

REAL ANALYSIS NOTES

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1. ALGEBRA AND ANALYSIS OF SETS

1.1. Limits.

Definition 1.1. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. We define

$$\inf \mathcal{A} = \bigcap_{A \in \mathcal{A}} A, \quad \sup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

Definition 1.2. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. We define

$$\liminf_{n \rightarrow \infty} A_n = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} A_k \right), \quad \limsup_{n \rightarrow \infty} A_n = \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} A_k \right)$$

Note 1.3.

- (1) $\liminf_{n \rightarrow \infty} A_n$ is the set of elements that are in all A_n except for finitely many.
- (2) $\limsup_{n \rightarrow \infty} A_n$ is the set of elements that are in infinitely many A_n .

Exercise 1.4. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then

- (1) $\liminf_{n \rightarrow \infty} A_n = \left\{ x \in X : \liminf_{n \rightarrow \infty} \chi_{A_n}(x) = 1 \right\}$
- (2) $\limsup_{n \rightarrow \infty} A_n = \left\{ x \in X : \limsup_{n \rightarrow \infty} \chi_{A_n}(x) = 1 \right\}$

Proof.

- (1) Let $x \in \liminf_{n \rightarrow \infty} A_n$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x \in A_k$. So for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\chi_{A_k}(x) = 1$. Then $\inf_{k \geq n^*} \chi_{A_k}(x) = 1$ and thus

$$1 = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \chi_{A_k}(x) \right) = \liminf_{n \rightarrow \infty} \chi_{A_n}(x)$$

Conversely, if $1 = \liminf_{n \rightarrow \infty} \chi_{A_n}(x)$, then choosing $\epsilon = \frac{1}{2}$, there exists $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n$ implies that $\chi_{A_k}(x) > 1 - \epsilon$. Hence for each $k \in \mathbb{N}$, $k \geq n$ implies that $\chi_{A_k}(x) = 1$. So for each $k \in \mathbb{N}$, $k \geq n$ implies that $x \in A_k$. So $x \in \liminf_{n \rightarrow \infty} A_n$.

- (2) Similar to (1).

□

Exercise 1.5. Let $A_k = [0, \frac{k}{k+1})$. Then

- (1) $\inf_{k \geq n} A_k = [0, \frac{n}{n+1})$
- (2) $\sup_{k \geq n} A_k = [0, 1)$
- (3) $\liminf_{n \rightarrow \infty} A_n = [0, 1)$
- (4) $\limsup_{n \rightarrow \infty} A_n = [0, 1)$

Proof. Straightforward.

□

Exercise 1.6. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then

$$\liminf_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} A_n$$

Proof. Let $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq n^*$, then $x \in A_k$. Let $n \in \mathbb{N}$. Choose $k = \max\{n^*, n\} \geq n^*$. Then $x \in A_k$. Hence for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x \in A_k$. So $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Thus $\liminf_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} A_n$. \square

Definition 1.7. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. If

$$\liminf_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} A_n$$

then we define

$$\lim_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} A_n$$

Exercise 1.8. Let X be a set and $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Suppose that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$ and $B_{n+1} \subset B_n$. Then

$$\begin{aligned} (1) \quad \lim_{n \rightarrow \infty} A_n &= \sup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n \\ (2) \quad \lim_{n \rightarrow \infty} B_n &= \inf_{n \in \mathbb{N}} B_n = \bigcap_{n=1}^{\infty} B_n \end{aligned}$$

Proof.

(1) Let $n \in \mathbb{N}$. Then

$$\begin{aligned} \inf_{k \geq n} A_k &= \bigcap_{k=n}^{\infty} A_k \\ &= A_n \end{aligned}$$

Thus

$$\begin{aligned} \liminf_{n \rightarrow \infty} A_n &= \bigcup_{n=1}^{\infty} \inf_{k \geq n} A_k \\ &= \bigcup_{n=1}^{\infty} A_n \end{aligned}$$

In addition,

$$\begin{aligned} \sup_{n \geq k} A_k &= \bigcup_{k=n}^{\infty} A_k \\ &= \bigcup_{k=1}^{\infty} A_k \end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{n \rightarrow \infty} A_n &= \bigcap_{n=1}^{\infty} \inf_{k \geq n} A_k \\ &= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_k \\ &= \bigcup_{n=1}^{\infty} A_n\end{aligned}$$

So

$$\lim_{n \rightarrow \infty} A_n = \sup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$$

(2) Similar

□

Exercise 1.9. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets and $(A_{n_k})_{k \in \mathbb{N}}$ a subsequence of $(A_n)_{n \in \mathbb{N}}$. Then

- (1) $\limsup_{k \rightarrow \infty} A_{n_k} \subset \limsup_{n \rightarrow \infty} A_n$
- (2) $\liminf_{n \rightarrow \infty} A_n \subset \limsup_{k \rightarrow \infty} A_{n_k}$

Proof.

- (1) The elements that are in A_{n_k} for infinitely many k are in A_n for infinitely many n .
- (2) Similar.

□

Exercise 1.10. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets, $(A_{n_k})_{k \in \mathbb{N}}$ a subsequence of $(A_n)_{n \in \mathbb{N}}$ and $A \subset X$. If $A_{n_k} \rightarrow A$, then

$$\liminf_{n \rightarrow \infty} A_n \subset A \subset \limsup_{n \rightarrow \infty} A_n$$

Proof. The previous exercises tells us that

$$\begin{aligned}\liminf_{n \rightarrow \infty} A_n &\subset \liminf_{k \rightarrow \infty} A_{n_k} \\ &= A \\ &= \limsup_{k \rightarrow \infty} A_{n_k} \\ &\subset \limsup_{n \rightarrow \infty} A_n\end{aligned}$$

□

Exercise 1.11. Let X be a set and $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Suppose that for each $n \in \mathbb{N}$, $A_n \subset B_n$. Then

- (1) $\limsup_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} B_n$
- (2) $\liminf_{n \rightarrow \infty} A_n \subset \liminf_{n \rightarrow \infty} B_n$

Proof.

- (1) Let $x \in \limsup_{n \rightarrow \infty} A_n$. Then for infinitely many $n \in \mathbb{N}$, $x \in A_n \subset B_n$. So for infinitely many $n \in \mathbb{N}$, $x \in B_n$. Hence $x \in \limsup_{n \rightarrow \infty} B_n$. Therefore $\limsup_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} B_n$.
- (2) Similar.

□

Exercise 1.12. *Let*

Exercise 1.13. *Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then*

- (1) $\limsup_{n \rightarrow \infty} A_n = \left(\liminf_{n \rightarrow \infty} A_n^c \right)^c$
- (2) $\liminf_{n \rightarrow \infty} A_n = \left(\limsup_{n \rightarrow \infty} A_n^c \right)^c$

Proof.

(1)

$$\begin{aligned} \left(\liminf_{n \rightarrow \infty} A_n^c \right)^c &= \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c \right)^c \\ &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\ &= \limsup_{n \rightarrow \infty} A_n \end{aligned}$$

(2) Similar.

□

Exercise 1.14. For $n \in \mathbb{N}$, define

$$A_n = \left\{ \frac{m}{n} : m \in \mathbb{N} \right\}$$

Then

- (1) $\liminf_{n \rightarrow \infty} A_n = \mathbb{N}$
- (2) $\limsup_{n \rightarrow \infty} A_n = \mathbb{Q} \cap (0, \infty)$

Proof.

- (1) For each $x \in \mathbb{N}$ and $n \in \mathbb{N}$, $x = \frac{nx}{n} \in A_n$. Hence $\mathbb{N} \subset \liminf_{n \rightarrow \infty} A_n$. Conversely, let $x \in \liminf_{n \rightarrow \infty} A_n$. Then there exists $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq n$, then $x \in A_k$. In particular, $x \in A_n$. Hence there exists $m_n \in \mathbb{N}$ such that $x = \frac{m_n}{n}$. Choose $s, t \in \mathbb{N}$ such that $x = \frac{s}{t}$ and $\gcd(s, t) = 1$. Suppose that $t \neq 1$. Then choose a prime $p > n$. By assumption, $x \in A_p$. Then there exist $m_p \in \mathbb{N}$ such that $x = \frac{m_p}{p}$. Hence $\frac{s}{t} = \frac{m_p}{p}$ and $tm_p = sp$. Since $t|sp$ and $\gcd(s, t) = 1$, we see that $t|p$. If $t \geq 1$, then p is not prime, a contradiction. So $t = 1$. Hence $x \in \mathbb{N}$. Thus $\liminf_{n \rightarrow \infty} A_n \subset \mathbb{N}$.
- (2) Let $x \in \mathbb{Q} \cap (0, \infty)$. Then there exist $s, t \in \mathbb{N}$ such that $x = \frac{s}{t}$. Define the subsequence $(A_{n_k})_{k \in \mathbb{N}}$ by $A_{n_k} = A_{tk}$. Then for each $k \in \mathbb{N}$, $x = \frac{sk}{tk} \in A_{tk} = A_{n_k}$. Thus $x \in \limsup_{n \rightarrow \infty} A_n$. Conversely, clearly $\limsup_{n \rightarrow \infty} A_n \subset \mathbb{Q} \cap (0, \infty)$.

□

Exercise 1.15. Let X be a set and $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Then

$$\limsup_{n \rightarrow \infty} A_n \cup B_n = \limsup_{n \rightarrow \infty} A_n \cup \limsup_{n \rightarrow \infty} B_n$$

Proof. Let $x \in \limsup_{n \rightarrow \infty} A_n \cup B_n$. Suppose that $x \notin \limsup_{n \rightarrow \infty} A_n$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ if $k \geq n^*$, then $x \notin A_k$. Let $n \in \mathbb{N}$. Then there exists k such that $k \geq \max\{n, n^*\}$ and $x \in A_k \cup B_k$. Since $k \geq n^*$, $x \notin A_k$. Thus $x \in B_k$. So for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x \in B_k$. Therefore $x \in \limsup_{n \rightarrow \infty} B_n$ and

$$\limsup_{n \rightarrow \infty} A_n \cup B_n \subset \limsup_{n \rightarrow \infty} A_n \cup \limsup_{n \rightarrow \infty} B_n$$

Conversely, a previous exercise tells us that $\limsup_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} A_n \cup B_n$ and $\limsup_{n \rightarrow \infty} B_n \subset \limsup_{n \rightarrow \infty} A_n \cup B_n$. Thus

$$\limsup_{n \rightarrow \infty} A_n \cup \limsup_{n \rightarrow \infty} B_n \subset \limsup_{n \rightarrow \infty} A_n \cup B_n$$

□

Exercise 1.16. Let X be a set and $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Then

$$\liminf_{n \rightarrow \infty} A_n \cap B_n = \liminf_{n \rightarrow \infty} A_n \cap \liminf_{n \rightarrow \infty} B_n$$

Proof. A previous exercise tells us that

$$\begin{aligned} \liminf_{n \rightarrow \infty} A_n \cap B_n &= \left(\limsup_{n \rightarrow \infty} A_n^c \cup B_n^c \right)^c \\ &= \left(\limsup_{n \rightarrow \infty} A_n^c \cup \limsup_{n \rightarrow \infty} B_n^c \right)^c \\ &= \left(\limsup_{n \rightarrow \infty} A_n^c \right)^c \cap \left(\limsup_{n \rightarrow \infty} B_n^c \right)^c \\ &= \liminf_{n \rightarrow \infty} A_n \cap \liminf_{n \rightarrow \infty} B_n \end{aligned}$$

□

1.2. Classes of sets.

Definition 1.17. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. Then \mathcal{A} is said to be an **algebra** on X if

- (1) $\mathcal{A} \neq \emptyset$
- (2) for each $A \in \mathcal{A}$, $A^c \in \mathcal{A}$
- (3) for each $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$

Definition 1.18. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. Then \mathcal{A} is said to be a **σ -algebra** on X if

- (1) $\mathcal{A} \neq \emptyset$
- (2) for each $A \in \mathcal{A}$, $A^c \in \mathcal{A}$
- (3) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

Exercise 1.19. Let X be a set and \mathcal{A} a σ -algebra on X . Then

- (1) $X, \emptyset \in \mathcal{A}$
- (2) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{A}$
- (3) For each $A, B \in \mathcal{A}$, $A \setminus B \in \mathcal{A}$

Proof.

- (1) Since $\mathcal{A} \neq \emptyset$, there exists $A \in \mathcal{A}$. Then $A^c \in \mathcal{A}$. Hence $X = A \cup A^c \in \mathcal{A}$ and $\emptyset = X^c \in \mathcal{A}$.
- (2) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Then $(A_n^c)_{n \in \mathbb{N}} \subset \mathcal{A}$. So $\bigcup_{n \in \mathbb{N}} A_n^c \in \mathcal{A}$. Therefore

$$\bigcap_{n \in \mathbb{N}} A_n = \left(\bigcup_{n \in \mathbb{N}} A_n^c \right)^c \in \mathcal{A}$$

- (3) Let $A, B \in \mathcal{A}$. Then $A \setminus B = A \cap B^c \in \mathcal{A}$.

□

Exercise 1.20. Let X be a set and $(\mathcal{A}_i)_{i \in I}$ a collection of σ -algebras (resp. algebra) on X . Then $\bigcap_{i \in I} \mathcal{A}_i$ is a σ -algebra (resp. algebra) on X .

Proof.

- (1) For each $i \in I$, $X \in \mathcal{A}_i$. Thus $X \in \bigcap_{i \in I} \mathcal{A}_i$ and $\bigcap_{i \in I} \mathcal{A}_i \neq \emptyset$.

- (2) Let $A \in \bigcap_{i \in I} \mathcal{A}_i$. Then for each $i \in I$, $A \in \mathcal{A}_i$. Hence for each $i \in I$, $A^c \in \mathcal{A}_i$. Thus $A^c \in \bigcap_{i \in I} \mathcal{A}_i$.
- (3) Let $(A_n)_{n \in \mathbb{N}} \subset \bigcap_{i \in I} \mathcal{A}_i$. Then for each $i \in I$, $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}_i$. Thus for each $i \in I$, $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_i$. So $\bigcup_{n \in \mathbb{N}} A_n \in \bigcap_{i \in I} \mathcal{A}_i$.

□

Definition 1.21. Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$. Put

$$\mathcal{S} = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ and } \mathcal{C} \subset \mathcal{A}\}$$

We define the **σ -algebra generated by \mathcal{C} on X** , $\sigma(\mathcal{C})$, by

$$\sigma(\mathcal{C}) = \bigcap_{\mathcal{A} \in \mathcal{S}} \mathcal{A}$$

Note 1.22. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and \mathcal{A} a σ -alg on X . By definition, if $\mathcal{C} \subset \mathcal{A}$, then $\sigma(\mathcal{C}) \subset \mathcal{A}$.

Note 1.23. Let X be a set, \mathcal{T} an ordered set and $(\mathcal{A}_t)_{t \in \mathcal{T}}$ a collection of σ -algebras on X . Suppose that for each $s, t \in \mathcal{T}$, if $s \leq t$, then $\mathcal{A}_s \subset \mathcal{A}_t$. If there exists $t \in \mathcal{T}$ such that $\mathcal{A}_t = \bigcup_{s \in \mathcal{T}} \mathcal{A}_s$, then $\bigcup_{t \in \mathcal{T}} \mathcal{A}_t$ is a σ -algebra on X . So if \mathcal{T} is finite or if $(\mathcal{A}_t)_{t \in \mathcal{T}}$ terminates, the union is σ -algebra.

Definition 1.24. Let (X, \mathcal{T}) be a topological space. We define the **Borel σ -algebra** on X , $\mathcal{B}(X)$, to be

$$\mathcal{B}(X) = \sigma(\mathcal{T})$$

The sets of $\mathcal{B}(X)$ are called **Borel sets**.

Exercise 1.25. The Borel σ -algebra on \mathbb{R} with the standard topology is given by

$$\mathcal{B}(\mathbb{R}) = \begin{cases} \sigma(\{(a, b] : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{[a, b] : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{[a, b) : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{(a, b) : a, b \in \mathbb{R} \text{ and } a < b\}) \end{cases}$$

Proof. Define

$$(1) \mathcal{C}_{lo} = \{(a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$$

$$(2) \mathcal{C}_c = \{[a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$$

$$(3) \mathcal{C}_{ro} = \{[a, b) : a, b \in \mathbb{R} \text{ and } a < b\}$$

$$(4) \mathcal{C}_o = \{(a, b) : a, b \in \mathbb{R} \text{ and } a < b\}$$

Recall that for each open set $A \subset \mathbb{R}$, there exist $(a_i)_{i \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ such that for each $i \in \mathbb{N}$, $a_i < b_i$, for each $i, j \in \mathbb{N}$, if $i \neq j$, then $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ and $A = \bigcup_{i \in \mathbb{N}} (a_i, b_i)$. This implies that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o)$.

Now, let $a, b \in \mathbb{R}$. Suppose that $a < b$. Then

- (1) $[a, b] = \bigcap_{n \in \mathbb{N}} (a - \frac{1}{n}, b]$, so $\sigma(\mathcal{C}_c) \subset \sigma(\mathcal{C}_{lo})$
- (2) $[a, b] = \bigcup_{n \in \mathbb{N}} [a, b - \frac{1}{n}]$, so $\sigma(\mathcal{C}_{ro}) \subset \sigma(\mathcal{C}_c)$
- (3) $(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b)$, so $\sigma(\mathcal{C}_o) \subset \sigma(\mathcal{C}_{ro})$
- (4) $(a, b] = \bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n})$, so $\sigma(\mathcal{C}_{lo}) \subset \sigma(\mathcal{C}_o)$

Hence $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o) = \sigma(\mathcal{C}_{ro}) = \sigma(\mathcal{C}_c) = \sigma(\mathcal{C}_{lo}) = \sigma(\mathcal{C}_o)$. \square

Exercise 1.26. Let X be a set. Define $\mathcal{A} = \{A \in \mathcal{A} : A \text{ is countable or } A^c \text{ is countable}\}$. Then \mathcal{A} is a σ -algebra on X .

Proof.

- (1) Since $X^c = \emptyset$ is countable, $X \in \mathcal{A}$.
- (2) Let $A \in \mathcal{A}$. Suppose that A^c is uncountable. Then by assumption, $A = (A^c)^c$ is countable. Hence $A^c \in \mathcal{A}$.
- (3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Then for each $n \in \mathbb{N}$, A_n is countable or A_n^c is countable. Suppose that $\bigcup_{n \in \mathbb{N}} A_n$ is uncountable. Then there exists $N \in \mathbb{N}$ such that A_N is uncountable. Hence A_N^c is countable. Thus

$$\left(\bigcup_{n \in \mathbb{N}} A_n \right)^c = \bigcap_{n \in \mathbb{N}} A_n^c \subset A_N^c$$

So $\left(\bigcup_{n \in \mathbb{N}} A_n \right)^c$ is countable and $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$. \square

Definition 1.27. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. We define

$$\mathcal{C} \cap A := \{S \cap A : S \in \mathcal{C}\}$$

Exercise 1.28. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. Then $\sigma(\mathcal{C}) \cap A$ is a σ -algebra on A .

Proof.

- (1) Clearly $\emptyset, A \in \sigma(\mathcal{C}) \cap A$.
- (2) Let $B \in \sigma(\mathcal{C}) \cap A$. Then there exists $S \in \sigma(\mathcal{C})$ such that $B = S \cap A$. Then $S^c \in \sigma(\mathcal{C})$. Thus

$$A \setminus B = S^c \cap A \in \sigma(\mathcal{C}) \cap A$$

- (3) Let $(B_n)_{n \in \mathbb{N}} \subset \sigma(\mathcal{C}) \cap A$. Then for each $n \in \mathbb{N}$, there exists $S_n \in \sigma(\mathcal{C})$ such that $B_n = S_n \cap A$. So $\bigcup_{n \in \mathbb{N}} S_n \in \sigma(\mathcal{C})$. Hence

$$\begin{aligned} \bigcup_{n \in \mathbb{N}} (B_n) &= \bigcup_{n \in \mathbb{N}} (S_n \cap A) \\ &= \left(\bigcup_{n \in \mathbb{N}} S_n \right) \cap A \\ &\in \sigma(\mathcal{C}) \cap A \end{aligned}$$

□

Exercise 1.29. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. Let $\sigma_A(\mathcal{C} \cap A)$ be the σ -algebra on A generated by $\mathcal{C} \cap A$. Define

$$\mathcal{G} = \{S \subset X : S \cap A \in \sigma_A(\mathcal{C} \cap A)\}$$

Then \mathcal{G} is a σ -algebra on X .

Proof. (1) Clearly $\emptyset, X \in \mathcal{G}$.

(2) Let $S \in \mathcal{G}$. Then $S \cap A \in \sigma_A(\mathcal{C} \cap A)$. Hence $A \setminus (S \cap A) = S^c \cap A \in \sigma_A(\mathcal{C} \cap A)$. So $S^c \in \mathcal{G}$.

(3) Let $(S_n)_{n \in \mathbb{N}} \subset \mathcal{G}$. Then for each $n \in \mathbb{N}$, $S_n \cap A \in \sigma_A(\mathcal{C} \cap A)$. Thus

$$\left(\bigcup_{n \in \mathbb{N}} S_n \right) \cap A = \bigcup_{n \in \mathbb{N}} (S_n \cap A) \in \sigma_A(\mathcal{C} \cap A)$$

Thus $\bigcup_{n \in \mathbb{N}} S_n \in \mathcal{G}$.

□

Exercise 1.30. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. Then

$$\sigma(\mathcal{C}) \cap A = \sigma_A(\mathcal{C} \cap A)$$

Proof. Clearly $\mathcal{C} \cap A \subset \sigma(\mathcal{C}) \cap A$. A previous exercise tells us that $\sigma(\mathcal{C}) \cap A$ is a σ -algebra on A . Thus $\sigma_A(\mathcal{C} \cap A) \subset \sigma(\mathcal{C}) \cap A$.

Conversely, from the previous exercise, we have that $\mathcal{G} = \{S \subset X : S \cap A \in \sigma_A(\mathcal{C} \cap A)\}$ is a σ -algebra on X . Clearly $\mathcal{C} \subset \mathcal{G}$. Then $\sigma(\mathcal{C}) \subset \mathcal{G}$. The definition of \mathcal{G} implies that $\sigma(\mathcal{C}) \cap A \subset \sigma_A(\mathcal{C} \cap A)$. Hence $\sigma(\mathcal{C}) \cap A = \sigma_A(\mathcal{C} \cap A)$. □

Definition 1.31. Let X be a set and \mathcal{A} be a σ -algebra on X . Then (X, \mathcal{A}) is called a **measurable space**.

2. MEASURES

2.1. Measures.

Definition 2.1. Let (X, \mathcal{A}) be a measurable space and $\mu : \mathcal{A} \rightarrow [0, \infty]$. Then μ is said to be a **measure** on (X, \mathcal{A}) if

- (1) there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$
- (2) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. If $(A_n)_{n \in \mathbb{N}}$ is disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

Definition 2.2. Let (X, \mathcal{A}) be a measurable space and μ a measure on (X, \mathcal{A}) . Then (X, \mathcal{A}, μ) is called a **measure space**.

Exercise 2.3. Let (X, \mathcal{A}, μ) be a measure space. Then

- (1) (monotonicity): for each $A, B \in \mathcal{A}$, if $A \subset B$, then $\mu(A) \leq \mu(B)$.
- (2) (subadditivity): for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$,

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$$

(3) (continuity from below): for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, if for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$, then

$$\mu\left(\sup_{n \in \mathbb{N}} A_n\right) = \sup_{n \in \mathbb{N}} \mu(A_n)$$

(4) (continuity from above): for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, if for each $n \in \mathbb{N}$, $A_{n+1} \subset A_n$ and $\mu(A_1) < \infty$, then

$$\mu\left(\inf_{n \in \mathbb{N}} A_n\right) = \inf_{n \in \mathbb{N}} \mu(A_n)$$

Proof.

(1) Let $A, B \in \mathcal{A}$. Suppose that $A \subset B$. Then

$$\begin{aligned} \mu(B) &= \mu\left((B \cap A) \cup (B \cap A^c)\right) \\ &= \mu(B \cap A) + \mu(B \cap A^c) \\ &= \mu(A) + \mu(B \cap A^c) \\ &\geq \mu(A) \end{aligned}$$

(2) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Define $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right)$. Then $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n$, $(B_n)_{n \in \mathbb{N}}$ disjoint and for each $n \in \mathbb{N}$, $B_n \subset A_n$. Thus

$$\begin{aligned} \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) \\ &= \sum_{n \in \mathbb{N}} \mu(B_n) \\ &\leq \sum_{n \in \mathbb{N}} \mu(A_n) \end{aligned}$$

(3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$. Then for each $n \in \mathbb{N}$, $\mu(A_n) \leq \mu(A_{n+1})$ and $\lim_{n \rightarrow \infty} \mu(A_n) = \sup_{n \in \mathbb{N}} \mu(A_n)$. Recall that $\sup A_n = \bigcup_{n \in \mathbb{N}} A_n$. Define $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus A_{n-1}$. Then $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $(B_n)_{n \in \mathbb{N}}$ is disjoint,

$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$ and for each $n \in \mathbb{N}$, $\bigcup_{n=1}^k B_n = A_k$. Then

$$\begin{aligned}
 \mu\left(\sup_{n \in \mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \\
 &= \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) \\
 &= \sum_{n \in \mathbb{N}} \mu(B_n) \\
 &= \lim_{k \rightarrow \infty} \sum_{n=1}^k \mu(B_n) \\
 &= \lim_{k \rightarrow \infty} \mu\left(\bigcup_{n=1}^k B_n\right) \\
 &= \lim_{k \rightarrow \infty} \mu(A_k) \\
 &= \sup_{n \in \mathbb{N}} \mu(A_n)
 \end{aligned}$$

- (4) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that for each $n \in \mathbb{N}$, $A_{n+1} \subset A_n$ and $\mu(A_1) < \infty$. Then for each $n \in \mathbb{N}$ $\mu(A_{n+1}) \leq \mu(A_n) \leq \mu(A_1) < \infty$ and the arithmetic that follows is well defined. Recall that $\inf_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} A_n$. For each $n \in \mathbb{N}$, define $B_n = A_1 \cap A_n$. Then for each $n \in \mathbb{N}$, $B_n \subset B_{n+1}$ and

$$\begin{aligned}
 \sup_{n \in \mathbb{N}} B_n &= \bigcup_{n \in \mathbb{N}} B_n \\
 &= A_1 \setminus \bigcap_{n \in \mathbb{N}} A_n \\
 &= A_1 \setminus \inf_{n \in \mathbb{N}} A_n
 \end{aligned}$$

So (3) implies that

$$\begin{aligned}
 \sup_{n \in \mathbb{N}} \mu(B_n) &= \mu\left(\sup_{n \in \mathbb{N}} B_n\right) \\
 &= \mu\left(A_1 \setminus \inf_{n \in \mathbb{N}} A_n\right) \\
 &= \mu(A_1) - \mu\left(\inf_{n \in \mathbb{N}} A_n\right)
 \end{aligned}$$

On the other hand,

$$\begin{aligned}\sup_{n \in \mathbb{N}} \mu(B_n) &= \sup_{n \in \mathbb{N}} \mu(A_1 \setminus A_n) \\ &= \sup_{n \in \mathbb{N}} \left[\mu(A_1) - \mu(A_n) \right] \\ &= \mu(A_1) - \inf_{n \in \mathbb{N}} \mu(A_n)\end{aligned}$$

Therefore

$$\mu\left(\inf_{n \in \mathbb{N}} A_n\right) = \inf_{n \in \mathbb{N}} \mu(A_n)$$

□

Exercise 2.4. Let (X, \mathcal{A}, μ) be a measure space, $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$. Then

- (1) $\mu\left(\liminf_{n \rightarrow \infty} A_n\right) \leq \liminf_{n \rightarrow \infty} \mu(A_n)$
- (2) If $\mu\left(\sup_{n \in \mathbb{N}} A_n\right) < \infty$, then $\limsup_{n \rightarrow \infty} \mu(A_n) \leq \mu\left(\liminf_{n \rightarrow \infty} A_n\right)$

Proof.

- (1) Since $\left(\inf_{k \geq n} A_k\right)_{n \in \mathbb{N}}$ is an increasing sequence and for each $n \in \mathbb{N}$ $\inf_{k \geq n} A_k \subset A_n$, we have that

$$\begin{aligned}\mu\left(\liminf_{n \rightarrow \infty} A_n\right) &= \mu\left[\sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} A_k\right)\right] \\ &= \sup_{n \in \mathbb{N}} \mu\left(\inf_{k \geq n} A_k\right) \\ &= \liminf_{n \rightarrow \infty} \mu\left(\inf_{k \geq n} A_k\right) \\ &\leq \liminf_{n \rightarrow \infty} \mu(A_n)\end{aligned}$$

- (2) Since $\mu\left(\sup_{k \geq 1} A_k\right) < \infty$, $\left(\sup_{k \geq n} A_k\right)_{n \in \mathbb{N}}$ is a decreasing sequence and for each $n \in \mathbb{N}$, $A_n \subset \sup_{k \geq n} A_k$, we have that

$$\begin{aligned}\mu\left(\limsup_{n \rightarrow \infty} A_n\right) &= \mu\left[\inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} A_k\right)\right] \\ &= \inf_{n \in \mathbb{N}} \mu\left(\sup_{k \geq n} A_k\right) \\ &= \limsup_{n \rightarrow \infty} \mu\left(\sup_{k \geq n} A_k\right) \\ &\geq \limsup_{n \rightarrow \infty} \mu(A_n)\end{aligned}$$

□

Exercise 2.5. Let (X, \mathcal{A}, μ) be a measure space, $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$. Suppose that $\mu\left(\sup_{n \in \mathbb{N}} A_n\right) < \infty$. Then $A_n \rightarrow A$ implies that $\mu(A_n) \rightarrow \mu(A)$.

