NOTES

Let μ be the Haar measure on $\mathbb{Z}_p^\times$.
For $x \in \mathbb{Q}_p$, $p^{-\nu(x)}$, so $\nu(x) = -\log_p |x|_p$
For $\alpha \in \mathbb{Q}_p$, define

$$\begin{aligned} m(\alpha) &= -\int_{\mathbb{Z}_p^\times} \nu(x-\alpha) d\mu(x) \\ &= \int_{\mathbb{Z}_p^\times} \log_p |x-\alpha|_p d\mu(x) \end{aligned}$$

Lemma 0.1. *For each* $\alpha \in \mathbb{Q}_p$,

$$m(\alpha) = \begin{cases} 0 & |\alpha|_p < 1 \\ -\frac{p}{(p-1)^2} & p = 1 \\ -\nu(\alpha) & |\alpha|_p > 1 \end{cases}$$

.

Proof. Let $\alpha \in \mathbb{Q}_p$. First assume $|\alpha|_p = 1$. Then $\alpha = \alpha_0 + \alpha_1 p + \cdots$ Define $Z_0 = \{x \in \mathbb{Z}_p^\times : x_0 \neq \alpha_0\}$ and $Z_n = \left(\bigcap_{i=0}^{n-1} \{x \in \mathbb{Z}_p^\times : x_i = \alpha_i\}\right) \cap \{x \in \mathbb{Z}_p^\times : x_n \neq \alpha_n\}$.
Then $(Z_i)_{i=1}^\infty$ are disjoint, $\mathbb{Z}_p^\times = \{\alpha\} \cup \left(\bigcup_{i=0}^\infty Z_i\right)$, for each $x \in Z_n$, $|x-\alpha|_p = p^{-n}$ and

$$\mu(Z_n) = \begin{cases} (p-1)^{-1} & n = 0 \\ p^{-n} & n \geq 1 \end{cases}$$

Thus for each $x \in \mathbb{Z}_p^\times$, $\log_p^+ |x-\alpha|_p = 0$ and so

$$\begin{aligned} \int_{\mathbb{Z}_p^\times} \log_p |x-\alpha|_p d\mu(x) &= \sum_{i=1}^\infty \int_{Z_i} log_p |x-\alpha| d\mu(x) \\ &= \sum_{i=1}^\infty -np^{-n} \\ &= -\frac{p}{(p-1)^2} \end{aligned}$$

Now assume that $|\alpha|_p < 1$. Then for each $x \in \mathbb{Z}_p^\times$, $\log_p |x-\alpha|_p = \log_p |x|_p = 0$. Thus

$$\int_{\mathbb{Z}_p^\times} \log_p |x-\alpha|_p d\mu(x) = 0$$

Finally assume that $|\alpha|_p > 1$. Then for each $x \in \mathbb{Z}_p^\times$, $\log_p |x-\alpha|_p = \log_p |\alpha|_p$ and thus

$$\begin{aligned}\int_{\mathbb{Z}_p^\times} \log_p |x-\alpha|_p d\mu(x) &= \log_p |\alpha|_p \\ &= -\nu(\alpha)\end{aligned}$$

$\square$