STOCHASTIC PROCESSES NOTES

Contents

1. Preliminaries

1.1. Basic Properties.

Definition 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(S, \mathcal{G})$ a measurable space, T an index set and $X : T \times \Omega \to S$. Then X is said to be a* ***stochastic processes*** *if for each $t \in T$, $X(t, \cdot)$ is $\mathcal{F}$-$\mathcal{G}$ measurable (i.e. X is just a collection of random variables indexed by time).*

Note 1.2. *We will work with $T = [0, \infty)$ and $(S, \mathcal{G}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and we often write $X_t(\omega)$ to mean $X(t, \omega)$ as well as the function $X(\cdot, \omega)$.*

Definition 1.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(S, \mathcal{G})$ a measurable space, T an index set and $X : T \times \Omega \to S$ a stochastic process. We can define a function $\Phi_X : \Omega \to S^T$ by $\Phi_X(\omega)(t) = X(t, \omega)$. The* ***law*** *of X is defined to be the pushforward measure of $\mathbb{P}$ by Φ_X and is denoted by $\mathcal{L}_X$. (i.e. $\mathcal{L}_X(A) = \mathbb{P}(\Phi_X^{-1}(A))$ for $A \in \mathcal{G}$)*

Definition 1.4. *Let X be a process and $\omega \in \Omega$. The function $X(\cdot, \omega)$ is called the* ***sample path*** *of ω. It is common to define properties of a process in terms of its sample paths.*

Definition 1.5. *Let X be a process. Then X is said to be* ***measurable*** *if $X : [0, \infty) \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0, \infty)) \bigotimes \mathcal{F}$-measurable.*

Definition 1.6. *Let X be a process. Then X is said to be* ***bounded*** *if there exists $K > 0$ such that for each $(t, \omega) \in [0, \infty) \times \Omega$, $|X(t, \omega)| < K$ (i.e. $X : [0, \infty) \times \Omega \to \mathbb{R}$ is bounded).*

Note 1.7. *The definition above changes from book to book, but in these notes, this is how we define bounded and it is very important to use this one instead of the path-wise a.s. defintion.*

Definition 1.8. *Let X be a process. Then X is said to be* ***continous*** *if for each $\omega \in \Omega$, $X(\cdot, \omega)$ is continous.*

Note 1.9. *We may similarly define* ***right continuous***, ***left continuous***, ***cadlag*** *and* ***caglad***.

Note 1.10. *There are a few notions of equivalence among processes. Below we mention two.*

Definition 1.11. *Let X and Y be processes. Then X is said to be a* ***modification*** *of Y if for each $t \geq 0$, $N_t = \{\omega \in \Omega : X_t(\omega) \neq Y_t(\omega)\}$ is a null set.*

Definition 1.12. *Let X and Y be processes. Then X is said to be* ***indistinguishable*** *from Y if $N = \{\omega \in \Omega : \text{ for some } t \geq 0, X_t(\omega) \neq Y_t(\omega)\}$ is a null set.*

Note 1.13. *These definitions clearly generate an equivalence relation and obviously indistinguishable implies modification.*

Exercise 1. *Let X and Y be processes. Suppose that X and Y are modifications and that a.s. (i.e. except on a null set) X and Y are right continuous. Then X and Y are indistinguishable.*

Proof. By assumption, $N_r = \{\omega \in \Omega : X_t(\omega) \text{ is not right continuous or } Y_t(\omega) \text{ is not right continuous}\}$ is a null set . Let $\omega \in \Omega \cap N_r^c$. Right continuity tells us that for each $t \geq 0$, $X_t(\omega) = Y_t(\omega)$ if and only if for each $t \in [0, \infty) \cap \mathbb{Q}$, $X_t(\omega) = Y_t(\omega)$. Since X and Y are modifications, for each $t \in [0, \infty) \cap \mathbb{Q}$, $N_t = \{\omega \in \Omega : X_t(\omega) \neq Y_t(\omega)\}$ is a null set. Thus

$$\begin{aligned} N &= \{\omega \in \Omega : \text{ for some } t \geq 0, X_t(\omega) \neq Y_t(\omega)\} \\ &= (N \cap N_r) \cup (N \cap N_r^c) \\ &= (N \cap N_r) \cup \{\omega \in \Omega \cap N_r^c : \text{ for some } t \in [0, \infty) \cap \mathbb{Q}, X_t(\omega) \neq Y_t(\omega)\} \\ &= (N \cap N_r) \cup \bigcup_{t \in [0,\infty) \cap \mathbb{Q}} N_t \end{aligned}$$

is a null set. Hence X and Y are indistinguishable. $\square$

Note 1.14. *Observe that if we have a have a process X that is right continuous a.s., say for $\omega \in N^c$, we can define an indistinguishable, right continuous process Y by*

$$Y(\cdot, \omega) = \begin{cases} X(\cdot, \omega) & \omega \in N^c \\ 0 & \omega \in N \end{cases}$$

Definition 1.15. *Let X be a stochastic process and $1 \leq p < \infty$. Then X is said to be* ***p-integrable*** *if*

$$\sup_{t \geq 0} \mathbb{E}|X_t|^p < \infty$$

Note 1.16. *When $p = 1$, we say integrable and when $p = 2$, we say square integrable.*

Definition 1.17. *Let $\mathcal{X}$ be a set of random variables. Then $\mathcal{X}$ is said to be* ***uniformly integrable*** *if*

$$\lim_{k \to \infty} \left[\sup_{X \in \mathcal{X}} \mathbb{E}(|X| \mathbf{1}_{\{|X| > k\}}) \right] = 0$$

Exercise 2. *Let $\mathcal{X}$ be a set of random variables. Then $\mathcal{X}$ is uniformly integrable iff*

(1) *there exists* $M > 0$ *such that for each* $X \in \mathcal{X}$, $\mathbb{E}[|X|] \leq M$
(2) *for each* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for each* $E \in \mathcal{F}$, *if* $\mathbb{P}(E) < \delta$, *then* $\sup\limits_{X \in \mathcal{X}} \mathbb{E}[|X|\mathbf{1}_E] < \epsilon$

Proof. ($\Rightarrow$): (1) Suppose that $\mathcal{X}$ is uniformly integrable. Then by definition, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $X \in \mathcal{X}$, $\mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] < 1$. Choose $M = K + 1$. Let $X \in \mathcal{X}$. Then

$$\begin{aligned}\mathbb{E}[|X|] &= \mathbb{E}[|X|\mathbf{1}_{\{X \leq K\}}] + \mathbb{E}[|X|\mathbf{1}_{\{X > K\}}] \\ &\leq K + 1 \\ &= M\end{aligned}$$

(2) Let $\epsilon > 0$. Then there exists $K_\epsilon \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K_\epsilon$, then for each $X \in X$, $\mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] < \epsilon/2$. Choose $\delta = \epsilon/2K_\epsilon$. Let $E \in \mathcal{F}$. Suppose that $\mathbb{P}(E) < \delta$. Then for each $X \in \mathcal{X}$,

$$\begin{aligned}\mathbb{E}[|X|\mathbf{1}_E] &= \mathbb{E}[|X|\mathbf{1}_{E \cap \{|X| > K_\epsilon\}}] + \mathbb{E}[|X|\mathbf{1}_{E \cap \{|X| \leq K_{ep}\}}] \\ &< \epsilon/2 + K_\epsilon \delta \\ &= \epsilon/2 + \epsilon/2 \\ &= \epsilon\end{aligned}$$

