Random Numbers

Random Variables are Useful

- We’ve seen we needed to generate lots of random variables for MCMC
- It is very common to not be able to calculate things analytically. Common to do some sort of Monte Carlo simulation.
- As you try to simulate rare events and/or have correlations, the quality of the random number generator (RNG) can be critical.
- There are MANY old, broken ones that don’t past statistical tests. (One of the NR authors spent an extra year in grad school due to a bad RNG).
- Please never use the built-in RNGs in e.g. C unless you really don’t care about your answer.
- NB - NR covers this stuff (and much more!) reasonably well in Chapter 7.

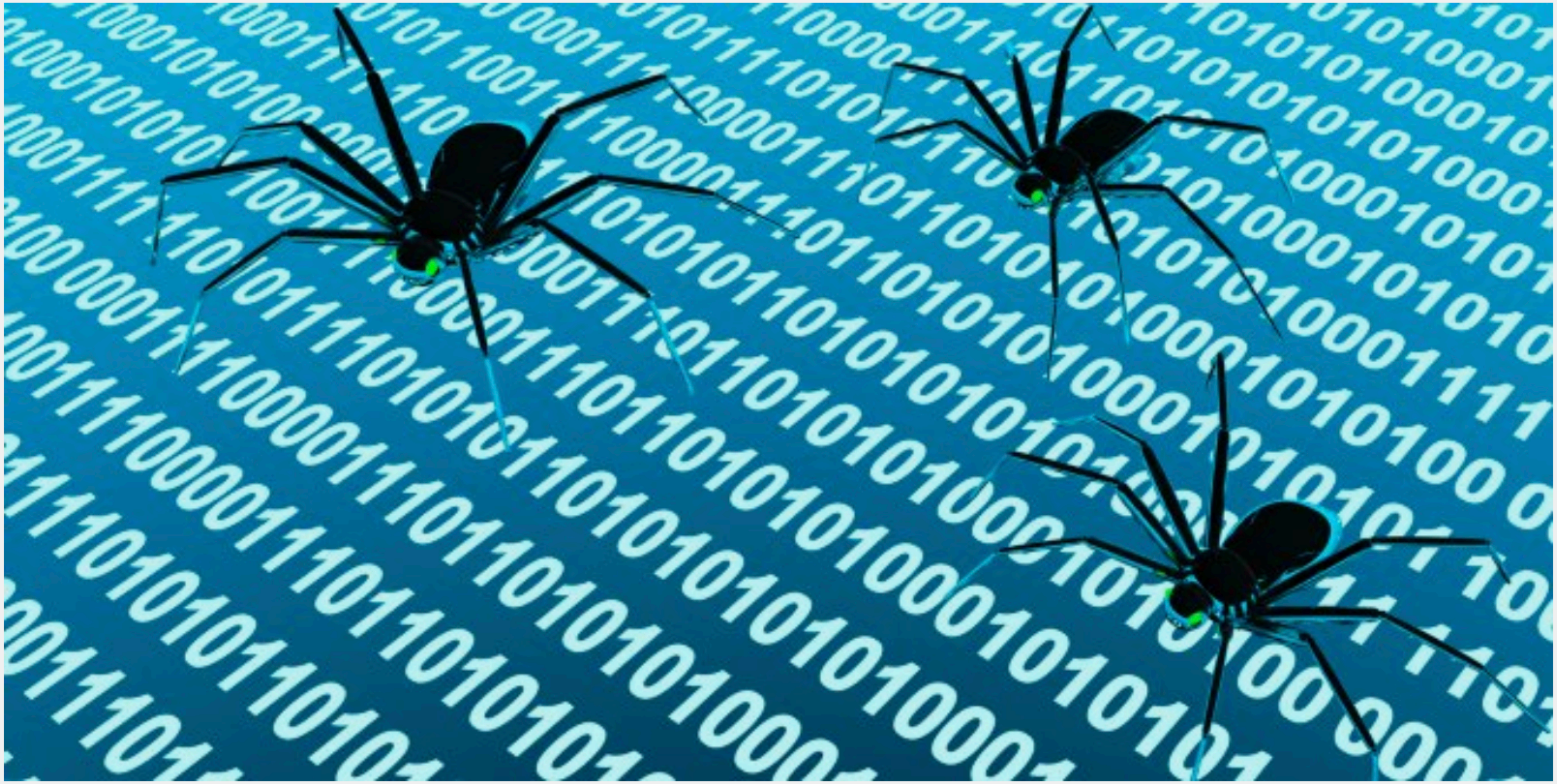
TOUR OF ACCOUNTING
OVER HERE WE HAVE OUR RANDOM NUMBER GENERATOR.
www.dilbert.com scottadams@aol.com
NINE NINE NINE NINE NINE NINE
10/25/01 © 2001 United Feature Syndicate, Inc.
ARE YOU SURE THAT'S RANDOM?
THAT'S THE PROBLEM WITH RANDOMNESS: YOU CAN NEVER BE SURE.

