

# REAL ANALYSIS NOTES

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## 1. MEASURE

### 1.1. Product Measures.

**Definition 1.1.** Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be measurable spaces. Put  $\mathcal{E} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$ . Then  $\mathcal{E}$  is an elementary family and thus  $\mathcal{M}_0 = \{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n \subset \mathcal{E} \text{ are disjoint}\}$  is an algebra on  $X \times Y$ . We define  $\pi_0 : \mathcal{M}_0 \rightarrow [0, \infty]$  by

$$\pi_0\left(\bigcup_{i=1}^n A_i \times B_i\right) = \sum_{i=1}^n \mu(A_i) \nu(B_i)$$

Since  $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{M}_0)$ , we define a product measure  $\mu \times \nu$  on  $(X \times Y, \mathcal{A} \otimes \mathcal{B})$  to be an extension of  $\pi_0$  to  $\mathcal{A} \otimes \mathcal{B}$ . The existence of which is guaranteed by Caratheodory's theorem and on  $\mathcal{A} \otimes \mathcal{B}$ ,

$$\begin{aligned}
\mu \times \nu(E) &= \inf \left\{ \sum_{n \in \mathbb{N}} \pi_0(E_i) : (E_i)_{i \in \mathbb{N}} \subset \mathcal{M}_0 \text{ and } E \subset \bigcup_{i \in \mathbb{N}} E_i \right\} \\
&= \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_i) \nu(B_i) : (A_i \times B_i)_{i \in \mathbb{N}} \subset \mathcal{E} \text{ and } E \subset \bigcup_{i \in \mathbb{N}} A_i \times B_i \right\}
\end{aligned}$$

If  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  are both sigma finite, then so is  $\pi_0$  and thus  $\mu \times \nu$  is unique.

## 2. INTEGRATION

### 2.1. Measurable Functions.

**Definition 2.1.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . Then  $f$  is said to be  **$\mathcal{A}$ - $\mathcal{B}$  measurable** if for each  $B \in \mathcal{B}$ ,  $f^{-1}(B) \in \mathcal{A}$ . When  $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  we say that  $f$  is  **$\mathcal{A}$ -measurable**. If  $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and  $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  or  $(\mathbb{R}, \mathcal{L})$ , then we say that  $f$  is **Borel measurable** or **Lebesgue measurable** respectively.

**Lemma 2.2.** Let  $(X, \mathcal{A}), (Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . Then

- (1)  $\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$  is a  $\sigma$ -algebra on  $Y$
- (2)  $\{f^{-1}(B) : B \in \mathcal{B}\}$  is a  $\sigma$ -algebra on  $X$

**Lemma 2.3.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. Suppose that there exists  $\mathcal{E} \subset Y$  such that  $\sigma(\mathcal{E}) = \mathcal{B}$ . Let  $f : X \rightarrow Y$ . If for each  $B \in \mathcal{E}$ ,  $f^{-1}(B) \in \mathcal{A}$ , then  $f$  is  $\mathcal{A}$ - $\mathcal{B}$  measurable.

*Proof.* The previous lemma tells us that  $\mathcal{L} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$  is a  $\sigma$ -algebra on  $Y$ . Since  $\mathcal{E} \subset \mathcal{L}$ , we have that  $\mathcal{B} = \sigma(\mathcal{E}) \subset \mathcal{L}$ .  $\square$

**Corollary 2.4.** Let  $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$  be topological spaces and  $f : X \rightarrow Y$ . If  $f$  is continuous, then  $f$  is  $\mathcal{B}(X)$ - $\mathcal{B}(Y)$  measurable.

*Proof.* Recall that  $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$  and continuity tells us that for each  $U \in \mathcal{T}_2$ ,  $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$ .  $\square$

**Definition 2.5.** Let  $X$  be a set and  $f : X \rightarrow \mathbb{C}$ . The  $f$  is said to be **simple** if  $f(X)$  is finite.

**Definition 2.6.** Let  $(X, \mathcal{A})$  be a measurable space. We define  $S^+(X, \mathcal{A}) = \{f : X \rightarrow [0, \infty) : f \text{ is simple, measurable}\}$  and  $S(X, \mathcal{A}) = \{f : X \rightarrow \mathbb{C} : f \text{ is simple, measurable}\}$

**Theorem 2.7.** Let  $(X, \mathcal{A})$  be a measurable space. Then

- (1) If  $f : X \rightarrow [0, \infty]$  is measurable, then there exists a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \rightarrow f$  pointwise and  $\phi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.
- (2) If  $f : X \rightarrow \mathbb{C}$  is measurable, then there exists a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset S$  such that for each  $n \in \mathbb{N}$ ,  $|\phi_n| \leq |\phi_{n+1}| \leq |f|$  and  $\phi_n \rightarrow f$  pointwise and  $\phi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.

## 2.2. Integration of Nonnegative Functions.

**Definition 2.8.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Define  $L^+(X, \mathcal{A}, \mu) = \{f : X \rightarrow [0, \infty] : f \text{ is measurable}\}$ . We will typically just write  $L^+$ .

**Theorem 2.9.** *Monotone Convergence Theorem* Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$ . Then

$$\sup_{n \in \mathbb{N}} \int f_n = \int \sup_{n \in \mathbb{N}} f_n$$

.

**Exercise 2.10.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$  and  $f \in L^+$ . Then

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

.

*Proof.* Suppose that  $f$  is simple. Then there exist  $(a_n)_{n=1}^n \subset [0, \infty)$  and  $(E_i)_{i=1}^n \subset \mathcal{A}$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for general  $f$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that  $\phi_n \rightarrow f$  pointwise and for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1} \leq f$ . Then monotone convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

**Exercise 2.11.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$ . Suppose that  $\mu_1 \leq \mu_2$ . Then for each  $f \in L^+$ ,

$$\int f d\mu_1 \leq \int f d\mu_2$$

*Proof.* First suppose that  $f$  is simple. Then there exist  $(a_n)_{n=1}^\infty \subset [0, \infty)$  and  $(E_i)_{i=1}^\infty \subset \mathcal{A}$  such that  $f = \sum_{i=1}^\infty a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d\mu_1 &= \sum_{i=1}^\infty a_i \mu_1(E_i) \\ &\leq \sum_{i=1}^\infty a_i \mu_2(E_i) \\ &= \int f d\mu_2 \end{aligned}$$

for general  $f$ ,

$$\begin{aligned} \int f d\mu_1 &= \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_1 \\ &\leq \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_2 \\ &= \int f d\mu_2 \end{aligned}$$

□

**Theorem 2.12.** *Fatou's Lemma* Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

**Theorem 2.13.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n.$$

**Exercise 2.14.** Let  $f \in L^+$  and suppose that  $\int f < \infty$ . Put  $N = \{x \in X : f(x) = \infty\}$  and  $S = \{x \in X : f(x) > 0\}$ . Then  $\mu(N) = 0$  and  $S$  is  $\sigma$ -finite.

*Proof.* Suppose that  $\mu(N) > 0$ . Define  $f_n = n \chi_N \in L^+$ . Then for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1} \leq f$  on  $N$ . So

$$\begin{aligned} \int f &\geq \int_N f \\ &= \lim_{n \rightarrow \infty} \int_N f_n \\ &= \lim_{n \rightarrow \infty} n \mu(N) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

Hence  $N$  is a null set. Now, put  $S_n = \{x \in X : f(x) > 1/n\}$ . Then  $S = \bigcup_{n \in \mathbb{N}} S_n$ . Suppose that there exists some  $n \in \mathbb{N}$  such that  $\mu(S_n) = \infty$ . Then

$$\begin{aligned} \int f &\geq \int_{S_n} f \\ &\geq \frac{1}{n} \mu(S_n) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

So for each  $n \in \mathbb{N}$ ,  $\mu(S_n) < \infty$  and  $S$  is  $\sigma$ -finite. □

**Exercise 2.15.** Let  $f \in L^+$ . Then  $f = 0$  a.e. iff for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$ .

*Proof.*  $f = 0$  a.e. implies that for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$  is clear. Conversely, suppose that for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$ . For  $n \in \mathbb{N}$  put  $N_n = \{x \in X : f(x) > 1/n\}$  and define  $N = \{x \in X : f(x) > 0\}$ . So  $N = \bigcup_{n \in \mathbb{N}} N_n$ . Let  $n \in \mathbb{N}$ . Then our assumption tells us that

$$\begin{aligned} 0 &= \int_{N_n} f \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

Hence for each  $n \in \mathbb{N}$ ,  $\mu(N_n) = 0$ . Thus  $\mu(N) = 0$  and  $f = 0$  a.e. as required. □

**Exercise 2.16.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$  and  $f \in L^+$ . Suppose that  $f_n \xrightarrow{p.w.} f$ ,  $\lim_{n \rightarrow \infty} \int f_n = \int f$  and  $\int f < \infty$ . Then for each  $E \in \mathcal{A}$ ,  $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$ . This result may fail to be true if  $\int f = \infty$

*Proof.* Let  $E \in \mathcal{A}$ . By Fatou's lemma,  $\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$ . Note that since  $\int f < \infty$ , we have that  $\int_{E^c} f \leq \int f < \infty$ . Thus we may write

$$\begin{aligned} \int_E f &= \int f - \int_{E^c} f \\ &\geq \int f - \liminf_{n \rightarrow \infty} \int_{E^c} f_n \\ &= \int f - \liminf_{n \rightarrow \infty} \left( \int f_n - \int_E f_n \right) \\ &= \int f - \int f + \limsup_{n \rightarrow \infty} \int_E f_n \\ &= \limsup_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Hence

$$\limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$$

and therefore

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

If we drop the assumption that  $\int f < \infty$ , then the result would fail to be true for the functions  $f = \infty \chi_{(0,1)}$  and  $f_n = \infty \chi_{(0,1)} + n \chi_{(1,1+1/n)}$ . Here  $f_n \xrightarrow{\text{p.w.}} f$ ,  $\lim_{n \rightarrow \infty} \int f_n = \int f = \infty$  and  $\lim_{n \rightarrow \infty} \int_{(1,\infty)} f_n = 1$  while  $\int_{(1,\infty)} f = 0$ .

□

**Exercise 2.17.** Let  $f \in L^+$ . Define  $\lambda : \mathcal{A} \rightarrow [0, \infty]$  by  $\lambda(E) = \int_E f d\mu$  for  $E \in \mathcal{A}$ . Then  $\lambda$  is a measure on  $(X, \mathcal{A})$  and for each  $g \in L^+$ ,  $\int g d\lambda = \int g f d\mu$ .

*Proof.* Clearly  $\lambda(\emptyset) = 0$ . Let  $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$  and suppose that for each  $i, j \in \mathbb{N}$ , if  $i \neq j$ , then  $A_i \cap A_j = \emptyset$ . For now, suppose that  $f$  is simple. Then there exist  $E_1, E_2, \dots, E_n \in \mathcal{A}$  and  $a_1, a_2, \dots, a_n \in [0, \infty)$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Then

$$\begin{aligned} \lambda\left(\bigcup_{j \in \mathbb{N}} A_j\right) &= \int_{\bigcup_{j \in \mathbb{N}} A_j} f \\ &= \sum_{i=1}^n a_i \mu\left(E_i \cap \left(\bigcup_{j \in \mathbb{N}} A_j\right)\right) \\ &= \sum_{i=1}^n a_i \mu\left(\bigcup_{j \in \mathbb{N}} E_i \cap A_j\right) \\ &= \sum_{i=1}^n a_i \sum_{j \in \mathbb{N}} \mu(E_i \cap A_j) \\ &= \sum_{j \in \mathbb{N}} \sum_{i=1}^n a_i \mu(E_i \cap A_j) \\ &= \sum_{j \in \mathbb{N}} \int_{A_j} f \\ &= \sum_{j \in \mathbb{N}} \lambda(A_j) \end{aligned}$$

Hence  $\lambda$  is a measure on  $(X, \mathcal{A})$ . Now, for a general  $f$ , there exist  $(\phi_n)_{n \in \mathbb{N}} \subset L^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n$  is simple,  $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \xrightarrow{\text{p.w.}} f$ . Put  $A = \bigcup_{j \in \mathbb{N}} A_j$  and define the measures  $\lambda_n$  by  $\lambda_n(E) = \int_E \phi_n$ . Note that we may define a monotonically increasing sequence of functions  $g_n : \mathbb{N} \rightarrow [0, \infty]$  by  $g_n(j) = \int_{A_j} \phi_n$ . Using monotone convergence three times and a nice application of the counting measure on  $\mathbb{N}$ , we may write

$$\begin{aligned}
 \lambda(A) &= \int_A f \\
 &= \lim_{n \rightarrow \infty} \int_A \phi_n \\
 &= \lim_{n \rightarrow \infty} \sum_{j \in \mathbb{N}} \int_{A_j} \phi_n \\
 &= \sum_{j \in \mathbb{N}} \lim_{n \rightarrow \infty} \int_{A_j} \phi_n \quad (\text{by the above}) \\
 &= \sum_{j \in \mathbb{N}} \int_{A_j} f \\
 &= \sum_{j \in \mathbb{N}} \lambda(A_j).
 \end{aligned}$$

Hence  $\lambda$  is a measure on  $(X, \mathcal{A})$ . Let  $g \in L^+$ . First assume that  $g$  is simple. Then there exist  $E_1, E_2, \dots, E_n \in \mathcal{A}$  and  $a_1, a_2, \dots, a_n \in [0, \infty)$  such that  $g = \sum_{i=1}^n a_i \chi_{E_i}$ . In this case, we have that

$$\begin{aligned}
 \int g d\lambda &= \sum_{i=1}^n a_i \lambda(E_i) \\
 &= \sum_{i=1}^n a_i \int_{E_i} f d\mu \\
 &= \int \left( \sum_{i=1}^n a_i \chi_{E_i} \right) f d\mu \\
 &= \int g f d\mu.
 \end{aligned}$$

Now for a general  $g \in L^+$ , there exist  $(\psi_n)_{n \in \mathbb{N}} \subset L^+$  such that for each  $n \in \mathbb{N}$ ,  $\psi_n$  is simple,  $\psi_n \leq \psi_{n+1} \leq f$  and  $\psi_n \xrightarrow{\text{p.w.}} g$ . Monotone convergence then gives us

$$\begin{aligned}
 \int g d\lambda &= \lim_{n \rightarrow \infty} \int \psi_n d\lambda \\
 &= \lim_{n \rightarrow \infty} \int \psi_n f d\mu \\
 &= \int g f d\mu \text{ as required.}
 \end{aligned}$$

□

**Exercise 2.18.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$  and  $f \in L^+$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \geq f_{n+1}$ ,  $f_n \xrightarrow{\text{p.w.}} f$  and  $\int f_1 < \infty$ . Then  $\lim_{n \rightarrow \infty} \int f_n = \int f$ .

*Proof.* First we note that since  $\int f_1 < \infty$ ,  $f_1 < \infty$  a.e., for each  $n \in \mathbb{N}$ ,  $f_1 - f_n$  and  $\int f_1 - \int f_n$  are well defined and  $\int f_n \leq \int f_1 < \infty$ . Also, for  $n \in \mathbb{N}$ ,  $f_1 - f_n \in L^+$ . So we may write

$$\begin{aligned} \int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n \\ &= \int [(f_1 - f_n) + f_n] - \int f_n \\ &= \int f_1 - \int f_n \end{aligned}$$

Put  $g_n = f + (f_1 - f_n)$ . Then  $g_n \in L^+$ , for each  $n \in \mathbb{N}$ ,  $g_n \leq g_{n+1}$  and  $g_n \xrightarrow{\text{p.w.}} f_1$ . Monotone convergence tells us that

$$\begin{aligned} \int f_1 &= \lim_{n \rightarrow \infty} \int g_n \\ &= \lim_{n \rightarrow \infty} \left[ \int f + (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \int f + \int (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \int f + \int f_1 - \int f_n \right] \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \int f$  and  $\lim_{n \rightarrow \infty} \int f_1$  exist,  $\lim_{n \rightarrow \infty} \int f_n = \int f$  as required. □

### 2.3. Integration of Complex Valued Functions.

**Definition 2.19.** Let  $f : X \rightarrow \mathbb{C}$  be measurable. Then  $f$  is said to be **integrable** if

$$\int |f| d\mu < \infty$$

**Definition 2.20.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Define  $L^1(X, \mathcal{A}, \mu) = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable and } \int |f| < \infty\}$

**Lemma 2.21.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is integrable iff  $f^+$  and  $f^-$  are integrable.

*Proof.*  $f^+, f^- \leq |f| = f^+ + f^-$  □

**Definition 2.22.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is said to be **extended integrable** if

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

**Lemma 2.23.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is integrable iff  $\text{Re}(f)$  and  $\text{Im}(f)$  are integrable.

*Proof.*  $|\text{Re}(f)|, |\text{Im}(f)| \leq |f| \leq |\text{Re}(f)| + |\text{Im}(f)|$  □

**Theorem 2.24.** Dominated Convergence Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$ ,  $f$  measurable and  $g \in L^1$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$  and for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then  $f \in L^1$  and  $\int f_n \rightarrow \int f$ .



**Exercise 2.25.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$ . Then

- (1)  $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$   
 (2) for each  $f \in L^1(\mu_1 + \mu_2)$ , we have that

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

*Proof.* (1) The first part is clear since similar exercise from the section on nonnegative functions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f| d\mu_1 + \int |f| d\mu_2$$

- (2) Suppose that  $f$  is simple. Then there exist  $(a_n)_{n=1}^n \subset \mathbb{C}$  and  $(E_i)_{i=1}^n \subset \mathcal{A}$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for general  $f$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S$  such that  $\phi_n \rightarrow f$  pointwise and for each  $n \in \mathbb{N}$ ,  $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ . Then dominated convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

**Theorem 2.26.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$ . Suppose that

$$\sum_{n \in \mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero,  $\sum_{n \in \mathbb{N}} f_n \in L^1$  and

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

**Theorem 2.27.** Let  $f \in L^1$ . Then for each  $\epsilon > 0$ , there exists  $\phi \in L^1$  such that  $\phi$  is simple and  $\int |f - \phi| < \epsilon$ .

**Exercise 2.28.** *Generalized Fatou's Lemma:* Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of measurable real valued functions. Suppose that there exists  $g \in L^1$  such that  $g \geq 0$  and for each  $n \in \mathbb{N}$ ,  $f_n \geq -g$ . Then  $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$ . What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?

*Proof.* First note that for each  $n \in \mathbb{N}$ ,  $\int f_n$  is well defined since  $f_n^- \leq g \in L^1$ . Since  $g + f_n \geq 0$ , we may use Fatou's lemma to write

$$\begin{aligned} \int g + \int \liminf_{n \rightarrow \infty} f_n &= \int \liminf_{n \rightarrow \infty} (g + f_n) \\ &\leq \liminf_{n \rightarrow \infty} \int (g + f_n) \\ &= \int g + \liminf_{n \rightarrow \infty} \int f_n \end{aligned}$$

Since  $\int g < \infty$ ,  $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$  as required. The analogue is as follows: Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of measurable real valued functions. Suppose that there exists  $g \in L^1$  such that  $g \geq 0$  and for each  $n \in \mathbb{N}$ ,  $f_n \leq g$ . Then  $\limsup_{n \rightarrow \infty} \int f_n \leq \int \limsup_{n \rightarrow \infty} f_n$ . To show this, just use the result from above with the sequence  $(g_n)_{n \in \mathbb{N}}$  given by  $g_n = -f_n$ . □

**Exercise 2.29.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$  and  $f : X \rightarrow \mathbb{C}$ . Suppose that  $f_n \xrightarrow{\text{uni}} f$ . Then

- (1) if  $\mu(X) < \infty$ , then  $f \in L^1(X, \mathcal{A}, \mu)$  and  $\lim_{n \rightarrow \infty} \int f_n = \int f$
- (2) if  $\mu(X) = \infty$ , then the conclusion of (1) may fail (find an example on  $\mathbb{R}$  with Lebesgue measure).

*Proof.* Choose  $N \in \mathbb{N}$  such that for  $n \geq N$  and  $x \in X$ ,  $|f(x) - f_n(x)| < 1$ . Then  $||f| - |f_N|| < 1$  and so  $|f| < |f_N| + 1$ . Thus  $\int |f| \leq \int |f_N| + \mu(X) < \infty$  and  $f \in L^1$ . Similarly for  $n \geq N$ ,  $|f_n| < |f| + 1$ . Dominated convergence then gives us that  $\lim_{n \rightarrow \infty} \int f_n = \int f$  as required. To see the necessity that  $\mu(X) < \infty$ , consider  $f \equiv 0$  and  $f_n = (1/n)\chi_{(0,n)}$ . Then  $f_n \xrightarrow{\text{uni}} f$ , but  $1 = \lim_{n \rightarrow \infty} \int f_n \neq \int f = 0$ . □

**Exercise 2.30.** *Generalized Dominated Convergence* Let  $f_n, g_n, f, g \in L^1$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$ ,  $g_n \xrightarrow{\text{a.e.}} g$ ,  $|f_n| \leq g_n$  and  $\int g_n \rightarrow \int g$ . Then  $\int f_n \rightarrow \int f$ .

