Supplementary material of the "Bivariate Simplex Distribution"

Abstract

In this supplementary material, we provide the full simulation results.

Table 1: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.5, 0.5, 2, 2, 1)^{\top}$, Scenario 1.

n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.501	0.501	1.954	1.950	0.850
	Bias	0.001	0.001	-0.046	-0.050	-0.150
	RMSE	0.027	0.027	0.202	0.208	0.284
	Coverage	93.6	93.4	92.0	91.2	98.0
100	Mean	0.500	0.500	1.970	1.975	0.902
	Bias	0.000	-0.000	-0.030	-0.025	-0.098
	RMSE	0.019	0.019	0.138	0.143	0.185
	Coverage	94.9	94.8	94.3	93.2	98.1
150	Mean	0.499	0.500	1.982	1.980	0.925
	Bias	-0.001	-0.000	-0.018	-0.020	-0.075
	RMSE	0.015	0.015	0.116	0.114	0.142
	Coverage	94.8	94.5	93.7	94.7	98.5
200	Mean	0.500	0.500	1.990	1.986	0.929
	Bias	0.000	0.000	-0.010	-0.014	-0.071
	RMSE	0.013	0.013	0.093	0.100	0.133
	Coverage	95.9	93.9	95.3	94.0	97.6
1000	Mean	0.501	0.500	1.995	1.996	0.971
	Bias	0.001	0.000	-0.005	-0.004	-0.029
	RMSE	0.006	0.006	0.046	0.044	0.053
	Coverage	95.2	95.1	93.3	94.5	98.3
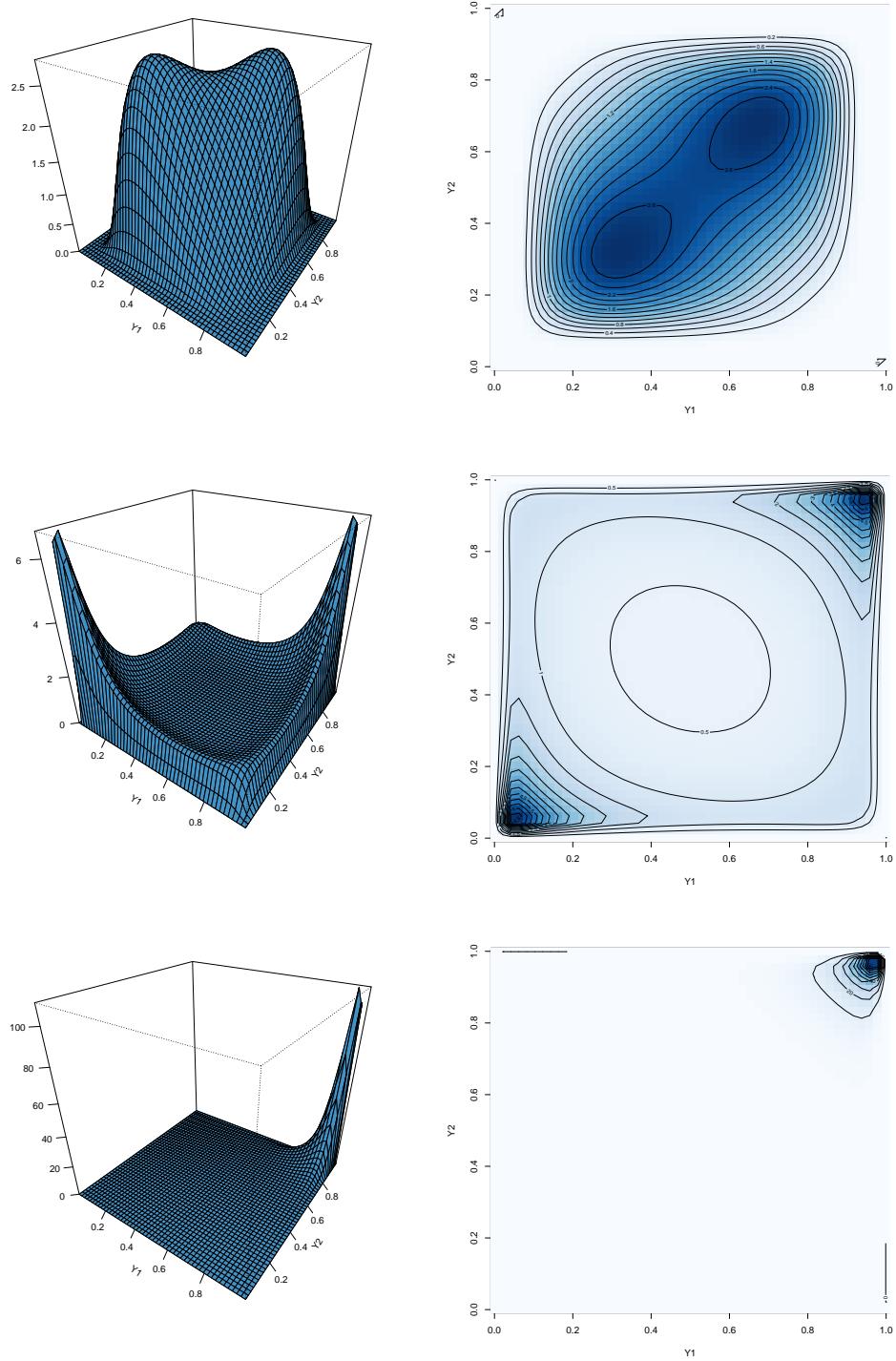
Figure 1: Surface and contour graphs for $\boldsymbol{\theta_1} = (0.5, 0.5, 2, 2, 1)$, $\boldsymbol{\theta_2} = (0.5, 0.5, 5, 5, 1)$ and $\boldsymbol{\theta_3} = (0.9, 0.9, \sqrt{11}, \sqrt{11}, 1)$, scenario 1.

