ECE59500RL HW2

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Problem 1

Problem 2

Problem 3

Problem 4

TODO: check all max vs inf norm

First, we need to assign labels to the states, which are the different spaces on the board. We will use a zero-indexed x-y coordinate system to refer to the different states $s_{x,y} \in \mathcal{S}$, with the origin at the bottom left square, $s_{0,0}$. Moving horizontally will increase the x-component and vertically the y-component, so that our state space is

$$\mathcal{S} = \{s_{ij} : i, j \in \mathbb{N}_0, \quad i, j \leq 5\}$$

> **Note**
>
> Although we are using two "dimensions" to identify each state, we still treat our state-space as a one-dimensional vector, so that we have one row for each $s \in \mathcal{S}$ in $\vec{v}$, P^π, and so forth. The order of the states for this vector will always be in row-major order using the same x-y coordinate system described above:
>
> $$(0,0), (1,0), (2,0), (3,0), (4,0), (0,1), (1,1), ..., (3,4), (4,4)$$

4.1.a

The policy can be evaluated analytically using the following equation:

$$\vec{v}^{\pi} = (I - \gamma P^{\pi})^{-1} \vec{R}^{\pi} \tag{1}$$

We have $\gamma = 0.95$ from the problem statement. P^{π} and $\vec{R}^{\pi}$ each need to be evaluated by going over each state in $\mathcal{S}$ and using the information given to us to evaluate them. Beginning with P^{π}:

$$P^{\pi}_{ij} = P(s_j | s_i, a) = P(s_j | s_i, \pi(s_i))$$

We can use Python to encode the logic described in the problem statement to programatically calculate P^{π} for each state transition:

```
from enum import Enum, StrEnum, auto
import numpy as np
import itertools
from IPython.display import Markdown


class BoardSpace(StrEnum):
    NORMAL = "N"
    MOUNTAIN = "M"
    LIGHTNING = "L"
    TREASURE = "T"


class Action(StrEnum):
    UP = "U"
    RIGHT = "R"
    DOWN = "D"
    LEFT = "L"


width = 5

board = np.full([width, width], BoardSpace.NORMAL)

board[2, 1] = BoardSpace.MOUNTAIN
board[3, 1] = BoardSpace.MOUNTAIN
board[1, 3] = BoardSpace.MOUNTAIN
```

```python
board[2, 3] = BoardSpace.LIGHTNING
board[4, 4] = BoardSpace.TREASURE

policy = np.array(
    [
        list("URRUU"),
        list("UDDDU"),
        list("UURRR"),
        list("LULLU"),
        list("RRRRU"),
    ]
).T


def i_1d(x, y):
    return np.ravel_multi_index([y, x], dims=[width, width])


def is_blocked(x, y):
    return (
        x < 0
        or y < 0
        or x >= width
        or y >= width
        or board[x, y] == BoardSpace.MOUNTAIN
    )


def get_trans_prob(x: int, y: int, a: Action):
    if x < 0 or x >= width or y < 0 or y >= width:
        raise RuntimeError("Invalid coordinates")

    if a not in Action:
        raise RuntimeError("Invalid action type")

    p_vec = np.zeros(width**2)
    i_state_1d = i_1d(x, y)

    if board[x, y] != BoardSpace.NORMAL:
        p_vec[i_state_1d] = 1
        return p_vec
```

```python
    direction_coordinates = {
        Action.LEFT: [x - 1, y],
        Action.RIGHT: [x + 1, y],
        Action.UP: [x, y + 1],
        Action.DOWN: [x, y - 1],
    }

    for direction, (x_next, y_next) in direction_coordinates.items():
        prob = 0.85 if direction == a else 0.05
        if is_blocked(x_next, y_next):
            p_vec[i_state_1d] += prob
        else:
            p_vec[i_1d(x_next, y_next)] += prob

    if p_vec.sum() != 1:
        raise RuntimeError("Probability vector did not add to 1")

    return p_vec


# (0, 0), (1, 0), (2, 0) ... (3, 4), (4, 4)
each_state = [(x, y) for y in range(5) for x in range(5)]

p = np.zeros([width**2, width**2])

for x, y in each_state:
    i = i_1d(x, y)
    p[i] = get_trans_prob(x, y, policy[x, y])

state_text = []
for i in range(width**2):
    y1, x1 = np.unravel_index(i, [width, width])
    a = policy[x1, y1]
    for j in np.argwhere(p[i]).flatten():
        y2, x2 = np.unravel_index(j, [width, width])
        state_text.append(
            rf"P^{{\pi}}(s_{{ {x2}, {y2} }} | s_{{ {x1}, {y1} }}, \pi(s_{{"
            rf" {x1},"
            rf" {y1} }}) = \text{{{a}}}) &= {p[i, j]:g} \\"
        )
    state_text.append(r"\\")
```

```python
state_text = "\n".join(state_text)


def vecfmt(v):
    return Markdown(", ".join([f"{e:g}" for e in v]))
```

