QUANTUM MECHANICS NOTES

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Contents

1. Hilbert Spaces

Note 1.1. *In the notes we will consider a Hilbert Space $\mathbb{H}$ with an inner product $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ which is linear in the first argument and antilinear in the second argument.*

Definition 1.2. *Let $\mathbb{H}$ be a Hilbert space and $A \in L(\mathbb{H})$. For each $x \in \mathbb{H}$, define $\phi_x \in \mathbb{H}^*$ by $\phi_x(y) = \langle Ay, x \rangle$. The Riesz representation theorem tells us that there exists $x' \in \mathbb{H}$ such that $\phi_x = \langle \cdot, x' \rangle$. Define $A^* \in L(\mathbb{H})$ by $A^* x = x'$. Thus for each $x, y \in \mathbb{H}$, we have that*

$$\langle Ay, x \rangle = \langle y, A^* x \rangle$$

*The linear operator A^* is called the **adjoint** of A.*

Exercise 1.3. *Let $A \in L(\mathbb{H})$. Then*

(1) for each $x, y \in \mathbb{H}$, $\langle x, Ay \rangle = \langle A^* x, y \rangle$.
(2) $A^{**} = A$.

Proof.

(1) Let $x, y \in \mathbb{H}$. Then

$$\begin{aligned} \langle x, Ay \rangle &= \overline{\langle Ay, x \rangle} \\ &= \overline{\langle y, A^* x \rangle} \\ &= \langle A^* x, y \rangle \end{aligned}$$

(2) We have that for each $x, y \in \mathbb{H}$, $\langle Ay, x \rangle = \langle y, A^* x \rangle = \langle A^{**} y, x \rangle \rangle$.
Hence for each $x, y \in \mathbb{H}$, $\langle (A - A^{**}) y, x \rangle = 0$. This implies that $A - A^{**} = 0$ and thus $A = A^{**}$.

$\square$

Definition 1.4. *Let $A \in L(\mathbb{H})$. Then A is said to be **self-adjoint** if $A = A^*$.*

Exercise 1.5. *Let $A \in L(\mathbb{H})$. Suppose that A is self adjoint. Then*

(1) the eigenvalues of A are real.
(2) the eigenvectors of distinct eigenvalues are orthogonal.

Proof.

(1) Let $\lambda \in \mathbb{C}$, $x \in \mathbb{H}$. Suppose that $x \neq 0$ and $Ax = \lambda x$. Then

$$\begin{aligned}\lambda\langle x, x\rangle &= \langle Ax, x\rangle \\ &= \langle x, A^*x\rangle \\ &= \langle x, Ax\rangle \\ &= \overline{\langle Ax, x\rangle} \\ &= \overline{\lambda\langle x, x\rangle} \\ &= \overline{\lambda}\langle x, x\rangle\end{aligned}$$

So $(\lambda - \overline{\lambda})\langle x, x\rangle = 0$. Since $x \neq 0$, $\langle x, x\rangle \neq 0$. Hence $\lambda = \overline{\lambda}$.

(2) Let $\lambda_1, \lambda_2 \in \mathbb{C}$ and $x_1, x_2 \in \mathbb{H}$. Suppose that $\lambda_1 \neq \lambda_2$, $x_1, x_2 \neq 0$, $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$. Then

$$\begin{aligned}\lambda_1\langle x_1, x_2\rangle &= \langle \lambda_1 x_1, x_2\rangle \\ &= \langle Ax_1, x_2\rangle \\ &= \langle x_1, A^*x_2\rangle \\ &= \langle x_1, Ax_2\rangle \\ &= \langle x_1, \lambda_2 x_2\rangle \\ &= \overline{\lambda_2}\langle x_1, x_2\rangle \\ &= \lambda_2\langle x_1, x_2\rangle \qquad \text{by (1)}\end{aligned}$$

So $(\lambda_1 - \lambda_2)\langle x_1, x_2\rangle = 0$. Since $\lambda_1 \neq \lambda_2$, we have that $\langle x_1, x_2\rangle = 0$.

