REAL ANALYSIS NOTES

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Contents

1. Measure 1
1.1. Product Measures 1
2. Integration 2
2.1. Measurable Functions 2
2.2. Integration of Nonnegative Functions 2
2.3. Integration of Complex Valued Functions 3
2.4. Integration on Product Spaces 5
2.5. Convergence 7
3. Differentiation 8
3.1. Signed Measures 8
3.2. The Lebesgue-Radon-Nikodym Theorem 9
3.3. Complex Measures 10
3.4. Differentiation 11
3.5. Functions of Bounded Variation 13
4. Topology 15
5. L^p Spaces 15
6. Functional Analysis 15
6.1. Normed Vector Spaces 15
6.2. Linear Functionals 17
6.3. The Baire Category Theorem and Consequences 18
7. Appendix 20
7.1. Summation 20

1. Measure

1.1. **Product Measures.**

Definition 1.1. *Let $(X,\mathcal{A},\mu),(Y,\mathcal{B},\nu)$ be measurable spaces. Put $\mathcal{E}=\{A\times B : A\in\mathcal{A} \text{ and } B\in\mathcal{B}\}$. Then $\mathcal{E}$ is an elementary family and thus $\mathcal{M}_0=\{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n\subset\mathcal{E} \text{ are disjoint}\}$ is an algebra on $X\times Y$. We define $\pi_0:\mathcal{M}_0\to[0,\infty]$ by*

$$\pi_0(\bigcup_{i=1}^n A_i\times B_i)=\sum_{i=1}^n \mu(A_i)\nu(B_i)$$

Since $\mathcal{A}\otimes\mathcal{B}=\sigma(\mathcal{M}_0)$, we define a product measure $\mu\times\nu$ on $(X\times Y,\mathcal{A}\otimes\mathcal{B})$ to be an extension of π_0 to $\mathcal{A}\otimes\mathcal{B}$. The existence of which is guaranteed by Caratheodory's theorem and on $\mathcal{A}\otimes\mathcal{B}$,

$$\mu \times \nu(E) = \inf\{\sum_{n\in\mathbb{N}} \pi_0(E_i) : (E_i)_{i\in\mathbb{N}} \subset \mathcal{M}_0 \text{ and } E \subset \bigcup_{i\in\mathbb{N}} E_i\}$$
$$= \inf\{\sum_{n\in\mathbb{N}} \mu(A_i)\nu(B_i) : (A_i \times B_i)_{i\in\mathbb{N}} \subset \mathcal{E} \text{ and } E \subset \bigcup_{i\in\mathbb{N}} A_i \times B_i\}$$

If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are both sigma finite, then so is π_0 and thus $\mu \times \nu$ is unique.

2. Integration

2.1. Measurable Functions.

Definition 2.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then f is said to be $\mathcal{A}$-$\mathcal{B}$ **measurable** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. When $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say that f is $\mathcal{A}$-**measurbale**. If $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}, \mathcal{L})$, then we say that f is **Borel measurable** or **Lebsgue measurbale** respectively.*

Lemma 2.2. *Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then*
1. *$\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y*
2. *$\{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra on X*

Lemma 2.3. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Suppose that there exists $\mathcal{E} \subset Y$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Let $f : X \to Y$. If for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$, then f is $\mathcal{A}$-$\mathcal{B}$ measurable.*

Corollary 2.4. *Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X \to Y$. If f is continuous, then f is $\mathcal{B}(X)$-$\mathcal{B}(Y)$ measurable.*

Definition 2.5. *Let X be a set and $f : X \to \mathbb{C}$. Then f is said to be **simple** if $f(X)$ is finite.*

Definition 2.6. *Let $(X, \mathcal{A})$ be a measurable space. We define $S^+(X, \mathcal{A}) = \{f : X \to [0, \infty) : f \text{ is simple, measurable}\}$ and $S(X, \mathcal{A}) = \{f : X \to \mathbb{C} : f \text{ is simple, measurable}\}$*

Theorem 2.7. *Let $(X, \mathcal{A})$ be a measurable space. Then*
1. *If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.*
2. *If $f : X \to \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset S$ such that for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.*

2.2. Integration of Nonnegative Functions.

Definition 2.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^+(X, \mathcal{A}, \mu) = \{f : X \to [0, \infty] : f \text{ is measurable}\}$ We will typically just write L^+.*

Theorem 2.9. *Monotone Convergence Theorem Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then*

$$\sup_{n\in\mathbb{N}} \int f_n = \int \sup_{n\in\mathbb{N}} f_n$$

.

Exercise 2.10. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$ and $f \in L^+$. Then*

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

.

Exercise 2.11. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Suppose that $\mu_1 \leq \mu_2$. Then for each $f \in L^+$,*

$$\int f d\mu_1 \leq \int f d\mu_2$$

Theorem 2.12. *Fatou's Lemma Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Theorem 2.13. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then*

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n.$$

Exercise 2.14. *Let $f \in L^+$ and suppose that $\int f < \infty$. Put $N = \{x \in X : f(x) = \infty\}$ and $S = \{x \in X : f(x) > 0\}$. Then $\mu(N) = 0$ and S is σ-finite.*

Exercise 2.15. *Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.*

Exercise 2.16. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that $f_n \xrightarrow{p.w.} f$, $\lim\limits_{n\to\infty} \int f_n = \int f$ and $\int f < \infty$. Then for each $E \in \mathcal{A}$, $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$. This result may fail to be true if $\int f = \infty$*

Exercise 2.17. *Let $f \in L^+$. Define $\lambda : \mathcal{A} \to [0, \infty]$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{A}$ Then λ is a measure on $(X, \mathcal{A})$ and for each $g \in L^+$, $\int g d\lambda = \int g f d\mu$.*

Exercise 2.18. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$, $f_n \xrightarrow{p.w.} f$ and $\int f_1 < \infty$. Then $\lim\limits_{n\to\infty} \int f_n = \int f$.*

2.3. **Integration of Complex Valued Functions.**

Definition 2.19. *Let $f : X \to \mathbb{C}$ be measurable. Then f is said to be **integrable** if*

$$\int |f| d\mu < \infty$$

Definition 2.20. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^1(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} : f \text{ is measurable and } \int |f| < \infty\}$*

Lemma 2.21. *Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff f^+ and f^- are integrable.*

Definition 2.22. *Let $f : X \to \mathbb{R}$ be measurable. Then f is said to be **extended integrable** if*

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

Lemma 2.23. *Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff $Re(f)$ and $Im(f)$ are integrable.*

Theorem 2.24. *Dominated Convergence Let $(f_n)_{n\in\mathbb{N}} \subset L^1$, f measurable and $g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. Then $f \in L^1$ and $\int f_n \to \int f$.*

Exercise 2.25. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Then*

(1) $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$
(2) for each $f \in L^1(\mu_1 + \mu_2)$, we have that

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

Theorem 2.26. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$. Suppose that*

$$\sum_{n\in\mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n\in\mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n$$

Theorem 2.27. *Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$.*

Exercise 2.28. *Generalized Fatou's Lemma: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \geq -g$. Then $\int \liminf\limits_{n\to\infty} f_n \leq \liminf\limits_{n\to\infty} \int f_n$. What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?*

Exercise 2.29. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \to \mathbb{C}$. Suppose that $f_n \xrightarrow{uni} f$. Then*

(1) if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim\limits_{n\to\infty} \int f_n = \int f$
(2) if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on $\mathbb{R}$ with Lebesgue measure).

