REAL ANALYSIS NOTES

Contents

1. Integration

1.1. Nonnegative Functions.

Definition 1.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^+(X, \mathcal{A}, \mu) = \{f : X \to [0, \infty] : f \text{ is measurable}\}$ We will typically just write L^+.*

Theorem 1.2. *Monotone Convergence Theorem Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then*

$$\sup_{n\in\mathbb{N}} \int f_n = \int \sup_{n\in\mathbb{N}} f_n$$

.

Theorem 1.3. *Fatou's Lemma Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Theorem 1.4. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then*

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n.$$

Exercise 1.5. *Let $f \in L^+$ and suppose that $\int f < \infty$. Put $N = \{x \in X : f(x) = \infty\}$ and $S = \{x \in X : f(x) > 0\}$. Then $\mu(N) = 0$ and S is σ-finite.*

Proof. Suppose that $\mu(N) > 0$. Define $f_n = n\chi_N \in L^+$. Then for each $n \in \mathbb{N}$, $f_n \leq f_{n+1} \leq f$ on N. So

$$\begin{aligned}\int f &\geq \int_N f \\ &= \lim_{n\to\infty} \int_N f_n \\ &= \lim_{n\to\infty} n\mu(N) \\ &= \infty, \text{ a contradiction.}\end{aligned}$$

Hence N is a null set. Now, put $S_n = \{x \in X : f(x) > 1/n\}$. Then $S = \bigcup_{n\in\mathbb{N}} S_n$. Suppose that there exists some $n \in \mathbb{N}$ such that $\mu(S_n) = \infty$. Then

$$\begin{aligned}\int f &\geq \int_{S_n} f \\ &\geq \frac{1}{n}\mu(S_n) \\ &= \infty, \text{ a contradiction.}\end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(S_n) < \infty$ and S is σ-finite.

□

Exercise 1.6. *Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.*

Proof. $f = 0$ a.e. implies that for each $E \in \mathcal{A}$, $\int_E f = 0$ is clear. Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f = 0$. For $n \in \mathbb{N}$ put $N_n = \{x \in X : f(x) > 1/n\}$ and define $N = \{x \in X : f(x) > 0\}$. So $N = \bigcup_{n\in\mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned}0 &= \int_{N_n} f \\ &\geq \frac{1}{n}\mu(N_n) \\ &\geq 0.\end{aligned}$$

Hence for each $n \in \mathbb{N}$, $\mu(N_n) = 0$. Thus $\mu(N) = 0$ and $f = 0$ a.e. as required.

□

Exercise 1.7. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that $f_n \xrightarrow{p.w.} f$, $\lim_{n\to\infty} \int f_n = \int f$ and $\int f < \infty$. Then for each $E \in \mathcal{A}$, $\lim_{n\to\infty} \int_E f_n = \int_E f$. This result may fail to be true if $\int f = \infty$*

Proof. Let $E \in \mathcal{A}$. By Fatou's lemma, $\int_E f \leq \liminf_{n\to\infty} \int_E f_n$. Note that since $\int f < \infty$, we have that $\int_{E^c} f \leq \int f < \infty$. Thus we may write

$$\begin{aligned}\int_E f &= \int f - \int_{E^c} f \\ &\geq \int f - \liminf_{n\to\infty} \int_{E^c} f_n \\ &= \int f - \liminf_{n\to\infty} \left(\int f_n - \int_E f_n \right) \\ &= \int f - \int f + \limsup_{n\to\infty} \int_E f_n \\ &= \limsup_{n\to\infty} \int_E f_n.\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \int_E f_n \leq \int_E f \leq \liminf_{n\to\infty} \int_E f_n$$

and therefore

$$\lim_{n\to\infty}\int_E f_n = \int_E f.$$

If we drop the assumption that $\int f < \infty$, then the result would fail to be true for the functions $f = \infty\chi_{(0,1)}$ and $f_n = \infty\chi_{(0,1)} + n\chi_{(1,1+1/n)}$. Here $f_n \xrightarrow{\text{p.w.}} f$, $\lim\limits_{n\to\infty}\int f_n = \int f = \infty$ and $\lim\limits_{n\to\infty}\int_{(1,\infty)} f_n = 1$ while $\int_{(1,\infty)} f = 0$.

