_ix_jx_k$ and y_ix_j, and the differential is $d(x_ix_jx_k) = 0$ and $d(y_ix_j) = 2x_ix_j \otimes u$. The latter can be viewed as the standard projection $H \otimes H \to \Lambda^2 H$, whose kernel is $\mathrm{Sym}^2 H$, that is the first two summands in the definition of $\bar{E}$. Now, the differential is $\Gamma_{g,1}$-equivariant, so cellular cycles form a subrepresentation proving the first assertion.

The only differential hitting $\mathbb{Z}e_{1^{n-3}} \otimes \mathrm{UMor}_3$ comes from $\mathbb{Z}e_{1^{n-3}2} \otimes \mathrm{UMor}_1$ and evaluates as

$$d(e_{1^{n-3}2}x) = 2e_{1^{n-3}} \otimes \omega x \otimes 1 + 2(-1)^{n-2} e_{1^{n-3}} \otimes 1 \otimes u.$$

Quotienting by cellular boundaries, we see that the quotient $\mathbb{Z}e_{1^{n-3}} \otimes E$ injects $\Gamma_{g,1}$-equivariantly into the quotient of $\widetilde{H}_{2n-3}(C_n(\mathcal{N})^\infty)$ by all terms involving u. In particular, E is a Γ_g-representation because $\widetilde{H}_{2n-3}(C_n(\mathcal{N})^\infty;\mathbb{Z}/p) \cong H^3(C_n(\Sigma_g);\mathbb{Z}/p)$ is, and it fits in a short exact sequence of the desired form.

□

Proposition 177. *$\mathcal{K}_g(p)$ acts trivially on $H^*(C_n(\Sigma_g);\mathbb{Z}/p)$ and on $E \otimes \mathbb{Z}/p$.*

Proof. Combining Propositions 156 and 166, $\mathcal{K}_{g,1}(p)$ acts trivially on the cellular chains $\widetilde{\mathrm{Ch}}_*\left(\widetilde{C}_\bullet(\mathcal{N})^\infty\right)$ and therefore on $H^*(C_n(\Sigma_g);\mathbb{Z}/p)$ and its subquotient $E \otimes \mathbb{Z}/p$, by Proposition 176. As these are Γ_g-representations, then $\mathcal{K}_g(p) = \eta(\mathcal{K}_{g,1}(p))$ acts trivially as required. □

Lemma 178. *For $g \geq 3$, the kernel of the Γ_g-action on $E \otimes \mathbb{Z}/p$ is precisely $\mathcal{K}_g(p)$.*

Remark 179. Before we see the long and tedious proof, let us remark that over $\mathbb{Z}$ there is a much shorter one based on the fact that the image of τ_g is (rationally) an irreducible Γ_g-representation. Over p, we have not made this fact to work.

Proof of Lemma 178. In light of Proposition 177, it suffices to fix $\phi \in \Gamma_g$ acting trivially and prove that $\phi \in \mathcal{K}_g(p)$. We will denote by $\phi(-)$ to alleviate the notation.

We first show that $\phi \in \mathcal{T}_g(p)$: by expressing $\phi(x_i) = \sum_j A_{ik} x_k$, where A is a $2g \times 2g$-matrix with entries in $\mathbb{Z}/p$, we will show that A is the identity matrix. The mapping class ϕ acts trivially on the associated graded of $E \otimes \mathbb{Z}/p$, which can be viewed as the group $E \otimes \mathbb{Z}/p$ with the naïve action from Section 3.4.2. Acting on $y_i x_i$ by the naïve action we have

$$y_i \otimes x_i \equiv \phi(y_i) \otimes \phi(x_i) = \sum_{1 \leq k \leq 2g} (A_{ik})^2 \, y_k x_k + \sum_{1 \leq k < l \leq 2g} A_{ik} A_{il} \, (y_k x_l + y_l x_k),$$

from which we deduce that $A_{ii}^2 \equiv 1 \pmod p$, so $A_{ii} \equiv \pm 1 \pmod p$, for all i, and $A_{ik} A_{il} \equiv 0$ for all $k \neq l$; combining the two statements, we have that $A_{ik} \equiv 0$ for all $k \neq i$. Thus A is a diagonal matrix with ± 1 on the diagonal; say $\lambda_i := A_{ii} \in \{\pm 1\}$. Now, from the trivial action of ϕ on $\Lambda^3 H / \omega H$, we have

$$x_i \wedge x_j \wedge x_k + \omega \wedge H \equiv \lambda_i \lambda_j \lambda_k \, x_i \wedge x_j \wedge x_k + \omega \wedge H \pmod p,$$

and thus $\lambda_i \lambda_j \lambda_k \equiv 1$ for all pairwise distinct i, j, k. This means that for any three $\lambda_i, \lambda_j, \lambda_k$ either precisely one or precisely three are ± 1; as $2g \geq 6$, it is not hard to see that all the λ_i must be $+1$, and A is the identity, as desired.

We proceed to show that $\zeta_I^p(\phi) = 0$ from which will follow that $\phi \in \mathcal{K}_g(p)$. Let $\bar{\phi} \in \mathcal{T}_{g,1}(p)$ be a lift of ϕ, and write $\xi_{\bar{\phi}} := \xi(\bar{\phi}) \in \mathrm{Hom}(H, \Lambda^2 H)$, to alleviate notation. From the triviality of the action on ϕ as prescribed by $\mathrm{UMor}(\tau \circ \bar{\phi})$, we have

$$(\phi - \mathrm{id})(y_i x_j + y_j x_i) = \xi_{\bar{\phi}}(y_i) \wedge x_j + \xi_{\bar{\phi}}(y_j) \wedge x_i \in p\Lambda^3 H + 2\omega H \subset \Lambda^3 H \subset \mathrm{UMor}_3, \tag{3.5.2}$$

for all $i, j = 1, \ldots, 2g$. We ignore the factor 2 of $\omega \wedge H$: it is invertible modulo $p > 2$, and it only makes the relation coarser over $\mathbb{Z}$.

We will extract a nice expression for $\xi_{\bar{\phi}}(y_i)$. Set $i = j$ in equation (3.5.2) to obtain

$$2\xi_{\bar{\phi}}(y_i) \wedge x_i \in p\Lambda^3 H + \omega \wedge H \tag{3.5.3}$$

for all $i = 1, \ldots, 2g$, which means we can express

$$\xi_{\bar{\phi}}(y_i) \wedge x_i = p\, w_i + \omega \wedge v_i, \tag{3.5.4}$$

for some $w_i \in \Lambda^3 H$ and $v_i \in H$.

Up to absorbing multiples of p into $p\,w_i$, v_i is a multiple of x_i. To see this, fix i, and write $v_i = \sum_j \lambda_j x_j$ with $\lambda_j \in \mathbb{Z}$. Substituting in (3.5.4) and wedging with x_i, we have

$$0 \equiv \sum_j \lambda_i\, \omega \wedge x_j \wedge x_i \pmod{p}. \tag{3.5.5}$$

As $g \geq 3$, for $j \neq i$, there is $k \in \{1,...,g\}$ with $i,j \notin \{2k-1, 2k\}$. The coefficient of $x_{2k-1} \wedge x_{2k} \wedge x_j \wedge x_i$ on the right hand side of (3.5.5) is precisely λ_j and $\lambda_i \equiv 0 \pmod{p}$. We deduce that $v_i \equiv \lambda_i x_i \pmod{p}$, and combing with (3.5.4), we may express

$$\xi_{\bar{\phi}}(y_i) = e_i \wedge x_i + p\,\bar{w}_i + \lambda_i\, \omega, \tag{3.5.6}$$

for some $e_i \in H$ and $\bar{w}_i \in \Lambda^2 H$, for $i = 1,...,2g$.

We now prove that e_i can be chosen to be independent of i. Substitute (3.5.6) back into equation (3.5.2) to obtain that, for all $i \neq j$,

$$\begin{aligned} 0 \equiv & e_i \wedge x_i \wedge x_j + p\,\bar{w}_i \wedge x_j + \lambda_i\, \omega \wedge x_j \\ & + e_j \wedge x_j \wedge x_i + p\,\bar{w}_j \wedge x_i + \lambda_j\, \omega \wedge x_i \\ \equiv & (e_i - e_j) \wedge x_i \wedge x_j \pmod{p\Lambda^3 H + \omega H}. \end{aligned}$$

Keep i,j fixed and write $f = e_i - e_j$. The above expression means that $f \wedge x_i \wedge x_j = pw + \omega \wedge v$ for some $v = \sum_{i=1}^{g} \alpha_i x_i \in H$ and $w \in \Lambda^3 H$, but we will show that in fact $f \wedge x_i \wedge x_j \in p\Lambda^3 H$. Wedging the expression of f with x_i, we have

$$\sum_{k=1}^{2g} \omega \wedge (\lambda_k\, x_k) \wedge x_i \equiv 0 \pmod{p}, \tag{3.5.7}$$

which is a summation of monomials of the form $x_{2l-1} \wedge x_{2l} \wedge x_k \wedge x_i$. These monomials vanish precisely when $k = i$ or either $i \in \{2l-1, 2l\}$ or $k \in \{2l-1, 2l\}$, and the surviving monomials are distinct basis elements of $\Lambda^4 H$. As $g \geq 3$, if $i \neq k$, there is always an l so that $i,k \notin \{2l-1, 2l\}$; thus equality (3.5.7) implies that $\lambda_k \equiv 0 \pmod{p}$ for all $k \neq i$. Repeating this argument with x_j instead of x_i, we have $\lambda_k \equiv 0 \pmod{p}$ for all $k \neq j$, and we conclude that $f \wedge x_i \wedge x_j \equiv 0 \pmod{p}$. This means for some coefficients $\mu_{ij}, \mu_{ji} \in \mathbb{Z}$, we have $(e_i - \mu_{ij} x_i) \equiv (e_j - \mu_{ji}) \pmod{p}$. Varying i,j, we find that μ_{ij} does not depend on j, so setting $\mu_i := \mu_{ij}$, there is an invariant $e \in H$ so that $e \equiv e_i - \mu_i x_i$, and $e_i \wedge x_i \equiv e \wedge x_i \pmod{p}$. Absorbing the difference of $e \wedge x_i$ and $e_i \wedge x_i$

into $p\,\bar{w}_i$ in (3.5.6), we have

$$\xi_{\bar{\phi}}(y_i) = e \wedge x_i + p\,\bar{w}_i' + \lambda_i\,\omega. \tag{3.5.8}$$

Finally, using Johnson's identification of the codomain of $\tau_{g,1}$ [23, p. 230], $\xi_{\bar{\phi}} \in \mathrm{Hom}(H, \Lambda^2 H)$ corresponds to the element $H \otimes \Lambda^2 H$ given by

$$\xi_{\bar{\phi}}' := \sum_{k=1}^{g} x_{2k-1} \otimes \xi_{\bar{\phi}}(y_{2g}) - x_{2k} \otimes \xi_{\bar{\phi}}(y_{2g-1}),$$

and using our expressions (3.5.8), we have

$$\xi_{\bar{\phi}}' = p\,W + \sum_{k=1}^{g} (e \wedge x_{2k-1} \otimes x_{2k} - e \wedge x_{2k} \otimes x_{2k-1} + \lambda_{2k-1}\omega \otimes x_{2k} - \lambda\omega \otimes x_{2k-1}), \tag{3.5.9}$$

where all terms divisible by p have been collected into $p\,W$. This element $\xi_{\bar{\phi}}' \in H \otimes \Lambda^2 H$ lies in the image of the Johnson homomorphism $\tau_{g,1}$ which is the subspace $\Lambda^3 H \subset H \otimes \Lambda^2 H$ described in the statement of Theorem 137. We project $P : H \otimes \Lambda^2 H \to \Lambda^3 H$ sending $a \wedge b \otimes c \mapsto a \wedge b \wedge c$; P is injective on the Johnson image (in fact, it is multiplication by 3 on $\Lambda^3 H$). The projection $P(\xi_{\bar{\phi}}' - pW)$ equals

$$2e \wedge \omega + \omega \wedge \sum_{k=1}^{g} (\lambda_{2k-1} x_{2k} - \lambda_{2k} x_{2k-1})$$

which is evidently in $\omega \wedge H \subset \Lambda^3 H$. Thus $\xi_{\bar{\phi}}' = pW \in \Lambda^3 H / \omega \wedge H$, or in other words $\zeta_I^p(\phi) \equiv 0 \pmod{p}$, as desired. □

Theorem 180. *For p an odd prime or 0, and $g \geq 3$, the kernel of the Γ_g-action on $H^3(C_n(\Sigma_g); \mathbb{Z}/p)$, and also on $H^*(C_n(\Sigma_g); \mathbb{Z}/p)$, is $\mathcal{K}_g(p)$.*

Proof. That the kernel contains $\mathcal{K}_g(p)$ is by Proposition 177. The kernel is precisely $\mathcal{K}_g(p)$ because of Lemma 178 and the Γ_g-equivariant embedding of E from Proposition 176. □

3.5.7 Degenerate cases

We deal now with some cases not accounted for by Theorem 180.

Proposition 181. *For any $g \geq 2$, the kernel of the Γ_g-action on $H_*(C_2(\Sigma_g);\mathbb{Z})$ is the Torelli group $\mathcal{T}_g$.*

Proof. From the decomposition (3.5.1) the $\Gamma_{g,1}$-representation $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty)$ is a direct sum of terms

$$\widetilde{\mathrm{Ch}}^B_{i,n} \otimes \mathrm{UMor}_{j,j} \otimes V, \tag{3.5.10}$$

where V is either $\mathbb{Z}$ in bidegree $(0,0)$ or $\mathbb{Z}u$ in bidegree $(0,1)$. The $\Gamma_{g,1}$-representation $H^*(C_2(\Sigma_g))$ is a subquotient of the weight $\bullet = 2$ part of this direct sum, whose only non-symplectic summand is $\mathbb{Z}\otimes\mathrm{UMor}_2\otimes\mathbb{Z}$. The cellular differential on $\mathrm{UMor}_2 = \mathbb{Z}\langle y_i, x_ix_j\rangle$ is $d(y_i) = 2x_i \otimes u$, so it maps $\mathbb{Z}\langle y_i\rangle$ injectively to $\mathbb{Z}\langle x_i \otimes u\rangle$, and vanishes on the x_ix_j. Therefore the subspace of UMor_2 lying in the cocycles is $\mathbb{Z}\langle x_ix_j\rangle$, which lies in the subring generated by UMor_1 and is thus symplectic. It follows that $\mathcal{T}_g$ acts trivially on $H_*(C_2(\Sigma_g)^\infty) \cong H^*(C_2(\Sigma_g))$. As $H^1(C_2(\Sigma_g)) \cong H^1(\Sigma_g)$, the result is sharp. □

Proposition 182. *For $n \geq 1$, the kernel of the Γ_2 action on $H_*(C_n(\Sigma_2);\mathbb{Z})$ is the Torelli group $\mathcal{T}_2$.*

Proof. It is well known (see e.g. Powell [43]) that there are no bounding pairs on Σ_g, and so $\mathcal{T}_2(p) = \mathcal{K}_2(p)$ acts trivially on $H_*(C_n(\Sigma_2))$. This is sharp by looking at $n = 1$. □

3.5.8 The $\mathbb{Q}$-homology of $C_\bullet(\Sigma_g)$

Theorem 183. *For $g \geq 3$, the kernel of the Γ_g action on $H^3(C_n(\Sigma_g);\mathbb{Q})$, and also on $H^*(C_n(\Sigma_g);\mathbb{Q})$, is $\mathcal{K}_g$.*

Proof. The proof of Theorem 180 goes through over $\mathbb{Q}$. □

Let $(\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_*, d_B) := (\widetilde{\mathrm{Ch}}^B_*, d_B) \otimes \mathbb{Q}$.

Lemma 184. *There is an isomomorphism $H_*(\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_*, d_B) \cong \mathbb{Q}[w] \otimes \Lambda[v]$ with bidegrees $|w| = (2,1)$ and $|v| = (3,2)$. Furthermore, $(\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_*, d_B)$ is quasi-isomomorphic to its homology via the dga morphism*

$$f : (\mathbb{Q}[w] \otimes \Lambda[v], 0) \to (\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_*, d_B)$$

sending $f(w) = e_{(1,(1))}$ and $f(v) = e_{(1,(2))}$, and extended as a map of algebras.

Proof. We will use input from Chapter 2 to compute the homology of $\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_{*,\bullet}$ in a way that gives rise to the map f.

The algebra $\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_{*,\bullet}$ coincides with $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{M})^\infty)$ for $g=0$, that is, with $\mathcal{M} \cong D^2$, so Proposition 103 gives isomomorphisms

$$H(\widetilde{\mathrm{Ch}}^{B}_{*,\bullet}, d_B) \cong \widetilde{H}_*(C_\bullet(D^2)^\infty, \mathbb{Q}) \cong H^{2\bullet - *}(C_\bullet(D^2); \mathbb{Q}). \tag{3.5.11}$$

By Theorem 18, for $i \geq 0, n \geq 0$ and arbitrary fixed $l \geq 1$, we also have

$$H^i(C_n(D^2);\mathbb{Q}) \cong H^{i+2l}(\mathrm{map}_\partial(D^2, S^{2+2l});\mathbb{Q}) \cong H^{i+2l}(\Omega^2 S^{2+2l};\mathbb{Q}). \tag{3.5.12}$$

From Theorem 76, isomorphisms (3.5.11) and (3.5.12) combine to an isomomorphism of algebras

$$(H_{*,\bullet}(\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_{*}, d_B), \mu) \xrightarrow{\cong} (H^{*+2l\bullet}(\Omega^2 S^{2+2l};\mathbb{Q}), \smile). \tag{3.5.13}$$

The cohomology ring of $\Omega^2 S^{2+2l}$ was computed in the proof of Lemma 53 to be $H^*(\Omega^2 S^{2+2l};\mathbb{Q}) = \Lambda[v'] \otimes \mathbb{Q}[w']$. The generators v' and w' are in bidegrees $|w'| = (2l,1)$ and $|w'| = (4l+1,2)$, and their preimages under isomorphism (3.5.13) live in bidegrees $(2,1)$ and $(3,2)$ of $H(\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_{*,\bullet}, d_B)$. In fact, $\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_{2,1} = \mathbb{Q}e_{(1,(1))}$ and $\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_{3,2} = \mathbb{Q}e_{(1,(2))}$, and so it must be that $e_{(1,(1))}$ and $e_{(1,(2))}$ survive to non-trivial cycles that map to w' and v', respectively, up to non-trivial scalar factors. By setting $w = [e_{(1,(1))}]$ and $v = [e_{(1,(2))}]$, we conclude that $H_*(\widetilde{\mathrm{Ch}}^{B,\mathbb{Q}}_{*,\bullet}, d_B) = \mathbb{Q}[w] \otimes \Lambda[v]$, as required.

The map f as defined in the statement is well-defined: both domain and target of f are graded commutative, if we forget the $\bullet$-grading, and the domain is free graded commutative. The differential vanishes on $f(w)$ and $f(v)$ as a consequence of the previous paragraph, so f is a map of dgas. It is clear that in homology it descends to an isomorphism. □

Over $\mathbb{Q}$, a divided power algebra is a polynomial algebra, so that $\mathrm{UMor}^{\mathbb{Q}}_* := \mathrm{UMor}_* \otimes \mathbb{Q} \cong \Lambda[x_1, ..., x_{2g}] \otimes \mathbb{Q}[y_1, ..., y_{2g}]$. We wish to state the following theorem analogously to Theorem 66, so we refer the reader to Definition 36 for the meaning of symplectic and Johnson action on a $\Gamma_{g,1}$-algebra. The definitions go through for a Γ_g-algebra via the surjection $\eta : \Gamma_{g,1} \to \Gamma_g$.

