

REAL ANALYSIS NOTES

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1. MEASURE

1.1. Product Measures.

Definition 1.1. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measurable spaces. Put $\mathcal{E} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$. Then \mathcal{E} is an elementary family and thus $\mathcal{M}_0 = \{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n \subset \mathcal{E} \text{ are disjoint}\}$ is an algebra on $X \times Y$. We define $\pi_0 : \mathcal{M}_0 \rightarrow [0, \infty]$ by

$$\pi_0\left(\bigcup_{i=1}^n A_i \times B_i\right) = \sum_{i=1}^n \mu(A_i)\nu(B_i)$$

Since $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{M}_0)$, we define a product measure $\mu \times \nu$ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ to be an extension of π_0 to $\mathcal{A} \otimes \mathcal{B}$. The existence of which is guaranteed by Caratheodory's theorem and on $\mathcal{A} \otimes \mathcal{B}$,

$$\begin{aligned}
\mu \times \nu(E) &= \inf \left\{ \sum_{n \in \mathbb{N}} \pi_0(E_i) : (E_i)_{i \in \mathbb{N}} \subset \mathcal{M}_0 \text{ and } E \subset \bigcup_{i \in \mathbb{N}} E_i \right\} \\
&= \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_i) \nu(B_i) : (A_i \times B_i)_{i \in \mathbb{N}} \subset \mathcal{E} \text{ and } E \subset \bigcup_{i \in \mathbb{N}} A_i \times B_i \right\}
\end{aligned}$$

If (X, \mathcal{A}, μ) and (Y, \mathcal{B}, ν) are both sigma finite, then so is π_0 and thus $\mu \times \nu$ is unique.

2. INTEGRATION

2.1. Measurable Functions.

Definition 2.1. Let (X, \mathcal{A}) and (Y, \mathcal{B}) be measurable spaces and $f : X \rightarrow Y$. Then f is said to be **\mathcal{A} - \mathcal{B} measurable** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. When $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say that f is **\mathcal{A} -measurable**. If $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}, \mathcal{L})$, then we say that f is **Borel measurable** or **Lebesgue measurable** respectively.

Lemma 2.2. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \rightarrow Y$. Then

- (1) $\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ -algebra on Y
- (2) $\{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ -algebra on X

Lemma 2.3. Let (X, \mathcal{A}) and (Y, \mathcal{B}) be measurable spaces. Suppose that there exists $\mathcal{E} \subset Y$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Let $f : X \rightarrow Y$. If for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$, then f is \mathcal{A} - \mathcal{B} measurable.

Proof. The previous lemma tells us that $\mathcal{L} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ -algebra on Y . Since $\mathcal{E} \subset \mathcal{L}$, we have that $\mathcal{B} = \sigma(\mathcal{E}) \subset \mathcal{L}$. \square

Corollary 2.4. Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X \rightarrow Y$. If f is continuous, then f is $\mathcal{B}(X)$ - $\mathcal{B}(Y)$ measurable.

Proof. Recall that $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$ and continuity tells us that for each $U \in \mathcal{T}_2$, $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$. \square

Definition 2.5. Let X be a set and $f : X \rightarrow \mathbb{C}$. The f is said to be **simple** if $f(X)$ is finite.

Definition 2.6. Let (X, \mathcal{A}) be a measurable space. We define $S^+(X, \mathcal{A}) = \{f : X \rightarrow [0, \infty) : f \text{ is simple, measurable}\}$ and $S(X, \mathcal{A}) = \{f : X \rightarrow \mathbb{C} : f \text{ is simple, measurable}\}$

Theorem 2.7. Let (X, \mathcal{A}) be a measurable space. Then

- (1) If $f : X \rightarrow [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \rightarrow f$ pointwise and $\phi_n \rightarrow f$ uniformly on any set on which f is bounded.
- (2) If $f : X \rightarrow \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S$ such that for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ and $\phi_n \rightarrow f$ pointwise and $\phi_n \rightarrow f$ uniformly on any set on which f is bounded.

2.2. Integration of Nonnegative Functions.

Definition 2.8. Let (X, \mathcal{A}, μ) be a measure space. Define $L^+(X, \mathcal{A}, \mu) = \{f : X \rightarrow [0, \infty] : f \text{ is measurable}\}$. We will typically just write L^+ .

Theorem 2.9. *Monotone Convergence Theorem* Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then

$$\sup_{n \in \mathbb{N}} \int f_n = \int \sup_{n \in \mathbb{N}} f_n$$

.

Exercise 2.10. Let μ_1, μ_2 be measures on (X, \mathcal{A}) and $f \in L^+$. Then

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

.

Proof. Suppose that f is simple. Then there exist $(a_n)_{n=1}^n \subset [0, \infty)$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for general f , choose $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that $\phi_n \rightarrow f$ pointwise and for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$. Then monotone convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

Exercise 2.11. Let μ_1, μ_2 be measures on (X, \mathcal{A}) . Suppose that $\mu_1 \leq \mu_2$. Then for each $f \in L^+$,

$$\int f d\mu_1 \leq \int f d\mu_2$$

Proof. First suppose that f is simple. Then there exist $(a_n)_{n=1}^\infty \subset [0, \infty)$ and $(E_i)_{i=1}^\infty \subset \mathcal{A}$ such that $f = \sum_{i=1}^\infty a_i \chi_{E_i}$. Then

$$\begin{aligned} \int f d\mu_1 &= \sum_{i=1}^\infty a_i \mu_1(E_i) \\ &\leq \sum_{i=1}^\infty a_i \mu_2(E_i) \\ &= \int f d\mu_2 \end{aligned}$$

for general f ,

$$\begin{aligned} \int f d\mu_1 &= \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_1 \\ &\leq \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_2 \\ &= \int f d\mu_2 \end{aligned}$$

□

Theorem 2.12. *Fatou's Lemma* Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Theorem 2.13. Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n.$$

Exercise 2.14. Let $f \in L^+$ and suppose that $\int f < \infty$. Put $N = \{x \in X : f(x) = \infty\}$ and $S = \{x \in X : f(x) > 0\}$. Then $\mu(N) = 0$ and S is σ -finite.

Proof. Suppose that $\mu(N) > 0$. Define $f_n = n \chi_N \in L^+$. Then for each $n \in \mathbb{N}$, $f_n \leq f_{n+1} \leq f$ on N . So

$$\begin{aligned} \int f &\geq \int_N f \\ &= \lim_{n \rightarrow \infty} \int_N f_n \\ &= \lim_{n \rightarrow \infty} n \mu(N) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

Hence N is a null set. Now, put $S_n = \{x \in X : f(x) > 1/n\}$. Then $S = \bigcup_{n \in \mathbb{N}} S_n$. Suppose that there exists some $n \in \mathbb{N}$ such that $\mu(S_n) = \infty$. Then

$$\begin{aligned} \int f &\geq \int_{S_n} f \\ &\geq \frac{1}{n} \mu(S_n) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(S_n) < \infty$ and S is σ -finite. □

Exercise 2.15. Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.

Proof. $f = 0$ a.e. implies that for each $E \in \mathcal{A}$, $\int_E f = 0$ is clear. Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f = 0$. For $n \in \mathbb{N}$ put $N_n = \{x \in X : f(x) > 1/n\}$ and define $N = \{x \in X : f(x) > 0\}$. So $N = \bigcup_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned} 0 &= \int_{N_n} f \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

Hence for each $n \in \mathbb{N}$, $\mu(N_n) = 0$. Thus $\mu(N) = 0$ and $f = 0$ a.e. as required. □

Exercise 2.16. Let $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that $f_n \xrightarrow{p.w.} f$, $\lim_{n \rightarrow \infty} \int f_n = \int f$ and $\int f < \infty$. Then for each $E \in \mathcal{A}$, $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$. This result may fail to be true if $\int f = \infty$

Proof. Let $E \in \mathcal{A}$. By Fatou's lemma, $\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$. Note that since $\int f < \infty$, we have that $\int_{E^c} f \leq \int f < \infty$. Thus we may write

$$\begin{aligned} \int_E f &= \int f - \int_{E^c} f \\ &\geq \int f - \liminf_{n \rightarrow \infty} \int_{E^c} f_n \\ &= \int f - \liminf_{n \rightarrow \infty} \left(\int f_n - \int_E f_n \right) \\ &= \int f - \int f + \limsup_{n \rightarrow \infty} \int_E f_n \\ &= \limsup_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Hence

$$\limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$$

and therefore

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

If we drop the assumption that $\int f < \infty$, then the result would fail to be true for the functions $f = \infty \chi_{(0,1)}$ and $f_n = \infty \chi_{(0,1)} + n \chi_{(1,1+1/n)}$. Here $f_n \xrightarrow{\text{p.w.}} f$, $\lim_{n \rightarrow \infty} \int f_n = \int f = \infty$ and $\lim_{n \rightarrow \infty} \int_{(1,\infty)} f_n = 1$ while $\int_{(1,\infty)} f = 0$.

□

Exercise 2.17. Let $f \in L^+$. Define $\lambda : \mathcal{A} \rightarrow [0, \infty]$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{A}$. Then λ is a measure on (X, \mathcal{A}) and for each $g \in L^+$, $\int g d\lambda = \int g f d\mu$.

Proof. Clearly $\lambda(\emptyset) = 0$. Let $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ and suppose that for each $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j = \emptyset$. For now, suppose that f is simple. Then there exist $E_1, E_2, \dots, E_n \in \mathcal{A}$ and $a_1, a_2, \dots, a_n \in [0, \infty)$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned} \lambda\left(\bigcup_{j \in \mathbb{N}} A_j\right) &= \int_{\bigcup_{j \in \mathbb{N}} A_j} f \\ &= \sum_{i=1}^n a_i \mu\left(E_i \cap \left(\bigcup_{j \in \mathbb{N}} A_j\right)\right) \\ &= \sum_{i=1}^n a_i \mu\left(\bigcup_{j \in \mathbb{N}} E_i \cap A_j\right) \\ &= \sum_{i=1}^n a_i \sum_{j \in \mathbb{N}} \mu(E_i \cap A_j) \\ &= \sum_{j \in \mathbb{N}} \sum_{i=1}^n a_i \mu(E_i \cap A_j) \\ &= \sum_{j \in \mathbb{N}} \int_{A_j} f \\ &= \sum_{j \in \mathbb{N}} \lambda(A_j) \end{aligned}$$

Hence λ is a measure on (X, \mathcal{A}) . Now, for a general f , there exist $(\phi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \xrightarrow{\text{p.w.}} f$. Put $A = \bigcup_{j \in \mathbb{N}} A_j$ and define the measures λ_n by $\lambda_n(E) = \int_E \phi_n$. Note that we may define a monotonically increasing sequence of functions $g_n : \mathbb{N} \rightarrow [0, \infty]$ by $g_n(j) = \int_{A_j} \phi_n$. Using monotone convergence three times and a nice application of the counting measure on \mathbb{N} , we may write

$$\begin{aligned}
 \lambda(A) &= \int_A f \\
 &= \lim_{n \rightarrow \infty} \int_A \phi_n \\
 &= \lim_{n \rightarrow \infty} \sum_{j \in \mathbb{N}} \int_{A_j} \phi_n \\
 &= \sum_{j \in \mathbb{N}} \lim_{n \rightarrow \infty} \int_{A_j} \phi_n \quad (\text{by the above}) \\
 &= \sum_{j \in \mathbb{N}} \int_{A_j} f \\
 &= \sum_{j \in \mathbb{N}} \lambda(A_j).
 \end{aligned}$$

Hence λ is a measure on (X, \mathcal{A}) . Let $g \in L^+$. First assume that g is simple. Then there exist $E_1, E_2, \dots, E_n \in \mathcal{A}$ and $a_1, a_2, \dots, a_n \in [0, \infty)$ such that $g = \sum_{i=1}^n a_i \chi_{E_i}$. In this case, we have that

$$\begin{aligned}
 \int g d\lambda &= \sum_{i=1}^n a_i \lambda(E_i) \\
 &= \sum_{i=1}^n a_i \int_{E_i} f d\mu \\
 &= \int \left(\sum_{i=1}^n a_i \chi_{E_i} \right) f d\mu \\
 &= \int g f d\mu.
 \end{aligned}$$

Now for a general $g \in L^+$, there exist $(\psi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ψ_n is simple, $\psi_n \leq \psi_{n+1} \leq f$ and $\psi_n \xrightarrow{\text{p.w.}} g$. Monotone convergence then gives us

$$\begin{aligned}
 \int g d\lambda &= \lim_{n \rightarrow \infty} \int \psi_n d\lambda \\
 &= \lim_{n \rightarrow \infty} \int \psi_n f d\mu \\
 &= \int g f d\mu \text{ as required.}
 \end{aligned}$$

□

Exercise 2.18. Let $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$, $f_n \xrightarrow{\text{p.w.}} f$ and $\int f_1 < \infty$. Then $\lim_{n \rightarrow \infty} \int f_n = \int f$.

Proof. First we note that since $\int f_1 < \infty$, $f_1 < \infty$ a.e., for each $n \in \mathbb{N}$, $f_1 - f_n$ and $\int f_1 - \int f_n$ are well defined and $\int f_n \leq \int f_1 < \infty$. Also, for $n \in \mathbb{N}$, $f_1 - f_n \in L^+$. So we may write

$$\begin{aligned} \int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n \\ &= \int [(f_1 - f_n) + f_n] - \int f_n \\ &= \int f_1 - \int f_n \end{aligned}$$

Put $g_n = f + (f_1 - f_n)$. Then $g_n \in L^+$, for each $n \in \mathbb{N}$, $g_n \leq g_{n+1}$ and $g_n \xrightarrow{\text{p.w.}} f_1$. Monotone convergence tells us that

$$\begin{aligned} \int f_1 &= \lim_{n \rightarrow \infty} \int g_n \\ &= \lim_{n \rightarrow \infty} \left[\int f + (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[\int f + \int (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[\int f + \int f_1 - \int f_n \right] \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \int f$ and $\lim_{n \rightarrow \infty} \int f_1$ exist, $\lim_{n \rightarrow \infty} \int f_n = \int f$ as required. □

2.3. Integration of Complex Valued Functions.

Definition 2.19. Let $f : X \rightarrow \mathbb{C}$ be measurable. Then f is said to be **integrable** if

$$\int |f| d\mu < \infty$$

Definition 2.20. Let (X, \mathcal{A}, μ) be a measure space. Define $L^1(X, \mathcal{A}, \mu) = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable and } \int |f| < \infty\}$

Lemma 2.21. Let $f : X \rightarrow \mathbb{R}$ be measurable. Then f is integrable iff f^+ and f^- are integrable.

Proof. $f^+, f^- \leq |f| = f^+ + f^-$ □

Definition 2.22. Let $f : X \rightarrow \mathbb{R}$ be measurable. Then f is said to be **extended integrable** if

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

Lemma 2.23. Let $f : X \rightarrow \mathbb{R}$ be measurable. Then f is integrable iff $\text{Re}(f)$ and $\text{Im}(f)$ are integrable.

Proof. $|\text{Re}(f)|, |\text{Im}(f)| \leq |f| \leq |\text{Re}(f)| + |\text{Im}(f)|$ □

Theorem 2.24. Dominated Convergence Let $(f_n)_{n \in \mathbb{N}} \subset L^1$, f measurable and $g \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. Then $f \in L^1$ and $\int f_n \rightarrow \int f$.

Exercise 2.25. Let μ_1, μ_2 be measures on (X, \mathcal{A}) . Then

- (1) $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$
 (2) for each $f \in L^1(\mu_1 + \mu_2)$, we have that

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

Proof. (1) The first part is clear since similar exercise from the section on nonnegative functions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f| d\mu_1 + \int |f| d\mu_2$$

- (2) Suppose that f is simple. Then there exist $(a_n)_{n=1}^n \subset \mathbb{C}$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for general f , choose $(\phi_n)_{n \in \mathbb{N}} \subset S$ such that $\phi_n \rightarrow f$ pointwise and for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$. Then dominated convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

Theorem 2.26. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$. Suppose that

$$\sum_{n \in \mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n \in \mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

Theorem 2.27. Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$.

Exercise 2.28. *Generalized Fatou's Lemma: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \geq -g$. Then $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$. What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?*

Proof. First note that for each $n \in \mathbb{N}$, $\int f_n$ is well defined since $f_n^- \leq g \in L^1$. Since $g + f_n \geq 0$, we may use Fatou's lemma to write

$$\begin{aligned} \int g + \int \liminf_{n \rightarrow \infty} f_n &= \int \liminf_{n \rightarrow \infty} (g + f_n) \\ &\leq \liminf_{n \rightarrow \infty} \int (g + f_n) \\ &= \int g + \liminf_{n \rightarrow \infty} \int f_n \end{aligned}$$

Since $\int g < \infty$, $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$ as required. The analogue is as follows: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \leq g$. Then $\limsup_{n \rightarrow \infty} \int f_n \leq \int \limsup_{n \rightarrow \infty} f_n$. To show this, just use the result from above with the sequence $(g_n)_{n \in \mathbb{N}}$ given by $g_n = -f_n$. \square

Exercise 2.29. *Let $(f_n)_{n \in \mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \rightarrow \mathbb{C}$. Suppose that $f_n \xrightarrow{\text{uni}} f$. Then*

- (1) *if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim_{n \rightarrow \infty} \int f_n = \int f$*
- (2) *if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on \mathbb{R} with Lebesgue measure).*

Proof. Choose $N \in \mathbb{N}$ such that for $n \geq N$ and $x \in X$, $|f(x) - f_n(x)| < 1$. Then $||f| - |f_N|| < 1$ and so $|f| < |f_N| + 1$. Thus $\int |f| \leq \int |f_N| + \mu(X) < \infty$ and $f \in L^1$. Similarly for $n \geq N$, $|f_n| < |f| + 1$. Dominated convergence then gives us that $\lim_{n \rightarrow \infty} \int f_n = \int f$ as required. To see the necessity that $\mu(X) < \infty$, consider $f \equiv 0$ and $f_n = (1/n)\chi_{(0,n)}$. Then $f_n \xrightarrow{\text{uni}} f$, but $1 = \lim_{n \rightarrow \infty} \int f_n \neq \int f = 0$. \square

Exercise 2.30. *Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$, $g_n \xrightarrow{\text{a.e.}} g$, $|f_n| \leq g_n$ and $\int g_n \rightarrow \int g$. Then $\int f_n \rightarrow \int f$.*

Proof. We simply use Fatou's lemma. Put $h_n = (g + g_n) - |f_n - f|$. Since for each $n \in \mathbb{N}$, $|f_n| \leq g_n$, we know that $|f| \leq g$. So $h_n \geq 0$ and $h_n \xrightarrow{\text{p.w.}} 2g$. Thus

$$\begin{aligned} 2 \int g &= \int \liminf_{n \rightarrow \infty} h_n \\ &\leq \liminf_{n \rightarrow \infty} \left[\left(\int g + \int g_n \right) - \int |f_n - f| \right] \\ &= 2 \int g + \liminf_{n \rightarrow \infty} \left(- \int |f_n - f| \right) \\ &= 2 \int g - \limsup_{n \rightarrow \infty} \int |f_n - f| \end{aligned}$$

Hence $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \rightarrow 0$ and $\int f_n \rightarrow \int f$ as required. \square

Exercise 2.31. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$. Then $\int |f_n - f| \rightarrow 0$ iff $\int |f_n| \rightarrow \int |f|$.

