

# REAL ANALYSIS NOTES

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## 1. MEASURE

### 1.1. Product Measures.

**Definition 1.1.** Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be measurable spaces. Put  $\mathcal{E} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$ . Then  $\mathcal{E}$  is an elementary family and thus  $\mathcal{M}_0 = \{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n \subset \mathcal{E} \text{ are disjoint}\}$  is an algebra on  $X \times Y$ . We define  $\pi_0 : \mathcal{M}_0 \rightarrow [0, \infty]$  by

$$\pi_0\left(\bigcup_{i=1}^n A_i \times B_i\right) = \sum_{i=1}^n \mu(A_i)\nu(B_i)$$

Since  $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{M}_0)$ , we define a product measure  $\mu \times \nu$  on  $(X \times Y, \mathcal{A} \otimes \mathcal{B})$  to be an extension of  $\pi_0$  to  $\mathcal{A} \otimes \mathcal{B}$ . The existence of which is guaranteed by Caratheodory's theorem and on  $\mathcal{A} \otimes \mathcal{B}$ ,

$$\begin{aligned} \mu \times \nu(E) &= \inf\left\{\sum_{n \in \mathbb{N}} \pi_0(E_i) : (E_i)_{i \in \mathbb{N}} \subset \mathcal{M}_0 \text{ and } E \subset \bigcup_{i \in \mathbb{N}} E_i\right\} \\ &= \inf\left\{\sum_{n \in \mathbb{N}} \mu(A_i)\nu(B_i) : (A_i \times B_i)_{i \in \mathbb{N}} \subset \mathcal{E} \text{ and } E \subset \bigcup_{i \in \mathbb{N}} A_i \times B_i\right\} \end{aligned}$$

If  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  are both sigma finite, then so is  $\pi_0$  and thus  $\mu \times \nu$  is unique.

## 2. INTEGRATION

## 2.1. Measurable Functions.

**Definition 2.1.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . Then  $f$  is said to be  **$\mathcal{A}$ - $\mathcal{B}$  measurable** if for each  $B \in \mathcal{B}$ ,  $f^{-1}(B) \in \mathcal{A}$ . When  $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  we say that  $f$  is  **$\mathcal{A}$ -measurable**. If  $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and  $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  or  $(\mathbb{R}, \mathcal{L})$ , then we say that  $f$  is **Borel measurable** or **Lebesgue measurable** respectively.

**Lemma 2.2.** Let  $(X, \mathcal{A}), (Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . Then

- (1)  $\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$  is a  $\sigma$ -algebra on  $Y$
- (2)  $\{f^{-1}(B) : B \in \mathcal{B}\}$  is a  $\sigma$ -algebra on  $X$

**Lemma 2.3.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. Suppose that there exists  $\mathcal{E} \subset Y$  such that  $\sigma(\mathcal{E}) = \mathcal{B}$ . Let  $f : X \rightarrow Y$ . If for each  $B \in \mathcal{E}$ ,  $f^{-1}(B) \in \mathcal{A}$ , then  $f$  is  $\mathcal{A}$ - $\mathcal{B}$  measurable.

*Proof.* The previous lemma tells us that  $\mathcal{L} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$  is a  $\sigma$ -algebra on  $Y$ . Since  $\mathcal{E} \subset \mathcal{L}$ , we have that  $\mathcal{B} = \sigma(\mathcal{E}) \subset \mathcal{L}$ .  $\square$

**Corollary 2.4.** Let  $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$  be topological spaces and  $f : X \rightarrow Y$ . If  $f$  is continuous, then  $f$  is  $\mathcal{B}(X)$ - $\mathcal{B}(Y)$  measurable.

*Proof.* Recall that  $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$  and continuity tells us that for each  $U \in \mathcal{T}_2$ ,  $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$ .  $\square$

**Definition 2.5.** Let  $X$  be a set and  $f : X \rightarrow \mathbb{C}$ . The  $f$  is said to be **simple** if  $f(X)$  is finite.

**Definition 2.6.** Let  $(X, \mathcal{A})$  be a measurable space. We define  $S^+(X, \mathcal{A}) = \{f : X \rightarrow [0, \infty) : f \text{ is simple, measurable}\}$  and  $S(X, \mathcal{A}) = \{f : X \rightarrow \mathbb{C} : f \text{ is simple, measurable}\}$

**Theorem 2.7.** Let  $(X, \mathcal{A})$  be a measurable space. Then

- (1) If  $f : X \rightarrow [0, \infty]$  is measurable, then there exists a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \rightarrow f$  pointwise and  $\phi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.
- (2) If  $f : X \rightarrow \mathbb{C}$  is measurable, then there exists a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset S$  such that for each  $n \in \mathbb{N}$ ,  $|\phi_n| \leq |\phi_{n+1}| \leq |f|$  and  $\phi_n \rightarrow f$  pointwise and  $\phi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.

## 2.2. Integration of Nonnegative Functions.

**Definition 2.8.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Define  $L^+(X, \mathcal{A}, \mu) = \{f : X \rightarrow [0, \infty] : f \text{ is measurable}\}$ . We will typically just write  $L^+$ .

**Theorem 2.9.** Monotone Convergence Theorem Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$ . Then

$$\sup_{n \in \mathbb{N}} \int f_n = \int \sup_{n \in \mathbb{N}} f_n$$

.

**Exercise 2.10.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$  and  $f \in L^+$ . Then

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

.

*Proof.* Suppose that  $f$  is simple. Then there exist  $(a_n)_{n=1}^\infty \subset [0, \infty)$  and  $(E_i)_{i=1}^\infty \subset \mathcal{A}$  such that  $f = \sum_{i=1}^\infty a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^\infty a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^\infty a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^\infty a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for general  $f$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that  $\phi_n \rightarrow f$  pointwise and for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1} \leq f$ . Then monotone convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

**Exercise 2.11.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$ . Suppose that  $\mu_1 \leq \mu_2$ . Then for each  $f \in L^+$ ,

$$\int f d\mu_1 \leq \int f d\mu_2$$

*Proof.* First suppose that  $f$  is simple. Then there exist  $(a_n)_{n=1}^\infty \subset [0, \infty)$  and  $(E_i)_{i=1}^\infty \subset \mathcal{A}$  such that  $f = \sum_{i=1}^\infty a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d\mu_1 &= \sum_{i=1}^\infty a_i \mu_1(E_i) \\ &\leq \sum_{i=1}^\infty a_i \mu_2(E_i) \\ &= \int f d\mu_2 \end{aligned}$$

for general  $f$ ,

$$\begin{aligned} \int f d\mu_1 &= \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_1 \\ &\leq \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_2 \\ &= \int f d\mu_2 \end{aligned}$$

□

**Theorem 2.12.** *Fatou's Lemma* Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

**Theorem 2.13.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n.$$

**Exercise 2.14.** Let  $f \in L^+$  and suppose that  $\int f < \infty$ . Put  $N = \{x \in X : f(x) = \infty\}$  and  $S = \{x \in X : f(x) > 0\}$ . Then  $\mu(N) = 0$  and  $S$  is  $\sigma$ -finite.

*Proof.* Suppose that  $\mu(N) > 0$ . Define  $f_n = n\chi_N \in L^+$ . Then for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1} \leq f$  on  $N$ . So

$$\begin{aligned} \int f &\geq \int_N f \\ &= \lim_{n \rightarrow \infty} \int_N f_n \\ &= \lim_{n \rightarrow \infty} n\mu(N) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

Hence  $N$  is a null set. Now, put  $S_n = \{x \in X : f(x) > 1/n\}$ . Then  $S = \bigcup_{n \in \mathbb{N}} S_n$ . Suppose that there exists some  $n \in \mathbb{N}$  such that  $\mu(S_n) = \infty$ . Then

$$\begin{aligned} \int f &\geq \int_{S_n} f \\ &\geq \frac{1}{n} \mu(S_n) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

So for each  $n \in \mathbb{N}$ ,  $\mu(S_n) < \infty$  and  $S$  is  $\sigma$ -finite.

□

**Exercise 2.15.** Let  $f \in L^+$ . Then  $f = 0$  a.e. iff for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$ .

*Proof.*  $f = 0$  a.e. implies that for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$  is clear. Conversely, suppose that for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$ . For  $n \in \mathbb{N}$  put  $N_n = \{x \in X : f(x) > 1/n\}$  and define  $N = \{x \in X : f(x) > 0\}$ . So  $N = \bigcup_{n \in \mathbb{N}} N_n$ . Let  $n \in \mathbb{N}$ . Then our assumption tells us that

$$\begin{aligned} 0 &= \int_{N_n} f \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

Hence for each  $n \in \mathbb{N}$ ,  $\mu(N_n) = 0$ . Thus  $\mu(N) = 0$  and  $f = 0$  a.e. as required.  $\square$

**Exercise 2.16.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$  and  $f \in L^+$ . Suppose that  $f_n \xrightarrow{p.w.} f$ ,  $\lim_{n \rightarrow \infty} \int f_n = \int f$  and  $\int f < \infty$ . Then for each  $E \in \mathcal{A}$ ,  $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$ . This result may fail to be true if  $\int f = \infty$

*Proof.* Let  $E \in \mathcal{A}$ . By Fatou's lemma,  $\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$ . Note that since  $\int f < \infty$ , we have that  $\int_{E^c} f \leq \int f < \infty$ . Thus we may write

$$\begin{aligned} \int_E f &= \int f - \int_{E^c} f \\ &\geq \int f - \liminf_{n \rightarrow \infty} \int_{E^c} f_n \\ &= \int f - \liminf_{n \rightarrow \infty} \left( \int f_n - \int_E f_n \right) \\ &= \int f - \int f + \limsup_{n \rightarrow \infty} \int_E f_n \\ &= \limsup_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Hence

$$\limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$$

and therefore

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

If we drop the assumption that  $\int f < \infty$ , then the result would fail to be true for the functions  $f = \infty \chi_{(0,1)}$  and  $f_n = \infty \chi_{(0,1)} + n \chi_{(1,1+1/n)}$ . Here  $f_n \xrightarrow{p.w.} f$ ,  $\lim_{n \rightarrow \infty} \int f_n = \int f = \infty$  and  $\lim_{n \rightarrow \infty} \int_{(1,\infty)} f_n = 1$  while  $\int_{(1,\infty)} f = 0$ .  $\square$

**Exercise 2.17.** Let  $f \in L^+$ . Define  $\lambda : \mathcal{A} \rightarrow [0, \infty]$  by  $\lambda(E) = \int_E f d\mu$  for  $E \in \mathcal{A}$ . Then  $\lambda$  is a measure on  $(X, \mathcal{A})$  and for each  $g \in L^+$ ,  $\int g d\lambda = \int g f d\mu$ .

*Proof.* Clearly  $\lambda(\emptyset) = 0$ . Let  $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$  and suppose that for each  $i, j \in \mathbb{N}$ , if  $i \neq j$ , then  $A_i \cap A_j = \emptyset$ . For now, suppose that  $f$  is simple. Then there exist  $E_1, E_2, \dots, E_n \in \mathcal{A}$  and  $a_1, a_2, \dots, a_n \in [0, \infty)$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Then

$$\begin{aligned}
 \lambda\left(\bigcup_{j \in \mathbb{N}} A_j\right) &= \int_{\bigcup_{j \in \mathbb{N}} A_j} f \\
 &= \sum_{i=1}^n a_i \mu\left(E_i \cap \left(\bigcup_{j \in \mathbb{N}} A_j\right)\right) \\
 &= \sum_{i=1}^n a_i \mu\left(\bigcup_{j \in \mathbb{N}} E_i \cap A_j\right) \\
 &= \sum_{i=1}^n a_i \sum_{j \in \mathbb{N}} \mu(E_i \cap A_j) \\
 &= \sum_{j \in \mathbb{N}} \sum_{i=1}^n a_i \mu(E_i \cap A_j) \\
 &= \sum_{j \in \mathbb{N}} \int_{A_j} f \\
 &= \sum_{j \in \mathbb{N}} \lambda(A_j)
 \end{aligned}$$

Hence  $\lambda$  is a measure on  $(X, \mathcal{A})$ . Now, for a general  $f$ , there exist  $(\phi_n)_{n \in \mathbb{N}} \subset L^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n$  is simple,  $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \xrightarrow{\text{p.w.}} f$ . Put  $A = \bigcup_{j \in \mathbb{N}} A_j$  and define the measures  $\lambda_n$  by  $\lambda_n(E) = \int_E \phi_n$ . Note that we may define a monotonically increasing sequence of functions  $g_n : \mathbb{N} \rightarrow [0, \infty]$  by  $g_n(j) = \int_{A_j} \phi_n$ . Using monotone convergence three times and a nice application of the counting measure on  $\mathbb{N}$ , we may write

$$\begin{aligned}
 \lambda(A) &= \int_A f \\
 &= \lim_{n \rightarrow \infty} \int_A \phi_n \\
 &= \lim_{n \rightarrow \infty} \sum_{j \in \mathbb{N}} \int_{A_j} \phi_n \\
 &= \sum_{j \in \mathbb{N}} \lim_{n \rightarrow \infty} \int_{A_j} \phi_n \quad (\text{by the above}) \\
 &= \sum_{j \in \mathbb{N}} \int_{A_j} f \\
 &= \sum_{j \in \mathbb{N}} \lambda(A_j).
 \end{aligned}$$

Hence  $\lambda$  is a measure on  $(X, \mathcal{A})$ . Let  $g \in L^+$ . First assume that  $g$  is simple. Then there exist  $E_1, E_2, \dots, E_n \in \mathcal{A}$  and  $a_1, a_2, \dots, a_n \in [0, \infty)$  such that  $g = \sum_{i=1}^n a_i \chi_{E_i}$ . In this case, we have that

$$\begin{aligned} \int g d\lambda &= \sum_{i=1}^n a_i \lambda(E_i) \\ &= \sum_{i=1}^n a_i \int_{E_i} f d\mu \\ &= \int \left( \sum_{i=1}^n a_i \chi_{E_i} \right) f d\mu \\ &= \int g f d\mu. \end{aligned}$$

Now for a general  $g \in L^+$ , there exist  $(\psi_n)_{n \in \mathbb{N}} \subset L^+$  such that for each  $n \in \mathbb{N}$ ,  $\psi_n$  is simple,  $\psi_n \leq \psi_{n+1} \leq f$  and  $\psi_n \xrightarrow{\text{p.w.}} g$ . Monotone convergence then gives us

$$\begin{aligned} \int g d\lambda &= \lim_{n \rightarrow \infty} \int \psi_n d\lambda \\ &= \lim_{n \rightarrow \infty} \int \psi_n f d\mu \\ &= \int g f d\mu \text{ as required.} \end{aligned}$$

□

**Exercise 2.18.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$  and  $f \in L^+$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \geq f_{n+1}$ ,  $f_n \xrightarrow{\text{p.w.}} f$  and  $\int f_1 < \infty$ . Then  $\lim_{n \rightarrow \infty} \int f_n = \int f$ .