□

Note 1.6. *To show that $A \in L(\mathbb{H})$ is self-adjoint. It suffices to show that for each $x_1, x_2 \in \mathbb{H}$, $\langle Ax_1, x_2\rangle = \langle x_1, Ax_2\rangle$*

2. Wave Mechanics

2.1. Schrodinger Equation.

Note 2.1. *In what follows, we will take $\mathcal{R} = \mathbb{R}$ or $\mathcal{R} = \mathbb{R}^3$ and $\mathbb{H} = L^2(\mathcal{R}) \cap \mathcal{N}(\mathcal{R})$ where $\mathcal{N}(\mathcal{R})$ signifies the "nice" functions on $\mathcal{R}$. The inner product on $\mathbb{H}$ is given by*

$$\langle f, g\rangle = \int_{\mathcal{R}} f\overline{g}\,dm_{\mathcal{R}}$$

We will typically discuss functions (states of the system) $\psi \in \mathbb{H}$ given by $x \mapsto \psi(x)$. The evolution of the state of a system will be given by $\Psi : \mathcal{R} \times \mathbb{R} \to \mathbb{C}$ given by $(x, t) \mapsto \Psi(x, t)$ where for each $t \in \mathbb{R}$, $\Psi(\cdot, t) \in \mathbb{H}$. In these notes, the Laplace operator Δ is only spatial.

Exercise 2.2. *Let $A \in L(\mathbb{H})$, $\lambda \in \mathbb{C}$ and $f = g + ih \in H \setminus \{0\}$. Suppose that $Af = \lambda f$. Then*

(1) $Ag = \lambda g$ and $Ah = \lambda h$.
(2) $A\overline{f} = \lambda \overline{f}$

Proof.

(1) Since A is linear,

$$\begin{aligned} Ag + iAh &= Af \\ &= \lambda f \\ &= \lambda g + i\lambda h \end{aligned}$$

Thus $Ag = \lambda g$ and $Ah = \lambda h$.

(2)

$$\begin{aligned} A\overline{f} &= A(g - ih) \\ &= Ag - iAh \\ &= \lambda g - i\lambda h \\ &= \lambda \overline{f} \end{aligned}$$

□

Definition 2.3. *For $j = 1, 2, 3$, define the j^{th} **position operator** $X_j \in L(\mathbb{H})$ by*

$$[X_j f](x) = x_j f(x)$$

.

Exercise 2.4. *The j^{th} position operator X_j is self-adjoint.*

Proof. Let $f, g \in \mathbb{H}$. Then and

$$\begin{aligned} \langle X_j f, g \rangle &= \int_{\mathcal{R}} x_j f(x) \overline{g(x)} dm_{\mathcal{R}}(x) \\ &= \int_{\mathcal{R}} f(x) x_j \overline{g(x)} dm_{\mathcal{R}}(x) \\ &= \int_{\mathcal{R}} f(x) \overline{x_j g(x)} dm_{\mathcal{R}}(x) \\ &= \langle f, X_j g \rangle \end{aligned}$$

□

Definition 2.5. *For $j = 1, 2, 3$, define the j^{th} **momentum operator** $P_j \in L(\mathbb{H})$ by*

$$P_j f = -i\hbar \frac{\partial f}{\partial x_j}$$

Exercise 2.6. *The j^{th} momentum operator P_j is self-adjoint.*

Proof. We will assume the case $\mathcal{R} = \mathbb{R}^3$ since the case $\mathcal{R} = \mathbb{R}$ uses the same method. Let $f, g \in \mathbb{H}$. Then

$$\begin{aligned}
\langle P_j f, g\rangle &= \int_{\mathbb{R}^3} -i\hbar \frac{\partial f}{\partial x_j}\overline{g} dm^3 \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}} -i\hbar \frac{\partial f}{\partial x_j}(x)\overline{g(x)} dm(x_j) dm^2(x_{j^-}) \\
&= \int_{\mathbb{R}^2} \left(- i\hbar f(x)\overline{g(x)}\right]_{x_j=-\infty}^{x_j=\infty} - \int_{\mathbb{R}} -i\hbar f(x)\frac{\partial \overline{g}}{\partial x_j} dm(x_j) dm^2(x_{j^-}) \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}} f(x) i\hbar \frac{\partial \overline{g}}{\partial x_j} dm(x_j) dm^2(x_{j^-}) \\
&= \int_{\mathbb{R}^3} f \overline{\left(-i\hbar \frac{\partial g}{\partial x_j}\right)} dm^3 \\
&= \langle f, P_j g\rangle
\end{aligned}$$

Hence P_j is self-adjoint.

