Singularity Solutions for Particles in Stokes Flow

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1 Introduction

This is a short collection of notes on singularity solutions to the Stokes equations. Most of the information here is derived from the Microhydrodynamics books by Graham [1] and Kim & Karilla [2]. Other references that I've found useful for Stokes flow problems include Acheson [3] and Batchelor [4].

2 Arriving at the Stokes Equations

Before we analyze the Stokes equations it's fair to ask where the equations arise in fluid dynamics. So let us begin with the Navier-Stokes equations,

$$\begin{aligned} \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) &= -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{1}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, and ρ and μ are the density and dynamic viscosity, respectively. The first equation is essentially Newton's second law for a parcel of fluid: the LHS is the mass times acceleration of the fluid and the RHS constitutes the forces acting on the fluid. The vector $\mathbf{f}$ denotes any additional forcing such as gravity. Note that the first equation has units of force per volume. The second equation is simply the incompressibility condition for the fluid flow.

Getting from the Navier-Stokes equations to the Stokes equations is a matter of making the equations dimensionless and throwing away terms that are negligible in the correct environments. If we know the general properties of the flow which we are investigating then we can rewrite the velocity, time, and pressure variables as

$$\mathbf{u} = \mathbf{U}\mathbf{u}', \quad t = \frac{L}{U}t', \quad p = \frac{\mu U}{L}p', \tag{2}$$

where $\mathbf{U}$ and L are the characteristic velocity and length while $\mathbf{u}'$, t', and p' are dimensionless quantities.

Substituting these expressions into (1) provides

$$
\begin{aligned}
\rho\left(\mathbf{U}(\partial_t\mathbf{u}') + \mathbf{U}^2(\mathbf{u}'\cdot\nabla\mathbf{u}')\right) &= -\frac{\mu\mathbf{U}}{L}\nabla p' + \mu\mathbf{U}\nabla^2\mathbf{u}' + \frac{\rho\mathbf{U}^2}{L}\mathbf{f}',\\
\rho\left(\frac{\mathbf{U}^2}{L}\mathbf{u}'_t + \frac{\mathbf{U}^2}{L}(\mathbf{u}'\cdot\nabla'\mathbf{u}')\right) &= -\frac{\mu\mathbf{U}}{L^2}\nabla' p' + \frac{\mu\mathbf{U}}{L^2}(\nabla')^2\mathbf{u}' + \frac{\rho\mathbf{U}^2}{L}\mathbf{f}',\\
\frac{\rho\mathbf{U}^2}{L}\left(\mathbf{u}'_t + \mathbf{u}'\cdot\nabla'\mathbf{u}'\right) &= -\frac{\mu\mathbf{U}}{L^2}\nabla' p' + \frac{\mu\mathbf{U}}{L^2}(\nabla')^2\mathbf{u}' + \frac{\rho\mathbf{U}^2}{L}\mathbf{f}',\\
\frac{\rho UL}{\mu}\left(\mathbf{u}'_t + \mathbf{u}'\cdot\nabla'\mathbf{u}'\right) &= -\nabla' p' + (\nabla')^2\mathbf{u}' + \frac{\rho\mathbf{U}L}{\mu}\mathbf{f}',\\
Re\left(\mathbf{u}'_t + \mathbf{u}'\cdot\nabla'\mathbf{u}'\right) &= -\nabla' p' + (\nabla')^2\mathbf{u}' + Re\mathbf{f}',
\end{aligned}
\tag{3}
$$

where $Re = (\rho UL)/\mu$ is the dimensionless Reynolds number. The primes indicate dimensionless quantities. The key here is the Reynolds number describing the system. If the characteristic length and velocity scales are very small compared to the viscosity, then the LHS of the dimensionless equations goes to zero while the RHS stays on the order of one. In the limit as $Re \to 0$ the LHS vanishes and we find that

$$
\begin{aligned}
-\nabla p + \mu\nabla^2\mathbf{u} &= \mathbf{0},\\
\nabla\cdot\mathbf{u} &= 0,
\end{aligned}
\tag{4}
$$

which are the Stokes equations. One way to interpret the Stokes equations is that there is no net force on the fluid leading to no acceleration.

3 General Solution to the Stokes Equations

Again consider the Stokes equations in an unbounded region,

$$
-\nabla p + \mu\nabla^2\mathbf{u} = \mathbf{0}, \tag{5}
$$

$$
\nabla\cdot\mathbf{u} = 0. \tag{6}
$$

Take the divergence of the first equation,

$$
\begin{aligned}
\partial_j\partial_j p &= \mu\left(\partial_j\partial_k\partial_k u_j\right)\\
&= \mu\left(\partial_k\partial_k\partial_j u_j\right)\\
&= 0 \implies \nabla^2 p = 0.
\end{aligned}
\tag{7}
$$

