

# CALCULUS ON MANIFOLDS NOTES

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## 1. REVIEW OF MULTIVARIABLE CALCULUS AND LINEAR ALGEBRA

### 1.1. The derivative.

## 2. CALCULUS ON MANIFOLDS

### 2.1. Submanifolds of $\mathbb{R}^n$ .

**Definition 2.1.1.** Let  $\Omega \subset \mathbb{R}^n$  and  $U \subset M$ . Then  $U$  is said to be **open** if there exists  $U' \subset \mathbb{R}^n$  such that  $U'$  is open in  $\mathbb{R}^n$  and  $U = M \cap U'$ .

**Definition 2.1.2.** Define the **upper half space** of  $\mathbb{R}^n$ , denoted  $\mathbb{H}_n$ , by

$$\mathbb{H}_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\partial\mathbb{H}_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$$

**Definition 2.1.3.** Let  $\Omega \subset \mathbb{R}^n, U \subset \Omega, V \subset \mathbb{H}_k$  and  $\phi : U \rightarrow V$ . Then  $\phi$  is said to be a **coordinate chart** from  $\Omega$  to  $\mathbb{H}_k$  if  $U$  is open in  $\Omega$ ,  $V$  is open in  $\mathbb{H}_k$  and  $\phi$  is a homeomorphism (that is,  $\phi$  is a bijection, continuous and  $\phi^{-1}$  is continuous). We will typically denote a chart from  $\Omega$  to  $\mathbb{H}_k$  by the pair  $(\phi, U)$ . Let  $\mathcal{A} = \{(\phi_\alpha, U_\alpha) : \alpha \in A\}$  be a set of coordinate charts from  $\Omega$  to  $\mathbb{H}_k$  indexed by  $A$ . Then  $\mathcal{A}$  is said to be a **smooth  $k$ -atlas** on  $\Omega$  if

$$(1) \Omega = \bigcup_{\alpha \in A} U_\alpha$$

(2) for each  $\alpha, \beta \in A$ ,  $U_\alpha \cap U_\beta \neq \emptyset$  implies that

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_\alpha \cap U_\beta) \rightarrow \phi_2(U_\alpha \cap U_\beta)$$

is smooth

If  $\mathcal{A}$  is a smooth  $k$ -atlas on  $\Omega$ , and  $(\phi, U) \in \mathcal{A}$ , then  $\phi$  is said to be a **smooth coordinate chart** from  $\Omega$  to  $\mathbb{H}_k$ .

**Definition 2.1.4.** Let  $\Omega \subset \mathbb{R}^n$  and  $\mathcal{A}$  a smooth  $k$ -atlas on  $\Omega$ . Then  $(\Omega, \mathcal{A})$  is said to be a  $k$ -dimensional smooth submanifold of  $\mathbb{R}^n$ . Define the **boundary** of  $\Omega$ , denoted  $\partial\Omega$ , by

$$\partial\Omega = \bigcup_{\substack{\phi \in \mathcal{A} \\ \phi: U \rightarrow V}} \phi^{-1}(V \cap \partial\mathbb{H}_k)$$

**Exercise 2.1.5.** Let  $\Omega$  be a  $k$ -dimensional smooth submanifold of  $\mathbb{R}^n$ . Then  $\partial\Omega$  is a  $k - 1$ -dimensional smooth manifold of  $\mathbb{R}^n$ .

*Proof.* Straightforward. □

**Definition 2.1.6.** Let  $\Omega$  be a smooth  $k$ -dimensional smooth submanifold of  $\mathbb{R}^n$ ,  $U \subset \Omega$  open in  $\Omega$ ,  $V \subset \mathbb{H}_k$  open in  $\mathbb{H}_k$  and  $\phi : U \rightarrow V$  a smooth coordinate chart on  $\Omega$ . Then  $\phi^{-1} : V \rightarrow U$  is called a **local smooth parametrization** of  $\Omega$ .

## 2.2. Differential Forms.

**Note 2.2.1.** The definitions in this section will introduce a very slick book-keeping device for doing calculus on manifolds. Since we are not developing the theory from the ground up, it may feel abstract. Hopefully the many exercises facilitate becoming accustomed to this book-keeping tool.

**Definition 2.2.2.** Define  $\mathcal{I}_{k,n} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_n \leq n\}$ . Let  $I \in \mathcal{I}_{k,n}$ . Then  $I$  is called a **multi-index**. Recall that  $\#\mathcal{I}_{k,n} = \binom{n}{k}$ .

