Disclaimer: These notes are derived from the following textbooks by John D. Anderson Jr:
1) Modern Compressible Flow, 2003, McGraw Hill
2) Hypersonic and High Temperature Gas Dynamics, 2000, AIAA

Talk Outline

1) What is Compressibility? When does it arise?

2) Approach / Equations

3) Shocks Application

4) All the Scary monsters

Motivation

→ We have to solve problems actual problems in the real world. That means analyzing the world as it actually exists, and not some idealized version. The most important problems are the ones that present themselves to us, not the ones we pick out of our own interest.

What We know

- Incompressible NS:

$$\begin{cases} \rho(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}) = -\nabla P + \mu\nabla^2\vec{u} \\ \nabla\cdot\vec{u} = 0 \end{cases}$$

→ Conservation of momentum, from Cauchy momentum eqn.

→ Conservation of mass

- Bernoulli: $P + \frac{1}{2}\rho|\vec{u}|^2 = \text{Const}$ → Conservation of energy

→ Notice these equations assume that density and viscosity are Constant
↳ temp dependent

* → These no longer hold when the density is non-Constant, which is the rule in life, not the exception
→ Rockets, gas engines, turbine engines, high speed, air planes > 250 mph, internal Combustion, ...

What Flow regimes are Compressible?

Def: Compressibility of a fluid: $\tau = -\frac{1}{V}\frac{dv}{dp}$, $[\tau] = \left[\frac{L^2}{F}\right]$
↳ volume

→ we know that fluids heat up w/ pressure and other transport (diffusion) thus we need to define this isentropically (reversible)

$$\tau = -\frac{1}{V}\frac{\partial v}{\partial p}$$ → Constant entropy

adiabatic: no change in heat
isentropic: reversible, no diffusion

Water: $\tau = 5\times 10^{-10}\ m^2/N$ at 1 atm
Air: $\tau = 10^{-5}\ m^2/N$ at 1 atm

→ Flows must be modeled as Compressible when Change in pressure results in Change in density too large to ignore

Ballpark: Density Change > 5% ⇒ Compressible ~ 250 mph for air

Basic Chemistry & Thermodynamics

↳ Science of energy

Perfect Gas: Intermolecular forces ignored, Ideal gas law $pv = nRT$
dependent on dist between particles
Eqn of state $p = \rho RT$

Calorically perfect: Constant C_v and C_p, ideal gas law holds

Thermally perfect: $C_v(T)$, $C_p(T)$, ideal gas law holds

Real gas: $C_v(T,P)$, $C_p(T,P)$, No ideal gas law

Van der Waals: $(p + \frac{a}{v^2})(v - b) = RT$, a and b depend on the gas
→ Real gases assumption requires high pressures (≈ 1000 atm) or low temp ($\approx 30K$)

$T < 1{,}000\ K$: Calorically perfect
$T > 1{,}000\ K$: Vibration of molecules makes it thermally perfect
$T > 2{,}500\ K$: Dissasociation, Chemically reacting perfect gases
$T > 9{,}000\ K$: Oxygen and Nitrogen ionization
$P > 1{,}000$ atm: Real gas

Dissassociation
($O_2 \rightarrow O$ at $> 2{,}500\ K$)
($N_2 \rightarrow N$ at $> 4{,}000\ K$)

Ex: $\approx 11{,}000\ K$ at stagnation point on Apollo re-entry

Real gas internal energy $e = e(T,v)$, enthalpy $h = e + pv = h(T,p)$

Thermally perfect: $e = e(T)$, $h = h(T)$

Calorically perfect: $e = C_v T$, $h = C_p T$

Most compressible flows are calorically perfect

Def: $\gamma = C_p / C_v$ ratio of specific heats

C_p: At constant pressure, the amount of heat needed to be supplied to produce a unit change in temperature

$$C_p = \left.\frac{\delta Q}{dT}\right|_{p = const}$$

Adiabatic: No heat added or removed
Reversible: No dissipative phenomena, zero diffusion, thermal conductivity, mass diffusion
Isentropic: Adiabatic and reversible

Units: heat $[Q] = \left[\frac{L^2 m}{T^2}\right]$, $[F] = \left[\frac{mL}{T^2}\right]$

Compressible Flows Approach

1) Write down Fundamental laws
   - → Mass is conserved
   - → $F = ma$ (time rate of Change of momentum)
   - → Energy is conserved

2) Apply to model of flow (Finite control volume, Infinitesimal fluid element, or Molecular approach)

3) Extract equations which embody such physical principles

Solving for $(P, \rho, T, \vec{v}) \to 6$ unknowns, Control Volume V with Surface S