The last equality comes from the divergence free condition on $\mathbf{u}$. Thus the pressure is harmonic. This motivates us to seek solutions $\mathbf{u}$ to the Stokes equations as the sum of homogeneous and particular solutions to the vector Laplace equation, i.e. $\mathbf{u} = \mathbf{u}^H + \mathbf{u}^P$ where

$$
\nabla^2\mathbf{u}^H = \mathbf{0}, \tag{8}
$$

$$
\nabla^2\mathbf{u}^P = \frac{1}{\mu}\nabla p. \tag{9}
$$

Let p_∞ be the pressure at infinity. Then the particular solution is given as

$$
\mathbf{u}^P = \frac{1}{2\mu}\mathbf{x}(p - p_\infty). \tag{10}
$$

We can check this,

$$
\begin{aligned}
\partial_j u_i^P &= \frac{1}{2\mu}\left(\delta_{ij}(p-p_\infty) + x_i\partial_j p\right) \\
\partial_j\partial_j u_i^P &= \frac{1}{2\mu}\left[\delta_{ij}\partial_j p + \delta_{ij}\partial_j p + x_i\partial_j\partial_j p\right] \\
&= \frac{1}{2\mu}\left[\partial_i p + \partial_i p + x_i\nabla^2 p\right] \\
&= \frac{1}{\mu}\partial_i p = \frac{1}{\mu}\nabla p
\end{aligned}
\tag{11}
$$

Incompressibility requires that $\nabla \cdot \mathbf{u}^H = -\nabla \cdot \mathbf{u}^P$, so we have

$$
\nabla \cdot \mathbf{u}^H = -\frac{1}{2\mu}\left[\partial_i x_i(p-p_\infty) + x_i\partial_i p\right] = -\frac{1}{2\mu}(3(p-p_\infty) + \mathbf{x}\cdot\nabla p). \tag{12}
$$

As $\mathbf{u}^H$ and p both satisfy Laplace's equation we feel that we can use general solutions to Laplace's equation to build our solution for the Stokes equations.

4 Spherical Harmonics

Above we saw that harmonic functions will provide solutions to the Stokes equations. Lets investigate the properties of harmonic functions. Laplace's equation in spherical coordinates without ϕ or θ dependence reduces to

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) = 0, \tag{13}
$$

which produces the solution

$$
F = \frac{A}{r} + B. \tag{14}
$$

The first term in the solution is the Green's function for Laplace's equation in three dimensions,

$$
\nabla^2 f(\mathbf{x}) = \delta(\mathbf{x}) \implies f(\mathbf{x}) = \frac{1}{4\pi r}. \tag{15}
$$

The Green's function tells us how Laplace's equation responds to a delta forcing at a given point. Now consider solving Poisson's equation, $\nabla^2 u = \varphi(x)$, for some function $\varphi(x)$. We can simply integrate the Green's function over the support of $\varphi(x)$ to recover the solution,

$$
u(x) = \int \frac{\varphi(x')}{4\pi|x-x'|}dx'. \tag{16}
$$

A key idea that we will leverage to derive solutions to the Stokes equations is that if f is a solution to Laplace's equation for $r > 0$ then so is $\nabla^n f$ for $n = 0, 1, 2, ...$ since

$$
\nabla^2\nabla f = \partial_i\partial_i\partial_j f = \partial_j(\partial_i\partial_i f) = \nabla\delta(\mathbf{x}). \tag{17}
$$

This thinking yields the decaying vector spherical harmonics as repeated gradient operations on $1/r$.

$$\nabla \frac{1}{r} = \partial_i \frac{1}{r} = -\frac{x_i}{r^3}, \tag{18}$$

$$\nabla\nabla \frac{1}{r} = \partial_j \partial_i \frac{1}{r} = \frac{3x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3}, \tag{19}$$

$$\nabla\nabla\nabla \frac{1}{r} = \partial_k \partial_j \partial_i \frac{1}{r} = \frac{3}{r^5}(\delta_{ik} x_j + \delta_{kj} x_i + \delta_{ij} x_k) - 15\frac{x_i x_j x_k}{r^7}. \tag{20}$$

Note that the repeated gradient operations are invariable under exchange of any two indices. All of the proceeding solutions decay as $r \to \infty$, and in general we have the asymptotic relationship

$$\nabla^n \frac{1}{r} \sim r^{-(n+1)}. \tag{21}$$

5 The Stokeslet

We will now utilize this class of decaying vector harmonics to derive the Stokeslet solution. Lets return to the Stokes equations in an unbounded fluid subjected to a point force at the origin,

$$\begin{gathered} -\nabla p + \mu \nabla^2 \mathbf{u} = -\mathbf{F}\delta(\mathbf{x}), \\ \nabla \cdot \mathbf{u} = 0. \end{gathered} \tag{22}$$

As the Stokes equation is a linear PDE, the velocity and pressure fields must depend linearly on the the force $\mathbf{F}$ exerted at the origin. Using our decaying vector harmonic solutions to Laplace's equation with the assumption that the pressure p decays to to p_∞ at infinity provides that

$$p = p_\infty + \alpha \left(\nabla \frac{1}{r}\right) \cdot \mathbf{F} = p_\infty - \alpha \frac{x_i}{r^3} F_i \tag{23}$$

for some unknown scalar α. Again splitting the velocity field $\mathbf{u} = \mathbf{u}^H + \mathbf{u}^P$, we take the homogeneous part to be scalar multiples of the first and third decaying vector harmonics multiplied by the force $\mathbf{F}$,

$$u_i^H = \beta \frac{1}{r} F_i + \gamma(\partial_i \partial_j \frac{1}{r}) F_j. \tag{24}$$