**Definition 2.2.3.** When working in  $\mathbb{R}^n$ , we introduce the formal objects  $dx_1, dx_2, \dots, dx_n$ . Let  $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_{k,n}$  and  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ . Write  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ . We formally define  $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$  and  $\phi_I = (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$ .

**Definition 2.2.4.** Let  $k \in \{0, 1, \dots, n\}$ . We define a  $C^\infty(\mathbb{R}^n)$ -module of dimension  $\binom{n}{k}$ , denoted  $\Gamma^k(\mathbb{R}^n)$  to be

$$\Gamma^k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx_I : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each  $\omega \in \Gamma^k(\mathbb{R}^n)$  and  $\chi \in \Gamma^l(\mathbb{R}^n)$ , we may form their **exterior product**, denoted by  $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$ . Thus the exterior product is a map  $\wedge : \Gamma^k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \rightarrow \Gamma^{k+l}(\mathbb{R}^n)$ . The exterior product is characterized by the following properties:

- (1) the exterior product is bilinear
- (2) for each  $\omega \in \Gamma^k(\mathbb{R}^n)$  and  $\chi \in \Gamma^l(\mathbb{R}^n)$ ,  $\omega \wedge \chi = -\chi \wedge \omega$
- (3) for each  $\omega \in \Gamma^k(\mathbb{R}^n)$ ,  $\omega \wedge \omega = 0$
- (4) for each  $f \in C^\infty(\mathbb{R}^n)$  and  $\omega \in \Gamma^k(\mathbb{R}^n)$ ,  $f \wedge \omega = f\omega$

We call  $\Gamma^k(\mathbb{R}^n)$  the differential  $k$ -forms on  $\mathbb{R}^n$ . Let  $\omega$  be a  $k$ -form on  $\mathbb{R}^n$ . If  $k \geq 1$ , then for each  $I \in \mathcal{I}_{k,n}$ , there exists  $f_I \in C^\infty(\mathbb{R}^n)$  such that  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$

**Note 2.2.5.** The terms  $dx_1, dx_2, \dots, dx_n$  are a sort of place holder for the coordinates of a point  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ . When we work with functions  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ , we will have different coordinates and to avoid confusion, we will write  $\{du_1, du_2, \dots, du_k\}$  when referencing the coordinates on  $\mathbb{R}^k$  and  $\{dx_1, dx_2, \dots, dx_n\}$  when referencing the coordinates on  $\mathbb{R}^n$ .

**Exercise 2.2.6.** Let  $B = (b_{i,j})$  be an  $n \times n$  matrix. Then

$$\bigwedge_{i=1}^n \left( \sum_{j=1}^n b_{i,j} dx_j \right) = (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

*Proof.* First we have

$$(*) \quad \bigwedge_{i=1}^n \left( \sum_{j=1}^n b_{i,j} dx_j \right) = \left( \sum_{j=1}^n b_{1,j} dx_j \right) \wedge \left( \sum_{j=1}^n b_{2,j} dx_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^n b_{n,j} dx_j \right)$$

The expression on the right side of  $(*)$  is just the sum all terms of the form

$$b_{1,j_1} b_{2,j_2} \cdots b_{n,j_n} dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n}$$

where  $j_k \in \{1, 2, \dots, n\}$ . The terms in which for some  $j$ ,  $dx_j$  appears more than once in the exterior product are zero. Thus the expression on the right of  $(*)$  is just the sum of all terms of the form

$$b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{n,\sigma(n)} dx_{\sigma(1)} \wedge dx_{\sigma(2)} \wedge \cdots \wedge dx_{\sigma(n)} = \text{sgn}(\sigma) b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{n,\sigma(n)} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where  $\sigma \in S_n$ . Explicitly writing this out, we see that

$$\begin{aligned} \bigwedge_{i=1}^n \left( \sum_{j=1}^n b_{i,j} dx_j \right) &= \left( \sum_{j=1}^n b_{1,j} dx_j \right) \wedge \left( \sum_{j=1}^n b_{2,j} dx_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^n b_{n,j} dx_j \right) \\ &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{n,\sigma(n)} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\ &= \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{n,\sigma(n)} \right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\ &= (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \end{aligned}$$

□

**Definition 2.2.7.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a 0-form on  $\mathbb{R}^n$ . We define a 1-form, denoted  $df$ , on  $\mathbb{R}^n$  by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

