STOCHASTIC PROCESSES NOTES

Contents

1. Basics

Definition 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(S, \mathcal{G})$ a measurable space, T an index set and $X : T \times \Omega \to S$. Then X is said to be a* ***stochastic processes*** *if for each $t \in T$, $X(t, \cdot)$ is $\mathcal{F}$-$\mathcal{G}$ measurable (i.e. X is just a collection of random variables indexed by time).*

Note 1.2. *We will work primarily with $T = [0, \infty)$ and $(S, \mathcal{G}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and we often write $X_t(\omega)$ to mean $X(t, \omega)$ as well as the function $X(\cdot, \omega)$.*

Definition 1.3. *Let X be a process and $\omega \in \Omega$. The function $X(\cdot, \omega)$ is called the* ***sample path*** *of ω.*

Definition 1.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t \geq 0}$ a collection of sub σ-algebras of $\mathcal{F}$ such that for each $s, t \in [0, \infty)$, $s < t$ implies that $\mathcal{F}_s \subset \mathcal{F}_t$. Then $\{\mathcal{F}_t\}_{t \geq 0}$ is said to be a* ***filtration*** *of $\mathcal{F}$.*

Definition 1.5. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t \geq 0}$ is said to be* ***complete*** *if for each $t \in [0, \infty)$, $\mathcal{F}_t$ contains all the null sets of $\mathcal{F}$. (i.e. $\mathcal{F}_0$ contains all the null sets of $\mathcal{F}$)*

Definition 1.6. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t \geq 0}$ is said to be* ***right continuous*** *if for each $t \in [0, \infty)$, $\mathcal{F}_t = \bigcap\limits_{s > t} \mathcal{F}_s$*

Definition 1.7. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t \geq 0}$ is said to satisfy the* ***usual conditions*** *if $\{\mathcal{F}_t\}_{t \geq 0}$ is complete and right continuous.*

Definition 1.8. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Define $\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \geq 0)$.*

Definition 1.9. *Let X be a process and $\{\mathcal{F}_t\}_{t \geq 0}$ a filtration. Then X is said to be* ***adapted*** *to $\{\mathcal{F}_t\}_{t \geq 0}$ if for each $t \in [0, \infty)$, X_t is $\mathcal{F}_t$ measurable*

Definition 1.10. *Let X be a process. Then the* ***minimal augmented filtration*** *of X is defined to be the smallest filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that X is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions. (here, "smallest" means that if $\{\mathcal{G}_t\}$ is filtration satisfying the usual conditions to which X is adapted, then for each $t \in [0, \infty)$, $\mathcal{F}_t \subset \mathcal{G}_t$)*

Definition 1.11. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(S, \mathcal{G})$ a measurable space, T an index set and $X : T \times \Omega \to S$ a stochastic process. We can define a function $\Phi_X : \Omega \to S^T$ by $\Phi_X(\omega)(t) = X(t, \omega)$. The* ***law*** *of X is defined to be the pushforward measure of $\mathbb{P}$ by Φ_X and is denoted by $\mathcal{L}_X$. (i.e. $\mathcal{L}_X(A) = \mathbb{P}(\Phi_X^{-1}(A))$ for $A \in \mathcal{G}$)*

Definition 1.12. *Let X and Y be processes. Then X and Y are said to be **modifications** if for each $t \geq 0$, $N_t = \{\omega \in \Omega : X_t(\omega) \neq Y_t(\omega)\}$ is a null set.*

Definition 1.13. *Let X and Y be processes. Then X and Y are said to be **indistinguishable** if $N = \{\omega \in \Omega : \text{ for some } t \geq 0, X_t(\omega) \neq Y_t(\omega)\}$ is a null set.*

Exercise 1. *Let X and Y be processes. Suppose that X and Y are modifications and that a.s. (i.e. except on a null set) X and Y have right continuous paths. Then X and Y are indistinguishable.*