Table 2: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.5, 0.5, 5, 5, 1)^\top$, Scenario 1.

n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.502	0.500	4.884	4.926	0.864
	Bias	0.002	−0.000	−0.116	−0.074	−0.136
	RMSE	0.037	0.036	0.530	0.504	0.261
	Coverage	93.8	95.5	85.9	88.1	98.5
100	Mean	0.500	0.500	4.950	4.970	0.907
	Bias	0.000	0.000	−0.050	−0.030	−0.093
	RMSE	0.026	0.025	0.353	0.351	0.173
	Coverage	94.2	95.5	86.8	87.2	99.0
150	Mean	0.499	0.499	4.959	4.970	0.924
	Bias	−0.001	−0.001	−0.041	−0.030	−0.076
	RMSE	0.021	0.021	0.286	0.292	0.139
	Coverage	94.6	95.1	87.7	84.8	98.6
200	Mean	0.500	0.499	4.971	4.976	0.932
	Bias	−0.000	−0.001	−0.029	−0.024	−0.068
	RMSE	0.019	0.019	0.254	0.246	0.125
	Coverage	94.1	94.4	85.4	85.9	98.3
1000	Mean	0.500	0.500	4.990	4.995	0.971
	Bias	−0.000	−0.000	−0.010	−0.005	−0.029
	RMSE	0.008	0.008	0.113	0.110	0.052
	Coverage	95.6	95.2	86.7	88.6	98.8

Table 3: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.9, 0.9, \sqrt{(11)}, \sqrt{(11)}, 1)^\top$, Scenario 1.

