



THE UNIVERSITY OF TEXAS AT AUSTIN  
McCOMBS SCHOOL OF BUSINESS

# Diagnostics & Transformations

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## Lecture 12

STA 371G

Predicting the fuel economy (miles per gallon) for different car models of the 70s.



Predicting the fuel economy (miles per gallon) for different car models of the 70s.



“LINE” assumptions:

- Linearity
- Normally distributed errors
- Independent errors
- Equal Variance (Homoscedasticity)

## Predicting MPG from Horsepower

```
model<-lm(MPG ~ HP, data=auto_mpg)
summary(model)
```

Call:

```
lm(formula = MPG ~ HP, data = auto_mpg)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-13.5710	-3.2592	-0.3435	2.7630	16.9240

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	39.935861	0.717499	55.66	<2e-16	***
HP	-0.157845	0.006446	-24.49	<2e-16	***

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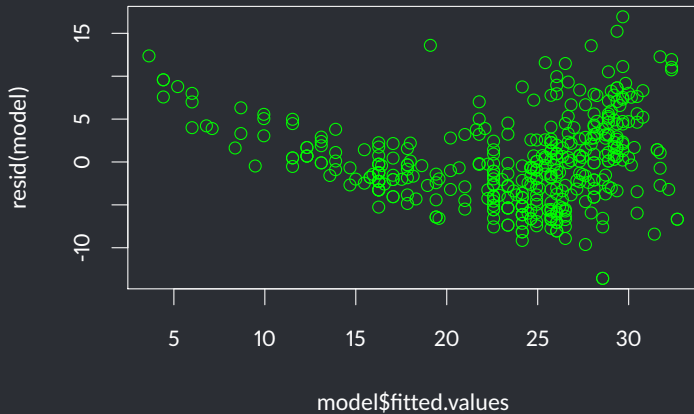
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.906 on 390 degrees of freedom

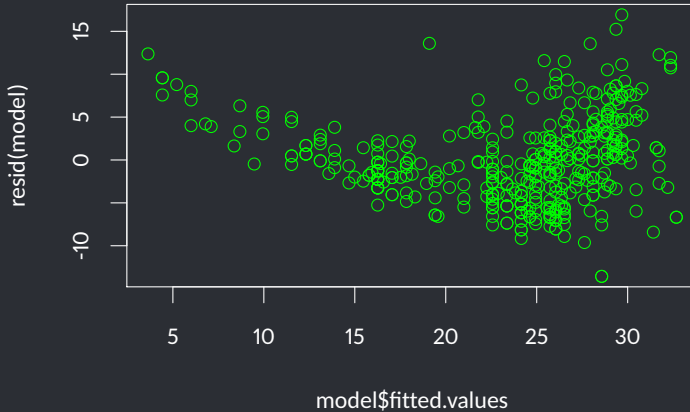
Multiple R-squared: 0.6059, Adjusted R-squared: 0.6049

F-statistic: 599.7 on 1 and 390 DF, p-value: < 2.2e-16

```
plot(model$fitted.values, resid(model), col='green')
```



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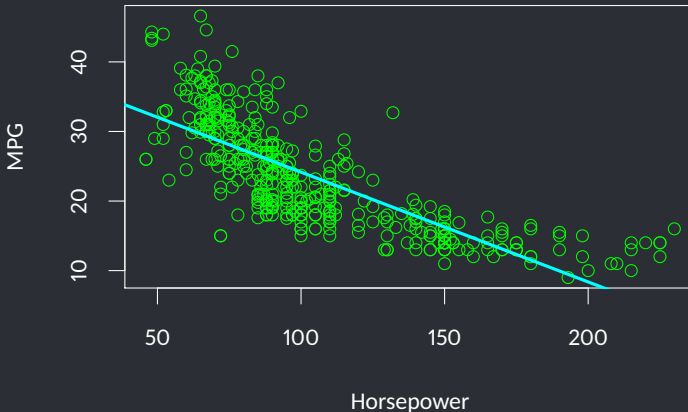


The trend in the residuals implies linearity issues.  
The “funnel” implies equal variance issues.

## Addressing the linearity issue

The relation between MPG and horsepower does not seem to be linear.