Proof. Suppose that $\int |f_n - f| \rightarrow 0$. Since

$$\begin{aligned} \left| \int |f_n| - \int |f| \right| &= \left| \int (|f_n| - |f|) \right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|, \end{aligned}$$

we see that $\int |f_n| \rightarrow \int |f|$. Conversely, suppose that $\int |f_n| \rightarrow \int |f|$. Put $h_n = |f_n - f|$, $g_n = |f_n| + |f|$, $h \equiv 0$ and $g = 2f$. Then $h_n \xrightarrow{a.e.} h$, $g_n \xrightarrow{a.e.} g$ and for each $n \in \mathbb{N}$, $h_n \leq g_n$. Our assumption implies that $\int g_n \rightarrow \int g$. Thus the last exercise tells us that $\int h_n \rightarrow \int h$ as required. \square

Exercise 2.32. Let $(r_n)_{n \in \mathbb{N}}$ be an enumeration of the rationals. Define $f : \mathbb{R} \rightarrow [0, \infty)$ by

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}$$

and define $g : X \rightarrow [0, \infty]$ by

$$g(x) = \sum_{n \in \mathbb{N}} 2^{-n} f(x - r_n).$$

Then

- (1) $g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.
- (2) $g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of \mathbb{R}
- (3) Taking $g \in L^1$, g is unbounded on each subinterval of \mathbb{R} and discontinuous everywhere and remains so after redefinition on a null set

Proof. For convenience, define $f_n : \mathbb{R} \rightarrow [0, \infty)$ by $f_n(x) = f(x - r_n)$ for $x \in \mathbb{R}$. To show (1) we note that for each $n \in \mathbb{N}$, $f_n \in L^1$ and

$$\begin{aligned} \int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1} \end{aligned}$$

Hence

$$\sum_{n \in \mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set, $g \in L^1$. In particular $\int |g| < \infty$ and so $|g|$ (and hence g) are finite almost everywhere. For (2), since $g < \infty$ a.e., so too is g^2 . Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Choose $N \in \mathbb{N}$ such that $r_N \in (a, b)$. Since all the terms in the sum are nonnegative, $g^2 \geq \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2$ and so

$$\begin{aligned}
\int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2 \\
&= \sum_{n \in \mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\
&\geq 2^{-2N} \int_{(a,b)} f_N^2 \\
&\geq 2^{-2N} \int_{r_N}^{b \wedge (r_N+1)} \frac{1}{x - r_N} dx \\
&= \infty
\end{aligned}$$

So g^2 is not integrable on any subinterval of \mathbb{R} . For (3), note that redefining g on a null set does not change the result of (2). Suppose that there is a finite subinterval $I \subset \mathbb{R}$ such that g is bounded on I . Hence there exists $M > 0$ such that for each $x \in I$, $g(x)^2 \leq M$. Then

$$\begin{aligned}
\int_I g^2 &\leq M^2 m(I) \\
&< \infty
\end{aligned}$$

which is a contradiction. So g is not bounded on any subinterval of \mathbb{R} . Now, suppose that there exists $x_0 \in \mathbb{R}$ such that g is continuous at x_0 . Choose $\delta > 0$ such that for each $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then $|g(x) - g(x_0)| < 1$. The reverse triangle inequality tells us that for each $x \in (x_0 - \delta, x_0 + \delta)$, $|g(x)| < 1 + |g(x_0)|$. Hence g is bounded on $(x_0 - \delta, x_0 + \delta)$ which is a contradiction. So g is discontinuous everywhere. \square

Exercise 2.33. Let $f \in L^1$.

- (1) If f is bounded, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\int_E |f| < \epsilon$.
- (2) The same conclusion holds for f unbounded.

Proof. (1) Since f is bounded, there exists $M > 0$ such that $|f| \leq M$. Let $\epsilon > 0$. Choose $\delta = \epsilon/2M$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then

$$\begin{aligned}
\int_E |f| &\leq M\mu(E) \\
&= M \frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

(2) Suppose that f is unbounded. Let $\epsilon > 0$. Then there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon/2$. Since ϕ is bounded, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$,

if $\mu(E) < \delta$, then $\int_E |\phi| < \epsilon/2$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then

$$\begin{aligned} \int_E |f| &\leq \int_E |f - \phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

Exercise 2.34. Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Define $F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x) = \int_{(-\infty, x]} f dm.$$

Then F is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Since $f \in L^1$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| dm < \epsilon.$$

Let $x \in \mathbb{R}$. Suppose that $|x - x_0| < \delta$. Then

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f| dm \\ &< \epsilon \end{aligned}$$

So F is continuous.

□

Exercise 2.35. Denote by δ_x the point mass measure at $x \in X$ on measurable space $(X, \mathcal{P}(X))$. Let $f : X \rightarrow \mathbb{C}$. Then

$$\int f d\delta_x = f(x)$$

Proof. First assume that f is simple. Then there exist $a_1, a_2, \dots, a_n \in \mathbb{C}$ and $E_1, E_2, \dots, E_n \in \mathcal{P}(X)$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Thus $\int f d\delta_x = f(x)$. Now assume that f , which is measurable by choice of σ -algebra, satisfies $f(X) \subset [0, \infty)$. Choose a sequence $(\phi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1}$ and $\phi_n \xrightarrow{p.w.} f$. From before, we see that for each $n \in \mathbb{N}$, $\int \phi_n d\delta_x = \phi_n(x)$. Monotone convergence tells us that $\int f d\delta_x = f(x)$. Now just extend to complex valued functions.

□

Exercise 2.36. Denote by $\#$ the counting measure on the measurable space $(X, \mathcal{P}(X))$. Let $f : X \rightarrow \mathbb{C}$ and suppose that $f \in L^1$. Then

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if f is integrable, then $\{x \in X : f(x) \neq 0\}$ is countable.

Proof. Please refer to the definition of the sum in the appendix. First suppose that $f(X) \subset [0, \infty)$. For $n \in \mathbb{N}$, put $X_n = \{x \in X : f(x) > 1/n\}$ and define $X^* = \{x \in X : f(x) > 0\}$, $X_0 = \{x \in X : f(x) = 0\}$. Then $X^* = \bigcup_{n \in \mathbb{N}} X_n$. Since $f \in L^1$, we have that for each $n \in \mathbb{N}$,

$$\begin{aligned} \infty &> \int f d\# \\ &\geq \int_{X_n} f d\# \\ &\geq \frac{1}{n} \#(X_n). \end{aligned}$$

Thus for each $n \in \mathbb{N}$, X_n is finite and X^* is countable. Thus there exists $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that $X^* = \{x_n\}_{n \in \mathbb{N}}$. For $n \in \mathbb{N}$, define $E_n = \{x_1, x_2, \dots, x_n\}$ and

$$\begin{aligned} f_n &= f \chi_{E_n} \\ &= \sum_{i=1}^n f(x_i) \chi_{\{x_i\}} \end{aligned}$$

Then $f_n \xrightarrow{\text{p.w.}} f \chi_{X^*} = f$ and for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. So

$$\begin{aligned} \int f &= \sup_{n \in \mathbb{N}} \int f_n \\ &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n f(x_i) \\ &= \sum_{x \in X^*} f(x) \\ &= \sum_{x \in X} f(x). \end{aligned}$$

For $f : X \rightarrow \mathbb{C}$, our L^1 assumption and the result above tell us that

$$\sum_{x \in X} |f(x)| < \infty.$$

Thus writing $f = g + ih$, we see that the same is true for f^+, f^-, g^+, g^- . Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f d\# = \sum_{x \in X} f(x).$$

□

Exercise 2.37. Let $f, g : X \rightarrow \mathbb{R}$. Suppose that $f, g \in L^1$. Then $f \leq g$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$.

Proof. Suppose $f \leq g$ a.e. Put $N = \{x \in X : f(x) > g(x)\} \subset N$. Then $\mu(N) = 0$ and $g - f \geq 0$ on N^c . So for each $E \in \mathcal{A}$,

$$\begin{aligned} \int_E g - \int_E f &= \int_E (g - f) \\ &= \int_{E \cap N^c} (g - f) \\ &\geq 0 \end{aligned}$$

Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$. Put $N_n = \{x \in X : f(x) - g(x) > 1/n\}$ and $N = \{x \in X : f(x) > g(x)\}$. Then $N = \bigcup_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned} 0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

So that $\mu(N_n) = 0$. Thus for each $n \in \mathbb{N}$, $\mu(N_n) = 0$ which implies $\mu(N) = 0$. Therefore $f \leq g$ a.e. as required. \square

Definition 2.38. Let $\mathcal{F} \subset L^1$. Then \mathcal{F} is said to be **uniformly integrable** if for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$. (i.e.

$$\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0).$$

Exercise 2.39. Suppose that μ is finite. Let $\mathcal{F} \subset L^1$. Then \mathcal{F} is uniformly integrable iff

- (1) there exists $M > 0$ such that $\sup_{f \in \mathcal{F}} \int |f| \leq M$
- (2) for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\sup_{f \in \mathcal{F}} \int_E |f| < \epsilon$.

Proof. (\Rightarrow): (1) Suppose that \mathcal{F} is uniformly integrable. Then there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < 1$. Choose $M = \mu(X)K + 1$. Then for each $f \in \mathcal{F}$,

$$\begin{aligned} \int |f| &= \int_{\{|f| > K\}} |f| + \int_{\{|f| \leq K\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M \end{aligned}$$

(2) Let $\epsilon > 0$. Then choose $K \in \mathbb{N}$ such that $\sup_{f \in \mathcal{F}} \int_{\{|f| > K\}} |f| < \epsilon/2$ and choose $\delta = \epsilon/2K$.

Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then for $f \in \mathcal{F}$,

$$\begin{aligned} \int_E |f| &= \int_{E \cap \{|f| > K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon \end{aligned}$$

(\Leftarrow): Choose $M > 0$ as in (1). Suppose that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $f \in \mathcal{F}$ such that $\mu(\{|f| > K\}) \geq \epsilon$. Choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $f_K \in \mathcal{F}$ such that $\mu(\{|f_K| > K\}) \geq \epsilon$. Then

$$\begin{aligned} \int |f_K| &\geq \int_{\{|f_K| > K\}} |f_K| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M, \end{aligned}$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $f \in \mathcal{F}$, $\mu(\{|f| > K\}) < \epsilon$. Since $\mu(\{|f| > k\})$ is a decreasing sequence in k , we have that $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \mu(\{|f| > k\}) = 0$. Now, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Choose $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$, $\mu(\{|f| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$,

$$\int_{\{|f| > k\}} |f| < \epsilon.$$

Thus $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$ as required. □

2.4. Integration on Product Spaces.

Definition 2.40. Let X , Y , and Z be sets, $E \subset X \times Y$ and $f : X \times Y \rightarrow Z$. For each $x \in X$, define $E_x = \{y \in Y : (x, y) \in E\}$ and $f_x : Y \rightarrow Z$ by $f_x(y) = f(x, y)$. For each $y \in Y$, define $E^y = \{x \in X : (x, y) \in E\}$ and $f^y : X \rightarrow Z$ by $f^y(x) = f(x, y)$.

Note 2.41. It is often helpful to observe that $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.

Lemma 2.42. Let (X, \mathcal{A}) , (Y, \mathcal{B}) be measurable spaces, $Z = [0, \infty]$ or \mathbb{C} and $f : X \times Y \rightarrow Z$.

- (1) For each $E \in \mathcal{A} \otimes \mathcal{B}$, $x \in X$, $y \in Y$, we have that $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$
- (2) If f is $\mathcal{A} \otimes \mathcal{B}$ -measurable, then for each $x \in X$, $y \in Y$, we have that f_x is \mathcal{B} -measurable and f^y is \mathcal{A} -measurable.

Theorem 2.43. Let (X, \mathcal{A}, μ) , (Y, \mathcal{B}, ν) be σ -finite measure spaces. Then for each $E \in \mathcal{A} \otimes \mathcal{B}$, the maps $\phi : X \rightarrow [0, \infty]$ and $\psi : Y \rightarrow [0, \infty]$ defined by $\phi(x) = \nu(E_x)$ and $\psi(y) = \mu(E^y)$ are \mathcal{A} -measurable and \mathcal{B} -measurable, respectively and

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

Theorem 2.44. *Fubini, Tonelli:* Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ -finite measure spaces.

- (1) (Tonelli) For each $f \in L^+(X \times Y)$, the functions $g : X \rightarrow [0, \infty]$, $h : Y \rightarrow [0, \infty]$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x)$ are \mathcal{A} -measurable and \mathcal{B} -measurable respectively and

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

- (2) (Fubini) For each $f \in L^1(X \times Y)$, $f_x \in L^1(\nu)$ for μ -a.e. $x \in X$ and $f^y \in L^1(\mu)$ for ν -a.e. $y \in Y$, respectively and the functions (after redefinition of f on a null set) $g : X \rightarrow \mathbb{C}$, $h : Y \rightarrow \mathbb{C}$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Furthermore

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

Note 2.45. We usually just write $\int \int f d\mu d\nu$ and $\int \int f d\nu d\mu$ instead of $\int h d\nu$ and $\int g d\mu$ respectively. We have a similar result for complete product measure spaces. See

Exercise 2.46. Take $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1])$, $\mathcal{B} = \mathcal{P}([0, 1])$ and μ, ν to be Lebesgue measure and counting measure respectively. Define $D = \{(x, y) \in [0, 1]^2 : x = y\}$ Show that

$$\int \chi_D d\mu \times \nu, \int \int \chi_D d\mu d\nu \text{ and } \int \int \chi_D d\nu d\mu$$

are all different. (Hint: for the first integral, use the definition of $\mu \times \nu$)

Proof. Let $x, y \in [0, 1]$. Then $(\chi_D)_x = \chi_{D_x} = \chi_x$ and $(\chi_D)^y = \chi_{D^y} = \chi_y$. Thus

$$\begin{aligned} \int \int \chi_D d\mu d\nu &= \int \mu(\{y\}) d\nu \\ &= \int 0 d\nu \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \int \int \chi_D d\mu d\nu &= \int \nu(\{x\}) d\mu \\ &= \int 1 d\mu \\ &= 1 \end{aligned}$$

Now, Observe that $\int \chi_D d\mu \times \nu = \mu \times \nu(D)$. Recall from the section on product measures that $\mu \times \nu(D) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) : (A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n\}$. Let $(A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$. Suppose that $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n$. Then for each $x \in [0, 1]$, $(x, x) \in \bigcup_{n \in \mathbb{N}} A_n \times B_n$. So for each $x \in [0, 1]$, there exists $n \in \mathbb{N}$, such that $x \in A_n \cap B_n$. Thus $[0, 1] \subset \bigcup_{n \in \mathbb{N}} A_n \cap B_n$. Since $1 = \mu([0, 1]) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap B_n)$, we know that there exists $n \in \mathbb{N}$ such that $0 < \mu(A_n \cap B_n)$. Thus $\mu(A_n) > 0$ and $\mu(B_n) > 0$. Since $\mu(B_n) > 0$, B_n must be infinite and therefore $\nu(B_n) = \infty$. So $\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) = \infty$.

□

Exercise 2.47. Let (X, \mathcal{A}, μ) be a σ -finite measure space and $f : X \rightarrow [0, \infty) \in L^+$. Show that $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ and $\mu \times m(G) = \int_X f d\mu$. The same is true if we replace " \geq " with " $>$ ". (Hint: to show that G is measurable, split up $(x, y) \mapsto f(x) - y$ into the composition of measurable functions.

Proof. Define $\phi : X \times [0, \infty) \rightarrow [0, \infty)^2$ and $\psi : [0, \infty)^2 \rightarrow [0, \infty)$ by $\phi(x, y) = (f(x), y)$ and $\psi(z, y) = z - y$. Then $G = \{(x, y) \in X \times [0, \infty) : \psi \circ \phi(x, y) \geq 0\}$. Let $A, B \in \mathcal{B}([0, \infty))$. Then $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0, \infty))$. Since $\mathcal{B}([0, \infty)^2) = \mathcal{B}([0, \infty)) \otimes \mathcal{B}([0, \infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0, \infty))\})$, we have that ϕ is $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty)^2)$ measurable. Since ψ is continuous, we have that ψ is $\mathcal{B}([0, \infty)^2)$ - $\mathcal{B}([0, \infty))$ measurable. This implies that $\psi \circ \phi$ is $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty))$ measurable. Thus $G = \psi \circ \phi^{-1}([0, \infty)) \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$. Now for $x \in X$, $G_x = \{y \in [0, \infty) : f(x) \geq y\} = [0, f(x)]$. Thus

$$\begin{aligned} \mu \times m(G) &= \int \chi_G d\mu \times m \\ &= \int_X \int_{[0, \infty)} \chi_{G_x} dm d\mu(x) \\ &= \int_X f(x) d\mu(x) \end{aligned}$$

The same reasoning holds if we replace " \geq " with " $>$ ". □

Exercise 2.48. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ -finite measure spaces and $f : X \rightarrow \mathbb{C}, g : Y \rightarrow \mathbb{C}$. Define $h : X \times Y \rightarrow \mathbb{C}$ by $h(x, y) = f(x)g(y)$.

- (1) If f is \mathcal{A} -measurable and g is \mathcal{B} -measurable, then h is $\mathcal{A} \otimes \mathcal{B}$ -measurable.
- (2) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and

$$\int_{X \times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu$$

Proof. (1) First suppose that f, g are simple. Then there exist $(A_i)_{i=1}^n \subset \mathcal{A}, (B_j)_{j=1}^m \subset \mathcal{B}$ and $(a_i)_{i=1}^n, (b_j)_{j=1}^m \subset \mathbb{C}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$ and $g = \sum_{j=1}^m b_j \chi_{B_j}$. Then $h = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{A_i \times B_j}$. So h is $\mathcal{A} \otimes \mathcal{B}$ -measurable. For general f, g , there exist $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$ and $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$ such that $f_n \rightarrow f$ pointwise, $g_n \rightarrow g$ pointwise and for each $n \in \mathbb{N}$, $|f_n| \leq |f_{n+1}| \leq |f|$ and $|g_n| \leq |g_{n+1}| \leq |g|$. For $n \in \mathbb{N}$, define $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$ by $h_n = f_n g_n$. Then $h_n \rightarrow h$ pointwise and for each $n \in \mathbb{N}$, $|h_n| \leq |h_{n+1}| \leq |h|$. Thus h is $\mathcal{A} \otimes \mathcal{B}$ -measurable.

- (2) First suppose f and g are simple as before. Then

$$\begin{aligned} \int_{X \times Y} |h| d\mu \times \nu &\leq \sum_{i=1}^n \sum_{j=1}^m |a_i b_j| \mu(A_i) \nu(B_j) \\ &= \left(\sum_{i=1}^n |a_i| \mu(A_i) \right) \left(\sum_{j=1}^m |b_j| \nu(B_j) \right) \\ &= \int_X |f| d\mu \int_Y |g| d\nu \\ &< \infty \end{aligned}$$

So $h \in L^1(\mu \times \nu)$. Furthermore,

$$\begin{aligned} \int_{X \times Y} h d\mu \times \nu &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mu(A_i) \nu(B_j) \\ &= \left(\sum_{i=1}^n a_i \mu(A_i) \right) \left(\sum_{j=1}^m b_j \nu(B_j) \right) \\ &= \int_X f d\mu \int_Y g d\nu \end{aligned}$$

For general $f \in L^1(\mu), g \in L^1(\nu)$, take $(h_n)_{n \in \mathbb{N}}$ as before. Monotone convergence and the result above say that

$$\begin{aligned} \int_{X \times Y} |h| d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} |h_n| d\mu \times \nu \\ &= \lim_{n \rightarrow \infty} \left(\int_X |f_n| d\mu \int_Y |g_n| d\nu \right) \\ &= \int_X |f| d\mu \int_Y |g| d\nu \\ &< \infty \end{aligned}$$

So $h \in L^1(\mu \times \nu)$. Dominated convergence and the result above then tell us that

$$\begin{aligned} \int_{X \times Y} h d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} h_n d\mu \times d\nu \\ &= \lim_{n \rightarrow \infty} \left(\int_X f_n d\mu \int_Y g_n d\nu \right) \\ &= \int_X f d\mu \int_Y g d\nu \end{aligned}$$

□

Note 2.49. In the above exercise part (2), we can replace L^1 with L^+ and get the same result by the same method.

Exercise 2.50. Let $f : \mathbb{R} \rightarrow [0, \infty) \in L^+$. Show that

$$\int_{\mathbb{R}} f dm = \int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) dm(t)$$

Proof. Note that

$$\int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) = \int_{[0, \infty)} \left[\int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}} dm \right] dm(t)$$

Comparing this with Tonelli's theorem, we can put $\chi_{\{x \in \mathbb{R} : f(x) \geq t\}} = (\chi_E)^t = \chi_{E^t}$. Then $E = \{(x, t) \in \mathbb{R} \times [0, \infty) : f(x) \geq t\}$ and $E_x = \{t \in [0, \infty) : f(x) \geq t\} = [0, f(x)]$. Tonelli's

theorem tells us that

$$\begin{aligned} \int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}}(x) dm(x) \right] dm(t) &= \int_{\mathbb{R}} \left[\int_{[0,\infty)} \chi_{[0,f(x)]}(t) dm(t) \right] dm(x) \\ &= \int_{\mathbb{R}} f(x) dm(x) \end{aligned}$$

□

2.5. Convergence.

Definition 2.51. Let (X, \mathcal{A}) be a measurable space. For convenience we will define $L^0 = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable}\}$.