*Proof.* First we note that since  $\int f_1 < \infty$ ,  $f_1 < \infty$  a.e., for each  $n \in \mathbb{N}$ ,  $f_1 - f_n$  and  $\int f_1 - \int f_n$  are well defined and  $\int f_n \leq \int f_1 < \infty$ . Also, for  $n \in \mathbb{N}$ ,  $f_1 - f_n \in L^+$ . So we may write

$$\begin{aligned} \int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n \\ &= \int [(f_1 - f_n) + f_n] - \int f_n \\ &= \int f_1 - \int f_n \end{aligned}$$

Put  $g_n = f + (f_1 - f_n)$ . Then  $g_n \in L^+$ , for each  $n \in \mathbb{N}$ ,  $g_n \leq g_{n+1}$  and  $g_n \xrightarrow{\text{p.w.}} f_1$ . Monotone convergence tells us that

$$\begin{aligned} \int f_1 &= \lim_{n \rightarrow \infty} \int g_n \\ &= \lim_{n \rightarrow \infty} \left[ \int f + (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \int f + \int (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \int f + \int f_1 - \int f_n \right] \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \int f$  and  $\lim_{n \rightarrow \infty} \int f_1$  exist,  $\lim_{n \rightarrow \infty} \int f_n = \int f$  as required. □

### 2.3. Integration of Complex Valued Functions.

**Definition 2.19.** Let  $f : X \rightarrow \mathbb{C}$  be measurable. Then  $f$  is said to be **integrable** if

$$\int |f| d\mu < \infty$$

**Definition 2.20.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Define  $L^1(X, \mathcal{A}, \mu) = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable and } \int |f| < \infty\}$

**Lemma 2.21.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is integrable iff  $f^+$  and  $f^-$  are integrable.

*Proof.*  $f^+, f^- \leq |f| = f^+ + f^-$  □

**Definition 2.22.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is said to be **extended integrable** if

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

**Lemma 2.23.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is integrable iff  $\text{Re}(f)$  and  $\text{Im}(f)$  are integrable.

*Proof.*  $|\text{Re}(f)|, |\text{Im}(f)| \leq |f| \leq |\text{Re}(f)| + |\text{Im}(f)|$  □

**Theorem 2.24.** Dominated Convergence Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$ ,  $f$  measurable and  $g \in L^1$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$  and for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then  $f \in L^1$  and  $\int f_n \rightarrow \int f$ .

**Exercise 2.25.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$ . Then

- (1)  $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$
- (2) for each  $f \in L^1(\mu_1 + \mu_2)$ , we have that

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

*Proof.* (1) The first part is clear since similar exercise from the section on nonnegative functions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f| d\mu_1 + \int |f| d\mu_2$$



(2) Suppose that  $f$  is simple. Then there exist  $(a_n)_{n=1}^\infty \subset \mathbb{C}$  and  $(E_i)_{i=1}^\infty \subset \mathcal{A}$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for general  $f$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S$  such that  $\phi_n \rightarrow f$  pointwise and for each  $n \in \mathbb{N}$ ,  $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ . Then dominated convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

**Theorem 2.26.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$ . Suppose that

$$\sum_{n \in \mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero,  $\sum_{n \in \mathbb{N}} f_n \in L^1$  and

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

**Theorem 2.27.** Let  $f \in L^1$ . Then for each  $\epsilon > 0$ , there exists  $\phi \in L^1$  such that  $\phi$  is simple and  $\int |f - \phi| < \epsilon$ .

**Exercise 2.28.** Generalized Fatou's Lemma: Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of measurable real valued functions. Suppose that there exists  $g \in L^1$  such that  $g \geq 0$  and for each  $n \in \mathbb{N}$ ,  $f_n \geq -g$ . Then  $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$ . What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?

*Proof.* First note that for each  $n \in \mathbb{N}$ ,  $\int f_n$  is well defined since  $f_n^- \leq g \in L^1$ . Since  $g + f_n \geq 0$ , we may use Fatou's lemma to write

$$\begin{aligned} \int g + \int \liminf_{n \rightarrow \infty} f_n &= \int \liminf_{n \rightarrow \infty} (g + f_n) \\ &\leq \liminf_{n \rightarrow \infty} \int (g + f_n) \\ &= \int g + \liminf_{n \rightarrow \infty} \int f_n \end{aligned}$$

Since  $\int g < \infty$ ,  $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$  as required. The analogue is as follows: Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of measurable real valued functions. Suppose that there exists  $g \in L^1$  such that  $g \geq 0$  and for each  $n \in \mathbb{N}$ ,  $f_n \leq g$ . Then  $\limsup_{n \rightarrow \infty} \int f_n \leq \int \limsup_{n \rightarrow \infty} f_n$ . To show this, just use the result from above with the sequence  $(g_n)_{n \in \mathbb{N}}$  given by  $g_n = -f_n$ .  $\square$

**Exercise 2.29.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$  and  $f : X \rightarrow \mathbb{C}$ . Suppose that  $f_n \xrightarrow{\text{uni}} f$ . Then

- (1) if  $\mu(X) < \infty$ , then  $f \in L^1(X, \mathcal{A}, \mu)$  and  $\lim_{n \rightarrow \infty} \int f_n = \int f$
- (2) if  $\mu(X) = \infty$ , then the conclusion of (1) may fail (find an example on  $\mathbb{R}$  with Lebesgue measure).

*Proof.* Choose  $N \in \mathbb{N}$  such that for  $n \geq N$  and  $x \in X$ ,  $|f(x) - f_n(x)| < 1$ . Then  $||f| - |f_N|| < 1$  and so  $|f| < |f_N| + 1$ . Thus  $\int |f| \leq \int |f_N| + \mu(X) < \infty$  and  $f \in L^1$ . Similarly for  $n \geq N$ ,  $|f_n| < |f| + 1$ . Dominated convergence then gives us that  $\lim_{n \rightarrow \infty} \int f_n = \int f$  as required. To see the necessity that  $\mu(X) < \infty$ , consider  $f \equiv 0$  and  $f_n = (1/n)\chi_{(0,n)}$ . Then  $f_n \xrightarrow{\text{uni}} f$ , but  $1 = \lim_{n \rightarrow \infty} \int f_n \neq \int f = 0$ .  $\square$

**Exercise 2.30.** Generalized Dominated Convergence Let  $f_n, g_n, f, g \in L^1$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$ ,  $g_n \xrightarrow{\text{a.e.}} g$ ,  $|f_n| \leq g_n$  and  $\int g_n \rightarrow \int g$ . Then  $\int f_n \rightarrow \int f$ .

*Proof.* We simply use Fatou's lemma. Put  $h_n = (g + g_n) - |f_n - f|$ . Since for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g_n$ , we know that  $|f| \leq g$ . So  $h_n \geq 0$  and  $h_n \xrightarrow{\text{p.w.}} 2g$ . Thus

$$\begin{aligned} 2 \int g &= \int \liminf_{n \rightarrow \infty} h_n \\ &\leq \liminf_{n \rightarrow \infty} \left[ \left( \int g + \int g_n \right) - \int |f_n - f| \right] \\ &= 2 \int g + \liminf_{n \rightarrow \infty} \left( - \int |f_n - f| \right) \\ &= 2 \int g - \limsup_{n \rightarrow \infty} \int |f_n - f| \end{aligned}$$

Hence  $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$  which implies that  $\int |f_n - f| \rightarrow 0$  and  $\int f_n \rightarrow \int f$  as required.  $\square$

**Exercise 2.31.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$  and  $f \in L^1$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$ . Then  $\int |f_n - f| \rightarrow 0$  iff  $\int |f_n| \rightarrow \int |f|$ .

*Proof.* Suppose that  $\int |f_n - f| \rightarrow 0$ . Since

$$\begin{aligned} \left| \int |f_n| - \int |f| \right| &= \left| \int (|f_n| - |f|) \right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|, \end{aligned}$$

we see that  $\int |f_n| \rightarrow \int |f|$ . Conversely, suppose that  $\int |f_n| \rightarrow \int |f|$ . Put  $h_n = |f_n - f|$ ,  $g_n = |f_n| + |f|$ ,  $h \equiv 0$  and  $g = 2f$ . Then  $h_n \xrightarrow{\text{a.e.}} h$ ,  $g_n \xrightarrow{\text{a.e.}} g$  and for each  $n \in \mathbb{N}$ ,  $h_n \leq g_n$ . Our assumption implies that  $\int g_n \rightarrow \int g$ . Thus the last exercise tells us that  $\int h_n \rightarrow \int h$  as required.  $\square$

**Exercise 2.32.** Let  $(r_n)_{n \in \mathbb{N}}$  be an enumeration of the rationals. Define  $f : \mathbb{R} \rightarrow [0, \infty)$  by

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}$$

and define  $g : X \rightarrow [0, \infty]$  by

$$g(x) = \sum_{n \in \mathbb{N}} 2^{-n} f(x - r_n).$$

Then

- (1)  $g \in L^1$  (perhaps after redefinition on a null set) and particularly  $g < \infty$  a.e.
- (2)  $g^2 < \infty$  a.e., but  $g^2$  is not integrable on any subinterval of  $\mathbb{R}$
- (3) Taking  $g \in L^1$ ,  $g$  is unbounded on each subinterval of  $\mathbb{R}$  and discontinuous everywhere and remains so after redefinition on a null set

*Proof.* For convenience, define  $f_n : \mathbb{R} \rightarrow [0, \infty)$  by  $f_n(x) = f(x - r_n)$  for  $x \in \mathbb{R}$ . To show (1) we note that for each  $n \in \mathbb{N}$ ,  $f_n \in L^1$  and

$$\begin{aligned} \int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1} \end{aligned}$$

Hence

$$\sum_{n \in \mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set,  $g \in L^1$ . In particular  $\int |g| < \infty$  and so  $|g|$  (and hence  $g$ ) are finite almost everywhere. For (2), since  $g < \infty$  a.e., so too is  $g^2$ . Let  $a, b \in \mathbb{R}$  and suppose that  $a < b$ . Choose  $N \in \mathbb{N}$  such that  $r_N \in (a, b)$ . Since all the terms in the sum are nonnegative,  $g^2 \geq \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2$  and so

$$\begin{aligned}
\int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2 \\
&= \sum_{n \in \mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\
&\geq 2^{-2N} \int_{(a,b)} f_N^2 \\
&\geq 2^{-2N} \int_{r_N}^{b \wedge (r_N+1)} \frac{1}{x - r_N} dx \\
&= \infty
\end{aligned}$$

So  $g^2$  is not integrable on any subinterval of  $\mathbb{R}$ . For (3), note that redefining  $g$  on a null set does not change the result of (2). Suppose that there is a finite subinterval  $I \subset \mathbb{R}$  such that  $g$  is bounded on  $I$ . Hence there exists  $M > 0$  such that for each  $x \in I$ ,  $g(x)^2 \leq M$ . Then

$$\begin{aligned}
\int_I g^2 &\leq M^2 m(I) \\
&< \infty
\end{aligned}$$

which is a contradiction. So  $g$  is not bounded on any subinterval of  $\mathbb{R}$ . Now, suppose that there exists  $x_0 \in \mathbb{R}$  such that  $g$  is continuous at  $x_0$ . Choose  $\delta > 0$  such that for each  $x \in \mathbb{R}$ , if  $|x - x_0| < \delta$ , then  $|g(x) - g(x_0)| < 1$ . The reverse triangle inequality tells us that for each  $x \in (x_0 - \delta, x_0 + \delta)$ ,  $|g(x)| < 1 + |g(x_0)|$ . Hence  $g$  is bounded on  $(x_0 - \delta, x_0 + \delta)$  which is a contradiction. So  $g$  is discontinuous everywhere.  $\square$

**Exercise 2.33.** Let  $f \in L^1$ .

- (1) If  $f$  is bounded, then for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ , if  $\mu(E) < \delta$ , then  $\int_E |f| < \epsilon$ .
- (2) The same conclusion holds for  $f$  unbounded.

*Proof.* (1) Since  $f$  is bounded, there exists  $M > 0$  such that  $|f| \leq M$ . Let  $\epsilon > 0$ . Choose  $\delta = \epsilon/2M$ . Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \delta$ . Then

$$\begin{aligned}
\int_E |f| &\leq M\mu(E) \\
&= M \frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

(2) Suppose that  $f$  is unbounded. Let  $\epsilon > 0$ . Then there exists  $\phi \in L^1$  such that  $\phi$  is simple and  $\int |f - \phi| < \epsilon/2$ . Since  $\phi$  is bounded, there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ ,

if  $\mu(E) < \delta$ , then  $\int_E |\phi| < \epsilon/2$ . Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \delta$ . Then

$$\begin{aligned} \int_E |f| &\leq \int_E |f - \phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

**Exercise 2.34.** Let  $f \in L^1(\mathbb{R}, \mathcal{L}, m)$ . Define  $F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x) = \int_{(-\infty, x]} f dm.$$

Then  $F$  is continuous.

*Proof.* Let  $x_0 \in \mathbb{R}$  and  $\epsilon > 0$ . Since  $f \in L^1$ , there exists  $\delta > 0$  such that for  $x \in \mathbb{R}$ , if  $|x - x_0| < \delta$ , then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| dm < \epsilon.$$

Let  $x \in \mathbb{R}$ . Suppose that  $|x - x_0| < \delta$ . Then

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f| dm \\ &< \epsilon \end{aligned}$$

So  $F$  is continuous.

□

**Exercise 2.35.** Denote by  $\delta_x$  the point mass measure at  $x \in X$  on measurable space  $(X, \mathcal{P}(X))$ . Let  $f : X \rightarrow \mathbb{C}$ . Then

$$\int f d\delta_x = f(x)$$

*Proof.* First assume that  $f$  is simple. Then there exist  $a_1, a_2, \dots, a_n \in \mathbb{C}$  and  $E_1, E_2, \dots, E_n \in \mathcal{P}(X)$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Thus  $\int f d\delta_x = f(x)$ . Now assume that  $f$ , which is measurable by choice of  $\sigma$ -algebra, satisfies  $f(X) \subset [0, \infty)$ . Choose a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset L^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n$  is simple,  $\phi_n \leq \phi_{n+1}$  and  $\phi_n \xrightarrow{p.w.} f$ . From before, we see that for each  $n \in \mathbb{N}$ ,  $\int \phi_n d\delta_x = \phi_n(x)$ . Monotone convergence tells us that  $\int f d\delta_x = f(x)$ . Now just extend to complex valued functions.

□

**Exercise 2.36.** Denote by  $\#$  the counting measure on the measurable space  $(X, \mathcal{P}(X))$ . Let  $f : X \rightarrow \mathbb{C}$  and suppose that  $f \in L^1$ . Then

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if  $f$  is integrable, then  $\{x \in X : f(x) \neq 0\}$  is countable.