($\Leftarrow$): Choose M as in (1). Suppose for the sake of contradiction that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $X \in \mathcal{X}$ such that $\mathbb{P}(\{|X| > K\}) \geq \epsilon$. Then choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $X_K \in \mathcal{X}$ such that $\mathbb{P}(\{|X_K| > K\}) \geq \epsilon$. So

$$\begin{aligned}\mathbb{E}[|X_K|] &\geq \mathbb{E}[|X_K|\mathbf{1}_{\{|X_K|>K\}}] \\ &\geq K\epsilon \\ &> M\end{aligned}$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $X \in \mathcal{X}$, $\mathbb{P}(\{|X| > K\}) < \epsilon$. Now, let $\epsilon > 0$. Choose K such that for each $X \in \mathcal{X}$, $\mathbb{P}(\{|X| > K\}) < \epsilon$. Since $(\{|X| > k\})_{k \in \mathbb{N}}$ is decreasing, for each $k \in \mathbb{N}$, if $k \geq K$, then for each $X \in \mathcal{X}$, $\mathbb{P}(\{|X| > k\}) < \epsilon$. Thus

$$\lim_{k \to \infty} \sup_{X \in \mathcal{X}} \mathbb{P}(\{|X| > k\}) = 0.$$

Finally, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Then Choose $K \in \mathbb{N}$ such that for $k \in \mathbb{N}$, if $k \geq K$, then for each $X \in \mathcal{X}$, $\mathbb{P}(\{|X| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $X \in \mathcal{X}$,

$$\mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] < \epsilon \text{ and therefore } \lim_{k \to \infty} \sup_{X \in \mathcal{X}} \mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] = 0.$$

□

Exercise 3. *Let* $1 < p < \infty$ *and* $\mathcal{X} \subset L^p(\Omega, \mathcal{F}, \mathbb{P})$. *If* $\sup\limits_{X \in \mathcal{X}} \mathbb{E}[|X|^p] < \infty$, *then* $\mathcal{X}$ *is uniformly integrable.*

Proof. Let $k \in \mathbb{N}$ and $X \in \mathcal{X}$. Then

$$\begin{aligned}\mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] &\leq \mathbb{E}\left[\frac{|X|^p}{k^{p-1}}\mathbf{1}_{\{|X|>k\}}\right] \\ &\leq \frac{\mathbb{E}[|X|^p]}{k^{p-1}}\end{aligned}$$

Therefore

$$\sup_{X\in\mathcal{X}} \mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] \leq \frac{1}{k^{p-1}} \sup_{X\in\mathcal{X}} \mathbb{E}[|X|^p]$$

which implies that

$$\lim_{k\to\infty}\left[\sup_{X\in\mathcal{X}} \mathbb{E}(|X|\mathbf{1}_{\{|X|>k\}})\right] = 0$$

Hence $\mathcal{X}$ is uniformly integrable. □

1.2. **Filtrations.**

Definition 1.18. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ a collection of sub σ-algebras of $\mathcal{F}$ such that for each $s, t \in [0,\infty)$, $s < t$ implies that $\mathcal{F}_s \subset \mathcal{F}_t$. Then $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be a* ***filtration*** *of $\mathcal{F}$.*

Definition 1.19. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be* ***complete*** *if for each $t \in [0,\infty)$, $\mathcal{F}_t$ contains all the null sets of $\mathcal{F}$. (i.e. $\mathcal{F}_0$ contains all the null sets of $\mathcal{F}$)*

Definition 1.20. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t\geq 0}$ is said to be* ***right continuous*** *if for each $t \in [0,\infty)$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$*

Definition 1.21. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t\geq 0}$ is said to satisfy the* ***usual conditions*** *if $\{\mathcal{F}_t\}_{t\geq 0}$ is complete and right continuous.*

Definition 1.22. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Define $\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \geq 0)$.*

Definition 1.23. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. Then X is said to be* ***adapted*** *to $\{\mathcal{F}_t\}_{t\geq 0}$ if for each $t \in [0,\infty)$, X_t is $\mathcal{F}_t$ measurable*

Definition 1.24. *Let X be a process. Then the* ***minimal augmented filtration*** *of X is defined to be the smallest filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. (here, "smallest" means that if $\{\mathcal{G}_t\}$ is filtration satisfying the usual conditions to which X is adapted, then for each $t \in [0,\infty)$, $\mathcal{F}_t \subset \mathcal{G}_t$)*

Definition 1.25. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Then X is said to be progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if for each $t \geq 0$, $X|_{[0,t]\times\Omega}$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable.*

Exercise 4. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. Suppose that X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. If X is right continuous or X is left continuous, then X is progressively measurable.*

Proof. Fist suppose that each path of X is right continuous. Let $t \geq 0$. Define $X_n : [0,t] \times \Omega \to \mathbb{R}$ by $X_n(s,\omega) = X_0(\omega)\mathbf{1}_{\{0\}}(s) + \sum_{k=1}^{2^n} X_{tk/2^n}(\omega)\mathbf{1}_{(t(k-1)/2^n, tk/2^n]}(s)$. Since X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, X_n is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable. Let $(s,\omega) \in [0,t] \times \Omega$. Suppose that $s > 0$. Then for each $n \in \mathbb{N}$, there exists a unique $k_n \in \mathbb{N}$ such that $k_n \leq 2^n$ and $t(k_n - 1)/2^n < s \leq tk_n/2^n$. Thus $X_n(s,\omega) = X_{tk_n/2^n}(\omega)$, $tk_n/2^n \geq s$ and $tk_n/2^n \to s$. Since each path of

X is right continuous, $X_n(s,\omega) \to X_s(\omega)$. If $s=0$, then $X_n(s,w) = X_0(\omega) = X_s(\omega)$. Hence $X_n(s,\omega) \to X_s(\omega)$. Since $X_n \to X|_{[0,t]\times\Omega}$ pointwise, $X|_{[0,t]\times\Omega}$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable. So X is progressively measurable.
Now suppose that each path of X is left continuous. Let $t \geq 0$. Define $X_n : [0,t] \times \Omega \to \mathbb{R}$ by $X_n(s,\omega) = \sum_{k=1}^{2^n} X_{t(k-1)/2^n}(\omega)\mathbf{1}_{[t(k-1)/2^n, tk/2^n)}(s) + X_t(\omega)\mathbf{1}_{\{t\}}(s)$. The rest of the proof is similar to the other case, just this time utilizing left continuity. □

Definition 1.26. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Define the **predictable σ-algebra** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on $[0,\infty) \times \Omega$, denoted by $\mathcal{P}$, by*

$$\mathcal{P} = \sigma(X : X \textit{ is left continuous and adapted to } \{\mathcal{F}_t\}_{t\geq 0})$$

Definition 1.27. *Let X be a process. Then X is said to be **predictable** if X is $\mathcal{P}$-measurbable.*

Note 1.28. *Let X be a process. If X is predictable, then X is adapted.*

Definition 1.29. *Let X be a process and $T : \Omega \to [0,\infty]$ a random variable. Define the **stopped process of** X **at** T, denoted X^T, by $X^T_t(\omega) = X_{t\wedge T(\omega)}(\omega)$.*

Note 1.30. *It is clear that if X is right continuous or left continuous, then so is X^T.*

2. Stopping Times

2.1. Basic Properites.

Definition 2.1. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and $T : \Omega \to [0,\infty]$ a random variable. Then T is said to be a **stopping time** for $\{\mathcal{F}_t\}_{t\geq 0}$ if for each $t \in [0,\infty]$, $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$.*

Note that if $t = \infty$, then

$$\{\omega \in \Omega : T(\omega) \leq t\} = \Omega \in \mathcal{F}_\infty$$

We will typically just say that T is a stopping time when the filtration is clear.

Proposition 2.2. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Then we have the following:*

1. *if S and T are stopping times, then $S+T$ is a stopping time.*
2. *if S is a stopping time, then for $\alpha \geq 1$, αS is a stopping time.*
3. *if there exists $c \in [0,\infty]$ such that $T \equiv c$, then T is a stopping time.*
4. *If $(T_n)_{n\in\mathbb{N}}$ is a sequence of stopping times, then $\sup\limits_{n\in\mathbb{N}} T_n$ and $\inf\limits_{n\in\mathbb{N}} T_n$ are stopping times.*

Note that the third statement holds for $T \equiv c$ almost surely if $\{\mathcal{F}_t\}_{t\geq 0}$ is complete and the last statement implies that mins $\limsup T_n$ and $\liminf T_n$ and that for stopping times S and T, $S \wedge T$ and $S \vee T$ are stopping times.