n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.901	0.900	3.248	3.247	0.858
	Bias	0.001	0.000	−0.069	−0.070	−0.142
	RMSE	0.011	0.011	0.335	0.342	0.274
	Coverage	93.4	93.0	93.5	91.7	98.3
100	Mean	0.901	0.900	3.278	3.283	0.896
	Bias	0.001	0.000	−0.038	−0.034	−0.104
	RMSE	0.008	0.008	0.237	0.235	0.189
	Coverage	93.7	95.1	92.1	94.4	98.5
150	Mean	0.900	0.900	3.291	3.290	0.925
	Bias	0.000	0.000	−0.026	−0.027	−0.075
	RMSE	0.006	0.006	0.192	0.185	0.141
	Coverage	94.6	93.9	93.7	93.8	98.2
200	Mean	0.900	0.900	3.292	3.290	0.937
	Bias	0.000	−0.000	−0.025	−0.027	−0.063
	RMSE	0.005	0.006	0.168	0.161	0.121
	Coverage	94.5	95.4	92.8	95.2	98.4
1000	Mean	0.900	0.900	3.310	3.306	0.972
	Bias	0.000	0.000	−0.007	−0.010	−0.028
	RMSE	0.002	0.002	0.072	0.072	0.051
	Coverage	95.4	96.2	95.2	94.6	98.5
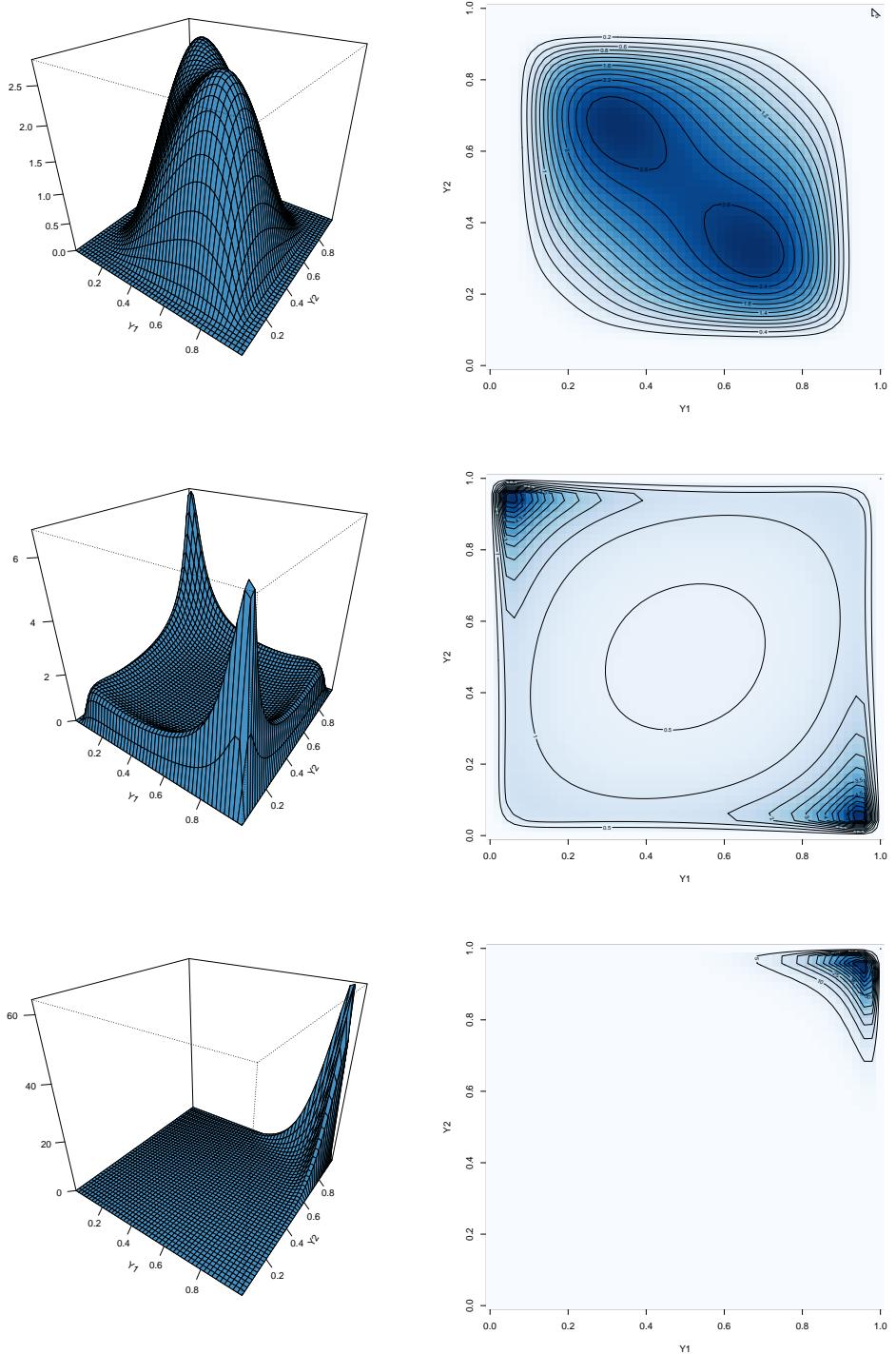
Figure 2: Surface and contour graphs for $\boldsymbol{\theta_1} = (0.5, 0.5, 2, 2, -1)$, $\boldsymbol{\theta_2} = (0.5, 0.5, 5, 5, -1)$ and $\boldsymbol{\theta_3} = (0.9, 0.9, \sqrt{11}, \sqrt{11}, -1)$, scenario 2.