Theorem 185. *There is an Γ_g-equivariant isomorphism of bigraded algebras*

$$H^*(C_\bullet(\Sigma_g);\mathbb{Q}) \cong H^*(\Lambda[v,x_1,...,x_{2g}] \otimes \mathbb{Q}[w,y_1,...,y_{2g},u]/(u^2),d)$$

where

- *the bidegrees are $|w| = (0,1)$, $|v| = (1,2)$, $|x_i| = (1,1)$, $|y_i| = (2,2)$ and $|u| = (2,1)$,*
- *the differential maps $d(v) = 2\omega - 2uw$ and $d(y_i) = 2ux_i$, and vanishes on the other generators,*
- *the Γ_g-action is trivial on u,v,w, symplectic on the x_i and the Johnson on the y_i.*

Proof. By writing $(R,d_R) := (\mathrm{UMor}_*^{\mathbb{Q}} \otimes \mathbb{Q}[u]/(u^2), d(y_i) = 2x_i \otimes u)$, the decomposition of bigraded algebras $\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty) \otimes \mathbb{Q} = \widetilde{\mathrm{Ch}}_*^{B,\mathbb{Q}} \otimes R_*$ gives the right hand side the differential $d|_{1\otimes R} = 1 \otimes d_R$ and $d|_{\widetilde{\mathrm{Ch}}_*^{B,\mathbb{Q}} \otimes 1} = (d_B + d_M + d_N) \otimes 1$. Denoting for brevity the bigraded algebra $A_* := \Lambda[v] \otimes \mathbb{Q}[w]$ from Lemma 184, we endow the tensor product $A_* \otimes R_*$ with the augmented differential d_A given by

$$d_A|_{1\otimes R} = 1 \otimes d_R, d_A(w \otimes 1) = 0, d_A(v \otimes 1) = 1 \otimes 2\omega - 2w \otimes u,$$

so that $H(A_* \otimes R_*, d_A)$ is the right hand side in the statement of the theorem. With the $\Gamma_{g,1}$-action on $A_* \otimes R_*$ inherited from R_* and trivial on A_*, the map

$$f \otimes \mathrm{id}_{R_*} : (A_* \otimes R_*, d_{AR}) \to (\widetilde{\mathrm{Ch}}_*^{B,\mathbb{Q}} \otimes R_*, d) = (\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty), d) \otimes \mathbb{Q}$$

is a $\Gamma_{g,1}$-equivariant morphism of algebras. It is furthermore a morphism of differential algebras from the easy computations that $d(f(w)) = 0$ and $d(f(v)) = 2\omega \otimes 1 - 2w \otimes u$. To conclude the theorem it suffices to show that $f \otimes \mathrm{id}_{R_*}$ induces an isomorphism on homology.

We filter R_* by the decreasing filtration $F^pR := R_{\geq p}$, and extend this to decreasing filtrations $F^p(A \otimes R) := A \otimes F^pR$ and $F^p(\widetilde{\mathrm{Ch}}_*^B \otimes R) := \widetilde{\mathrm{Ch}}_*^B \otimes F^pR$, compatible with both differentials d_A and d, and preserved by $f \otimes \mathrm{id}_R$. We thus obtain a morphism of the associated spectral sequences which on the first page is precisely

$$f \otimes \mathrm{gr}(\mathrm{id}_R) = \mathrm{gr}(f \otimes \mathrm{id}_R) : \Big(\mathrm{gr}(A \otimes R), \mathrm{gr}(d_A)\Big) \longrightarrow \Big(\mathrm{gr}(\widetilde{\mathrm{Ch}}_*^B \otimes R), \mathrm{gr}(d)\Big). \quad (3.5.14)$$

The induced map $H^*(\mathrm{gr}(f\otimes \mathrm{id}_R))$ is the morphism induced by $f\otimes \mathrm{id}_R$ on 2nd pages of the spectral sequences. The associated graded differentials simplify to $\mathrm{gr}(d_A) = d_A \otimes 1 + 1\otimes d_R$ and $\mathrm{gr}(d) = d_B \otimes 1 + 1\otimes d_R$, so by the Künneth theorem, $H^*(\mathrm{gr}(f\otimes \mathrm{id}_R)) = H^*(f)\otimes H^*(\mathrm{id}_R) = H^*(f)\otimes \mathrm{id}_{H^*(R,d_R)}$. As $H^*(f)$ is an isomorphism by Lemma 184, then $H^*(\mathrm{gr}(f\otimes \mathrm{id}_R))$ is also an isomomorphism, and Theorem 3.5 of McCleary [34] allows us to conclude that $H^*(f\otimes \mathrm{id}_R)$ is an isomorphism, as required. □

3.5.9 The $\mathbb{Z}/2$-homology of $C_\bullet(\Sigma_g)$

The following is the closed analogue of Bianchi's Theorem 3.2 from [2].

Theorem 186. *There is a Γ_g-equivariant isomorphism of bigraded algebras*

$$H^*(C_\bullet(\Sigma_g);\mathbb{Z}/2) \cong H^*(C_\bullet(D^2);\mathbb{Z}/2)\otimes \mathbb{Z}/2[x_i, y_{i,m}, u]/(x_i^2, y_{i,m}^2, u^2),$$

where the indices range in $1 \le i \le 2g$, $m \ge 1$, the bidegrees are $|x_i| = (1,1)$, $|y_{i,m}| = (2m,2m)$ $|u| = (2,1)$, and the Γ_g-action is trivial on u and symplectic on the x_i and $y_{i,m}$ for $m \ge 1$.

Proof. The desired cohomology is computed by the complex $(\widetilde{\mathrm{Ch}}_*(C_\bullet(\mathcal{N})^\infty), d)\otimes \mathbb{Z}/2$. Modulo 2, both the differential summands d_M and d_N, and the cross-contribution ξ vanish, therefore

$$H_*(C_\bullet(\mathcal{N})^\infty;\mathbb{Z}/2) \cong H_*(\widetilde{\mathrm{Ch}}^B_*(\mathcal{N}), d_B)\otimes \mathrm{UMor}_* \otimes \mathbb{Z}/2[u]/(u^2),$$

and the first factor reduces to $H_*(\widetilde{\mathrm{Ch}}^B_*(\mathcal{N}), d_B) = H_*(C_\bullet(D^2)^\infty) \cong H^*(C_\bullet(D^2);\mathbb{Z}/2)$ via Proposition 103. The second factor has trivial $\mathcal{T}_g$-action, and the mod-2 divided power algebra takes the form

$$\Gamma_{\mathbb{Z}}[y_i : 1 \le i \le 2g]\otimes \mathbb{Z}/2 = \mathbb{Z}/2[y_{i,m} : 1 \le i \le 2g, m \ge 1]/(x_i^2, y_{i,m}).$$

□

Chapter 4

Foggy roller coasters

This chapter is adapted from the preprint [4] of the author and Andrea Bianchi, appearing on the Arxiv.

4.1 Introduction

Let $g \geq 2$. We shall denote by $\mathcal{M} = \Sigma_{g,1}$ and $\Gamma_{g,1} = \pi_0(\mathrm{Diff}^+_\partial(\mathcal{M}))$. Recall the Johnson filtration of the mapping class group

$$\Gamma_{g,1} = J(0) \supseteq J(1) \supseteq J(2) \supseteq J(3) \ldots$$

defined in (1.4.1).

In [3], Bianchi–Miller–Wilson proved that $J(n)$ acts trivially on $H_*(F_n(\mathcal{M}); \mathbb{Z})$, and was conjectured that:

Conjecture 187. *The kernel of the $\Gamma_{g,1}$-action on $H_*(F_n(\mathcal{M}); \mathbb{Z})$ is generated by $J(n)$ and the Dehn twist $T_{\partial\mathcal{M}}$ along the boundary of $\mathcal{M}$.*

In support of this conjecture, in this chapter we prove the following theorem.

Theorem 188. *For $g \geq 2$ and $n \geq 1$, the $(n-1)^{st}$ stage of the Johnson filtration $J(n-1)$ acts non-trivially on $H_n(F_n(\mathcal{M}))$.*

We remark that the statement of Theorem 188 is false for $g = 1$ where the subgroup $J(2) \subset \Gamma_{1,1}$ is generated by the Dehn twist $T_{\partial\mathcal{M}}$ which acts trivially on $H_*(F_n(\Sigma_{1,1}))$ for all $n \geq 1$. Furthermore, we note that for $g \geq 2$, in the case $n = 1$ Theorem 188 reduces to the well-known fact that $\Gamma_{g,1}$ acts non-trivially on $H_1(\mathcal{M})$, and the case $n = 2$ is proved by Bianchi [2]. In the body of the chapter, we will assume $n \geq 3$.

4.1.1 Structure of the chapter and strategy of the proof

We prove Theorem 188 by generalising a method used in work by Bianchi, Miller, Wilson and the author [2, 3, 48]. We begin by constructing a triple (ϕ, x, y) of a mapping class $\phi \in J(n-1)$, a homology class $x \in H_n(F_n(\mathcal{M}))$ and a cohomology class $y \in H^n(F_n(\mathcal{M}))$, the latter two representing a closed and proper submanifold of $F_n(\mathcal{M})$, respectively. Our choice of (ϕ, x, y) satisfies

$$\langle \phi_*(x) - x, y \rangle = -1 \neq 0,$$

where $\langle -, - \rangle$ is the Kronecker pairing. This exhibits that $\phi_*(x) \neq x$ and thus that $\phi \in J(n-1)$ acts non-trivially on $H_n(F_n(\mathcal{M}))$ concluding Theorem 188. The triple (ϕ, x, y) is defined in Section 4.3.

In section 4.4, we reduce the Kronecker pairing $\langle \phi_*(x) - x, y \rangle$ to a combinatorial formula capitulated in Proposition 216 in Section 4.4.8. The reduction takes place through a series of excisions: by interpreting the Kronecker Pairing as submanifold-interesection, we zoom in to smaller open subsets of $F_n(\mathcal{M})$ that contains this intersection. We would suggest that a first reading of Section 4.4 should cover the motivational subsection 4.4.1, the figures of the section, and the concluding subsections 4.4.7 and 4.4.8.

Finally, the combinatorial formulas involve a quantity called the *content of a word*, which is similar to one used by the author in [49]. Roughly speaking given two words w, w' in an alphabet, the w-content of w', $C(w, w')$, is the number of words that "look like" w appearing as (not necessarily contiguous) subsequences of the word w', counted with appropriate signs. Finding the right contents to compute and completing the computation is the content of Section 4.5.

The proof is summarised at the end of Section 4.5.1 on page 157.

4.2 Preliminaries

In the entire chapter, unless stated otherwise, homology and cohomology are taken with coefficients in $\mathbb{Z}$; in fact, all arguments work for homology and cohomology over any commutative ring R.

4.2.1 The Johnson filtration

Let $* \in \partial\mathcal{M}$ be a fixed basepoint, and let $\pi = \pi_1(\mathcal{M}, *)$. The lower central series of π is defined by $\pi^{(0)} = \pi$ and $\pi^{(i+1)} = [\pi, \pi^{(i)}]$, for $i \geq 0$: here $[\pi, \pi^{(i)}]$ is the subgroup of π generated by all commutators of an element of π and an element of $\pi^{(i)}$. The *Johnson subgroup* $J(i) \subset \Gamma_{g,1}$ the kernel of the action of $\Gamma_{g,1}$ on $\pi/\pi^{(i)}$, i.e. the subgroup of mapping classes acting trivially on $\pi/\pi^{(i)}$.

For example $J(1)$ is the Torelli group, i.e. the subgroup of $\Gamma_{g,1}$ acting trivially on $\pi/[\pi,\pi] \cong H_1(\mathcal{M})$. A typical example of a mapping class in $J(1)$ is a *bounding pair* (see Figure 4.4): if α, α' are disjoint, non-isotopic, non-separating simple closed curves in $\mathcal{M}$ that together bound a subsurface of $\mathcal{M}$, then the bounding pair $BP_{\alpha,\alpha'}$ is defined as the product $D_\alpha D_{\alpha'}^{-1} \in \Gamma_{g,1}$, where D_α and $D_{\alpha'}$ are the Dehn twists around α and α'. In fact $J(1)$ is generated by bounding pairs and by Dehn twists around separating simple closed curves in $\mathcal{M}$, see Birman, and Powell [5, 43] (also Putman [44, Corollary 1.4]).

In order to obtain elements in $J(n)$ for $n \geq 2$ we will use that the Johnson filtration of $\Gamma_{g,1}$ is a *central filtration*: for all $i, j \geq 0$, the subgroup $[J(i), J(j)] \subset \Gamma_{g,1}$, generated by commutators of an element in $J(i)$ and an element in $J(j)$, is contained in $J(i+j)$, see Johnson [24] (also Church and Putman [9, Proposition 2.9]).

4.2.2 Action of diffeomorphisms and Poincare duality

This section is the ordered configuration analogue of Proposition 103.

We denote by $\mathring{\mathcal{M}} = \mathcal{M} \setminus \partial\mathcal{M}$ the interior of $\mathcal{M}$. The space $F_n(\mathcal{M})$ is a non-compact manifold with corners of dimension $2n$. A configuration $(z_1, \dots, z_n)$ lies in $\partial F_n(\mathring{\mathcal{M}})$ if and only if at least one z_i lies on $\partial\mathcal{M}$. Thus $F_n(\mathring{\mathcal{M}})$ is the interior of $F_n(\mathcal{M})$, and hence the inclusion $F_n(\mathring{\mathcal{M}}) \subset F_n(\mathcal{M})$ is a homotopy equivalence. We will henceforth focus on $F_n(\mathring{\mathcal{M}})$, which is an orientable $2n$-manifold without boundary, equipped with a canonical smooth structure. We fix once and for all an orientation on $\mathcal{M}$, inducing a canonical orientation on $\mathring{\mathcal{M}}^m$ and on each open subspace thereof.

Let $\mathrm{Diff}^+_\partial(\mathcal{M})$ be the topological group of diffeomorphisms of $\mathcal{M}$ fixing $\partial\mathcal{M}$ pointwise: it acts naturally on $F_n(\mathring{\mathcal{M}})$, and hence also on the homology groups $H_*(F_n(\mathring{\mathcal{M}}))$; isotopic diffeomorphisms of $\mathcal{M}$ induce isotopic self maps of $F_n(\mathring{\mathcal{M}}))$, and hence the same map in homology. This gives rise to an action of $\Gamma_{g,1}$ on $H_*(F_n(\mathring{\mathcal{M}}))$. Similarly, there is an action of $\Gamma_{g,1}$ on the cohomology of $F_n(\mathring{\mathcal{M}})$. The Kronecker pairing is balanced with respect to these actions: for all $x \in H_i(F_n(\mathring{\mathcal{M}}))$, $y \in H^i(F_n(\mathring{\mathcal{M}}))$ and $\phi \in \Gamma_{g,1}$ we have the equality

$$\langle \phi_*(x), y\rangle = \left\langle x, \phi^*(y)\right\rangle.$$

We can also consider the diagonal action of $\mathrm{Diff}^+_\partial(\mathcal{M})$ on the cartesian power $\mathcal{M}^n$; this action preserves the following subspaces of $\mathcal{M}^n$:[1]

- $A'_n(\mathcal{M}) = \{(z_1, \dots, z_n) \in \mathcal{M}^n \mid z_i \in \partial\mathcal{M} \text{ for some } i\}$;
- $\Delta_n(\mathcal{M}) = \left\{(z_1, \dots, z_n) \in \mathcal{M}^n \mid z_i = z_j \text{ for some } i \neq j\right\}$.

We have an induced action of the mapping class group $\Gamma_{g,1}$, on the relative homology $H_*(\mathcal{M}^n, \Delta_n(\mathcal{M}) \cup A'_n(\mathcal{M}))$. Poincare duality for the oriented $2n$-manifold $F_n(\mathring{\mathcal{M}})$, together with the identification of the Borel-Moore homology of $F_n(\mathring{\mathcal{M}})$ with the relative homology $H_*(\mathcal{M}^n, \Delta_n(\mathcal{M}) \cup A'_n(\mathcal{M}))$, gives a $\Gamma_{g,1}$-isomorphism

$$H^*(F_n(\mathring{\mathcal{M}})) \cong H_{2n-*}(\mathcal{M}^n, \Delta_n(\mathcal{M}) \cup A'_n(\mathcal{M})).$$

4.2.3 Properties of the Kronecker Pairing

We are going to use also the following standard principles governing the Kronecker pairing.

- Let $N_1 \subset F_n(\mathring{\mathcal{M}})$ be an oriented, compact i-submanifold, and let $N_2 \subset F_n(\mathring{\mathcal{M}})$ be an oriented, proper $(2n-i)$-submanifold (i.e. N_2 is a closed subset of $F_n(\mathring{\mathcal{M}})$). Suppose that N_1 and N_2 intersect transversely, and let $k \in \mathbb{Z}$ be the num-

[1] The notation is taken from Bianchi–Miller–Wilson [3], and inspired by the notation of Moriyama [37].

ber of intersection points, counted with sign.[2] Then $\langle [N_1], [N_2] \rangle$ is equal to k, where $[N_1] \in H_i(F_n(\mathring{\mathcal{M}}))$ is the fundamental homology class of N_1, and $[N_2] \in H^i(F_n(\mathring{\mathcal{M}}))$ is the fundamental Borel-Moore homology class of N_2, regarded as a cohomology class by virtue of Poincare duality.

- Let $\mathscr{U} \subset F_n(\mathring{\mathcal{M}})$ be an open subspace, and let N_1 and N_2 be as above, with $N_1 \subset \mathscr{U}$. Then $N_2 \cap \mathscr{U}$ is a proper submanifold of U, giving a class $[N_2 \cap \mathscr{U}] \in H^i(U)$ which is the restriction of $[N_2] \in H^i(F_n(\mathring{\mathcal{M}}))$. In particular there is an equality $\langle [N_1], [N_2] \rangle_{F_n(\mathring{\mathcal{M}})} = \langle [N_1], [N_2 \cap \mathscr{U}] \rangle_{\mathscr{U}}$.
- Let N_1, N_2 and $\mathscr{U}$ be as above, and let $\mathscr{V} \subset \mathscr{U}$ be a subspace such that $N_2 \cap \mathscr{V} = \emptyset$. Let $\mathrm{Th}(N_2 \cap \mathscr{U}) \in H^i(\mathscr{U}, \mathscr{U} \setminus N_2)$ be the Thom class associated with the properly embedded manifold $N_2 \cap \mathscr{U} \subset \mathscr{U}$, and let $[N_2 \cap \mathscr{U}]_{\mathrm{rel}.\mathscr{V}}$ be its image along the natural map $H^i(\mathscr{U}, \mathscr{U} \setminus N_2) \to H^i(\mathscr{U}, \mathscr{V})$. Then further mapping to $H^i(\mathscr{U})$ sends the previous classes to $[N_2 \cap \mathscr{U}] \in H^i(\mathscr{U})$. In particular there is an equality of Kronecker pairings $\langle [N_1], [N_2 \cap \mathscr{U}] \rangle_{\mathscr{U}} = \langle [N_1]_{\mathrm{rel}.\mathscr{V}}, [N_2 \cap \mathscr{U}]_{\mathrm{rel}.\mathscr{V}} \rangle_{\mathscr{U}\,\mathrm{rel}.\mathscr{V}}$, where $[N_1]_{\mathrm{rel}.\mathscr{V}} \in H_i(\mathscr{U}, \mathscr{V})$ is the homology class represented by N_1.

