REAL ANALYSIS NOTES

Contents

1. Measure

1.1. Product Measures.

Definition 1.1. *Let $(X,\mathcal{A},\mu),(Y,\mathcal{B},\nu)$ be measurable spaces. Put $\mathcal{E}=\{A\times B : A\in\mathcal{A}$ and $B\in\mathcal{B}\}$. Then $\mathcal{E}$ is an elementary family and thus $\mathcal{M}_0=\{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n\subset\mathcal{E}$ are disjoint$\}$ is an algebra on $X\times Y$. We define $\pi_0:\mathcal{M}_0\to[0,\infty]$ by*

$$\pi_0(\bigcup_{i=1}^n A_i\times B_i)=\sum_{i=1}^n \mu(A_i)\nu(B_i)$$

Since $\mathcal{A}\otimes\mathcal{B}=\sigma(\mathcal{M}_0)$, we define a product measure $\mu\times\nu$ on $(X\times Y,\mathcal{A}\otimes\mathcal{B})$ to be an extension of π_0 to $\mathcal{A}\otimes\mathcal{B}$. The existence of which is guaranteed by Caratheodory's theorem and on $\mathcal{A}\otimes\mathcal{B}$,

$$\begin{aligned}\mu\times\nu(E)&=\inf\{\sum_{n\in\mathbb{N}}\pi_0(E_i):(E_i)_{i\in\mathbb{N}}\subset\mathcal{M}_0 \text{ and } E\subset\bigcup_{i\in\mathbb{N}}E_i\}\\&=\inf\{\sum_{n\in\mathbb{N}}\mu(A_i)\nu(B_i):(A_i\times B_i)_{i\in\mathbb{N}}\subset\mathcal{E} \text{ and } E\subset\bigcup_{i\in\mathbb{N}}A_i\times B_i\}\end{aligned}$$

If $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are both sigma finite, then so is π_0 and thus $\mu\times\nu$ is unique.

2. Integration

2.1. Measurable Functions.

Definition 2.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then f is said to be $\mathcal{A}$-$\mathcal{B}$* ***measurable*** *if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. When $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say that f is $\mathcal{A}$-**measurbale**. If $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}, \mathcal{L})$, then we say that f is* ***Borel measurable*** *or* ***Lebsgue measurbale*** *respectively.*

Lemma 2.2. *Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then*
(1) *$\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y*
(2) *$\{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra on X*

Lemma 2.3. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Suppose that there exists $\mathcal{E} \subset Y$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Let $f : X \to Y$. If for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$, then f is $\mathcal{A}$-$\mathcal{B}$ measurable.*

Proof. The previous lemma tells us that $\mathcal{L} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y. Since $\mathcal{E} \subset \mathcal{L}$, we have that $\mathcal{B} = \sigma(\mathcal{E}) \subset \mathcal{L}$. $\square$

Corollary 2.4. *Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X \to Y$. If f is continuous, then f is $\mathcal{B}(X)$-$\mathcal{B}(Y)$ measurable.*

Proof. Recall that $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$ and continuity tells us that for each $U \in \mathcal{T}_2$, $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$. $\square$

Definition 2.5. *Let X be a set and $f : X \to \mathbb{C}$. The f is said to be* ***simple*** *if $f(X)$ is finite.*

Definition 2.6. *Let $(X, \mathcal{A})$ be a measurable space. We define $S^+(X, \mathcal{A}) = \{f : X \to [0, \infty) : f \text{ is simple, measurable}\}$ and $S(X, \mathcal{A}) = \{f : X \to \mathbb{C} : f \text{ is simple, measurable}\}$*

Theorem 2.7. *Let $(X, \mathcal{A})$ be a measurable space. Then*
(1) *If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.*
(2) *If $f : X \to \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S$ such that for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.*

2.2. Integration of Nonnegative Functions.

Definition 2.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^+(X, \mathcal{A}, \mu) = \{f : X \to [0, \infty] : f \text{ is measurable}\}$ We will typically just write L^+.*

Theorem 2.9. *Monotone Convergence Theorem Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then*

$$\sup_{n \in \mathbb{N}} \int f_n = \int \sup_{n \in \mathbb{N}} f_n$$

.

Exercise 2.10. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$ and $f \in L^+$. Then*

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

.

Proof. Suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset [0,\infty)$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}\int f d(\mu_1+\mu_2) &= \sum_{i=1}^n a_i(\mu_1+\mu_2)(E_i)\\ &= \sum_{i=1}^n a_i(\mu_1(E_i)+\mu_2(E_i))\\ &= \sum_{i=1}^n a_i\mu_1(E_i)+a_i\mu_2(E_i)\\ &= \int f d\mu_1 + \int f d\mu_2\end{aligned}$$

Now for general f, choose $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that $\phi_n \to f$ pointwise and for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$. Then monotone convergence tells us that

$$\begin{aligned}\int f d(\mu_1+\mu_2) &= \lim_{n\to\infty}\int \phi_n d(\mu_1+\mu_2)\\ &= \lim_{n\to\infty}\int \phi_n d\mu_1 + \lim_{n\to\infty}\int \phi_n d\mu_2\\ &= \int f d\mu_1 + \int f d\mu_2\end{aligned}$$

□

Exercise 2.11. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Suppose that $\mu_1 \leq \mu_2$. Then for each $f \in L^+$,*

$$\int f d\mu_1 \leq \int f d\mu_2$$

Proof. First suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset [0,\infty)$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}\int f d\mu_1 &= \sum_{i=1}^n a_i\mu_1(E_i)\\ &\leq \sum_{i=1}^n a_i\mu_2(E_i)\\ &= \int f d\mu_2\end{aligned}$$

for general f,

$$\begin{aligned}\int f d\mu_1 &= \sup_{\substack{s\in S^+\\ s\leq f}} \int s d\mu_1 \\ &\leq \sup_{\substack{s\in S^+\\ s\leq f}} \int s d\mu_2 \\ &= \int f d\mu_2\end{aligned}$$

□

Theorem 2.12. *Fatou's Lemma Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Theorem 2.13. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then*

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n.$$

Exercise 2.14. *Let $f \in L^+$ and suppose that $\int f < \infty$. Put $N = \{x \in X : f(x) = \infty\}$ and $S = \{x \in X : f(x) > 0\}$. Then $\mu(N) = 0$ and S is σ-finite.*

Proof. Suppose that $\mu(N) > 0$. Define $f_n = n\chi_N \in L^+$. Then for each $n \in \mathbb{N}$, $f_n \leq f_{n+1} \leq f$ on N. So

$$\begin{aligned}\int f &\geq \int_N f \\ &= \lim_{n\to\infty} \int_N f_n \\ &= \lim_{n\to\infty} n\mu(N) \\ &= \infty, \text{ a contradiction.}\end{aligned}$$

Hence N is a null set. Now, put $S_n = \{x \in X : f(x) > 1/n\}$. Then $S = \bigcup_{n\in\mathbb{N}} S_n$. Suppose that there exists some $n \in \mathbb{N}$ such that $\mu(S_n) = \infty$. Then

$$\begin{aligned}\int f &\geq \int_{S_n} f \\ &\geq \frac{1}{n}\mu(S_n) \\ &= \infty, \text{ a contradiction.}\end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(S_n) < \infty$ and S is σ-finite.

□

Exercise 2.15. *Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.*

Proof. $f = 0$ a.e. implies that for each $E \in \mathcal{A}$, $\int_E f = 0$ is clear. Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f = 0$. For $n \in \mathbb{N}$ put $N_n = \{x \in X : f(x) > 1/n\}$ and define $N = \{x \in X : f(x) > 0\}$. So $N = \bigcup_{n\in\mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned} 0 &= \int_{N_n} f \\ &\geq \frac{1}{n}\mu(N_n) \\ &\geq 0. \end{aligned}$$

Hence for each $n \in \mathbb{N}$, $\mu(N_n) = 0$. Thus $\mu(N) = 0$ and $f = 0$ a.e. as required.

□

Exercise 2.16. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that $f_n \xrightarrow{p.w.} f$, $\lim_{n\to\infty} \int f_n = \int f$ and $\int f < \infty$. Then for each $E \in \mathcal{A}$, $\lim_{n\to\infty} \int_E f_n = \int_E f$. This result may fail to be true if $\int f = \infty$*

Proof. Let $E \in \mathcal{A}$. By Fatou's lemma, $\int_E f \leq \liminf_{n\to\infty} \int_E f_n$. Note that since $\int f < \infty$, we have that $\int_{E^c} f \leq \int f < \infty$. Thus we may write

$$\begin{aligned} \int_E f &= \int f - \int_{E^c} f \\ &\geq \int f - \liminf_{n\to\infty} \int_{E^c} f_n \\ &= \int f - \liminf_{n\to\infty} \left(\int f_n - \int_E f_n \right) \\ &= \int f - \int f + \limsup_{n\to\infty} \int_E f_n \\ &= \limsup_{n\to\infty} \int_E f_n. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} \int_E f_n \leq \int_E f \leq \liminf_{n\to\infty} \int_E f_n$$

and therefore

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

If we drop the assumption that $\int f < \infty$, then the result would fail to be true for the functions $f = \infty\chi_{(0,1)}$ and $f_n = \infty\chi_{(0,1)} + n\chi_{(1,1+1/n)}$. Here $f_n \xrightarrow{\text{p.w.}} f$, $\lim_{n\to\infty} \int f_n = \int f = \infty$ and $\lim_{n\to\infty} \int_{(1,\infty)} f_n = 1$ while $\int_{(1,\infty)} f = 0$.

