STOCHASTIC PROCESSES NOTES

Contents

1. Preliminaries

1.1. Basic Properties.

Definition 1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(S, \mathcal{G})$ a measurable space, T an index set and $X : T \times \Omega \to S$. Then X is said to be a* ***stochastic processes*** *if for each $t \in T$, $X(t, \cdot)$ is $\mathcal{F}$-$\mathcal{G}$ measurable (i.e. X is just a collection of random variables indexed by time).*

Note 1.2. *We will work with $T = [0, \infty)$ and $(S, \mathcal{G}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and we often write $X_t(\omega)$ to mean $X(t, \omega)$ as well as the function $X(\cdot, \omega)$.*

Definition 1.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(S, \mathcal{G})$ a measurable space, T an index set and $X : T \times \Omega \to S$ a stochastic process. We can define a function $\Phi_X : \Omega \to S^T$ by $\Phi_X(\omega)(t) = X(t, \omega)$. The* ***law*** *of X is defined to be the pushforward measure of $\mathbb{P}$ by Φ_X and is denoted by $\mathcal{L}_X$. (i.e. $\mathcal{L}_X(A) = \mathbb{P}(\Phi_X^{-1}(A))$ for $A \in \mathcal{G}$)*

Definition 1.4. *Let X be a process and $\omega \in \Omega$. The function $X(\cdot, \omega)$ is called the* ***sample path*** *of ω. It is common to define properties of a process in terms of its sample paths.*

Definition 1.5. *Let X be a process. Then X is said to be* ***measurable*** *if $X : [0, \infty) \times \Omega \to \mathbb{R}$ is $\mathcal{B}([0, \infty)) \bigotimes \mathcal{F}$-measurable.*

Definition 1.6. *Let X be a process. Then X is said to be* ***bounded*** *if there exists $K > 0$ such that for each $(t, \omega) \in [0, \infty) \times \Omega$, $|X(t, \omega)| < K$ (i.e. $X : [0, \infty) \times \Omega \to \mathbb{R}$ is bounded).*

Note 1.7. *The definition above changes from book to book, but in these notes, this is how we define bounded and it is very important to use this one instead of the path-wise a.s. defintion.*

Definition 1.8. *Let X be a process. Then X is said to be* ***continous*** *if for each $\omega \in \Omega$, $X(\cdot, \omega)$ is continous.*

Note 1.9. *We may similarly define right continuous, left continuous, cadlag, caglad, increasing and being of bounded variation.*

Note 1.10. *There are a few notions of equivalence among processes. Below we mention two.*

Definition 1.11. *Let X and Y be processes. Then X is said to be a* ***modification*** *of Y if for each $t \geq 0$, $N_t = \{\omega \in \Omega : X_t(\omega) \neq Y_t(\omega)\}$ is a null set.*

Definition 1.12. *Let X and Y be processes. Then X is said to be* ***indistinguishable*** *from Y if $N = \{\omega \in \Omega : \text{ for some } t \geq 0, X_t(\omega) \neq Y_t(\omega)\}$ is a null set.*

Note 1.13. *These definitions clearly generate an equivalence relation and obviously indistinguishable implies modification.*

Exercise 1. *Let X and Y be processes. Suppose that X and Y are modifications and that a.s. (i.e. except on a null set) X and Y are right continuous. Then X and Y are indistinguishable.*

Proof. By assumption, $N_r = \{\omega \in \Omega : X_t(\omega) \text{ is not right continuous or } Y_t(\omega) \text{ is not right continuous}\}$ is a null set . Let $\omega \in \Omega \cap N_r^c$. Right continuity tells us that for each $t \geq 0$, $X_t(\omega) = Y_t(\omega)$ if and only if for each $t \in [0, \infty) \cap \mathbb{Q}$, $X_t(\omega) = Y_t(\omega)$. Since X and Y are modifications, for each $t \in [0, \infty) \cap \mathbb{Q}$, $N_t = \{\omega \in \Omega : X_t(\omega) \neq Y_t(\omega)\}$ is a null set. Thus

