PORTFOLIO THEORY NOTES

Contents

Note 0.1. *In these notes we will mostly consider a probability space* $(\Omega, \mathcal{F}, P)$ *and a random variable* $X : \Omega \to \mathbb{R}$. *We assume that* $X \in L^1(P)$ *and* $F_X : \mathbb{R} \to (0,1)$ *is strictly increasing and continuous. We will call such a random variable "nice". The random variable* X *will usually refer to the return on some portfolio. As such, we will define the loss of* X *to be* $L_X = -X$.

1. Risk Measures

1.1. Value at Risk.

Definition 1.1. *Let* X *be a nice random variable and* $\alpha \in (0,1)$. *We define the* ***value at risk of*** X ***at confidence level*** α, *denoted by* $VaR_\alpha(X)$, *to be*

$$VaR_\alpha(X) = F_{L_X}^{-1}(\alpha)$$

Thus

$$P(L_X > VaR_\alpha(X)) = 1 - \alpha$$

Note 1.2. *In practice, we take* $\alpha = .95$ *or* $\alpha = .99$.

1.2. Expected Shortfall.

Definition 1.3. *Let* X *be a nice random variable and* $\alpha \in (0,1)$. *We define the* ***expected shortfall of*** X ***with tail probability*** $1 - \alpha$, *denoted by* $ES_\alpha(X)$, *to be*

$$ES_\alpha(X) = \frac{1}{1-\alpha} \int_{[\alpha,1)} VaR_p(X) dm(p)$$

Note 1.4. *If* X *represents the return on a portfolio, then* $ES_\alpha(X)$ *is just the average of the* $VaR_p(X)$ *on the interval* $(\alpha, 1]$.

Exercise 1.5. *Let* X *be a nice random variable and* $\alpha \in (0,1)$. *Then*

$$ES_\alpha(X) = E[L_X | L_X \geq Var_\alpha(X)]$$

Proof. Recall that for measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a measurable function $f : X \to Y$ and a measure $\mu : \mathcal{A} \to [0, \infty]$, we may form the push-foreward measure of μ by f, $f_*\mu : \mathcal{B} \to [0, \infty]$ with the following property: for each $g : Y \to \mathbb{C}$, $g \in L^1(f_*\mu)$ iff $g \circ f \in L^1(\mu)$ and for each $B \in \mathcal{B}$,

$$\int_{f^{-1}(B)} g \circ f d\mu = \int_B g df_*\mu$$

Also recall that for an increasing continuous, bijective $F : \mathbb{R} \to (0, 1)$, we may form the Borel measure μ_F with $\mu_F((a, b]) = F(b) - F(a)$. Then observe that $F_*\mu_F = m$ because

$$\begin{aligned} F_*\mu_F((a, b]) &= \mu_F(F^{-1}((a, b])) \\ &= \mu_F((F^{-1}(a), F^{-1}(b)]) \\ &= F(F^{-1}(b)) - F(F^{-1}(a)) \\ &= b - a \end{aligned}$$

Then

$$\begin{aligned} E[L_X | L_X \geq VaR_\alpha(X)] &= E[L_X | L_X \geq F_{L_X}^{-1}(\alpha)] \\ &= \frac{1}{1-\alpha} E[L_X I_{\{L_X \geq F_{L_X}^{-1}(\alpha)\}}] \\ &= \frac{1}{1-\alpha} \int_{\{L_X \geq F_{L_X}^{-1}(\alpha)\}} L_X dP \\ &= \frac{1}{1-\alpha} \int_{[F_{L_X}^{-1}(\alpha), \infty)} p dF_{L_X}(p) \\ &= \frac{1}{1-\alpha} \int_{[F_{L_X}^{-1}(\alpha), \infty)} p dF_{L_X}(p) \\ &= \frac{1}{1-\alpha} \int_{[F_{L_X}^{-1}(\alpha), \infty)} (F_{L_X}^{-1} \circ F_{L_X})(p) dF_{L_X}(p) \\ &= \frac{1}{1-\alpha} \int_{[\alpha, 1)} F_{L_X}^{-1}(p) dm(p) \\ &= \frac{1}{1-\alpha} \int_{[\alpha, 1)} VaR_p(X) dm(p) \\ &= ES_\alpha(X) \end{aligned}$$