□

Exercise 2.17. *Let $f \in L^+$. Define $\lambda : \mathcal{A} \to [0, \infty]$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{A}$ Then λ is a measure on $(X, \mathcal{A})$ and for each $g \in L^+$, $\int g d\lambda = \int g f d\mu$.*

Proof. Clearly $\lambda(\varnothing) = 0$. Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ and suppose that for each $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j = \varnothing$. For now, suppose that f is simple. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0, \infty)$ such that $f = \sum\limits_{i=1}^{n} a_i \chi_{E_i}$. Then

$$\begin{aligned}
\lambda\bigg(\bigcup_{j\in\mathbb{N}} A_j\bigg) &= \int_{\bigcup_{j\in\mathbb{N}} A_j} f \\
&= \sum_{i=1}^{n} a_i \mu\bigg(E_i \cap \bigg(\bigcup_{j\in\mathbb{N}} A_j\bigg)\bigg) \\
&= \sum_{i=1}^{n} a_i \mu\bigg(\bigcup_{j\in\mathbb{N}} E_i \cap A_j\bigg) \\
&= \sum_{i=1}^{n} a_i \sum_{j\in\mathbb{N}} \mu(E_i \cap A_j) \\
&= \sum_{j\in\mathbb{N}} \sum_{i=1}^{n} a_i \mu(E_i \cap A_j) \\
&= \sum_{j\in\mathbb{N}} \int_{A_j} f \\
&= \sum_{j\in\mathbb{N}} \lambda(A_j)
\end{aligned}$$

Hence λ is a measure on $(X, \mathcal{A})$. Now, for a general f, there exist $(\phi_n)_{n\in\mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \xrightarrow{\text{p.w.}} f$. Put $A = \bigcup_{j\in\mathbb{N}} A_j$ and define the measures λ_n by $\lambda_n(E) = \int_E \phi_n$. Note that we may define a monotonically increasing sequence of functions $g_n : \mathbb{N} \to [0, \infty]$ by $g_n(j) = \int_{A_j} \phi_n$. Using monotone convergence three times and a nice application of the counting measure on $\mathbb{N}$, we may write

$$\begin{aligned}
\lambda(A) &= \int_A f \\
&= \lim_{n\to\infty} \int_A \phi_n \\
&= \lim_{n\to\infty} \sum_{j\in\mathbb{N}} \int_{A_j} \phi_n \\
&= \sum_{j\in\mathbb{N}} \lim_{n\to\infty} \int_{A_j} \phi_n \quad \text{(by the above)} \\
&= \sum_{j\in\mathbb{N}} \int_{A_j} f \\
&= \sum_{j\in\mathbb{N}} \lambda(A_j).
\end{aligned}$$

Hence λ is a measure on $(X, \mathcal{A})$. Let $g \in L^+$. First assume that g is simple. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0, \infty)$ such that $g = \sum\limits_{i=1}^{n} a_i \chi_{E_i}$. In this case, we have that

$$\begin{aligned}
\int g d\lambda &= \sum_{i=1}^{n} a_i \lambda(E_i) \\
&= \sum_{i=1}^{n} a_i \int_{E_i} f d\mu \\
&= \int \left(\sum_{i=1}^{n} a_i \chi_{E_i} \right) f d\mu \\
&= \int g f d\mu.
\end{aligned}$$

Now for a general $g \in L^+$, there exist $(\psi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ψ_n is simple, $\psi_n \leq \psi_{n+1} \leq f$ and $\psi_n \xrightarrow{\text{p.w.}} g$. Monotone convergence then gives us

$$\begin{aligned}
\int g d\lambda &= \lim_{n \to \infty} \int \psi_n d\lambda \\
&= \lim_{n \to \infty} \int \psi_n f d\mu \\
&= \int g f d\mu \text{ as required.}
\end{aligned}$$

□

Exercise 2.18. *Let $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$, $f_n \xrightarrow{p.w.} f$ and $\int f_1 < \infty$. Then $\lim\limits_{n \to \infty} \int f_n = \int f$.*

Proof. First we note that since $\int f_1 < \infty$, $f_1 < \infty$ a.e., for each $n \in \mathbb{N}$, $f_1 - f_n$ and $\int f_1 - \int f_n$ are well defined and $\int f_n \leq \int f_1 < \infty$. Also, for $n \in \mathbb{N}$, $f_1 - f_n \in L^+$. So we may write

$$\begin{aligned}
\int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n \\
&= \int [(f_1 - f_n) + f_n] - \int f_n \\
&= \int f_1 - \int f_n
\end{aligned}$$

Put $g_n = f + (f_1 - f_n)$. Then $g_n \in L^+$, for each $n \in \mathbb{N}$, $g_n \leq g_{n+1}$ and $g_n \xrightarrow{\text{p.w.}} f_1$. Monotone convergence tells us that

$$\begin{aligned}\int f_1 &= \lim_{n\to\infty} \int g_n \\ &= \lim_{n\to\infty} \left[\int f + (f_1 - f_n)\right] \\ &= \lim_{n\to\infty} \left[\int f + \int (f_1 - f_n)\right] \\ &= \lim_{n\to\infty} \left[\int f + \int f_1 - \int f_n\right]\end{aligned}$$

Since $\lim\limits_{n\to\infty} \int f$ and $\lim\limits_{n\to\infty} \int f_1$ exist, $\lim\limits_{n\to\infty} \int f_n = \int f$ as required.

□

2.3. **Integration of Complex Valued Functions.**

Definition 2.19. *Let $f : X \to \mathbb{C}$ be measurable. Then f is said to be **integrable** if*

$$\int |f| d\mu < \infty$$

Definition 2.20. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^1(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} :$ f is measurable and $\int |f| < \infty\}$*

Lemma 2.21. *Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff f^+ and f^- are integrable.*

Proof. $f^+, f^- \leq |f| = f^+ + f^-$ □

Definition 2.22. *Let $f : X \to \mathbb{R}$ be measurable. Then f is said to be **extended integrable** if*

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

Lemma 2.23. *Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff $Re(f)$ and $Im(f)$ are integrable.*

Proof. $|Re(f)|, |Im(f)| \leq |f| \leq |Re(f)| + |Im(f)|$ □

Theorem 2.24. *Dominated Convergence Let $(f_n)_{n\in\mathbb{N}} \subset L^1$, f measurable and $g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. Then $f \in L^1$ and $\int f_n \to \int f$.*

Exercise 2.25. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Then*

1. *$L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$*
2. *for each $f \in L^1(\mu_1 + \mu_2)$, we have that*

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

Proof. (1) The firt part is clear since similar exercise from the section on nonnegative funtions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f| d\mu_1 + \int |f| d\mu_2$$

(2) Suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset \mathbb{C}$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum\limits_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}\int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2\end{aligned}$$

Now for general f, choose $(\phi_n)_{n\in\mathbb{N}} \subset S$ such that $\phi_n \to f$ pointwise and for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$. Then dominated convergence tells us that

$$\begin{aligned}\int f d(\mu_1 + \mu_2) &= \lim_{n\to\infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n\to\infty} \int \phi_n d\mu_1 + \lim_{n\to\infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2\end{aligned}$$

□

Theorem 2.26. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$. Suppose that*

$$\sum_{n\in\mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n\in\mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n$$

Theorem 2.27. *Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$.*

Exercise 2.28. *Generalized Fatou's Lemma: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \geq -g$. Then $\int \liminf\limits_{n\to\infty} f_n \leq \liminf\limits_{n\to\infty} \int f_n$. What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?*

Proof. First note that for each $n \in \mathbb{N}$, $\int f_n$ is well defined since $f_n^- \leq g \in L^1$. Since $g + f_n \geq 0$, we may use Fatou's lemma to write

$$\begin{aligned} \int g + \int \liminf_{n\to\infty} f_n &= \int \liminf_{n\to\infty}(g + f_n) \\ &\leq \liminf_{n\to\infty} \int (g + f_n) \\ &= \int g + \liminf_{n\to\infty} \int f_n \end{aligned}$$

Since $\int g < \infty$, $\int \liminf\limits_{n\to\infty} f_n \leq \liminf\limits_{n\to\infty} \int f_n$ as required. The analogue is as follows: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \leq g$. Then $\limsup\limits_{n\to\infty} \int f_n \leq \int \limsup\limits_{n\to\infty} f_n$. To show this, just use the result from above with the sequence $(g_n)_{n\in\mathbb{N}}$ given by $g_n = -f_n$. □

Exercise 2.29. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \to \mathbb{C}$. Suppose that $f_n \xrightarrow{uni} f$. Then*

1. *if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim\limits_{n\to\infty} \int f_n = \int f$*
2. *if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on $\mathbb{R}$ with Lebesgue measure).*

Proof. Choose $N \in \mathbb{N}$ such that for $n \geq N$ and $x \in X$, $|f(x) - f_n(x)| < 1$. Then $||f| - |f_N|| < 1$ and so $|f| < |f_N| + 1$. Thus $\int |f| \leq \int |f_N| + \mu(X) < \infty$ and $f \in L^1$. Similarly for $n \geq N$, $|f_n| < |f| + 1$. Dominated convergence then gives us that $\lim\limits_{n\to\infty} \int f_n = \int f$ as required. To see the necessity that $\mu(X) < \infty$, consider $f \equiv 0$ and $f_n = (1/n)\chi_{(0,n)}$. Then $f_n \xrightarrow{\text{uni}} f$, but $1 = \lim\limits_{n\to\infty} \int f_n \neq \int f = 0$. □

Exercise 2.30. *Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$, $g_n \xrightarrow{a.e.} g$, $|f_n| \leq g_n$ and $\int g_n \to \int g$. Then $\int f_n \to \int f$.*

Proof. We simply use Fatou's lemma. Put $h_n = (g + g_n) - |f_n - f|$. Since for each $n \in \mathbb{N}$, $|f_n| \leq g_n$, we know that $|f| \leq g$. So $h_n \geq 0$ and $h_n \xrightarrow{\text{p.w.}} 2g$. Thus

$$\begin{aligned} 2\int g &= \int \liminf_{n\to\infty} h_n \\ &\leq \liminf_{n\to\infty} \left[\left(\int g + \int g_n \right) - \int |f_n - f| \right] \\ &= 2\int g + \liminf_{n\to\infty} \left(- \int |f_n - f| \right) \\ &= 2\int g - \limsup_{n\to\infty} \int |f_n - f| \end{aligned}$$

Hence $\limsup\limits_{n\to\infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $\int f_n \to \int f$ as required. □

Exercise 2.31. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.*

Proof. Suppose that $\int |f_n - f| \to 0$. Since

$$\begin{aligned}\left|\int |f_n| - \int |f|\right| &= \left|\int (|f_n| - |f|)\right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|,\end{aligned}$$

we see that $\int |f_n| \to \int |f|$. Conversely, suppose that $\int |f_n| \to \int |f|$. Put $h_n = |f_n - f|$, $g_n = |f_n| + |f|$, $h \equiv 0$ and $g = 2f$. Then $h_n \xrightarrow{\text{a.e.}} h$, $g_n \xrightarrow{\text{a.e.}} g$ and for each $n \in \mathbb{N}$, $h_n \leq g_n$. Our assumption implies that $\int g_n \to \int g$. Thus the last exercise tells us that $\int h_n \to \int h$ as required.