The proof is similar to showing measurbaility of the same functions.

Proposition 2.3. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a right continuous filtration and $T : \Omega \to [0,\infty]$ a random variable. Then T is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$ if and only if for each $t \geq 0$, $\{\omega \in \Omega : T(\omega) < t\} \in \mathcal{F}_t$.*

Proof. Suppose that for each $s \geq 0$, $\{\omega \in \Omega : T(\omega) < s\} \in \mathcal{F}_s$. Let $t \geq 0$. Let $m \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $n \geq m$ implies that $\{T < t + 1/n\} \in \mathcal{F}_{t+1/n} \subset \mathcal{F}_{t+1/m}$. So $\bigcap\limits_{n\in\mathbb{N}} \{T < t + 1/n\} \in \mathcal{F}_{t+1/m}$. Since $m \in \mathbb{N}$ is arbitrary, $\bigcap\limits_{n\in\mathbb{N}} \{T < t + 1/n\} \in \bigcap\limits_{m\in\mathbb{N}} \mathcal{F}_{t+1/m}$. Then

$$\begin{aligned}
\{\omega \in \Omega : T(\omega) \leq t\} &= \bigcap_{n\in\mathbb{N}} \{\omega \in \Omega : T(\omega) < t + 1/n\} \\
&\in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{t+1/n} \\
&= \mathcal{F}_{t^+} \\
&= \mathcal{F}_t \qquad \text{(right continuity)}
\end{aligned}$$

Hence T is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Conversely, suppose that T is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Let $t \geq 0$. Then

$$\begin{aligned}
\{\omega \in \Omega : T(\omega) < t\} &= \bigcup_{n\in\mathbb{N}} \{\omega \in \Omega : T(\omega) \leq t - 1/n\} \\
&\in \mathcal{F}_t
\end{aligned}$$

□

Exercise 5. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be right continuous and $T : \Omega \to [0, \infty)$ a finite stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Define $T_n : \Omega \to [0, \infty)$ by $T_n(\omega) = k_\omega/2^n$ where $k_\omega \in \mathbb{N}$ is the unique positive integer such that $(k_\omega - 1)/2^n \leq T(\omega) < k_\omega/2^n$. Then for each $n \in \mathbb{N}$, T_n is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.*

Proof. Let $k \in \mathbb{N}$. Then we observe that

$$\begin{aligned}
\{\omega \in \Omega : T_n(\omega) = k/2^n\} &= \{\omega \in \Omega : (k-1)/2^n \leq T(\omega) < k/2^n\} \\
&= T^{-1}([0, k/2^n)) \setminus T^{-1}([0, (k-1)/2^n)) \\
&\in \mathcal{F}_{k/2^n} \quad \text{(right continuity)}
\end{aligned}$$

Now let $t \in [0, \infty)$. Choose $k \in \mathbb{N}$ such that $(k-1)/2^n \leq t < k/2^n$. Then

$$\begin{aligned}
T_n^{-1}([0, t]) &= \bigcup_{i=1}^{k-1} T_n^{-1}(\{i/2^n\}) \\
&\in \mathcal{F}_{(k-1)/2^n} \\
&\subset \mathcal{F}_t
\end{aligned}$$

Hence T_n is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.

□

Definition 2.4. *Let X be a process and $\Lambda \in \mathcal{B}(\mathbb{R})$. Then the* ***hitting time*** *of Λ is defined to be the random variable $T_\Lambda : \Omega \to [0, \infty]$ given by $T_\Lambda(\omega) = \inf\{t > 0 : X_t(\omega) \in \Lambda\}$.*

Theorem 2.5. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a complete filtration, X an a.s. right continuous process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and $\Lambda \subset \mathbb{R}$ an open set. Then T_Λ is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.*

Proof. Let $t \geq 0$. Note that for $\omega \in \Omega$, if $X(\cdot, \omega)$ is right continuous, then since Λ is open, we have that $T_\Lambda(\omega) < t$ if and only if there exists $s \in (0, t)$ such that $X_s(\omega) \in \Lambda$ if and only if there exists $s \in \mathbb{Q} \cap (0, t)$ such that $X_s(\omega) \in \Lambda$. The completeness of $\{\mathcal{F}_t\}_{t\geq 0}$ tells us that for each $s \in [0, \infty)$, $\mathcal{F}_s$ has all the null sets of $\mathcal{F}$. Finally, Λ being open and X being adapted tell us that for each $s < t$, $X_s^{-1}(\Lambda) \in \mathcal{F}_s \subset \mathcal{F}_t$.

By assumption $N = \{\omega \in \Omega : X(\cdot, \omega) \text{ is not right continuous}\}$ is a null set, so for each $s \in [0, \infty)$, $N \in \mathcal{F}_s$ and $N^c \in \mathcal{F}_s$. The previous note implies that

$$\begin{aligned}
\{\omega \in \Omega : T_\Lambda(\omega) < t\} &= (\{\omega \in \Omega : T_\Lambda(\omega) < t\} \cap N^c) \cup (\{\omega \in \Omega : T_\Lambda(\omega) < t\} \cap N) \\
&= \left(\bigcup_{s \in \mathbb{Q} \cap (0,t)} \{\omega \in \Omega : X_s(\omega) \in \Lambda\} \cap N^c \right) \cup \left(\{\omega \in \Omega : T(\omega) < t\} \cap N \right) \\
&\in \mathcal{F}_t
\end{aligned}$$

Hence T_Λ is a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$. $\square$

Note that the same result is true if X is an a.s. left continuous process. Also, as usual, we could get rid of the assumption that $\{\mathcal{F}_t\}_{t \geq 0}$ is complete if each sample path of X were right or left continuous instead of the paths being almost surely right or left continuous.

2.2. **Stopping Algebra.**

Definition 2.6. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$. The **stopping algebra** of T to be $\mathcal{F}_T = \{A \in \mathcal{F} : \text{ for each } t \geq 0, A \cap T^{-1}([0, t]) \in \mathcal{F}_t\}$*

Note that $\mathcal{F}_T$ is a σ-algebra and that if $\{\mathcal{F}_t\}_{t \geq 0}$ is complete, then so is $\mathcal{F}_T$.
Note that for $T \equiv t$, $\mathcal{F}_T = \mathcal{F}_t$.

Exercise 6. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be right continous and T a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$. Then $\mathcal{F}_T = \{A \in \mathcal{F} : \text{ for each } t > 0, A \cap \{T < t\} \in \mathcal{F}_t\}$.*

Proof. Define $\mathcal{F}_{T_<} = \{A \in \mathcal{F} : \text{ for each } t > 0, A \cap \{T < t\} \in \mathcal{F}_t\}$. Let $A \in \mathcal{F}_T$ and $t > 0$. Since for each $n \in \mathbb{N}$, $A \cap \{T \leq t - 1/n\} \in \mathcal{F}_{t-1/n} \subset \mathcal{F}_t$, we know that $\bigcup_{n \in \mathbb{N}} \big(A \cap \{T \leq t - 1/n\}\big) \in \mathcal{F}_t$. Thus

$$\begin{aligned}
A \cap \{T < t\} &= A \cap \left(\bigcup_{n \in \mathbb{N}} \{T \leq t - 1/n\} \right) \\
&= \bigcup_{n \in \mathbb{N}} \big(A \cap \{T \leq t - 1/n\}\big) \\
&\in \mathcal{F}_t
\end{aligned}$$

and $A \in \mathcal{F}_{T_<}$. Conversely, suppose that $A \in \mathcal{F}_{T_<}$ and let $t \geq 0$. Let $m \in \mathbb{N}$. Then for $n \in \mathbb{N}$, $n \geq m$ implies that $A \cap \{T < t + 1/n\} \in \mathcal{F}_{t+1/n} \subset \mathcal{F}_{t+1/m}$. Then $\bigcap_{n \in \mathbb{N}} \big(A \cap \{T <$