Let  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$  be a  $k$ -form on  $\mathbb{R}^n$ . We can define a differential  $k+1$ -form, denoted  $d\omega$ , on  $\mathbb{R}^n$  by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx_I$$

**Exercise 2.2.8.** On  $\mathbb{R}^3$ , put

- (1)  $\omega_0 = f_0$ ,
- (2)  $\omega_1 = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ ,
- (3)  $\omega_2 = f_1 dx_2 \wedge dx_3 - f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2$

Show that

- (1)  $d\omega_0 = \frac{\partial f_0}{\partial x_1} dx_1 + \frac{\partial f_0}{\partial x_2} dx_2 + \frac{\partial f_0}{\partial x_3} dx_3$
- (2)  $d\omega_1 = \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left( \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \right) dx_1 \wedge dx_3 + \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 \wedge dx_2$

$$(3) \quad d\omega_2 = \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3$$

*Proof.* Straightforward. □

**Exercise 2.2.9.** Let  $I \in \mathcal{I}_{k,n}$ . Then there is a unique  $I_* \in \mathcal{I}_{n-k,n}$  such that  $dx_I \wedge dx_{I_*} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ .

**Definition 2.2.10.** We define a linear map  $*$  :  $\Gamma^k(\mathbb{R}^n) \rightarrow \Gamma^{n-k}(\mathbb{R}^n)$  called the **Hodge \*-operator** by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

**Definition 2.2.11.** Let  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$  be smooth. Write  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ . We define  $\phi^* : \Gamma^k(\mathbb{R}^n) \rightarrow \Gamma^k(\mathbb{R}^k)$  via the following properties:

- (1) for each 0-form  $f$  on  $\mathbb{R}^n$ ,  $\phi^* f = f \circ \phi$
- (2) for  $i = 1, \dots, n$ ,  $\phi^* dx_i = d\phi_i$
- (3) for an  $s$ -form  $\omega$ , and a  $t$ -form  $\chi$  on  $\mathbb{R}^n$ ,  $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
- (4) for  $l$ -forms  $\omega, \chi$  on  $\mathbb{R}^n$ ,  $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

**Exercise 2.2.12.** Let  $\Omega \subset \mathbb{R}^n$  be a  $k$ -dimensional smooth submanifold of  $\mathbb{R}^n$ ,  $\phi : U \rightarrow V$  a smooth parametrization of  $\Omega$ ,  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$  an  $k$ -form on  $\mathbb{R}^n$ . Then

$$\phi^*\omega = \left( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det D\phi_I) \right) du_1 \wedge du_2 \wedge \cdots \wedge du_k$$

*Proof.* Using the definitions, we see that

$$\begin{aligned} \phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \end{aligned}$$

A previous exercise tells us that for each  $I \in \mathcal{I}_{k,n}$ ,

$$\begin{aligned} d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\ &= \left( \sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u_j} du_j \right) \wedge \left( \sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u_j} du_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u_j} du_j \right) \\ &= (\det D\phi_I) du_1 \wedge du_2 \wedge \cdots \wedge du_k \end{aligned}$$

Therefore

$$\begin{aligned}
 \phi^* \omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\
 &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det D\phi_I) du_1 \wedge du_2 \wedge \cdots \wedge du_k \\
 &= \left( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det D\phi_I) \right) du_1 \wedge du_2 \wedge \cdots \wedge du_k
 \end{aligned}$$

□

### 2.3. Integration of Differential Forms.

**Definition 2.3.1.** Let  $U \subset \mathbb{R}^k$  be open and  $\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$  a  $k$ -form on  $\mathbb{R}^k$ . Define

$$\int_U \omega = \int_U f dx$$

**Definition 2.3.2.** Let  $\Omega \subset \mathbb{R}^n$  be a  $k$ -dimensional oriented smooth submanifold of  $\mathbb{R}^n$ ,  $\omega$  a  $k$ -form on  $\mathbb{R}^n$  and  $\phi : U \rightarrow V$  a local smooth, orientation-preserving parametrization of  $\Omega$ . Define

$$\int_V \omega = \int_U \phi^* \omega$$

#### Exercise 2.3.3.

**Theorem 2.3.4. (Stokes)** Let  $\Omega \subset \mathbb{R}^n$  be a  $k$ -dimensional oriented smooth submanifold of  $\mathbb{R}^n$  and  $\omega$  a  $k-1$ -form on  $\mathbb{R}^n$ . Then

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$