PORTFOLIO THEORY NOTES

Contents

Note 0.1. *In these notes we will mostly consider random variables X that model returns. As such we may assume that $X \in L^1(\mathbb{P})$ and $F_X : \mathbb{R} \to (0,1)$ is bijective and continuous. We will call such random variabels "nice".*

1. Risk Measures

1.1. Value at Risk.

Definition 1.1. *Let X be a nice random variable and $\epsilon \in (0,1)$. We define the **value at risk of X at confidence level** ϵ, denoted by $VaR_\epsilon(X)$, to be*

$$VaR_\epsilon(X) = F_{-X}^{-1}(\epsilon)$$

Note 1.2. *If X represents the return of a portfolio, then $Var_\epsilon(X)$ is just a bound such that with probability ϵ, the loss of the portfolio is not less than the bound.*

1.2. Estimating the Value at Risk.

1.3. Average Value at Risk.

Definition 1.3. *Let X be a nice random variable and $\epsilon \in (0,1)$. We define the **average value at risk of X with tail probability** ϵ, denoted by $AVaR_\epsilon(X)$, to be*

$$AVaR_\epsilon(X) = \frac{1}{1-\epsilon} \int_{[\epsilon,\infty)} VaR_p(X)dm(p)$$

Note 1.4. *If X represents the return on a portfolio, then $AVaR_\epsilon(X)$ is just the average of the $VaR_p(X)$ over all $p < \epsilon$.*

Exercise 1.5. *Let X be a nice random variable and $\epsilon \in (0,1)$. Then $AVaR_\epsilon(X) = \mathbb{E}[-X| - X \geq VaR_\epsilon(X)]$.*

Proof. Recall that for measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a measurable function $f : X \to Y$ and a measure $\mu : \mathcal{A} \to [0,\infty]$, we may form the push-foreward measure of μ by f, $f_*\mu : \mathcal{B} \to [0,\infty]$ with the following property: for each $g : Y \to \mathbb{C}$, $g \in L^1(f_*\mu)$ iff $g \circ f \in L^1(\mu)$ and for each $B \in \mathcal{B}$,

$$\int_{f^{-1}(B)} g \circ f d\mu = \int_B g df_* \mu$$

Put $L = -X$. Note that

$$\begin{aligned}
\mathbb{E}[L|L \geq F_L^{-1}(\epsilon)] &= \mathbb{E}[L|L \geq F_L^{-1}(\epsilon)] \\
&= \frac{1}{1-\epsilon}\mathbb{E}[L\mathbf{1}_{\{L \geq F_L^{-1}(\epsilon)\}}] \\
&= \frac{1}{1-\epsilon}\int_{\{L \geq F_L^{-1}(\epsilon)\}} LdP \\
&= \frac{1}{1-\epsilon}\int_{[F_L^{-1}(\epsilon),\infty)} xdF_L(x)
\end{aligned}$$

Let μ be the Lebesgue-Stieltjes measure obtained from F_L (i.e. $d\mu = dF_L$). Consider $F_L : \mathbb{R} \to (0,1)$ as in the theorem recalled above. Then for each $(a,b] \subset [0,1]$ with $a' = F_L^{-1}(a)$ (could be $-\infty$) and $b' = F_L^{-1}(b)$, we have that

$$\begin{aligned}
F_{L*}\mu((a,b]) &= \mu(F_L^{-1}((a,b])) \\
&= \mu((a',b']) \\
&= F_L(b') - F_L(a') \\
&= b - a
\end{aligned}$$

So $F_{L*}\mu = m$. Hence

$$\begin{aligned}
\int_{[F_L^{-1}(\epsilon),\infty)} xdF_L(x) &= \int_{[F_L^{-1}(\epsilon),\infty)} (F_L^{-1} \circ F_L)(x)dF_L(x) \\
&= \int_{[\epsilon,\infty)} F_L^{-1}(x)dm(x) \\
&= \int_{[\epsilon,\infty)} VaR_\epsilon(X)dm(x)
\end{aligned}$$

□

Note 1.6. *If X represents the return of a portfolio. We may define the **loss of** X, denoted by L_X, to be $L_X = -X$. Then $AVaR_\epsilon(X) = \mathbb{E}[L_X|L_X > VaR_\epsilon(X)]$.*

Theorem 1.7. *Let X be a nice random variable and $\epsilon \in (0,1)$. Then*

$$AVaR_\epsilon(X) = \min_{\theta \in \mathbb{R}} \left(\theta + \frac{1}{1-\epsilon}\mathbb{E}[(-X-\theta)^+]\right)$$

Proof. For $\omega \in \Omega, \theta \in \mathbb{R}$, put $g_\omega(\theta) = (-X(\omega) - \theta)^+$ and for $\theta \in \mathbb{R}, \epsilon \in (0,1)$, put $f_\epsilon(\theta) = \theta + \frac{1}{1-\epsilon}\mathbb{E}[g(\theta)]$. Then for each $\omega \in \Omega$, g_ω is convex. This implies that for each $\epsilon \in (0,1)$, f_ϵ is convex and therefore continuous.