The resulting state transition probabilities are listed as follows:

$$
\begin{aligned}
P^\pi(s_{0,0}|s_{0,0}, \pi(s_{0,0}) = \mathrm{U}) = 0.1 \\
P^\pi(s_{1,0}|s_{0,0}, \pi(s_{0,0}) = \mathrm{U}) = 0.05 \\
P^\pi(s_{0,1}|s_{0,0}, \pi(s_{0,0}) = \mathrm{U}) = 0.85 \\
\\
P^\pi(s_{0,0}|s_{1,0}, \pi(s_{1,0}) = \mathrm{R}) = 0.05 \\
P^\pi(s_{1,0}|s_{1,0}, \pi(s_{1,0}) = \mathrm{R}) = 0.05 \\
P^\pi(s_{2,0}|s_{1,0}, \pi(s_{1,0}) = \mathrm{R}) = 0.85 \\
P^\pi(s_{1,1}|s_{1,0}, \pi(s_{1,0}) = \mathrm{R}) = 0.05 \\
\\
P^\pi(s_{1,0}|s_{2,0}, \pi(s_{2,0}) = \mathrm{R}) = 0.05 \\
P^\pi(s_{2,0}|s_{2,0}, \pi(s_{2,0}) = \mathrm{R}) = 0.1 \\
P^\pi(s_{3,0}|s_{2,0}, \pi(s_{2,0}) = \mathrm{R}) = 0.85 \\
\\
P^\pi(s_{2,0}|s_{3,0}, \pi(s_{3,0}) = \mathrm{U}) = 0.05 \\
P^\pi(s_{3,0}|s_{3,0}, \pi(s_{3,0}) = \mathrm{U}) = 0.9 \\
P^\pi(s_{4,0}|s_{3,0}, \pi(s_{3,0}) = \mathrm{U}) = 0.05 \\
\\
P^\pi(s_{3,0}|s_{4,0}, \pi(s_{4,0}) = \mathrm{U}) = 0.05 \\
P^\pi(s_{4,0}|s_{4,0}, \pi(s_{4,0}) = \mathrm{U}) = 0.1 \\
P^\pi(s_{4,1}|s_{4,0}, \pi(s_{4,0}) = \mathrm{U}) = 0.85 \\
\\
P^\pi(s_{0,0}|s_{0,1}, \pi(s_{0,1}) = \mathrm{U}) = 0.05 \\
P^\pi(s_{0,1}|s_{0,1}, \pi(s_{0,1}) = \mathrm{U}) = 0.05 \\
P^\pi(s_{1,1}|s_{0,1}, \pi(s_{0,1}) = \mathrm{U}) = 0.05 \\
P^\pi(s_{0,2}|s_{0,1}, \pi(s_{0,1}) = \mathrm{U}) = 0.85
\end{aligned}
$$

$$P^\pi(s_{1,0}|s_{1,1},\pi(s_{1,1})=\mathrm{D})=0.85$$
$$P^\pi(s_{0,1}|s_{1,1},\pi(s_{1,1})=\mathrm{D})=0.05$$
$$P^\pi(s_{1,1}|s_{1,1},\pi(s_{1,1})=\mathrm{D})=0.05$$
$$P^\pi(s_{1,2}|s_{1,1},\pi(s_{1,1})=\mathrm{D})=0.05$$

$$P^\pi(s_{2,1}|s_{2,1},\pi(s_{2,1})=\mathrm{D})=1$$

$$P^\pi(s_{3,1}|s_{3,1},\pi(s_{3,1})=\mathrm{D})=1$$

$$P^\pi(s_{4,0}|s_{4,1},\pi(s_{4,1})=\mathrm{U})=0.05$$
$$P^\pi(s_{4,1}|s_{4,1},\pi(s_{4,1})=\mathrm{U})=0.1$$
$$P^\pi(s_{4,2}|s_{4,1},\pi(s_{4,1})=\mathrm{U})=0.85$$

$$P^\pi(s_{0,1}|s_{0,2},\pi(s_{0,2})=\mathrm{U})=0.05$$
$$P^\pi(s_{0,2}|s_{0,2},\pi(s_{0,2})=\mathrm{U})=0.05$$
$$P^\pi(s_{1,2}|s_{0,2},\pi(s_{0,2})=\mathrm{U})=0.05$$
$$P^\pi(s_{0,3}|s_{0,2},\pi(s_{0,2})=\mathrm{U})=0.85$$