The second term behaves as $1/r^3$, which is overly singular at the origin, corresponding to $\nabla\nabla\delta(\mathbf{x})$. Since we only have a regular delta forcing at the origin we set $\gamma = 0$. Recalling our particular solution $\mathbf{u}^P$ from earlier we then have the velocity field as

$$u_i = \frac{1}{2\mu} x_i (p - p_\infty) + v_i^H = -\frac{\alpha}{2\mu} \frac{x_i x_j}{r^3} F_j + \frac{\beta}{r} F_i. \tag{25}$$

Enforcing incompressibility provides that

$$\begin{aligned}
\partial_i u_i &= -\frac{\alpha}{2\mu}\partial_i\left(\frac{x_i x_j}{r^3}\right)F_j + \partial_i\frac{\beta}{r}F_i \\
&= -\frac{\alpha}{2\mu}\left(3\frac{x_j}{r^3} + \frac{x_i\delta_{ij}}{r^3} - 3\frac{x_i x_j x_i}{r^5}\right)F_j - \frac{\beta x_i F_i}{r^3} \\
&= -\frac{\alpha}{2\mu}\left(3\frac{x_j}{r^3} + \frac{x_j}{r^3} - 3\frac{x_j}{r^3}\right)F_j - \frac{\beta x_i F_i}{r^3} \\
&= -\frac{\alpha}{2\mu}\frac{x_j}{r^3}F_j - \beta\frac{x_j}{r^3}F_j = 0 \implies \beta = -\frac{\alpha}{2\mu}
\end{aligned} \tag{26}$$

With β we can now write the velocity field as

$$u_i = -\frac{\alpha}{2\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right)F_j \tag{27}$$

This is starting to look familiar, and we just need to find α now. The last piece of the puzzle is given by integrating the Stokes equation over an arbitrary volume containing the delta forcing.

$$\begin{aligned}
\int_V \nabla\cdot\sigma + \delta(\mathbf{x})\mathbf{F}\ d\mathbf{x} &= 0 \\
\int_V \nabla\cdot\sigma\ d\mathbf{x} &= -\mathbf{F} \\
\int_S \hat{\mathbf{n}}\cdot(-p\delta_{ij} + \mu(\partial_i u_j + \partial_j u_i))dS(\mathbf{x}) &= -\mathbf{F}
\end{aligned} \tag{28}$$

Now we need the gradient of $\mathbf{u}$,

$$\begin{aligned}
\nabla\mathbf{u} = \partial_i u_j &= -\frac{\alpha}{2\mu}\left(-\frac{\delta_{jk}x_i}{r^3} + \frac{\delta_{ij}x_k}{r^3} + \frac{\delta_{ki}x_j}{r^3} - 3\frac{x_i x_j x_k}{r^5}\right)F_k \\
\nabla\mathbf{u}^T = \partial_j u_i &= -\frac{\alpha}{2\mu}\left(-\frac{\delta_{ij}x_j}{r^3} + \frac{\delta_{ij}x_k}{r^3} + \frac{\delta_{kj}x_i}{r^3} - 3\frac{x_i x_j x_k}{r^5}\right)F_k
\end{aligned} \tag{29}$$

Therefore we have

$$\begin{aligned}
\hat{\mathbf{n}}\cdot\sigma &= -n_i\delta_{ij}p - \frac{\alpha}{2}\left(-n_i\frac{\delta_{jk}x_i}{r^3} + n_i\frac{\delta_{ji}x_k}{r^3} + n_i\frac{\delta_{ki}x_j}{r^3} - 3n_i\frac{x_i x_j x_k}{r^5}\right)F_k \\
&- \frac{\alpha}{2}\left(-n_i\frac{\delta_{ik}x_j}{r^3} + n_i\frac{\delta_{ij}x_k}{r^3} + n_i\frac{\delta_{jk}x_i}{r^3} - 3n_i\frac{x_i x_k x_j}{r^5}\right)F_k \\
&= -n_j p - \frac{\alpha}{2}\left(-n_i\frac{\delta_{jk}x_i}{r^3} + n_j\frac{x_k}{r^3} + n_k\frac{x_j}{r^3} - 3n_i\frac{x_i x_j x_k}{r^5}\right)F_k \\
&- \frac{\alpha}{2}\left(-n_k\frac{x_j}{r^3} + n_j\frac{x_k}{r^3} + n_i\frac{\delta_{jk}x_i}{r^3} - 3n_i\frac{x_i x_k x_j}{r^5}\right)F_k \\
&= -n_j p - \frac{\alpha}{2}\left(n_j\frac{x_k}{r^3} - 3n_i\frac{x_i x_j x_k}{r^5}\right)F_k - \frac{\alpha}{2}\left(n_j\frac{x_k}{r^3} - 3n_i\frac{x_i x_k x_j}{r^5}\right)F_k \\
&= -n_j p - \alpha\left(n_j\frac{x_k}{r^3} - 3n_i\frac{x_i x_j x_k}{r^5}\right)F_k
\end{aligned} \tag{30}$$