Exercise 2.30. *Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$, $g_n \xrightarrow{a.e.} g$, $|f_n| \leq g_n$ and $\int g_n \to \int g$. Then $\int f_n \to \int f$.*

Exercise 2.31. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.*

Exercise 2.32. *Let $(r_n)_{n\in\mathbb{N}}$ be an enummeration of the rationals. Define $f : \mathbb{R} \to [0, \infty)$ by*

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

and define $g : X \to [0, \infty]$ by

$$g(x) = \sum_{n\in\mathbb{N}} 2^{-n} f(x - r_n).$$

Then

(1) $g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.
(2) $g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of $\mathbb{R}$

(3) *Taking* $g \in L^1$, *g is unbounded on each subinterval of* $\mathbb{R}$ *and discontinuous everywhere and remains so after redefinition on a null set*

Exercise 2.33. *Let* $f \in L^1$.

(1) *If f is bounded, then for each* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for each* $E \in \mathcal{A}$, *if* $\mu(E) < \delta$, *then* $\int_E |f| < \epsilon$.
(2) *The same conclusion holds for f unbounded.*

Exercise 2.34. *Let* $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. *Define* $F : \mathbb{R} \to \mathbb{R}$ *by*

$$F(x) = \int_{(-\infty,x]} f dm.$$

Then F is continuous.

Exercise 2.35. *Denote by* δ_x *the point mass measure at* $x \in X$ *on measurable space* $(X, \mathcal{P}(X))$. *Let* $f : X \to \mathbb{C}$. *Then*

$$\int f d\delta_x = f(x)$$

Exercise 2.36. *Denote by* $\#$ *the counting measure on the measurable space* $(X, \mathcal{P}(X))$. *Let* $f : X \to \mathbb{C}$ *and suppose that* $f \in L^1$. *Then*

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if f is integrable, then $\{x \in X : f(x) \neq 0\}$ *is countable.*

Exercise 2.37. *Let* $f, g : X \to \mathbb{R}$. *Suppose that* $f, g \in L^1$. *Then* $f \leq g$ *a.e. iff for each* $E \in \mathcal{A}$, $\int_E f \leq \int_E g$.

Definition 2.38. *Let* $\mathcal{F} \subset L^1$. *Then* $\mathcal{F}$ *is said to be* ***uniformly integrable*** *if for each* $\epsilon > 0$, *there exists* $K \in \mathbb{N}$ *such that for each* $k \in \mathbb{N}$, *if* $k \geq K$, *then* $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| < \epsilon$. *(i.e.* $\lim\limits_{k\to\infty} \sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| = 0$*).*

Exercise 2.39. *Suppose that μ is finite. Let* $\mathcal{F} \subset L^1$. *Then* $\mathcal{F}$ *is uniformly integrable iff*

(1) *there exists* $M > 0$ *such that* $\sup\limits_{f \in \mathcal{F}} \int |f| \leq M$
(2) *for each* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for each* $E \in \mathcal{A}$, *if* $\mu(E) < \delta$, *then* $\sup\limits_{f \in \mathcal{F}} \int_E |f| < \epsilon$.

2.4. Integration on Product Spaces.

Definition 2.40. *Let X, Y, and Z be sets,* $E \subset X \times Y$ *and* $f : X \times Y \to Z$. *For each* $x \in X$, *define* $E_x = \{y \in Y : (x, y) \in E\}$ *and* $f_x : Y \to Z$ *by* $f_x(y) = f(x, y)$. *For each* $y \in Y$, *define* $E^y = \{x \in X : (x, y) \in E\}$ *and* $f^y : X \to Z$ *by* $f^y(x) = f(x, y)$.

Note 2.41. *It is often helpful to observe that* $(\chi_E)_x = \chi_{E_x}$ *and* $(\chi_E)^y = \chi_{E^y}$.

Lemma 2.42. *Let* $(X, \mathcal{A}), (Y, \mathcal{B})$ *be measurable spaces,* $Z = [0, \infty]$ *or* $\mathbb{C}$ *and* $f : X \times Y \to Z$.

(1) *For each* $E \in \mathcal{A} \otimes \mathcal{B}$, $x \in X$, $y \in Y$, *we have that* $E_x \in \mathcal{B}$ *and* $E^y \in \mathcal{A}$

(2) *If f is $\mathcal{A}\otimes\mathcal{B}$-measurable, then for each $x \in X$, $y \in Y$, we have that f_x is $\mathcal{B}$-measurable and f^y is $\mathcal{A}$-measurable.*

Theorem 2.43. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then for each $E \in \mathcal{A}\otimes\mathcal{B}$, the maps $\phi : X \to [0, \infty]$ and $\psi : Y \to [0, \infty]$ defined by $\phi(x) = \nu(E_x)$ and $\psi(y) = \mu(E^y)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, respectively and*

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

Theorem 2.44. *Fubini, Tonelli: Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.*

(1) *(Tonelli) For each $f \in L^+(X \times Y)$, the functions $g : X \to [0, \infty]$, $h : Y \to [0, \infty]$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable respectively and*

$$\int_{X\times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

(2) *(Fubini) For each $f \in L^1(X \times Y)$, $f_x \in L^1(\nu)$ for μ-a.e. $x \in X$ and $f^y \in L^1(\mu)$ for ν-a.e. $y \in Y$, respectively and the functions (after redefinition of f on a null set) $g : X \to \mathbb{C}$, $h : Y \to \mathbb{C}$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Furthermore*

$$\int_{X\times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

Note 2.45. *We usually just write $\int\int f d\mu d\nu$ and $\int\int f d\nu d\mu$ instead of $\int h d\nu$ and $\int g d\mu$ respectively. We have a similar result for complete product measure spaces. See*

