CALCULUS ON MANIFOLDS NOTES

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2.1. Submanifolds of $\mathbb{R}^n$.

Definition 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ and $U \subset M$. Then U is said to be **open** if there exists $U' \subset \mathbb{R}^n$ such that U' is open in $\mathbb{R}^n$ and $U = M \cap U'$.*

Definition 2.1.2. *Define the **upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}_n$, by*

$$\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\partial\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$$

Definition 2.1.3. *Let $\Omega \subset \mathbb{R}^n, U \subset \Omega, V \subset \mathbb{H}_k$ and $\phi : U \to V$. Then ϕ is said to be a **coordinate chart** from Ω to $\mathbb{H}_k$ if U is open in Ω, V is open in $\mathbb{H}_k$ and ϕ is a homeomorphism (that is, ϕ is a bijection, continuous and ϕ^{-1} is continuous). We will typically denote a chart from Ω to $\mathbb{H}_k$ by the pair (ϕ, U). Let $\mathcal{A} = \{(\phi_\alpha, U_\alpha) : \alpha \in A\}$ be a set of coordinate charts from Ω to $\mathbb{H}_k$ indexed by A. Then $\mathcal{A}$ is said to be a **smooth k-atlas** on Ω if*

(1) $\Omega = \bigcup\limits_{\alpha \in A} U_\alpha$

(2) *for each $\alpha, \beta \in A$, $U_\alpha \cap U_\beta \neq \varnothing$ implies that*

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_\alpha \cap U_\beta) \to \phi_2(U_\alpha \cap U_\beta)$$

is smooth

*If $\mathcal{A}$ is a smooth k-atlas on Ω, and $(\phi, U) \in \mathcal{A}$, then ϕ is said to be a **smooth coordinate chart** from Ω to $\mathbb{H}_k$.*

Definition 2.1.4. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A}$ a smooth k-atlas on Ω. Then $(\Omega, \mathcal{A})$ is said to be a k-dimensional smooth submanifold of $\mathbb{R}^n$. Define the **boundary** of Ω, denoted $\partial\Omega$, by*

$$\partial\Omega = \bigcup_{\substack{\phi \in \mathcal{A} \\ \phi : U \to V}} \phi^{-1}(V \cap \partial\mathbb{H}_k)$$

Exercise 2.1.5. *Let Ω be a k-dimensional smooth submanifold of $\mathbb{R}^n$. Then $\partial\Omega$ is a $k-1$-dimensional smooth manifold of $\mathbb{R}^n$.*

Definition 2.1.6. *Let Ω be a smooth k-dimensional smooth submanifold of $\mathbb{R}^n$, $U \subset \Omega$ open in Ω, $V \subset \mathbb{H}_k$ open in $\mathbb{H}_k$ and $\phi : U \to V$ a smooth coordinate chart on Ω. Then $\phi^{-1} : V \to U$ is called a **smooth parametrization** of Ω.*

2.2. Differential Forms.

Note 2.2.1. *The definitions in this section will introduce a very slick book-keeping device for doing calculus on manifolds. Since we are not developing the theory from the ground up, it may feel abstract. Hopefully the many exercises facilitate becoming accostumed to this book-keeping tool.*

Definition 2.2.2. *Define $\mathcal{I}_{k,n} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_n \le n\}$. Let $I \in \mathcal{I}_{k,n}$. Then I is called a **multi-index**. Recall that $\#\mathcal{I}_{k,n} = \binom{n}{k}$.*

Definition 2.2.3. *When working in $\mathbb{R}^n$, we introduce the formal objects $dx_1, dx_2, \cdots, dx_n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.*

Definition 2.2.4. *Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be*

$$\Gamma^k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \operatorname{span}\{dx_I : I \in \mathcal{I}_{k,n}\} & k \ge 1 \end{cases}$$

*For each $\omega \in \Gamma^k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Gamma^k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:*

(1) *the exterior product is bilinear*
(2) *for each $\omega \in \Gamma^k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$*
(3) *for each $\omega \in \Gamma^k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$*
(4) *for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Gamma^k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$*

We call $\Gamma^k(\mathbb{R}^n)$ the differential k-forms on $\mathbb{R}^n$. Let ω be a k-form on $\mathbb{R}^n$. If $k \ge 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx_I$