4.3 The set up

In this section, we define our triple of classes $x \in H_n(F_n(\mathring{\mathcal{M}}))$ and $y \in H^n(F_n(\mathring{\mathcal{M}}))$, and an element $\phi \in J(n-1)$, together with an apparatus of subspaces of $F_n(\mathcal{M})$ which we will use in the next sections. We also check that $\langle x, y \rangle = 0$. Recall from the introduction that we assume $n \geq 3$.

4.3.1 A list of subspaces of $\mathring{\mathcal{M}}$

We first introduce a few subspaces of $\mathring{\mathcal{M}}$, see Figure 4.1. All figures in this and in the next two sections depict surfaces of genus 2, but the arguments are valid for every genus $g \geq 2$. We fix:

- an open annulus $A \subset \mathring{\mathcal{M}}$, supporting a non-trivial first homology class, and $n-2$ oriented simple closed curves $a_1, \ldots, a_{n-2} \subset A$, representing the same generator

[2]The sign of an intersection point is positive if the concatenation of the orientations of N_1 and N_2 coincides with the orientation of $F_n(\mathring{\mathcal{M}})$, and is negative otherwise.

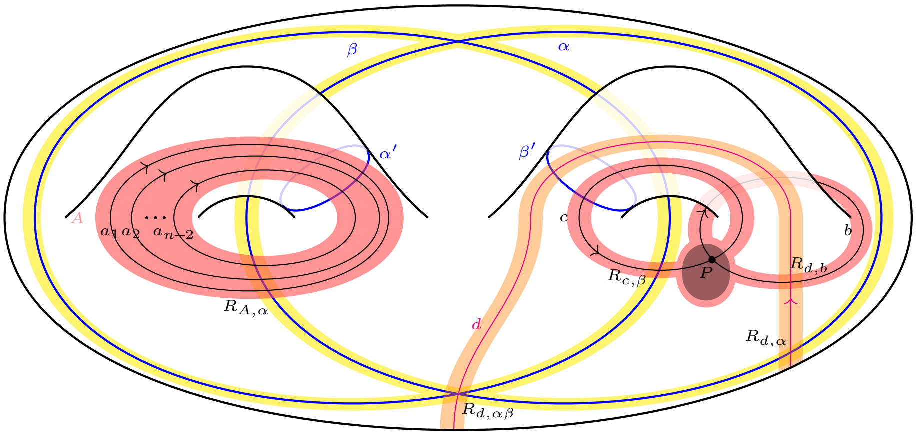


Figure 4.1 The surface $\mathcal{M}$ and some relevant subspaces of it. The regions $U_\alpha, U_\beta, U_b, U_c$ and U_d (shaded but not labelled) are tubular neighbourhoods of their corresponding (labelled) curves. Each region labelled R is a rectangle of intersection. The yellow region is $\mathcal{F}$ and the union of the yellow and red regions is $\mathcal{E}$.

of $H_1(A)$ and occurring in this order on A, from left to right according to the orientation of the curves and of $A \subset \mathring{\mathcal{M}}$;

- a pair of oriented simple closed curves b, c that are disjoint from A and intersect once transversely at a point P;
- a simple, properly embedded oriented arc d, whose closure $\bar{d} \subset \mathcal{M}$ is an embedded arc, transverse to $\partial\mathcal{M}$; d intersects b once, transversely in a point, and is disjoint from the other subspaces mentioned so far;
- a pair of disjoint simple closed curves α, α', bounding a subsurface of $\mathring{\mathcal{M}}$ of genus 1 that contains b and c; α intersects d twice, transversely; α' intersects A in an open segment, and intersects each a_i once, transversely; all other intersections of α and α' with the subspaces mentioned so far are empty;
- a pair of disjoint simple closed curves β, β', bounding a subsurface of $\mathring{\mathcal{M}}$ of genus 1 that contains A and α'; β' is parallel to b, it intersects c and d transversely in a point, and is disjoint from all other subspaces mentioned so far; β intersects c, d and α transversely, respectively in one, one and two points; β is disjoint from all other subspaces mentioned so far.

We assume that d, α and β share one intersection point, and that otherwise no three of the curves and arcs mentioned have a common intersection point. We fix tubular neighbourhoods $U_\alpha, U_\beta, U_b, U_c, U_d$ of α, β, b, c, d, respectively, in $\mathring{\mathcal{M}}$. We assume that each of $U_\alpha \cap A$, $U_\alpha \cap U_b$, $U_\beta \cap U_c$ and $U_\beta \cap U_d$ consists of a single rectangle, that each of $U_\alpha \cap U_\beta$ and $U_d \cap (U_\alpha \cup U_\beta)$ consists of two rectangles, and that any other pair of open sets chosen among $A, U_\alpha, U_\beta, U_b, U_c, U_d$ are disjoint.

We denote by $\mathcal{E} \subset \mathring{\mathcal{M}}$ the open subsurface $A \cup U_\alpha \cup U_\beta \cup U_b \cup U_c$, which is the union of the green and pink areas in Figure 4.1. We denote by $\mathcal{F} \subset \mathcal{E}$ the open subsurface $U_\alpha \cup U_\beta$, which is the green area in Figure 4.1, and by $\bar{\mathcal{F}}$ its closure: the latter is a surface of genus 0 with 4 boundary curves. The *outer* boundary curve of $\mathcal{F}$, i.e. the one that is parallel to $\partial\mathcal{M}$, is denoted $\partial^{\mathrm{out}}\mathcal{F}$.

We further give names to the following rectangles:

- $R_{A,\alpha} = A \cap \bar{U}_\alpha$; this is contained in $\bar{\mathcal{F}}$;
- $R_{c,\beta} = U_c \cap \bar{U}_\beta$; this is contained in $\bar{\mathcal{F}}$;
- the intersection $U_d \cap \bar{\mathcal{F}}$ consists of two rectangles $R_{d,\alpha}$ and $R_{d,\alpha\beta}$, satisfying $R_{d,\alpha\beta} \cap \beta \neq \emptyset = R_{d,\alpha} \cap \beta$;
- $R_{d,b} = U_b \cap \bar{U}_d$.

All these rectangles are homeomorphic to $[0,1] \times (0,1)$, and have thus boundary homeomorphic to $\{0,1\} \times (0,1)$. We denote by $\mathring{R}_{A,\alpha}$ the interior of $R_{A,\alpha}$ and by $\partial\mathring{R}_{A,\alpha}$ its boundary; similarly for the other rectangles. Finally, we fix a small open disc $P \in D_P \subset U_b \cap U_c$, which is the pink-grey area in Figure 4.1.

We use the following convention regarding signs of intersection points of oriented curves, with particular reference to Figure 4.1: let $\gamma_1, \gamma_2 \subset \mathring{\mathcal{M}}$ be two oriented curves or arcs that are transverse to each other and represent, respectively, a first homology and a first cohomology class of a subsurface of $\mathring{\mathcal{M}}$; then the Kronecker pairing $\langle \gamma_1, \gamma_2 \rangle$ is equal to the number of points in $\gamma_1 \cap \gamma_2$, counted with sign, where $Q \in \mathring{\mathcal{M}}$ contributes $+1$ if γ_1 meets γ_2 from right (according to the orientation of $\mathring{\mathcal{M}}$), and -1 otherwise. For instance, b represents a class in $H_1(\mathring{\mathcal{M}})$, and c and d classes in $H^1(\mathring{\mathcal{M}})$; then $\langle b, c \rangle = +1$, as b meets d once, from right; similarly $\langle b, d \rangle = +1$.

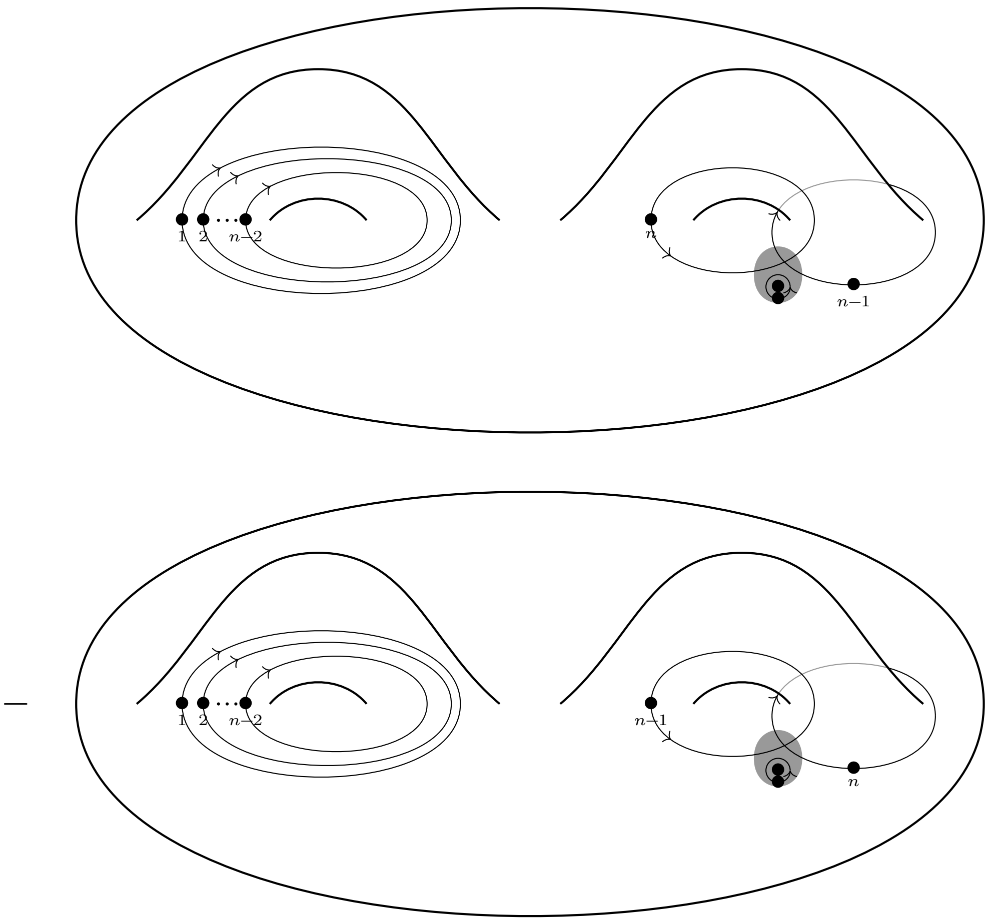


Figure 4.2 The homology class $x \in H_n(F_n(\mathring{\mathcal{M}}))$ is represented by a closed n-dimensional submanifold N_1 obtained by gluing the two drawn n-submanifolds along their equal boundary in the grey region. N_1 is the product of an $(n-2)$-torus $\mathbb{T}^{n-2}$ on the left genus and a $\Sigma_2 = \mathbb{T}^2 \# \mathbb{T}^2$ on the right genus.

4.3.2 Definition of x

We refer to Figure 4.2. The class $x \in H_n(F_n(\mathring{\mathcal{M}}))$ is defined as the image of the fundamental homology class of a certain oriented manifold N_1 along a map $\iota_{N_1}\colon N_1 \to F_n(\mathring{\mathcal{M}})$. To define N_1, we first need to introduce a certain family of configurations of two points, parametrised by a (closed) surface of genus 2.

Definition 189. Let $S^1 \subset \mathbb{C}$ denote the unit circle, and let $\mathcal{I} = \left\{ z \in S^1 \,|\, \Re(z) \geq 1/2 \right\}$. Let $\mathcal{S}_{1,1}$ denote the space $(S^1 \times S^1) \setminus (\mathring{\mathcal{I}} \times \mathring{\mathcal{I}})$, and note that $\mathcal{S}_{1,1}$ is a compact surface of genus 1 with one boundary curve $\partial\mathcal{S}_{1,1}$; $\mathcal{S}_{1,1}$ inherits an orientation from $S^1 \times S^1$. Fix a homeomorphism $\rho\colon S^1 \xrightarrow{\cong} \partial\mathcal{S}_{1,1}$, and let $\mathcal{S}_2$ be the space obtained as quotient of $\mathcal{S}_{1.1} \sqcup S^1 \times [0,1] \sqcup \mathcal{S}_{1,1}$ by identifying along ρ the boundary of the first copy of $\mathcal{S}_{1,1}$ with $S^1 \times \{0\}$, and the boundary of the second copy of $\mathcal{S}_{1,1}$ with $S^1 \times \{1\}$.

Note that $\mathcal{S}_2$ is a topological closed, orientable surface of genus 2; we fix the orientation on $\mathcal{S}_2$ for which the first embedding $\mathcal{S}_{1,1} \hookrightarrow \mathcal{S}_2$ is orientation-preserving, whereas the other is orientation-reversing. Our next aim is to define a map $\iota\colon \mathcal{S}_2 \to \mathring{\mathcal{M}}^2$. Let $I_b \subset b \cap D_P$ and $I_c \subset c \cap D_P$ be two small, closed interval neighbourhoods of P in b and c respectively, and fix homeomorphisms of pairs $\iota_b\colon (S^1, \mathcal{I}) \cong (b, I_b)$ and $\iota_c\colon (S^1, \mathcal{I}) \cong (c, I_c)$. Denote by $\bar{\iota}_c\colon S^1 \to c$ the composition of complex conjugation $S^1 \xrightarrow{\cong} S^1$ and $\iota_c\colon S^1 \xrightarrow{\cong} c$. Consider the maps

$$\iota_b \times \iota_c \;,\; \bar{\iota}_c \times \iota_b \;:\; S^1 \times S^1 \to \mathring{\mathcal{M}}^2,$$

and note that the preimage of the diagonal of $\mathring{\mathcal{M}}^2$ consists, for each of the maps, only of the point $(1,1) \in S^1 \times S^1$. Thus we can define a map $\mathring{\iota}_{\mathcal{S}_2}\colon \mathcal{S}_{1,1} \sqcup \mathcal{S}_{1,1} \to F_2(\mathring{\mathcal{M}})$ by taking the restriction of $\iota_b \times \iota_c$ and the restriction of $\bar{\iota}_c \times \iota_b$, respectively, on the two copies of $\mathcal{S}_{1,1}$.

We then note that both compositions $(\iota_b \times \iota_c) \circ \rho$ and $(\bar{\iota}_c \times \iota_b) \circ \rho$ are maps $S^1 \to \mathring{\mathcal{M}}^2$ with image inside $F_2(D_P)$, and that the two maps

$$(\iota_b \times \iota_c) \circ \rho \;,\; (\bar{\iota}_c \times \iota_b) \circ \rho\colon S^1 \to F_2(D_P),$$

are homotopic. We fix a homotopy $S^1 \times [0,1] \to F_2(D_P)$ between the two maps, and use this homotopy to complete $\mathring{\iota}_{\mathcal{S}_2}$ to a map $\iota_{\mathcal{S}_2}\colon \mathcal{S}_2 \to F_2(\mathring{\mathcal{M}})$. With a little care one

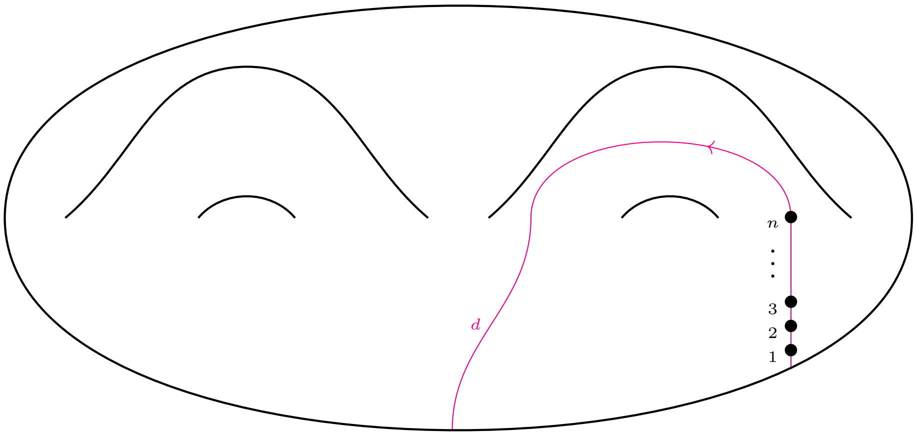


Figure 4.3 The cohomology class $y \in H^n(F_n(\mathring{\mathcal{M}}))$ is represented by the proper embedded n-dimensional submanifold of $F_n(\mathring{\mathcal{M}})$ where the particles $1,\ldots,n$ move along d in increasing order.

can achieve that $\iota_{\mathcal{S}_2}$ is injective, and thus consider $\mathcal{S}_2$ as a subspace of $F_2(\mathring{\mathcal{M}})$; we will however not make use of this property in the following.

By construction ι has image in $F_2(b \cup c \cup D_P)$. We define $N_1 = (S^1)^{n-2} \times \mathcal{S}_2$, and orient it as a product of oriented manifolds. We fix parametrisations $\iota_{a_i}\colon S^1 \xrightarrow{\cong} a_i$ that are compatible with the orientations of the curves a_i, and define

$$\iota_{N_1} := \iota_{a_1} \times \cdots \times \iota_{a_{n-2}} \times \iota_{\mathcal{S}_2}\colon N_1 \to \mathring{\mathcal{M}}^n.$$

We note that the image of ι_{N_1} is contained in $F_n(\mathring{\mathcal{M}})$. Loosely speaking, $\iota(N_1)$ is the subspace of configuration of n ordered particles $(z_1,\ldots,z_n)$ in $\mathring{\mathcal{M}}$, such that for $1 \leq i \leq n-2$ the particle z_i lies on a_i, and the particles z_{n-1} and z_{n-2} assemble into a configuration in the image of $\iota_{\mathcal{S}_2}$. We let $x \in H_n(F_n(\mathring{\mathcal{M}}))$ be the image of the fundamental class of N_1 along ι_{N_1}.

We remark that ι_{N_1} restricts to an injective map on $(S^1)^{n-1} \times (\mathcal{S}_{1,1} \sqcup \mathcal{S}_{1,1})$.