Definition 2.52. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **in measure** if for each $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \rightarrow 0$. This is written $f_n \xrightarrow{\mu} f$.

Definition 2.53. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **almost uniformly** if for each $\epsilon > 0$, there exists $N \in \mathcal{A}$ such that $\mu(N) < \epsilon$ and $f_n \xrightarrow{\text{uni}} f$ on N^c . This is written $f_n \xrightarrow{\text{a.u.}} f$.

Theorem 2.54. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$.

Theorem 2.55. (Egoroff): Suppose that $\mu(X) < \infty$. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{\text{a.e.}}$. Then $f_n \xrightarrow{\text{a.u.}} f$.

Exercise 2.56. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.

Proof. Let $\epsilon > 0$. for $n \in \mathbb{N}$, define $E_{\epsilon,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$. Then for $n \in \mathbb{N}$,

$$\begin{aligned} \int |f - f_n| &\geq \int_{E_{\epsilon,n}} |f - f_n| \\ &\geq \epsilon \mu(E_{\epsilon,n}). \end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$. Since $\int |f - f_n| \rightarrow 0$, we have that $\mu(E_{\epsilon,n}) \rightarrow 0$. Since $\epsilon > 0$ is arbitrary, $f_n \xrightarrow{\mu} f$ as required. □

Exercise 2.57. Suppose $\mu(X) < \infty$. Define $d : L^0 \times L^0 \rightarrow [0, \infty)$ by

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \quad f, g \in L^0$$

Then d is a metric on L^0 if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each $f, g \in L^0$, $d(f, g) \leq \mu(X)$.

Proof. Let $f, g \in L^0$. Clearly $d(f, g) = d(g, f)$. If $f = g$ a.e. then clearly $d(f, g) = 0$. Conversely, if $d(f, g) = 0$, then $\frac{|f - g|}{1 + |f - g|} = 0$ a.e and so $|f - g| = 0$ a.e. which implies $f = g$ a.e. It is not hard to show that $\phi : [0, \infty) \rightarrow [0, \infty)$ given by $\phi(x) = \frac{x}{1+x}$ satisfies $\phi(x + y) \leq \phi(x) + \phi(y)$. Thus satisfies the triangle inequality. Now, let $(f_n)_{n \in \mathbb{N}} \subset L^0$. Suppose that $f_n \not\xrightarrow{\mu} f$. Then there exists $\epsilon > 0, \delta > 0$ and a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that

for each $k \in \mathbb{N}$, $\mu(E_{\epsilon, n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$. It is not hard to show that ϕ from earlier is increasing. Thus for each $k \in \mathbb{N}$,

$$\begin{aligned} d(f_{n_k}, f) &= \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{\epsilon}{1 + \epsilon} \\ &\geq \frac{\epsilon \delta}{1 + \epsilon} \end{aligned}$$

So $f_{n_k} \not\stackrel{d}{\rightarrow} f$. Hence $f_{n_k} \stackrel{d}{\rightarrow} f$ implies that $f_{n_k} \stackrel{\mu}{\rightarrow} f$. Conversely, suppose that $f_{n_k} \stackrel{\mu}{\rightarrow} f$. Let $\epsilon > 0$. Then $\delta = \frac{\epsilon}{1 + \mu(X)} > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then $\mu(E_{\delta, n}) < \frac{\delta}{1 + \delta}$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Since ϕ is increasing and $\phi \leq 1$, we have that

$$\begin{aligned} d(f_n, f) &= \int \frac{|f_n - f|}{1 + |f_n - f|} \\ &= \int_{E_{\delta, n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E_{\delta, n}^c} \frac{|f_n - f|}{1 + |f_n - f|} \\ &\leq \mu(E_{\delta, n}) + \mu(X) \frac{\delta}{1 + \delta} \\ &< \frac{\delta}{1 + \delta} (1 + \mu(X)) \\ &\leq \delta (1 + \mu(X)) \\ &= \epsilon \end{aligned}$$

□

Exercise 2.58. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \stackrel{\mu}{\rightarrow} f$. Then $f \geq 0$ a.e. and $\int f \leq \liminf_{n \rightarrow \infty} \int f_n$.

Proof. Since $f_n \stackrel{\mu}{\rightarrow} f$, there is a subsequence converging to f a.e. So clearly $f \geq 0$ a.e. Now, choose a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $\int f_{n_k} \rightarrow \liminf_{n \rightarrow \infty} \int f_n$. Since $f_n \stackrel{\mu}{\rightarrow} f$ so does $(f_{n_k})_{k \in \mathbb{N}}$. Therefore there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ of $(f_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Thus $f \geq 0$ a.e. and Fatou's lemma tells us that

$$\begin{aligned} \int f &\leq \liminf_{j \in \mathbb{N}} \int f_{n_{k_j}} \\ &= \liminf_{n \rightarrow \infty} \int f_n. \end{aligned}$$

□

Exercise 2.59. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \stackrel{\mu}{\rightarrow} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.

Proof. Clearly $(f_n)_{n \in \mathbb{N}} \subset L^1$. Since $f_n \xrightarrow{\mu} f$, there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$. This implies that $|f| \leq g$ a.e. and so $f \in L^1$. For $n \in \mathbb{N}$, put $h_n = 2g - |f_n - f|$. Then for each $n \in \mathbb{N}$, $h_n \geq 0$ and $h_n \xrightarrow{\mu} 2g$. By the previous exercise

$$\begin{aligned} \int 2g &\leq \liminf_{n \rightarrow \infty} \int (2g - |f_n - f|) \\ &= \int 2g - \limsup_{n \rightarrow \infty} \int |f_n - f|. \end{aligned}$$

So $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \rightarrow 0$ and $f_n \xrightarrow{L^1} f$ as required. \square

Exercise 2.60. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \rightarrow \mathbb{C}$.

- (1) If ϕ is continuous, and $f_n \xrightarrow{\text{a.e.}} f$ then $\phi \circ f_n \xrightarrow{\text{a.e.}} \phi \circ f$.
- (2) If ϕ is uniformly continuous and $f_n \rightarrow f$ uniformly, almost uniformly or in measure, then $\phi \circ f_n \rightarrow \phi \circ f$ uniformly, almost uniformly or in measure, respectively.
- (3) Find a counter example to (2) if we drop the word "uniform".

Proof. (1) Clear

(2) Suppose that ϕ is uniformly continuous.

(uniform conv.) Suppose that $f_n \xrightarrow{\text{uni}} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Now choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq N$ then for each $x \in X$, $|f_n(x) - f(x)| < \delta$. Let $n \in \mathbb{N}$, suppose $n \geq N$, Let $x \in X$. Then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Thus $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$.

(almost uni.) Suppose that $f_n \xrightarrow{\text{a.u.}} f$. Let $\epsilon > 0$. Choose $N \in \mathcal{A}$ such $\mu(N) < \epsilon$ and $f_n \xrightarrow{\text{uni}} f$ on N^c . Then from above, we know that $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$ on N^c . Thus $\phi \circ f_n \xrightarrow{\text{a.u.}} \phi \circ f$.

(measure) Suppose that $f_n \xrightarrow{\mu} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Observe that for $x \in X$, if $|f_n(x) - f(x)| < \delta$, then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Hence $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$. By definition of convergence in measure, $\mu(F_{n,\delta}) \rightarrow 0$. Thus $\mu(E_{n,\epsilon}) \rightarrow 0$. Hence $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.

(3)

\square

Exercise 2.61. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{\text{a.u.}} f$. Then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\text{a.e.}} f$.

Proof. (measure) Let $\epsilon > 0$, $\delta > 0$. Choose $M \in \mathcal{A}$ such that $\mu(M) < \delta$ and $f_n \xrightarrow{\text{uni}} f$ on M^c . Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then for each $x \in M^c$, $|f_n(x) - f(x)| < \epsilon$. Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then $E_{\epsilon,n} \subset M$ and $\mu(E_{\epsilon,n}) < \delta$. Thus $\mu(E_{\epsilon,n}) \rightarrow 0$ and $f_n \xrightarrow{\mu} f$.

(a.e.) For each $n \in \mathbb{N}$, Choose $N_n \in \mathcal{A}$ such that $\mu(N_n) < 1/n$ and $f_n \xrightarrow{\text{uni}} f$ on N_n^c . Observe that for $x \in X$, if $x \in \bigcup_{n \in \mathbb{N}} N_n^c$, then $f_n(x) \rightarrow f(x)$. Thus $N = \{x \in X : f_n(x) \not\rightarrow f(x)\} \subset \bigcap_{n \in \mathbb{N}} N_n$. Therefore $\mu(N) = 0$ and $f_n \xrightarrow{\text{a.e.}} f$. \square

Exercise 2.62. Let $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then

- (1) $f_n + g_n \xrightarrow{\mu} f + g$
- (2) if $\mu(X) < \infty$, then $f_n g_n \xrightarrow{\mu} f g$

Proof. (1) Let $\epsilon > 0$. For convenience, put $F_{n,\epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$, $G_{n,\epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$, and $(F + G)_{n,\epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g(x))| \geq \epsilon\}$. Observe that for $x \in X$, $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$. Thus $(F + G)_{n,\epsilon} \subset F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$. Since $\mu(F_{n,\epsilon/2} \cup G_{n,\epsilon/2}) \leq \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}) \rightarrow 0$, we have that $\mu((F + G)_{n,\epsilon}) \rightarrow 0$. Hence $f_n + g_n \xrightarrow{\mu} f + g$.

- (2) Suppose that $\mu(X) < \infty$. Let $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(f_n g_n)_{n \in \mathbb{N}}$. Choose a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ and $g_{n_{k_j}} \xrightarrow{\text{a.e.}} g$. Then $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.e.}} f g$. Egoroff's theorem tells us that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.u.}} f g$, which implies that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} f g$. Thus for each subsequence $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ of $(f_n g_n)_{n \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ of $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} f g$. Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that $f_n g_n \xrightarrow{\mu} f g$. □

Exercise 2.63. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $\epsilon > 0$,

$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then $f_n \xrightarrow{\text{a.e.}} f$.

Proof. Let $\epsilon > 0$. By assumption we know that

$$\begin{aligned} \int \left[\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} \right] d\mu &= \sum_{n \in \mathbb{N}} \int \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} d\mu \\ &= \sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \\ &< \infty \end{aligned}$$

Thus we also know that $\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} < \infty$ a.e. Equivalently, we could say that for a.e. $x \in X$, $|\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}| < \infty$. For $k \in \mathbb{N}$, define $N_k = \{x \in X : \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > 1/k\}} = \infty\}$. Then for each $k \in \mathbb{N}$, $\mu(N_k) = 0$. Define $N = \bigcup_{k \in \mathbb{N}} N_k$. Then $\mu(N) = 0$. Let $x \in N^c$ and $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \epsilon$. Then $\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\} \subset \{n \in \mathbb{N} : |f_n(x) - f(x)| > 1/k\}$ which is finite because $x \in N_k^c$. Put $M = \max\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}$. Then for $m \geq M$, $|f_m(x) - f(x)| \leq \epsilon$. Thus $f_n(x) \rightarrow f(x)$. Hence $f_n \xrightarrow{\text{a.e.}} f$. □

3. DIFFERENTIATION

3.1. Signed Measures.

Definition 3.1. Let (X, \mathcal{A}) be a measurable space and $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$. Then ν is said to be a **signed measure** if

- (1) for each $E \in \mathcal{A}$, $\nu(E) < \infty$ or for each $E \in \mathcal{A}$, $\nu(E) > -\infty$.
- (2) $\nu(\emptyset) = 0$

(3) for each $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ if $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is disjoint, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$ and if $|\sum_{n \in \mathbb{N}} \nu(E_n)| < \infty$, then $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely.

Exercise 3.2. Let $\nu : \mathcal{A} \rightarrow [0, \infty]$ be a signed measure and $(E_n)_{n \in \mathbb{N}}, (F_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. If $(E_n)_{n \in \mathbb{N}}$ is increasing, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \rightarrow \infty} \nu(E_n)$. If $(F_n)_{n \in \mathbb{N}}$ is decreasing and $|\nu(E_1)| < \infty$, then $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$.

Proof. Put $E'_1 = E_1$, $F'_1 = F_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $E'_n = E_n \setminus E_{n-1}$ and $F'_n = F_1 \setminus F_n$. Then $(E'_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is disjoint. Thus

$$\begin{aligned} \nu(\bigcup_{n \in \mathbb{N}} E_n) &= \nu(\bigcup_{n \in \mathbb{N}} E'_n) \\ &= \sum_{n \in \mathbb{N}} \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \sum_{n=1}^n \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \nu(E_n) \end{aligned}$$

Since $(F'_n)_{n \in \mathbb{N}}$ is increasing, we now know that

$$\begin{aligned} \nu(F_1) - \nu(\bigcap_{n \in \mathbb{N}} F_n) &= \nu(F_1 \setminus \bigcap_{n \in \mathbb{N}} F_n) \\ &= \nu(\bigcup_{n \in \mathbb{N}} F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F_1 \setminus F_n) \\ &= \nu(F_1) - \lim_{n \rightarrow \infty} \nu(F_n) \end{aligned}$$

Since $|\nu(F_1)| < \infty$, we see that $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$. □

Definition 3.3. Let (X, \mathcal{A}) be a measurable space and $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$ a signed measure and $E \in \mathcal{A}$. Then E is said to be ν -**positive**, ν -**negative** and ν -**null** if for each $F \in \mathcal{A}$, $F \subset E$ implies that $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ respectively.

Exercise 3.4. Let $E \subset \mathcal{A}$. If E is positive, negative or null, then for each $F \in \mathcal{A}$, if $F \subset E$, then F is positive, negative or null respectively.

Proof. Clear □

Exercise 3.5. Let $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ be positive, negative or null. Then $\bigcup_{n \in \mathbb{N}} E_n$ is positive, negative or null respectively.

Proof. Suppose that $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is positive. Let $F \in \mathcal{A}$. Suppose that $F \subset \bigcup_{n \in \mathbb{N}} E_n$. Put

$P_1 = E_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $P_n = E_n \setminus (\bigcup_{j=1}^{n-1} E_j)$. So $\bigcup_{n \in \mathbb{N}} P_n = \bigcup_{n \in \mathbb{N}} E_n$ and $(P_n)_{n \in \mathbb{N}}$ is

disjoint. Thus

$$\begin{aligned}\nu(F) &= \nu\left(F \cap \bigcup_{n \in \mathbb{N}} P_n\right) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} (F \cap P_n)\right) \\ &= \sum_{n \in \mathbb{N}} \nu(F \cap P_n) \\ &\geq 0\end{aligned}$$

The process is the same if $(E_n)_{n \in \mathbb{N}}$ is negative and null. \square

Theorem 3.6. *Hahn Decomposition:* Let ν be a signed measure on (X, \mathcal{A}) . Then there exist $P, N \in \mathcal{A}$ such that P is positive, N is negative, $X = N \cup P$ and $N \cap P = \emptyset$. Furthermore, these two sets are unique in the following sense: For any $P', N' \in \mathcal{A}$, if N, P satisfy the properties above, $P' \Delta P = N' \Delta N$ is null.

Definition 3.7. Let ν be a signed measure on (X, \mathcal{A}) and $P, N \in \mathcal{A}$. Then P and N are said to form a **Hahn decomposition** of X with respect to ν if P, N satisfy the results in the above theorem.

Definition 3.8. Let μ, ν be signed measures on (X, \mathcal{A}) . Then μ and ν are said to be **mutually singular** if there exist $E, F \in \mathcal{A}$ such that $X = E \cup F$, $E \cap F = \emptyset$ and E is μ -null and F is ν -null. We will denote this by $\mu \perp \nu$.

Theorem 3.9. *Jordan Decomposition:* Let ν be a signed measure on (X, \mathcal{A}) . Then there exist unique positive measures ν^+ and ν^- on (X, \mathcal{A}) such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof. Choose a Hahn decomposition P, N of X with respect to ν . Define ν^+, ν^- by $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$. \square

Definition 3.10. Let ν be a signed measure on (X, \mathcal{A}) . Then ν^+ and ν^- from the last theorem are called the **positive** and **negative variations** of ν respectively. We define the **total variation** measure $|\nu|$ on (X, \mathcal{A}) by $|\nu| = \nu^+ + \nu^-$.

Definition 3.11. Let ν be a signed measure on (X, \mathcal{A}) . Then ν is said to be σ -finite if $|\nu|$ is σ -finite.

Exercise 3.12. Let ν be a signed measure and λ, μ positive measures on (X, \mathcal{A}) . Suppose that $\nu = \lambda - \mu$. Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. Choose a Hahn decomposition P, N of X with respect to ν . Let $E \in \mathcal{A}$. Then

$$\begin{aligned}\lambda(E \cap P) - \mu(E \cap P) &= \nu(E \cap P) \\ &= \nu^+(E \cap P)\end{aligned}$$

So $\lambda(E \cap P) \geq \nu^+(E \cap P)$ and therefore

$$\begin{aligned}\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) \\ &= \nu^+(E)\end{aligned}$$

Similarly $\mu(E \cap N) \geq \nu^-(E \cap N)$ and $\mu(E) \geq \nu^-(E)$. \square

Exercise 3.13. Let ν_1, ν_2 be signed measures on (X, \mathcal{A}) . Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Hint: use the last exercise)

Proof. Since

$$\begin{aligned}\nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\ &= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)\end{aligned}$$

the previous exercise tells us that $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$ and $\mu = \nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$. Therefore

$$\begin{aligned}|\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ &\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\ &= |\nu_1| + |\nu_2|\end{aligned}$$

□

Note 3.14. Recall that a previous exercise from the section on complex valued functions tells us that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.

Definition 3.15. Let ν be a signed measure on (X, \mathcal{A}) . Then we define $L^1(\nu) = L^1(|\nu|)$. For $f \in L^1(\nu)$, we define

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

Exercise 3.16. Let ν_1, ν_2 be signed measures on (X, \mathcal{A}) . Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$

Proof. The previous exercise tells us that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. Two previous exercises from the section on nonnegative functions tells us that

$$\begin{aligned}\int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\ &= \int |f| d|\nu_1| + \int |f| d|\nu_2|\end{aligned}$$

□

Exercise 3.17. Let ν, μ be signed measures on (X, \mathcal{A}) and $E \in \mathcal{A}$. Then

- (1) E is ν -null iff $|\nu|(E) = 0$
- (2) $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. (1) Suppose that E is ν -null. Choose a Hahn decomposition P, N of X with respect to ν . Then $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = \nu(E \cap N) = 0$. Therefore $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Conversely, suppose that $|\nu|(E) = 0$. Then $\nu^+(E) = \nu^-(E) = 0$. Let $F \in \mathcal{A}$. Suppose that $F \subset E$. Then $\nu^+(F) = 0$ and $\nu^-(F) = 0$. Therefore $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. So E is ν -null.

- (2) Suppose that $\nu \perp \mu$. Then there exist $E, F \in \mathcal{A}$ such that $E \cup F = X$, $E \cap F = \emptyset$, E is μ -null and F is ν -null. By (1), F is $|\nu|$ -null and thus $|\nu| \perp \mu$. If $|\nu| \perp \mu$, choose $E, F \in \mathcal{A}$ as before. Since F is $|\nu|$ -null, we know that $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$. This implies that F is ν^+ -null and F is ν^- -null. So $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Finally assume that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. **FINISH!!!!**

□

Exercise 3.18. Let ν be a signed measure on (X, \mathcal{A}) . Then

- (1) for $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$
- (2) if ν is finite, then for each $E \in \mathcal{A}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : f \text{ is measurable and } |f| \leq 1\}$

Proof. (1) Let $f \in L^1(\nu)$. Then

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f d\nu^+ - \int f d\nu^- \right| \\ &\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \\ &= \int |f| d(\nu^+ + \nu^-) \\ &= \int |f| d|\nu| \end{aligned}$$

- (2) Let $E \in \mathcal{A}$. Let $f : X \rightarrow \mathbb{R}$ be measurable and suppose that $|f| \leq 1$. Since ν is finite, so is $|\nu|$ and thus $f \in L^1(\nu)$. Then (1) tells us that

$$\begin{aligned} \left| \int_E f d\nu \right| &\leq \int_E |f| d|\nu| \\ &\leq |\nu|(E) \end{aligned}$$

Now, choose a Hahn decomposition P, N of X with respect to ν . Define $f = \chi_P - \chi_N$. Then $|f| \leq 1$, f is measurable and

$$\begin{aligned} \left| \int_E f d\nu \right| &= \left| \int_E f d\nu^+ - \int_E f d\nu^- \right| \\ &= |\nu^+(E \cap P) + \nu^-(E \cap N)| \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E). \end{aligned}$$

□

Exercise 3.19. Let μ be a positive measure on (X, \mathcal{A}) and $f \in L^0(X, \mathcal{A})$ extended μ -integrable. Define ν on (X, \mathcal{A}) by $\nu(E) = \int_E f d\mu$. Then

- (1) ν is a signed measure
- (2) for each $E \in \mathcal{A}$, $|\nu|(E) = \int_E |f| d\mu$.