Substituting back in the equation for the pressure and using that $\hat{\mathbf{n}} = \mathbf{x}/r$ on the surface of the sphere provides

$$\begin{aligned}\hat{\mathbf{n}} \cdot \sigma &= -n_j p_\infty + \alpha \frac{x_j x_k}{r^4} F_k - \alpha \left(\frac{x_j x_k}{r^4} - 3 \frac{x_i x_i x_j x_k}{r^6} \right) F_k \\ &= -n_j p_\infty + 3\alpha \frac{x_j x_k}{r^4} F_k\end{aligned} \tag{31}$$

Now our surface integral takes the form

$$\begin{aligned}\int_S \hat{\mathbf{n}} \cdot \sigma &= \int_S \left(-n_j p_\infty + 3\alpha \frac{x_j x_k}{r^4} F_k \right) dS(x) \\ &= -p_\infty \int_S \frac{x_j}{r} \, dS(x) + 3\alpha \int_S \frac{x_j x_k}{r^4} F_k \, dS(x)\end{aligned} \tag{32}$$

The first integral is zero since we are integrating over the surface of a sphere with radius ε thus $r = \varepsilon$ constant. The second integral we can rewrite with the divergence theorem

$$\begin{aligned}\int_S \hat{\mathbf{n}} \cdot \sigma &= 3\alpha \int_S \frac{x_j x_k}{r^4} F_k \, dS(x) \\ &= 3\alpha F_k \int_S \frac{r n_j x_k}{r^4} \, dS(x) \\ &= 3\alpha F_k \int_V \frac{\partial_j x_k}{r^3} \, dV \\ &= 3\alpha F_k \int_V \frac{\delta_{jk}}{r^3} \, dV \\ &= 3\alpha F_j \int_V \frac{1}{r^3} \, dV \\ &= 3\alpha F_j \left(\frac{4\pi}{3} \right) = 4\pi\alpha F_j\end{aligned} \tag{33}$$

Recalling that we need this integral to equal $-\mathbf{F}$ gives us $\alpha = -1/(4\pi)$. Therefore we arrive at

$$u_i = \frac{1}{8\pi\mu} \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) F_j, \tag{34}$$

which is the familiar formula for the Stokeslet. Lastly, we can write the Stokeslet in terms of the vector spherical harmonics,

$$G_{ij}(\mathbf{x}) = \frac{1}{8\pi\mu} \left(\delta_{ij} - x_i \partial_j \right) \frac{1}{r} \tag{35}$$

6 Spheres in Flow

Consider a sphere centered at the origin with radius a translating with velocity $\mathbf{U}$ in an infinite Stokes flow. As with deriving the Stokeslet, the pressure and velocity fields must depend linearly on a vector, which is $\mathbf{U}$ this time (last time it was F from the delta forcing). We will again look for a pressure that goes to p_∞ as $r \to \infty$ and split the velocity field as $\mathbf{u} = \mathbf{u}^H + \mathbf{u}^P$. Following the same idea as before, for the pressure field we have

$$p = p_\infty + \alpha U \cdot \nabla \frac{1}{r} = p_\infty - \alpha U_i \frac{x_i}{r^3}. \tag{36}$$

The corresponding particular solution to the velocity is then

$$u_i^P = -\frac{\alpha}{2\mu}\frac{x_i x_j}{r^3}U_j \tag{37}$$

We take the homogeneous solution to the velocity field as a linear combination of the first and third vector spherical harmonics seen earlier,

$$u_i^H = \beta\frac{1}{r}U_i + \gamma\left(\partial_i\partial_j\frac{1}{r}\right)U_j. \tag{38}$$

The origin is no longer part of the flow domain, so γ may be nonzero now. Thus we have the general form for the velocity field

$$u_i = -\frac{\alpha}{2\mu}\frac{x_i x_j}{r^3}U_j + \beta\frac{1}{r}U_i + \gamma\left(\partial_i\partial_j\frac{1}{r}\right)U_j, \tag{39}$$

where we need to solve for the constants α, β, and γ using the incompressibility and no slip conditions. Taking the divergence of the velocity,

$$\begin{aligned}\partial_i u_i &= -\frac{\alpha}{2\mu}\left(3\frac{x_j}{r^3}U_j + \frac{\delta_{ij}x_i}{r^3}U_j - 3\frac{x_i x_j x_i}{r^5}U_j\right) - \beta\frac{x_i}{r^3}U_i + \gamma\left(\partial_j\partial_i\partial_i\frac{1}{r}\right)U_j \\ &= -\frac{\alpha}{2\mu}\left(3\frac{x_j}{r^3}U_j + \frac{x_j}{r^3}U_j - 3\frac{x_j}{r^3}U_j\right) - \beta\frac{x_i}{r^3}U_i \\ &= -\frac{\alpha}{2\mu}\frac{x_j}{r^3}U_j - \beta\frac{x_i}{r^3}U_i = 0 \implies \alpha = -2\mu\beta\end{aligned} \tag{40}$$

Now we have

$$u_i = \frac{\beta}{r}\left(\delta_{ij} + \frac{x_i x_j}{r^2}\right)U_j + \gamma\left(\partial_i\partial_j\frac{1}{r}\right)U_j. \tag{41}$$