Exercise 2.46. *Take $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1]), \mathcal{B} = \mathcal{P}([0, 1])$ and μ, ν to be Lebesgue measure and counting measure respectively. Define $D = \{(x, y) \in [0, 1]^2 : x = y\}$ Show that*

$$\int \chi_D d\mu \times \nu, \int\int \chi_D d\mu d\nu \text{ and } \int\int \chi_D d\nu d\mu$$

are all different. (Hint: for the first integral, use the definition of $\mu \times \nu$)

Exercise 2.47. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $f : X \to [0, \infty) \in L^+$. Show that $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ and $\mu \times m(G) = \int_X f d\mu$. The same is true if we replace "$\geq$" with "$>$". (Hint: to show that G is measurable, split up $(x, y) \mapsto f(x) - y)$ into the composition of measurable functions.*

Exercise 2.48. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$. Define $h : X \times Y \to \mathbb{C}$ by $h(x, y) = f(x)g(y)$.*

(1) *If f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable, then h is $\mathcal{A} \otimes \mathcal{B}$-measurable.*
(2) *If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and*

$$\int_{X\times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu$$

Note 2.49. *In the above exercise part (2), we can replace L^1 with L^+ and get the same result by the same method.*

Exercise 2.50. *Let $f : \mathbb{R} \to [0, \infty) \in L^+$. Show that*

$$\int_{\mathbb{R}} f dm = \int_{[0,\infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) dm(t)$$

2.5. Convergence.

Definition 2.51. *Let $(X, \mathcal{A})$ be a measurable space. For convencience we will define $L^0 = \{f : X \to \mathbb{C} : f \text{ is measurable}\}$.*

Definition 2.52. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **in measure** if for each $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$. This is written $f_n \xrightarrow{\mu} f$.*

Definition 2.53. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **almost uniformly** if for each $\epsilon > 0$, there exists $N \in \mathcal{A}$ such that $\mu(N) < \epsilon$ and $f_n \xrightarrow{uni} f$ on N^c. This is written $f_n \xrightarrow{a.u.} f$.*

Theorem 2.54. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{a.e.} f$.*

Theorem 2.55. *(Egoroff): Suppose that $\mu(X) < \infty$. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.e.}$. Then $f_n \xrightarrow{a.u.} f$.*

Exercise 2.56. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.*

Exercise 2.57. *Suppose $\mu(X) < \infty$. Define $d : L^0 \times L^0 \to [0, \infty)$ by*

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \qquad f, g \in L^0$$

Then d is a metric on L^0 if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each $f, g \in L^0$, $d(f, g) \leq \mu(X)$.

Exercise 2.58. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \xrightarrow{\mu} f$. Then $f \geq 0$ a.e. and $\int f \leq \liminf_{n\to\infty} \int f_n$.*

Exercise 2.59. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \xrightarrow{\mu} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.*

Exercise 2.60. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$.*

1. *If ϕ is continuous, and $f_n \xrightarrow{a.e.} f$ then $\phi \circ f_n \xrightarrow{a.e.} \phi \circ f$.*
2. *If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly or in measure, respectively.*
3. *Find a counter example to (2) if we drop the word "uniform".*

Exercise 2.61. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.u} f$. Then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{a.e.} f$.*

Exercise 2.62. *Let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then*

1. *$f_n + g_n \xrightarrow{\mu} f + g$*
2. *if $\mu(X) < \infty$, then $f_n g_n \xrightarrow{\mu} fg$*

Exercise 2.63. *Let $(f_n)_{n\in\mathbb{N}} L^0$ and $f \in L^0$. Suppose that for each $\epsilon > 0$,*

$$\sum_{n\in\mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then $f_n \xrightarrow{a.e.} f$.

3. Differentiation

3.1. Signed Measures.

Definition 3.1. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$. Then ν is said to be a* ***signed measure*** *if*

(1) *for each $E \in \mathcal{A}$, $\nu(E) < \infty$ or for each $E \in \mathcal{A}$, $\nu(E) > -\infty$.*
(2) *$\nu(\varnothing) = 0$*
(3) *for each $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ if $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is disjoint, then $\nu(\bigcup\limits_{n\in\mathbb{N}} E_n) = \sum\limits_{n\in\mathbb{N}} \nu(E_n)$ and if $|\sum\limits_{n\in\mathbb{N}} \nu(E_n)| < \infty$, then $\sum\limits_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely.*

Exercise 3.2. *Let $\nu : \mathcal{A} \to [0, \infty]$ be a signed measure and $(E_n)_{n\in\mathbb{N}}, (F_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. If $(E_n)_{n\in\mathbb{N}}$ is increasing, then $\nu(\bigcup\limits_{n\in\mathbb{N}} E_n) = \lim\limits_{n\to\infty} \nu(E_n)$. If $(F_n)_{n\in\mathbb{N}}$ is decreasing and $|\nu(E_1)| < \infty$, then $\nu(\bigcap\limits_{n\in\mathbb{N}} F_n) = \lim\limits_{n\to\infty} \nu(F_n)$.*

Definition 3.3. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$ a signed measure and $E \in \mathcal{A}$. Then E is said to be ν-**positive**, ν-**negative** and ν-**null** if for each $F \in \mathcal{A}$, $F \subset E$ implies that $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ respectively.*

Exercise 3.4. *Let $E \subset \mathcal{A}$. If E is positive, negative or null, then for each $F \in \mathcal{A}$, if $F \subset E$, then F is positive, negative or null respectively.*

Exercise 3.5. *Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ be positive, negative or null. Then $\bigcup\limits_{n\in\mathbb{N}} E_n$ is positive, negative or null respectively.*

Theorem 3.6. *Hahn Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist $P, N \in \mathcal{A}$ such that P is positive, N is negative, $X = N \cup P$ and $N \cap P = \varnothing$. Furthermore, these two sets are unique in the following sense: For any $P', N' \in \mathcal{A}$, if N, P satisfy the properties above, $P' \Delta P = N' \Delta N$ is null.*

Definition 3.7. *Let ν be a signed measure on $(X, \mathcal{A})$ and $P, N \in \mathcal{A}$. Then P and N are said to form a* ***Hahn decomposition*** *of X with respect to ν if P, N satisfy the results in the above theorem.*

Definition 3.8. *Let μ, ν be signed measures on $(X, \mathcal{A})$. Then μ and ν are said to be* ***mutually singular*** *if there exist $E, F \in \mathcal{A}$ such that $X = E \cup F$, $E \cap F = \varnothing$ and E is μ-null and F is ν-null. We will denote this by $\mu \perp \nu$.*

Theorem 3.9. *Jordan Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{A})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Definition 3.10. *Let ν be a signed measure on $(X, \mathcal{A})$. Then ν^+ and ν^- from the last theorem are called the* ***positive*** *and* ***negative variations*** *of ν respectively. We define the* ***total variation*** *measure $|\nu|$ on $(X, \mathcal{A})$ by $|\nu| = \nu^+ + \nu^-$.*