Proof. (1) Clearly $\nu(\emptyset) = 0$ and ν is finite by assumption. Let $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_n)_{n \in \mathbb{N}}$ is disjoint. Then

$$\begin{aligned}
 \nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \\
 &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu - \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
 &= \sum_{n \in \mathbb{N}} \left[\int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right] \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n} f d\mu \\
 &= \sum_{n \in \mathbb{N}} \nu(E_n)
 \end{aligned}$$

If $|\nu(\bigcup_{n \in \mathbb{N}} E_n)| < \infty$, then $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu < \infty$ and $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu < \infty$ because

$$\begin{aligned}
 |\nu(\bigcup_{n \in \mathbb{N}} E_n)| &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \right| \\
 &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \right|
 \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
 \sum_{n \in \mathbb{N}} |\nu(E_n)| &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f d\mu \right| \\
 &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right| \\
 &\leq \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu + \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
 &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu + \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
 &< \infty
 \end{aligned}$$

So the sum $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely and ν is a signed measure.

- (2) Put $P = \{x \in X : f(x) \geq 0\}$ and $N = \{x \in X : f(x) < 0\}$. Then P, N form a Hahn decomposition of X with respect to ν . Thus for $E \in \mathcal{A}$,

$$\nu^+(E) = \int_{E \cap P} f d\mu = \int_E f^+ d\mu$$

and

$$\nu^-(E) = \int_{E \cap N} f d\mu = \int_E f^- d\mu$$

. So for $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$$

□

3.2. The Lebesgue-Radon-Nikodym Theorem.

Definition 3.20. Let (X, \mathcal{A}) be a measurable space, ν be a signed measure on (X, \mathcal{A}) and μ a measure on (X, \mathcal{A}) . Then ν is said to be **absolutely continuous** with respect to μ , denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.

Note 3.21. If there exists an extended μ -integrable $f \in L^0(X, \mathcal{A})$ such that for each $E \in \mathcal{A}$, $\nu(E) = \int_E f d\mu$, then we write $d\nu = f d\mu$.

Theorem 3.22. Let (X, \mathcal{A}) be a measurable space, ν be a σ -finite signed measure on (X, \mathcal{A}) and μ a σ -finite measure on (X, \mathcal{A}) . Then there exist unique σ -finite signed measures λ, ρ on (X, \mathcal{A}) such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists an extended μ -integrable $f \in L^0(X, \mathcal{A})$ such that $\rho = f d\mu$ and f is unique μ -a.e.

Definition 3.23. The decomposition $\nu = \lambda + \rho$ is referred to as the **Lebesgue decomposition of ν with respect to μ** . In the case $\nu \ll \mu$, we have $\lambda = 0$ and $\rho = \nu$ and we define the **Radon-Nikodym derivative of ν with respect to μ** , denoted by $d\nu/d\mu$, to be $d\nu/d\mu = f$ where $d\nu = f d\mu$.

Theorem 3.24. Let ν be a σ -finite signed measure on (X, \mathcal{A}) and μ, λ σ -finite measures on (X, \mathcal{A}) . Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then

(1) for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2) $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Exercise 3.25. Let $(\nu_n)_{n \in \mathbb{N}}$ be a sequence of measures and μ a measure.

(1) If for each $n \in \mathbb{N}$, $\nu_n \ll \mu$, then $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$.

(2) If for each $n \in \mathbb{N}$, $\nu_n \perp \mu$, then $\sum_{n \in \mathbb{N}} \nu_n \perp \mu$.

Proof. (1) Let $E \in \mathcal{A}$. Suppose that $\mu(E) = 0$. Then for each $n \in \mathbb{N}$, $\nu_i(E) = 0$ and thus $\sum_{n \in \mathbb{N}} \nu_n(E) = 0$. Hence $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$.

(2) For each $n \in \mathbb{N}$, there exist $N_i, M_i \in \mathcal{A}$ such that $N_i \cap M_i = \emptyset$, $N_i \cup M_i = X$ and $\nu_i(M_i) = \mu(N_i) = 0$. Put $N = \bigcup_{n \in \mathbb{N}} N_i$ and $M = N^c$. Note that for each $n \in \mathbb{N}$, $M \subset N_i^c = M_i$. So $\mu(N) \leq \sum_{n \in \mathbb{N}} \mu(N_i) = 0$ and $(\sum_{n \in \mathbb{N}} \nu_i)(M) \leq \sum_{n \in \mathbb{N}} \nu_i(M_i) = 0$. Thus $\sum_{n \in \mathbb{N}} \nu_i \perp \mu$.

□

Exercise 3.26. Choose $X = [0, 1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$. Let m be Lebesgue measure and μ the counting measure.

Then

- (1) $m \ll \mu$ but for each $f \in L^+$, $dm \neq f d\mu$
- (2) There is no Lebesgue decomposition of μ with respect to m .

Proof. (1) Let $E \in \mathcal{A}$. If $\mu(E) = 0$, then $E = \emptyset$ and $m(E) = 0$. So $m \ll \mu$. Suppose for the sake of contradiction that there exists $f \in L^+$ such that $dm = f d\mu$. Then

$$\begin{aligned} 1 &= m(X) \\ &= \sum_{x \in X} f(x) \end{aligned}$$

Put $Z = \{x \in X : f(x) \neq 0\}$. Then Z is countable. So

$$\begin{aligned} 1 &= m(X \setminus Z) \\ &= \sum_{x \in X \setminus Z} f(x) \\ &= 0 \end{aligned}$$

This is a contradiction, so no such f exists.

- (2) Suppose for the sake of contradiction that there is a Lebesgue decomposition for μ with respect to m given by $\mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho \ll m$. We may assume λ and ρ are positive. Then for each $x \in X$, $m(\{x\}) = 0$ which implies that $\rho(\{x\}) = 0$. Let $E \subset X$, if E is countable, then $\lambda(E) = \mu(E)$. If E is uncountable, choose $F \subset E$ such that F is countable. Then

$$\begin{aligned} \lambda(E) &\geq \lambda(F) \\ &= \mu(F) \\ &= \infty \end{aligned}$$

So $\lambda = \mu$. This is a contradiction since $\mu \not\ll m$. □

Exercise 3.27. Let (X, \mathcal{F}, μ) be a measure space and \mathcal{E} a sub σ -alg of \mathcal{F} and $f \in L^1(\mu)$. Define $\nu : \mathcal{E} \rightarrow [0, \infty]$ by $\nu(E) = \int_E f d\mu$. Let $\bar{\mu}$ be the restriction of μ to \mathcal{E} . Define the **expectation of f given \mathcal{E}** to be $E[f|\mathcal{E}] = d\nu/d\bar{\mu}$. Then for each $E \in \mathcal{E}$,

$$\int_E E[f|\mathcal{E}] d\mu = \int_E f d\mu$$

Proof. Let $E \in \mathcal{E}$. By definition,

$$\begin{aligned} \int_E E[f|\mathcal{E}] d\mu &= \int_E d\nu/d\bar{\mu} d\mu \\ &= \int_E d\nu/d\bar{\mu} d\bar{\mu} \quad (\text{since } E \in \mathcal{E}) \\ &= \nu(E) \\ &= \int_E f d\mu \end{aligned}$$

□

3.3. Complex Measures.

Definition 3.28. Let (X, \mathcal{A}) be a measurable space and $\nu : \mathcal{A} \rightarrow \mathbb{C}$. Then ν is said to be a **complex measure** if

- (1) $\nu(\emptyset) = 0$
- (2) for each sequence $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, if $(E_n)_{n \in \mathbb{N}}$ is disjoint, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$ and $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely.

Note 3.29. We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.

Definition 3.30. Let (X, \mathcal{A}) be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on (X, \mathcal{A}) . We define $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$. For $f \in L^1(\nu)$, we define

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

Theorem 3.31. Let (X, \mathcal{A}) be a measurable space, ν a complex measure on (X, \mathcal{A}) and μ a σ -finite measure on (X, \mathcal{A}) . Then there exists a complex measure λ on (X, \mathcal{A}) and $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$ and such that for each complex measure λ' on (X, \mathcal{A}) , $f' \in L^1(\mu)$, if $\nu = d\lambda' + f' d\mu$, then $\lambda = \lambda'$ and $f = f'$ μ -a.e.

Theorem 3.32. Let ν be a complex measure on (X, \mathcal{A}) and μ, λ σ -finite measures on (X, \mathcal{A}) . Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then

- (1) for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

- (2) $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Definition 3.33. Let (X, \mathcal{A}) be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on (X, \mathcal{A}) . Define $\mu = |\nu_1| + |\nu_2|$. Then $\nu \ll \mu$ and thus There exists $f \in L^1(\mu)$ such that $d\nu = f d\mu$. Define $|\nu| : \mathcal{A} \rightarrow [0, \infty)$ by $|\nu|(E) = \int_E |f| d\mu$ for each $E \in \mathcal{A}$. We call $|\nu|$ the **total variation of ν** .

Exercise 3.34. Let ν be a complex measure on (X, \mathcal{A}) and μ a σ -finite measures on (X, \mathcal{A}) . If $\nu \ll \mu$, then $\{x \in X : d\nu/d\mu(x) = 0\}$ is ν -null.

Proof. Define $f = d\nu/d\mu$ and $E = \{x : f(x) = 0\}$. Let $A \in \mathcal{A}$ and suppose that $A \subset E$. Then

$$\begin{aligned} \nu(A) &= \int_A f d\mu \\ &= 0 \end{aligned}$$

□

Exercise 3.35. Let (X, \mathcal{A}) be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on (X, \mathcal{A}) . Then $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu_2|$.

Proof. Let μ and f be as in the definition of $|\nu|$. Since for each $E \in \mathcal{A}$, we have

$$\begin{aligned}\nu(E) &= \int_E f d\mu \\ &= \int_E f_1 d\mu + i \int_E f_2 d\mu\end{aligned}$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E)$$

we know that $\nu_1 = f_1 d\mu$ and $\nu_2 = f_2 d\mu$.

A previous exercise tells us that $d|\nu_1| = |f_1| d\mu$ and $d|\nu_2| = |f_2| d\mu$. Since $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$, we have that

$$\begin{aligned}|\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2|\end{aligned}$$

□

Exercise 3.36. Let (X, \mathcal{A}) be a measurable space, ν a complex measure on (X, \mathcal{A}) and $c \in \mathbb{C}$. Then $|c\nu| = |c||\nu|$.

Proof. Define μ and f as before so that $d\nu = f d\mu$. Then $d(c\nu) = cf d\mu$. Hence

$$\begin{aligned}d|c\nu| &= |cf| d\mu \\ &= |c||f| d\mu \\ &= |c| d|\nu|\end{aligned}$$

So $|c\nu| = |c||\nu|$.

□

Exercise 3.37. Let (X, \mathcal{A}) be a measurable space and ν a complex measure on (X, \mathcal{A}) . Then

- (1) for each $E \in \mathcal{A}$, $|\nu(E)| \leq |\nu|(E)$.
- (2) $\nu \ll |\nu|$ and $|d\nu/d|\nu|| = 1$ $|\nu|$ -a.e.
- (3) $L^1(\nu) = L^1(|\nu|)$ and for each $g \in L^1(\nu)$, $|\int g d\nu| \leq \int |g| d|\nu|$

Proof. Let $\mu, f \in L^1(\mu)$ be as in the definition of $|\nu|$.

- (1) Let $E \in \mathcal{A}$. Then

$$\begin{aligned}|\nu(E)| &= \left| \int_E f d\mu \right| \\ &\leq \int_E |f| d\mu \\ &= |\nu|(E)\end{aligned}$$

- (2) Let $E \in \mathcal{A}$ and suppose that $|\nu|(E) = 0$. The previous part implies $|\nu(E)| = 0$ and $\nu \ll |\nu|$. Put $g = d\nu/d|\nu|$. Then

$$\begin{aligned}f &= \frac{d\nu}{d\mu} \\ &= g|f| \quad \mu\text{-a.e.}\end{aligned}$$

Hence $|f| = |g||f|$ μ -a.e. Since $|\nu| \ll \mu$, $|f| = |g||f|$ $|\nu|$ -a.e.

A previous exercise tells us that $|f| \neq 0$ $|\nu|$ -a.e. Thus $|g| = 1$ $|\nu|$ -a.e.

(3) Write $\nu = \nu_1 + i\nu_2$ and $f = f_1 + if_2$. First we observe that

$$\begin{aligned} L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\ &= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\ &= L^1(|\nu_1| + |\nu_2|) \\ &= L^1(\mu) \end{aligned}$$

The previous exercise tells us that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \\ &= \mu \end{aligned}$$

Let $g \in L^1(\mu)$. Then

$$\begin{aligned} \int |g|d|\nu| &\leq \int |g|d\mu \\ &< \infty \end{aligned}$$

So $g \in L^1(|\nu|)$.

Conversely, let $g \in L^1(|\nu|)$. Then

$$\begin{aligned} \int |g|d|\nu_1|, \int |g|d|\nu_2| &\leq \int |g|d|\nu| \\ &< \infty \end{aligned}$$

So

$$\begin{aligned} \int |g|d\mu &= \int |g|d|\nu_1| + \int |g|d|\nu_2| \\ &< \infty \end{aligned}$$

and $g \in L^1(\mu)$. Hence $L^1(\nu) = L^1(|\nu|)$.

Now, let $g \in L^1(\nu) = L^1(|\nu|)$, then

$$\begin{aligned} \left| \int g d\nu \right| &= \left| \int g f d\mu \right| \\ &\leq \int |g| |f| d\mu \\ &= \int |g| d|\nu| \end{aligned}$$

□

3.4. Differentiation.

Definition 3.38. Let $f : \mathbb{R}^n \rightarrow \mathbb{C}$. Then f is said to be **locally integrable** (with respect to Lebesgue measure) if f is measurable and for each $K \subset \mathbb{R}^n$, K compact implies $\int_K |f| dm < \infty$. We define $L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{C} : f \text{ is locally integrable}\}$

Definition 3.39. For $f \in L^1_{loc}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$, we define the **average of f over $B(x, r)$** , denoted by $Af(x, r)$, to be

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f dm$$

Exercise 3.40. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Define

$$H^*f(x) = \sup \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\} \quad (x \in \mathbb{R}^n)$$

Then $Hf \leq H^*f \leq 2^n Hf$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\left\{ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f| dm : r > 0 \right\} \subset \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\}$$

So $Hf(x) \leq H^*f(x)$. Let B be a ball. Then there exists $y \in \mathbb{R}^n$, $R > 0$ such that $B = B(y, R)$. Suppose that $x \in B$. Then $B \subset B(x, 2R)$. Since $m(B(x, 2R)) = 2^n m(B(y, R))$, we have that

$$\begin{aligned} \frac{1}{m(B)} \int_B |f| dm &\leq \frac{1}{m(B)} \int_{m(B(x, 2R))} |f| dm \\ &= \frac{2^n}{m(B(x, 2R))} \int_{m(B(x, 2R))} |f| dm \end{aligned}$$

Thus $H^*f(x) \leq 2^n Hf(x)$. □

Lemma 3.41. Let $f \in L^1_{loc}(\mathbb{R}^n)$, then $Af : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ is continuous.

Definition 3.42. Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define its **Hardy Littlewood maximal function**, denoted by Hf to be

$$Hf(x) = \sup_{r>0} A|f|(x, r) \quad x \in \mathbb{R}^n$$

Theorem 3.43. There exists $C > 0$ such that for each $f \in L^1(m)$ and $\alpha > 0$,

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f| dm$$

Exercise 3.44. Let $f \in L^1(\mathbb{R}^n)$. Suppose that $\|f\|_1 > 0$. Then there exist $C, R > 0$ such that for each $x \in \mathbb{R}^n$, if $|x| > R$, then $Hf(x) \geq C|x|^{-n}$. Hence there exists $C' > 0$ such that for each $\alpha > 0$, $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$ when α is small.

Proof. Since $\|f\|_1 > 0$, there exists $R > 0$ such that $\int_{B(0, R)} |f| dm > 0$. Recall that there exists $K > 0$ such that for each $x \in \mathbb{R}^n$ and $r > 0$, $m(B(x, r)) = Kr^n$. Choose

$$C = \frac{\int_{B(0, R)} |f| dm}{K2^n}$$

. Let $x \in \mathbb{R}^n$. Suppose that $|x| > R$. Then $B(0, R) \subset B(x, 2|x|)$. Thus

$$\begin{aligned} Hf(x) &\geq \frac{1}{m(B(x, 2|x|))} \int_{B(x, 2|x|)} |f| dm \\ &= \frac{1}{K2^n|x|^n} \int_{B(x, 2|x|)} |f| dm \\ &\geq \frac{1}{K2^n|x|^n} \int_{B(0, R)} |f| dm \\ &= \frac{C}{|x|^n} \end{aligned}$$

Let $a < \frac{C}{2R^n}$. Then $R^n < \frac{C}{2a}$. Choose $C' = \frac{KC}{2}$. Let $A = \{x \in \mathbb{R}^n : R < |x| < (\frac{C}{a})^{\frac{1}{n}}\}$. For $x \in A$,

$$\begin{aligned} Hf(x) &\geq \frac{C}{|x|^n} \\ &> a \end{aligned}$$

Thus $A \subset m(\{x \in \mathbb{R}^n : Hf(x) > a\})$ and therefore

$$\begin{aligned} m(\{x \in \mathbb{R}^n : Hf(x) > a\}) &\geq m(A) \\ &= m(B(0, (C/a)^{1/n})) - m(B(0, R)) \\ &= K \left[\frac{C}{a} - R^n \right] \\ &> K \left[\frac{C}{a} - \frac{C}{2a} \right] \\ &= \frac{KC}{2a} \\ &= \frac{C'}{a} \end{aligned}$$

□

Theorem 3.45. Let $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \rightarrow 0} Af(x, r) = f(x)$$

. Equivalently, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \rightarrow 0} \left[\frac{1}{m(B(x, r))} \int_{B(x, r)} [f(y) - f(x)] dm(y) \right] = 0$$

Note 3.46. We can a stronger result of the same flavor.

Definition 3.47. Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define the **Lebesgue set of f** , denoted by L_f , to be

$$\begin{aligned} L_f &= \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0} A|f - f(x)|(x, r) = 0\} \\ &= \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \left[\frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0 \right\} \end{aligned}$$

Exercise 3.48. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. If f is continuous at x , then $x \in L_f$.

Proof. Suppose that f is continuous at x . Let $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that for each $y \in \mathbb{R}^n$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Let $r > 0$. Suppose that $r < \delta$. Then for each $y \in \mathbb{R}^n$, $y \in B(x, r)$ implies that $|f(x) - f(y)| < \epsilon$ and thus

$$\begin{aligned} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(B(x, r))} \epsilon m(B(x, r)) \\ &= \epsilon \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0} \left[\frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0$$

and $x \in L_f$. □

Theorem 3.49. Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then $m((L_f)^c) = 0$

Definition 3.50. Let $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then $(E_r)_{r>0}$ is said to **shrink nicely** to x if

- (1) for each $r > 0$, $E_r \subset B(x, r)$
- (2) there exists $\alpha > 0$ such that for each $r > 0$, $m(E_r) > \alpha m(B(x, r))$

Theorem 3.51. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then for each $x \in L_f$,

$$\lim_{r \rightarrow 0} \left[\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \right] = 0$$

and

$$\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

Definition 3.52. Let $\mu : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, \infty]$ be a Borel measure. Then μ is said to be **regular** if

- (1) for each $K \subset \mathbb{R}^n$, if K is compact, then $\mu(K) < \infty$
- (2) for each $E \in \mathcal{B}(\mathbb{R}^n)$, $\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subset U\}$

Let ν be a signed or complex Borel measure on \mathbb{R}^n . Then ν is said to be regular if $|\nu|$ is regular.

