QUANTUM MECHANICS NOTES

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Contents

1. Introduction

1.1. Schrödinger Equation.

Definition 1.1. *A particle with potential energy $V(x,t)$ is completely decribed by its* ***position wavefunction*** $\Psi(x,t)$*, which satisfies the* ***Schrödinger equation****:*

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi$$

Interpretation 1.2. *We interpret $|\Psi(x,t)|^2$ to be the* ***probability density*** *for the position, x, of the particle at time t. Therefore, we require that for each $t \in \mathbb{R}$,*

$$\int_{\mathbb{R}^n} \Psi(x,t)^*\Psi(x,t)dx = 1$$

1.2. Operators.

Definition 1.3. *We define the j^{th}* ***position*** *and* ***momentum coordinate operators*** *X_j, P_j, (in position space) by*

$$X_j\Psi(x,t) = x_j\Psi(x,t)$$

and

$$P_j\Psi(x,t) = -i\hbar\frac{\partial}{\partial x_j}\Psi(x,t)$$

We define the ***position*** *and* ***momentum*** *operators, X and P, by*

$$X = (X_1, X_2, \cdots, X_n)$$

and

$$P = (P_1, P_2, \cdots, P_n)$$

We denote $P \cdot P$ by P^2. Note that

$$P^2 = -\hbar^2 \Delta$$

If the partical has potential energy $V(x,t)$, we define the ***Hamiltonian*** *operator, H, by*

$$H = \frac{P^2}{2m} + V$$

Thus the Schrödinger equation reads

$$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi$$

Note 1.4. *If the potential energy doesn't depend on time, we may write*

$$H = \frac{P^2}{2m} + V(X)$$

meaning Hamiltonian only depends on the position and momentum operators, X and P. For the rest of these notes, we assume that the potential energy V does not depend on time.

Definition 1.5. *Let A and B be operators. Then B is said to be the* ***adjoint*** *of A if for each Ψ_1, Ψ_2,*

$$\langle \Psi_1 | A\Psi_2 \rangle = \langle B\Psi_1 | \Psi_2 \rangle$$

i.e.

$$\int_{\mathbb{R}^n} \Psi_1^* (A\Psi_2) dx = \int_{\mathbb{R}^n} (B\Psi_1)^* \Psi_2 dx$$

If B is the adjoint of A, we write

$$B = A^\dagger$$

Exercise 1.6. *Let A be an operator, then*

(1) for each Ψ_1, Ψ_2, $\langle A\Psi_1 | \Psi_2 \rangle = \langle \Psi_1 | A^\dagger \Psi_2 \rangle$
(2) $(A^\dagger)^\dagger = A$

Proof. (1) For wavefunctions Ψ_1, Ψ_2, we have

$$\begin{aligned} \langle A\Psi_1 | \Psi_2 \rangle &= \langle \Psi_2 | A\Psi_1 \rangle^* \\ &= \langle A^\dagger \Psi_2 | \Psi_1 \rangle^* \quad \text{(by definition)} \\ &= \langle \Psi_1 | A^\dagger \Psi_2 \rangle \end{aligned}$$

(2) For each Ψ_1, Ψ_2, we have that

$$\begin{aligned} \langle A\Psi_1 | \Psi_2 \rangle &= \langle \Psi_1 | A^\dagger \Psi_2 \rangle \\ &= \langle (A^\dagger)^\dagger \Psi_1 | \Psi_2 \rangle \end{aligned}$$

This implies that for each Ψ_1, Ψ_2,

$$\left\langle \left[A - (A^\dagger)^\dagger\right] \Psi_1, \Psi_2 \right\rangle = 0$$

Therefore for each Ψ_1,

$$\left[A - (A^\dagger)^\dagger\right] \Psi_1 = 0$$

Hence $\langle A - (A^\dagger)^\dagger = 0$ and $A = (A^\dagger)^\dagger$.