$$P^\pi(s_{1,1}|s_{1,2},\pi(s_{1,2})=\mathrm{U})=0.05$$
$$P^\pi(s_{0,2}|s_{1,2},\pi(s_{1,2})=\mathrm{U})=0.05$$
$$P^\pi(s_{1,2}|s_{1,2},\pi(s_{1,2})=\mathrm{U})=0.85$$
$$P^\pi(s_{2,2}|s_{1,2},\pi(s_{1,2})=\mathrm{U})=0.05$$

$$P^\pi(s_{1,2}|s_{2,2},\pi(s_{2,2})=\mathrm{R})=0.05$$
$$P^\pi(s_{2,2}|s_{2,2},\pi(s_{2,2})=\mathrm{R})=0.05$$
$$P^\pi(s_{3,2}|s_{2,2},\pi(s_{2,2})=\mathrm{R})=0.85$$
$$P^\pi(s_{2,3}|s_{2,2},\pi(s_{2,2})=\mathrm{R})=0.05$$

$$P^\pi(s_{2,2}|s_{3,2},\pi(s_{3,2})=\mathrm{R})=0.05$$
$$P^\pi(s_{3,2}|s_{3,2},\pi(s_{3,2})=\mathrm{R})=0.05$$
$$P^\pi(s_{4,2}|s_{3,2},\pi(s_{3,2})=\mathrm{R})=0.85$$
$$P^\pi(s_{3,3}|s_{3,2},\pi(s_{3,2})=\mathrm{R})=0.05$$

$$P^\pi(s_{4,1}|s_{4,2},\pi(s_{4,2})=\mathrm{R})=0.05$$
$$P^\pi(s_{3,2}|s_{4,2},\pi(s_{4,2})=\mathrm{R})=0.05$$
$$P^\pi(s_{4,2}|s_{4,2},\pi(s_{4,2})=\mathrm{R})=0.85$$
$$P^\pi(s_{4,3}|s_{4,2},\pi(s_{4,2})=\mathrm{R})=0.05$$

$$P^\pi(s_{0,2}|s_{0,3},\pi(s_{0,3})=\mathrm{L})=0.05$$
$$P^\pi(s_{0,3}|s_{0,3},\pi(s_{0,3})=\mathrm{L})=0.9$$
$$P^\pi(s_{0,4}|s_{0,3},\pi(s_{0,3})=\mathrm{L})=0.05$$

$$P^\pi(s_{1,3}|s_{1,3},\pi(s_{1,3})=\mathrm{U})=1$$

$$P^\pi(s_{2,3}|s_{2,3},\pi(s_{2,3})=\mathrm{L})=1$$

$$P^\pi(s_{3,2}|s_{3,3},\pi(s_{3,3})=\mathrm{L})=0.05$$
$$P^\pi(s_{2,3}|s_{3,3},\pi(s_{3,3})=\mathrm{L})=0.85$$
$$P^\pi(s_{4,3}|s_{3,3},\pi(s_{3,3})=\mathrm{L})=0.05$$
$$P^\pi(s_{3,4}|s_{3,3},\pi(s_{3,3})=\mathrm{L})=0.05$$

$$P^\pi(s_{4,2}|s_{4,3},\pi(s_{4,3})=\mathrm{U})=0.05$$
$$P^\pi(s_{3,3}|s_{4,3},\pi(s_{4,3})=\mathrm{U})=0.05$$
$$P^\pi(s_{4,3}|s_{4,3},\pi(s_{4,3})=\mathrm{U})=0.05$$
$$P^\pi(s_{4,4}|s_{4,3},\pi(s_{4,3})=\mathrm{U})=0.85$$

$$P^\pi(s_{0,3}|s_{0,4},\pi(s_{0,4})=\mathrm{R})=0.05$$
$$P^\pi(s_{0,4}|s_{0,4},\pi(s_{0,4})=\mathrm{R})=0.1$$
$$P^\pi(s_{1,4}|s_{0,4},\pi(s_{0,4})=\mathrm{R})=0.85$$

$$P^\pi(s_{0,4}|s_{1,4},\pi(s_{1,4})=\mathrm{R})=0.05$$
$$P^\pi(s_{1,4}|s_{1,4},\pi(s_{1,4})=\mathrm{R})=0.1$$
$$P^\pi(s_{2,4}|s_{1,4},\pi(s_{1,4})=\mathrm{R})=0.85$$