*Proof.* Please refer to the definition of the sum in the appendix. First suppose that  $f(X) \subset [0, \infty)$ . For  $n \in \mathbb{N}$ , put  $X_n = \{x \in X : f(x) > 1/n\}$  and define  $X^* = \{x \in X : f(x) > 0\}$ ,  $X_0 = \{x \in X : f(x) = 0\}$ . Then  $X^* = \bigcup_{n \in \mathbb{N}} X_n$ . Since  $f \in L^1$ , we have that for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} \infty &> \int f d\# \\ &\geq \int_{X_n} f d\# \\ &\geq \frac{1}{n} \#(X_n). \end{aligned}$$

Thus for each  $n \in \mathbb{N}$ ,  $X_n$  is finite and  $X^*$  is countable. Thus there exists  $\{x_n\}_{n \in \mathbb{N}} \subset X$  such that  $X^* = \{x_n\}_{n \in \mathbb{N}}$ . For  $n \in \mathbb{N}$ , define  $E_n = \{x_1, x_2, \dots, x_n\}$  and

$$\begin{aligned} f_n &= f \chi_{E_n} \\ &= \sum_{i=1}^n f(x_i) \chi_{\{x_i\}} \end{aligned}$$

Then  $f_n \xrightarrow{\text{p.w.}} f \chi_{X^*} = f$  and for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$ . So

$$\begin{aligned} \int f &= \sup_{n \in \mathbb{N}} \int f_n \\ &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n f(x_i) \\ &= \sum_{x \in X^*} f(x) \\ &= \sum_{x \in X} f(x). \end{aligned}$$

For  $f : X \rightarrow \mathbb{C}$ , our  $L^1$  assumption and the result above tell us that

$$\sum_{x \in X} |f(x)| < \infty.$$

Thus writing  $f = g + ih$ , we see that the same is true for  $f^+, f^-, g^+, g^-$ . Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f d\# = \sum_{x \in X} f(x).$$

□

**Exercise 2.37.** Let  $f, g : X \rightarrow \mathbb{R}$ . Suppose that  $f, g \in L^1$ . Then  $f \leq g$  a.e. iff for each  $E \in \mathcal{A}$ ,  $\int_E f \leq \int_E g$ .

*Proof.* Suppose  $f \leq g$  a.e. Put  $N = \{x \in X : f(x) > g(x)\} \subset N$ . Then  $\mu(N) = 0$  and  $g - f \geq 0$  on  $N^c$ . So for each  $E \in \mathcal{A}$ ,

$$\begin{aligned} \int_E g - \int_E f &= \int_E (g - f) \\ &= \int_{E \cap N^c} (g - f) \\ &\geq 0 \end{aligned}$$

Conversely, suppose that for each  $E \in \mathcal{A}$ ,  $\int_E f \leq \int_E g$ . Put  $N_n = \{x \in X : f(x) - g(x) > 1/n\}$  and  $N = \{x \in X : f(x) > g(x)\}$ . Then  $N = \bigcup_{n \in \mathbb{N}} N_n$ . Let  $n \in \mathbb{N}$ . Then our assumption tells us that

$$\begin{aligned} 0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

So that  $\mu(N_n) = 0$ . Thus for each  $n \in \mathbb{N}$ ,  $\mu(N_n) = 0$  which implies  $\mu(N) = 0$ . Therefore  $f \leq g$  a.e. as required.  $\square$

**Definition 2.38.** Let  $\mathcal{F} \subset L^1$ . Then  $\mathcal{F}$  is said to be **uniformly integrable** if for each  $\epsilon > 0$ , there exists  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then  $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$ . (i.e.

$$\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0).$$

**Exercise 2.39.** Suppose that  $\mu$  is finite. Let  $\mathcal{F} \subset L^1$ . Then  $\mathcal{F}$  is uniformly integrable iff

- (1) there exists  $M > 0$  such that  $\sup_{f \in \mathcal{F}} \int |f| \leq M$
- (2) for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ , if  $\mu(E) < \delta$ , then  $\sup_{f \in \mathcal{F}} \int_E |f| < \epsilon$ .

*Proof.* ( $\Rightarrow$ ): (1) Suppose that  $\mathcal{F}$  is uniformly integrable. Then there exists  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then  $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < 1$ . Choose  $M = \mu(X)K + 1$ . Then for each  $f \in \mathcal{F}$ ,

$$\begin{aligned} \int |f| &= \int_{\{|f| > K\}} |f| + \int_{\{|f| \leq K\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M \end{aligned}$$

(2) Let  $\epsilon > 0$ . Then choose  $K \in \mathbb{N}$  such that  $\sup_{f \in \mathcal{F}} \int_{\{|f| > K\}} |f| < \epsilon/2$  and choose  $\delta = \epsilon/2K$ .

Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \delta$ . Then for  $f \in \mathcal{F}$ ,

$$\begin{aligned} \int_E |f| &= \int_{E \cap \{|f| > K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon \end{aligned}$$

( $\Leftarrow$ ): Choose  $M > 0$  as in (1). Suppose that there exists  $\epsilon > 0$  such that for each  $K \in \mathbb{N}$ , there exists  $f \in \mathcal{F}$  such that  $\mu(\{|f| > K\}) \geq \epsilon$ . Choose  $K \in \mathbb{N}$  such that  $K > M/\epsilon$ . Then choose  $f_K \in \mathcal{F}$  such that  $\mu(\{|f_K| > K\}) \geq \epsilon$ . Then

$$\begin{aligned} \int |f_K| &\geq \int_{\{|f_K| > K\}} |f_K| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M, \end{aligned}$$

which is a contradiction. Hence for each  $\epsilon > 0$ , there exists  $K \in \mathbb{N}$  such that for each  $f \in \mathcal{F}$ ,  $\mu(\{|f| > K\}) < \epsilon$ . Since  $\mu(\{|f| > k\})$  is a decreasing sequence in  $k$ , we have that  $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \mu(\{|f| > k\}) = 0$ . Now, let  $\epsilon > 0$ . Choose  $\delta > 0$  as in (2). Choose  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then for each  $f \in \mathcal{F}$ ,  $\mu(\{|f| > k\}) < \delta$ . Then for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then for each  $f \in \mathcal{F}$ ,

$$\int_{\{|f| > k\}} |f| < \epsilon.$$

Thus  $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$  as required. □

## 2.4. Integration on Product Spaces.

**Definition 2.40.** Let  $X$ ,  $Y$ , and  $Z$  be sets,  $E \subset X \times Y$  and  $f : X \times Y \rightarrow Z$ . For each  $x \in X$ , define  $E_x = \{y \in Y : (x, y) \in E\}$  and  $f_x : Y \rightarrow Z$  by  $f_x(y) = f(x, y)$ . For each  $y \in Y$ , define  $E^y = \{x \in X : (x, y) \in E\}$  and  $f^y : X \rightarrow Z$  by  $f^y(x) = f(x, y)$ .

**Note 2.41.** It is often helpful to observe that  $(\chi_E)_x = \chi_{E_x}$  and  $(\chi_E)^y = \chi_{E^y}$ .

**Lemma 2.42.** Let  $(X, \mathcal{A})$ ,  $(Y, \mathcal{B})$  be measurable spaces,  $Z = [0, \infty]$  or  $\mathbb{C}$  and  $f : X \times Y \rightarrow Z$ .

- (1) For each  $E \in \mathcal{A} \otimes \mathcal{B}$ ,  $x \in X$ ,  $y \in Y$ , we have that  $E_x \in \mathcal{B}$  and  $E^y \in \mathcal{A}$
- (2) If  $f$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable, then for each  $x \in X$ ,  $y \in Y$ , we have that  $f_x$  is  $\mathcal{B}$ -measurable and  $f^y$  is  $\mathcal{A}$ -measurable.

**Theorem 2.43.** Let  $(X, \mathcal{A}, \mu)$ ,  $(Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces. Then for each  $E \in \mathcal{A} \otimes \mathcal{B}$ , the maps  $\phi : X \rightarrow [0, \infty]$  and  $\psi : Y \rightarrow [0, \infty]$  defined by  $\phi(x) = \nu(E_x)$  and  $\psi(y) = \mu(E^y)$  are  $\mathcal{A}$ -measurable and  $\mathcal{B}$ -measurable, respectively and

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$



**Theorem 2.44.** *Fubini, Tonelli:* Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces.

- (1) (Tonelli) For each  $f \in L^+(X \times Y)$ , the functions  $g : X \rightarrow [0, \infty]$ ,  $h : Y \rightarrow [0, \infty]$  defined by  $g(x) = \int_Y f(x, y) d\nu(y)$  and  $h(y) = \int_X f(x, y) d\mu(x)$  are  $\mathcal{A}$ -measurable and  $\mathcal{B}$ -measurable respectively and

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

- (2) (Fubini) For each  $f \in L^1(X \times Y)$ ,  $f_x \in L^1(\nu)$  for  $\mu$ -a.e.  $x \in X$  and  $f^y \in L^1(\mu)$  for  $\nu$ -a.e.  $y \in Y$ , respectively and the functions (after redefinition of  $f$  on a null set)  $g : X \rightarrow \mathbb{C}$ ,  $h : Y \rightarrow \mathbb{C}$  defined by  $g(x) = \int_Y f(x, y) d\nu(y)$  and  $h(y) = \int_X f(x, y) d\mu(x)$  are in  $L^1(\mu)$  and  $L^1(\nu)$  respectively. Furthermore

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

**Note 2.45.** We usually just write  $\int \int f d\mu d\nu$  and  $\int \int f d\nu d\mu$  instead of  $\int h d\nu$  and  $\int g d\mu$  respectively. We have a similar result for complete product measure spaces. See

**Exercise 2.46.** Take  $X = Y = [0, 1]$ ,  $\mathcal{A} = \mathcal{B}([0, 1])$ ,  $\mathcal{B} = \mathcal{P}([0, 1])$  and  $\mu, \nu$  to be Lebesgue measure and counting measure respectively. Define  $D = \{(x, y) \in [0, 1]^2 : x = y\}$  Show that

$$\int \chi_D d\mu \times \nu, \int \int \chi_D d\mu d\nu \text{ and } \int \int \chi_D d\nu d\mu$$

are all different. (Hint: for the first integral, use the definition of  $\mu \times \nu$ )

*Proof.* Let  $x, y \in [0, 1]$ . Then  $(\chi_D)_x = \chi_{D_x} = \chi_x$  and  $(\chi_D)^y = \chi_{D^y} = \chi_y$ . Thus

$$\begin{aligned} \int \int \chi_D d\mu d\nu &= \int \mu(\{y\}) d\nu \\ &= \int 0 d\nu \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \int \int \chi_D d\mu d\nu &= \int \nu(\{x\}) d\mu \\ &= \int 1 d\mu \\ &= 1 \end{aligned}$$

Now, Observe that  $\int \chi_D d\mu \times \nu = \mu \times \nu(D)$ . Recall from the section on product measures that  $\mu \times \nu(D) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) : (A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n\}$ . Let  $(A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ . Suppose that  $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n$ . Then for each  $x \in [0, 1]$ ,  $(x, x) \in \bigcup_{n \in \mathbb{N}} A_n \times B_n$ . So for each  $x \in [0, 1]$ , there exists  $n \in \mathbb{N}$ , such that  $x \in A_n \cap B_n$ . Thus  $[0, 1] \subset \bigcup_{n \in \mathbb{N}} A_n \cap B_n$ . Since  $1 = \mu([0, 1]) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap B_n)$ , we know that there exists  $n \in \mathbb{N}$  such that  $0 < \mu(A_n \cap B_n)$ . Thus  $\mu(A_n) > 0$  and  $\mu(B_n) > 0$ . Since  $\mu(B_n) > 0$ ,  $B_n$  must be infinite and therefore  $\nu(B_n) = \infty$ . So  $\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) = \infty$ .

□

**Exercise 2.47.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $f : X \rightarrow [0, \infty) \in L^+$ . Show that  $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$  and  $\mu \times m(G) = \int_X f d\mu$ . The same is true if we replace " $\geq$ " with " $>$ ". (Hint: to show that  $G$  is measurable, split up  $(x, y) \mapsto f(x) - y$  into the composition of measurable functions.

*Proof.* Define  $\phi : X \times [0, \infty) \rightarrow [0, \infty)^2$  and  $\psi : [0, \infty)^2 \rightarrow [0, \infty)$  by  $\phi(x, y) = (f(x), y)$  and  $\psi(z, y) = z - y$ . Then  $G = \{(x, y) \in X \times [0, \infty) : \psi \circ \phi(x, y) \geq 0\}$ . Let  $A, B \in \mathcal{B}([0, \infty))$ . Then  $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0, \infty))$ . Since  $\mathcal{B}([0, \infty)^2) = \mathcal{B}([0, \infty)) \otimes \mathcal{B}([0, \infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0, \infty))\})$ , we have that  $\phi$  is  $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty)^2)$  measurable. Since  $\psi$  is continuous, we have that  $\psi$  is  $\mathcal{B}([0, \infty)^2)$ - $\mathcal{B}([0, \infty))$  measurable. This implies that  $\psi \circ \phi$  is  $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty))$  measurable. Thus  $G = \psi \circ \phi^{-1}([0, \infty)) \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ . Now for  $x \in X$ ,  $G_x = \{y \in [0, \infty) : f(x) \geq y\} = [0, f(x)]$ . Thus

$$\begin{aligned} \mu \times m(G) &= \int \chi_G d\mu \times m \\ &= \int_X \int_{[0, \infty)} \chi_{G_x} dm d\mu(x) \\ &= \int_X f(x) d\mu(x) \end{aligned}$$

The same reasoning holds if we replace " $\geq$ " with " $>$ ". □

**Exercise 2.48.** Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces and  $f : X \rightarrow \mathbb{C}, g : Y \rightarrow \mathbb{C}$ . Define  $h : X \times Y \rightarrow \mathbb{C}$  by  $h(x, y) = f(x)g(y)$ .

- (1) If  $f$  is  $\mathcal{A}$ -measurable and  $g$  is  $\mathcal{B}$ -measurable, then  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable.
- (2) If  $f \in L^1(\mu)$  and  $g \in L^1(\nu)$ , then  $h \in L^1(\mu \times \nu)$  and

$$\int_{X \times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu$$

*Proof.* (1) First suppose that  $f, g$  are simple. Then there exist  $(A_i)_{i=1}^n \subset \mathcal{A}, (B_j)_{j=1}^m \subset \mathcal{B}$  and  $(a_i)_{i=1}^n, (b_j)_{j=1}^m \subset \mathbb{C}$  such that  $f = \sum_{i=1}^n a_i \chi_{A_i}$  and  $g = \sum_{j=1}^m b_j \chi_{B_j}$ . Then  $h = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{A_i \times B_j}$ . So  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable. For general  $f, g$ , there exist  $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$  and  $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$  such that  $f_n \rightarrow f$  pointwise,  $g_n \rightarrow g$  pointwise and for each  $n \in \mathbb{N}$ ,  $|f_n| \leq |f_{n+1}| \leq |f|$  and  $|g_n| \leq |g_{n+1}| \leq |g|$ . For  $n \in \mathbb{N}$ , define  $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$  by  $h_n = f_n g_n$ . Then  $h_n \rightarrow h$  pointwise and for each  $n \in \mathbb{N}$ ,  $|h_n| \leq |h_{n+1}| \leq |h|$ . Thus  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable.