□

Lemma 1.6. *Let $\alpha \in (0, 1)$. Define $f_\alpha : \mathbb{R} \to \mathbb{R}$ by*

$$f_\alpha(\theta) = \theta + \frac{1}{1-\alpha} E[(L_X - \theta)^+]$$

Then f_α is convex and

$$\frac{df_\alpha}{d\theta}(\theta) = \frac{F_{L_X}(\theta) - \alpha}{1-\alpha}$$

Proof. Recall that given $g : \Omega \times \mathbb{R} \to \mathbb{R}$, if for each $\omega \in \Omega$, $g(\omega, \theta)$ is convex in θ, then $E[g(\cdot, \theta)]$ is convex in theta. For $\omega \in \Omega, \theta \in \mathbb{R}$, put

$$g(\omega, \theta) = (L_X(\omega) - \theta)^+$$

So

$$f_\alpha(\theta) = \theta + \frac{1}{1-\alpha} E[g(\cdot, \theta)]$$

Then for each $\omega \in \Omega$, $g(\omega, \cdot)$ is convex. This implies that for each $\alpha \in (0,1)$, f_α is convex and therefore continuous.

Now Let $\theta, \theta' \in \mathbb{R}$. Suppose that $\theta < \theta'$. Then

$$\frac{f_\alpha(\theta') - f_\alpha(\theta)}{\theta' - \theta} = 1 + \frac{1}{1-\alpha} E\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\right]$$

For $\omega \in \Omega$, we have that

$$\frac{(L_X(\omega) - \theta')^+ - (L_X(\omega) - \theta)^+}{\theta' - \theta} = \begin{cases} -1 & \theta' \leq L_X(\omega) \\ 0 & L_X(\omega) \leq \theta \\ \epsilon \in (-1, 0) & \theta < L_X(\omega) < \theta' \end{cases}$$

This implies that

$$\begin{aligned} -(F_{L_X}(\theta') - F_{L_X}(\theta)) &= -P(\theta < L_X < \theta') \\ &\leq E\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta} I_{L_X \in (\theta, \theta')}\right] \\ &< 0 \end{aligned}$$

Thus there exists $c \in (0,1)$ such that

$$E\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta} I_{L_X \in (\theta, \theta')}\right] = -c(F_{L_X}(\theta') - F_{L_X}(\theta))$$

In addition, $P(\theta' \leq L_X) = 1 - F_{L_X}(\theta')$. Therefore

$$E\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\right] = -(1 - F_{L_X}(\theta')) - c[F_{L_X}(\theta') - F_{L_X}(\theta)]$$

This implies that

$$\lim_{\theta' \to \theta^+} E\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\right] = F_{L_X}(\theta) - 1$$

Finally we have that

$$\lim_{\theta' \to \theta^+} \frac{f_\alpha(\theta') - f_\alpha(\theta)}{\theta' - \theta} = 1 + \frac{1}{1-\alpha} \lim_{\theta' \to \theta^+} E\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\right]$$
$$= 1 + \frac{F_{L_X}(\theta) - 1}{1-\alpha}$$
$$= \frac{F_{L_X}(\theta) - \alpha}{1-\alpha}$$

The case is similar for the lefthand limit.

□

Theorem 1.7. *Let X be a nice random variable and $\alpha \in (0,1)$. Then*

$$VaR_\alpha(X) = \arg\min_{\theta \in \mathbb{R}} \left(\theta + \frac{1}{1-\alpha} E[(L_X - \theta)^+]\right)$$

and

$$ES_\alpha(X) = \min_{\theta \in \mathbb{R}} \left(\theta + \frac{1}{1-\alpha} E[(L_X - \theta)^+]\right)$$

Proof. Define f_α as before. The previous lemma tells us that

$$\frac{df_\alpha}{d\theta}(\theta) = \frac{F_{L_X}(\theta) - \alpha}{1-\alpha}$$

Thus

$$\lim_{\theta \to \infty} \frac{df_\alpha}{d\theta}(\theta) = 1$$

and

$$\lim_{\theta \to -\infty} \frac{df_\alpha}{d\theta}(\theta) = -\frac{\alpha}{1-\alpha} < 0$$