4.3.3 Definition of y

We refer to Figure 4.3. Fix a parametrisation $\iota_d\colon [0,1] \cong \bar{d}$ of the closure in $\mathcal{M}$ of the arc d, compatible with the orientation on d. We denote by $\Delta^n \subset [0,1]^n$ the

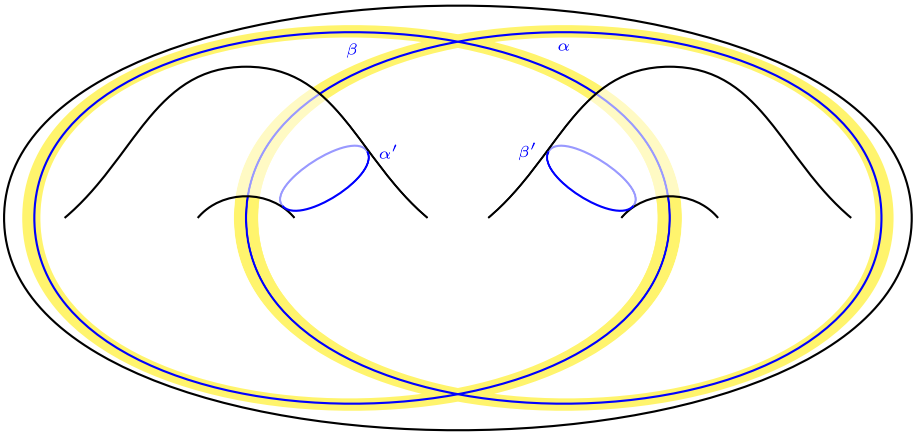


Figure 4.4 The mapping class ϕ, obtained as iterated commutator of the bounding pairs $\phi_\alpha = D_\alpha D_{\alpha'}^{-1}$ and $\phi_\beta = D_\beta D_{\beta'}^{-1}$, can be represented by a diffeomorphism Φ supported on the subsurface $\mathcal{F}$ (the yellow region).

standard simplex, consisting of all $(t_1,\dots,t_n) \in [0,1]^n$ with $t_1 \le \dots \le t_n$. Restricting ι_d^n gives a closed embedding $\iota_{\Delta^n} \colon \Delta^n \to \mathring{\mathcal{M}}^n$; the preimage of $F_n(\mathring{\mathcal{M}})$ is the interior $\mathring{\Delta}^n$ of Δ^n. We define $N_2 = \iota_{\Delta^n}(\mathring{\Delta}^n)$: it is an oriented, proper submanifold of $F_n(\mathring{\mathcal{M}})$ of dimension n, and it contains all configurations of n distinct points $z_1,\dots,z_n \in \mathring{\mathcal{M}}$ such that all z_i lie on d and occur on d in the same order as their indices prescribe. We let $y \in H^n(F_n(\mathring{\mathcal{M}}))$ be the cohomology class represented by $N_2 \subset F_n(\mathring{\mathcal{M}})$:

Lemma 190. *The Kronecker pairing $\langle x,y\rangle$ vanishes.*

Proof. The image of ι_{N_1} is disjoint from N_2 inside $F_n(\mathring{\mathcal{M}})$. $\square$

4.3.4 Definition of ϕ

We refer to Figure 4.4. Consider the bounding pairs $\phi_\alpha := D_\alpha D_{\alpha'}^{-1}$ and $\phi_\beta := D_\beta D_{\beta'}^{-1}$, belonging to $J(1)$. Take the commutator

$$\phi = [\phi_\alpha,[\phi_\alpha,[\dots[\phi_\alpha,\phi_\beta]\dots]]]^{-1}$$

where ϕ_α appears $n-2$ times. By centrality of the Johnson filtration, $\phi \in J(n-1)$, since it is an iterated commutator of $n-1$ elements in $J(1)$. Moreover $D_{\alpha'}$ and $D_{\beta'}$

commute with D_α, with D_β and with each other; hence ϕ can be rewritten as

$$\phi = [D_\alpha, [D_\alpha, [\dots [D_\alpha, D_\beta] \dots]]]^{-1}.$$

Thanks to Lemma 190, we only need to check that the Kronecker pairing $\langle x, \phi^*(y) \rangle$ does not vanish, in order to prove Theorem 188.

Notation 191. We fix a diffeomorphism $\Phi \in \mathrm{Diff}^+_\partial(\mathcal{M})$ representing ϕ and supported on $\mathcal{F}$, i.e. Φ fixes pointwise the complement of $\mathcal{F}$, and denote $e = \Phi^{-1}(d) \subset \mathcal{M}$.

We assume that the arc e is transverse to all a_i, ∂A, c and ∂U_c, that $\bar{A} \cap e$ is a union of closed segments intersecting each a_i and each component of ∂A in a single point, and similarly $\bar{U}_c$ is a union of closed segments intersecting c and each component of ∂U_c in a single point.

4.4 Putting fog

This and the next subsection are dedicated to proving that $\langle x, \phi^*(y) \rangle = 1$. In this section, we make a first step to this computation, by reducing the Kronecker Pairing to the combinatorial formula of Proposition 216.

4.4.1 Motivation

We compute the intersection of (the submanifolds representing)[3] x and $\phi^*(y)$, by performing a series of excisions, i.e. restricting to smaller open submanifolds of $F_n(\mathcal{M})$ which contain the intersection of x and $\phi^*(y)$. We poetically called this process fog, because it was helpful to think of every excised part of $\mathcal{M}$ as fading into fog.

For example, $\phi^*(y)$ is the submanifold where particles $1, \dots, n$ move along the curve $\Phi^{-1}(d)$, which is supported in $\mathcal{F} \cup U_d$, so we can cover its complement by fog. The upshot is that $\mathcal{F} \cup U_d$ is disjoint from the region D_p which supported the surgery defining x, so, as far as $\phi^*(y)$ can see, x is the disjoint union of two submanifolds: this will split $\langle x, \phi^*(y) \rangle$ into a sum of intersections.

[3]For the purpose of this motivation, submanifolds are identified with the classes they represent, and Kronecker pairings with intersections. By a submanifold $X \subset F_n(\mathcal{M})$ *living in* $U \subset \mathcal{M}$, we mean $X \subset F_n(U)$.

A closer look reveals that the intersection $\langle x, \phi^*(y)\rangle$ lives in the region of $F_n(\mathcal{M})$ where particles $1, \ldots n-2$ lie inside $R_{A,\alpha}$, and particles $n-1$ and n inside $R_{c,\beta}$ and $R_{d,b}$ in some order. This recurring division between the first $n-2$ and last 2 particles comes from the definition of x as a product in $F_n(\mathcal{M}) \subset \mathcal{M}^n$. This, on the one hand, allows us to write our intersection as a product of intersections. On the other hand, it allows to put fog everywhere except at these rectangular regions.

In the body of the section, which carefully formalises these steps, it would be instructive to the reader to keep in mind and compare Figures 4.2 and 4.3 with Figures 4.5 and 4.6.

4.4.2 Putting fog away from $\mathcal{F}$

We define certain Φ-invariant subspaces $\mathscr{V} \subset \mathscr{U} \subset F_n(\mathring{\mathcal{M}})$, with $\mathscr{U}$ open. We then replace x and y, respectively, by classes $x' \in H_i(\mathscr{U}, \mathscr{V})$ and $y' \in H^i(\mathscr{U}, \mathscr{V})$. The computation of $\langle x, \phi^*(y)\rangle$ will be reduced to the computation of $\left\langle x', \phi^*(y')\right\rangle_{\mathscr{U}\,\mathrm{rel.}\mathscr{V}}$.

Definition 192. Recall the notation from Subsection 4.3.1. We define $\mathscr{U} \subset F_n(\mathring{\mathcal{M}})$ as the open subspace of configurations $(z_1, \ldots, z_n)$ satisfying the following:

($\mathscr{U}$1) all points z_i lie in $\mathcal{E}$;

($\mathscr{U}$2) the points $z_1, \ldots, z_{n-2}$ lie in the union $\mathcal{F} \cup A$;

($\mathscr{U}$3) at least one among z_{n-1} and z_n lies in the union $\mathcal{F} \cup U_c$;

($\mathscr{U}$4) at least one among z_{n-1} and z_n lies in U_b.

We define $\mathscr{V} \subset \mathscr{U}$ as the subspace of configurations satisfying in addition the following:

($\mathscr{V}$1) at least one point z_i lies in the subspace

$$\mathcal{E} \setminus (\mathcal{F} \cup U_d) = (A \setminus \mathring{R}_{A,\alpha}) \cup (U_c \setminus \mathring{R}_{c,\beta}) \cup (U_b \setminus \mathring{R}_{d,b}).$$

We observe the following facts.

- Recall from Subsection 4.3.4 that we have fixed a diffeomorphism $\Phi\colon \mathcal{M} \to \mathcal{M}$ fixing pointwise the complement of $\mathcal{F}$; the action of Φ on $F_n(\mathring{\mathcal{M}})$ preserves both subspaces $\mathscr{U}$ and $\mathscr{V}$.

- Recall from Subsection 4.3.2 the map $\iota_{N_1}\colon N_1 \to F_n(\mathring{\mathcal{M}})$; then ι_{N_1} takes values inside $\mathscr{U}$. Define $x' \in H_n(\mathscr{U})$ as the image along ι_{N_1} of the fundamental homology class of N_1; then the natural map $H_n(\mathscr{U}) \to H_n(F_n(\mathring{\mathcal{M}}))$ sends $x' \mapsto x$ and, by the previous remark, $(\Phi|_{\mathscr{U}})_*(x') \mapsto \phi_*(x)$.

- Recall from Subsection 4.3.3 the submanifold $N_2 \subset F_n(\mathring{\mathcal{M}})$; then N_2 is disjoint from $\mathscr{V}$. We let $y' \in H^n(\mathscr{U},\mathscr{V})$ be the class $[N_2 \cap \mathscr{U}]_{\mathrm{rel.}\mathscr{V}}$; then the natural map $H^n(\mathscr{U},\mathring{\mathscr{V}}) \to H^n(\mathscr{U})$ sends $y' \mapsto y|_{\mathscr{U}}$ and $(\Phi|_{\mathscr{U}})^*(y') \mapsto (\phi^*(y))|_{\mathscr{U}}$, where $(-)|_{\mathscr{U}}\colon H^n(F_n(\mathring{\mathcal{M}})) \to H^n(\mathscr{U})$ is the natural map.

From now on, by abuse of notation, we will denote the image of x' along the map $H_n(\mathscr{U}) \to H_n(\mathscr{U},\mathscr{V})$ also by $x' \in H_n(\mathscr{U},\mathscr{V})$. By the principles listed in Subsection 4.2.2, we have the equality $\langle x, \phi^*(y)\rangle = \left\langle x', (\Phi|_{\mathscr{U}})^*(y')\right\rangle_{\mathscr{U}\,\mathrm{rel.}\mathscr{V}}$.

Notation 193. We denote by d_α, $d_{\alpha\beta}$ and d_b the closed segments $d \cap R_{d,\alpha}$, $d \cap R_{d,\alpha\beta}$ and $d \cap R_{d,b}$, respectively; we denote by $\mathring{d}_\alpha$, $\mathring{d}_{\alpha\beta}$ and $\mathring{d}_b$ their interiors, respectively.

Consider a configuration $(z_1, \dots, z_n)$ in the intersection $N_2 \cap \mathscr{U}$. By definition of N_2, both z_{n-1} and z_n must lie on d, and should occur on d in their natural order; condition ($\mathscr{U}$4) imposes then that at least one of z_{n-1}, z_n lies on $\mathring{d}_b$, whereas ($\mathscr{U}$3) imposes that at least one of z_{n-1}, z_n lies on $\mathring{d}_\alpha \cup \mathring{d}_{\alpha\beta}$. We conclude that $N_2 \cap \mathscr{U}$ splits as the disjoint union $Q_n \sqcup Q_{n-1}$ of two closed submanifolds of $\mathscr{U}$:

- Q_n contains configurations $(z_1, \dots, z_n) \in N_2 \cap \mathscr{U}$ for which $z_1, \dots, z_{n-1} \in \mathring{d}_\alpha$ and $z_n \in \mathring{d}_b$;

- Q_{n-1} contains configurations $(z_1, \dots, z_n) \in N_2 \cap \mathscr{U}$ for which $z_1, \dots, z_{n-2} \in \mathring{d}_\alpha$, $z_{n-1} \in \mathring{d}_b$ and $z_n \in \mathring{d}_{\alpha,\beta}$.

The notation for Q_{n-1} and Q_n depends on which point lies on $\mathring{d}_b$.

Definition 194. For $i \geq 0$ we denote by $W_i \subset F_i(\mathring{d}_\alpha)$ the subspace of configurations $(z_1, \dots, z_i)$ such that, according to the orientation of $\mathring{d}_\alpha \subset d$, we have $z_1 < \cdots < z_i$.

We have natural homeomorphisms $Q_n \cong W_{n-1} \times \mathring{d}_b$ and $Q_{n-1} \cong W_{n-2} \times \mathring{d}_b \times \mathring{d}_{\alpha\beta}$. The above discussion also shows that $y' \in H^n(\mathscr{U},\mathscr{V})$ can be written as a sum of two cohomology classes, and each of these summands is represented by a submanifold of $\mathscr{U}$ which is disjoint from $\mathscr{V}$ and has a product decomposition.

4.4.3 Excision of $\mathring{\mathscr{V}}$

Our next goal is to use excision in order to replace the pair $(\mathscr{U},\mathscr{V})$ by another pair $(\mathscr{U}',\mathscr{V}')$ of subspaces of $F_n(\mathring{\mathcal{M}})$, such that $(\mathscr{U}',\mathscr{V}')$ splits as a disjoint union of products of pairs of spaces, in such a way that the two summands of y' discussed above take the form of cohomology cross products.

Definition 195. Recall Definition 192. We let $\mathscr{U}' = \mathscr{U} \setminus \mathring{\mathscr{V}}$ and $\mathscr{V}' = \partial\mathscr{V} = \mathscr{V} \setminus \mathring{\mathscr{V}}$.

A configuration $(z_1,\dots,z_n)$ lies in $\mathring{\mathscr{V}} \subset \mathscr{U}$ if it satisfies conditions ($\mathscr{U}$1-4) and the following condition, which is the "open version" of ($\mathscr{V}$1):

($\mathring{\mathscr{V}}$1) at least one point z_i lies in the subspace

$$\mathcal{E} \setminus (\bar{\mathcal{F}} \cup \bar{U}_d) = (A \setminus R_{A,\alpha}) \cup (U_c \setminus R_{c,\beta}) \cup (U_b \setminus R_{d,b}).$$

Notation 196. We denote $\mathcal{F}^R \subset \mathring{\mathcal{M}}$ the union of $\mathcal{F} \cup R_{A,\alpha} \cup R_{c,\beta}$, and by $\partial\mathcal{F}^R = \partial R_{A,\alpha} \cup \partial R_{c,\beta}$ its boundary.

Note that $\mathcal{F}^R$ is obtained from $\mathcal{F}$ by adding four boundary segments.

Lemma 197. *The space $\mathscr{U}' \subset F_n(\mathring{\mathcal{M}})$ contains all configurations $(z_1,\dots,z_n)$ satisfying the following conditions:*

($\mathscr{U}'$1) all points z_i lie in $\mathcal{F}^R \sqcup R_{d,b}$;

($\mathscr{U}'$2) the points $z_1,\dots,z_{n-2}$ lie in $\mathcal{F} \cup R_{A,\alpha} \subset \mathcal{F}^R$

($\mathscr{U}'$3) exactly one of z_{n-1}, z_n lies in $\mathcal{F} \cup R_{c,\beta} \subset \mathcal{F}^R$;

($\mathscr{U}'$4) exactly one of z_{n-1}, z_n lies in $R_{d,b}$.

The subspace $\mathscr{V}' \subset \mathscr{U}'$ can be characterised by the following additional condition:

($\mathscr{V}'$1) at least one among $z_1,\dots,z_{n-2}$ lies on $\partial R_{A,\alpha}$, or at least one of z_{n-1}, z_n lies on $\partial R_{c,\beta} \cup \partial R_{d,b}$.

Proof. Condition ($\mathscr{U}'$1) is equivalent to ($\mathscr{U}$1) together with the negation of ($\mathring{\mathscr{V}}$1), since $\mathcal{E} \setminus \big((A \setminus R_{A,\alpha}) \cup (U_c \setminus R_{c,\beta}) \cup (U_b \setminus R_{d,b})\big) = \mathcal{F}^R \sqcup R_{d,b}$. Similarly, condition ($\mathscr{U}'$2) is equivalent to $(\mathscr{U}2) \wedge \neg(\mathring{\mathscr{V}}1)$. Moreover, the following equivalences hold:

- $\big((\mathscr{U}3)\wedge\neg(\mathring{\mathscr{V}}1)\big)\iff\big(\{z_{n-1},z_n\}\cap(\mathcal{F}\cup R_{c,\beta})\neq\emptyset\big)$;
- $\big((\mathscr{U}4)\wedge\neg(\mathring{\mathscr{V}}1)\big)\iff\big(\{z_{n-1},z_n\}\cap R_{d,b}\neq\emptyset\big)$.

Using that $\mathcal{F}\cup R_{c,\beta}$ and $R_{d,b}$ are disjoint, we have that $(\mathscr{U}'3)\wedge(\mathscr{U}'4)$ is equivalent to $\big((\mathscr{U}3)\wedge\neg(\mathring{\mathscr{V}}1)\big)\wedge\big((\mathscr{U}4)\wedge\neg(\mathring{\mathscr{V}}1)\big)$, which is equivalent to $(\mathscr{U}3)\wedge(\mathscr{U}4)\wedge\neg(\mathring{\mathscr{V}}1)$. Finally, condition $(\mathscr{V}'1)$ is equivalent to $(\mathscr{V}1)\wedge\neg(\mathring{\mathscr{V}}1)$. $\square$

Notation 198. For a finite set S and a space Z we denote by Z^S the space of all functions $z\colon S\to Z$, and by $F_S(Z)\subset Z^S$ the subspace of injective functions. For $i\in S$ and $z\in Z^S$ we usually denote $z_i=z(i)\in Z$; we regard elements of $F_S(Z)$ as S-labeled configurations of distinct points in Z.

We denote $[n]:=\{1,\dots,n\}$ and identify $Z^{[n]}\cong Z^n$ and $F_{[n]}(Z)\cong F_n(Z)$.

We note that the pair $(\mathscr{U}',\mathscr{V}')$ decomposes as a disjoint union of two pairs $(\mathscr{U}'_{n-1},\mathscr{V}'_{n-1})$ and $(\mathscr{U}'_n,\mathscr{V}'_n)$, where for $i=n-1,n$ we let $(\mathscr{U}'_i,\mathscr{V}'_i)$ contain those configurations $(z_1,\dots,z_n)\in(\mathscr{U}',\mathscr{V}')$ for which $z_i\in R_{d,b}$.

Definition 199. For $i=n-1,n$, we let $\mathscr{X}_i\subset F_{[n]\setminus\{i\}}(\mathcal{F}^R)$ be the subspace of configurations satisfying $(\mathscr{U}'2)$ and $z_{2n-1-i}\in\mathcal{F}\cup R_{c,\beta}$ (the latter condition is a specification of $(\mathscr{U}'3)$), and we let $\mathscr{Y}_i\subset\mathscr{X}_i$ be the subspace of configurations satisfying $z_{2n-1-i}\in\partial R_{c,\beta}$ (the latter condition is a specification of $(\mathscr{V}'1)$).

The following is a straightforward corollary of Lemma 197

Corollary 200. *For $i=n-1,n$,*

- *we have a canonical homeomorphism of pairs*

$$(\mathscr{U}'_i,\mathscr{V}'_i)\cong(\mathscr{X}_i,\mathscr{Y}_i)\times(F_{\{i\}}(R_{d,b}),F_{\{i\}}(\partial R_{d,b}))$$

- *the action of the diffeomorphism Φ on $F_n(\mathring{\mathcal{M}})$ preserves each of the subspaces $\mathscr{U}'_i$ and $\mathscr{V}'_i$; the action of Φ on $\mathscr{U}'_i$ is the product of the action on $\mathscr{X}_i$ (as Φ-invariant subspace of $F_{[n]\setminus\{i\}}(\mathring{\mathcal{M}})$) with the identity of $R_{d,b}$.*

Definition 201. For $i=n-1,n$ we let $y'_i\in H^n(\mathscr{U}'_i,\mathscr{V}'_i)$ be the class represented by the oriented submanifold Q_i, which is disjoint from $\mathscr{V}'_i$. See Figure 4.5.