Continuity (1 eqn)

$$\underbrace{-\int_S (\rho\vec{v})\cdot\hat{n}\,ds}_{\text{mass entering V}} = \partial_t \int_V \rho\,dv \qquad \left(\text{Incomp: } -\int_S \rho\vec{v}\cdot n = -\int_V \nabla\cdot(\rho\vec{v}) = -\rho\int_V \nabla\cdot\vec{v} = 0\right)$$

Momentum ($F = ma$ = time rate change in momentum) (3 eqn)

RHS: Forces acting on V, $F = ma$

$$\underbrace{\int_S (\rho\vec{v}\cdot\hat{n})\vec{v}\,ds}_{\text{momentum leaving V}} + \underbrace{\int_V \partial_t(\rho\vec{v})\,dV}_{\text{time change in total momentum in V}} = \underbrace{\int_V \rho\vec{F}\,dV}_{\text{Body forces}} - \underbrace{\int_S p\hat{n}\,ds}_{\text{pressure on Surface}} + F_{\text{viscous}}$$

F_{viscous}: Viscous forces integrated over surface $-\int_S \hat{n}\cdot\mu(\frac{1}{2}\nabla\vec{u}+\frac{1}{2}\nabla\vec{u}^T)$

For Incompressible, all we need is mass and momentum equations

Tie it all together!

Energy Conservation (1 eqn) (Internal: e, kinetic: $\frac{1}{2}|\vec{v}|^2$)

Work = Force · Distance

$$\int_V \dot{q}\rho\,dv - \int_S (p\vec{v}\cdot\hat{n})\,ds + \int_V \rho(\vec{f}\cdot\vec{v})\,dv = \int_V \partial_t\left[\rho\left(e+\tfrac{1}{2}|\vec{v}|^2\right)\right] - \int_S \rho\left(e+\tfrac{1}{2}|V|^2\right)(\vec{v}\cdot\hat{n})\,ds$$

- $\dot{q}$ ↳ rate heat added per unit mass
- $\int_V \dot{q}\rho\,dv$: Rate heat added to Volume from surroundings
- $\int_S (p\vec{v}\cdot\hat{n})\,ds$: Rate Work done on V from Pressure forces on S.
- $\int_V \rho(\vec{f}\cdot\vec{v})\,dv$: Rate work done on V due to body forces
- $\int_V \partial_t[\dots]$: Time rate change of total energy in V
- $\int_S \rho(\dots)\,ds$: Energy leaving V

Tie it all together!

Equation of State (to relate density, pressure, Temperature) (1 eqn)
→ $P = \rho RT$ (ideal) and $e = e(T, v)$

Enthalpy $h = e + pv = h(T, p)$
→ What gas regime we are in changes our eqn of state!

Dimensionless Numbers

$Re = \frac{\rho U L}{\mu} \sim \frac{\text{inertial forces}}{\text{Viscous forces}}$ $\qquad M_\infty^2 \sim \frac{\text{Flow kinetic energy}}{\text{Flow internal energy}}$

$\text{Prandtl} = \frac{\mu C_p}{k} \sim \frac{\text{Frictional dissipation}}{\text{Thermal Conduction}}$ $\qquad$ Ratio Specific heats $= \frac{C_p}{C_v}$

Shocks (Oblique)

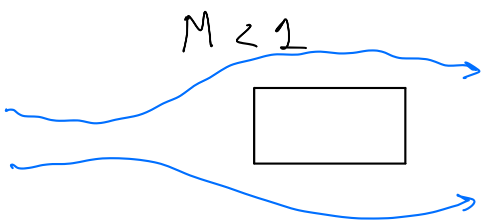
Static pressure: P, dynamic: $\frac{1}{2}\rho|\vec{v}|^2$, total energy $= P + \frac{1}{2}\rho|\vec{v}|^2$

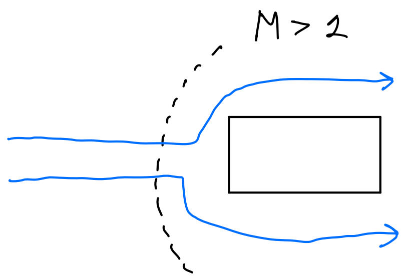
→ very thin region (10^{-5} cm std air), over which flow switches from supersonic to subsonic
→ Static pressure, temp, and density increase across it

→ Mach 1 = 343 m/s, 767 mph, dependent on Temp and the medium
→ Mach number = $\frac{\text{Speed}}{\text{Speed of Sound}}$

→ When a body is placed in > Mach 1 flow, the presence of the body cannot be propogated upstream by sound waves (since they are slower than the flow)
→ When V < Mach 1, the streamlines begin to change and compensate for the body far upstream
→ When V > Mach 1, sound waves cannot propogate upstream and instead coalesce near body at shock wave
→ In front of the shockwave there is no knowledge of the shock
→ Behind the shockwave the flow is subsonic