Definition 3.11. *Let ν be a signed measure on $(X, \mathcal{A})$. Then ν is said to be σ-finite if $|\nu|$ is σ-finite.*

Exercise 3.12. *Let ν be a signed measure and λ, μ positive measures on $(X, \mathcal{A})$. Suppose that $\nu = \lambda - \mu$. Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.*

Exercise 3.13. *Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Hint: use the last exercise)*

Note 3.14. *Recall that a previous exercise from the section on complex valued functions tells us that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.*

Definition 3.15. *Let ν be a signed measure on $(X, \mathcal{A})$. Then we define $L^1(\nu) = L^1(|\nu|)$. For $f \in L^1(\nu)$, we define*

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

Exercise 3.16. *Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$*

Exercise 3.17. *Let ν, μ be signed measures on $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then*

(1) E is ν-null iff $|\nu|(E) = 0$
(2) $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Exercise 3.18. *Let ν be a signed measure on $(X, \mathcal{A})$. Then*

(1) for $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$
(2) if ν is finite, then for each $E \in \mathcal{A}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : f$ is measurable and $|f| \leq 1\}$

Exercise 3.19. *Let μ be a positive measure on $(X, \mathcal{A})$ and $f \in L^0(X, \mathcal{A})$ extended μ-integrable. Define ν on $(X, \mathcal{A})$ by $\nu(E) = \int_E f d\mu$. Then*

(1) ν is a signed measure
(2) for each $E \in \mathcal{A}$, $|\nu|(E) = \int_E |f| d\mu$.

3.2. **The Lebesgue-Radon-Nikodym Theorem.**

Definition 3.20. *Let $(X, \mathcal{A})$ be a measureable space, ν be a signed measure on $(X, \mathcal{A})$ and μ a measure on $(X, \mathcal{A})$. Then ν is said to be **absolutely continuous** with respect to μ, denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.*

Note 3.21. *If there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that for each $E \in \mathcal{A}$, $\nu(E) = \int_E f d\mu$, then we write $d\nu = f d\mu$.*

Theorem 3.22. *Let $(X, \mathcal{A})$ be a measureable space, ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that $\rho = f d\mu$ and f is unique μ-a.e.*

Definition 3.23. *The decomposition $\nu = \lambda + \rho$ is referred to as the **Lebesgue decomposition of ν with respect to** μ. In the case $\nu \ll \mu$, we have $\lambda = 0$ and $\rho = \nu$ and we define the **Radon-Nikodym derivative of ν with respect to** μ, denoted by $d\nu/d\mu$, to be $d\nu/d\mu = f$ where $d\nu = f d\mu$.*

Theorem 3.24. *Let ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then*

(1) for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2) $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Exercise 3.25. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures and μ a measure.*

(1) *If for each $n \in \mathbb{N}$, $\nu_n \ll \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.*
(2) *If for each $n \in \mathbb{N}$, $\nu_n \perp \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \perp \mu$.*

Exercise 3.26. *Choose $X = [0,1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$. Let m be Lebesgue measure and μ the counting measure.*

Then

(1) *$m \ll \mu$ but for each $f \in L^+$, $dm \neq f d\mu$*
(2) *There is no Lebesgue decomposition of μ with respect to m.*

Exercise 3.27. *Let $(X, \mathcal{F}, \mu)$ be a measure space and $\mathcal{E}$ a sub σ-alg of $\mathcal{F}$ and $f \in L^1(\mu)$. Define $\nu : \mathcal{E} \to [0,\infty]$ by $\nu(E) = \int_E f d\mu$. Let $\overline{\mu}$ be the restriction of μ to $\mathcal{E}$. Define the* ***expectation of** f **given** $\mathcal{E}$ to be $E[f|\mathcal{E}] = d\nu/d\overline{\mu}$. Then for each $E \in \mathcal{E}$,*

$$\int_E E[f|\mathcal{E}]d\mu = \int_E f d\mu$$

3.3. Complex Measures.

Definition 3.28. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to \mathbb{C}$. Then ν is said to be a* ***complex measure*** *if*

(1) *$\nu(\varnothing) = 0$*
(2) *for each sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if $(E_n)_{n\in\mathbb{N}}$ is disjoint, then $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \nu(E_n)$ and $\sum_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely.*

Note 3.29. *We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.*

Definition 3.30. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. We define $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$. For $f \in L^1(\nu)$, we define*

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

Theorem 3.31. *Let $(X, \mathcal{A})$ be a measurable space, ν a complex measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exists a complex measure λ on $(X, \mathcal{A})$ and $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$ and such that for each complex measure λ' on $(X, \mathcal{A})$, $f' \in L^1(\mu)$, if $\nu = d\lambda' + f' d\mu$, then $\lambda = \lambda'$ and $f = f'$ μ-a.e.*

Theorem 3.32. *Let ν be a complex measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then*

(1) *for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and*

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2) *$\nu \ll \lambda$ and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Definition 3.33. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. Define $\mu = |\nu_1| + |\nu_2|$. Then $\nu \ll \mu$ and thus There exists $f \in L^1(\mu)$ such that $d\nu = f d\mu$. Define $|\nu| : \mathcal{A} \to [0, \infty)$ by $|\nu|(E) = \int_E |f| d\mu$ for each $E \in \mathcal{A}$. We call $|\nu|$ the* ***total variation of*** *ν.*

Exercise 3.34. *Let ν be a complex measure on $(X, \mathcal{A})$ and μ a σ-finite measures on $(X, \mathcal{A})$. If $\nu \ll \mu$, then $\{x \in X : d\nu/d\mu(x) = 0\}$ is ν-null.*

Exercise 3.35. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. Then $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu 2|$.*

Exercise 3.36. *Let $(X, \mathcal{A})$ be a measurable space, ν a complex measure on $(X, \mathcal{A})$ and $c \in \mathbb{C}$. Then $|c\nu| = |c||\nu|$.*

Exercise 3.37. *Let $(X, \mathcal{A})$ be a measurable space and ν a complex measure on $(X, \mathcal{A})$. Then*

(1) for each $E \in \mathcal{A}$, $|\nu(E)| \leq |\nu|(E)$.
(2) $\nu \ll |\nu|$ and $\big|d\nu/d|\nu|\big| = 1$ $|\nu|$-a.e.
(3) $L^1(\nu) = L^1(|\nu|)$ and for each $g \in L^1(\nu)$, $|\int g d\nu| \leq \int |g| d|\nu|$

3.4. **Differentiation.**

Definition 3.38. *Let $f : \mathbb{R}^n \to \mathbb{C}$. Then f is said to be* ***locally integrable*** *(with respect to Lebesgue measure) if f is measurable and for each $K \subset \mathbb{R}$, K is compact implies $\int_K |f| dm < \infty$. We define $L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is locally integrable}\}$*

Definition 3.39. *For $f \in L^1_{loc}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$, we define the* ***average of f over*** *$B(x, r)$, denoted by $Af(x, r)$, to be*

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x,r)} f dm$$

Exercise 3.40. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Define*

$$H^* f(x) = \sup\{\frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B\} \quad (x \in \mathbb{R}^n)$$

Then $Hf \leq H^* f \leq 2^n Hf$.