*Proof.* We simply use Fatou's lemma. Put  $h_n = (g + g_n) - |f_n - f|$ . Since for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g_n$ , we know that  $|f| \leq g$ . So  $h_n \geq 0$  and  $h_n \xrightarrow{\text{p.w.}} 2g$ . Thus

$$\begin{aligned} 2 \int g &= \int \liminf_{n \rightarrow \infty} h_n \\ &\leq \liminf_{n \rightarrow \infty} \left[ \left( \int g + \int g_n \right) - \int |f_n - f| \right] \\ &= 2 \int g + \liminf_{n \rightarrow \infty} \left( - \int |f_n - f| \right) \\ &= 2 \int g - \limsup_{n \rightarrow \infty} \int |f_n - f| \end{aligned}$$

Hence  $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$  which implies that  $\int |f_n - f| \rightarrow 0$  and  $\int f_n \rightarrow \int f$  as required.  $\square$

**Exercise 2.31.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$  and  $f \in L^1$ . Suppose that  $f_n \xrightarrow{a.e.} f$ . Then  $\int |f_n - f| \rightarrow 0$  iff  $\int |f_n| \rightarrow \int |f|$ .

*Proof.* Suppose that  $\int |f_n - f| \rightarrow 0$ . Since

$$\begin{aligned} \left| \int |f_n| - \int |f| \right| &= \left| \int (|f_n| - |f|) \right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|, \end{aligned}$$

we see that  $\int |f_n| \rightarrow \int |f|$ . Conversely, suppose that  $\int |f_n| \rightarrow \int |f|$ . Put  $h_n = |f_n - f|$ ,  $g_n = |f_n| + |f|$ ,  $h \equiv 0$  and  $g = 2|f|$ . Then  $h_n \xrightarrow{a.e.} h$ ,  $g_n \xrightarrow{a.e.} g$  and for each  $n \in \mathbb{N}$ ,  $h_n \leq g_n$ . Our assumption implies that  $\int g_n \rightarrow \int g$ . Thus the last exercise tells us that  $\int h_n \rightarrow \int h$  as required.  $\square$

**Exercise 2.32.** Let  $(r_n)_{n \in \mathbb{N}}$  be an enumeration of the rationals. Define  $f : \mathbb{R} \rightarrow [0, \infty)$  by

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}$$

and define  $g : X \rightarrow [0, \infty]$  by

$$g(x) = \sum_{n \in \mathbb{N}} 2^{-n} f(x - r_n).$$

Then

- (1)  $g \in L^1$  (perhaps after redefinition on a null set) and particularly  $g < \infty$  a.e.
- (2)  $g^2 < \infty$  a.e., but  $g^2$  is not integrable on any subinterval of  $\mathbb{R}$
- (3) Taking  $g \in L^1$ ,  $g$  is unbounded on each subinterval of  $\mathbb{R}$  and discontinuous everywhere and remains so after redefinition on a null set

*Proof.* For convenience, define  $f_n : \mathbb{R} \rightarrow [0, \infty)$  by  $f_n(x) = f(x - r_n)$  for  $x \in \mathbb{R}$ . To show (1) we note that for each  $n \in \mathbb{N}$ ,  $f_n \in L^1$  and

$$\begin{aligned} \int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1} \end{aligned}$$

Hence

$$\sum_{n \in \mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set,  $g \in L^1$ . In particular  $\int |g| < \infty$  and so  $|g|$  (and hence  $g$ ) are finite almost everywhere. For (2), since  $g < \infty$  a.e., so too is  $g^2$ . Let  $a, b \in \mathbb{R}$  and suppose that  $a < b$ . Choose  $N \in \mathbb{N}$  such that  $r_N \in (a, b)$ . Since all the terms in the sum are nonnegative,  $g^2 \geq \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2$  and so

$$\begin{aligned}
\int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2 \\
&= \sum_{n \in \mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\
&\geq 2^{-2N} \int_{(a,b)} f_N^2 \\
&\geq 2^{-2N} \int_{r_N}^{b \wedge (r_N+1)} \frac{1}{x - r_N} dx \\
&= \infty
\end{aligned}$$

So  $g^2$  is not integrable on any subinterval of  $\mathbb{R}$ . For (3), note that redefining  $g$  on a null set does not change the result of (2). Suppose that there is a finite subinterval  $I \subset \mathbb{R}$  such that  $g$  is bounded on  $I$ . Hence there exists  $M > 0$  such that for each  $x \in I$ ,  $g(x)^2 \leq M$ . Then

$$\begin{aligned}
\int_I g^2 &\leq M^2 m(I) \\
&< \infty
\end{aligned}$$

which is a contradiction. So  $g$  is not bounded on any subinterval of  $\mathbb{R}$ . Now, suppose that there exists  $x_0 \in \mathbb{R}$  such that  $g$  is continuous at  $x_0$ . Choose  $\delta > 0$  such that for each  $x \in \mathbb{R}$ , if  $|x - x_0| < \delta$ , then  $|g(x) - g(x_0)| < 1$ . The reverse triangle inequality tells us that for each  $x \in (x_0 - \delta, x_0 + \delta)$ ,  $|g(x)| < 1 + |g(x_0)|$ . Hence  $g$  is bounded on  $(x_0 - \delta, x_0 + \delta)$  which is a contradiction. So  $g$  is discontinuous everywhere.  $\square$

**Exercise 2.33.** Let  $f \in L^1$ .

- (1) If  $f$  is bounded, then for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ , if  $\mu(E) < \delta$ , then  $\int_E |f| < \epsilon$ .
- (2) The same conclusion holds for  $f$  unbounded.

*Proof.* (1) Since  $f$  is bounded, there exists  $M > 0$  such that  $|f| \leq M$ . Let  $\epsilon > 0$ . Choose  $\delta = \epsilon/2M$ . Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \delta$ . Then

$$\begin{aligned}
\int_E |f| &\leq M\mu(E) \\
&= M \frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

(2) Suppose that  $f$  is unbounded. Let  $\epsilon > 0$ . Then there exists  $\phi \in L^1$  such that  $\phi$  is simple and  $\int |f - \phi| < \epsilon/2$ . Since  $\phi$  is bounded, there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ ,

if  $\mu(E) < \delta$ , then  $\int_E |\phi| < \epsilon/2$ . Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \delta$ . Then

$$\begin{aligned} \int_E |f| &\leq \int_E |f - \phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

**Exercise 2.34.** Let  $f \in L^1(\mathbb{R}, \mathcal{L}, m)$ . Define  $F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x) = \int_{(-\infty, x]} f dm.$$

Then  $F$  is continuous.

*Proof.* Let  $x_0 \in \mathbb{R}$  and  $\epsilon > 0$ . Since  $f \in L^1$ , there exists  $\delta > 0$  such that for  $x \in \mathbb{R}$ , if  $|x - x_0| < \delta$ , then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| dm < \epsilon.$$

Let  $x \in \mathbb{R}$ . Suppose that  $|x - x_0| < \delta$ . Then

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f| dm \\ &< \epsilon \end{aligned}$$

So  $F$  is continuous.

□

**Exercise 2.35.** Denote by  $\delta_x$  the point mass measure at  $x \in X$  on measurable space  $(X, \mathcal{P}(X))$ . Let  $f : X \rightarrow \mathbb{C}$ . Then

$$\int f d\delta_x = f(x)$$

*Proof.* First assume that  $f$  is simple. Then there exist  $a_1, a_2, \dots, a_n \in \mathbb{C}$  and  $E_1, E_2, \dots, E_n \in \mathcal{P}(X)$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Thus  $\int f d\delta_x = f(x)$ . Now assume that  $f$ , which is measurable by choice of  $\sigma$ -algebra, satisfies  $f(X) \subset [0, \infty)$ . Choose a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset L^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n$  is simple,  $\phi_n \leq \phi_{n+1}$  and  $\phi_n \xrightarrow{p.w.} f$ . From before, we see that for each  $n \in \mathbb{N}$ ,  $\int \phi_n d\delta_x = \phi_n(x)$ . Monotone convergence tells us that  $\int f d\delta_x = f(x)$ . Now just extend to complex valued functions.

□

**Exercise 2.36.** Denote by  $\#$  the counting measure on the measurable space  $(X, \mathcal{P}(X))$ . Let  $f : X \rightarrow \mathbb{C}$  and suppose that  $f \in L^1$ . Then

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if  $f$  is integrable, then  $\{x \in X : f(x) \neq 0\}$  is countable.

*Proof.* Please refer to the definition of the sum in the appendix. First suppose that  $f(X) \subset [0, \infty)$ . For  $n \in \mathbb{N}$ , put  $X_n = \{x \in X : f(x) > 1/n\}$  and define  $X^* = \{x \in X : f(x) > 0\}$ ,  $X_0 = \{x \in X : f(x) = 0\}$ . Then  $X^* = \bigcup_{n \in \mathbb{N}} X_n$ . Since  $f \in L^1$ , we have that for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} \infty &> \int f d\# \\ &\geq \int_{X_n} f d\# \\ &\geq \frac{1}{n} \#(X_n). \end{aligned}$$

Thus for each  $n \in \mathbb{N}$ ,  $X_n$  is finite and  $X^*$  is countable. Thus there exists  $\{x_n\}_{n \in \mathbb{N}} \subset X$  such that  $X^* = \{x_n\}_{n \in \mathbb{N}}$ . For  $n \in \mathbb{N}$ , define  $E_n = \{x_1, x_2, \dots, x_n\}$  and

$$\begin{aligned} f_n &= f \chi_{E_n} \\ &= \sum_{i=1}^n f(x_i) \chi_{\{x_i\}} \end{aligned}$$

Then  $f_n \xrightarrow{\text{p.w.}} f \chi_{X^*} = f$  and for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$ . So

$$\begin{aligned} \int f &= \sup_{n \in \mathbb{N}} \int f_n \\ &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n f(x_i) \\ &= \sum_{x \in X^*} f(x) \\ &= \sum_{x \in X} f(x). \end{aligned}$$

For  $f : X \rightarrow \mathbb{C}$ , our  $L^1$  assumption and the result above tell us that

$$\sum_{x \in X} |f(x)| < \infty.$$

Thus writing  $f = g + ih$ , we see that the same is true for  $f^+, f^-, g^+, g^-$ . Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f d\# = \sum_{x \in X} f(x).$$

□

**Exercise 2.37.** Let  $f, g : X \rightarrow \mathbb{R}$ . Suppose that  $f, g \in L^1$ . Then  $f \leq g$  a.e. iff for each  $E \in \mathcal{A}$ ,  $\int_E f \leq \int_E g$ .

*Proof.* Suppose  $f \leq g$  a.e. Put  $N = \{x \in X : f(x) > g(x)\} \subset N$ . Then  $\mu(N) = 0$  and  $g - f \geq 0$  on  $N^c$ . So for each  $E \in \mathcal{A}$ ,

$$\begin{aligned} \int_E g - \int_E f &= \int_E (g - f) \\ &= \int_{E \cap N^c} (g - f) \\ &\geq 0 \end{aligned}$$

Conversely, suppose that for each  $E \in \mathcal{A}$ ,  $\int_E f \leq \int_E g$ . Put  $N_n = \{x \in X : f(x) - g(x) > 1/n\}$  and  $N = \{x \in X : f(x) > g(x)\}$ . Then  $N = \bigcup_{n \in \mathbb{N}} N_n$ . Let  $n \in \mathbb{N}$ . Then our assumption tells us that

$$\begin{aligned} 0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

So that  $\mu(N_n) = 0$ . Thus for each  $n \in \mathbb{N}$ ,  $\mu(N_n) = 0$  which implies  $\mu(N) = 0$ . Therefore  $f \leq g$  a.e. as required.  $\square$

**Definition 2.38.** Let  $\mathcal{F} \subset L^1$ . Then  $\mathcal{F}$  is said to be **uniformly integrable** if for each  $\epsilon > 0$ , there exists  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then  $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$ . (i.e.

$$\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0).$$

**Exercise 2.39.** Suppose that  $\mu$  is finite. Let  $\mathcal{F} \subset L^1$ . Then  $\mathcal{F}$  is uniformly integrable iff

- (1) there exists  $M > 0$  such that  $\sup_{f \in \mathcal{F}} \int |f| \leq M$
- (2) for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ , if  $\mu(E) < \delta$ , then  $\sup_{f \in \mathcal{F}} \int_E |f| < \epsilon$ .

*Proof.* ( $\Rightarrow$ ): (1) Suppose that  $\mathcal{F}$  is uniformly integrable. Then there exists  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then  $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < 1$ . Choose  $M = \mu(X)K + 1$ . Then for each  $f \in \mathcal{F}$ ,

$$\begin{aligned} \int |f| &= \int_{\{|f| > K\}} |f| + \int_{\{|f| \leq K\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M \end{aligned}$$

(2) Let  $\epsilon > 0$ . Then choose  $K \in \mathbb{N}$  such that  $\sup_{f \in \mathcal{F}} \int_{\{|f| > K\}} |f| < \epsilon/2$  and choose  $\delta = \epsilon/2K$ .

Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \delta$ . Then for  $f \in \mathcal{F}$ ,

$$\begin{aligned} \int_E |f| &= \int_{E \cap \{|f| > K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon \end{aligned}$$

( $\Leftarrow$ ): Choose  $M > 0$  as in (1). Suppose that there exists  $\epsilon > 0$  such that for each  $K \in \mathbb{N}$ , there exists  $f \in \mathcal{F}$  such that  $\mu(\{|f| > K\}) \geq \epsilon$ . Choose  $K \in \mathbb{N}$  such that  $K > M/\epsilon$ . Then choose  $f_K \in \mathcal{F}$  such that  $\mu(\{|f_K| > K\}) \geq \epsilon$ . Then

$$\begin{aligned} \int |f_K| &\geq \int_{\{|f_K| > K\}} |f_K| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M, \end{aligned}$$

which is a contradiction. Hence for each  $\epsilon > 0$ , there exists  $K \in \mathbb{N}$  such that for each  $f \in \mathcal{F}$ ,  $\mu(\{|f| > K\}) < \epsilon$ . Since  $\mu(\{|f| > k\})$  is a decreasing sequence in  $k$ , we have that  $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \mu(\{|f| > k\}) = 0$ . Now, let  $\epsilon > 0$ . Choose  $\delta > 0$  as in (2). Choose  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then for each  $f \in \mathcal{F}$ ,  $\mu(\{|f| > k\}) < \delta$ . Then for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then for each  $f \in \mathcal{F}$ ,

$$\int_{\{|f| > k\}} |f| < \epsilon.$$

Thus  $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$  as required. □

## 2.4. Integration on Product Spaces.

**Definition 2.40.** Let  $X$ ,  $Y$ , and  $Z$  be sets,  $E \subset X \times Y$  and  $f : X \times Y \rightarrow Z$ . For each  $x \in X$ , define  $E_x = \{y \in Y : (x, y) \in E\}$  and  $f_x : Y \rightarrow Z$  by  $f_x(y) = f(x, y)$ . For each  $y \in Y$ , define  $E^y = \{x \in X : (x, y) \in E\}$  and  $f^y : X \rightarrow Z$  by  $f^y(x) = f(x, y)$ .

**Note 2.41.** It is often helpful to observe that  $(\chi_E)_x = \chi_{E_x}$  and  $(\chi_E)^y = \chi_{E^y}$ .

**Lemma 2.42.** Let  $(X, \mathcal{A})$ ,  $(Y, \mathcal{B})$  be measurable spaces,  $Z = [0, \infty]$  or  $\mathbb{C}$  and  $f : X \times Y \rightarrow Z$ .

- (1) For each  $E \in \mathcal{A} \otimes \mathcal{B}$ ,  $x \in X$ ,  $y \in Y$ , we have that  $E_x \in \mathcal{B}$  and  $E^y \in \mathcal{A}$
- (2) If  $f$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable, then for each  $x \in X$ ,  $y \in Y$ , we have that  $f_x$  is  $\mathcal{B}$ -measurable and  $f^y$  is  $\mathcal{A}$ -measurable.

**Theorem 2.43.** Let  $(X, \mathcal{A}, \mu)$ ,  $(Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces. Then for each  $E \in \mathcal{A} \otimes \mathcal{B}$ , the maps  $\phi : X \rightarrow [0, \infty]$  and  $\psi : Y \rightarrow [0, \infty]$  defined by  $\phi(x) = \nu(E_x)$  and  $\psi(y) = \mu(E^y)$  are  $\mathcal{A}$ -measurable and  $\mathcal{B}$ -measurable, respectively and

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$



**Theorem 2.44.** *Fubini, Tonelli:* Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces.

- (1) (Tonelli) For each  $f \in L^+(X \times Y)$ , the functions  $g : X \rightarrow [0, \infty]$ ,  $h : Y \rightarrow [0, \infty]$  defined by  $g(x) = \int_Y f_x(y) d\nu(y)$  and  $h(y) = \int_X f^y(x) d\mu(x)$  are  $\mathcal{A}$ -measurable and  $\mathcal{B}$ -measurable respectively and

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

- (2) (Fubini) For each  $f \in L^1(X \times Y)$ ,  $f_x \in L^1(\nu)$  for  $\mu$ -a.e.  $x \in X$  and  $f^y \in L^1(\mu)$  for  $\nu$ -a.e.  $y \in Y$ , respectively and the functions (after redefinition of  $f$  on a null set)  $g : X \rightarrow \mathbb{C}$ ,  $h : Y \rightarrow \mathbb{C}$  defined by  $g(x) = \int_Y f_x(y) d\nu(y)$  and  $h(y) = \int_X f^y(x) d\mu(x)$  are in  $L^1(\mu)$  and  $L^1(\nu)$  respectively. Furthermore

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

**Note 2.45.** We usually just write  $\int \int f d\mu d\nu$  and  $\int \int f d\nu d\mu$  instead of  $\int h d\nu$  and  $\int g d\mu$  respectively. We have a similar result for complete product measure spaces. See

**Exercise 2.46.** Take  $X = Y = [0, 1]$ ,  $\mathcal{A} = \mathcal{B}([0, 1])$ ,  $\mathcal{B} = \mathcal{P}([0, 1])$  and  $\mu, \nu$  to be Lebesgue measure and counting measure respectively. Define  $D = \{(x, y) \in [0, 1]^2 : x = y\}$  Show that

$$\int \chi_D d\mu \times \nu, \int \int \chi_D d\mu d\nu \text{ and } \int \int \chi_D d\nu d\mu$$

are all different. (Hint: for the first integral, use the definition of  $\mu \times \nu$ )

*Proof.* Let  $x, y \in [0, 1]$ . Then  $(\chi_D)_x = \chi_{D_x} = \chi_x$  and  $(\chi_D)^y = \chi_{D^y} = \chi_y$ . Thus

$$\begin{aligned} \int \int \chi_D d\mu d\nu &= \int \mu(\{y\}) d\nu \\ &= \int 0 d\nu \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \int \int \chi_D d\mu d\nu &= \int \nu(\{x\}) d\mu \\ &= \int 1 d\mu \\ &= 1 \end{aligned}$$

Now, Observe that  $\int \chi_D d\mu \times \nu = \mu \times \nu(D)$ . Recall from the section on product measures that  $\mu \times \nu(D) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) : (A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n\}$ . Let  $(A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ . Suppose that  $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n$ . Then for each  $x \in [0, 1]$ ,  $(x, x) \in \bigcup_{n \in \mathbb{N}} A_n \times B_n$ . So for each  $x \in [0, 1]$ , there exists  $n \in \mathbb{N}$ , such that  $x \in A_n \cap B_n$ . Thus  $[0, 1] \subset \bigcup_{n \in \mathbb{N}} A_n \cap B_n$ . Since  $1 = \mu([0, 1]) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap B_n)$ , we know that there exists  $n \in \mathbb{N}$  such that  $0 < \mu(A_n \cap B_n)$ . Thus  $\mu(A_n) > 0$  and  $\mu(B_n) > 0$ . Since  $\mu(B_n) > 0$ ,  $B_n$  must be infinite and therefore  $\nu(B_n) = \infty$ . So  $\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) = \infty$ .

□

**Exercise 2.47.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $f : X \rightarrow [0, \infty) \in L^+$ . Show that  $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$  and  $\mu \times m(G) = \int_X f d\mu$ . The same is true if we replace " $\geq$ " with " $>$ ". (Hint: to show that  $G$  is measurable, split up  $(x, y) \mapsto f(x) - y$  into the composition of measurable functions.

*Proof.* Define  $\phi : X \times [0, \infty) \rightarrow [0, \infty)^2$  and  $\psi : [0, \infty)^2 \rightarrow [0, \infty)$  by  $\phi(x, y) = (f(x), y)$  and  $\psi(z, y) = z - y$ . Then  $G = \{(x, y) \in X \times [0, \infty) : \psi \circ \phi(x, y) \geq 0\}$ . Let  $A, B \in \mathcal{B}([0, \infty))$ . Then  $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0, \infty))$ . Since  $\mathcal{B}([0, \infty)^2) = \mathcal{B}([0, \infty)) \otimes \mathcal{B}([0, \infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0, \infty))\})$ , we have that  $\phi$  is  $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty)^2)$  measurable. Since  $\psi$  is continuous, we have that  $\psi$  is  $\mathcal{B}([0, \infty)^2)$ - $\mathcal{B}([0, \infty))$  measurable. This implies that  $\psi \circ \phi$  is  $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty))$  measurable. Thus  $G = \psi \circ \phi^{-1}([0, \infty)) \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ . Now for  $x \in X$ ,  $G_x = \{y \in [0, \infty) : f(x) \geq y\} = [0, f(x)]$ . Thus

$$\begin{aligned} \mu \times m(G) &= \int \chi_G d\mu \times m \\ &= \int_X \int_{[0, \infty)} \chi_{G_x} dm d\mu(x) \\ &= \int_X f(x) d\mu(x) \end{aligned}$$

The same reasoning holds if we replace " $\geq$ " with " $>$ ". □

**Exercise 2.48.** Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces and  $f : X \rightarrow \mathbb{C}, g : Y \rightarrow \mathbb{C}$ . Define  $h : X \times Y \rightarrow \mathbb{C}$  by  $h(x, y) = f(x)g(y)$ .