Table 4: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.5, 0.5, 2, 2, -1)^\top$, scenario 2.

n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.500	0.501	1.956	1.964	−0.855
	Bias	0.000	0.001	−0.044	−0.036	0.145
	RMSE	0.027	0.026	0.193	0.198	0.272
	Coverage	94.6	94.5	92.8	93.0	98.4
100	Mean	0.500	0.499	1.985	1.973	−0.910
	Bias	−0.000	−0.001	−0.015	−0.027	0.090
	RMSE	0.019	0.018	0.138	0.141	0.171
	Coverage	94.2	94.6	95.1	93.6	98.6
150	Mean	0.500	0.500	1.984	1.987	−0.924
	Bias	−0.000	−0.000	−0.016	−0.013	0.076
	RMSE	0.016	0.015	0.115	0.115	0.137
	Coverage	95.1	94.9	93.9	93.4	98.8
200	Mean	0.500	0.500	1.986	1.984	−0.940
	Bias	−0.000	0.000	−0.014	−0.016	0.060
	RMSE	0.012	0.013	0.102	0.099	0.112
	Coverage	96.1	96.1	93.5	94.1	99.0
1000	Mean	0.500	0.500	1.994	1.997	−0.975
	Bias	0.000	−0.000	−0.006	−0.003	0.025
	RMSE	0.006	0.006	0.046	0.046	0.047
	Coverage	93.7	94.1	94.0	93.8	99.3

Table 5: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.5, 0.5, 5, 5, -1)^\top$, scenario 2.

n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.500	0.499	4.870	4.888	−0.843
	Bias	0.000	−0.001	−0.130	−0.112	0.157
	RMSE	0.039	0.037	0.523	0.521	0.285
	Coverage	93.3	94.1	86.8	86.0	97.6
100	Mean	0.501	0.499	4.957	4.936	−0.903
	Bias	0.001	−0.001	−0.043	−0.064	0.097
	RMSE	0.027	0.025	0.343	0.355	0.186
	Coverage	94.2	95.4	88.1	85.6	97.8
150	Mean	0.499	0.500	4.968	4.944	−0.922
	Bias	−0.001	0.000	−0.032	−0.056	0.078
	RMSE	0.022	0.021	0.297	0.297	0.146
	Coverage	93.6	94.8	85.4	83.8	98.4
200	Mean	0.500	0.499	4.969	4.961	−0.935
	Bias	0.000	−0.001	−0.031	−0.039	0.065
	RMSE	0.019	0.018	0.252	0.255	0.118
	Coverage	93.7	95.1	86.8	84.2	98.9
1000	Mean	0.500	0.500	4.990	4.982	−0.970
	Bias	0.000	0.000	−0.010	−0.018	0.030
	RMSE	0.008	0.008	0.111	0.112	0.053
	Coverage	94.1	94.1	88.9	87.5	98.4

Table 6: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.9, 0.9, \sqrt{(11)}, \sqrt{(11)}, -1)^\top$, scenario 2.

n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.900	0.901	3.238	3.252	-0.867
	Bias	0.000	0.001	-0.079	-0.064	0.133
	RMSE	0.011	0.011	0.344	0.336	0.251
	Coverage	93.9	91.3	91.3	92.0	98.6
100	Mean	0.900	0.900	3.269	3.275	-0.907
	Bias	0.000	0.000	-0.047	-0.041	0.093
	RMSE	0.008	0.008	0.234	0.238	0.173
	Coverage	93.4	94.3	93.6	93.2	98.5
150	Mean	0.900	0.900	3.298	3.278	-0.922
	Bias	0.000	0.000	-0.019	-0.039	0.078
	RMSE	0.006	0.006	0.188	0.196	0.143
	Coverage	93.5	94.1	95.0	91.6	98.7
200	Mean	0.900	0.900	3.303	3.304	-0.933
	Bias	0.000	0.000	-0.014	-0.013	0.067
	RMSE	0.006	0.005	0.164	0.163	0.121
	Coverage	94.2	94.1	94.5	94.7	98.9
1000	Mean	0.900	0.900	3.309	3.310	-0.971
	Bias	0.000	0.000	-0.008	-0.006	0.029
	RMSE	0.003	0.002	0.074	0.074	0.053
	Coverage	94.0	94.8	94.5	95.4	97.8

Table 7: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.5, 0.5, 2, 2, 0)^\top$, scenario 3.