Lemma 3.41. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then $Af : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ is continuous.*

Definition 3.42. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define its* ***Hardy Littlewood maximal function****, denoted by Hf to be*

$$Hf(x) = \sup_{r>0} A|f|(x, r) \quad x \in \mathbb{R}^n$$

Theorem 3.43. *There exists $C > 0$ such that for each $f \in L^1(m)$ and $\alpha > 0$,*

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{a} \int |f| dm$$

Exercise 3.44. *Let $f \in Ł^1(\mathbb{R}^n)$. Suppose that $||f||_1 > 0$. Then there exist $C, R > 0$ such that for each $x \in \mathbb{R}^n$, if $|x| > R$, then $Hf(x) \geq C|x|^{-n}$. Hence there exists $C' > 0$ such that for each $\alpha > 0$, $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$ when α is small.*

Theorem 3.45. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{r \to 0} Af(x,r) = f(x)$$

. Equivalently, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} [f(y) - f(x)] dm(y) \right] = 0$$

Note 3.46. *We can a stronger result of the same flavor.*

Definition 3.47. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define the* ***Lebesgue set of*** *f, denoted by L_f, to be*

$$\begin{aligned} L_f &= \{x \in \mathbb{R}^n : \lim_{r \to 0} A|f - f(x)|(x,r) = 0\} \\ &= \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) \right] = 0 \right\} \end{aligned}$$

Exercise 3.48. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. If f is continuous at x, then $x \in L_f$.*

Theorem 3.49. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then $m((L_f)^c) = 0$*

Definition 3.50. *Let $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then $(E_r)_{r>0}$ is said to* ***shrink nicely to*** *x if*

(1) *for each $r > 0$, $E_r \subset B(x,r)$*
(2) *there exists $\alpha > 0$ such that for each $r > 0$, $m(E_r) > \alpha m(B(x,r))$*

Theorem 3.51. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then for each $x \in L_f$,*

$$\lim_{r \to 0} \left[\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \right] = 0$$

and

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

Definition 3.52. *Let $\mu : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$ be a Borel measure. Then μ is said to be* ***regular*** *if*

(1) *for each $K \subset \mathbb{R}^n$, if K is compact, then $\mu(K) < \infty$*
(2) *for each $E \in \mathcal{B}(\mathbb{R}^n)$, $\mu(E) = \inf\{\mu(U) : U$ is open and $E \subset U\}$*

Let ν be a signed or complex Borel measure on $\mathbb{R}^n$. Then ν is said to be regular if $|\nu|$ is regular.

Theorem 3.53. *Let ν be a regular signed or complex measure on $\mathbb{R}^n$. Let $d\nu = d\lambda + f dm$ be the Lebesgue decomposition of ν with respect to m. Then for m-a.e. $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(R^n)$, if $(E_r)_{r>0}$ shrinks nicely to x, then*

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

3.5. Functions of Bounded Variation.

Definition 3.54. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Define $F_+ : \mathbb{R} \to \mathbb{R}$ by*

$$F_+(x) = \lim_{t \to x^+} F(t) = \inf\{F(t) : t > x\}$$

Note 3.55. *Observe that $F \leq F_+$ and F_+ is increasing.*

Exercise 3.56. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then for each $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in (x, x+\delta)$, $0 \leq F_+(y) - F(y) \leq \epsilon$.*

Exercise 3.57. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then F_+ is right continuous.*

Theorem 3.58. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then*
(1) $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$ is countable
(2) F and F_+ are differentiable a.e. and $F' = F'_+$ a.e.

Definition 3.59. *Let $F : \mathbb{R} \to \mathbb{C}$. Define $T_F : \mathbb{R} \to \mathbb{R}$ by*

$$T_F(x) = \sup\left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \quad (x \in \mathbb{R})$$

T_F is called the ***total variation function of*** *F.*

Exercise 3.60. *Let $F : \mathbb{R} \to \mathbb{C}$. Then T_F is increasing.*

Lemma 3.61. *Let $F : \mathbb{R} \to \mathbb{R}$. Then $T_F + F$ and $T_F - F$ are increasing.*

Exercise 3.62. *For each $F : \mathbb{R} \to \mathbb{C}$, $T_{|F|} \leq T_F$.*

Definition 3.63. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to have* ***bounded variation*** *if $\lim\limits_{x \to \infty} T_F(x) < \infty$. The* ***total variation of*** *F, denoted by $TV(F)$, is defined to be $TV(F) = \lim\limits_{x \to \infty} T_F(x)$. We define $BV = \{F : \mathbb{R} \to \mathbb{C} : TV(F) < \infty\}$.*

Definition 3.64. *Let $a, b \in \mathbb{R}$ and $F : [a,b] \to \mathbb{C}$. Define $G_F : \mathbb{R} \to \mathbb{C}$ by $G_F = F(a)\chi_{(-\infty,a)} + F\chi_{[a,b]} + F(b)\chi_{(b,\infty)}$. Then F is said to have* ***bounded variation on*** *$[a,b]$ if $G_F \in BV$. The* ***total variation of*** *F* ***on*** *$[a,b]$, denoted by $TV(F, [a,b])$, is defined to be $TV(F,[a,b]) = TV(G_F)$ We define $BV([a,b]) = \{F : [a,b] \to \mathbb{C} : TV(F,[a,b]) < \infty\}$.*

Note 3.65. *Equivalently, $TV(F,[a,b]) = \sup\left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset [a,b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$ and $F \in BV([a,b])$ iff $TV(F,[a,b]) < \infty$. In general,*

Exercise 3.66. *Let $F \in BV$. Then F is bounded.*

Exercise 3.67. *Let $F : \mathbb{R} \to \mathbb{R}$. If F is bounded and increasing, then $F \in BV$.*

Exercise 3.68. *Let $F : \mathbb{R} \to \mathbb{C}$. If F is differentiable and F' is bounded on $[a,b]$, then, $F \in BV([a,b])$.*

Exercise 3.69. *Define $F, G : \mathbb{R} \to \mathbb{R}$ by*

$$F(x) = \begin{cases} x^2 sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then F and G are differentiable, $F \in BV([-1,1])$ and $G \notin BV([-1,1])$.