Theorem 3.53. Let ν be a regular signed or complex measure on \mathbb{R}^n . Let $d\nu = d\lambda + f dm$ be the Lebesgue decomposition of ν with respect to m . Then for m -a.e. $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$, if $(E_r)_{r>0}$ shrinks nicely to x , then

$$\lim_{r \rightarrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

3.5. Functions of Bounded Variation.

Definition 3.54. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Define $F_+ : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F_+(x) = \lim_{t \rightarrow x^+} F(t) = \inf\{F(t) : t > x\}$$

Note 3.55. Observe that $F \leq F_+$ and F_+ is increasing.

Exercise 3.56. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Then for each $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in (x, x + \delta)$, $0 \leq F_+(y) - F(y) \leq \epsilon$.

Proof. For the sake of contradiction, suppose not. Then there exists $x \in \mathbb{R}$ and $\epsilon > 0$ such that for each $\delta > 0$, there exist $y \in (x, x + \delta)$ such that $F_+(y) - F(y) > \epsilon$. Then there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that for each $n \in \mathbb{N}$, $y_n \in (x, x + \frac{1}{n})$, $y_n > y_{n+1}$ and $F_+(y_n) - F(y_n) > \epsilon$. Choose $N \in \mathbb{N}$ such that $(N - 1)\epsilon > F(y_1) - F(x)$. Then

$$\begin{aligned} F(y_1) - F(x) &= \sum_{i=1}^{N-1} \left[F(y_i) - F_+(y_{i+1}) + F_+(y_{i+1}) - F(y_{i+1}) \right] + F(y_N) - F(x) \\ &= \sum_{i=1}^{N-1} \left[F(y_i) - F_+(y_{i+1}) \right] + \sum_{i=1}^{N-1} \left[F_+(y_{i+1}) - F(y_{i+1}) \right] + F(y_N) - F(x) \\ &\geq (N - 1)\epsilon \\ &> F(y_1) - F(x) \end{aligned}$$

This is a contradiction, so the claim holds. \square

Exercise 3.57. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Then F_+ is right continuous.

Proof. Let $x \in \mathbb{R}$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that for each $y \in (x, x + \delta_1)$ $0 \leq F(y) - F_+(x) < \epsilon/2$. There exists $\delta_2 > 0$ such that for each $y \in (x, x + \delta_2)$, $0 \leq F_+(y) - F(y) < \epsilon/2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Let $y \in (x, x + \delta)$.

$$\begin{aligned} |F_+(x) - F_+(y)| &\leq |F_+(x) - F(y)| + |F(y) - F_+(y)| \\ &= (F(y) - F_+(x)) + (F_+(y) - F(y)) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So $\lim_{t \rightarrow x^+} F_+(t) = F_+(x)$ and F_+ is right continuous. \square

Theorem 3.58. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Then

- (1) $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$ is countable
- (2) F and F_+ are differentiable a.e. and $F' = F'_+$ a.e.

Definition 3.59. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Define $T_F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$T_F(x) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \quad (x \in \mathbb{R})$$

T_F is called the **total variation function of F** .

Exercise 3.60. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Then T_F is increasing.

Proof. Let $x, y \in \mathbb{R}$. Suppose that $x < y$.

Define $A_x = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$ and $A_y = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = y \right\}$. Let $z \in A_x$. Then there exists $(x_i)_{i=0}^n \subset \mathbb{R}$ such that $(x_i)_{i=0}^n$ is increasing, $x_n = x$ and $z = \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$. Then

$$\begin{aligned}
z &\leq z + |F(y) - F(x)| \\
&= \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \\
&\in A_y
\end{aligned}$$

So $z \leq \sup A_y = T_F(y)$ and thus $F_T(x) = \sup A_x \leq T_F(y)$ \square

Lemma 3.61. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$. Then $T_F + F$ and $T_F - F$ are increasing.*

Exercise 3.62. *For each $F : \mathbb{R} \rightarrow \mathbb{C}$, $T_{|F|} \leq T_F$.*

Proof. Let $F : \mathbb{R} \rightarrow \mathbb{C}$, $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then by the reverse triangle inequality,

$$\sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

Thus

$$\begin{aligned}
T_{|F|}(x) &= \sup \left\{ \sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&\leq \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&= T_F(x)
\end{aligned}$$

Hence $T_{|F|} \leq T_F$ \square

Definition 3.63. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Then F is said to have **bounded variation** if $\lim_{x \rightarrow \infty} T_F(x) < \infty$. The **total variation of F** , denoted by $TV(F)$, is defined to be $TV(F) = \lim_{x \rightarrow \infty} T_F(x)$. We define $BV = \{F : \mathbb{R} \rightarrow \mathbb{C} : TV(F) < \infty\}$.*

Definition 3.64. *Let $a, b \in \mathbb{R}$ and $F : [a, b] \rightarrow \mathbb{C}$. Define $G_F : \mathbb{R} \rightarrow \mathbb{C}$ by $G_F = F(a)\chi_{(-\infty, a)} + F\chi_{[a, b]} + F(b)\chi_{(b, \infty)}$. Then F is said to have **bounded variation on $[a, b]$** if $G_F \in BV$. The **total variation of F on $[a, b]$** , denoted by $TV(F, [a, b])$, is defined to be $TV(F, [a, b]) = TV(G_F)$. We define $BV([a, b]) = \{F : [a, b] \rightarrow \mathbb{C} : TV(F, [a, b]) < \infty\}$.*

Note 3.65. *Equivalently, $TV(F, [a, b]) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset [a, b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$ and $F \in BV([a, b])$ iff $TV(F, [a, b]) < \infty$. In general,*

Exercise 3.66. *Let $F \in BV$. Then F is bounded.*

Proof. If F is unbounded, then the supremum in the previous definition is clearly infinite. \square

Exercise 3.67. *Let $F : \mathbb{R} \rightarrow \mathbb{R}$. If F is bounded and increasing, then $F \in BV$.*

Proof. Suppose that F is bounded and increasing. Then $-\infty < \inf_{x \in \mathbb{R}} F(x) \leq \sup_{x \in \mathbb{R}} F(x) < \infty$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n F(x_i) - F(x_{i-1}) \\ &= F(x) - F(x_0) \end{aligned}$$

Thus

$$T_F(x) = F(x) - \inf_{x \in \mathbb{R}} F(x)$$

. This implies that

$$\begin{aligned} TV(F) &= \sup_{x \in \mathbb{R}} F(x) - \inf_{x \in \mathbb{R}} F(x) \\ &< \infty \end{aligned}$$

Hence $F \in BV$. □

Exercise 3.68. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. If F is differentiable and F' is bounded on $[a, b]$, then, $F \in BV([a, b])$.

Proof. Suppose that F is differentiable and F' is bounded on $[a, b]$. Then there exists $M > 0$ such that for each $x \in [a, b]$, $|F'(x)| \leq M$. Let $(x_i)_{i=1}^n \subset [a, b]$. Suppose that $(x_i)_{i=1}^n$ is strictly increasing, $x_0 = a$ and $x_n = b$. By the mean value theorem, for each $i = 1, 2, \dots, n$, there exists $c_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$. Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n |F'(c_i)(x_i - x_{i-1})| \\ &\leq \sum_{i=1}^n M(x_i - x_{i-1}) \\ &= M(b - a) \end{aligned}$$

Hence $TV(F, [a, b]) \leq M(b - a)$. □

Exercise 3.69. Define $F, G : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x) = \begin{cases} x^2 \sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 \sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then F and G are differentiable, $F \in BV([-1, 1])$ and $G \notin BV([-1, 1])$.

Proof. On $\mathbb{R} \setminus \{0\}$,

$$\begin{aligned} F'(x) &= 2x \sin(x^{-1}) - \sin(x^{-1}) \\ &= \sin(x^{-1})(2x - 1) \end{aligned}$$

We see that F is also differentiable at $x = 0$ since

$$\begin{aligned} F'(0) &= \lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-1})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-1}) \\ &= 0 \end{aligned}$$

Therefore for each $x \in [-1, 1]$, $|F'(x)| \leq 3$. Which by a previous exercise implies that $F \in BV([-1, 1])$.

On $\mathbb{R} \setminus \{0\}$,

$$\begin{aligned} G'(x) &= 2x \sin(x^{-2}) - \frac{2 \sin(x^{-2})}{x} \\ &= \sin(x^{-2}) \left(2x - \frac{2}{x}\right) \end{aligned}$$

We see that G is also differentiable at $x = 0$ since

$$\begin{aligned} G'(0) &= \lim_{x \rightarrow 0} \frac{G(x) - G(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-2})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-2}) \\ &= 0 \end{aligned}$$

For $n \in \mathbb{N}$, define $(x_i)_{i=0}^n \subset [-1, 1]$ by

$$x_i = \frac{-1}{\sqrt{\frac{\pi}{2} + i\pi}}$$

Then for each $n \in \mathbb{N}$, $(x_i)_{i=1}^n$ is strictly increasing and for each $i = 1, 2, \dots, n$ we have that

$$\begin{aligned} |G(x_i) - G(x_{i-1})| &= \frac{1}{\frac{\pi}{2} + i\pi} + \frac{1}{\frac{\pi}{2} + (i-1)\pi} \\ &= \frac{2}{\pi} \left[\frac{(2i-1) + (2i+1)}{(2i+1)(2i-1)} \right] \\ &= \frac{2}{\pi} \left[\frac{4i}{4i^2 - 1} \right] \\ &> \frac{2}{i\pi} \end{aligned}$$

Hence for each $n \in \mathbb{N}$,

$$\begin{aligned} TV(G, [-1, 1]) &\geq \sum_{i=1}^n |G(x_i) - G(x_{i-1})| \\ &> \frac{2}{\pi} \sum_{i=1}^n \frac{1}{i} \end{aligned}$$

Therefore $G \notin BV([-1, 1])$. □

Exercise 3.70. *The following is stated for BV , but is also true for $BV([a, b])$.*

- (1) *For each $F, G \in BV$, $T_{F+G} \leq T_F + T_G$ and therefore BV is a vector space.*
- (2) *For each $F : \mathbb{R} \rightarrow \mathbb{C}$, $F \in BV$ iff $\operatorname{Re}(f) \in BV$ and $\operatorname{Im}(F) \in BV$.*
- (3) *For each $F : \mathbb{R} \rightarrow \mathbb{R}$, $F \in BV$ iff there exist functions $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$*
- (4) *For each $F \in BV$ and $x \in \mathbb{R}$, $\lim_{t \rightarrow x^+} F(t)$ and $\lim_{t \rightarrow x^-} F(t)$ exist.*
- (5) *For each $F \in BV$, $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$ is countable.*
- (6) *For each $F \in BV$, F and F_+ are differentiable a.e. and $F' = (F_+)'$ a.e.*
- (7) *For each $F \in BV, c \in \mathbb{R}$, $F - c \in BV$*

Proof. (1) Let $F, G \in BV$, $x \in \mathbb{R}$ and $\epsilon > 0$. Since $T_{F+G}(x) < \infty$, $T_{F+G}(x) - \epsilon < T_{F+G}(x)$. Thus there exists $(x_i)_{i=0}^n \subset \mathbb{R}$ such that $(x_i)_{i=0}^n$ is increasing, $x_n = x$ and $T_{F+G}(x) < \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1})| + \epsilon$. Therefore

$$\begin{aligned} T_{F+G}(x) &< \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1})| + \epsilon \\ &\leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + \sum_{i=1}^n |G(x_i) - G(x_{i-1})| + \epsilon \\ &\leq T_F(x) + T_G(x) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $T_{F+G}(x) \leq T_F(x) + T_G(x)$. Therefore $TV(F+G) \leq TV(F) + TV(G) < \infty$. Thus $F+G \in BV$. It is straight forward to verify the other requirements needed to show that BV is a vector space.

- (2) Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Write $F = F_1 + iF_2$ with $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$. Suppose that $F \in BV$. Note that for each $x_1, x_2 \in \mathbb{R}$ and $j = 1, 2$, $|F_j(x_1) - F_j(x_2)| \leq |F(x_1) - F(x_2)|$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then for $j = 1, 2$

$$\sum_{i=1}^n |F_j(x_i) - F_j(x_{i-1})| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

. Thus for $j = 1, 2$ we have that $T_{F_j}(x) \leq T_F(x)$ which implies that $\operatorname{Re}(f), \operatorname{Im}(F) \in BV$. Conversely, Suppose that $\operatorname{Re}(f), \operatorname{Im}(F) \in BV$. Then $F = \operatorname{Re}(f) + i\operatorname{Im}(f) \in BV$ by (1).

- (3) Suppose that $F \in BV$. Choose $F_1 = \frac{1}{2}(T_F - F)$ and $F_2 = \frac{1}{2}(T_F + F)$. Then F_1, F_2 are bounded, increasing and $F = F_1 + F_2$. Conversely, if there exist $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$, then $F_1, F_2 \in BV$. By (1) $F \in BV$.
- (4) This is clear by previous results and (3)

- (5) This is clear by previous results and (3)
- (6) This is clear by previous results and (3)
- (7) Clearly constant functions have zero total variation. The rest is implied by (1). \square

Lemma 3.71. *Let $F \in BV$. Then $\lim_{x \rightarrow -\infty} T_F(x) = 0$ and if F is right continuous, then T_F is right continuous.*

Definition 3.72. *Define $NBV = \{F \in BV : F \text{ is right continuous and } \lim_{x \rightarrow -\infty} F(x) = 0\}$.*

Theorem 3.73. *Let $M(\mathbb{R})$ be the set of complex Borel measures on \mathbb{R} . For $F \in NBV$, define $\mu_F \in M(\mathbb{R})$ by $\mu_F((-\infty, x]) = F(x)$. Then $F \mapsto \mu_F$ defines a bijection $NBV \rightarrow M(\mathbb{R})$. In addition, $|\mu_F| = \mu_{T_F}$*

Theorem 3.74. *Let $F \in NBV$. Then $F' \in L^1(m)$, $\mu_F \perp m$ iff $F' = 0$ a.e. and $\mu_F \ll m$ iff for each $x \in \mathbb{R}$, $\int_{(-\infty, x]} F' dm = F(x)$*

Definition 3.75. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Then F is said to be **absolutely continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Definition 3.76. *Let $F : [a, b] \rightarrow \mathbb{C}$. Then F is said to be **absolutely continuous on** $[a, b]$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Proposition 3.77. *Let $F : [a, b] \rightarrow \mathbb{C}$. If F is absolutely continuous on $[a, b]$, then $F \in BV[a, b]$.*

Exercise 3.78. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Suppose that there exists $f \in L^1(m)$ such that $F(x) = \int_{(-\infty, x]} f dm$. Then $F \in NBV$.*

Proof. Let $x \in \mathbb{R}$ and $(x_i)_{i=1}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=1}^n$ is increasing and $x_n = x$. Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \left| \int_{(x_{i-1}, x_i]} f dm \right| \\ &\leq \sum_{i=1}^n \int_{(x_{i-1}, x_i]} |f| dm \\ &= \int_{(x_0, x]} |f| dm \\ &< \int |f| dm \end{aligned}$$

Hence $T_F(x) \leq \int |f| dm$. Since $x \in \mathbb{R}$ is arbitrary, $TV(F) \leq \int |f| dm$. Therefore $F \in BV$. By the continuity from above and below for measures and the fact that $m(x) = 0$ for each $x \in \mathbb{R}$, F is continuous. By continuity from above for measures, $\lim_{x \rightarrow -\infty} F(x) = 0$. So $F \in NBV$. \square

Lemma 3.79. *Let $F \in NBV$. Then F is absolutely continuous iff $\mu_F \ll m$.*

Exercise 3.80. *Fundamental Theorem of Calculus: Let $F : [a, b] \rightarrow \mathbb{C}$. The following are equivalent:*

- (1) F is absolutely continuous on $[a, b]$.
 (2) there exists $f \in L^1([a, b], m)$ such that for each $x \in [a, b]$, $F(x) - F(a) = \int_{(a, x]} f dm$
 (3) F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$ and for each $x \in [a, b]$, $F(x) - F(a) = \int_{(a, x]} F' dm$

Proof. (1) \implies (3)

Suppose that F is absolutely continuous on $[a, b]$. Then $F \in BV[a, b]$. Extend F to \mathbb{R} by setting $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$. Then $G = F - F(a) \in NBV$ and is absolutely continuous. The previous lemma implies that there exists $f \in L^1(m)$ such that $\mu_G = f dm$. A previous theorem implies that for a.e. $x \in [a, b]$

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow x} \frac{\mu_G((x, x+r])}{m((x, x+r])} \\ &= f(x) \end{aligned}$$

So F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$ and by construction, for each $x \in [a, b]$, we have that

$$\begin{aligned} F(x) - F(a) &= \mu_G((a, x]) \\ &= \int_{(a, x]} f dm \\ &= \int_{(a, x]} F' dm \end{aligned}$$

(3) \implies (2)

Trivial.

(2) \implies (1)

Suppose that there exists $f \in L^1([a, b], m)$ such that for each $x \in [a, b]$, $F(x) - F(a) = \int_{(a, x]} f dm$. Extend F as before and obtain G as before. Note that a previous exercise implies that $G \in NBV$. Since $\mu_G \ll m$, the previous lemma implies that G is absolutely continuous. \square

Exercise 3.81. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. If F is absolutely continuous. Then F is differentiable a.e.

Proof. Let $n \in \mathbb{N}$. Since F is absolutely continuous on \mathbb{R} , F is absolutely continuous on $[-n, n]$. The FTC implies that F is differentiable a.e. on $[-n, n]$. Since $n \in \mathbb{N}$ is arbitrary, F is differentiable a.e. on \mathbb{R} . \square

Exercise 3.82. Let $F : \mathbb{R} \rightarrow \mathbb{C}$. Then F is Lipschitz continuous iff F is absolutely continuous and F' is bounded a.e.

Proof. Suppose that F is Lipschitz continuous. Then there exists $M > 0$ such that for each $x, y \in \mathbb{R}$, $|F(x) - F(y)| \leq M|x - y|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{M}$. Let $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, Suppose that $\sum_{i=1}^n b_i - a_i < \delta$. Then

$$\begin{aligned} \sum_{i=1}^n |F(b_i) - F(a_i)| &\leq \sum_{i=1}^n M(b_i - a_i) \\ &< M\delta \\ &= \epsilon \end{aligned}$$

Hence F is absolutely continuous. For each $x, y \in \mathbb{R}$, if $x \neq y$, then $\left| \frac{F(x) - F(y)}{x - y} \right| \leq M$. Hence for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Conversely, suppose that F is absolutely continuous and F' is bounded a.e. Then there exists $M > 0$ such that for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Let $x, y \in \mathbb{R}$. Suppose $x < y$. Then the FTC implies that

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_{(x,y]} F' dm \right| \\ &\leq \int_{(x,y]} |F'| dm \\ &= M|y - x| \end{aligned}$$

and F is Lipschitz continuous. □

Exercise 3.83. Construct an increasing function $F : \mathbb{R} \rightarrow \mathbb{R}$ whose discontinuities is \mathbb{Q} .

Proof. Let $(q_n)_{n \in \mathbb{N}}$ be an enumeration of \mathbb{Q} . Define $F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F = \sum_{n \in \mathbb{N}} 2^{-n} \chi_{[q_n, \infty)}$$

. Equivalently, if we define $S_x = \{n \in \mathbb{N} : q_n \leq x\}$, then we may write

$$F(x) = \sum_{n \in S_x} 2^{-n}$$

Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Then $S_x \subsetneq S_y$. So $F(x) < F(y)$ and therefore F is strictly increasing.