Proof. Suppose that $A_n \rightarrow A$. Then the previous exercise tells us that

$$\begin{aligned} \mu(A) &= \mu\left(\liminf_{n \rightarrow \infty} A_n\right) \\ &\leq \liminf_{n \rightarrow \infty} \mu(A_n) \\ &\leq \limsup_{n \rightarrow \infty} \mu(A_n) \\ &\leq \mu(\limsup_{n \rightarrow \infty} A_n) \\ &= \mu(A) \end{aligned}$$

Thus $\mu(A) = \limsup_{n \rightarrow \infty} \mu(A_n) = \liminf_{n \rightarrow \infty} \mu(A_n)$ and $\mu(A_n) \rightarrow \mu(A)$ □

2.2. Outer Measures.

Definition 2.6. Let X be a set and $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$. Then μ^* is said to be an **outer measure on X** if

- (1) $\mu^*(\emptyset) = 0$
- (2) for each $A, B \subset X$, if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$.
- (3) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$,

$$\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$$

Theorem 2.7. Construction of Outer Measures:

Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \rightarrow [0, \infty]$. Suppose that $\emptyset, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. Define $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \rho(E_n) : (E_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } A \subset \bigcup_{n \in \mathbb{N}} E_n \right\}$$

Then μ^* is an outer measure on X .

Note 2.8. In particular, for each $A \in \mathcal{E}$, $\mu^*(A) = \rho(A)$.

Definition 2.9. Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \rightarrow [0, \infty]$. Suppose that $\emptyset, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$. Let μ^* be the outer measure on X defined as in the last theorem. Then μ^* is called the **outer measure on X induced by ρ** .

Definition 2.10. Let X be a set, μ^* an outer measure on X and $A \subset X$. Then A is said to be μ^* -**outer measurable** if for each $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Theorem 2.11. Let X be a set and μ^* an outer measure on X . Define $\mathcal{A} = \{A \subset X : A \text{ is } \mu^*\text{-measurable}\}$. Then \mathcal{A} is a σ -algebra on X and $\mu^*|_{\mathcal{A}}$ is a complete measure on (X, \mathcal{A}) .

Definition 2.12. Let X be a set, \mathcal{A}_0 be an algebra on X and $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty]$. Then μ_0 is said to be a **premeasure on (X, \mathcal{A}_0)** if

- (1) there exists $A \in \mathcal{A}_0$ such that $\mu_0(A) < \infty$

(2) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}_0$, if $(A_n)_{n \in \mathbb{N}}$ is disjoint and $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_0$, then

$$\mu_0\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu_0(A_n)$$

Note 2.13. The same reasoning applied to measures shows that $\mu_0(\emptyset) = 0$.

Theorem 2.14. Let X be a set, \mathcal{A}_0 an algebra on X , μ_0 a premeasure on (X, \mathcal{A}_0) and μ^* the outer measure on X induced by μ_0 . Put $\mathcal{A} = \sigma(\mathcal{A}_0)$. If μ_0 is σ -finite, then there exists a unique measure μ on (X, \mathcal{A}) such that $\mu|_{\mathcal{A}_0} = \mu^*|_{\mathcal{A}_0} = \mu_0$.

2.3. Product Measures.

Definition 2.15. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ -finite measurable spaces. Put $\mathcal{E} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$. Then \mathcal{E} is an elementary family and thus $\mathcal{M}_0 = \{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n \subset \mathcal{E} \text{ are disjoint}\}$ is an algebra on $X \times Y$. We define $\pi_0 : \mathcal{M}_0 \rightarrow [0, \infty]$ by

$$\pi_0\left(\bigcup_{i=1}^n A_i \times B_i\right) = \sum_{i=1}^n \mu(A_i) \nu(B_i)$$

Then π_0 is a premeasure on $(X \times Y, \mathcal{M}_0)$. Since $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{M}_0)$, we define the **product measure**, $\mu \times \nu$ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$, to be the unique extension of π_0 to $\mathcal{A} \otimes \mathcal{B}$. The existence of which is guaranteed by a theorem in the previous section. In particular,

$$\begin{aligned} \mu \times \nu(E) &= \inf \left\{ \sum_{n \in \mathbb{N}} \pi_0(E_n) : (E_n)_{n \in \mathbb{N}} \subset \mathcal{M}_0 \text{ and } E \subset \bigcup_{n \in \mathbb{N}} E_n \right\} \\ &= \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) : (A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } E \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n \right\} \end{aligned}$$

3. INTEGRATION

3.1. Measurable Functions.

Definition 3.1. Let (X, \mathcal{A}) and (Y, \mathcal{B}) be measurable spaces and $f : X \rightarrow Y$. Then f is said to be **\mathcal{A} - \mathcal{B} measurable** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. When $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say that f is **\mathcal{A} -measurable**. If $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}, \mathcal{L})$, then we say that f is **Borel measurable** or **Lebesgue measurable** respectively.

Exercise 3.2. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \rightarrow Y$. Then

- (1) $\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ -algebra on Y
- (2) $\{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ -algebra on X

Proof.

- (1) Define $\mathcal{L} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$. Clearly $Y \in \mathcal{L}$. Let $B \in \mathcal{L}$. Then $f^{-1}(B) \in \mathcal{A}$. Hence

$$f^{-1}(B^c) = (f^{-1}(B))^c \in \mathcal{A}$$

Thus $B^c \in \mathcal{L}$. Now, let $(B_n)_{n \in \mathbb{N}} \subset \mathcal{L}$. Then for each $n \in \mathbb{N}$, $f^{-1}(B_n) \in \mathcal{A}$. Thus

$$f^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \bigcup_{n \in \mathbb{N}} f^{-1}(B_n) \in \mathcal{A}$$

Hence $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{L}$.

(2) Similar to (1). □

Exercise 3.3. Let (X, \mathcal{A}) and (Y, \mathcal{B}) be measurable spaces. Suppose that there exists $\mathcal{E} \subset Y$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Let $f : X \rightarrow Y$. Then f is \mathcal{A} - \mathcal{B} measurable iff for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$.

Proof. By definition, if f is \mathcal{A} - \mathcal{B} measurable, then for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. Conversely, suppose that for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. The previous lemma tells us that $\mathcal{L} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ -algebra on Y . Since $\mathcal{E} \subset \mathcal{L}$, we have that $\mathcal{B} = \sigma(\mathcal{E}) \subset \mathcal{L}$. So f is \mathcal{A} - \mathcal{B} measurable. □

Exercise 3.4. Let X, Y be sets, $f : X \rightarrow Y$ and $\mathcal{E} \subset \mathcal{P}(Y)$. Then $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$.

Proof. Clearly $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$. Since $f^{-1}(\sigma(\mathcal{E}))$ is a σ -algebra, we have that $\sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. Since $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$, the previous exercise tells us that f is $f^{-1}(\sigma(\mathcal{E}))$ - $\sigma(\mathcal{E})$ measurable. Then $f^{-1}(\sigma(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. So $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$. □

Exercise 3.5. Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X \rightarrow Y$. If f is continuous, then f is $\mathcal{B}(X)$ - $\mathcal{B}(Y)$ measurable.

Proof. Recall that $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$ and continuity tells us that for each $U \in \mathcal{T}_2$, $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$. □

Definition 3.6. Let X be a set and $f : X \rightarrow \mathbb{C}$. Then f is said to be **simple** if $f(X)$ is finite.

Definition 3.7. Let (X, \mathcal{A}) be a measurable space. We define $S^+(X, \mathcal{A}) = \{f : X \rightarrow [0, \infty) : f \text{ is simple, measurable}\}$ and $S(X, \mathcal{A}) = \{f : X \rightarrow \mathbb{C} : f \text{ is simple, measurable}\}$

Theorem 3.8. Let (X, \mathcal{A}) be a measurable space. Then

- (1) If $f : X \rightarrow [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \rightarrow f$ pointwise and $\phi_n \rightarrow f$ uniformly on any set on which f is bounded.
- (2) If $f : X \rightarrow \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S$ such that for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ and $\phi_n \rightarrow f$ pointwise and $\phi_n \rightarrow f$ uniformly on any set on which f is bounded.

3.2. Integration of Nonnegative Functions.

Definition 3.9. Let (X, \mathcal{A}, μ) be a measure space. Define $L^+(X, \mathcal{A}, \mu) = \{f : X \rightarrow [0, \infty] : f \text{ is measurable}\}$. We will typically just write L^+ .

Theorem 3.10. Monotone Convergence Theorem Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then

$$\sup_{n \in \mathbb{N}} \int f_n = \int \sup_{n \in \mathbb{N}} f_n$$

Exercise 3.11. Let μ_1, μ_2 be measures on (X, \mathcal{A}) and $f \in L^+$. Then

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

Proof. Suppose that f is simple. Then there exist $(a_n)_{n=1}^\infty \subset [0, \infty)$ and $(E_i)_{i=1}^\infty \subset \mathcal{A}$ such that $f = \sum_{i=1}^\infty a_i \chi_{E_i}$. Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^\infty a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^\infty a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^\infty a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for a general f , choose $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that $\phi_n \rightarrow f$ pointwise and for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$. Then monotone convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

Exercise 3.12. Let μ_1, μ_2 be measures on (X, \mathcal{A}) . Suppose that $\mu_1 \leq \mu_2$. Then for each $f \in L^+$,

$$\int f d\mu_1 \leq \int f d\mu_2$$

Proof. First suppose that f is simple. Then there exist $(a_n)_{n=1}^\infty \subset [0, \infty)$ and $(E_i)_{i=1}^\infty \subset \mathcal{A}$ such that $f = \sum_{i=1}^\infty a_i \chi_{E_i}$. Then

$$\begin{aligned} \int f d\mu_1 &= \sum_{i=1}^\infty a_i \mu_1(E_i) \\ &\leq \sum_{i=1}^\infty a_i \mu_2(E_i) \\ &= \int f d\mu_2 \end{aligned}$$

for general f ,

$$\begin{aligned} \int f d\mu_1 &= \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_1 \\ &\leq \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_2 \\ &= \int f d\mu_2 \end{aligned}$$

□

Theorem 3.13. *Fatou's Lemma* Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Theorem 3.14. Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n.$$

Exercise 3.15. Let $f \in L^+$ and suppose that $\int f < \infty$. Put $N = \{x \in X : f(x) = \infty\}$ and $S = \{x \in X : f(x) > 0\}$. Then $\mu(N) = 0$ and S is σ -finite.

Proof. Suppose that $\mu(N) > 0$. Define $f_n = n\chi_N \in L^+$. Then for each $n \in \mathbb{N}$, $f_n \leq f_{n+1} \leq f$ on N . So

$$\begin{aligned} \int f &\geq \int_N f \\ &= \lim_{n \rightarrow \infty} \int_N f_n \\ &= \lim_{n \rightarrow \infty} n\mu(N) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

Hence N is a null set. Now, put $S_n = \{x \in X : f(x) > 1/n\}$. Then $S = \bigcup_{n \in \mathbb{N}} S_n$. Suppose that there exists some $n \in \mathbb{N}$ such that $\mu(S_n) = \infty$. Then

$$\begin{aligned} \int f &\geq \int_{S_n} f \\ &\geq \frac{1}{n} \mu(S_n) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(S_n) < \infty$ and S is σ -finite.

□

Exercise 3.16. Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.

Proof. $f = 0$ a.e. implies that for each $E \in \mathcal{A}$, $\int_E f = 0$ is clear. Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f = 0$. For $n \in \mathbb{N}$ put $N_n = \{x \in X : f(x) > 1/n\}$ and define $N = \{x \in X : f(x) > 0\}$. So $N = \bigcup_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned} 0 &= \int_{N_n} f \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

Hence for each $n \in \mathbb{N}$, $\mu(N_n) = 0$. Thus $\mu(N) = 0$ and $f = 0$ a.e. as required. \square

Exercise 3.17. Let $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that $f_n \xrightarrow{p.w.} f$, $\lim_{n \rightarrow \infty} \int f_n = \int f$ and $\int f < \infty$. Then for each $E \in \mathcal{A}$, $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$. This result may fail to be true if $\int f = \infty$

Proof. Let $E \in \mathcal{A}$. By Fatou's lemma, $\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$. Note that since $\int f < \infty$, we have that $\int_{E^c} f \leq \int f < \infty$. Thus we may write

$$\begin{aligned} \int_E f &= \int f - \int_{E^c} f \\ &\geq \int f - \liminf_{n \rightarrow \infty} \int_{E^c} f_n \\ &= \int f - \liminf_{n \rightarrow \infty} \left(\int f_n - \int_E f_n \right) \\ &= \int f - \int f + \limsup_{n \rightarrow \infty} \int_E f_n \\ &= \limsup_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Hence

$$\limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$$

and therefore

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

If we drop the assumption that $\int f < \infty$, then the result would fail to be true for the functions $f = \infty \chi_{(0,1)}$ and $f_n = \infty \chi_{(0,1)} + n \chi_{(1,1+1/n)}$. Here $f_n \xrightarrow{p.w.} f$, $\lim_{n \rightarrow \infty} \int f_n = \int f = \infty$ and $\lim_{n \rightarrow \infty} \int_{(1,\infty)} f_n = 1$ while $\int_{(1,\infty)} f = 0$. \square

Exercise 3.18. Let $f \in L^+$. Define $\lambda : \mathcal{A} \rightarrow [0, \infty]$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{A}$. Then λ is a measure on (X, \mathcal{A}) and for each $g \in L^+$, $\int g d\lambda = \int g f d\mu$.

Proof. Clearly $\lambda(\emptyset) = 0$. Let $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ and suppose that for each $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j = \emptyset$. For now, suppose that f is simple. Then there exist $E_1, E_2, \dots, E_n \in \mathcal{A}$ and $a_1, a_2, \dots, a_n \in [0, \infty)$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}
\lambda\left(\bigcup_{j \in \mathbb{N}} A_j\right) &= \int_{\bigcup_{j \in \mathbb{N}} A_j} f \\
&= \sum_{i=1}^n a_i \mu\left(E_i \cap \left(\bigcup_{j \in \mathbb{N}} A_j\right)\right) \\
&= \sum_{i=1}^n a_i \mu\left(\bigcup_{j \in \mathbb{N}} E_i \cap A_j\right) \\
&= \sum_{i=1}^n a_i \sum_{j \in \mathbb{N}} \mu(E_i \cap A_j) \\
&= \sum_{j \in \mathbb{N}} \sum_{i=1}^n a_i \mu(E_i \cap A_j) \\
&= \sum_{j \in \mathbb{N}} \int_{A_j} f \\
&= \sum_{j \in \mathbb{N}} \lambda(A_j)
\end{aligned}$$

Hence λ is a measure on (X, \mathcal{A}) . Now, for a general f , there exist $(\phi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \xrightarrow{\text{p.w.}} f$. Put $A = \bigcup_{j \in \mathbb{N}} A_j$ and define the measures λ_n by $\lambda_n(E) = \int_E \phi_n$. Note that we may define a monotonically increasing sequence of functions $g_n : \mathbb{N} \rightarrow [0, \infty]$ by $g_n(j) = \int_{A_j} \phi_n$. Using monotone convergence three times and a nice application of the counting measure on \mathbb{N} , we may write

$$\begin{aligned}
\lambda(A) &= \int_A f \\
&= \lim_{n \rightarrow \infty} \int_A \phi_n \\
&= \lim_{n \rightarrow \infty} \sum_{j \in \mathbb{N}} \int_{A_j} \phi_n \\
&= \sum_{j \in \mathbb{N}} \lim_{n \rightarrow \infty} \int_{A_j} \phi_n \quad (\text{by the above}) \\
&= \sum_{j \in \mathbb{N}} \int_{A_j} f \\
&= \sum_{j \in \mathbb{N}} \lambda(A_j).
\end{aligned}$$

Hence λ is a measure on (X, \mathcal{A}) . Let $g \in L^+$. First assume that g is simple. Then there exist $E_1, E_2, \dots, E_n \in \mathcal{A}$ and $a_1, a_2, \dots, a_n \in [0, \infty)$ such that $g = \sum_{i=1}^n a_i \chi_{E_i}$. In this case, we have that

$$\begin{aligned} \int g d\lambda &= \sum_{i=1}^n a_i \lambda(E_i) \\ &= \sum_{i=1}^n a_i \int_{E_i} f d\mu \\ &= \int \left(\sum_{i=1}^n a_i \chi_{E_i} \right) f d\mu \\ &= \int g f d\mu. \end{aligned}$$

Now for a general $g \in L^+$, there exist $(\psi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ψ_n is simple, $\psi_n \leq \psi_{n+1} \leq f$ and $\psi_n \xrightarrow{\text{p.w.}} g$. Monotone convergence then gives us

$$\begin{aligned} \int g d\lambda &= \lim_{n \rightarrow \infty} \int \psi_n d\lambda \\ &= \lim_{n \rightarrow \infty} \int \psi_n f d\mu \\ &= \int g f d\mu \text{ as required.} \end{aligned}$$

□

Exercise 3.19. Let $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$, $f_n \xrightarrow{\text{p.w.}} f$ and $\int f_1 < \infty$. Then $\lim_{n \rightarrow \infty} \int f_n = \int f$.

Proof. First we note that since $\int f_1 < \infty$, $f_1 < \infty$ a.e., for each $n \in \mathbb{N}$, $f_1 - f_n$ and $\int f_1 - \int f_n$ are well defined and $\int f_n \leq \int f_1 < \infty$. Also, for $n \in \mathbb{N}$, $f_1 - f_n \in L^+$. So we may write

$$\begin{aligned} \int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n \\ &= \int [(f_1 - f_n) + f_n] - \int f_n \\ &= \int f_1 - \int f_n \end{aligned}$$

Put $g_n = f + (f_1 - f_n)$. Then $g_n \in L^+$, for each $n \in \mathbb{N}$, $g_n \leq g_{n+1}$ and $g_n \xrightarrow{\text{p.w.}} f_1$. Monotone convergence tells us that

$$\begin{aligned} \int f_1 &= \lim_{n \rightarrow \infty} \int g_n \\ &= \lim_{n \rightarrow \infty} \left[\int f + (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[\int f + \int (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[\int f + \int f_1 - \int f_n \right] \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \int f$ and $\lim_{n \rightarrow \infty} \int f_1$ exist, $\lim_{n \rightarrow \infty} \int f_n = \int f$ as required. □

3.3. Integration of Complex Valued Functions.

Definition 3.20. Let $f : X \rightarrow \mathbb{C}$ be measurable. Then f is said to be **integrable** if

$$\int |f| d\mu < \infty$$

Definition 3.21. Let (X, \mathcal{A}, μ) be a measure space. Define $L^1(X, \mathcal{A}, \mu) = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable and } \int |f| < \infty\}$

Lemma 3.22. Let $f : X \rightarrow \mathbb{R}$ be measurable. Then f is integrable iff f^+ and f^- are integrable.

Proof. $f^+, f^- \leq |f| = f^+ + f^-$ □

Definition 3.23. Let $f : X \rightarrow \mathbb{R}$ be measurable. Then f is said to be **extended integrable** if

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

Lemma 3.24. Let $f : X \rightarrow \mathbb{R}$ be measurable. Then f is integrable iff $\text{Re}(f)$ and $\text{Im}(f)$ are integrable.

Proof. $|\text{Re}(f)|, |\text{Im}(f)| \leq |f| \leq |\text{Re}(f)| + |\text{Im}(f)|$ □

Theorem 3.25. Dominated Convergence Let $(f_n)_{n \in \mathbb{N}} \subset L^1$, f measurable and $g \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. Then $f \in L^1$ and $\int f_n \rightarrow \int f$.

Exercise 3.26. Let μ_1, μ_2 be measures on (X, \mathcal{A}) . Then

- (1) $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$
- (2) for each $f \in L^1(\mu_1 + \mu_2)$, we have that

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

Proof. (1) The first part is clear since similar exercise from the section on nonnegative functions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f| d\mu_1 + \int |f| d\mu_2$$

- (2) Suppose that f is simple. Then there exist $(a_n)_{n=1}^\infty \subset \mathbb{C}$ and $(E_i)_{i=1}^\infty \subset \mathcal{A}$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for general f , choose $(\phi_n)_{n \in \mathbb{N}} \subset S$ such that $\phi_n \rightarrow f$ pointwise and for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$. Then dominated convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

Theorem 3.27. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$. Suppose that

$$\sum_{n \in \mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n \in \mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

Theorem 3.28. Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$.

Exercise 3.29. Generalized Fatou's Lemma: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \geq -g$. Then $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$. What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?

Proof. First note that for each $n \in \mathbb{N}$, $\int f_n$ is well defined since $f_n^- \leq g \in L^1$. Since $g + f_n \geq 0$, we may use Fatou's lemma to write

$$\begin{aligned} \int g + \int \liminf_{n \rightarrow \infty} f_n &= \int \liminf_{n \rightarrow \infty} (g + f_n) \\ &\leq \liminf_{n \rightarrow \infty} \int (g + f_n) \\ &= \int g + \liminf_{n \rightarrow \infty} \int f_n \end{aligned}$$

Since $\int g < \infty$, $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$ as required. The analogue is as follows: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \leq g$. Then $\limsup_{n \rightarrow \infty} \int f_n \leq \int \limsup_{n \rightarrow \infty} f_n$. To show this, just use the result from above with the sequence $(g_n)_{n \in \mathbb{N}}$ given by $g_n = -f_n$. \square

Exercise 3.30. Let $(f_n)_{n \in \mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \rightarrow \mathbb{C}$. Suppose that $f_n \xrightarrow{\text{uni}} f$. Then

- (1) if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim_{n \rightarrow \infty} \int f_n = \int f$
- (2) if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on \mathbb{R} with Lebesgue measure).

Proof. Choose $N \in \mathbb{N}$ such that for $n \geq N$ and $x \in X$, $|f(x) - f_n(x)| < 1$. Then $||f| - |f_N|| < 1$ and so $|f| < |f_N| + 1$. Thus $\int |f| \leq \int |f_N| + \mu(X) < \infty$ and $f \in L^1$. Similarly for $n \geq N$, $|f_n| < |f| + 1$. Dominated convergence then gives us that $\lim_{n \rightarrow \infty} \int f_n = \int f$ as required. To see the necessity that $\mu(X) < \infty$, consider $f \equiv 0$ and $f_n = (1/n)\chi_{(0,n)}$. Then $f_n \xrightarrow{\text{uni}} f$, but $1 = \lim_{n \rightarrow \infty} \int f_n \neq \int f = 0$. \square

Exercise 3.31. Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$, $g_n \xrightarrow{\text{a.e.}} g$, $|f_n| \leq g_n$ and $\int g_n \rightarrow \int g$. Then $\int f_n \rightarrow \int f$.

Proof. We simply use Fatou's lemma. Put $h_n = (g + g_n) - |f_n - f|$. Since for each $n \in \mathbb{N}$, $|f_n| \leq g_n$, we know that $|f| \leq g$. So $h_n \geq 0$ and $h_n \xrightarrow{\text{p.w.}} 2g$. Thus

$$\begin{aligned} 2 \int g &= \int \liminf_{n \rightarrow \infty} h_n \\ &\leq \liminf_{n \rightarrow \infty} \left[\left(\int g + \int g_n \right) - \int |f_n - f| \right] \\ &= 2 \int g + \liminf_{n \rightarrow \infty} \left(- \int |f_n - f| \right) \\ &= 2 \int g - \limsup_{n \rightarrow \infty} \int |f_n - f| \end{aligned}$$

Hence $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \rightarrow 0$ and $\int f_n \rightarrow \int f$ as required. \square

Exercise 3.32. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$. Then $\int |f_n - f| \rightarrow 0$ iff $\int |f_n| \rightarrow \int |f|$.

Proof. Suppose that $\int |f_n - f| \rightarrow 0$. Since

$$\begin{aligned} \left| \int |f_n| - \int |f| \right| &= \left| \int (|f_n| - |f|) \right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|, \end{aligned}$$

we see that $\int |f_n| \rightarrow \int |f|$. Conversely, suppose that $\int |f_n| \rightarrow \int |f|$. Put $h_n = |f_n - f|$, $g_n = |f_n| + |f|$, $h \equiv 0$ and $g = 2f$. Then $h_n \xrightarrow{\text{a.e.}} h$, $g_n \xrightarrow{\text{a.e.}} g$ and for each $n \in \mathbb{N}$, $h_n \leq g_n$. Our assumption implies that $\int g_n \rightarrow \int g$. Thus the last exercise tells us that $\int h_n \rightarrow \int h$ as required. \square

Exercise 3.33. Let $(r_n)_{n \in \mathbb{N}}$ be an enumeration of the rationals. Define $f : \mathbb{R} \rightarrow [0, \infty)$ by

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}$$

and define $g : X \rightarrow [0, \infty]$ by

$$g(x) = \sum_{n \in \mathbb{N}} 2^{-n} f(x - r_n).$$

Then

- (1) $g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.
- (2) $g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of \mathbb{R}
- (3) Taking $g \in L^1$, g is unbounded on each subinterval of \mathbb{R} and discontinuous everywhere and remains so after redefinition on a null set

Proof. For convenience, define $f_n : \mathbb{R} \rightarrow [0, \infty)$ by $f_n(x) = f(x - r_n)$ for $x \in \mathbb{R}$. To show (1) we note that for each $n \in \mathbb{N}$, $f_n \in L^1$ and

$$\begin{aligned} \int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1} \end{aligned}$$

Hence

$$\sum_{n \in \mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set, $g \in L^1$. In particular $\int |g| < \infty$ and so $|g|$ (and hence g) are finite almost everywhere. For (2), since $g < \infty$ a.e., so too is g^2 . Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Choose $N \in \mathbb{N}$ such that $r_N \in (a, b)$. Since all the terms in the sum are nonnegative, $g^2 \geq \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2$ and so

$$\begin{aligned}
\int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2 \\
&= \sum_{n \in \mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\
&\geq 2^{-2N} \int_{(a,b)} f_N^2 \\
&\geq 2^{-2N} \int_{r_N}^{b \wedge (r_N+1)} \frac{1}{x - r_N} dx \\
&= \infty
\end{aligned}$$

So g^2 is not integrable on any subinterval of \mathbb{R} . For (3), note that redefining g on a null set does not change the result of (2). Suppose that there is a finite subinterval $I \subset \mathbb{R}$ such that g is bounded on I . Hence there exists $M > 0$ such that for each $x \in I$, $g(x)^2 \leq M$. Then

$$\begin{aligned}
\int_I g^2 &\leq M^2 m(I) \\
&< \infty
\end{aligned}$$

which is a contradiction. So g is not bounded on any subinterval of \mathbb{R} . Now, suppose that there exists $x_0 \in \mathbb{R}$ such that g is continuous at x_0 . Choose $\delta > 0$ such that for each $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then $|g(x) - g(x_0)| < 1$. The reverse triangle inequality tells us that for each $x \in (x_0 - \delta, x_0 + \delta)$, $|g(x)| < 1 + |g(x_0)|$. Hence g is bounded on $(x_0 - \delta, x_0 + \delta)$ which is a contradiction. So g is discontinuous everywhere. \square

Exercise 3.34. Let $f \in L^1$.

- (1) If f is bounded, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\int_E |f| < \epsilon$.
- (2) The same conclusion holds for f unbounded.

Proof. (1) Since f is bounded, there exists $M > 0$ such that $|f| \leq M$. Let $\epsilon > 0$. Choose $\delta = \epsilon/2M$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then

$$\begin{aligned}
\int_E |f| &\leq M\mu(E) \\
&= M \frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

(2) Suppose that f is unbounded. Let $\epsilon > 0$. Then there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon/2$. Since ϕ is bounded, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$,

if $\mu(E) < \delta$, then $\int_E |\phi| < \epsilon/2$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then

$$\begin{aligned} \int_E |f| &\leq \int_E |f - \phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

Exercise 3.35. Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Define $F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x) = \int_{(-\infty, x]} f dm.$$

Then F is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Since $f \in L^1$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| dm < \epsilon.$$

Let $x \in \mathbb{R}$. Suppose that $|x - x_0| < \delta$. Then

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f| dm \\ &< \epsilon \end{aligned}$$

So F is continuous.