$\square$

Exercise 2.32. *Let $(r_n)_{n\in\mathbb{N}}$ be an enummeration of the rationals. Define $f : \mathbb{R} \to [0, \infty)$ by*

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

and define $g : X \to [0, \infty]$ by

$$g(x) = \sum_{n\in\mathbb{N}} 2^{-n} f(x - r_n).$$

Then

(1) *$g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.*
(2) *$g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of $\mathbb{R}$*
(3) *Taking $g \in L^1$, g is unbounded on each subinterval of $\mathbb{R}$ and discontinuous everywhere and remains so after redefinition on a null set*

Proof. For convenience, define $f_n : \mathbb{R} \to [0, \infty)$ by $f_n(x) = f(x - r_n)$ for $x \in \mathbb{R}$. To show (1) we note that for each $n \in \mathbb{N}$, $f_n \in L^1$ and

$$\begin{aligned}\int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1}\end{aligned}$$

Hence

$$\sum_{n\in\mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set, $g \in L^1$. In particular $\int |g| < \infty$ and so $|g|$ (and hence g) are finite almost everywhere. For (2), since $g < \infty$ a.e., so too is g^2. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Choose $N \in \mathbb{N}$ such that $r_N \in (a, b)$. Since all the terms in the sum are nonnegative, $g^2 \geq \sum_{n\in\mathbb{N}} 2^{-2n} f_n^2$ and so

$$\begin{aligned}
\int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n\in\mathbb{N}} 2^{-2n} f_n^2 \\
&= \sum_{n\in\mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\
&\geq 2^{-2N} \int_{(a,b)} f_N^2 \\
&\geq 2^{-2N} \int_{r_N}^{b\wedge(r_N+1)} \frac{1}{x-r_N} dx \\
&= \infty
\end{aligned}$$

So g^2 is not integrable on any subinterval of $\mathbb{R}$. For (3), note that redefining g on a null set does not change the result of (2). Suppose that there is a finite subinterval $I \subset \mathbb{R}$ such that g is bounded on I. Hence there exists $M > 0$ such that for each $x \in I$, $g(x)^2 \leq M$. Then

$$\begin{aligned}
\int_I g^2 &\leq M^2 m(I) \\
&< \infty
\end{aligned}$$

which is a contradiction. So g is not bounded on any subinterval of $\mathbb{R}$. Now, suppose that there exists $x_0 \in \mathbb{R}$ such that g is continuous at x_0. Choose $\delta > 0$ such that for each $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then $|g(x) - g(x_0)| < 1$. The reverse triangle inequality tells us that for each $x \in (x_0 - \delta, x_0 + \delta)$, $|g(x)| < 1 + |g(x_0)|$. Hence g is bounded on $(x_0 - \delta, x_0 + \delta)$ which is a contradiction. So g is discontinuous everywhere. $\square$

Exercise 2.33. *Let $f \in L^1$.*

1. *If f is bounded, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\int_E |f| < \epsilon$.*
2. *The same conclusion holds for f unbounded.*

Proof. (1) Since f is bounded, there exists $M > 0$ such that $|f| \leq M$. Let $\epsilon > 0$. Choose $\delta = \epsilon/2M$. Let $E \in \mathcal{A}$. Suppose that $\mu(A) < \delta$. Then

$$\begin{aligned}
\int_E |f| &\leq M\mu(E) \\
&= M\frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

(2) Suppose that f is unbounded. Let $\epsilon > 0$. Then there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon/2$. Since ϕ is bounded, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$,

if $\mu(E) < \delta$, then $\int_E |\phi| < \epsilon/2$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then

$$\begin{aligned}\int_E |f| &\leq \int_E |f-\phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon\end{aligned}$$

□

Exercise 2.34. *Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Define $F : \mathbb{R} \to \mathbb{R}$ by*

$$F(x) = \int_{(-\infty, x]} f dm.$$

Then F is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Since $f \in L^1$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| dm < \epsilon.$$

Let $x \in \mathbb{R}$. Suppose that $|x - x_0| < \delta$. Then

$$\begin{aligned}|F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f| dm \\ &< \epsilon\end{aligned}$$

So F is continuous.

□

Exercise 2.35. *Denote by δ_x the point mass measure at $x \in X$ on measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$. Then*

$$\int f d\delta_x = f(x)$$

Proof. First assume that f is simple. Then there exist $a_1, a_2, \cdots, a_n \in \mathbb{C}$ and $E_1, E_2, \cdots, E_n \in \mathcal{P}(X)$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$ Thus $\int f d\delta_x = f(x)$. Now assume that f, which is measurable by choice of σ-algebra, satisfies $f(X) \subset [0, \infty)$. Choose a sequence $(\phi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1}$ and $\phi_n \xrightarrow{\text{p.w}} f$. From before, we see that for each $n \in \mathbb{N}$, $\int \phi_n d\delta_x = \phi_n(x)$. Monotone convergence tells us that $\int f d\delta_x = f(x)$. Now just extend to complex valued functions.

□

Exercise 2.36. *Denote by $\#$ the counting measure on the measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$ and suppose that $f \in L^1$. Then*

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if f is integrable, then $\{x \in X : f(x) \neq 0\}$ is countable.

Proof. Please refer to the definition of the sum in the appendix. First suppose that $f(X) \subset [0,\infty)$. For $n \in \mathbb{N}$, put $X_n = \{x \in X : f(x) > 1/n\}$ and define $X^* = \{x \in X : f(x) > 0\}$, $X_0 = \{x \in X : f(x) = 0\}$ Then $X^* = \bigcup_{n\in\mathbb{N}} X_n$. Since $f \in L^1$, we have that for each $n \in \mathbb{N}$,

$$\begin{aligned}\infty &> \int f d\# \\ &\geq \int_{X_n} f d\# \\ &\geq \frac{1}{n}\#(X_n).\end{aligned}$$

Thus for each $n \in \mathbb{N}$, X_n is finite and X^* is countable. Thus there exists $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $X^* = \{x_n\}_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$, define $E_n = \{x_1, x_2, \cdots, x_n\}$ and

$$\begin{aligned}f_n &= f\chi_{E_n} \\ &= \sum_{i=1}^{n} f(x_i)\chi_{\{x_i\}}\end{aligned}$$

Then $f_n \xrightarrow{\text{p.w.}} f\chi_{X^*} = f$ and for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. So

$$\begin{aligned}\int f &= \sup_{n\in\mathbb{N}} \int f_n \\ &= \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} f(x_i) \\ &= \sum_{x\in X^*} f(x) \\ &= \sum_{x\in X} f(x).\end{aligned}$$

For $f : X \to \mathbb{C}$, our L^1 assumption and the result above tell us that

$$\sum_{x\in X} |f(x)| < \infty.$$

Thus writing $f = g + ih$, we see that the same is true for f^+, f^-, g^+, g^-. Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f d\# = \sum_{x\in X} f(x).$$

$\square$

Exercise 2.37. *Let $f, g : X \to \mathbb{R}$. Suppose that $f, g \in L^1$. Then $f \leq g$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$.*

Proof. Suppose $f \leq g$ a.e. Put $N = \{x \in X : f(x) > g(x)\} \subset N$. Then $\mu(N) = 0$ and $g - f \geq 0$ on N^c. So for each $E \in \mathcal{A}$,

$$\begin{aligned} \int_E g - \int_E f &= \int_E (g - f) \\ &= \int_{E \cap N^c} (g - f) \\ &\geq 0 \end{aligned}$$

Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$. Put $N_n = \{x \in X : f(x) - g(x) > 1/n\}$ and $N = \{x \in X : f(x) > g(x)\}$. Then $N = \bigcup_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned} 0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n}\mu(N_n) \\ &\geq 0. \end{aligned}$$

So that $\mu(N_n) = 0$. Thus for each $n \in \mathbb{N}$, $\mu(N_n) = 0$ which implies $\mu(N) = 0$. Therefore $f \leq g$ a.e. as required. $\square$

Definition 2.38. *Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is said to be **uniformly integrable** if for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup\limits_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$. (i.e. $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| = 0$).*

Exercise 2.39. *Suppose that μ is finite. Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is uniformly integrable iff*

(1) *there exists $M > 0$ such that $\sup\limits_{f \in \mathcal{F}} \int |f| \leq M$*
(2) *for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\sup\limits_{f \in \mathcal{F}} \int_E |f| < \epsilon$.*

Proof. ($\Rightarrow$): (1) Suppose that $\mathcal{F}$ is uniformly integrable. Then there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| < 1$. Choose $M = \mu(X)K + 1$. Then for each $f \in \mathcal{F}$,

$$\begin{aligned} \int |f| &= \int_{\{|f| > K\}} |f| + \int_{\{|f| \leq K|\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M \end{aligned}$$

(2) Let $\epsilon > 0$. Then choose $K \in \mathbb{N}$ such that $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>K\}} |f| < \epsilon/2$ and choose $\delta = \epsilon/2K$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then for $f \in \mathcal{F}$,

$$\begin{aligned}\int_E |f| &= \int_{E \cap \{|f|>K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon\end{aligned}$$

($\Leftarrow$): Choose $M > 0$ as in (1). Suppose that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $f \in \mathcal{F}$ such that $\mu(\{|f| > K\}) \geq \epsilon$. Choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $f_K \in \mathcal{F}$ such that $\mu(\{|f_K| > K\}) \geq \epsilon$. Then

$$\begin{aligned}\int |f_K| &\geq \int_{\{|f_K|>K\}} |f| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M,\end{aligned}$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $f \in \mathcal{F}$, $\mu(\{|f| > K\}) < \epsilon$. Since $\mu(\{|f| > k\})$ is a decreasing sequence in k, we have that $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \mu(\{|f| > k\}) = 0$. Now, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Choose $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$, $\mu(\{|f| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$,

$$\int_{\{|f|>k\}} |f| < \epsilon.$$

Thus $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| = 0$ as required.