The no slip condition at the boundary of the sphere requires that $\mathbf{u} = \mathbf{U}$ when $r = a$. Evaluating the velocity at $r = a$ provides

$$\begin{aligned}u_i|_{r=a} &= \left[\frac{\beta}{r}\left(\delta_{ij} + \frac{x_i x_j}{r^2}\right)U_j + \gamma\left(-\frac{\delta_{ij}}{r^3} + 3\frac{x_i x_j}{r^5}\right)U_j\right]\Bigg|_{r=a} \\ &= \frac{\beta}{a}\left(\delta_{ij} + \frac{x_i x_j}{a^2}\right)U_j + \gamma\left(-\frac{\delta_{ij}}{a^3} + 3\frac{x_i x_j}{a^5}\right)U_j \\ &= \left(\frac{\beta}{a} - \frac{\gamma}{a^3}\right)U_i + \left(\frac{\beta}{a^3} + \frac{3\gamma}{a^5}\right)x_i x_j U_j = U_i\end{aligned} \tag{42}$$

Since the first term is agnostic to the location on the surface of the sphere while the second term depends nonlinearly on the location on the sphere we set the second coefficient to zero and the first to one. This provides two equations,

$$\begin{aligned}\frac{\beta}{a} - \frac{\gamma}{a^3} &= 1, \\ \frac{\beta}{a^3} + \frac{3\gamma}{a^5} &= 0.\end{aligned} \tag{43}$$

Solving these equations provides that $\beta = 3a/4$ and $\gamma = -a^3/4$. Thus the final velocity and pressure fields are

$$u_i = \frac{3a}{4r}\left(\delta_{ij} + \frac{x_i x_j}{r^2}\right)U_j - \frac{a^3}{4}\left(\partial_i\partial_j\frac{1}{r}\right)U_j, \tag{44}$$

$$p = p_\infty + \frac{x_i}{4\pi r^3}F_i. \tag{45}$$

Expanding the second term,

$$u_i = \frac{3a}{4r}\left(\delta_{ij} + \frac{x_i x_j}{r^2}\right)U_j - \frac{a^3}{4r^3}\left(-\delta_{ij} + 3\frac{x_i x_j}{r^2}\right)U_j. \tag{46}$$

The first term in the velocity can be recognized as the Stokeslet, $\mathbf{G}\cdot\mathbf{F}$, corresponding to the force $\mathbf{F} = 6\pi\mu a\mathbf{U}$. Therefore we have recovered Stokes's law for the drag (force) on a sphere. If the background flow is steady $\mathbf{U}_\infty$ then the drag force on the sphere is $\mathbf{F} = -6\pi\mu a(\mathbf{U} - \mathbf{U}_\infty)$. For the second term, we claim that it is equal to a multiple of the Laplacian of the Stokeslet which corresponds to the flow form a source dipole. The term arises from the force produced by the fluid as it is moved by the sphere, and decays as $1/r^3$. Let's prove the claim,

$$\begin{aligned}
\mathbf{G} &= \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right)\\
\nabla\mathbf{G} = \partial_k G_{ij} &= \frac{1}{8\pi\mu}\left(-\frac{\delta_{ij}x_k}{r^3} + \frac{\delta_{ik}x_j}{r^3} + \frac{\delta_{jk}x_i}{r^3} - \frac{3x_i x_j x_k}{r^5}\right)\\
\nabla^2\mathbf{G} = \partial_k\partial_k G_{ij} &= \frac{1}{8\pi\mu}\Bigg(-\frac{3\delta_{ij}}{r^3} + \frac{3\delta_{ij}x_k x_k}{r^5} + \frac{\delta_{ik}\delta_{jk}}{r^3} - \frac{3\delta_{ik}x_j x_k}{r^5} + \frac{\delta_{jk}\delta_{ik}}{r^3}\\
&\quad - \frac{3\delta_{jk}x_i x_k}{r^5} - \frac{3\delta_{ik}x_j x_k}{r^5} - \frac{3x_i\delta_{jk}x_k}{r^5} - \frac{9x_i x_j}{r^5} + \frac{15x_i x_j x_k x_k}{r^7}\Bigg)\\
&= \frac{1}{8\pi\mu}\Bigg(-\frac{3\delta_{ij}}{r^3} + \frac{3\delta_{ij}}{r^3} + \frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5} + \frac{\delta_{ij}}{r^3}\\
&\quad - \frac{3x_i x_j}{r^5} - \frac{3x_j x_i}{r^5} - \frac{3x_i x_j}{r^5} - \frac{9x_i x_j}{r^5} + \frac{15x_i x_j}{r^5}\Bigg)\\
&= \frac{1}{8\pi\mu}\left(\frac{2\delta_{ij}}{r^3} - \frac{6x_i x_j}{r^5}\right) = -\frac{1}{4\pi\mu r^3}\left(-\delta_{ij} + \frac{3x_i x_j}{r^2}\right)
\end{aligned} \tag{47}$$

With our drag force $\mathbf{F} = 6\pi\mu a\mathbf{U}$, we can write the above result as

$$\begin{aligned}
\nabla^2\mathbf{G}\cdot\mathbf{F} &= -\frac{6a}{4r^3}\left(-\delta_{ij} + \frac{3x_i x_j}{r^2}\right)U_j\\
\frac{a^2}{6}\nabla^2\mathbf{G}\cdot\mathbf{F} &= -\frac{a^3}{4r^3}\left(-\delta_{ij} + \frac{3x_i x_j}{r^2}\right)U_j
\end{aligned} \tag{48}$$