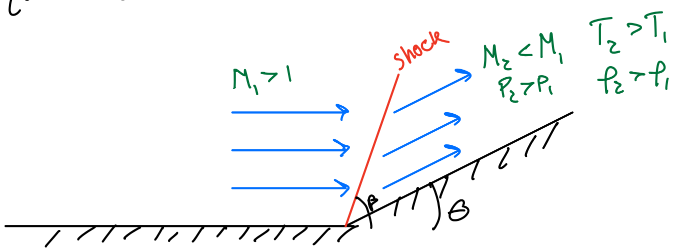
* Exchanging kinetic energy into internal energy *





→ Oblique shocks are when the flow is "turned into itself" over a wedge, vehicle, object.
→ Inherintly two dimensional

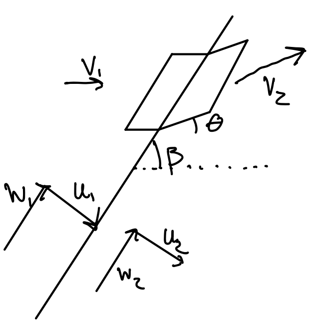
Oblique Shock Relations



Want: u, v, T, P, ρ behind shock

→ We know the incoming freestream conditions, how do we find the values of u, v, T, ρ, P behind the shock?

→ Apply integral forms of conservation equations to control volume



w_1, u_1 are tangential and normal components of velocity wrt. shock orientation

We essentially transform the shockwave to be vertical

$M_{n_1} = M_1 \sin(\beta)$ → Normal component of upstream mach number

⇒ Continuity: $\rho_1 u_1 = \rho_2 u_2$

⇒ Momentum: $\begin{cases} w_1 = w_2 & \text{(Tangential Component of velocity is Conserved!)} \\ p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \end{cases}$

⇒ Energy equation: $h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$

Enthalpy: $h = e + pv = h(T, p)$

⇒ Take away: Changes across free stream shock are governed by <u>Normal</u> Component

⇒ Assuming Calorically perfect gas and adiabatic flow, with $p = \rho RT$ and $h = C_p T$
(Constant $C_p, C_v, \gamma = C_p/C_v$)

Functions of M_1 and γ only!

$$\begin{cases} \frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_{n_1}^2}{(\gamma-1) M_{n_1}^2 + 2} \\ \frac{p_2}{p_1} = 1 + \left(\frac{2\gamma}{\gamma+1}\right)\left(M_{n_1}^2 - 1\right) \\ M_{n_2}^2 = \frac{M_{n_1}^2 + [2/(\gamma-1)]}{[2\gamma/(\gamma-1)] M_{n_1}^2 - 1}, \quad M_2 = \frac{M_{n_2}}{\sin(\beta-\theta)} \\ \frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \end{cases}$$

<u>θ-β-M relation</u>: $\tan\theta = 2\cot\beta\left[\frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + \cos(2\beta)) + 2}\right]$

→ For a given M_1, there is a maximum deflection angle θ_{max}, if $\theta > \theta_{max}$ the shock is detached and curved

→ $\theta < \theta_{max}$ produces 2 values of β, nature prefers the smaller one (determined by back pressure)

→ β varies inversely with M_1

Awesome: For Calorically perfect, M_2, ρ_2/ρ_1, p_2/p_1, T_2/T_1 as functions of M_1 only (Good up to $M_1 > 5$)

For Thermally perfect: M_1 and T_1 dependence

For Chemically reacting: M_1, T_1, p_1

Very High temp: No closed form solutions

Entropy increases across the shock

→ Very large velocity and temp gradients drive viscous diffusion

Scary Monsters

A list of all the physically relevant, complex phenomena in supersonic and hypersonic flow fields past bodies.

- → Boundary layers
- → Boundary layer non-linear interaction with inviscid flow
- → Boundary layer laminar - turbulence transition
- → Thin shock layers and boundary layer - shock interaction at high mach number
- → High temperature effects at high velocities
  - → Stagnation point heating, chemically reacting boundary layer
  - → Gas vibrational excitement, dissociation, ionization, molecular radiation
  - → Non-equilibrium flows at low density
  - → Vehicle ablation
- → Loss of continuum equations at high altitude, low density
- → Entropy layer with strong vorticity interaction

Coefficients determining forces and heating

Dynamic pressure $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$, Body lift $= L$, Body drag $= D$, Body surface area $= S$

Lift Coeff: $C_L = \dfrac{L}{q_\infty S}$

Drag Coeff: $C_D = \dfrac{D}{q_\infty S}$

Normal-force Coeff: $C_N = \dfrac{N}{q_\infty S}$, $N =$ normal force on body

Axial-force Coeff: $C_A = \dfrac{A}{q_\infty S}$, $A =$ Axial force on body