0XFFFFFFFF EVERY TIME IS 0XDEADBEEF —

How a months-old AMD microcode bug destroyed my weekend [UPDATED]

AMD shipped Ryzen 3000 with a serious microcode bug in its random number generator.

JIM SALTER - 10/29/2019, 7:00 AM



Adobe

Enlarge / Ryzen 3000's RDRAND function—what should be a high-quality pseudo-random number generator—just returns 0xFFFFFFFF every time, until its microcode is patched.

PRNG

- Computers very bad at generating truly random numbers
- Instead, starting from some state (*seed*), generate another number that is statistically uncorrelated from previous ones.
- Usual output is uniformly distributed integer, possibly rescaled to be a float between 0,1.
- np.random.rand(om) will return floats from 0,1.
- Important - starting from same state, PRNGs will produce same sequence. If you want reproducibility, make sure you know/set initial state (np.random.seed, or fancier)
- Virtually all fancier random numbers work based off of uniform PRNG. *Don't write your own! Do check you're using a good one!*
- Modern implementations very fast - often ~dozen operations. Will affect how we make choices in more complicated situations.

Non-Uniform

- We'll assume you have access to a good uniform PRNG.
- Let's say you want to simulate the waiting time for a radioactive decay. Distribution proportional to exp(-t).
- Can we remap a uniform deviate into an exponential one?
- What is the probability that we have to wait longer than time t? exp(-t).
- If I told you my waiting time was longer than 30% of the samples, what was it? Well, exp(-t)=0.3, or t=-log(0.3)
- I know that, represented as a probability vs. number of events, my waiting time is a uniform deviate. i.e. half the time, I wait longer than 50% of samples, 10% of the time I wait longer than 90% of the samples etc.
- So, replace 30% by uniform, and t=-log(rand) should be exponentially distributed.

Power Law

- Power law distributions are another common case.
- $dN/ds=s^{-\alpha}$.
- How do we do this?