□

Note 2.7. *Often instead of the operators X_1, X_2, X_3 and P_1, P_2, P_3 acting on functions f with $(x_1, x_2, x_3) \mapsto f(x_1, x_2, x_3)$ we write X, Y, Z and P_x, P_y, P_z acting on functions f with $(x, y, z) \mapsto f(x, y, z)$*

Definition 2.8. *Consider a particle of mass m with a "nice" potential energy $V : \mathcal{R} \to \mathbb{R}$ where $V(x)$ signifies the potential energy of a particle at position x. We define the* ***Hamiltonian operator****, $H \in L(\mathbb{H})$, by*

$$H = -\frac{\hbar^2}{2m}\Delta + VI$$

where I is the identity operator. Thus for each $x \in \mathcal{R}$,

$$[H\psi](x) = -\frac{\hbar^2}{2m}\Delta\psi(x) + V(x)\psi(x)$$

Exercise 2.9. *The Hamiltonian operator H is self-adjoint. Hint: use Green's identity.*

Proof. Let $f, g \in \mathbb{H}$. Then

$$\langle Hf, g\rangle = \int_{\mathcal{R}} \left[- \frac{\hbar^2}{2m}\Delta f(x) + V(x)f(x)\right]\overline{g}(x) dm_{\mathcal{R}}(x)$$

and

$$\langle f, Hg\rangle = \int_{\mathcal{R}} f(x)\left[- \frac{\hbar^2}{2m}\Delta \overline{g}(x) + V(x)\overline{g}(x)\right] dm_{\mathcal{R}}(x)$$

So

$$\begin{aligned}\langle Hf, g\rangle - \langle f, Hg\rangle &= -\frac{\hbar^2}{2m}\int_{\mathcal{R}} \overline{g}\Delta f - f\Delta\overline{g} dm_{\mathcal{R}} \\ &= -\frac{\hbar^2}{2m}\int_{\partial\mathcal{R}} \overline{g}\nabla_n f - f\nabla_n\overline{g} ds_R \quad \text{(Green's identity)} \\ &= 0\end{aligned}$$

So $\langle Hf, g\rangle = \langle f, Hg\rangle$ and H is self-adjoint. $\square$

Definition 2.10. *Suppose we have a system containing a particle with potential energy $V : \mathcal{R} \to \mathbb{R}$. Let $\psi \in \mathbb{H}$ and $\Psi : \mathcal{R} \times \mathbb{R} \to \mathbb{C}$. We say that ψ is a **state** (of the system) if*

$$\langle \psi, \psi\rangle = \int_{\mathcal{R}} \psi(x)\overline{\psi(x)} dm_{\mathcal{R}}(x) = 1$$

*We say that Ψ is a **wave function** (of the system) if for each $t \in \mathbb{R}$, $\Psi(\cdot, t) \in \mathbb{H}$ is a state of the system and for each $t \in \mathbb{R}$, Ψ satisfies the **Schrodinger equation**:*

$$i\hbar\frac{\partial\Psi}{\partial t}(\cdot, t) = H\Psi(\cdot, t)$$

Note 2.11. *Our interpretation of this model is the following: The wave function tells us the state of the system at each time $t \in \mathbb{R}$ and each state ψ of the system gives us the probability density of the position X of the particle via the relation:*

$$\mathbb{P}(X \in A) = \int_A \psi\overline{\psi} dm_{\mathcal{R}}$$

The act of measuring a physical quantity of the state of the system, such as the energy, the momentum in the x direction, etc, correspond to applying a self-adjoint operator to the state. The eigenvalues of self-adjoint operators tell us the possible measurements of that physical quantity. We assume that the eigenvectors of our self-adjoint operator form an orthonormal basis for $\mathbb{H}$. If a system has a wavefunction Ψ and at time t, we measure a physical quantity corresponding to a self-adjoint operator A with distinct eigenvalues $(\lambda_j)_{j\in J}$ and corresponding eigen states (normalized eigenvectors) $(\psi_j)_{j\in J}$, then the probability of measuring the quantity λ_j, $\mathbb{P}(m(\Psi, A, t) = \lambda_j)$ is given by

$$\mathbb{P}(m(\Psi, A, t) = \lambda_j) = |\langle \Psi(\cdot, t), \psi_j\rangle|^2 = \left|\int_{\mathcal{R}} \Psi(x, t)\overline{\psi_j(x)} dm_{\mathcal{R}}(x)\right|^2$$

If for $j \in J$, we define $c_j : \mathbb{R} \to \mathbb{C}$ by

$$c_j(t) = \int_{\mathcal{R}} \Psi(x, t)\overline{\psi_j(x)} dm_{\mathcal{R}}(x)$$

then the expected value of the measurement at time t, $\langle A\rangle_t$, is given by

$$\langle A\rangle_t = \sum_{j\in J} \lambda_j |c_j(t)|^2$$