n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.501	0.500	1.963	1.984	-0.017
	Bias	0.001	0.000	-0.037	-0.016	-0.017
	RMSE	0.026	0.027	0.197	0.204	0.439
	Coverage	94.3	94.0	92.9	93.1	93.0
100	Mean	0.499	0.500	1.978	1.984	0.001
	Bias	-0.001	0.000	-0.022	-0.016	0.001
	RMSE	0.019	0.019	0.143	0.137	0.303
	Coverage	94.9	95.6	93.8	94.7	93.2
150	Mean	0.499	0.500	1.991	1.994	0.001
	Bias	-0.001	0.000	-0.009	-0.006	0.001
	RMSE	0.015	0.015	0.114	0.115	0.244
	Coverage	95.3	95.4	93.5	95.8	93.0
200	Mean	0.500	0.500	1.991	1.987	-0.009
	Bias	0.000	0.000	-0.009	-0.013	-0.009
	RMSE	0.014	0.013	0.098	0.097	0.222
	Coverage	95.0	94.7	94.2	94.9	93.1
1000	Mean	0.500	0.500	1.997	1.999	-0.006
	Bias	0.000	0.000	-0.003	-0.001	-0.006
	RMSE	0.006	0.006	0.044	0.044	0.098
	Coverage	94.8	95.6	95.2	94.6	93.6
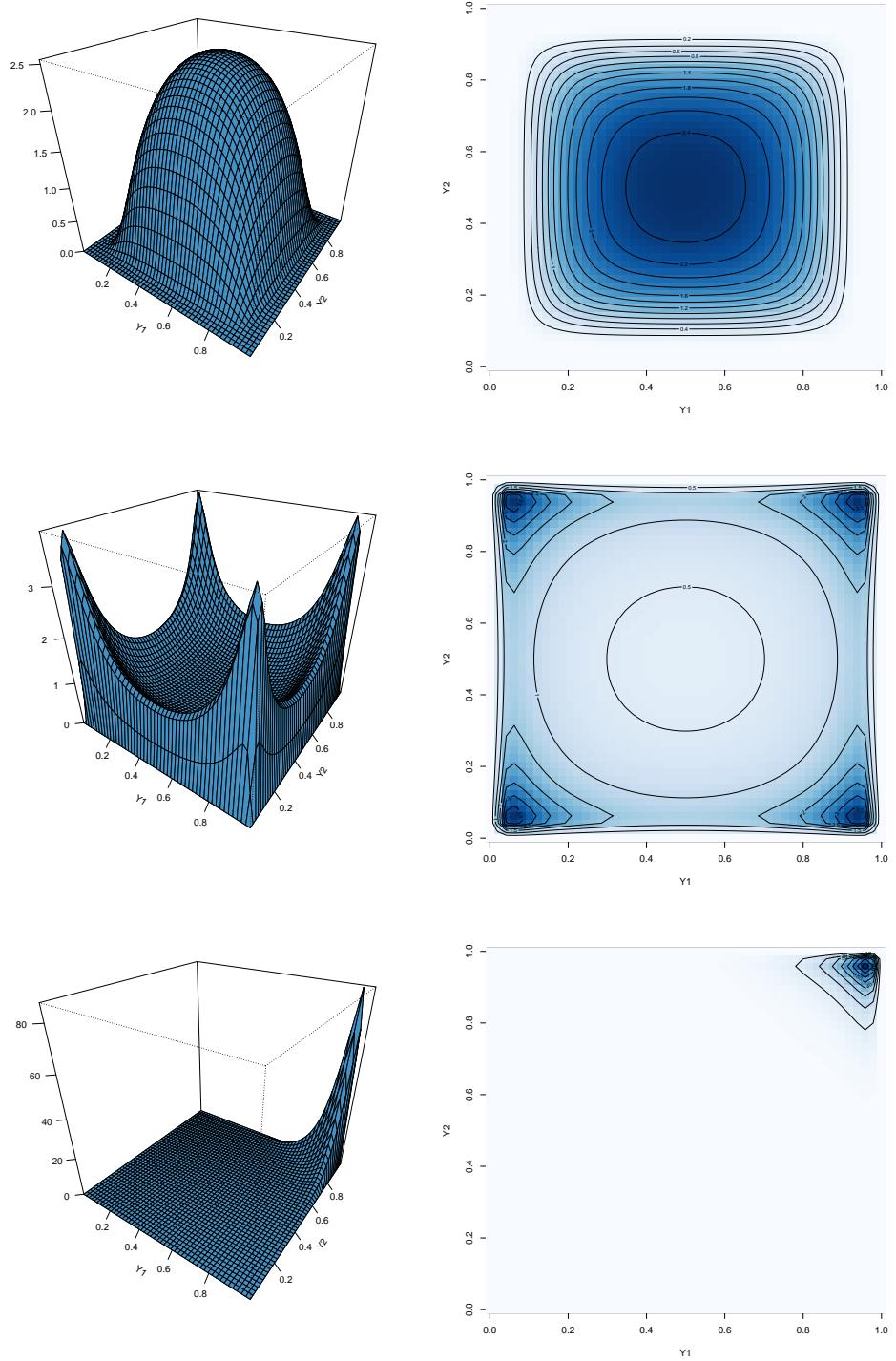
Figure 3: Surface and contour graphs for $\boldsymbol{\theta_1} = (0.5, 0.5, 2, 2, 0)$, $\boldsymbol{\theta_2} = (0.5, 0.5, 5, 5, 0)$ and $\boldsymbol{\theta_3} = (0.9, 0.9, \sqrt{11}, \sqrt{11}, 0)$, scenario 3.

