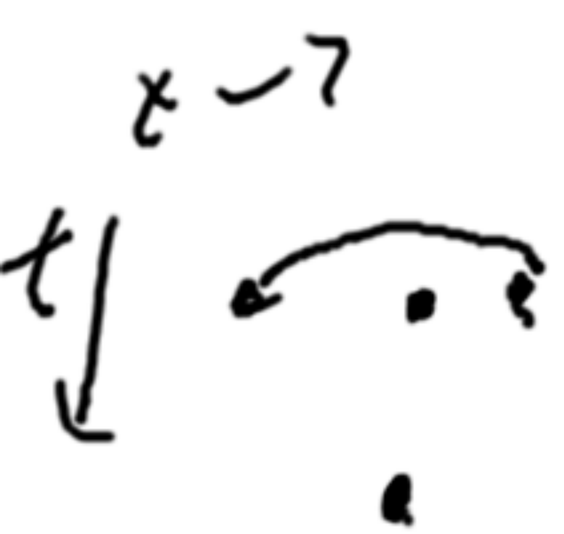
$$\frac{\partial f}{\partial t} = -u \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial t} = \frac{f(x, t+\delta t) - f(x, t)}{dt}$$

$$\frac{\partial f}{\partial x} = \frac{f(x, t) - f(x-dx, t)}{dx} = \text{works}$$

$$\frac{\partial f}{\partial x} = \frac{f(x+dx, t) - f(x-dx, t)}{2\Delta x} = \text{unconditionally unstable}$$

$$\frac{\partial}{\partial t} \Rightarrow \frac{f(x, t+\delta t) - f(x, t)}{dt} = \frac{f(x, t+dt) - \frac{[f(x-dx, t) + f(x+dx, t)]}{2}}{dt}$$



$$f(x, t+dt) = \frac{1}{2}\left(f(x+dx, t) + f(x-dx, t)\right) - \frac{dt\, v}{2\, dx}\left[f(x+dx, t) - f(x-dx, t)\right]$$

$$f(x) = e^{ikx}$$

$$f(x, t+dt) = \frac{1}{2}\left(e^{ik(x+dx)} + e^{ik(x-dx)}\right) - \frac{v\,dt}{2dx}\left(e^{ik(x+dx)} - e^{ik(x-dx)}\right)$$

$$= e^{ikx}\left(\frac{1}{2}\left(e^{ikdx} + e^{-ikdx}\right) - \frac{v\,dt}{2dx}\left(e^{ikdx} - e^{-ikdx}\right)\right)$$

$$= e^{ikx}\left(1 \cos(k\,dx) - \frac{v\,dt}{dx}\, i \sin(k\,dx)\right)$$

$$\cos^2 + \left(\frac{v\,dt}{dx}\right)^2 \sin^2 \leq 1 \quad \text{which} \leq 1 \text{ when } \frac{v\,dt}{dx} \leq 1$$

or $dt \leq dx/v$

= same CFL condition

Lax update $\dfrac{f(x,t+dt) - \frac{1}{2}\left(f(x-dx,t)+f(x+dx,t)\right)}{\underbrace{dt}} = v\,dx\left(f(x+dx,t)-f(x-dx,t)\right)$

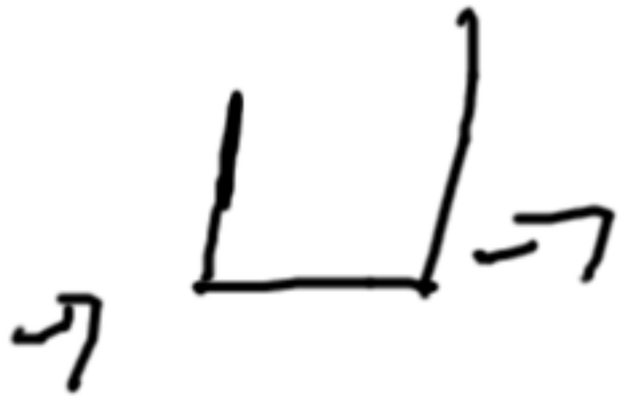
$\Longrightarrow$ rhs

subtract $f(x,t)$

$$\frac{f(x,t+dt)-f(x,t)}{dt} = v\,dx\left(f(x+dx,t)-f(x-dx,t)\right) + \underbrace{\frac{1}{2dt}\left(f(x+dx,t)-2f(x,t)+f(x-dx,t)\right)}$$