$\square$

Exercise 1.8. *Let $f \in L^+$. Define $\lambda : \mathcal{A} \to [0,\infty]$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{A}$ Then λ is a measure on $(X,\mathcal{A})$ and for each $g \in L^+$, $\int g d\lambda = \int g f d\mu$.*

Proof. Clearly $\lambda(\varnothing) = 0$. Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ and suppose that for each $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j = \varnothing$. For now, suppose that f is simple. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0,\infty)$ such that $f = \sum\limits_{i=1}^{n} a_i\chi_{E_i}$. Then

$$\begin{aligned}
\lambda\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) &= \int_{\bigcup_{j\in\mathbb{N}} A_j} f \\
&= \sum_{i=1}^{n} a_i\mu\Big(E_i \cap \Big(\bigcup_{j\in\mathbb{N}} A_j\Big)\Big) \\
&= \sum_{i=1}^{n} a_i\mu\Big(\bigcup_{j\in\mathbb{N}} E_i \cap A_j\Big) \\
&= \sum_{i=1}^{n} a_i \sum_{j\in\mathbb{N}} \mu(E_i \cap A_j) \\
&= \sum_{j\in\mathbb{N}}\sum_{i=1}^{n} a_i\mu(E_i \cap A_j) \\
&= \sum_{j\in\mathbb{N}} \int_{A_j} f \\
&= \sum_{j\in\mathbb{N}} \lambda(A_j)
\end{aligned}$$

Hence λ is a measure on $(X,\mathcal{A})$. Now, for a general f, there exist $(\phi_n)_{n\in\mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \le \phi_{n+1} \le f$ and $\phi_n \xrightarrow{\text{p.w.}} f$. Put $A = \bigcup_{j\in\mathbb{N}} A_j$ and define the measures λ_n by $\lambda_n(E) = \int_E \phi_n$. Note that we may define a monotonically increasing sequence of functions $g_n : \mathbb{N} \to [0,\infty]$ by $g_n(j) = \int_{A_j} \phi_n$. Using monotone convergence three times and a nice application of the counting measure on $\mathbb{N}$, we may write

$$\begin{aligned}\lambda(A) &= \int_A f \\ &= \lim_{n\to\infty}\int_A \phi_n \\ &= \lim_{n\to\infty}\sum_{j\in\mathbb{N}}\int_{A_j}\phi_n \\ &= \sum_{j\in\mathbb{N}}\lim_{n\to\infty}\int_{A_j}\phi_n \quad \text{(by the above)} \\ &= \sum_{j\in\mathbb{N}}\int_{A_j} f \\ &= \sum_{j\in\mathbb{N}}\lambda(A_j).\end{aligned}$$

Hence λ is a measure on $(X,\mathcal{A})$. Let $g \in L^+$. First assume that g is simple. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0,\infty)$ such that $g = \sum\limits_{i=1}^{n} a_i \chi_{E_i}$. In this case, we have that

$$\begin{aligned}\int g d\lambda &= \sum_{i=1}^{n} a_i\lambda(E_i) \\ &= \sum_{i=1}^{n} a_i \int_{E_i} f d\mu \\ &= \int\left(\sum_{i=1}^{n} a_i\chi_{E_i}\right) f d\mu \\ &= \int g f d\mu.\end{aligned}$$

Now for a general $g \in L^+$, there exist $(\psi_n)_{n\in\mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ψ_n is simple, $\psi_n \leq \psi_{n+1} \leq f$ and $\psi_n \xrightarrow{\text{p.w.}} g$. Monotone convergence then gives us

$$\begin{aligned}\int g d\lambda &= \lim_{n\to\infty}\int \psi_n d\lambda \\ &= \lim_{n\to\infty}\int \psi_n f d\mu \\ &= \int g f d\mu \text{ as required.}\end{aligned}$$

□

Exercise 1.9. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$, $f_n \xrightarrow{p.w.} f$ and $\int f_1 < \infty$. Then $\lim\limits_{n\to\infty}\int f_n = \int f$.*

Proof. First we note that since $\int f_1 < \infty$, $f_1 < \infty$ a.e., for each $n \in \mathbb{N}$, $f_1 - f_n$ and $\int f_1 - \int f_n$ are well defined and $\int f_n \leq \int f_1 < \infty$. Also, for $n \in \mathbb{N}$, $f_1 - f_n \in L^+$. So we may write

$$\begin{aligned}\int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n \\ &= \int [(f_1 - f_n) + f_n] - \int f_n \\ &= \int f_1 - \int f_n\end{aligned}$$