- (1) If  $f$  is  $\mathcal{A}$ -measurable and  $g$  is  $\mathcal{B}$ -measurable, then  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable.
- (2) If  $f \in L^1(\mu)$  and  $g \in L^1(\nu)$ , then  $h \in L^1(\mu \times \nu)$  and

$$\int_{X \times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu$$

*Proof.* (1) First suppose that  $f, g$  are simple. Then there exist  $(A_i)_{i=1}^n \subset \mathcal{A}, (B_j)_{j=1}^m \subset \mathcal{B}$  and  $(a_i)_{i=1}^n, (b_j)_{j=1}^m \subset \mathbb{C}$  such that  $f = \sum_{i=1}^n a_i \chi_{A_i}$  and  $g = \sum_{j=1}^m b_j \chi_{B_j}$ . Then  $h = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{A_i \times B_j}$ . So  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable. For general  $f, g$ , there exist  $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$  and  $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$  such that  $f_n \rightarrow f$  pointwise,  $g_n \rightarrow g$  pointwise and for each  $n \in \mathbb{N}$ ,  $|f_n| \leq |f_{n+1}| \leq |f|$  and  $|g_n| \leq |g_{n+1}| \leq |g|$ . For  $n \in \mathbb{N}$ , define  $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$  by  $h_n = f_n g_n$ . Then  $h_n \rightarrow h$  pointwise and for each  $n \in \mathbb{N}$ ,  $|h_n| \leq |h_{n+1}| \leq |h|$ . Thus  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable.

- (2) First suppose  $f$  and  $g$  are simple as before. Then

$$\begin{aligned} \int_{X \times Y} |h| d\mu \times \nu &\leq \sum_{i=1}^n \sum_{j=1}^m |a_i b_j| \mu(A_i) \nu(B_j) \\ &= \left( \sum_{i=1}^n |a_i| \mu(A_i) \right) \left( \sum_{j=1}^m |b_j| \nu(B_j) \right) \\ &= \int_X |f| d\mu \int_Y |g| d\nu \\ &< \infty \end{aligned}$$

So  $h \in L^1(\mu \times \nu)$ . Furthermore,

$$\begin{aligned} \int_{X \times Y} h d\mu \times \nu &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mu(A_i) \nu(B_j) \\ &= \left( \sum_{i=1}^n a_i \mu(A_i) \right) \left( \sum_{j=1}^m b_j \nu(B_j) \right) \\ &= \int_X f d\mu \int_Y g d\nu \end{aligned}$$

For general  $f \in L^1(\mu), g \in L^1(\nu)$ , take  $(h_n)_{n \in \mathbb{N}}$  as before. Monotone convergence and the result above say that

$$\begin{aligned} \int_{X \times Y} |h| d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} |h_n| d\mu \times \nu \\ &= \lim_{n \rightarrow \infty} \left( \int_X |f_n| d\mu \int_Y |g_n| d\nu \right) \\ &= \int_X |f| d\mu \int_Y |g| d\nu \\ &< \infty \end{aligned}$$

So  $h \in L^1(\mu \times \nu)$ . Dominated convergence and the result above then tell us that

$$\begin{aligned} \int_{X \times Y} h d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} h_n d\mu \times d\nu \\ &= \lim_{n \rightarrow \infty} \left( \int_X f_n d\mu \int_Y g_n d\nu \right) \\ &= \int_X f d\mu \int_Y g d\nu \end{aligned}$$

□

**Note 2.49.** In the above exercise part (2), we can replace  $L^1$  with  $L^+$  and get the same result by the same method.

**Exercise 2.50.** Let  $f : \mathbb{R} \rightarrow [0, \infty) \in L^+$ . Show that

$$\int_{\mathbb{R}} f dm = \int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) dm(t)$$

*Proof.* Note that

$$\int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) = \int_{[0, \infty)} \left[ \int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}} dm \right] dm(t)$$

Comparing this with Tonelli's theorem, we can put  $\chi_{\{x \in \mathbb{R} : f(x) \geq t\}} = (\chi_E)^t = \chi_{E^t}$ . Then  $E = \{(x, t) \in \mathbb{R} \times [0, \infty) : f(x) \geq t\}$  and  $E_x = \{t \in [0, \infty) : f(x) \geq t\} = [0, f(x)]$ . Tonelli's

theorem tells us that

$$\begin{aligned} \int_{[0,\infty)} \left[ \int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}}(x) dm(x) \right] dm(t) &= \int_{\mathbb{R}} \left[ \int_{[0,\infty)} \chi_{[0,f(x)]}(t) dm(t) \right] dm(x) \\ &= \int_{\mathbb{R}} f(x) dm(x) \end{aligned}$$

□

## 2.5. Convergence.

**Definition 2.51.** Let  $(X, \mathcal{A})$  be a measurable space. For convenience we will define  $L^0 = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable}\}$ .

**Definition 2.52.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Then  $f_n$  converges to  $f$  **in measure** if for each  $\epsilon > 0$ ,  $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \rightarrow 0$ . This is written  $f_n \xrightarrow{\mu} f$ .

**Definition 2.53.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Then  $f_n$  converges to  $f$  **almost uniformly** if for each  $\epsilon > 0$ , there exists  $N \in \mathcal{A}$  such that  $\mu(N) < \epsilon$  and  $f_n \xrightarrow{\text{uni}} f$  on  $N^c$ . This is written  $f_n \xrightarrow{\text{a.u.}} f$ .

**Theorem 2.54.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . If  $f_n \xrightarrow{\mu} f$ , then there exists a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  such that  $f_{n_k} \xrightarrow{\text{a.e.}} f$ .

**Theorem 2.55.** (Egoroff): Suppose that  $\mu(X) < \infty$ . Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that  $f_n \xrightarrow{\text{a.e.}}$ . Then  $f_n \xrightarrow{\text{a.u.}} f$ .

**Exercise 2.56.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$  and  $f \in L^1$ . If  $f_n \xrightarrow{L^1} f$ , then  $f_n \xrightarrow{\mu} f$ .

*Proof.* Let  $\epsilon > 0$ . for  $n \in \mathbb{N}$ , define  $E_{\epsilon,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$ . Then for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \int |f - f_n| &\geq \int_{E_{\epsilon,n}} |f - f_n| \\ &\geq \epsilon \mu(E_{\epsilon,n}). \end{aligned}$$

So for each  $n \in \mathbb{N}$ ,  $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$ . Since  $\int |f - f_n| \rightarrow 0$ , we have that  $\mu(E_{\epsilon,n}) \rightarrow 0$ . Since  $\epsilon > 0$  is arbitrary,  $f_n \xrightarrow{\mu} f$  as required. □

**Exercise 2.57.** Suppose  $\mu(X) < \infty$ . Define  $d : L^0 \times L^0 \rightarrow [0, \infty)$  by

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \quad f, g \in L^0$$

Then  $d$  is a metric on  $L^0$  if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each  $f, g \in L^0$ ,  $d(f, g) \leq \mu(X)$ .

*Proof.* Let  $f, g \in L^0$ . Clearly  $d(f, g) = d(g, f)$ . If  $f = g$  a.e. then clearly  $d(f, g) = 0$ . Conversely, if  $d(f, g) = 0$ , then  $\frac{|f - g|}{1 + |f - g|} = 0$  a.e and so  $|f - g| = 0$  a.e. which implies  $f = g$  a.e. It is not hard to show that  $\phi : [0, \infty) \rightarrow [0, \infty)$  given by  $\phi(x) = \frac{x}{1+x}$  satisfies  $\phi(x + y) \leq \phi(x) + \phi(y)$ . Thus satisfies the triangle inequality. Now, let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ . Suppose that  $f_n \not\xrightarrow{\mu} f$ . Then there exists  $\epsilon > 0, \delta > 0$  and a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  such that

for each  $k \in \mathbb{N}$ ,  $\mu(E_{\epsilon, n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$ . It is not hard to show that  $\phi$  from earlier is increasing. Thus for each  $k \in \mathbb{N}$ ,

$$\begin{aligned} d(f_{n_k}, f) &= \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{\epsilon}{1 + \epsilon} \\ &\geq \frac{\epsilon \delta}{1 + \epsilon} \end{aligned}$$

So  $f_{n_k} \not\stackrel{d}{\rightarrow} f$ . Hence  $f_{n_k} \stackrel{d}{\rightarrow} f$  implies that  $f_{n_k} \stackrel{\mu}{\rightarrow} f$ . Conversely, suppose that  $f_{n_k} \stackrel{\mu}{\rightarrow} f$ . Let  $\epsilon > 0$ . Then  $\delta = \frac{\epsilon}{1 + \mu(X)} > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then  $\mu(E_{\delta, n}) < \frac{\delta}{1 + \delta}$ . Let  $n \in \mathbb{N}$ . Suppose that  $n \geq N$ . Since  $\phi$  is increasing and  $\phi \leq 1$ , we have that

$$\begin{aligned} d(f_n, f) &= \int \frac{|f_n - f|}{1 + |f_n - f|} \\ &= \int_{E_{\delta, n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E_{\delta, n}^c} \frac{|f_n - f|}{1 + |f_n - f|} \\ &\leq \mu(E_{\delta, n}) + \mu(X) \frac{\delta}{1 + \delta} \\ &< \frac{\delta}{1 + \delta} (1 + \mu(X)) \\ &\leq \delta (1 + \mu(X)) \\ &= \epsilon \end{aligned}$$

□

**Exercise 2.58.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \geq 0$  and  $f_n \stackrel{\mu}{\rightarrow} f$ . Then  $f \geq 0$  a.e. and  $\int f \leq \liminf_{n \rightarrow \infty} \int f_n$ .

*Proof.* Since  $f_n \stackrel{\mu}{\rightarrow} f$ , there is a subsequence converging to  $f$  a.e. So clearly  $f \geq 0$  a.e. Now, choose a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  such that  $\int f_{n_k} \rightarrow \liminf_{n \rightarrow \infty} \int f_n$ . Since  $f_n \stackrel{\mu}{\rightarrow} f$  so does  $(f_{n_k})_{k \in \mathbb{N}}$ . Therefore there exists a subsequence  $(f_{n_{k_j}})_{j \in \mathbb{N}}$  of  $(f_{n_k})_{k \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ . Thus  $f \geq 0$  a.e. and Fatou's lemma tells us that

$$\begin{aligned} \int f &\leq \liminf_{j \in \mathbb{N}} \int f_{n_{k_j}} \\ &= \liminf_{n \rightarrow \infty} \int f_n. \end{aligned}$$

□

**Exercise 2.59.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that there exists  $g \in L^1$  such that for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then  $f_n \stackrel{\mu}{\rightarrow} f$  implies that  $f \in L^1$  and  $f_n \xrightarrow{L^1} f$ .

*Proof.* Clearly  $(f_n)_{n \in \mathbb{N}} \subset L^1$ . Since  $f_n \xrightarrow{\mu} f$ , there exists a subsequence  $(f_{n_k})_{k \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$  such that  $f_{n_k} \xrightarrow{\text{a.e.}} f$ . This implies that  $|f| \leq g$  a.e. and so  $f \in L^1$ . For  $n \in \mathbb{N}$ , put  $h_n = 2g - |f_n - f|$ . Then for each  $n \in \mathbb{N}$ ,  $h_n \geq 0$  and  $h_n \xrightarrow{\mu} 2g$ . By the previous exercise

$$\begin{aligned} \int 2g &\leq \liminf_{n \rightarrow \infty} \int (2g - |f_n - f|) \\ &= \int 2g - \limsup_{n \rightarrow \infty} \int |f_n - f|. \end{aligned}$$

So  $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$  which implies that  $\int |f_n - f| \rightarrow 0$  and  $f_n \xrightarrow{L^1} f$  as required.  $\square$

**Exercise 2.60.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ ,  $f \in L^0$  and  $\phi : \mathbb{C} \rightarrow \mathbb{C}$ .

- (1) If  $\phi$  is continuous, and  $f_n \xrightarrow{\text{a.e.}} f$  then  $\phi \circ f_n \xrightarrow{\text{a.e.}} \phi \circ f$ .
- (2) If  $\phi$  is uniformly continuous and  $f_n \rightarrow f$  uniformly, almost uniformly or in measure, then  $\phi \circ f_n \rightarrow \phi \circ f$  uniformly, almost uniformly or in measure, respectively.
- (3) Find a counter example to (2) if we drop the word "uniform".

*Proof.* (1) Clear

(2) Suppose that  $\phi$  is uniformly continuous.

(uniform conv.) Suppose that  $f_n \xrightarrow{\text{uni}} f$ . Let  $\epsilon > 0$ . Choose  $\delta > 0$  such that for each  $z, w \in \mathbb{C}$ , if  $|z - w| < \delta$ , then  $|\phi(z) - \phi(w)| < \epsilon$ . Now choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$  if  $n \geq N$  then for each  $x \in X$ ,  $|f_n(x) - f(x)| < \delta$ . Let  $n \in \mathbb{N}$ , suppose  $n \geq N$ . Let  $x \in X$ . Then  $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$ . Thus  $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$ .

(almost uni.) Suppose that  $f_n \xrightarrow{\text{a.u.}} f$ . Let  $\epsilon > 0$ . Choose  $N \in \mathcal{A}$  such  $\mu(N) < \epsilon$  and  $f_n \xrightarrow{\text{uni}} f$  on  $N^c$ . Then from above, we know that  $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$  on  $N^c$ . Thus  $\phi \circ f_n \xrightarrow{\text{a.u.}} \phi \circ f$ .

(measure) Suppose that  $f_n \xrightarrow{\mu} f$ . Let  $\epsilon > 0$ . Choose  $\delta > 0$  such that for each  $z, w \in \mathbb{C}$ , if  $|z - w| < \delta$ , then  $|\phi(z) - \phi(w)| < \epsilon$ . Observe that for  $x \in X$ , if  $|f_n(x) - f(x)| < \delta$ , then  $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$ . Hence  $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$ . By definition of convergence in measure,  $\mu(F_{n,\delta}) \rightarrow 0$ . Thus  $\mu(E_{n,\epsilon}) \rightarrow 0$ . Hence  $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$ .

(3)

$\square$

**Exercise 2.61.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that  $f_n \xrightarrow{\text{a.u.}} f$ . Then  $f_n \xrightarrow{\mu} f$  and  $f_n \xrightarrow{\text{a.e.}} f$ .

*Proof.* (measure) Let  $\epsilon > 0$ ,  $\delta > 0$ . Choose  $M \in \mathcal{A}$  such that  $\mu(M) < \delta$  and  $f_n \xrightarrow{\text{uni}} f$  on  $M^c$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then for each  $x \in M^c$ ,  $|f_n(x) - f(x)| < \epsilon$ . Let  $n \in \mathbb{N}$ . Suppose  $n \geq N$ . Then  $E_{\epsilon,n} \subset M$  and  $\mu(E_{\epsilon,n}) < \delta$ . Thus  $\mu(E_{\epsilon,n}) \rightarrow 0$  and  $f_n \xrightarrow{\mu} f$ .

(a.e.) For each  $n \in \mathbb{N}$ , Choose  $N_n \in \mathcal{A}$  such that  $\mu(N_n) < 1/n$  and  $f_n \xrightarrow{\text{uni}} f$  on  $N_n^c$ . Observe that for  $x \in X$ , if  $x \in \bigcup_{n \in \mathbb{N}} N_n^c$ , then  $f_n(x) \rightarrow f(x)$ . Thus  $N = \{x \in X : f_n(x) \not\rightarrow f(x)\} \subset \bigcap_{n \in \mathbb{N}} N_n$ . Therefore  $\mu(N) = 0$  and  $f_n \xrightarrow{\text{a.e.}} f$ .  $\square$

**Exercise 2.62.** Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset L^0$  and  $f, g \in L^0$ . Suppose that  $f_n \xrightarrow{\mu} f$  and  $g_n \xrightarrow{\mu} g$ . Then

- (1)  $f_n + g_n \xrightarrow{\mu} f + g$
- (2) if  $\mu(X) < \infty$ , then  $f_n g_n \xrightarrow{\mu} f g$

*Proof.* (1) Let  $\epsilon > 0$ . For convenience, put  $F_{n,\epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$ ,  $G_{n,\epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$ , and  $(F + G)_{n,\epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g(x))| \geq \epsilon\}$ . Observe that for  $x \in X$ ,  $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$ . Thus  $(F + G)_{n,\epsilon} \subset F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$ . Since  $\mu(F_{n,\epsilon/2} \cup G_{n,\epsilon/2}) \leq \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}) \rightarrow 0$ , we have that  $\mu((F + G)_{n,\epsilon}) \rightarrow 0$ . Hence  $f_n + g_n \xrightarrow{\mu} f + g$ .

- (2) Suppose that  $\mu(X) < \infty$ . Let  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  be a subsequence of  $(f_n g_n)_{n \in \mathbb{N}}$ . Choose a subsequence  $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$  and  $g_{n_{k_j}} \xrightarrow{\text{a.e.}} g$ . Then  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.e.}} f g$ . Egoroff's theorem tells us that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.u.}} f g$ , which implies that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} f g$ . Thus for each subsequence  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  of  $(f_n g_n)_{n \in \mathbb{N}}$ , there exists a subsequence  $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$  of  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  such that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} f g$ . Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that  $f_n g_n \xrightarrow{\mu} f g$ . □

**Exercise 2.63.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that for each  $\epsilon > 0$ ,

$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then  $f_n \xrightarrow{\text{a.e.}} f$ .

*Proof.* Let  $\epsilon > 0$ . By assumption we know that

$$\begin{aligned} \int \left[ \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} \right] d\mu &= \sum_{n \in \mathbb{N}} \int \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} d\mu \\ &= \sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \\ &< \infty \end{aligned}$$

Thus we also know that  $\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} < \infty$  a.e. Equivalently, we could say that for a.e.  $x \in X$ ,  $|\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}| < \infty$ . For  $k \in \mathbb{N}$ , define  $N_k = \{x \in X : \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > 1/k\}} = \infty\}$ . Then for each  $k \in \mathbb{N}$ ,  $\mu(N_k) = 0$ . Define  $N = \bigcup_{k \in \mathbb{N}} N_k$ . Then  $\mu(N) = 0$ . Let  $x \in N^c$  and  $\epsilon > 0$ . Choose  $k \in \mathbb{N}$  such that  $1/k < \epsilon$ . Then  $\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\} \subset \{n \in \mathbb{N} : |f_n(x) - f(x)| > 1/k\}$  which is finite because  $x \in N_k^c$ . Put  $M = \max\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}$ . Then for  $m \geq M$ ,  $|f_m(x) - f(x)| \leq \epsilon$ . Thus  $f_n(x) \rightarrow f(x)$ . Hence  $f_n \xrightarrow{\text{a.e.}} f$ . □

### 3. DIFFERENTIATION

#### 3.1. Signed Measures.

**Definition 3.1.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$ . Then  $\nu$  is said to be a **signed measure** if

- (1) for each  $E \in \mathcal{A}$ ,  $\nu(E) < \infty$  or for each  $E \in \mathcal{A}$ ,  $\nu(E) > -\infty$ .
- (2)  $\nu(\emptyset) = 0$

(3) for each  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  if  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is disjoint, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$  and if  $|\sum_{n \in \mathbb{N}} \nu(E_n)| < \infty$ , then  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely.

**Exercise 3.2.** Let  $\nu : \mathcal{A} \rightarrow [0, \infty]$  be a signed measure and  $(E_n)_{n \in \mathbb{N}}, (F_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . If  $(E_n)_{n \in \mathbb{N}}$  is increasing, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \rightarrow \infty} \nu(E_n)$ . If  $(F_n)_{n \in \mathbb{N}}$  is decreasing and  $|\nu(E_1)| < \infty$ , then  $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$ .

*Proof.* Put  $E'_1 = E_1$ ,  $F'_1 = F_1$  and for  $n \in \mathbb{N}$ ,  $n \geq 2$ , put  $E'_n = E_n \setminus E_{n-1}$  and  $F'_n = F_1 \setminus F_n$ . Then  $(E'_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is disjoint. Thus

$$\begin{aligned} \nu(\bigcup_{n \in \mathbb{N}} E_n) &= \nu(\bigcup_{n \in \mathbb{N}} E'_n) \\ &= \sum_{n \in \mathbb{N}} \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \sum_{n=1}^n \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \nu(E_n) \end{aligned}$$

Since  $(F'_n)_{n \in \mathbb{N}}$  is increasing, we now know that

$$\begin{aligned} \nu(F_1) - \nu(\bigcap_{n \in \mathbb{N}} F_n) &= \nu(F_1 \setminus \bigcap_{n \in \mathbb{N}} F_n) \\ &= \nu(\bigcup_{n \in \mathbb{N}} F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F_1 \setminus F_n) \\ &= \nu(F_1) - \lim_{n \rightarrow \infty} \nu(F_n) \end{aligned}$$

Since  $|\nu(F_1)| < \infty$ , we see that  $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$ . □

**Definition 3.3.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$  a signed measure and  $E \in \mathcal{A}$ . Then  $E$  is said to be  $\nu$ -**positive**,  $\nu$ -**negative** and  $\nu$ -**null** if for each  $F \in \mathcal{A}$ ,  $F \subset E$  implies that  $\nu(F) \geq 0$ ,  $\nu(F) \leq 0$ ,  $\nu(F) = 0$  respectively.

**Exercise 3.4.** Let  $E \subset \mathcal{A}$ . If  $E$  is positive, negative or null, then for each  $F \in \mathcal{A}$ , if  $F \subset E$ , then  $F$  is positive, negative or null respectively.