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} \Rightarrow \frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \nabla^2 f\,\frac{v\,dx}{dt}$$

$\frac{\partial f}{\partial t} = D\nabla^2 f$ = diffusion equation



net rate of flow = time derivative

So for conserved flow $\boxed{\frac{\partial f}{\partial t} = -\frac{\partial (uf)}{\partial x}}$ exactly conserve with no input

$$\frac{\partial f}{\partial t} + \frac{\partial (uf)}{\partial t} = \text{rate of creation } q$$

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conservation of mass : ρ is conserved $\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial x} = 0$

conservation of momentum : $\frac{\partial p}{\partial t} + \frac{\partial (up)}{\partial x} \Rightarrow \frac{\partial (u\rho)}{dt} + \frac{\partial (u^2\rho)}{\partial x} = ?$

$$F = \frac{dp}{dt} \left(= \frac{d}{dt}(mv) = ma \right)$$

Force on a surface $= P \cdot a$

$$\Rightarrow f \sim -\frac{dP}{dx} \quad \left(-\nabla P = f/\text{volume} \right)$$

$\Rightarrow$ momentum conservation becomes $\frac{\partial u\rho}{\partial t} + \frac{\partial u^2\rho}{\partial x} V = -\nabla P$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial \left(u\rho + P \right)}{\partial t} = 0$$

Conservation of energy

Power = Force · velocity

$$= \frac{du}{\partial t}$$

$$\frac{\partial E}{\partial t} + \frac{\partial uE}{\partial t} = -\frac{\partial (uP)}{\partial x}$$

Energy/unit mass

$$\Rightarrow \frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u E + uP)}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (u\rho)}{\partial t} = 0 \qquad \frac{\partial (u\rho)}{\partial t} + \frac{\partial (u^2\rho + P)}{\partial t} = 0$$

$$\frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u E + uP)}{\partial t} = 0$$

Euler equations

3 equations 4 unknowns, need 1 more equation

some relation between E & P $\Rightarrow$ equation of state

$E = a\underset{\#}{n}kT$, $a = \frac{3}{2}$ for ideal gas ($= \frac{5}{2}$ for air)

$P = \underset{\substack{\uparrow \\ \text{\# density}}}{n}kT$, so $PV = \underset{\substack{\uparrow \\ \text{\# of things}}}{n}kT$

$dE = -PdV = -ankT$
$= -aPV$

$d(PV) = VdP + P\cdot dV =$

$adE = ad(PV) = -PdV$

$aPdV + aVdP = -PdV$

$\frac{a\,dP}{P} = -(1+a)\frac{dV}{V} =$ $a\ln(P) = -(1+a)\log(V) + C$

$P = P_0 (V/V_0)^{-\frac{(1+a)}{a}}$

$$\frac{\rho}{\rho_0} = \frac{1}{V/V_0} \Rightarrow P = P_0 (\rho/\rho_0)^{1+1/a} \quad \text{for } a = \frac{3}{2}$$

$$P = P_0 (\rho/\rho_0)^{1+\frac{2}{3}} = P_0 (\rho/\rho_0)^{5/3} \quad \text{for ideal gas}$$

$$P = P_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma} \text{ where } \gamma \sim 5/3 \Rightarrow 4/3$$

$$\Rightarrow a = 5/2 \Rightarrow P \sim \rho^{1+2/5} \sim \rho^{7/5} \text{ diatomic atmosphere}$$

replace in Euler equations to get closed form

2D system: $\frac{\partial f_1}{\partial t} + a_{11}\frac{\partial f_1}{\partial x} + a_{12}\frac{\partial f_2}{\partial x} = 0$ or $\frac{\partial \vec{f}}{\partial t} + A\frac{\partial \vec{f}}{\partial x} = 0$

$$\frac{\partial f_2}{\partial t} + a_{21}\frac{\partial f_1}{\partial x} + a_{22}\frac{\partial f_2}{\partial x} = 0$$

$f_{12} = b_{12} f(x - vt) \Rightarrow$ plug in

$\frac{\partial f}{\partial t} = v \frac{\partial f}{\partial \xi} = -v f'$

$-v f' + A f' = 0$

$A f' = v f' \Rightarrow$ eigen problem $A x = v x$

v will be eigenvalues of A

$\Rightarrow$ what do you think CFL is going to be?

will have to use largest eigenvalue for CFL

in hydro, we get bulk velocity, bulk $\pm$ sound speed