Put $g_n = f + (f_1 - f_n)$. Then $g_n \in L^+$, for each $n \in \mathbb{N}$, $g_n \leq g_{n+1}$ and $g_n \xrightarrow{\text{p.w.}} f_1$. Monotone convergence tells us that

$$\begin{aligned}\int f_1 &= \lim_{n\to\infty} \int g_n \\ &= \lim_{n\to\infty} \left[\int f + (f_1 - f_n) \right] \\ &= \lim_{n\to\infty} \left[\int f + \int (f_1 - f_n) \right] \\ &= \lim_{n\to\infty} \left[\int f + \int f_1 - \int f_n \right]\end{aligned}$$

Since $\lim\limits_{n\to\infty} \int f$ and $\lim\limits_{n\to\infty} \int f_1$ exist, $\lim\limits_{n\to\infty} \int f_n = \int f$ as required.

□

1.2. **Complex Valued Functions.**

Definition 1.10. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^1(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} : f$ is measurable and $\int |f| < \infty\}$*

Theorem 1.11. *Dominated Convergence Let $(f_n)_{n\in\mathbb{N}} \subset L^1$, f measurable and $g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. Then $f \in L^1$ and $\int f_n \to \int f$.*

Theorem 1.12. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$. Suppose that*

$$\sum_{n\in\mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n\in\mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n$$

Theorem 1.13. *Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$.*

Exercise 1.14. *Generalized Fatou's Lemma: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \geq -g$. Then $\int \liminf\limits_{n\to\infty} f_n \leq \liminf\limits_{n\to\infty} \int f_n$. What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?*

Proof. First note that for each $n \in \mathbb{N}$, $\int f_n$ is well defined since $f_n^- \leq g \in L^1$. Since $g + f_n \geq 0$, we may use Fatou's lemma to write

$$\begin{aligned}\int g + \int \liminf_{n\to\infty} f_n &= \int \liminf_{n\to\infty}(g + f_n)\\ &\leq \liminf_{n\to\infty} \int (g + f_n)\\ &= \int g + \liminf_{n\to\infty} \int f_n\end{aligned}$$

Since $\int g < \infty$, $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$ as required. The analogue is as follows: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \leq g$. Then $\limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n$. To show this, just use the result from above with the sequence $(g_n)_{n\in\mathbb{N}}$ given by $g_n = -f_n$. $\square$

Exercise 1.15. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \to \mathbb{C}$. Suppose that $f_n \xrightarrow{uni} f$. Then*

1. *if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim_{n\to\infty} \int f_n = \int f$*
2. *if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on $\mathbb{R}$ with Lebesgue measure).*

Proof. Choose $N \in \mathbb{N}$ such that for $n \geq N$ and $x \in X$, $|f(x) - f_n(x)| < 1$. Then $||f| - |f_N|| < 1$ and so $|f| < |f_N| + 1$. Thus $\int |f| \leq \int |f_N| + \mu(X) < \infty$ and $f \in L^1$. Similarly for $n \geq N$, $|f_n| < |f| + 1$. Dominated convergence then gives us that $\lim_{n\to\infty} \int f_n = \int f$ as required. To see the necessity that $\mu(X) < \infty$, consider $f \equiv 0$ and $f_n = (1/n)\chi_{(0,n)}$. Then $f_n \xrightarrow{\text{uni}} f$, but $1 = \lim_{n\to\infty} \int f_n \neq \int f = 0$. $\square$

Exercise 1.16. *Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$, $g_n \xrightarrow{a.e.} g$, $|f_n| \leq g_n$ and $\int g_n \to \int g$. Then $\int f_n \to \int f$.*

Proof. We simply use Fatou's lemma. Put $h_n = (g + g_n) - |f_n - f|$. Since for each $n \in \mathbb{N}$, $|f_n| \leq g_n$, we know that $|f| \leq g$. So $h_n \geq 0$ and $h_n \xrightarrow{\text{p.w.}} 2g$. Thus

$$\begin{aligned}2\int g &= \int \liminf_{n\to\infty} h_n\\ &\leq \liminf_{n\to\infty} \left[\left(\int g + \int g_n\right) - \int |f_n - f|\right]\\ &= 2\int g + \liminf_{n\to\infty}\left(-\int |f_n - f|\right)\\ &= 2\int g - \limsup_{n\to\infty} \int |f_n - f|\end{aligned}$$

Hence $\limsup_{n\to\infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $\int f_n \to \int f$ as required. $\square$