□

Exercise 3.36. Denote by δ_x the point mass measure at $x \in X$ on measurable space $(X, \mathcal{P}(X))$. Let $f : X \rightarrow \mathbb{C}$. Then

$$\int f d\delta_x = f(x)$$

Proof. First assume that f is simple. Then there exist $a_1, a_2, \dots, a_n \in \mathbb{C}$ and $E_1, E_2, \dots, E_n \in \mathcal{P}(X)$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Thus $\int f d\delta_x = f(x)$. Now assume that f , which is measurable by choice of σ -algebra, satisfies $f(X) \subset [0, \infty)$. Choose a sequence $(\phi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1}$ and $\phi_n \xrightarrow{p.w.} f$. From before, we see that for each $n \in \mathbb{N}$, $\int \phi_n d\delta_x = \phi_n(x)$. Monotone convergence tells us that $\int f d\delta_x = f(x)$. Now just extend to complex valued functions.

□

Exercise 3.37. Denote by $\#$ the counting measure on the measurable space $(X, \mathcal{P}(X))$. Let $f : X \rightarrow \mathbb{C}$ and suppose that $f \in L^1$. Then

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if f is integrable, then $\{x \in X : f(x) \neq 0\}$ is countable.

Proof. Please refer to the definition of the sum in the appendix. First suppose that $f(X) \subset [0, \infty)$. For $n \in \mathbb{N}$, put $X_n = \{x \in X : f(x) > 1/n\}$ and define $X^* = \{x \in X : f(x) > 0\}$, $X_0 = \{x \in X : f(x) = 0\}$. Then $X^* = \bigcup_{n \in \mathbb{N}} X_n$. Since $f \in L^1$, we have that for each $n \in \mathbb{N}$,

$$\begin{aligned} \infty &> \int f d\# \\ &\geq \int_{X_n} f d\# \\ &\geq \frac{1}{n} \#(X_n). \end{aligned}$$

Thus for each $n \in \mathbb{N}$, X_n is finite and X^* is countable. Thus there exists $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that $X^* = \{x_n\}_{n \in \mathbb{N}}$. For $n \in \mathbb{N}$, define $E_n = \{x_1, x_2, \dots, x_n\}$ and

$$\begin{aligned} f_n &= f \chi_{E_n} \\ &= \sum_{i=1}^n f(x_i) \chi_{\{x_i\}} \end{aligned}$$

Then $f_n \xrightarrow{\text{p.w.}} f \chi_{X^*} = f$ and for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. So

$$\begin{aligned} \int f &= \sup_{n \in \mathbb{N}} \int f_n \\ &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n f(x_i) \\ &= \sum_{x \in X^*} f(x) \\ &= \sum_{x \in X} f(x). \end{aligned}$$

For $f : X \rightarrow \mathbb{C}$, our L^1 assumption and the result above tell us that

$$\sum_{x \in X} |f(x)| < \infty.$$

Thus writing $f = g + ih$, we see that the same is true for f^+, f^-, g^+, g^- . Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f d\# = \sum_{x \in X} f(x).$$

□

Exercise 3.38. Let $f, g : X \rightarrow \mathbb{R}$. Suppose that $f, g \in L^1$. Then $f \leq g$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$.

Proof. Suppose $f \leq g$ a.e. Put $N = \{x \in X : f(x) > g(x)\} \subset N$. Then $\mu(N) = 0$ and $g - f \geq 0$ on N^c . So for each $E \in \mathcal{A}$,

$$\begin{aligned} \int_E g - \int_E f &= \int_E (g - f) \\ &= \int_{E \cap N^c} (g - f) \\ &\geq 0 \end{aligned}$$

Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$. Put $N_n = \{x \in X : f(x) - g(x) > 1/n\}$ and $N = \{x \in X : f(x) > g(x)\}$. Then $N = \bigcup_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned} 0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

So that $\mu(N_n) = 0$. Thus for each $n \in \mathbb{N}$, $\mu(N_n) = 0$ which implies $\mu(N) = 0$. Therefore $f \leq g$ a.e. as required. \square

Definition 3.39. Let $\mathcal{F} \subset L^1$. Then \mathcal{F} is said to be **uniformly integrable** if for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$. (i.e.

$$\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0).$$

Exercise 3.40. Suppose that μ is finite. Let $\mathcal{F} \subset L^1$. Then \mathcal{F} is uniformly integrable iff

- (1) there exists $M > 0$ such that $\sup_{f \in \mathcal{F}} \int |f| \leq M$
- (2) for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\sup_{f \in \mathcal{F}} \int_E |f| < \epsilon$.

Proof. (\Rightarrow): (1) Suppose that \mathcal{F} is uniformly integrable. Then there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < 1$. Choose $M = \mu(X)K + 1$. Then for each $f \in \mathcal{F}$,

$$\begin{aligned} \int |f| &= \int_{\{|f| > K\}} |f| + \int_{\{|f| \leq K\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M \end{aligned}$$

(2) Let $\epsilon > 0$. Then choose $K \in \mathbb{N}$ such that $\sup_{f \in \mathcal{F}} \int_{\{|f| > K\}} |f| < \epsilon/2$ and choose $\delta = \epsilon/2K$.

Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then for $f \in \mathcal{F}$,

$$\begin{aligned} \int_E |f| &= \int_{E \cap \{|f| > K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon \end{aligned}$$

(\Leftarrow): Choose $M > 0$ as in (1). Suppose that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $f \in \mathcal{F}$ such that $\mu(\{|f| > K\}) \geq \epsilon$. Choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $f_K \in \mathcal{F}$ such that $\mu(\{|f_K| > K\}) \geq \epsilon$. Then

$$\begin{aligned} \int |f_K| &\geq \int_{\{|f_K| > K\}} |f_K| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M, \end{aligned}$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $f \in \mathcal{F}$, $\mu(\{|f| > K\}) < \epsilon$. Since $\mu(\{|f| > k\})$ is a decreasing sequence in k , we have that $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \mu(\{|f| > k\}) = 0$. Now, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Choose $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$, $\mu(\{|f| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$,

$$\int_{\{|f| > k\}} |f| < \epsilon.$$

Thus $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$ as required. □

3.4. Integration on Product Spaces.

Definition 3.41. Let X , Y , and Z be sets, $E \subset X \times Y$ and $f : X \times Y \rightarrow Z$. For each $x \in X$, define $E_x = \{y \in Y : (x, y) \in E\}$ and $f_x : Y \rightarrow Z$ by $f_x(y) = f(x, y)$. For each $y \in Y$, define $E^y = \{x \in X : (x, y) \in E\}$ and $f^y : X \rightarrow Z$ by $f^y(x) = f(x, y)$.

Note 3.42. It is often helpful to observe that $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.

Lemma 3.43. Let (X, \mathcal{A}) , (Y, \mathcal{B}) be measurable spaces, $Z = [0, \infty]$ or \mathbb{C} and $f : X \times Y \rightarrow Z$.

- (1) For each $E \in \mathcal{A} \otimes \mathcal{B}$, $x \in X$, $y \in Y$, we have that $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$
- (2) If f is $\mathcal{A} \otimes \mathcal{B}$ -measurable, then for each $x \in X$, $y \in Y$, we have that f_x is \mathcal{B} -measurable and f^y is \mathcal{A} -measurable.

Theorem 3.44. Let (X, \mathcal{A}, μ) , (Y, \mathcal{B}, ν) be σ -finite measure spaces. Then for each $E \in \mathcal{A} \otimes \mathcal{B}$, the maps $\phi : X \rightarrow [0, \infty]$ and $\psi : Y \rightarrow [0, \infty]$ defined by $\phi(x) = \nu(E_x)$ and $\psi(y) = \mu(E^y)$ are \mathcal{A} -measurable and \mathcal{B} -measurable, respectively and

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

Theorem 3.45. *Fubini, Tonelli:* Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ -finite measure spaces.

- (1) (Tonelli) For each $f \in L^+(X \times Y)$, the functions $g : X \rightarrow [0, \infty]$, $h : Y \rightarrow [0, \infty]$ defined by $g(x) = \int_Y f(x, y) d\nu(y)$ and $h(y) = \int_X f(x, y) d\mu(x)$ are \mathcal{A} -measurable and \mathcal{B} -measurable respectively and

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

- (2) (Fubini) For each $f \in L^1(X \times Y)$, $f_x \in L^1(\nu)$ for μ -a.e. $x \in X$ and $f^y \in L^1(\mu)$ for ν -a.e. $y \in Y$, respectively and the functions (after redefinition of f on a null set) $g : X \rightarrow \mathbb{C}$, $h : Y \rightarrow \mathbb{C}$ defined by $g(x) = \int_Y f(x, y) d\nu(y)$ and $h(y) = \int_X f(x, y) d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Furthermore

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

Note 3.46. We usually just write $\int \int f d\mu d\nu$ and $\int \int f d\nu d\mu$ instead of $\int h d\nu$ and $\int g d\mu$ respectively. We have a similar result for complete product measure spaces. See

Exercise 3.47. Take $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1])$, $\mathcal{B} = \mathcal{P}([0, 1])$ and μ, ν to be Lebesgue measure and counting measure respectively. Define $D = \{(x, y) \in [0, 1]^2 : x = y\}$ Show that

$$\int \chi_D d\mu \times \nu, \int \int \chi_D d\mu d\nu \text{ and } \int \int \chi_D d\nu d\mu$$

are all different. (Hint: for the first integral, use the definition of $\mu \times \nu$)

Proof. Let $x, y \in [0, 1]$. Then $(\chi_D)_x = \chi_{D_x} = \chi_x$ and $(\chi_D)^y = \chi_{D^y} = \chi_y$. Thus

$$\begin{aligned} \int \int \chi_D d\mu d\nu &= \int \mu(\{y\}) d\nu \\ &= \int 0 d\nu \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \int \int \chi_D d\mu d\nu &= \int \nu(\{x\}) d\mu \\ &= \int 1 d\mu \\ &= 1 \end{aligned}$$

Now, Observe that $\int \chi_D d\mu \times \nu = \mu \times \nu(D)$. Recall from the section on product measures that $\mu \times \nu(D) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) : (A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n\}$. Let $(A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$. Suppose that $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n$. Then for each $x \in [0, 1]$, $(x, x) \in \bigcup_{n \in \mathbb{N}} A_n \times B_n$. So for each $x \in [0, 1]$, there exists $n \in \mathbb{N}$, such that $x \in A_n \cap B_n$. Thus $[0, 1] \subset \bigcup_{n \in \mathbb{N}} A_n \cap B_n$. Since $1 = \mu([0, 1]) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap B_n)$, we know that there exists $n \in \mathbb{N}$ such that $0 < \mu(A_n \cap B_n)$. Thus $\mu(A_n) > 0$ and $\mu(B_n) > 0$. Since $\mu(B_n) > 0$, B_n must be infinite and therefore $\nu(B_n) = \infty$. So $\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) = \infty$.

□

Exercise 3.48. Let (X, \mathcal{A}, μ) be a σ -finite measure space and $f : X \rightarrow [0, \infty) \in L^+$. Show that $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ and $\mu \times m(G) = \int_X f d\mu$. The same is true if we replace " \geq " with " $>$ ". (Hint: to show that G is measurable, split up $(x, y) \mapsto f(x) - y$ into the composition of measurable functions.

Proof. Define $\phi : X \times [0, \infty) \rightarrow [0, \infty)^2$ and $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ by $\phi(x, y) = (f(x), y)$ and $\psi(z, y) = z - y$. Then $G = \{(x, y) \in X \times [0, \infty) : \psi \circ \phi(x, y) \geq 0\}$. Let $A, B \in \mathcal{B}([0, \infty))$. Then $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0, \infty))$. Since $\mathcal{B}([0, \infty)^2) = \mathcal{B}([0, \infty)) \otimes \mathcal{B}([0, \infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0, \infty))\})$, we have that ϕ is $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty)^2)$ measurable. Since ψ is continuous, we have that ψ is $\mathcal{B}([0, \infty)^2)$ - $\mathcal{B}([0, \infty))$ measurable. This implies that $\psi \circ \phi$ is $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty))$ measurable. Thus $G = \psi \circ \phi^{-1}([0, \infty)) \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$. Now for $x \in X$, $G_x = \{y \in [0, \infty) : f(x) \geq y\} = [0, f(x)]$. Thus

$$\begin{aligned} \mu \times m(G) &= \int \chi_G d\mu \times m \\ &= \int_X \int_{[0, \infty)} \chi_{G_x} dm d\mu(x) \\ &= \int_X f(x) d\mu(x) \end{aligned}$$

The same reasoning holds if we replace " \geq " with " $>$ ". □

Exercise 3.49. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ -finite measure spaces and $f : X \rightarrow \mathbb{C}, g : Y \rightarrow \mathbb{C}$. Define $h : X \times Y \rightarrow \mathbb{C}$ by $h(x, y) = f(x)g(y)$.

- (1) If f is \mathcal{A} -measurable and g is \mathcal{B} -measurable, then h is $\mathcal{A} \otimes \mathcal{B}$ -measurable.
- (2) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and

$$\int_{X \times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu$$

Proof. (1) First suppose that f, g are simple. Then there exist $(A_i)_{i=1}^n \subset \mathcal{A}, (B_j)_{j=1}^m \subset \mathcal{B}$ and $(a_i)_{i=1}^n, (b_j)_{j=1}^m \subset \mathbb{C}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$ and $g = \sum_{j=1}^m b_j \chi_{B_j}$. Then $h = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{A_i \times B_j}$. So h is $\mathcal{A} \otimes \mathcal{B}$ -measurable. For general f, g , there exist $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$ and $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$ such that $f_n \rightarrow f$ pointwise, $g_n \rightarrow g$ pointwise and for each $n \in \mathbb{N}$, $|f_n| \leq |f_{n+1}| \leq |f|$ and $|g_n| \leq |g_{n+1}| \leq |g|$. For $n \in \mathbb{N}$, define $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$ by $h_n = f_n g_n$. Then $h_n \rightarrow h$ pointwise and for each $n \in \mathbb{N}$, $|h_n| \leq |h_{n+1}| \leq |h|$. Thus h is $\mathcal{A} \otimes \mathcal{B}$ -measurable.

- (2) First suppose f and g are simple as before. Then

$$\begin{aligned} \int_{X \times Y} |h| d\mu \times \nu &\leq \sum_{i=1}^n \sum_{j=1}^m |a_i b_j| \mu(A_i) \nu(B_j) \\ &= \left(\sum_{i=1}^n |a_i| \mu(A_i) \right) \left(\sum_{j=1}^m |b_j| \nu(B_j) \right) \\ &= \int_X |f| d\mu \int_Y |g| d\nu \\ &< \infty \end{aligned}$$

So $h \in L^1(\mu \times \nu)$. Furthermore,

$$\begin{aligned} \int_{X \times Y} h d\mu \times \nu &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mu(A_i) \nu(B_j) \\ &= \left(\sum_{i=1}^n a_i \mu(A_i) \right) \left(\sum_{j=1}^m b_j \nu(B_j) \right) \\ &= \int_X f d\mu \int_Y g d\nu \end{aligned}$$

For general $f \in L^1(\mu), g \in L^1(\nu)$, take $(h_n)_{n \in \mathbb{N}}$ as before. Monotone convergence and the result above say that

$$\begin{aligned} \int_{X \times Y} |h| d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} |h_n| d\mu \times \nu \\ &= \lim_{n \rightarrow \infty} \left(\int_X |f_n| d\mu \int_Y |g_n| d\nu \right) \\ &= \int_X |f| d\mu \int_Y |g| d\nu \\ &< \infty \end{aligned}$$

So $h \in L^1(\mu \times \nu)$. Dominated convergence and the result above then tell us that

$$\begin{aligned} \int_{X \times Y} h d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} h_n d\mu \times d\nu \\ &= \lim_{n \rightarrow \infty} \left(\int_X f_n d\mu \int_Y g_n d\nu \right) \\ &= \int_X f d\mu \int_Y g d\nu \end{aligned}$$

□

Note 3.50. In the above exercise part (2), we can replace L^1 with L^+ and get the same result by the same method.

Exercise 3.51. Let $f : \mathbb{R} \rightarrow [0, \infty) \in L^+$. Show that

$$\int_{\mathbb{R}} f dm = \int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) dm(t)$$

Proof. Note that

$$\int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) = \int_{[0, \infty)} \left[\int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}} dm \right] dm(t)$$

Comparing this with Tonelli's theorem, we can put $\chi_{\{x \in \mathbb{R} : f(x) \geq t\}} = (\chi_E)^t = \chi_{E^t}$. Then $E = \{(x, t) \in \mathbb{R} \times [0, \infty) : f(x) \geq t\}$ and $E_x = \{t \in [0, \infty) : f(x) \geq t\} = [0, f(x)]$. Tonelli's

theorem tells us that

$$\begin{aligned} \int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}}(x) dm(x) \right] dm(t) &= \int_{\mathbb{R}} \left[\int_{[0,\infty)} \chi_{[0,f(x)]}(t) dm(t) \right] dm(x) \\ &= \int_{\mathbb{R}} f(x) dm(x) \end{aligned}$$

□

3.5. Convergence.

Definition 3.52. Let (X, \mathcal{A}) be a measurable space. For convenience we will define $L^0 = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable}\}$.

Definition 3.53. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **in measure** if for each $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \rightarrow 0$. This is written $f_n \xrightarrow{\mu} f$.

Definition 3.54. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **almost uniformly** if for each $\epsilon > 0$, there exists $N \in \mathcal{A}$ such that $\mu(N) < \epsilon$ and $f_n \xrightarrow{\text{uni}} f$ on N^c . This is written $f_n \xrightarrow{\text{a.u.}} f$.

Theorem 3.55. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$.

Theorem 3.56. (Egoroff): Suppose that $\mu(X) < \infty$. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{\text{a.e.}}$. Then $f_n \xrightarrow{\text{a.u.}} f$.

Exercise 3.57. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.

Proof. Let $\epsilon > 0$. for $n \in \mathbb{N}$, define $E_{\epsilon,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$. Then for $n \in \mathbb{N}$,

$$\begin{aligned} \int |f - f_n| &\geq \int_{E_{\epsilon,n}} |f - f_n| \\ &\geq \epsilon \mu(E_{\epsilon,n}). \end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$. Since $\int |f - f_n| \rightarrow 0$, we have that $\mu(E_{\epsilon,n}) \rightarrow 0$. Since $\epsilon > 0$ is arbitrary, $f_n \xrightarrow{\mu} f$ as required. □

Exercise 3.58. Suppose $\mu(X) < \infty$. Define $d : L^0 \times L^0 \rightarrow [0, \infty)$ by

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \quad f, g \in L^0$$

Then d is a metric on L^0 if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each $f, g \in L^0$, $d(f, g) \leq \mu(X)$.

Proof. Let $f, g \in L^0$. Clearly $d(f, g) = d(g, f)$. If $f = g$ a.e. then clearly $d(f, g) = 0$. Conversely, if $d(f, g) = 0$, then $\frac{|f - g|}{1 + |f - g|} = 0$ a.e and so $|f - g| = 0$ a.e. which implies $f = g$ a.e. It is not hard to show that $\phi : [0, \infty) \rightarrow [0, \infty)$ given by $\phi(x) = \frac{x}{1+x}$ satisfies $\phi(x + y) \leq \phi(x) + \phi(y)$. Thus satisfies the triangle inequality. Now, let $(f_n)_{n \in \mathbb{N}} \subset L^0$. Suppose that $f_n \not\xrightarrow{\mu} f$. Then there exists $\epsilon > 0, \delta > 0$ and a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that

for each $k \in \mathbb{N}$, $\mu(E_{\epsilon, n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$. It is not hard to show that ϕ from earlier is increasing. Thus for each $k \in \mathbb{N}$,

$$\begin{aligned} d(f_{n_k}, f) &= \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{\epsilon}{1 + \epsilon} \\ &\geq \frac{\epsilon \delta}{1 + \epsilon} \end{aligned}$$

So $f_{n_k} \not\stackrel{d}{\rightarrow} f$. Hence $f_{n_k} \stackrel{d}{\rightarrow} f$ implies that $f_{n_k} \stackrel{\mu}{\rightarrow} f$. Conversely, suppose that $f_{n_k} \stackrel{\mu}{\rightarrow} f$. Let $\epsilon > 0$. Then $\delta = \frac{\epsilon}{1 + \mu(X)} > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then $\mu(E_{\delta, n}) < \frac{\delta}{1 + \delta}$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Since ϕ is increasing and $\phi \leq 1$, we have that

$$\begin{aligned} d(f_n, f) &= \int \frac{|f_n - f|}{1 + |f_n - f|} \\ &= \int_{E_{\delta, n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E_{\delta, n}^c} \frac{|f_n - f|}{1 + |f_n - f|} \\ &\leq \mu(E_{\delta, n}) + \mu(X) \frac{\delta}{1 + \delta} \\ &< \frac{\delta}{1 + \delta} (1 + \mu(X)) \\ &\leq \delta (1 + \mu(X)) \\ &= \epsilon \end{aligned}$$

□

Exercise 3.59. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \stackrel{\mu}{\rightarrow} f$. Then $f \geq 0$ a.e. and $\int f \leq \liminf_{n \rightarrow \infty} \int f_n$.

Proof. Since $f_n \stackrel{\mu}{\rightarrow} f$, there is a subsequence converging to f a.e. So clearly $f \geq 0$ a.e. Now, choose a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $\int f_{n_k} \rightarrow \liminf_{n \rightarrow \infty} \int f_n$. Since $f_n \stackrel{\mu}{\rightarrow} f$ so does $(f_{n_k})_{k \in \mathbb{N}}$. Therefore there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ of $(f_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Thus $f \geq 0$ a.e. and Fatou's lemma tells us that

$$\begin{aligned} \int f &\leq \liminf_{j \in \mathbb{N}} \int f_{n_{k_j}} \\ &= \liminf_{n \rightarrow \infty} \int f_n. \end{aligned}$$

□

Exercise 3.60. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \stackrel{\mu}{\rightarrow} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.

Proof. Clearly $(f_n)_{n \in \mathbb{N}} \subset L^1$. Since $f_n \xrightarrow{\mu} f$, there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$. This implies that $|f| \leq g$ a.e. and so $f \in L^1$. For $n \in \mathbb{N}$, put $h_n = 2g - |f_n - f|$. Then for each $n \in \mathbb{N}$, $h_n \geq 0$ and $h_n \xrightarrow{\mu} 2g$. By the previous exercise

$$\begin{aligned} \int 2g &\leq \liminf_{n \rightarrow \infty} \int (2g - |f_n - f|) \\ &= \int 2g - \limsup_{n \rightarrow \infty} \int |f_n - f|. \end{aligned}$$

So $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \rightarrow 0$ and $f_n \xrightarrow{L^1} f$ as required. \square

Exercise 3.61. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \rightarrow \mathbb{C}$.

- (1) If ϕ is continuous, and $f_n \xrightarrow{\text{a.e.}} f$ then $\phi \circ f_n \xrightarrow{\text{a.e.}} \phi \circ f$.
- (2) If ϕ is uniformly continuous and $f_n \rightarrow f$ uniformly, almost uniformly or in measure, then $\phi \circ f_n \rightarrow \phi \circ f$ uniformly, almost uniformly or in measure, respectively.
- (3) Find a counter example to (2) if we drop the word "uniform".

Proof. (1) Clear

(2) Suppose that ϕ is uniformly continuous.

(uniform conv.) Suppose that $f_n \xrightarrow{\text{uni}} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Now choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq N$ then for each $x \in X$, $|f_n(x) - f(x)| < \delta$. Let $n \in \mathbb{N}$, suppose $n \geq N$. Let $x \in X$. Then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Thus $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$.

(almost uni.) Suppose that $f_n \xrightarrow{\text{a.u.}} f$. Let $\epsilon > 0$. Choose $N \in \mathcal{A}$ such $\mu(N) < \epsilon$ and $f_n \xrightarrow{\text{uni}} f$ on N^c . Then from above, we know that $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$ on N^c . Thus $\phi \circ f_n \xrightarrow{\text{a.u.}} \phi \circ f$.

(measure) Suppose that $f_n \xrightarrow{\mu} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Observe that for $x \in X$, if $|f_n(x) - f(x)| < \delta$, then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Hence $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$. By definition of convergence in measure, $\mu(F_{n,\delta}) \rightarrow 0$. Thus $\mu(E_{n,\epsilon}) \rightarrow 0$. Hence $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.

(3)

\square

Exercise 3.62. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{\text{a.u.}} f$. Then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\text{a.e.}} f$.

Proof. (measure) Let $\epsilon > 0$, $\delta > 0$. Choose $M \in \mathcal{A}$ such that $\mu(M) < \delta$ and $f_n \xrightarrow{\text{uni}} f$ on M^c . Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then for each $x \in M^c$, $|f_n(x) - f(x)| < \epsilon$. Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then $E_{\epsilon,n} \subset M$ and $\mu(E_{\epsilon,n}) < \delta$. Thus $\mu(E_{\epsilon,n}) \rightarrow 0$ and $f_n \xrightarrow{\mu} f$.

(a.e.) For each $n \in \mathbb{N}$, Choose $N_n \in \mathcal{A}$ such that $\mu(N_n) < 1/n$ and $f_n \xrightarrow{\text{uni}} f$ on N_n^c . Observe that for $x \in X$, if $x \in \bigcup_{n \in \mathbb{N}} N_n^c$, then $f_n(x) \rightarrow f(x)$. Thus $N = \{x \in X : f_n(x) \not\rightarrow f(x)\} \subset \bigcap_{n \in \mathbb{N}} N_n$. Therefore $\mu(N) = 0$ and $f_n \xrightarrow{\text{a.e.}} f$. \square

Exercise 3.63. Let $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then

- (1) $f_n + g_n \xrightarrow{\mu} f + g$
- (2) if $\mu(X) < \infty$, then $f_n g_n \xrightarrow{\mu} f g$

Proof. (1) Let $\epsilon > 0$. For convenience, put $F_{n,\epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$, $G_{n,\epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$, and $(F + G)_{n,\epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g(x))| \geq \epsilon\}$. Observe that for $x \in X$, $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$. Thus $(F + G)_{n,\epsilon} \subset F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$. Since $\mu(F_{n,\epsilon/2} \cup G_{n,\epsilon/2}) \leq \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}) \rightarrow 0$, we have that $\mu((F + G)_{n,\epsilon}) \rightarrow 0$. Hence $f_n + g_n \xrightarrow{\mu} f + g$.

- (2) Suppose that $\mu(X) < \infty$. Let $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(f_n g_n)_{n \in \mathbb{N}}$. Choose a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ and $g_{n_{k_j}} \xrightarrow{\text{a.e.}} g$. Then $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.e.}} f g$. Egoroff's theorem tells us that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.u.}} f g$, which implies that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} f g$. Thus for each subsequence $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ of $(f_n g_n)_{n \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ of $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} f g$. Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that $f_n g_n \xrightarrow{\mu} f g$. □

Exercise 3.64. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $\epsilon > 0$,

$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then $f_n \xrightarrow{\text{a.e.}} f$.

Proof. Let $\epsilon > 0$. By assumption we know that

$$\begin{aligned} \int \left[\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} \right] d\mu &= \sum_{n \in \mathbb{N}} \int \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} d\mu \\ &= \sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \\ &< \infty \end{aligned}$$

Thus we also know that $\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} < \infty$ a.e. Equivalently, we could say that for a.e. $x \in X$, $|\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}| < \infty$. For $k \in \mathbb{N}$, define $N_k = \{x \in X : \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > 1/k\}} = \infty\}$. Then for each $k \in \mathbb{N}$, $\mu(N_k) = 0$. Define $N = \bigcup_{k \in \mathbb{N}} N_k$. Then $\mu(N) = 0$. Let $x \in N^c$ and $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \epsilon$. Then $\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\} \subset \{n \in \mathbb{N} : |f_n(x) - f(x)| > 1/k\}$ which is finite because $x \in N_k^c$. Put $M = \max\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}$. Then for $m \geq M$, $|f_m(x) - f(x)| \leq \epsilon$. Thus $f_n(x) \rightarrow f(x)$. Hence $f_n \xrightarrow{\text{a.e.}} f$. □

4. DIFFERENTIATION

4.1. Signed Measures.

Definition 4.1. Let (X, \mathcal{A}) be a measurable space and $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$. Then ν is said to be a **signed measure** if

- (1) for each $E \in \mathcal{A}$, $\nu(E) < \infty$ or for each $E \in \mathcal{A}$, $\nu(E) > -\infty$.
- (2) $\nu(\emptyset) = 0$

(3) for each $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ if $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is disjoint, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$ and if $|\sum_{n \in \mathbb{N}} \nu(E_n)| < \infty$, then $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely.

Exercise 4.2. Let $\nu : \mathcal{A} \rightarrow [0, \infty]$ be a signed measure and $(E_n)_{n \in \mathbb{N}}, (F_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. If $(E_n)_{n \in \mathbb{N}}$ is increasing, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \rightarrow \infty} \nu(E_n)$. If $(F_n)_{n \in \mathbb{N}}$ is decreasing and $|\nu(E_1)| < \infty$, then $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$.