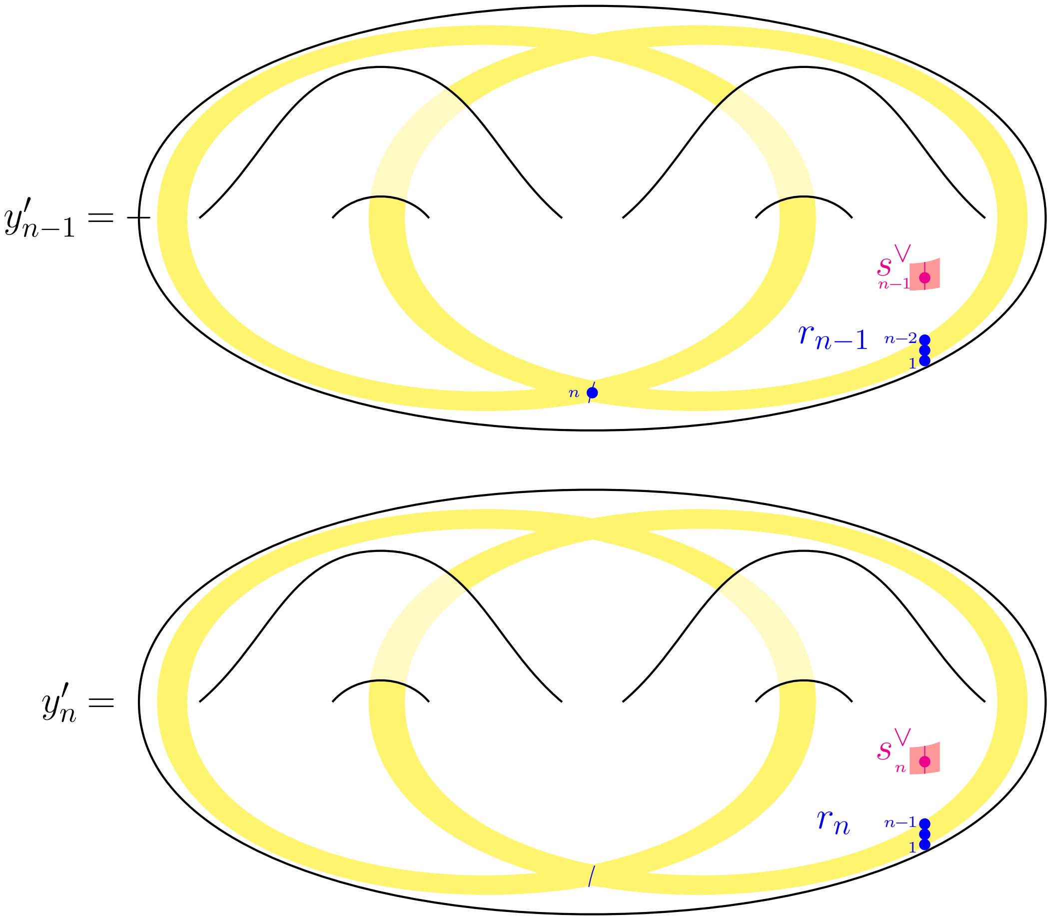


Figure 4.5 The cohomology class y' splits as a sum $y'_{n-1} + y'_n$, and each summand further factors as a cross product so that $y'_{n-1} = -r_{n-1} \times s^\vee$ and $y'_n = r_n \times s^\vee$.

The disjoint union and excision isomorphisms combine as an isomorphism

$$H^n(\mathscr{U}'_n, \mathscr{V}'_n) \oplus H^n(\mathscr{U}'_{n-1}, \mathscr{V}'_{n-1}) \cong H^n(\mathscr{U}', \mathscr{V}') \cong H^n(\mathscr{U}, \mathscr{V}),$$

and the images of y'_n and y'_{n-1} are two cohomology classes in $H^n(\mathscr{U}, \mathscr{V})$ whose sum is y'. Therefore, in order to compute $\langle x, \phi^* y\rangle = \left\langle x', \Phi^*(y')\right\rangle_{\mathscr{U}\,\mathrm{rel.}\,\mathscr{V}}$, we can first compute the homology class corresponding to x' in $H_n(\mathscr{U}'_n, \mathscr{V}'_n) \oplus H_n(\mathscr{U}'_{n-1}, \mathscr{V}'_{n-1})$, and then take the Kronecker pairing of this class with $\Phi^*(y'_n) + \Phi^*(y'_{n-1})$.

4.4.4 Replacing x' by an excised homology class

Definition 202. Let $I = [0,1]$, and let $N'_1 = I^n$. Fix orientation-preserving parametrisations

$$\sigma_{a_1}, \dots, \sigma_{a_{n-2}} \ , \ \sigma_c \ , \ \sigma_b \ : I \to \mathring{\mathcal{M}}$$

of each closed segment $a_1 \cap R_{A,\alpha}, \dots, a_{n-2} \cap R_{A,\alpha}$, $c \cap R_{c,\beta}$ and $b \cap R_{d,b}$, where each of the latter segments inherits an orientation from the curve or arc containing it.

We define a map $\iota_{N'_1} \colon \{n-1, n\} \times N'_1 \to \mathring{\mathcal{M}}^n$ as follows:

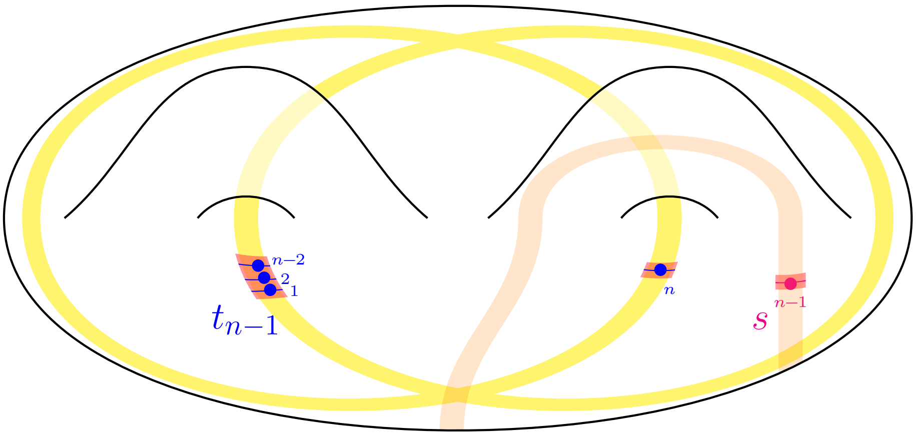
- on $\{n-1\} \times N'_1 \cong N'_1$ we take the cartesian product $\sigma_{a_1} \times \cdots \times \sigma_{a_{n-2}} \times \sigma_b \times \sigma_c$;
- on $\{n\} \times N'_1 \cong N'_1$ we take the cartesian product $\sigma_{a_1} \times \cdots \times \sigma_{a_{n-2}} \times \bar{\sigma}_c \times \sigma_b$, where $\bar{\sigma}_c(t) = \sigma_c(1-t)$.

Note that the restriction of $\iota_{N'_1}$ on $\{i\} \times N'_1$ uses σ_b on the i^{th} coordinate. We observe that $\iota_{N'_1}$ has image inside $\mathscr{U}'$, and sends $\partial N'_1$ inside $\mathscr{V}'$. Moreover $\iota_{N'_1}$ factors as the composition of an orientation-preserving embedding $\varepsilon \colon \{n-1, n\} \times N'_1 \hookrightarrow N_1$ followed by the map $\iota_{N_1} \colon N_1 \to \mathring{\mathcal{M}}^n$, and the difference $N_1 \setminus \varepsilon(\{n-1, n\} \times \mathring{N}'_1)$ is sent inside $\mathscr{V}$ by ι_{N_1}. By excision we obtain the following lemma.

Lemma 203. *The relative fundamental class of $\{n-1, n\} \times (N'_1, \partial N'_1)$ is sent along $\iota_{N'_1}$ to the class $x' \in H_n(\mathscr{U}, \mathscr{V})$, i.e. to the image of the fundamental class of N_1 along $\iota_{N_1} \colon N_1 \to \mathscr{U}$.*

Definition 204. For $i = n-1, n$ we denote by $x'_i \in H_n(\mathscr{U}'_i, \mathscr{V}'_i)$ the image along $\iota_{N'_1}$ of the relative fundamental class of $\{i\} \times (N'_1, \partial N'_1)$. See Figure 4.6.

$x'_{n-1} =$



$x'_n = -$

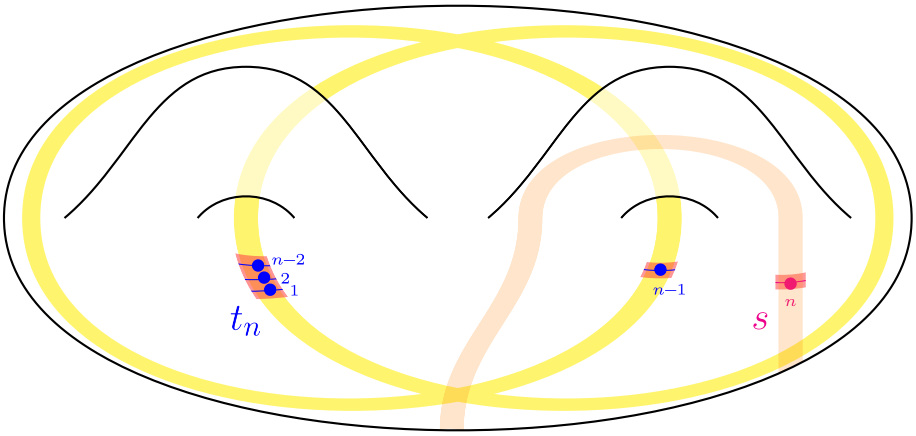


Figure 4.6 The homology class x' splits as a sum $x'_{n-1} + x'_n$, and each summand further factors as a cross product so that $x'_{n-1} = -t_{n-1} \times s$ and $x'_n = t_n \times s$.

By Lemma 203 the sum $x'_{n-1}+x'_n$ corresponds to x' along the composite isomorphism $H_n(\mathscr{U}'_{n-1},\mathscr{V}'_{n-1})\oplus H_n(\mathscr{U}'_n,\mathscr{V}'_n)\cong H_n(\mathscr{U}',\mathscr{V}')\cong H_n(\mathscr{U},\mathscr{V})$. Together with the discussion of the previous subsection, we obtain the following lemma.

Lemma 205. *The Kronecker pairing $\langle x,\phi^* y\rangle$ can be computed as*

$$\left\langle x'_{n-1},\Phi^*(y'_{n-1})\right\rangle_{\mathscr{U}'_{n-1}\mathrm{rel.}\mathscr{V}'_{n-1}}+\left\langle x'_n,\Phi^*(y'_n)\right\rangle_{\mathscr{U}'_n\mathrm{rel.}\mathscr{V}'_n}.$$

4.4.5 Decomposing the classes x'_i as products

We can make a further step. We identify the cube $\{n-1\}\times N'_1$ with the product $I^{[n]\setminus\{n-1\}}\times I^{\{n-1\}}$ and the cube $\{n\}\times N'_1$ with the product $I^{[n-1]}\times I^{\{n\}}$. Note that the first identification is orientation-reversing, as we swap the coordinates relative to the indices $n-1$ and n, whereas the second is orientation-preserving.

The map $\iota_{N'_1}$ restricts to the following product maps on $\{n-1\}\times N'_1$ and $\{n\}\times N'_1$, with images in $\mathscr{U}'_{n-1}$ and $\mathscr{U}'_n$, respectively:

$$(\sigma_{a_1}\times\cdots\times\sigma_{a_{n-2}}\times\sigma_c)\times\sigma_b\colon I^{[n]\setminus\{n-1\}}\times I^{\{n-1\}}\to\mathscr{X}_{n-1}\times F_{\{n-1\}}(R_b,d);$$

$$(\sigma_{a_1}\times\cdots\times\sigma_{a_{n-2}}\times\bar\sigma_c)\times\sigma_b\colon I^{[n-1]}\times I^{\{n\}}\to\mathscr{X}_n\times F_{\{n\}}(R_b,d);$$

Definition 206. For $i=n-1,n$ we denote by $t_i\in H_{n-1}(\mathscr{X}_i,\mathscr{Y}_i)$ the image of the relative fundamental class of $I^{[n]\setminus\{i\}}$ along the product map $\sigma_{a_1}\times\cdots\times\sigma_{a_{n-2}}\times\sigma_c$ (for $i=n$ we replace the last factor by $\bar\sigma_c$). We denote by $s\in H_1(R_{d,b},\partial R_{d,b})$ the image of the relative fundamental class of I along σ_b.

For $i=n-1,n$, up to regarding s as a class in $H_1(F_{\{i\}}(R_{d,b}),F_{\{i\}}(\partial R_{d,b}))$, the previous discussion shows that $x'_i=(-1)^{n-i}t_i\times s$, where the homology cross product is with respect to the product decomposition of the couple $(\mathscr{U}'_i,\mathscr{V}'_i)$ from Corollary 200. The difference in sign comes from the above remark about (non-)compatibility with the orientation of the two identifications $N'_1\cong I^{n-1}\times I$ considered above.

Similarly, for $i=n-1,n$ we have inclusions $Q_i\subset\mathscr{U}'_i$. For $i=n$, consider the product decompositions $Q_n=W_{n-1}\times\mathring{d}_b$ and $\mathscr{U}'_n=\mathscr{X}_n\times F_{\{n\}}(R_{d,b})$; then the inclusion $Q_n\subset\mathscr{U}'_n$ is the product of the inclusions $W_{n-1}\subset\mathscr{X}_n$ and $\mathring{d}_b\subset R_{d,b}\cong F_{\{n\}}(R_{d,b})$. In the same way we have a product inclusion of Q_{n-1} in $\mathscr{U}'_{n-1}$: more precisely, we have a canonical, orientation-reversing homeomorphism $Q_{n-1}\cong(W_{n-2}\times\mathring{d}_{\alpha\beta})\times\mathring{d}_b$ and a canonical, orientation-preserving homeomorphism $\mathscr{U}'_{n-1} \cong \mathscr{X}_{n-1} \times F_{\{n-1\}}(R_{d,b})$, and the inclusion $Q_{n-1} \subset \mathscr{U}'_{n-1}$ is the product of the inclusions $(W_{n-2} \times \mathring{d}_{\alpha\beta}) \subset \mathscr{X}_{n-1}$ and $\mathring{d}_b \subset F_{\{n-1\}}(R_{d,b})$.

Definition 207. We denote by $s^\vee \in H^1(R_{d,b}, \partial R_{d,b})$ the cohomology class represented by the oriented submanifold $\mathring{d}_b$, which is disjoint from $\partial R_{d,b}$. We have $\left\langle s, s^\vee \right\rangle_{R_{d,b}\text{rel.}\partial R_{d,b}} = 1$, using the convention from Subsection 4.3.1.

Similarly, we denote by $r_n \in H^{n-1}(\mathscr{X}_n, \mathscr{Y}_n)$ the class represented by the submanifold W_{n-1}, and by $r_{n-1} \in H^{n-1}(\mathscr{X}_{n-1}, \mathscr{Y}_{n-1})$ the class represented by the submanifold $W_{n-2} \times \mathring{d}_{\alpha\beta}$.

For $i = n-1, n$, up to regarding $s^\vee$ as a class in $H^1(F_{\{i\}}(R_{d,b}), F_{\{i\}}(\partial R_{d,b}))$, the previous discussion shows that $y'_i = (-1)^{n-i} r_i \times s^\vee$, where the cohomology cross product is with respect to the product decomposition of $(\mathscr{U}'_i, \mathscr{V}'_i)$ from Corollary 200. Again, the difference of sign is due to the (non-)compatibility with the orientation in the identifications $Q_n \cong W_{n-1} \times \mathring{d}_b$ and $Q_{n-1} \cong W_{n-2} \times \mathring{d}_{\alpha\beta} \times \mathring{d}_b$. Using the compatibility of Kronecker pairing and cross product, together with the second part of Corollary 200, we get the following proposition, which complements Lemma 205

Proposition 208. *The Kronecker pairing $\langle x, \phi^* y \rangle$ can be computed as*

$$\left\langle t_{n-1}, \Phi^*(r_{n-1}) \right\rangle_{\mathscr{X}_{n-1}\text{rel.}\mathscr{Y}_{n-1}} + \left\langle t_n, \Phi^*(r_n) \right\rangle_{\mathscr{X}_n\text{rel.}\mathscr{Y}_n}.$$

In the following, we will prove that the first summand vanishes, whereas the second summand is -1. We remark that at this point only the spaces $\mathscr{X}_i$ and $\mathscr{Y}_i$ are involved in our computations, for $i = n-1, n$: these are subspaces of $F_{[n]\setminus\{i\}}(\mathcal{F}^R)$. Our problem has been simplified, as we are now considering ordered configurations of $n-1$ points (instead of n) on the surface $\mathcal{F}^R$ from Notation 196 (instead of $\mathring{\mathcal{M}}$).

4.4.6 Putting fog away from $R_{A,\alpha}$ and $R_{c,\beta}$

We apply another excision argument to compute the Kronecker pairings from Proposition 208.

Notation 209. Recall Notation 191. We denote by $e_\alpha, e_{\alpha\beta} \subset \bar{\mathcal{F}}$ the images of d_α and $d_{\alpha\beta}$ along Φ^{-1}, respectively, and by $\mathring{e}_\alpha, \mathring{e}_{\alpha\beta}$ their interiors.

Recall Definitions 194 and 207: the cohomology class $\Phi^*(r_n) \in H^{n-1}(\mathscr{X}_n, \mathscr{Y}_n)$ is represented by the submanifold $\Phi^{-1}(W_{n-1}) \subset \mathscr{X}_n \setminus \mathscr{Y}_n$, containing all configurations $(z_1, \ldots, z_{n-1})$ such that $z_1, \ldots, z_{n-1}$ occur in this order on the open segment $\mathring{e}_\alpha$. Similarly, $\Phi^*(r_{n-1}) \in H^{n-1}(\mathscr{X}_{n-1}, \mathscr{Y}_{n-1})$ is represented by $\Phi^{-1}(W_{n-2} \times F_{\{n\}}(\mathring{d}_{\alpha\beta}))$, containing configurations $(z_1, \ldots, z_{n-2}, z_n)$ such that $z_1, \ldots, z_{n-2}$ occur in this order on $\mathring{e}_\alpha$, and $z_n \in \mathring{e}_{\alpha\beta}$. We next refine the notation from Subsection 4.3.1.

Notation 210. We fix small, disjoint tubular neighbourhoods $U_{a_1}, \ldots, U_{a_{n-2}} \subset A$ of $a_1, \ldots, a_{n-2}$, and denote $R_{a_i,\alpha} = U_{a_i} \cap R_{A,\alpha} \cong [0,1] \times (0,1)$. We let the image of the relative fundamental class of I along σ_{a_i} (respectively, σ_c) be the preferred generator of $H_1(R_{a_i,\alpha}, \partial R_{a_i,\alpha}) \cong \mathbb{Z}$ (respectively, $H_1(R_{c,\beta}, \partial R_{c,\beta}) \cong \mathbb{Z}$).

Note that also that each inclusion $(R_{a_i,\alpha}, \partial R_{a_i,\alpha}) \subset (R_{A,\alpha}, \partial R_{A,\alpha})$ is a homotopy equivalence, inducing *the same* isomorphism $H_1(R_{A,\alpha}, \partial R_{A,\alpha}) \cong \mathbb{Z}$.

Definition 211. For $i = n-1, n$ we define $\mathscr{X}_i' \subset \mathscr{X}_i$ as the open subspace of configurations $(z_1, \ldots, z_{n-2}, z_{2n-1-i})$ such that $z_j \in R_{a_j,\alpha}$ for all $1 \leq j \leq n-2$, and such that $z_{2n-1-i} \in R_{c,\beta}$. We let $\mathscr{Y}_i' = \mathscr{X}_i' \cap \mathscr{Y}_i$.

