Assignment 5 CMPUT 272 Winter 2012 Posted: March 20
Due before 10:00am, *Thursday* April 12
in box labeled "CMPUT 272 Winter 2012" (opposite room CSC 145)

Write solutions in your own words. Include an acknowledgment statement citing (i) all reference materials (other than the textbook and handouts) which you used and (ii) persons with whom you have discussed your solution.

The following information must appear in the top right corner of the first page: CMPUT 272, Assignment 5, *your name*, and the *id* of your lecture section (B1, EB1, B2, or EB2). Please staple the pages together.

1. Prove that any two consecutive terms of the Fibonacci sequence are relatively prime.

2. During the execution of Mergesort (a divide-and-conquer sorting algorithm) on an input of size n, the algorithm recursively calls itself twice on inputs of size $n/2$. After the recursive calls return, n additional time units are required to combine the results into a solution for the input of size n. The execution of this algorithm on an input of size one requires one unit of time. For any natural $n > 0$, the time taken by this algorithm on an input of size n can be expressed as a function $T(n)$ on positive integers as follows:
$$T(n) = \begin{cases} 1, & n = 1 \\ 2T(n/2) + n, & n > 1 \end{cases}$$
Prove that for all natural k and natural n such that $n = 2^k$ we have $T(n) = n + n \log_2 n$.

3. Use induction to prove the binomial theorem: $\forall a, b \in \mathbb{R}, \forall n \in \mathbb{N}, (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$. (Note, define $0^0 = 0$.)

4. Prove that $\forall n \in \mathbb{N}^+, \forall p \in \mathbb{P}, \forall a \in (\mathbb{N}^+)^n, p \mid (\prod_{i=1}^{n} a_i) \Rightarrow (\exists k \in \{1, 2, \ldots, n\}, p \mid a_k)$.

5. Which of the following relations on $\{1,2,3,4\} \times \{1,2,3,4\}$ are equivalence relations on $\{1,2,3,4\}$? Determine the properties of an equivalence relation that each non-equivalence relation lacks.
   (a) $\{(1,1),(2,2),(3,3),(4,4)\}$;
   (b) $\{(1,1),(1,2),(2,1),(3,3),(4,4)\}$;
   (c) $\{(1,1),(1,2),(1,4),(2,2),(2,4),(3,3),(4,1),(4,2),(4,4)\}$.

6. For each relation defined below, prove or disprove that it is a partial order.
   (a) Define a relation R on the set $\mathbb{Z}$ of integers as follows: for all $m, n \in \mathbb{Z}$,
   $$m \ R \ n \text{ if and only if every prime factor of } m \text{ is a prime factor of } n;$$
   (b) Define a relation R on the set $\mathbb{Z}$ of integers as follows: for all $m, n \in \mathbb{Z}$,
   $$m \ R \ n \text{ if and only if } m + n \text{ is even.}$$

7. The following binary relations R are defined on $\mathbb{R}$. Determine for each of them whether or not it is an equivalence or a partial order. If not, state which properties are violated:
   (a) $x \ R \ y$ if and only if $x = y$;
   (b) $x \ R \ y$ if and only if $x < y$;
   (c) $x \ R \ y$ if and only if $x \leq y$;
   (d) $x \ R \ y$ if and only if $x^2 = y^2$;

(e) $x\ R\ y$ if and only if $x^2 \leq y^2$.

8. Given two posets $(A, \preceq_1)$ and $(B, \preceq_2)$, and a function $f : A \to B$. f is *order-preserving* if for any two elements $a_1, a_2 \in A$, $a_1 \preceq_1 a_2$ implies $f(a_1) \preceq_2 f(a_2)$. How many such order-preserving functions are there for each of the following, where both $\preceq_1$ and $\preceq_2$ are the usual "less than and equal to" relation on $\mathbb{R}$:

   (a) $A = \{1, 2, 3, 4\}$, $B = \{1, 2\}$;

   (b) $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$;

   (c) $A = \{1, 2, \ldots, n\}$ for some $n \geq 1$, $B = \{1, 2\}$;

   (d) $A = \{1, 2\}$, $B = \{1, 2, \ldots, n\}$ for some $n \geq 1$.

9. Sort the following list into lexicographic order $(\{1, 2, 3, 4\}^*, \leq_L)$ using a four-pass bucket sort:

$$1234, 3112, 2144, 1212, 424, 3, 2313, 441, 3123, 1432, 31, 3141, 1243, 23, 1111.$$

   You need to show the contents of the buckets after each pass.

10. The following is a version of the binary search algorithm from Question 9 in Assignment 4. The pre- and post- conditions specify what the program is expected to do.

Binary search

$s \in \mathbb{N}^+,\ A \in \mathbb{Z}^s,\ x \in \mathbb{Z},\ \mathit{found} \in \{F, T\}$;
  PRECONDITIONS: $\forall i \in \{1, 2, 3, \ldots, s-1\}, A[i] < A[i+1]$.
$h, \ell, r \in \mathbb{N}$;
  $\ell, r, \mathit{found} := 1, s, F$;
  while $(\sim\mathit{found} \wedge (\ell \leq r))$ **do**
    if $((x < A[\ell]) \vee (A[r] < x))$ **then**
      $\ell := r + 1$
    else if $(\ell = r)$ **then**
      $h, \mathit{found} := \ell, T$
    else
      $h := \lfloor(\ell + r)/2\rfloor$;
      if $(A[h] = x)$ **then**
        $\mathit{found} := T$
      else if $(A[h] > x)$ **then**
        $r := h - 1$
      else
        $\ell := h + 1$
      endif
    endif
  endwhile
  POSTCONDITIONS: $(\mathit{found} \Rightarrow (A[h] = x)) \wedge (\sim\mathit{found} \Rightarrow (\forall i \in \{1, 2, 3, \ldots, s\}, A[i] \neq x))$.

End of binary search

   (a) (Done in Assignment 4, to find a variant expression for the **while**-loop and prove its properties.)

   (b) Find such an invariant for the **while**-loop which combined with the loop exit condition implies the postconditions. Prove the formula you found is an invariant for the **while**-loop.

   (c) Prove that the postconditions are satisfied when the **while**-loop exits.