Proof. By assumption, $N_r = \{\omega \in \Omega : X_t(\omega) \text{ is not right continuous or } Y_t(\omega) \text{ is not right continuous}\}$ is a null set . Let $\omega \in \Omega \cap N_r^c$. Right continuity tells us that for each $t \geq 0$, $X_t(\omega) = Y_t(\omega)$ if and only if for each $t \in [0, \infty) \cap \mathbb{Q}$, $X_t(\omega) = Y_t(\omega)$. Since X and Y are modifications, for each $t \in [0, \infty) \cap \mathbb{Q}$, $N_t = \{\omega \in \Omega : X_t(\omega) \neq Y_t(\omega)\}$ is a null set. Thus

$$\begin{aligned}
N &= \{\omega \in \Omega : \text{ for some } t \geq 0, X_t(\omega) \neq Y_t(\omega)\} \\
&= (N \cap N_r) \cup (N \cap N_r^c) \\
&= (N \cap N_r) \cup \{\omega \in \Omega \cap N_r^c : \text{ for some } t \in [0,\infty) \cap \mathbb{Q}, X_t(\omega) \neq Y_t(\omega)\} \\
&= (N \cap N_r) \cup \bigcup_{t \in [0,\infty) \cap \mathbb{Q}} N_t
\end{aligned}$$

is a null set. Hence X and Y are indistinguishable. □

Definition 1.14. *Let X be a process. Then X is said to be **cadlag** if $N = \{\omega \in \Omega : X_t(\omega) \text{ is not cadlag}\}$ is a null set.*

Definition 1.15. *Let X be a process and $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Then X is said to be progressively measurable with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ if for each $t \geq 0$, $X|_{[0,t] \times \Omega}$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable.*

Exercise 2. *Let X be a process and $\{\mathcal{F}_t\}_{t \geq 0}$ a filtration. Suppose that X is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. If each sample path of X is right continuous or each path of X is left continuous, then X is progressively measurable.*

Proof. Fist suppose that each path of X is right continuous. Let $t \geq 0$. Define $X_n : [0,t] \times \Omega \to \mathbb{R}$ by $X_n(s, \omega) = X_0(\omega)\mathbf{1}_{\{0\}}(s) + \sum_{k=1}^{2^n} X_{tk/2^n}(\omega)\mathbf{1}_{(t(k-1)/2^n, tk/2^n]}(s)$. Since X is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, X_n is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable. Let $(s, \omega) \in [0,t] \times \Omega$. Suppose that $s > 0$. Then for each $n \in \mathbb{N}$, there exists a unique $k_n \in \mathbb{N}$ such that $k_n \leq 2^n$ and $t(k_n - 1)/2^n < s \leq tk_n/2^n$. Thus $X_n(s, \omega) = X_{tk_n/2^n}(\omega)$, $tk_n/2^n \geq s$ and $tk_n/2^n \to s$. Since each path of X is right continuous, $X_n(s, \omega) \to X_s(\omega)$. If $s = 0$, then $X_n(s, w) = X_0(\omega) = X_s(\omega)$. Hence $X_n(s, \omega) \to X_s(\omega)$. Since $X_n \to X|_{[0,t] \times \Omega}$ pointwise, $X|_{[0,t] \times \Omega}$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable. So X is progressively measurable.
Now suppose that each path of X is left continuous. Let $t \geq 0$. Define $X_n : [0,t] \times \Omega \to \mathbb{R}$ by $X_n(s, \omega) = \sum_{k=1}^{2^n} X_{t(k-1)/2^n}(\omega)\mathbf{1}_{[t(k-1)/2^n, tk/2^n)}(s) + X_t(\omega)\mathbf{1}_{\{t\}}(s)$. The rest of the proof is similar to the other case, just this time utilizing left continuity. □

Exercise 3. *Let X be a process and $\{\mathcal{F}_t\}_{t \geq 0}$ a complete filtration. Suppose that X is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and that X is right continuous or X is left continuous (a.s. of course). Then X is progressively measurable.*

Proof. First we assume that X is right continuous. Let $N = \{\omega \in \Omega : X(\cdot, \omega) \text{ is not right continuous}\}$. By assumption, N is null and therefore for each $t \geq 0$, $N \in \mathcal{F}_t$. Define $\widetilde{X} : [0, \infty) \times \Omega \to [0, \infty)$ by $\widetilde{X}(t, \omega) = X(t, \omega)\mathbf{1}_{N^c}(\omega)$

Since for each $t \geq 0$, X_t is $\mathcal{F}_t$ measurable and $N^c \in \mathcal{F}_t$, we have that $\widetilde{X}_t$ is $\mathcal{F}_t$ measurable and thus $\widetilde{X}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. By construction, each sample path of $\widetilde{X}$ is right continuous. By the previous exercise, $\widetilde{X}$ is porgressively measurable.