Now we see that we can write the fluid velocity as

$$\mathbf{u} = \left(1 + \frac{a^2}{6}\nabla^2\right)\mathbf{G}\cdot\mathbf{F}, \tag{49}$$

with the Stokes drag $\mathbf{F} = 6\pi\mu a\mathbf{U}$. Lastly, lets check the traction on the surface of the sphere given by $\mathbf{t} = \hat{\mathbf{n}} \cdot \sigma|_{r=a}$. For the gradient of the velocity we have

$$
\begin{aligned}
\partial_k u_i &= -\frac{3ax_k}{4r^3}\left(\delta_{ij} + \frac{x_i x_j}{r^2}\right) U_j + \frac{3a}{4r}\left(\frac{\delta_{ik}x_j}{r^2} + \frac{\delta_{kj}x_i}{r^2} - \frac{2x_i x_j x_k}{r^4}\right) \\
&+ \frac{3a^3 x_k}{4r^5}\left(-\delta_{ij} + 3\frac{x_i x_j}{r^2}\right) U_j - \frac{a^3}{4r^3}\left(\frac{3\delta_{ik}x_j}{r^2} + \frac{3\delta_{jk}x_i}{r^2} - \frac{6x_i x_j x_k}{r^4}\right) \\
\partial_k u_i|_{r=a} &= -\frac{3x_k}{4a^2}\left(\delta_{ij} + \frac{x_i x_j}{r^2}\right) U_j + \frac{3}{4}\left(\frac{\delta_{ik}x_j}{a^2} + \frac{\delta_{kj}x_i}{a^2} - \frac{2x_i x_j x_k}{a^4}\right) \\
&+ \frac{3x_k}{4a^2}\left(-\delta_{ij} + 3\frac{x_i x_j}{a^2}\right) U_j - \frac{3}{4}\left(\frac{\delta_{ik}x_j}{a^2} + \frac{\delta_{jk}x_i}{a^2} - \frac{2x_i x_j x_k}{r^4}\right) \\
&= -\frac{3x_k}{4a^2}\left(2\delta_{ij}\right) U_j + \frac{6x_i x_j x_k}{4a^4} U_j \\
&= -\frac{3x_k}{2a^2} U_i + \frac{6x_i x_j x_k}{4a^4} U_j \\
\partial_i u_k &= -\frac{3ax_i}{4r^3}\left(\delta_{kj} + \frac{x_k x_j}{r^2}\right) U_j + \frac{3a}{4r}\left(\frac{\delta_{ik}x_j}{r^2} + \frac{\delta_{ij}x_k}{r^2} - \frac{2x_k x_j x_i}{r^4}\right) \\
&+ \frac{3a^3 x_i}{4r^5}\left(-\delta_{kj} + 3\frac{x_k x_j}{r^2}\right) U_j - \frac{a^3}{4r^3}\left(\frac{3\delta_{ik}x_j}{r^2} + \frac{3\delta_{ji}x_k}{r^2} - \frac{6x_i x_j x_k}{r^4}\right) \\
\partial_i u_k|_{r=a} &= -\frac{3x_i}{4a^2}\left(\delta_{kj} + \frac{x_k x_j}{a^2}\right) U_j + \frac{3}{4}\left(\frac{\delta_{ik}x_j}{a^2} + \frac{\delta_{ij}x_k}{a^2} - \frac{2x_k x_j x_i}{a^4}\right) \\
&+ \frac{3x_i}{4a^2}\left(-\delta_{kj} + 3\frac{x_k x_j}{a^2}\right) U_j - \frac{3}{4}\left(\frac{\delta_{ik}x_j}{a^2} + \frac{\delta_{ji}x_k}{a^2} - \frac{2x_i x_j x_k}{a^4}\right) \\
&= -\frac{3x_i}{4a^2}\left(2\delta_{kj} - \frac{2x_k x_j}{a^2}\right) U_j \\
&= -\frac{3x_i}{2a^2}\delta_{kj} U_j + \frac{6x_i x_j x_k}{4a^4} U_j \\
&= -\frac{3x_i}{2a^2} U_k + \frac{6x_i x_j x_k}{4a^4} U_j
\end{aligned}
\tag{50}
$$

Plugging this into the stress tensor, $\sigma = -p\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ and using $\hat{\mathbf{n}} = \mathbf{x}/a$ on the surface of the sphere,

$$
\begin{aligned}
\hat{\mathbf{n}} \cdot \sigma|_{r=a} &= n_k\left(-p\delta_{ki} + \mu\partial_k u_i + \mu\partial_i u_k\right) \\
&= -pn_i + -\frac{3\mu x_k x_k}{2a^3} U_i + \frac{6\mu x_i x_j x_k x_k}{4a^5} U_j - \frac{3\mu x_i x_k}{2a^3} U_k + \frac{6\mu x_i x_j x_k x_k}{4a^5} U_j \\
&= -pn_i + -\frac{3\mu}{2a} U_i + \frac{6\mu x_i x_j}{4a^3} U_j - \frac{3\mu x_i x_k}{2a^3} U_k + \frac{6\mu x_i x_j}{4a^3} U_j
\end{aligned}
\tag{51}
$$