□

Definition 1.7. *An linear operator Q is* ***self-adjoint*** *if*

$$Q = Q^\dagger$$

Interpretation 1.8. *For each measurable, observable quantity $\hat{Q}$, there is a self-adjoint operator Q whose eigenvalues are the possible measurment values and whose eigenfunctions are the possible states of the system at measurment.*

Exercise 1.9. *The operators X_j, P_j and H are self adjoint.*

Proof. Since x_j is real, clearly

$$\langle \Psi_1 | X_j \Psi_2 \rangle = \langle X_j \Psi_1 | \Psi_2 \rangle$$

Similarly, we have that

$$\begin{aligned}
\langle \Psi_1 | P_j \Psi_2 \rangle &= \int_{\mathbb{R}^n} \Psi_1^* \bigg(-i\hbar \frac{\partial}{\partial x_j} \Psi_2 \bigg) dx \\
&= -i\hbar \int_{\mathbb{R}^n} \Psi_1^* \bigg(\frac{\partial}{\partial x_j} \Psi_2 \bigg) dx \\
&= i\hbar \int_{\mathbb{R}_n} \bigg(\frac{\partial}{\partial x_j} \Psi_1^* \bigg) \Psi_2 dx \qquad \text{(integration by parts)} \\
&= \int_{\mathbb{R}^n} \bigg(-i\hbar \frac{\partial}{\partial x_j} \Psi_1 \bigg)^* \Psi_2 dx \\
&= \langle P \Psi_1 | \Psi_2 \rangle
\end{aligned}$$

Finally

$$\begin{aligned}
\langle \Psi_1 | H \Psi_2 \rangle - \langle H \Psi_1 | \Psi_2 \rangle &= \int_{\mathbb{R}^n} \Psi_1^* \bigg(-\frac{\hbar^2}{2m} \Delta \Psi_2 + V \Psi_2 \bigg) dx - \int_{\mathbb{R}^n} \bigg(-\frac{\hbar^2}{2m} \Delta \Psi_1 + V \Psi_1 \bigg)^* \Psi_2 dx \\
&= \frac{\hbar^2}{2m} \int_{\mathbb{R}^n} (\Delta \Psi_1^*) \Psi_2 - \Psi_1^* (\Delta \Psi_2) dx \\
&= 0 \qquad \text{(Green's second identity)}
\end{aligned}$$

□

Exercise 1.10. *Let Q be a self-adjoint operator. Then*

(1) the eigenvalues of Q are real.
(2) the eigenfunctions of Q corresponding to distinct eigenvalues are orthogonal.

Proof.

(1) Let λ be an eigenvalue of Q with corresponding eigenfunction Ψ. Then

$$\begin{aligned}
\lambda \langle \Psi | \Psi \rangle &= \langle \Psi | Q \Psi \rangle \\
&= \langle Q \Psi | \Psi \rangle \\
&= \lambda^* \langle \Psi | \Psi \rangle
\end{aligned}$$

Thus $\lambda = \lambda^*$ and is real

(2) Let λ_1 and λ_2 be eigenvalues of Q with corresponding eigenfunctions Ψ_1 and Ψ_2. Suppose that $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned}\lambda_2\langle\Psi_1|\Psi_2\rangle &= \langle\Psi_1|Q\Psi_2\rangle \\ &= \langle Q\Psi_1|\Psi_2\rangle \\ &= \lambda_1\langle\Psi_1|\Psi_2\rangle\end{aligned}$$

So $(\lambda_2-\lambda_1)\langle\Psi_1|\Psi_2\rangle = 0$. Which implies that $\langle\Psi_1|\Psi_2\rangle = 0$

□

Definition 1.11. *Let A and B be operators. The **commutator** of A and B, $[A,B]$, is defined by*

$$[A,B] = AB - BA$$

Exercise 1.12. *We have $[X_j, P_j] = i\hbar$.*

Proof. For a position wave function Ψ,

$$\begin{aligned}[X_j,P_j]\Psi(x,t) &= [x_j, -i\hbar\frac{\partial}{\partial x_j}]\Psi(x,t) \\ &= (-i\hbar)\left[x_j\frac{\partial}{\partial x_j}\Psi(x,t) - \frac{\partial}{\partial x_j}x_j\Psi(x,t)\right] \\ &= (-i\hbar)\left[x_j\frac{\partial}{\partial x_j}\Psi(x,t) - \Psi(x,t) - x_j\frac{\partial}{\partial x_j}\Psi(x,t)\right] \\ &= i\hbar\Psi(x,t)\end{aligned}$$

