PORTFOLIO THEORY NOTES

Contents

Note 0.1. *In these notes we will mostly consider random variables X that model returns. As such we may assume that $F_X : \mathbb{R} \to (0,1)$ is bijective and continuous.*

1. Risk Measures

1.1. **Value at Risk.**

Definition 1.1. *Let X be a random variable and $\epsilon > 0$. Assume that F_X is bijective. We define the* ***value at risk of X at confidence level*** *$1-\epsilon$, denoted by $VaR_\epsilon(X)$, to be*

$$VaR_\epsilon(X) = -F_X^{-1}(\epsilon)$$

Note 1.2. *If X represents the return of a portfolio, then $Var_\epsilon(X)$ is just a bound such that with probability ϵ, the loss of the portfolio is not less than the bound.*

1.2. **Sampling the Value at Risk.**

1.3. **Average Value at Risk.**

Definition 1.3. *Let X be a random variable and $\epsilon > 0$. Assume that F_X is bijective. We define the* ***average value at risk of X with tail probability*** *ϵ, denoted by $AVaR_\epsilon(X)$, to be*

$$AVaR_\epsilon(X) = \frac{1}{\epsilon}\int_{(0,\epsilon]} VaR_p(X)dm(p)$$

Note 1.4. *If X represents the return on a portfolio, then $AVaR_\epsilon(X)$ is just the average of the $VaR_p(X)$ over all $p < \epsilon$.*

Exercise 1.5. *Let X be a random variable and $\epsilon > 0$. Suppose that $F_X : \mathbb{R} \to (0,1)$ is continuous and bijective. Then $AVaR_\epsilon(X) = \mathbb{E}[-X| - X \geq VaR_\epsilon(X)]$.*

Proof. Recall that for measurable spaces $(X, \mathcal{A}), (Y, \mathcal{B})$, measurable $f : X \to Y$, measure $\mu : \mathcal{A} \to [0,\infty]$, we may form the push-foreward measure of μ by f, $f_*\mu : \mathcal{B} \to [0,\infty]$ with the folling property: for each $g : Y \to \mathbb{C}$, $g \in L^1(f_*\mu)$ iff $g \circ f \in L^1(\mu)$ and for each $B \in \mathcal{B}$,

$$\int_{f^{-1}(B)} g \circ f d\mu = \int_B g df_*\mu$$

Note that

$$\begin{aligned}
\mathbb{E}[-X| - X \geq -F_X^{-1}(\epsilon)] &= -\mathbb{E}[X|X \leq F_X^{-1}(\epsilon)] \\
&= -\frac{1}{\epsilon}\mathbb{E}[X\mathbf{1}_{\{X\leq F_X^{-1}(\epsilon)\}}] \\
&= -\frac{1}{\epsilon}\int_{\{X\leq F_X^{-1}(\epsilon)\}} XdP \\
&= -\frac{1}{\epsilon}\int_{(-\infty,F_X^{-1}(\epsilon)]} xdF_X(x)
\end{aligned}$$

Let μ be the Lebesgue-Stieltjes measure obtained from F_X (i.e. $d\mu = dF_X$). Consider $F_X : \mathbb{R} \to (0,1)$ as in the theorem recalled above. Then for each $(a,b] \subset [0,1]$ with $a' = F_X^{-1}(a)$ (could be $-\infty$) and $b' = F_X^{-1}(b)$, we have that

$$\begin{aligned}
F_{X*}\mu((a,b]) &= \mu(F_X^{-1}((a,b])) \\
&= \mu((a',b']) \\
&= F_X(b') - F_X(a') \\
&= b - a
\end{aligned}$$

So $F_{X*}\mu = m$. Hence

$$\begin{aligned}
\int_{(-\infty,F_X^{-1}(\epsilon)]} xdF_X(x) &= \int_{(-\infty,F_X^{-1}(\epsilon)]} (F_X^{-1}\circ F_X)(x)dF_X(x) \\
&= \int_{(0,\epsilon]} F_X^{-1}(x)dm(x)
\end{aligned}$$

□

Note 1.6. *If X represents the return of a portfolio. We may define the* ***loss of*** *X, denoted by L_X, to be $L_X = -X$. Then $AVaR_\epsilon(X) = \mathbb{E}[L|L > VaR_\epsilon(X)]$.*

Theorem 1.7. *Let X be random variable and $\epsilon > 0$. Suppose that X is "nice". Then*

$$AVaR_\epsilon(X) = \min_{\theta\in\mathbb{R}}(\theta + \frac{1}{\epsilon}\mathbb{E}[(-X-\theta)^+])$$

Proof. ??? I have no clue □

1.4. **Sampling the Average Value at Risk.**