□

2.4. **Integration on Product Spaces.**

Definition 2.40. *Let X, Y, and Z be sets, $E \subset X \times Y$ and $f : X \times Y \to Z$. For each $x \in X$, define $E_x = \{y \in Y : (x, y) \in E\}$ and $f_x : Y \to Z$ by $f_x(y) = f(x, y)$. For each $y \in Y$, define $E^y = \{x \in X : (x, y) \in E\}$ and $f^y : X \to Z$ by $f^y(x) = f(x, y)$.*

Note 2.41. *It is often helpful to observe that $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.*

Lemma 2.42. *Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces, $Z = [0, \infty]$ or $\mathbb{C}$ and $f : X \times Y \to Z$.*
1. *For each $E \in \mathcal{A} \otimes \mathcal{B}$, $x \in X$, $y \in Y$, we have that $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$*
2. *If f is $\mathcal{A} \otimes \mathcal{B}$-measurable, then for each $x \in X$, $y \in Y$, we have that f_x is $\mathcal{B}$-measurable and f^y is $\mathcal{A}$-measurable.*

Theorem 2.43. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then for each $E \in \mathcal{A} \otimes \mathcal{B}$, the maps $\phi : X \to [0, \infty]$ and $\psi : Y \to [0, \infty]$ defined by $\phi(x) = \nu(E_x)$ and $\psi(y) = \mu(E^y)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, respectively and*

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

Theorem 2.44. *Fubini, Tonelli: Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.*

(1) *(Tonelli) For each $f \in L^+(X \times Y)$, the functions $g : X \to [0, \infty]$, $h : Y \to [0, \infty]$ defined by $g(x) = \int_Y f_x(y)d\nu(y)$ and $h(y) = \int_X f^y(x)d\mu(x)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable respectively and*
$$\int_{X\times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$
(2) *(Fubini) For each $f \in L^1(X \times Y)$, $f_x \in L^1(\nu)$ for μ-a.e. $x \in X$ and $f^y \in L^1(\mu)$ for ν-a.e. $y \in Y$, respectively and the functions (after redefinition of f on a null set) $g : X \to \mathbb{C}$, $h : Y \to \mathbb{C}$ defined by $g(x) = \int_Y f_x(y)d\nu(y)$ and $h(y) = \int_X f^y(x)d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Furthermore*
$$\int_{X\times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

Note 2.45. *We usually just write $\int\int f d\mu d\nu$ and $\int\int f d\nu d\mu$ instead of $\int h d\nu$ and $\int g d\mu$ respectively. We have a similar result for complete product measure spaces. See*

Exercise 2.46. *Take $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1]), \mathcal{B} = \mathcal{P}([0, 1])$ and μ, ν to be Lebesgue measure and counting measure respectively. Define $D = \{(x, y) \in [0, 1]^2 : x = y\}$ Show that*
$$\int \chi_D d\mu \times \nu, \int\int \chi_D d\mu d\nu \text{ and } \int\int \chi_D d\nu d\mu$$
are all different. (Hint: for the first integral, use the definition of $\mu \times \nu$)

Proof. Let $x, y \in [0, 1]$. Then $(\chi_D)_x = \chi_{D_x} = \chi_x$ and $(\chi_D)^y = \chi_{D^y} = \chi_y$. Thus
$$\begin{aligned}\int\int \chi_D d\mu d\nu &= \int \mu(\{y\})d\nu \\ &= \int 0 d\nu \\ &= 0\end{aligned}$$
and
$$\begin{aligned}\int\int \chi_D d\mu d\nu &= \int \nu(\{x\})d\mu \\ &= \int 1 d\mu \\ &= 1\end{aligned}$$
Now, Observe that $\int \chi_D d\mu \times \nu = \mu \times \nu(D)$. Recall from the section on product measures that $\mu \times \nu(D) = \inf\{\sum_{n\in\mathbb{N}} \mu(A_n)\nu(B_n) : (A_n \times B_n)_{n\in\mathbb{N}} \subset \mathcal{E} \text{ and } D \subset \bigcup_{n\in\mathbb{N}} A_n \times B_n\}$. Let $(A_n \times B_n)_{n\in\mathbb{N}} \subset \mathcal{E}$. Suppose that $D \subset \bigcup_{n\in\mathbb{N}} A_n \times B_n$. Then for each $x \in [0, 1]$, $(x, x) \in \bigcup_{n\in\mathbb{N}} A_n \times B_n$. So for each $x \in [0, 1]$, there exists $n \in \mathbb{N}$, such that $x \in A_n \cap B_n$. Thus $[0, 1] \subset \bigcup_{n\in\mathbb{N}} A_n \cap B_n$. Since $1 = \mu([0, 1]) \leq \sum_{n\in\mathbb{N}} \mu(A_n \cap B_n)$, we know that there exists $n \in \mathbb{N}$ such that $0 < \mu(A_n \cap B_n)$. Thus $\mu(A_n) > 0$ and $\mu(B_n) > 0$. Since $\mu(B_n) > 0$, B_n must be infinite and therefore $\nu(B_n) = \infty$. So $\sum_{n\in\mathbb{N}} \mu(A_n)\nu(B_n) = \infty$.

□

Exercise 2.47. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $f : X \to [0, \infty) \in L^+$. Show that $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ and $\mu \times m(G) = \int_X f d\mu$. The same is true if we replace "$\geq$" with "$>$". (Hint: to show that G is measurable, split up $(x, y) \mapsto f(x) - y$) into the composition of measurable functions.*

Proof. Define $\phi : X \times [0, \infty) \to [0, \infty)^2$ and $\psi : [0, \infty)^2 \to [0, \infty)$ by $\phi(x, y) = (f(x), y)$ and $\psi(z, y) = z - y$. Then $G = \{(x, y) \in X \times [0, \infty) : \psi \circ \phi(x, y) \geq 0\}$. Let $A, B \in \mathcal{B}([0, \infty))$. Then $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0, \infty))$. Since $\mathcal{B}([0, \infty)^2) = \mathcal{B}([0, \infty)) \otimes \mathcal{B}([0, \infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0, \infty))\})$, we have that ϕ is $\mathcal{A} \otimes \mathcal{B}([0, \infty))$-$\mathcal{B}([0, \infty)^2)$ measurable. Since ψ is continuous, we have that ψ is $\mathcal{B}([0, \infty)^2)$-$\mathcal{B}([0, \infty))$ measurable. This implies that $\psi \circ \phi$ is $\mathcal{A} \otimes \mathcal{B}([0, \infty))$-$\mathcal{B}([0, \infty))$ measurable. Thus $G = \psi \circ \phi^{-1}([0, \infty)) \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$. Now for $x \in X$, $G_x = \{y \in [0, \infty) : f(x) \geq y\} = [0, f(x)]$. Thus

$$
\begin{aligned}
\mu \times m(G) &= \int \chi_G d\mu \times m \\
&= \int_X \int_{[0,\infty)} \chi_{G_x} dm d\mu(x) \\
&= \int_X f(x) d\mu(x)
\end{aligned}
$$

The same reasoning holds if we replace "$\geq$" with "$>$". □

Exercise 2.48. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$. Define $h : X \times Y \to \mathbb{C}$ by $h(x, y) = f(x)g(y)$.*

1. *If f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable, then h is $\mathcal{A} \otimes \mathcal{B}$-measurable.*
2. *If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and*

$$
\int_{X \times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu
$$

Proof.

1. First suppose that f, g are simple. Then there exist $(A_i)_{i=1}^n \subset \mathcal{A}$, $(B_j)_{j=1}^m \subset \mathcal{B}$ and $(a_i)_{i=1}^n, (b_i)_{j=1}^m \subset \mathbb{C}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$ and $g = \sum_{j=1}^m b_j \chi_{B_j}$. Then $h = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{A_i \times B_j}$. So h is $\mathcal{A} \otimes \mathcal{B}$-measurable. For general f, g, there exist $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$ and $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$ such that $f_n \to f$ pointwise, $g_n \to g$ pointwise and for each $n \in \mathbb{N}$, $|f_n| \leq |f_{n+1}| \leq |f|$ and $|g_n| \leq |g_{n+1}| \leq |g|$. For $n \in \mathbb{N}$, define $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$ by $h_n = f_n g_n$. Then $h_n \to h$ pointwise and for each $n \in \mathbb{N}$, $|h_n| \leq |h_{n+1}| \leq |h|$. Thus h is $\mathcal{A} \otimes \mathcal{B}$-measurable.
2. First suppose f and g are simple as before. Then

$$
\begin{aligned}
\int_{X \times Y} |h| d\mu \times \nu &\leq \sum_{i=1}^n \sum_{j=1}^m |a_i b_j| \mu(A_i) \nu(B_j) \\
&= \big(\sum_{i=1}^n |a_i| \mu(A_i)\big)\big(\sum_{j=1}^m |b_j| \nu(B_j)\big) \\
&= \int_X |f| d\mu \int_Y |g| d\nu \\
&< \infty
\end{aligned}
$$

So $h \in L^1(\mu \times \nu)$. Furthermore,

$$\begin{aligned}\int_{X\times Y} h d\mu \times \nu &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \mu(A_i)\nu(B_j)\\ &= \big(\sum_{i=1}^{n} a_i\mu(A_i)\big)\big(\sum_{j=1}^{m} b_j\nu(B_j)\big)\\ &= \int_X f d\mu \int_Y g d\nu\end{aligned}$$