For each $x, y \in \mathbb{R}$ with $x < y$, define $S_{x,y} = \{n \in \mathbb{N} : x < q_n \leq y\}$. Note that $\lim_{y \rightarrow x^+} \min(S_{x,y}) = \infty$ and if $y \in \mathbb{R} \setminus \mathbb{Q}$, then $\lim_{x \rightarrow y^-} \min(S_{x,y}) = \infty$.

Now, let $x \in \mathbb{R}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$. Choose $\delta > 0$ such that $\min(S_{x, x+\delta}) \geq N$. Let $y \in [x, \infty)$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_y} 2^{-n} - \sum_{n \in S_x} 2^{-n} \\ &= \sum_{n \in S_{x,y}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence F is right continuous. Now let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ as before and $\delta > 0$ such that $\min(S_{x-\delta, x}) \geq N$. Let $y \in (-\infty, x]$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_x} 2^{-n} - \sum_{n \in S_y} 2^{-n} \\ &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence F is left continuous on $\mathbb{R} \setminus \mathbb{Q}$.

Now, let $x \in \mathbb{Q}$. Then there exists $j \in \mathbb{N}$ such that $q_j = x$. Choose $\epsilon = 2^{-j}$. Let $\delta > 0$. Choose $y = x - \frac{\delta}{2}$. Then $|x - y| < \delta$ and

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\geq 2^{-j} \\ &= \epsilon \end{aligned}$$

Hence F is discontinuous from the left at x . Since $x \in \mathbb{Q}$ is arbitrary, F is discontinuous from the left on \mathbb{Q} . \square

Exercise 3.84. Let $(F_n)_{n \in \mathbb{N}} \in NBV$ be a sequence of nonnegative, increasing functions. If for each $x \in \mathbb{R}$, $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$, then for a.e. $x \in \mathbb{R}$, F is differentiable at x and $F'(x) = \sum_{n \in \mathbb{N}} F'_n(x)$.

Proof. Define $\mu = \sum_{n \in \mathbb{N}} \mu_{F_n}$. Note that

$$\begin{aligned} \mu((-\infty, x]) &= \sum_{n \in \mathbb{N}} \mu_{F_n}((-\infty, x]) \\ &= \sum_{n \in \mathbb{N}} F_n(x) \\ &= F(x) \end{aligned}$$

Hence $F \in NBV$ and $\mu = \mu_F$. For each $n \in \mathbb{N}$, there exist $\lambda_n \in M(\mathbb{R})$ and $f_n \in L^1(\mathbb{R})$ such that $d\mu_{F_n} = d\lambda_n + f_n dm$ and $\lambda \perp m$. Since for each $n \in \mathbb{N}$, λ_n, f_n are nonnegative, we have that $d\mu_F = \sum_{n \in \mathbb{N}} d\lambda_n + (\sum_{n \in \mathbb{N}} f_n) dm$. By a previous theorem, for a.e. $x \in \mathbb{R}$,

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow 0} \frac{\mu_F((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} f_n(x) \\ &= \sum_{n \in \mathbb{N}} \lim_{r \rightarrow 0} \frac{\mu_{F_n}((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} F'_n(x) \end{aligned}$$

□

Exercise 3.85. Let $F : [0, 1] \rightarrow [0, 1]$ be the Cantor function. Extend F to \mathbb{R} by setting $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $([a_n, b_n])_{n \in \mathbb{N}}$ be an enumeration of the closed subintervals of $[0, 1]$ with rational endpoints. For $n \in \mathbb{N}$, define $F_n : \mathbb{R} \rightarrow [0, 1]$ by $F_n(x) = F(\frac{x-a_n}{b_n-a_n})$. Define $G : \mathbb{R} \rightarrow \mathbb{R}$ by $G = \sum_{n \in \mathbb{N}} 2^{-n} F_n$. Then G is continuous, strictly increasing on $[0, 1]$ and $G' = 0$ a.e.

Proof. Since F is continuous on \mathbb{R} , we have that for each $n \in \mathbb{N}$, F_n is continuous on \mathbb{R} . We observe that for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $|2^{-n} F_n(x)| \leq 2^{-n}$. Thus the Weierstrass M-test implies that G converges uniformly on \mathbb{R} and is therefore continuous. Since F is increasing, for each $n \in \mathbb{N}$, F_n is increasing. Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Choose $j \in \mathbb{N}$ such that $x < a_j < y < b_j$. Then

$$\begin{aligned} G(x) &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(x) \\ &= \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(x) + 0 \\ &< \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(y) + 2^{-j} F_j(y) \\ &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(y) \\ &= G(y) \end{aligned}$$

So G is strictly increasing.

Now we observe that for each $n \in \mathbb{N}$, $F_n \in NBV$. The previous exercise implies that

$$G' = \sum 2^{-n} F'_n = 0 \text{ a.e.}$$

□

4. TOPOLOGY

Definition 4.1. Let (X, \mathcal{A}) and (Y, \mathcal{B}) be topological spaces and $f : X \rightarrow Y$. Then

- (1) f is said to be **continuous** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$.
- (2) f is said to be **open** if for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$.
- (3) f is said to be **closed** if for each $A \subset X$, if $A^c \in \mathcal{A}$, then $f(A)^c \in \mathcal{B}$.

5. L^p SPACES

6. FUNCTIONAL ANALYSIS

6.1. Normed Vector Spaces.

Note 6.1. In the following, we will consider vector spaces over \mathbb{C} . There are analogous results for real vector spaces as well, just replace every \mathbb{C} with \mathbb{R} .

Definition 6.2. Let X be a normed vector space. Then X is said to be a **Banach space** if X is complete.

Definition 6.3. Let X be a normed vector space and $(x_i)_{i=1}^n \subset X$. The series $\sum_{i=1}^{\infty} x_i$ is said to **converge** if the sequence $s_n := \sum_{i=1}^n x_i$ converges. The series $\sum_{i=1}^{\infty} x_i$ is said to **converge absolutely** if $\sum_{i \in \mathbb{N}} \|x_i\| < \infty$.

Theorem 6.4. Let X be a normed vector space. Then X is complete iff for each $(x_i)_{i \in \mathbb{N}} \subset X$, $\sum_{i=1}^{\infty} x_i$ converges absolutely implies that $\sum_{i=1}^{\infty} x_i$ converges.

Proof. Suppose that X is complete. Let $(x_i)_{i \in \mathbb{N}} \subset X$. Suppose that $\sum_{i=1}^{\infty} x_i$ converges absolutely. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq N$ and $m < n$, then $\sum_{m+1}^n \|x_i\| < \epsilon$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right\| \\ &= \left\| \sum_{i=m+1}^n x_i \right\| \\ &\leq \sum_{i=m+1}^n \|x_i\| \\ &< \epsilon \end{aligned}$$

Thus $(s_n)_{n \in \mathbb{N}}$ is Cauchy. Since X is complete, $\sum_{i=1}^{\infty} x_i$ converges. Conversely, Suppose that for each $(x_i)_{i \in \mathbb{N}} \subset X$, $\sum_{i=1}^{\infty} x_i$ converges absolutely implies that $\sum_{i=1}^{\infty} x_i$ converges. Let $(x_i)_{i \in \mathbb{N}} \subset X$ be Cauchy. Proceed inductively to create a strictly increasing sequence $(n_i)_{i \in \mathbb{N}} \subset \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq n_i$, then $\|x_m - x_n\| < 2^{-i}$. Define $(y_i)_{i \in \mathbb{N}} \subset X$ by

$$y_i = \begin{cases} x_{n_1} & i = 1 \\ x_{n_i} - x_{n_{i-1}} & i \geq 2 \end{cases}$$

Then $\sum_{i=1}^k y_i = x_{n_k}$ and

$$\begin{aligned} \sum_{i \in \mathbb{N}} \|y_i\| &= \|x_{n_1}\| + \sum_{i \in \mathbb{N}} \|x_{n_i} - x_{n_{i-1}}\| \\ &\leq \|x_{n_1}\| + \sum_{i \in \mathbb{N}} 2^{-i} \\ &= \|x_{n_1}\| + 1 \end{aligned}$$

Hence $(x_{n_k})_{k \in \mathbb{N}} = (\sum_{i=1}^k y_i)_{i \in \mathbb{N}}$ converges. Since $(x_i)_{i \in \mathbb{N}}$ is Cauchy and has a convergent subsequence, it converges. So X is complete. \square

Definition 6.5. Let X, Y be normed vector spaces. A linear map $T : X \rightarrow Y$ is said to be **bounded** if there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$.

Exercise 6.6. Let X, Y be normed vector spaces and $T : X \rightarrow Y$ a linear map. Then T is bounded iff there exists $r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$

Proof. Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Thus $T(B(0, 1)) \subset B(0, C + 1)$. Conversely. Suppose that there exists

$r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$. Define $C = \frac{2s}{r}$. Let $x \in X$. Put $\alpha = \frac{r}{2\|x\|}$. Then $\alpha x \in B(0, r)$. So $T(\alpha x) = \alpha T(x) \in B(0, s)$. Hence

$$\begin{aligned}\|T(\alpha x)\| &= \|\alpha T(x)\| \\ &= |\alpha| \|T(x)\| \\ &= \frac{r}{2\|x\|} \|T(x)\| \\ &< s.\end{aligned}$$

Thus

$$\|Tx\| < \frac{2s}{r} \|x\| = C\|x\|$$

So T is bounded. □

Theorem 6.7. *Let X, Y be normed vector spaces and $T : X \rightarrow Y$ a linear map. Then the following are equivalent:*

- (1) T is continuous
- (2) T is continuous at $x = 0$
- (3) T is bounded

Proof. (1) \implies (2):

Trivial

(2) \implies (3):

Suppose that T is continuous at $x = 0$. Then there exists $\delta > 0$ such that for each $x \in X$, if $\|x\| < \delta$, then $\|Tx\| < 1$. Choose $C = \frac{2}{\delta}$. If $x = 0$, then $\|Tx\| \leq C\|x\|$. Suppose that $\|x\| \neq 0$. Define $y = \frac{\delta}{2\|x\|}x$. Then $\|y\| < \delta$. So

$$\|Ty\| = \frac{\delta}{2\|x\|} \|Tx\| < 1$$

Thus

$$\begin{aligned}\|Tx\| &< \frac{2}{\delta} \|x\| \\ &= C\|x\|\end{aligned}$$

Hence T is bounded.

(3) \implies (1)

Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{C+1}$. Let $x, y \in X$. Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned}\|Tx - Ty\| &= \|T(x - y)\| \\ &\leq C\|x - y\| \\ &< (C + 1)\delta \\ &= \epsilon\end{aligned}$$

So T is continuous. □

Definition 6.8. Let X, Y be normed vector spaces. Define $L(X, Y) = \{T : X \rightarrow Y : T \text{ is bounded}\}$. Define $\|\cdot\| : L(X, Y) \rightarrow [0, \infty)$ by

$$\|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

We call $\|\cdot\|$ the **operator norm on $L(X, Y)$**

Exercise 6.9. Let X, Y be normed vector spaces. If $X \neq \{0\}$, then the operator norm on $L(X, Y)$ is given by:

$$(1) \|T\| = \sup_{\|x\|=1} \|Tx\|$$

$$(2) \|T\| = \sup_{x \neq 0} \|x\|^{-1} \|Tx\|$$

$$(3) \|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

Proof. Since $X \neq \{0\}$, the supremums in (1) and (2) are well defined. Let $T \in L(X, Y)$. By linearity of T , the sets over which the supremums are taken in (1) and (2) are the same. So (1) and (2) are equal.

Now, put $M = \sup_{\|x\|=1} \|Tx\|$, $m = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$ and let $x \in X$. If $\|x\| = 0$, then $\|Tx\| \leq M\|x\|$. Suppose that $\|x\| \neq 0$. Then

$$\begin{aligned} \|Tx\| &= \left(\|T(x/\|x\|)\| \right) \|x\| \\ &\leq M\|x\| \end{aligned}$$

Hence $M \in \{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$. Therefore $m \leq M$

Let $C \in \{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$. Suppose that $\|x\| = 1$. Then $\|Tx\| \leq C\|x\| = C$. So $M \leq C$. Therefore $M \leq m$. So $M = m$ and the supremum in (1) is the same as the infimum in (3). \square

Note 6.10. From here on, unless stated otherwise, we assume $X \neq 0$.

Exercise 6.11. Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then for each $x \in X$, $\|Tx\| \leq \|T\|\|x\|$

Proof. This is just part of the previous exercise. Let $x \in X$. If $x = 0$, then $\|Tx\| \leq \|T\|\|x\|$. Suppose that $x \neq 0$. Then $\|Tx\| = \|T(x/\|x\|)\|\|x\| \leq \|T\|\|x\|$ \square

Exercise 6.12. Let X, Y be normed vector spaces. Then the operator norm is a norm on $L(X, Y)$.

Proof. Let $S, T \in L(X, Y)$ and $\alpha \in \mathbb{C}$. For each $x \in X$, we have that

$$\begin{aligned} \|(S + T)x\| &= \|Sx + Tx\| \\ &\leq \|Sx\| + \|Tx\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= (\|S\| + \|T\|)\|x\| \end{aligned}$$

So $\|S + T\| \leq \|S\| + \|T\|$.

Using the definition of $\|T\|$, we see that

$$\begin{aligned}
 \|\alpha T\| &= \sup_{\|x\|=1} \|(\alpha T)x\| \\
 &= \sup_{\|x\|=1} |\alpha| \|Tx\| \\
 &= |\alpha| \sup_{\|x\|=1} \|Tx\| \\
 &= |\alpha| \|T\|
 \end{aligned}$$

$$\text{So } \|\alpha S\| = |\alpha| \|S\|.$$

Suppose that $\|T\| = 0$. Let $x \in X$. Then $\|Tx\| \leq \|T\| \|x\| = 0$. So $Tx = 0$. Since $x \in X$ is arbitrary, we have that $T = 0$. \square

Exercise 6.13. *Let X be a normed vector space. Then addition and scalar multiplication are continuous on $X \times X$ and $\|\cdot\| : X \rightarrow [0, \infty)$ is continuous.*

Proof. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Let $(x_1, y_1), (x_2, y_2) \in X \times X$. Suppose that $\|(x_1, y_1) - (x_2, y_2)\| = \max\{\|x_1 - x_2\|, \|y_1 - y_2\|\} < \delta$. Then

$$\begin{aligned}
 \|(x_1 + y_1) - (x_2 + y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\
 &\leq \|x_1 - x_2\| + \|y_1 - y_2\| \\
 &< 2\delta \\
 &= \epsilon
 \end{aligned}$$

Hence addition is uniformly continuous.

Let $(\lambda_1, x_1) \in \mathbb{C} \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(|\lambda_1| + \|x_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(\lambda_2, x_2) \in \mathbb{C} \times X$. Suppose that $\|(\lambda_1, x_1) - (\lambda_2, x_2)\| = \max\{|\lambda_1 - \lambda_2|, \|x_1 - x_2\|\} < \delta$. Then

$$\begin{aligned}
 \|\lambda_1 x_1 - \lambda_2 x_2\| &= \|\lambda_1 x_1 - \lambda_1 x_2 + \lambda_1 x_2 - \lambda_2 x_2\| \\
 &= \|\lambda_1(x_1 - x_2) + (\lambda_1 - \lambda_2)x_2\| \\
 &\leq |\lambda_1| \|x_1 - x_2\| + |\lambda_1 - \lambda_2| \|x_2\| \\
 &\leq |\lambda_1| \|x_1 - x_2\| + |\lambda_1 - \lambda_2| (\|x_1 - x_2\| + \|x_1\|) \\
 &< |\lambda_1| \delta + \delta(\delta + \|x_1\|) \\
 &= (|\lambda_1| + \|x_1\|) \delta + \delta^2 \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon
 \end{aligned}$$

Since $(\lambda_1, x_1) \in \mathbb{C} \times X$ is arbitrary, scalar multiplication is continuous.

Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x, y \in X$. Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned} \left| \|x\| - \|y\| \right| &\leq \|x - y\| \\ &< \delta \\ &= \epsilon \end{aligned}$$

So $\|\cdot\| : X \rightarrow [0, \infty)$ is uniformly continuous. \square

Exercise 6.14. Let X, Y be normed vector spaces. If Y is complete, then so is $L(X, Y)$.

Proof. Suppose that Y is complete. Let $(T_n)_{n \in \mathbb{N}} \subset L(X, Y)$. Suppose that $(T_n)_{n \in \mathbb{N}}$ is Cauchy. Since for each $m, n \in \mathbb{N}$, $|\|T_m\| - \|T_n\|| \leq \|T_m - T_n\|$, we have that $(\|T_n\|)_{n \in \mathbb{N}} \subset [0, \infty)$ is Cauchy. Hence $\lim_{n \rightarrow \infty} \|T_n\|$ exists.

Let $x \in X$ and $m, n \in \mathbb{N}$. Then

$$\begin{aligned} \|T_m x - T_n x\| &= \|(T_m - T_n)x\| \\ &\leq \|T_m - T_n\| \|x\| \end{aligned}$$

So $(T_n x)_{n \in \mathbb{N}} \subset Y$ is Cauchy and hence converges. Define $T : X \rightarrow Y$ by $Tx = \lim_{n \rightarrow \infty} T_n x$.

Since addition and scalar multiplication are continuous, T is linear. Let $x \in X$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then $\|Tx - T_n x\| < \epsilon$. Then for each $n \in \mathbb{N}$, if $n \geq N$ we have that

$$\begin{aligned} \|Tx\| &\leq \|Tx - T_n x\| + \|T_n x\| \\ &< \epsilon + \|T_n x\| \\ &\leq \epsilon + \|T_n\| \|x\| \end{aligned}$$

Thus $\|Tx\| \leq \epsilon + (\lim_{n \rightarrow \infty} \|T_n\|) \|x\|$. Since $\epsilon > 0$ is arbitrary, $\|Tx\| \leq (\lim_{n \rightarrow \infty} \|T_n\|) \|x\|$. Thus $T \in L(X, Y)$ and $\|T\| \leq \lim_{n \rightarrow \infty} \|T_n\|$.

Note that since addition, scalar multiplication and $\|\cdot\|$ are continuous, we have that for each $n \in \mathbb{N}$ and $x \in X$, $\|(T_n - T_m)x\|$ converges to $\|(T_n - T)x\|$ because

$$\begin{aligned} \lim_{m \rightarrow \infty} \|(T_n - T_m)x\| &= \lim_{m \rightarrow \infty} \|T_n x - T_m x\| \\ &= \|T_n x - \lim_{m \rightarrow \infty} T_m x\| \\ &= \|T_n x - Tx\| \\ &= \|(T_n - T)x\| \end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ if $n, m \geq N$, then $\|T_n - T_m\| < \epsilon$. Then for each $n \in \mathbb{N}$ if $n \geq N$, then for each $x \in X$,

$$\|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\| < \epsilon \|x\|$$

Combining this with the previous fact, we see that for each $n \in N$, if $n \geq N$, then for each $x \in X$,

$$\|(T_n - T)x\| \leq \epsilon \|x\|$$

In particular, for each $n \in \mathbb{N}$, if $n \geq N$, then

$$\|T_n - T\| = \sup_{\|x\|=1} \|(T_n - T)x\| \leq \epsilon$$

This implies that T_n converges to T in $L(X, Y)$. Since

$$|\|T_n\| - \|T\|| \leq \|T_n - T\|$$

It is clear that $\lim_{n \rightarrow \infty} \|T_n\| = \|T\|$ □

Definition 6.15. Let X be a normed vector space and $M \subset X$ a closed subspace. Define $\|\cdot\| : X/M \rightarrow [0, \infty)$ by

$$\|x + M\| := \inf_{y \in M} \|x + y\|$$

We call $\|\cdot\|$ the **subspace norm on X/M**

Exercise 6.16. Let X be a normed vector space and $M \subsetneq X$ a proper, closed subspace of X . Then

- (1) The previously defined subspace norm on X/M is well defined and is a norm.
- (2) For each $\epsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.
- (3) The projection map $\pi : X \rightarrow X/M$ defined by $\pi(x) = x + M$ is continuous and $\|\pi\| = 1$.
- (4) If X is complete, then X/M is complete.