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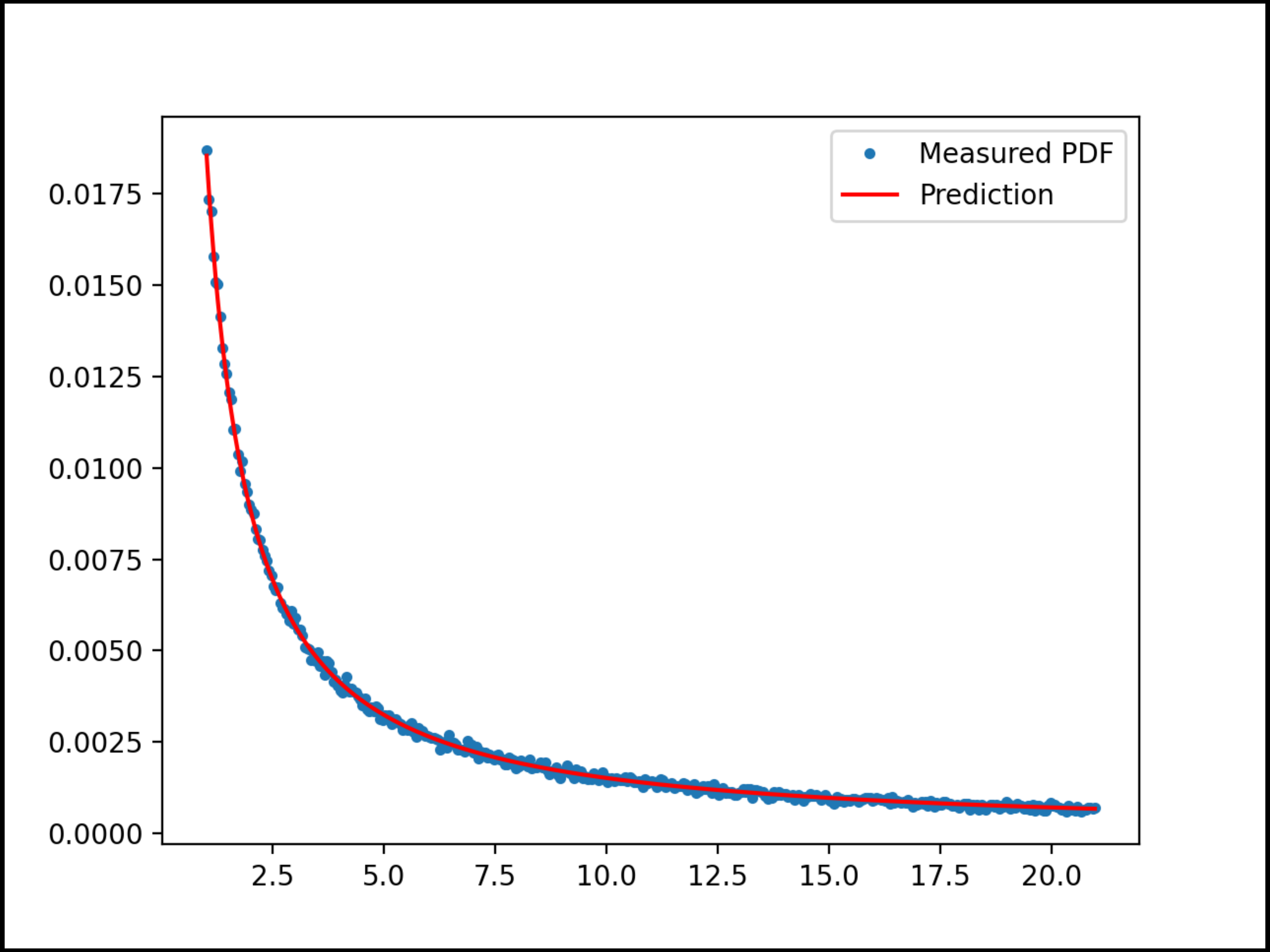
- How do we do this?
- $\int s^{-\alpha} = s^{(1-\alpha)}/(1-\alpha) = x$, $s=((1-\alpha)x)^{(1/(1-\alpha))}$. Need to normalize PDF, cancels $(1-\alpha)$, leaves $s=x^{(1/(1-\alpha))}$ where x is uniform on (0,1).
- NB - area unbounded towards 0 for $\alpha>1$, towards infinity for $\alpha<1$.

See plot_powlaw.py



General Technique

- More generally, if we can integrate the PDF to a CDF and invert that, we can generate deviates drawn from the PDF.
- Sometimes you can do this, sometimes you can't.
- Clever techniques can sometimes help.

Gaussian Deviates (Box-Muller)

- Can we analytically integrate a Gaussian? Sadly, no.
- Can we do so in 2-d? Happily, yes.
- So, we can convert the unit circle to *two* Gaussian numbers.