- (2) First suppose  $f$  and  $g$  are simple as before. Then

$$\begin{aligned} \int_{X \times Y} |h| d\mu \times \nu &\leq \sum_{i=1}^n \sum_{j=1}^m |a_i b_j| \mu(A_i) \nu(B_j) \\ &= \left( \sum_{i=1}^n |a_i| \mu(A_i) \right) \left( \sum_{j=1}^m |b_j| \nu(B_j) \right) \\ &= \int_X |f| d\mu \int_Y |g| d\nu \\ &< \infty \end{aligned}$$

So  $h \in L^1(\mu \times \nu)$ . Furthermore,

$$\begin{aligned} \int_{X \times Y} h d\mu \times \nu &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mu(A_i) \nu(B_j) \\ &= \left( \sum_{i=1}^n a_i \mu(A_i) \right) \left( \sum_{j=1}^m b_j \nu(B_j) \right) \\ &= \int_X f d\mu \int_Y g d\nu \end{aligned}$$

For general  $f \in L^1(\mu), g \in L^1(\nu)$ , take  $(h_n)_{n \in \mathbb{N}}$  as before. Monotone convergence and the result above say that

$$\begin{aligned} \int_{X \times Y} |h| d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} |h_n| d\mu \times \nu \\ &= \lim_{n \rightarrow \infty} \left( \int_X |f_n| d\mu \int_Y |g_n| d\nu \right) \\ &= \int_X |f| d\mu \int_Y |g| d\nu \\ &< \infty \end{aligned}$$

So  $h \in L^1(\mu \times \nu)$ . Dominated convergence and the result above then tell us that

$$\begin{aligned} \int_{X \times Y} h d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} h_n d\mu \times d\nu \\ &= \lim_{n \rightarrow \infty} \left( \int_X f_n d\mu \int_Y g_n d\nu \right) \\ &= \int_X f d\mu \int_Y g d\nu \end{aligned}$$

□

**Note 2.49.** In the above exercise part (2), we can replace  $L^1$  with  $L^+$  and get the same result by the same method.

**Exercise 2.50.** Let  $f : \mathbb{R} \rightarrow [0, \infty) \in L^+$ . Show that

$$\int_{\mathbb{R}} f dm = \int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) dm(t)$$

*Proof.* Note that

$$\int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) = \int_{[0, \infty)} \left[ \int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}} dm \right] dm(t)$$

Comparing this with Tonelli's theorem, we can put  $\chi_{\{x \in \mathbb{R} : f(x) \geq t\}} = (\chi_E)^t = \chi_{E^t}$ . Then  $E = \{(x, t) \in \mathbb{R} \times [0, \infty) : f(x) \geq t\}$  and  $E_x = \{t \in [0, \infty) : f(x) \geq t\} = [0, f(x)]$ . Tonelli's

theorem tells us that

$$\begin{aligned} \int_{[0,\infty)} \left[ \int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}}(x) dm(x) \right] dm(t) &= \int_{\mathbb{R}} \left[ \int_{[0,\infty)} \chi_{[0,f(x)]}(t) dm(t) \right] dm(x) \\ &= \int_{\mathbb{R}} f(x) dm(x) \end{aligned}$$

□

## 2.5. Convergence.

**Definition 2.51.** Let  $(X, \mathcal{A})$  be a measurable space. For convenience we will define  $L^0 = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable}\}$ .

**Definition 2.52.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Then  $f_n$  converges to  $f$  **in measure** if for each  $\epsilon > 0$ ,  $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \rightarrow 0$ . This is written  $f_n \xrightarrow{\mu} f$ .

**Definition 2.53.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Then  $f_n$  converges to  $f$  **almost uniformly** if for each  $\epsilon > 0$ , there exists  $N \in \mathcal{A}$  such that  $\mu(N) < \epsilon$  and  $f_n \xrightarrow{\text{uni}} f$  on  $N^c$ . This is written  $f_n \xrightarrow{\text{a.u.}} f$ .

**Theorem 2.54.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . If  $f_n \xrightarrow{\mu} f$ , then there exists a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  such that  $f_{n_k} \xrightarrow{\text{a.e.}} f$ .

**Theorem 2.55.** (Egoroff): Suppose that  $\mu(X) < \infty$ . Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that  $f_n \xrightarrow{\text{a.e.}}$ . Then  $f_n \xrightarrow{\text{a.u.}} f$ .

**Exercise 2.56.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$  and  $f \in L^1$ . If  $f_n \xrightarrow{L^1} f$ , then  $f_n \xrightarrow{\mu} f$ .

*Proof.* Let  $\epsilon > 0$ . for  $n \in \mathbb{N}$ , define  $E_{\epsilon,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$ . Then for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \int |f - f_n| &\geq \int_{E_{\epsilon,n}} |f - f_n| \\ &\geq \epsilon \mu(E_{\epsilon,n}). \end{aligned}$$

So for each  $n \in \mathbb{N}$ ,  $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$ . Since  $\int |f - f_n| \rightarrow 0$ , we have that  $\mu(E_{\epsilon,n}) \rightarrow 0$ . Since  $\epsilon > 0$  is arbitrary,  $f_n \xrightarrow{\mu} f$  as required. □

**Exercise 2.57.** Suppose  $\mu(X) < \infty$ . Define  $d : L^0 \times L^0 \rightarrow [0, \infty)$  by

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \quad f, g \in L^0$$

Then  $d$  is a metric on  $L^0$  if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each  $f, g \in L^0$ ,  $d(f, g) \leq \mu(X)$ .

*Proof.* Let  $f, g \in L^0$ . Clearly  $d(f, g) = d(g, f)$ . If  $f = g$  a.e. then clearly  $d(f, g) = 0$ . Conversely, if  $d(f, g) = 0$ , then  $\frac{|f - g|}{1 + |f - g|} = 0$  a.e and so  $|f - g| = 0$  a.e. which implies  $f = g$  a.e. It is not hard to show that  $\phi : [0, \infty) \rightarrow [0, \infty)$  given by  $\phi(x) = \frac{x}{1+x}$  satisfies  $\phi(x + y) \leq \phi(x) + \phi(y)$ . Thus satisfies the triangle inequality. Now, let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ . Suppose that  $f_n \not\xrightarrow{\mu} f$ . Then there exists  $\epsilon > 0, \delta > 0$  and a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  such that

for each  $k \in \mathbb{N}$ ,  $\mu(E_{\epsilon, n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$ . It is not hard to show that  $\phi$  from earlier is increasing. Thus for each  $k \in \mathbb{N}$ ,

$$\begin{aligned} d(f_{n_k}, f) &= \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{\epsilon}{1 + \epsilon} \\ &\geq \frac{\epsilon \delta}{1 + \epsilon} \end{aligned}$$

So  $f_{n_k} \not\stackrel{d}{\rightarrow} f$ . Hence  $f_{n_k} \stackrel{d}{\rightarrow} f$  implies that  $f_{n_k} \stackrel{\mu}{\rightarrow} f$ . Conversely, suppose that  $f_{n_k} \stackrel{\mu}{\rightarrow} f$ . Let  $\epsilon > 0$ . Then  $\delta = \frac{\epsilon}{1 + \mu(X)} > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then  $\mu(E_{\delta, n}) < \frac{\delta}{1 + \delta}$ . Let  $n \in \mathbb{N}$ . Suppose that  $n \geq N$ . Since  $\phi$  is increasing and  $\phi \leq 1$ , we have that

$$\begin{aligned} d(f_n, f) &= \int \frac{|f_n - f|}{1 + |f_n - f|} \\ &= \int_{E_{\delta, n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E_{\delta, n}^c} \frac{|f_n - f|}{1 + |f_n - f|} \\ &\leq \mu(E_{\delta, n}) + \mu(X) \frac{\delta}{1 + \delta} \\ &< \frac{\delta}{1 + \delta} (1 + \mu(X)) \\ &\leq \delta (1 + \mu(X)) \\ &= \epsilon \end{aligned}$$

□

**Exercise 2.58.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \geq 0$  and  $f_n \stackrel{\mu}{\rightarrow} f$ . Then  $f \geq 0$  a.e. and  $\int f \leq \liminf_{n \rightarrow \infty} \int f_n$ .

*Proof.* Since  $f_n \stackrel{\mu}{\rightarrow} f$ , there is a subsequence converging to  $f$  a.e. So clearly  $f \geq 0$  a.e. Now, choose a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  such that  $\int f_{n_k} \rightarrow \liminf_{n \rightarrow \infty} \int f_n$ . Since  $f_n \stackrel{\mu}{\rightarrow} f$  so does  $(f_{n_k})_{k \in \mathbb{N}}$ . Therefore there exists a subsequence  $(f_{n_{k_j}})_{j \in \mathbb{N}}$  of  $(f_{n_k})_{k \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ . Thus  $f \geq 0$  a.e. and Fatou's lemma tells us that

$$\begin{aligned} \int f &\leq \liminf_{j \in \mathbb{N}} \int f_{n_{k_j}} \\ &= \liminf_{n \rightarrow \infty} \int f_n. \end{aligned}$$

□

**Exercise 2.59.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that there exists  $g \in L^1$  such that for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then  $f_n \stackrel{\mu}{\rightarrow} f$  implies that  $f \in L^1$  and  $f_n \xrightarrow{L^1} f$ .

*Proof.* Clearly  $(f_n)_{n \in \mathbb{N}} \subset L^1$ . Since  $f_n \xrightarrow{\mu} f$ , there exists a subsequence  $(f_{n_k})_{k \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$  such that  $f_{n_k} \xrightarrow{\text{a.e.}} f$ . This implies that  $|f| \leq g$  a.e. and so  $f \in L^1$ . For  $n \in \mathbb{N}$ , put  $h_n = 2g - |f_n - f|$ . Then for each  $n \in \mathbb{N}$ ,  $h_n \geq 0$  and  $h_n \xrightarrow{\mu} 2g$ . By the previous exercise

$$\begin{aligned} \int 2g &\leq \liminf_{n \rightarrow \infty} \int (2g - |f_n - f|) \\ &= \int 2g - \limsup_{n \rightarrow \infty} \int |f_n - f|. \end{aligned}$$

So  $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$  which implies that  $\int |f_n - f| \rightarrow 0$  and  $f_n \xrightarrow{L^1} f$  as required.  $\square$

**Exercise 2.60.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ ,  $f \in L^0$  and  $\phi : \mathbb{C} \rightarrow \mathbb{C}$ .

- (1) If  $\phi$  is continuous, and  $f_n \xrightarrow{\text{a.e.}} f$  then  $\phi \circ f_n \xrightarrow{\text{a.e.}} \phi \circ f$ .
- (2) If  $\phi$  is uniformly continuous and  $f_n \rightarrow f$  uniformly, almost uniformly or in measure, then  $\phi \circ f_n \rightarrow \phi \circ f$  uniformly, almost uniformly or in measure, respectively.
- (3) Find a counter example to (2) if we drop the word "uniform".

*Proof.* (1) Clear

(2) Suppose that  $\phi$  is uniformly continuous.

(uniform conv.) Suppose that  $f_n \xrightarrow{\text{uni}} f$ . Let  $\epsilon > 0$ . Choose  $\delta > 0$  such that for each  $z, w \in \mathbb{C}$ , if  $|z - w| < \delta$ , then  $|\phi(z) - \phi(w)| < \epsilon$ . Now choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$  if  $n \geq N$  then for each  $x \in X$ ,  $|f_n(x) - f(x)| < \delta$ . Let  $n \in \mathbb{N}$ , suppose  $n \geq N$ . Let  $x \in X$ . Then  $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$ . Thus  $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$ .

(almost uni.) Suppose that  $f_n \xrightarrow{\text{a.u.}} f$ . Let  $\epsilon > 0$ . Choose  $N \in \mathcal{A}$  such  $\mu(N) < \epsilon$  and  $f_n \xrightarrow{\text{uni}} f$  on  $N^c$ . Then from above, we know that  $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$  on  $N^c$ . Thus  $\phi \circ f_n \xrightarrow{\text{a.u.}} \phi \circ f$ .

(measure) Suppose that  $f_n \xrightarrow{\mu} f$ . Let  $\epsilon > 0$ . Choose  $\delta > 0$  such that for each  $z, w \in \mathbb{C}$ , if  $|z - w| < \delta$ , then  $|\phi(z) - \phi(w)| < \epsilon$ . Observe that for  $x \in X$ , if  $|f_n(x) - f(x)| < \delta$ , then  $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$ . Hence  $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$ . By definition of convergence in measure,  $\mu(F_{n,\delta}) \rightarrow 0$ . Thus  $\mu(E_{n,\epsilon}) \rightarrow 0$ . Hence  $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$ .

(3)

$\square$

**Exercise 2.61.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that  $f_n \xrightarrow{\text{a.u.}} f$ . Then  $f_n \xrightarrow{\mu} f$  and  $f_n \xrightarrow{\text{a.e.}} f$ .

*Proof.* (measure) Let  $\epsilon > 0$ ,  $\delta > 0$ . Choose  $M \in \mathcal{A}$  such that  $\mu(M) < \delta$  and  $f_n \xrightarrow{\text{uni}} f$  on  $M^c$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then for each  $x \in M^c$ ,  $|f_n(x) - f(x)| < \epsilon$ . Let  $n \in \mathbb{N}$ . Suppose  $n \geq N$ . Then  $E_{\epsilon,n} \subset M$  and  $\mu(E_{\epsilon,n}) < \delta$ . Thus  $\mu(E_{\epsilon,n}) \rightarrow 0$  and  $f_n \xrightarrow{\mu} f$ .

(a.e.) For each  $n \in \mathbb{N}$ , Choose  $N_n \in \mathcal{A}$  such that  $\mu(N_n) < 1/n$  and  $f_n \xrightarrow{\text{uni}} f$  on  $N_n^c$ . Observe that for  $x \in X$ , if  $x \in \bigcup_{n \in \mathbb{N}} N_n^c$ , then  $f_n(x) \rightarrow f(x)$ . Thus  $N = \{x \in X : f_n(x) \not\rightarrow f(x)\} \subset \bigcap_{n \in \mathbb{N}} N_n$ . Therefore  $\mu(N) = 0$  and  $f_n \xrightarrow{\text{a.e.}} f$ .  $\square$

**Exercise 2.62.** Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset L^0$  and  $f, g \in L^0$ . Suppose that  $f_n \xrightarrow{\mu} f$  and  $g_n \xrightarrow{\mu} g$ . Then

- (1)  $f_n + g_n \xrightarrow{\mu} f + g$
- (2) if  $\mu(X) < \infty$ , then  $f_n g_n \xrightarrow{\mu} f g$

*Proof.* (1) Let  $\epsilon > 0$ . For convenience, put  $F_{n,\epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$ ,  $G_{n,\epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$ , and  $(F + G)_{n,\epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g(x))| \geq \epsilon\}$ . Observe that for  $x \in X$ ,  $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$ . Thus  $(F + G)_{n,\epsilon} \subset F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$ . Since  $\mu(F_{n,\epsilon/2} \cup G_{n,\epsilon/2}) \leq \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}) \rightarrow 0$ , we have that  $\mu((F + G)_{n,\epsilon}) \rightarrow 0$ . Hence  $f_n + g_n \xrightarrow{\mu} f + g$ .

- (2) Suppose that  $\mu(X) < \infty$ . Let  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  be a subsequence of  $(f_n g_n)_{n \in \mathbb{N}}$ . Choose a subsequence  $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$  and  $g_{n_{k_j}} \xrightarrow{\text{a.e.}} g$ . Then  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.e.}} f g$ . Egoroff's theorem tells us that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.u.}} f g$ , which implies that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} f g$ . Thus for each subsequence  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  of  $(f_n g_n)_{n \in \mathbb{N}}$ , there exists a subsequence  $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$  of  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  such that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} f g$ . Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that  $f_n g_n \xrightarrow{\mu} f g$ . □

**Exercise 2.63.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that for each  $\epsilon > 0$ ,

$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then  $f_n \xrightarrow{\text{a.e.}} f$ .