$$\begin{aligned} N &= \{\omega \in \Omega : \text{ for some } t \geq 0, X_t(\omega) \neq Y_t(\omega)\} \\ &= (N \cap N_r) \cup (N \cap N_r^c) \\ &= (N \cap N_r) \cup \{\omega \in \Omega \cap N_r^c : \text{ for some } t \in [0, \infty) \cap \mathbb{Q}, X_t(\omega) \neq Y_t(\omega)\} \\ &= (N \cap N_r) \cup \bigcup_{t \in [0,\infty) \cap \mathbb{Q}} N_t \end{aligned}$$

is a null set. Hence X and Y are indistinguishable. □

1.2. **Filtrations.**

Definition 1.14. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t \geq 0}$ a collection of sub σ-algebras of $\mathcal{F}$ such that for each $s, t \in [0, \infty)$, $s < t$ implies that $\mathcal{F}_s \subset \mathcal{F}_t$. Then $\{\mathcal{F}_t\}_{t \geq 0}$ is said to be a* ***filtration*** *of $\mathcal{F}$.*

Definition 1.15. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t \geq 0}$ is said to be* ***complete*** *if for each $t \in [0, \infty)$, $\mathcal{F}_t$ contains all the null sets of $\mathcal{F}$. (i.e. $\mathcal{F}_0$ contains all the null sets of $\mathcal{F}$)*

Definition 1.16. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t \geq 0}$ is said to be* ***right continuous*** *if for each $t \in [0, \infty)$, $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$*

Definition 1.17. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Then $\{\mathcal{F}_t\}_{t \geq 0}$ is said to satisfy the* ***usual conditions*** *if $\{\mathcal{F}_t\}_{t \geq 0}$ is complete and right continuous.*

Definition 1.18. *Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. Define $\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \geq 0)$.*

Definition 1.19. *Let X be a process and $\{\mathcal{F}_t\}_{t \geq 0}$ a filtration. Then X is said to be* ***adapted*** *to $\{\mathcal{F}_t\}_{t \geq 0}$ if for each $t \in [0, \infty)$, X_t is $\mathcal{F}_t$ measurable*

Definition 1.20. *Let X be a process. Then the* ***minimal augmented filtration*** *of X is defined to be the smallest filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. (here, "smallest" means that if $\{\mathcal{G}_t\}$ is filtration satisfying the usual conditions to which X is adapted, then for each $t \in [0,\infty)$, $\mathcal{F}_t \subset \mathcal{G}_t$)*

Definition 1.21. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Then X is said to be progressively measurable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if for each $t \geq 0$, $X|_{[0,t]\times\Omega}$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable.*

Exercise 2. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. Suppose that X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. If X is right continuous or X is left continuous, then X is progressively measurable.*

Proof. Fist suppose that each path of X is right continuous. Let $t \geq 0$. Define $X_n : [0,t] \times \Omega \to \mathbb{R}$ by $X_n(s,\omega) = X_0(\omega)\mathbf{1}_{\{0\}}(s) + \sum_{k=1}^{2^n} X_{tk/2^n}(\omega)\mathbf{1}_{(t(k-1)/2^n, tk/2^n]}(s)$. Since X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, X_n is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable. Let $(s,\omega) \in [0,t] \times \Omega$. Suppose that $s > 0$. Then for each $n \in \mathbb{N}$, there exists a unique $k_n \in \mathbb{N}$ such that $k_n \leq 2^n$ and $t(k_n-1)/2^n < s \leq tk_n/2^n$. Thus $X_n(s,\omega) = X_{tk_n/2^n}(\omega)$, $tk_n/2^n \geq s$ and $tk_n/2^n \to s$. Since each path of X is right continuous, $X_n(s,\omega) \to X_s(\omega)$. If $s = 0$, then $X_n(s,w) = X_0(\omega) = X_s(\omega)$. Hence $X_n(s,\omega) \to X_s(\omega)$. Since $X_n \to X|_{[0,t]\times\Omega}$ pointwise, $X|_{[0,t]\times\Omega}$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ measurable. So X is progressively measurable.
Now suppose that each path of X is left continuous. Let $t \geq 0$. Define $X_n : [0,t] \times \Omega \to \mathbb{R}$ by $X_n(s,\omega) = \sum_{k=1}^{2^n} X_{t(k-1)/2^n}(\omega)\mathbf{1}_{[t(k-1)/2^n, tk/2^n)}(s) + X_t(\omega)\mathbf{1}_{\{t\}}(s)$. The rest of the proof is similar to the other case, just this time utilizing left continuity. ☐