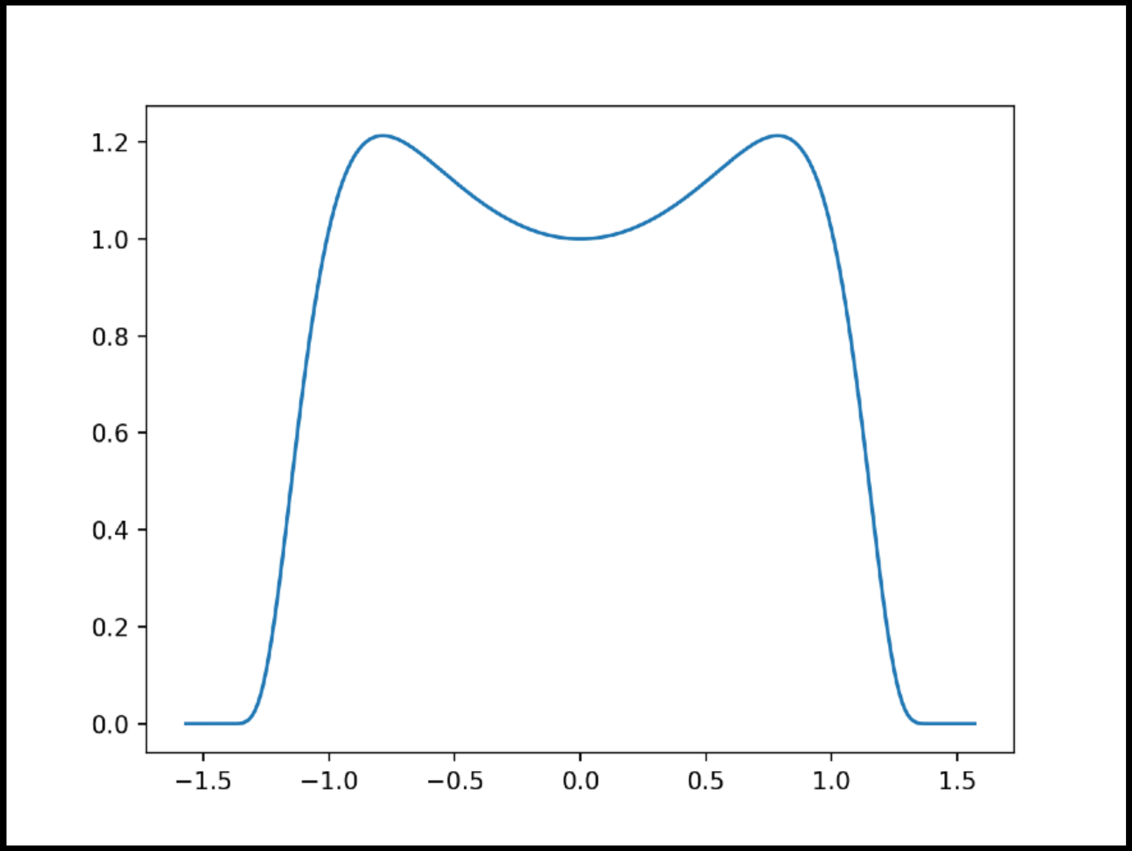
Rejection

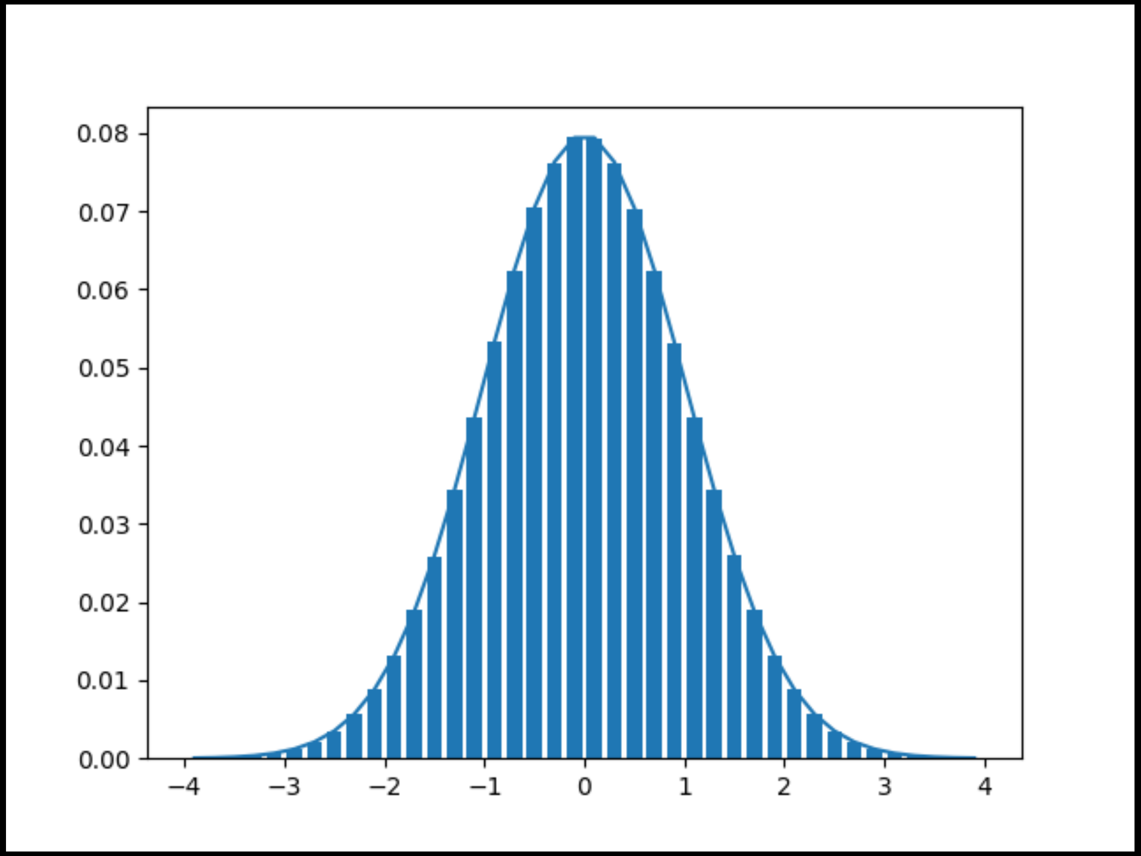
- Say I have a distribution I can analytically draw from.
- Then if that is always larger than the distribution I care about, I can ask if a sample from the first falls within the second.
- If yes, that is a random deviate following the second.
- The closer the first distribution is to the second, the more efficient this technique is.

Transforms Revisited

- Rejection can be tricky - need distribution we can analytically sample that is alway larger than desired. Easier if range is finite.
- Change of variables: $P'(y)dy=P(x)dx$, $P'(y)=P(x)dx/dy$.
- Pick say $x=\mathrm{atan}(y)$. $dx/dy=\sec^2(y)$. Then $P'(y)=P(\mathrm{atan}(y))/\cos^2(y)$. But y bounded in $(-\pi/2,\pi/2)$, so any sufficiently tall box works as bounding region.
- Generate samples in box. If sample falls under $P'(y)$, return $\mathrm{atan}(y)$.