For general $f \in L^1(\mu), g \in L^1(\nu)$, take $(h_n)_{n\in\mathbb{N}}$ as before. Monotone convergence and the result above say that

$$\begin{aligned}\int_{X\times Y} |h| d\mu \times d\nu &= \lim_{n\to\infty}\int_{X\times Y} |h_n| d\mu \times \nu\\ &= \lim_{n\to\infty}\left(\int_X |f_n| d\mu \int_Y |g_n| d\nu\right)\\ &= \int_X |f| d\mu \int_Y |g| d\nu\\ &< \infty\end{aligned}$$

So $h \in L^1(\mu \times \nu)$. Dominated convergence and the result above then tell us that

$$\begin{aligned}\int_{X\times Y} h d\mu \times d\nu &= \lim_{n\to\infty}\int_{X\times Y} h_n d\mu \times d\nu\\ &= \lim_{n\to\infty}\left(\int_X f_n d\mu \int_Y g_n d\nu\right)\\ &= \int_X f d\mu \int_Y g d\nu\end{aligned}$$

□

Note 2.49. *In the above exercise part (2), we can replace L^1 with L^+ and get the same result by the same method.*

Exercise 2.50. *Let $f : \mathbb{R} \to [0,\infty) \in L^+$. Show that*

$$\int_{\mathbb{R}} f dm = \int_{[0,\infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) dm(t)$$

Proof. Note that

$$\int_{[0,\infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) = \int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x\in\mathbb{R}: f(x)\geq t\}} dm\right] dm(t)$$

Comparing this with Tonelli's theorem, we can put $\chi_{\{x\in\mathbb{R}:f(x)\geq t\}} = (\chi_E)^t = \chi_{E^t}$. Then $E = \{(x,t) \in \mathbb{R} \times [0,\infty) : f(x) \geq t\}$ and $E_x = \{t \in [0,\infty) : f(x) \geq t\} = [0, f(x)]$. Tonelli's theorem tells us that

$$\int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x\in\mathbb{R}: f(x)\geq t\}}(x) dm(x) \right] dm(t) = \int_{\mathbb{R}} \left[\int_{[0,\infty)} \chi_{[0,f(x)]}(t) dm(t) \right] dm(x)$$
$$= \int_{\mathbb{R}} f(x) dm(x)$$

□

2.5. Convergence.

Definition 2.51. *Let $(X, \mathcal{A})$ be a measurable space. For convencience we will define $L^0 = \{f : X \to \mathbb{C} : f$ is measurable$\}$.*

Definition 2.52. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **in measure** if for each $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$. This is written $f_n \xrightarrow{\mu} f$.*

Definition 2.53. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **almost uniformly** if for each $\epsilon > 0$, there exists $N \in \mathcal{A}$ such that $\mu(N) < \epsilon$ and $f_n \xrightarrow{uni} f$ on N^c. This is written $f_n \xrightarrow{a.u.} f$.*

Theorem 2.54. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{a.e.} f$.*

Theorem 2.55. *(Egoroff): Suppose that $\mu(X) < \infty$. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.e.}$. Then $f_n \xrightarrow{a.u.} f$.*

Exercise 2.56. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.*

Proof. Let $\epsilon > 0$. for $n \in \mathbb{N}$, define $E_{\epsilon,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$. Then for $n \in \mathbb{N}$,

$$\int |f - f_n| \geq \int_{E_{\epsilon,n}} |f - f_n|$$
$$\geq \epsilon\mu(E_{\epsilon,n}).$$

So for each $n \in \mathbb{N}$, $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$. Since $\int |f - f_n| \to 0$, we have that $\mu(E_{\epsilon,n}) \to 0$. Since $\epsilon > 0$ is arbitrary, $f_n \xrightarrow{\mu} f$ as required. □

Exercise 2.57. *Suppose $\mu(X) < \infty$. Define $d : L^0 \times L^0 \to [0, \infty)$ by*

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \qquad f, g \in L^0$$

Then d is a metric on L^0 if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each $f, g \in L^0$, $d(f, g) \leq \mu(X)$.

Proof. Let $f, g \in L^0$. Clearly $d(f, g) = d(g, f)$. If $f = g$ a.e. then clearly $d(f, g) = 0$. Conversely, if $d(f, g) = 0$, then $\frac{|f-g|}{1+|f-g|} = 0$ a.e and so $|f - g| = 0$ a.e. which implies $f = g$ a.e. It is not hard to show that $\phi : [0, \infty) \to [0, \infty)$ given by $\phi(x) = \frac{x}{1+x}$ satisfies $\phi(x + y) \leq \phi(x) + \phi(y)$. Thus satisfies the triangle inequality. Now, let $(f_n)_{n\in\mathbb{N}} \subset L^0$. Suppose that $f_n \stackrel{\mu}{\not\to} f$. Then there exists $\epsilon > 0, \delta > 0$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$, $\mu(E_{\epsilon,n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$. It is not hard to show that ϕ from earlier is increasing. Thus for each $k \in \mathbb{N}$,

$$\begin{aligned}
d(f_{n_k}, f) &= \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\
&\geq \int_{E_{\epsilon,n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\
&\geq \int_{E_{\epsilon,n_k}} \frac{\epsilon}{1+\epsilon} \\
&\geq \frac{\epsilon\delta}{1+\epsilon}
\end{aligned}$$

So $f_{n_k} \overset{d}{\not\to} f$. Hence $f_{n_k} \xrightarrow{d} f$ implies that $f_{n_k} \xrightarrow{\mu} f$. Conversely, suppose that $f_{n_k} \xrightarrow{\mu} f$. Let $\epsilon > 0$. Then $\delta = \frac{\epsilon}{1+\mu(X)} > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then $\mu(E_{\delta,n}) < \frac{\delta}{1+\delta}$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Since ϕ is increasing and $\phi \leq 1$, we have that

$$\begin{aligned}
d(f_n, f) &= \int \frac{|f_n - f|}{1 + |f_n - f|} \\
&= \int_{E_{\delta,n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E^c_{\delta,n}} \frac{|f_n - f|}{1 + |f_n - f|} \\
&\leq \mu(E_{\delta,n}) + \mu(X)\frac{\delta}{1+\delta} \\
&< \frac{\delta}{1+\delta}(1 + \mu(X)) \\
&\leq \delta(1 + \mu(X)) \\
&= \epsilon
\end{aligned}$$

□

Exercise 2.58. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \xrightarrow{\mu} f$. Then $f \geq 0$ a.e. and $\int f \leq \liminf\limits_{n\to\infty} \int f_n$.*

Proof. Since $f_n \xrightarrow{\mu} f$, there is a subsequence converging to f a.e. So clearly $f \geq 0$ a.e. Now, choose a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $\int f_{n_k} \to \liminf\limits_{n\to\infty} \int f_n$. Since $f_n \xrightarrow{\mu} f$ so does $(f_{n_k})_{k\in N}$. Therefore there exists a subsequence $(f_{n_{k_j}})_{k\in\mathbb{N}}$ of $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Thus $f \geq 0$ a.e. and Fatou's lemma tells us that

$$\begin{aligned}
\int f &\leq \liminf_{j\in\mathbb{N}} \int f_{n_{k_j}} \\
&= \liminf_{n\to\infty} \int f_n.
\end{aligned}$$

□

Exercise 2.59. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \xrightarrow{\mu} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.*

Proof. Clearly $(f_n)_{n\in\mathbb{N}} \subset L^1$. Since $f_n \xrightarrow{\mu} f$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$. This implies that $|f| \leq g$ a.e. and so $f \in L^1$. For $n \in \mathbb{N}$, put $h_n = 2g - |f_n - f|$. Then for each $n \in \mathbb{N}$, $h_n \geq 0$ and $h_n \xrightarrow{\mu} 2g$. By the previous exercise

$$\begin{aligned}\int 2g &\leq \liminf_{n\to\infty} \int (2g - |f_n - f|) \\ &= \int 2g - \limsup_{n\to\infty} \int |f_n - f|.\end{aligned}$$

So $\limsup\limits_{n\to\infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $f_n \xrightarrow{L^1} f$ as required. □

Exercise 2.60. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$.*

(1) *If ϕ is continuous, and $f_n \xrightarrow{a.e.} f$ then $\phi \circ f_n \xrightarrow{a.e.} \phi \circ f$.*
(2) *If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly or in measure, respectively.*
(3) *Find a counter example to (2) if we drop the word "uniform".*

Proof. (1) Clear

(2) Suppose that ϕ is unifomly continuous.

(uniform conv.) Suppose that $f_n \xrightarrow{\text{uni}} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Now choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq n$ then for each $x \in X$, $|f_n(x) - f(x)| < \delta$. Let $n \in \mathbb{N}$, suppose $n \geq N$, Let $x \in X$. Then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Thus $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$.

(almost uni.) Suppose that $f_n \xrightarrow{\text{a.u.}} f$. Let $\epsilon > 0$. Choose $N \in \mathcal{A}$ such $\mu(N) < \epsilon$ and $f_n \xrightarrow{\text{uni}} f$ on N^c. Then from above, we know that $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$ on N^c. Thus $\phi \circ f_n \xrightarrow{\text{a.u.}} \phi \circ f$.

(measure) Suppose that $f_n \xrightarrow{\mu} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Observe that for $x \in X$, if $|f_n(x) - f(x)| < \delta$, then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Hence $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$. By definition of convergence in measure, $\mu(F_{n,\delta}) \to 0$. Thus $\mu(E_{n,\epsilon}) \to 0$. Hence $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.

(3)

□

Exercise 2.61. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.u} f$. Then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{a.e.} f$.*

Proof. (measure) Let $\epsilon > 0$, $\delta > 0$. Choose $M \in \mathcal{A}$ such that $\mu(M) < \delta$ and $f_n \xrightarrow{\text{uni}} f$ on M^c. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then for each $x \in M^c$, $|f_n(x) - f(x)| < \epsilon$. Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then $E_{\epsilon,n} \subset M$ and $\mu(E_{\epsilon,n}) < \delta$. Thus $\mu(E_{\epsilon,n}) \to 0$ and $f_n \xrightarrow{\mu} f$.