Definition 1.22. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Define the* ***predictable σ-algebra*** *with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ on $[0,\infty) \times \Omega$, denoted by $\mathcal{P}$, by*

$$\mathcal{P} = \sigma(X : X \textit{ is left continuous and adapted to } \{\mathcal{F}_t\}_{t\geq 0})$$

Definition 1.23. *Let X be a process. Then X is said to be* ***predictable*** *if X is $\mathcal{P}$-measurbable.*

2. Stopping Times

2.1. Basic Properites.

Definition 2.1. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and $T : \Omega \to [0,\infty]$ a random variable. Then T is said to be a* ***stopping time*** *for $\{\mathcal{F}_t\}_{t\geq 0}$ if for each $t \in [0,\infty]$, $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$.*

Note that if $t = \infty$, then

$$\{\omega \in \Omega : T(\omega) \leq t\} = \Omega \in \mathcal{F}_\infty$$

We will typically just say that T is a stopping time when the filtration is clear.

Proposition 2.2. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. Then we have the following:*

1. *if S and T are stopping times, then $S + T$ is a stopping time.*
2. *if S is a stopping time, then for $\alpha \geq 1$, αS is a stopping time.*
3. *if there exists $c \in [0,\infty]$ such that $T \equiv c$, then T is a stopping time.*
4. *If $(T_n)_{n\in\mathbb{N}}$ is a sequence of stopping times, then $\sup\limits_{n\in\mathbb{N}} T_n$ and $\inf\limits_{n\in\mathbb{N}} T_n$ are stopping times.*

Note that the third statement holds for $T \equiv c$ almost surely if $\{\mathcal{F}_t\}_{t\geq 0}$ is complete and the last statement implies that mins $\limsup T_n$ and $\liminf T_n$ and that for stopping times S and T, $S \wedge T$ and $S \vee T$ are stopping times.

The proof is similar to showing measurbaility of the same functions.

Proposition 2.3. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a right continuous filtration and $T : \Omega \to [0, \infty]$ a random variable. Then T is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$ if and only if for each $t \geq 0$, $\{\omega \in \Omega : T(\omega) < t\} \in \mathcal{F}_t$.*

Proof. Suppose that for each $s \geq 0$, $\{\omega \in \Omega : T(\omega) < s\} \in \mathcal{F}_s$. Let $t \geq 0$. Then

$$\begin{aligned}
\{\omega \in \Omega : T(\omega) \leq t\} &= \bigcap_{n\in\mathbb{N}} \{\omega \in \Omega : T(\omega) < t + 1/n\} \\
&\in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{t+1/n} \\
&= \mathcal{F}_t \qquad \text{(right continuity)}
\end{aligned}$$

Hence T is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Conversely, suppose that T is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Let $t \geq 0$. Then

$$\begin{aligned}
\{\omega \in \Omega : T(\omega) < t\} &= \bigcup_{n\in\mathbb{N}} \{\omega \in \Omega : T(\omega) \leq t - 1/n\} \\
&\in \mathcal{F}_t
\end{aligned}$$

□

Exercise 3. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be right continuous and $T : \Omega \to [0, \infty)$ a finite stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Define $T_n : \Omega \to [0, \infty)$ by $T_n(\omega) = k_\omega/2^n$ where $k_\omega \in \mathbb{N}$ is the unique positive integer such that $(k_\omega - 1)/2^n \leq T(\omega) < k_\omega/2^n$. Then for each $n \in \mathbb{N}$, T_n is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.*