```
plot(auto_mpg$HP, auto_mpg$MPG, col='green',  
      xlab='Horsepower', ylab='MPG')  
abline(model, col='cyan', lwd=3)
```



## Addressing the linearity issue

If we could horizontally shift the data on the far right towards the left, the plot would look more linear.



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So let's predict the MPG of a car not from the horsepower, but from a “transformation” of the horsepower.

## Addressing the linearity issue

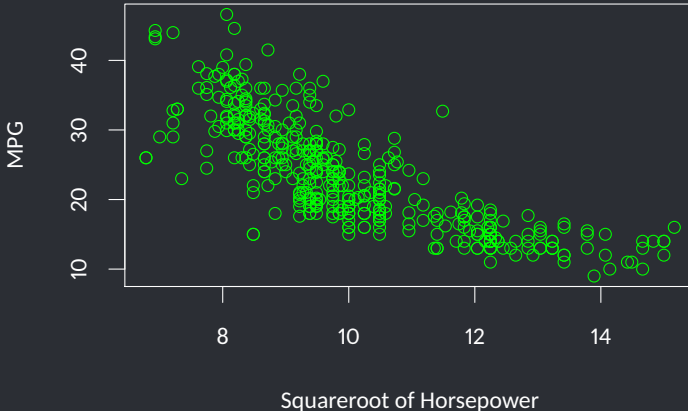
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So let's predict the MPG of a car not from the horsepower, but from a “transformation” of the horsepower.

For example, the relation between MPG and HP is not linear, but the one between MPG and  $\sqrt{\text{HP}}$  could be!

## Addressing the linearity issue

```
auto_mpg$HP_sqrt <- sqrt(auto_mpg$HP)
plot(auto_mpg$HP_sqrt, auto_mpg$MPG, col='green',
      xlab='Squareroot of Horsepower', ylab='MPG')
```



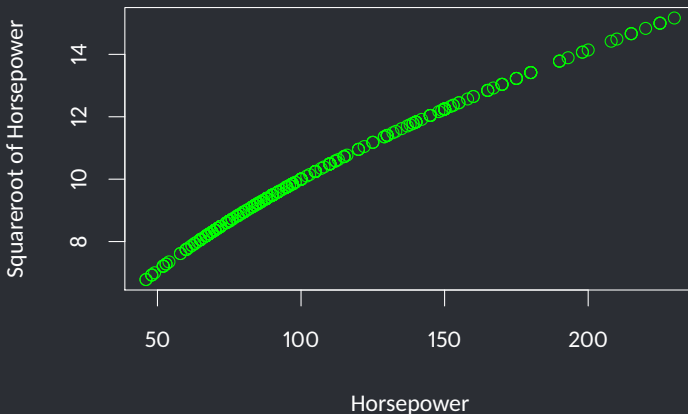
## Addressing the linearity issue

It indeed seems a bit better. Notice the change in the range of the horizontal axis.

## Addressing the linearity issue

It indeed seems a bit better. Notice the change in the range of the horizontal axis. It has changed from [49,225] to [7,15]. The shift is larger for the data on the far right.

```
plot(auto_mpg$HP, auto_mpg$HP_sqrt, col='green',  
      xlab='Horsepower', ylab='Squareroot of Horsepower')
```



## Addressing the linearity issue

```
model2 <- lm(MPG ~ HP_sqrt, data=auto_mpg)
summary(model2)
```

Call:

```
lm(formula = MPG ~ HP_sqrt, data = auto_mpg)
```

Residuals:

Min	1Q	Median	3Q	Max
-13.9768	-3.2239	-0.2252	2.6881	16.1411

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )	
(Intercept)	58.705	1.349	43.52	<2e-16	***
HP_sqrt	-3.503	0.132	-26.54	<2e-16	***

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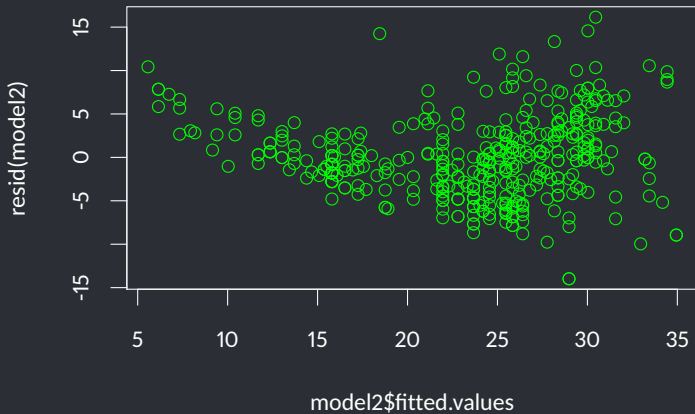
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.665 on 390 degrees of freedom

Multiple R-squared: 0.6437, Adjusted R-squared: 0.6428

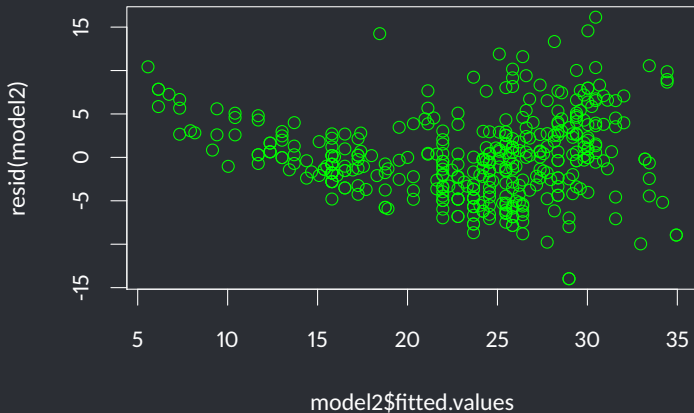
F-statistic: 704.6 on 1 and 390 DF, p-value: < 2.2e-16