Inserting the equation for the pressure expands this to

$$
\begin{aligned}
\hat{\mathbf{n}} \cdot \sigma|_{r=a} &= -n_i\left(p_\infty\delta_{ij} + \frac{x_k F_k}{4\pi a^3}\right) + -\frac{3\mu}{2a} U_i + \frac{6\mu x_i x_j}{4a^3} U_j - \frac{3\mu x_i x_k}{2a^3} U_k + \frac{6\mu x_i x_j}{4a^3} U_j \\
&= -\hat{\mathbf{n}} p_\infty + \frac{x_i x_k F_k}{4\pi a^4} + -\frac{3\mu}{2a} U_i + \frac{6\mu x_i x_j}{4a^3} U_j - \frac{3\mu x_i x_k}{2a^3} U_k + \frac{6\mu x_i x_j}{4a^3} U_j \\
&= -\hat{\mathbf{n}} p_\infty + \frac{x_i x_k F_k}{4\pi a^4} + -\frac{3\mu}{2a} U_i + \frac{6\mu x_i x_j}{4a^3} U_j - \frac{3\mu x_i x_k}{2a^3} U_k + \frac{6\mu x_i x_j}{4a^3} U_j
\end{aligned}
\tag{52}
$$

The force on the sphere is the Stokes drag, $F = 6\pi\mu a\mathbf{U}$,

$$\begin{aligned}\hat{\mathbf{n}}\cdot\sigma|_{r=a} &= -\hat{\mathbf{n}}p_\infty + \frac{3\mu x_i x_k U_k}{2a^3} + -\frac{3\mu}{2a}U_i + \frac{6\mu x_i x_j}{4a^3}U_j - \frac{3\mu x_i x_k}{2a^3}U_k + \frac{6\mu x_i x_j}{4a^3}U_j \\ &= -\hat{\mathbf{n}}p_\infty - \frac{3\mu}{2a}U_i \\ &= -\hat{\mathbf{n}}p_\infty - \frac{3\mu\mathbf{U}}{2a}\end{aligned} \tag{53}$$

Now we can validate the total force on the sphere, $\mathbf{F} = \int_S \mathbf{t} = \int_S \hat{\mathbf{n}}\cdot\sigma$. Integrating the constant pressure scalar against the normal vector about the surface returns zero and for the second term we are just integrating a constant vector on the surface. Thus the total force is

$$\mathbf{F} = -\frac{3\mu\mathbf{U}}{2a}4\pi a^2 = -6\pi\mu a\mathbf{U} \tag{54}$$

as desired.

7 Spherical Drops with Interior Flows

Now consider a spherical droplet of radius a in translational motion with velocity $\mathbf{U}$. The drop has its own interior flow which alters the drag and the resulting exterior flow field. Lets solve for both the interior and exterior flow fields.

Both fluids satisfy the Stokes equations and the interior fluid must be free of singularities. Let $\mathbf{u}^i$ and $\mathbf{u}^o$ denote the interior and exterior flow fields, respectively. Again we only have the vector $\mathbf{U}$ for the velocity fields to depend linearly on. We assume that the exterior flow field is a linear combination of a Stokeslet and its Laplacian as before,

$$\mathbf{u}^o = \frac{3a}{4}\left(\alpha + \beta a^2\nabla^2\right)\mathbf{G}\cdot\mathbf{U}, \tag{55}$$

where α and β are to be solved for. There cannot be any singularities in the interior flow since the domain contains the origin. The solution is given as a linear combination of the normalized droplet velocity $\mathbf{U}$ and the Stokeson, $\mathbf{H}$, as

$$\begin{aligned}\mathbf{u}^i &= \gamma\hat{\mathbf{U}} + \frac{\eta}{a^2}\mathbf{H}\cdot\mathbf{U}, \\ \mathbf{H}_{ij} &= 2r^2\delta_{ij} - x_i x_j.\end{aligned} \tag{56}$$

The pressure corresponding to the flow produced by a Stokeson is $p = 10\mu\mathbf{x}\cdot\mathbf{U}$. The Stokeson is useful for interior problems as it is finite at the origin and singular at infinity. Now, the constants $\alpha, \beta, \gamma, \eta$ will be solved for by considering the boundary conditions at the surface of the drop. The kinematic condition at $r = a$ is

$$\hat{\mathbf{n}}\cdot\mathbf{u}^i = \hat{\mathbf{n}}\cdot\mathbf{U} = \hat{\mathbf{n}}\cdot\mathbf{u}^o. \tag{57}$$

Since the sphere in this case is a drop, there is not a no slip condition at the interface, but instead a requirement of no relative motion between the two fluids. Additionally, the tangential stress exerted by each fluid at the surface must be equal and opposite. These conditions can be written as

$$\begin{aligned}
(\mathbf{I}-\hat{\mathbf{n}}\hat{\mathbf{n}})\cdot\mathbf{u}^o &= (\mathbf{I}-\hat{\mathbf{n}}\hat{\mathbf{n}})\cdot\mathbf{u}^i,\\
\hat{\mathbf{n}}\cdot\left(\mu^o(\nabla\mathbf{u}^o+(\nabla\mathbf{u}^o)^T)\right)\cdot(\mathbf{I}-\hat{\mathbf{n}}\hat{\mathbf{n}}) &= \hat{\mathbf{n}}\cdot\left(\mu^i(\nabla\mathbf{u}^i+(\nabla\mathbf{u}^i)^T)\right)\cdot(\mathbf{I}-\hat{\mathbf{n}}\hat{\mathbf{n}}).
\end{aligned} \tag{58}$$

Solving these equations provides

$$\begin{aligned}
\alpha &= \frac{2+3\lambda}{3(1+\lambda)},\\
\beta &= \frac{\lambda}{6(1+\lambda)},
\end{aligned} \tag{59}$$

where $\lambda = \mu^i/\mu^o$. We see that as $\lambda \to \infty$ we recover the solution for a rigid sphere while $\lambda \to 0$ retrieves the solution for a bubble.