*Proof.* Clear □

**Exercise 3.5.** Let  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  be positive, negative or null. Then  $\bigcup_{n \in \mathbb{N}} E_n$  is positive, negative or null respectively.

*Proof.* Suppose that  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is positive. Let  $F \in \mathcal{A}$ . Suppose that  $F \subset \bigcup_{n \in \mathbb{N}} E_n$ . Put

$P_1 = E_1$  and for  $n \in \mathbb{N}$ ,  $n \geq 2$ , put  $P_n = E_n \setminus (\bigcup_{j=1}^{n-1} E_j)$ . So  $\bigcup_{n \in \mathbb{N}} P_n = \bigcup_{n \in \mathbb{N}} E_n$  and  $(P_n)_{n \in \mathbb{N}}$  is



disjoint. Thus

$$\begin{aligned}\nu(F) &= \nu\left(F \cap \bigcup_{n \in \mathbb{N}} P_n\right) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} (F \cap P_n)\right) \\ &= \sum_{n \in \mathbb{N}} \nu(F \cap P_n) \\ &\geq 0\end{aligned}$$

The process is the same if  $(E_n)_{n \in \mathbb{N}}$  is negative and null.  $\square$

**Theorem 3.6.** *Hahn Decomposition:* Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then there exist  $P, N \in \mathcal{A}$  such that  $P$  is positive,  $N$  is negative,  $X = N \cup P$  and  $N \cap P = \emptyset$ . Furthermore, these two sets are unique in the following sense: For any  $P', N' \in \mathcal{A}$ , if  $N, P$  satisfy the properties above,  $P' \Delta P = N' \Delta N$  is null.

**Definition 3.7.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$  and  $P, N \in \mathcal{A}$ . Then  $P$  and  $N$  are said to form a **Hahn decomposition** of  $X$  with respect to  $\nu$  if  $P, N$  satisfy the results in the above theorem.

**Definition 3.8.** Let  $\mu, \nu$  be signed measures on  $(X, \mathcal{A})$ . Then  $\mu$  and  $\nu$  are said to be **mutually singular** if there exist  $E, F \in \mathcal{A}$  such that  $X = E \cup F$ ,  $E \cap F = \emptyset$  and  $E$  is  $\mu$ -null and  $F$  is  $\nu$ -null. We will denote this by  $\mu \perp \nu$ .

**Theorem 3.9.** *Jordan Decomposition:* Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then there exist unique positive measures  $\nu^+$  and  $\nu^-$  on  $(X, \mathcal{A})$  such that  $\nu = \nu^+ - \nu^-$  and  $\nu^+ \perp \nu^-$ .

*Proof.* Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Define  $\nu^+, \nu^-$  by  $\nu^+(E) = \nu(E \cap P)$  and  $\nu^-(E) = \nu(E \cap N)$ .  $\square$

**Definition 3.10.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then  $\nu^+$  and  $\nu^-$  from the last theorem are called the **positive** and **negative variations** of  $\nu$  respectively. We define the **total variation** measure  $|\nu|$  on  $(X, \mathcal{A})$  by  $|\nu| = \nu^+ + \nu^-$ .

**Definition 3.11.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then  $\nu$  is said to be  $\sigma$ -finite if  $|\nu|$  is  $\sigma$ -finite.

**Exercise 3.12.** Let  $\nu$  be a signed measure and  $\lambda, \mu$  positive measures on  $(X, \mathcal{A})$ . Suppose that  $\nu = \lambda - \mu$ . Then  $\lambda \geq \nu^+$  and  $\mu \geq \nu^-$ .

*Proof.* Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Let  $E \in \mathcal{A}$ . Then

$$\begin{aligned}\lambda(E \cap P) - \mu(E \cap P) &= \nu(E \cap P) \\ &= \nu^+(E \cap P)\end{aligned}$$

So  $\lambda(E \cap P) \geq \nu^+(E \cap P)$  and therefore

$$\begin{aligned}\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) \\ &= \nu^+(E)\end{aligned}$$

Similarly  $\mu(E \cap N) \geq \nu^-(E \cap N)$  and  $\mu(E) \geq \nu^-(E)$ .  $\square$

**Exercise 3.13.** Let  $\nu_1, \nu_2$  be signed measures on  $(X, \mathcal{A})$ . Suppose that  $\nu_1 + \nu_2$  is a signed measure. Then  $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ . (Hint: use the last exercise)

*Proof.* Since

$$\begin{aligned}\nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\ &= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)\end{aligned}$$

the previous exercise tells us that  $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$  and  $\mu = \nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$ . Therefore

$$\begin{aligned}|\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ &\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\ &= |\nu_1| + |\nu_2|\end{aligned}$$

□

**Note 3.14.** Recall that a previous exercise from the section on complex valued functions tells us that  $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$ .

**Definition 3.15.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then we define  $L^1(\nu) = L^1(|\nu|)$ . For  $f \in L^1(\nu)$ , we define

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

**Exercise 3.16.** Let  $\nu_1, \nu_2$  be signed measures on  $(X, \mathcal{A})$ . Suppose that  $\nu_1 + \nu_2$  is a signed measure. Then  $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$

*Proof.* The previous exercise tells us that  $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ . Two previous exercises from the section on nonnegative functions tells us that

$$\begin{aligned}\int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\ &= \int |f| d|\nu_1| + \int |f| d|\nu_2|\end{aligned}$$

□

**Exercise 3.17.** Let  $\nu, \mu$  be signed measures on  $(X, \mathcal{A})$  and  $E \in \mathcal{A}$ . Then

- (1)  $E$  is  $\nu$ -null iff  $|\nu|(E) = 0$
- (2)  $\nu \perp \mu$  iff  $|\nu| \perp \mu$  iff  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ .

*Proof.* (1) Suppose that  $E$  is  $\nu$ -null. Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Then  $\nu^+(E) = \nu(E \cap P) = 0$  and  $\nu^-(E) = \nu(E \cap N) = 0$ . Therefore  $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$ . Conversely, suppose that  $|\nu|(E) = 0$ . Then  $\nu^+(E) = \nu^-(E) = 0$ . Let  $F \in \mathcal{A}$ . Suppose that  $F \subset E$ . Then  $\nu^+(F) = 0$  and  $\nu^-(F) = 0$ . Therefore  $\nu(F) = \nu^+(F) - \nu^-(F) = 0$ . So  $E$  is  $\nu$ -null.

- (2) Suppose that  $\nu \perp \mu$ . Then there exist  $E, F \in \mathcal{A}$  such that  $E \cup F = X$ ,  $E \cap F = \emptyset$ ,  $E$  is  $\mu$ -null and  $F$  is  $\nu$ -null. By (1),  $F$  is  $|\nu|$ -null and thus  $|\nu| \perp \mu$ . If  $|\nu| \perp \mu$ , choose  $E, F \in \mathcal{A}$  as before. Since  $F$  is  $|\nu|$ -null, we know that  $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$ . This implies that  $F$  is  $\nu^+$ -null and  $F$  is  $\nu^-$ -null. So  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ . Finally assume that  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ . **FINISH!!!!**

□

**Exercise 3.18.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then

- (1) for  $f \in L^1(\nu)$ ,  $|\int f d\nu| \leq \int |f| d|\nu|$
- (2) if  $\nu$  is finite, then for each  $E \in \mathcal{A}$ ,  $|\nu|(E) = \sup\{|\int_E f d\nu| : f \text{ is measurable and } |f| \leq 1\}$

*Proof.* (1) Let  $f \in L^1(\nu)$ . Then

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f d\nu^+ - \int f d\nu^- \right| \\ &\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \\ &= \int |f| d(\nu^+ + \nu^-) \\ &= \int |f| d|\nu| \end{aligned}$$

- (2) Let  $E \in \mathcal{A}$ . Let  $f : X \rightarrow \mathbb{R}$  be measurable and suppose that  $|f| \leq 1$ . Since  $\nu$  is finite, so is  $|\nu|$  and thus  $f \in L^1(\nu)$ . Then (1) tells us that

$$\begin{aligned} \left| \int_E f d\nu \right| &\leq \int_E |f| d|\nu| \\ &\leq |\nu|(E) \end{aligned}$$

Now, choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Define  $f = \chi_P - \chi_N$ . Then  $|f| \leq 1$ ,  $f$  is measurable and

$$\begin{aligned} \left| \int_E f d\nu \right| &= \left| \int_E f d\nu^+ - \int_E f d\nu^- \right| \\ &= |\nu^+(E \cap P) + \nu^-(E \cap N)| \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E). \end{aligned}$$

□

**Exercise 3.19.** Let  $\mu$  be a positive measure on  $(X, \mathcal{A})$  and  $f \in L^0(X, \mathcal{A})$  extended  $\mu$ -integrable. Define  $\nu$  on  $(X, \mathcal{A})$  by  $\nu(E) = \int_E f d\mu$ . Then

- (1)  $\nu$  is a signed measure
- (2) for each  $E \in \mathcal{A}$ ,  $|\nu|(E) = \int_E |f| d\mu$ .

*Proof.* (1) Clearly  $\nu(\emptyset) = 0$  and  $\nu$  is finite by assumption. Let  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Suppose that  $(E_n)_{n \in \mathbb{N}}$  is disjoint. Then

$$\begin{aligned}
 \nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \\
 &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu - \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
 &= \sum_{n \in \mathbb{N}} \left[ \int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right] \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n} f d\mu \\
 &= \sum_{n \in \mathbb{N}} \nu(E_n)
 \end{aligned}$$

If  $|\nu(\bigcup_{n \in \mathbb{N}} E_n)| < \infty$ , then  $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu < \infty$  and  $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu < \infty$  because

$$\begin{aligned}
 |\nu(\bigcup_{n \in \mathbb{N}} E_n)| &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \right| \\
 &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \right|
 \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
 \sum_{n \in \mathbb{N}} |\nu(E_n)| &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f d\mu \right| \\
 &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right| \\
 &\leq \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu + \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
 &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu + \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
 &< \infty
 \end{aligned}$$

So the sum  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely and  $\nu$  is a signed measure.

- (2) Put  $P = \{x \in X : f(x) \geq 0\}$  and  $N = \{x \in X : f(x) < 0\}$ . Then  $P, N$  form a Hahn decomposition of  $X$  with respect to  $\nu$ . Thus for  $E \in \mathcal{A}$ ,

$$\nu^+(E) = \int_{E \cap P} f d\mu = \int_E f^+ d\mu$$

and

$$\nu^-(E) = \int_{E \cap N} f d\mu = \int_E f^- d\mu$$

. So for  $E \in \mathcal{A}$ ,

$$|\nu|(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$$

□

### 3.2. The Lebesgue-Radon-Nikodym Theorem.

**Definition 3.20.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  be a signed measure on  $(X, \mathcal{A})$  and  $\mu$  a measure on  $(X, \mathcal{A})$ . Then  $\nu$  is said to be **absolutely continuous** with respect to  $\mu$ , denoted  $\nu \ll \mu$ , if for each  $E \in \mathcal{A}$ ,  $\mu(E) = 0$  implies that  $\nu(E) = 0$ .

**Note 3.21.** If there exists an extended  $\mu$ -integrable  $f \in L^0(X, \mathcal{A})$  such that for each  $E \in \mathcal{A}$ ,  $\nu(E) = \int_E f d\mu$ , then we write  $d\nu = f d\mu$ .

**Theorem 3.22.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  be a  $\sigma$ -finite signed measure on  $(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measure on  $(X, \mathcal{A})$ . Then there exist unique  $\sigma$ -finite signed measures  $\lambda, \rho$  on  $(X, \mathcal{A})$  such that  $\lambda \perp \mu$ ,  $\rho \ll \mu$  and  $\nu = \lambda + \rho$ , and there exists an extended  $\mu$ -integrable  $f \in L^0(X, \mathcal{A})$  such that  $\rho = f d\mu$  and  $f$  is unique  $\mu$ -a.e.

**Definition 3.23.** The decomposition  $\nu = \lambda + \rho$  is referred to as the **Lebesgue decomposition of  $\nu$  with respect to  $\mu$** . In the case  $\nu \ll \mu$ , we have  $\lambda = 0$  and  $\rho = \nu$  and we define the **Radon-Nikodym derivative of  $\nu$  with respect to  $\mu$** , denoted by  $d\nu/d\mu$ , to be  $d\nu/d\mu = f$  where  $d\nu = f d\mu$ .

**Theorem 3.24.** Let  $\nu$  be a  $\sigma$ -finite signed measure on  $(X, \mathcal{A})$  and  $\mu, \lambda$   $\sigma$ -finite measures on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \mu$  and  $\mu \ll \lambda$ . Then

(1) for each  $g \in L^1(\nu)$ ,  $g(d\nu/d\mu) \in L^1(\mu)$  and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2)  $\nu \ll \lambda$  and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

**Exercise 3.25.** Let  $(\nu_n)_{n \in \mathbb{N}}$  be a sequence of measures and  $\mu$  a measure.

(1) If for each  $n \in \mathbb{N}$ ,  $\nu_n \ll \mu$ , then  $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$ .

(2) If for each  $n \in \mathbb{N}$ ,  $\nu_n \perp \mu$ , then  $\sum_{n \in \mathbb{N}} \nu_n \perp \mu$ .

*Proof.* (1) Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) = 0$ . Then for each  $n \in \mathbb{N}$ ,  $\nu_n(E) = 0$  and thus  $\sum_{n \in \mathbb{N}} \nu_n(E) = 0$ . Hence  $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$ .

(2) For each  $n \in \mathbb{N}$ , there exist  $N_i, M_i \in \mathcal{A}$  such that  $N_i \cap M_i = \emptyset$ ,  $N_i \cup M_i = X$  and  $\nu_i(M_i) = \mu(N_i) = 0$ . Put  $N = \bigcup_{n \in \mathbb{N}} N_i$  and  $M = N^c$ . Note that for each  $n \in \mathbb{N}$ ,  $M \subset N_i^c = M_i$ . So  $\mu(N) \leq \sum_{n \in \mathbb{N}} \mu(N_i) = 0$  and  $(\sum_{n \in \mathbb{N}} \nu_i)(M) \leq \sum_{n \in \mathbb{N}} \nu_i(M_i) = 0$ . Thus  $\sum_{n \in \mathbb{N}} \nu_i \perp \mu$ .

□

**Exercise 3.26.** Choose  $X = [0, 1]$ ,  $\mathcal{A} = \mathcal{B}_{[0,1]}$ . Let  $m$  be Lebesgue measure and  $\mu$  the counting measure.

Then

- (1)  $m \ll \mu$  but for each  $f \in L^+$ ,  $dm \neq f d\mu$
- (2) There is no Lebesgue decomposition of  $\mu$  with respect to  $m$ .

*Proof.* (1) Let  $E \in \mathcal{A}$ . If  $\mu(E) = 0$ , then  $E = \emptyset$  and  $m(E) = 0$ . So  $m \ll \mu$ . Suppose for the sake of contradiction that there exists  $f \in L^+$  such that  $dm = f d\mu$ . Then

$$\begin{aligned} 1 &= m(X) \\ &= \sum_{x \in X} f(x) \end{aligned}$$

Put  $Z = \{x \in X : f(x) \neq 0\}$ . Then  $Z$  is countable. So

$$\begin{aligned} 1 &= m(X \setminus Z) \\ &= \sum_{x \in X \setminus Z} f(x) \\ &= 0 \end{aligned}$$

This is a contradiction, so no such  $f$  exists.

- (2) Suppose for the sake of contradiction that there is a Lebesgue decomposition for  $\mu$  with respect to  $m$  given by  $\mu = \lambda + \rho$  where  $\lambda \perp m$  and  $\rho \ll m$ . We may assume  $\lambda$  and  $\rho$  are positive. Then for each  $x \in X$ ,  $m(\{x\}) = 0$  which implies that  $\rho(\{x\}) = 0$ . Let  $E \subset X$ , if  $E$  is countable, then  $\lambda(E) = \mu(E)$ . If  $E$  is uncountable, choose  $F \subset E$  such that  $F$  is countable. Then

$$\begin{aligned} \lambda(E) &\geq \lambda(F) \\ &= \mu(F) \\ &= \infty \end{aligned}$$

So  $\lambda = \mu$ . This is a contradiction since  $\mu \not\ll m$ . □

**Exercise 3.27.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and  $\mathcal{E}$  a sub  $\sigma$ -alg of  $\mathcal{F}$  and  $f \in L^1(\mu)$ . Define  $\nu : \mathcal{E} \rightarrow [0, \infty]$  by  $\nu(E) = \int_E f d\mu$ . Let  $\bar{\mu}$  be the restriction of  $\mu$  to  $\mathcal{E}$ . Define the **expectation of  $f$  given  $\mathcal{E}$**  to be  $E[f|\mathcal{E}] = d\nu/d\bar{\mu}$ . Then for each  $E \in \mathcal{E}$ ,

$$\int_E E[f|\mathcal{E}] d\mu = \int_E f d\mu$$

*Proof.* Let  $E \in \mathcal{E}$ . By definition,

$$\begin{aligned} \int_E E[f|\mathcal{E}] d\mu &= \int_E d\nu/d\bar{\mu} d\mu \\ &= \int_E d\nu/d\bar{\mu} d\bar{\mu} \quad (\text{since } E \in \mathcal{E}) \\ &= \nu(E) \\ &= \int_E f d\mu \end{aligned}$$

□

### 3.3. Complex Measures.

**Definition 3.28.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow \mathbb{C}$ . Then  $\nu$  is said to be a **complex measure** if

- (1)  $\nu(\emptyset) = 0$
- (2) for each sequence  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ , if  $(E_n)_{n \in \mathbb{N}}$  is disjoint, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$  and  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely.

**Note 3.29.** We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.

**Definition 3.30.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu = \nu_1 + i\nu_2$  a complex measure on  $(X, \mathcal{A})$ . We define  $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$ . For  $f \in L^1(\nu)$ , we define

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

**Theorem 3.31.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  a complex measure on  $(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measure on  $(X, \mathcal{A})$ . Then there exists a complex measure  $\lambda$  on  $(X, \mathcal{A})$  and  $f \in L^1(\mu)$  such that  $\lambda \perp \mu$  and  $d\nu = d\lambda + f d\mu$  and such that for each complex measure  $\lambda'$  on  $(X, \mathcal{A})$ ,  $f' \in L^1(\mu)$ , if  $\nu = d\lambda' + f' d\mu$ , then  $\lambda = \lambda'$  and  $f = f'$   $\mu$ -a.e.

**Theorem 3.32.** Let  $\nu$  be a complex measure on  $(X, \mathcal{A})$  and  $\mu, \lambda$   $\sigma$ -finite measures on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \mu$  and  $\mu \ll \lambda$ . Then

- (1) for each  $g \in L^1(\nu)$ ,  $g(d\nu/d\mu) \in L^1(\mu)$  and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

- (2)  $\nu \ll \lambda$  and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

**Definition 3.33.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu = \nu_1 + i\nu_2$  a complex measure on  $(X, \mathcal{A})$ . Define  $\mu = |\nu_1| + |\nu_2|$ . Then  $\nu \ll \mu$  and thus There exists  $f \in L^1(\mu)$  such that  $d\nu = f d\mu$ . Define  $|\nu| : \mathcal{A} \rightarrow [0, \infty)$  by  $|\nu|(E) = \int_E |f| d\mu$  for each  $E \in \mathcal{A}$ . We call  $|\nu|$  the **total variation of  $\nu$** .

**Exercise 3.34.** Let  $\nu$  be a complex measure on  $(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measures on  $(X, \mathcal{A})$ . If  $\nu \ll \mu$ , then  $\{x \in X : d\nu/d\mu(x) = 0\}$  is  $\nu$ -null.

*Proof.* Define  $f = d\nu/d\mu$  and  $E = \{x : f(x) = 0\}$ . Let  $A \in \mathcal{A}$  and suppose that  $A \subset E$ . Then

$$\begin{aligned} \nu(A) &= \int_A f d\mu \\ &= 0 \end{aligned}$$

□

**Exercise 3.35.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu = \nu_1 + i\nu_2$  a complex measure on  $(X, \mathcal{A})$ . Then  $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu_2|$ .

*Proof.* Let  $\mu$  and  $f$  be as in the definition of  $|\nu|$ . Since for each  $E \in \mathcal{A}$ , we have

$$\begin{aligned}\nu(E) &= \int_E f d\mu \\ &= \int_E f_1 d\mu + i \int_E f_2 d\mu\end{aligned}$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E)$$

we know that  $\nu_1 = f_1 d\mu$  and  $\nu_2 = f_2 d\mu$ .

A previous exercise tells us that  $d|\nu_1| = |f_1| d\mu$  and  $d|\nu_2| = |f_2| d\mu$ . Since  $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$ , we have that

$$\begin{aligned}|\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2|\end{aligned}$$

□

**Exercise 3.36.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  a complex measure on  $(X, \mathcal{A})$  and  $c \in \mathbb{C}$ . Then  $|c\nu| = |c||\nu|$ .

*Proof.* Define  $\mu$  and  $f$  as before so that  $d\nu = f d\mu$ . Then  $d(c\nu) = cf d\mu$ . Hence

$$\begin{aligned}d|c\nu| &= |cf| d\mu \\ &= |c||f| d\mu \\ &= |c| d|\nu|\end{aligned}$$

So  $|c\nu| = |c||\nu|$ .

□

**Exercise 3.37.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu$  a complex measure on  $(X, \mathcal{A})$ . Then

- (1) for each  $E \in \mathcal{A}$ ,  $|\nu(E)| \leq |\nu|(E)$ .
- (2)  $\nu \ll |\nu|$  and  $|d\nu/d|\nu|| = 1$   $|\nu|$ -a.e.
- (3)  $L^1(\nu) = L^1(|\nu|)$  and for each  $g \in L^1(\nu)$ ,  $|\int g d\nu| \leq \int |g| d|\nu|$

*Proof.* Let  $\mu, f \in L^1(\mu)$  be as in the definition of  $|\nu|$ .

