PORTFOLIO THEORY NOTES

Contents

Note 0.1. *In these notes we will mostly consider random variables X that model returns. As such we may assume that $X \in L^1(\mathbb{P})$ and $F_X : \mathbb{R} \to (0,1)$ is bijective and continuous. We will call such random variabels "nice". The random variable X will usually be taken to mean the return on some portfolio. As such, we will define the loss of X to be $L_X = -X$.*

1. Risk Measures

1.1. Value at Risk.

Definition 1.1. *Let X be a nice random variable and $\epsilon \in (0,1)$. We define the* ***value at risk of X at confidence level*** *ϵ, denoted by $VaR_\epsilon(X)$, to be*

$$\begin{aligned} VaR_\epsilon(X) &= F_{-X}^{-1}(\epsilon) \\ &= F_{L_X}^{-1}(\epsilon) \end{aligned}$$

Note 1.2. *If X represents the return of a portfolio, then $Var_\epsilon(X)$ is just a bound such that with probability ϵ, the loss of the portfolio is not less than the bound.*

1.2. Estimating the Value at Risk.

1.3. Average Value at Risk.

Definition 1.3. *Let X be a nice random variable and $\epsilon \in (0,1)$. We define the* ***average value at risk of X with tail probability*** *ϵ, denoted by $AVaR_\epsilon(X)$, to be*

$$AVaR_\epsilon(X) = \frac{1}{1-\epsilon}\int_{[\epsilon,\infty)} VaR_p(X)dm(p)$$

Note 1.4. *If X represents the return on a portfolio, then $AVaR_\epsilon(X)$ is just the average of the $VaR_p(X)$ over all $p < \epsilon$.*

Exercise 1.5. *Let X be a nice random variable and $\epsilon \in (0,1)$. Then*

$$\begin{aligned} AVaR_\epsilon(X) &= \mathbb{E}[-X| - X \geq VaR_\epsilon(X)] \\ &= \mathbb{E}[L_X | L_X \geq Var_\epsilon(X)] \end{aligned}$$

Proof. Recall that for measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a measurable function $f : X \to Y$ and a measure $\mu : \mathcal{A} \to [0, \infty]$, we may form the push-foreward measure of μ by f, $f_*\mu : \mathcal{B} \to [0, \infty]$ with the following property: for each $g : Y \to \mathbb{C}$, $g \in L^1(f_*\mu)$ iff $g \circ f \in L^1(\mu)$ and for each $B \in \mathcal{B}$,

$$\int_{f^{-1}(B)} g \circ f d\mu = \int_B g df_*\mu$$

Also recall that for an increasing continuous, bijective $F : \mathbb{R} \to (0, 1)$, we may form the Borel measure μ_F with $\mu_F((a, b]) = F(b) - F(a)$. Then observe that $F_*\mu_F = m$ because

$$\begin{aligned} F_*\mu_F((a, b]) &= \mu_F(F^{-1}((a, b])) \\ &= \mu_F((F^{-1}(a), F^{-1}(b)]) \\ &= F(F^{-1}(a)) - F(F^{-1}(b)) \\ &= b - a \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[L_X | L_X \geq F_{L_X}^{-1}(\epsilon)] &= \mathbb{E}[L_X | L_X \geq F_{L_X}^{-1}(\epsilon)] \\ &= \frac{1}{1-\epsilon} \mathbb{E}[L_X \mathbf{1}_{\{L_X \geq F_{L_X}^{-1}(\epsilon)\}}] \\ &= \frac{1}{1-\epsilon} \int_{\{L_X \geq F_{L_X}^{-1}(\epsilon)\}} L_X dP \\ &= \frac{1}{1-\epsilon} \int_{[F_{L_X}^{-1}(\epsilon), \infty)} x dF_{L_X}(x) \end{aligned}$$

Using the facts recalled earilier, we have

$$\begin{aligned} \int_{[F_{L_X}^{-1}(\epsilon), \infty)} x dF_{L_X}(x) &= \int_{[F_{L_X}^{-1}(\epsilon), \infty)} (F_{L_X}^{-1} \circ F_{L_X})(x) dF_{L_X}(x) \\ &= \int_{[\epsilon, \infty)} F_{L_X}^{-1}(x) dm(x) \\ &= \int_{[\epsilon, \infty)} VaR_\epsilon(X) dm(x) \end{aligned}$$

□

Note 1.6. *If X represents the return of a portfolio. We may define the* ***loss of*** *X, denoted by L_X, to be $L_X = -X$. Then $AVaR_\epsilon(X) = \mathbb{E}[L_X | L_X > VaR_\epsilon(X)]$.*

Theorem 1.7. *Let X be a nice random variable and $\epsilon \in (0, 1)$. Then*

$$AVaR_\epsilon(X) = \min_{\theta \in \mathbb{R}} \left(\theta + \frac{1}{1-\epsilon} \mathbb{E}[(-X - \theta)^+] \right)$$

Proof. For $\omega \in \Omega, \theta \in \mathbb{R}$, put $g_\omega(\theta) = (-X(\omega) - \theta)^+$ and for $\theta \in \mathbb{R}, \epsilon \in (0, 1)$, put $f_\epsilon(\theta) = \theta + \frac{1}{1-\epsilon}\mathbb{E}[g(\theta)]$. Then for each $\omega \in \Omega$, g_ω is convex. This implies that for each $\epsilon \in (0, 1)$, f_ϵ is convex and therefore continuous.