Proof. Put $E'_1 = E_1$, $F'_1 = F_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $E'_n = E_n \setminus E_{n-1}$ and $F'_n = F_1 \setminus F_n$. Then $(E'_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is disjoint. Thus

$$\begin{aligned} \nu(\bigcup_{n \in \mathbb{N}} E_n) &= \nu(\bigcup_{n \in \mathbb{N}} E'_n) \\ &= \sum_{n \in \mathbb{N}} \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \sum_{n=1}^n \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \nu(E_n) \end{aligned}$$

Since $(F'_n)_{n \in \mathbb{N}}$ is increasing, we now know that

$$\begin{aligned} \nu(F_1) - \nu(\bigcap_{n \in \mathbb{N}} F_n) &= \nu(F_1 \setminus \bigcap_{n \in \mathbb{N}} F_n) \\ &= \nu(\bigcup_{n \in \mathbb{N}} F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F_1 \setminus F_n) \\ &= \nu(F_1) - \lim_{n \rightarrow \infty} \nu(F_n) \end{aligned}$$

Since $|\nu(F_1)| < \infty$, we see that $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$. □

Definition 4.3. Let (X, \mathcal{A}) be a measurable space and $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$ a signed measure and $E \in \mathcal{A}$. Then E is said to be ν -**positive**, ν -**negative** and ν -**null** if for each $F \in \mathcal{A}$, $F \subset E$ implies that $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ respectively.

Exercise 4.4. Let $E \subset \mathcal{A}$. If E is positive, negative or null, then for each $F \in \mathcal{A}$, if $F \subset E$, then F is positive, negative or null respectively.

Proof. Clear □

Exercise 4.5. Let $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ be positive, negative or null. Then $\bigcup_{n \in \mathbb{N}} E_n$ is positive, negative or null respectively.

Proof. Suppose that $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is positive. Let $F \in \mathcal{A}$. Suppose that $F \subset \bigcup_{n \in \mathbb{N}} E_n$. Put

$P_1 = E_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $P_n = E_n \setminus (\bigcup_{j=1}^{n-1} E_j)$. So $\bigcup_{n \in \mathbb{N}} P_n = \bigcup_{n \in \mathbb{N}} E_n$ and $(P_n)_{n \in \mathbb{N}}$ is

disjoint. Thus

$$\begin{aligned}\nu(F) &= \nu\left(F \cap \bigcup_{n \in \mathbb{N}} P_n\right) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} (F \cap P_n)\right) \\ &= \sum_{n \in \mathbb{N}} \nu(F \cap P_n) \\ &\geq 0\end{aligned}$$

The process is the same if $(E_n)_{n \in \mathbb{N}}$ is negative and null. \square

Theorem 4.6. *Hahn Decomposition:* Let ν be a signed measure on (X, \mathcal{A}) . Then there exist $P, N \in \mathcal{A}$ such that P is positive, N is negative, $X = N \cup P$ and $N \cap P = \emptyset$. Furthermore, these two sets are unique in the following sense: For any $P', N' \in \mathcal{A}$, if N, P satisfy the properties above, $P' \Delta P = N' \Delta N$ is null.

Definition 4.7. Let ν be a signed measure on (X, \mathcal{A}) and $P, N \in \mathcal{A}$. Then P and N are said to form a **Hahn decomposition** of X with respect to ν if P, N satisfy the results in the above theorem.

Definition 4.8. Let μ, ν be signed measures on (X, \mathcal{A}) . Then μ and ν are said to be **mutually singular** if there exist $E, F \in \mathcal{A}$ such that $X = E \cup F$, $E \cap F = \emptyset$ and E is μ -null and F is ν -null. We will denote this by $\mu \perp \nu$.

Theorem 4.9. *Jordan Decomposition:* Let ν be a signed measure on (X, \mathcal{A}) . Then there exist unique positive measures ν^+ and ν^- on (X, \mathcal{A}) such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof. Choose a Hahn decomposition P, N of X with respect to ν . Define ν^+, ν^- by $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$. \square

Definition 4.10. Let ν be a signed measure on (X, \mathcal{A}) . Then ν^+ and ν^- from the last theorem are called the **positive** and **negative variations** of ν respectively. We define the **total variation** measure $|\nu|$ on (X, \mathcal{A}) by $|\nu| = \nu^+ + \nu^-$.

Definition 4.11. Let ν be a signed measure on (X, \mathcal{A}) . Then ν is said to be σ -finite if $|\nu|$ is σ -finite.

Exercise 4.12. Let ν be a signed measure and λ, μ positive measures on (X, \mathcal{A}) . Suppose that $\nu = \lambda - \mu$. Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. Choose a Hahn decomposition P, N of X with respect to ν . Let $E \in \mathcal{A}$. Then

$$\begin{aligned}\lambda(E \cap P) - \mu(E \cap P) &= \nu(E \cap P) \\ &= \nu^+(E \cap P)\end{aligned}$$

So $\lambda(E \cap P) \geq \nu^+(E \cap P)$ and therefore

$$\begin{aligned}\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) \\ &= \nu^+(E)\end{aligned}$$

Similarly $\mu(E \cap N) \geq \nu^-(E \cap N)$ and $\mu(E) \geq \nu^-(E)$. \square

Exercise 4.13. Let ν_1, ν_2 be signed measures on (X, \mathcal{A}) . Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Hint: use the last exercise)

Proof. Since

$$\begin{aligned}\nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\ &= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)\end{aligned}$$

the previous exercise tells us that $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$ and $\mu = \nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$. Therefore

$$\begin{aligned}|\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ &\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\ &= |\nu_1| + |\nu_2|\end{aligned}$$

□

Note 4.14. Recall that a previous exercise from the section on complex valued functions tells us that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.

Definition 4.15. Let ν be a signed measure on (X, \mathcal{A}) . Then we define $L^1(\nu) = L^1(|\nu|)$. For $f \in L^1(\nu)$, we define

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

Exercise 4.16. Let ν_1, ν_2 be signed measures on (X, \mathcal{A}) . Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$

Proof. The previous exercise tells us that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. Two previous exercises from the section on nonnegative functions tells us that

$$\begin{aligned}\int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\ &= \int |f| d|\nu_1| + \int |f| d|\nu_2|\end{aligned}$$

□

Exercise 4.17. Let ν, μ be signed measures on (X, \mathcal{A}) and $E \in \mathcal{A}$. Then

- (1) E is ν -null iff $|\nu|(E) = 0$
- (2) $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. (1) Suppose that E is ν -null. Choose a Hahn decomposition P, N of X with respect to ν . Then $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = \nu(E \cap N) = 0$. Therefore $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Conversely, suppose that $|\nu|(E) = 0$. Then $\nu^+(E) = \nu^-(E) = 0$. Let $F \in \mathcal{A}$. Suppose that $F \subset E$. Then $\nu^+(F) = 0$ and $\nu^-(F) = 0$. Therefore $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. So E is ν -null.

- (2) Suppose that $\nu \perp \mu$. Then there exist $E, F \in \mathcal{A}$ such that $E \cup F = X$, $E \cap F = \emptyset$, E is μ -null and F is ν -null. By (1), F is $|\nu|$ -null and thus $|\nu| \perp \mu$. If $|\nu| \perp \mu$, choose $E, F \in \mathcal{A}$ as before. Since F is $|\nu|$ -null, we know that $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$. This implies that F is ν^+ -null and F is ν^- -null. So $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Finally assume that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. **FINISH!!!!**

□

Exercise 4.18. Let ν be a signed measure on (X, \mathcal{A}) . Then

- (1) for $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$
- (2) if ν is finite, then for each $E \in \mathcal{A}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : f \text{ is measurable and } |f| \leq 1\}$

Proof. (1) Let $f \in L^1(\nu)$. Then

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f d\nu^+ - \int f d\nu^- \right| \\ &\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \\ &= \int |f| d(\nu^+ + \nu^-) \\ &= \int |f| d|\nu| \end{aligned}$$

- (2) Let $E \in \mathcal{A}$. Let $f : X \rightarrow \mathbb{R}$ be measurable and suppose that $|f| \leq 1$. Since ν is finite, so is $|\nu|$ and thus $f \in L^1(\nu)$. Then (1) tells us that

$$\begin{aligned} \left| \int_E f d\nu \right| &\leq \int_E |f| d|\nu| \\ &\leq |\nu|(E) \end{aligned}$$

Now, choose a Hahn decomposition P, N of X with respect to ν . Define $f = \chi_P - \chi_N$. Then $|f| \leq 1$, f is measurable and

$$\begin{aligned} \left| \int_E f d\nu \right| &= \left| \int_E f d\nu^+ - \int_E f d\nu^- \right| \\ &= |\nu^+(E \cap P) + \nu^-(E \cap N)| \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E). \end{aligned}$$

□

Exercise 4.19. Let μ be a positive measure on (X, \mathcal{A}) and $f \in L^0(X, \mathcal{A})$ extended μ -integrable. Define ν on (X, \mathcal{A}) by $\nu(E) = \int_E f d\mu$. Then

- (1) ν is a signed measure
- (2) for each $E \in \mathcal{A}$, $|\nu|(E) = \int_E |f| d\mu$.

Proof. (1) Clearly $\nu(\emptyset) = 0$ and ν is finite by assumption. Let $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_n)_{n \in \mathbb{N}}$ is disjoint. Then

$$\begin{aligned}
 \nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \\
 &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu - \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
 &= \sum_{n \in \mathbb{N}} \left[\int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right] \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n} f d\mu \\
 &= \sum_{n \in \mathbb{N}} \nu(E_n)
 \end{aligned}$$

If $|\nu(\bigcup_{n \in \mathbb{N}} E_n)| < \infty$, then $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu < \infty$ and $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu < \infty$ because

$$\begin{aligned}
 |\nu(\bigcup_{n \in \mathbb{N}} E_n)| &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \right| \\
 &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \right|
 \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
 \sum_{n \in \mathbb{N}} |\nu(E_n)| &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f d\mu \right| \\
 &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right| \\
 &\leq \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu + \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
 &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu + \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
 &< \infty
 \end{aligned}$$

So the sum $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely and ν is a signed measure.

- (2) Put $P = \{x \in X : f(x) \geq 0\}$ and $N = \{x \in X : f(x) < 0\}$. Then P, N form a Hahn decomposition of X with respect to ν . Thus for $E \in \mathcal{A}$,

$$\nu^+(E) = \int_{E \cap P} f d\mu = \int_E f^+ d\mu$$

and

$$\nu^-(E) = \int_{E \cap N} f d\mu = \int_E f^- d\mu$$

. So for $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$$

□

4.2. The Lebesgue-Radon-Nikodym Theorem.

Definition 4.20. Let (X, \mathcal{A}) be a measurable space, ν be a signed measure on (X, \mathcal{A}) and μ a measure on (X, \mathcal{A}) . Then ν is said to be **absolutely continuous** with respect to μ , denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.

Note 4.21. If there exists an extended μ -integrable $f \in L^0(X, \mathcal{A})$ such that for each $E \in \mathcal{A}$, $\nu(E) = \int_E f d\mu$, then we write $d\nu = f d\mu$.

Theorem 4.22. Let (X, \mathcal{A}) be a measurable space, ν be a σ -finite signed measure on (X, \mathcal{A}) and μ a σ -finite measure on (X, \mathcal{A}) . Then there exist unique σ -finite signed measures λ, ρ on (X, \mathcal{A}) such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists an extended μ -integrable $f \in L^0(X, \mathcal{A})$ such that $d\rho = f d\mu$ and f is unique μ -a.e.

Definition 4.23. The decomposition $\nu = \lambda + \rho$ is referred to as the **Lebesgue decomposition of ν with respect to μ** . In the case $\nu \ll \mu$, we have $\lambda = 0$ and $\rho = \nu$ and we define the **Radon-Nikodym derivative of ν with respect to μ** , denoted by $d\nu/d\mu$, to be $d\nu/d\mu = f$ where $d\nu = f d\mu$.

Theorem 4.24. Let ν be a σ -finite signed measure on (X, \mathcal{A}) and μ, λ σ -finite measures on (X, \mathcal{A}) . Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then

(1) for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2) $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Exercise 4.25. Let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence of measures and μ a measure.

(1) If for each $n \in \mathbb{N}$, $\nu_n \ll \mu$, then $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$.

(2) If for each $n \in \mathbb{N}$, $\nu_n \perp \mu$, then $\sum_{n \in \mathbb{N}} \nu_n \perp \mu$.

Proof. (1) Let $E \in \mathcal{A}$. Suppose that $\mu(E) = 0$. Then for each $n \in \mathbb{N}$, $\nu_n(E) = 0$ and thus $\sum_{n \in \mathbb{N}} \nu_n(E) = 0$. Hence $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$.

(2) For each $n \in \mathbb{N}$, there exist $N_i, M_i \in \mathcal{A}$ such that $N_i \cap M_i = \emptyset$, $N_i \cup M_i = X$ and $\nu_i(M_i) = \mu(N_i) = 0$. Put $N = \bigcup_{n \in \mathbb{N}} N_i$ and $M = N^c$. Note that for each $n \in \mathbb{N}$, $M \subset N_i^c = M_i$. So $\mu(N) \leq \sum_{n \in \mathbb{N}} \mu(N_i) = 0$ and $(\sum_{n \in \mathbb{N}} \nu_i)(M) \leq \sum_{n \in \mathbb{N}} \nu_i(M_i) = 0$. Thus $\sum_{n \in \mathbb{N}} \nu_i \perp \mu$.

□

Exercise 4.26. Choose $X = [0, 1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$. Let m be Lebesgue measure and μ the counting measure.

Then

- (1) $m \ll \mu$ but for each $f \in L^+$, $dm \neq fd\mu$
- (2) There is no Lebesgue decomposition of μ with respect to m .

Proof. (1) Let $E \in \mathcal{A}$. If $\mu(E) = 0$, then $E = \emptyset$ and $m(E) = 0$. So $m \ll \mu$. Suppose for the sake of contradiction that there exists $f \in L^+$ such that $dm = fd\mu$. Then

$$\begin{aligned} 1 &= m(X) \\ &= \sum_{x \in X} f(x) \end{aligned}$$

Put $Z = \{x \in X : f(x) \neq 0\}$. Then Z is countable. So

$$\begin{aligned} 1 &= m(X \setminus Z) \\ &= \sum_{x \in X \setminus Z} f(x) \\ &= 0 \end{aligned}$$

This is a contradiction, so no such f exists.

- (2) Suppose for the sake of contradiction that there is a Lebesgue decomposition for μ with respect to m given by $\mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho \ll m$. We may assume λ and ρ are positive. Then for each $x \in X$, $m(\{x\}) = 0$ which implies that $\rho(\{x\}) = 0$. Let $E \subset X$, if E is countable, then $\lambda(E) = \mu(E)$. If E is uncountable, choose $F \subset E$ such that F is countable. Then

$$\begin{aligned} \lambda(E) &\geq \lambda(F) \\ &= \mu(F) \\ &= \infty \end{aligned}$$

So $\lambda = \mu$. This is a contradiction since $\mu \not\ll m$.

□

Exercise 4.27. Let (X, \mathcal{F}, μ) be a measure space and \mathcal{E} a sub σ -alg of \mathcal{F} and $f \in L^1(\mu)$. Define $\nu : \mathcal{E} \rightarrow [0, \infty]$ by $\nu(E) = \int_E f d\mu$. Then ν is σ -finite. Let $\bar{\mu}$ be the restriction of μ to \mathcal{E} . So $\nu \ll \bar{\mu}$. Define the **expectation of f given \mathcal{E}** to be $E[f|\mathcal{E}] = d\nu/d\bar{\mu} \in L^1(X, \mathcal{F}, \bar{\mu})$. Then for each $E \in \mathcal{E}$,

$$\int_E E[f|\mathcal{E}] d\mu = \int_E f d\mu$$

Proof. Let $E \in \mathcal{E}$. By definition,

$$\begin{aligned} \int_E E[f|\mathcal{E}] d\mu &= \int_E d\nu/d\bar{\mu} d\mu \\ &= \int_E d\nu/d\bar{\mu} d\bar{\mu} \quad (\text{since } E \in \mathcal{E}) \\ &= \nu(E) \\ &= \int_E f d\mu \end{aligned}$$

□

4.3. Complex Measures.

Definition 4.28. Let (X, \mathcal{A}) be a measurable space and $\nu : \mathcal{A} \rightarrow \mathbb{C}$. Then ν is said to be a **complex measure** if

- (1) $\nu(\emptyset) = 0$
- (2) for each sequence $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, if $(E_n)_{n \in \mathbb{N}}$ is disjoint, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$ and $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely.

Note 4.29. We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.

Definition 4.30. Let (X, \mathcal{A}) be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on (X, \mathcal{A}) . We define $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$. For $f \in L^1(\nu)$, we define

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

Theorem 4.31. Let (X, \mathcal{A}) be a measurable space, ν a complex measure on (X, \mathcal{A}) and μ a σ -finite measure on (X, \mathcal{A}) . Then there exists a complex measure λ on (X, \mathcal{A}) and $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$ and such that for each complex measure λ' on (X, \mathcal{A}) , $f' \in L^1(\mu)$, if $\nu = d\lambda' + f' d\mu$, then $\lambda = \lambda'$ and $f = f'$ μ -a.e.

Theorem 4.32. Let ν be a complex measure on (X, \mathcal{A}) and μ, λ σ -finite measures on (X, \mathcal{A}) . Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then

- (1) for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

- (2) $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Definition 4.33. Let (X, \mathcal{A}) be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on (X, \mathcal{A}) . Define $\mu = |\nu_1| + |\nu_2|$. Then $\nu \ll \mu$ and thus There exists $f \in L^1(\mu)$ such that $d\nu = f d\mu$. Define $|\nu| : \mathcal{A} \rightarrow [0, \infty)$ by $|\nu|(E) = \int_E |f| d\mu$ for each $E \in \mathcal{A}$. We call $|\nu|$ the **total variation of ν** .

Exercise 4.34. Let ν be a complex measure on (X, \mathcal{A}) and μ a σ -finite measures on (X, \mathcal{A}) . If $\nu \ll \mu$, then $\{x \in X : d\nu/d\mu(x) = 0\}$ is ν -null.

Proof. Define $f = d\nu/d\mu$ and $E = \{x : f(x) = 0\}$. Let $A \in \mathcal{A}$ and suppose that $A \subset E$. Then

$$\begin{aligned} \nu(A) &= \int_A f d\mu \\ &= 0 \end{aligned}$$

□

Exercise 4.35. Let (X, \mathcal{A}) be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on (X, \mathcal{A}) . Then $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu_2|$.

Proof. Let μ and f be as in the definition of $|\nu|$. Since for each $E \in \mathcal{A}$, we have

$$\begin{aligned}\nu(E) &= \int_E f d\mu \\ &= \int_E f_1 d\mu + i \int_E f_2 d\mu\end{aligned}$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E)$$

we know that $\nu_1 = f_1 d\mu$ and $\nu_2 = f_2 d\mu$.

A previous exercise tells us that $d|\nu_1| = |f_1| d\mu$ and $d|\nu_2| = |f_2| d\mu$. Since $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$, we have that

$$\begin{aligned}|\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2|\end{aligned}$$

□

Exercise 4.36. Let (X, \mathcal{A}) be a measurable space, ν a complex measure on (X, \mathcal{A}) and $c \in \mathbb{C}$. Then $|c\nu| = |c||\nu|$.

Proof. Define μ and f as before so that $d\nu = f d\mu$. Then $d(c\nu) = cf d\mu$. Hence

$$\begin{aligned}d|c\nu| &= |cf| d\mu \\ &= |c||f| d\mu \\ &= |c| d|\nu|\end{aligned}$$

So $|c\nu| = |c||\nu|$.

□

Exercise 4.37. Let (X, \mathcal{A}) be a measurable space and ν a complex measure on (X, \mathcal{A}) . Then

- (1) for each $E \in \mathcal{A}$, $|\nu(E)| \leq |\nu|(E)$.
- (2) $\nu \ll |\nu|$ and $|d\nu/d|\nu|| = 1$ $|\nu|$ -a.e.
- (3) $L^1(\nu) = L^1(|\nu|)$ and for each $g \in L^1(\nu)$, $|\int g d\nu| \leq \int |g| d|\nu|$

Proof. Let $\mu, f \in L^1(\mu)$ be as in the definition of $|\nu|$.

- (1) Let $E \in \mathcal{A}$. Then

$$\begin{aligned}|\nu(E)| &= \left| \int_E f d\mu \right| \\ &\leq \int_E |f| d\mu \\ &= |\nu|(E)\end{aligned}$$

- (2) Let $E \in \mathcal{A}$ and suppose that $|\nu|(E) = 0$. The previous part implies $|\nu(E)| = 0$ and $\nu \ll |\nu|$. Put $g = d\nu/d|\nu|$. Then

$$\begin{aligned}f &= \frac{d\nu}{d\mu} \\ &= g|f| \quad \mu\text{-a.e.}\end{aligned}$$

Hence $|f| = |g||f|$ μ -a.e. Since $|\nu| \ll \mu$, $|f| = |g||f|$ $|\nu|$ -a.e.

A previous exercise tells us that $|f| \neq 0$ $|\nu|$ -a.e. Thus $|g| = 1$ $|\nu|$ -a.e.

(3) Write $\nu = \nu_1 + i\nu_2$ and $f = f_1 + if_2$. First we observe that

$$\begin{aligned} L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\ &= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\ &= L^1(|\nu_1| + |\nu_2|) \\ &= L^1(\mu) \end{aligned}$$

The previous exercise tells us that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \\ &= \mu \end{aligned}$$

Let $g \in L^1(\mu)$. Then

$$\begin{aligned} \int |g|d|\nu| &\leq \int |g|d\mu \\ &< \infty \end{aligned}$$

So $g \in L^1(|\nu|)$.

Conversely, let $g \in L^1(|\nu|)$. Then

$$\begin{aligned} \int |g|d|\nu_1|, \int |g|d|\nu_2| &\leq \int |g|d|\nu| \\ &< \infty \end{aligned}$$

So

$$\begin{aligned} \int |g|d\mu &= \int |g|d|\nu_1| + \int |g|d|\nu_2| \\ &< \infty \end{aligned}$$

and $g \in L^1(\mu)$. Hence $L^1(\nu) = L^1(|\nu|)$.

Now, let $g \in L^1(\nu) = L^1(|\nu|)$, then

$$\begin{aligned} \left| \int g d\nu \right| &= \left| \int g f d\mu \right| \\ &\leq \int |g| |f| d\mu \\ &= \int |g| d|\nu| \end{aligned}$$

□

4.4. Differentiation.

Definition 4.38. Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$. Then f is said to be **locally integrable** (with respect to Lebesgue measure) if f is measurable and for each $K \subset \mathbb{R}^n$, K is compact implies $\int_K |f| dm < \infty$. We define $L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{C} : f \text{ is locally integrable}\}$

Definition 4.39. For $f \in L^1_{loc}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$, we define the **average of f over $B(x, r)$** , denoted by $Af(x, r)$, to be

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f dm$$

Exercise 4.40. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Define

$$H^*f(x) = \sup \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\} \quad (x \in \mathbb{R}^n)$$

Then $Hf \leq H^*f \leq 2^n Hf$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\left\{ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f| dm : r > 0 \right\} \subset \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\}$$

So $Hf(x) \leq H^*f(x)$. Let B be a ball. Then there exists $y \in \mathbb{R}^n$, $R > 0$ such that $B = B(y, R)$. Suppose that $x \in B$. Then $B \subset B(x, 2R)$. Since $m(B(x, 2R)) = 2^n m(B(y, R))$, we have that

$$\begin{aligned} \frac{1}{m(B)} \int_B |f| dm &\leq \frac{1}{m(B)} \int_{m(B(x, 2R))} |f| dm \\ &= \frac{2^n}{m(B(x, 2R))} \int_{m(B(x, 2R))} |f| dm \end{aligned}$$

Thus $H^*f(x) \leq 2^n Hf(x)$. □

Lemma 4.41. Let $f \in L^1_{loc}(\mathbb{R}^n)$, then $Af : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ is continuous.

Definition 4.42. Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define its **Hardy Littlewood maximal function**, denoted by Hf to be

$$Hf(x) = \sup_{r>0} A|f|(x, r) \quad x \in \mathbb{R}^n$$

Theorem 4.43. There exists $C > 0$ such that for each $f \in L^1(m)$ and $\alpha > 0$,

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f| dm$$

Exercise 4.44. Let $f \in L^1(\mathbb{R}^n)$. Suppose that $\|f\|_1 > 0$. Then there exist $C, R > 0$ such that for each $x \in \mathbb{R}^n$, if $|x| > R$, then $Hf(x) \geq C|x|^{-n}$. Hence there exists $C' > 0$ such that for each $\alpha > 0$, $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$ when α is small.

Proof. Since $\|f\|_1 > 0$, there exists $R > 0$ such that $\int_{B(0, R)} |f| dm > 0$. Recall that there exists $K > 0$ such that for each $x \in \mathbb{R}^n$ and $r > 0$, $m(B(x, r)) = Kr^n$. Choose

$$C = \frac{\int_{B(0, R)} |f| dm}{K2^n}$$

. Let $x \in \mathbb{R}^n$. Suppose that $|x| > R$. Then $B(0, R) \subset B(x, 2|x|)$. Thus

$$\begin{aligned} Hf(x) &\geq \frac{1}{m(B(x, 2|x|))} \int_{B(x, 2|x|)} |f| dm \\ &= \frac{1}{K2^n|x|^n} \int_{B(x, 2|x|)} |f| dm \\ &\geq \frac{1}{K2^n|x|^n} \int_{B(0, R)} |f| dm \\ &= \frac{C}{|x|^n} \end{aligned}$$

Let $a < \frac{C}{2R^n}$. Then $R^n < \frac{C}{2a}$. Choose $C' = \frac{KC}{2}$. Let $A = \{x \in \mathbb{R}^n : R < |x| < (\frac{C}{a})^{\frac{1}{n}}\}$. For $x \in A$,

$$\begin{aligned} Hf(x) &\geq \frac{C}{|x|^n} \\ &> a \end{aligned}$$

Thus $A \subset m(\{x \in \mathbb{R}^n : Hf(x) > a\})$ and therefore

$$\begin{aligned} m(\{x \in \mathbb{R}^n : Hf(x) > a\}) &\geq m(A) \\ &= m(B(0, (C/a)^{1/n})) - m(B(0, R)) \\ &= K \left[\frac{C}{a} - R^n \right] \\ &> K \left[\frac{C}{a} - \frac{C}{2a} \right] \\ &= \frac{KC}{2a} \\ &= \frac{C'}{a} \end{aligned}$$

□

Theorem 4.45. Let $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \rightarrow 0} Af(x, r) = f(x)$$

. Equivalently, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \rightarrow 0} \left[\frac{1}{m(B(x, r))} \int_{B(x, r)} [f(y) - f(x)] dm(y) \right] = 0$$

Note 4.46. We can a stronger result of the same flavor.

Definition 4.47. Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define the **Lebesgue set of f** , denoted by L_f , to be

$$\begin{aligned} L_f &= \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0} A|f - f(x)|(x, r) = 0\} \\ &= \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \left[\frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0 \right\} \end{aligned}$$

Exercise 4.48. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. If f is continuous at x , then $x \in L_f$.

Proof. Suppose that f is continuous at x . Let $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that for each $y \in \mathbb{R}^n$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Let $r > 0$. Suppose that $r < \delta$. Then for each $y \in \mathbb{R}^n$, $y \in B(x, r)$ implies that $|f(x) - f(y)| < \epsilon$ and thus

$$\begin{aligned} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(B(x, r))} \epsilon m(B(x, r)) \\ &= \epsilon \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0} \left[\frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0$$

and $x \in L_f$. □

Theorem 4.49. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then $m((L_f)^c) = 0$

Definition 4.50. Let $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then $(E_r)_{r>0}$ is said to **shrink nicely** to x if

- (1) for each $r > 0$, $E_r \subset B(x, r)$
- (2) there exists $\alpha > 0$ such that for each $r > 0$, $m(E_r) > \alpha m(B(x, r))$

Theorem 4.51. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then for each $x \in L_f$,

$$\lim_{r \rightarrow 0} \left[\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \right] = 0$$

and

$$\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

Definition 4.52. Let $\mu : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, \infty]$ be a Borel measure. Then μ is said to be **regular** if

- (1) for each $K \subset \mathbb{R}^n$, if K is compact, then $\mu(K) < \infty$
- (2) for each $E \in \mathcal{B}(\mathbb{R}^n)$, $\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subset U\}$

Let ν be a signed or complex Borel measure on \mathbb{R}^n . Then ν is said to be regular if $|\nu|$ is regular.

Theorem 4.53. Let ν be a regular signed or complex measure on \mathbb{R}^n . Let $d\nu = d\lambda + f dm$ be the Lebesgue decomposition of ν with respect to m . Then for m -a.e. $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$, if $(E_r)_{r>0}$ shrinks nicely to x , then

$$\lim_{r \rightarrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

4.5. Functions of Bounded Variation.

Definition 4.54. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Define $F_+ : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F_+(x) = \lim_{t \rightarrow x^+} F(t) = \inf\{F(t) : t > x\}$$

Note 4.55. Observe that $F \leq F_+$ and F_+ is increasing.