$$P^\pi(s_{2,3}|s_{2,4},\pi(s_{2,4})=\mathrm{R})=0.05$$
$$P^\pi(s_{1,4}|s_{2,4},\pi(s_{2,4})=\mathrm{R})=0.05$$

$$P^\pi(s_{2,4}|s_{2,4}, \pi(s_{2,4}) = \mathrm{R}) = 0.05$$
$$P^\pi(s_{3,4}|s_{2,4}, \pi(s_{2,4}) = \mathrm{R}) = 0.85$$

$$P^\pi(s_{3,3}|s_{3,4}, \pi(s_{3,4}) = \mathrm{R}) = 0.05$$
$$P^\pi(s_{2,4}|s_{3,4}, \pi(s_{3,4}) = \mathrm{R}) = 0.05$$
$$P^\pi(s_{3,4}|s_{3,4}, \pi(s_{3,4}) = \mathrm{R}) = 0.05$$
$$P^\pi(s_{4,4}|s_{3,4}, \pi(s_{3,4}) = \mathrm{R}) = 0.85$$

$$P^\pi(s_{4,4}|s_{4,4}, \pi(s_{4,4}) = \mathrm{U}) = 1$$

All other possible state transitions have probability 0.

Moving onto $\vec{R}^\pi$:

$$\vec{R}^\pi = \begin{bmatrix} R(s_{1,1}, \pi(s_{1,1}) \\ R(s_{1,2}, \pi(s_{1,2}) \\ \dots \\ R(s_{4,4}, \pi(s_{4,4})) \end{bmatrix}_{|\mathcal{S}|\times 1}$$

So given the reward function described, we just have a simple vector with two nonzero elements:

```
r = np.zeros(width * width, dtype=int)
r[i_1d(*np.argwhere(board == BoardSpace.LIGHTNING).squeeze())] = -1
r[i_1d(*np.argwhere(board == BoardSpace.TREASURE).squeeze())] = 1
```

$$\vec{R}^\pi = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 1]^T$$

So, now we can substitute in our values into Equation 1 and solve:

```
gamma = 0.95
v = np.linalg.inv(np.identity(width**2) - gamma * p) @ r
value_text = []
for i, val in enumerate(v):
    y, x = np.unravel_index(i, [width, width])
    value_text.append(rf"v^{{\pi}}(s_{{ {x},{y} }}) &= {val:g} \\")
```

```python
value_text = "\n".join(value_text)
```

$$
\begin{aligned}
v^\pi(s_{0,0}) &= 4.33896\\
v^\pi(s_{1,0}) &= 2.64128\\
v^\pi(s_{2,0}) &= 2.70644\\
v^\pi(s_{3,0}) &= 2.87785\\
v^\pi(s_{4,0}) &= 6.07858\\
v^\pi(s_{0,1}) &= 4.70749\\
v^\pi(s_{1,1}) &= 2.61613\\
v^\pi(s_{2,1}) &= 0\\
v^\pi(s_{3,1}) &= 0\\
v^\pi(s_{4,1}) &= 6.64324\\
v^\pi(s_{0,2}) &= 5.14368\\
v^\pi(s_{1,2}) &= 2.85117\\
v^\pi(s_{2,2}) &= 3.79493\\
v^\pi(s_{3,2}) &= 5.48513\\
v^\pi(s_{4,2}) &= 7.08781\\
v^\pi(s_{0,3}) &= 5.62269\\
v^\pi(s_{1,3}) &= 0\\
v^\pi(s_{2,3}) &= -20\\
v^\pi(s_{3,3}) &= -14.2964\\
v^\pi(s_{4,3}) &= 16.5959\\
v^\pi(s_{0,4}) &= 12.0203\\
v^\pi(s_{1,4}) &= 13.1409\\
v^\pi(s_{2,4}) &= 14.0205\\
v^\pi(s_{3,4}) &= 16.9416\\
v^\pi(s_{4,4}) &= 20
\end{aligned}
$$

4.1.b

Let $\vec{v}_0 = \mathbf{0}$. Then for each step in the iteration,

$$\vec{v}_{t+1} = \vec{R}^\pi + \gamma P^\pi \vec{v}_t$$

To determine T, the number of iterations we need to make in order to obtain $\|v_T - v^\pi\|_\infty \leq 0.01$, we can use the following theorem:

$$T \geq \frac{\log\left(\frac{\|\vec{v}_0 - \vec{v}^\pi\|_\infty}{\varepsilon}\right)}{\log \frac{1}{\gamma}} \tag{2}$$