Proof. (1) Let $x, y \in X$ and $\alpha \in \mathbb{C}$. Suppose that $x + M = y + M$. Then there exists $m \in M$ such that $x = y + m$. Since M is a subspace, the map $T : M \rightarrow M$ given by $Tx = x + m$ is a bijection. So

$$\inf_{z \in M} \|y + m + z\| = \inf_{z \in M} \|y + z\|$$

which implies that

$$\begin{aligned} \|x + M\| &= \inf_{z \in M} \|x + z\| \\ &= \inf_{z \in M} \|y + m + z\| \\ &= \inf_{z \in M} \|y + z\| \\ &= \|y + M\| \end{aligned}$$

So $\|\cdot\| : X/M \rightarrow [0, \infty)$ is well defined.

We observe that for each $z, w \in M$,

$$\|x + y + z\| \leq \|x + w\| + \|y + w + z\|$$

Taking infimums over M with respect to z in this inequality implies that for each $w \in M$,

$$\begin{aligned} \inf_{z \in M} \|x + y + z\| &\leq \inf_{z \in M} \left(\|x + w\| + \|y + w + z\| \right) \\ &= \|x + w\| + \inf_{z \in M} \|y + w + z\| \end{aligned}$$

Again we use the fact that for each $w \in M$,

$$\inf_{z \in M} \|y + w + z\| = \inf_{z \in M} \|y + z\|$$

This implies that for each $w \in M$,

$$\inf_{z \in M} \|x + y + z\| \leq \|x + w\| + \inf_{z \in M} \|y + z\|$$

Therefore, taking infimums over M with respect to w in this inequality yields

$$\begin{aligned} \|x + y + M\| &= \inf_{z \in M} \|x + y + z\| \\ &\leq \inf_{w \in M} \left(\|x + w\| + \inf_{z \in M} \|y + z\| \right) \\ &= \inf_{w \in M} \|x + w\| + \inf_{z \in M} \|y + z\| \\ &= \|x + M\| + \|y + M\| \end{aligned}$$

If $\alpha = 0$, then $\alpha x = 0$. Choosing $z = 0 \in M$ gives $\|\alpha x + M\| = 0 = |\alpha| \|x + M\|$. Suppose that $\alpha \neq 0$. Then the map $T : M \rightarrow M$ given by $Tx = \alpha^{-1}x$ is a bijection and thus $\inf_{z \in M} \|x + \alpha^{-1}z\| = \inf_{z \in M} \|x + z\|$. Hence we have that

$$\begin{aligned} \|\alpha x + M\| &= \inf_{z \in M} \|\alpha x + z\| \\ &= \inf_{z \in M} |\alpha| \|x + \alpha^{-1}z\| \\ &= |\alpha| \inf_{z \in M} \|x + \alpha^{-1}z\| \\ &= |\alpha| \inf_{z \in M} \|x + z\| \\ &= |\alpha| \|x + M\| \end{aligned}$$

Suppose that $\|x\| = 0$. Choose a sequence $(z_n)_{n \in \mathbb{N}} \subset M$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x - z_n\| &= \inf_{z \in M} \|x + z\| \\ &= 0 \end{aligned}$$

Then $\lim_{n \rightarrow \infty} z_n = x$. Since M is closed, $x \in M$. Hence $x + M = 0 + M$.

- (2) Since M is a proper subspace, there exists $v \in X$ such that $v \notin M$. Then $\|v + M\| \neq 0$. Let $\epsilon > 0$. Then $(1 - \epsilon)^{-1} \|v + M\| > \|v + M\|$. So there exists $z \in M$ such that

$$0 < \|v + M\| \leq \|v + z\| < (1 - \epsilon)^{-1} \|v + M\|$$

Choose $x = \|v + z\|^{-1}(v + z)$. Then $\|x\| = 1$ and

$$\begin{aligned}\|x + M\| &= \|v + z\|^{-1}\|v + z + M\| \\ &= \|v + z\|^{-1}\|v + M\| \\ &> 1 - \epsilon\end{aligned}$$

- (3) Let $x \in X$. Taking $z = 0$, we see that $\|\pi(x)\| = \|x + M\| \leq \|x + z\| = \|x\|$. So π is bounded and in particular,

$$\sup_{\|x\|=1} \|\pi(x)\| \leq 1$$

From (2) we see that

$$\sup_{\|x\|=1} \|\pi(x)\| \geq 1$$

Hence $\|\pi\| = 1$.

- (4) Suppose that X is complete. Let $(x_i + M)_{i \in \mathbb{N}} \subset X/M$. Suppose that $\sum_{i \in \mathbb{N}} \|x_i + M\| < \infty$. Let $\epsilon > 0$. Then for each $i \in \mathbb{N}$, there exists $z_i \in M$ such that $\|x_i + z_i\| < \|x_i + M\| + \epsilon 2^{-i}$. Define the sequence $(a_i)_{i \in \mathbb{N}} \subset X$ by $a_i = x_i + z_i$. Then we have

$$\begin{aligned}\sum_{i \in \mathbb{N}} \|a_i\| &= \sum_{i \in \mathbb{N}} \|x_i + z_i\| \\ &\leq \sum_{i \in \mathbb{N}} \left(\|x_i + M\| + \epsilon 2^{-i} \right) \\ &= \sum_{i \in \mathbb{N}} \|x_i + M\| + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\sum_{i \in \mathbb{N}} \|a_i\| \leq \sum_{i \in \mathbb{N}} \|x_i + M\| < \infty$$

Since X is complete, $\sum_{i=1}^{\infty} a_i$ converges in X . Define $(s_n)_{n \in \mathbb{N}} \subset X$ and $s \in X$ by $s_n = \sum_{i=1}^n a_i$ and $s = \sum_{i=1}^{\infty} a_i$. Since $\lim_{n \rightarrow \infty} s_n = s$, and $\pi : X \rightarrow X/M$ is continuous, it follows that $\lim_{n \rightarrow \infty} \pi(s_n) = \pi(s)$. Since

$$\begin{aligned}\pi(s_n) &= \sum_{i=1}^n a_i + M \\ &= \sum_{i=1}^n x_i + M\end{aligned}$$

We have that $\sum_{i=1}^{\infty} x_i + M$ converges which implies that X/M is complete. □

Exercise 6.17. Let X, Y be normed vector spaces. Define $\phi : L(X, Y) \times X \rightarrow Y$ by $\phi(T, x) = Tx$. Then ϕ is continuous.

Proof. Let $(T_1, x_1) \in L(X, Y) \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|x_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(t_2, x_2) \in L(X, Y) \times X$. Suppose that

$$\|(T_1, x_1) - (T_2, x_2)\| = \max\{\|T_1 - T_2\|, \|x_1 - x_2\|\} < \delta$$

. Then

$$\begin{aligned} \|\phi(T_1, x_1) - \phi(T_2, x_2)\| &= \|T_1x_1 - T_2x_2\| \\ &= \|T_1x_1 - T_2x_1 + T_2x_1 - T_2x_2\| \\ &\leq \|(T_1 - T_2)x_1\| + \|T_2(x_1 - x_2)\| \\ &\leq \|T_1 - T_2\|\|x_1\| + \|T_2\|\|x_1 - x_2\| \\ &\leq \|T_1 - T_2\|\|x_1\| + (\|T_1 - T_2\| + \|T_1\|)\|x_1 - x_2\| \\ &< \delta\|x_1\| + (\delta + \|T_1\|)\delta \\ &= \delta(\|T_1\| + \|x_1\|) + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So ϕ is continuous. □

Exercise 6.18. Let X be a normed vector space and $M \subset X$ a subspace. Then \overline{M} is a subspace.

Proof. Let $x, y \in \overline{M}$ and $\alpha \in \mathbb{C}$. Then there exist sequences $(x_n)_{n \in \mathbb{N}} \subset M$ and $(y_n)_{n \in \mathbb{N}} \subset M$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$. Since M is a subspace, $(x_n + y_n)_{n \in \mathbb{N}} \subset M$ and $(\alpha x_n)_{n \in \mathbb{N}} \subset M$. Since addition and scalar multiplication are continuous, we have that $x_n + y_n \rightarrow x + y$ and $\alpha x_n \rightarrow \alpha x$. Thus $x + y \in \overline{M}$ and $\alpha x \in \overline{M}$ and hence \overline{M} is a subspace. □

Exercise 6.19. Let X, Y, Z be normed vector spaces, $T \in L(X, Y)$ and $S \in L(Y, Z)$. Define $ST : X \rightarrow Z$ by $STx = S(Tx)$. Then $ST \in L(X, Z)$ and $\|ST\| \leq \|S\|\|T\|$.

Proof. Clearly ST is linear. Let $x \in X$. Then

$$\begin{aligned} \|STx\| &= \|S(Tx)\| \\ &\leq \|S\|\|Tx\| \\ &\leq \|S\|\|T\|\|x\| \end{aligned}$$

So $\|ST\| \leq \|S\|\|T\|$. □

Definition 6.20. Let X be a Banach space and an associative algebra. Then X is said to be a Banach algebra if for each $S, T \in X$, $\|ST\| \leq \|S\|\|T\|$. If there exists $I \in X$ such that $I \neq 0$ and for each $T \in X$, $IT = TI = T$, then X is said to be **unital** with identity I . An element $T \in X$ is said to be **invertible** if there exists $S \in X$ such that $TS = ST = I$.

Exercise 6.21. Let X be a unital Banach algebra. Then $\|I\| \leq 1$.

Proof. Since $I \neq 0$, $\|I\| \neq 0$. By definition,

$$\|I\| = \|II\| \leq \|I\|\|I\|$$

Hence $1 \leq \|I\|$. □

Note 6.22. *If X is a Banach space, then a previous exercise implies that $L(X, X)$ equipped with composition is a unital Banach algebra where I is the identity operator. It is easy to see that $\|I\| = 1$.*

Note 6.23. *Let X be a Banach algebra. Then the set of invertible elements in X is a group.*

Exercise 6.24. *Let X be a Banach algebra. Then multiplication is continuous.*

Proof. Let $(S_1, T_1) \in X \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|S_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(S_2, T_2) \in X \times X$. Suppose that

$$\|(S_1, T_1) - (S_2, T_2)\| = \max\{\|S_1 - S_2\|, \|T_1 - T_2\|\} < \delta$$

. Then

$$\begin{aligned} \|S_1T_1 - S_2T_2\| &= \|S_1T_1 - S_2T_1 + S_2T_1 - S_2T_2\| \\ &\leq \|S_1 - S_2\|\|T_1\| + \|S_2\|\|T_1 - T_2\| \\ &\leq \|S_1 - S_2\|\|T_1\| + (\|S_1 - S_2\| + \|S_1\|)\|T_1 - T_2\| \\ &\leq \delta\|T_1\| + (\delta + \|S_1\|)\delta \\ &= \delta(\|S_1\| + \|T_1\|) + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

Definition 6.25. *Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then T is said to be **invertible** or an **isomorphism** if T is a bijection and $T^{-1} \in L(Y, X)$.*

Definition 6.26. *Let X be a Banach space. Define $GL(X) := \{T \in L(X, X) : T \text{ is invertible}\}$.*

Exercise 6.27. *Let X be a Banach space. Then*

(1) *For each $T \in L(X, X)$, if $\|I - T\| < 1$, then T is invertible and*

$$T^{-1} = \sum_{n=0}^{\infty} (I - T)^n$$

(2) *For each $S, T \in L(X, X)$, if S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$, then T is invertible.*

(3) *$GL(X)$ is open.*

Proof. (1) Let $T \in L(X, X)$. Suppose that $\|I - T\| < 1$. Then

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n < \infty$$

. Since X is complete, so is $L(X, X)$ and thus $\sum_{n=0}^{\infty} (I - T)^n$ converges in $L(X, X)$.

Define $(S_k)_{k=0}^\infty \subset L(X, X)$ and $S \in L(X, X)$ by $S_k = \sum_{n=0}^k (I - T)^n$ and $S = \sum_{n=0}^\infty (I - T)^n$. Then for each $k \in \mathbb{N}$,

$$\begin{aligned} S_k T &= S_k - S_k(I - T) \\ &= (I - T)^0 - (I - T)^{k+1} \\ &= I - (I - T)^{k+1} \end{aligned}$$

and $\|S_k T - I\| \leq \|I - T\|^{k+1}$. Since multiplication on Banach algebras is continuous, we have that

$$ST = \left(\lim_{k \rightarrow \infty} S_k \right) T = \lim_{k \rightarrow \infty} S_k T = I$$

Similarly $TS = I$. Thus T is invertible and $T^{-1} = S \in L(X, X)$.

(2) Let $S, T \in L(X, X)$. Suppose that S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$. Then

$$\begin{aligned} \|I - S^{-1}T\| &= \|S^{-1}(S - T)\| \\ &\leq \|S^{-1}\| \|S - T\| \\ &< 1 \end{aligned}$$

So $S^{-1}T$ is invertible. Thus $T = S(S^{-1}T)$ is invertible.

(3) Let $T \in GL(X)$. Choose $\delta = \|T^{-1}\|^{-1}$. By (2), $B(T, \delta) \subset GL(X)$. □

Exercise 6.28. Let $M(X, \mathcal{A})$ denote the set of complex measures on the measurable space (X, \mathcal{A}) . Define $\|\cdot\| : M(X, \mathcal{A}) \rightarrow [0, \infty)$ by $\|\mu\| = |\mu|(X)$. Then $\|\cdot\|$ is a norm on $M(X, \mathcal{A})$.

Proof. Let $\mu, \nu \in M(X, \mathcal{A})$ and $\alpha \in \mathbb{C}$. Exercises in a previous section tell us that $|\mu + \nu| \leq |\mu| + |\nu|$ and $|\alpha\mu| = |\alpha||\mu|$. So clearly $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ and $\|c\mu\| = |c|\|\mu\|$. If $\|\mu\| = 0$, then X is μ -null and μ is the zero measure. □

6.2. Linear Functionals.

Definition 6.29. Let X be a normed vector space and $T : X \rightarrow \mathbb{C}$. Then T is said to be a **linear functional on X** if T is linear and T is said to be a **bounded linear functional on X** if $T \in L(X, \mathbb{C})$. We define the **dual space of X** , denoted X^* , by $X^* = L(X, \mathbb{C})$.

Definition 6.30. Let X be a normed vector space and $p : X \rightarrow \mathbb{R}$. Then p is said to be a **sublinear functional** if for each $x, y \in X$, $\lambda \geq 0$,

- (1) $p(x + y) \leq p(x) + p(y)$
- (2) $p(\lambda x) = \lambda p(x)$

Note 6.31. Let X be a vector space and $\|\cdot\| : X \rightarrow [0, \infty)$ be a seminorm, then $\|\cdot\|$ is a sublinear functional.

Theorem 6.32. *Hahn-Banach Theorem: Let X be a vector space, $p : X \rightarrow \mathbb{R}$ a sublinear functional, $M \subset X$ a subspace and $f : M \rightarrow \mathbb{C}$ a linear functional. If for each $x \in M$, $|f(x)| \leq p(x)$, then there exists a linear functional $F : X \rightarrow \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x)$ and $F|_M = f$.*

Exercise 6.33. *Let X be a normed vector space, $M \subset X$ a subspace and $f \in M^*$. Then there exists $F \in X^*$ such that $\|F\| = \|f\|$ and $F|_M = f$.*

Proof. If $f = 0$, Choose $F = 0$. Suppose $f \neq 0$. Then $\|f\| \neq 0$ and there exists $x_0 \in M$ such that $x_0 \neq 0$. Thus $\|f\| = \sup\{|f(x)| : x \in M \text{ and } \|x\| = 1\}$. Define $p : X \rightarrow [0, \infty)$ by $p(x) = \|f\|\|x\|$. Then p is a sublinear functional on X and for each $x \in M$, $|f(x)| \leq p(x)$. So there exists a linear functional $F : X \rightarrow \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x) = \|f\|\|x\|$ and $F|_M = f$. Thus $F \in X^*$ with $\|F\| \leq \|f\|$. Also

$$\|F\| = \sup_{\substack{x \in X \\ \|x\|=1}} |F(x)| \geq \sup_{\substack{x \in M \\ \|x\|=1}} |F(x)| = \sup_{\substack{x \in M \\ \|x\|=1}} |f(x)| = \|f\|$$

So $\|F\| = \|f\|$. □

Exercise 6.34. *Let X be a normed vector space, $M \subsetneq X$ a proper closed subspace and $x \in X \setminus M$. Then there exists $F \in X^*$ such that $F|_M = 0$, $\|F\| = 1$ and $F(x) = \|x + M\| \neq 0$. (Hint: Consider $f : M + \mathbb{C}x \rightarrow \mathbb{C}$ defined by $f(m + \lambda x) = \lambda\|x + M\|$.)*

Proof. Define $f : M + \mathbb{C}x \rightarrow \mathbb{C}$ as above. Clearly f is linear and $f|_M = 0$. Let $m \in M$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $|f(m + \lambda x)| = 0 \leq \|m + \lambda x\|$. Suppose that $\lambda \neq 0$. Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \inf_{z \in M} \|z + \lambda x\| \\ &\leq \|m + \lambda x\| \end{aligned}$$

So $f \in (M + \mathbb{C}x)^*$ and $\|f\| \leq 1$. Let $\epsilon > 0$. A previous exercise tells us that there exist $m \in M, \lambda \in \mathbb{C}$ such that $\|m + \lambda x\| = 1$ and $\|m + \lambda x + M\| > 1 - \epsilon$. Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \|m + \lambda x + M\| \\ &> 1 - \epsilon \end{aligned}$$

So

$$\|f\| = \sup_{\substack{z \in M + \mathbb{C}x \\ \|z\|=1}} |f(z)| \geq 1$$

Hence $\|f\| = 1$. The same exercise also tells us that $f(x) = \|x + M\| \neq 0$. Using the previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{M + \mathbb{C}x} = f$. □

Exercise 6.35. *Let X be a normed vector space and $x \in X$. If $x \neq 0$, then there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$.*

Proof. Define $f : \mathbb{C}x \rightarrow \mathbb{C}$ by $f(\lambda x) = \lambda\|x\|$. Then f is linear and $f(x) = \|x\|$. Clearly

$$\sup_{\substack{z \in \mathbb{C}x \\ \|z\|=1}} |f(z)| = 1$$

So $f \in (\mathbb{C}x)^*$ and $\|f\| = 1$. By a previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{\mathbb{C}x} = f$. \square

Exercise 6.36. Let X be a normed vector space. Then X^* separates the points of X .

Proof. Let $x, y \in X$. Suppose that $x \neq y$. Then $x - y \neq 0$. The previous exercise implies that there exists $F \in X^*$ such that $\|F\| = 1$ and

$$F(x) - F(y) = F(x - y) = \|x - y\| \neq 0$$

Thus $F(x) \neq F(y)$ and X^* separates the points of X . \square

Definition 6.37. Let X, Y be metric spaces and $T : X \rightarrow Y$. Then T is said to be an **isometry** if for each $x_1, x_2 \in X$, $d(Tx_1, Tx_2) = d(x_1, x_2)$.

Exercise 6.38. Let X, Y be metric spaces and $T : X \rightarrow Y$ an isometry. Then T is injective.

Proof. Let $x_1, x_2 \in X$. Suppose that $Tx_1 = Tx_2$. Then $0 = d(Tx_1, Tx_2) = d(x_1, x_2)$. So $x_1 = x_2$. Hence T is injective. \square

Note 6.39. Let X, Y be metric spaces and $T : X \rightarrow Y$ an isometry. Then T is clearly continuous. If T is surjective, then T^{-1} is an isometry and therefore continuous. Hence T is a homeomorphism.

Exercise 6.40. Let X be a normed vector space and $x \in X$. Define $\hat{x} : X^* \rightarrow \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| = \|x\|$.

Proof. Let $f, g \in X^*$ and $\lambda \in \mathbb{C}$. Then

$$\hat{x}(f + \lambda g) = (f + \lambda g)(x) = f(x) + \lambda g(x) = \hat{x}(f) + \lambda \hat{x}(g)$$

So \hat{x} is linear. For each $f \in X^*$,

$$|\hat{x}(f)| = |f(x)| \leq \|x\| \|f\|$$

Hence $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leq \|x\|$. If $x = 0$, then $\hat{x} = 0$ and $\|\hat{x}\| = \|x\|$. Suppose that $x \neq 0$. Then a previous exercise implies that there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$. Then we have that

$$\sup_{\substack{f \in X^* \\ \|f\|=1}} |\hat{x}(f)| = \sup_{\substack{f \in X^* \\ \|f\|=1}} |f(x)| \geq |F(x)| = \|x\|$$

Hence $\|\hat{x}\| = \|x\|$. \square

Exercise 6.41. Let X be a normed vector space. Define $\phi : X \rightarrow X^{**}$ by $\phi(x) = \hat{x}$. Then ϕ is a linear isometry.