- (1) Let  $E \in \mathcal{A}$ . Then

$$\begin{aligned}|\nu(E)| &= \left| \int_E f d\mu \right| \\ &\leq \int_E |f| d\mu \\ &= |\nu|(E)\end{aligned}$$

- (2) Let  $E \in \mathcal{A}$  and suppose that  $|\nu|(E) = 0$ . The previous part implies  $|\nu(E)| = 0$  and  $\nu \ll |\nu|$ . Put  $g = d\nu/d|\nu|$ . Then

$$\begin{aligned}f &= \frac{d\nu}{d\mu} \\ &= g|f| \quad \mu\text{-a.e.}\end{aligned}$$

Hence  $|f| = |g||f|$   $\mu$ -a.e. Since  $|\nu| \ll \mu$ ,  $|f| = |g||f|$   $|\nu|$ -a.e.



A previous exercise tells us that  $|f| \neq 0$   $|\nu|$ -a.e. Thus  $|g| = 1$   $|\nu|$ -a.e.

(3) Write  $\nu = \nu_1 + i\nu_2$  and  $f = f_1 + if_2$ . First we observe that

$$\begin{aligned} L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\ &= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\ &= L^1(|\nu_1| + |\nu_2|) \\ &= L^1(\mu) \end{aligned}$$

The previous exercise tells us that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \\ &= \mu \end{aligned}$$

Let  $g \in L^1(\mu)$ . Then

$$\begin{aligned} \int |g|d|\nu| &\leq \int |g|d\mu \\ &< \infty \end{aligned}$$

So  $g \in L^1(|\nu|)$ .

Conversely, let  $g \in L^1(|\nu|)$ . Then

$$\begin{aligned} \int |g|d|\nu_1|, \int |g|d|\nu_2| &\leq \int |g|d|\nu| \\ &< \infty \end{aligned}$$

So

$$\begin{aligned} \int |g|d\mu &= \int |g|d|\nu_1| + \int |g|d|\nu_2| \\ &< \infty \end{aligned}$$

and  $g \in L^1(\mu)$ . Hence  $L^1(\nu) = L^1(|\nu|)$ .

Now, let  $g \in L^1(\nu) = L^1(|\nu|)$ , then

$$\begin{aligned} \left| \int g d\nu \right| &= \left| \int g f d\mu \right| \\ &\leq \int |g| |f| d\mu \\ &= \int |g| d|\nu| \end{aligned}$$

□

### 3.4. Differentiation.

**Definition 3.38.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ . Then  $f$  is said to be **locally integrable** (with respect to Lebesgue measure) if  $f$  is measurable and for each  $K \subset \mathbb{R}^n$ ,  $K$  compact implies  $\int_K |f| dm < \infty$ . We define  $L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{C} : f \text{ is locally integrable}\}$

**Definition 3.39.** For  $f \in L^1_{loc}(\mathbb{R}^n)$ ,  $r > 0$ ,  $x \in \mathbb{R}^n$ , we define the **average of  $f$  over  $B(x, r)$** , denoted by  $Af(x, r)$ , to be

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f dm$$

**Exercise 3.40.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . Define

$$H^*f(x) = \sup \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\} \quad (x \in \mathbb{R}^n)$$

Then  $Hf \leq H^*f \leq 2^n Hf$ .

*Proof.* Let  $x \in \mathbb{R}^n$ . Then

$$\left\{ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f| dm : r > 0 \right\} \subset \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\}$$

So  $Hf(x) \leq H^*f(x)$ . Let  $B$  be a ball. Then there exists  $y \in \mathbb{R}^n$ ,  $R > 0$  such that  $B = B(y, R)$ . Suppose that  $x \in B$ . Then  $B \subset B(x, 2R)$ . Since  $m(B(x, 2R)) = 2^n m(B(y, R))$ , we have that

$$\begin{aligned} \frac{1}{m(B)} \int_B |f| dm &\leq \frac{1}{m(B)} \int_{m(B(x, 2R))} |f| dm \\ &= \frac{2^n}{m(B(x, 2R))} \int_{m(B(x, 2R))} |f| dm \end{aligned}$$

Thus  $H^*f(x) \leq 2^n Hf(x)$ . □

**Lemma 3.41.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ , then  $Af : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  is continuous.

**Definition 3.42.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . We define its **Hardy Littlewood maximal function**, denoted by  $Hf$  to be

$$Hf(x) = \sup_{r>0} A|f|(x, r) \quad x \in \mathbb{R}^n$$

**Theorem 3.43.** There exists  $C > 0$  such that for each  $f \in L^1(m)$  and  $\alpha > 0$ ,

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f| dm$$

**Exercise 3.44.** Let  $f \in L^1(\mathbb{R}^n)$ . Suppose that  $\|f\|_1 > 0$ . Then there exist  $C, R > 0$  such that for each  $x \in \mathbb{R}^n$ , if  $|x| > R$ , then  $Hf(x) \geq C|x|^{-n}$ . Hence there exists  $C' > 0$  such that for each  $\alpha > 0$ ,  $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$  when  $\alpha$  is small.

*Proof.* Since  $\|f\|_1 > 0$ , there exists  $R > 0$  such that  $\int_{B(0, R)} |f| dm > 0$ . Recall that there exists  $K > 0$  such that for each  $x \in \mathbb{R}^n$  and  $r > 0$ ,  $m(B(x, r)) = Kr^n$ . Choose

$$C = \frac{\int_{B(0, R)} |f| dm}{K2^n}$$

. Let  $x \in \mathbb{R}^n$ . Suppose that  $|x| > R$ . Then  $B(0, R) \subset B(x, 2|x|)$ . Thus

$$\begin{aligned} Hf(x) &\geq \frac{1}{m(B(x, 2|x|))} \int_{B(x, 2|x|)} |f| dm \\ &= \frac{1}{K2^n|x|^n} \int_{B(x, 2|x|)} |f| dm \\ &\geq \frac{1}{K2^n|x|^n} \int_{B(0, R)} |f| dm \\ &= \frac{C}{|x|^n} \end{aligned}$$

Let  $a < \frac{C}{2R^n}$ . Then  $R^n < \frac{C}{2a}$ . Choose  $C' = \frac{KC}{2}$ . Let  $A = \{x \in \mathbb{R}^n : R < |x| < (\frac{C}{a})^{\frac{1}{n}}\}$ . For  $x \in A$ ,

$$\begin{aligned} Hf(x) &\geq \frac{C}{|x|^n} \\ &> a \end{aligned}$$

Thus  $A \subset m(\{x \in \mathbb{R}^n : Hf(x) > a\})$  and therefore

$$\begin{aligned} m(\{x \in \mathbb{R}^n : Hf(x) > a\}) &\geq m(A) \\ &= m(B(0, (C/a)^{1/n})) - m(B(0, R)) \\ &= K \left[ \frac{C}{a} - R^n \right] \\ &> K \left[ \frac{C}{a} - \frac{C}{2a} \right] \\ &= \frac{KC}{2a} \\ &= \frac{C'}{a} \end{aligned}$$

□

**Theorem 3.45.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ , then for a.e.  $x \in \mathbb{R}^n$ ,

$$\lim_{r \rightarrow 0} Af(x, r) = f(x)$$

. Equivalently, for a.e.  $x \in \mathbb{R}^n$ ,

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} [f(y) - f(x)] dm(y) \right] = 0$$

**Note 3.46.** We can a stronger result of the same flavor.

**Definition 3.47.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . We define the **Lebesgue set of  $f$** , denoted by  $L_f$ , to be

$$\begin{aligned} L_f &= \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0} A|f - f(x)|(x, r) = 0\} \\ &= \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0 \right\} \end{aligned}$$

**Exercise 3.48.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ . If  $f$  is continuous at  $x$ , then  $x \in L_f$ .

*Proof.* Suppose that  $f$  is continuous at  $x$ . Let  $\epsilon > 0$ . By assumption, there exists  $\delta > 0$  such that for each  $y \in \mathbb{R}^n$ , if  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ . Let  $r > 0$ . Suppose that  $r < \delta$ . Then for each  $y \in \mathbb{R}^n$ ,  $y \in B(x, r)$  implies that  $|f(x) - f(y)| < \epsilon$  and thus

$$\begin{aligned} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(B(x, r))} \epsilon m(B(x, r)) \\ &= \epsilon \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0$$

and  $x \in L_f$ . □

**Theorem 3.49.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . Then  $m((L_f)^c) = 0$

**Definition 3.50.** Let  $x \in \mathbb{R}^n$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ . Then  $(E_r)_{r>0}$  is said to **shrink nicely** to  $x$  if

- (1) for each  $r > 0$ ,  $E_r \subset B(x, r)$
- (2) there exists  $\alpha > 0$  such that for each  $r > 0$ ,  $m(E_r) > \alpha m(B(x, r))$

**Theorem 3.51.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ . Then for each  $x \in L_f$ ,

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \right] = 0$$

and

$$\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

**Definition 3.52.** Let  $\mu : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, \infty]$  be a Borel measure. Then  $\mu$  is said to be **regular** if

- (1) for each  $K \subset \mathbb{R}^n$ , if  $K$  is compact, then  $\mu(K) < \infty$
- (2) for each  $E \in \mathcal{B}(\mathbb{R}^n)$ ,  $\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subset U\}$

Let  $\nu$  be a signed or complex Borel measure on  $\mathbb{R}^n$ . Then  $\nu$  is said to be regular if  $|\nu|$  is regular.

**Theorem 3.53.** Let  $\nu$  be a regular signed or complex measure on  $\mathbb{R}^n$ . Let  $d\nu = d\lambda + f dm$  be the Lebesgue decomposition of  $\nu$  with respect to  $m$ . Then for  $m$ -a.e.  $x \in \mathbb{R}^n$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ , if  $(E_r)_{r>0}$  shrinks nicely to  $x$ , then

$$\lim_{r \rightarrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

### 3.5. Functions of Bounded Variation.

**Definition 3.54.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Define  $F_+ : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F_+(x) = \lim_{t \rightarrow x^+} F(t) = \inf\{F(t) : t > x\}$$

**Note 3.55.** Observe that  $F \leq F_+$  and  $F_+$  is increasing.

**Exercise 3.56.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then for each  $x \in \mathbb{R}$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $y \in (x, x + \delta)$ ,  $0 \leq F_+(y) - F(y) \leq \epsilon$ .

*Proof.* For the sake of contradiction, suppose not. Then there exists  $x \in \mathbb{R}$  and  $\epsilon > 0$  such that for each  $\delta > 0$ , there exist  $y \in (x, x + \delta)$  such that  $F_+(y) - F(y) > \epsilon$ . Then there exists a sequence  $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  such that for each  $n \in \mathbb{N}$ ,  $y_n \in (x, x + \frac{1}{n})$ ,  $y_n > y_{n+1}$  and  $F_+(y_n) - F(y_n) > \epsilon$ . Choose  $N \in \mathbb{N}$  such that  $(N - 1)\epsilon > F(y_1) - F(x)$ . Then

$$\begin{aligned} F(y_1) - F(x) &= \sum_{i=1}^{N-1} \left[ F(y_i) - F_+(y_{i+1}) + F_+(y_{i+1}) - F(y_{i+1}) \right] + F(y_N) - F(x) \\ &= \sum_{i=1}^{N-1} \left[ F(y_i) - F_+(y_{i+1}) \right] + \sum_{i=1}^{N-1} \left[ F_+(y_{i+1}) - F(y_{i+1}) \right] + F(y_N) - F(x) \\ &\geq (N - 1)\epsilon \\ &> F(y_1) - F(x) \end{aligned}$$

This is a contradiction, so the claim holds.  $\square$

**Exercise 3.57.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then  $F_+$  is right continuous.

*Proof.* Let  $x \in \mathbb{R}$ . Let  $\epsilon > 0$ . Then there exists  $\delta_1 > 0$  such that for each  $y \in (x, x + \delta_1)$   $0 \leq F(y) - F_+(x) < \epsilon/2$ . There exists  $\delta_2 > 0$  such that for each  $y \in (x, x + \delta_2)$ ,  $0 \leq F_+(y) - F(y) < \epsilon/2$ . Choose  $\delta = \min\{\delta_1, \delta_2\}$ . Let  $y \in (x, x + \delta)$ .

$$\begin{aligned} |F_+(x) - F_+(y)| &\leq |F_+(x) - F(y)| + |F(y) - F_+(y)| \\ &= (F(y) - F_+(x)) + (F_+(y) - F(y)) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So  $\lim_{t \rightarrow x^+} F_+(t) = F_+(x)$  and  $F_+$  is right continuous.  $\square$

**Theorem 3.58.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then

- (1)  $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$  is countable
- (2)  $F$  and  $F_+$  are differentiable a.e. and  $F' = F'_+$  a.e.

**Definition 3.59.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Define  $T_F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$T_F(x) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \quad (x \in \mathbb{R})$$

$T_F$  is called the **total variation function of  $F$** .

**Exercise 3.60.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $T_F$  is increasing.

*Proof.* Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ .

Define  $A_x = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$  and  $A_y = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = y \right\}$ . Let  $z \in A_x$ . Then there exists  $(x_i)_{i=0}^n \subset \mathbb{R}$  such that  $(x_i)_{i=0}^n$  is increasing,  $x_n = x$  and  $z = \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$ . Then

$$\begin{aligned}
z &\leq z + |F(y) - F(x)| \\
&= \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \\
&\in A_y
\end{aligned}$$

So  $z \leq \sup A_y = T_F(y)$  and thus  $F_T(x) = \sup A_x \leq T_F(y)$   $\square$

**Lemma 3.61.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $T_F + F$  and  $T_F - F$  are increasing.*

**Exercise 3.62.** *For each  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $T_{|F|} \leq T_F$ .*

*Proof.* Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then by the reverse triangle inequality,

$$\sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

Thus

$$\begin{aligned}
T_{|F|}(x) &= \sup \left\{ \sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&\leq \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&= T_F(x)
\end{aligned}$$

Hence  $T_{|F|} \leq T_F$   $\square$

**Definition 3.63.** *Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is said to have **bounded variation** if  $\lim_{x \rightarrow \infty} T_F(x) < \infty$ . The **total variation of  $F$** , denoted by  $TV(F)$ , is defined to be  $TV(F) = \lim_{x \rightarrow \infty} T_F(x)$ . We define  $BV = \{F : \mathbb{R} \rightarrow \mathbb{C} : TV(F) < \infty\}$ .*

**Definition 3.64.** *Let  $a, b \in \mathbb{R}$  and  $F : [a, b] \rightarrow \mathbb{C}$ . Define  $G_F : \mathbb{R} \rightarrow \mathbb{C}$  by  $G_F = F(a)\chi_{(-\infty, a)} + F\chi_{[a, b]} + F(b)\chi_{(b, \infty)}$ . Then  $F$  is said to have **bounded variation on  $[a, b]$**  if  $G_F \in BV$ . The **total variation of  $F$  on  $[a, b]$** , denoted by  $TV(F, [a, b])$ , is defined to be  $TV(F, [a, b]) = TV(G_F)$ . We define  $BV([a, b]) = \{F : [a, b] \rightarrow \mathbb{C} : TV(F, [a, b]) < \infty\}$ .*

**Note 3.65.** *Equivalently,  $TV(F, [a, b]) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset [a, b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$  and  $F \in BV([a, b])$  iff  $TV(F, [a, b]) < \infty$ . In general,*

**Exercise 3.66.** *Let  $F \in BV$ . Then  $F$  is bounded.*

*Proof.* If  $F$  is unbounded, then the supremum in the previous definition is clearly infinite.  $\square$

**Exercise 3.67.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . If  $F$  is bounded and increasing, then  $F \in BV$ .*

*Proof.* Suppose that  $F$  is bounded and increasing. Then  $-\infty < \inf_{x \in \mathbb{R}} F(x) \leq \sup_{x \in \mathbb{R}} F(x) < \infty$ . Let  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n F(x_i) - F(x_{i-1}) \\ &= F(x) - F(x_0) \end{aligned}$$

Thus

$$T_F(x) = F(x) - \inf_{x \in \mathbb{R}} F(x)$$

. This implies that

$$\begin{aligned} TV(F) &= \sup_{x \in \mathbb{R}} F(x) - \inf_{x \in \mathbb{R}} F(x) \\ &< \infty \end{aligned}$$

Hence  $F \in BV$ . □

**Exercise 3.68.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . If  $F$  is differentiable and  $F'$  is bounded on  $[a, b]$ , then,  $F \in BV([a, b])$ .

*Proof.* Suppose that  $F$  is differentiable and  $F'$  is bounded on  $[a, b]$ . Then there exists  $M > 0$  such that for each  $x \in [a, b]$ ,  $|F'(x)| \leq M$ . Let  $(x_i)_{i=1}^n \subset [a, b]$ . Suppose that  $(x_i)_{i=1}^n$  is strictly increasing,  $x_0 = a$  and  $x_n = b$ . By the mean value theorem, for each  $i = 1, 2, \dots, n$ , there exists  $c_i \in (x_{i-1}, x_i)$  such that  $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n |F'(c_i)(x_i - x_{i-1})| \\ &\leq \sum_{i=1}^n M(x_i - x_{i-1}) \\ &= M(b - a) \end{aligned}$$

Hence  $TV(F, [a, b]) \leq M(b - a)$ . □

**Exercise 3.69.** Define  $F, G : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x) = \begin{cases} x^2 \sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 \sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then  $F$  and  $G$  are differentiable,  $F \in BV([-1, 1])$  and  $G \notin BV([-1, 1])$ .

*Proof.* On  $\mathbb{R} \setminus \{0\}$ ,

$$\begin{aligned} F'(x) &= 2x \sin(x^{-1}) - \sin(x^{-1}) \\ &= \sin(x^{-1})(2x - 1) \end{aligned}$$

We see that  $F$  is also differentiable at  $x = 0$  since

$$\begin{aligned} F'(0) &= \lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-1})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-1}) \\ &= 0 \end{aligned}$$

Therefore for each  $x \in [-1, 1]$ ,  $|F'(x)| \leq 3$ . Which by a previous exercise implies that  $F \in BV([-1, 1])$ .

On  $\mathbb{R} \setminus \{0\}$ ,

$$\begin{aligned} G'(x) &= 2x \sin(x^{-2}) - \frac{2 \sin(x^{-2})}{x} \\ &= \sin(x^{-2}) \left(2x - \frac{2}{x}\right) \end{aligned}$$

We see that  $G$  is also differentiable at  $x = 0$  since

$$\begin{aligned} G'(0) &= \lim_{x \rightarrow 0} \frac{G(x) - G(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-2})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-2}) \\ &= 0 \end{aligned}$$

For  $n \in \mathbb{N}$ , define  $(x_i)_{i=0}^n \subset [-1, 1]$  by

$$x_i = \frac{-1}{\sqrt{\frac{\pi}{2} + i\pi}}$$

Then for each  $n \in \mathbb{N}$ ,  $(x_i)_{i=1}^n$  is strictly increasing and for each  $i = 1, 2, \dots, n$  we have that

$$\begin{aligned} |G(x_i) - G(x_{i-1})| &= \frac{1}{\frac{\pi}{2} + i\pi} + \frac{1}{\frac{\pi}{2} + (i-1)\pi} \\ &= \frac{2}{\pi} \left[ \frac{(2i-1) + (2i+1)}{(2i+1)(2i-1)} \right] \\ &= \frac{2}{\pi} \left[ \frac{4i}{4i^2 - 1} \right] \\ &> \frac{2}{i\pi} \end{aligned}$$



Hence for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} TV(G, [-1, 1]) &\geq \sum_{i=1}^n |G(x_i) - G(x_{i-1})| \\ &> \frac{2}{\pi} \sum_{i=1}^n \frac{1}{i} \end{aligned}$$

Therefore  $G \notin BV([-1, 1])$ . □

**Exercise 3.70.** *The following is stated for  $BV$ , but is also true for  $BV([a, b])$ .*

- (1) *For each  $F, G \in BV$ ,  $T_{F+G} \leq T_F + T_G$  and therefore  $BV$  is a vector space.*
- (2) *For each  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $F \in BV$  iff  $\operatorname{Re}(f) \in BV$  and  $\operatorname{Im}(F) \in BV$ .*
- (3) *For each  $F : \mathbb{R} \rightarrow \mathbb{R}$ ,  $F \in BV$  iff there exist functions  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $F_1, F_2$  are bounded, increasing and  $F = F_1 - F_2$*
- (4) *For each  $F \in BV$  and  $x \in \mathbb{R}$ ,  $\lim_{t \rightarrow x^+} F(t)$  and  $\lim_{t \rightarrow x^-} F(t)$  exist.*
- (5) *For each  $F \in BV$ ,  $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$  is countable.*
- (6) *For each  $F \in BV$ ,  $F$  and  $F_+$  are differentiable a.e. and  $F' = (F_+)'$  a.e.*
- (7) *For each  $F \in BV, c \in \mathbb{R}$ ,  $F - c \in BV$*

*Proof.* (1) Let  $F, G \in BV$ ,  $x \in \mathbb{R}$  and  $\epsilon > 0$ . Since  $T_{F+G}(x) < \infty$ ,  $T_{F+G}(x) - \epsilon < T_{F+G}(x)$ . Thus there exists  $(x_i)_{i=0}^n \subset \mathbb{R}$  such that  $(x_i)_{i=0}^n$  is increasing,  $x_n = x$  and  $T_{F+G}(x) < \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1})| + \epsilon$ . Therefore

$$\begin{aligned} T_{F+G}(x) &< \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1})| + \epsilon \\ &\leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + \sum_{i=1}^n |G(x_i) - G(x_{i-1})| + \epsilon \\ &\leq T_F(x) + T_G(x) + \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary,  $T_{F+G}(x) \leq T_F(x) + T_G(x)$ . Therefore  $TV(F+G) \leq TV(F) + TV(G) < \infty$ . Thus  $F+G \in BV$ . It is straight forward to verify the other requirements needed to show that  $BV$  is a vector space.