Note 2.2.5. *The terms $dx_1, dx_2, \cdots, dx_n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du_1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx_1, dx_2, \cdots, dx_n\}$ when referencing the coordinates on $\mathbb{R}^n$.*

Exercise 2.2.6. *Let $B = (b_{i,j})$ be an $n \times n$ matrix. Then*

$$\bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx_j \right) = (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

Proof. First we have

$$(*) \qquad \bigwedge_{i=1}^{n}\left(\sum_{j=1}^{n}b_{i,j}dx_j\right) = \left(\sum_{j=1}^{n}b_{1,j}dx_j\right)\wedge\left(\sum_{j=1}^{n}b_{2,j}dx_j\right)\wedge\cdots\wedge\left(\sum_{j=1}^{n}b_{n,j}dx_j\right)$$

The expression on the right side of $(*)$ is just the sum all terms of the form

$$b_{1,j_1}b_{2,j_2}\cdots b_{n,j_n}dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}$$

where $j_k \in \{1, 2, \cdots, n\}$. The terms in which for some j, dx_j appears more than once in the exterior product are zero. Thus the expression on the right of $(*)$ is just the sum of all terms of the form

$$b_{1,\sigma(1)}b_{2,\sigma(2)}\cdots b_{n,\sigma(n)}dx_{\sigma(1)}\wedge dx_{\sigma(2)}\wedge\cdots\wedge dx_{\sigma(n)} = \operatorname{sgn}(\sigma)b_{1,\sigma(1)}b_{2,\sigma(2)}\cdots b_{n,\sigma(n)}dx_1\wedge dx_2\wedge\cdots\wedge dx_n$$

where $\sigma \in S_n$. Explicitely writing this out, we see that

$$\bigwedge_{i=1}^{n}\left(\sum_{j=1}^{n}b_{i,j}dx_j\right) = \left(\sum_{j=1}^{n}b_{1,j}dx_j\right)\wedge\left(\sum_{j=1}^{n}b_{2,j}dx_j\right)\wedge\cdots\wedge\left(\sum_{j=1}^{n}b_{n,j}dx_j\right)$$

$$= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)b_{1,\sigma(1)}b_{2,\sigma(2)}\cdots b_{n,\sigma(n)}dx_1\wedge dx_2\wedge\cdots\wedge dx_n$$

$$= \left(\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)b_{1,\sigma(1)}b_{2,\sigma(2)}\cdots b_{n,\sigma(n)}\right)dx_1\wedge dx_2\wedge\cdots\wedge dx_n$$

$$= (\det B)dx_1\wedge dx_2\wedge\cdots\wedge dx_n$$

$\square$

Definition 2.2.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted df, on $\mathbb{R}^n$ by*

$$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i$$

Let $\omega = \sum\limits_{I\in\mathcal{I}_{k,n}} f_I dx_I$ be a k-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega = \sum_{I\in\mathcal{I}_{k,n}}df_I\wedge dx_I$$

Exercise 2.2.8. *Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx_I \wedge dx_{I_*} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.*

Definition 2.2.9. *We define a linear map $* : \Gamma^k(\mathbb{R}^n) \to \Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge $*$-operator** by*

$$* \sum_{I\in\mathcal{I}_{k,n}} f_I dx_I = \sum_{I\in\mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 2.2.10. *Let $\phi : \mathbb{R}^k \to \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We define $\phi^* : \Gamma^k(\mathbb{R}^n) \to \Gamma^k(\mathbb{R}^k)$ via the following properties:*

    (1) *for each 0-form f on $\mathbb{R}^n$, $\phi^* f = f \circ \phi$*
    (2) *for $i = 1, \cdots, n$, $\phi^* dx_i = d\phi_i$*
    (3) *for an s-form ω, and a t-form χ on $\mathbb{R}^n$, $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$*

(4) *for l-forms ω, χ on $\mathbb{R}^n$, $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$*

Exercise 2.2.11. *Let $\Omega \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold of $\mathbb{R}^n$, $\phi : U \to V$ a smooth parametrization of Ω, $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$ an k-form on $\mathbb{R}^n$. Then*

$$\phi^*\omega = \left(\sum_{I \in \mathcal{I}_{k,n}} f_I \circ \phi \det(D\phi_I) \right) du_1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof.   □