$t+1/n\}\big) \in \mathcal{F}_{t+1/m}$. Since $m \in \mathbb{N}$ is arbitrary, $\bigcap\limits_{n\in\mathbb{N}} \big(A \cap \{T < t+1/n\}\big) \in \bigcap\limits_{m\in\mathbb{N}} \mathcal{F}_{t+1/m}$. Thus

$$\begin{aligned} A \cap \{T \le t\} &= A \cap \left(\bigcap_{n\in\mathbb{N}} \{T < t+1/n\}\right) \\ &= \bigcap_{n\in\mathbb{N}} \big(A \cap \{T < t+1/n\}\big) \\ &= \bigcap_{m\in\mathbb{N}} \mathcal{F}_{t+1/m} \\ &= \mathcal{F}_{t^+} \\ &= \mathcal{F}_t \quad \text{(right continuity)} \end{aligned}$$

Hence $\mathcal{F}_{T_<} = \mathcal{F}_T$ as required. $\square$

Lemma 2.7. *Let $\{\mathcal{F}_t\}_{t\ge0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\ge0}$. Then T is $\mathcal{F}_T$ measurable.*

Proof. Let $t, s \in [0,\infty]$. Then

$$\begin{aligned} T^{-1}([0,s]) \cap T^{-1}([0,t]) &= T^{-1}([0, t\wedge s]) \\ &\in \mathcal{F}_{t\wedge s} \\ &\subset \mathcal{F}_t. \end{aligned}$$

So for each $s \in [0,\infty]$, $T^{-1}([0,s]) \in \mathcal{F}_T$. Hence T is $\mathcal{F}_T measurable.$ $\square$

Lemma 2.8. *Let S, T be stopping times for $\{\mathcal{F}_t\}_{t\ge0}$. If $S \le T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.*

Proof. Suppose that $S \le T$. Let $\Lambda \in \mathcal{F}_S$ and let $t \ge 0$. Then by definition, $\Lambda \cap S^{-1}([0,t]) \in \mathcal{F}_t$. Since $\Lambda \cap T^{-1}([0,t]) = [\Lambda \cap S^{-1}([0,t])] \cap (T^{-1}([0,t]) \in \mathcal{F}_t$, we have that $\Lambda \in \mathcal{F}_T$. $\square$

Lemma 2.9. *Let T be a stopping time for a right continuous filtration $\{\mathcal{F}_t\}_{t\ge0}$. Define $\mathcal{F}_{T^+} = \bigcap_{\epsilon>0} \mathcal{F}_{T+\epsilon}$. Then $\mathcal{F}_T = \mathcal{F}_{T^+}$.*

Proof. From the last lemma, $F_T \subset F_{T^+}$. Let $\Lambda \in \mathcal{F}_{T^+}, \epsilon > 0$, and $t \ge 0$. By definition, $\Lambda \in \mathcal{F}_{T+\epsilon}$. So $\Lambda \cap (T+\epsilon)^{-1}([0,t]) \in \mathcal{F}_t$. Then $\Lambda \cap T^{-1}([0,t-\epsilon]) \in \mathcal{F}_t$. Since $t \ge 0$ is arbitrary, for each $s \ge 0$, $\Lambda \cap T^{-1}([0,s]) \in \mathcal{F}_{s+\epsilon}$. Since $\epsilon > 0$ is arbitrary, right continuity tells us that for each $s \ge 0$, $\Lambda \cap T^{-1}([0,s]) \in \mathcal{F}_{s+} = \mathcal{F}_s$. Hence $\Lambda \in \mathcal{F}_T$ and $\mathcal{F}_T = \mathcal{F}_{T^+}$ $\square$

Lemma 2.10. *Let $\{\mathcal{F}_t\}_{t\ge0}$ be a filtration satisfying the usual conditions, X an a.s. right continuous process adapted $\{\mathcal{F}_t\}_{t\ge0}$ and $T : \Omega \to [0,\infty)$ a finite stopping time for $\{\mathcal{F}_t\}_{t\ge0}$. Then X_T is $\mathcal{F}_T$-measurable.*

Note 2.11. *The same result is achieved if we assume only that $\{\mathcal{F}_t\}_{t\ge0}$ is right continuous and each sample path of X is right continuous.*

Proof. Let $B \in \mathcal{B}([0,\infty))$. For $n \in \mathbb{N}$, define T_n as in the previous exercise. Let $k \in \mathbb{N}$. Since X is adapted to $\{\mathcal{F}_t\}_{t\ge0}$, we have that

$$\begin{aligned} X_{T_n}^{-1}(B) \cap T_n^{-1}(k/2^n) &= X_{k/2^n}^{-1}(B) \cap T_n^{-1}(k/2^n) \\ &\in \mathcal{F}_{k/2^n} \quad \text{(previous exercise)} \end{aligned}$$

Now let $t \geq 0$. Choose $k \in \mathbb{N}$ such that $(k-1)/2^n \leq t < k/2^n$. Then

$$\begin{aligned} X_{T_n}^{-1}(B) \cap T_n^{-1}([0,t]) &= \bigcup_{i=1}^{k-1} [X_{T_n}^{-1}(B) \cap T_n^{-1}(k/2^n)] \\ &\in \mathcal{F}_{(k-1)/2^n} \\ &\subset \mathcal{F}_t \end{aligned}$$

Thus $X_{T_n}^{-1}(B) \in \mathcal{F}_{T_n}$ and X_{T_n} is $\mathcal{F}_{T_n}$-measurable and therefore $\mathcal{F}_{T+1/2^n}$-measurable. Now, let $m \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, if $n \geq m$, X_{T_n} is $\mathcal{F}_{T+1/2^m}$-measurable. Right continuity almost surely tells us that $X_{T_n} \xrightarrow{\text{a.e.}} X_T$. Since $\{\mathcal{F}_t\}_{t\geq 0}$ is complete, so is $\mathcal{F}_{T+1/2^m}$ and we know that X_T is $\mathcal{F}_{T+1/2^m}$-measurable. Since $m \in \mathbb{N}$ is arbitrary and $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous, X_T is measurable with respect to $\mathcal{F}_{T^+} = \mathcal{F}_T$.

□

Exercise 7. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a right continuous filtration, X a right continuous, adapted process and T a finite stopping time. Then the stopped process X^T is right continuous and adapted.*

Note that the same is true if X is a.e. right continuous as long as $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions.

Proof. Clearly X^T is right continuous. Let $t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$. Since $\Omega = \{T \leq t\} \cup \{T > t\}$, we have that

$$\begin{aligned} \{X_{T\wedge t} \in B\} &= \Big(\{X_{T\wedge t} \in B\} \cap \{T \leq t\}\Big) \cup \Big(\{X_{T\wedge t} \in B\} \cap \{T > t\}\Big) \\ &= \Big(\{X_T \in B\} \cap \{T \leq t\}\Big) \cup \Big(\{X_t \in B\} \cap \{T > t\}\Big) \end{aligned}$$

The previous lemma tells us that X_T is $\mathcal{F}_T$ measurable, so $\{X_{T\wedge t} \in B\} \in \mathcal{F}_t$. Hence X^T is adapted.

□

Exercise 8. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a right continuous filtration, X a right continuous, adapted process, $S \leq T$ finite stopping times, and Z an $\mathcal{F}_S$-measurable random variable. Then $Z(X^T - X^S)$ is right continuous and adapted.*

Note that the same is true with "cadlag" instead of "right continuous".

