PORTFOLIO THEORY NOTES

Contents

Note 0.1. *In these notes we will mostly consider random variables X that model returns. As such we may assume that $X \in L^1(\mathbb{P})$ and $F_X : \mathbb{R} \to (0,1)$ is bijective and continuous. We will call such random variabels "nice".*

1. Risk Measures

1.1. Value at Risk.

Definition 1.1. *Let X be a nice random variable and $\epsilon \in (0,1)$. We define the* ***value at risk of*** X ***at confidence level*** $1-\epsilon$*, denoted by $VaR_\epsilon(X)$, to be*

$$VaR_\epsilon(X) = F_{-X}^{-1}(\epsilon)$$

Note 1.2. *If X represents the return of a portfolio, then $Var_\epsilon(X)$ is just a bound such that with probability ϵ, the loss of the portfolio is not less than the bound.*

1.2. Estimating the Value at Risk.

1.3. Average Value at Risk.

Definition 1.3. *Let X be a nice random variable and $\epsilon \in (0,1)$. We define the* ***average value at risk of*** X ***with tail probability*** ϵ*, denoted by $AVaR_\epsilon(X)$, to be*

$$AVaR_\epsilon(X) = \frac{1}{\epsilon}\int_{(0,\epsilon]} VaR_p(X)dm(p)$$

Note 1.4. *If X represents the return on a portfolio, then $AVaR_\epsilon(X)$ is just the average of the $VaR_p(X)$ over all $p < \epsilon$.*

Exercise 1.5. *Let X be a nice random variable and $\epsilon \in (0,1)$. Then $AVaR_\epsilon(X) = \mathbb{E}[-X| - X \geq VaR_\epsilon(X)]$.*

Proof. Recall that for measurable spaces $(X,\mathcal{A}), (Y,\mathcal{B})$, measurable $f : X \to Y$, measure $\mu : \mathcal{A} \to [0,\infty]$, we may form the push-foreward measure of μ by f, $f_*\mu : \mathcal{B} \to [0,\infty]$ with the following property: for each $g : Y \to \mathbb{C}$, $g \in L^1(f_*\mu)$ iff $g \circ f \in L^1(\mu)$ and for each $B \in \mathcal{B}$,

$$\int_{f^{-1}(B)} g \circ f d\mu = \int_B g df_*\mu$$

Note that

$$\begin{aligned}\mathbb{E}[-X| -X \geq -F_X^{-1}(\epsilon)] &= -\mathbb{E}[X|X \leq F_X^{-1}(\epsilon)] \\ &= -\frac{1}{\epsilon}\mathbb{E}[X\mathbf{1}_{\{X \leq F_X^{-1}(\epsilon)\}}] \\ &= -\frac{1}{\epsilon}\int_{\{X \leq F_X^{-1}(\epsilon)\}} XdP \\ &= -\frac{1}{\epsilon}\int_{(-\infty, F_X^{-1}(\epsilon)]} xdF_X(x)\end{aligned}$$

Let μ be the Lebesgue-Stieltjes measure obtained from F_X (i.e. $d\mu = dF_X$). Consider $F_X : \mathbb{R} \to (0,1)$ as in the theorem recalled above. Then for each $(a,b] \subset [0,1]$ with $a' = F_X^{-1}(a)$ (could be $-\infty$) and $b' = F_X^{-1}(b)$, we have that

$$\begin{aligned}F_{X*}\mu((a,b]) &= \mu(F_X^{-1}((a,b])) \\ &= \mu((a',b']) \\ &= F_X(b') - F_X(a') \\ &= b - a\end{aligned}$$

So $F_{X*}\mu = m$. Hence

$$\begin{aligned}\int_{(-\infty, F_X^{-1}(\epsilon)]} xdF_X(x) &= \int_{(-\infty, F_X^{-1}(\epsilon)]} (F_X^{-1} \circ F_X)(x)dF_X(x) \\ &= \int_{(0,\epsilon]} F_X^{-1}(x)dm(x)\end{aligned}$$

□

Note 1.6. *If X represents the return of a portfolio. We may define the* ***loss of*** *X, denoted by L_X, to be $L_X = -X$. Then $AVaR_\epsilon(X) = \mathbb{E}[L_X|L_X > VaR_\epsilon(X)]$.*

Theorem 1.7. *Let X be a nice random variable and $\epsilon \in (0,1)$. Then*

$$AVaR_\epsilon(X) = \min_{\theta \in \mathbb{R}} \left(\theta + \frac{1}{\epsilon}\mathbb{E}[(-X-\theta)^+]\right)$$

Proof. For $\omega \in \Omega, \theta \in \mathbb{R}$, put $g_\omega(\theta) = (-X(\omega) - \theta)^+$ and for $\theta \in \mathbb{R}, \epsilon \in (0,1)$, put $f_\epsilon(\theta) = \theta + \frac{1}{\epsilon}\mathbb{E}[g(\theta)]$. Then for each $\omega \in \Omega$, g_ω is convex. This implies that for each $\epsilon \in (0,1)$, f_ϵ is convex and therefore continuous with right and left.