Table 8: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.5, 0.5, 5, 5, 0)^{\top}$, scenario 3.

n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.501	0.500	4.923	4.936	0.018
	Bias	0.001	0.000	-0.077	-0.064	0.018
	RMSE	0.039	0.039	0.537	0.478	0.428
	Coverage	93.3	92.5	88.7	91.9	93.2
100	Mean	0.500	0.501	4.983	4.953	-0.005
	Bias	0.000	0.001	-0.017	-0.047	-0.005
	RMSE	0.026	0.027	0.358	0.355	0.302
	Coverage	94.8	93.9	91.7	91.5	94.4
150	Mean	0.501	0.499	4.974	4.978	-0.004
	Bias	0.001	-0.001	-0.026	-0.022	-0.004
	RMSE	0.023	0.022	0.295	0.298	0.247
	Coverage	94.0	93.7	91.0	91.7	94.2
200	Mean	0.500	0.500	4.986	4.986	-0.004
	Bias	0.000	0.000	-0.014	-0.014	-0.004
	RMSE	0.019	0.018	0.255	0.240	0.210
	Coverage	94.9	95.5	93.1	93.9	94.9
1000	Mean	0.500	0.500	4.995	4.997	-0.006
	Bias	0.000	0.000	-0.005	-0.003	-0.006
	RMSE	0.008	0.008	0.112	0.113	0.093
	Coverage	95.8	94.1	94.5	93.8	94.9

Table 9: Mean, Bias, RMSE and Coverage of 95% confidence intervals (%) for $\boldsymbol{\theta} = (0.9, 0.9, \sqrt{11}, \sqrt{11}, 0)^{\top}$, scenario 3.

| n	Medida	μ_1	μ_2	σ_1^2	σ_2^2	λ
50	Mean	0.901	0.900	3.270	3.274	-0.008
	Bias	0.001	0.000	-0.046	-0.043	-0.008
	RMSE	0.011	0.012	0.330	0.348	0.423
	Coverage	93.5	93.1	94.1	91.6	92.7
100	Mean	0.900	0.900	3.297	3.290	-0.011
	Bias	0.000	0.000	-0.020	-0.027	-0.011
	RMSE	0.008	0.008	0.237	0.240	0.298
	Coverage	93.7	93.9	93.2	93.6	94.2
150	Mean	0.900	0.900	3.311	3.300	-0.016
	Bias	0.000	0.000	-0.006	-0.016	-0.016
	RMSE	0.007	0.006	0.192	0.186	0.253
	Coverage	94.6	94.7	94.2	95.2	93.0
200	Mean	0.900	0.900	3.301	3.312	-0.013
	Bias	0.000	0.000	-0.015	-0.005	-0.013
	RMSE	0.005	0.006	0.168	0.164	0.207
	Coverage	95.0	93.4	94.7	95.4	94.4
1000	Mean	0.900	0.900	3.312	3.310	-0.007
	Bias	0.000	0.000	-0.004	-0.007	-0.007
	RMSE	0.003	0.003	0.072	0.072	0.096
	Coverage	94.7	94.6	96.1	95.6	94.3
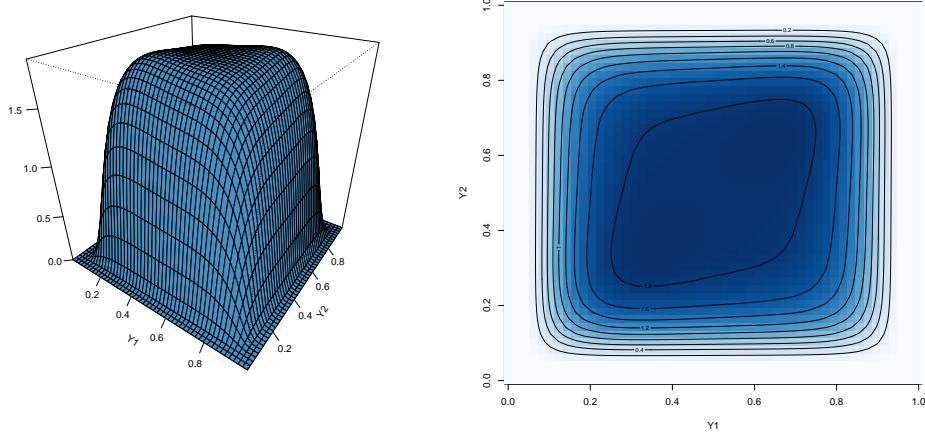
Figure 4: Surface and contour graphs for $\boldsymbol{\theta}_1 = (0.5, 0.5, \sqrt{5.5}, \sqrt{5.5}, 0.1)^\top$, sub-scenario 1.