Let $L = -X$ be the loss of X. One can show that

$$\frac{\partial f_\epsilon}{\partial \theta}(\theta) = \frac{F_L(\theta) - \epsilon}{1 - \epsilon}$$

The details can be found in [?], but will be omitted here. Thus

$$\lim_{\theta \to \infty} \frac{\partial f_\epsilon}{\partial \theta}(\theta) = 1$$

and

$$\lim_{\theta \to -\infty} \frac{\partial f_\epsilon}{\partial \theta}(\theta) = -\frac{\epsilon}{1-\epsilon} < 0$$

This implies that there exists $\theta^* \in \mathbb{R}$ such that $f_\epsilon(\theta^*) = \inf_{\theta \in \mathbb{R}} f_\epsilon(\theta)$

Thus

$$\frac{\partial f_\epsilon}{\partial \theta}(\theta^*) = 0$$

which implies that

$$F_L(\theta^*) = \epsilon$$

This implies that $\theta^* = VaR_\epsilon(X)$ Finally, evaluating f_ϵ at θ^* shows us that

$$\begin{aligned}
f_\epsilon(\theta^*) &= \theta^* + \frac{1}{1-\epsilon}\mathbb{E}[(L-\theta^*)^+] \\
&= \theta^* + \frac{1}{\mathbb{P}(L > \theta^*)}\mathbb{E}[(L-\theta^*)\mathbf{1}_{\{L>\theta^*\}}] \\
&= \theta^* + \frac{1}{\mathbb{P}(L > \theta^*)}\mathbb{E}[L\mathbf{1}_{\{L>\theta^*\}}] - \frac{1}{\mathbb{P}(L > \theta^*)}\mathbb{E}[\theta^*\mathbf{1}_{\{L>\theta^*\}}] \\
&= \theta^* + \frac{1}{\mathbb{P}(L > \theta^*)}\mathbb{E}[L\mathbf{1}_{\{L>\theta^*\}}] - \theta^* \\
&= \mathbb{E}[L | L > \theta^*] \\
&= \mathbb{E}[L | L > VaR_\epsilon(X)] \\
&= AVaR_\epsilon(X)
\end{aligned}$$

□

1.4. Estimating the Value at Risk.

Definition 1.8. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\epsilon \in (0,1)$. Define*

$$\widehat{VaR_\epsilon}(X) =$$

1.5. Estimating the Average Value at Risk.

Definition 1.9. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\epsilon \in (0,1)$. Define*

$$A\widehat{VaR_\epsilon}(X) =$$

Lemma 1.10. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\epsilon \in (0,1)$. Then $A\widehat{VarR_\epsilon}(X)$ is an unbiased estimator for $AVaR_\epsilon(X)$.*

Proof. For each $\epsilon \in (0,1), \omega \in \Omega$ and $\theta \in \mathbb{R}$, define

$$f_\epsilon(\omega)(\theta) = \theta + \frac{1}{n(1-\epsilon)} \sum_{i=1}^{n} \max(-X_i(\omega) - \theta, 0)$$

Note that for each $\epsilon \in (0,1)$ and $\omega \in \Omega$, $f_\epsilon(\omega)$ is convex and continuous. In this case with no expectation, it is easy to show that

$$\lim_{\theta \to \infty} \frac{\partial f_\epsilon(\omega)}{\partial \theta}(\theta) = 1$$

and

$$\lim_{\theta \to -\infty} \frac{\partial f_\epsilon(\omega)}{\partial \theta}(\theta) = -\frac{\epsilon}{1-\epsilon} < 0$$

So for each $\epsilon \in (0,1)$ and $\omega \in \Omega$, $f_\epsilon(\omega)$ achieves its minimum at . Then $\{\theta \in \mathbb{R} : f_\epsilon(\omega)(\theta) \leq m + 1\}$ is bounded

Since f_ϵ is continuous, we have that

$$\inf_{\theta \in \mathbb{R}} f_\epsilon(\theta) = \inf_{\theta \in \mathbb{Q}} f_\epsilon(\theta)$$

which is measurable.

□

References