We have a homeomorphism of couples

$$(\mathscr{X}_i', \mathscr{Y}_i') \cong (R_{a_1,\alpha}, \partial R_{a_1,\alpha}) \times \cdots \times (R_{a_{n-2},\alpha}, \partial R_{a_{n-2},\alpha}) \times (R_{c,\beta}, \partial R_{c,\beta}).$$

Recall that, for $i = n-1, n$, the class $t_i \in H_{n-1}(\mathscr{X}_i, \mathscr{Y}_i)$ is the image of the relative fundamental class of $I^{[n]\setminus\{i\}}$ along $\sigma_{a_1} \times \cdots \times \sigma_{a_{n-2}} \times \sigma_c$ (for $i = n$, the last factor being $\bar{\sigma}_c$); the image of the latter product map is however contained in $\mathscr{X}_i'$, and thus we can define the class $t_i' \in H_{n-1}(\mathscr{X}_i', \mathscr{Y}_i')$ in the same way, so that $t_i' \mapsto t_i$ along the natural map $H_{n-1}(\mathscr{X}_i', \mathscr{Y}_i') \to H_{n-1}(\mathscr{X}_i, \mathscr{Y}_i)$. If we define $r_i' \in H^{n-1}(\mathscr{X}_i', \mathscr{Y}_i')$ as the restriction of $\Phi^*(r_i)$, we have an equality

$$\left\langle t_i, \Phi^*(r_i) \right\rangle_{\mathscr{X}_i \mathrm{rel.} \mathscr{Y}_i} = \left\langle t_i', r_i' \right\rangle_{\mathscr{X}_i' \mathrm{rel.} \mathscr{Y}_i'}.$$

Combining with Proposition 208, we deduce

Proposition 212. *We have the equality*

$$\langle x, \phi^* y \rangle = \langle t_n', r_n' \rangle + \langle t_{n-1}', r_{n-1}' \rangle.$$

The previous discussion shows that t'_i is equal to the homology cross product $\sigma_{a_1}[I,\partial I]\times\cdots\times\sigma_{a_{n-2}}[I,\partial I]\times\sigma_c[I,\partial I]$ (for $i=n$ replace σ_c by $\bar{\sigma}_c$, which accounts on a change of sign); since t'_i splits as a homology cross product, we will seek a similar product decomposition for the cohomology class r'_i, for $i=n-1,n$.

4.4.7 A combinatorial formula

Definition 213. Recall Notation 209. We let $f_1,\dots,f_\mu\subset\mathring{e}_\alpha$ be the sequence of segments in $\mathring{e}_\alpha\cap(R_{A,\alpha}\sqcup R_{c,\beta})$, listed in the order in which they appear along $\mathring{e}_\alpha$; each segment inherits an orientation from $\mathring{e}_\alpha$. Similarly, we let $g_1,\dots,g_\nu\subset\mathring{e}_{\alpha\beta}$ be the sequence of segments in $\mathring{e}_{\alpha\beta}\cap(R_{A,\alpha}\sqcup R_{c,\beta})$.

We define $\theta(f_i)\in\{\mathbf{a},\mathbf{b}\}$ so that $f_i\subset R_{A,\alpha}$ if $\theta(f_i)=\mathbf{a}$, and $f_i\subset R_{c,\beta}$ if $\theta(f_i)=\mathbf{b}$; similarly we define $\theta(g_j)\in\{\mathbf{a},\mathbf{b}\}$.

For all i such that $\theta(f_i)=\mathbf{a}$, we define $\epsilon(f_i)\in\{\pm1\}$ so that f_i represents the cohomology class $\epsilon(f_i)$ in $H^1(R_{A,\alpha},\partial R_{A,\alpha})\cong\mathbb{Z}$, where the last identification is dual to the one from Notation 210. Similarly, for all i such that $\theta(f_i)=\mathbf{b}$, we define $\epsilon(f_i)\in\{\pm1\}$ by considering $H^1(R_{c,\beta},\partial R_{c,\beta})\cong\mathbb{Z}$. And analogously, we define $\epsilon(g_j)\in\{\pm1\}$ for all j.

We first focus on the case $i=n$: the intersection of $\Phi^{-1}(W_{n-1})$ with $\mathscr{X}'_n$ splits as a disjoint union of products of segments: the disjoint union is parametrised by the set of all sequences $(i_1,\dots,i_{n-1})$ of indices $1\le i_j\le\mu$ satisfying the following:

- $i_1\le\cdots\le i_{n-1}$, and each equality $i_j=i_{j+1}$ implies $\epsilon(f_{i_j})=+1$;
- $\theta(f_{i_j})=\mathbf{a}$ for $1\le j\le n-2$;
- $\theta(f_{i_{n-1}})=\mathbf{b}$.

The product of segments corresponding to $(i_1,\dots,i_{n-1})$ is

$$(f_{i_1}\cap R_{a_1,\alpha})\times\cdots\times(f_{i_{n-2}}\cap R_{a_{n-2},\alpha})\times f_{i_{n-1}}.$$

The previous product, seen as a proper $(n-1)$-submanifold of $\mathscr{X}'_n$ which is disjoint from $\mathscr{Y}'_n$, supports a cohomology class whose Kronecker pairing with t'_n is

$$\epsilon(f_{i_1})\cdot\cdots\cdot\epsilon(f_{i_{n-2}})\cdot(-\epsilon(f_{i_{n-1}}))=\pm1\in\mathbb{Z},\tag{4.4.1}$$

where the sign in the last factor is due to the fact that t'_n is described in terms of $\bar{\sigma}_c$ rather than σ_c. Summing over all $(i_1,\dots,i_{n-1})$ yields $\left\langle t'_n, r'_n \right\rangle_{\mathscr{X}'_n \mathrm{rel.} \mathscr{Y}'_n}$.

Let us now consider the case $i = n-1$: the intersection of $\Phi^{-1}(W_{n-2} \times \mathring{d}_{\alpha\beta})$ with $\mathscr{X}'_{n-1}$ splits as a disjoint union of products of segments: the disjoint union is parametrised by the set of all sequences $(i_1,\dots,i_{n-2};l)$ of indices $1 \le i_j \le \mu$ and $1 \le l \le \nu$ satisfying the following:

- $i_1 \le \dots \le i_{n-2}$, and each equality $i_j = i_{j+1}$ implies $\epsilon(f_{i_j}) = +1$;
- $\theta(f_{i_j}) = \mathbf{a}$ for $1 \le j \le n-2$;
- $\theta(g_l) = \mathbf{b}$.

The product of segments corresponding to $(i_1,\dots,i_{n-2};l)$ is

$$(f_{i_1} \cap R_{a_1,\alpha}) \times \dots \times (f_{i_{n-2}} \cap R_{a_{n-2},\alpha}) \times g_l.$$

The previous product, seen as a proper $(n-1)$-submanifold of $\mathscr{X}'_{n-1}$ which is disjoint from $\mathscr{Y}'_{n-1}$, supports a cohomology class whose Kronecker pairing with t'_{n-1} is

$$\epsilon(f_{i_1}) \cdot \dots \cdot \epsilon(f_{i_{n-2}}) \cdot \epsilon(g_l) = \pm 1 \in \mathbb{Z}.$$

Summing over all $(i_1,\dots,i_{n-2};l)$ yields $\left\langle t'_{n-1}, r'_{n-1} \right\rangle_{\mathscr{X}'_{n-1} \mathrm{rel.} \mathscr{Y}'_{n-1}}$. Note that the conditions on $(i_1,\dots,i_{n-2};l)$ split as a list of conditions only on $(i_1,\dots,i_{n-2})$ and one condition only on l; the sum computing $\left\langle t'_{n-1}, r'_{n-1} \right\rangle_{\mathscr{X}'_{n-1} \mathrm{rel.} \mathscr{Y}'_{n-1}}$ is thus equal to the product of the following two sums:

- the sum $\sum_{(i_1,\dots,i_{n-2})} \epsilon(f_{i_1}) \cdot \dots \cdot \epsilon(f_{i_{n-2}})$, for $(i_1,\dots,i_{n-2})$ satisfying the first two conditions in the previous list;
- the sum $\sum_{l\colon \theta(g_l)=\mathbf{b}} \epsilon(g_l)$.

4.4.8 Reinterpretation in terms of contents

It will be helpful for computations to record the above findings in the context of words in free monoids and groups.

Definition 214. We denote by $\mathbb{M}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1} \rangle$ the free monoid in the letters $\mathbf{a}$, $\mathbf{a}^{-1}$, $\mathbf{b}$ and $\mathbf{b}^{-1}$; it contains unreduced words $\underline{w} = (w_1^{\epsilon_1}, w_2^{\epsilon_2}, \dots, w_h^{\epsilon_h})$, where $h \ge 0$, and for all

$1 \leq i \leq h$ we have $w_i \in \{\mathbf{a}, \mathbf{b}\}$ and $\epsilon_i \in \{\pm 1\}$; composition is given by concatenation. We denote by $\mathbb{F}\langle \mathbf{a}, \mathbf{b}\rangle$ the free group generated by $\mathbf{a}, \mathbf{b}$, and by $\iota\colon \mathbb{M}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1}\rangle \to \mathbb{F}\langle \mathbf{a}, \mathbf{b}\rangle$ the surjective monoid homomorphism sending $(w^\epsilon) \mapsto w^\epsilon$.

Definition 215 (Contents)**.** Fix a word $\underline{w} = (w_1^{\epsilon_1}, w_2^{\epsilon_2}, \dots, w_h^{\epsilon_h}) \in \mathbb{M}\langle \mathbf{a}, \mathbf{b}\rangle$. For $k \geq 0$, an $\mathbf{a}^k\mathbf{b}$-*occurence in* $\underline{w}$ is a sequence $\mathfrak{s} = (i_1, \dots, i_{k+1})$ of indices $1 \leq i_j \leq h$ satisfying the following:

- $i_1 \leq \cdots \leq i_{k+1}$ and equality $i_j = i_{j+1}$ implies $\epsilon_{i_j} = +1$;
- $w_{i_j} = \mathbf{a}$ for $1 \leq j \leq k$;
- $w_{i_{k+1}} = \mathbf{b}$.

The *sign of* $\mathfrak{s}$ is the product $\sigma(\mathfrak{s}) = \prod_{1 \leq j \leq k+1} \epsilon_{i_j} = \pm 1$, and the $\mathbf{a}^k\mathbf{b}$-*content of* $\underline{w}$ is

$$\mathfrak{c}(\mathbf{a}^k\mathbf{b}; \underline{w}) := \sum_{\mathbf{a}^k\mathbf{b}\text{-occurences } \mathfrak{s} \text{ in } \underline{w}} \sigma(\mathfrak{s}).$$

Analogously, for $k \geq 0$, an $\mathbf{a}^k$-*occurence in* $\underline{w}$ is a sequence $\mathfrak{s} = (i_1, \dots, i_k)$ satisfying conditions (1) and (2) above. The *sign* $\sigma(\mathfrak{s})$ of an occurrence $\mathfrak{s}$, and the $\mathbf{a}^k$-*content of* $\underline{w}$, denoted $\mathfrak{c}(\mathbf{a}^k, \underline{w})$, are analogously defined.

We observe that $\mathfrak{c}(\mathbf{a}^0, \underline{w}) = 1$ for all words $\underline{w}$, since there exists precisely one occurrence $\mathfrak{s} = ()$ of $\mathbf{a}^0$ in $\underline{w}$, carrying sign 1. In the light of Definition 215, the previous discussion translates into the following proposition.

Proposition 216. *The two summands of Proposition 212 can be computed via the equalities*

$$\langle t'_n, r'_n\rangle = -\mathfrak{c}\Big(\mathbf{a}^{n-2}\mathbf{b}\ ,\ (\theta(f_1)^{\epsilon(f_1)}, \dots, \theta(f_\mu)^{\epsilon(f_\mu)})\Big);$$
$$\langle t'_{n-1}, r'_{n-1}\rangle = \mathfrak{c}\Big(\mathbf{a}^{n-2}\ ,\ (\theta(f_1)^{\epsilon(f_1)}, \dots, \theta(f_\mu)^{\epsilon(f_\mu)})\Big) \cdot \Big(\theta(g_1)^{\epsilon(g_1)}, \dots, \theta(g_\nu)^{\epsilon(g_\nu)}\Big)_{\mathbf{b}}.$$

Here $(-)_{\mathbf{b}} : \mathbb{M}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1}\rangle \to \mathbb{Z}$ *is the composition of* $\iota\colon \mathbb{M}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1}\rangle \to \mathbb{F}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1}\rangle$ *and the group homomorphism* $\mathbb{F}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1}\rangle \to \mathbb{Z}$ *sending* $\mathbf{a} \mapsto 0$ *and* $\mathbf{b} \mapsto 1$.

In the next section we will use the previous results to prove that $\langle t'_n, r'_n\rangle = -1$, whereas $\langle t'_{n-1}, r'_{n-1}\rangle = 0$ because $\Big(\theta(g_1)^{\epsilon(g_1)}, \dots, \theta(g_\nu)^{\epsilon(g_\nu)}\Big)_{\mathbf{b}}$ vanishes.

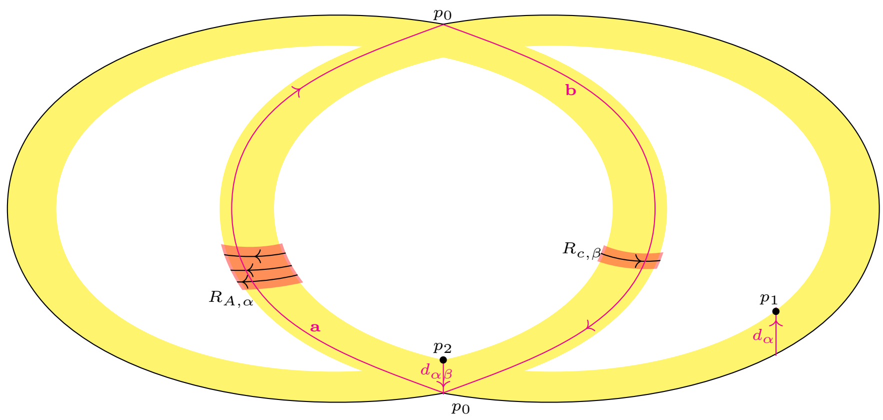


Figure 4.7 The surface $\mathcal{F}_+$.

4.5 Computing contents

Having previously reduced the computation of $\langle x, \phi^*(y)\rangle$ through the equality $\langle x, \phi^*(y)\rangle = \langle t'_{n-1}, r'_{n-1}\rangle + \langle t'_n, r'_n\rangle$, in this section, we will calculate the two summands by computing the contents from Proposition 216. To do so, we compactify the subspace $\mathcal{F}$, pick a basis for its fundamental groupoid, and express all relevant paths in terms of this basis.

Definition 217. Recall Subsection 4.3.1 and Notation 196. We consider the subspace $\mathcal{F}^R \cup (d \cap \bar{\mathcal{F}}) \cup \partial^{\text{out}}\mathcal{F} \subset \mathring{\mathcal{M}}$, and define

$$\mathcal{F}_+ := \left(\mathcal{F}^R \cup (d \cap \bar{\mathcal{F}}) \cup \partial^{\text{out}}\mathcal{F}\right)/\partial^{\text{out}}\mathcal{F}$$

as the quotient of the space $\mathcal{F}^R \cup (d \cap \bar{\mathcal{F}}) \cup \partial^{\text{out}}\mathcal{F}$ by $\partial^{\text{out}}\mathcal{F}$. See Figure 4.7.

We denote by $\Phi_+ : \mathcal{F}_+ \to \mathcal{F}_+$ the homeomorphism induced on the quotient by the restriction of Φ on $\mathcal{F}^R \cup (d \cap \bar{\mathcal{F}}) \cup \partial^{\text{out}}\mathcal{F}$.

We notice that the quotient $\bar{\mathcal{F}}/\partial^{\text{out}}\mathcal{F}$ is a compact surface of genus 0 with 3 boundary curves; the space $\mathcal{F}_+$ is obtained from the latter by removing parts of the boundary, leaving only the following portions:

- $\partial R_{A,\alpha}$ and $\partial R_{c,\beta}$;
- the unique point in $\partial d_\alpha \setminus \partial^{\text{out}}\mathcal{F}$, which we call p_2.

- the unique point in $\partial d_{\alpha\beta} \setminus \partial^{\mathrm{out}}\mathcal{F}$, which we call p_1.

We moreover call $p_0 \in \mathcal{F}_+$ the quotient point of $\partial^{\mathrm{out}}\mathcal{F}$. We consider the fundamental groupoid $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$, i.e. the category whose three objects are p_0,p_1,p_2 and whose morphisms from p_i to p_j are the homotopy classes of paths from p_i to p_j. In particular, all morphisms of this groupoid are generated by the following morphisms (and their inverses):

- the paths $d_\alpha\colon p_0 \to p_1$ and $d_{\alpha\beta}\colon p_2 \to p_0$;
- the loops $\mathbf{a},\mathbf{b}\colon p_0 \to p_0$ from Figure 4.7: $\mathbf{a}$ is contained in the closure of U_α in $\mathcal{F}_+$ and intersects all segments $a_i \cap R_{A,\alpha}$ once, transversely and from right; $\mathbf{b}$ is contained in the closure of U_β in $\mathcal{F}_+$ and intersects the segment $c \cap R_{c,\beta}$ once, transversely and from right.

We compose morphisms in $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$ according to the convention for which the following holds: $\mathbf{a}d_\alpha$ is a defined morphism $p_0 \to p_1$, whereas the composition $d_\alpha\mathbf{a}$ is not defined.

Note that $\pi_1(\mathcal{F}_+,p_0) \cong \mathbb{F}\langle\mathbf{a},\mathbf{b}\rangle$ is free on the generators $\mathbf{a}$ and $\mathbf{b}$. Moreover, the set of morphisms in $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$ from p_0 to p_1 is $\mathbb{F}\langle\mathbf{a},\mathbf{b}\rangle d_\alpha$, and the set of morphisms from p_2 to p_0 is $d_{\alpha\beta}\mathbb{F}\langle\mathbf{a},\mathbf{b}\rangle$.

Since the homeomorphism $\Phi_+\colon \mathcal{F}_+ \to \mathcal{F}_+$ fixes the points p_0,p_1,p_2, it acts on $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$. The images of d_α and $d_{\alpha\beta}$ along Φ_+^{-1} are the classes of the paths $e_\alpha\colon p_0 \to p_1$ and $e_{\alpha\beta}\colon p_2 \to p_0$. By Definition 213 we have equalities of morphisms in $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$

$$e_\alpha = \theta(f_1)^{\epsilon(f_1)}\dots\theta(f_\mu)^{\epsilon(f_\mu)}d_\alpha\colon p_0 \to p_1; \tag{4.5.1}$$

$$e_{\alpha\beta} = d_{\alpha\beta}\theta(g_1)^{\epsilon(g_1)}\dots\theta(g_\nu)^{\epsilon(g_\nu)}\colon p_2 \to p_0. \tag{4.5.2}$$

We remind that, in the previous expression, the words $\theta(f_1)^{\epsilon(f_1)}\dots\theta(f_\mu)^{\epsilon(f_\mu)}$ and $\theta(g_1)^{\epsilon(g_1)}\dots\theta(g_\nu)^{\epsilon(g_\nu)}$, representing elements in $\mathbb{F}\langle\mathbf{a},\mathbf{b}\rangle$, may not be reduced.