Exercise 4.56. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Then for each $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in (x, x + \delta)$, $0 \leq F_+(y) - F(y) \leq \epsilon$.

Proof. For the sake of contradiction, suppose not. Then there exists $x \in \mathbb{R}$ and $\epsilon > 0$ such that for each $\delta > 0$, there exist $y \in (x, x + \delta)$ such that $F_+(y) - F(y) > \epsilon$. Then there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that for each $n \in \mathbb{N}$, $y_n \in (x, x + \frac{1}{n})$, $y_n > y_{n+1}$ and $F_+(y_n) - F(y_n) > \epsilon$. Choose $N \in \mathbb{N}$ such that $(N - 1)\epsilon > F(y_1) - F(x)$. Then

$$\begin{aligned} F(y_1) - F(x) &= \sum_{i=1}^{N-1} \left[F(y_i) - F_+(y_{i+1}) + F_+(y_{i+1}) - F(y_{i+1}) \right] + F(y_N) - F(x) \\ &= \sum_{i=1}^{N-1} \left[F(y_i) - F_+(y_{i+1}) \right] + \sum_{i=1}^{N-1} \left[F_+(y_{i+1}) - F(y_{i+1}) \right] + F(y_N) - F(x) \\ &\geq (N - 1)\epsilon \\ &> F(y_1) - F(x) \end{aligned}$$

This is a contradiction, so the claim holds. \square

Exercise 4.57. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Then F_+ is right continuous.

Proof. Let $x \in \mathbb{R}$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that for each $y \in (x, x + \delta_1)$ $0 \leq F(y) - F_+(x) < \epsilon/2$. There exists $\delta_2 > 0$ such that for each $y \in (x, x + \delta_2)$, $0 \leq F_+(y) - F(y) < \epsilon/2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Let $y \in (x, x + \delta)$.

$$\begin{aligned} |F_+(x) - F_+(y)| &\leq |F_+(x) - F(y)| + |F(y) - F_+(y)| \\ &= (F(y) - F_+(x)) + (F_+(y) - F(y)) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So $\lim_{t \rightarrow x^+} F_+(t) = F_+(x)$ and F_+ is right continuous. \square

Theorem 4.58. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Then

- (1) $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$ is countable
- (2) F and F_+ are differentiable a.e. and $F' = F'_+$ a.e.

Definition 4.59. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Define $T_F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_F(x) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \quad (x \in \mathbb{R})$$

T_F is called the **total variation function of F** .

Exercise 4.60. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Then T_F is increasing.

Proof. Let $x, y \in \mathbb{R}$. Suppose that $x < y$.

Define $A_x = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$ and $A_y = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = y \right\}$. Let $z \in A_x$. Then there exists $(x_i)_{i=0}^n \subset \mathbb{R}$ such that $(x_i)_{i=0}^n$ is increasing, $x_n = x$ and $z = \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$. Then

$$\begin{aligned}
z &\leq z + |F(y) - F(x)| \\
&= \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \\
&\in A_y
\end{aligned}$$

So $z \leq \sup A_y = T_F(y)$ and thus $F_T(x) = \sup A_x \leq T_F(y)$ \square

Lemma 4.61. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$. Then $T_F + F$ and $T_F - F$ are increasing.*

Exercise 4.62. *For each $F : \mathbb{R} \rightarrow \mathbb{C}$, $T_{|F|} \leq T_F$.*

Proof. Let $F : \mathbb{R} \rightarrow \mathbb{C}$, $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then by the reverse triangle inequality,

$$\sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

Thus

$$\begin{aligned}
T_{|F|}(x) &= \sup \left\{ \sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&\leq \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&= T_F(x)
\end{aligned}$$

Hence $T_{|F|} \leq T_F$ \square

Definition 4.63. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Then F is said to have **bounded variation** if $\lim_{x \rightarrow \infty} T_F(x) < \infty$. The **total variation of F** , denoted by $TV(F)$, is defined to be $TV(F) = \lim_{x \rightarrow \infty} T_F(x)$. We define $BV = \{F : \mathbb{R} \rightarrow \mathbb{C} : TV(F) < \infty\}$.*

Definition 4.64. *Let $a, b \in \mathbb{R}$ and $F : [a, b] \rightarrow \mathbb{C}$. Define $G_F : \mathbb{R} \rightarrow \mathbb{C}$ by $G_F = F(a)\chi_{(-\infty, a)} + F\chi_{[a, b]} + F(b)\chi_{(b, \infty)}$. Then F is said to have **bounded variation on $[a, b]$** if $G_F \in BV$. The **total variation of F on $[a, b]$** , denoted by $TV(F, [a, b])$, is defined to be $TV(F, [a, b]) = TV(G_F)$. We define $BV([a, b]) = \{F : [a, b] \rightarrow \mathbb{C} : TV(F, [a, b]) < \infty\}$.*

Note 4.65. *Equivalently, $TV(F, [a, b]) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset [a, b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$ and $F \in BV([a, b])$ iff $TV(F, [a, b]) < \infty$. In general,*

Exercise 4.66. *Let $F \in BV$. Then F is bounded.*

Proof. If F is unbounded, then the supremum in the previous definition is clearly infinite. \square

Exercise 4.67. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$. If F is bounded and increasing, then $F \in BV$.*

Proof. Suppose that F is bounded and increasing. Then $-\infty < \inf_{x \in \mathbb{R}} F(x) \leq \sup_{x \in \mathbb{R}} F(x) < \infty$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n F(x_i) - F(x_{i-1}) \\ &= F(x) - F(x_0) \end{aligned}$$

Thus

$$T_F(x) = F(x) - \inf_{x \in \mathbb{R}} F(x)$$

. This implies that

$$\begin{aligned} TV(F) &= \sup_{x \in \mathbb{R}} F(x) - \inf_{x \in \mathbb{R}} F(x) \\ &< \infty \end{aligned}$$

Hence $F \in BV$. □

Exercise 4.68. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. If F is differentiable and F' is bounded on $[a, b]$, then, $F \in BV([a, b])$.

Proof. Suppose that F is differentiable and F' is bounded on $[a, b]$. Then there exists $M > 0$ such that for each $x \in [a, b]$, $|F'(x)| \leq M$. Let $(x_i)_{i=1}^n \subset [a, b]$. Suppose that $(x_i)_{i=1}^n$ is strictly increasing, $x_0 = a$ and $x_n = b$. By the mean value theorem, for each $i = 1, 2, \dots, n$, there exists $c_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$. Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n |F'(c_i)(x_i - x_{i-1})| \\ &\leq \sum_{i=1}^n M(x_i - x_{i-1}) \\ &= M(b - a) \end{aligned}$$

Hence $TV(F, [a, b]) \leq M(b - a)$. □

Exercise 4.69. Define $F, G : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x) = \begin{cases} x^2 \sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 \sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then F and G are differentiable, $F \in BV([-1, 1])$ and $G \notin BV([-1, 1])$.

Proof. On $\mathbb{R} \setminus \{0\}$,

$$\begin{aligned} F'(x) &= 2x \sin(x^{-1}) - \sin(x^{-1}) \\ &= \sin(x^{-1})(2x - 1) \end{aligned}$$

We see that F is also differentiable at $x = 0$ since

$$\begin{aligned} F'(0) &= \lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-1})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-1}) \\ &= 0 \end{aligned}$$

Therefore for each $x \in [-1, 1]$, $|F'(x)| \leq 3$. Which by a previous exercise implies that $F \in BV([-1, 1])$.

On $\mathbb{R} \setminus \{0\}$,

$$\begin{aligned} G'(x) &= 2x \sin(x^{-2}) - \frac{2 \sin(x^{-2})}{x} \\ &= \sin(x^{-2}) \left(2x - \frac{2}{x} \right) \end{aligned}$$

We see that G is also differentiable at $x = 0$ since

$$\begin{aligned} G'(0) &= \lim_{x \rightarrow 0} \frac{G(x) - G(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-2})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-2}) \\ &= 0 \end{aligned}$$

For $n \in \mathbb{N}$, define $(x_i)_{i=0}^n \subset [-1, 1]$ by

$$x_i = \frac{-1}{\sqrt{\frac{\pi}{2} + i\pi}}$$

Then for each $n \in \mathbb{N}$, $(x_i)_{i=1}^n$ is strictly increasing and for each $i = 1, 2, \dots, n$ we have that

$$\begin{aligned} |G(x_i) - G(x_{i-1})| &= \frac{1}{\frac{\pi}{2} + i\pi} + \frac{1}{\frac{\pi}{2} + (i-1)\pi} \\ &= \frac{2}{\pi} \left[\frac{(2i-1) + (2i+1)}{(2i+1)(2i-1)} \right] \\ &= \frac{2}{\pi} \left[\frac{4i}{4i^2 - 1} \right] \\ &> \frac{2}{i\pi} \end{aligned}$$

Hence for each $n \in \mathbb{N}$,

$$\begin{aligned} TV(G, [-1, 1]) &\geq \sum_{i=1}^n |G(x_i) - G(x_{i-1})| \\ &> \frac{2}{\pi} \sum_{i=1}^n \frac{1}{i} \end{aligned}$$

Therefore $G \notin BV([-1, 1])$. □

Exercise 4.70. *The following is stated for BV , but is also true for $BV([a, b])$.*

- (1) *For each $F, G \in BV$, $T_{F+G} \leq T_F + T_G$ and therefore BV is a vector space.*
- (2) *For each $F : \mathbb{R} \rightarrow \mathbb{C}$, $F \in BV$ iff $\operatorname{Re}(f) \in BV$ and $\operatorname{Im}(F) \in BV$.*
- (3) *For each $F : \mathbb{R} \rightarrow \mathbb{R}$, $F \in BV$ iff there exist functions $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$*
- (4) *For each $F \in BV$ and $x \in \mathbb{R}$, $\lim_{t \rightarrow x^+} F(t)$ and $\lim_{t \rightarrow x^-} F(t)$ exist.*
- (5) *For each $F \in BV$, $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$ is countable.*
- (6) *For each $F \in BV$, F and F_+ are differentiable a.e. and $F' = (F_+)'$ a.e.*
- (7) *For each $F \in BV, c \in \mathbb{R}$, $F - c \in BV$*

Proof. (1) Let $F, G \in BV$, $x \in \mathbb{R}$ and $\epsilon > 0$. Since $T_{F+G}(x) < \infty$, $T_{F+G}(x) - \epsilon < T_{F+G}(x)$. Thus there exists $(x_i)_{i=0}^n \subset \mathbb{R}$ such that $(x_i)_{i=0}^n$ is increasing, $x_n = x$ and $T_{F+G}(x) < \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1})| + \epsilon$. Therefore

$$\begin{aligned} T_{F+G}(x) &< \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1})| + \epsilon \\ &\leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + \sum_{i=1}^n |G(x_i) - G(x_{i-1})| + \epsilon \\ &\leq T_F(x) + T_G(x) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $T_{F+G}(x) \leq T_F(x) + T_G(x)$. Therefore $TV(F+G) \leq TV(F) + TV(G) < \infty$. Thus $F+G \in BV$. It is straight forward to verify the other requirements needed to show that BV is a vector space.

- (2) Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Write $F = F_1 + iF_2$ with $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$. Suppose that $F \in BV$. Note that for each $x_1, x_2 \in \mathbb{R}$ and $j = 1, 2$, $|F_j(x_1) - F_j(x_2)| \leq |F(x_1) - F(x_2)|$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then for $j = 1, 2$

$$\sum_{i=1}^n |F_j(x_i) - F_j(x_{i-1})| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

. Thus for $j = 1, 2$ we have that $T_{F_j}(x) \leq T_F(x)$ which implies that $\operatorname{Re}(f), \operatorname{Im}(F) \in BV$. Conversely, Suppose that $\operatorname{Re}(f), \operatorname{Im}(F) \in BV$. Then $F = \operatorname{Re}(f) + i\operatorname{Im}(f) \in BV$ by (1).

- (3) Suppose that $F \in BV$. Choose $F_1 = \frac{1}{2}(T_F - F)$ and $F_2 = \frac{1}{2}(T_F + F)$. Then F_1, F_2 are bounded, increasing and $F = F_1 + F_2$. Conversely, if there exist $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$, then $F_1, F_2 \in BV$. By (1) $F \in BV$.
- (4) This is clear by previous results and (3)

- (5) This is clear by previous results and (3)
- (6) This is clear by previous results and (3)
- (7) Clearly constant functions have zero total variation. The rest is implied by (1). \square

Lemma 4.71. *Let $F \in BV$. Then $\lim_{x \rightarrow -\infty} T_F(x) = 0$ and if F is right continuous, then T_F is right continuous.*

Definition 4.72. *Define $NBV = \{F \in BV : F \text{ is right continuous and } \lim_{x \rightarrow -\infty} F(x) = 0\}$.*

Theorem 4.73. *Let $M(\mathbb{R})$ be the set of complex Borel measures on \mathbb{R} . For $F \in NBV$, define $\mu_F \in M(\mathbb{R})$ by $\mu_F((-\infty, x]) = F(x)$. Then $F \mapsto \mu_F$ defines a bijection $NBV \rightarrow M(\mathbb{R})$. In addition, $|\mu_F| = \mu_{T_F}$*

Theorem 4.74. *Let $F \in NBV$. Then $F' \in L^1(m)$, $\mu_F \perp m$ iff $F' = 0$ a.e. and $\mu_F \ll m$ iff for each $x \in \mathbb{R}$, $\int_{(-\infty, x]} F' dm = F(x)$*

Definition 4.75. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Then F is said to be **absolutely continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Definition 4.76. *Let $F : [a, b] \rightarrow \mathbb{C}$. Then F is said to be **absolutely continuous on** $[a, b]$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Proposition 4.77. *Let $F : [a, b] \rightarrow \mathbb{C}$. If F is absolutely continuous on $[a, b]$, then $F \in BV[a, b]$.*

Exercise 4.78. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Suppose that there exists $f \in L^1(m)$ such that $F(x) = \int_{(-\infty, x]} f dm$. Then $F \in NBV$.*

Proof. Let $x \in \mathbb{R}$ and $(x_i)_{i=1}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=1}^n$ is increasing and $x_n = x$. Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \left| \int_{(x_{i-1}, x_i]} f dm \right| \\ &\leq \sum_{i=1}^n \int_{(x_{i-1}, x_i]} |f| dm \\ &= \int_{(x_0, x]} |f| dm \\ &< \int |f| dm \end{aligned}$$

Hence $T_F(x) \leq \int |f| dm$. Since $x \in \mathbb{R}$ is arbitrary, $TV(F) \leq \int |f| dm$. Therefore $F \in BV$. By the continuity from above and below for measures and the fact that $m(x) = 0$ for each $x \in \mathbb{R}$, F is continuous. By continuity from above for measures, $\lim_{x \rightarrow -\infty} F(x) = 0$. So $F \in NBV$. \square

Lemma 4.79. *Let $F \in NBV$. Then F is absolutely continuous iff $\mu_F \ll m$.*

Exercise 4.80. *Fundamental Theorem of Calculus: Let $F : [a, b] \rightarrow \mathbb{C}$. The following are equivalent:*

- (1) F is absolutely continuous on $[a, b]$.
 (2) there exists $f \in L^1([a, b], m)$ such that for each $x \in [a, b]$, $F(x) - F(a) = \int_{(a, x]} f dm$
 (3) F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$ and for each $x \in [a, b]$, $F(x) - F(a) = \int_{(a, x]} F' dm$

Proof. (1) \implies (3)

Suppose that F is absolutely continuous on $[a, b]$. Then $F \in BV[a, b]$. Extend F to \mathbb{R} by setting $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$. Then $G = F - F(a) \in NBV$ and is absolutely continuous. The previous lemma implies that there exists $f \in L^1(m)$ such that $\mu_G = f dm$. A previous theorem implies that for a.e. $x \in [a, b]$

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow x} \frac{\mu_G((x, x+r])}{m((x, x+r])} \\ &= f(x) \end{aligned}$$

So F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$ and by construction, for each $x \in [a, b]$, we have that

$$\begin{aligned} F(x) - F(a) &= \mu_G((a, x]) \\ &= \int_{(a, x]} f dm \\ &= \int_{(a, x]} F' dm \end{aligned}$$

(3) \implies (2)

Trivial.

(2) \implies (1)

Suppose that there exists $f \in L^1([a, b], m)$ such that for each $x \in [a, b]$, $F(x) - F(a) = \int_{(a, x]} f dm$. Extend F as before and obtain G as before. Note that a previous exercise implies that $G \in NBV$. Since $\mu_G \ll m$, the previous lemma implies that G is absolutely continuous. \square

Exercise 4.81. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. If F is absolutely continuous. Then F is differentiable a.e.

Proof. Let $n \in \mathbb{N}$. Since F is absolutely continuous on \mathbb{R} , F is absolutely continuous on $[-n, n]$. The FTC implies that F is differentiable a.e. on $[-n, n]$. Since $n \in \mathbb{N}$ is arbitrary, F is differentiable a.e. on \mathbb{R} . \square

Exercise 4.82. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Then F is Lipschitz continuous iff F is absolutely continuous and F' is bounded a.e.

Proof. Suppose that F is Lipschitz continuous. Then there exists $M > 0$ such that for each $x, y \in \mathbb{R}$, $|F(x) - F(y)| \leq M|x - y|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{M}$. Let $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, Suppose that $\sum_{i=1}^n b_i - a_i < \delta$. Then

$$\begin{aligned} \sum_{i=1}^n |F(b_i) - F(a_i)| &\leq \sum_{i=1}^n M(b_i - a_i) \\ &< M\delta \\ &= \epsilon \end{aligned}$$

Hence F is absolutely continuous. For each $x, y \in \mathbb{R}$, if $x \neq y$, then $\left| \frac{F(x)-F(y)}{x-y} \right| \leq M$. Hence for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Conversely, suppose that F is absolutely continuous and F' is bounded a.e. Then there exists $M > 0$ such that for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Let $x, y \in \mathbb{R}$. Suppose $x < y$. Then the FTC implies that

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_{(x,y]} F' dm \right| \\ &\leq \int_{(x,y]} |F'| dm \\ &= M|y - x| \end{aligned}$$

and F is Lipschitz continuous. □

Exercise 4.83. Construct an increasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ whose discontinuities is \mathbb{Q} .

Proof. Let $(q_n)_{n \in \mathbb{N}}$ be an enumeration of \mathbb{Q} . Define $F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F = \sum_{n \in \mathbb{N}} 2^{-n} \chi_{[q_n, \infty)}$$

. Equivalently, if we define $S_x = \{n \in \mathbb{N} : q_n \leq x\}$, then we may write

$$F(x) = \sum_{n \in S_x} 2^{-n}$$

Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Then $S_x \subsetneq S_y$. So $F(x) < F(y)$ and therefore F is strictly increasing.

For each $x, y \in \mathbb{R}$ with $x < y$, define $S_{x,y} = \{n \in \mathbb{N} : x < q_n \leq y\}$. Note that $\lim_{y \rightarrow x^+} \min(S_{x,y}) = \infty$ and if $y \in \mathbb{R} \setminus \mathbb{Q}$, then $\lim_{x \rightarrow y^-} \min(S_{x,y}) = \infty$.

Now, let $x \in \mathbb{R}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$. Choose $\delta > 0$ such that $\min(S_{x, x+\delta}) \geq N$. Let $y \in [x, \infty)$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_y} 2^{-n} - \sum_{n \in S_x} 2^{-n} \\ &= \sum_{n \in S_{x,y}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence F is right continuous. Now let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ as before and $\delta > 0$ such that $\min(S_{x-\delta, x}) \geq N$. Let $y \in (-\infty, x]$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_x} 2^{-n} - \sum_{n \in S_y} 2^{-n} \\ &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence F is left continuous on $\mathbb{R} \setminus \mathbb{Q}$.

Now, let $x \in \mathbb{Q}$. Then there exists $j \in \mathbb{N}$ such that $q_j = x$. Choose $\epsilon = 2^{-j}$. Let $\delta > 0$. Choose $y = x - \frac{\delta}{2}$. Then $|x - y| < \delta$ and

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\geq 2^{-j} \\ &= \epsilon \end{aligned}$$

Hence F is discontinuous from the left at x . Since $x \in \mathbb{Q}$ is arbitrary, F is discontinuous from the left on \mathbb{Q} . \square

Exercise 4.84. Let $(F_n)_{n \in \mathbb{N}} \in NBV$ be a sequence of nonnegative, increasing functions. If for each $x \in \mathbb{R}$, $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$, then for a.e. $x \in \mathbb{R}$, F is differentiable at x and $F'(x) = \sum_{n \in \mathbb{N}} F'_n(x)$.

Proof. Define $\mu = \sum_{n \in \mathbb{N}} \mu_{F_n}$. Note that

$$\begin{aligned} \mu((-\infty, x]) &= \sum_{n \in \mathbb{N}} \mu_{F_n}((-\infty, x]) \\ &= \sum_{n \in \mathbb{N}} F_n(x) \\ &= F(x) \end{aligned}$$

Hence $F \in NBV$ and $\mu = \mu_F$. For each $n \in \mathbb{N}$, there exist $\lambda_n \in M(\mathbb{R})$ and $f_n \in L^1(\mathbb{R})$ such that $d\mu_{F_n} = d\lambda_n + f_n dm$ and $\lambda \perp m$. Since for each $n \in \mathbb{N}$, λ_n, f_n are nonnegative, we have that $d\mu_F = \sum_{n \in \mathbb{N}} d\lambda_n + (\sum_{n \in \mathbb{N}} f_n) dm$. By a previous theorem, for a.e. $x \in \mathbb{R}$,

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow 0} \frac{\mu_F((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} f_n(x) \\ &= \sum_{n \in \mathbb{N}} \lim_{r \rightarrow 0} \frac{\mu_{F_n}((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} F'_n(x) \end{aligned}$$

□

Exercise 4.85. Let $F : [0, 1] \rightarrow [0, 1]$ be the Cantor function. Extend F to \mathbb{R} by setting $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $([a_n, b_n])_{n \in \mathbb{N}}$ be an enumeration of the closed subintervals of $[0, 1]$ with rational endpoints. For $n \in \mathbb{N}$, define $F_n : \mathbb{R} \rightarrow [0, 1]$ by $F_n(x) = F(\frac{x-a_n}{b_n-a_n})$. Define $G : \mathbb{R} \rightarrow \mathbb{R}$ by $G = \sum_{n \in \mathbb{N}} 2^{-n} F_n$. Then G is continuous, strictly increasing on $[0, 1]$ and $G' = 0$ a.e.

Proof. Since F is continuous on \mathbb{R} , we have that for each $n \in \mathbb{N}$, F_n is continuous on \mathbb{R} . We observe that for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $|2^{-n} F_n(x)| \leq 2^{-n}$. Thus the Weierstrass M-test implies that G converges uniformly on \mathbb{R} and is therefore continuous. Since F is increasing, for each $n \in \mathbb{N}$, F_n is increasing. Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Choose $j \in \mathbb{N}$ such that $x < a_j < y < b_j$. Then

$$\begin{aligned} G(x) &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(x) \\ &= \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(x) + 0 \\ &< \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(y) + 2^{-j} F_j(y) \\ &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(y) \\ &= G(y) \end{aligned}$$

So G is strictly increasing.

Now we observe that for each $n \in \mathbb{N}$, $F_n \in NBV$. The previous exercise implies that

$$G' = \sum 2^{-n} F'_n = 0 \text{ a.e.}$$

□

5. TOPOLOGY

Definition 5.1. Let (X, \mathcal{A}) and (Y, \mathcal{B}) be topological spaces and $f : X \rightarrow Y$. Then

- (1) f is said to be **continuous** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$.
- (2) f is said to be **open** if for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$.
- (3) f is said to be **closed** if for each $A \subset X$, if $A^c \in \mathcal{A}$, then $f(A)^c \in \mathcal{B}$.

6. L^p SPACES

Definition 6.1. Let (X, \mathcal{A}, μ) be a measure space and $p \in (0, \infty]$. Define $\|\cdot\|_p : L^0(X, \mathcal{A}, \mu) \rightarrow [0, \infty]$ by

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{\frac{1}{p}} \quad (p < \infty)$$

and

$$\|f\|_\infty = \inf \left\{ \lambda > 0 : \mu(\{x \in X : \lambda < |f(x)|\}) = 0 \right\}$$

We define

$$L^p(X, \mathcal{A}, \mu) = \{f \in L^0(X, \mathcal{A}, \mu) : \|f\|_p < \infty\}$$

Theorem 6.2. Hölder's Inequality: Let (X, \mathcal{A}, μ) be a measure space, $p, q \in [1, \infty)$ and $f, g \in L^0$. Suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Exercise 6.3. Minkowski Inequality: Let (X, \mathcal{A}, μ) be a measure space, $p \in [1, \infty)$ and $f, g \in L^p$. Then $f + g \in L^p$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. Define $\phi : \mathbb{R} \rightarrow [0, \infty)$ by $\phi(x) = |x|^p$. Then ϕ is convex because it is the composition of an increasing convex function with a convex function. By Jensen's inequality, we have that

$$\phi\left(\frac{1}{2}[f + g]\right) \leq \frac{1}{2}[\phi(f) + \phi(g)]$$

This implies that

$$\frac{1}{2^p}|f + g|^p \leq \frac{1}{2}(|f|^p + |g|^p)$$

Hence

$$\begin{aligned} \int |f + g|^p d\mu &\leq 2^{p-1} \int |f|^p + |g|^p d\mu \\ &= 2^{p-1} \left(\int |f|^p d\mu + \int |g|^p d\mu \right) \\ &= 2^{p-1} \left(\|f\|_p^p + \|g\|_p^p \right) \\ &< \infty \end{aligned}$$

So $f + g \in L^p$. Now, it is not hard to see that $|f + g|^p \leq (|f| + |g|)|f + g|^{p-1}$. Let q be the conjugate of p , so that $\frac{1}{p} + \frac{1}{q} = 1$. Then $q(p-1) = p$. We use Hölder's inequality to show that

$$\begin{aligned} \|f + g\|_p^p &= \int |f + g|^p d\mu \\ &\leq \int |f| |f + g|^{p-1} d\mu + \int |g| |f + g|^{p-1} d\mu \\ &\leq \|f\|_p \left(\int |f + g|^{(p-1)q} d\mu \right)^{\frac{1}{q}} + \|g\|_p \left(\int |f + g|^{(p-1)q} d\mu \right)^{\frac{1}{q}} \\ &= \|f\|_p \left(\int |f + g|^p d\mu \right)^{\frac{1}{q}} + \|g\|_p \left(\int |f + g|^p d\mu \right)^{\frac{1}{q}} \\ &= (\|f\|_p + \|g\|_p) \left(\int |f + g|^p d\mu \right)^{\frac{1}{q}} \\ &= (\|f\|_p + \|g\|_p) \|f + g\|_p^{p/q} \end{aligned}$$

Since $\|f + g\|_p < \infty$, we see that

$$\begin{aligned}\|f\|_p + \|g\|_p &\geq \|f + g\|_p^{p-p/q} \\ &= \|f + g\|_p^{p(1-1/q)} \\ &= \|f + g\|_p^{p/p} \\ &= \|f + g\|_p\end{aligned}$$

□

Exercise 6.4. Let (X, \mathcal{A}, μ) be a measure space, $p, q \in (0, \infty]$. Suppose that $\mu(X) < \infty$ and $p < q$. Then $L^q \subset L^p$. In particular, if $\mu(X) = 1$, then for each $f \in L^q$, $\|f\|_p \leq \|f\|_q$.

Proof. Suppose that $q = \infty$. Let $f \in L^q$. Then

$$\begin{aligned}\|f\|_p &= \left(\int |f|^p d\mu \right)^{\frac{1}{p}} \\ &\leq \left(\int \|f\|_\infty^p d\mu \right)^{\frac{1}{p}} \\ &= \|f\|_\infty \mu(X)^{\frac{1}{p}}\end{aligned}$$

If $q < \infty$, then $\frac{q}{p} > 1$ and the conjugate of $\frac{q}{p}$ is $\frac{1}{1-p/q}$. By Hölder's inequality, we have that

$$\begin{aligned}\|f\|_p^p &= \|f^p\|_1 \\ &\leq \|f^p\|_{\frac{q}{p}} \|1\|_{\frac{1}{1-p/q}} \\ &= \left(\int |f|^{\frac{pq}{p}} d\mu \right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}} \\ &= \left(\int |f|^q d\mu \right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}} \\ &= \|f\|_q^p \mu(X)^{1-\frac{p}{q}}\end{aligned}$$

Hence

$$\begin{aligned}\|f\|_p &\leq \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}} \\ &< \infty\end{aligned}$$

□

7. FUNCTIONAL ANALYSIS

7.1. Normed Vector Spaces.

Note 7.1. In the following, we will consider vector spaces over \mathbb{C} . There are analogous results for real vector spaces as well, just replace every \mathbb{C} with \mathbb{R} .