Exercise 3.70. *The following is stated for BV, but is also true for $BV([a,b])$.*

1. *For each $F, G \in BV$, $T_{F+G} \leq T_F + T_G$ and therefore BV is a vector space.*
2. *For each $F : \mathbb{R} \to \mathbb{C}$, $F \in BV$ iff $Re(f) \in BV$ and $Im(F) \in BV$.*
3. *For each $F : \mathbb{R} \to \mathbb{R}$, $F \in BV$ iff there exist functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$*
4. *For each $F \in BV$ and $x \in \mathbb{R}$, $\lim_{t \to x^+} F(t)$ and $\lim_{t \to x^-} F(t)$ exist.*
5. *For each $F \in BV$, $\{x \in R : F \text{ is not continuous at } x\}$ is countable.*
6. *For each $F \in BV$, F and F_+ are differentiable a.e. and $F' = (F_+)'$ a.e.*
7. *For each $F \in BV, c \in \mathbb{R}$, $F - c \in BV$*

Lemma 3.71. *Let $F \in BV$. Then $\lim_{x \to -\infty} T_F(x) = 0$ and if F is right continuous, then T_F is right continuous.*

Definition 3.72. *Define $NBV = \{F \in BV : F \text{ is right continuous and } \lim_{x \to -\infty} F(x) = 0\}$.*

Theorem 3.73. *Let $M(\mathbb{R})$ be the set of complex Borel measures on $\mathbb{R}$. For $F \in NBV$, define $\mu_F \in M(\mathbb{R})$ by $\mu_F((-\infty, x]) = F(x)$. Then $F \mapsto \mu_F$ defines a bijection $NBV \to M(\mathbb{R})$. In addition, $|\mu_F| = \mu_{T_F}$*

Theorem 3.74. *Let $F \in NBV$. Then $F' \in L^1(m)$, $\mu_F \perp m$ iff $F' = 0$ a.e. and $\mu_F \ll m$ iff for each $x \in \mathbb{R}$, $\int_{(-\infty,x]} F' dm = F(x)$*

Definition 3.75. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to be* ***absolutely continuous*** *if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Definition 3.76. *Let $F : [a,b] \to \mathbb{C}$. Then F is said to be* ***absolutely continuous on*** *$[a,b]$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a,b])$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Proposition 3.77. *Let $F : [a,b] \to \mathbb{C}$. If F is absolutely continuous on $[a,b]$, then $F \in BV[a,b]$.*

Exercise 3.78. *Let $F : \mathbb{R} \to \mathbb{C}$. Suppose that there exists $f \in L^1(m)$ such that $F(x) = \int_{(-\infty,x} f dm$. Then $F \in NBV$.*

Lemma 3.79. *Let $F \in NBV$. Then F is absolutely continuous iff $\mu_F \ll m$.*

Exercise 3.80. *Fundamental Theorem of Calculus: Let $F : [a,b] \to \mathbb{C}$. The following are equivalent:*

1. *F is absolutely continuous on $[a,b]$.*
2. *there exists $f \in L^1([a,b], m)$ such that for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} f dm$*
3. *F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b], m)$ and for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} F' dm$*

Exercise 3.81. *Let $F : \mathbb{R} \to \mathbb{C}$. If F is absolutely continuous. Then F is differentiable a.e.*

Exercise 3.82. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is Lipschitz continuous iff F is absolutely continuous and F' is bounded a.e.*

Exercise 3.83. *Construct an increasing function $F : \mathbb{R} \to \mathbb{R}$ whose discontinuities is $\mathbb{Q}$.*

Exercise 3.84. *Let $(F_n)_{n\in\mathbb{N}} \in NBV$ be a sequence of nonnegative, increasing functions. If for each $x \in \mathbb{R}$, $F(x) = \sum_{n\in\mathbb{N}} F_n(x) < \infty$, then for a.e. $x \in \mathbb{R}$, F is differentiable at x and $F'(x) = \sum_{n\in\mathbb{N}} F_n'(x)$.*

Exercise 3.85. *Let $F : [0,1] \to [0,1]$ be the Cantor function. Extend F to $\mathbb{R}$ by setting $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $([a_n, b_n])_{n\in\mathbb{N}}$ be an ennumeration of the closed subintervals of $[0,1]$ with rational endpoints. For $n \in \mathbb{N}$, define $F_n : \mathbb{R} \to [0,1]$ by $F_n(x) = F(\frac{x-a_n}{b_n-a_n})$. Define $G : \mathbb{R} \to \mathbb{R}$ by $G = \sum_{n\in\mathbb{N}} 2^{-n} F_n$. Then G is continuous, strictly increasing on $[0,1]$ and $G' = 0$ a.e.*

4. Topology

Definition 4.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then*

(1) f is said to be ***continuous*** *if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$.*
(2) f is said to be open if for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$.
(3) f is said to be ***closed*** *if for each $A \subset X$, if $A^c \in \mathcal{A}$, then $f(A)^c \in \mathcal{B}$.*

5. L^p Spaces

6. Functional Analysis

6.1. Normed Vector Spaces.

Note 6.1. *In the following, we will consider vector spaces over $\mathbb{C}$. There are analogous results for real vector spaces as well, just replace every $\mathbb{C}$ with $\mathbb{R}$.*

Definition 6.2. *Let X be a normed vector space. Then X is said to be a* ***Banach space*** *if X is complete.*

Definition 6.3. *Let X be a normed vector space and $(x_i)_{i=1}^n \subset X$. The series $\sum_{i=1}^\infty x_i$ is said to* ***converge*** *if the sequence $s_n := \sum_{i=1}^n x_i$ converges. The series $\sum_{i=1}^\infty x_i$ is said to* ***converge absolutely*** *if $\sum_{i\in\mathbb{N}} \|x_i\| < \infty$.*

Theorem 6.4. *Let X be a normed vector space. Then X is complete iff for each $(x_i)_{i\in\mathbb{N}} \subset X$, $\sum_{i=1}^\infty x_i$ converges absolutely implies that $\sum_{i=1}^\infty x_i$ converges.*

Definition 6.5. *Let X, Y be a normed vector spaces. A linear map $T : X \to Y$ is said to be* ***bounded*** *if there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$.*

Exercise 6.6. *Let X, Y be a normed vector spaces and $T : X \to Y$ a linear map. Then T is bounded iff there exists $r, s > 0$ such that $T(B(0,r)) \subset B(0,s)$*

Theorem 6.7. *Let X, Y be normed vector spaces and $T : X \to Y$ a linear map. Then the following are equivalent:*