However, we cannot use the analytical solution of v^π, so we need to form some kind of bound on it instead and use that. We know that

$$v^\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t) | s_0 = s\right], \quad \forall s \in \mathcal{S}$$

by definition, and we know that our reward function is bounded on the interval $[-1, 1]$. Therefore, we can say that $|R(s,a)| \leq 1$ so that

$$\|v^\pi\|_\infty \leq 1 \cdot \sum_{t=0}^{\infty} \gamma^t = \frac{1}{1-\gamma}$$

And since $v_0 = \mathbf{0}$, we can say that

$$\|\vec{v}_0 - \vec{v}^\pi\|_\infty \leq \|v^\pi\|_\infty \leq \frac{1}{1-\gamma}$$

So, finally:

$$\frac{\log\left(\frac{1}{\varepsilon(1-\gamma)}\right)}{\log \frac{1}{\gamma}}$$

will give us a conservative estimate of T which is more than (or equal to) the number of iterations actually required to obtain our desired error.

We can evaluate this result using Python, and then perform T iterations by implementing the iteration algorithm above.

```python
from numpy.linalg import norm

epsilon = 0.01
v0 = np.zeros(width**2)
n_iterations = int(
    np.ceil(np.log(1 / (epsilon * (1 - gamma))) / np.log(1 / gamma))
)

v_t = v0
v_history = [v0]
for t in range(n_iterations):
    v_t = r + gamma * p @ v_t
    v_history.append(v_t)

v_history = np.array(v_history)

value_text_iter = []
for i, val in enumerate(v):
    y, x = np.unravel_index(i, [width, width])
    value_text_iter.append(
        rf"v_{{T = {n_iterations}}} (s_{{ {x},{y} }}) &= {val:g} \\"
    )

value_text_iter = "\n".join(value_text_iter)
```

We perform $T = 149$ iterations, and our final v_T is:

$$
\begin{aligned}
v_{T=149}(s_{0,0}) &= 4.33896 \\
v_{T=149}(s_{1,0}) &= 2.64128 \\
v_{T=149}(s_{2,0}) &= 2.70644 \\
v_{T=149}(s_{3,0}) &= 2.87785 \\
v_{T=149}(s_{4,0}) &= 6.07858 \\
v_{T=149}(s_{0,1}) &= 4.70749 \\
v_{T=149}(s_{1,1}) &= 2.61613 \\
v_{T=149}(s_{2,1}) &= 0 \\
v_{T=149}(s_{3,1}) &= 0 \\
v_{T=149}(s_{4,1}) &= 6.64324 \\
v_{T=149}(s_{0,2}) &= 5.14368
\end{aligned}
$$

$$
\begin{aligned}
v_{T=149}(s_{1,2}) &= 2.85117 \\
v_{T=149}(s_{2,2}) &= 3.79493 \\
v_{T=149}(s_{3,2}) &= 5.48513 \\
v_{T=149}(s_{4,2}) &= 7.08781 \\
v_{T=149}(s_{0,3}) &= 5.62269 \\
v_{T=149}(s_{1,3}) &= 0 \\
v_{T=149}(s_{2,3}) &= -20 \\
v_{T=149}(s_{3,3}) &= -14.2964 \\
v_{T=149}(s_{4,3}) &= 16.5959 \\
v_{T=149}(s_{0,4}) &= 12.0203 \\
v_{T=149}(s_{1,4}) &= 13.1409 \\
v_{T=149}(s_{2,4}) &= 14.0205 \\
v_{T=149}(s_{3,4}) &= 16.9416 \\
v_{T=149}(s_{4,4}) &= 20
\end{aligned}
$$

We can verify that our desired conditon holds:

```
max_error = norm(v_t - v, ord=np.inf)
```