(a.e.) For each $n \in \mathbb{N}$, Choose $N_n \in \mathcal{A}$ such that $\mu(N_n) < 1/n$ and $f_n \xrightarrow{\text{uni}} f$ on N_n^c. Observe that for $x \in X$, if $x \in \bigcup_{n\in\mathbb{N}} N_n^c$, then $f_n(x) \to f(x)$. Thus $N = \{x \in X : f_n(x) \not\to f(x)\} \subset \bigcap_{n\in\mathbb{N}} N_n$. Therefor $\mu(N) = 0$ and $f_n \xrightarrow{\text{a.e.}} f$. □

Exercise 2.62. *Let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then*

(1) $f_n + g_n \xrightarrow{\mu} f + g$
(2) *if* $\mu(X) < \infty$, *then* $f_n g_n \xrightarrow{\mu} fg$

Proof. (1) Let $\epsilon > 0$. For convenience, put $F_{n,\epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$, $G_{n,\epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$, and $(F+G)_{n,\epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g_n(x))| \geq \epsilon\}$ Observe that for $x \in X$, $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$. Thus $(F+G)_{n,\epsilon} \subset F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$. Since $\mu(F_{n,\epsilon/2} \cup G_{n,\epsilon/2}) \leq \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}) \to 0$, we have that $\mu((F+G)_{n,\epsilon}) \to 0$. Hence $f_n + g_n \xrightarrow{\mu} f + g$.

(2) Suppose that $\mu(X) < \infty$. Let $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(f_n g_n)_{n \in \mathbb{N}}$. Choose a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e}} f$ and $g_{n_{k_j}} \xrightarrow{\text{a.e}} g$. Then $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.e.}} fg$. Egoroff's theorem tells us that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.u.}} fg$, which implies that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$. Thus for each subsequence $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ of $(f_n g_n)_{n \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ of $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$. Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that $f_n g_n \xrightarrow{\mu} fg$.

□

Exercise 2.63. *Let* $(f_n)_{n \in \mathbb{N}} L^0$ *and* $f \in L^0$. *Suppose that for each* $\epsilon > 0$,

$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then $f_n \xrightarrow{a.e.} f$.

Proof. Let $\epsilon > 0$. By assumption we know that

$$\begin{aligned}
\int \left[\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} \right] d\mu &= \sum_{n \in \mathbb{N}} \int \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} d\mu \\
&= \sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \\
&< \infty
\end{aligned}$$

Thus we also know that $\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} < \infty$ a.e. Equivalently, we could say that for a.e. $x \in X$, $|\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\}| < \infty$. For $k \in \mathbb{N}$, define $N_k = \{x \in X : \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > 1/k\}} = \infty\}$. Then for each $k \in \mathbb{N}$, $\mu(N_k) = 0$. Define $N = \bigcup_{k \in \mathbb{N}} N_k$. Then $\mu(N) = 0$. Let $x \in N^c$ and $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \epsilon$. Then $\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\} \subset \{n \in \mathbb{N} : f_n(x) - f(x) > 1/k\}$ which is finite because $x \in N_k^c$. Put $M = \max\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\}$. Then for $m \geq M$, $|f_m(x) - f(x) \leq \epsilon|$. Thus $f_n(x) \to f(x)$. Hence $f_n \xrightarrow{\text{a.e.}} f$. □

3. Differentiation

3.1. **Signed Measures.**

Definition 3.1. *Let* $(X, \mathcal{A})$ *be a measurable space and* $\nu : \mathcal{A} \to [-\infty, \infty]$. *Then* ν *is said to be a* ***signed measure*** *if*

(1) *for each* $E \in \mathcal{A}$, $\nu(E) < \infty$ *or for each* $E \in \mathcal{A}$, $\nu(E) > -\infty$.
(2) $\nu(\varnothing) = 0$

(3) *for each* $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ *if* $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ *is disjoint, then* $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \nu(E_n)$ *and if* $|\sum_{n\in\mathbb{N}} \nu(E_n)| < \infty$, *then* $\sum_{n\in\mathbb{N}} \nu(E_n)$ *converges absolutely.*

Exercise 3.2. *Let* $\nu : \mathcal{A} \to [0,\infty]$ *be a signed measure and* $(E_n)_{n\in\mathbb{N}}, (F_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. *If* $(E_n)_{n\in\mathbb{N}}$ *is increasing, then* $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \nu(E_n)$. *If* $(F_n)_{n\in\mathbb{N}}$ *is decreasing and* $|\nu(E_1)| < \infty$, *then* $\nu(\bigcap_{n\in\mathbb{N}} F_n) = \lim_{n\to\infty} \nu(F_n)$.

Proof. Put $E_1' = E_1$, $F_1' = F_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $E_n' = E_n \setminus E_{n-1}$ and $F_n' = F_1 \setminus F_n$. Then $(E_n')_{n\in\mathbb{N}} \subset \mathcal{A}$ is disjoint. Thus

$$\begin{aligned}\nu(\bigcup_{n\in\mathbb{N}} E_n) &= \nu(\bigcup_{n\in\mathbb{N}} E_n') \\ &= \sum_{n\in\mathbb{N}} \nu(E_n') \\ &= \lim_{n\to\infty} \sum_{n=1}^{n} \nu(E_n') \\ &= \lim_{n\to\infty} \nu(E_n)\end{aligned}$$

Since $(F_n')_{n\in\mathbb{N}}$ is increasing, we now know that

$$\begin{aligned}\nu(F_1) - \nu(\bigcap_{n\in\mathbb{N}} F_n) &= \nu(F_1 \setminus \bigcap_{n\in\mathbb{N}} F_n) \\ &= \nu(\bigcup_{n\in\mathbb{N}} F_n') \\ &= \lim_{n\to\infty} \nu(F_n') \\ &= \lim_{n\to\infty} \nu(F_1 \setminus F_n) \\ &= \nu(F_1) - \lim_{n\to\infty} \nu(F_n)\end{aligned}$$

Since $|\nu(F_1)| < \infty$, we see that $\nu(\bigcap_{n\in\mathbb{N}} F_n) = \lim_{n\to\infty} \nu(F_n)$. □

Definition 3.3. *Let* $(X, \mathcal{A})$ *be a measurable space and* $\nu : \mathcal{A} \to [-\infty, \infty]$ *a signed measure and* $E \in \mathcal{A}$. *Then* E *is said to be* ν-***positive***, ν-***negative*** *and* ν-***null*** *if for each* $F \in \mathcal{A}$, $F \subset E$ *implies that* $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ *respectively.*

Exercise 3.4. *Let* $E \subset \mathcal{A}$. *If* E *is positive, negative or null, then for each* $F \in \mathcal{A}$, *if* $F \subset E$, *then* F *is positive, negative or null respectively.*

Proof. Clear □

Exercise 3.5. *Let* $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ *be positive, negative or null. Then* $\bigcup_{n\in\mathbb{N}} E_n$ *is positive, negative or null respectively.*

Proof. Suppose that $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is positive. Let $F \in \mathcal{A}$. Suppose that $F \subset \bigcup_{n\in\mathbb{N}} E_n$. Put $P_1 = E_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $P_n = E_n \setminus (\bigcup_{j=1}^{n-1} E_j)$. So $\bigcup_{n\in\mathbb{N}} P_n = \bigcup_{n\in\mathbb{N}} E_n$ and $(P_n)_{n\in\mathbb{N}}$ is disjoint. Thus

$$\begin{aligned}\nu(F) &= \nu(F \cap \bigcup_{n\in\mathbb{N}} P_n) \\ &= \nu(\bigcup_{n\in\mathbb{N}} (F \cap P_n)) \\ &= \sum_{n\in\mathbb{N}} \nu(F \cap P_n) \\ &\geq 0\end{aligned}$$

The process is the same if $(E_n)_{n\in\mathbb{N}}$ is negative and null. $\square$

Theorem 3.6. *Hahn Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist $P, N \in \mathcal{A}$ such that P is positive, N is negative, $X = N \cup P$ and $N \cap P = \varnothing$. Furthermore, these two sets are unique in the following sense: For any $P', N' \in \mathcal{A}$, if N, P satisfy the properties above, $P' \Delta P = N' \Delta N$ is null.*

Definition 3.7. *Let ν be a signed measure on $(X, \mathcal{A})$ and $P, N \in \mathcal{A}$. Then P and N are said to form a **Hahn decomposition** of X with respect to ν if P, N satisfy the results in the above theorem.*

Definition 3.8. *Let μ, ν be signed measures on $(X, \mathcal{A})$. Then μ and ν are said to be **mutually singular** if there exist $E, F \in \mathcal{A}$ such that $X = E \cup F$, $E \cap F = \varnothing$ and E is μ-null and F is ν-null. We will denote this by $\mu \perp \nu$.*

Theorem 3.9. *Jordan Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{A})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. Choose a Hahn decomposition P, N of X with respect to ν. Define ν^+, ν^- by $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$. $\square$

Definition 3.10. *Let ν be a signed measure on $(X, \mathcal{A})$. Then ν^+ and ν^- from the last theorem are called the **positive** and **negative variations** of ν respectively. We define the **total variation** measure $|\nu|$ on $(X, \mathcal{A})$ by $|\nu| = \nu^+ + \nu^-$.*

Definition 3.11. *Let ν be a signed measure on $(X, \mathcal{A})$. Then ν is said to be σ-finite if $|\nu|$ is σ-finite.*

Exercise 3.12. *Let ν be a signed measure and λ, μ positive measures on $(X, \mathcal{A})$. Suppose that $\nu = \lambda - \mu$. Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.*

Proof. Choose a Hahn decomposition P, N of X with respect to ν. Let $E \in \mathcal{A}$. Then

$$\begin{aligned}\lambda(E \cap P) - \mu(E \cap P) &= \nu(E \cap P) \\ &= \nu^+(E \cap P)\end{aligned}$$

So $\lambda(E \cap P) \geq \nu^+(E \cap P)$ and therefore

$$\begin{aligned}\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) \\ &= \nu^+(E)\end{aligned}$$