Hence $[X_j,P_j] = i\hbar$ □

1.3. **Continuity Equation.**

Exercise 1.13. *If V is real and Ψ satisfies the Schrödinger equation, then*

$$i\hbar\frac{\partial}{\partial t}\Psi^* = -H\Psi^*$$

Proof. We have that

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\Psi^* &= \left(-i\hbar\frac{\partial}{\partial t}\Psi\right)^* \\ &= \left(-\left[-\frac{\hbar^2}{2m}\Delta\Psi + V\Psi\right]\right)^* \\ &= -\left[-\frac{\hbar^2}{2m}\Delta\Psi^* + V\Psi^*\right] \\ &= -H\Psi^*\end{aligned}$$

□

Exercise 1.14. *We have that*

$$\frac{\partial}{\partial t}(\Psi^*\Psi) + \frac{\hbar}{2mi}\nabla\cdot\left[\Psi^*(\nabla\Psi) - (\nabla\Psi^*)\Psi\right] = 0$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}(\Psi^*\Psi) &= \left(\frac{\partial}{\partial t}\Psi^*\right)\Psi + \Psi^*\left(\frac{\partial}{\partial t}\Psi\right) \\
&= \left(\frac{\hbar}{2mi}(\Delta\Psi^*)\Psi - \frac{1}{i\hbar}V\Psi^*\Psi\right) + \left(-\frac{\hbar}{2mi}\Psi^*(\Delta\Psi) + \frac{1}{i\hbar}V\Psi^*\Psi\right) \\
&= \frac{\hbar}{2mi}\Big[(\Delta\Psi^*)\Psi - \Psi^*(\Delta\Psi)\Big] \\
&= -\frac{\hbar}{2mi}\Big[\Psi^*(\Delta\Psi) - (\Delta\Psi^*)\Psi\Big] \\
&= -\frac{\hbar}{2mi}\nabla\cdot\Big[\Psi^*(\nabla\Psi) - (\nabla\Psi^*)\Psi\Big]
\end{aligned}$$

Therefore

$$\frac{\partial}{\partial t}(\Psi^*\Psi) + \frac{\hbar}{2mi}\nabla\cdot\Big[\Psi^*(\nabla\Psi) - (\nabla\Psi^*)\Psi\Big] = 0$$

□

Definition 1.15. *We define the* ***probability current density****, j, of the particle to be*

$$j = \frac{\hbar}{2mi}\Big[\Psi^*(\nabla\Psi) - (\nabla\Psi^*)\Psi\Big]$$

1.4. Position and Momentum Space.

Definition 1.16. *We define the* ***momemtum wavefunction****, Φ, of the particle to be the Fourier transform of the position wavefunction:*

$$\begin{aligned}
\Phi(p,t) &= F[\Psi](p,t) \\
&= \frac{1}{(2\pi\hbar)^{n/2}}\int_{\mathbb{R}^n}\Psi(x,t)e^{-i\frac{p\cdot x}{\hbar}}dx
\end{aligned}$$

Note 1.17. *We recall the following facts about Fourier transforms:*

(1)

$$\Phi(p,t) = \frac{1}{(2\pi\hbar)^{n/2}}\int_{\mathbb{R}^n}\Psi(x,t)e^{-i\frac{p\cdot x}{\hbar}}dx$$

and

$$\Psi(x,t) = \frac{1}{(2\pi\hbar)^{n/2}}\int_{\mathbb{R}^n}\Phi(p,t)e^{i\frac{p\cdot x}{\hbar}}dp$$

(2)

$$F\left[\frac{\partial}{\partial x_j}\Psi\right] = \frac{ip_j}{\hbar}F[\Psi]$$

and

$$F^{-1}\left[\frac{\partial}{\partial p_j}\Phi\right] = -\frac{ix_j}{\hbar}F[\Psi]$$

(3)

$$\int_{\mathbb{R}^n}\Psi_1^*\Psi_2dx = \int_{\mathbb{R}^n}F[\Psi_1]^*F[\Psi_2]dx$$

Note 1.18. *Let $Q(X,P)$ be a self-adjoint operator. Then the properties of the Fourier transform inmply that:*

$$Q(X,P) = \begin{cases} Q(x, -i\hbar\nabla) & \textit{(position space)} \\ Q(i\hbar\nabla, p) & \textit{(momentum space)} \end{cases}$$

Exercise 1.19. *If Ψ satisfies the Schrödinger equation, then Φ satisfies*

$$i\hbar\frac{\partial}{\partial t}\Phi = \frac{p^2}{2m}\Phi + V(i\hbar\nabla)\Phi$$