Proof. Let $k \in \mathbb{N}$. Then we observe that

$$\begin{aligned}
\{\omega \in \Omega : T_n(\omega) = k/2^n\} &= \{\omega \in \Omega : (k-1)/2^n \leq T(\omega) < k/2^n\} \\
&= T^{-1}([0, k/2^n)) \setminus T^{-1}([0, (k-1)/2^n)) \\
&\in \mathcal{F}_{k/2^n} \quad \text{(right continuity)}
\end{aligned}$$

Now let $t \in [0, \infty)$. Choose $k \in \mathbb{N}$ such that $(k-1)/2^n \leq t < k/2^n$. Then

$$\begin{aligned}
T_n^{-1}([0, t]) &= \bigcup_{i=1}^{k-1} T_n^{-1}(\{i/2^n\}) \\
&\in \mathcal{F}_{(k-1)/2^n} \\
&\subset \mathcal{F}_t
\end{aligned}$$

Hence T_n is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.

□

Definition 2.4. *Let X be a process and $\Lambda \in \mathcal{B}(\mathbb{R})$. Then the* ***hitting time*** *of Λ is defined to be the random variable $T_\Lambda : \Omega \to [0, \infty]$ given by $T_\Lambda(\omega) = \inf\{t > 0 : X_t(\omega) \in \Lambda\}$.*

Theorem 2.5. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a complete filtration, X an a.s. right continuous process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and $\Lambda \subset \mathbb{R}$ an open set. Then T_Λ is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$.*

Proof. Let $t \geq 0$. Note that for $\omega \in \Omega$, if $X(\cdot, \omega)$ is right continuous, then since Λ is open, we have that $T_\Lambda(\omega) < t$ if and only if there exists $s \in (0,t)$ such that $X_s(\omega) \in \Lambda$ if and only if there exists $s \in \mathbb{Q} \cap (0,t)$ such that $X_s(\omega) \in \Lambda$. The completeness of $\{\mathcal{F}_t\}_{t\geq 0}$ tells us that for each $s \in [0,\infty)$, $\mathcal{F}_s$ has all the null sets of $\mathcal{F}$. Finally, Λ being open and X being adapted tell us that for each $s < t$, $X_s^{-1}(\Lambda) \in \mathcal{F}_s \subset \mathcal{F}_t$.

By assumption $N = \{\omega \in \Omega : X(\cdot,\omega) \text{ is not right continuous}\}$ is a null set, so for each $s \in [0,\infty)$, $N \in \mathcal{F}_s$ and $N^c \in \mathcal{F}_s$. The previous note implies that

$$\begin{aligned}\{\omega \in \Omega : T_\Lambda(\omega) < t\} &= (\{\omega \in \Omega : T_\Lambda(\omega) < t\} \cap N^c) \cup (\{\omega \in \Omega : T_\Lambda(\omega) < t\} \cap N) \\ &= \left(\bigcup_{s \in \mathbb{Q}\cap(0,t)} \{\omega \in \Omega : X_s(\omega) \in \Lambda\} \cap N^c\right) \cup \left(\{\omega \in \Omega : T(\omega) < t\} \cap N\right) \\ &\in \mathcal{F}_t\end{aligned}$$

Hence T_Λ is a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. □

Note that the same result is true if X is an a.s. left continuous process. Also, as usual, we could get rid of the assumption that $\{\mathcal{F}_t\}_{t\geq 0}$ is complete if each sample path of X were right or left continuous instead of the paths being almost surely right or left continuous.

2.2. **Stopping Algebra.**

Definition 2.6. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. The **stopping algebra** of T to be $\mathcal{F}_T = \{A \in \mathcal{F} : \text{ for each } t \geq 0, A \cap T^{-1}([0,t]) \in \mathcal{F}_t\}$*

Note that $\mathcal{F}_T$ is a σ-algebra and that if $\{\mathcal{F}_t\}_{t\geq 0}$ is complete, then so is $\mathcal{F}_T$.