Now, let $t \geq 0$ and $B \in \mathcal{B}([0, t])$. If $0 \notin B$, then

$$X|_{[0,t]\times\Omega}^{-1}(B) = \widetilde{X}|_{[0,t]\times\Omega}^{-1}(B)$$

. If $0 \in B$, then

$$\begin{aligned}
X|_{[0,t]\times\Omega}^{-1}(B) &= \left[X|_{[0,t]\times\Omega}^{-1}(B) \cap ([0, t] \times N)\right] \cup \left[X|_{[0,t]\times\Omega}^{-1}(B) \cap ([0, t] \times N^c)\right] \\
&= X|_{[0,t]\times\Omega}^{-1}(B \setminus \{0\}) \cup X|_{[0,t]\times\Omega}^{-1}(\{0\}) \\
&= \widetilde{X}|_{[0,t]\times\Omega}^{-1}(B \setminus \{0\}) \cup X^{-1}(\{0\}) \cup ([0, t] \times N)
\end{aligned}$$

$\square$

Definition 1.16. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and $T : \Omega \to [0, \infty]$ a random variable. Then T is said to be a **stopping time** for $\{\mathcal{F}_t\}_{t\geq 0}$ if for each $t \in [0, \infty]$, $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$.*

Note that if $t = \infty$, then

$$\{\omega \in \Omega : T(\omega) \leq t\} = \Omega \in \mathcal{F}_\infty$$

We will typically just say that T is a stopping time when the filtration is clear.

Proposition 1.17. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Then we have the following:*

1. *if S and T are stopping times, then $S + T$ is a stopping time.*
2. *if S is a stopping time, then for $\alpha \geq 1$, αS is a stopping time.*
3. *if there exists $c \in [0, \infty]$ such that $T \equiv c$, then T is a stopping time.*
4. *If $(T_n)_{n\in\mathbb{N}}$ is a sequence of stopping times, then $\sup\limits_{n\in\mathbb{N}} T_n$ and $\inf\limits_{n\in\mathbb{N}} T_n$ are stopping times.*

Note that the third statement holds for $T \equiv c$ almost surely if $\{\mathcal{F}_t\}_{t\geq 0}$ is complete and the last statement implies that mins $\limsup T_n$ and $\liminf T_n$ and that for stopping times S and T, $S \wedge T$ and $S \vee T$ are stopping times.

The proof is similar to showing measurbaility of the same functions.

Proposition 1.18. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a right continuous filtration and $T : \Omega \to [0, \infty]$ a random variable. Then T is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$ if and only if for each $t \geq 0$, $\{\omega \in \Omega : T(\omega) < t\} \in \mathcal{F}_t$.*

Proof. Suppose that for each $s \geq 0$, $\{\omega \in \Omega : T(\omega) < s\} \in \mathcal{F}_s$. Let $t \geq 0$. Then

$$\begin{aligned}
\{\omega \in \Omega : T(\omega) \leq t\} &= \bigcap_{n\in\mathbb{N}} \{\omega \in \Omega : T(\omega) < t + 1/n\} \\
&\in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{t+1/n} \\
&= \mathcal{F}_t \qquad \text{(right continuity)}
\end{aligned}$$

Conversely, suppose that T is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Let $t \geq 0$. Then

$$\begin{aligned}\{\omega \in \Omega : T(\omega) < t\} &= \bigcup_{n\in\mathbb{N}}\{\omega \in \Omega : T(\omega) \leq t - 1/n\} \\ &\in \mathcal{F}_t\end{aligned}$$