Proof. Starting with the Schrödinger equation, we have

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi &= \left[\frac{P^2}{2m} + V(X)\right]\Psi \\ &= \left[\frac{-\hbar^2}{2m}\Delta + V(x)\right]\Psi \qquad \text{(position space)} \end{aligned}$$

Taking Fourier transforms of both sides, we see that

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Phi &= \left[\frac{P^2}{2m} + V(X)\right]\Phi \\ &= \left[\frac{p^2}{2m} + V(i\hbar\nabla)\right]\Phi \qquad \text{(position space)} \end{aligned}$$

□

Interpretation 1.20. *We interpret $|\Phi(p,t)|^2$ to be the probability density for the momentum, p, of the particle at time t.*

Note 1.21. *For a self-adjoint operator $Q(X,P)$, the expected value of Q, is given by*

$$\langle Q\rangle = \begin{cases} \langle \Psi(x,t)|Q(x,-i\hbar\nabla)\Psi(x,t)\rangle & \textit{(position space)} \\ \langle \Phi(p,t)|Q(i\hbar\nabla,p)\Phi(p,t)\rangle & \textit{(momentum space)} \end{cases}$$

1.5. **Stationary States.**

Definition 1.22. *When the potential energy V doesn't depend on time, we look for solutions to the Schrödinger equation of the form*

$$\Psi(x,t) = \psi(x)\varphi(t)$$

With a closer look, we find that

(1) $H\psi = E\psi$
(2) $\varphi(t) = e^{-i\frac{E}{\hbar}t}$

Statement (1) is referred to as the ***time-independent Schrödinger equation****. Eigenfuntions of the Hamiltonian operator are called* ***stationary states****. If the possible eigenvalues for the Hamiltonian operator are discreet $(E_n)_{n\in\mathbb{N}}$ with stationary states $(\psi_n)_{n\in\mathbb{N}}$, then the general solution to the Schrödinger equation is*

$$\Psi(x,t) = \sum_{n\in\mathbb{N}} c_n\psi_n(x)e^{-i\frac{E_n}{\hbar}t}$$

where

$$c_n = \int_{\mathbb{R}^n} \psi_n^*(x)\Psi(x,0)dx$$

Definition 1.23. *If the spectrum of the Hamiltonian is discreet, the stationary state with the least energy is called the* ***ground state****. The stationary states that are not the ground state are called* ***excited states****.*

2. Fundamental Examples in One Dimension

2.1. The Harmonic Oscillator.

Definition 2.1. *The* ***harmonic oscillator*** *in one dimension is defined by the potential energy:*

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

We define the ***lowering operator****, a, by*

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\Big(m\omega X + iP\Big)$$

Exercise 2.2. *The adjoint of the lowering operator is*

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\Big(m\omega X - iP\Big)$$

Proof. For a wave functions Ψ_1, Ψ_2,

$$\begin{aligned}\int_{\mathbb{R}} \left[\frac{1}{\sqrt{2\hbar m\omega}}\Big(m\omega X - iP\Big)\Psi_1\right]^* \Psi_2 dx &= \frac{1}{\sqrt{2\hbar m\omega}}\int_{\mathbb{R}} (m\omega x\Psi_1(x,t)^*\Psi_2(x,t) - \hbar\Big(\frac{\partial}{\partial x}\Psi_1(x,t)^*\Big)\Psi_2(x,t)dx \\ &= \frac{1}{\sqrt{2\hbar m\omega}}\int_{\mathbb{R}} (m\omega x\Psi_1(x,t)^*\Psi_2(x,t) + \hbar\Psi_1(x,t)^*\Big(\frac{\partial}{\partial x}\Psi_2(x,t)\Big)dx \\ &= \int_{\mathbb{R}} \Psi_1^*\left[\frac{1}{\sqrt{2\hbar m\omega}}\Big(m\omega X + iP\Big)\Psi_2\right]dx\end{aligned}$$

□

Definition 2.3. *We call $a^\dagger$ the* ***raising operator*** *and together, a and $a^\dagger$ are called the ladder operators.*