```
plot(model2$fitted.values, resid(model2), col='green')
```



The trend flattened a bit.

```
plot(model2$fitted.values, resid(model2), col='green')
```



The trend flattened a bit.

Can we do better? Let's try some other transformation.



# Logarithmic transformation

One of the most common transformations is the logarithmic transformation with base  $e$  (natural logarithm).

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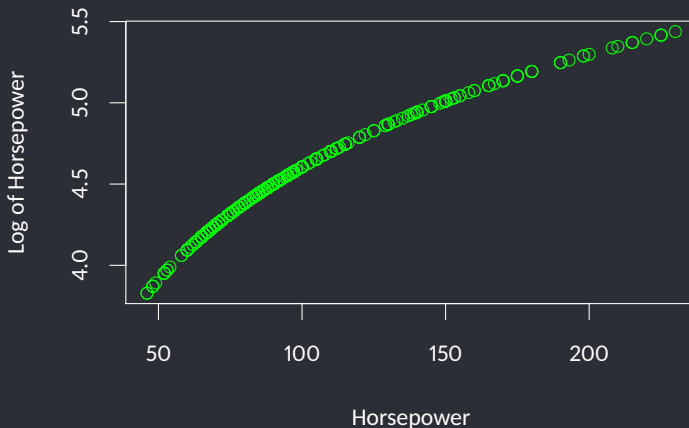
$$\log(e^2) = 2$$

In general:

$$y = e^x \quad \longleftrightarrow \quad \log y = x.$$

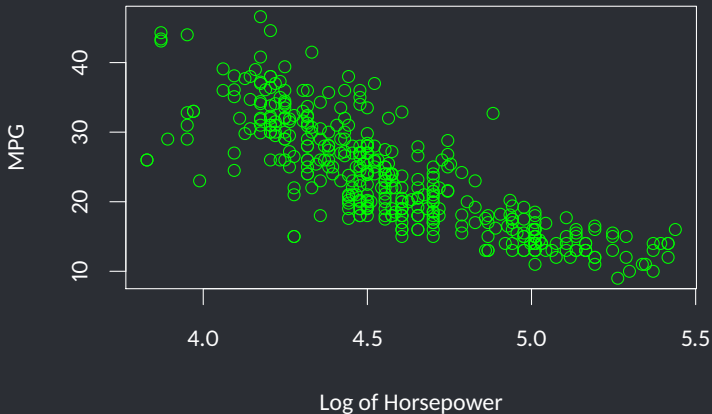
## Logarithmic transformation

```
auto_mpg$HP_ln <- log(auto_mpg$HP)
plot(auto_mpg$HP, auto_mpg$HP_ln, col='green',
      xlab='Horsepower', ylab='Log of Horsepower')
```



## Logarithmic transformation

```
plot(auto_mpg$HP_ln, auto_mpg$MPG, col='green',  
      xlab='Log of Horsepower', ylab='MPG')
```



```
model3 <- lm(MPG ~ HP_ln, data=auto_mpg)
summary(model3)
```

Call:

```
lm(formula = MPG ~ HP_ln, data = auto_mpg)
```

Residuals:

Min	1Q	Median	3Q	Max
-14.2299	-2.7818	-0.2322	2.6661	15.4695

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	108.6997	3.0496	35.64	<2e-16 ***
HP_ln	-18.5822	0.6629	-28.03	<2e-16 ***

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Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

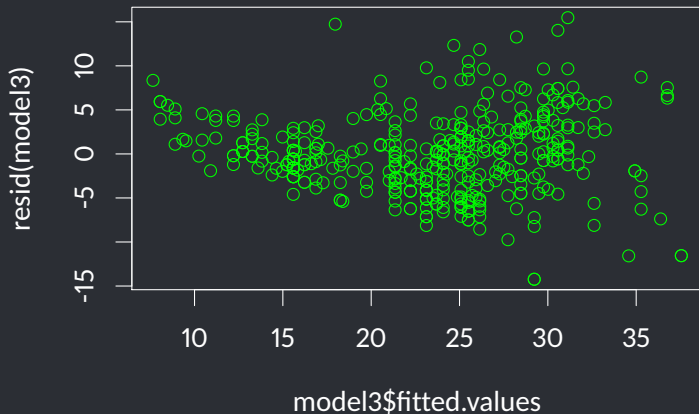
Residual standard error: 4.501 on 390 degrees of freedom

Multiple R-squared: 0.6683, Adjusted R-squared: 0.6675

F-statistic: 785.9 on 1 and 390 DF, p-value: < 2.2e-16



```
plot(model3$fitted.values, resid(model3), col='green')
```



The trend flattened even more.

## Transforming a Predictor

It is equivalent to “cutting the distribution of  $X$  into vertical slices and changing the spacing of the slices.”

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When the nonlinearity is the biggest issue in the model, transforming the predictor is a good start.

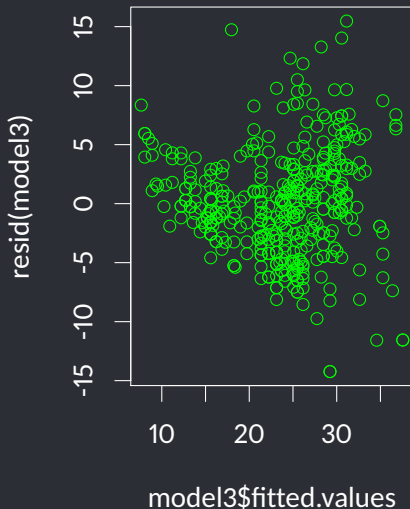
Finding the right transformation is a bit of art, field knowledge and trial and error.

## Transforming a Predictor

$x$	$\sqrt{x}$	$\log x$
1	1	0
10	3.16	2.3
100	10	4.61
1000	31.62	6.91
10000	100	9.21
100000	316.23	11.51

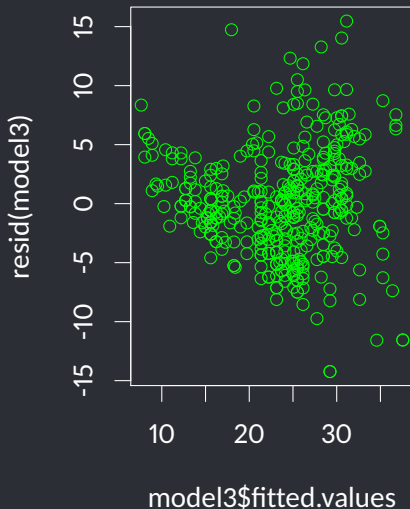
## Addressing the equal variance issue

- The (unexplained) variance in the response is higher in some regions.



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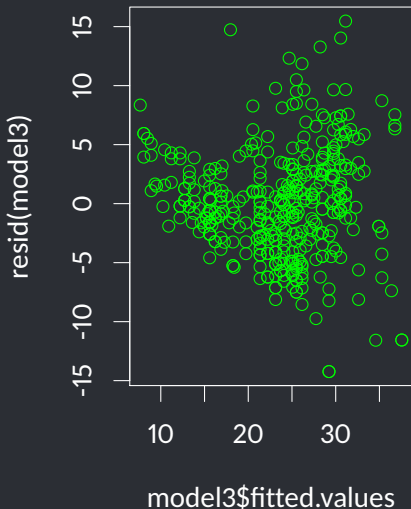
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- Remember how log-transformation shrinks larger numbers more than it shrinks smaller numbers.





## Addressing the equal variance issue

- The (unexplained) variance in the response is higher in some regions.
- Remember how log-transformation shrinks larger numbers more than it shrinks smaller numbers.
- Log-transformation of the response often helps with fixing heteroscedasticity (and non-normality)!