(1) T is continuous
(2) T is continuous at $x = 0$
(3) T is bounded

Definition 6.8. *Let X, Y be normed vector spaces. Define $L(X,Y) = \{T : X \to Y : T \text{ is bounded}\}$. Define $\|\cdot\| : L(X,Y) \to [0,\infty)$ by*

$$\|T\| = \inf\{C \geq 0 : \text{for each } x \in X,\ \|Tx\| \leq C\|x\|\}$$

We call $\|\cdot\|$ the ***operator norm on*** *$L(X,Y)$*

Exercise 6.9. *Let X, Y be normed vector spaces. If $X \neq \{0\}$, then the operater norm on $L(X,Y)$ is given by:*

(1) $\|T\| = \sup\limits_{\|x\|=1} \|Tx\|$
(2) $\|T\| = \sup\limits_{x \neq 0} \|x\|^{-1}\|Tx\|$
(3) $\|T\| = \inf\{C \geq 0 : \text{for each } x \in X,\ \|Tx\| \leq C\|x\|\}$

Note 6.10. *From here on, unless stated otherwise, we assume $X \neq 0$.*

Exercise 6.11. *Let X, Y be normed vector spaces and $T \in L(X,Y)$. Then for each $x \in X$, $\|Tx\| \leq \|T\|\|x\|$*

Exercise 6.12. *Let X, Y be normed vector spaces. Then the operator norm is a norm on $L(X,Y)$.*

Exercise 6.13. *Let X be a normed vector space. Then addition and scalar multiplication are continuous on $X \times X$ and $\|\cdot\| : X \to [0,\infty)$ is continuous.*

Exercise 6.14. *Let X, Y be normed vector spaces. If Y is complete, then so is $L(X,Y)$.*

Definition 6.15. *Let X be a normed vector space and $M \subset X$ a closed subspace. Define $\|\cdot\| : X/M \to [0,\infty)$ by*

$$\|x + M\| := \inf_{y \in M} \|x + y\|$$

We call $\|\cdot\|$ the ***subspace norm on*** *X/M*

Exercise 6.16. *Let X be a normed vector space and $M \subsetneq X$ a proper, closed subspace of M. Then*

(1) The previously defined subspace norm on X/M is well defined and is a norm.
(2) For each $\epsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.
(3) The projection map $\pi : X \to X/M$ defined by $\pi(x) = x + M$ is continuous and $\|\pi\| = 1$.
(4) If X is complete, then X/M is complete.

Exercise 6.17. *Let X, Y be normed vector spaces and $T \in L(X,Y)$. Then*

(1) $\ker T$ *is closed*
(2) there exists a unique map $S : X/\ker T \to T(X)$ such that $T = S \circ \pi$. Furthermore S is a bounded linear bijection and $\|S\| = \|T\|$.

Exercise 6.18. *Let X, Y be normed vector spaces. Define $\phi : L(X,Y) \times X \to Y$ by $\phi(T,x) = Tx$. Then ϕ is continuous.*

Exercise 6.19. *Let X be a normed vector space and $M \subset X$ a subspace. Then $\overline{M}$ is a subspace.*

Exercise 6.20. *Let X, Y, Z be normed vector spaces, $T \in L(X,Y)$ and $S \in L(Y,Z)$. Define $ST : X \to Z$ by $STx = S(Tx)$. Then $ST \in L(X,Z)$ and $\|ST\| \leq \|S\|\|T\|$.*

Definition 6.21. *Let X be a Banach space and an associative algebra. Then X is said to be a Banach algebra if for each $S, T \in X$, $\|ST\| \leq \|S\|\|T\|$. If there exists $I \in X$ such that $I \neq 0$ and for each $T \in X$, $IT = TI = T$, then X is said to be* ***unital*** *with identity I. An element $T \in X$ is said to be* ***invertible*** *if there exists $S \in X$ such that $TS = ST = I$.*

Exercise 6.22. *Let X be a unital Banach algebra. Then $\|I\| \leq 1$.*

Note 6.23. *If X is a Banach space, then a previous exercise implies that $L(X,X)$ equipped with composition is a unital Banach algebra where I is the identity operator. It is easy to see that $\|I\| = 1$.*

Note 6.24. *Let X be a Banach algebra. Then the set of invertible elements in X is a group.*

Exercise 6.25. *Let X be a Banach algebra. Then mulitplication is continuous.*

Definition 6.26. *Let X, Y be a normed vector spaces and $T \in L(X,Y)$. Then T is said to be* ***invertible*** *or an* ***isomorphism*** *if T is a bijection and $T^{-1} \in L(Y,X)$.*

Definition 6.27. *Let X be a Banach space. Define $GL(X) := \{T \in L(X,X) : T \text{ is invertible}\}$.*

Exercise 6.28. *Let X be a Banach space. Then*

(1) For each $T \in L(X,X)$, if $\|I - T\| < 1$, then T is invertible and

$$T^{-1} = \sum_{n=0}^{\infty} (I - T)^n$$

(2) For each $S, T \in L(X,X)$, if S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$, then T is invertible.

(3) $GL(X)$ is open.

Exercise 6.29. *Let $M(X, \mathcal{A})$ denote the set of complex measures on the measurable space $(X, \mathcal{A})$. Define $\|\cdot\| : M(X, \mathcal{A}) \to [0, \infty)$ by $\|\mu\| = |\mu|(X)$. Then $\|\cdot\|$ is a norm on $M(X, \mathcal{A})$.*

6.2. **Linear Functionals.**

Definition 6.30. *Let X be a normed vector space and $T : X \to \mathbb{C}$. Then T is said to be a* ***linear functional on*** *X if T is linear and T is said to be a* ***bounded linear functional on*** *X if $T \in L(X, \mathbb{C})$. We define the* ***dual space of*** *X, denoted X^*, by $X^* = L(X, \mathbb{C})$.*

Definition 6.31. *Let X be a normed vector space and $p : X \to \mathbb{R}$. Then p is said to be a* ***sublinear functional*** *if for each $x, y \in X$, $\lambda \geq 0$,*

(1) $p(x + y) \leq p(x) + p(y)$

(2) $p(\lambda x) = \lambda p(x)$

Note 6.32. *Let X be a vector space and $\|\cdot\| : X \to [0, \infty)$ be a seminorm, then $\|\cdot\|$ is a sublinear functional.*

Theorem 6.33. *Hahn-Banach Theorem: Let X be a vector space, $p : X \to \mathbb{R}$ a sublinear functional, $M \subset X$ a subspace and $f : M \to C$ a linear functional. If for each $x \in M$, $|f(x)| \leq p(x)$, then there exists a linear functional $F : X \to \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x)$ and $F|_M = f$.*

Exercise 6.34. *Let X be a normed vector space, $M \subset X$ a subspace and $f \in M^*$. Then there exists $F \in X^*$ such that $\|F\| = \|f\|$ and $F|_M = f$.*