- (2) Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Write  $F = F_1 + iF_2$  with  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$ . Suppose that  $F \in BV$ . Note that for each  $x_1, x_2 \in \mathbb{R}$  and  $j = 1, 2$ ,  $|F_j(x_1) - F_j(x_2)| \leq |F(x_1) - F(x_2)|$ . Let  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then for  $j = 1, 2$

$$\sum_{i=1}^n |F_j(x_i) - F_j(x_{i-1})| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

. Thus for  $j = 1, 2$  we have that  $T_{F_j}(x) \leq T_F(x)$  which implies that  $\operatorname{Re}(f), \operatorname{Im}(F) \in BV$ . Conversely, Suppose that  $\operatorname{Re}(f), \operatorname{Im}(F) \in BV$ . Then  $F = \operatorname{Re}(f) + i\operatorname{Im}(f) \in BV$  by (1).

- (3) Suppose that  $F \in BV$ . Choose  $F_1 = \frac{1}{2}(T_F - F)$  and  $F_2 = \frac{1}{2}(T_F + F)$ . Then  $F_1, F_2$  are bounded, increasing and  $F = F_1 + F_2$ . Conversely, if there exist  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $F_1, F_2$  are bounded, increasing and  $F = F_1 - F_2$ , then  $F_1, F_2 \in BV$ . By (1)  $F \in BV$ .
- (4) This is clear by previous results and (3)

- (5) This is clear by previous results and (3)
- (6) This is clear by previous results and (3)
- (7) Clearly constant functions have zero total variation. The rest is implied by (1).  $\square$

**Lemma 3.71.** *Let  $F \in BV$ . Then  $\lim_{x \rightarrow -\infty} T_F(x) = 0$  and if  $F$  is right continuous, then  $T_F$  is right continuous.*

**Definition 3.72.** *Define  $NBV = \{F \in BV : F \text{ is right continuous and } \lim_{x \rightarrow -\infty} F(x) = 0\}$ .*

**Theorem 3.73.** *Let  $M(\mathbb{R})$  be the set of complex Borel measures on  $\mathbb{R}$ . For  $F \in NBV$ , define  $\mu_F \in M(\mathbb{R})$  by  $\mu_F((-\infty, x]) = F(x)$ . Then  $F \mapsto \mu_F$  defines a bijection  $NBV \rightarrow M(\mathbb{R})$ . In addition,  $|\mu_F| = \mu_{T_F}$*

**Theorem 3.74.** *Let  $F \in NBV$ . Then  $F' \in L^1(m)$ ,  $\mu_F \perp m$  iff  $F' = 0$  a.e. and  $\mu_F \ll m$  iff for each  $x \in \mathbb{R}$ ,  $\int_{(-\infty, x]} F' dm = F(x)$*

**Definition 3.75.** *Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is said to be **absolutely continuous** if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$ ,  $\sum_{i=1}^n b_i - a_i < \delta$  implies that  $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$ .*

**Definition 3.76.** *Let  $F : [a, b] \rightarrow \mathbb{C}$ . Then  $F$  is said to be **absolutely continuous on**  $[a, b]$  if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$ ,  $\sum_{i=1}^n b_i - a_i < \delta$  implies that  $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$ .*

**Proposition 3.77.** *Let  $F : [a, b] \rightarrow \mathbb{C}$ . If  $F$  is absolutely continuous on  $[a, b]$ , then  $F \in BV[a, b]$ .*

**Exercise 3.78.** *Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Suppose that there exists  $f \in L^1(m)$  such that  $F(x) = \int_{(-\infty, x]} f dm$ . Then  $F \in NBV$ .*

*Proof.* Let  $x \in \mathbb{R}$  and  $(x_i)_{i=1}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=1}^n$  is increasing and  $x_n = x$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \left| \int_{(x_{i-1}, x_i]} f dm \right| \\ &\leq \sum_{i=1}^n \int_{(x_{i-1}, x_i]} |f| dm \\ &= \int_{(x_0, x]} |f| dm \\ &< \int |f| dm \end{aligned}$$

Hence  $T_F(x) \leq \int |f| dm$ . Since  $x \in \mathbb{R}$  is arbitrary,  $TV(F) \leq \int |f| dm$ . Therefore  $F \in BV$ . By the continuity from above and below for measures and the fact that  $m(x) = 0$  for each  $x \in \mathbb{R}$ ,  $F$  is continuous. By continuity from above for measures,  $\lim_{x \rightarrow -\infty} F(x) = 0$ . So  $F \in NBV$ .  $\square$

**Lemma 3.79.** *Let  $F \in NBV$ . Then  $F$  is absolutely continuous iff  $\mu_F \ll m$ .*

**Exercise 3.80.** *Fundamental Theorem of Calculus: Let  $F : [a, b] \rightarrow \mathbb{C}$ . The following are equivalent:*

- (1)  $F$  is absolutely continuous on  $[a, b]$ .  
 (2) there exists  $f \in L^1([a, b], m)$  such that for each  $x \in [a, b]$ ,  $F(x) - F(a) = \int_{(a, x]} f dm$   
 (3)  $F$  is differentiable a.e. on  $[a, b]$ ,  $F' \in L^1([a, b], m)$  and for each  $x \in [a, b]$ ,  $F(x) - F(a) = \int_{(a, x]} F' dm$

*Proof.* (1)  $\implies$  (3)

Suppose that  $F$  is absolutely continuous on  $[a, b]$ . Then  $F \in BV[a, b]$ . Extend  $F$  to  $\mathbb{R}$  by setting  $F(x) = F(a)$  for  $x < a$  and  $F(x) = F(b)$  for  $x > b$ . Then  $G = F - F(a) \in NBV$  and is absolutely continuous. The previous lemma implies that there exists  $f \in L^1(m)$  such that  $\mu_G = f dm$ . A previous theorem implies that for a.e.  $x \in [a, b]$

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow x} \frac{\mu_G((x, x+r])}{m((x, x+r])} \\ &= f(x) \end{aligned}$$

So  $F$  is differentiable a.e. on  $[a, b]$ ,  $F' \in L^1([a, b], m)$  and by construction, for each  $x \in [a, b]$ , we have that

$$\begin{aligned} F(x) - F(a) &= \mu_G((a, x]) \\ &= \int_{(a, x]} f dm \\ &= \int_{(a, x]} F' dm \end{aligned}$$

(3)  $\implies$  (2)

Trivial.

(2)  $\implies$  (1)

Suppose that there exists  $f \in L^1([a, b], m)$  such that for each  $x \in [a, b]$ ,  $F(x) - F(a) = \int_{(a, x]} f dm$ . Extend  $F$  as before and obtain  $G$  as before. Note that a previous exercise implies that  $G \in NBV$ . Since  $\mu_G \ll m$ , the previous lemma implies that  $G$  is absolutely continuous.  $\square$

**Exercise 3.81.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . If  $F$  is absolutely continuous. Then  $F$  is differentiable a.e.

*Proof.* Let  $n \in \mathbb{N}$ . Since  $F$  is absolutely continuous on  $\mathbb{R}$ ,  $F$  is absolutely continuous on  $[-n, n]$ . The FTC implies that  $F$  is differentiable a.e. on  $[-n, n]$ . Since  $n \in \mathbb{N}$  is arbitrary,  $F$  is differentiable a.e. on  $\mathbb{R}$ .  $\square$

**Exercise 3.82.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is Lipschitz continuous iff  $F$  is absolutely continuous and  $F'$  is bounded a.e.

*Proof.* Suppose that  $F$  is Lipschitz continuous. Then there exists  $M > 0$  such that for each  $x, y \in \mathbb{R}$ ,  $|F(x) - F(y)| \leq M|x - y|$ . Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{M}$ . Let  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$ , Suppose that  $\sum_{i=1}^n b_i - a_i < \delta$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(b_i) - F(a_i)| &\leq \sum_{i=1}^n M(b_i - a_i) \\ &< M\delta \\ &= \epsilon \end{aligned}$$

Hence  $F$  is absolutely continuous. For each  $x, y \in \mathbb{R}$ , if  $x \neq y$ , then  $\left| \frac{F(x) - F(y)}{x - y} \right| \leq M$ . Hence for a.e.  $x \in \mathbb{R}$ ,  $|F'(x)| \leq M$ . Conversely, suppose that  $F$  is absolutely continuous and  $F'$  is bounded a.e. Then there exists  $M > 0$  such that for a.e.  $x \in \mathbb{R}$ ,  $|F'(x)| \leq M$ . Let  $x, y \in \mathbb{R}$ . Suppose  $x < y$ . Then the FTC implies that

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_{(x,y]} F' dm \right| \\ &\leq \int_{(x,y]} |F'| dm \\ &= M|y - x| \end{aligned}$$

and  $F$  is Lipschitz continuous. □

**Exercise 3.83.** Construct an increasing function  $F : \mathbb{R} \rightarrow \mathbb{R}$  whose discontinuities is  $\mathbb{Q}$ .

*Proof.* Let  $(q_n)_{n \in \mathbb{N}}$  be an enumeration of  $\mathbb{Q}$ . Define  $F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F = \sum_{n \in \mathbb{N}} 2^{-n} \chi_{[q_n, \infty)}$$

. Equivalently, if we define  $S_x = \{n \in \mathbb{N} : q_n \leq x\}$ , then we may write

$$F(x) = \sum_{n \in S_x} 2^{-n}$$

Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ . Then  $S_x \subsetneq S_y$ . So  $F(x) < F(y)$  and therefore  $F$  is strictly increasing.

For each  $x, y \in \mathbb{R}$  with  $x < y$ , define  $S_{x,y} = \{n \in \mathbb{N} : x < q_n \leq y\}$ . Note that  $\lim_{y \rightarrow x^+} \min(S_{x,y}) = \infty$  and if  $y \in \mathbb{R} \setminus \mathbb{Q}$ , then  $\lim_{x \rightarrow y^-} \min(S_{x,y}) = \infty$ .

Now, let  $x \in \mathbb{R}$  and  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that  $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$ . Choose  $\delta > 0$  such that  $\min(S_{x, x+\delta}) \geq N$ . Let  $y \in [x, \infty)$ . Suppose that  $|x - y| < \delta$ . Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_y} 2^{-n} - \sum_{n \in S_x} 2^{-n} \\ &= \sum_{n \in S_{x,y}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence  $F$  is right continuous. Now let  $x \in \mathbb{R} \setminus \mathbb{Q}$  and  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  as before and  $\delta > 0$  such that  $\min(S_{x-\delta, x}) \geq N$ . Let  $y \in (-\infty, x]$ . Suppose that  $|x - y| < \delta$ . Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_x} 2^{-n} - \sum_{n \in S_y} 2^{-n} \\ &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence  $F$  is left continuous on  $\mathbb{R} \setminus \mathbb{Q}$ .

Now, let  $x \in \mathbb{Q}$ . Then there exists  $j \in \mathbb{N}$  such that  $q_j = x$ . Choose  $\epsilon = 2^{-j}$ . Let  $\delta > 0$ . Choose  $y = x - \frac{\delta}{2}$ . Then  $|x - y| < \delta$  and

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\geq 2^{-j} \\ &= \epsilon \end{aligned}$$

Hence  $F$  is discontinuous from the left at  $x$ . Since  $x \in \mathbb{Q}$  is arbitrary,  $F$  is discontinuous from the left on  $\mathbb{Q}$ .  $\square$

**Exercise 3.84.** Let  $(F_n)_{n \in \mathbb{N}} \in NBV$  be a sequence of nonnegative, increasing functions. If for each  $x \in \mathbb{R}$ ,  $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$ , then for a.e.  $x \in \mathbb{R}$ ,  $F$  is differentiable at  $x$  and  $F'(x) = \sum_{n \in \mathbb{N}} F'_n(x)$ .

*Proof.* Define  $\mu = \sum_{n \in \mathbb{N}} \mu_{F_n}$ . Note that

$$\begin{aligned} \mu((-\infty, x]) &= \sum_{n \in \mathbb{N}} \mu_{F_n}((-\infty, x]) \\ &= \sum_{n \in \mathbb{N}} F_n(x) \\ &= F(x) \end{aligned}$$

Hence  $F \in NBV$  and  $\mu = \mu_F$ . For each  $n \in \mathbb{N}$ , there exist  $\lambda_n \in M(\mathbb{R})$  and  $f_n \in L^1(\mathbb{R})$  such that  $d\mu_{F_n} = d\lambda_n + f_n dm$  and  $\lambda \perp m$ . Since for each  $n \in \mathbb{N}$ ,  $\lambda_n, f_n$  are nonnegative, we have that  $d\mu_F = \sum_{n \in \mathbb{N}} d\lambda_n + (\sum_{n \in \mathbb{N}} f_n) dm$ . By a previous theorem, for a.e.  $x \in \mathbb{R}$ ,

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow 0} \frac{\mu_F((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} f_n(x) \\ &= \sum_{n \in \mathbb{N}} \lim_{r \rightarrow 0} \frac{\mu_{F_n}((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} F'_n(x) \end{aligned}$$

□

**Exercise 3.85.** Let  $F : [0, 1] \rightarrow [0, 1]$  be the Cantor function. Extend  $F$  to  $\mathbb{R}$  by setting  $F(x) = 0$  for  $x < 0$  and  $F(x) = 1$  for  $x > 1$ . Let  $([a_n, b_n])_{n \in \mathbb{N}}$  be an enumeration of the closed subintervals of  $[0, 1]$  with rational endpoints. For  $n \in \mathbb{N}$ , define  $F_n : \mathbb{R} \rightarrow [0, 1]$  by  $F_n(x) = F(\frac{x-a_n}{b_n-a_n})$ . Define  $G : \mathbb{R} \rightarrow \mathbb{R}$  by  $G = \sum_{n \in \mathbb{N}} 2^{-n} F_n$ . Then  $G$  is continuous, strictly increasing on  $[0, 1]$  and  $G' = 0$  a.e.

*Proof.* Since  $F$  is continuous on  $\mathbb{R}$ , we have that for each  $n \in \mathbb{N}$ ,  $F_n$  is continuous on  $\mathbb{R}$ . We observe that for each  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ ,  $|2^{-n} F_n(x)| \leq 2^{-n}$ . Thus the Weierstrass M-test implies that  $G$  converges uniformly on  $\mathbb{R}$  and is therefore continuous. Since  $F$  is increasing, for each  $n \in \mathbb{N}$ ,  $F_n$  is increasing. Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ . Choose  $j \in \mathbb{N}$  such that  $x < a_j < y < b_j$ . Then

$$\begin{aligned} G(x) &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(x) \\ &= \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(x) + 0 \\ &< \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(y) + 2^{-j} F_n(y) \\ &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(y) \\ &= G(y) \end{aligned}$$

So  $G$  is strictly increasing.

Now we observe that for each  $n \in \mathbb{N}$ ,  $F_n \in NBV$ . The previous exercise implies that

$$G' = \sum 2^{-n} F'_n = 0 \text{ a.e.}$$

□

#### 4. TOPOLOGY

#### 5. $L^p$ SPACES

#### 6. FUNCTIONAL ANALYSIS

##### 6.1. Normed Vector Spaces.

**Note 6.1.** In the following, we will consider vector spaces over  $\mathbb{C}$ . There are analogous results for real vector spaces as well, just replace every  $\mathbb{C}$  with  $\mathbb{R}$ .

**Definition 6.2.** Let  $X$  be a normed vector space. Then  $X$  is said to be a **Banach space** if  $X$  is complete.

**Definition 6.3.** Let  $X$  be a normed vector space and  $(x_i)_{i=1}^n \subset X$ . The series  $\sum_{i=1}^{\infty} x_i$  is said to **converge** if the sequence  $s_n := \sum_{i=1}^n x_i$  converges. The series  $\sum_{i=1}^{\infty} x_i$  is said to **converge absolutely** if  $\sum_{i \in \mathbb{N}} \|x_i\| < \infty$ .

**Theorem 6.4.** Let  $X$  be a normed vector space. Then  $X$  is complete iff for each  $(x_i)_{i \in \mathbb{N}} \subset X$ ,  $\sum_{i=1}^{\infty} x_i$  converges absolutely implies that  $\sum_{i=1}^{\infty} x_i$  converges.

*Proof.* Suppose that  $X$  is complete. Let  $(x_i)_{i \in \mathbb{N}} \subset X$ . Suppose that  $\sum_{i=1}^{\infty} x_i$  converges absolutely. Let  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $m, n \in \mathbb{N}$ , if  $m, n \geq N$  and  $m < n$ , then  $\sum_{m+1}^n \|x_i\| < \epsilon$ . Let  $m, n \in \mathbb{N}$ . Suppose that  $m < n$ . Then

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right\| \\ &= \left\| \sum_{i=m+1}^n x_i \right\| \\ &\leq \sum_{i=m+1}^n \|x_i\| \\ &< \epsilon \end{aligned}$$

Thus  $(s_n)_{n \in \mathbb{N}}$  is cauchy. Since  $X$  is complete,  $\sum_{i=1}^{\infty} x_i$  converges. Conversely, Suppose that for each  $(x_i)_{i \in \mathbb{N}} \subset X$ ,  $\sum_{i=1}^{\infty} x_i$  converges absolutely implies that  $\sum_{i=1}^{\infty} x_i$  converges. Let  $(x_i)_{i \in \mathbb{N}} \subset X$  be cauchy. Proceed inductively to create a strictly increasing sequence  $(n_i)_{i \in \mathbb{N}} \subset \mathbb{N}$  such that for each  $m, n \in \mathbb{N}$ , if  $m, n \geq n_i$ , then  $\|x_m - x_n\| < 2^{-i}$ . Define  $(y_i)_{i \in \mathbb{N}} \subset X$  by

$$y_i = \begin{cases} x_{n_1} & i = 1 \\ x_{n_i} - x_{n_{i-1}} & i \geq 2 \end{cases}$$

Then  $\sum_{i=1}^k y_i = x_{n_k}$  and

$$\begin{aligned} \sum_{i \in \mathbb{N}} \|y_i\| &= \|x_{n_1}\| + \sum_{i \in \mathbb{N}} \|x_{n_i} - x_{n_{i-1}}\| \\ &\leq \|x_{n_1}\| + \sum_{i \in \mathbb{N}} 2^{-i} \\ &= \|x_{n_1}\| + 1 \end{aligned}$$

Hence  $(x_{n_k})_{k \in \mathbb{N}} = (\sum_{i=1}^k y_i)_{i \in \mathbb{N}}$  converges. Since  $(x_i)_{i \in \mathbb{N}}$  is cauchy and has a convergent subsequence, it converges. So  $X$  is complete.  $\square$

**Definition 6.5.** Let  $X, Y$  be a normed vector spaces. A linear map  $T : X \rightarrow Y$  is said to be **bounded** if there exists  $C \geq 0$  such that for each  $x \in X$ ,  $\|Tx\| \leq C\|x\|$ .

**Exercise 6.6.** Let  $X, Y$  be a normed vector spaces and  $T : X \rightarrow Y$  a linear map. Then  $T$  is bounded iff there exists  $r, s > 0$  such that  $T(B(0, r)) \subset B(0, s)$

*Proof.* Suppose that  $T$  is bounded. Then there exists  $C \geq 0$  such that for each  $x \in X$ ,  $\|Tx\| \leq C\|x\|$ . Thus  $T(B(0, 1)) \subset B(0, C + 1)$ . Conversely. Suppose that there exists  $r, s > 0$  such that  $T(B(0, r)) \subset B(0, s)$ . Define  $C = \frac{2s}{r}$ . Let  $x \in X$ . Put  $\alpha = \frac{r}{2\|x\|}$ . Then  $\alpha x \in B(0, r)$ . So  $T(\alpha x) = \alpha T(x) \in B(0, s)$ . Hence

$$\begin{aligned} \|T(\alpha x)\| &= \|\alpha T(x)\| \\ &= |\alpha| \|T(x)\| \\ &= \frac{r}{2\|x\|} \|T(x)\| \\ &< s. \end{aligned}$$

Thus

$$\|Tx\| < \frac{2s}{r}\|x\| = C\|x\|$$

So  $T$  is bounded. □

**Theorem 6.7.** *Let  $X, Y$  be normed vector spaces and  $T : X \rightarrow Y$  a linear map. Then the following are equivalent:*

- (1)  $T$  is continuous
- (2)  $T$  is continuous at  $x = 0$
- (3)  $T$  is bounded

*Proof.* (1)  $\implies$  (2):

Trivial

(2)  $\implies$  (3):

Suppose that  $T$  is continuous at  $x = 0$ . Then there exists  $\delta > 0$  such that for each  $x \in X$ , if  $\|x\| < \delta$ , then  $\|Tx\| < 1$ . Choose  $C = \frac{2}{\delta}$ . If  $x = 0$ , then  $\|Tx\| \leq C\|x\|$ . Suppose that  $\|x\| \neq 0$ . Define  $y = \frac{\delta}{2\|x\|}x$ . Then  $\|y\| < \delta$ . So

$$\|Ty\| = \frac{\delta}{2\|x\|}\|Tx\| < 1$$

Thus

$$\begin{aligned}\|Tx\| &< \frac{2}{\delta}\|x\| \\ &= C\|x\|\end{aligned}$$

Hence  $T$  is bounded.