Definition 7.2. Let X be a normed vector space. Then X is said to be a **Banach space** if X is complete.

Definition 7.3. Let X be a normed vector space and $(x_i)_{i=1}^n \subset X$. The series $\sum_{i=1}^\infty x_i$ is said to **converge** if the sequence $s_n := \sum_{i=1}^n x_i$ converges. The series $\sum_{i=1}^\infty x_i$ is said to **converge absolutely** if $\sum_{i \in \mathbb{N}} \|x_i\| < \infty$.

Theorem 7.4. *Let X be a normed vector space. Then X is complete iff for each $(x_i)_{i \in \mathbb{N}} \subset X$, $\sum_{i=1}^{\infty} x_i$ converges absolutely implies that $\sum_{i=1}^{\infty} x_i$ converges.*

Proof. Suppose that X is complete. Let $(x_i)_{i \in \mathbb{N}} \subset X$. Suppose that $\sum_{i=1}^{\infty} x_i$ converges absolutely. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq N$ and $m < n$, then $\sum_{m+1}^n \|x_i\| < \epsilon$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right\| \\ &= \left\| \sum_{i=m+1}^n x_i \right\| \\ &\leq \sum_{i=m+1}^n \|x_i\| \\ &< \epsilon \end{aligned}$$

Thus $(s_n)_{n \in \mathbb{N}}$ is Cauchy. Since X is complete, $\sum_{i=1}^{\infty} x_i$ converges. Conversely, Suppose that for each $(x_i)_{i \in \mathbb{N}} \subset X$, $\sum_{i=1}^{\infty} x_i$ converges absolutely implies that $\sum_{i=1}^{\infty} x_i$ converges. Let $(x_i)_{i \in \mathbb{N}} \subset X$ be Cauchy. Proceed inductively to create a strictly increasing sequence $(n_i)_{i \in \mathbb{N}} \subset \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq n_i$, then $\|x_m - x_n\| < 2^{-i}$. Define $(y_i)_{i \in \mathbb{N}} \subset X$ by

$$y_i = \begin{cases} x_{n_1} & i = 1 \\ x_{n_i} - x_{n_{i-1}} & i \geq 2 \end{cases}$$

Then $\sum_{i=1}^k y_i = x_{n_k}$ and

$$\begin{aligned} \sum_{i \in \mathbb{N}} \|y_i\| &= \|x_{n_1}\| + \sum_{i \in \mathbb{N}} \|x_{n_i} - x_{n_{i-1}}\| \\ &\leq \|x_{n_1}\| + \sum_{i \in \mathbb{N}} 2^{-i} \\ &= \|x_{n_1}\| + 1 \end{aligned}$$

Hence $(x_{n_k})_{k \in \mathbb{N}} = (\sum_{i=1}^k y_i)_{i \in \mathbb{N}}$ converges. Since $(x_i)_{i \in \mathbb{N}}$ is Cauchy and has a convergent subsequence, it converges. So X is complete. \square

Definition 7.5. *Let X, Y be normed vector spaces. A linear map $T : X \rightarrow Y$ is said to be **bounded** if there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$.*

Exercise 7.6. *Let X, Y be normed vector spaces and $T : X \rightarrow Y$ a linear map. Then T is bounded iff there exists $r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$*

Proof. Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Thus $T(B(0, 1)) \subset B(0, C + 1)$. Conversely, Suppose that there exists $r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$. Define $C = \frac{2s}{r}$. Let $x \in X$. Put $\alpha = \frac{r}{2\|x\|}$. Then

$\alpha x \in B(0, r)$. So $T(\alpha x) = \alpha T(x) \in B(0, s)$. Hence

$$\begin{aligned}\|T(\alpha x)\| &= \|\alpha T(x)\| \\ &= |\alpha| \|T(x)\| \\ &= \frac{r}{2\|x\|} \|T(x)\| \\ &< s.\end{aligned}$$

Thus

$$\|Tx\| < \frac{2s}{r} \|x\| = C\|x\|$$

So T is bounded. □

Theorem 7.7. *Let X, Y be normed vector spaces and $T : X \rightarrow Y$ a linear map. Then the following are equivalent:*

- (1) T is continuous
- (2) T is continuous at $x = 0$
- (3) T is bounded

Proof. (1) \implies (2):

Trivial

(2) \implies (3):

Suppose that T is continuous at $x = 0$. Then there exists $\delta > 0$ such that for each $x \in X$, if $\|x\| < \delta$, then $\|Tx\| < 1$. Choose $C = \frac{2}{\delta}$. If $x = 0$, then $\|Tx\| \leq C\|x\|$. Suppose that $\|x\| \neq 0$. Define $y = \frac{\delta}{2\|x\|}x$. Then $\|y\| < \delta$. So

$$\|Ty\| = \frac{\delta}{2\|x\|} \|Tx\| < 1$$

Thus

$$\begin{aligned}\|Tx\| &< \frac{2}{\delta} \|x\| \\ &= C\|x\|\end{aligned}$$

Hence T is bounded.

(3) \implies (1)

Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{C+1}$. Let $x, y \in X$. Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned}\|Tx - Ty\| &= \|T(x - y)\| \\ &\leq C\|x - y\| \\ &< (C + 1)\delta \\ &= \epsilon\end{aligned}$$

So T is continuous. □

Definition 7.8. Let X, Y be normed vector spaces. Define $L(X, Y) = \{T : X \rightarrow Y : T \text{ is bounded}\}$. Define $\|\cdot\| : L(X, Y) \rightarrow [0, \infty)$ by

$$\|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

We call $\|\cdot\|$ the **operator norm on $L(X, Y)$**

Exercise 7.9. Let X, Y be normed vector spaces. If $X \neq \{0\}$, then the operator norm on $L(X, Y)$ is given by:

$$(1) \|T\| = \sup_{\|x\|=1} \|Tx\|$$

$$(2) \|T\| = \sup_{x \neq 0} \|x\|^{-1} \|Tx\|$$

$$(3) \|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

Proof. Since $X \neq \{0\}$, the supremums in (1) and (2) are well defined. Let $T \in L(X, Y)$. By linearity of T , the sets over which the supremums are taken in (1) and (2) are the same. So (1) and (2) are equal.

Now, put $M = \sup_{\|x\|=1} \|Tx\|$, $m = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$ and let $x \in X$. If $\|x\| = 0$, then $\|Tx\| \leq M\|x\|$. Suppose that $\|x\| \neq 0$. Then

$$\begin{aligned} \|Tx\| &= \left(\|T(x/\|x\|)\| \right) \|x\| \\ &\leq M\|x\| \end{aligned}$$

Hence $M \in \{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$. Therefore $m \leq M$

Let $C \in \{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$. Suppose that $\|x\| = 1$. Then $\|Tx\| \leq C\|x\| = C$. So $M \leq C$. Therefore $M \leq m$. So $M = m$ and the supremum in (1) is the same as the infimum in (3). \square

Note 7.10. From here on, unless stated otherwise, we assume $X \neq 0$.

Exercise 7.11. Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then for each $x \in X$, $\|Tx\| \leq \|T\|\|x\|$

Proof. This is just part of the previous exercise. Let $x \in X$. If $x = 0$, then $\|Tx\| \leq \|T\|\|x\|$. Suppose that $x \neq 0$. Then $\|Tx\| = \|T(x/\|x\|)\|\|x\| \leq \|T\|\|x\|$ \square

Exercise 7.12. Let X, Y be normed vector spaces. Then the operator norm is a norm on $L(X, Y)$.

Proof. Let $S, T \in L(X, Y)$ and $\alpha \in \mathbb{C}$. For each $x \in X$, we have that

$$\begin{aligned} \|(S + T)x\| &= \|Sx + Tx\| \\ &\leq \|Sx\| + \|Tx\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= (\|S\| + \|T\|)\|x\| \end{aligned}$$

So $\|S + T\| \leq \|S\| + \|T\|$.

Using the definition of $\|T\|$, we see that

$$\begin{aligned}
 \|\alpha T\| &= \sup_{\|x\|=1} \|(\alpha T)x\| \\
 &= \sup_{\|x\|=1} |\alpha| \|Tx\| \\
 &= |\alpha| \sup_{\|x\|=1} \|Tx\| \\
 &= |\alpha| \|T\|
 \end{aligned}$$

$$\text{So } \|\alpha S\| = |\alpha| \|S\|.$$

Suppose that $\|T\| = 0$. Let $x \in X$. Then $\|Tx\| \leq \|T\| \|x\| = 0$. So $Tx = 0$. Since $x \in X$ is arbitrary, we have that $T = 0$. \square

Exercise 7.13. *Let X be a normed vector space. Then addition and scalar multiplication are continuous on $X \times X$ and $\|\cdot\| : X \rightarrow [0, \infty)$ is continuous.*

Proof. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Let $(x_1, y_1), (x_2, y_2) \in X \times X$. Suppose that $\|(x_1, y_1) - (x_2, y_2)\| = \max\{\|x_1 - x_2\|, \|y_1 - y_2\|\} < \delta$. Then

$$\begin{aligned}
 \|(x_1 + y_1) - (x_2 + y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\
 &\leq \|x_1 - x_2\| + \|y_1 - y_2\| \\
 &< 2\delta \\
 &= \epsilon
 \end{aligned}$$

Hence addition is uniformly continuous.

Let $(\lambda_1, x_1) \in \mathbb{C} \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(|\lambda_1| + \|x_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(\lambda_2, x_2) \in \mathbb{C} \times X$. Suppose that $\|(\lambda_1, x_1) - (\lambda_2, x_2)\| = \max\{|\lambda_1 - \lambda_2|, \|x_1 - x_2\|\} < \delta$. Then

$$\begin{aligned}
 \|\lambda_1 x_1 - \lambda_2 x_2\| &= \|\lambda_1 x_1 - \lambda_1 x_2 + \lambda_1 x_2 - \lambda_2 x_2\| \\
 &= \|\lambda_1(x_1 - x_2) + (\lambda_1 - \lambda_2)x_2\| \\
 &\leq |\lambda_1| \|x_1 - x_2\| + |\lambda_1 - \lambda_2| \|x_2\| \\
 &\leq |\lambda_1| \|x_1 - x_2\| + |\lambda_1 - \lambda_2| (\|x_1 - x_2\| + \|x_1\|) \\
 &< |\lambda_1| \delta + \delta(\delta + \|x_1\|) \\
 &= (|\lambda_1| + \|x_1\|) \delta + \delta^2 \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon
 \end{aligned}$$

Since $(\lambda_1, x_1) \in \mathbb{C} \times X$ is arbitrary, scalar multiplication is continuous.

Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x, y \in X$. Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned} \left| \|x\| - \|y\| \right| &\leq \|x - y\| \\ &< \delta \\ &= \epsilon \end{aligned}$$

So $\|\cdot\| : X \rightarrow [0, \infty)$ is uniformly continuous. \square

Exercise 7.14. Let X, Y be normed vector spaces. If Y is complete, then so is $L(X, Y)$.

Proof. Suppose that Y is complete. Let $(T_n)_{n \in \mathbb{N}} \subset L(X, Y)$. Suppose that $(T_n)_{n \in \mathbb{N}}$ is Cauchy. Since for each $m, n \in \mathbb{N}$, $|\|T_m\| - \|T_n\|| \leq \|T_m - T_n\|$, we have that $(\|T_n\|)_{n \in \mathbb{N}} \subset [0, \infty)$ is Cauchy. Hence $\lim_{n \rightarrow \infty} \|T_n\|$ exists.

Let $x \in X$ and $m, n \in \mathbb{N}$. Then

$$\begin{aligned} \|T_m x - T_n x\| &= \|(T_m - T_n)x\| \\ &\leq \|T_m - T_n\| \|x\| \end{aligned}$$

So $(T_n x)_{n \in \mathbb{N}} \subset Y$ is Cauchy and hence converges. Define $T : X \rightarrow Y$ by $Tx = \lim_{n \rightarrow \infty} T_n x$.

Since addition and scalar multiplication are continuous, T is linear. Let $x \in X$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then $\|Tx - T_n x\| < \epsilon$. Then for each $n \in \mathbb{N}$, if $n \geq N$ we have that

$$\begin{aligned} \|Tx\| &\leq \|Tx - T_n x\| + \|T_n x\| \\ &< \epsilon + \|T_n x\| \\ &\leq \epsilon + \|T_n\| \|x\| \end{aligned}$$

Thus $\|Tx\| \leq \epsilon + (\lim_{n \rightarrow \infty} \|T_n\|) \|x\|$. Since $\epsilon > 0$ is arbitrary, $\|Tx\| \leq (\lim_{n \rightarrow \infty} \|T_n\|) \|x\|$. Thus $T \in L(X, Y)$ and $\|T\| \leq \lim_{n \rightarrow \infty} \|T_n\|$.

Note that since addition, scalar multiplication and $\|\cdot\|$ are continuous, we have that for each $n \in \mathbb{N}$ and $x \in X$, $\|(T_n - T_m)x\|$ converges to $\|(T_n - T)x\|$ because

$$\begin{aligned} \lim_{m \rightarrow \infty} \|(T_n - T_m)x\| &= \lim_{m \rightarrow \infty} \|T_n x - T_m x\| \\ &= \|T_n x - \lim_{m \rightarrow \infty} T_m x\| \\ &= \|T_n x - Tx\| \\ &= \|(T_n - T)x\| \end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ if $n, m \geq N$, then $\|T_n - T_m\| < \epsilon$. Then for each $n \in \mathbb{N}$ if $n \geq N$, then for each $x \in X$,

$$\|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\| < \epsilon \|x\|$$

Combining this with the previous fact, we see that for each $n \in N$, if $n \geq N$, then for each $x \in X$,

$$\|(T_n - T)x\| \leq \epsilon \|x\|$$

In particular, for each $n \in \mathbb{N}$, if $n \geq N$, then

$$\|T_n - T\| = \sup_{\|x\|=1} \|(T_n - T)x\| \leq \epsilon$$

This implies that T_n converges to T in $L(X, Y)$. Since

$$|\|T_n\| - \|T\|| \leq \|T_n - T\|$$

It is clear that $\lim_{n \rightarrow \infty} \|T_n\| = \|T\|$ □

Definition 7.15. Let X be a normed vector space and $M \subset X$ a closed subspace. Define $\|\cdot\| : X/M \rightarrow [0, \infty)$ by

$$\|x + M\| := \inf_{y \in M} \|x + y\|$$

We call $\|\cdot\|$ the **subspace norm on X/M**

Exercise 7.16. Let X be a normed vector space and $M \subsetneq X$ a proper, closed subspace of X . Then

- (1) The previously defined subspace norm on X/M is well defined and is a norm.
- (2) For each $\epsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.
- (3) The projection map $\pi : X \rightarrow X/M$ defined by $\pi(x) = x + M$ is continuous and $\|\pi\| = 1$.
- (4) If X is complete, then X/M is complete.

Proof. (1) Let $x, y \in X$ and $\alpha \in \mathbb{C}$. Suppose that $x + M = y + M$. Then there exists $m \in M$ such that $x = y + m$. Since M is a subspace, the map $T : M \rightarrow M$ given by $Tx = x + m$ is a bijection. So

$$\inf_{z \in M} \|y + m + z\| = \inf_{z \in M} \|y + z\|$$

which implies that

$$\begin{aligned} \|x + M\| &= \inf_{z \in M} \|x + z\| \\ &= \inf_{z \in M} \|y + m + z\| \\ &= \inf_{z \in M} \|y + z\| \\ &= \|y + M\| \end{aligned}$$

So $\|\cdot\| : X/M \rightarrow [0, \infty)$ is well defined.

We observe that for each $z, w \in M$,

$$\|x + y + z\| \leq \|x + w\| + \|y + w + z\|$$

Taking infimums over M with respect to z in this inequality implies that for each $w \in M$,

$$\begin{aligned} \inf_{z \in M} \|x + y + z\| &\leq \inf_{z \in M} \left(\|x + w\| + \|y + w + z\| \right) \\ &= \|x + w\| + \inf_{z \in M} \|y + w + z\| \end{aligned}$$

Again we use the fact that for each $w \in M$,

$$\inf_{z \in M} \|y + w + z\| = \inf_{z \in M} \|y + z\|$$

This implies that for each $w \in M$,

$$\inf_{z \in M} \|x + y + z\| \leq \|x + w\| + \inf_{z \in M} \|y + z\|$$

Therefore, taking infimums over M with respect to w in this inequality yields

$$\begin{aligned} \|x + y + M\| &= \inf_{z \in M} \|x + y + z\| \\ &\leq \inf_{w \in M} \left(\|x + w\| + \inf_{z \in M} \|y + z\| \right) \\ &= \inf_{w \in M} \|x + w\| + \inf_{z \in M} \|y + z\| \\ &= \|x + M\| + \|y + M\| \end{aligned}$$

If $\alpha = 0$, then $\alpha x = 0$. Choosing $z = 0 \in M$ gives $\|\alpha x + M\| = 0 = |\alpha| \|x + M\|$. Suppose that $\alpha \neq 0$. Then the map $T : M \rightarrow M$ given by $Tx = \alpha^{-1}x$ is a bijection and thus $\inf_{z \in M} \|x + \alpha^{-1}z\| = \inf_{z \in M} \|x + z\|$. Hence we have that

$$\begin{aligned} \|\alpha x + M\| &= \inf_{z \in M} \|\alpha x + z\| \\ &= \inf_{z \in M} |\alpha| \|x + \alpha^{-1}z\| \\ &= |\alpha| \inf_{z \in M} \|x + \alpha^{-1}z\| \\ &= |\alpha| \inf_{z \in M} \|x + z\| \\ &= |\alpha| \|x + M\| \end{aligned}$$

Suppose that $\|x\| = 0$. Choose a sequence $(z_n)_{n \in \mathbb{N}} \subset M$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x - z_n\| &= \inf_{z \in M} \|x + z\| \\ &= 0 \end{aligned}$$

Then $\lim_{n \rightarrow \infty} z_n = x$. Since M is closed, $x \in M$. Hence $x + M = 0 + M$.

- (2) Since M is a proper subspace, there exists $v \in X$ such that $v \notin M$. Then $\|v + M\| \neq 0$. Let $\epsilon > 0$. Then $(1 - \epsilon)^{-1} \|v + M\| > \|v + M\|$. So there exists $z \in M$ such that

$$0 < \|v + M\| \leq \|v + z\| < (1 - \epsilon)^{-1} \|v + M\|$$

Choose $x = \|v + z\|^{-1}(v + z)$. Then $\|x\| = 1$ and

$$\begin{aligned}\|x + M\| &= \|v + z\|^{-1}\|v + z + M\| \\ &= \|v + z\|^{-1}\|v + M\| \\ &> 1 - \epsilon\end{aligned}$$

- (3) Let $x \in X$. Taking $z = 0$, we see that $\|\pi(x)\| = \|x + M\| \leq \|x + z\| = \|x\|$. So π is bounded and in particular,

$$\sup_{\|x\|=1} \|\pi(x)\| \leq 1$$

From (2) we see that

$$\sup_{\|x\|=1} \|\pi(x)\| \geq 1$$

Hence $\|\pi\| = 1$.

- (4) Suppose that X is complete. Let $(x_i + M)_{i \in \mathbb{N}} \subset X/M$. Suppose that $\sum_{i \in \mathbb{N}} \|x_i + M\| < \infty$. Let $\epsilon > 0$. Then for each $i \in \mathbb{N}$, there exists $z_i \in M$ such that $\|x_i + z_i\| < \|x_i + M\| + \epsilon 2^{-i}$. Define the sequence $(a_i)_{i \in \mathbb{N}} \subset X$ by $a_i = x_i + z_i$. Then we have

$$\begin{aligned}\sum_{i \in \mathbb{N}} \|a_i\| &= \sum_{i \in \mathbb{N}} \|x_i + z_i\| \\ &\leq \sum_{i \in \mathbb{N}} \left(\|x_i + M\| + \epsilon 2^{-i} \right) \\ &= \sum_{i \in \mathbb{N}} \|x_i + M\| + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\sum_{i \in \mathbb{N}} \|a_i\| \leq \sum_{i \in \mathbb{N}} \|x_i + M\| < \infty$$

Since X is complete, $\sum_{i=1}^{\infty} a_i$ converges in X . Define $(s_n)_{n \in \mathbb{N}} \subset X$ and $s \in X$ by $s_n = \sum_{i=1}^n a_i$ and $s = \sum_{i=1}^{\infty} a_i$. Since $\lim_{n \rightarrow \infty} s_n = s$, and $\pi : X \rightarrow X/M$ is continuous, it follows that $\lim_{n \rightarrow \infty} \pi(s_n) = \pi(s)$. Since

$$\begin{aligned}\pi(s_n) &= \sum_{i=1}^n a_i + M \\ &= \sum_{i=1}^n x_i + M\end{aligned}$$

We have that $\sum_{i=1}^{\infty} x_i + M$ converges which implies that X/M is complete. □

Exercise 7.17. Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then

- (1) $\ker T$ is closed
- (2) there exists a unique map $S : X/\ker T \rightarrow T(X)$ such that $T = S \circ \pi$. Furthermore S is a bounded linear bijection and $\|S\| = \|T\|$.

Proof. (1) Since T is continuous and $\ker T = T^{-1}(\{0\})$, we have that $\ker T$ is closed.
 (2) Suppose that there exists $S_1, S_2 \in L(X/\ker T, T(X))$ such that $T = S_1 \circ \pi$ and $T = S_2 \circ \pi$. Let $x \in X$. Then

$$S_1(x + \ker T) = S_1(\pi(x)) = T(x) = S_2(\pi(x)) = S_2(x + \ker T)$$

So $S_1 = S_2$. Therefore such a map is unique.

Define $S : X/\ker T \rightarrow T(X)$ by $S(x + \ker T) = T(x)$. Then S is clearly a linear bijection that satisfies $T = S \circ \pi$. Let $x \in X$ and $z \in \ker T$. Then

$$\begin{aligned} \|S(x + \ker T)\| &= \|T(x)\| \\ &= \|T(x + z)\| \\ &\leq \|T\| \|x + z\| \end{aligned}$$

Thus

$$\|S(x + \ker T)\| \leq \|T\| \inf_{z \in \ker T} \|x + z\| = \|T\| \|x + \ker T\|$$

So S is bounded and $\|S\| \leq \|T\|$. This implies that

$$\|T\| = \|S \circ \pi\| \leq \|S\| \|\pi\| = \|S\|$$

Thus $\|S\| = \|T\|$. □

Exercise 7.18. Let X, Y be normed vector spaces. Define $\phi : L(X, Y) \times X \rightarrow Y$ by $\phi(T, x) = Tx$. Then ϕ is continuous.

Proof. Let $(T_1, x_1) \in L(X, Y) \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|x_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(t_2, x_2) \in L(X, Y) \times X$. Suppose that

$$\|(T_1, x_1) - (T_2, x_2)\| = \max\{\|T_1 - T_2\|, \|x_1 - x_2\|\} < \delta$$

. Then

$$\begin{aligned}
\|\phi(T_1, x_1) - \phi(T_2, x_2)\| &= \|T_1x_1 - T_2x_2\| \\
&= \|T_1x_1 - T_2x_1 + T_2x_1 - T_2x_2\| \\
&\leq \|(T_1 - T_2)x_1\| + \|T_2(x_1 - x_2)\| \\
&\leq \|T_1 - T_2\|\|x_1\| + \|T_2\|\|x_1 - x_2\| \\
&\leq \|T_1 - T_2\|\|x_1\| + (\|T_1 - T_2\| + \|T_1\|)\|x_1 - x_2\| \\
&< \delta\|x_1\| + (\delta + \|T_1\|)\delta \\
&= \delta(\|T_1\| + \|x_1\|) + \delta^2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So ϕ is continuous. \square

Exercise 7.19. Let X be a normed vector space and $M \subset X$ a subspace. Then \overline{M} is a subspace.

Proof. Let $x, y \in \overline{M}$ and $\alpha \in \mathbb{C}$. Then there exist sequences $(x_n)_{n \in \mathbb{N}} \subset M$ and $(y_n)_{n \in \mathbb{N}} \subset M$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Since M is a subspace, $(x_n + y_n)_{n \in \mathbb{N}} \subset M$ and $(\alpha x_n)_{n \in \mathbb{N}} \subset M$. Since addition and scalar multiplication are continuous, we have that $x_n + y_n \rightarrow x + y$ and $\alpha x_n \rightarrow \alpha x$. Thus $x + y \in \overline{M}$ and $\alpha x \in \overline{M}$ and hence \overline{M} is a subspace. \square

Exercise 7.20. Let X, Y, Z be normed vector spaces, $T \in L(X, Y)$ and $S \in L(Y, Z)$. Define $ST : X \rightarrow Z$ by $STx = S(Tx)$. Then $ST \in L(X, Z)$ and $\|ST\| \leq \|S\|\|T\|$.

Proof. Clearly ST is linear. Let $x \in X$. Then

$$\begin{aligned}
\|STx\| &= \|S(Tx)\| \\
&\leq \|S\|\|Tx\| \\
&\leq \|S\|\|T\|\|x\|
\end{aligned}$$

So $\|ST\| \leq \|S\|\|T\|$. \square

Definition 7.21. Let X be a Banach space and an associative algebra. Then X is said to be a Banach algebra if for each $S, T \in X$, $\|ST\| \leq \|S\|\|T\|$. If there exists $I \in X$ such that $I \neq 0$ and for each $T \in X$, $IT = TI = T$, then X is said to be **unital** with identity I . An element $T \in X$ is said to be **invertible** if there exists $S \in X$ such that $TS = ST = I$.

Exercise 7.22. Let X be a unital Banach algebra. Then $\|I\| \leq 1$.

Proof. Since $I \neq 0$, $\|I\| \neq 0$. By definition,

$$\|I\| = \|II\| \leq \|I\|\|I\|$$

Hence $1 \leq \|I\|$. \square

Note 7.23. If X is a Banach space, then a previous exercise implies that $L(X, X)$ equipped with composition is a unital Banach algebra where I is the identity operator. It is easy to see that $\|I\| = 1$.

Note 7.24. Let X be a Banach algebra. Then the set of invertible elements in X is a group.

Exercise 7.25. Let X be a Banach algebra. Then multiplication is continuous.

Proof. Let $(S_1, T_1) \in X \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|S_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(S_2, T_2) \in X \times X$. Suppose that

$$\|(S_1, T_1) - (S_2, T_2)\| = \max\{\|S_1 - S_2\|, \|T_1 - T_2\|\} < \delta$$

. Then

$$\begin{aligned} \|S_1 T_1 - S_2 T_2\| &= \|S_1 T_1 - S_2 T_1 + S_2 T_1 - S_2 T_2\| \\ &\leq \|S_1 - S_2\| \|T_1\| + \|S_2\| \|T_1 - T_2\| \\ &\leq \|S_1 - S_2\| \|T_1\| + (\|S_1 - S_2\| + \|S_1\|) \|T_1 - T_2\| \\ &\leq \delta \|T_1\| + (\delta + \|S_1\|) \delta \\ &= \delta (\|S_1\| + \|T_1\|) + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

Definition 7.26. Let X, Y be a normed vector spaces and $T \in L(X, Y)$. Then T is said to be **invertible** or an **isomorphism** if T is a bijection and $T^{-1} \in L(Y, X)$.

Definition 7.27. Let X be a Banach space. Define $GL(X) := \{T \in L(X, X) : T \text{ is invertible}\}$.

Exercise 7.28. Let X be a Banach space. Then

(1) For each $T \in L(X, X)$, if $\|I - T\| < 1$, then T is invertible and

$$T^{-1} = \sum_{n=0}^{\infty} (I - T)^n$$

(2) For each $S, T \in L(X, X)$, if S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$, then T is invertible.

(3) $GL(X)$ is open.

Proof. (1) Let $T \in L(X, X)$. Suppose that $\|I - T\| < 1$. Then

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n < \infty$$

. Since X is a complete, so is $L(X, X)$ and thus $\sum_{n=0}^{\infty} (I - T)^n$ converges in $L(X, X)$.

Define $(S_k)_{k=0}^{\infty} \subset L(X, X)$ and $S \in L(X, X)$ by $S_k = \sum_{n=0}^k (I - T)^n$ and

$S = \sum_{n=0}^{\infty} (I - T)^n$. Then for each $k \in \mathbb{N}$,

$$\begin{aligned} S_k T &= S_k - S_k (I - T) \\ &= (I - T)^0 - (I - T)^{k+1} \\ &= I - (I - T)^{k+1} \end{aligned}$$

and $\|S_k T - I\| \leq \|I - T\|^{k+1}$. Since multiplication on Banach algebras is continuous, we have that

$$ST = \left(\lim_{k \rightarrow \infty} S_k\right)T = \lim_{k \rightarrow \infty} S_k T = I$$

Similarly $TS = I$. Thus T is invertible and $T^{-1} = S \in L(X, X)$.

(2) Let $S, T \in L(X, X)$. Suppose that S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$. Then

$$\begin{aligned} \|I - S^{-1}T\| &= \|S^{-1}(S - T)\| \\ &\leq \|S^{-1}\| \|S - T\| \\ &< 1 \end{aligned}$$

So $S^{-1}T$ is invertible. Thus $T = S(S^{-1}T)$ is invertible.