Similarly $\mu(E \cap N) \geq \nu^-(E \cap N)$ and $\mu(E) \geq \nu^-(E)$. $\square$

Exercise 3.13. *Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Hint: use the last exercise)*

Proof. Since

$$\begin{aligned} \nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\ &= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-) \end{aligned}$$

the previous exercise tells us that $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$ and $\mu = \nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$. Therefore

$$\begin{aligned} |\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ &\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\ &= |\nu_1| + |\nu_2| \end{aligned}$$

□

Note 3.14. *Recall that a previous exercise from the section on complex valued functions tells us that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.*

Definition 3.15. *Let ν be a signed measure on $(X, \mathcal{A})$. Then we define $L^1(\nu) = L^1(|\nu|)$. For $f \in L^1(\nu)$, we define*

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

Exercise 3.16. *Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$*

Proof. The previous exercise tells us that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. Two previous exercises from the section on nonnegative functions tells us that

$$\begin{aligned} \int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\ &= \int |f| d|\nu_1| + \int |f| d|\nu_2| \end{aligned}$$

□

Exercise 3.17. *Let ν, μ be signed measures on $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then*

(1) *E is ν-null iff $|\nu|(E) = 0$*
(2) *$\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.*

Proof. (1) Suppose that E is ν-null. Choose a Hahn decomposition P, N of X with respect to ν. Then $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = \nu(E \cap N) = 0$. Therefore $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Conversely, suppose that $|\nu|(E) = 0$. Then $\nu^+(E) = \nu^-(E) = 0$. Let $F \in \mathcal{A}$. Suppose that $F \subset E$. Then $\nu^+(F) = 0$ and $\nu^-(F) = 0$. Therefore $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. So E is ν-null.

(2) Suppose that $\nu \perp \mu$. Then there exist $E, F \in \mathcal{A}$ such that $E \cup F = X$, $E \cap F = \varnothing$, E is μ-null and F is ν-null. By (1), F is $|\nu|$-null and thus $|\nu| \perp \mu$. If $|\nu| \perp \mu$, choose $E, F \in \mathcal{A}$ as before. Since F is $|\nu|$-null, we know that $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$. This implies that F is ν^+-null and F is ν^--null. So $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Finally assume that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. **FINISH!!!!**

$\square$

Exercise 3.18. *Let ν be a signed measure on $(X, \mathcal{A})$. Then*

(1) *for $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$*
(2) *if ν is finite, then for each $E \in \mathcal{A}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : f$ is measurable and $|f| \leq 1\}$*

Proof. (1) Let $f \in L^1(\nu)$. Then

$$\begin{aligned}\left|\int f d\nu\right| &= \left|\int f d\nu^+ - \int f d\nu^-\right| \\ &\leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \\ &= \int |f| d(\nu^+ + \nu^-) \\ &= \int |f| d|\nu|\end{aligned}$$

(2) Let $E \in \mathcal{A}$. Let $f : X \to \mathbb{R}$ be measurable and suppose that $|f| \leq 1$. Since ν is finite, so is $|\nu|$ and thus $f \in L^1(\nu)$. Then (1) tells us that

$$\begin{aligned}|\int_E f d\nu| &\leq \int_E |f| d|\nu| \\ &\leq |\nu|(E)\end{aligned}$$

Now, choose a Hahn decomposition P, N of X with respect to ν. Define $f = \chi_P - \chi_N$. Then $|f| \leq 1$, f is measurable and

$$\begin{aligned}\left|\int_E f d\nu\right| &= \left|\int_E f d\nu^+ - \int_E f d\nu^-\right| \\ &= |\nu^+(E \cap P) + \nu^-(E \cap N)| \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E).\end{aligned}$$

$\square$

Exercise 3.19. *Let μ be a positive measure on $(X, \mathcal{A})$ and $f \in L^0(X, \mathcal{A})$ extended μ-integrable. Define ν on $(X, \mathcal{A})$ by $\nu(E) = \int_E f d\mu$. Then*

(1) *ν is a signed measure*
(2) *for each $E \in \mathcal{A}$, $|\nu|(E) = \int_E |f| d\mu$.*

Proof. (1) Clearly $\nu(\varnothing) = 0$ and ν is finte by assumption. Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_n)_{n\in\mathbb{N}}$ is disjoint. Then

$$\begin{aligned}
\nu(\bigcup_{n\in\mathbb{N}} E_n) &= \int_{\bigcup_{n\in\mathbb{N}} E_n} f d\mu \\
&= \int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu \\
&= \sum_{n\in\mathbb{N}} \int_{E_n} f^+ d\mu - \sum_{n\in\mathbb{N}} \int_{E_n} f^- d\mu \\
&= \sum_{n\in\mathbb{N}} \left[\int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right] \\
&= \sum_{n\in\mathbb{N}} \int_{E_n} f d\mu \\
&= \sum_{n\in\mathbb{N}} \nu(E_n)
\end{aligned}$$

If $|\nu(\bigcup_{n\in\mathbb{N}} E_n)| < \infty$, then $\int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu < \infty$ and $\int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu < \infty$ because

$$\begin{aligned}
|\nu(\bigcup_{n\in\mathbb{N}} E_n)| &= \left| \int_{\bigcup_{n\in\mathbb{N}} E_n} f d\mu \right| \\
&= \left| \int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu \right|
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |\nu(E_n)| &= \sum_{n\in\mathbb{N}} \left| \int_{E_n} f d\mu \right| \\
&= \sum_{n\in\mathbb{N}} \left| \int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right| \\
&\leq \sum_{n\in\mathbb{N}} \int_{E_n} f^+ d\mu + \sum_{n\in\mathbb{N}} \int_{E_n} f^- d\mu \\
&= \int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu + \int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu \\
&< \infty
\end{aligned}$$

So the sum $\sum_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely and ν is a signed measure.

(2) Put $P = \{x \in X : f(x) \geq 0\}$ and $N = \{x \in X : f(x) < 0\}$. Then P, N form a Hahn decomposition of X with respect to ν. Thus for $E \in \mathcal{A}$,

$$\nu^+(E) = \int_{E\cap P} f d\mu = \int_E f^+ d\mu$$

and

$$\nu^-(E) = \int_{E \cap N} f d\mu = \int_E f^- d\mu$$

. So for $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$$

□

3.2. **The Lebesgue-Radon-Nikodym Theorem.**

Definition 3.20. *Let $(X, \mathcal{A})$ be a measureable space, ν be a signed measure on $(X, \mathcal{A})$ and μ a measure on $(X, \mathcal{A})$. Then ν is said to be* ***absolutely continuous*** *with respect to μ, denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.*

Note 3.21. *If there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that for each $E \in \mathcal{A}$, $\nu(E) = \int_E f d\mu$, then we write $d\nu = f d\mu$.*

Theorem 3.22. *Let $(X, \mathcal{A})$ be a measureable space, ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that $\rho = f d\mu$ and f is unique μ-a.e.*

Definition 3.23. *The decomposition $\nu = \lambda + \rho$ is referred to as the* ***Lebesgue decomposition of*** *ν* ***with respect to*** *μ. In the case $\nu \ll \mu$, we have $\lambda = 0$ and $\rho = \nu$ and we define the* ***Radon-Nikodym derivative of*** *ν* ***with respect to*** *μ, denoted by $d\nu/d\mu$, to be $d\nu/d\mu = f$ where $d\nu = f d\mu$.*

Theorem 3.24. *Let ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then*

1. *for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and*
$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$
2. *$\nu \ll \lambda$ and*
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Exercise 3.25. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures and μ a measure.*

1. *If for each $n \in \mathbb{N}$, $\nu_n \ll \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.*
2. *If for each $n \in \mathbb{N}$, $\nu_n \perp \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \perp \mu$.*

Proof.
(1) Let $E \in \mathcal{A}$. Suppose that $\mu(E) = 0$. Then for each $n \in \mathbb{N}$, $\nu_i(E) = 0$ and thus $\sum_{n\in\mathbb{N}} \nu_n(E) = 0$. Hence $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.
(2) For each $n \in \mathbb{N}$, there exist $N_i, M_i \in \mathcal{A}$ such that $N_i \cap M_i = \varnothing$, $N_i \cup M_i = X$ and $\nu_i(M_i) = \mu(N_i) = 0$. Put $N = \bigcup_{n\in\mathbb{N}} N_i$ and $M = N^c$. Note that for each $n \in \mathbb{N}$, $M \subset N_i^c = M_i$. So $\mu(N) \leq \sum_{n\in\mathbb{N}} \mu(N_i) = 0$ and $(\sum_{n\in\mathbb{N}} \nu_i)(M) \leq \sum_{n\in\mathbb{N}} \nu_i(M_i) = 0$. Thus $\sum_{n\in\mathbb{N}} \nu_i \perp \mu$.

□

Exercise 3.26. *Choose $X = [0,1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$. Let m be Lebesgue measure and μ the counting measure.*

Then

1. $m \ll \mu$ *but for each* $f \in L^+$, $dm \neq f d\mu$
2. *There is no Lebesgue decomposition of* μ *with respect to* m.

Proof. (1) Let $E \in \mathcal{A}$. If $\mu(E) = 0$, then $E = \varnothing$ and $m(E) = 0$. So $m \ll \mu$. Suppose for the sake of contradiction that there exists $f \in L^+$ such that $dm = f d\mu$. Then

$$\begin{aligned} 1 &= m(X) \\ &= \sum_{x \in X} f(x) \end{aligned}$$

Put $Z = \{x \in X : f(x) \neq 0\}$. Then Z is countable. So

$$\begin{aligned} 1 &= m(X \setminus Z) \\ &= \sum_{x \in X \setminus Z} f(x) \\ &= 0 \end{aligned}$$

This is a contradiction, so no such f exists.