Lemma 2.7. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then T is $\mathcal{F}_T$ measurable.*

Proof. Let $t, s \in [0,\infty]$. Then

$$\begin{aligned}T^{-1}([0,s]) \cap T^{-1}([0,t]) &= T^{-1}([0,t\wedge s]) \\ &\in \mathcal{F}_{t\wedge s} \\ &\subset \mathcal{F}_t.\end{aligned}$$

So for each $s \in [0,\infty]$, $T^{-1}([0,s]) \in \mathcal{F}_T$. Hence T is $\mathcal{F}_T$*measurable.* □

Lemma 2.8. *Let S, T be stopping times for $\{\mathcal{F}_t\}_{t\geq 0}$. If $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.*

Proof. Suppose that $S \leq T$. Let $\Lambda \in \mathcal{F}_S$ and let $t \geq 0$. Then by definition, $\Lambda \cap S^{-1}([0,t]) \in \mathcal{F}_t$. Since $\Lambda \cap T^{-1}([0,t]) = [\Lambda \cap S^{-1}([0,t])] \cap (T^{-1}([0,t]) \in \mathcal{F}_t$, we have that $\Lambda \in \mathcal{F}_T$. □

Lemma 2.9. *Let T be a stopping time for a right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Define $\mathcal{F}_{T^+} = \bigcap_{\epsilon>0} \mathcal{F}_{T+\epsilon}$. Then $\mathcal{F}_T = \mathcal{F}_{T^+}$.*

Proof. From the last lemma, $F_T \subset F_{T^+}$. Let $\Lambda \in \mathcal{F}_{T^+}, \epsilon > 0$, and $t \geq 0$. By definition, $\Lambda \in \mathcal{F}_{T+\epsilon}$. So $\Lambda \cap (T+\epsilon)^{-1}([0,t]) \in \mathcal{F}_t$. Then $\Lambda \cap T^{-1}([0,t-\epsilon]) \in \mathcal{F}_t$. Since $t \geq 0$ is arbitrary, for each $s \geq 0$, $\Lambda \cap T^{-1}([0,s]) \in \mathcal{F}_{s+\epsilon}$. Since $\epsilon > 0$ is arbitrary, right continuity tells us that for each $s \geq 0$, $\Lambda \cap T^{-1}([0,s]) \in \mathcal{F}_{s+} = \mathcal{F}_s$. Hence $\Lambda \in \mathcal{F}_T$ and $\mathcal{F}_T = \mathcal{F}_{T^+}$ □

Lemma 2.10. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration satisfying the usual conditions, X an a.s. right continuous process adapted $\{\mathcal{F}_t\}_{t\geq 0}$ and $T : \Omega \to [0,\infty)$ a finite stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then X_T is $\mathcal{F}_T$-measurable.*

Note 2.11. *The same result is achieved if we assume only that $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and each sample path of X is right continuous.*

Proof. Let $B \in \mathcal{B}([0,\infty))$. For $n \in \mathbb{N}$, define T_n as in the previous exercise. Let $k \in \mathbb{N}$. Since X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, we have that

$$\begin{aligned} X_{T_n}^{-1}(B) \cap T_n^{-1}(k/2^n) &= X_{k/2^n}^{-1}(B) \cap T_n^{-1}(k/2^n) \\ &\in \mathcal{F}_{k/2^n} \quad \text{(previous exercise)} \end{aligned}$$

Now let $t \geq 0$. Choose $k \in \mathbb{N}$ such that $(k-1)/2^n \leq t < k/2^n$. Then

$$\begin{aligned} X_{T_n}^{-1}(B) \cap T_n^{-1}([0,t]) &= \bigcup_{i=1}^{k-1} [X_{T_n}^{-1}(B) \cap T_n^{-1}(k/2^n)] \\ &\in \mathcal{F}_{(k-1)/2^n} \\ &\subset \mathcal{F}_t \end{aligned}$$

Thus $X_{T_n}^{-1}(B) \in \mathcal{F}_{T_n}$ and X_{T_n} is $\mathcal{F}_{T_n}$-measurable and therefore $\mathcal{F}_{T+1/2^n}$-measurable. Now, let $m \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, if $n \geq m$, X_{T_n} is $\mathcal{F}_{T+1/2^m}$-measurable. Right continuity almost surely tells us that $X_{T_n} \xrightarrow{\text{a.e.}} X_T$. Since $\{\mathcal{F}_t\}_{t\geq 0}$ is complete, so is $\mathcal{F}_{T+1/2^m}$ and we know that X_T is $\mathcal{F}_{T+1/2^m}$-measurable. Since $m \in \mathbb{N}$ is arbitrary and $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous, X_T is measurable with respect to $\mathcal{F}_{T^+} = \mathcal{F}_T$.