8 The Multipole Expansion

Spheres and droplets have nice analytical solutions, but what can we say about arbitrary bodies in unbounded Stokes flow? Each infinitesimal element of the particle surface exerts a delta force on the surrounding fluid and these forces can be summed up by integrating the Stokeslet over the surface of the particle,

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty(\mathbf{x}) - \int_{S_p} \mathbf{G}_{ij}(\mathbf{x}-\mathbf{x}')(\hat{\mathbf{n}}_k\sigma_{kj}(\mathbf{x}'))dS(\mathbf{x}'). \tag{60}$$

Here $\hat{\mathbf{n}}$ in the unit normal pointing inwards to get the force on the particle surface. For $\mathbf{x}$ far away from the particle surface, we can Taylor expand the Stokeslet about the center of the particle:

$$\begin{aligned}
G_{ij}(\mathbf{x}-\mathbf{x}') &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\mathbf{x}'\cdot\nabla)^n\mathbf{G}(\mathbf{x})\\
&= G_{ij}|_{\mathbf{x}'=\mathbf{0}} - x'_l\partial_l G_{ij}|_{\mathbf{x}'=\mathbf{0}} + \frac{1}{2}x'_l x'_m\partial_m\partial_l G_{ij}|_{\mathbf{x}'=\mathbf{0}} + O\left(\left(\frac{a}{|\mathbf{x}|}\right)^4\right).
\end{aligned} \tag{61}$$

Note that we take a to be the characteristic length of the particle. We can substitute this into our equation for $\mathbf{u}$ to yield

$$\begin{aligned}
u_i(\mathbf{x}) = u_{\infty,i}(x) &- G_{ij}\big|_{\mathbf{x}'=\mathbf{0}}\int_{S_p} n_k\sigma_{kj}(\mathbf{x}')\,dS(\mathbf{x}')\\
&+ \partial_l G_{ij}\big|_{\mathbf{x}'=\mathbf{0}}\int_{S_p} x'_l(n_k\sigma_{kj}(\mathbf{x}'))dS(\mathbf{x}')\\
&- \frac{1}{2}\partial_m\partial_l G_{ij}\big|_{\mathbf{x}'=\mathbf{0}}\int_{S_p} x'_l x'_m(n_k\sigma_{kj}(\mathbf{x}'))dS(\mathbf{x}')\\
&+ O\left(\left(\frac{a}{|\mathbf{x}|}\right)^4\right).
\end{aligned} \tag{62}$$

The first integral on the RHS is the total stress exerted on the particle by the fluid, i.e. the drag,

$$F_j^{\text{drag}} = \int_{S_p} n_k \sigma_{kj}(\mathbf{x}')\, dS(\mathbf{x}'), \tag{63}$$

which decays as $|\mathbf{x}|^{-1}$. The second term contains the gradient of the Stokeslet which produces point stresses and torques exerted on the fluid, known as a force (or Stokeslet) dipole. The integrand is the position vector on the surface multiplied by the traction $\hat{\mathbf{n}} \cdot \sigma$. We can rewrite the second term as

$$\begin{aligned} D_{lj} G_{ij,l} &= \partial_l G_{ij}\big|_{\mathbf{x}'=\mathbf{0}} \int_{S_p} x_l'(n_k \sigma_{kj}(\mathbf{x}')) dS(\mathbf{x}'), \\ D_{lj} &= \int_{S_p} x_l'(n_k \sigma_{kj}(\mathbf{x}')) dS(\mathbf{x}') \end{aligned} \tag{64}$$

where the $,l$ subscript denotes the gradient of G_{ij} indexed by l. Further, D_{lj} can be decomposed into its symmetric and antisymmetric portions,

$$\begin{aligned} S_{lj} &= \frac{1}{2}(D_{lj} + D_{jl}) = \frac{1}{2} \int_{S_p} \left(x_l'(\hat{\mathbf{n}} \cdot \sigma)_j + x_j'(\hat{\mathbf{n}} \cdot \sigma)_l \right) dS(\mathbf{x}'), \\ A_{lj} &= \frac{1}{2}(D_{lj} - D_{jl}) = \frac{1}{2} \int_{S_p} \left(x_l'(\hat{\mathbf{n}} \cdot \sigma)_j - x_j'(\hat{\mathbf{n}} \cdot \sigma)_l \right) dS(\mathbf{x}'). \end{aligned} \tag{65}$$

The symmetric portion S_{lj} is called the Stresslet and is the pure straining motion arising from the force dipole. The antisymmetric portion A_{lj} is termed the Rotlet and constitutes the flow field produced by a point torque.

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