Gaussian Deviates from atan



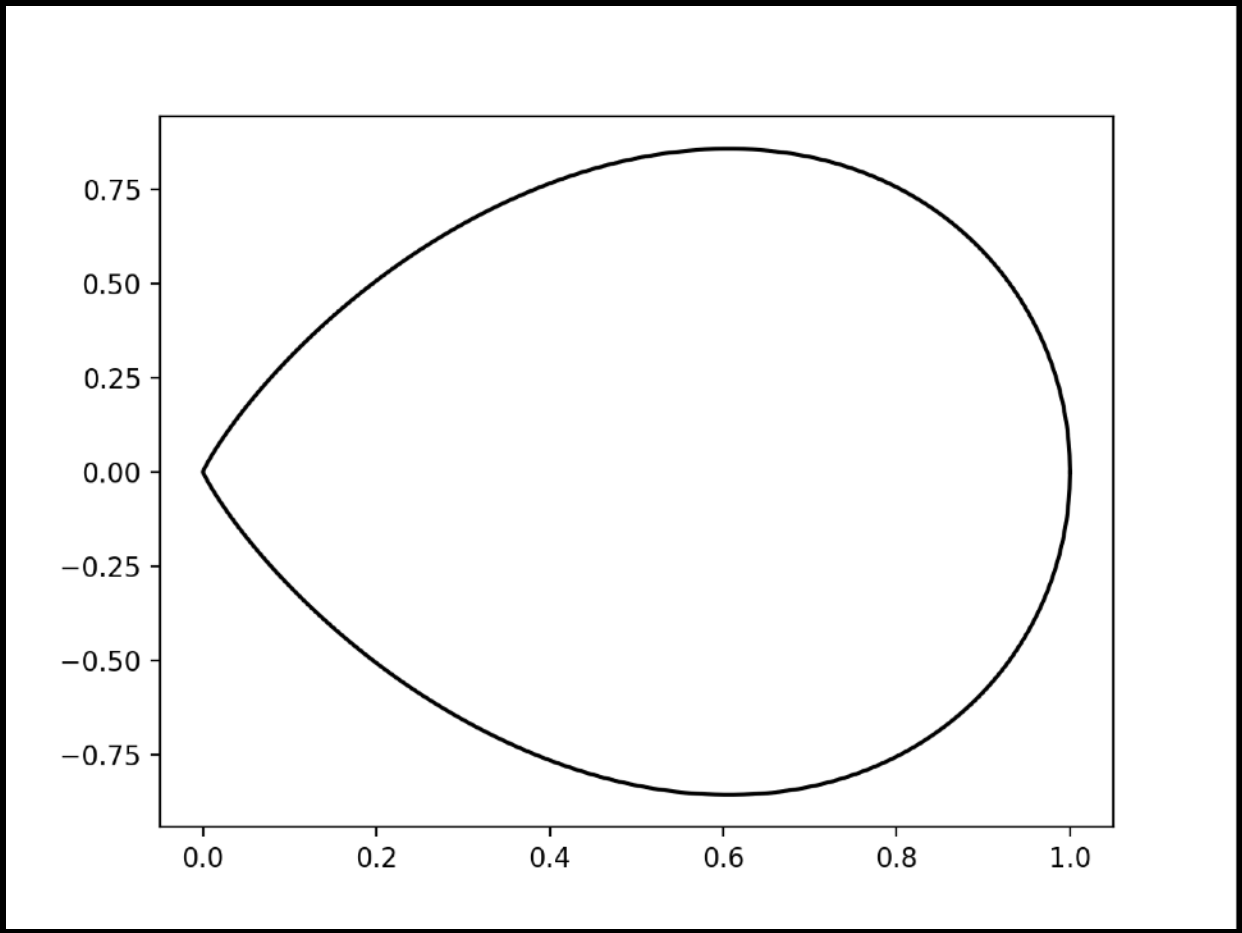


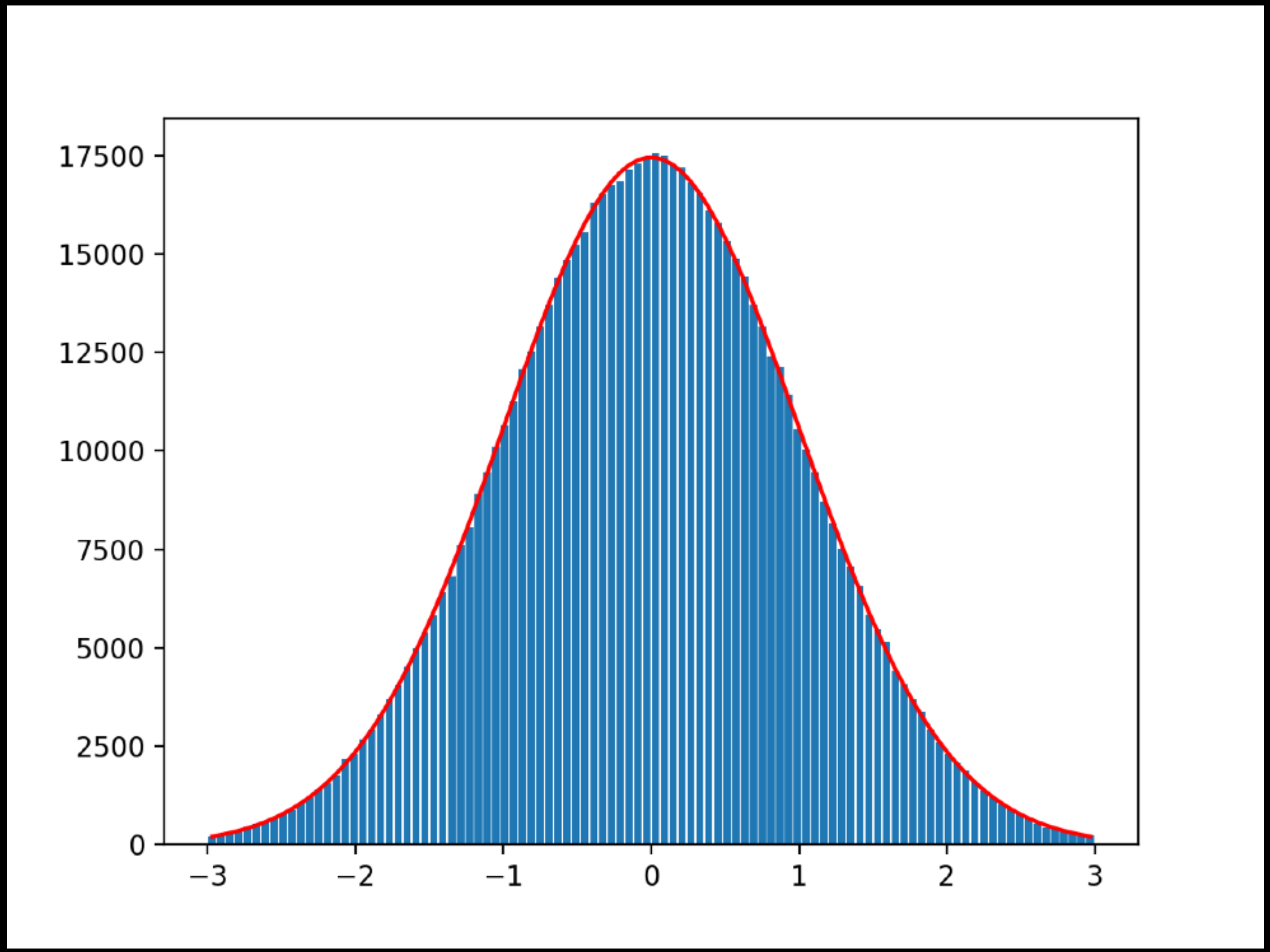
- Left: $P'(y)=\exp(-0.5\ \mathrm{atan}(y)^2)/\cos(y)^2$.
- Right: distribution of deviates w/Gaussian prediction plotted.

Ratio of Uniforms

- Previous method not-ideal - requires atan, cos calculations. Better mapping is $x=v/u$ where v,u uniform.
- Math is:
  - take a (u,v) plane where $0<u<\mathrm{sqrt}(p(v/u))$
  - sample u,v uniformly in this region
  - return v/u(!)
- This works because Jacobian is constant, so this is a remapping of the full number line.
- In practice: draw a box in u,v big enough to cover the probability region. Draw a random sample, if it falls inside the probability region, return v/u.

Results





- Left: bounding region for Gaussian in ratio-of-uniforms
- Right: histogram + prediction.

Squeeze

- Slowest part of ratio of uniforms usually evaluation of pdf.

- If we can write curve that are easy to check inside & outside true PDF, we can save time

- The closer interior/exterior bound ("squeeze") the answer, the better off we are

- Any sufficiently fast curves help. The better they are, the fewer times we need to check exact PDF, but answers will still be correct for not great curves.

Brute Force

- Of course, if you have the PDF, you can tabulate it and make a sum to get the CDF.
- You can then just numerically invert the CDF to get random deviates. Accuracy will only be as good as your interpolation/inversion, but efficiency is very high and can handle arbitrary distributions.
- Can always combine with e.g. transform to get fit onto compact region