(3)  $\implies$  (1)

Suppose that  $T$  is bounded. Then there exists  $C \geq 0$  such that for each  $x \in X$ ,  $\|Tx\| \leq C\|x\|$ . Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{C+1}$ . Let  $x, y \in X$ . Suppose that  $\|x - y\| < \delta$ . Then

$$\begin{aligned}\|Tx - Ty\| &= \|T(x - y)\| \\ &\leq C\|x - y\| \\ &< (C + 1)\delta \\ &= \epsilon\end{aligned}$$

So  $T$  is continuous. □

**Definition 6.8.** *Let  $X, Y$  be normed vector spaces. Define  $L(X, Y) = \{T : X \rightarrow Y : T \text{ is bounded}\}$ . Define  $\|\cdot\| : L(X, Y) \rightarrow [0, \infty)$  by*

$$\|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

*We call  $\|\cdot\|$  the **operator norm on  $L(X, Y)$***

**Exercise 6.9.** *Let  $X, Y$  be normed vector spaces. If  $X \neq \{0\}$ , then the operator norm on  $L(X, Y)$  is given by:*



$$(1) \|T\| = \sup_{\|x\|=1} \|Tx\|$$

$$(2) \|T\| = \sup_{x \neq 0} \|x\|^{-1} \|Tx\|$$

$$(3) \|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

*Proof.* Since  $X \neq \{0\}$ , the supremums in (1) and (2) are well defined. Let  $T \in L(X, Y)$ . By linearity of  $T$ , the sets over which the supremums are taken in (1) and (2) are the same. So (1) and (2) are equal.

Now, put  $M = \sup_{\|x\|=1} \|Tx\|$ ,  $m = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$  and let  $x \in X$ . If  $\|x\| = 0$ , then  $\|Tx\| \leq M\|x\|$ . Suppose that  $\|x\| \neq 0$ . Then

$$\begin{aligned} \|Tx\| &= \left( \|T(x/\|x\|)\| \right) \|x\| \\ &\leq M\|x\| \end{aligned}$$

Hence  $M \in \{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$ . Therefore  $m \leq M$

Let  $C \in \{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$ . Suppose that  $\|x\| = 1$ . Then  $\|Tx\| \leq C\|x\| = C$ . So  $M \leq C$ . Therefore  $M \leq m$ . So  $M = m$  and the supremum in (1) is the same as the infimum in (3).  $\square$

**Note 6.10.** From here on, unless stated otherwise, we assume  $X \neq 0$ .

**Exercise 6.11.** Let  $X, Y$  be normed vector spaces and  $T \in L(X, Y)$ . Then for each  $x \in X$ ,  $\|Tx\| \leq \|T\|\|x\|$

*Proof.* This is just part of the previous exercise. Let  $x \in X$ . If  $x = 0$ , then  $\|Tx\| \leq \|T\|\|x\|$ . Suppose that  $x \neq 0$ . Then  $\|Tx\| = \|T(x/\|x\|)\|\|x\| \leq \|T\|\|x\|$   $\square$

**Exercise 6.12.** Let  $X, Y$  be normed vector spaces. Then the operator norm is a norm on  $L(X, Y)$ .

*Proof.* Let  $S, T \in L(X, Y)$  and  $\alpha \in \mathbb{C}$ . For each  $x \in X$ , we have that

$$\begin{aligned} \|(S + T)x\| &= \|Sx + Tx\| \\ &\leq \|Sx\| + \|Tx\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= (\|S\| + \|T\|)\|x\| \end{aligned}$$

So  $\|S + T\| \leq \|S\| + \|T\|$ .

Using the definition of  $\|T\|$ , we see that

$$\begin{aligned}
 \|\alpha T\| &= \sup_{\|x\|=1} \|(\alpha T)x\| \\
 &= \sup_{\|x\|=1} |\alpha| \|Tx\| \\
 &= |\alpha| \sup_{\|x\|=1} \|Tx\| \\
 &= |\alpha| \|T\|
 \end{aligned}$$

$$\text{So } \|\alpha S\| = |\alpha| \|S\|.$$

Suppose that  $\|T\| = 0$ . Let  $x \in X$ . Then  $\|Tx\| \leq \|T\|\|x\| = 0$ . So  $Tx = 0$ . Since  $x \in X$  is arbitrary, we have that  $T = 0$ .  $\square$

**Exercise 6.13.** *Let  $X$  be a normed vector space. Then addition and scalar multiplication are continuous on  $X \times X$  and  $\|\cdot\| : X \rightarrow [0, \infty)$  is continuous.*

*Proof.* Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{2}$ . Let  $(x_1, y_1), (x_2, y_2) \in X \times X$ . Suppose that  $\|(x_1, y_1) - (x_2, y_2)\| = \max\{\|x_1 - x_2\|, \|y_1 - y_2\|\} < \delta$ . Then

$$\begin{aligned}
 \|(x_1 + y_1) - (x_2 + y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\
 &\leq \|x_1 - x_2\| + \|y_1 - y_2\| \\
 &< 2\delta \\
 &= \epsilon
 \end{aligned}$$

Hence addition is uniformly continuous.

Let  $(\lambda_1, x_1) \in \mathbb{C} \times X$  and  $\epsilon > 0$ . Choose  $\delta = \min\{\frac{\epsilon}{2(|\lambda_1| + \|x_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$ . Let  $(\lambda_2, x_2) \in \mathbb{C} \times X$ . Suppose that  $\|(\lambda_1, x_1) - (\lambda_2, x_2)\| = \max\{|\lambda_1 - \lambda_2|, \|x_1 - x_2\|\} < \delta$ . Then

$$\begin{aligned}
 \|\lambda_1 x_1 - \lambda_2 x_2\| &= \|\lambda_1 x_1 - \lambda_1 x_2 + \lambda_1 x_2 - \lambda_2 x_2\| \\
 &= \|\lambda_1(x_1 - x_2) + (\lambda_1 - \lambda_2)x_2\| \\
 &\leq |\lambda_1| \|x_1 - x_2\| + |\lambda_1 - \lambda_2| \|x_2\| \\
 &\leq |\lambda_1| \|x_1 - x_2\| + |\lambda_1 - \lambda_2| (\|x_1 - x_2\| + \|x_1\|) \\
 &< |\lambda_1| \delta + \delta(\delta + \|x_1\|) \\
 &= (|\lambda_1| + \|x_1\|) \delta + \delta^2 \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon
 \end{aligned}$$

Since  $(\lambda_1, x_1) \in \mathbb{C} \times X$  is arbitrary, scalar multiplication is continuous.

Let  $\epsilon > 0$ . Choose  $\delta = \epsilon$ . Let  $x, y \in X$ . Suppose that  $\|x - y\| < \delta$ . Then

$$\begin{aligned} \left| \|x\| - \|y\| \right| &\leq \|x - y\| \\ &< \delta \\ &= \epsilon \end{aligned}$$

So  $\|\cdot\| : X \rightarrow [0, \infty)$  is uniformly continuous.  $\square$

**Exercise 6.14.** Let  $X, Y$  be normed vector spaces. If  $Y$  is complete, then so is  $L(X, Y)$ .

*Proof.* Suppose that  $Y$  is complete. Let  $(T_n)_{n \in \mathbb{N}} \subset L(X, Y)$ . Suppose that  $(T_n)_{n \in \mathbb{N}}$  is Cauchy. Since for each  $m, n \in \mathbb{N}$ ,  $|\|T_m\| - \|T_n\|| \leq \|T_m - T_n\|$ , we have that  $(\|T_n\|)_{n \in \mathbb{N}} \subset [0, \infty)$  is Cauchy. Hence  $\lim_{n \rightarrow \infty} \|T_n\|$  exists.

Let  $x \in X$  and  $m, n \in \mathbb{N}$ . Then

$$\begin{aligned} \|T_m x - T_n x\| &= \|(T_m - T_n)x\| \\ &\leq \|T_m - T_n\| \|x\| \end{aligned}$$

So  $(T_n x)_{n \in \mathbb{N}} \subset Y$  is Cauchy and hence converges. Define  $T : X \rightarrow Y$  by  $Tx = \lim_{n \rightarrow \infty} T_n x$ .

Since addition and scalar multiplication are continuous,  $T$  is linear. Let  $x \in X$  and  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then  $\|Tx - T_n x\| < \epsilon$ . Then for each  $n \in \mathbb{N}$ , if  $n \geq N$  we have that

$$\begin{aligned} \|Tx\| &\leq \|Tx - T_n x\| + \|T_n x\| \\ &< \epsilon + \|T_n x\| \\ &\leq \epsilon + \|T_n\| \|x\| \end{aligned}$$

Thus  $\|Tx\| \leq \epsilon + (\lim_{n \rightarrow \infty} \|T_n\|) \|x\|$ . Since  $\epsilon > 0$  is arbitrary,  $\|Tx\| \leq (\lim_{n \rightarrow \infty} \|T_n\|) \|x\|$ . Thus  $T \in L(X, Y)$  and  $\|T\| \leq \lim_{n \rightarrow \infty} \|T_n\|$ .

Note that since addition, scalar multiplication and  $\|\cdot\|$  are continuous, we have that for each  $n \in \mathbb{N}$  and  $x \in X$ ,  $\|(T_n - T_m)x\|$  converges to  $\|(T_n - T)x\|$  because

$$\begin{aligned} \lim_{m \rightarrow \infty} \|(T_n - T_m)x\| &= \lim_{m \rightarrow \infty} \|T_n x - T_m x\| \\ &= \|T_n x - \lim_{m \rightarrow \infty} T_m x\| \\ &= \|T_n x - Tx\| \\ &= \|(T_n - T)x\| \end{aligned}$$

Let  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $m, n \in \mathbb{N}$  if  $n, m \geq N$ , then  $\|T_n - T_m\| < \epsilon$ . Then for each  $n \in \mathbb{N}$  if  $n \geq N$ , then for each  $x \in X$ ,

$$\|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\| < \epsilon \|x\|$$

Combining this with the previous fact, we see that for each  $n \in N$ , if  $n \geq N$ , then for each  $x \in X$ ,

$$\|(T_n - T)x\| \leq \epsilon \|x\|$$

In particular, for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then

$$\|T_n - T\| = \sup_{\|x\|=1} \|(T_n - T)x\| \leq \epsilon$$

This implies that  $T_n$  converges to  $T$  in  $L(X, Y)$ . Since

$$|\|T_n\| - \|T\|| \leq \|T_n - T\|$$

It is clear that  $\lim_{n \rightarrow \infty} \|T_n\| = \|T\|$  □

**Definition 6.15.** Let  $X$  be a normed vector space and  $M \subset X$  a closed subspace. Define  $\|\cdot\| : X/M \rightarrow [0, \infty)$  by

$$\|x + M\| := \inf_{y \in M} \|x + y\|$$

We call  $\|\cdot\|$  the **subspace norm on  $X/M$**

**Exercise 6.16.** Let  $X$  be a normed vector space and  $M \subsetneq X$  a proper, closed subspace of  $X$ . Then

- (1) The previously defined subspace norm on  $X/M$  is well defined and is a norm.
- (2) For each  $\epsilon > 0$ , there exists  $x \in X$  such that  $\|x\| = 1$  and  $\|x + M\| \geq 1 - \epsilon$ .
- (3) The projection map  $\pi : X \rightarrow X/M$  defined by  $\pi(x) = x + M$  is continuous and  $\|\pi\| = 1$ .
- (4) If  $X$  is complete, then  $X/M$  is complete.

*Proof.* (1) Let  $x, y \in X$  and  $\alpha \in \mathbb{C}$ . Suppose that  $x + M = y + M$ . Then there exists  $m \in M$  such that  $x = y + m$ . Since  $M$  is a subspace, the map  $T : M \rightarrow M$  given by  $Tx = x + m$  is a bijection. So

$$\inf_{z \in M} \|y + m + z\| = \inf_{z \in M} \|y + z\|$$

which implies that

$$\begin{aligned} \|x + M\| &= \inf_{z \in M} \|x + z\| \\ &= \inf_{z \in M} \|y + m + z\| \\ &= \inf_{z \in M} \|y + z\| \\ &= \|y + M\| \end{aligned}$$

So  $\|\cdot\| : X/M \rightarrow [0, \infty)$  is well defined.

We observe that for each  $z, w \in M$ ,

$$\|x + y + z\| \leq \|x + w\| + \|y + w + z\|$$

Taking infimums over  $M$  with respect to  $z$  in this inequality implies that for each  $w \in M$ ,

$$\begin{aligned} \inf_{z \in M} \|x + y + z\| &\leq \inf_{z \in M} \left( \|x + w\| + \|y + w + z\| \right) \\ &= \|x + w\| + \inf_{z \in M} \|y + w + z\| \end{aligned}$$

Again we use the fact that for each  $w \in M$ ,

$$\inf_{z \in M} \|y + w + z\| = \inf_{z \in M} \|y + z\|$$

This implies that for each  $w \in M$ ,

$$\inf_{z \in M} \|x + y + z\| \leq \|x + w\| + \inf_{z \in M} \|y + z\|$$

Therefore, taking infimums over  $M$  with respect to  $w$  in this inequality yields

$$\begin{aligned} \|x + y + M\| &= \inf_{z \in M} \|x + y + z\| \\ &\leq \inf_{w \in M} \left( \|x + w\| + \inf_{z \in M} \|y + z\| \right) \\ &= \inf_{w \in M} \|x + w\| + \inf_{z \in M} \|y + z\| \\ &= \|x + M\| + \|y + M\| \end{aligned}$$

If  $\alpha = 0$ , then  $\alpha x = 0$ . Choosing  $z = 0 \in M$  gives  $\|\alpha x + M\| = 0 = |\alpha| \|x + M\|$ . Suppose that  $\alpha \neq 0$ . Then the map  $T : M \rightarrow M$  given by  $Tx = \alpha^{-1}x$  is a bijection and thus  $\inf_{z \in M} \|x + \alpha^{-1}z\| = \inf_{z \in M} \|x + z\|$ . Hence we have that

$$\begin{aligned} \|\alpha x + M\| &= \inf_{z \in M} \|\alpha x + z\| \\ &= \inf_{z \in M} |\alpha| \|x + \alpha^{-1}z\| \\ &= |\alpha| \inf_{z \in M} \|x + \alpha^{-1}z\| \\ &= |\alpha| \inf_{z \in M} \|x + z\| \\ &= |\alpha| \|x + M\| \end{aligned}$$

Suppose that  $\|x\| = 0$ . Choose a sequence  $(z_n)_{n \in \mathbb{N}} \subset M$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x - z_n\| &= \inf_{z \in M} \|x + z\| \\ &= 0 \end{aligned}$$

Then  $\lim_{n \rightarrow \infty} z_n = x$ . Since  $M$  is closed,  $x \in M$ . Hence  $x + M = 0 + M$ .

- (2) Since  $M$  is a proper subspace, there exists  $v \in X$  such that  $v \notin M$ . Then  $\|v + M\| \neq 0$ . Let  $\epsilon > 0$ . Then  $(1 - \epsilon)^{-1} \|v + M\| > \|v + M\|$ . So there exists  $z \in M$  such that

$$0 < \|v + M\| \leq \|v + z\| < (1 - \epsilon)^{-1} \|v + M\|$$

Choose  $x = \|v + z\|^{-1}(v + z)$ . Then  $\|x\| = 1$  and

$$\begin{aligned}\|x + M\| &= \|v + z\|^{-1}\|v + z + M\| \\ &= \|v + z\|^{-1}\|v + M\| \\ &> 1 - \epsilon\end{aligned}$$

- (3) Let  $x \in X$ . Taking  $z = 0$ , we see that  $\|\pi(x)\| = \|x + M\| \leq \|x + z\| = \|x\|$ . So  $\pi$  is bounded and in particular,

$$\sup_{\|x\|=1} \|\pi(x)\| \leq 1$$

From (2) we see that

$$\sup_{\|x\|=1} \|\pi(x)\| \geq 1$$

Hence  $\|\pi\| = 1$ .

- (4) Suppose that  $X$  is complete. Let  $(x_i + M)_{i \in \mathbb{N}} \subset X/M$ . Suppose that  $\sum_{i \in \mathbb{N}} \|x_i + M\| < \infty$ . Let  $\epsilon > 0$ . Then for each  $i \in \mathbb{N}$ , there exists  $z_i \in M$  such that  $\|x_i + z_i\| < \|x_i + M\| + \epsilon 2^{-i}$ . Define the sequence  $(a_i)_{i \in \mathbb{N}} \subset X$  by  $a_i = x_i + z_i$ . Then we have

$$\begin{aligned}\sum_{i \in \mathbb{N}} \|a_i\| &= \sum_{i \in \mathbb{N}} \|x_i + z_i\| \\ &\leq \sum_{i \in \mathbb{N}} \left( \|x_i + M\| + \epsilon 2^{-i} \right) \\ &= \sum_{i \in \mathbb{N}} \|x_i + M\| + \epsilon\end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, it follows that

$$\sum_{i \in \mathbb{N}} \|a_i\| \leq \sum_{i \in \mathbb{N}} \|x_i + M\| < \infty$$

Since  $X$  is complete,  $\sum_{i=1}^{\infty} a_i$  converges in  $X$ . Define  $(s_n)_{n \in \mathbb{N}} \subset X$  and  $s \in X$  by  $s_n = \sum_{i=1}^n a_i$  and  $s = \sum_{i=1}^{\infty} a_i$ . Since  $\lim_{n \rightarrow \infty} s_n = s$ , and  $\pi : X \rightarrow X/M$  is continuous, it follows that  $\lim_{n \rightarrow \infty} \pi(s_n) = \pi(s)$ . Since

$$\begin{aligned}\pi(s_n) &= \sum_{i=1}^n a_i + M \\ &= \sum_{i=1}^n x_i + M\end{aligned}$$

We have that  $\sum_{i=1}^{\infty} x_i + M$  converges which implies that  $X/M$  is complete. □

**Exercise 6.17.** Let  $X, Y$  be normed vector spaces. Define  $\phi : L(X, Y) \times X \rightarrow Y$  by  $\phi(T, x) = Tx$ . Then  $\phi$  is continuous.

*Proof.* Let  $(T_1, x_1) \in L(X, Y) \times X$  and  $\epsilon > 0$ . Choose  $\delta = \min\{\frac{\epsilon}{2(\|x_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$ . Let  $(t_2, x_2) \in L(X, Y) \times X$ . Suppose that

$$\|(T_1, x_1) - (T_2, x_2)\| = \max\{\|T_1 - T_2\|, \|x_1 - x_2\|\} < \delta$$

. Then

$$\begin{aligned} \|\phi(T_1, x_1) - \phi(T_2, x_2)\| &= \|T_1x_1 - T_2x_2\| \\ &= \|T_1x_1 - T_2x_1 + T_2x_1 - T_2x_2\| \\ &\leq \|(T_1 - T_2)x_1\| + \|T_2(x_1 - x_2)\| \\ &\leq \|T_1 - T_2\|\|x_1\| + \|T_2\|\|x_1 - x_2\| \\ &\leq \|T_1 - T_2\|\|x_1\| + (\|T_1 - T_2\| + \|T_1\|)\|x_1 - x_2\| \\ &< \delta\|x_1\| + (\delta + \|T_1\|)\delta \\ &= \delta(\|T_1\| + \|x_1\|) + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So  $\phi$  is continuous. □

**Exercise 6.18.** Let  $X$  be a normed vector space and  $M \subset X$  a subspace. Then  $\overline{M}$  is a subspace.

*Proof.* Let  $x, y \in \overline{M}$  and  $\alpha \in \mathbb{C}$ . Then there exist sequences  $(x_n)_{n \in \mathbb{N}} \subset M$  and  $(y_n)_{n \in \mathbb{N}} \subset M$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . Since  $M$  is a subspace,  $(x_n + y_n)_{n \in \mathbb{N}} \subset M$  and  $(\alpha x_n)_{n \in \mathbb{N}} \subset M$ . Since addition and scalar multiplication are continuous, we have that  $x_n + y_n \rightarrow x + y$  and  $\alpha x_n \rightarrow \alpha x$ . Thus  $x + y \in \overline{M}$  and  $\alpha x \in \overline{M}$  and hence  $\overline{M}$  is a subspace. □

**Exercise 6.19.** Let  $X, Y, Z$  be normed vector spaces,  $T \in L(X, Y)$  and  $S \in L(Y, Z)$ . Define  $ST : X \rightarrow Z$  by  $STx = S(Tx)$ . Then  $ST \in L(X, Z)$  and  $\|ST\| \leq \|S\|\|T\|$ .

*Proof.* Clearly  $ST$  is linear. Let  $x \in X$ . Then

$$\begin{aligned} \|STx\| &= \|S(Tx)\| \\ &\leq \|S\|\|Tx\| \\ &\leq \|S\|\|T\|\|x\| \end{aligned}$$

So  $\|ST\| \leq \|S\|\|T\|$ . □

**Definition 6.20.** Let  $X$  be a Banach space and an associative algebra. Then  $X$  is said to be a Banach algebra if for each  $S, T \in X$ ,  $\|ST\| \leq \|S\|\|T\|$ . If there exists  $I \in X$  such that  $I \neq 0$  and for each  $T \in X$ ,  $IT = TI = T$ , then  $X$  is said to be **unital** with identity  $I$ . An element  $T \in X$  is said to be **invertible** if there exists  $S \in X$  such that  $TS = ST = I$ .