Proof. Clearly $Z(X^T - X^S)$ is right continuous. Define the processes $G_t = \mathbf{1}_{\{t>S\}}$ and $G^c_t = \mathbf{1}_{\{t\leq S\}}$. Then we may rewrite

$$\begin{aligned} Z(X^T - X^S) &= GZ(X^T - X^S) + G^c Z(X^T - X^S) \\ &= GZ(X^T - X^S) \end{aligned}$$

The last exercise tells us that $X^T - X^S$ is adapted, so we need only show that GZ is adapted. Let $t \geq 0$ and $B \in \mathcal{B}(\mathbb{R})$. We can rewrite $\Omega = \{t \leq S\} \cup \{t > S\}$. Define $A_1 = \mathbf{1}_{\{t>S\}} Z \in B\} \cap \{t \leq S\}$ and $A_2 = \{\mathbf{1}_{\{t>S\}} Z \in B\} \cap \{t > S\}$. Then

$$\begin{aligned} \{\mathbf{1}_{\{t>S\}} Z \in B\} &= \Big(\{\mathbf{1}_{\{t>S\}} Z \in B\} \cap \{t \leq S\}\Big) \cup \Big(\{\mathbf{1}_{\{t>S\}} Z \in B\} \cap \{t > S\}\Big) \\ &= A_1 \cup A_2 \end{aligned}$$

If $0 \in B$, then $A_1 = \{t \leq S\}$. If $0 \notin B$, then $A_1 = \varnothing$. Hence $A_1 \in \mathcal{F}_t$. Now, $A_2 = \{Z \in B\} \cap \{t > S\}$. Since Z is $\mathcal{F}_S$-measurable, a previous exercise tells us that $A_2 \in \mathcal{F}_t$. Therefore GZ is adapted.

□

Theorem 2.12. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration satisfying the usual conditions and $T : \Omega \to [0, \infty)$ a finite stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.*
Then $\mathcal{F}_T = \sigma(X_T : X$ is an a.s. right continuous process adapted to $\{\mathcal{F}_t\}_{t\geq 0})$.

Note that the above is true if we assume only the right continuity of $\{\mathcal{F}_t\}_{t\geq 0}$ and right continuity everywhere.

Proof. Put $\mathcal{G} = \sigma(X_T : X$ is a right continuous process adapted to $\{\mathcal{F}_t\}_{t\geq 0})$. Let $\Lambda \in \mathcal{F}_T$. Define $X_t(\omega) = \mathbf{1}_\Lambda(\omega)\mathbf{1}_{T^{-1}([0,t])}(\omega) = \mathbf{1}_{\Lambda\cap T^{-1}([0,t])}(\omega)$. Then each sample path of X is clearly right continuous. Moreover, since $\Lambda \in \mathcal{F}_T$, for each $t \geq 0$, $\Lambda \cap T^{-1}([0, t]) \in \mathcal{F}_t$. So for each $t \geq 0$, X_t is $\mathcal{F}_t$ measurable and thus X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. So by definition, X_T is $\mathcal{G}$ measurable. It is easy to see that $X_T = \mathbf{1}_\Lambda$ and therefore $\Lambda \in \mathcal{G}$. Hence $\mathcal{F}_T \subset \mathcal{G}$.

Now, let X be an a.s. right continuous process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ By the previous lemma, X_T is $\mathcal{F}_T$-measurable. So $\mathcal{G} \subset \mathcal{F}_T$. Hence $\mathcal{F}_T = \mathcal{G}$ as required. □

2.3. Predictability and Accessibility.

Definition 2.13. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then T is said to be* ***predictable*** *if there exists a sequence $(T_n)_{n\in N}$ of stopping times such that $T_n \nearrow T$ a.s and for each $n \in \mathbb{N}$, $T_n < T$ a.s. on $\{T > 0\}$.*

Definition 2.14. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then T is said to be* ***totally inaccessible*** *if for each predictable stopping time S for $\{\mathcal{F}_t\}_{t\geq 0}$,*

$$\mathbb{P}(\{T = S\} \cap \{T < \infty\}) = 0.$$

Definition 2.15. *Let S, T be stopping times. Define $(S, T] = \{(t, \omega) \in [0, \infty) \times \Omega : S(\omega) < t \leq T(\omega)\}$. We define $[S, T], (S, T), [S, T)$ similarly.*

Lemma 2.16. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a right continuous filtration and $\mathcal{P}$ the predictable σ-algebra with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\sigma((S, T] : S, T$ are stopping times$) \subset \mathcal{P}$.*

Proof. Let S, T be stopping times. Then the process (indicator function) $\mathbf{1}_{(S,T]}$ is clearly left continuous. Let $t \in [0, \infty)$. Define $h : \Omega \to \mathbb{R}$ by $h(\omega) = \mathbf{1}_{\{S<t\}\cap\{t\leq T\}}$. Then for each $\omega \in \Omega$, $h(\omega) = \mathbf{1}_{(S,T]}(t, \omega)$. Since $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous, $\{S < t\} \cap \{t \leq T\} \in \mathcal{F}_t$ and thus h is $\mathcal{F}_t$-measurable. So $\mathbf{1}_{(S,T]}$ is adapted. So by definition $\mathbf{1}_{(S,T]}$ is $\mathcal{P}$-measurable. □

Definition 2.17. *Let X be a process. Then X is said to be* ***simple*** *if there exists bounded random variables $Z_0, Z_1, \cdots, Z_n$ and finite stopping times $S_1 \leq T_1, S_2 \leq T_2, \cdots, S_n \leq T_n$ such that*

$$X = Z_0\mathbf{1}_{\{0\}} + \sum_{k=1}^{n} Z_k\mathbf{1}_{(S_k,T_k]}$$

Exercise 9. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be right continous and S, T be finite stopping times and Z a bounded random variable. Suppose that $S \leq T$ and Z is $\mathcal{F}_S$-measurable. Then $Z\mathbf{1}_{(S,T]}$ is adapted.*

Proof. Recall that since $S \leq T$, $\mathcal{F}_S \subset \mathcal{F}_T$. Thus Z is also $\mathcal{F}_T$-measurable. Let $t \geq 0$. Define $g_t(\omega) = Z(\omega)\mathbf{1}_{(S,T]}(t,\omega)$. First assume that $t > 0$. Then $\Omega = \{t \in (0,S]\} \cup \{t \in (S,T]\} \cup \{t \in (T,\infty)\}$. Let $B \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}
g^{-1}(B) &= g^{-1}(B) \cap \Omega \\
&= \Big(g^{-1}(B) \cap \{t \in (0,S]\}\Big) \cup \Big(g^{-1}(B) \cap \{t \in (S,T]\}\Big) \cup \Big(g^{-1}(B) \cap \{t \in (T,\infty)\}\Big) \\
&= \Big(g^{-1}(B) \cap \{t \leq S\}\Big) \cup \Big(\big[g^{-1}(B) \cap \{S < t\}\big] \cap \{t \leq T\}\Big) \cup \Big(g^{-1}(B) \cap \{T < t\}\Big) \\
&\in \mathcal{F}_t \qquad \text{(right continuity of } \{\mathcal{F}_t\}_{t\geq 0}, \text{ earlier exercise)}
\end{aligned}$$

So g_t is $\mathcal{F}_t$-measurable. If $t = 0$, then $g_t \equiv 0$ is $\mathcal{F}_t$-measurable. Thus $Z\mathbf{1}_{(S,T]}$ is adapted. □

Corollary 2.18. *Let $X = Z_0\mathbf{1}_{\{0\}} + \sum_{k=1}^n Z_k\mathbf{1}_{(S_k,T_k]}$ be a simple process. If Z_0 is $\mathcal{F}_0$-measurable and for $k = 1,2,\dots,n$, Z_k is $\mathcal{F}_{S_k}$-measurable, then X is adapted.*

Corollary 2.19. *Let X be a simple process. Then X is predictable iff X is adapted*

Lemma 2.20. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a right continuous filtration. Then $\mathcal{P} = \sigma(X : X \text{ is simple and predictable})$.*

Proof. Let $\mathcal{P}^* = \sigma(X : X \text{ is simple and predictable})$. Clearly $\mathcal{P}^* \subset \mathcal{P}$. Let X be a left continuous adapted process. Define a sequence $X^{(n)}$ of simple, predictable processes by $X^{(n)}(t,\omega) = \mathbf{1}_{\{|X_0|\leq n\}}(\omega)X_0\mathbf{1}_{\{0\}} + \sum\limits_{k=0}^{n2^n-1} \mathbf{1}_{\{|X_{k/2^n}|\leq n\}}(\omega)X_{k/2^n}(\omega)\mathbf{1}_{(k/2^n,(k+1)/2^n]}(t)$. Then left continuity implies that $X^{(n)} \xrightarrow{\text{p.w.}} X$. So X is $\mathcal{P}^*$-measurable and thus $\mathcal{P} \subset \mathcal{P}^*$. □