$$\|v_T - v^\pi\|_\infty = 0.009591 \leq 0.01$$

4.1.c

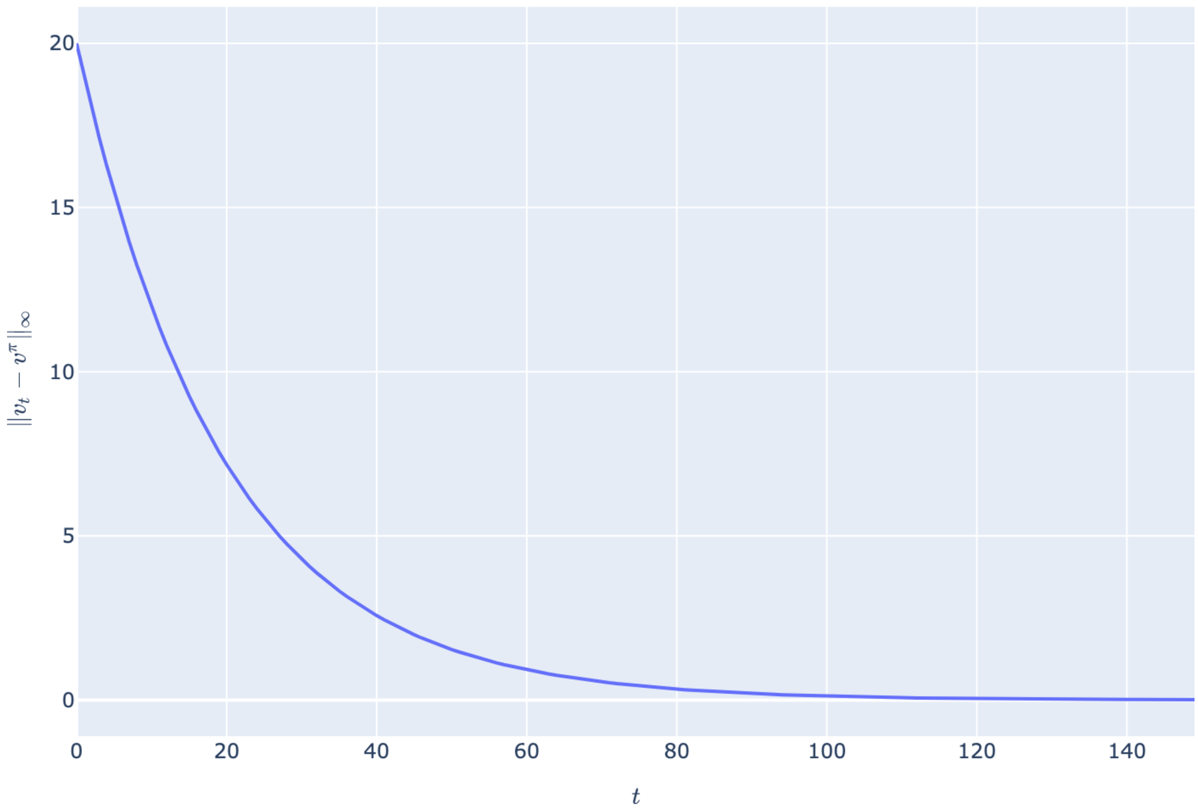
We kept track of the full history of v_t, so we can calculate the error for each timestep and plot it:

```
import plotly.graph_objects as go
import plotly.io as pio

pio.renderers.default = "png"
pio.kaleido.scope.default_scale = 2

error_history = norm(v_history - v, ord=np.inf, axis=1)
go.Figure(
    data=[go.Scatter(y=error_history, mode="lines")],
```

```
    layout=dict(
        xaxis_title="$t$", yaxis_title=r"$\Vert v_t - v^\pi \Vert_{\infty}$"
    ),
)
```



4.2

To perform value iteration, we initialize $\vec{v}_0 = \mathbf{0}$ and then for each iteration which increases t by one, we perform the operation:

$$v_{t+1}(s) = \max_{a \in \mathcal{A}} \left[R(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[v_t(s')] \right], \quad \forall s \in \mathcal{S}$$

However, since our

We can use the following theorem to determine when we have completed a sufficient number of iterations:

$$T \geq \frac{\log\left(\frac{\|\vec{v}_0 - \vec{v}^*\|_\infty}{\varepsilon}\right)}{\log \frac{1}{\gamma}}$$

This is the same as Equation 2 but with v^* instead of v^π. We can use the same logic as before to bound T since the only difference is that v^* now considers all possible policies, but all the value functions for all policies can still be bound by the reward function as before, such that:

$$\|v^*\|_\infty \leq 1 \cdot \sum_{t=0}^{\infty} \gamma^t = \frac{1}{1-\gamma}$$

Therefore, we can use the same $T = 149$ as before.

```
v0 = np.zeros(width**2)

v_t = v0

epsilon = 0.01

n_iterations = int(
    np.ceil(np.log(1 / (epsilon * (1 - gamma))) / np.log(1 / gamma))
)

# FIXME
for t in range(n_iterations):
    # for t in range(10000):
    v_t = np.array(
        [
            np.max(
                [
                    r[i_1d(x, y)]
                    + gamma * np.sum(get_trans_prob(x, y, a) * v_t)
                    for a in Action
                ]
            )
            for x, y in each_state
        ]
    )
```