```
auto_mpg$MPG_ln <- log(auto_mpg$MPG)
model4<-lm(MPG_ln ~ HP_ln, data=auto_mpg)
summary(model4)
```

Call:

```
lm(formula = MPG_ln ~ HP_ln, data = auto_mpg)
```

Residuals:

	Min	1Q	Median	3Q	Max
	-0.6523	-0.1218	0.0079	0.1163	0.6373

Coefficients:

	Estimate	Std. Error	t value	Pr(> t )
(Intercept)	6.9606	0.1215	57.3	<2e-16 ***
HP_ln	-0.8418	0.0264	-31.9	<2e-16 ***

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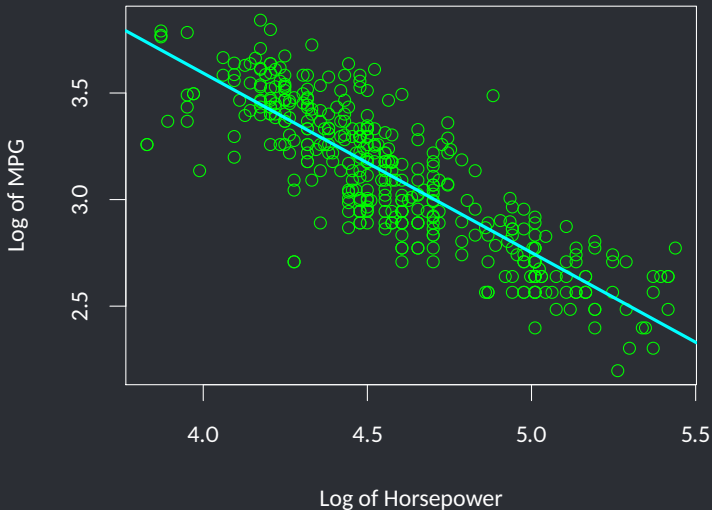
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.18 on 390 degrees of freedom

Multiple R-squared: 0.723, Adjusted R-squared: 0.722

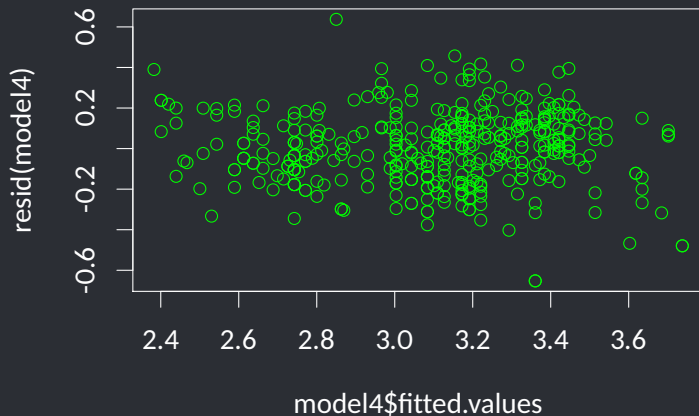
F-statistic: 1.02e+03 on 1 and 390 DF, p-value: <2e-16

```
plot(auto_mpg$HP_ln, auto_mpg$MPG_ln, col='green',  
      xlab='Log of Horsepower', ylab='Log of MPG')  
abline(model4, col='cyan', lwd=3)
```



Beautiful!

```
plot(model4$fitted.values, resid(model4), col='green')
```



Look how awesome our models are getting

Predictor (X)	Response (Y)	$R^2$	Residual SE
HP	MPG	0.61	4.91
$\sqrt{\text{HP}}$	MPG	0.64	4.66
log HP	MPG	0.67	4.5
log HP	log MPG	0.72	0.18

## Interpretation of $\beta$ values

The model before the transformations was:

$$\widehat{\text{MPG}} = 39.94 - 0.16 \cdot \text{HP}$$

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The interpretation of  $-0.16$  is “for each unit of increase in the horsepower, the MPG estimate reduces by 0.16”.

After transforming the predictor or the response, this interpretation does not hold!



## Interpretation of $\beta$ values

When the square root of HP is used:

$$\widehat{\text{MPG}} = 58.71 - 3.5 \cdot \sqrt{\text{HP}}$$



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$$\widehat{\text{MPG}} = 58.71 - 3.5 \cdot \sqrt{\text{HP}}$$

The interpretation of  $-3.5$  is “for each unit of increase in the square root of the horsepower, the MPG estimate reduces by 3.5”.



## Interpretation of $\beta$ values

Similarly, in the following model:

$$\widehat{\text{MPG}} = 108.7 - 18.58 \cdot \log \text{HP}$$



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Similarly, in the following model:

$$\widehat{\text{MPG}} = 108.7 - 18.58 \cdot \log \text{HP}$$

The interpretation of  $-18.58$  is “for each unit of increase in the natural logarithm of the horsepower, the MPG estimate reduces by 18.58”.



## Where to start

- If the model has two or three of the equal variance, normality and linearity issues, try transforming  $Y$ .

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- After transforming the response, if the nonlinearity is not fixed, try transforming the predictor(s) as well.
- There is no rule for which transformations will work in all cases; trial and error may be required.
- Remember, the interpretations of the coefficients will change after you transform one or more variables!