3. Martingales

3.1. Localization.

Note 3.1. *For the remainder of these notes, we will assume all filtrations satisfy the usual conditions.*

Definition 3.2. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and $\mathcal{D}$ be a set of stochastic processes. Define the **localized class of** $\mathcal{D}$ (with respect to $\{\mathcal{F}_t\}_{t\geq 0}$), denoted $\mathcal{D}_{loc}$, to be the set of processes such that $X \in \mathcal{D}_{loc}$ iff there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times for $\{\mathcal{F}_t\}_{t\geq 0}$ such that $T_n \nearrow \infty$ a.s. and for each $n \in \mathbb{N}$, $X^{T_n} \in \mathcal{D}$. The sequence $(T_n)_{n\in\mathbb{N}}$ is called a **localizing sequence** for X. If $X \in \mathcal{D}_{loc}$, we say that X is **locally** in $\mathcal{D}$*

Note 3.3. *A process X is often referred to as a "local" object instead of locally an object. The difference is subtle but the former expression typically takes place in some assumed subuniverse of objects. For example, in the later definition of a local (sub,super)martingale, we only consider processes that are adapted because not doing so would allow processes that are only locally adapted. When the difference needs to be made clear, the definition of a local object will be given. Otherwise we assume that a local object is just locally an object.*

Note 3.4. *Taking $T_n \equiv \infty$, it is easy to see that $\mathcal{D} \subset \mathcal{D}_{loc}$.*

Definition 3.5. *Let $\mathcal{D}$ be a set of processes, T a sopping time and X a process. Then T is said to **reduce** X **with respect to** $\mathcal{D}$ if $X^T \in \mathcal{D}$.*

Definition 3.6. *Let $\mathcal{D}$ be a set of processes. Then $\mathcal{D}$ is said to be **stable under stopping** if for each $X \in \mathcal{D}$ and stopping time T, $X^T \in \mathcal{D}$.*

3.2. **Finite Variation Processes.**

Definition 3.7. *Let A be a process. Then A is said to be* ***increasing*** *if A is right continous and for each $\omega \in \Omega$, $A(\cdot, \omega)$ is increasing.*

Definition 3.8. *Let A be a process, then A is said to have* ***finite variation*** *if there exist increasing processes A_1 and A_2 such that $A = A_1 - A_2$.*

Note 3.9. *Let A be a process with finite variation. Then A is cadlag. Also, for each stopping time T, A^T has finite variation.*

Definition 3.10. *Let A be a process of finite variation. Define the process A^d by*

$$A_t^d = \begin{cases} 0 & t = 0 \\ \sum\limits_{0<s\leq t} \Delta A_s & t > 0 \end{cases}$$

and define the process A^c by

$$A^c = A - A^d$$

We call A^c the ***continuous part*** *of A and we call A^d the* ***discontinuous part*** *of A.*

Exercise 10. *Let A be an process of finite variation. Then A^c and A^d have finite variation and A^c is continuous. In particular, if A is increasing, then so are A^c and A^d.*

Proof. First suppose that A is increasing. Let $\omega \in \Omega$, $s \in [0, \infty)$ and $\epsilon > 0$. Put $S = \{t \in [0, \infty) : s < t \leq s + 1 \text{ and } \Delta A(t, \omega) > 0\}$. If S is finite, then clearly $A^d(\cdot, \omega)$ is right continuous at s. Suppose that S is infinite. Since A is cadlag, S is countable. Let $(s_n)_{n\in\mathbb{N}}$ be an enumeration of S. Since $A(\cdot, \omega)$ is increasing, the total variation of $A(\cdot, \omega)$ on $[s, s+1]$ is just $v = A(s + 1, \omega) - A(s, \omega)$. Since $\sum\limits_{n\in\mathbb{N}} \Delta A(s_n, \omega) \leq v < \infty$ we know that the sequence $(a_k)_{k\in\mathbb{N}}$ defined by $a_k = \sum\limits_{n=k}^{\infty} \Delta A(s_n, \omega)$ satisfies $a_k \to 0$ as $k \to \infty$. So there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq K$ implies that $a_k < \epsilon$. Choose $\delta = \min\{s_1, s_2, \cdots, s_{K-1}\}$. Let $t \in [0, \infty)$. Suppose that $s < t$ and $|s - t| < \delta$. Then

$$\begin{aligned} |A^d(t, \omega) - A^d(s, \omega)| &= |\sum_{s<x\leq t} \Delta A(x, \omega)| \\ &< |\sum_{\substack{k\geq K \\ s_k\in(s,\delta)}} \Delta A(s_k, \omega)| \\ &\leq |\sum_{k=K}^{\infty} A(s_k, \omega)| \\ &< \epsilon \end{aligned}$$

So A^d and hence A^c is right continuous. Since each sample path of A is increasing, so is each sample path of A^d. Thus A^d is increasing and so has finite variation. Let $\omega \in \Omega$ and $s, t \in [0, \infty)$. Suppose that $s \leq t$. Then

$$\begin{aligned} \Delta A^c(s, \omega) &= \Delta A(s, \omega) - \Delta A^d(s, \omega) \\ &= \Delta A(s, \omega) - \Delta A(s, \omega) \\ &= 0 \end{aligned}$$

So A^c is continous. Let $v = A(t, \omega) - A(s, \omega)$ be to total variation of $A(\cdot, \omega)$ on $[s, t]$. Again using the fact that $A^d(t, \omega) - A^d(s, \omega) \leq v$, we see that

$$\begin{aligned} A^c(t, \omega) - A^c(s, \omega) &= (A(t, \omega) - A^d(t, \omega)) - (A(s, \omega) - A^d(s, \omega)) \\ &= (A(t, \omega) - A(s, \omega)) - (A^d(t, \omega) - A^d(s, \omega)) \\ &\geq 0 \end{aligned}$$

So A^c is increasing and has finite variation. In general, if A has finite variation, then there exist increasing processes A_1, A_2 such that $A = A_1 - A_2$. Note that the linearity of Δ implies that

$$\begin{aligned} A_1^d - A_2^d &= (A_1 - A_2)^d \\ &= A^d \end{aligned}$$

Hence $A^c = A - (A_1^d - A_2^d)$. Since

$$\begin{aligned} A &= (A_1^c + A_1^d) - (A_2^c + A_2^d) \\ &= (A_1^c - A_2^c) + (A_1^d - A_2^d) \end{aligned}$$

we know that $A^c = A_1^c - A_2^c$ The results from before tell us that $A_1^c, A_2^c, A_1^d, A_2^d$ are all increasing and A_1^c, A_2^c are continuous. So we have that A^c and A^d have finite variation and A^c is continuous. □

Exercise 11. *Let A be an increasing process. If A is integrable, then there exists a random variable $A_\infty : \Omega \to \mathbb{R}$ such that $A_t \to A_\infty$ a.s. and A_∞ is integrable.*