*Proof.* Let  $\epsilon > 0$ . By assumption we know that

$$\begin{aligned} \int \left[ \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} \right] d\mu &= \sum_{n \in \mathbb{N}} \int \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} d\mu \\ &= \sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \\ &< \infty \end{aligned}$$

Thus we also know that  $\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} < \infty$  a.e. Equivalently, we could say that for a.e.  $x \in X$ ,  $|\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}| < \infty$ . For  $k \in \mathbb{N}$ , define  $N_k = \{x \in X : \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > 1/k\}} = \infty\}$ . Then for each  $k \in \mathbb{N}$ ,  $\mu(N_k) = 0$ . Define  $N = \bigcup_{k \in \mathbb{N}} N_k$ . Then  $\mu(N) = 0$ . Let  $x \in N^c$  and  $\epsilon > 0$ . Choose  $k \in \mathbb{N}$  such that  $1/k < \epsilon$ . Then  $\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\} \subset \{n \in \mathbb{N} : |f_n(x) - f(x)| > 1/k\}$  which is finite because  $x \in N_k^c$ . Put  $M = \max\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}$ . Then for  $m \geq M$ ,  $|f_m(x) - f(x)| \leq \epsilon$ . Thus  $f_n(x) \rightarrow f(x)$ . Hence  $f_n \xrightarrow{\text{a.e.}} f$ . □

### 3. DIFFERENTIATION

#### 3.1. Signed Measures.

**Definition 3.1.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$ . Then  $\nu$  is said to be a **signed measure** if

- (1) for each  $E \in \mathcal{A}$ ,  $\nu(E) < \infty$  or for each  $E \in \mathcal{A}$ ,  $\nu(E) > -\infty$ .
- (2)  $\nu(\emptyset) = 0$

(3) for each  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  if  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is disjoint, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$  and if  $|\sum_{n \in \mathbb{N}} \nu(E_n)| < \infty$ , then  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely.

**Exercise 3.2.** Let  $\nu : \mathcal{A} \rightarrow [0, \infty]$  be a signed measure and  $(E_n)_{n \in \mathbb{N}}, (F_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . If  $(E_n)_{n \in \mathbb{N}}$  is increasing, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \rightarrow \infty} \nu(E_n)$ . If  $(F_n)_{n \in \mathbb{N}}$  is decreasing and  $|\nu(E_1)| < \infty$ , then  $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$ .

*Proof.* Put  $E'_1 = E_1$ ,  $F'_1 = F_1$  and for  $n \in \mathbb{N}$ ,  $n \geq 2$ , put  $E'_n = E_n \setminus E_{n-1}$  and  $F'_n = F_1 \setminus F_n$ . Then  $(E'_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is disjoint. Thus

$$\begin{aligned} \nu(\bigcup_{n \in \mathbb{N}} E_n) &= \nu(\bigcup_{n \in \mathbb{N}} E'_n) \\ &= \sum_{n \in \mathbb{N}} \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \sum_{n=1}^n \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \nu(E_n) \end{aligned}$$

Since  $(F'_n)_{n \in \mathbb{N}}$  is increasing, we now know that

$$\begin{aligned} \nu(F_1) - \nu(\bigcap_{n \in \mathbb{N}} F_n) &= \nu(F_1 \setminus \bigcap_{n \in \mathbb{N}} F_n) \\ &= \nu(\bigcup_{n \in \mathbb{N}} F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F_1 \setminus F_n) \\ &= \nu(F_1) - \lim_{n \rightarrow \infty} \nu(F_n) \end{aligned}$$

Since  $|\nu(F_1)| < \infty$ , we see that  $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$ . □

**Definition 3.3.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$  a signed measure and  $E \in \mathcal{A}$ . Then  $E$  is said to be  $\nu$ -**positive**,  $\nu$ -**negative** and  $\nu$ -**null** if for each  $F \in \mathcal{A}$ ,  $F \subset E$  implies that  $\nu(F) \geq 0$ ,  $\nu(F) \leq 0$ ,  $\nu(F) = 0$  respectively.

**Exercise 3.4.** Let  $E \subset \mathcal{A}$ . If  $E$  is positive, negative or null, then for each  $F \in \mathcal{A}$ , if  $F \subset E$ , then  $F$  is positive, negative or null respectively.

*Proof.* Clear □

**Exercise 3.5.** Let  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  be positive, negative or null. Then  $\bigcup_{n \in \mathbb{N}} E_n$  is positive, negative or null respectively.

*Proof.* Suppose that  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is positive. Let  $F \in \mathcal{A}$ . Suppose that  $F \subset \bigcup_{n \in \mathbb{N}} E_n$ . Put

$P_1 = E_1$  and for  $n \in \mathbb{N}$ ,  $n \geq 2$ , put  $P_n = E_n \setminus (\bigcup_{j=1}^{n-1} E_j)$ . So  $\bigcup_{n \in \mathbb{N}} P_n = \bigcup_{n \in \mathbb{N}} E_n$  and  $(P_n)_{n \in \mathbb{N}}$  is



disjoint. Thus

$$\begin{aligned}\nu(F) &= \nu\left(F \cap \bigcup_{n \in \mathbb{N}} P_n\right) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} (F \cap P_n)\right) \\ &= \sum_{n \in \mathbb{N}} \nu(F \cap P_n) \\ &\geq 0\end{aligned}$$

The process is the same if  $(E_n)_{n \in \mathbb{N}}$  is negative and null.  $\square$

**Theorem 3.6.** *Hahn Decomposition:* Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then there exist  $P, N \in \mathcal{A}$  such that  $P$  is positive,  $N$  is negative,  $X = N \cup P$  and  $N \cap P = \emptyset$ . Furthermore, these two sets are unique in the following sense: For any  $P', N' \in \mathcal{A}$ , if  $N, P$  satisfy the properties above,  $P' \Delta P = N' \Delta N$  is null.

**Definition 3.7.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$  and  $P, N \in \mathcal{A}$ . Then  $P$  and  $N$  are said to form a **Hahn decomposition** of  $X$  with respect to  $\nu$  if  $P, N$  satisfy the results in the above theorem.

**Definition 3.8.** Let  $\mu, \nu$  be signed measures on  $(X, \mathcal{A})$ . Then  $\mu$  and  $\nu$  are said to be **mutually singular** if there exist  $E, F \in \mathcal{A}$  such that  $X = E \cup F$ ,  $E \cap F = \emptyset$  and  $E$  is  $\mu$ -null and  $F$  is  $\nu$ -null. We will denote this by  $\mu \perp \nu$ .

**Theorem 3.9.** *Jordan Decomposition:* Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then there exist unique positive measures  $\nu^+$  and  $\nu^-$  on  $(X, \mathcal{A})$  such that  $\nu = \nu^+ - \nu^-$  and  $\nu^+ \perp \nu^-$ .

*Proof.* Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Define  $\nu^+, \nu^-$  by  $\nu^+(E) = \nu(E \cap P)$  and  $\nu^-(E) = \nu(E \cap N)$ .  $\square$

**Definition 3.10.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then  $\nu^+$  and  $\nu^-$  from the last theorem are called the **positive** and **negative variations** of  $\nu$  respectively. We define the **total variation** measure  $|\nu|$  on  $(X, \mathcal{A})$  by  $|\nu| = \nu^+ + \nu^-$ .

**Definition 3.11.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then  $\nu$  is said to be  $\sigma$ -finite if  $|\nu|$  is  $\sigma$ -finite.

**Exercise 3.12.** Let  $\nu$  be a signed measure and  $\lambda, \mu$  positive measures on  $(X, \mathcal{A})$ . Suppose that  $\nu = \lambda - \mu$ . Then  $\lambda \geq \nu^+$  and  $\mu \geq \nu^-$ .

*Proof.* Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Let  $E \in \mathcal{A}$ . Then

$$\begin{aligned}\lambda(E \cap P) - \mu(E \cap P) &= \nu(E \cap P) \\ &= \nu^+(E \cap P)\end{aligned}$$

So  $\lambda(E \cap P) \geq \nu^+(E \cap P)$  and therefore

$$\begin{aligned}\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) \\ &= \nu^+(E)\end{aligned}$$

Similarly  $\mu(E \cap N) \geq \nu^-(E \cap N)$  and  $\mu(E) \geq \nu^-(E)$ .  $\square$

**Exercise 3.13.** Let  $\nu_1, \nu_2$  be signed measures on  $(X, \mathcal{A})$ . Suppose that  $\nu_1 + \nu_2$  is a signed measure. Then  $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ . (Hint: use the last exercise)

*Proof.* Since

$$\begin{aligned}\nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\ &= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)\end{aligned}$$

the previous exercise tells us that  $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$  and  $\mu = \nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$ . Therefore

$$\begin{aligned}|\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ &\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\ &= |\nu_1| + |\nu_2|\end{aligned}$$

□

**Note 3.14.** Recall that a previous exercise from the section on complex valued functions tells us that  $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$ .

**Definition 3.15.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then we define  $L^1(\nu) = L^1(|\nu|)$ . For  $f \in L^1(\nu)$ , we define

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

**Exercise 3.16.** Let  $\nu_1, \nu_2$  be signed measures on  $(X, \mathcal{A})$ . Suppose that  $\nu_1 + \nu_2$  is a signed measure. Then  $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$

*Proof.* The previous exercise tells us that  $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ . Two previous exercises from the section on nonnegative functions tells us that

$$\begin{aligned}\int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\ &= \int |f| d|\nu_1| + \int |f| d|\nu_2|\end{aligned}$$

□

**Exercise 3.17.** Let  $\nu, \mu$  be signed measures on  $(X, \mathcal{A})$  and  $E \in \mathcal{A}$ . Then

- (1)  $E$  is  $\nu$ -null iff  $|\nu|(E) = 0$
- (2)  $\nu \perp \mu$  iff  $|\nu| \perp \mu$  iff  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ .

*Proof.* (1) Suppose that  $E$  is  $\nu$ -null. Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Then  $\nu^+(E) = \nu(E \cap P) = 0$  and  $\nu^-(E) = \nu(E \cap N) = 0$ . Therefore  $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$ . Conversely, suppose that  $|\nu|(E) = 0$ . Then  $\nu^+(E) = \nu^-(E) = 0$ . Let  $F \in \mathcal{A}$ . Suppose that  $F \subset E$ . Then  $\nu^+(F) = 0$  and  $\nu^-(F) = 0$ . Therefore  $\nu(F) = \nu^+(F) - \nu^-(F) = 0$ . So  $E$  is  $\nu$ -null.

- (2) Suppose that  $\nu \perp \mu$ . Then there exist  $E, F \in \mathcal{A}$  such that  $E \cup F = X$ ,  $E \cap F = \emptyset$ ,  $E$  is  $\mu$ -null and  $F$  is  $\nu$ -null. By (1),  $F$  is  $|\nu|$ -null and thus  $|\nu| \perp \mu$ . If  $|\nu| \perp \mu$ , choose  $E, F \in \mathcal{A}$  as before. Since  $F$  is  $|\nu|$ -null, we know that  $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$ . This implies that  $F$  is  $\nu^+$ -null and  $F$  is  $\nu^-$ -null. So  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ . Finally assume that  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ . **FINISH!!!!**

□

**Exercise 3.18.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then

- (1) for  $f \in L^1(\nu)$ ,  $|\int f d\nu| \leq \int |f| d|\nu|$
- (2) if  $\nu$  is finite, then for each  $E \in \mathcal{A}$ ,  $|\nu|(E) = \sup\{|\int_E f d\nu| : f \text{ is measurable and } |f| \leq 1\}$

*Proof.* (1) Let  $f \in L^1(\nu)$ . Then

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f d\nu^+ - \int f d\nu^- \right| \\ &\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \\ &= \int |f| d(\nu^+ + \nu^-) \\ &= \int |f| d|\nu| \end{aligned}$$

- (2) Let  $E \in \mathcal{A}$ . Let  $f : X \rightarrow \mathbb{R}$  be measurable and suppose that  $|f| \leq 1$ . Since  $\nu$  is finite, so is  $|\nu|$  and thus  $f \in L^1(\nu)$ . Then (1) tells us that

$$\begin{aligned} \left| \int_E f d\nu \right| &\leq \int_E |f| d|\nu| \\ &\leq |\nu|(E) \end{aligned}$$

Now, choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Define  $f = \chi_P - \chi_N$ . Then  $|f| \leq 1$ ,  $f$  is measurable and

$$\begin{aligned} \left| \int_E f d\nu \right| &= \left| \int_E f d\nu^+ - \int_E f d\nu^- \right| \\ &= |\nu^+(E \cap P) + \nu^-(E \cap N)| \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E). \end{aligned}$$

□

**Exercise 3.19.** Let  $\mu$  be a positive measure on  $(X, \mathcal{A})$  and  $f \in L^0(X, \mathcal{A})$  extended  $\mu$ -integrable. Define  $\nu$  on  $(X, \mathcal{A})$  by  $\nu(E) = \int_E f d\mu$ . Then

- (1)  $\nu$  is a signed measure
- (2) for each  $E \in \mathcal{A}$ ,  $|\nu|(E) = \int_E |f| d\mu$ .

*Proof.* (1) Clearly  $\nu(\emptyset) = 0$  and  $\nu$  is finite by assumption. Let  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Suppose that  $(E_n)_{n \in \mathbb{N}}$  is disjoint. Then

$$\begin{aligned}
 \nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \\
 &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu - \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
 &= \sum_{n \in \mathbb{N}} \left[ \int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right] \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n} f d\mu \\
 &= \sum_{n \in \mathbb{N}} \nu(E_n)
 \end{aligned}$$

If  $|\nu(\bigcup_{n \in \mathbb{N}} E_n)| < \infty$ , then  $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu < \infty$  and  $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu < \infty$  because

$$\begin{aligned}
 |\nu(\bigcup_{n \in \mathbb{N}} E_n)| &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \right| \\
 &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \right|
 \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
 \sum_{n \in \mathbb{N}} |\nu(E_n)| &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f d\mu \right| \\
 &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right| \\
 &\leq \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu + \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
 &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu + \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
 &< \infty
 \end{aligned}$$

So the sum  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely and  $\nu$  is a signed measure.

- (2) Put  $P = \{x \in X : f(x) \geq 0\}$  and  $N = \{x \in X : f(x) < 0\}$ . Then  $P, N$  form a Hahn decomposition of  $X$  with respect to  $\nu$ . Thus for  $E \in \mathcal{A}$ ,

$$\nu^+(E) = \int_{E \cap P} f d\mu = \int_E f^+ d\mu$$

and

$$\nu^-(E) = \int_{E \cap N} f d\mu = \int_E f^- d\mu$$

. So for  $E \in \mathcal{A}$ ,

$$|\nu|(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$$

□

### 3.2. The Lebesgue-Radon-Nikodym Theorem.

**Definition 3.20.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  be a signed measure on  $(X, \mathcal{A})$  and  $\mu$  a measure on  $(X, \mathcal{A})$ . Then  $\nu$  is said to be **absolutely continuous** with respect to  $\mu$ , denoted  $\nu \ll \mu$ , if for each  $E \in \mathcal{A}$ ,  $\mu(E) = 0$  implies that  $\nu(E) = 0$ .