(2) Suppose for the sake of contradiction that there is a Lebesgue decomposition for μ with respect to m given by $\mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho \ll m$. We may assume λ and ρ are positive. Then for each $x \in X$, $m(\{x\}) = 0$ which implies that $\rho(\{x\}) = 0$. Let $E \subset X$, if E is countable, then $\lambda(E) = \mu(E)$. If E is uncountable, choose $F \subset E$ such that F is countable. Then

$$\begin{aligned} \lambda(E) &\geq \lambda(F) \\ &= \mu(F) \\ &= \infty \end{aligned}$$

So $\lambda = \mu$. This is a contradiction since $\mu \not\perp m$.

□

Exercise 3.27. *Let $(X, \mathcal{F}, \mu)$ be a measure space and $\mathcal{E}$ a sub σ-alg of $\mathcal{F}$ and $f \in L^1(\mu)$. Define $\nu : \mathcal{E} \to [0, \infty]$ by $\nu(E) = \int_E f d\mu$. Let $\overline{\mu}$ be the restriction of μ to $\mathcal{E}$. Define the **expectation of f given $\mathcal{E}$** to be $E[f|\mathcal{E}] = d\nu/d\overline{\mu}$. Then for each $E \in \mathcal{E}$,*

$$\int_E E[f|\mathcal{E}] d\mu = \int_E f d\mu$$

Proof. Let $E \in \mathcal{E}$. By definition,

$$\begin{aligned} \int_E E[f|\mathcal{E}] d\mu &= \int_E d\nu/d\overline{\mu} d\mu \\ &= \int_E d\nu/d\overline{\mu} d\overline{\mu} \qquad \text{(since } E \in \mathcal{E}\text{)} \\ &= \nu(E) \\ &= \int_E f d\mu \end{aligned}$$

□

3.3. Complex Measures.

Definition 3.28. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to \mathbb{C}$. Then ν is said to be a* ***complex measure*** *if*

(1) $\nu(\varnothing) = 0$
(2) *for each sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if $(E_n)_{n\in\mathbb{N}}$ is disjoint, then $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \nu(E_n)$ and $\sum_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely.*

Note 3.29. *We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.*

Definition 3.30. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. We define $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$. For $f \in L^1(\nu)$, we define*

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

Theorem 3.31. *Let $(X, \mathcal{A})$ be a measurable space, ν a complex measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exists a complex measure λ on $(X, \mathcal{A})$ and $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$ and such that for each complex measure λ' on $(X, \mathcal{A})$, $f' \in L^1(\mu)$, if $\nu = d\lambda' + f' d\mu$, then $\lambda = \lambda'$ and $f = f'$ μ-a.e.*

Theorem 3.32. *Let ν be a complex measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then*

(1) *for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and*
$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$
(2) *$\nu \ll \lambda$ and*
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Definition 3.33. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. Define $\mu = |\nu_1| + |\nu_2|$. Then $\nu \ll \mu$ and thus There exists $f \in L^1(\mu)$ such that $d\nu = f d\mu$. Define $|\nu| : \mathcal{A} \to [0, \infty)$ by $|\nu|(E) = \int_E |f| d\mu$ for each $E \in \mathcal{A}$. We call $|\nu|$ the* ***total variation of*** *ν.*

Exercise 3.34. *Let ν be a complex measure on $(X, \mathcal{A})$ and μ a σ-finite measures on $(X, \mathcal{A})$. If $\nu \ll \mu$, then $\{x \in X : d\nu/d\mu(x) = 0\}$ is ν-null.*

Proof. Define $f = d\nu/d\mu$ and $E = \{x : f(x) = 0\}$. Let $A \in \mathcal{A}$ and suppose that $A \subset E$. Then

$$\begin{aligned} \nu(A) &= \int_A f d\mu \\ &= 0 \end{aligned}$$

□

Exercise 3.35. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. Then $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu 2|$.*

Proof. Let μ and f be as in the definition of $|\nu|$. Since for each $E \in \mathcal{A}$, we have

$$\begin{aligned}\nu(E) &= \int_E f d\mu \\ &= \int_E f_1 d\mu + i \int_E f_2 d\mu\end{aligned}$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E)$$

we know that $\nu_1 = f_1 d\mu$ and $\nu_2 = f_2 d\mu$.

A previous exercise tells us that $d|v_1| = |f_1| d\mu$ and $d|v_2| = |f_2| d\mu$. Since $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$, we have that

$$\begin{aligned}|\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2|\end{aligned}$$

□

Exercise 3.36. *Let $(X, \mathcal{A})$ be a measurable space and ν a complex measure on $(X, \mathcal{A})$. Then*
1. *for each $E \in \mathcal{A}$, $|\nu(E)| \leq |\nu|(E)$.*
2. *$\nu \ll |\nu|$ and $\big|d\nu/d|\nu|\big| = 1$ $|\nu|$-a.e.*
3. *$L^1(\nu) = L^1(|\nu|)$ and for each $g \in L^1(\nu)$, $|\int g d\nu| \leq \int |g| d|\nu|$*

Proof. Let μ, $f \in L^1(\mu)$ be as in the definition of $|\nu|$.

1. Let $E \in \mathcal{A}$. Then

$$\begin{aligned}|\nu(E)| &= \left| \int_E f d\mu \right| \\ &\leq \int_E |f| d\mu \\ &= |\nu|(E)\end{aligned}$$

2. Let $E \in \mathcal{A}$ and suppose that $|\nu|(E) = 0$. The previous part implies $|\nu(E)| = 0$ and $\nu \ll |\nu|$. Put $g = d\nu/d|\nu|$. Then

$$\begin{aligned}f &= \frac{d\nu}{d\mu} \\ &= g|f| \quad \mu\text{-a.e.}\end{aligned}$$

   Hence $|f| = |g||f|$ μ-a.e. Since $|\nu| \ll \mu$, $|f| = |g||f|$ $|\nu|$-a.e.
   A previous exercise tells us that $|f| \neq 0$ $|\nu|$-a.e. Thus $|g| = 1$ $|\nu|$-a.e.

3. Write $\nu = \nu_1 + i\nu_2$ and $f = f_1 + if_2$. First we observe that

$$\begin{aligned}L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\ &= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\ &= L^1(|\nu_1| + |\nu_2|) \\ &= L^1(\mu)\end{aligned}$$

The previous exercise tells us that

$$\begin{aligned}|\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \\ &= \mu\end{aligned}$$

Let $g \in L^1(\mu)$. Then

$$\begin{aligned}\int |g| d|\nu| &\leq \int |g| d\mu \\ &< \infty\end{aligned}$$

So $g \in L^1(|\nu|)$.

Conversely, let $g \in L^1(|\nu|)$. Then

$$\begin{aligned}\int |g| d|\nu_1|, \int |g| d|\nu_2| &\leq \int |g| d|\nu| \\ &< \infty\end{aligned}$$

So

$$\begin{aligned}\int |g| d\mu &= \int |g| d|\nu_1| + \int |g| d|\nu_2| \\ &< \infty\end{aligned}$$

and $g \in L^1(\mu)$. Hence $L^1(\nu) = L^1(|\nu|)$.

Now, let $g \in L^1(\nu) = L^1(|\nu|)$, then

$$\begin{aligned}\left|\int g d\nu\right| &= \left|\int g f d\mu\right| \\ &\leq \int |g||f| d\mu \\ &= \int |g| d|\nu|\end{aligned}$$

□

3.4. Differentiation.

Definition 3.37. *Let $f : \mathbb{R}^n \to \mathbb{C}$. Then f is said to be* ***locally integrable*** *(with respect to Lebesgue measure) if f is measurable and for each $K \subset \mathbb{R}$, K is compact implies $\int_K |f| dm < \infty$. We define $L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is locally integrable}\}$*

Definition 3.38. *For $f \in L^1_{loc}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$, we define the* ***average of f over*** *$B(x, r)$, denoted by $Af(x, r)$, to be*

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x,r)} f dm$$

Lemma 3.39. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then $Af : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ is continuous.*

Definition 3.40. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define its* ***Hardy Littlewood maximal function****, denoted by Hf to be*

$$Hf(x) = \sup_{r>0} A|f|(x, r) \quad x \in \mathbb{R}^n$$

Theorem 3.41. *There exists $C > 0$ such that for each $f \in L^1(m)$ and $\alpha > 0$,*

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{a} \int |f| dm$$

Theorem 3.42. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{r \to 0} Af(x, r) = f(x)$$

. Equivalently, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} [f(y) - f(x)] dm(y)\right] = 0$$

Note 3.43. *We can a stronger result of the same flavor.*

Definition 3.44. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define the* ***Lebesgue set of*** *f, denoted by L_f, to be*

$$\begin{aligned} L_f &= \{x \in \mathbb{R}^n : \lim_{r \to 0} A|f - f(x)|(x, r) = 0\} \\ &= \left\{x \in \mathbb{R}^n : \lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y)\right] = 0\right\} \end{aligned}$$

Theorem 3.45. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then $m((L_f)^c) = 0$*

Definition 3.46. *Let $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then $(E_r)_{r>0}$ is said to* ***shrink nicely to*** *x if*

1. *for each $r > 0$, $E_r \subset B(x, r)$*
2. *there exists $\alpha > 0$ such that for each $r > 0$, $m(E_r) > \alpha m(B(x, r))$*

Theorem 3.47. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then for each $x \in L_f$,*

$$\lim_{r \to 0} \left[\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y)\right] = 0$$

and

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

Definition 3.48. *Let $\mu : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$ be a Borel measure. Then μ is said to be* ***regular*** *if*

1. *for each $K \subset \mathbb{R}^n$, if K is compact, then $\mu(K) < \infty$*
2. *for each $E \in \mathcal{B}(\mathbb{R}^n)$, $\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subset U\}$*

Let ν be a signed or complex Borel measure on $\mathbb{R}^n$. Then ν is said to be regular if $|\nu|$ is regular.

Theorem 3.49. *Let ν be a regular signed or complex measure on $\mathbb{R}^n$. Let $d\nu = d\lambda + f dm$ be the Lebesgue decomposition of ν with respect to m. Then for m-a.e. $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(R^n)$, if $(E_r)_{r>0}$ shrinks nicely to x, then*

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

4. Appendix

4.1. Summation.

Definition 4.1. *Let $f : X \to [0, \infty)$, Then we define*

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.