Exercise 2.4. *We have that*

(1) $aa^\dagger = \frac{1}{\hbar\omega}H + \frac{1}{2}$
(2) $a^\dagger a = \frac{1}{\hbar\omega}H - \frac{1}{2}$
(3) $[a, a^\dagger] = 1$

Proof. (1)

$$\begin{aligned} aa^\dagger &= \frac{1}{2\hbar m\omega}\big(m\omega X + iP\big)\big(m\omega X - iP\big) \\ &= \frac{1}{2\hbar m\omega}\Big[\big(m^2\omega^2X^2 + P^2\big) - m\omega i\big(XP - PX\big)\Big] \\ &= \frac{1}{\hbar\omega}\big(\frac{1}{2m}P^2 + \frac{1}{2}m\omega^2X^2\big) - \frac{i}{2\hbar}\big[X, P\big] \\ &= \frac{1}{\hbar\omega}H + \frac{1}{2} \end{aligned}$$

(2) Similar
(3) Trivial

□

Exercise 2.5. *If $H\psi = E\psi$, then*

(1) $Ha\psi = (E - \hbar\omega)a\psi$
(2) $Ha^\dagger\psi = (E + \hbar\omega)a^\dagger\psi$

Proof.

(1)

$$\begin{aligned} Ha\psi &= \hbar\omega\left(aa^\dagger - \frac{1}{2}\right)a\psi \\ &= \hbar\omega\left(aa^\dagger a - \frac{1}{2}a\right)\psi \\ &= \hbar\omega a\left(a^\dagger a - \frac{1}{2}\right)\psi \\ &= \hbar\omega a\left(a^\dagger a + \frac{1}{2} - 1\right)\psi \\ &= \hbar\omega a\left(\frac{1}{\hbar\omega}H - 1\right)\psi \\ &= aH\psi - \hbar\omega a\psi \\ &= (E - \hbar\omega)a\psi \end{aligned}$$

(2) Similar

□

Interpretation 2.6. *The lowering operator "lowers" a stationary state ψ with energy E to a stationary state $a\psi$ with energy $E - \hbar\omega$ and the raising operator "raises" a stationary state ψ with energy E to a stationary state $a^\dagger\psi$ with energy $E + \hbar\omega$.*

Definition 2.7. *Since the zero function is a solution to the time-independent Schrödinger equation, we define the ground state, ψ_0 of the harmonic oscillator to be the stationary state that satisfies $a\psi_0 = 0$. The excited states ψ_n, for $n \geq 1$, are obtained by applying the rasing operator n times and then normalizing.*

Exercise 2.8. *We have that*

(1) $\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$

(2) $E_0 = \frac{1}{2}\hbar\omega$
(3) $\psi_n = c_n(a^\dagger)^n\psi_0$ *(for some constant c_n)*
(4) $E_n = \hbar\omega(n + \frac{1}{2})$

Proof.

(1) The simple differential equation $a\psi_0 = 0$ has the solution
$$\psi_0 = Ae^{-\frac{m\omega}{2\hbar}x^2}$$
Thus
$$|\psi_0|^2 = |A|^2 e^{-\frac{m\omega}{\hbar}x^2}$$
If we normalize this function, we obtain
$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$
(2) It is tedious but straightforward to show that
$$H\psi_0 = \frac{1}{2}\hbar\omega\psi_0$$
(3) Clear by definition.
(4) Clear by previous exercise.

□

Exercise 2.9.
(1) $\psi_{n+1} = \frac{1}{\sqrt{n+1}}a^\dagger\psi_n$
(2) $\psi_{n-1} = \frac{1}{\sqrt{n}}a\psi_n$
Hint: use the adjoint-ness of a and $a^\dagger$

Proof.

(1)
$$\begin{aligned} aa^\dagger\psi_n &= \left(\frac{1}{\hbar\omega}H + \frac{1}{2}\right)\psi_n \\ &= \frac{1}{\hbar\omega}E_n\psi_n + \frac{1}{2}\psi_n \\ &= (n+1)\psi_n \end{aligned}$$
Since $\psi_{n+1} = ca^\dagger\psi_n$, we have that
$$\begin{aligned} 1 &= \langle\psi_{n+1}|\psi_{n+1}\rangle \\ &= \langle ca^\dagger\psi_n|ca^\dagger\psi_n\rangle \\ &= |c|^2\langle a^\dagger\psi_n|a^\dagger\psi_n\rangle \\ &= |c|^2\langle aa^\dagger\psi_n|\psi_n\rangle \\ &= |c|^2\langle(n+1)\psi_n|\psi_n\rangle \\ &= |c|^2(n+1)\langle\psi_n|\psi_n\rangle \\ &= |c|^2(n+1) \end{aligned}$$

So $c = \frac{1}{\sqrt{n+1}}$

(2) Similar to (1).