**Note 3.21.** If there exists an extended  $\mu$ -integrable  $f \in L^0(X, \mathcal{A})$  such that for each  $E \in \mathcal{A}$ ,  $\nu(E) = \int_E f d\mu$ , then we write  $d\nu = f d\mu$ .

**Theorem 3.22.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  be a  $\sigma$ -finite signed measure on  $(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measure on  $(X, \mathcal{A})$ . Then there exist unique  $\sigma$ -finite signed measures  $\lambda, \rho$  on  $(X, \mathcal{A})$  such that  $\lambda \perp \mu$ ,  $\rho \ll \mu$  and  $\nu = \lambda + \rho$ , and there exists an extended  $\mu$ -integrable  $f \in L^0(X, \mathcal{A})$  such that  $\rho = f d\mu$  and  $f$  is unique  $\mu$ -a.e.

**Definition 3.23.** The decomposition  $\nu = \lambda + \rho$  is referred to as the **Lebesgue decomposition of  $\nu$  with respect to  $\mu$** . In the case  $\nu \ll \mu$ , we have  $\lambda = 0$  and  $\rho = \nu$  and we define the **Radon-Nikodym derivative of  $\nu$  with respect to  $\mu$** , denoted by  $d\nu/d\mu$ , to be  $d\nu/d\mu = f$  where  $d\nu = f d\mu$ .

**Theorem 3.24.** Let  $\nu$  be a  $\sigma$ -finite signed measure on  $(X, \mathcal{A})$  and  $\mu, \lambda$   $\sigma$ -finite measures on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \mu$  and  $\mu \ll \lambda$ . Then

(1) for each  $g \in L^1(\nu)$ ,  $g(d\nu/d\mu) \in L^1(\mu)$  and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2)  $\nu \ll \lambda$  and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

**Exercise 3.25.** Let  $(\nu_n)_{n \in \mathbb{N}}$  be a sequence of measures and  $\mu$  a measure.

(1) If for each  $n \in \mathbb{N}$ ,  $\nu_n \ll \mu$ , then  $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$ .

(2) If for each  $n \in \mathbb{N}$ ,  $\nu_n \perp \mu$ , then  $\sum_{n \in \mathbb{N}} \nu_n \perp \mu$ .

*Proof.* (1) Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) = 0$ . Then for each  $n \in \mathbb{N}$ ,  $\nu_i(E) = 0$  and thus  $\sum_{n \in \mathbb{N}} \nu_n(E) = 0$ . Hence  $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$ .

(2) For each  $n \in \mathbb{N}$ , there exist  $N_i, M_i \in \mathcal{A}$  such that  $N_i \cap M_i = \emptyset$ ,  $N_i \cup M_i = X$  and  $\nu_i(M_i) = \mu(N_i) = 0$ . Put  $N = \bigcup_{n \in \mathbb{N}} N_i$  and  $M = N^c$ . Note that for each  $n \in \mathbb{N}$ ,  $M \subset N_i^c = M_i$ . So  $\mu(N) \leq \sum_{n \in \mathbb{N}} \mu(N_i) = 0$  and  $(\sum_{n \in \mathbb{N}} \nu_i)(M) \leq \sum_{n \in \mathbb{N}} \nu_i(M_i) = 0$ . Thus  $\sum_{n \in \mathbb{N}} \nu_i \perp \mu$ .

□

**Exercise 3.26.** Choose  $X = [0, 1]$ ,  $\mathcal{A} = \mathcal{B}_{[0,1]}$ . Let  $m$  be Lebesgue measure and  $\mu$  the counting measure.

Then

- (1)  $m \ll \mu$  but for each  $f \in L^+$ ,  $dm \neq f d\mu$
- (2) There is no Lebesgue decomposition of  $\mu$  with respect to  $m$ .

*Proof.* (1) Let  $E \in \mathcal{A}$ . If  $\mu(E) = 0$ , then  $E = \emptyset$  and  $m(E) = 0$ . So  $m \ll \mu$ . Suppose for the sake of contradiction that there exists  $f \in L^+$  such that  $dm = f d\mu$ . Then

$$\begin{aligned} 1 &= m(X) \\ &= \sum_{x \in X} f(x) \end{aligned}$$

Put  $Z = \{x \in X : f(x) \neq 0\}$ . Then  $Z$  is countable. So

$$\begin{aligned} 1 &= m(X \setminus Z) \\ &= \sum_{x \in X \setminus Z} f(x) \\ &= 0 \end{aligned}$$

This is a contradiction, so no such  $f$  exists.

- (2) Suppose for the sake of contradiction that there is a Lebesgue decomposition for  $\mu$  with respect to  $m$  given by  $\mu = \lambda + \rho$  where  $\lambda \perp m$  and  $\rho \ll m$ . We may assume  $\lambda$  and  $\rho$  are positive. Then for each  $x \in X$ ,  $m(\{x\}) = 0$  which implies that  $\rho(\{x\}) = 0$ . Let  $E \subset X$ , if  $E$  is countable, then  $\lambda(E) = \mu(E)$ . If  $E$  is uncountable, choose  $F \subset E$  such that  $F$  is countable. Then

$$\begin{aligned} \lambda(E) &\geq \lambda(F) \\ &= \mu(F) \\ &= \infty \end{aligned}$$

So  $\lambda = \mu$ . This is a contradiction since  $\mu \not\ll m$ . □

**Exercise 3.27.** Let  $(X, \mathcal{F}, \mu)$  be a measure space and  $\mathcal{E}$  a sub  $\sigma$ -alg of  $\mathcal{F}$  and  $f \in L^1(\mu)$ . Define  $\nu : \mathcal{E} \rightarrow [0, \infty]$  by  $\nu(E) = \int_E f d\mu$ . Let  $\bar{\mu}$  be the restriction of  $\mu$  to  $\mathcal{E}$ . Define the **expectation of  $f$  given  $\mathcal{E}$**  to be  $E[f|\mathcal{E}] = d\nu/d\bar{\mu}$ . Then for each  $E \in \mathcal{E}$ ,

$$\int_E E[f|\mathcal{E}] d\mu = \int_E f d\mu$$

*Proof.* Let  $E \in \mathcal{E}$ . By definition,

$$\begin{aligned} \int_E E[f|\mathcal{E}] d\mu &= \int_E d\nu/d\bar{\mu} d\mu \\ &= \int_E d\nu/d\bar{\mu} d\bar{\mu} \quad (\text{since } E \in \mathcal{E}) \\ &= \nu(E) \\ &= \int_E f d\mu \end{aligned}$$

□

### 3.3. Complex Measures.

**Definition 3.28.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow \mathbb{C}$ . Then  $\nu$  is said to be a **complex measure** if

- (1)  $\nu(\emptyset) = 0$
- (2) for each sequence  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ , if  $(E_n)_{n \in \mathbb{N}}$  is disjoint, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$  and  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely.

**Note 3.29.** We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.

**Definition 3.30.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu = \nu_1 + i\nu_2$  a complex measure on  $(X, \mathcal{A})$ . We define  $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$ . For  $f \in L^1(\nu)$ , we define

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

**Theorem 3.31.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  a complex measure on  $(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measure on  $(X, \mathcal{A})$ . Then there exists a complex measure  $\lambda$  on  $(X, \mathcal{A})$  and  $f \in L^1(\mu)$  such that  $\lambda \perp \mu$  and  $d\nu = d\lambda + f d\mu$  and such that for each complex measure  $\lambda'$  on  $(X, \mathcal{A})$ ,  $f' \in L^1(\mu)$ , if  $\nu = d\lambda' + f' d\mu$ , then  $\lambda = \lambda'$  and  $f = f'$   $\mu$ -a.e.

**Theorem 3.32.** Let  $\nu$  be a complex measure on  $(X, \mathcal{A})$  and  $\mu, \lambda$   $\sigma$ -finite measures on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \mu$  and  $\mu \ll \lambda$ . Then

- (1) for each  $g \in L^1(\nu)$ ,  $g(d\nu/d\mu) \in L^1(\mu)$  and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

- (2)  $\nu \ll \lambda$  and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

**Definition 3.33.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu = \nu_1 + i\nu_2$  a complex measure on  $(X, \mathcal{A})$ . Define  $\mu = |\nu_1| + |\nu_2|$ . Then  $\nu \ll \mu$  and thus There exists  $f \in L^1(\mu)$  such that  $d\nu = f d\mu$ . Define  $|\nu| : \mathcal{A} \rightarrow [0, \infty)$  by  $|\nu|(E) = \int_E |f| d\mu$  for each  $E \in \mathcal{A}$ . We call  $|\nu|$  the **total variation of  $\nu$** .

**Exercise 3.34.** Let  $\nu$  be a complex measure on  $(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measures on  $(X, \mathcal{A})$ . If  $\nu \ll \mu$ , then  $\{x \in X : d\nu/d\mu(x) = 0\}$  is  $\nu$ -null.

*Proof.* Define  $f = d\nu/d\mu$  and  $E = \{x : f(x) = 0\}$ . Let  $A \in \mathcal{A}$  and suppose that  $A \subset E$ . Then

$$\begin{aligned} \nu(A) &= \int_A f d\mu \\ &= 0 \end{aligned}$$

□

**Exercise 3.35.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu = \nu_1 + i\nu_2$  a complex measure on  $(X, \mathcal{A})$ . Then  $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu_2|$ .

*Proof.* Let  $\mu$  and  $f$  be as in the definition of  $|\nu|$ . Since for each  $E \in \mathcal{A}$ , we have

$$\begin{aligned}\nu(E) &= \int_E f d\mu \\ &= \int_E f_1 d\mu + i \int_E f_2 d\mu\end{aligned}$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E)$$

we know that  $\nu_1 = f_1 d\mu$  and  $\nu_2 = f_2 d\mu$ .

A previous exercise tells us that  $d|\nu_1| = |f_1| d\mu$  and  $d|\nu_2| = |f_2| d\mu$ . Since  $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$ , we have that

$$\begin{aligned}|\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2|\end{aligned}$$

□

**Exercise 3.36.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu$  a complex measure on  $(X, \mathcal{A})$ . Then

- (1) for each  $E \in \mathcal{A}$ ,  $|\nu(E)| \leq |\nu|(E)$ .
- (2)  $\nu \ll |\nu|$  and  $|d\nu/d|\nu|| = 1$   $|\nu|$ -a.e.
- (3)  $L^1(\nu) = L^1(|\nu|)$  and for each  $g \in L^1(\nu)$ ,  $|\int g d\nu| \leq \int |g| d|\nu|$

*Proof.* Let  $\mu, f \in L^1(\mu)$  be as in the definition of  $|\nu|$ .

- (1) Let  $E \in \mathcal{A}$ . Then

$$\begin{aligned}|\nu(E)| &= \left| \int_E f d\mu \right| \\ &\leq \int_E |f| d\mu \\ &= |\nu|(E)\end{aligned}$$

- (2) Let  $E \in \mathcal{A}$  and suppose that  $|\nu|(E) = 0$ . The previous part implies  $|\nu(E)| = 0$  and  $\nu \ll |\nu|$ . Put  $g = d\nu/d|\nu|$ . Then

$$\begin{aligned}f &= \frac{d\nu}{d\mu} \\ &= g|f| \quad \mu\text{-a.e.}\end{aligned}$$

Hence  $|f| = |g||f|$   $\mu$ -a.e. Since  $|\nu| \ll \mu$ ,  $|f| = |g||f|$   $|\nu|$ -a.e.

A previous exercise tells us that  $|f| \neq 0$   $|\nu|$ -a.e. Thus  $|g| = 1$   $|\nu|$ -a.e.

- (3) Write  $\nu = \nu_1 + i\nu_2$  and  $f = f_1 + if_2$ . First we observe that

$$\begin{aligned}L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\ &= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\ &= L^1(|\nu_1| + |\nu_2|) \\ &= L^1(\mu)\end{aligned}$$



The previous exercise tells us that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \\ &= \mu \end{aligned}$$

Let  $g \in L^1(\mu)$ . Then

$$\begin{aligned} \int |g|d|\nu| &\leq \int |g|d\mu \\ &< \infty \end{aligned}$$

So  $g \in L^1(|\nu|)$ .

Conversely, let  $g \in L^1(|\nu|)$ . Then

$$\begin{aligned} \int |g|d|\nu_1|, \int |g|d|\nu_2| &\leq \int |g|d|\nu| \\ &< \infty \end{aligned}$$

So

$$\begin{aligned} \int |g|d\mu &= \int |g|d|\nu_1| + \int |g|d|\nu_2| \\ &< \infty \end{aligned}$$

and  $g \in L^1(\mu)$ . Hence  $L^1(\nu) = L^1(|\nu|)$ .

Now, let  $g \in L^1(\nu) = L^1(|\nu|)$ , then

$$\begin{aligned} \left| \int g d\nu \right| &= \left| \int g f d\mu \right| \\ &\leq \int |g| |f| d\mu \\ &= \int |g| d|\nu| \end{aligned}$$

□

### 3.4. Differentiation.

**Definition 3.37.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ . Then  $f$  is said to be **locally integrable** (with respect to Lebesgue measure) if  $f$  is measurable and for each  $K \subset \mathbb{R}^n$ ,  $K$  is compact implies  $\int_K |f| dm < \infty$ . We define  $L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{C} : f \text{ is locally integrable}\}$

**Definition 3.38.** For  $f \in L^1_{loc}(\mathbb{R}^n)$ ,  $r > 0$ ,  $x \in \mathbb{R}^n$ , we define the **average of  $f$  over  $B(x, r)$** , denoted by  $Af(x, r)$ , to be

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f dm$$

**Exercise 3.39.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . Define

$$H^*f(x) = \sup \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\} \quad (x \in \mathbb{R}^n)$$

Then  $Hf \leq H^*f \leq 2^n Hf$ .

*Proof.* Let  $x \in \mathbb{R}^n$ . Then

$$\left\{ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f| dm : r > 0 \right\} \subset \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\}$$

So  $Hf(x) \leq H^*f(x)$ . Let  $B$  be a ball. Then there exists  $y \in \mathbb{R}^n$ ,  $R > 0$  such that  $B = B(y, R)$ . Suppose that  $x \in B$ . Then  $B \subset B(x, 2R)$ . Since  $m(B(x, 2R)) = 2^n m(B(y, R))$ , we have that

$$\begin{aligned} \frac{1}{m(B)} \int_B |f| dm &\leq \frac{1}{m(B)} \int_{m(B(x, 2R))} |f| dm \\ &= \frac{2^n}{m(B(x, 2R))} \int_{m(B(x, 2R))} |f| dm \end{aligned}$$

Thus  $H^*f(x) \leq 2^n Hf(x)$ . □

**Lemma 3.40.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ , then  $Af : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  is continuous.