(3) Let $T \in GL(X)$. Choose $\delta = \|T^{-1}\|^{-1}$. By (2), $B(T, \delta) \subset GL(X)$. □

Exercise 7.29. Let $M(X, \mathcal{A})$ denote the set of complex measures on the measurable space (X, \mathcal{A}) . Define $\|\cdot\| : M(X, \mathcal{A}) \rightarrow [0, \infty)$ by $\|\mu\| = |\mu|(X)$. Then $\|\cdot\|$ is a norm on $M(X, \mathcal{A})$.

Proof. Let $\mu, \nu \in M(X, \mathcal{A})$ and $\alpha \in \mathbb{C}$. Exercises in a previous section tell us that $|\mu + \nu| \leq |\mu| + |\nu|$ and $|\alpha\mu| = |\alpha||\mu|$. So clearly $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ and $\|c\mu\| = |c|\|\mu\|$. If $\|\mu\| = 0$, then X is μ -null and μ is the zero measure. □

7.2. Linear Functionals.

Definition 7.30. Let X be a normed vector space and $T : X \rightarrow \mathbb{C}$. Then T is said to be a **linear functional on X** if T is linear and T is said to be a **bounded linear functional on X** if $T \in L(X, \mathbb{C})$. We define the **dual space of X** , denoted X^* , by $X^* = L(X, \mathbb{C})$.

Definition 7.31. Let X be a normed vector space and $p : X \rightarrow \mathbb{R}$. Then p is said to be a **sublinear functional** if for each $x, y \in X$, $\lambda \geq 0$,

- (1) $p(x + y) \leq p(x) + p(y)$
- (2) $p(\lambda x) = \lambda p(x)$

Note 7.32. Let X be a vector space and $\|\cdot\| : X \rightarrow [0, \infty)$ be a seminorm, then $\|\cdot\|$ is a sublinear functional.

Theorem 7.33. *Hahn-Banach Theorem:* Let X be a vector space, $p : X \rightarrow \mathbb{R}$ a sublinear functional, $M \subset X$ a subspace and $f : M \rightarrow \mathbb{C}$ a linear functional. If for each $x \in M$, $|f(x)| \leq p(x)$, then there exists a linear functional $F : X \rightarrow \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x)$ and $F|_M = f$.

Exercise 7.34. Let X be a normed vector space, $M \subset X$ a subspace and $f \in M^*$. Then there exists $F \in X^*$ such that $\|F\| = \|f\|$ and $F|_M = f$.

Proof. If $f = 0$, Choose $F = 0$. Suppose $f \neq 0$. Then $\|f\| \neq 0$ and there exists $x_0 \in M$ such that $x_0 \neq 0$. Thus $\|f\| = \sup\{|f(x)| : x \in M \text{ and } \|x\| = 1\}$. Define $p : X \rightarrow [0, \infty)$ by $p(x) = \|f\|\|x\|$. Then p is a sublinear functional on X and for each $x \in M$, $|f(x)| \leq p(x)$. So

there exists a linear functional $F : X \rightarrow \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x) = \|f\|\|x\|$ and $F|_M = f$. Thus $F \in X^*$ with $\|F\| \leq \|f\|$. Also

$$\|F\| = \sup_{\substack{x \in X \\ \|x\|=1}} |F(x)| \geq \sup_{\substack{x \in M \\ \|x\|=1}} |F(x)| = \sup_{\substack{x \in M \\ \|x\|=1}} |f(x)| = \|f\|$$

So $\|F\| = \|f\|$. □

Exercise 7.35. Let X be a normed vector space, $M \subsetneq X$ a proper closed subspace and $x \in X \setminus M$. Then there exists $F \in X^*$ such that $F|_M = 0$, $\|F\| = 1$ and $F(x) = \|x + M\| \neq 0$. (Hint: Consider $f : M + \mathbb{C}x \rightarrow \mathbb{C}$ defined by $f(m + \lambda x) = \lambda\|x + M\|$.)

Proof. Define $f : M + \mathbb{C}x \rightarrow \mathbb{C}$ as above. Clearly f is linear and $f|_M = 0$. Let $m \in M$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $|f(m + \lambda x)| = 0 \leq \|m + \lambda x\|$. Suppose that $\lambda \neq 0$. Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \inf_{z \in M} \|z + \lambda x\| \\ &\leq \|m + \lambda x\| \end{aligned}$$

So $f \in (M + \mathbb{C}x)^*$ and $\|f\| \leq 1$. Let $\epsilon > 0$. A previous exercise tells us that there exist $m \in M, \lambda \in \mathbb{C}$ such that $\|m + \lambda x\| = 1$ and $\|m + \lambda x + M\| > 1 - \epsilon$. Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \|m + \lambda x + M\| \\ &> 1 - \epsilon \end{aligned}$$

So

$$\|f\| = \sup_{\substack{z \in M + \mathbb{C}x \\ \|z\|=1}} |f(z)| \geq 1$$

Hence $\|f\| = 1$. The same exercise also tells us that $f(x) = \|x + M\| \neq 0$. Using the previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{M + \mathbb{C}x} = f$. □

Exercise 7.36. Let X be a normed vector space and $x \in X$. If $x \neq 0$, then there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$.

Proof. Define $f : \mathbb{C}x \rightarrow \mathbb{C}$ by $f(\lambda x) = \lambda\|x\|$. Then f is linear and $f(x) = \|x\|$. Clearly

$$\sup_{\substack{z \in \mathbb{C}x \\ \|z\|=1}} |f(z)| = 1$$

So $f \in (\mathbb{C}x)^*$ and $\|f\| = 1$. By a previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{\mathbb{C}x} = f$. □

Exercise 7.37. Let X be a normed vector space. Then X^* separates the points of X .

Proof. Let $x, y \in X$. Suppose that $x \neq y$. Then $x - y \neq 0$. The previous exercise implies that there exists $F \in X^*$ such that $\|F\| = 1$ and

$$F(x) - F(y) = F(x - y) = \|x - y\| \neq 0$$

Thus $F(x) \neq F(y)$ and X^* separates the points of X . □

Definition 7.38. Let X, Y be metric spaces and $T : X \rightarrow Y$. Then T is said to be an **isometry** if for each $x_1, x_2 \in X$, $d(Tx_1, Tx_2) = d(x_1, x_2)$.

Exercise 7.39. Let X, Y be metric spaces and $T : X \rightarrow Y$ an isometry. Then T is injective.

Proof. Let $x_1, x_2 \in X$. Suppose that $Tx_1 = Tx_2$. Then $0 = d(Tx_1, Tx_2) = d(x_1, x_2)$. So $x_1 = x_2$. Hence T is injective. \square

Note 7.40. Let X, Y be metric spaces and $T : X \rightarrow Y$ an isometry. Then T is clearly continuous. If T is surjective, then T^{-1} is an isometry and therefore continuous. Hence T is a homeomorphism.

Exercise 7.41. Let X be a normed vector space and $x \in X$. Define $\hat{x} : X^* \rightarrow \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| = \|x\|$.

Proof. Let $f, g \in X^*$ and $\lambda \in \mathbb{C}$. Then

$$\hat{x}(f + \lambda g) = (f + \lambda g)(x) = f(x) + \lambda g(x) = \hat{x}(f) + \lambda \hat{x}(g)$$

So \hat{x} is linear. For each $f \in X^*$,

$$|\hat{x}(f)| = |f(x)| \leq \|x\| \|f\|$$

Hence $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leq \|x\|$. If $x = 0$, then $\hat{x} = 0$ and $\|\hat{x}\| = \|x\|$. Suppose that $x \neq 0$. Then a previous exercise implies that there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$. Then we have that

$$\sup_{\substack{f \in X^* \\ \|f\|=1}} |\hat{x}(f)| = \sup_{\substack{f \in X^* \\ \|f\|=1}} |f(x)| \geq |F(x)| = \|x\|$$

Hence $\|\hat{x}\| = \|x\|$. \square

Exercise 7.42. Let X be a normed vector space. Define $\phi : X \rightarrow X^{**}$ by $\phi(x) = \hat{x}$. Then ϕ is a linear isometry.

Proof. Let $x, y \in X$ and $\lambda \in \mathbb{C}$. Then for each $f \in X^*$, we have that

$$\begin{aligned} \phi(x + \lambda y)(f) &= \widehat{x + \lambda y}(f) \\ &= f(x + \lambda y) \\ &= f(x) + \lambda f(y) \\ &= \hat{x}(f) + \lambda \hat{y}(f) \\ &= \phi(x)(f) + \lambda \phi(y)(f) \end{aligned}$$

So $\phi(x + \lambda y) = \phi(x) + \lambda \phi(y)$ and ϕ is linear. The previous exercise tells us that

$$\begin{aligned} \|\phi(x) - \phi(y)\| &= \|\phi(x - y)\| \\ &= \|\widehat{x - y}\| = \|x - y\| \end{aligned}$$

So ϕ is an isometry. \square

Definition 7.43. Let X be a normed vector space and define $\phi : X \rightarrow X^{**}$ as above. We define $\widehat{X} = \phi(X) \subset X^{**}$. Since \widehat{X} and X are isomorphic, we may identify X as a subset of X^{**} .

Definition 7.44. Let X be a normed vector space and define $\phi : X \rightarrow X^{**}$ as above. Then X is said to be **reflexive** if ϕ is surjective. In this case ϕ is then an isomorphism

Exercise 7.45. Let X be a normed vector space and $f : X \rightarrow \mathbb{C}$ a linear functional on X . Then f is bounded iff $\ker f$ is closed.

Proof. Suppose that f is continuous. Since $\{0\}$ is closed, we have that $\ker f = f^{-1}(\{0\})$ is closed. Conversely, suppose that $\ker f$ is closed. If $\ker f = X$, then $f = 0$ and f is continuous. Suppose that $\ker f \neq X$. Then $\ker f$ is a proper, closed subspace of X . A previous exercise tells us that there exists $x \in X$ such that $\|x\| = 1$ and $\|x + \ker f\| > \frac{1}{2}$. Let $y \in X$. Suppose that $\|y\| < \frac{1}{2}$. Then for each $z \in \ker f$,

$$\begin{aligned} \|z - (x + y)\| &= \|(z - x) - y\| \\ &\geq \|z - x\| - \|y\| \\ &> \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

So $x + y \notin \ker f$. Therefore $f(B(x, \frac{1}{2})) \cap \{0\} = \emptyset$. If $f(B(x, \frac{1}{2}))$ is unbounded, then $f(B(x, \frac{1}{2})) = \mathbb{C}$ by linearity. This is a contradiction since $0 \notin f(B(x, \frac{1}{2}))$. So There exists $s > 0$ such that $f(B(x, \frac{1}{2})) \subset B(0, s)$ and thus f is bounded. \square

Exercise 7.46. Let X be a normed vector space.

- (1) Let $M \subsetneq X$ be a proper closed subspace of X and $x \in X \setminus M$. Then $M + \mathbb{C}x$ is closed.
- (2) Let $M \subset X$ be a finite dimensional subspace of X . Then M is closed.

Proof. (1) Let $y \in X$ and $(y_n)_{n \in \mathbb{N}} \subset M + \mathbb{C}x$. Suppose that $y_n \rightarrow y$. If $y \in M$, then $y \in M + \mathbb{C}x$. Suppose that $y \notin M$. For each $n \in \mathbb{N}$, there exists $m_n \in M$ and $\lambda_n \in \mathbb{C}$ such that $y_n = m_n + \lambda_n x$. A previous exercise tells us that there exists $F \in X^*$ such that $\|F\| = 1$, $F|_M = 0$ and $F(x) = \|x + M\| \neq 0$. Since F is continuous, $F(y_n) \rightarrow F(y)$. Since for each $n \in \mathbb{N}$,

$$F(y_n) = F(m_n + \lambda_n x) = F(m_n) + \lambda_n(F(x)) = \lambda_n F(x)$$

we have that $\lambda_n F(x) \rightarrow F(y)$. Since $F(x) \neq 0$, this implies that $\lambda_n \rightarrow F(x)^{-1}F(y)$. It follows that $\lambda_n x \rightarrow F(x)^{-1}F(y)x$. Since for each $n \in \mathbb{N}$, $m_n = y_n - \lambda_n x$, we know that $m_n \rightarrow y - F(x)^{-1}F(y)x$. Since $(m_n)_{n \in \mathbb{N}} \subset M$ and M is closed, we have that $y - F(x)^{-1}F(y)x \in M$ and therefore $y \in M + \mathbb{C}x$. Hence $M + \mathbb{C}x$ is closed.

- (2) If $M = X$, then M is closed. Suppose that $M \neq X$. Let $(x_i)_{i=1}^n$ be a basis for M . Define $N_0 = \{0\}$ and for each $i = 1, 2, \dots, n$, define $N_i = N_{i-1} + \mathbb{C}x_i$. Since N_0 is a proper closed subspace of X and $x_1 \in X \setminus N_0$, (1) implies that N_1 is closed. Proceed inductively to obtain that $M = N_n$ is closed. \square

Exercise 7.47. Let X be an infinite-dimensional normed vector space.

- (1) There exists a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that for each $m, n \in \mathbb{N}$, $\|x_n\| = 1$ and if $m \neq n$, then $\|x_m - x_n\| > \frac{1}{2}$.
- (2) X is not locally compact.

Proof. (1) Define $N_0 = \{0\}$. Then N_0 is a closed proper subspace of X . Choose $x_1 \in X$ such that $\|x_1\| = 1$. Using the results of previous exercises, we proceed inductively.

For each $n \geq 2$ we define $N_{n-1} = \text{span}(x_1, x_2, \dots, x_{n-1})$. Then N_{n-1} is a closed proper subspace of X . Thus we may choose $x_n \in X$ such that $\|x_n\| = 1$ and $\|x_n + N_{n-1}\| > \frac{1}{2}$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then $x_m \in N_{n-1}$. Thus $\|x_n - x_m\| \geq \|x_n + N_{n-1}\| > \frac{1}{2}$.

- (2) Suppose that X is locally compact. Then $\overline{B(0, 1)}$ is compact and therefore sequentially compact. Using $(x_n)_{n \in \mathbb{N}} \subset \overline{B(0, 1)}$ defined in (1), we see that there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$, $x \in \overline{B(0, 1)}$ such that $x_{n_k} \rightarrow x$. Then $(x_{n_k})_{k \in \mathbb{N}}$ is Cauchy. So there exists $N \in \mathbb{N}$ such that for each $j, k \in \mathbb{N}$, if $j, k \geq N$, then $\|x_{n_j} - x_{n_k}\| < \frac{1}{2}$. Then $\|x_{n_N} - x_{n_{N+1}}\| < \frac{1}{2}$. This is a contradiction since by construction, $\|x_{n_N} - x_{n_{N+1}}\| > \frac{1}{2}$. Thus X is not locally compact. \square

Exercise 7.48. Let X, Y be normed vector spaces and $T \in L(X, Y)$.

- (1) Define the **adjoint of** T , $T^* : Y^* \rightarrow X^*$ by $T^*(f) = f \circ T$. Then $T^* \in L(Y^*, X^*)$.
- (2) Applying the result from (1) twice, we have that $T^{**} \in L(X^{**}, Y^{**})$. We have that for each $x \in X$, $T^{**}(\hat{x}) = \widehat{T(x)}$.
- (3) T^* is injective iff $T(X)$ is dense in Y .
- (4) If $T^*(Y^*)$ is dense in X^* , then T is injective. The converse is true if X is reflexive.

Proof. (1) Let $f \in Y^*$. Then $\|T^*(f)\| = \|f \circ T\| \leq \|T\|\|f\|$. So $T^* \in L(Y^*, X^*)$ with $\|T^*\| \leq \|T\|$.

- (2) Let $x \in X$. Let $f \in Y^*$. Then

$$\begin{aligned} T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= \widehat{T(x)}(f) \end{aligned}$$

Hence $T^{**}(\hat{x}) = \widehat{T(x)}$.

- (3) Suppose that $T(X)$ is not dense in Y . Then $\overline{T(X)} \neq Y$. So $T(X)$ is a proper closed subspace of Y and there exists $y \in Y$ such that $y \notin \overline{T(X)}$. By a previous exercise, there exists $f \in Y^*$ such that $f(y) = \|y + \overline{T(X)}\| \neq 0$, $\|f\| = 1$ and $f|_{\overline{T(X)}} = 0$. Let $x \in X$. Then $T^*(f)(x) = f \circ T(x) = 0$. Hence $T^*(f) = 0 = T^*(0)$. Since $f \neq 0$, T^* is not injective.

Now suppose that $T(X)$ is dense in Y . Let $f, g \in Y^*$. Define $h \in Y^*$ by $h = f - g$. Suppose that $T^*(f) = T^*(g)$. Then $T^*(h) = 0$. So for each $x \in X$, $h(T(x)) = 0$. Let $y \in Y$ and $\epsilon > 0$. By continuity, there exists $\delta > 0$ such that for each $y' \in Y$, if

$\|y - y'\| < \delta$, then $\|h(y) - h(y')\| < \epsilon$. Since $T(X)$ is dense in Y , there exists $x \in X$ such that $\|y - T(x)\| < \delta$. Thus

$$\begin{aligned}\|h(y)\| &\leq \|h(y) - h(T(x))\| + \|h(T(x))\| \\ &= \|h(y) - h(T(x))\| \\ &< \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|h(y)\| = 0$. This implies that $h(y) = 0$ and therefore $f(y) = g(y)$. Since $y \in Y$ is arbitrary, $f = g$ and T^* is injective.

- (4) For the sake of contradiction, suppose that $T^*(Y^*)$ is dense in X^* and T is not injective. Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $T(x_1) = T(x_2)$. Define $x = x_1 - x_2$. Then $x \neq 0$ and $T(x) = 0$. A previous exercise implies that there exists $F \in X^*$ such that $F(x) = \|x\| \neq 0$ and $\|F\| = 1$. Let $\epsilon > 0$. Choose $g \in Y^*$ such that $\|F - T^*(g)\| < \epsilon$. Then

$$\begin{aligned}\|x\| &= |F(x)| \\ &\leq |F(x) - T^*(g)(x)| + |T^*(g)(x)| \\ &< \epsilon\|x\| + |g(T(x))| \\ &= \epsilon\|x\|\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\|x\| = 0$ which is a contradiction. Hence if $T^*(Y^*)$ is dense in X^* , then T is injective.

Now, suppose that X is reflexive and T is injective. Let $\phi_1, \phi_2 \in X^{**}$. Suppose that $T^{**}(\phi_1) = T^{**}(\phi_2)$. Then $T^{**}(\phi_1 - \phi_2) = 0$. Since X is reflexive, there exist $x_1, x_2 \in X$ such that $\phi_1 = \hat{x}_1$ and $\phi_2 = \hat{x}_2$. Define $x = x_1 - x_2$. Then $T^{**}(\hat{x}) = 0$. So for each $f \in Y^*$,

$$\begin{aligned}T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= 0\end{aligned}$$

Suppose that $T(x) \neq 0$. Then a previous exercise implies that there exists $g \in Y^*$ such that $g(T(x)) = \|T(x)\| \neq 0$ and $\|g\| = 1$. This is a contradiction since $g(T(x)) = 0$. So $T(x) = 0$. Since T is injective, this implies that $x = 0$. Hence $\hat{x} = 0$ and thus $\phi_1 = \phi_2$. Thus T^{**} is injective. By (3), we have that $T^*(Y^*)$ is dense in X^* . □

Exercise 7.49. Let X be a normed vector space. Then X is reflexive iff X^* is reflexive.

Proof. Suppose that X is reflexive. Let $\alpha \in X^{***}$. Define $f : X \rightarrow \mathbb{C}$ by $f(x) = \alpha(\hat{x})$. Clearly f is linear and a previous exercise tells us that for each $x \in X$,

$$\begin{aligned} |f(x)| &\leq \|\alpha\| \|\hat{x}\| \\ &= \|\alpha\| \|x\| \end{aligned}$$

So $f \in X^*$. Let $\phi \in X^{**}$. Since X is reflexive, there exists $x \in X$ such that $\phi = \hat{x}$. Then

$$\begin{aligned} \alpha(\phi) &= \alpha(\hat{x}) \\ &= f(x) \\ &= \hat{x}(f) \\ &= \hat{f}(\hat{x}) \\ &= \hat{f}(\phi) \end{aligned}$$

Hence $\alpha = \hat{f}$. Thus the map $X^* \rightarrow X^{***}$ given by $f \mapsto \hat{f}$ is surjective and so X^* is reflexive.

Conversely, suppose that X^* is reflexive. Since $\phi : X \rightarrow X^{**}$ given by $\phi(x) = \hat{x}$ is an isometry, $\widehat{X} \subset X^{**}$ is closed. For the sake of contradiction, suppose that $\widehat{X} \neq X^{**}$. Then there exists $\alpha \in X^{**}$ such that $\alpha \notin \widehat{X}$. Thus there exists $F \in X^{***}$ such that $\|F\| = 1$, $F(\alpha) = \|\alpha + \widehat{X}\| \neq 0$ and $F|_{\widehat{X}} = 0$. Since X^* is reflexive, there exists $f \in X^*$ such that $F = \hat{f}$. A previous exercise tells us that $\|f\| = \|\hat{f}\| = \|F\| = 1$. Since for each $x \in X$, $f(x) = \hat{x}(f) = \hat{f}(\hat{x}) = F(\hat{x}) = 0$, we have that $f = 0$. Thus $\|f\| = 0$, a contradiction. So $\widehat{X} = X^{**}$ and X is reflexive. □

7.3. The Baire Category Theorem and Consequences.

Theorem 7.50. *Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is surjective, then T is open.*

Corollary 7.51. *Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is a bijection, then $T^{-1} \in L(X, Y)$.*

Definition 7.52. *Let X, Y be sets and $f : X \rightarrow Y$. We define the **graph of f** , $\Gamma(f)$, by $\Gamma(f) = \{(x, y) \in X \times Y : f(x) = y\}$.*

Theorem 7.53. *Let X, Y be Banach spaces and $T : X \rightarrow Y$ a linear map. If $\Gamma(T)$ is closed, then $T \in L(X, Y)$.*

Note 7.54. *We recall that $\Gamma(T)$ is closed iff for each $(x_n)_{n \in \mathbb{N}} \subset X$, $x \in X$ and $y \in Y$ if $x_n \rightarrow x$ and $T(x_n) \rightarrow y$, then $T(x) = y$.*

Theorem 7.55. *Let X, Y be Banach spaces and $S \subset L(X, Y)$. If for each $x \in X$,*

$$\sup_{T \in S} \|Tx\| < \infty$$

then

$$\sup_{T \in S} \|T\| < \infty$$

Exercise 7.56. Let μ be counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Define $h : \mathbb{N} \rightarrow \mathbb{N}$ and ν on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ by $h(n) = n$ and $d\nu = h d\mu$. Define $X = L^1(\nu)$ and $Y = L^1(\mu)$. Equip both X and Y with the L^1 norm with respect to μ .

- (1) We have that X is a proper subspace of Y and therefore X is not complete.
- (2) Define $T : X \rightarrow Y$ by $Tf(n) = nf(n)$. Then T is linear, $\Gamma(T)$ is closed, and T is unbounded.
- (3) Define $S : Y \rightarrow X$ by $Sg(n) = \frac{1}{n}g(n)$. Then $S \in L(Y, X)$, S is surjective and S is not open.

Proof. (1) Note that for each $f : \mathbb{N} \rightarrow \mathbb{C}$,

$$\begin{aligned} \|f\|_{\mu,1} &= \sum_{n=1}^{\infty} |f(n)| \\ &\leq \sum_{n=1}^{\infty} n|f(n)| \\ &= \|f\|_{\nu,1} \end{aligned}$$

Hence X is a subspace of Y . Define $f : \mathbb{N} \rightarrow \mathbb{C}$ by $f(n) = \frac{1}{n^2}$. Then

$$\|f\|_{\mu,1} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

So $f \in Y$. However

$$\|f\|_{\nu,1} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

So $f \notin X$. Thus X is a proper subspace of Y . Let $g \in Y$ and $\epsilon > 0$. Since the simple functions are dense in $L^1(\mu)$, there exists $\phi \in L^1(\mu)$ such that ϕ is simple and $\|g - \phi\|_{\mu,1} < \epsilon$. Then there exist $(c_i)_{i=1}^k \subset \mathbb{C}$ and $(E_i)_{i=1}^k \subset \mathcal{P}(\mathbb{N})$ such that for each $i = 1, 2, \dots, k$, E_i is finite and

$$\phi = \sum_{i=1}^k c_i \chi_{E_i}$$

Define $c = \max\{|c_i| : i = 1, 2, \dots, k\}$ and $m = \max \bigcup_{i=1}^k E_i$. Then

$$\begin{aligned} \|\phi\|_{\nu,1} &= \sum_{n=1}^m n|\phi(n)| \\ &\leq \sum_{n=1}^m mc \\ &= cm^2 \\ &< \infty \end{aligned}$$

Hence $\phi \in X$ and X is dense in Y . Since X is a dense, proper subspace, it is not closed. Since Y is complete and $X \subset Y$ is not closed, we have that X is not complete.

- (2) Clearly T is linear. Let $(f_j)_{j \in \mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_j \xrightarrow{L^1(\mu)} f$ and $Tf_j \xrightarrow{L^1(\mu)} g$.

Note that for each $j \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$|f_j(n) - f(n)| \leq \sum_{n=1}^{\infty} |f_j(n) - f(n)| = \|f_j - f\|_{\mu,1}$$

and

$$|nf_j(n) - g(n)| \leq \sum_{n=1}^{\infty} |nf_j(n) - g(n)| = \|Tf_j - g\|_{\mu,1}$$

Thus for each $n \in \mathbb{N}$, $f_j(n) \xrightarrow{j} f(n)$ and $nf_j(n) \xrightarrow{j} g(n)$. This implies that for each $n \in \mathbb{N}$, $nf(n) = g(n)$. Thus $Tf = g$ which implies that $\Gamma(T)$ is closed. Suppose, for the sake of contradiction, that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\|_{\mu,1} \leq C\|f\|_{\mu,1}$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f : \mathbb{N} \rightarrow \mathbb{C}$ by $f = \chi_{\{n\}}$. As established above, $S^+ \subset L^1(\mu)$. Then $\|f\|_{\mu,1} = 1$ and

$$\begin{aligned} \|Tf\|_{\mu,1} &= n \\ &> C \\ &= C\|f\|_{\mu,1} \end{aligned}$$

which is a contradiction. So T is unbounded.

- (3) Clearly S is linear. Let $g \in Y$. Then

$$\begin{aligned} \|Sg\|_{\mu,1} &= \sum_{n=1}^{\infty} \frac{1}{n} |g(n)| \\ &\leq \sum_{n=1}^{\infty} |g(n)| \\ &= \|g\|_{\mu,1} \end{aligned}$$

So S is bounded and $\|S\| \leq 1$. Thus $S \in L(Y, X)$. Let $f \in X$. Define $g : \mathbb{N} \rightarrow \mathbb{C}$ by $g(n) = nf(n)$. By definition, $g \in Y$ and we have that

$$\begin{aligned} Sg(n) &= \frac{1}{n} g(n) \\ &= f(n) \end{aligned}$$

Hence $Sg = f$ and thus S is surjective. Let $g \in Y$. Suppose that $Sg = 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n)| = \|Sg\| = 0$$

Thus for each $n \in \mathbb{N}$, $g(n) = 0$. Hence $\ker g = \{0\}$ and g is injective. Note that $S^{-1} = T$. If g is open, then T is continuous which as shown above is a contradiction. So g is not open. □

Exercise 7.57. Let $X = C^1([0, 1])$ and $Y = C([0, 1])$. Equip both X and Y with the uniform norm.

- (1) Then X is not complete

(2) Define $T : X \rightarrow Y$ by $Tf = f'$. Then $\Gamma(T)$ is closed and T is not bounded.

Proof. (1) Recall that for each $a, b \geq 0$ and $p \in \mathbb{N}$,

$$(a^{\frac{1}{p}} + b^{\frac{1}{p}})^p = \sum_{n=0}^p \binom{p}{n} a^{\frac{n}{p}} b^{\frac{p-n}{p}} \geq a + b$$

Thus $(a + b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$.

For each $n \in \mathbb{N}$, define $f_n : [0, 1] \rightarrow \mathbb{C}$ by $f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n^2}}$. Then $(f_n)_{n \in \mathbb{N}} \subset X$. Define $f : [0, 1] \rightarrow \mathbb{C}$ by $f(x) = |x - \frac{1}{2}|$. Then $f \in Y \cap X^c$. Note that for each $n \in \mathbb{N}$, $f \leq f_n$. Our observation above implies that for each $x \in X$,

$$\begin{aligned} f_n(x) &= \left[(x - \frac{1}{2})^2 + \frac{1}{n^2} \right]^{\frac{1}{2}} \\ &\leq |x - \frac{1}{2}| + \frac{1}{n} \end{aligned}$$

Thus $0 \leq f_n - f \leq \frac{1}{n}$. This implies that $f_n \xrightarrow{u} f$. Since $f \notin X$, X is not complete.