In the following we compute e_α and $e_{\alpha\beta}$ as morphisms in $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$, using that Φ^* is isotopic, as a homeomorphism of $\mathcal{F}_+$, to the iterated commutator of Dehn twists $[D_\alpha,[D_\alpha,[\dots[D_\alpha,D_\beta]]]\dots]$, where D_α and D_β are the Dehn twists along the curves α and β in $\mathcal{F}_+$, and D_α is repeated $n-2$ times.

A direct computation gives the following lemma.

Lemma 218. *The action of the Dehn twists D_α, D_β on $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$ is given on the generating morphisms by*

$$\begin{aligned} D_\alpha * \mathbf{a} &= \mathbf{a}, & D_\beta * \mathbf{a} &= \mathbf{b}\mathbf{a}\mathbf{b}^{-1}, \\ D_\alpha * \mathbf{b} &= \mathbf{a}\mathbf{b}\mathbf{a}^{-1}, & D_\beta * \mathbf{b} &= \mathbf{b}, \\ D_\alpha * d_\alpha &= \mathbf{a} d_\alpha, & D_\beta * d_\alpha &= d_\alpha, \\ D_\alpha * d_{\alpha\beta} &= d_{\alpha\beta}\mathbf{a}^{-1}, & D_\beta * d_{\alpha\beta} &= d_{\alpha\beta}\mathbf{b}^{-1}. \end{aligned}$$

In particular D_α and D_β act on $\pi_1(\mathcal{F}_+;p_0) = \langle \mathbf{a}, \mathbf{b}\rangle$ by the conjugations $\underline{w} \mapsto \mathbf{a}\underline{w}\mathbf{a}^{-1}$ and $\underline{w} \mapsto \mathbf{b}\underline{w}\mathbf{b}^{-1}$ respectively.

Proof. Let $\gamma \subset \mathcal{F}_+$ be an immersed arc in $\mathcal{F}_+$ representing a morphism $q \to q'$ in $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$, i.e. $q, q' \in \{p_0,p_1,p_2\}$, and let $\delta \subset \mathcal{F}_+$ be either α or β; assume that γ and δ are transverse. To compute the image of the morphism $\gamma\colon q \to q'$ along D_δ, we orient γ from q to q', and list the intersection points in $\gamma \cap \delta$ as $\{q_1,\dots,q_h\}$, so that, using the orientation of γ, we have $q < q_1 < \dots < q_h < q'$. We then have that $D_\delta * \gamma\colon q \to q'$ is homotopic to the concatenation

$$D_\delta * \gamma \simeq \gamma' := \gamma|_{[q,q_1]}\delta_{q_1}\gamma|_{[q_1,q_2]}\delta_{q_2}\cdot\dots\cdot\delta_{q_h}\gamma|_{[q_h,q']}.$$

Here $\gamma|_{[\hat{q}\tilde{q}]}$ is the segment of γ from $\hat{q}$ to $\tilde{q}$, and δ_{q_i} is the simple loop δ based at q_i, with orientation so that when arriving from γ to δ we turn left.

We draw two *barriers* $b_{\mathbf{a}}$ and $b_{\mathbf{b}}$ that are arcs with endpoints in $\partial\mathcal{F}_+$ so that

(i) $\mathcal{F}_+ - (b_{\mathbf{a}} \cup b_{\mathbf{b}})$ is simply connected;

(ii) $b_{\mathbf{a}}$ intersects $\mathbf{a}$ transversely once, and is disjoint from $\mathbf{b}$;

(iii) $b_{\mathbf{b}}$ intersects $\mathbf{b}$ transversely once, and is disjoint from $\mathbf{a}$.

We first assume $q = q' = p_0$, i.e. that γ is a loop based at p_0, and we further assume that γ is transverse to the barriers and that no intersection point of γ and δ lies on the barriers. As a consequence, also γ' is transverse to the barriers: here, since γ' may not be embedded, we consider it as a parametrised loop $[0,1] \to \mathcal{F}_+$. We may present the morphism $\gamma'\colon p_0 \to p_0$ as a word in $\mathbf{a}^{\pm 1}$ and $\mathbf{b}^{\pm 1}$ with the following recipe. For each $\mathbf{d} = \mathbf{a}, \mathbf{b}$, we say that an intersection time between γ' and $b_{\mathbf{d}}$ is an element $t \in [0,1]$ such that $\gamma'(t) \in b_{\mathbf{d}}$. We say that t is positive if γ' intersects $b_{\mathbf{d}}$ in the same direction as $\mathbf{d}$, and negative otherwise. We record the intersection times of γ' with the barriers, taking them in the order that corresponds to the orientation of γ'; we obtain a sequence $w_1,\dots,w_k \in \{\mathbf{a},\mathbf{b}\}$, capturing the labels of the barrier on which each intersection point lies, and a sequence of signs $\epsilon_1,\dots,\epsilon_k \in \{\pm 1\}$ corresponding to the sign of the intersections. We then have an equality

$$\gamma' = w_1^{\epsilon_1}\dots w_k^{\epsilon_k} \in \pi_1(\mathcal{F}_+;p_0).$$

If $\gamma\colon q \to q'$ is not a loop, then we concatenate $\gamma'\colon q \to q'$ with a morphism $q' \to q$ (typically d_α^{-1} or $d_{\alpha\beta}^{-1}$) that avoids the barriers in order to obtain a loop based at p_0, perform the same reasoning and deconcatenate in the end.

In Figure 4.8, we exhibit by the above method that $D_\beta * \mathbf{a} = \mathbf{bab}^{-1}$. The other cases are analogous. □

Corollary 219. *The morphisms e_α and $e_{\alpha\beta}$ in $\Pi_1(\mathcal{F}_+;p_0,p_1,p_2)$ are given by the following formulas*

$$e_\alpha = [\mathbf{a},\dots[\mathbf{a},\mathbf{b}]\dots]d_\alpha; \qquad e_{\alpha\beta} = d_{\alpha\beta}([\mathbf{a},\dots[\mathbf{a},\mathbf{b}]\dots])^{-1},$$

where $\mathbf{a}$ *is repeated* $n-2$ *times.*

Proof. Let $W(-,\star)$ be a word in the letters $-$ and $\star$, and consider $W(-,\star)$ as a rule to produce an element of a group by plugging into $-$ and $\star$ elements of that group. By Lemma 218, we have the equalities

$$D_\alpha * (W(\mathbf{a},\mathbf{b})d_\alpha) = \mathbf{a}W(\mathbf{a},\mathbf{b})\mathbf{a}^{-1}\mathbf{a}d_\alpha = \mathbf{a}W(\mathbf{a},\mathbf{b})d_\alpha;$$
$$D_\beta * (W(\mathbf{a},\mathbf{b})d_\alpha) = \mathbf{b}W(\mathbf{a},\mathbf{b})\mathbf{b}^{-1}d_\alpha.$$

Applying the above two rules we obtain, for another generic word $W'(-,\star)$, the equality

$$W'(D_\alpha,D_\beta) * (W(\mathbf{a},\mathbf{b})d_\alpha) = W'(\mathbf{a},\mathbf{b})W(\mathbf{a},\mathbf{b})(W'(\mathbf{a},1))^{-1}d_\alpha.$$

If the word W' is a commutator of two other words, $W'(\mathbf{a},1) = 1$ in $\pi_1(\mathcal{F}_+,p_0)$. Now recall that $\Phi_+^{-1} = [D_\alpha,\dots[D_\alpha,D_\beta]\dots]$ with D_α appearing $n-2$ times: we can apply the previous formula with the iterated commutator $W'(-,\star) = [-,[-,\dots[-,\star]\dots]]$, where

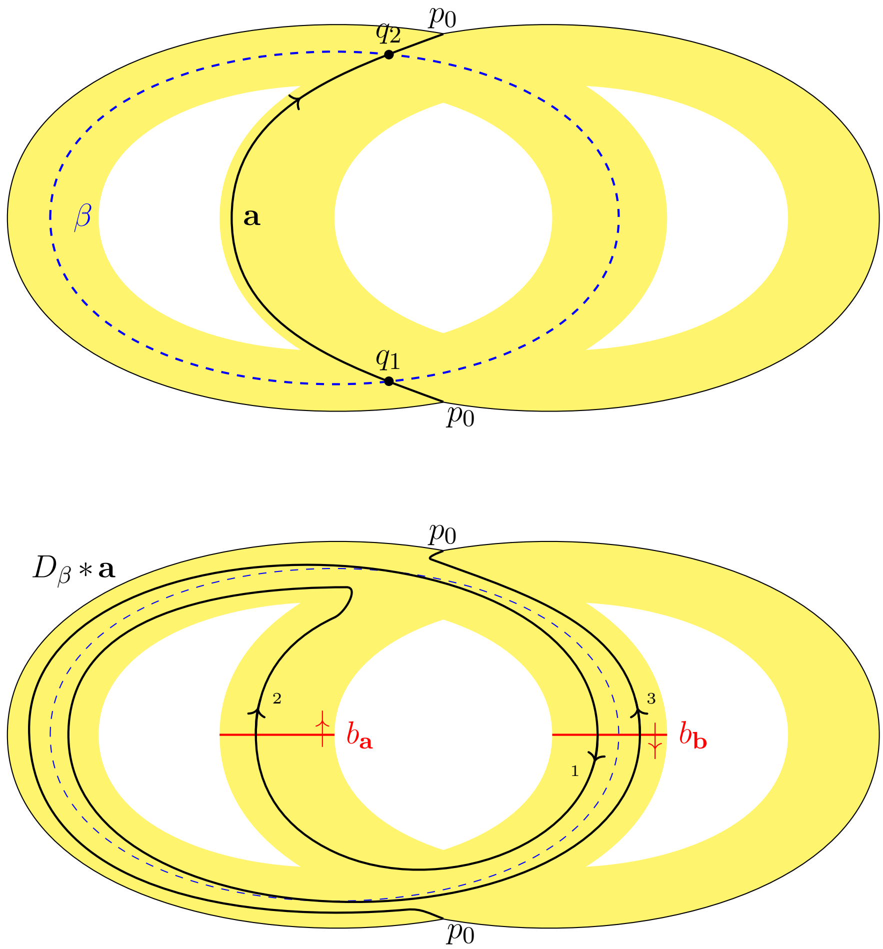


Figure 4.8 Action of the Dehn twist along β (blue curve) on **a** (black curve) . Since the curves intersect twice at q_1 and q_2, the loop $D_\beta * \mathbf{a}$ (thick black curve, below) is homotopic to the surgery of **a** with two copies of β. It intersects the (blue horizontal) barriers three times (labelled 1,2,3), in the order $b_{\mathbf{b}}$, $b_{\mathbf{a}}$, $b_{\mathbf{b}}$ with respective signs $+, +, -$, so $D_\beta * \mathbf{a} = \mathbf{bab}^{-1}$.

$-$ occurs $n-2$ times, together with the trivial word $W(-,\star) = 1$, to obtain

$$e_\alpha = \Phi_+^{-1} * d_\alpha = [D_\alpha, \dots [D_\alpha, D_\beta] \dots] * d_\alpha = [\mathbf{a}, \dots [\mathbf{a}, \mathbf{b}] \dots] d_\alpha,$$

where γ_α occurs $n-2$ times, as desired.

For $d_{\alpha\beta}$ a similar process yields

$$D_\delta * (d_{\alpha\beta} W(\mathbf{a}, \mathbf{b})) = d_{\alpha\beta} \gamma_\delta^{-1} \gamma_\delta W(\mathbf{a}, \mathbf{b}) \gamma_\delta^{-1} = d_{\alpha\beta} W(\mathbf{a}, \mathbf{b}) \gamma_\delta^{-1}$$

for $\delta = \alpha, \beta$, and consequently

$$W'(D_\alpha, D_\beta) * (d_{\alpha\beta} W(\mathbf{a}, \mathbf{b})) = d_{\alpha\beta} W(\gamma_\alpha, \gamma_\beta)(W'(\mathbf{a}, \mathbf{b}))^{-1},$$

from which the desired result again follows. $\square$

Putting together Corollary 219, Equations (4.5.1) and (4.5.2), and Proposition 216, we immediately obtain

Corollary 220. *We have the equalities*

$$\langle t'_n, r'_n \rangle = -\mathfrak{c}\Big(\mathbf{a}^{n-2}\mathbf{b}\ ,\ [\mathbf{a}, \dots [\mathbf{a}, \mathbf{b}] \dots]\Big);$$
$$\langle t'_{n-1}, r'_{n-1} \rangle = \mathfrak{c}\Big(\mathbf{a}^{n-2}\ , [\mathbf{a}, \dots [\mathbf{a}, \mathbf{b}] \dots]\Big) \cdot \Big([\mathbf{a}, \dots [\mathbf{a}, \mathbf{b}] \dots]^{-1}\Big)_{\mathbf{b}},$$

where, in each iterated commutator, $\mathbf{a}$ *appears* $n-2$ *times.*

By virtue of being a commutator, $[\mathbf{a}, \dots [\mathbf{a}, \mathbf{b}] \dots]^{-1}$ lies in the kernel of the homomorphism $(-)_{\mathbf{b}}$, that has codomain the abelian group $\mathbb{Z}$. We immediately deduce

Proposition 221. $\langle t'_{n-1}, r'_{n-1} \rangle = 0$.

4.5.1 The non-vanishing content

It remains to compute

$$\langle t'_n, r'_n \rangle = -\mathfrak{c}\Big(\mathbf{a}^{n-2}\mathbf{b}\ ,\ [\mathbf{a}, \dots [\mathbf{a}, \mathbf{b}] \dots]\Big),$$

where α appears $n-2$ times. To do so effectively, we introduce the following algebra.

Definition 222. The *non-commutative* algebra R is defined by starting with the quotient

$$\mathbb{Z}\langle \mathbf{x}, \mathbf{y} \rangle / (\mathbf{yx}, \mathbf{y}^2)$$

of the free associative algebra $\mathbb{Z}\langle \mathbf{x}, \mathbf{y} \rangle$ by the two-sided ideal generated by $\mathbf{yx}$ and $\mathbf{y}^2$, and by taking the completion with respect to the two-sided ideal generated by $\mathbf{x}$ and $\mathbf{y}$. In other words,

$$R = \{\mathbf{p}(\mathbf{x}) + \mathbf{q}(\mathbf{x})\mathbf{y}\ :\ \mathbf{p}(\mathbf{x}), \mathbf{q}(\mathbf{x}) \text{ are formal power series in } \mathbf{x}\},$$

and multiplication satisfies

$$(\mathbf{p}(\mathbf{x})+\mathbf{q}(\mathbf{x})\mathbf{y})(\mathbf{p}'(\mathbf{x})+\mathbf{q}'(\mathbf{x})\mathbf{y})=\mathbf{p}(\mathbf{x})\mathbf{p}'(\mathbf{x})+\Big(\mathbf{p}(\mathbf{x})\mathbf{q}'(\mathbf{x})+\mathbf{q}(\mathbf{x})\mathbf{p}'(0)\Big)\mathbf{y},$$

where $\mathbf{p}'(0)$ is the constant term of $\mathbf{p}'(\mathbf{x})$.

Write $R^{\geq 1}=\{\mathbf{p}(\mathbf{x})+\mathbf{q}(\mathbf{x})\mathbf{y}\in R\ :\ \mathbf{p}(0)=0\}$.

Lemma 223. *In the multiplicative monoid $1+R^{\geq 1}$, elements of the form*

$$1+\mathbf{x}\mathbf{p}(\mathbf{x}) \text{ and } 1+\mathbf{p}(\mathbf{x})\mathbf{y}$$

have unique two-sided inverses

$$\frac{1}{1+\mathbf{x}\mathbf{p}(\mathbf{x})}=\sum_{k\geq 0}(-\mathbf{x}\mathbf{p}(\mathbf{x}))^k \text{ and } 1-\mathbf{p}(\mathbf{x})\mathbf{y},$$

respectively.

Proof. Two-sided inverses are unique in a monoid as long as they exist, so it suffices to provide some two-sided inverses for $1+\mathbf{x}\mathbf{p}(\mathbf{x})$ and $1+\mathbf{p}(\mathbf{x})\mathbf{y}$. We have

$$(1+\mathbf{p}(\mathbf{x})\mathbf{y})(1-\mathbf{p}(\mathbf{x})\mathbf{y})=1=(1-\mathbf{p}(\mathbf{x})\mathbf{y})(1+\mathbf{p}(\mathbf{x})\mathbf{y}),$$

and for $1+\mathbf{x}\mathbf{p}(\mathbf{x})$ we merely observe that each coefficient of a power of $\mathbf{x}$ in the sum $\sum_{k\geq 0}(-\mathbf{x}\mathbf{p}(\mathbf{x}))^k$ can be computed by a finite sum. $\square$

Definition 224. The total content of $\underline{w}\in\mathbb{M}\langle\mathbf{a}^{\pm 1},\mathbf{b}^{\pm 1}\rangle$ is defined as

$$\mathfrak{C}(\underline{w}):=\sum_{k\geq 0}\mathfrak{c}(\mathbf{a}^k,\underline{w})\mathbf{x}^k+\mathfrak{c}(\mathbf{a}^k\mathbf{b},\underline{w})\mathbf{x}^k\mathbf{y}\in 1+R^{\geq 1}\subset R.$$

It is a straightforward calculation that $\mathfrak{C}(\mathbf{a}^{-1})=1+\mathbf{x}$ and $\mathfrak{C}(\mathbf{b})=1+\mathbf{y}$.