Now, with $\vec{v}_T$ determined, we need to find the policy corresponding to this value function, which is:

$$\pi_T(s) = \arg\max_{a \in \mathcal{A}} [R(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[v_T(s')], \quad \forall s \in \mathcal{S}$$

```
policy_opt = np.full_like(policy, "")

actions_list = list(Action)

for x, y in each_state:
    i_max = np.argmax(
        [
            r[i_1d(x, y)] + gamma * np.sum(get_trans_prob(x, y, a) * v_t)
            for a in actions_list
        ]
    )

    policy_opt[x, y] = actions_list[i_max].value

# Account for different coordinate systems between numpy and our board
print(np.flipud(policy_opt.T))
```

```
[['R' 'R' 'R' 'R' 'U']
 ['U' 'U' 'U' 'R' 'U']
 ['U' 'R' 'R' 'R' 'U']
 ['U' 'D' 'U' 'U' 'U']
 ['R' 'R' 'R' 'R' 'U']]
```

The learned policy is printed above and its representation corresponds one-to-one with the shape and orientation of the original board given in the problem statement.

Lastly, we can calculate the value function for this policy by re-calculating our P matrix and then using the analytical solution:

```
p_opt = np.array(
    [get_trans_prob(x, y, policy_opt[x, y]) for x, y in each_state],
)

v_opt = np.linalg.inv(np.identity(width**2) - gamma * p_opt) @ r

value_text = []
for val, (x, y) in zip(v_opt, each_state):
    value_text.append(rf"v^{{\pi_T}}(s_{{ {x},{y} }}) &= {val:g} \\")

value_text = "\n".join(value_text)
```

The $v^{\pi_T}(s)$ is shown for each state below:

$$
\begin{aligned}
v^{\pi_T}(s_{0,0}) &= 11.8594 \\
v^{\pi_T}(s_{1,0}) &= 12.6235 \\
v^{\pi_T}(s_{2,0}) &= 13.4931 \\
v^{\pi_T}(s_{3,0}) &= 14.3797 \\
v^{\pi_T}(s_{4,0}) &= 15.3223 \\
v^{\pi_T}(s_{0,1}) &= 11.3527 \\
v^{\pi_T}(s_{1,1}) &= 11.8921 \\
v^{\pi_T}(s_{2,1}) &= 0 \\
v^{\pi_T}(s_{3,1}) &= 0 \\
v^{\pi_T}(s_{4,1}) &= 16.3265 \\
v^{\pi_T}(s_{0,2}) &= 11.9941 \\
v^{\pi_T}(s_{1,2}) &= 12.5169 \\
v^{\pi_T}(s_{2,2}) &= 13.3594 \\
v^{\pi_T}(s_{3,2}) &= 16.1985 \\
v^{\pi_T}(s_{4,2}) &= 17.3965 \\
v^{\pi_T}(s_{0,3}) &= 12.7438 \\
v^{\pi_T}(s_{1,3}) &= 0 \\
v^{\pi_T}(s_{2,3}) &= -20 \\
v^{\pi_T}(s_{3,3}) &= 15.7238 \\
v^{\pi_T}(s_{4,3}) &= 18.607 \\
v^{\pi_T}(s_{0,4}) &= 13.577 \\
v^{\pi_T}(s_{1,4}) &= 14.4667 \\
v^{\pi_T}(s_{2,4}) &= 15.4148 \\
v^{\pi_T}(s_{3,4}) &= 18.5082 \\
v^{\pi_T}(s_{4,4}) &= 20
\end{aligned}
$$

4.3

To begin our policy iteration algorithm, we initialize the policy with a uniform random distribution over $\mathcal{A}$:

```python
seed = 12983198247918274912849821 74
rng = np.random.default_rng(seed)

int_to_action = np.vectorize(lambda i: actions_list[i])

policy_0 = int_to_action(rng.integers(low=0, high=4, size=(width, width)))

print(policy_0)
```

```
[['R' 'D' 'U' 'U' 'R']
 ['U' 'R' 'R' 'R' 'D']
 ['L' 'D' 'R' 'R' 'U']
 ['U' 'U' 'D' 'R' 'L']
 ['L' 'D' 'L' 'R' 'U']]
```

The policy is shown above using the original board shape and orientation.

Now, for each iteration, we evaluate π_t by computing v^{π_t} and then we improve the policy by updating its action for each state so that it yields the maximum return according to:

$$\pi_{t+1}(s) = \arg\max_{a \in \mathcal{A}} [R(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}[v^{\pi_t}(s')], \quad \forall s \in \mathcal{S}$$

To determine how many iterations are necessary, we use the following theorem:

$$\|v^{\pi_T} - v^*\|_\infty \leq \gamma^T \|v^{\pi_0} - v^*\|_\infty$$

We can substitute our desired accuracy ε in and solve for T:

$$T \geq \frac{\log\left(\frac{\|v^{\pi_0} - v^*\|_\infty}{\varepsilon}\right)}{\log(\frac{1}{\gamma})}$$

This is the same form we are used to seeing.