Proof. Let $x, y \in X$ and $\lambda \in \mathbb{C}$. Then for each $f \in X^*$, we have that

$$\begin{aligned}\phi(x + \lambda y)(f) &= \widehat{x + \lambda y}(f) \\ &= f(x + \lambda y) \\ &= f(x) + \lambda f(y) \\ &= \hat{x}(f) + \lambda \hat{y}(f) \\ &= \phi(x)(f) + \lambda \phi(y)(f)\end{aligned}$$

So $\phi(x + \lambda y) = \phi(x) + \lambda \phi(y)$ and ϕ is linear. The previous exercise tells us that

$$\begin{aligned}\|\phi(x) - \phi(y)\| &= \|\phi(x - y)\| \\ &= \|\widehat{x - y}\| = \|x - y\|\end{aligned}$$

So ϕ is an isometry. □

Definition 6.42. Let X be a normed vector space and define $\phi : X \rightarrow X^{**}$ as above. We define $\hat{X} = \phi(X) \subset X^{**}$. Since \hat{X} and X are isomorphic, we may identify X as a subset of X^{**} .

Definition 6.43. Let X be a normed vector space and define $\phi : X \rightarrow X^{**}$ as above. Then X is said to be reflexive if ϕ is surjective. In this case ϕ is then an isomorphism

Exercise 6.44. Let X be a normed vector space and $f : X \rightarrow \mathbb{C}$ a linear functional on X . Then f is bounded iff $\ker f$ is closed.

Proof. Suppose that f is continuous. Since $\{0\}$ is closed, we have that $\ker f = f^{-1}(\{0\})$ is closed. Conversely, suppose that $\ker f$ is closed. If $\ker f = X$, then $f = 0$ and f is continuous. Suppose that $\ker f \neq X$. Then $\ker f$ is a proper, closed subspace of X . A previous exercise tells us that there exists $x \in X$ such that $\|x\| = 1$ and $\|x + \ker f\| > \frac{1}{2}$. Let $y \in X$. Suppose that $\|y\| < \frac{1}{2}$. Then for each $z \in \ker f$,

$$\begin{aligned}\|z - (x + y)\| &= \|(z - x) - y\| \\ &\geq \|z - x\| - \|y\| \\ &> \frac{1}{2} - \frac{1}{2} \\ &= 0\end{aligned}$$

So $x + y \notin \ker f$. Therefore $f(B(x, \frac{1}{2})) \cap \{0\} = \emptyset$. If $f(B(x, \frac{1}{2}))$ is unbounded, then $f(B(x, \frac{1}{2})) = \mathbb{C}$ by linearity. This is a contradiction since $0 \notin f(B(x, \frac{1}{2}))$. So There exists $s > 0$ such that $f(B(x, \frac{1}{2})) \subset B(0, s)$ and thus f is bounded. □

Exercise 6.45. Let X be a normed vector space.

- (1) Let $M \subsetneq X$ be a proper closed subspace of X and $x \in X \setminus M$. Then $M + \mathbb{C}x$ is closed.
- (2) Let $M \subset X$ be a finite dimensional subspace of X . Then M is closed.

Proof. (1) Let $y \in X$ and $(y_n)_{n \in \mathbb{N}} \subset M + \mathbb{C}x$. Suppose that $y_n \rightarrow y$. If $y \in M$, then $y \in M + \mathbb{C}x$. Suppose that $y \notin M$. For each $n \in \mathbb{N}$, there exists $m_n \in M$ and $\lambda_n \in \mathbb{C}$ such that $y_n = m_n + \lambda_n x$. A previous exercise tells us that there exists $F \in X^*$ such that $\|F\| = 1$, $F|_M = 0$ and $F(x) = \|x + M\| \neq 0$. Since F is continuous, $F(y_n) \rightarrow F(y)$. Since for each $n \in \mathbb{N}$,

$$F(y_n) = F(m_n + \lambda_n x) = F(m_n) + \lambda_n(F(x)) = \lambda_n F(x)$$

we have that $\lambda_n F(x) \rightarrow F(y)$. Since $F(x) \neq 0$, this implies that $\lambda_n \rightarrow F(x)^{-1}F(y)$. It follows that $\lambda_n x \rightarrow F(x)^{-1}F(y)x$. Since for each $n \in \mathbb{N}$, $m_n = y_n - \lambda_n x$, we know that $m_n \rightarrow y - F(x)^{-1}F(y)x$. Since $(m_n)_{n \in \mathbb{N}} \subset M$ and M is closed, we have that $y - F(x)^{-1}F(y)x \in M$ and therefore $y \in M + \mathbb{C}x$. Hence $M + \mathbb{C}x$ is closed.

- (2) If $M = X$, then M is closed. Suppose that $M \neq X$. Let $(x_i)_{i=1}^n$ be a basis for M . Define $N_0 = \{0\}$ and for each $i = 1, 2, \dots, n$, define $N_i = N_{i-1} + \mathbb{C}x_i$. Since N_0 is a proper closed subspace of X and $x_1 \in X \setminus N_0$, (1) implies that N_1 is closed. Proceed inductively to obtain that $M = N_n$ is closed. □

Exercise 6.46. Let X be an infinite-dimensional normed vector space.

- (1) There exists a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that for each $m, n \in \mathbb{N}$, $\|x_n\| = 1$ and if $m \neq n$, then $\|x_m - x_n\| > \frac{1}{2}$.
 (2) X is not locally compact.

Proof. (1) Define $N_0 = \{0\}$. Then N_0 is a closed proper subspace of X . Choose $x_1 \in X$ such that $\|x_1\| = 1$. Using the results of previous exercises, we proceed inductively. For each $n \geq 2$ we define $N_{n-1} = \text{span}(x_1, x_2, \dots, x_{n-1})$. Then N_{n-1} is a closed proper subspace of X . Thus we may choose $x_n \in X$ such that $\|x_n\| = 1$ and $\|x_n + N_{n-1}\| > \frac{1}{2}$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then $x_m \in N_{n-1}$. Thus $\|x_n - x_m\| \geq \|x_n + N_{n-1}\| > \frac{1}{2}$.

- (2) Suppose that X is locally compact. Then $\overline{B(0, 1)}$ is compact and therefore sequentially compact. Using $(x_n)_{n \in \mathbb{N}} \subset \overline{B(0, 1)}$ defined in (1), we see that there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$, $x \in \overline{B(0, 1)}$ such that $x_{n_k} \rightarrow x$. Then $(x_{n_k})_{k \in \mathbb{N}}$ is Cauchy. So there exists $N \in \mathbb{N}$ such that for each $j, k \in \mathbb{N}$, if $j, k \geq N$, then $\|x_{n_j} - x_{n_k}\| < \frac{1}{2}$. Then $\|x_{n_N} - x_{n_{N+1}}\| < \frac{1}{2}$. This is a contradiction since by construction, $\|x_{n_N} - x_{n_{N+1}}\| > \frac{1}{2}$. Thus X is not locally compact. □

Exercise 6.47. Let X, Y be normed vector spaces and $T \in L(X, Y)$.

- (1) Define the **adjoint of T** , $T^* : Y^* \rightarrow X^*$ by $T^*(f) = f \circ T$. Then $T^* \in L(Y^*, X^*)$.
 (2) Applying the result from (1) twice, we have that $T^{**} \in L(X^{**}, Y^{**})$. We have that for each $x \in X$, $T^{**}(\hat{x}) = \widehat{T(x)}$.
 (3) T^* is injective iff $T(X)$ is dense in Y .
 (4) If $T^*(Y^*)$ is dense in X^* , then T is injective. The converse is true if X is reflexive.

Proof. (1) Let $f \in Y^*$. Then $\|T^*(f)\| = \|f \circ T\| \leq \|T\|\|f\|$. So $T^* \in L(Y^*, X^*)$ with $\|T^*\| \leq \|T\|$.

(2) Let $x \in X$. Let $f \in Y^*$. Then

$$\begin{aligned}
 T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\
 &= \hat{x}(T^*(f)) \\
 &= \hat{x}(f \circ T) \\
 &= f \circ T(x) \\
 &= f(T(x)) \\
 &= \widehat{T(x)}(f)
 \end{aligned}$$

Hence $T^{**}(\hat{x}) = \widehat{T(x)}$.

(3) Suppose that $T(X)$ is not dense in Y . Then $\overline{T(X)} \neq Y$. So $T(X)$ is a proper closed subspace of Y and there exists $y \in Y$ such that $y \notin \overline{T(X)}$. By a previous exercise, there exists $f \in Y^*$ such that $f(y) = \|y + \overline{T(X)}\| \neq 0$, $\|f\| = 1$ and $f|_{\overline{T(X)}} = 0$. Let $x \in X$. Then $T^*(f)(x) = f \circ T(x) = 0$. Hence $T^*(f) = 0 = T^*(0)$. Since $f \neq 0$, T^* is not injective.

Now suppose that $T(X)$ is dense in Y . Let $f, g \in Y^*$. Define $h \in Y^*$ by $h = f - g$. Suppose that $T^*(f) = T^*(g)$. Then $T^*(h) = 0$. So for each $x \in X$, $h(T(x)) = 0$. Let $y \in Y$ and $\epsilon > 0$. By continuity, there exists $\delta > 0$ such that for each $y' \in Y$, if $\|y - y'\| < \delta$, then $\|h(y) - h(y')\| < \epsilon$. Since $T(X)$ is dense in Y , there exists $x \in X$ such that $\|y - T(x)\| < \delta$. Thus

$$\begin{aligned}
 \|h(y)\| &\leq \|h(y) - h(T(x))\| + \|h(T(x))\| \\
 &= \|h(y) - h(T(x))\| \\
 &< \epsilon
 \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|h(y)\| = 0$. This implies that $h(y) = 0$ and therefore $f(y) = g(y)$. Since $y \in Y$ is arbitrary, $f = g$ and T^* is injective.

(4) For the sake of contradiction, suppose that $T^*(Y^*)$ is dense in X^* and T is not injective. Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $T(x_1) = T(x_2)$. Define $x = x_1 - x_2$. Then $x \neq 0$ and $T(x) = 0$. A previous exercise implies that there exists $F \in X^*$ such that $F(x) = \|x\| \neq 0$ and $\|F\| = 1$. Let $\epsilon > 0$. Choose $g \in Y^*$ such that $\|F - T^*(g)\| < \epsilon$. Then

$$\begin{aligned}
 \|x\| &= |F(x)| \\
 &\leq |F(x) - T^*(g)(x)| + |T^*(g)(x)| \\
 &< \epsilon\|x\| + |g(T(x))| \\
 &= \epsilon\|x\|
 \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\|x\| = 0$ which is a contradiction. Hence if $T^*(Y^*)$ is dense in X^* , then T is injective.

Now, suppose that X is reflexive and T is injective. Let $\phi_1, \phi_2 \in X^{**}$. Suppose

that $T^{**}(\phi_1) = T^{**}(\phi_2)$. Then $T^{**}(\phi_1 - \phi_2) = 0$. Since X is reflexive, there exist $x_1, x_2 \in X$ such that $\phi_1 = \hat{x}_1$ and $\phi_2 = \hat{x}_2$. Define $x = x_1 - x_2$. Then $T^{**}(\hat{x}) = 0$. So for each $f \in Y^*$,

$$\begin{aligned} T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= 0 \end{aligned}$$

Suppose that $T(x) \neq 0$. Then a previous exercise implies that there exists $g \in Y^*$ such that $g(T(x)) = \|T(x)\| \neq 0$ and $\|g\| = 1$. This is a contradiction since $g(T(x)) = 0$. So $T(x) = 0$. Since T is injective, this implies that $x = 0$. Hence $\hat{x} = 0$ and thus $\phi_1 = \phi_2$. Thus T^{**} is injective. By (3), we have that $T^*(Y^*)$ is dense in X^* . \square

Exercise 6.48. Let X be a normed vector space. Then X is reflexive iff X^* is reflexive.

Proof. Suppose that X is reflexive. Let $\alpha \in X^{***}$. Define $f : X \rightarrow \mathbb{C}$ by $f(x) = \alpha(\hat{x})$. Clearly f is linear and a previous exercise tells us that for each $x \in X$,

$$\begin{aligned} |f(x)| &\leq \|\alpha\| \|\hat{x}\| \\ &= \|\alpha\| \|x\| \end{aligned}$$

So $f \in X^*$. Let $\phi \in X^{**}$. Since X is reflexive, there exists $x \in X$ such that $\phi = \hat{x}$. Then

$$\begin{aligned} \alpha(\phi) &= \alpha(\hat{x}) \\ &= f(x) \\ &= \hat{x}(f) \\ &= \hat{f}(\hat{x}) \\ &= \hat{f}(\phi) \end{aligned}$$

Hence $\alpha = \hat{f}$. Thus the map $X^* \rightarrow X^{***}$ given by $f \mapsto \hat{f}$ is surjective and so X^* is reflexive.

Conversely, suppose that X^* is reflexive. Since $\phi : X \rightarrow X^{**}$ given by $\phi(x) = \hat{x}$ is an isometry, $\widehat{X} \subset X^{**}$ is closed. For the sake of contradiction, suppose that $\widehat{X} \neq X^{**}$. Then there exists $\alpha \in X^{**}$ such that $\alpha \notin \widehat{X}$. Thus there exists $F \in X^{***}$ such that $\|F\| = 1$, $F(\alpha) = \|\alpha + \widehat{X}\| \neq 0$ and $F|_{\widehat{X}} = 0$. Since X^* is reflexive, there exists $f \in X^*$ such that $F = \hat{f}$. A previous exercise tells us that $\|f\| = \|\hat{f}\| = \|F\| = 1$. Since for each $x \in X$, $f(x) = \hat{x}(f) = \hat{f}(\hat{x}) = F(\hat{x}) = 0$, we have that $f = 0$. Thus $\|f\| = 0$, a contradiction. So $\widehat{X} = X^{**}$ and X is reflexive. \square

6.3. The Baire Category Theorem and Consequences.

Theorem 6.49. Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is surjective, then T is open.

Corollary 6.50. Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is a bijection, then $T^{-1} \in L(X, Y)$.

Definition 6.51. Let X, Y be sets and $f : X \rightarrow Y$. We define the **graph of f** , $\Gamma(f)$, by $\Gamma(f) = \{(x, y) \in X \times Y : f(x) = y\}$.

Theorem 6.52. Let X, Y be Banach spaces and $T : X \rightarrow Y$ a linear map. If $\Gamma(T)$ is closed, then $T \in L(X, Y)$.

Note 6.53. We recall that $\Gamma(T)$ is closed iff for each $(x_n)_{n \in \mathbb{N}} \subset X$, $x \in X$ and $y \in Y$ if $x_n \rightarrow x$ and $T(x_n) \rightarrow y$, then $T(x) = y$.

Theorem 6.54. Let X, Y be Banach spaces and $S \subset L(X, Y)$. If for each $x \in X$,

$$\sup_{T \in S} \|Tx\| < \infty$$

then

$$\sup_{T \in S} \|T\| < \infty$$

Exercise 6.55. Let μ be counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Define $h : \mathbb{N} \rightarrow \mathbb{N}$ and ν on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ by $h(n) = n$ and $d\nu = h d\mu$. Define $X = L^1(\nu)$ and $Y = L^1(\mu)$. Equip both X and Y with the L^1 norm with respect to μ .

- (1) We have that X is a proper subspace of Y and therefore X is not complete.
- (2) Define $T : X \rightarrow Y$ by $Tf(n) = nf(n)$. Then T is linear, $\Gamma(T)$ is closed, and T is unbounded.
- (3) Define $S : Y \rightarrow X$ by $Sg(n) = \frac{1}{n}g(n)$. Then $S \in L(Y, X)$, S is surjective and S is not open.

Proof. (1) Note that for each $f : \mathbb{N} \rightarrow \mathbb{C}$,

$$\begin{aligned} \|f\|_{\mu,1} &= \sum_{n=1}^{\infty} |f(n)| \\ &\leq \sum_{n=1}^{\infty} n|f(n)| \\ &= \|f\|_{\nu,1} \end{aligned}$$

Hence X is a subspace of Y . Define $f : \mathbb{N} \rightarrow \mathbb{C}$ by $f(n) = \frac{1}{n^2}$. Then

$$\|f\|_{\mu,1} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

So $f \in Y$. However

$$\|f\|_{\nu,1} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

So $f \notin X$. Thus X is a proper subspace of Y . Let $g \in Y$ and $\epsilon > 0$. Since the simple functions are dense in $L^1(\mu)$, there exists $\phi \in L^1(\mu)$ such that ϕ is simple and

$\|g - \phi\|_{\mu,1} < \epsilon$. Then there exist $(c_i)_{i=1}^k \subset \mathbb{C}$ and $(E_i)_{i=1}^k \subset \mathcal{P}(\mathbb{N})$ such that for each $i = 1, 2, \dots, k$, E_i is finite and

$$\phi = \sum_{i=1}^k c_i \chi_{E_i}$$

Define $c = \max\{|c_i| : i = 1, 2, \dots, k\}$ and $m = \max \bigcup_{i=1}^k E_i$. Then

$$\begin{aligned} \|\phi\|_{\nu,1} &= \sum_{n=1}^m n |\phi(n)| \\ &\leq \sum_{n=1}^m mc \\ &= cm^2 \\ &< \infty \end{aligned}$$

Hence $\phi \in X$ and X is dense in Y . Since X is a dense, proper subspace, it is not closed. Since Y is complete and $X \subset Y$ is not closed, we have that X is not complete.

- (2) Clearly T is linear. Let $(f_j)_{j \in \mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_j \xrightarrow{L^1(\mu)} f$ and $Tf_j \xrightarrow{L^1(\mu)} g$.

Note that for each $j \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$|f_j(n) - f(n)| \leq \sum_{n=1}^{\infty} |f_j(n) - f(n)| = \|f_j - f\|_{\mu,1}$$

and

$$|nf_j(n) - g(n)| \leq \sum_{n=1}^{\infty} |nf_j(n) - g(n)| = \|Tf_j - g\|_{\mu,1}$$

Thus for each $n \in \mathbb{N}$, $f_j(n) \xrightarrow{j} f(n)$ and $nf_j(n) \xrightarrow{j} g(n)$. This implies that for each $n \in \mathbb{N}$, $nf(n) = g(n)$. Thus $Tf = g$ which implies that $\Gamma(T)$ is closed. Suppose, for the sake of contradiction, that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\|_{\mu,1} \leq C\|f\|_{\mu,1}$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f : \mathbb{N} \rightarrow \mathbb{C}$ by $f = \chi_{\{n\}}$. As established above, $S^+ \subset L^1(\mu)$. Then $\|f\|_{\mu,1} = 1$ and

$$\begin{aligned} \|Tf\|_{\mu,1} &= n \\ &> C \\ &= C\|f\|_{\mu,1} \end{aligned}$$

which is a contradiction. So T is unbounded.

(3) Clearly S is linear. Let $g \in Y$. Then

$$\begin{aligned}\|Sg\|_{\mu,1} &= \sum_{n=1}^{\infty} \frac{1}{n} |g(n)| \\ &\leq \sum_{n=1}^{\infty} |g(n)| \\ &= \|g\|_{\mu,1}\end{aligned}$$

So S is bounded and $\|S\| \leq 1$. Thus $S \in L(Y, X)$. Let $f \in X$. Define $g : \mathbb{N} \rightarrow \mathbb{C}$ by $g(n) = nf(n)$. By definition, $g \in Y$ and we have that

$$\begin{aligned}Sg(n) &= \frac{1}{n} g(n) \\ &= f(n)\end{aligned}$$

Hence $Sg = f$ and thus S is surjective.

□

7. APPENDIX

7.1. Summation.

Definition 7.1. Let $f : X \rightarrow [0, \infty)$, Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \rightarrow \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \rightarrow \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^- . In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \rightarrow \mathbb{C}$.

Note 7.2. Let $f : X \rightarrow \mathbb{C}$ and $\alpha : X \rightarrow X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.