□

Theorem 2.12. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration satisfying the usual conditions and $T : \Omega \to [0,\infty)$ a finite stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathcal{F}_T = \sigma(X_T : X$ is an a.s. right continuous process adapted to $\{\mathcal{F}_t$*

Note that the above is true if we assume only the right continuity of $\{\mathcal{F}_t\}_{t\geq 0}$ and right continuity everywhere.

Proof. Put $\mathcal{G} = \sigma(X_T : X$ is a cadlag process adapted to $\{\mathcal{F}_t\}_{t\geq 0})$. Let $\Lambda \in \mathcal{F}_T$. Define $X_t(\omega) = \mathbf{1}_\Lambda(\omega)\mathbf{1}_{T^{-1}([0,t])}(\omega) = \mathbf{1}_{\Lambda\cap T^{-1}([0,t])}(\omega)$. Then each sample path of X is clearly right continuous. Moreover, since $\Lambda \in \mathcal{F}_T$, for each $t \geq 0$, $\Lambda \cap T^{-1}([0,t]) \in \mathcal{F}_t$. So for each $t \geq 0$, X_t is $\mathcal{F}_t$ measurable and thus X is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. So by definition, X_T is $\mathcal{G}$ measurable. It is easy to see that $X_T = \mathbf{1}_\Lambda$ and therefore $\Lambda \in \mathcal{G}$. Hence $\mathcal{F}_T \subset \mathcal{G}$.

Now, let X be an a.s. right continuous process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ By the previous lemma, X_T is $\mathcal{F}_T$-measurable. So $\mathcal{G} \subset \mathcal{F}_T$. Hence $\mathcal{F}_T = \mathcal{G}$ as required. □

2.3. **Predictability and Accessibility.**

Definition 2.13. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then T is said to be **predictable** if there exists a sequence $(T_n)_{n\in N}$ of stopping times such that $T_n \nearrow T$ a.s and for each $n \in \mathbb{N}$, $T_n < T$ a.s. on $\{T > 0\}$.*

Definition 2.14. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and T a stopping time for $\{\mathcal{F}_t\}_{t\geq 0}$. Then T is said to be **totally inaccessible** if for each predictable stopping time S for $\{\mathcal{F}_t\}_{t\geq 0}$,*

$$\mathbb{P}(\{T = S\} \cap \{T < \infty\}) = 0.$$

3. Martingales

Note 3.1. *For the remainder of these notes, we will assume all filtrations satisfy the usual conditions.*

Definition 3.2. *Let X be a process and $T : \Omega \to [0, \infty]$ a random variable. Define the* ***stopped process of*** *X* ***at*** *T, denoted X^T, by $X^T_t(\omega) = X_{t \wedge T(\omega)}(\omega)$.*

Note 3.3. *Taking $T_n \equiv \infty$, it is easy to see that $\mathcal{D} \subset \mathcal{D}_{loc}$.*

Definition 3.4. *Let $\mathcal{X}$ be a set of random variables. Then $\mathcal{X}$ is said to be* ***uniformly integrable*** *if*

$$\lim_{k \to \infty} \left[\sup_{X \in \mathcal{X}} \mathbb{E}(|X| \mathbf{1}_{\{|X| > k\}}) \right] = 0$$

Exercise 4. *Let $\mathcal{X}$ be a set of random variables. Then $\mathcal{X}$ is uniformly integrable iff*

(1) *there exists $M > 0$ such that for each $X \in \mathcal{X}$, $\mathbb{E}[|X|] \leq M$*
(2) *for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{F}$, if $\mathbb{P}(E) < \delta$, then $\sup\limits_{X \in \mathcal{X}} \mathbb{E}[|X| \mathbf{1}_E] < \epsilon$*