Thus $\lim_{\theta \to \infty} f_\alpha(\theta) = \lim_{\theta \to -\infty} f_\alpha(\theta) = \infty$. The convexity of f_α implies that there exists a unique $\theta^* \in \mathbb{R}$ such that $f_\alpha(\theta^*) = \inf_{\theta \in \mathbb{R}} f_\alpha(\theta)$ Thus

$$\frac{df_\alpha}{d\theta}(\theta^*) = 0$$

which implies that

$$F_{L_X}(\theta^*) = \alpha$$

By definition, $\theta^* = VaR_\alpha(X)$. Finally, evaluating f_α at θ^* shows us that

$$
\begin{aligned}
f_\alpha(\theta^*) &= \theta^* + \frac{1}{1-\alpha}E[(L_X-\theta^*)^+] \\
&= \theta^* + \frac{1}{P(L_X>\theta^*)}E[(L_X-\theta^*)I_{\{L_X>\theta^*\}}] \\
&= \theta^* + \frac{1}{P(L_X>\theta^*)}E[L_X I_{\{L_X>\theta^*\}}] - \frac{1}{P(L_X>\theta^*)}E[\theta^* I_{\{L_X>\theta^*\}}] \\
&= \theta^* + \frac{1}{P(L_X>\theta^*)}E[L_X I_{\{L_X>\theta^*\}}] - \theta^* \\
&= E[L_X|L_X>\theta^*] \\
&= E[L_X|L_X>VaR_\alpha(X)] \\
&= ES_\alpha(X)
\end{aligned}
$$

□

2. Estimation of Risk

2.1. Estimating the Value at Risk in the IID Case.

Definition 2.1. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\alpha \in (0,1)$. Define*

$$\widehat{v}_\alpha =$$

2.2. Estimating the Expected Shortfall in the IID Case.

Definition 2.2. *Let X be a nice random random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\alpha \in (0,1)$. Define*

$$\widehat{e}_{\alpha,n} = \frac{\sum_{i=1}^n L_{X_i} I_{L_{X_i} \geq \widehat{v}_\alpha}}{\sum_{i=1}^n I_{L_X \geq \widehat{v}_\alpha}}$$

Lemma 2.3. *Let X be a nice random random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\alpha \in (0,1)$. Then $\widehat{e}_{\alpha,n} \xrightarrow{a.e.} ES_\alpha(X)$.*

Proof. Since $(L_X)_{i=1}^n \subset L^1$ are iid, the SLLN tells us that for each $v \in \mathbb{R}$,

$$\frac{1}{n}\sum_{i=1}^n L_{X_i} I_{\{L_{X_i} \geq v\}} \xrightarrow{\text{a.e.}} E[L_X I_{\{X>v\}}]$$

□

Proof. For each $\alpha \in (0,1), \omega \in \Omega$ and $\theta \in \mathbb{R}$, define

$$f_\alpha(\omega)(\theta) = \theta + \frac{1}{n(1-\alpha)}\sum_{i=1}^n \max(-X_i(\omega) - \theta, 0)$$

Note that for each $\alpha \in (0,1)$ and $\omega \in \Omega$, $f_\alpha(\omega)$ is convex and continuous. In this case with no expectation, it is easy to show that

$$\lim_{\theta \to \infty} \frac{\partial f_\alpha(\omega)}{\partial \theta}(\theta) = 1$$

and

$$\lim_{\theta \to -\infty} \frac{\partial f_\alpha(\omega)}{\partial \theta}(\theta) = -\frac{\alpha}{1-\alpha} < 0$$

So for each $\alpha \in (0,1)$ and $\omega \in \Omega$, $f_\alpha(\omega)$ achieves its minimum at . Then $\{\theta \in \mathbb{R} : f_\alpha(\omega)(\theta) \leq m+1\}$ is bounded

Since f_α is continuous, we have that

$$\inf_{\theta \in \mathbb{R}} f_\alpha(\theta) = \inf_{\theta \in \mathbb{Q}} f_\alpha(\theta)$$

which is measurable.

$\square$

References