**Exercise 6.21.** Let  $X$  be a unital Banach algebra. Then  $\|I\| \leq 1$ .

*Proof.* Since  $I \neq 0$ ,  $\|I\| \neq 0$ . By definition,

$$\|I\| = \|II\| \leq \|I\|\|I\|$$

Hence  $1 \leq \|I\|$ . □

**Note 6.22.** *If  $X$  is a Banach space, then a previous exercise implies that  $L(X, X)$  equipped with composition is a unital Banach algebra where  $I$  is the identity operator. It is easy to see that  $\|I\| = 1$ .*



**Note 6.23.** Let  $X$  be a Banach algebra. Then the set of invertible elements in  $X$  is a group.

**Exercise 6.24.** Let  $X$  be a Banach algebra. Then multiplication is continuous.

*Proof.* Let  $(S_1, T_1) \in X \times X$  and  $\epsilon > 0$ . Choose  $\delta = \min\{\frac{\epsilon}{2(\|S_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$ . Let  $(S_2, T_2) \in X \times X$ . Suppose that

$$\|(S_1, T_1) - (S_2, T_2)\| = \max\{\|S_1 - S_2\|, \|T_1 - T_2\|\} < \delta$$

. Then

$$\begin{aligned} \|S_1T_1 - S_2T_2\| &= \|S_1T_1 - S_2T_1 + S_2T_1 - S_2T_2\| \\ &\leq \|S_1 - S_2\|\|T_1\| + \|S_2\|\|T_1 - T_2\| \\ &\leq \|S_1 - S_2\|\|T_1\| + (\|S_1 - S_2\| + \|S_1\|)\|T_1 - T_2\| \\ &\leq \delta\|T_1\| + (\delta + \|S_1\|)\delta \\ &= \delta(\|S_1\| + \|T_1\|) + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

**Definition 6.25.** Let  $X, Y$  be normed vector spaces and  $T \in L(X, Y)$ . Then  $T$  is said to be **invertible** or an **isomorphism** if  $T$  is a bijection and  $T^{-1} \in L(Y, X)$ .

**Definition 6.26.** Let  $X$  be a Banach space. Define  $GL(X) := \{T \in L(X, X) : T \text{ is invertible}\}$ .

**Exercise 6.27.** Let  $X$  be a Banach space. Then

(1) For each  $T \in L(X, X)$ , if  $\|I - T\| < 1$ , then  $T$  is invertible and

$$T^{-1} = \sum_{n=0}^{\infty} (I - T)^n$$

(2) For each  $S, T \in L(X, X)$ , if  $S$  is invertible and  $\|S - T\| < \|S^{-1}\|^{-1}$ , then  $T$  is invertible.

(3)  $GL(X)$  is open.

*Proof.* (1) Let  $T \in L(X, X)$ . Suppose that  $\|I - T\| < 1$ . Then

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n < \infty$$

. Since  $X$  is complete, so is  $L(X, X)$  and thus  $\sum_{n=0}^{\infty} (I - T)^n$  converges in  $L(X, X)$ .

Define  $(S_k)_{k=0}^{\infty} \subset L(X, X)$  and  $S \in L(X, X)$  by  $S_k = \sum_{n=0}^k (I - T)^n$  and

$S = \sum_{n=0}^{\infty} (I - T)^n$ . Then for each  $k \in \mathbb{N}$ ,

$$\begin{aligned} S_k T &= S_k - S_k(I - T) \\ &= (I - T)^0 - (I - T)^{k+1} \\ &= I - (I - T)^{k+1} \end{aligned}$$

and  $\|S_k T - I\| \leq \|I - T\|^{k+1}$ . Since multiplication on Banach algebras is continuous, we have that

$$ST = \left(\lim_{k \rightarrow \infty} S_k\right)T = \lim_{k \rightarrow \infty} S_k T = I$$

Similarly  $TS = I$ . Thus  $T$  is invertible and  $T^{-1} = S \in L(X, X)$ .

(2) Let  $S, T \in L(X, X)$ . Suppose that  $S$  is invertible and  $\|S - T\| < \|S^{-1}\|^{-1}$ . Then

$$\begin{aligned} \|I - S^{-1}T\| &= \|S^{-1}(S - T)\| \\ &\leq \|S^{-1}\| \|S - T\| \\ &< 1 \end{aligned}$$

So  $S^{-1}T$  is invertible. Thus  $T = S(S^{-1}T)$  is invertible.

(3) Let  $T \in GL(X)$ . Choose  $\delta = \|T^{-1}\|^{-1}$ . By (2),  $B(T, \delta) \subset GL(X)$ . □

**Exercise 6.28.** Let  $M(X, \mathcal{A})$  denote the set of complex measures on the measurable space  $(X, \mathcal{A})$ . Define  $\|\cdot\| : M(X, \mathcal{A}) \rightarrow [0, \infty)$  by  $\|\mu\| = |\mu|(X)$ . Then  $\|\cdot\|$  is a norm on  $M(X, \mathcal{A})$ .

*Proof.* Let  $\mu, \nu \in M(X, \mathcal{A})$  and  $\alpha \in \mathbb{C}$ . Exercises in a previous section tell us that  $|\mu + \nu| \leq |\mu| + |\nu|$  and  $|\alpha\mu| = |\alpha||\mu|$ . So clearly  $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$  and  $\|c\mu\| = |c|\|\mu\|$ . If  $\|\mu\| = 0$ , then  $X$  is  $\mu$ -null and  $\mu$  is the zero measure. □

## 6.2. Linear Functionals.

**Definition 6.29.** Let  $X$  be a normed vector space and  $T : X \rightarrow \mathbb{C}$ . Then  $T$  is said to be a **linear functional on  $X$**  if  $T$  is linear and  $T$  is said to be a **bounded linear functional on  $X$**  if  $T \in L(X, \mathbb{C})$ . We define the **dual space of  $X$** , denoted  $X^*$ , by  $X^* = L(X, \mathbb{C})$ .

**Definition 6.30.** Let  $X$  be a normed vector space and  $p : X \rightarrow \mathbb{R}$ . Then  $p$  is said to be a **sublinear functional** if for each  $x, y \in X$ ,  $\lambda \geq 0$ ,

- (1)  $p(x + y) \leq p(x) + p(y)$
- (2)  $p(\lambda x) = \lambda p(x)$

**Note 6.31.** Let  $X$  be a vector space and  $\|\cdot\| : X \rightarrow [0, \infty)$  be a seminorm, then  $\|\cdot\|$  is a sublinear functional.

**Theorem 6.32.** *Hahn-Banach Theorem:* Let  $X$  be a vector space,  $p : X \rightarrow \mathbb{R}$  a sublinear functional,  $M \subset X$  a subspace and  $f : M \rightarrow \mathbb{C}$  a linear functional. If for each  $x \in M$ ,  $|f(x)| \leq p(x)$ , then there exists a linear functional  $F : X \rightarrow \mathbb{C}$  such that for each  $x \in X$ ,  $|F(x)| \leq p(x)$  and  $F|_M = f$ .

**Exercise 6.33.** Let  $X$  be a normed vector space,  $M \subset X$  a subspace and  $f \in M^*$ . Then there exists  $F \in X^*$  such that  $\|F\| = \|f\|$  and  $F|_M = f$ .

*Proof.* If  $f = 0$ , Choose  $F = 0$ . Suppose  $f \neq 0$ . Then  $\|f\| \neq 0$  and there exists  $x_0 \in M$  such that  $x_0 \neq 0$ . Thus  $\|f\| = \sup\{|f(x)| : x \in M \text{ and } \|x\| = 1\}$ . Define  $p : X \rightarrow [0, \infty)$  by  $p(x) = \|f\|\|x\|$ . Then  $p$  is a sublinear functional on  $X$  and for each  $x \in M$ ,  $|f(x)| \leq p(x)$ . So

there exists a linear functional  $F : X \rightarrow \mathbb{C}$  such that for each  $x \in X$ ,  $|F(x)| \leq p(x) = \|f\|\|x\|$  and  $F|_M = f$ . Thus  $F \in X^*$  with  $\|F\| \leq \|f\|$ . Also

$$\|F\| = \sup_{\substack{x \in X \\ \|x\|=1}} |F(x)| \geq \sup_{\substack{x \in M \\ \|x\|=1}} |F(x)| = \sup_{\substack{x \in M \\ \|x\|=1}} |f(x)| = \|f\|$$

So  $\|F\| = \|f\|$ . □

**Exercise 6.34.** Let  $X$  be a normed vector space,  $M \subsetneq X$  a proper closed subspace and  $x \in X \setminus M$ . Then there exists  $F \in X^*$  such that  $F|_M = 0$ ,  $\|F\| = 1$  and  $F(x) = \|x + M\| \neq 0$ . (Hint: Consider  $f : M + \mathbb{C}x \rightarrow \mathbb{C}$  defined by  $f(m + \lambda x) = \lambda\|x + M\|$ .)

*Proof.* Define  $f : M + \mathbb{C}x \rightarrow \mathbb{C}$  as above. Clearly  $f$  is linear and  $f|_M = 0$ . Let  $m \in M$  and  $\lambda \in \mathbb{C}$ . If  $\lambda = 0$ , then  $|f(m + \lambda x)| = 0 \leq \|m + \lambda x\|$ . Suppose that  $\lambda \neq 0$ . Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \inf_{z \in M} \|z + \lambda x\| \\ &\leq \|m + \lambda x\| \end{aligned}$$

So  $f \in (M + \mathbb{C}x)^*$  and  $\|f\| \leq 1$ . Let  $\epsilon > 0$ . A previous exercise tells us that there exist  $m \in M, \lambda \in \mathbb{C}$  such that  $\|m + \lambda x\| = 1$  and  $\|m + \lambda x + M\| > 1 - \epsilon$ . Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \|m + \lambda x + M\| \\ &> 1 - \epsilon \end{aligned}$$

So

$$\|f\| = \sup_{\substack{z \in M + \mathbb{C}x \\ \|z\|=1}} |f(z)| \geq 1$$

Hence  $\|f\| = 1$ . The same exercise also tells us that  $f(x) = \|x + M\| \neq 0$ . Using the previous exercise, there exists  $F \in X^*$  such that  $\|F\| = \|f\| = 1$  and  $F|_{M + \mathbb{C}x} = f$ . □

**Exercise 6.35.** Let  $X$  be a normed vector space and  $x \in X$ . If  $x \neq 0$ , then there exists  $F \in X^*$  such that  $\|F\| = 1$  and  $F(x) = \|x\|$ .

*Proof.* Define  $f : \mathbb{C}x \rightarrow \mathbb{C}$  by  $f(\lambda x) = \lambda\|x\|$ . Then  $f$  is linear and  $f(x) = \|x\|$ . Clearly

$$\sup_{\substack{z \in \mathbb{C}x \\ \|z\|=1}} |f(z)| = 1$$

So  $f \in (\mathbb{C}x)^*$  and  $\|f\| = 1$ . By a previous exercise, there exists  $F \in X^*$  such that  $\|F\| = \|f\| = 1$  and  $F|_{\mathbb{C}x} = f$ . □

**Exercise 6.36.** Let  $X$  be a normed vector space. Then  $X^*$  separates the points of  $X$ .

*Proof.* Let  $x, y \in X$ . Suppose that  $x \neq y$ . Then  $x - y \neq 0$ . The previous exercise implies that there exists  $F \in X^*$  such that  $\|F\| = 1$  and

$$F(x) - F(y) = F(x - y) = \|x - y\| \neq 0$$

Thus  $F(x) \neq F(y)$  and  $X^*$  separates the points of  $X$ . □

**Exercise 6.37.** Let  $X$  be a normed vector space and  $x \in X$ . Define  $\hat{x} : X^* \rightarrow \mathbb{C}$  by  $\hat{x}(f) = f(x)$ . Then  $\hat{x} \in X^{**}$  and  $\|\hat{x}\| = \|x\|$ .

*Proof.* Let  $f, g \in X^*$  and  $\lambda \in \mathbb{C}$ . Then

$$\hat{x}(f + \lambda g) = (f + \lambda g)(x) = f(x) + \lambda g(x) = \hat{x}(f) + \lambda \hat{x}(g)$$

So  $\hat{x}$  is linear. For each  $f \in X^*$ ,

$$|\hat{x}(f)| = |f(x)| \leq \|x\| \|f\|$$

Hence  $\hat{x} \in X^{**}$  with  $\|\hat{x}\| \leq \|x\|$ . If  $x = 0$ , then  $\hat{x} = 0$  and  $\|\hat{x}\| = \|x\|$ . Suppose that  $x \neq 0$ . Then a previous exercise implies that there exists  $F \in X^*$  such that  $\|F\| = 1$  and  $F(x) = \|x\|$ . Then we have that

$$\sup_{\substack{f \in X^* \\ \|f\|=1}} |\hat{x}(f)| = \sup_{\substack{f \in X^* \\ \|f\|=1}} |f(x)| \geq |F(x)| = \|x\|$$

Hence  $\|\hat{x}\| = \|x\|$ . □

**Definition 6.38.** Let  $X, Y$  be metric spaces and  $T : X \rightarrow Y$ . Then  $T$  is said to be an **isometry** if for each  $x_1, x_2 \in X$ ,  $d(Tx_1, Tx_2) = d(x_1, x_2)$ .

**Exercise 6.39.** Let  $X, Y$  be metric spaces and  $T : X \rightarrow Y$  an isometry. Then  $T$  is injective.

*Proof.* Let  $x_1, x_2 \in X$ . Suppose that  $Tx_1 = Tx_2$ . Then  $0 = d(Tx_1, Tx_2) = d(x_1, x_2)$ . So  $x_1 = x_2$ . Hence  $T$  is injective. □

**Exercise 6.40.** Let  $X$  be a normed vector space. Define  $\phi : X \rightarrow X^{**}$  by  $\phi(x) = \hat{x}$ . Then  $\phi$  is a linear isometry.

*Proof.* Let  $x, y \in X$  and  $\lambda \in \mathbb{C}$ . Then for each  $f \in X^*$ , we have that

$$\begin{aligned} \phi(x + \lambda y)(f) &= f(x + \lambda y) \\ &= f(x) + \lambda f(y) \\ &= \hat{x}(f) + \lambda \hat{y}(f) \\ &= \phi(x)(f) + \lambda \phi(y)(f) \end{aligned}$$

So  $\phi(x + \lambda y) = \phi(x) + \lambda \phi(y)$  and  $\phi$  is linear. A previous exercise tells us that

$$\begin{aligned} \|\phi(x) - \phi(y)\| &= \|\phi(x - y)\| \\ &= \|\widehat{x - y}\| = \|x - y\| \end{aligned}$$

So  $\phi$  is an isometry. □

**Definition 6.41.** Let  $X$  be a normed vector space. Then  $X$  is said to be **reflexive** if  $X$  is isomorphic to  $X^{**}$ .

**Exercise 6.42.** Let  $X$  be a normed vector space and  $f : X \rightarrow \mathbb{C}$  a linear functional on  $X$ . Then  $f$  is bounded iff  $\ker f$  is closed.

*Proof.* Suppose that  $f$  is continuous. Since  $\{0\}$  is closed, we have that  $\ker f = f^{-1}(\{0\})$  is closed. Conversely, suppose that  $\ker f$  is closed. If  $\ker f = X$ , then  $f = 0$  and  $f$  is continuous. Suppose that  $\ker f \neq X$ . Then  $\ker f$  is a proper, closed subspace of  $X$ . A

previous exercise tells us that there exists  $x \in X$  such that  $\|x\| = 1$  and  $\|x + \ker f\| > \frac{1}{2}$ . Let  $y \in X$ . Suppose that  $\|y\| < \frac{1}{2}$ . Then for each  $z \in \ker f$ ,

$$\begin{aligned} \|z - (x + y)\| &= \|(z - x) - y\| \\ &\geq \|z - x\| - \|y\| \\ &> \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

So  $x + y \notin \ker f$ . Therefore  $f(B(x, \frac{1}{2})) \cap \{0\} = \emptyset$ . If  $f(B(x, \frac{1}{2}))$  is unbounded, then  $f(B(x, \frac{1}{2})) = \mathbb{C}$  by linearity. This is a contradiction since  $0 \notin f(B(x, \frac{1}{2}))$ . So There exists  $s > 0$  such that  $f(B(x, \frac{1}{2})) \subset B(0, s)$  and thus  $f$  is bounded.  $\square$

**Exercise 6.43.** Let  $X$  be a normed vector space.

- (1) Let  $M \subsetneq X$  be a proper closed subspace of  $X$  and  $x \in X \setminus M$ . Then  $M + \mathbb{C}x$  is closed.
- (2) Let  $M \subset X$  be a finite dimensional subspace of  $X$ . Then  $M$  is closed.

*Proof.* (1) Let  $y \in X$  and  $(y_n)_{n \in \mathbb{N}} \subset M + \mathbb{C}x$ . Suppose that  $y_n \rightarrow y$ . If  $y \in M$ , then  $y \in M + \mathbb{C}x$ . Suppose that  $y \notin M$ . For each  $n \in \mathbb{N}$ , there exists  $m_n \in M$  and  $\lambda_n \in \mathbb{C}$  such that  $y_n = m_n + \lambda_n x$ . A previous exercise tells us that there exists  $F \in X^*$  such that  $\|F\| = 1$ ,  $F|_M = 0$  and  $F(x) = \|x + M\| \neq 0$ . Since  $F$  is continuous,  $F(y_n) \rightarrow F(y)$ . Since for each  $n \in \mathbb{N}$ ,

$$F(y_n) = F(m_n + \lambda_n x) = F(m_n) + \lambda_n F(x) = \lambda_n F(x)$$

we have that  $\lambda_n F(x) \rightarrow F(y)$ . Since  $F(x) \neq 0$ , this implies that  $\lambda_n \rightarrow F(x)^{-1}F(y)$ . It follows that  $\lambda_n x \rightarrow F(x)^{-1}F(y)x$ . Since for each  $n \in \mathbb{N}$ ,  $m_n = y_n - \lambda_n x$ , we know that  $m_n \rightarrow y - F(x)^{-1}F(y)x$ . Since  $(m_n)_{n \in \mathbb{N}} \subset M$  and  $M$  is closed, we have that  $y - F(x)^{-1}F(y)x \in M$  and therefore  $y \in M + \mathbb{C}x$ . Hence  $M + \mathbb{C}x$  is closed.

- (2) If  $M = X$ , then  $M$  is closed. Suppose that  $M \neq X$ . Let  $(x_i)_{i=1}^n$  be a basis for  $M$ . Define  $N_0 = \{0\}$  and for each  $i = 1, 2, \dots, n$ , define  $N_i = N_{i-1} + \mathbb{C}x_i$ . Since  $N_0$  is a proper closed subspace of  $X$  and  $x_1 \in X \setminus N_0$ , (1) implies that  $N_1$  is closed. Proceed inductively to obtain that  $M = N_n$  is closed.

$\square$

**Exercise 6.44.** Let  $X$  be an infinite-dimensional normed vector space.

- (1) There exists a sequence  $(x_n)_{n \in \mathbb{N}} \subset X$  such that for each  $m, n \in \mathbb{N}$ ,  $\|x_n\| = 1$  and if  $m \neq n$ , then  $\|x_m - x_n\| > \frac{1}{2}$ .
- (2)  $X$  is not locally compact.

*Proof.* (1) Define  $N_0 = \{0\}$ . Then  $N_0$  is a closed proper subspace of  $X$ . Choose  $x_1 \in X$  such that  $\|x_1\| = 1$ . Using the results of previous exercises, we proceed inductively. For each  $n \geq 2$  we define  $N_{n-1} = \text{span}(x_1, x_2, \dots, x_{n-1})$ . Then  $N_{n-1}$  is a closed proper subspace of  $X$ . Thus we may choose  $x_n \in X$  such that  $\|x_n\| = 1$  and  $\|x_n + N_{n-1}\| > \frac{1}{2}$ . Let  $m, n \in \mathbb{N}$ . Suppose that  $m < n$ . Then  $x_m \in N_{n-1}$ . Thus  $\|x_n - x_m\| \geq \|x_n + N_{n-1}\| > \frac{1}{2}$

- (2) Suppose that  $X$  is locally compact. Then  $\overline{B(0,1)}$  is compact and therefore sequentially compact. Using  $(x_n)_{n \in \mathbb{N}} \subset \overline{B(0,1)}$  defined in (1), we see that there exists a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$ ,  $x \in \overline{B(0,1)}$  such that  $x_{n_k} \rightarrow x$ . Then  $(x_{n_k})_{k \in \mathbb{N}}$  is Cauchy. So there exists  $N \in \mathbb{N}$  such that for each  $j, k \in \mathbb{N}$ , if  $j, k \geq N$ , then  $\|x_{n_j} - x_{n_k}\| < \frac{1}{2}$ . Then  $\|x_{n_N} - x_{n_{N+1}}\| < \frac{1}{2}$ . This is a contradiction since by construction,  $\|x_{n_N} - x_{n_{N+1}}\| > \frac{1}{2}$ . Thus  $X$  is not locally compact.

□

## 7. APPENDIX

## 7.1. Summation.

**Definition 7.1.** Let  $f : X \rightarrow [0, \infty)$ , Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when  $X$  is countable. For  $f : X \rightarrow \mathbb{C}$ , we can write  $f = g + ih$  where  $g, h : X \rightarrow \mathbb{R}$ . If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for  $g^+, g^-, h^+, h^-$ . In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation  $\sum_{x \in X} f(x)$  where  $f : X \rightarrow \mathbb{C}$ .

**Note 7.2.** Let  $f : X \rightarrow \mathbb{C}$  and  $\alpha : X \rightarrow X$  a bijection. If  $\sum_{x \in X} |f(x)| < \infty$ , then  $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$ .