Table 10: Simulation results for model comparison in Scenario 1, $\boldsymbol{\theta}_1 = (0.5, 0.5, \sqrt{5.5}, \sqrt{5.5}, 0.1)^\top$, with $R = 1000$ replicates.

| n	Model	Loglik	AIC	BIC	Correlation
50	Simplex-FGM	21.283	-32.567	-23.007	
	Simplex-Gaussian	21.267	-32.535	-22.974	
	Simplex-Frank	21.258	-32.517	-22.957	
	Simplex-Clayton	21.257	-32.515	-22.955	
	Simplex-Gumbel	20.667	-31.335	-21.775	0.131
	Beta-Clayton	18.613	-27.225	-17.665	
	Beta-FGM	18.573	-27.146	-17.586	
	Beta-Gaussian	18.572	-27.143	-17.583	
	Beta-Frank	18.559	-27.118	-17.558	
	Beta-Gumbel	17.969	-25.939	-16.378	
100	Simplex-FGM	39.648	-69.297	-56.271	
	Simplex-Frank	39.630	-69.261	-56.235	
	Simplex-Gaussian	39.621	-69.241	-56.215	
	Simplex-Clayton	39.550	-69.100	-56.074	
	Simplex-Gumbel	38.834	-67.668	-54.642	0.108
	Beta-Clayton	34.626	-59.252	-46.226	
	Beta-FGM	34.620	-59.240	-46.214	
	Beta-Gaussian	34.615	-59.230	-46.204	
	Beta-Frank	34.613	-59.226	-46.200	
	Beta-Gumbel	33.817	-57.633	-44.607	
150	Simplex-FGM	57.578	-105.156	-90.102	
	Simplex-Frank	57.572	-105.145	-90.091	
	Simplex-Gaussian	57.571	-105.143	-90.089	
	Simplex-Clayton	57.471	-104.942	-89.888	
	Simplex-Gumbel	56.796	-103.592	-88.539	0.086
	Beta-Gaussian	50.208	-90.416	-75.363	
	Beta-Frank	50.198	-90.396	-75.343	
	Beta-FGM	50.197	-90.395	-75.341	
	Beta-Clayton	50.187	-90.375	-75.322	
	Beta-Gumbel	49.424	-88.849	-73.795	
200	Simplex-FGM	75.691	-141.383	-124.891	
	Simplex-Frank	75.691	-141.382	-124.891	
	Simplex-Gaussian	75.652	-141.304	-124.812	
	Simplex-Clayton	75.563	-141.126	-124.635	
	Simplex-Gumbel	74.826	-139.651	-123.160	0.079
	Beta-Frank	66.090	-122.179	-105.688	
	Beta-FGM	66.082	-122.165	-105.673	
	Beta-Gaussian	66.071	-122.142	-105.650	
	Beta-Clayton	66.059	-122.118	-105.626	
	Beta-Gumbel	65.219	-120.437	-103.946	
500	Simplex-FGM	185.409	-360.817	-339.744	
	Simplex-Frank	185.406	-360.812	-339.739	
	Simplex-Gaussian	185.348	-360.695	-339.622	
	Simplex-Clayton	185.163	-360.326	-339.253	
	Simplex-Gumbel	184.309	-358.617	-337.544	0.057
	Beta-Frank	161.806	-313.613	-292.540	
	Beta-FGM	161.800	-313.600	-292.527	
	Beta-Gaussian	161.779	-313.557	-292.484	
	Beta-Clayton	161.684	-313.368	-292.295	
	Beta-Gumbel	160.700	-311.400	-290.327	
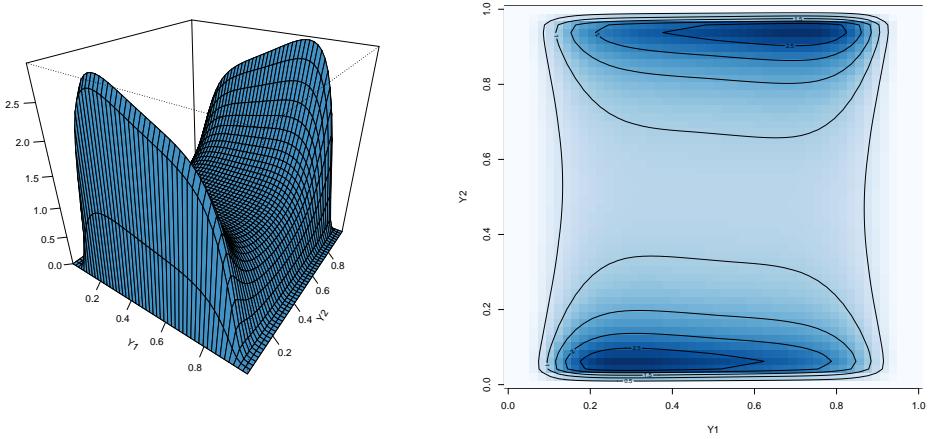
Figure 5: Surface and contour graphs for $\boldsymbol{\theta}_2 = (0.5, 0.5, \sqrt{5.5}, \sqrt{25}, 0.25)^{\top}$, sub-scenario 3.