Proof. ($\Rightarrow$): (1) Suppose that $\mathcal{X}$ is uniformly integrable. Then by definition, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $X \in \mathcal{X}$, $\mathbb{E}[|X| \mathbf{1}_{\{|X| > k\}}] < 1$. Choose $M = K + 1$. Let $X \in \mathcal{X}$. Then

$$\begin{aligned}
\mathbb{E}[|X|] &= \mathbb{E}[|X| \mathbf{1}_{\{X \leq K\}}] + \mathbb{E}[|X| \mathbf{1}_{\{X > K\}}] \\
&\leq K + 1 \\
&= M
\end{aligned}$$

(2) Let $\epsilon > 0$. Then there exists $K_\epsilon \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K_\epsilon$, then for each $X \in X$, $\mathbb{E}[|X| \mathbf{1}_{\{|X| > k\}}] < \epsilon/2$. Choose $\delta = \epsilon/2K_\epsilon$. Let $E \in \mathcal{F}$. Suppose that $\mathbb{P}(E) < \delta$. Then for each $X \in \mathcal{X}$,

$$\begin{aligned}
\mathbb{E}[|X| \mathbf{1}_E] &= \mathbb{E}[|X| \mathbf{1}_{E \cap \{|X| > K_\epsilon\}}] + \mathbb{E}[|X| \mathbf{1}_{E \cap \{|X| \leq K_{ep}\}}] \\
&< \epsilon/2 + K_\epsilon \delta \\
&= \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

($\Leftarrow$): Choose M as in (1). Suppose for the sake of contradiction that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $X \in \mathcal{X}$ such that $\mathbb{P}(\{|X| > K\}) \geq \epsilon$. Then choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $X_K \in \mathcal{X}$ such that $\mathbb{P}(\{|X_K| > K\}) \geq \epsilon$. So

$$\begin{aligned}
\mathbb{E}[|X_K|] &\geq \mathbb{E}[|X_K| \mathbf{1}_{\{|X_K| > K\}}] \\
&\geq K\epsilon \\
&> M
\end{aligned}$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $X \in \mathcal{X}$, $\mathbb{P}(\{|X| > K\}) < \epsilon$. Now, let $\epsilon > 0$. Choose K such that for each $X \in \mathcal{X}$, $\mathbb{P}(\{|X| > K\}) < \epsilon$. Since $(\{|X| > k\})_{k \in \mathbb{N}}$ is decreasing, for each $k \in \mathbb{N}$, if $k \geq K$, then for each $X \in \mathcal{X}$, $\mathbb{P}(\{|X| > k\}) < \epsilon$. Thus

$$\lim_{k \to \infty} \sup_{X \in \mathcal{X}} \mathbb{P}(\{|X| > k\}) = 0.$$

Finally, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Then Choose $K \in \mathbb{N}$ such that for $k \in \mathbb{N}$, if $k \geq K$, then for each $X \in \mathcal{X}$, $\mathbb{P}(\{|X| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $X \in \mathcal{X}$,

$$\mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] < \epsilon \text{ and therefore } \lim_{k\to\infty} \sup_{X\in\mathcal{X}} \mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] = 0.$$

□

Definition 3.5. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **submartingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if*

1. *M is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$*
2. *for each $t \in [0,\infty)$, $\mathbb{E}[|M_t|] < \infty$*
3. *for each $s,t \in [0,\infty)$, if $s < t$, then $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$ a.s.*

Note 3.6. *The third statement is techinally with respect to $\mathcal{F}_s$ because $\mathbb{E}[\cdot|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable. However, since $\mathcal{F}_s$ contains all the null sets of $\mathcal{F}$, to say a.s. is unambiguous.*

Definition 3.7. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **supermartingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if $-M$ is a submartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.*

Definition 3.8. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and M a process. The M is said to be a **martingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if M is a submartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ and M is a supermartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.*

3.1. Increasing and Finite Variation Processes.

Definition 3.9. *Let A be a process. Then A is said to be **increasing** if for each $\omega \in \Omega$, $A(\cdot,\omega)$ is right continuous and increasing.*