Proof. Suppose that A is integrable. Then $\sup_{t \geq 0} \mathbb{E}|A_t| < \infty$. Note that A_n^+ increases pointwise to $f = \sup_{n \in \mathbb{N}} A_n^+$ and A_n^- decreases pointwise to $g = \inf_{n \in \mathbb{N}} A_n^-$ which are both measurable. We apply monotone convergence twice, once to the increasing sequence and once to the decreasing sequence which we may do since we know that $\mathbb{E}A_1^- \leq \mathbb{E}|A_1| < \infty$. Thus $\lim_{n \to \infty} \mathbb{E}A_n^+ = \mathbb{E}f$ and $\lim_{n \to \infty} \mathbb{E}A_n^- = \mathbb{E}g$. Since for each $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}A_n^+, \mathbb{E}A_n^- &\leq \mathbb{E}|A_n| \\ &\leq \sup_{t \geq 0} \mathbb{E}|A_t| \\ &< \infty \end{aligned}$$

we know that $\mathbb{E}f, \mathbb{E}g < \infty$ and so f, g are finite a.s. Let $\Gamma = \{\omega \in \Omega : f(\omega) < \infty \text{ and } g(\omega) < \infty\}$ which is measurable. Put $A_\infty = \sup_{n \in \mathbb{N}} A_n \mathbf{1}_\Gamma$. Then A_∞ is measureable and $A_\infty = (f - g)\mathbf{1}_\Gamma$. Also, $E|A_\infty| = \mathbb{E}f + \mathbb{E}g < \infty$. So $A_\infty : \Omega \to \mathbb{R}$ is a random variable and $A_t \to A_\infty$ as $t \to \infty$. □

Note 3.11. *The converse is also true if $\mathbb{E}|A_0| < \infty$.*

Definition 3.12. *Let A be a process with finite variation. We define the* ***total variation process of*** *A to be the process*

$$V_A(t, \omega) = \int_{[0,t]} d|A^\omega|$$

where A^ω denotes the sample path of ω.

Note 3.13. *Each sample path of V_A is increasing and if A is right continuous, then so is V_A. Hence V_A is a nonnegative, increasing process and has finite variation.*

Definition 3.14. *Let A be a process with finite variation. Then A is said to have **integrable variation** if V_A is integrable.*

Exercise 12. *Let A be a finite variation process with (locally) integrable variation. If $\mathbb{E}|A_0| < \infty$, then A is (locally) integrable.*

Proof. First, suppose that A has integrable variation. Let $t \in [0, \infty)$. Note that $|A_t - A_0| \leq V_{At}$. Thus

$$\begin{aligned} \mathbb{E}|A_t| &\leq \mathbb{E}|A_t - A_0| + \mathbb{E}|A_0| \\ &\leq \mathbb{E}V_{At} + \mathbb{E}|A_0| \\ &\leq \sup_{t\geq 0} \mathbb{E}V_{At} + \mathbb{E}|A_0| \end{aligned}$$

So $\sup_{t\geq 0} \mathbb{E}|A_t| \leq \sup_{t\geq 0} \mathbb{E}V_{At} + \mathbb{E}|A_0| < \infty$ and A is integrable. Now suppose that A has locally integrable variation. Choose a localizing sequence $(T_n)_{n\in\mathbb{N}}$ for A. Let $n \in \mathbb{N}$. Then A^{T_n} has integrable variation. The previous case tells us that A^{T_n} is integrable. So A is locally integrable. □

3.3. **Martingales.**

Definition 3.15. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **submartingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if*

1. *M is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$*
2. *for each $t \in [0, \infty)$, $\mathbb{E}[|M_t|] < \infty$*
3. *for each $s, t \in [0, \infty)$, if $s < t$, then $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$ a.s.*

Note 3.16. *The third statement is techinally with respect to $\mathcal{F}_s$ because $\mathbb{E}[\cdot|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable. However, since $\mathcal{F}_s$ contains all the null sets of $\mathcal{F}$, to say a.s. is unambiguous.*

Definition 3.17. *Let X be process. Then X is said to be a **local submartingale** if X is adapted and locally X is a submartingale.*

Note 3.18. *Here we require that a local submartingale be adapted. The definition of a local (super)martingale is similar.*

Exercise 13. *Let A be an adapted, increasing process. Suppose that for each $t \in [0, \infty)$, $\mathbb{E}|A_t| < \infty$ Then A is a submartingale.*

Proof. Since the first and second conditions in the definition of a submartingale are satisfied by assumption, we need only verify the third condition. Let $s, t \in [0, \infty)$. Suppose that $s < t$. By assumption $A_t \geq A_s$. Thus

$$\mathbb{E}[A_t|\mathcal{F}_s] \geq \mathbb{E}[A_s|\mathcal{F}_s] = A_s \text{ a.s.}$$

So A is indeed a submartingale. □

Exercise 14. *Let A be an adapted, and increasing finite variation process. If $\mathbb{E}|A_0| < \infty$ and A has (locally) integrable variation, then A is a (local) submartingale.*

Proof. An exercise from the previous section tells us that A is (locally) integrable. If A is integrable, the previous exercise tells us that A is a submartingale. Now suppose that A is locally integrable. Choose a localizing sequence $(T_n)_{n\in\mathbb{N}}$ for A. Let $n \in \mathbb{N}$. Then A^{T_n} is integrable and hence a submartingale. So A is a local submartingale. □

Definition 3.19. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **supermartingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if $-M$ is a submartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.*

Definition 3.20. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **martingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if M is a submartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ and M is a supermartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.*

Exercise 15. *Let X be a local martingale. Then X is locally uniformly integrable.*

Proof. **FINISH!!!!** □

Theorem 3.21. *Doob's inequalities: Let X be a nonnegative, right continuous submartingale. Then for each $t \geq 0$ and $C > 0$,*

$$\mathbb{P}\Big(\sup_{0\leq s\leq t} X_s > C\Big) \leq \frac{\mathbb{E}[X_t]}{C}$$

and for each $p, q \geq 1$, if p, q are conjugate, then

$$||\sup_{t\geq 0} X_t||_p \leq q \sup_{t\geq 0} ||X_t||_q$$

Definition 3.22. *We define $\mathcal{M} = \{X : X$ is a unformly integrable, cadlag martingale$\}$.*

Note 3.23. *Observe that $\mathcal{M}$ is a vector space.*

Theorem 3.24. *Let M be a right continuous martingale. If M is integrable, then*

1. *M is cadlag a.s.*
2. *there exists a random variable $M_\infty : \Omega \to \mathbb{R}$ such that $M_t \to M_\infty$ a.s.*
3. *M_∞ is integrable*

Theorem 3.25. *Let M be a martingale. Then there exists a cadlag martingale N such that N is a modification of M. **Bass 3.13***

Theorem 3.26. *Let M be a right continuous martingale. If M is uniformly integrable, then $\mathbb{E}[M_\infty|\mathcal{F}_t] = M_t$ a.s.*

Theorem 3.27. *Let M be a right continuous martingale. If M is uniformly integrable, then $M_t \xrightarrow{L^1} M_\infty$.*

Definition 3.28. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. Then X is said to be **class** $\mathcal{D}$ if $\{X_T : T$ is a finite stopping time for $\{\mathcal{F}_t\}_{t\geq 0}\}$ is uniformly integrable.*

Theorem 3.29. *Doob-Meyer Decomposition:*

1. *Let X a right continuous local submartingale. Then there exists a unique increasing process A such that $A_0 = 0$ and $X - X_0 - A$ is a local martingale*
2. *Let X is a right continuous submartingale of class $\mathcal{D}$. Then there exists a unique increasing process A such that $A_0 = 0$, A is integrable and $X - X_0 - A$ is a uniformly integrable martingale*

3.4. Square Integrable Martingales.

Definition 3.30. *X is said to be* ***square integrable*** *if* $\sup_{t\geq 0}\mathbb{E}|X_t^2| < \infty$. *We define* $\mathcal{M}^2 = \{X : X \text{ is a square integrable, cadlag martingale}\}$.

Note 3.31. *Observe that $\mathcal{M}$ and $\mathcal{M}^2$ are vector spaces and from previous results on uniform integrability, if M is square integrable, then M is uniformly integrable. Thus $\mathcal{M}^2 \subset \mathcal{M}$.*

Theorem 3.32. *Let $M \in \mathcal{M}$. Then $M \in \mathcal{M}^2$ iff $\mathbb{E}[M_\infty^2] < \infty$. In this case,*

$$\mathbb{E}[M_\infty^2] = \sup_{t\geq 0}\mathbb{E}[M_t^2]$$

.