Lemma 225. *The total content is multiplicative, in the sense that, for $\underline{w},\underline{w}'\in\mathbb{M}\langle\mathbf{a}^{\pm 1},\mathbf{b}^{\pm 1}\rangle$, we have*

$$\mathfrak{C}(\underline{ww'})=\mathfrak{C}(\underline{w})\mathfrak{C}(\underline{w}'). \tag{4.5.3}$$

Proof. Write $\underline{w}=(w_1^{\epsilon_1},\dots,w_h^{\epsilon_h})$ and $\underline{w}'=(w_{h+1}^{\epsilon_{h+1}},\dots,w_{h+h'}^{\epsilon_{h+h'}})$. For every $\mathbf{a}^k\mathbf{b}$-occurence $s=(i_1,\dots,i_{k+1})$ in $\underline{ww'}$, there is an index $0\leq l\leq k+1$ s.t. $i_j\leq h$ for $j\leq l$ and

$i_j \geq h+1$ for $j \geq l+1$. If $l = k+1$, then s is an $\mathbf{a}^k\mathbf{b}$-occurence in $\underline{w}$. If instead $j < k+1$, then $s_1 := (i_1, \dots, i_l)$ is an $\mathbf{a}^l$-occurence in $\underline{w}$ and $s_2 := (i_{l+1}, \dots, i_{k+1})$ is an $\mathbf{a}^{k-l}\mathbf{b}$-occurence in $\underline{w}'$. Conversely for any $0 \leq j \leq k+1$, any $\mathbf{a}^l$-occurence s_1 in $\underline{w}$ and any $\mathbf{a}^{k-l}\mathbf{b}$-occurence s_2 in $\underline{w}'$, the concatenation (s_1, s_2) is an $\mathbf{a}^k\mathbf{b}$-occurence in $\underline{ww}'$. We have thus provided a bijection between $\mathbf{a}^k\mathbf{b}$-occurences in $\underline{ww'}$ and the disjoint union of $\mathbf{a}^k\mathbf{b}$-occurences in $\underline{w}$ with the set of triples consisting of an index $0 \leq l \leq k$, an $\mathbf{a}^l$-occurence s_1 in $\underline{w}$ and $\mathbf{a}^{k-l}\mathbf{b}$-occurence s_2 in $\underline{w}'$. Moreover, in the latter case, the sign is multiplicative, i.e. $\sigma(s_1 s_2) = \sigma(s)$. Therefore we have

$$\mathfrak{c}(\mathbf{a}^k\mathbf{b}, \underline{ww'}) = \mathfrak{c}(\mathbf{a}^k\mathbf{b}, \underline{w}) + \sum_{0 \leq l \leq k} \mathfrak{c}(\mathbf{a}^l, \underline{w})\mathfrak{c}(\mathbf{a}^{k-l}\mathbf{b}, \underline{w}'). \tag{4.5.4}$$

A similar argument on $\mathbf{a}^k$-occurences shows that

$$\mathfrak{c}(\mathbf{a}^k, \underline{ww'}) = \sum_{0 \leq l \leq k} \mathfrak{c}(\mathbf{a}^l \underline{w})\mathfrak{c}(\mathbf{a}^{k-l}, \underline{w}'). \tag{4.5.5}$$

Finally, we can compute $\mathfrak{C}(\underline{ww'})$ and $\mathfrak{C}(\underline{w})\mathfrak{C}(\underline{w}')$ using the multiplication rules of R: the equality $\mathfrak{C}(ww') = \mathfrak{C}(w)\mathfrak{C}(w')$ reduces to the equalities (4.5.4) and (4.5.5) for all $k \geq 0$. $\square$

Proposition 226. *The contents* $\mathfrak{c}(\mathbf{a}^k\mathbf{b}, -)$*,* $\mathfrak{c}(\mathbf{a}^k, -)$ *and* $\mathfrak{C}$ *are well-defined as functions from the free group* $\mathbb{F}\langle \mathbf{a}, \mathbf{b} \rangle$*, i.e. they factor along the surjection* $\iota \colon \mathbb{M}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1} \rangle \to \mathbb{F}\langle \mathbf{a}, \mathbf{b} \rangle$*.*

Proof. The length-2 word $(\mathbf{a}, \mathbf{a}^{-1}) \in \mathbb{M}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1} \rangle$ has no $\mathbf{a}^k\mathbf{b}$-occurences for any $k \geq 0$, and it has precisely two $\mathbf{a}^k$-occurences of opposite signs for all $k \geq 1$; therefore $\mathfrak{C}(\mathbf{a}, \mathbf{a}^{-1}) = 1$ which is equal to $\mathfrak{C}()$, the total content of the empty word. Similarly, $\mathfrak{C}(\underline{w}) = 1$ for each of the words $\underline{w} = (\mathbf{a}^{-1}, \mathbf{a}), (\mathbf{b}, \mathbf{b}^{-1})$ and $(\mathbf{b}^{-1}, \mathbf{b})$ in $\mathbb{M}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1} \rangle$. The multiplicativity of $\mathfrak{C}$ implies that cancelling a pair of opposite letters in an unreduced word of $\mathbb{M}\langle \mathbf{a}^{\pm 1}, \mathbf{b}^{\pm 1} \rangle$ does not change the total content; thus $\mathfrak{C}$ is a well defined function on $\mathbb{F}\langle \mathbf{a}, \mathbf{b} \rangle$. By reading the coefficients of $\mathfrak{C}$, the same holds for $\mathfrak{c}(\mathbf{a}^k\mathbf{b}, -)$ and $\mathfrak{c}(\mathbf{a}^k, -)$, for all $k \geq 0$. $\square$

Proposition 226 will be used in the following lemma.

Lemma 227. *Let* $\underline{w} \in \mathbb{F}\langle \mathbf{a}, \mathbf{b} \rangle$*. If* $\mathfrak{C}(\underline{w}) = 1 + \mathbf{p}(\mathbf{x})\mathbf{y}$*, then*

$$\mathfrak{C}([\mathbf{a}, \underline{w}]) = 1 + \frac{\mathbf{x}}{1-\mathbf{x}}\mathbf{p}(\mathbf{x})\mathbf{y}.$$

Proof. The word $\mathbf{a}$ has no $\mathbf{a}^k\mathbf{b}$-occurence and precisely one $\mathbf{a}^k$-occurence for every $k \geq 0$, carrying positive sign; thus $\mathfrak{C}(\mathbf{a}) = \sum_{k\geq 0} \mathbf{x}^k = \frac{1}{1-\mathbf{x}}$. By the multiplicativity of the total content, $\mathfrak{C}(\underline{w}^{-1})$ and $\mathfrak{C}(\mathbf{a}^{-1})$ are two-sided inverses of $\mathfrak{C}(\underline{w})$ and $\mathfrak{C}(\mathbf{a})$ in the monoid $1 + R^{\geq 1}$; thus by Lemma 223 we have $\mathfrak{C}(\mathbf{a}^{-1}) = 1 - \mathbf{x}$ and $\mathfrak{C}(\underline{w}^{-1}) = 1 - \mathbf{p}(\mathbf{x})\mathbf{y}$. We can then compute

$$\begin{aligned}
\mathfrak{C}([\mathbf{a}, \underline{w}]) &= \mathfrak{C}(\mathbf{a}\underline{w}\mathbf{a}^{-1}\underline{w}^{-1}) = \mathfrak{C}(\mathbf{a})\mathfrak{C}(\underline{w})\mathfrak{C}(\mathbf{a}^{-1})\mathfrak{C}(\underline{w}^{-1}) \\
&= \frac{1}{1-\mathbf{x}}(1 + \mathbf{p}(\mathbf{x})\mathbf{y})(1-\mathbf{x})(1 - \mathbf{p}(\mathbf{x})\mathbf{y}) \\
&= \frac{1}{1-\mathbf{x}}(1 + \mathbf{p}(\mathbf{x})\mathbf{y} - \mathbf{x})(1 - \mathbf{p}(\mathbf{x})\mathbf{y}) \\
&= \frac{1}{1-\mathbf{x}}(1 - \mathbf{x} + \mathbf{x}\mathbf{p}(\mathbf{x})\mathbf{y}) \\
&= 1 + \frac{\mathbf{x}}{1-\mathbf{x}}\mathbf{p}(\mathbf{x})\mathbf{y}.
\end{aligned}$$

□

Putting together Proposition 216, Corollary 219, Proposition 226 and Lemma 227, we obtain the following corollary.

Corollary 228. $\langle t'_n, r'_n \rangle = -1$.

Proof. Repeated applications of Lemma 227 show that

$$\mathfrak{C}([\mathbf{a}, [\mathbf{a}, \ldots [\mathbf{a}, \mathbf{b}] \ldots]]) = 1 + \left(\frac{\mathbf{x}}{1-\mathbf{x}}\right)^{n-2}\mathbf{y},$$

where $\mathbf{a}$ appears $n-2$ times in the iterated commutator. Modulo $\mathbf{x}^{n-1}\mathbf{y}$, this expression is equal to $1 + \mathbf{x}^{n-2}\mathbf{y}$, so that $\mathfrak{c}(\mathbf{a}^{n-2}\mathbf{b}, [\alpha, [\alpha, \ldots [\alpha, \beta] \ldots]]) = 1$. By Corollary 219 we have

$$[\mathbf{a}, [\mathbf{a}, \ldots [\mathbf{a}, \mathbf{b}] \ldots]] = \theta(f_1)^{\epsilon(f_1)} \cdot \cdots \cdot \theta(f_\mu)^{\epsilon(f_\mu)} \in \pi_1(\mathcal{F}_+, p_0) \cong \mathbb{F}\langle \mathbf{a}, \mathbf{b} \rangle,$$

and by Proposition 226 we obtain the equality

$$\mathfrak{c}(\mathbf{a}^{n-2}\mathbf{b},[\mathbf{a},[\mathbf{a},\ldots[\mathbf{a},\mathbf{b}]\ldots]]) = \mathfrak{c}(\mathbf{a}^{n-2}\mathbf{b},\theta(f_1)^{\epsilon(f_1)}\cdot\cdots\cdot\theta(f_\mu)^{\epsilon(f_\mu)}).$$

Finally, Proposition 216 gives the equality

$$\langle t'_n, r'_n\rangle = -\mathfrak{c}(\mathbf{a}^{n-2}\mathbf{b},\theta(f_1)^{\epsilon(f_1)}\cdot\cdots\cdot\theta(f_\mu)^{\epsilon(f_\mu)}).$$

□

Proof of Theorem 188. Summarising the previous results, we have

$$\langle \phi_* x - x, y\rangle = \langle \phi_* x, y\rangle - \langle x, y\rangle = \langle x, \phi^* y\rangle - 0 = \langle t'_{n-1}, r'_{n-1}\rangle + \langle t'_n, r'_n\rangle = -1,$$

where the second equality follows from Lemma 190, the third is by Proposition 212, and the fourth is by Proposition 221 and Corollary 228. As a result, we have exhibited an element $x \in H_n(F_n(\mathring{\mathcal{M}}))$ and a $\phi \in J(n-1)$ acting non-trivially on x, as desired. □

4.6 Configuration spaces of closed surfaces

Lastly, we shift our focus to the variant of Theorem 188 for closed surfaces. The closed n^{th} Johnson subgroup $J_g(n) \subset \Gamma_g$ coincides with the image of $J(n)$ under the surjective group homomorphism $\eta : \Gamma_{g,1} \to \Gamma_g$. As discussed in Section 1.2.3, it was proved independently by Looijenga and the author [31, 48] that the Torelli group $J_g(1) \subset \Gamma_g$ acts non-trivially on $H_*(F_3(\Sigma_g))$ for $g \geq 3$. On the other hand, the arguments of Bianchi–Miller–Wilson [3] can be adapted to show that $J_g(n)$ acts trivially on $H_*(F_n(\Sigma_g))$, providing a lower bound to the kernels of the actions. We now provide an upper bound.

Theorem 229. *For $g \geq 2$ and $n \geq 1$, the subgroup $J_{g+1}(n-1) \subset \Gamma_{g+1}$ acts non-trivially on $H_{n+1}(F_{n+1}(\Sigma_{g+1}))$.*

We remark that the statement of Theorem 229 fails for $g+1 = 1$, since we have that $J_1(n-1)$ is the trivial group for $n \geq 2$. Similarly, for $g+1 = 2$, we have the equality $J_2(2) = J_2(1)$, as both groups are generated by separating Dehn twists. Unable to prove non-triviality of the action of the Johnson filtration of Γ_2 on $H_*(F_n(\Sigma_2))$, we propose the following conjecture.

Conjecture 230. *The Torelli group* $J_2(1) \subset \Gamma_2$ *acts trivially on* $H_*(F_n(\Sigma_2))$ *for all* $n \geq 1$*; in other words,* $H_*(F_n(\Sigma_2))$ *are symplectic representations of* Γ_2.

It was communicated to us that Dan Petersen and Orsola Tommasi have a proof of this conjecture for homology with rational coefficients [40].

4.6.1 A list of subspaces of Σ_{g+1}

Let $g \geq 2$ and regard Σ_{g+1} as the closed surface obtained from $\mathcal{M} = \Sigma_{g,1}$ by glueing the surface $\mathcal{T} := \Sigma_{1,1}$ along the boundary. We denote by $\mathring{\mathcal{T}}$ the interior of $\mathcal{T}$, which is an open subset of Σ_{g+1}. Let $l \subset \mathring{\mathcal{T}}$ be an oriented simple closed curve; we fix an oriented, simple closed curve $\hat{d} \subset \Sigma_{g+1}$ intersecting l once, transversely and positively, and such that $\hat{d} \cap \mathcal{M} = d$. See Figure 4.9.

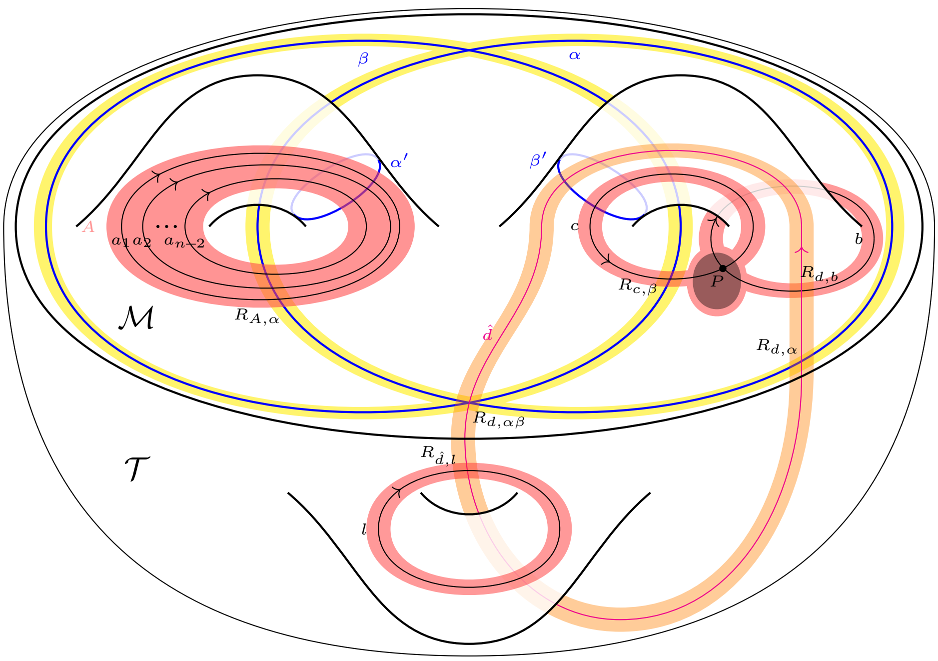


Figure 4.9 The surface Σ_{g+1} and some relevant subspaces of it, analogous to Figure 4.1.

We also fix small tubular neighbourhoods U_l and $U_{\hat{d}}$ of $l, \hat{d} \subset \Sigma_{g+1}$, respectively, and let $R_{d,l}$ be the intersection $U_l \cap \bar{U}_{\hat{d}}$.

The open inclusion $\mathring{\mathcal{M}} \sqcup \mathring{\mathcal{T}} \subset \Sigma_{g+1}$ gives rise to an open inclusion $F_n(\mathring{\mathcal{M}}) \times F_{\{n+1\}}(\mathring{\mathcal{T}}) \subset F_{n+1}(\Sigma_{g+1})$ (see Notation 198). We note that $F_{n+1}(\Sigma_{g+1})$ is an oriented $(2n+2)$-manifold, and thus we can define cohomology classes of $F_{n+1}(\Sigma_{g+1})$ by giving a properly embedded, oriented submanifold.

4.6.2 The homology, cohomology and mapping classes in the closed case

Definition 231. We let $\hat{x} \in H_{n+1}(F_{n+1}(\Sigma_{g+1}))$ be the image of the cross product homology class $x \times [l]$, where $x \in H_n(F_n(\mathcal{M}))$ is the homology class from Subsection 4.3.2, and $[l] \in H_1(F_1(\mathring{\mathcal{T}}))$ is the fundamental homology class of l.

Definition 232. We let $\hat{N}_2 \subset F_{n+1}(\Sigma_{g+1})$ be the subspace of configurations $(z_1, \ldots, z_{n+1})$ such that all z_i lie on $\hat{d}$, and up to cyclic permutations they occur in this order along $\hat{d}$. We note that $\hat{N}_2$ is an oriented $n+1$-submanifold of $F_{n+1}(\Sigma_{g+1})$, as it admits a proper parametrisation by $S^1 \times \mathring{\Delta}^n$. We let $\hat{y} \in H^{n+1}(F_{n+1}(\Sigma_{g+1}))$ be the cohomology class represented by $\hat{N}_2$.

Definition 233. We let $\hat{\Phi}\colon \Sigma_{g+1} \to \Sigma_{g+1}$ be the diffeomorphism extending Φ by the identity of $\mathcal{T}$, and $\hat{\phi} \in \Gamma_{g+1}$ be the corresponding mapping class.

Note that $\hat{\phi} \in J_{g+1}(n)$. We want to prove that $\hat{\phi}$ acts non-trivially on $\hat{x}$, and we will again prove that the Kronecker pairing $\langle \hat{\phi}_*(\hat{x}) - \hat{x}, \hat{y} \rangle$ is non-zero. We notice that $\hat{x}$ is supported on configurations $(z_1, \ldots, z_{n+1})$ such that $z_1 \in A$, and in particular all such configurations do not belong to $\hat{N}_2$: it follows that $\langle \hat{x}, \hat{y} \rangle = 0$.

4.6.3 Putting fog in the closed case

Definition 234. We let $\hat{\mathscr{U}} \subset F_{n+1}(\Sigma_{g+1})$ be the open subspace $F_n(\mathring{\mathcal{M}}) \times F_{\{n+1\}}(U_l)$, i.e. the subspace of configurations $(z_1, \ldots, z_{n+1})$ satisfying $z_1, \ldots, z_n \in \mathring{\mathcal{M}}$ and $z_{n+1} \in U_l$. We let $\hat{\mathscr{V}} \subset \mathscr{U}$ be the closed subspace of configurations satisfying in addition that $z_{n+1} \notin \mathring{R}_{\hat{d},l}$.

Both subspaces $\hat{\mathscr{V}} \subset \hat{\mathscr{U}} \subset F_{n+1}(\Sigma_{g+1})$ are invariant under the action of $\hat{\Phi}$; moreover $\hat{x}$ is the image of a homology class $\hat{x}' = x \times [l] \in H_{n+1}(\mathscr{U})$, so that we can compute $\langle \hat{\phi}_*(\hat{x}), \hat{y} \rangle_{F_{n+1}(\Sigma_{g+1})}$ by restricting $\hat{y}$ to the space $\hat{\mathscr{U}}$, i.e. as $\langle \hat{x}', \hat{y}|_{\hat{\mathscr{U}}} \rangle_{\hat{\mathscr{U}}}$.

We then note that the intersection of $\hat{N}_2$ with $\mathscr{U} = F_n(\mathring{\mathcal{M}}) \times F_{\{n+1\}}(U_l)$ is equal to the product $N_2 \times (\hat{d} \cap U_l)$, and is in particular disjoint from $\hat{\mathscr{V}}$. We can therefore denote by $\hat{y}' \in H^{n+1}(\hat{\mathscr{U}},\hat{\mathscr{V}})$ the class represented by $\hat{N}_2 \cap \hat{\mathscr{U}}$, project $\hat{x}'$ to a relative homology class in $H_{n+1}(\hat{\mathscr{U}},\hat{\mathscr{V}})$, that by abuse of notation we still call $\hat{x}'$, and identify $\langle \hat{x}', \hat{y}|_{\hat{\mathscr{U}}} \rangle_{\hat{\mathscr{U}}} = \langle \hat{x}', \hat{y}' \rangle_{\hat{\mathscr{U}}\,\mathrm{rel.}\hat{\mathscr{V}}}$.

Using the product decomposition of the couple $(\hat{\mathscr{U}},\hat{\mathscr{V}}) = F_n(\mathring{\mathcal{M}}) \times (U_l, U_l \setminus \mathring{R}_{\hat{d},l})$, we can compute $\langle \hat{x}', \hat{y}' \rangle_{\hat{\mathscr{U}}\,\mathrm{rel.}\hat{\mathscr{V}}}$ as $\langle x, y \rangle_{F_n(\mathring{\mathcal{M}})} \cdot \langle [l], \hat{d} \cap U_l \rangle_{U_l \mathrm{rel.} U_l \setminus \mathring{R}_{\hat{d},l}}$. And in the previous product the second factor is equal to 1, whereas the first factor is equal to -1 by the proof of Theorem 188 on page 157. This completes the proof of Theorem 229.

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