□

Exercise 2.10. *The n^{th} stationary state is given by $\psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\psi_0$*

Proof. Clear by induction. □

Exercise 2.11. *Show that*
(1) $\psi_1(x) = \left(\frac{4m^3\omega^3}{\hbar^3\pi}\right)xe^{-\frac{m\omega}{2\hbar}x^2}$
(2) $E_1 = \frac{3}{2}\hbar\omega$

Proof. Straightforward. □

Exercise 2.12. *If particle one is in state ψ_0 at time $t = 0$, then the momentum wave function is*

$$\Phi(p,t) = \left(\frac{1}{m\omega\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2m\omega\hbar}p^2}e^{-i\frac{\omega}{2}t}$$

Proof. By assumption

$$\Psi(x,t) = \psi_0(x)e^{-i\frac{\omega}{2}t}$$

Thus

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}} \Psi(x,t)e^{-i\frac{px}{\hbar}}\,dx$$

The rest is straightforward. □

3. Fundamental Examples in Three Dimensions

3.1. Spherical Coordinates.

Definition 3.1. *We now set $n = 3$, and work with spherical coordinates (r, θ, ϕ) where r is the distance in from the origin, $0 \le \theta \le \pi$ is the angle with initial side on the positive z-axis, and $0 \le \phi < 2\pi$ is the angle in the x-y plane with initial side on the positive x-axis going towards the positive y-axis.*

Proposition 3.2. *In spherical coordinates, the time independent Schrödinger equation becomes*

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi + V\psi = E\psi$$

Definition 3.3. *If the potential energy V only depends on r, then we can solve for stationary solutions of the form $\psi(r,\theta,\phi) = R(r), Y(\theta,\phi)$. It results that there is some constant l such that*

(1)
$$\frac{1}{R}\frac{d}{dr}r^2\frac{dR}{dr} - \frac{2m}{\hbar^2}r^2(V - E) = l(l+1)$$

(2)
$$\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = -l(l+1)$$

*The number l is called the **azimuthal quantum number**, equation (1) is called the **radial equation** and equation (2) is called the **angular equation**.*

Definition 3.4. *We can look for solutions to the angular equation of the form $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$. It results that there is some constant m such that*

$$\frac{1}{\Theta}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+l(l+1)\sin^2\theta=m^2 \tag{1}$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2 \tag{2}$$

Equation (2) has the solution

$$\Phi(\phi)=e^{im\phi}$$

Since (r,θ,ϕ) is the same point in space as $(r,\theta,\phi+2\pi)$, we require that $\Phi(\phi)=\Phi(\phi+2\pi)$. This implies that $m\in\mathbb{Z}$. The integer m is called the ***magnetic quantum number****.*

If $l\in\mathbb{N}_0$ and $m\le l$, then equation (1) has the solution

$$\Theta(\theta)=AP_l^m(\cos\theta)$$

where P_l^m is the ***associated Legendre*** *function given by*

$$P_l^m(x)=(1-x^2)^{\frac{|m|}{2}}\left(\frac{d}{dx}\right)^{|m|}P_l(x)$$

and $P_l(x)$ is the l^{th} ***Legendre polynomial*** *defined by*

$$P_l(x)=\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l$$

The angular function $Y_l^m(\theta,\phi)=A_l^mP_l^m(\cos\theta)e^{im\phi}$ may be normalized by setting

$$A_l^m=\epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}$$

where

$$\epsilon=\begin{cases}(-1)^m & m\ge 0\\ 1 & m<0\end{cases}$$

The normalized angular functions are called ***spherical harmonics****.*

Exercise 3.5. *Compute some spherical harmonics.*

Definition 3.6. *If we make the substitution $u(r)=rR(r)$, we may rewrite the radial equation as*

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}=\left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u=Eu$$

which looks like the one dimensional Schrödinger equation. The function

$$V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$

is called the ***effective potential****.*

3.2. Spherical Harmonic Oscillator.