Inverting a Matrix through Calculating to RREF

Given the matrix A:

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 1 & -2 & 3 \\ 5 & 1 & 2 \end{bmatrix}$$

We want to find A^{-1}.

Step 1

Start by augmenting A with the identity matrix:

$$\left[\begin{array}{ccc|ccc} -4 & 2 & 1 & 1 & 0 & 0 \\ 1 & -2 & 3 & 0 & 1 & 0 \\ 5 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Now, swap the rows:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 0 & 1 & 0 \\ -4 & 2 & 1 & 1 & 0 & 0 \\ 5 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Perform Gaussian elimination (row reduction) to convert the left side to the identity matrix. After performing RREF operations, we get:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{13} & -\frac{3}{13} & \frac{4}{13} \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\frac{9}{13} & \frac{14}{13} & -\frac{10}{13} \end{array}\right]$$

Conclusion

The left-hand side is now the identity matrix, and the right-hand side is the inverse of A:

$$A^{-1} = \begin{bmatrix} \frac{1}{13} & -\frac{3}{13} & \frac{4}{13} \\ 1 & -1 & 1 \\ -\frac{9}{13} & \frac{14}{13} & -\frac{10}{13} \end{bmatrix}$$

If we cannot reduce to RREF, the matrix is not invertible.

Find A^{-1}

We are solving the system $AX + B = CX$. We want to express it as:

$$B = CX - AX$$

$$B = (C - A)X$$

Thus,

$$(C - A)^{-1}B = X$$

We need to compute $C - A$.

$$C - A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -5 \end{bmatrix}$$

Now, we compute the inverse of $C - A$.
To find $(C - A)^{-1}$, we first compute the determinant:

$$\det(C - A) = (-1)(-5) - (2)(1) = 5 - 2 = 3$$

Thus,

$$(C - A)^{-1} = \frac{1}{3} \begin{bmatrix} -5 & -1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Next, we compute:

$$(C - A)^{-1} \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$$

Performing the matrix multiplication:

$$\begin{bmatrix} -\frac{5}{3}(0) + -\frac{1}{3}(2) & -\frac{5}{3}(3) + -\frac{1}{3}(1) \\ -\frac{2}{3}(0) + -\frac{1}{3}(2) & -\frac{2}{3}(3) + -\frac{1}{3}(1) \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{16}{3} \\ -\frac{2}{3} & -\frac{7}{3} \end{bmatrix}$$

Thus, the solution is:

$$X = \begin{bmatrix} -\frac{2}{3} & -\frac{16}{3} \\ -\frac{2}{3} & -\frac{7}{3} \end{bmatrix}$$

Calculating the Inverse of Matrices

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 10 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

Let A, B, and C represent the matrices on the left, middle, and right, respectively.

Getting Inverse of A using Minors

The formula for the inverse of a matrix is:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

For this matrix, $\det(A) = 1$, so:

$$A^{-1} = 1 \cdot \operatorname{adj}(A)$$

We create the submatrices of A:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For each cofactor:

$$+\begin{vmatrix} 1 & 5 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 5 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$-\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix}$$

$$+\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} - \begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

Getting the Determinants of Each Submatrix

The resulting matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 11 & -5 & 1 \end{bmatrix}$$

After transposing, we obtain the inverse of A:

$$A^{-1} = \begin{bmatrix} 1 & -3 & 11 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $X = A^{-1}CB^{-1}$

After a few steps of solving:

$$\begin{bmatrix} 1 & -3 & 11 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{30} & \frac{4}{15} \end{bmatrix} = \begin{bmatrix} \frac{77}{15} & -\frac{106}{15} \\ -\frac{61}{30} & \frac{49}{15} \\ \frac{7}{15} & -\frac{11}{15} \end{bmatrix}$$