(2) Let $(f_n)_{n \in \mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_n \xrightarrow{u} f$ and $Tf_n \xrightarrow{u} g$. Let $x \in [0, 1]$. Then $f_n(x) \rightarrow f(x)$ and $f_n(0) \rightarrow f(0)$ and $f'_n \xrightarrow{u} g$. Applying the DCT to this sequence of integrable functions that converges uniformly to an integrable function on a finite measure space (a previous exercise) we have that

$$\begin{aligned} f_n(x) - f_n(0) &= \int_{[0, x]} f'_n dm \\ &\rightarrow \int_{[0, x]} g dm \end{aligned}$$

Since $f_n(x) - f_n(0) \rightarrow f(x) - f(0)$, we know that

$$f(x) - f(0) = \int_{[0, x]} g dm$$

. Thus $Tf = g$ and $\Gamma(T)$ is closed.

Suppose for the sake of contradiction that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\| \leq C\|f\|$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f \in X$ by $f(x) = x^n$. Then $\|f\| = 1$ and

$$\begin{aligned} \|Tf\| &= \|f'\| \\ &= n \\ &> C \\ &= C\|f\| \end{aligned}$$

which is a contradiction. So T is not bounded. □

Exercise 7.58. Let X, Y be Banach spaces and $T \in L(X, Y)$. Then $X/\ker T \cong T(X)$ iff $T(X)$ is closed.

Proof. Since X is a Banach space and T is continuous, we have that $\ker T$ is closed and $X/\ker T$ is a Banach space. Suppose that $X/\ker T \cong T(X)$. Then $T(X)$ is complete. Since Y is complete, this implies that $T(X)$ is closed.

Conversely Suppose that $T(X)$ is closed. Then $T(X)$ is complete. Define $S : X/\ker T \rightarrow T(X)$ by $S(x + \ker T) = T(x)$. A previous exercise tells us that the map $S : X/\ker T \rightarrow T(X)$ defined by $S(x + \ker T) = T(x)$ is a bounded linear bijection. Since $T(X)$ is complete and S is surjective, S^{-1} is bounded and thus S is an isomorphism. \square

Exercise 7.59. Let X be a separable Banach space. Define $B_X = \{x \in X : \|x\| < 1\}$. Let $(x_n)_{n \in \mathbb{N}} \subset B_X$ a dense subset of the unit ball and μ the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Define $T : L^1(\mu) \rightarrow X$ by

$$Tf = \sum_{n=1}^{\infty} f(n)x_n$$

Then

- (1) T is well defined and $T \in L(L^1(\mu), X)$
- (2) T is surjective
- (3) There exists a closed subspace $K \subset L^1(\mu)$ such that $L^1(\mu)/K \cong X$

Proof. (1) Let $f \in L^1(\mu)$. Since X is complete and

$$\begin{aligned} \sum_{n=1}^{\infty} \|f(n)x_n\| &= \sum_{n=1}^{\infty} |f(n)|\|x_n\| \\ &\leq \sum_{n=1}^{\infty} |f(n)| \\ &< \infty \end{aligned}$$

we have that $\sum_{n=1}^{\infty} f(n)x_n$ converges and thus $Tf \in X$. Hence T is well defined.

Clearly T is linear. Let $f \in L^1(\mu)$. Then

$$\begin{aligned} \|Tf\| &= \left\| \sum_{n=1}^{\infty} f(n)x_n \right\| \\ &\leq \sum_{n=1}^{\infty} \|f(n)x_n\| \\ &\leq \sum_{n=1}^{\infty} |f(n)| \\ &= \|f\|_1 \end{aligned}$$

So T is bounded with $\|T\| \leq 1$.

- (2) Let $x \in X$. Suppose that $\|x\| < 1$. Then $x \in B_X$. So there exists $n_1 \in \mathbb{N}$ such that $\|x - x_{n_1}\| < \frac{1}{2}$. Then $2(x - x_{n_1}) \in B_X$. Since for each $j \in \mathbb{N}$, $B_X \setminus (x_n)_{n=1}^j$ is dense in B_X , there exists $n_2 \in \mathbb{N}$ such that $x_{n_2} \notin (x_n)_{n=1}^{n_1}$ and $\|2(x - x_{n_1}) - x_{n_2}\| < \frac{1}{2}$ which implies that $\|x - (x_{n_1} - \frac{1}{2}x_{n_2})\| < \frac{1}{4}$.

Proceed inductively to obtain a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that for each $k \geq 2$, $x_{n_k} \notin (x_n)_{n=1}^{n_{k-1}}$ and $\|x - \sum_{j=1}^k 2^{1-j}x_{n_j}\| < \frac{1}{2^k}$. Then $x = \sum_{k=1}^{\infty} 2^{1-k}x_{n_k}$.

Define $f : \mathbb{N} \rightarrow \mathbb{C}$ by $f = \sum_{k=1}^{\infty} 2^{1-k}\chi_{\{n_k\}}$. Then $\|f\|_1 = \sum_{k=1}^{\infty} 2^{1-k} < \infty$, so $f \in L^1(\mu)$ and $Tf = \sum_{k=1}^{\infty} 2^{1-k}x_{n_k} = x$. Now, suppose that $\|x\| \geq 1$, then $\frac{1}{2\|x\|}x \in B_X$.

The above argument shows that there exists $f \in L^1(\mu)$ such that $Tf = \frac{1}{2\|x\|}x$. Then $2\|x\|f \in L^1(\mu)$ and $T(2\|x\|f) = 2\|x\|Tf = x$.

So for each $x \in X$, there exists $f \in L^1(\mu)$ such that $Tf = x$ and thus T is surjective.

- (3) Since X is a Banach space and T is surjective, the previous exercise implies that $L^1(\mu)/\ker T \cong X$.

□

Exercise 7.60. Let X, Y be Banach spaces and $T : X \rightarrow Y$ a linear map. If for each $f \in Y^*$, $f \circ T \in X^*$, then $T \in L(X, Y)$.

Proof. Suppose that for each $f \in Y^*$, $f \circ T \in X^*$. Let $x \in X$,

□

8. PROBABILITY

8.1. Distributions.

Definition 8.1. Let Ω be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then \mathcal{P} is said to be a π -**system** on Ω if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Definition 8.2. Let Ω be a set and $\mathcal{L} \subset \mathcal{P}(\Omega)$. Then \mathcal{L} is said to be a λ -**system** on Ω if

- (1) $\mathcal{L} \neq \emptyset$
- (2) for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$
- (3) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}$, if $(A_n)_{n \in \mathbb{N}}$ is disjoint, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}$

Exercise 8.3. Let Ω be a set and \mathcal{L} a λ -system on Ω . Then

- (1) $\Omega, \emptyset \in \mathcal{L}$

Proof. Straightforward. □

Definition 8.4. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. Put

$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(\Omega) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the λ -**system on Ω generated by \mathcal{C}** , $\lambda(\mathcal{C})$, to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L} \in \mathcal{S}} \mathcal{L}$$

Exercise 8.5. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. If \mathcal{C} is a λ -system and \mathcal{C} is a π -system, then \mathcal{C} is a σ -algebra.

Proof. Suppose that \mathcal{C} is a λ -system and \mathcal{C} is a π -system. Then we need only verify the third axiom in the definition of a σ -algebra. Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$. Define $B_1 = A_1$ and for $n \geq 2$, define $B_n = A_n \cap \left(\bigcup_{k=1}^{n-1} A_k \right)^c = A_n \cap \left(\bigcap_{k=1}^{n-1} A_k^c \right) \in \mathcal{C}$. Then $(B_n)_{n \in \mathbb{N}}$ is disjoint and therefore $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{C}$. □

Theorem 8.6. (*Dynkin's Theorem*)

Let Ω be a set.

- (1) Let \mathcal{P} be a π -system on Ω and \mathcal{L} a λ -system on Ω . If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
- (2) Let \mathcal{P} be a π -system on Ω . Then $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

Exercise 8.7. Let (Ω, \mathcal{F}) be a measurable space and μ, ν probability measures on (Ω, \mathcal{F}) . Put $\mathcal{L}_{\mu, \nu} = \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$. Then $\mathcal{L}_{\mu, \nu}$ is a λ -system on Ω .

Proof.

- (1) $\emptyset \in \mathcal{L}_{\mu, \nu}$.
- (2) Let $A \in \mathcal{L}_{\mu, \nu}$. Then $\mu(A) = \nu(A)$. Thus

$$\begin{aligned} \mu(A^c) &= 1 - \mu(A) \\ &= 1 - \nu(A) \\ &= \nu(A^c) \end{aligned}$$

So $A^c \in \mathcal{L}_{\mu, \nu}$.

- (3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}_{\mu, \nu}$. So for each $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$. Suppose that $(A_n)_{n \in \mathbb{N}}$ is disjoint. Then

$$\begin{aligned} \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \sum_{n \in \mathbb{N}} \mu(A_n) \\ &= \sum_{n \in \mathbb{N}} \nu(A_n) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \end{aligned}$$

Hence $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}_{\mu, \nu}$.

□

Exercise 8.8. Let (Ω, \mathcal{F}) be a measurable space, μ, ν probability measures on (Ω, \mathcal{F}) and $\mathcal{P} \subset \mathcal{A}$ a π -system on Ω . Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Proof. Using the previous exercise, we see that $\mathcal{P} \subset \mathcal{L}_{\mu, \nu}$. Dynkin's theorem implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu, \nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$. □

Definition 8.9. Let $F : \mathbb{R} \rightarrow \mathbb{R}$. Then F is said to be a **probability distribution function** if

- (1) F is right continuous
- (2) F is increasing
- (3) $F(-\infty) = 0$ and $F(\infty) = 1$

Definition 8.10. Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We define $F_P : \mathbb{R} \rightarrow \mathbb{R}$, by

$$F_P(x) = P((-\infty, x])$$

We call F_P the **probability distribution function of P** .

Exercise 8.11. Let (Ω, \mathcal{F}, P) be a probability measure. Then F_P is a probability distribution function.

Proof. (1) Let $x \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}} \subset [x, \infty)$. Suppose that $x_n \rightarrow x$. Then $(x, x_n] \rightarrow \emptyset$ because $\limsup_{n \rightarrow \infty} (x, x_n] = \emptyset$. Thus

$$F(x_n) - F(x) = P((x, x_n]) \rightarrow P(\emptyset) = 0$$

This implies that

$$F(x_n) \rightarrow F(x)$$

. So F is right continuous.

- (2) Clearly F_P is increasing.
- (3) Continuity from below tells us that

$$F(-\infty) = \lim_{n \rightarrow -\infty} F(n) = \lim_{n \rightarrow -\infty} P((-\infty, n]) = 0$$

and continuity from above tell us that

$$F(\infty) = \lim_{n \rightarrow \infty} F(n) = \lim_{n \rightarrow \infty} P((-\infty, n]) = 1$$

□

Exercise 8.12. Let μ, ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $F_\mu = F_\nu$ iff $\mu = \nu$.

Proof. Clearly if $\mu = \nu$, then $F_\mu = F_\nu$. Conversely, suppose that $F_\mu = F_\nu$. Then for each $x \in \mathbb{R}$,

$$\begin{aligned}\mu((-\infty, x]) &= F_\mu(x) \\ &= F_\nu(x) \\ &= \nu((-\infty, x])\end{aligned}$$

Put $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then \mathcal{C} is a π -system and for each $A \in \mathcal{C}$, $\mu(A) = \nu(A)$. Hence for each $A \in \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, $\mu(A) = \nu(A)$. So $\mu = \nu$. \square

Definition 8.13. Let (Ω, \mathcal{F}) be a measurable space and $X : \Omega \rightarrow \mathbb{R}$. Then X is said to be a **random variable** on (Ω, \mathcal{F}) if X is \mathcal{F} - $\mathcal{B}(\mathbb{R})$ measurable.

Definition 8.14. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . We define the **probability distribution** of X , $P_X : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$, to be the measure

$$P_X = X_*P$$

so that for each $A \in \mathcal{B}(\mathbb{R})$,

$$P_X(A) = P(X^{-1}(A))$$

We define the **probability distribution function** of X , $F_X : \mathbb{R} \rightarrow [0, 1]$, to be

$$F_X = F_{P_X}$$

Definition 8.15. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . If $P_X \ll m$, we define the **probability density** of X , $f_X : \mathbb{R} \rightarrow \mathbb{R}$, by

$$f_X = \frac{dP_X}{dm}$$

Exercise 8.16. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on (Ω, \mathcal{F}) . Then for each $x \in \mathbb{R}$,

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} X_n > x\right) \leq \liminf_{n \rightarrow \infty} P(X_n > x)$$

Proof. Let $\omega \in \left\{\liminf_{n \rightarrow \infty} X_n > x\right\}$. Then $x < \liminf_{n \rightarrow \infty} X_n(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} X_k(\omega)\right)$. So there exists $n^* \in \mathbb{N}$ such that $x < \inf_{k \geq n^*} X_k(\omega)$. Then for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x < X_k(\omega)$. So there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Hence $\inf_{k \geq n^*} \mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Thus $\liminf_{n \rightarrow \infty} \mathbf{1}_{\{X_n > x\}}(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \mathbf{1}_{\{X_k > x\}}(\omega)\right) = 1$. Therefore $\omega \in \liminf_{n \rightarrow \infty} \{X_n > x\}$ and we have shown that

$$\left\{\liminf_{n \rightarrow \infty} X_n > x\right\} \subset \liminf_{n \rightarrow \infty} \{X_n > x\}$$

Then

$$\begin{aligned} P\left(\liminf_{n \rightarrow \infty} X_n > x\right) &\leq P\left(\liminf_{n \rightarrow \infty} \{X_k > x\}\right) \\ &\leq \liminf_{n \rightarrow \infty} P(\{X_k > x\}) \end{aligned}$$

□

Definition 8.17. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+(\Omega) \cup L^1$. Define the **expectation of X** , $E[X]$, to be

$$\mathbb{E}[X] = \int X dP$$

8.2. Independence.

Definition 8.18. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C} \subset \mathcal{F}$. Then \mathcal{C} is said to be **independent** if for each $(A_i)_{i=1}^n \subset \mathcal{C}$,

$$P\left(\bigcap_{k=1}^n A_k\right) = \prod_{k=1}^n P(A_k)$$

Definition 8.19. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Then $\mathcal{C}_1, \dots, \mathcal{C}_n$ are said to be **independent** if for each $A_1 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$, A_1, \dots, A_n are independent.

Note 8.20. We will explicitly say that for each $i = 1, \dots, n$, \mathcal{C}_i is independent when talking about the independence of the elements of \mathcal{C}_i to avoid ambiguity.

Definition 8.21. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are said to be **independent** if for each $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent.

Exercise 8.22. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff $\sigma(X_1), \dots, \sigma(X_n)$ are independent.

Proof. Suppose that X_1, \dots, X_n are independent. Let $A_1 \in \sigma(X_1), \dots, A_n \in \sigma(X_n)$. Then for each $i = 1, \dots, n$, there exists $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X_i^{-1}(B_i)$. Then A_1, \dots, A_n are independent. Hence $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Conversely, suppose that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$. Then for each $i = 1, \dots, n$, $X_i^{-1}B_i \in \sigma(X_i)$. Then $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent. Hence X_1, \dots, X_n are independent. □

Exercise 8.23. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{F}$ a collection of σ -algebras on Ω . Suppose that for each $i = 1, \dots, n$, X_i is \mathcal{F}_i -measurable. If $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent, then X_1, \dots, X_n are independent.

Proof. For each $i = 1, \dots, n$, $\sigma(X_i) \subset \mathcal{F}_i$. So $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Hence X_1, \dots, X_n are independent. □

Exercise 8.24. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Suppose that for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent, then $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent.

Proof. Let $A_2 \in \mathcal{C}_2$. Define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap A_2) = P(A)P(A_2)\}$. Then

$$(1) \quad \Omega \in \mathcal{L}$$

(2) If $A \in \mathcal{L}$, then

$$\begin{aligned} P(A^c \cap A_2) &= P(A_2) - P(A_2 \cap A) \\ &= P(A_2) - P(A_2)P(A) \\ &= (1 - P(A))P(A_2) \\ &= P(A^c)P(A_2) \end{aligned}$$

So $A^c \in \mathcal{L}$

(3) If $(B_n)_{n \in \mathbb{N}} \subset \mathcal{L}$ is disjoint, then

$$\begin{aligned} P\left(\left[\bigcup_{n \in \mathbb{N}} B_n\right] \cap A_2\right) &= P\left(\bigcup_{n \in \mathbb{N}} B_n \cap A_2\right) \\ &= \sum_{n \in \mathbb{N}} P(B_n \cap A_2) \\ &= \sum_{n \in \mathbb{N}} P(B_n)P(A_2) \\ &= \left[\sum_{n \in \mathbb{N}} P(B_n)\right]P(A_2) \\ &= P\left(\bigcup_{n \in \mathbb{N}} A_n\right)P(A_2) \end{aligned}$$

So $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{L}$.

Thus \mathcal{L} is a λ -system. Since $\mathcal{C}_1 \subset \mathcal{L}$ is a π -system, Dynkin's theorem tells us that $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Since $A_2 \in \mathcal{C}_2$ is arbitrary $\sigma(\mathcal{C}_1)$ and \mathcal{C}_2 are independent. The same reasoning implies that $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent. Let $A_2 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$. We may do the same process with

$$\mathcal{L} = \left\{ A \in \mathcal{F} : P\left(A \cap \left(\bigcap_{i=2}^n A_i\right)\right) = P(A) \prod_{i=2}^n P(A_i) \right\}$$

and conclude that $\sigma(\mathcal{C}_1), \mathcal{C}_2, \dots, \mathcal{C}_n$ are independent. Which, using the same reasoning would imply that $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent. \square

Exercise 8.25. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff for each $x_1, \dots, x_n \in \mathbb{R}$,

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$$

Proof. Suppose that X_1, \dots, X_n are independent. Then $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $x_1, \dots, x_n \in \mathbb{R}$. Then for each $i = 1, \dots, n$, $\{X_i \leq x_i\} \in \sigma(X_i)$. Hence

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i). \text{ Conversely, suppose that for each}$$

$x_1, \dots, x_n \in \mathbb{R}$, $P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Define $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$.

Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$. For each $i = 1, \dots, n$, define $\mathcal{C}_i = X_i^{-1}\mathcal{C}$. Then for each $i = 1, \dots, n$, \mathcal{C}_i

is a π -system and

$$\begin{aligned}\sigma(\mathcal{C}_i) &= \sigma(X^{-1}(\mathcal{C})) \\ &= X_i^{-1}(\sigma(\mathcal{C})) \\ &= X_i^{-1}(\mathcal{B}(\mathbb{R})) \\ &= \sigma(X_i)\end{aligned}$$

By assumption, $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent. The previous exercises tell us that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Then X_1, \dots, X_n are independent. \square

Exercise 8.26. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Define $X = (X_1, \dots, X_n)$. If X_1, \dots, X_n are independent, then

$$P_X = \prod_{i=1}^n P_{X_i}$$

.

Proof. Let $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}P_X(A_1 \times \dots \times A_n) &= P(X \in A_1 \times \dots \times A_n) \\ &= P(X_1 \in A_1, \dots, X_n \in A_n) \\ &= P(X_1 \in A_1) \cdots P(X_n \in A_n) \\ &= P_{X_1}(A_1) \cdots P_{X_n}(A_n) \\ &= \prod_{i=1}^n P_{X_i}(A_1 \times \dots \times A_n)\end{aligned}$$

Put

$$\mathcal{P} = \{A_1 \times \dots \times A_n : A_1 \in \mathcal{B}(\mathbb{R}), \dots, A_n \in \mathcal{B}(\mathbb{R})\}$$

Then \mathcal{P} is a π -system and

$$\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R}) \otimes \dots \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^n)$$

A previous exercise then tells us that $P_X = \prod_{i=1}^n P_{X_i}$ \square

Exercise 8.27. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R} \in L^0$. Suppose that $f_1 \circ X_1, \dots, f_n \circ X_n \in L^+(\Omega)$ or $f_1 \circ X_1, \dots, f_n \circ X_n \in L^1(\Omega)$. If X_1, \dots, X_n are independent, then

$$E[f_1(X_1) \cdots f_n(X_n)] = \prod_{i=1}^n E[f_i(X_i)]$$

Proof. Define the random vector $X : \Omega \rightarrow \mathbb{R}^n$ by $X = (X_1, \dots, X_n)$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ by $g(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$. Suppose that for each $i = 1, \dots, n$, $f_i \in L^+(\mathbb{R})$. Then

$g \in L^+(\mathbb{R}^n)$ and by change of variables,

$$\begin{aligned}
E[f_1(X_1) \cdots f_n(X_n)] &= E[g(X)] \\
&= \int_{\Omega} g \circ X dP \\
&= \int_{\mathbb{R}^n} g(x) dP_X(x) \\
&= \int_{\mathbb{R}^n} g(x) d \prod_{i=1}^n P_{X_i}(x) \\
&= \prod_{i=1}^n \int_{\mathbb{R}} f_i(x) dP_{X_i}(x) \\
&= \prod_{i=1}^n \int_{\Omega} f_i \circ X dP \\
&= \prod_{i=1}^n E[f_i(X_i)]
\end{aligned}$$

If for each $i = 1, \dots, n$, $f_i \in L^1(\mathbb{R}, P_{X_i})$, then following the above reasoning with $|g|$ tells us that $g \in L^1(\mathbb{R}^n, P_X)$ and we use change of variables and Fubini's theorem to get the same result. \square

8.3. L^p Spaces for Probability.

Note 8.28. Recall that for a probability space (Ω, \mathcal{F}, P) and $1 \leq p \leq q \leq \infty$ we have $L^q \subset L^p$ and for each $X \in L^q$, $\|X\|_p \leq \|X\|_q$. Also recall that for $X, Y \in L^2$, we have that $\|XY\|_1 \leq \|X\|_2 \|Y\|_2$.

Definition 8.29. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Define the **variance of X** , $\text{Var}(X)$, to be

$$\text{Var}(X) = \mathbb{E}[(X - E[X])^2]$$

.

Definition 8.30. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the

Definition 8.31. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the **covariance of X and Y** , $\text{Cov}(X, Y)$, to be

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Exercise 8.32. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Then the covariance is well defined and $\text{Cov}(X, Y)^2 \leq \text{Var}(X)\text{Var}(Y)$

Proof. By Holder's inequality,

$$\begin{aligned}
 |Cov(X, Y)| &= \left| \int (X - E[X])(Y - E[Y])dP \right| \\
 &\leq \int |(X - E[X])(Y - E[Y])|dP \\
 &= \|(X - E[X])(Y - E[Y])\|_1 \\
 &\leq \|X - E[X]\|_2 \|Y - E[Y]\|_2 \\
 &= \left(\int |X - E[X]|^2 dP \right)^{\frac{1}{2}} \left(\int |Y - E[Y]|^2 dP \right)^{\frac{1}{2}} \\
 &= Var(X)^{\frac{1}{2}} Var(Y)^{\frac{1}{2}}
 \end{aligned}$$

So $Cov(X, Y)^2 \leq Var(X)Var(Y)$. □

Exercise 8.33. Let (Ω, \mathcal{F}, P) be a measure space and $X, Y \in L^2$. Then

- (1) $Cov(X, Y) = E[XY] - E[X]E[Y]$
- (2) If X, Y are independent, then $Cov(X, Y) = 0$
- (3) $Var(X) = E[X^2] - E[X]^2$
- (4) for each $a, b \in \mathbb{R}$, $Var(aX + b) = a^2 Var(X)$.
- (5) $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Proof.

- (1) We have that

$$\begin{aligned}
 Cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
 &= E[XY - E[Y]X - E[X]Y + E[X]E[Y]] \\
 &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
 &= E[XY] - E[X]E[Y]
 \end{aligned}$$

- (2) Suppose that X, Y are independent. Then $E[XY] = E[X]E[Y]$. Hence

$$\begin{aligned}
 Cov(X, Y) &= E[XY] - E[X]E[Y] \\
 &= E[X]E[Y] - E[X]E[Y] \\
 &= 0
 \end{aligned}$$

- (3) Part (1) implies that

$$\begin{aligned}
 Var(X) &= Cov(X, X) \\
 &= E[X^2] - E[X]^2
 \end{aligned}$$

- (4) Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
 Var(aX + b) &= E[(aX + b)^2] - E[aX + b]^2 \\
 &= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\
 &= a^2E[X^2] + 2abE[X] + b^2 - (a^2E[X]^2 + 2abE[X] + b^2) \\
 &= a^2(E[X^2] - E[X]^2) \\
 &= a^2Var(X)
 \end{aligned}$$

(5) We have that

$$\begin{aligned}
 \text{Var}(X + Y) &= E[(X + Y)^2] - E[X + Y]^2 \\
 &= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
 &= E[X^2] + 2E[XY] + E[Y^2] - (E[X]^2 + 2E[X]E[Y] + E[Y]^2) \\
 &= (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2) + 2(E[XY] - E[X]E[Y]) \\
 &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)
 \end{aligned}$$

□

Definition 8.34. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. The **correlation of X and Y** , $\text{Cor}(X, Y)$, is defined to be

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Exercise 8.35.

Exercise 8.36. *Jensen's Inequality* Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^1$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$. If ϕ is convex, then

$$\phi(E[X]) \leq E[\phi(X)]$$

Proof. Put $x_0 = E[X]$. Since ϕ is convex, there exist $a, b \in \mathbb{R}$ such that $\phi(x_0) = ax_0 + b$ and for each $x \in \mathbb{R}$, $\phi(x) \geq ax + b$. Then

$$\begin{aligned}
 E[\phi(X)] &= \int \phi(X) dP \\
 &\geq \int [aX + b] dP \\
 &= a \int X dP + b \\
 &= aE[X] + b \\
 &= ax_0 + b \\
 &= \phi(x_0) \\
 &= \phi(E[X])
 \end{aligned}$$

□

Exercise 8.37. *Markov's Inequality:* Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+$. Then for each $a \in (0, \infty)$,

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Proof. Let $a \in (0, \infty)$. Then $a\mathbf{1}_{\{X \geq a\}} \leq X\mathbf{1}_{\{X \geq a\}}$. Thus

$$\begin{aligned} aP(X \geq a) &= \int a\mathbf{1}_{\{X \geq a\}} dP \\ &= \int X\mathbf{1}_{\{X \geq a\}} dP \\ &\leq \int X dP \\ &= E[X] \end{aligned}$$

Therefore

$$P(X \geq a) \leq \frac{E[X]}{a}$$

□

Exercise 8.38. *Chebychev's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a \in (0, \infty)$,*

$$P(|X - E[X]| \geq a) \leq \frac{\text{Var}(X)}{a^2}$$

Proof. Let $a \in (0, \infty)$. Then

$$\begin{aligned} P(|X - E[X]| \geq a) &= P((X - E[X])^2 \geq a^2) \\ &\leq \frac{E[(X - E[X])^2]}{a^2} \\ &= \frac{\text{Var}(X)}{a^2} \end{aligned}$$

□

Exercise 8.39. *Chernoff's Bound: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a, t \in (0, \infty)$,*

$$P(X \geq a) \leq e^{-ta} E[e^{tX}]$$

Proof. Let $a, t \in (0, \infty)$. Then

$$\begin{aligned} P(X \geq a) &= P(tX \geq ta) \\ &= P(e^{tX} \geq e^{ta}) \\ &\leq e^{-ta} E[e^{tX}] \end{aligned}$$

□

Exercise 8.40. *Weak Law of Large Numbers: Let (Ω, \mathcal{F}, P) be a probability space $(X_i)_{i \in \mathbb{N}} \subset L^2$. Suppose that $(X_i)_{i \in \mathbb{N}}$ are iid. Then*

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

Proof. Put $\mu = E[X_1]$ and $\sigma^2 = \text{Var}(X_1)$. Then

$$\begin{aligned} E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] &= \frac{1}{n} \sum_{i=1}^n E[X_i] \\ &= \frac{1}{n} \sum_{i=1}^n \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) &= \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) \\ &= \frac{1}{n^2} \sum_{i=1}^n \sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Let $\epsilon > 0$. Then

$$\begin{aligned} P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - E[X_1]\right| \geq \epsilon\right) &= P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) \\ &= P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n X_i\right]\right| \geq \epsilon\right) \\ &\leq \frac{\text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right)}{\epsilon^2} \\ &= \frac{\sigma^2/n}{\epsilon^2} \\ &= \frac{\sigma^2}{n\epsilon^2} \rightarrow 0 \end{aligned}$$

So

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

□

9. APPENDIX

9.1. Summation.

Definition 9.1. Let $f : X \rightarrow [0, \infty)$, Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \rightarrow \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \rightarrow \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^- . In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \rightarrow \mathbb{C}$.

Note 9.2. Let $f : X \rightarrow \mathbb{C}$ and $\alpha : X \rightarrow X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.