Exercise 6.35. *Let X be a normed vector space, $M \subsetneq X$ a proper closed subspace and $x \in X \setminus M$. Then there exists $F \in X^*$ such that $F|_M = 0$, $\|F\| = 1$ and $F(x) = \|x + M\| \neq 0$. (Hint: Consider $f : M + \mathbb{C}x \to \mathbb{C}$ defined by $f(m + \lambda x) = \lambda \|x + M\|$.)*

Exercise 6.36. *Let X be a normed vector space and $x \in X$. If $x \neq 0$, then there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$.*

Exercise 6.37. *Let X be a normed vector space. Then X^* separates the points of X.*

Definition 6.38. *Let X, Y be metric spaces and $T : X \to Y$. Then T is said to be an **isometry** if for each $x_1, x_2 \in X$, $d(Tx_1, Tx_2) = d(x_1, x_2)$.*

Exercise 6.39. *Let X, Y be metric spaces and $T : X \to Y$ and isometry. Then T is injective.*

Note 6.40. *Let X, Y be metric spaces and $T : X \to Y$ an isometry. Then T is clearly continuous. If T is surjective, then T^{-1} is an isometry and therefore continuous. Hence T is a homeomorphism.*

Exercise 6.41. *Let X be a normed vector space and $x \in X$. Define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| = \|x\|$.*

Exercise 6.42. *Let X be a normed vector space. Define $\phi : X \to X^{**}$ by $\phi(x) = \hat{x}$. Then ϕ is a linear isometry.*

Definition 6.43. *Let X be a normed vector space and define $\phi : X \to X^{**}$ as above. We define $\widehat{X} = \phi(X) \subset X^{**}$. Since $\widehat{X}$ and X are isomorphic, we may identify X as a subset of X^{**}.*

Definition 6.44. *Let X be a normed vector space and define $\phi : X \to X^{**}$ as above. Then X is said to be reflexive if ϕ is surjective. In this case ϕ is then an isomorphism*

Exercise 6.45. *Let X be a normed vector space and $f : X \to \mathbb{C}$ a linear functional on X. Then f is bounded iff $\ker f$ is closed.*

Exercise 6.46. *Let X be a normed vector space.*

(1) Let $M \subsetneq X$ be a proper closed subspace of X and $x \in X \setminus M$. Then $M + \mathbb{C}x$ is closed.
(2) Let $M \subset X$ be a finite dimensional subspace of X. Then M is closed.

Exercise 6.47. *Let X be an infinite-dimensional normed vector space.*

(1) There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that for each $m, n \in \mathbb{N}$, $\|x_n\| = 1$ and if $m \neq n$, then $\|x_m - x_n\| > \frac{1}{2}$.
(2) X is not locally compact.

Exercise 6.48. *Let X, Y be normed vector spaces and $T \in L(X, Y)$.*

*(1) Define the **adjoint of** T, $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$. Then $T^* \in L(Y^*, X^*)$.*
(2) Applying the result from (1) twice, we have that $T^{**} \in L(X^{**}, Y^{**})$. We have that for each $x \in X$, $T^{**}(\hat{x}) = \widehat{T(x)}$.
(3) T^* is injective iff $T(X)$ is dense in Y.
(4) If $T^*(Y^*)$ is dense in X^*, then T is injective. The converse is true if X is reflexive.

Exercise 6.49. *Let X be a normed vector space. Then X is reflexive iff X^* is reflexive.*

6.3. The Baire Category Theorem and Consequences.

Theorem 6.50. *Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is surjective, then T is open.*

Corollary 6.51. *Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is a bijection, then $T^{-1} \in L(X, Y)$.*

Definition 6.52. *Let X, Y be sets and $f : X \to Y$. We define the **graph of f**, $\Gamma(f)$, by $\Gamma(f) = \{(x, y) \in X \times Y : f(x) = y\}$.*

Theorem 6.53. *Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If $\Gamma(T)$ is closed, then $T \in L(X, Y)$.*

Note 6.54. *We recall that $\Gamma(T)$ is closed iff for each $(x_n)_{n\in\mathbb{N}} \subset X$, $x \in X$ and $y \in Y$ if $x_n \to x$ and $T(x_n) \to y$, then $T(x) = y$.*

Theorem 6.55. *Let X, Y be Banach spaces and $S \subset L(X, Y)$. If for each $x \in X$,*

$$\sup_{T\in S} \|Tx\| < \infty$$

then

$$\sup_{T\in S} \|T\| < \infty$$

Exercise 6.56. *Let μ be counting measure on $(N, \mathcal{P}(\mathbb{N}))$. Define $h : \mathbb{N} \to \mathbb{N}$ and ν on $(N, \mathcal{P}(\mathbb{N}))$ by $h(n) = n$ and $d\nu = hd\mu$. Define $X = L^1(\nu)$ and $Y = L^1(\mu)$. Equip both X and Y with the L^1 norm with respect to μ.*

1. *We have that X is a proper subspace of Y and therefore X is not complete.*
2. *Define $T : X \to Y$ by $Tf(n) = nf(n)$. Then T is linear, $\Gamma(T)$ is closed, and T is unbounded.*
3. *Define $S : Y \to X$ by $Sg(n) = \frac{1}{n}g(n)$. Then $S \in L(Y, X)$, S is surjective and S is not open.*

Exercise 6.57. *Let $X = C^1([0, 1])$ and $Y = C([0, 1])$. Equip both X and Y with the uniform norm.*

1. *Then X is not complete*
2. *Define $T : X \to Y$ by $Tf = f'$. Then $\Gamma(T)$ is closed and T is not bounded.*

Exercise 6.58. *Let X, Y be Banach spaces and $T \in L(X, Y)$. Then $X/\ker T \cong T(X)$ iff $T(X)$ is closed.*

Exercise 6.59. *Let X be a separable Banach space. Define $B_X = \{x \in X : \|x\| < 1\}$. Let $(x_n)_{n\in\mathbb{N}} \subset B_X$ a dense subset of the unit ball and μ the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Define $T : L^1(\mu) \to X$ by*

$$Tf = \sum_{n=1}^{\infty} f(n)x_n$$

Then

1. *T is well defined and $T \in L(L^1(\mu), X)$*
2. *T is surjective*
3. *There exists a closed subspace $K \subset L^1(\mu)$ such that $L^1(\mu)/K \cong X$*

Exercise 6.60. *Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If for each $f \in Y^*$, $f \circ T \in X^*$, then $T \in L(X, Y)$.*

7. Appendix

7.1. Summation.

Definition 7.1. *Let $f : X \to [0, \infty)$, Then we define*

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note 7.2. *Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.*