Table 11: Simulation results for model comparison in Scenario 3, $\boldsymbol{\theta}_3 = (0.5, 0.5, \sqrt{5.5}, \sqrt{25}, 0.25)^{\top}$, with $R = 1000$ replicates.

n	Model	Loglik	AIC	BIC	Correlation
50	Simplex-FGM	17.358	−24.716	−15.156	
	Simplex-Gaussian	17.348	−24.697	−15.137	
	Simplex-Frank	17.339	−24.677	−15.117	
	Simplex-Clayton	17.298	−24.596	−15.036	
	Simplex-Gumbel	16.583	−23.165	−13.605	
	Beta-Clayton	10.497	−10.993	−1.433	0.149
	Beta-Gaussian	10.449	−10.899	−1.339	
	Beta-FGM	10.434	−10.868	−1.308	
	Beta-Frank	10.431	−10.862	−1.302	
	Beta-Gumbel	9.683	−9.365	0.195	
100	Simplex-Clayton	32.354	−54.708	−41.682	
	Simplex-FGM	32.053	−54.106	−41.080	
	Simplex-Frank	32.043	−54.087	−41.061	
	Simplex-Gaussian	31.995	−53.991	−40.965	
	Simplex-Gumbel	30.915	−51.829	−38.803	0.130
	Beta-Clayton	18.971	−27.942	−14.916	
	Beta-Frank	18.970	−27.939	−14.913	
	Beta-Gaussian	18.964	−27.928	−14.903	
	Beta-FGM	18.960	−27.920	−14.894	
	Beta-Gumbel	17.832	−25.664	−12.638	
150	Simplex-FGM	46.886	−83.772	−68.718	
	Simplex-Frank	46.883	−83.766	−68.712	
	Simplex-Gaussian	46.822	−83.644	−68.591	
	Simplex-Clayton	46.618	−83.237	−68.184	
	Simplex-Gumbel	45.584	−81.169	−66.116	0.115
	Beta-Frank	27.504	−45.008	−29.955	
	Beta-Gaussian	27.500	−45.000	−29.947	
	Beta-FGM	27.488	−44.977	−29.924	
	Beta-Clayton	27.475	−44.950	−29.896	
	Beta-Gumbel	26.210	−42.421	−27.367	
200	Simplex-FGM	61.502	−113.004	−96.512	
	Simplex-Frank	61.488	−112.975	−96.484	
	Simplex-Gaussian	61.403	−112.805	−96.314	
	Simplex-Clayton	61.078	−112.156	−95.665	
	Simplex-Gumbel	59.962	−109.924	−93.433	0.108
	Beta-Frank	36.008	−62.017	−45.525	
	Beta-FGM	35.997	−61.994	−45.502	
	Beta-Gaussian	35.993	−61.985	−45.493	
	Beta-Clayton	35.912	−61.824	−45.332	
	Beta-Gumbel	34.468	−58.937	−42.445	
500	Simplex-FGM	149.083	−288.167	−267.094	
	Simplex-Frank	149.079	−288.157	−267.084	
	Simplex-Gaussian	148.951	−287.902	−266.829	
	Simplex-Clayton	148.354	−286.708	−265.635	
	Simplex-Gumbel	146.692	−283.384	−262.311	0.089
	Beta-Frank	86.264	−162.528	−141.455	
	Beta-Gaussian	86.253	−162.506	−141.433	
	Beta-FGM	86.233	−162.466	−141.393	
	Beta-Clayton	86.037	−162.074	−141.001	
	Beta-Gumbel	83.867	−157.733	−136.660	