□

Exercise 4. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be right continuous and $T : \Omega \to [0,\infty)$ be a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Define $T_n : \Omega \to [0,\infty)$ by $T_n(\omega) = k_\omega/2^n$ where $k_\omega \in \mathbb{N}$ is the unique positive integer such that $(k_\omega - 1)/2^n \leq T(\omega) < k_\omega/2^n$. Then for each $n \in \mathbb{N}$, T_n is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.* ***FINISH!!!!!!!!***

Proof. Let $t \geq 0$. □

Definition 1.19. *Let X be a process and $\Lambda \in \mathcal{B}(\mathbb{R})$. Then the **hitting time** of Λ is defined to be the random variable $T_\Lambda : \Omega \to [0,\infty]$ given by $T_\Lambda(\omega) = \inf\{t > 0 : X_t(\omega) \in \Lambda\}$.*

Theorem 1.20. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a complete filtration, X a right continuous process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and $\Lambda \subset \mathbb{R}$ an open set. Then T_Λ is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.*

Proof. Let $t \geq 0$. Note that for $\omega \in \Omega$, if $X(\cdot,\omega)$ is right continuous, then since Λ is open, we have that $T_\Lambda(\omega) < t$ if and only if there exists $s \in (0,t)$ such that $X_s(\omega) \in \Lambda$ if and only if there exists $s \in \mathbb{Q} \cap (0,t)$ such that $X_s(\omega) \in \Lambda$. The completeness of $\{\mathcal{F}_t\}_{t\geq 0}$ tells us that for each $s \in [0,\infty)$, $\mathcal{F}_s$ has all the null sets of $\mathcal{F}$. Finally, Λ being open and X being adapted tell us that for each $s < t$, $X_s^{-1}(\Lambda) \in \mathcal{F}_s \subset \mathcal{F}_t$.

By assumption $N = \{\omega \in \Omega : X(\cdot,\omega)$ is not right continuous$\}$ is a null set, so for each $s \in [0,\infty)$, $N \in \mathcal{F}_s$ and $N^c \in \mathcal{F}_s$. The previous note implies that

$$\begin{aligned}\{\omega \in \Omega : T_\Lambda(\omega) < t\} &= (\{\omega \in \Omega : T_\Lambda(\omega) < t\} \cap N^c) \cup (\{\omega \in \Omega : T_\Lambda(\omega) < t\} \cap N) \\ &= \left(\bigcup_{s\in\mathbb{Q}\cap(0,t)}\{\omega \in \Omega : X_s(\omega) \in \Lambda\} \cap N^c\right) \cup \left(\{\omega \in \Omega : T(\omega) < t\} \cap N\right) \\ &\in \mathcal{F}_t\end{aligned}$$

Hence T_Λ is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. □

Note that the same result is true if X is a left continuous process. Also, as usual, we could get rid of the assumption that $\{\mathcal{F}_t\}_{t\geq 0}$ is complete if each sample path of X were right or left continuous instead of the paths being almost surely right or left continuous.

Definition 1.21. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. The **stopping algebra** of T to be $\mathcal{F}_T = \{A \in \mathcal{F} :$ for each $t \geq 0, A \cap T^{-1}([0,t]) \in \mathcal{F}_t\}$*

Note that $\mathcal{F}_T$ is a σ-algebra

Lemma 1.22. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then T is $\mathcal{F}_T$ measurable.*

Proof. Let $t, s \in [0,\infty]$. Then

$$\begin{aligned}T^{-1}([0,s]) \cap T^{-1}([0,t]) &= T^{-1}([0,t\wedge s]) \\ &\in \mathcal{F}_{t\wedge s} \\ &\subset \mathcal{F}_t.\end{aligned}$$

So for each $s \in [0,\infty]$, $T^{-1}([0,s]) \in \mathcal{F}_T$. Hence T is $\mathcal{F}_T$*measurable*. □

Theorem 1.23. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and $T:\Omega\to[0,\infty)$ a finite stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathcal{F}_T=\sigma(X_T : X$ is an everywhere cadlag process adapted to $\{\mathcal{F}_t\}_{t\geq 0})$.*