Let $L = -X$ be the loss of X. One can show that

$$\frac{\partial f_\epsilon}{\partial \theta}(\theta) = \frac{F_L(\theta) - 1 + \epsilon}{\epsilon}$$

. The details can be found in [**?**], but will be omitted here. Thus

$$\lim_{\theta \to \infty} \frac{\partial f_\epsilon}{\partial \theta}(\theta) = 1$$

and

$$\lim_{\theta \to -\infty} \frac{\partial f_\epsilon}{\partial \theta}(\theta) = \frac{\epsilon - 1}{\epsilon} < 0$$

This implies that there exists $\theta^* \in \mathbb{R}$ such that $f(\theta^*) = \inf_{\theta \in \mathbb{R}}(f_\theta)$

Thus
$$\frac{\partial f_\epsilon}{\partial \theta}(\theta^*) = 0$$
which implies that
$$F_L(\theta^*) = 1 - \epsilon$$
This implies that $\theta^* = Var_{1-\epsilon}$ Finally, evaluating f_ϵ at θ^* shows us that

$$\begin{aligned}
f_\epsilon(\theta^*) &= \theta^* + \frac{1}{\epsilon}\mathbb{E}[(L-\theta^*)^+] \\
&= \theta^* + \frac{1}{\mathbb{P}(L>\theta^*)}\mathbb{E}[(L-\theta^*)\mathbf{1}_{\{L>\theta^*\}}] \\
&= \theta^* + \frac{1}{\mathbb{P}(L>\theta^*)}\mathbb{E}[L\mathbf{1}_{\{L>\theta^*\}}] - \frac{1}{\mathbb{P}(L>\theta^*)}\mathbb{E}[\theta^*\mathbf{1}_{\{L>\theta^*\}}] \\
&= \theta^* + \frac{1}{\mathbb{P}(L>\theta^*)}\mathbb{E}[L\mathbf{1}_{\{L>\theta^*\}}] - \theta^* \\
&= \mathbb{E}[L|L>\theta^*] \\
? &= \mathbb{E}[L|L>VaR_{1-\epsilon}(X)]\text{that}1-\epsilon \text{ is bad}
\end{aligned}$$

□

1.4. **Estimating the Average Value at Risk.**

Definition 1.8. *Let X be a random nice random variable and $X_1, \cdots, X_n \overset{iid}{\sim} X$. We define the **sample average value at risk of X with tail probability** ϵ, denoted by $\widehat{AVar_\epsilon}(X)$, to be*
$$\widehat{AVaR_\epsilon}(X) = \inf_{\theta \in \mathbb{R}} \left(\theta + \frac{1}{n\epsilon}\sum_{i=1}^n \max(-X_i - \theta, 0)\right)$$

Lemma 1.9. *Let X be a random nice random variable and $X_1, \cdots, X_n \overset{iid}{\sim} X$. Then $\widehat{AVarR_\epsilon}(X)$ is an unbiased estimator for $AVaR_\epsilon(X)$.*

Proof. For each $\theta \in \mathbb{R}$, define
$$f_\theta = \theta + \frac{1}{n\epsilon}\sum_{i=1}^n \max(-X_i - \theta, 0)$$
Note that for each $\theta \in \mathbb{R}$, $f_\theta \in L^1(\mathbb{P})$.

Since f_θ is continuous in θ, we have that
$$\inf_{\theta\in\mathbb{R}} f_\theta = \inf_{\theta \in \mathbb{Q}} f_\theta$$
which is measurable. For $\theta_1, \theta_2 \in [0,\infty)$, if $\theta_1 < \theta_2$, then $\max(-X_i - \theta_1, 0) \geq \max(-X_i - \theta_2, 0)$. So

$$f_0 \leq \frac{1}{n\epsilon}\sum_{i=1}^n \max(-X_i, 0)$$

□