Homework 1 (50 points) Due: September 13, 2024 11:59 pm
COMPSCI 733: Advanced Algorithms and Designs

Documentation: (5 points) Type your solutions using Latex (www.overleaf.com or https://www.latex-project.org/). Submit your solutions (pdf is enough) to Canvas.

Problem 1: (15 points) Assume we have a one dimensional array of real numbers with $A.length = n$ and indices, $1, 2, \ldots, n$. In the code for InsertionSort shown below, the outer loop index j goes from 2 to n, and the inner index i of the while loop goes "backward".

Algorithm 1 INSERTION-SORT-FORWARD(A)

for $j = 2$ to n **do**
  $key = A[j]$
  //Insert $A[j]$ into the sorted sequence $A[1, \ldots, j-1]$.
  $i = j - 1$
  while $i > 0$ and $A[i] > key$ **do** State$A[i+1] = A[i]$
    $i = i - 1$
  end while
  $A[i+1] = key$
end for

(a) Write a version of the pseusocode for InsertionSortBackward where the outer index j goes "backward", and the inner index i goes "forward".

Algorithm 2 INSERTION-SORT-BACKWARD(A)

Your pseudocode goes here.

(b) Prove that your program (InsertionSortBackward)is correct. See CLRS Chapter 2 and the given "ProgramCorrectness.pptx" slides.

Steps:

1. Write a loop invariant.
2. Show Initialization holds.
3. Show Maintenance holds.
4. Show Termination holds.

Problem 2: (15 points)

A tree is a simple connected graph with no cycles.

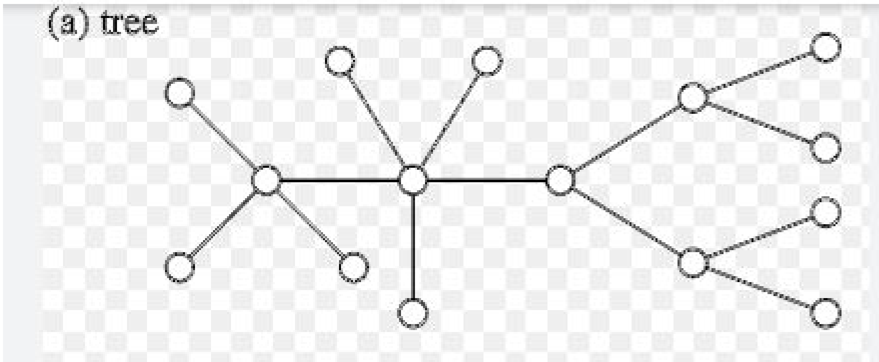

Figure 1:

Use the mathematical induction steps given in the following template to prove that a tree with n vertices has exactly $n-1$ edges for all $n \geq 1$.

Induction proof template:

$P(n)$: A tree with n vertices has exactly $n-1$ edges.

Prove: $P(n)$ is true for all $n \geq 1$.

Proof. (By induction on n)

- Prove the base case.
- Write the inductive hypothesis ($n = k$).
- Prove the inductive step. (Hint: Start with any tree with $n = k+1$ vertices.)
- Conclusion. (Given) Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

□

Problem 3: (15 points)

Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0, 2^1, 2^2 \ldots$, and so on i.e., $1 = 2^0, 2 = 2^1, 3 = 2^0 + 2^1, 4 = 2^2, \ldots.$ [Hint: For the inductive step, separately consider the cases where $k+1$ is even and where $k+1$ is odd. When it is even, note that $(k+1)/2$ is an integer.]

Strong induction proof template:

$P(n)$: Every positive integer n can be written as a sum of distinct powers of two.

Prove: $P(n)$ is true for all $n \geq 1$.

Proof. (By strong induction on n)

- Prove the base case.
- Write the strong inductive hypothesis ($n = 1, 2, \ldots, k$).
- Prove the inductive step.[Hint:Separately consider the cases where $k+1$ is even and where $k+1$ is odd. When it is even, note that $(k+1)/2$ is an integer.]
- Write the conclusion.

$\square$