Proof. Put $\mathcal{G}=\sigma(X_T : X$ is an everywhere cadlag process adapted to $\{\mathcal{F}_t\}_{t\geq 0})$. Let $\Lambda\in\mathcal{F}_T$. Define $X_t(\omega)=\mathbf{1}_\Lambda(\omega)\mathbf{1}_{T^{-1}([0,t])}(\omega)=\mathbf{1}_{\Lambda\cap T^{-1}([0,t])}(\omega)$. Then each sample path of X is clearly cadlag. Moreover, since $\Lambda\in\mathcal{F}_T$, for each $t\geq 0$, $\Lambda\cap T^{-1}([0,t])\in\mathcal{F}_t$. So for each $t\geq 0$, X_t is $\mathcal{F}_t$ measurable and thus X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. So by definition, X_T is $\mathcal{G}$ measurable. It is easy to see that $X_T=\mathbf{1}_\Lambda$ and therefore $\Lambda\in\mathcal{G}$. Hence $\mathcal{F}_T\subset\mathcal{G}$.

Now, let X be an everywhere cadlag process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and $t\geq 0$. Define $\phi:\Omega\to[0,\infty)\times\Omega$ by $\phi(\omega)=(T(\omega),\omega)$. Then $X_T=X\circ\phi$. For $t\geq 0$, define the σ-algebra $\mathcal{F}_t\cap T^{-1}([0,t])$ on $T^{-1}([0,t])$ by $\mathcal{F}_t\cap T^{-1}([0,t])=\{A\cap T^{-1}([0,t]) : A\in\mathcal{F}_t\}$. Let $U_1\in\mathcal{B}([0,\infty))$ and $U_2\in\mathcal{F}_t$. Then $\phi|^{-1}_{T^{-1}([0,t])}(U1\times U2)=T^{-1}(U_1)\cap(U_2)\cap T^{-1}([0,t])$. Since T is $\mathcal{F}_T$ measurable, $T^{-1}(U_1)\in\mathcal{F}_T$. By definition, $T^{-1}(U_1)\cap T^{-1}([0,t])\in\mathcal{F}_t\cap T^{-1}([0,t])$. Since $U_2\in\mathcal{F}_t$, we have that $\phi|^{-1}_{T^{-1}([0,t])}(U1\times U2)\in\mathcal{F}_t\cap T^{-1}([0,t])$. So $\phi|_{T^{-1}([0,t])}$ is $(\mathcal{F}_t\cap T^{-1}([0,t]),\mathcal{B}([0,\infty))\bigotimes\mathcal{F}_t)$ measurable.

Since each sample path of X is cadlag and X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, X is progressively measurable. In particular, $X|_{[0,t]\times\Omega}$ is $\mathcal{B}([0,\infty))\bigotimes\mathcal{F}_t)$ measurable.

Hence $X_T|_{T^{-1}([0,t])}$ is $F_t\cap T^{-1}([0,t])$ measurable and for $B\in\mathcal{B}([0,\infty))$,

$$\begin{aligned}X_T^{-1}(B)\cap T^{-1}([0,t])&=X_T|^{-1}_{T^{-1}([0,t])}(B)\\&\in\mathcal{F}_t\cap T^{-1}([0,t])\\&\subset\mathcal{F}_t\end{aligned}$$

So X_T is $\mathcal{F}_T$ measurable and thus $\mathcal{G}\subset\mathcal{F}_T$. □

2. Martingales

Definition 2.1. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **submartingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if*

1. *M is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$*
2. *for each $t\in[0,\infty)$, $\mathbb{E}[|M_t|]<\infty$*
3. *for each $s,t\in[0,\infty)$, if $s<t$, then $\mathbb{E}[M_t|\mathcal{F}_s]\geq M_s$*

Recall that the third condition means equal except on a null set.

Definition 2.2. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **supermartingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if*

1. *M is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$*
2. *for each $t\in[0,\infty)$, $\mathbb{E}[|M_t|]<\infty$*
3. *for each $s,t\in[0,\infty)$, if $s<t$, then $\mathbb{E}[M_t|\mathcal{F}_s]\leq M_s$*

Definition 2.3. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **submartingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if $-M$ is a supermartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$*

Definition 2.4. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **martingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if M is a submartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ and M is a supermartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.*

Theorem 2.5.

3. Brownian Motion

Definition 3.1.