Note 3.10. *It is clear that an increasing process A is cadlag and that for each $(t,\omega) \in [0,\infty)\times\Omega$, A admits a Lebesgue-Stiltjes measure on $[0,t]$ which we denote by $dA(\omega)$.*

3.2. Localization.

Definition 3.11. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and $\mathcal{D}$ be a set of stochastic processes. Define the localized class of $\mathcal{D}$ (with respect to $\{\mathcal{F}_t\}_{t\geq 0}$), denoted $\mathcal{D}_{loc}$, to be the set of processes such that $X \in \mathcal{D}_{loc}$ iff there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times for $\{\mathcal{F}_t\}_{t\geq 0}$ such that $T_n \nearrow \infty$ a.s. and for each $n \in \mathbb{N}$, $X^{T_n}\mathbf{1}_{\{T_n>0\}} \in \mathcal{D}$.*

3.3. Local Martingales and Semimartingales.

Definition 3.12. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. Then X is said to be a **local martingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if X is a locally uniformly integrable martingale.*

Definition 3.13. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. Then X is said to be a **semimartingale** with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if there exist processes M and A such that M is a local martingal with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, and A is locally of bounded variation with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.*

3.4. Doob-Meyer Decomposition.

Definition 3.14. *Let X be a process and $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration. Then X is said to be **of class** $\mathcal{D}$ if $\{X_T : T$ is a finite stopping time for $\{\mathcal{F}_t\}_{t\geq 0}\}$ is uniformly integrable.*

Theorem 3.15. *Doob-Meyer Decomposition: Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and X a cadlag submartingale (w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$) of class $\mathcal{D}$. If $X_0 = 0$ a.s., then there exist processes M and A such that M is a uniformly integrable, right continuous martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, $M_0 = 0$ a.s., A is a predictable, increasing process, $A_0 = 0$ a.s. and $X = M + A$ a.s. This decomposition is unique a.s.*

Exercise 5. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and A a bounded, and a.s. increasing process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. Then A is a submartingale of class $\mathcal{D}$.*

Proof. By assumption, the A is adapted so the first submartingale criterion is satisfied. Since A is a bounded process, the second submartingale criterion is satisfied. Let $s, t \in [0, \infty)$. Suppose that $s < t$. Then $A_s \leq A_t$ a.s. So

$$\begin{aligned} \mathbb{E}[A_t|\mathcal{F}_s] &\geq \mathbb{E}[A_s|\mathcal{F}_s] \text{ a.s.} \\ &= A_s \text{ a.s.} \end{aligned}$$

So the third submartingale criterion is satisfied and A is a submartingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Put $\mathcal{T} = \{T : T \text{ is a finite stopping time for } \{\mathcal{F}_t\}_{t\geq 0}\}$. Since A is a bounded process, there exists $M > 0$ such that for each $(t, \omega) \in [0, \infty) \times \Omega$, $|X(t, \omega)| \leq M$. In particular for each $T \in \mathcal{T}$, $|X_T| \leq M$ on Ω. Then it is easy to show that

(1) $\sup\limits_{T\in\mathcal{T}} \mathbb{E}[|X_T|] \leq M$
(2) for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{F}$, if $\mathbb{P}(E) < \delta$, then $\sup\limits_{T\in\mathcal{T}} \mathbb{E}[|X_T|\mathbf{1}_E] < \epsilon$

By a previous exercise, $\{X_T : T \in \mathcal{T}\}$ is uniformly integrable. So X is of class $\mathcal{D}$. □

4. Stochastic Integration

Definition 4.1. *Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration and A a bounded, increasing, and cadlag process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. Then by a previous exercise A is a submartingale of class $\mathcal{D}$. The Doob-Meyer decomposition tells us that there exist processes M and $\widetilde{A}$ such that M is a uniformly integrable, right continuous martingale, $M_0 = 0$ a.s., $\widetilde{A}$ is a predictable, increasing process, $\widetilde{A}_0 = 0$ a.s. and $A = A_0 + X + \widetilde{A}$ a.s. We define the **compensator** of A to be the process $\widetilde{A}$.*

5. Levi Processes