**Definition 3.41.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . We define its **Hardy Littlewood maximal function**, denoted by  $Hf$  to be

$$Hf(x) = \sup_{r>0} A|f|(x, r) \quad x \in \mathbb{R}^n$$

**Theorem 3.42.** There exists  $C > 0$  such that for each  $f \in L^1(m)$  and  $\alpha > 0$ ,

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f| dm$$

**Exercise 3.43.** Let  $f \in L^1(\mathbb{R}^n)$ . Suppose that  $\|f\|_1 > 0$ . Then there exist  $C, R > 0$  such that for each  $x \in \mathbb{R}^n$ , if  $|x| > R$ , then  $Hf(x) \geq C|x|^{-n}$ . Hence there exists  $C' > 0$  such that for each  $\alpha > 0$ ,  $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$  when  $\alpha$  is small.

*Proof.* Since  $\|f\|_1 > 0$ , there exists  $R > 0$  such that  $\int_{B(0, R)} |f| dm > 0$ . Recall that there exists  $K > 0$  such that for each  $x \in \mathbb{R}^n$  and  $r > 0$ ,  $m(B(x, r)) = Kr^n$ . Choose

$$C = \frac{\int_{B(0, R)} |f| dm}{K2^n}$$

. Let  $x \in \mathbb{R}^n$ . Suppose that  $|x| > R$ . Then  $B(0, R) \subset B(x, 2|x|)$ . Thus

$$\begin{aligned} Hf(x) &\geq \frac{1}{m(B(x, 2|x|))} \int_{B(x, 2|x|)} |f| dm \\ &= \frac{1}{K2^n|x|^n} \int_{B(x, 2|x|)} |f| dm \\ &\geq \frac{1}{K2^n|x|^n} \int_{B(0, R)} |f| dm \\ &= \frac{C}{|x|^n} \end{aligned}$$

Let  $a < \frac{C}{2R^n}$ . Then  $R^n < \frac{C}{2\alpha}$ . Choose  $C' = \frac{KC}{2}$ . Let  $A = \{x \in \mathbb{R}^n : R < |x| < (\frac{C}{\alpha})^{\frac{1}{n}}\}$ . For  $x \in A$ ,

$$\begin{aligned} Hf(x) &\geq \frac{C}{|x|^n} \\ &> \alpha \end{aligned}$$

Thus  $A \subset m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\})$  and therefore

$$\begin{aligned} m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) &\geq m(A) \\ &= m(B(0, (C/\alpha)^{1/n})) - m(B(0, R)) \\ &= K \left[ \frac{C}{\alpha} - R^n \right] \\ &> K \left[ \frac{C}{\alpha} - \frac{C}{2\alpha} \right] \\ &= \frac{KC}{2\alpha} \\ &= \frac{C'}{\alpha} \end{aligned}$$

□

**Theorem 3.44.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ , then for a.e.  $x \in \mathbb{R}^n$ ,

$$\lim_{r \rightarrow 0} Af(x, r) = f(x)$$

. Equivalently, for a.e.  $x \in \mathbb{R}^n$ ,

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} [f(y) - f(x)] dm(y) \right] = 0$$

**Note 3.45.** We can a stronger result of the same flavor.

**Definition 3.46.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . We define the **Lebesgue set of  $f$** , denoted by  $L_f$ , to be

$$\begin{aligned} L_f &= \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0} A|f - f(x)|(x, r) = 0\} \\ &= \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0 \right\} \end{aligned}$$

**Exercise 3.47.** Let  $f \in L^1_{loc}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ . If  $f$  is continuous at  $x$ , then  $x \in L_f$ .

*Proof.* Suppose that  $f$  is continuous at  $x$ . Let  $\epsilon > 0$ . By assumption, there exists  $\delta > 0$  such that for each  $y \in \mathbb{R}^n$ , if  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ . Let  $r > 0$ . Suppose that  $r < \delta$ . Then for each  $y \in \mathbb{R}^n$ ,  $y \in B(x, r)$  implies that  $|f(x) - f(y)| < \epsilon$  and thus

$$\begin{aligned} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(B(x, r))} \epsilon m(B(x, r)) \\ &= \epsilon \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0$$

and  $x \in L_f$ . □

**Theorem 3.48.** *Let  $f \in L^1_{loc}(\mathbb{R}^n)$ . Then  $m((L_f)^c) = 0$*

**Definition 3.49.** *Let  $x \in \mathbb{R}^n$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ . Then  $(E_r)_{r>0}$  is said to **shrink nicely** to  $x$  if*

- (1) *for each  $r > 0$ ,  $E_r \subset B(x, r)$*
- (2) *there exists  $\alpha > 0$  such that for each  $r > 0$ ,  $m(E_r) > \alpha m(B(x, r))$*

**Theorem 3.50.** *Let  $f \in L^1_{loc}(\mathbb{R}^n)$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ . Then for each  $x \in L_f$ ,*

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \right] = 0$$

and

$$\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

**Definition 3.51.** *Let  $\mu : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, \infty]$  be a Borel measure. Then  $\mu$  is said to be **regular** if*

- (1) *for each  $K \subset \mathbb{R}^n$ , if  $K$  is compact, then  $\mu(K) < \infty$*
- (2) *for each  $E \in \mathcal{B}(\mathbb{R}^n)$ ,  $\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subset U\}$*

*Let  $\nu$  be a signed or complex Borel measure on  $\mathbb{R}^n$ . Then  $\nu$  is said to be regular if  $|\nu|$  is regular.*

**Theorem 3.52.** *Let  $\nu$  be a regular signed or complex measure on  $\mathbb{R}^n$ . Let  $d\nu = d\lambda + f dm$  be the Lebesgue decomposition of  $\nu$  with respect to  $m$ . Then for  $m$ -a.e.  $x \in \mathbb{R}^n$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ , if  $(E_r)_{r>0}$  shrinks nicely to  $x$ , then*

$$\lim_{r \rightarrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

### 3.5. Functions of Bounded Variation.

**Definition 3.53.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Define  $F_+ : \mathbb{R} \rightarrow \mathbb{R}$  by*

$$F_+(x) = \lim_{t \rightarrow x^+} F(t) = \inf\{F(t) : t > x\}$$

**Note 3.54.** *Observe that  $F \leq F_+$  and  $F_+$  is increasing.*

**Exercise 3.55.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then for each  $x \in \mathbb{R}$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $y \in (x, x + \delta)$ ,  $0 \leq F_+(y) - F(y) \leq \epsilon$ .*

*Proof.* For the sake of contradiction, suppose not. Then there exists  $x \in \mathbb{R}$  and  $\epsilon > 0$  such that for each  $\delta > 0$ , there exist  $y \in (x, x + \delta)$  such that  $F_+(y) - F(y) > \epsilon$ . Then there

exists a sequence  $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  such that for each  $n \in \mathbb{N}$ ,  $y_n \in (x, x + \frac{1}{n})$ ,  $y_n > y_{n+1}$  and  $F_+(y_n) - F(y_n) > \epsilon$ . Choose  $N \in \mathbb{N}$  such that  $(N-1)\epsilon > F(y_1) - F(x)$ . Then

$$\begin{aligned} F(y_1) - F(x) &= \sum_{i=1}^{N-1} \left[ F(y_i) - F_+(y_{i+1}) + F_+(y_{i+1}) - F(y_{i+1}) \right] + F(y_N) - F(x) \\ &= \sum_{i=1}^{N-1} \left[ F(y_i) - F_+(y_{i+1}) \right] + \sum_{i=1}^{N-1} \left[ F_+(y_{i+1}) - F(y_{i+1}) \right] + F(y_N) - F(x) \\ &\geq (N-1)\epsilon \\ &> F(y_1) - F(x) \end{aligned}$$

This is a contradiction, so the claim holds.  $\square$

**Exercise 3.56.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then  $F_+$  is right continuous.

*Proof.* Let  $x \in \mathbb{R}$ . Let  $\epsilon > 0$ . Then there exists  $\delta_1 > 0$  such that for each  $y \in (x, x + \delta_1)$   $0 \leq F(y) - F_+(x) < \epsilon/2$ . There exists  $\delta_2 > 0$  such that for each  $y \in (x, x + \delta_2)$ ,  $0 \leq F_+(y) - F(y) < \epsilon/2$ . Choose  $\delta = \min\{\delta_1, \delta_2\}$ . Let  $y \in (x, x + \delta)$ .

$$\begin{aligned} |F_+(x) - F_+(y)| &\leq |F_+(x) - F(y)| + |F(y) - F_+(y)| \\ &= (F(y) - F_+(x)) + (F_+(y) - F(y)) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So  $\lim_{t \rightarrow x^+} F_+(t) = F_+(x)$  and  $F_+$  is right continuous.  $\square$

**Theorem 3.57.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then

- (1)  $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$  is countable
- (2)  $F$  and  $F_+$  are differentiable a.e. and  $F' = F'_+$  a.e.

**Definition 3.58.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Define  $T_F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$T_F(x) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \quad (x \in \mathbb{R})$$

$T_F$  is called the **total variation function of  $F$** .

**Exercise 3.59.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $T_F$  is increasing.

*Proof.* Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ .

Define  $A_x = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$  and  $A_y = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = y \right\}$ . Let  $z \in A_x$ . Then there exists  $(x_i)_{i=0}^n \subset \mathbb{R}$  such that  $(x_i)_{i=0}^n$  is increasing,  $x_n = x$  and  $z = \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$ . Then

$$\begin{aligned}
z &\leq z + |F(y) - F(x)| \\
&= \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \\
&\in A_y
\end{aligned}$$

So  $z \leq \sup A_y = T_F(y)$  and thus  $F_T(x) = \sup A_x \leq T_F(y)$   $\square$

**Lemma 3.60.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $T_F + F$  and  $T_F - F$  are increasing.*

**Exercise 3.61.** *For each  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $T_{|F|} \leq T_F$ .*

*Proof.* Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then by the reverse triangle inequality,

$$\sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

Thus

$$\begin{aligned}
T_{|F|}(x) &= \sup \left\{ \sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&\leq \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&= T_F(x)
\end{aligned}$$

Hence  $T_{|F|} \leq T_F$   $\square$

**Definition 3.62.** *Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is said to have **bounded variation** if  $\lim_{x \rightarrow \infty} T_F(x) < \infty$ . The **total variation of  $F$** , denoted by  $TV(F)$ , is defined to be  $TV(F) = \lim_{x \rightarrow \infty} T_F(x)$ . We define  $BV = \{F : \mathbb{R} \rightarrow \mathbb{C} : TV(F) < \infty\}$ .*

**Definition 3.63.** *Let  $a, b \in \mathbb{R}$  and  $F : [a, b] \rightarrow \mathbb{C}$ . Define  $G_F : \mathbb{R} \rightarrow \mathbb{C}$  by  $G_F = F(a)\chi_{(-\infty, a)} + F\chi_{[a, b]} + F(b)\chi_{(b, \infty)}$ . Then  $F$  is said to have **bounded variation on  $[a, b]$**  if  $G_F \in BV$ . The **total variation of  $F$  on  $[a, b]$** , denoted by  $TV(F, [a, b])$ , is defined to be  $TV(F, [a, b]) = TV(G_F)$ . We define  $BV([a, b]) = \{F : [a, b] \rightarrow \mathbb{C} : TV(F, [a, b]) < \infty\}$ .*

**Note 3.64.** *Equivalently,  $TV(F, [a, b]) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset [a, b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$  and  $F \in BV([a, b])$  iff  $TV(F, [a, b]) < \infty$ . In general,*

**Exercise 3.65.** *Let  $F \in BV$ . Then  $F$  is bounded.*

*Proof.* If  $F$  is unbounded, then the supremum in the previous definition is clearly infinite.  $\square$

**Exercise 3.66.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . If  $F$  is bounded and increasing, then  $F \in BV$ .*

*Proof.* Suppose that  $F$  is bounded and increasing. Then  $-\infty < \inf_{x \in \mathbb{R}} F(x) \leq \sup_{x \in \mathbb{R}} F(x) < \infty$ . Let  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n F(x_i) - F(x_{i-1}) \\ &= F(x) - F(x_0) \end{aligned}$$

Thus

$$T_F(x) = F(x) - \inf_{x \in \mathbb{R}} F(x)$$

. This implies that

$$\begin{aligned} TV(F) &= \sup_{x \in \mathbb{R}} F(x) - \inf_{x \in \mathbb{R}} F(x) \\ &< \infty \end{aligned}$$

Hence  $F \in BV$ . □

**Exercise 3.67.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . If  $F$  is differentiable and  $F'$  is bounded on  $[a, b]$ , then,  $F \in BV([a, b])$ .

*Proof.* Suppose that  $F$  is differentiable and  $F'$  is bounded on  $[a, b]$ . Then there exists  $M > 0$  such that for each  $x \in [a, b]$ ,  $|F'(x)| \leq M$ . Let  $(x_i)_{i=1}^n \subset [a, b]$ . Suppose that  $(x_i)_{i=1}^n$  is strictly increasing,  $x_0 = a$  and  $x_n = b$ . By the mean value theorem, for each  $i = 1, 2, \dots, n$ , there exists  $c_i \in (x_{i-1}, x_i)$  such that  $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n |F'(c_i)(x_i - x_{i-1})| \\ &\leq \sum_{i=1}^n M(x_i - x_{i-1}) \\ &= M(b - a) \end{aligned}$$

Hence  $TV(F, [a, b]) \leq M(b - a)$ . □

**Exercise 3.68.** Define  $F, G : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x) = \begin{cases} x^2 \sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 \sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then  $F$  and  $G$  are differentiable,  $F \in BV([-1, 1])$  and  $G \notin BV([-1, 1])$ .

*Proof.* On  $\mathbb{R} \setminus \{0\}$ ,

$$\begin{aligned} F'(x) &= 2x \sin(x^{-1}) - \sin(x^{-1}) \\ &= \sin(x^{-1})(2x - 1) \end{aligned}$$

We see that  $F$  is also differentiable at  $x = 0$  since

$$\begin{aligned} F'(0) &= \lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-1})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-1}) \\ &= 0 \end{aligned}$$

Therefore for each  $x \in [-1, 1]$ ,  $|F'(x)| \leq 3$ . Which by a previous exercise implies that  $F \in BV([-1, 1])$ .

