Homework 5 (50 points) Due: October 11, 2024 11:59 pm
COMPSCI 733: Advanced Algorithms and Designs
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Documentation: (5 points) Type your solutions using Latex

(www.overleaf.com or https://www.latex-project.org/). Submit your solutions (pdf is enough) to Canvas.

Problem 1: (15 points)

Prove that if an edge (u, v) is contained in some minimum spanning tree, then it is a light edge crossing some cut of the graph.

Proof. Let $G = (V, E)$ be a connected, weighted graph, and let T be a minimum spanning tree (MST) of G. Suppose (u, v) is an edge in T. We want to prove that (u, v) is a light edge crossing some cut of G.

1. Consider the MST T. Removing the edge (u, v) from T divides T into two disjoint subtrees, say A and B, where $u \in A$ and $v \in B$. Define the cut (A, B) as the set of all edges that have one endpoint in A and the other in B.

2. Since (u, v) is an edge of T, it must cross the cut (A, B).

3. By the definition of an MST, if we add any edge other than (u, v) that crosses the cut (A, B), it would create a cycle in T. Therefore, (u, v) must be the only edge in T that crosses this cut.

4. Now, suppose (u, v) were not the lightest edge crossing the cut (A, B). Then there would exist another edge (x, y) crossing the cut with a weight less than (u, v).

5. If (x, y) were added to T, replacing (u, v), it would form a spanning tree with a smaller total weight than T. This would contradict the assumption that T is an MST.

6. Therefore, (u, v) must be the lightest edge crossing the cut (A, B).

Hence, if an edge (u, v) is contained in a minimum spanning tree, it must be a light edge crossing some cut of the graph. $\square$

Problem 2: (15 points)

Let $G = (V, E, W)$ is a weighted connected (undirected) graph where all edges have distinct weights.Use proof by contradiction to prove that the Minimum Spanning Tree of G is unique.

Proof. (by contradiction) Suppose there are two different MSTs, T_1 and T_2, for the graph $G = (V, E, W)$. Since they are both MSTs of G, they are both both minimum spanning trees, they must have the same total weight.

1. Since T_1 and T_2 are different, there must be at least one edge e that is in T_1 but not in T_2. Let us assume that $e \in T_1$ but $e \notin T_2$.

2. Since a tree with n vertices has exactly $n-1$ edges, and adding one more edge forms a cycle, adding e to T_2 creates a cycle. Let this cycle be C.

3. Since $e \notin T_2$, there must be at least one other edge e' in the cycle C that is present in T_2 but not in T_1 (otherwise T_2 would have included e). Therefore, e' is an edge in both the cycle C and in T_2, but it is not in T_1.

4. Because all edge weights are distinct, one of the edges in C must be the lightest. Consider two cases:

   - If e is the lightest edge in C, then it should have been in T_2 instead of e', because MSTs choose the lightest edges to avoid cycles.
   - If e' is the lightest edge in C, then removing e from T_1 and replacing it with e' would give a tree with a smaller total weight than T_1, contradicting the fact that T_1 is an MST.

5. In either case, we reach a contradiction: the existence of two different MSTs implies that one of them is not actually an MST.

Therefore, if all edge weights are distinct, the MST of G must be unique. □

Problem 3: (15 points)

Let $G = (V, E, W)$ is a weighted connected (undirected) graph where all edges have distinct weights. Use induction on $|V| = n$, to prove that the Minimum Spanning Tree of G is unique for all $n \geq 1$.

Proof. (by induction on $n = |V|$)

- **Base case:** When $n = 1$, the graph G has only one vertex and no edges. Therefore, the minimum spanning tree is trivially unique.

- **Inductive hypothesis:** Assume that for any graph $G' = (V', E', W')$ with $|V'| = k$ vertices, where all edge weights are distinct, the minimum spanning tree of G' is unique.

- **Inductive step:** Now, consider any tree T_1 of a graph $G = (V, E, W)$ with $k+1$ vertices, where all edge weights are distinct.

   1. Remove a vertex v from T_1. The resulting subgraph G' has k vertices.
   2. By the inductive hypothesis, the minimum spanning tree of G' is unique. Let T' be this unique MST.
   3. Consider the edges that connect v to the vertices of T'. Since all edge weights are distinct, there is a unique lightest edge, say e, that connects v to T'.

4. Check if T_1 contains the edge e. If T_1 contains e, then it must be the MST because it includes the lightest edge connecting v to the rest of the tree. If T_1 does not contain e, then replacing any heavier edge in T_1 with e would produce a spanning tree with a smaller total weight, contradicting the assumption that T_1 is the MST.
5. Therefore, T_1 must include the edge e, and combined with the uniqueness of T' by the inductive hypothesis, this means that the minimum spanning tree of G is also unique.

- **Conclusion:** By the principle of mathematical induction, the minimum spanning tree of G is unique for all $n \geq 1$ when all edge weights are distinct.

$\square$