Let $L = -X$ be the loss of X. One can show that

$$\frac{\partial f_\epsilon}{\partial \theta}(\theta) = \frac{F_L(\theta) - \epsilon}{1-\epsilon}$$

The details can be found in [**?**], but will be omitted here. Thus

$$\lim_{\theta\to\infty} \frac{\partial f_\epsilon}{\partial \theta}(\theta) = 1$$

and

$$\lim_{\theta\to-\infty} \frac{\partial f_\epsilon}{\partial \theta}(\theta) = -\frac{\epsilon}{1-\epsilon} < 0$$

This implies that there exists $\theta^* \in \mathbb{R}$ such that $f_\epsilon(\theta^*) = \inf_{\theta\in\mathbb{R}} f_\epsilon(\theta)$

Thus

$$\frac{\partial f_\epsilon}{\partial \theta}(\theta^*) = 0$$

which implies that

$$F_L(\theta^*) = \epsilon$$

This implies that $\theta^* = VaR_\epsilon(X)$ Finally, evaluating f_ϵ at θ^* shows us that

$$\begin{aligned}
f_\epsilon(\theta^*) &= \theta^* + \frac{1}{1-\epsilon}\mathbb{E}[(L-\theta^*)^+] \\
&= \theta^* + \frac{1}{\mathbb{P}(L>\theta^*)}\mathbb{E}[(L-\theta^*)\mathbf{1}_{\{L>\theta^*\}}] \\
&= \theta^* + \frac{1}{\mathbb{P}(L>\theta^*)}\mathbb{E}[L\mathbf{1}_{\{L>\theta^*\}}] - \frac{1}{\mathbb{P}(L>\theta^*)}\mathbb{E}[\theta^*\mathbf{1}_{\{L>\theta^*\}}] \\
&= \theta^* + \frac{1}{\mathbb{P}(L>\theta^*)}\mathbb{E}[L\mathbf{1}_{\{L>\theta^*\}}] - \theta^* \\
&= \mathbb{E}[L|L>\theta^*] \\
&= \mathbb{E}[L|L>VaR_\epsilon(X)] \\
&= AVaR_\epsilon(X)
\end{aligned}$$

□

1.4. **Estimating the Average Value at Risk.**

Definition 1.8. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\epsilon \in (0,1)$. We define the **sample average value at risk of X with tail probability** ϵ, denoted by $\widehat{AVar_\epsilon}(X)$, to be*

$$\widehat{AVaR_\epsilon}(X) = \min_{\theta\in\mathbb{R}} \left(\theta + \frac{1}{n(1-\epsilon)}\sum_{i=1}^n \max(-X_i - \theta, 0)\right)$$

Lemma 1.9. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\epsilon \in (0,1)$. Then $\widehat{AVarR_\epsilon}(X)$ is an unbiased estimator for $AVaR_\epsilon(X)$.*

Proof. For each $\epsilon \in (0,1), \omega \in \Omega$ and $\theta \in \mathbb{R}$, define

$$f_\epsilon(\omega)(\theta) = \theta + \frac{1}{n(1-\epsilon)}\sum_{i=1}^n \max(-X_i(\omega) - \theta, 0)$$

Note that for each $\epsilon \in (0,1)$ and $\omega \in \Omega$, $f_\epsilon(\omega)$ is convex and continuous. In this case with no expectation, it is easy to show that

$$\lim_{\theta \to \infty} \frac{\partial f_\epsilon(\omega)}{\partial \theta}(\theta) = 1$$

and

$$\lim_{\theta \to -\infty} \frac{\partial f_\epsilon(\omega)}{\partial \theta}(\theta) = -\frac{\epsilon}{1-\epsilon} < 0$$

So for each $\epsilon \in (0,1)$ and $\omega \in \Omega$, $f_\epsilon(\omega)$ achieves its minimum at . Then $\{\theta \in \mathbb{R} : f_\epsilon(\omega)(\theta) \leq m+1\}$ is bounded

Since f_ϵ is continuous, we have that

$$\inf_{\theta \in \mathbb{R}} f_\epsilon(\theta) = \inf_{\theta \in \mathbb{Q}} f_\epsilon(\theta)$$

which is measurable.

$\square$

References