On  $\mathbb{R} \setminus \{0\}$ ,

$$\begin{aligned} G'(x) &= 2x \sin(x^{-2}) - \frac{2 \sin(x^{-2})}{x} \\ &= \sin(x^{-2}) \left( 2x - \frac{2}{x} \right) \end{aligned}$$

We see that  $G$  is also differentiable at  $x = 0$  since

$$\begin{aligned} G'(0) &= \lim_{x \rightarrow 0} \frac{G(x) - G(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-2})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-2}) \\ &= 0 \end{aligned}$$

For  $n \in \mathbb{N}$ , define  $(x_i)_{i=0}^n \subset [-1, 1]$  by

$$x_i = \frac{-1}{\sqrt{\frac{\pi}{2} + i\pi}}$$

Then for each  $n \in \mathbb{N}$ ,  $(x_i)_{i=1}^n$  is strictly increasing and for each  $i = 1, 2, \dots, n$  we have that

$$\begin{aligned} |G(x_i) - G(x_{i-1})| &= \frac{1}{\frac{\pi}{2} + i\pi} + \frac{1}{\frac{\pi}{2} + (i-1)\pi} \\ &= \frac{2}{\pi} \left[ \frac{(2i-1) + (2i+1)}{(2i+1)(2i-1)} \right] \\ &= \frac{2}{\pi} \left[ \frac{4i}{4i^2 - 1} \right] \\ &> \frac{2}{i\pi} \end{aligned}$$



Hence for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} TV(G, [-1, 1]) &\geq \sum_{i=1}^n |G(x_i) - G(x_{i-1})| \\ &> \frac{2}{\pi} \sum_{i=1}^n \frac{1}{i} \end{aligned}$$

Therefore  $G \notin BV([-1, 1])$ . □

**Exercise 3.69.** *The following is stated for  $BV$ , but is also true for  $BV([a, b])$ .*

- (1) *For each  $F, G \in BV$ ,  $T_{F+G} \leq T_F + T_G$  and therefore  $BV$  is a vector space.*
- (2) *For each  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $F \in BV$  iff  $\operatorname{Re}(f) \in BV$  and  $\operatorname{Im}(F) \in BV$ .*
- (3) *For each  $F : \mathbb{R} \rightarrow \mathbb{R}$ ,  $F \in BV$  iff there exist functions  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $F_1, F_2$  are bounded, increasing and  $F = F_1 - F_2$*
- (4) *For each  $F \in BV$  and  $x \in \mathbb{R}$ ,  $\lim_{t \rightarrow x^+} F(t)$  and  $\lim_{t \rightarrow x^-} F(t)$  exist.*
- (5) *For each  $F \in BV$ ,  $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$  is countable.*
- (6) *For each  $F \in BV$ ,  $F$  and  $F_+$  are differentiable a.e. and  $F' = (F_+)'$  a.e.*
- (7) *For each  $F \in BV, c \in \mathbb{R}$ ,  $F - c \in BV$*

*Proof.* (1) Let  $F, G \in BV$ ,  $x \in \mathbb{R}$  and  $\epsilon > 0$ . Since  $T_{F+G}(x) < \infty$ ,  $T_{F+G}(x) - \epsilon < T_{F+G}(x)$ . Thus there exists  $(x_i)_{i=0}^n \subset \mathbb{R}$  such that  $(x_i)_{i=0}^n$  is increasing,  $x_n = x$  and  $T_{F+G}(x) < \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1})| + \epsilon$ . Therefore

$$\begin{aligned} T_{F+G}(x) &< \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1})| + \epsilon \\ &\leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + \sum_{i=1}^n |G(x_i) - G(x_{i-1})| + \epsilon \\ &\leq T_F(x) + T_G(x) + \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary,  $T_{F+G}(x) \leq T_F(x) + T_G(x)$ . Therefore  $TV(F+G) \leq TV(F) + TV(G) < \infty$ . Thus  $F+G \in BV$ . It is straight forward to verify the other requirements needed to show that  $BV$  is a vector space.

- (2) Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Write  $F = F_1 + iF_2$  with  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$ . Suppose that  $F \in BV$ . Note that for each  $x_1, x_2 \in \mathbb{R}$  and  $j = 1, 2$ ,  $|F_j(x_1) - F_j(x_2)| \leq |F(x_1) - F(x_2)|$ . Let  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then for  $j = 1, 2$

$$\sum_{i=1}^n |F_j(x_i) - F_j(x_{i-1})| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

. Thus for  $j = 1, 2$  we have that  $T_{F_j}(x) \leq T_F(x)$  which implies that  $\operatorname{Re}(f), \operatorname{Im}(F) \in BV$ . Conversely, Suppose that  $\operatorname{Re}(f), \operatorname{Im}(F) \in BV$ . Then  $F = \operatorname{Re}(f) + i\operatorname{Im}(f) \in BV$  by (1).

- (3) Suppose that  $F \in BV$ . Choose  $F_1 = \frac{1}{2}(T_F - F)$  and  $F_2 = \frac{1}{2}(T_F + F)$ . Then  $F_1, F_2$  are bounded, increasing and  $F = F_1 + F_2$ . Conversely, if there exist  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $F_1, F_2$  are bounded, increasing and  $F = F_1 - F_2$ , then  $F_1, F_2 \in BV$ . By (1)  $F \in BV$ .
- (4) This is clear by previous results and (3)

- (5) This is clear by previous results and (3)
- (6) This is clear by previous results and (3)
- (7) Clearly constant functions have zero total variation. The rest is implied by (1).  $\square$

**Lemma 3.70.** *Let  $F \in BV$ . Then  $\lim_{x \rightarrow -\infty} T_F(x) = 0$  and if  $F$  is right continuous, then  $T_F$  is right continuous.*

**Definition 3.71.** *Define  $NBV = \{F \in BV : F \text{ is right continuous and } \lim_{x \rightarrow -\infty} F(x) = 0\}$ .*

**Theorem 3.72.** *Let  $M(\mathbb{R})$  be the set of complex Borel measures on  $\mathbb{R}$ . For  $F \in NBV$ , define  $\mu_F \in M(\mathbb{R})$  by  $\mu_F((-\infty, x]) = F(x)$ . Then  $F \mapsto \mu_F$  defines a bijection  $NBV \rightarrow M(\mathbb{R})$ . In addition,  $|\mu_F| = \mu_{T_F}$*

**Theorem 3.73.** *Let  $F \in NBV$ . Then  $F' \in L^1(m)$ ,  $\mu_F \perp m$  iff  $F' = 0$  a.e. and  $\mu_F \ll m$  iff for each  $x \in \mathbb{R}$ ,  $\int_{(-\infty, x]} F' dm = F(x)$*

**Definition 3.74.** *Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is said to be **absolutely continuous** if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$ ,  $\sum_{i=1}^n b_i - a_i < \delta$  implies that  $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$ .*

**Definition 3.75.** *Let  $F : [a, b] \rightarrow \mathbb{C}$ . Then  $F$  is said to be **absolutely continuous on**  $[a, b]$  if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$ ,  $\sum_{i=1}^n b_i - a_i < \delta$  implies that  $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$ .*

**Proposition 3.76.** *Let  $F : [a, b] \rightarrow \mathbb{C}$ . If  $F$  is absolutely continuous on  $[a, b]$ , then  $F \in BV$ .*

**Lemma 3.77.** *Let  $F \in NBV$ . Then  $F$  is absolutely continuous iff  $\mu_F \ll m$ .*

**Theorem 3.78.** *Fundamental Theorem of Calculus: Let  $F : [a, b] \rightarrow \mathbb{C}$ . The following are equivalent:*

- (1)  $F$  is absolutely continuous on  $[a, b]$ .
- (2) there exists  $f \in L^1(m)$  such that for each  $x \in [a, b]$ ,  $F(x) - F(a) = \int_{(a, x]} f dm$
- (3)  $F$  is differentiable a.e. on  $[a, b]$ ,  $F' \in L^1(m)$  and for each  $x \in [a, b]$ ,  $F(x) - F(a) = \int_{(a, x]} F' dm$

**Exercise 3.79.** *Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . If  $F$  is absolutely continuous. Then  $F$  is differentiable a.e.*

*Proof.* Let  $n \in \mathbb{N}$ . Since  $F$  is absolutely continuous on  $\mathbb{R}$ ,  $F$  is absolutely continuous on  $[-n, n]$ . The FTC implies that  $F$  is differentiable a.e. on  $[-n, n]$ . Since  $n \in \mathbb{N}$  is arbitrary,  $F$  is differentiable a.e. on  $\mathbb{R}$ .  $\square$

**Exercise 3.80.** *Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is Lipschitz continuous iff  $F$  is absolutely continuous and  $F'$  is bounded a.e.*

*Proof.* Suppose that  $F$  is Lipschitz continuous. Then there exists  $M > 0$  such that for each  $x, y \in \mathbb{R}$ ,  $|F(x) - F(y)| \leq M|x - y|$ . Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{M}$ . Let  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$ , Suppose that  $\sum_{i=1}^n b_i - a_i < \delta$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(b_i) - F(a_i)| &\leq \sum_{i=1}^n M(b_i - a_i) \\ &< M\delta \\ &= \epsilon \end{aligned}$$

Hence  $F$  is absolutely continuous. For each  $x, y \in \mathbb{R}$ , if  $x \neq y$ , then  $\left| \frac{F(x) - F(y)}{x - y} \right| \leq M$ . Hence for a.e.  $x \in \mathbb{R}$ ,  $|F'(x)| \leq M$ . Conversely, suppose that  $F$  is absolutely continuous and  $F'$  is bounded a.e. Then there exists  $M > 0$  such that for a.e.  $x \in \mathbb{R}$ ,  $|F'(x)| \leq M$ . Let  $x, y \in \mathbb{R}$ . Suppose  $x < y$ . Then the FTC implies that

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_{(x,y]} F' dm \right| \\ &\leq \int_{(x,y]} |F'| dm \\ &= M|y - x| \end{aligned}$$

and  $F$  is Lipschitz continuous. □

**Exercise 3.81.** Construct an increasing function  $F : \mathbb{R} \rightarrow \mathbb{R}$  whose discontinuities is  $\mathbb{Q}$ .

*Proof.* Let  $(q_n)_{n \in \mathbb{N}}$  be an enumeration of  $\mathbb{Q}$ . Define  $F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F = \sum_{n \in \mathbb{N}} 2^{-n} \chi_{[q_n, \infty)}$$

. Equivalently, if we define  $S_x = \{n \in \mathbb{N} : q_n \leq x\}$ , then we may write

$$F(x) = \sum_{n \in S_x} 2^{-n}$$

Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ . Then  $S_x \subsetneq S_y$ . So  $F(x) < F(y)$  and therefore  $F$  is strictly increasing.

For each  $x, y \in \mathbb{R}$  with  $x < y$ , define  $S_{x,y} = \{n \in \mathbb{N} : x < q_n \leq y\}$ . Note that  $\lim_{y \rightarrow x^+} \min(S_{x,y}) = \infty$  and if  $y \in \mathbb{R} \setminus \mathbb{Q}$ , then  $\lim_{x \rightarrow y^-} \min(S_{x,y}) = \infty$ .

Now, let  $x \in \mathbb{R}$  and  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that  $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$ . Choose  $\delta > 0$  such that  $\min(S_{x, x+\delta}) \geq N$ . Let  $y \in [x, \infty)$ . Suppose that  $|x - y| < \delta$ . Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_y} 2^{-n} - \sum_{n \in S_x} 2^{-n} \\ &= \sum_{n \in S_{x,y}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence  $F$  is right continuous. Now let  $x \in \mathbb{R} \setminus \mathbb{Q}$  and  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  as before and  $\delta > 0$  such that  $\min(S_{x-\delta, x}) \geq N$ . Let  $y \in (-\infty, x]$ . Suppose that  $|x - y| < \delta$ . Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_x} 2^{-n} - \sum_{n \in S_y} 2^{-n} \\ &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence  $F$  is left continuous on  $\mathbb{R} \setminus \mathbb{Q}$ .

Now, let  $x \in \mathbb{Q}$ . Then there exists  $j \in \mathbb{N}$  such that  $q_j = x$ . Choose  $\epsilon = 2^{-j}$ . Let  $\delta > 0$ . Choose  $y = x - \frac{\delta}{2}$ . Then  $|x - y| < \delta$  and

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\geq 2^{-j} \\ &= \epsilon \end{aligned}$$

Hence  $F$  is discontinuous from the left at  $x$ . Since  $x \in \mathbb{Q}$  is arbitrary,  $F$  is discontinuous from the left on  $\mathbb{Q}$ .  $\square$

**Exercise 3.82.** Let  $(F_n)_{n \in \mathbb{N}} \in NBV$  be a sequence of nonnegative, increasing functions. If for each  $x \in \mathbb{R}$ ,  $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$ , then for a.e.  $x \in \mathbb{R}$ ,  $F$  is differentiable at  $x$  and  $F'(x) = \sum_{n \in \mathbb{N}} F'_n(x)$ .

*Proof.* Define  $\mu = \sum_{n \in \mathbb{N}} \mu_{F_n}$ . Note that

$$\begin{aligned} \mu((-\infty, x]) &= \sum_{n \in \mathbb{N}} \mu_{F_n}((-\infty, x]) \\ &= \sum_{n \in \mathbb{N}} F_n(x) \\ &= F(x) \end{aligned}$$

Hence  $F \in NBV$  and  $\mu = \mu_F$ . For each  $n \in \mathbb{N}$ , there exist  $\lambda_n \in M(\mathbb{R})$  and  $f_n \in L^1(\mathbb{R})$  such that  $d\mu_{F_n} = d\lambda_n + f_n dm$  and  $\lambda \perp m$ . Since for each  $n \in \mathbb{N}$ ,  $\lambda_n, f_n$  are nonnegative, we have that  $d\mu_F = \sum_{n \in \mathbb{N}} d\lambda_n + (\sum_{n \in \mathbb{N}} f_n) dm$ . By a previous theorem, for a.e.  $x \in \mathbb{R}$ ,

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow 0} \frac{\mu_F((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} f_n(x) \\ &= \sum_{n \in \mathbb{N}} \lim_{r \rightarrow 0} \frac{\mu_{F_n}((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} F'_n(x) \end{aligned}$$

□

**Exercise 3.83.** Let  $F : [0, 1] \rightarrow [0, 1]$  be the Cantor function. Extend  $F$  to  $\mathbb{R}$  by setting  $F(x) = 0$  for  $x < 0$  and  $F(x) = 1$  for  $x > 1$ . Let  $([a_n, b_n])_{n \in \mathbb{N}}$  be an enumeration of the closed subintervals of  $[0, 1]$  with rational endpoints. For  $n \in \mathbb{N}$ , define  $F_n : \mathbb{R} \rightarrow [0, 1]$  by  $F_n(x) = F(\frac{x-a_n}{b_n-a_n})$ . Define  $G : \mathbb{R} \rightarrow \mathbb{R}$  by  $G = \sum_{n \in \mathbb{N}} 2^{-n} F_n$ . Then  $G$  is continuous, strictly increasing on  $[0, 1]$  and  $G' = 0$  a.e.

*Proof.* Since  $F$  is continuous on  $\mathbb{R}$ , we have that for each  $n \in \mathbb{N}$ ,  $F_n$  is continuous on  $\mathbb{R}$ . We observe that for each  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ ,  $|2^{-n} F_n(x)| \leq 2^{-n}$ . Thus the Weierstrass M-test implies that  $G$  converges uniformly on  $\mathbb{R}$  and is therefore continuous. Since  $F$  is increasing, for each  $n \in \mathbb{N}$ ,  $F_n$  is increasing. Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ . Choose  $j \in \mathbb{N}$  such that  $x < a_j < y < b_j$ . Then

$$\begin{aligned} G(x) &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(x) \\ &= \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(x) + 0 \\ &< \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(y) + 2^{-j} F_j(y) \\ &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(y) \\ &= G(y) \end{aligned}$$

So  $G$  is strictly increasing.

Now we observe that for each  $n \in \mathbb{N}$ ,  $F_n \in NBV$ . The previous exercise implies that

$$G' = \sum 2^{-n} F'_n = 0 \text{ a.e.}$$

□

#### 4. APPENDIX

##### 4.1. Summation.

**Definition 4.1.** Let  $f : X \rightarrow [0, \infty)$ , Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when  $X$  is countable. For  $f : X \rightarrow \mathbb{C}$ , we can write  $f = g + ih$  where  $g, h : X \rightarrow \mathbb{R}$ . If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for  $g^+, g^-, h^+, h^-$ . In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.