Proof. Suppose $M \in \mathcal{M}^2$. Fatou's lemma tells us that

$$\begin{aligned}\mathbb{E}[M_\infty^2] &\leq \sup_{t\geq 0}\mathbb{E}[M_t^2] \\ &< \infty\end{aligned}$$

Conversely, suppose that $\mathbb{E}[M_\infty^2] < \infty$. Let $t \geq 0$. Since $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$, Jensen's inequality tells us that $M_t^2 \leq \mathbb{E}[M_\infty^2|\mathcal{F}_t]$ and thus $\mathbb{E}[M_t^2] \leq \mathbb{E}[M_\infty^2]$. So $\sup_{t\geq 0}\mathbb{E}[M_t^2] \leq \mathbb{E}[M_\infty^2]$ and hence $M \in \mathcal{M}^2$. By the above, this implies $\mathbb{E}[M_\infty^2] = \sup_{t\geq 0}\mathbb{E}[M_t^2]$.

□

Note 3.33. *This fact means that for $M \in \mathcal{M}^2$, $M_\infty \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.*

Definition 3.34. *We define an inner product $\langle\cdot,\cdot\rangle : \mathcal{M}^2 \times \mathcal{M}^2 \to [0,\infty)$ by*

$$\langle M, N\rangle = \mathbb{E}[M_\infty N_\infty]$$

The norm induced by this inner product will be denoted by $||\cdot||_{\mathcal{M}^2}$

Note 3.35. *The above definition does not give a well defined inner product. Technically we should identify processes that are modifications. We assume this from now on. Also, recall that since these square integrable martingales are cadlag, any modifications are actually indistinguishable.*

Lemma 3.36. *With the inner product defined above, $\mathcal{M}^2$ is a hilbert space and $\mathcal{M}^2$ is isomprphic to $L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$.*

Proof. Define $\phi : \mathcal{M}^2 \to \mathbb{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ by $\phi(M) = M_\infty$. Then ϕ is linear. Let $M, N \in \mathcal{M}^2$. Suppose that $M_\infty = N_\infty$ a.s. Then for each $t \geq 0$,

$$\begin{aligned}M_t &= \mathbb{E}[M_\infty|\mathcal{F}_t] \\ &= \mathbb{E}[N_\infty|\mathcal{F}_t] \\ &= N_t \text{ a.s.}\end{aligned}$$

So ϕ is injective. Recall that since $\mathbb{P}(\Omega) = 1$, $\mathbb{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P}) \subset \mathbb{L}^1(\Omega, \mathcal{F}_\infty, \mathbb{P})$. Thus for each $M_\infty \in \mathbb{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, we can define a process M by $M_t = \mathbb{E}[M_\infty|\mathcal{F}_t]$ which would mean that for each $t \geq 0$, $\mathbb{E}[M_t] = \mathbb{E}[M_\infty] < \infty$. So M is a martingale. Jensen's inequality tells us that for each $t \geq 0$, $M_t^2 \leq \mathbb{E}[M_\infty^2|\mathcal{F}_t]$ which implies $\mathbb{E}[M_t^2] \leq \mathbb{E}[M_\infty^2]$. Hence M is a square integrable martingale. A previous result tells us that there exists a modification N of M such that N is a cadlag, square integrable martingale. So $N \in \mathcal{M}^2$, $\phi(N) = M_\infty$ and ϕ is surjective. Clearly ϕ preserves the inner product, so the lemma holds. □

Note 3.37. *Let* $(M^n)_{n\in\mathbb{N}} \subset \mathcal{M}^2 \subset \mathcal{M}^2$ *and* $M \in \mathcal{M}^2$. *By the lemma above,* $M^n \xrightarrow{\mathcal{M}^2} M$ *iff* $M^n_\infty \xrightarrow{L^2} M_\infty$

Theorem 3.38. *Let* $(M^n)_{n\in\mathbb{N}} \subset \mathcal{M}^2$ *and* $M \in \mathcal{M}^2$. *Suppose that* $M^n \xrightarrow{\mathcal{M}^2} M$. *Then there exists a subsequence* $(M^{n_k})_{k\in\mathbb{N}}$ *such that* $M^{n_k} \to M$ *uniformly in* t *a.s.*

Proof. Choose a subsequence $(M^{n_k})_{k\in\mathbb{N}}$ such that $\sum\limits_{k\in\mathbb{N}} ||M^{n_k}_\infty - M_\infty||_2 < \infty$. Then Jensen's and Doob's inequalities tell us that

$$\begin{aligned}
\mathbb{E}[\sum_{k\in\mathbb{N}} \sup_{t\geq 0} |M^{n_k}_t - M_t|] &= \sum_{k\in\mathbb{N}} \mathbb{E}[\sup_{t\geq 0} |M^{n_k}_t - M_t|] \\
&\leq \sum_{k\in\mathbb{N}} \mathbb{E}[\sup_{t\geq 0} |M^{n_k}_t - M_t|^2]^{\frac{1}{2}} \\
&\leq \sum_{k\in\mathbb{N}} 2 \sup_{t\geq 0} \mathbb{E}[|M^{n_k}_t - M_t|^2]^{\frac{1}{2}} \\
&= 2 \sum_{k\in\mathbb{N}} \mathbb{E}[|M^{n_k}_\infty - M_\infty|^2]^{\frac{1}{2}} \\
&< \infty
\end{aligned}$$

So $\sum\limits_{k\in\mathbb{N}} \sup_{t\geq 0} |M^{n_k}_t - M_t| < \infty$ a.s. Thus $\sup_{t\geq 0} |M^{n_k}_t - M_t| \to 0$ a.s. as required. □

Definition 3.39. *Define* $\mathcal{M}^{2,c} = \{X \in \mathcal{M}^2 : X \text{ is continuous}\}$

Corollary 3.40. $\mathcal{M}^{2,c}$ *is a closed subspace of* $\mathcal{M}^2$

Proof. Clearly $\mathcal{M}^{2,c}$ is a subspace. Let $M \in \overline{\mathcal{M}^{2,c}}$. Choose a sequence $(M^n)_{n\in\mathbb{N}} \subset M^{2,c}$ such that $M^n \xrightarrow{\mathcal{M}^2} M$. Then choose a subsequence $(M^{n_k})_{k\in\mathbb{N}}$ such that $M^{n_k} \to M$ uniformly in t a.s. Then the sample paths of M are continuous a.s. Since we can modify M to have continuous paths and we identify modifications, we have that $M \in \mathcal{M}^{2,c}$ and $\mathcal{M}^{2,c}$ is a closed subspace. □

4. Stochastic Integration

Definition 4.1. *Define* $\mathcal{S} = \{X : [0,\infty) \times \Omega \to \mathbb{R} : X \text{ is simple and predictable}\}$. *The processes in* S *will play the role that simple functions play in a real analysis course. We may define the norm* $||\cdot||_u : S \to [0,\infty)$ *by*

$$||X||_u = \sup_{t\geq 0} ||X_t||_\infty$$

To sprecify that we are using this norm, we will write $\mathcal{S}_u$.

Note 4.2. *Clearly* $\mathcal{S}$ *is a vector space.*

Definition 4.3. *Define* $L^0(\Omega, \mathcal{F}, \mathbb{P}) = \{H : \Omega \to \mathbb{R} : H \text{ is } \mathcal{F}\text{-measurable}\}$. *We will assume that* L^0 *is topologized by convergence in probability with respect to* $\mathbb{P}$.

Definition 4.4. *Let* $H = Z_0\mathbf{1}_{\{0\}} + \sum_{k=1}^n Z_k\mathbf{1}_{(S_k,T_k]} \in \mathcal{S}$ *and* X *a stochastic process. We define the* ***stochastic integral*** *of* H *with respect to* X, *denoted by* $\mathcal{I}_X(H)$, *to be the process* $Z_0X_0 + \sum_{k=1}^n Z_k(X^{T_k} - X^{S_k})$. *For* $t \geq 0$, *it is customary to write* $\int_0^t H_s dX_s$ *or* $\int_0^t H dX$ *instead of* $(\mathcal{I}_X(H))_t$.