Using the same logic as before, we can bound v^* by:

$$\|v^*\|_\infty \leq 1 \cdot \sum_{t=0}^{\infty} \gamma^t = \frac{1}{1-\gamma}$$

Therefore:

$$\|v^{\pi_0} - v^*\|_\infty \leq \|v^{\pi_0}\|_\infty + \frac{1}{1-\gamma}$$

So we can get a conservative upper bound on T by substituting this into the inequality above:

$$T \geq \frac{\log\left(\frac{\|v^{\pi_0}\|_\infty + (1-\gamma)^{-1}}{\varepsilon}\right)}{\log(\frac{1}{\gamma})}$$

Depending on how good our randomly generated initial policy is, T will vary.

We solve for T and perform our iterations:

```
def evaluate_policy(policy):
    p_policy = np.array(
        [get_trans_prob(x, y, policy[x, y]) for x, y in each_state],
    )

    v_policy = np.linalg.inv(np.identity(width**2) - gamma * p_policy) @ r

    return v_policy


def improve_policy(policy, v_policy):
    for x, y in each_state:
        i_max = np.argmax(
            [
                r[i_1d(x, y)]
                + gamma * np.sum(get_trans_prob(x, y, a) * v_policy)
                for a in actions_list
            ]
        )
        policy[x, y] = actions_list[i_max]


policy_t = policy_0

v_policy_0 = evaluate_policy(policy_0)

n_iterations = int(
    np.ceil(
        np.log((norm(v_policy_0, np.inf) + (1 / (1 - gamma))) / epsilon)
        / np.log(1 / gamma)
```

```
    )
)

for i in range(n_iterations):
    v_policy_t = evaluate_policy(policy_t)

    improve_policy(policy_t, v_policy_t)

policy_opt = policy_t

print(np.flipud(policy_opt.T))
```

```
[['R' 'R' 'R' 'R' 'U']
 ['U' 'U' 'U' 'R' 'U']
 ['U' 'R' 'R' 'R' 'U']
 ['U' 'D' 'U' 'U' 'U']
 ['R' 'R' 'R' 'R' 'U']]
```

The learned policy is shown above. The number of iterations was $T = 162$. We can evaluate the policy to view the value function:

```
v_policy_opt = evaluate_policy(policy_opt)
value_text = []
for val, (x, y) in zip(v_policy_opt, each_state):
    value_text.append(rf"v^{{\pi_T}}(s_{{ {x},{y} }}) &= {val:g} \\")

value_text = "\n".join(value_text)
```

The value $v^{\pi_T}(s)$ is shown for each state below:

$$
\begin{aligned}
v^{\pi_T}(s_{0,0}) &= 11.8594 \\
v^{\pi_T}(s_{1,0}) &= 12.6235 \\
v^{\pi_T}(s_{2,0}) &= 13.4931 \\
v^{\pi_T}(s_{3,0}) &= 14.3797 \\
v^{\pi_T}(s_{4,0}) &= 15.3223 \\
v^{\pi_T}(s_{0,1}) &= 11.3527 \\
v^{\pi_T}(s_{1,1}) &= 11.8921 \\
v^{\pi_T}(s_{2,1}) &= 0
\end{aligned}
$$

$$v^{\pi_T}(s_{3,1}) = 0$$
$$v^{\pi_T}(s_{4,1}) = 16.3265$$
$$v^{\pi_T}(s_{0,2}) = 11.9941$$
$$v^{\pi_T}(s_{1,2}) = 12.5169$$
$$v^{\pi_T}(s_{2,2}) = 13.3594$$
$$v^{\pi_T}(s_{3,2}) = 16.1985$$
$$v^{\pi_T}(s_{4,2}) = 17.3965$$
$$v^{\pi_T}(s_{0,3}) = 12.7438$$
$$v^{\pi_T}(s_{1,3}) = 0$$
$$v^{\pi_T}(s_{2,3}) = -20$$
$$v^{\pi_T}(s_{3,3}) = 15.7238$$
$$v^{\pi_T}(s_{4,3}) = 18.607$$
$$v^{\pi_T}(s_{0,4}) = 13.577$$
$$v^{\pi_T}(s_{1,4}) = 14.4667$$
$$v^{\pi_T}(s_{2,4}) = 15.4148$$
$$v^{\pi_T}(s_{3,4}) = 18.5082$$
$$v^{\pi_T}(s_{4,4}) = 20$$