Exercise 1.17. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.*

Proof. Suppose that $\int |f_n - f| \to 0$. Since

$$\begin{aligned}\left|\int |f_n| - \int |f|\right| &= \left|\int (|f_n| - |f|)\right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|,\end{aligned}$$

we see that $\int |f_n| \to \int |f|$. Conversely, suppose that $\int |f_n| \to \int |f|$. Put $h_n = |f_n - f|$, $g_n = |f_n| + |f|$, $h \equiv 0$ and $g = 2f$. Then $h_n \xrightarrow{\text{a.e.}} h$, $g_n \xrightarrow{\text{a.e.}} g$ and for each $n \in \mathbb{N}$, $h_n \leq g_n$. Our assumption implies that $\int g_n \to \int g$. Thus the last exercise tells us that $\int h_n \to \int h$ as required.

□

Exercise 1.18. *Let $(r_n)_{n\in\mathbb{N}}$ be an enummeration of the rationals. Define $f : \mathbb{R} \to [0, \infty)$ by*

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

and define $g : X \to [0, \infty]$ by

$$g(x) = \sum_{n\in\mathbb{N}} 2^{-n} f(x - r_n).$$

Then

(1) *$g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.*
(2) *$g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of $\mathbb{R}$*
(3) *Taking $g \in L^1$, g is unbounded on each subinterval of $\mathbb{R}$ and discontinuous everywhere and remains so after redefinition on a null set*

Proof. For convenience, define $f_n : \mathbb{R} \to [0, \infty)$ by $f_n(x) = f(x - r_n)$ for $x \in \mathbb{R}$. To show (1) we note that for each $n \in \mathbb{N}$, $f_n \in L^1$ and

$$\begin{aligned}\int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1}\end{aligned}$$

Hence

$$\sum_{n\in\mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set, $g \in L^1$. In particular $\int |g| < \infty$ and so $|g|$ (and hence g) are finite almost everywhere. For (2), since $g < \infty$ a.e., so too is g^2. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Choose $N \in \mathbb{N}$ such that $r_N \in (a, b)$. Since all the terms in the sum are nonnegative, $g^2 \geq \sum_{n\in\mathbb{N}} 2^{-2n} f_n^2$ and so

$$\begin{aligned}\int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n\in\mathbb{N}} 2^{-2n} f_n^2 \\ &= \sum_{n\in\mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\ &\geq 2^{-2N} \int_{(a,b)} f_N^2 \\ &\geq 2^{-2N} \int_{r_N}^{b\wedge(r_N+1)} \frac{1}{x-r_N} dx \\ &= \infty\end{aligned}$$

So g^2 is not integrable on any subinterval of $\mathbb{R}$. For (3), note that redefining g on a null set does not change the result of (2). Suppose that there is a finite subinterval $I \subset \mathbb{R}$ such that g is bounded on I. Hence there exists $M > 0$ such that for each $x \in I$, $g(x)^2 \leq M$. Then

$$\begin{aligned}\int_I g^2 &\leq M^2 m(I) \\ &< \infty\end{aligned}$$

which is a contradiction. So g is not bounded on any subinterval of $\mathbb{R}$. Now, suppose that there exists $x_0 \in \mathbb{R}$ such that g is continuous at x_0. Choose $\delta > 0$ such that for each $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then $|g(x) - g(x_0)| < 1$. The reverse triangle inequality tells us that for each $x \in (x_0 - \delta, x_0 + \delta)$, $|g(x)| < 1 + |g(x_0)|$. Hence g is bounded on $(x_0 - \delta, x_0 + \delta)$ which is a contradiction. So g is discontinuous everywhere. □

Exercise 1.19. *Let $f \in L^1$.*

(1) *If f is bounded, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\int_E |f| < \epsilon$.*
(2) *The same conclusion holds for f unbounded.*

Proof. (1) Since f is bounded, there exists $M > 0$ such that $|f| \leq M$. Let $\epsilon > 0$. Choose $\delta = \epsilon/2M$. Let $E \in \mathcal{A}$. Suppose that $\mu(A) < \delta$. Then

$$\begin{aligned}\int_E |f| &\leq M\mu(E) \\ &= M\frac{\epsilon}{2M} \\ &= \frac{\epsilon}{2} \\ &< \epsilon\end{aligned}$$

(2) Suppose that f is unbounded. Let $\epsilon > 0$. Then there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon/2$. Since ϕ is bounded, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$,

if $\mu(E) < \delta$, then $\int_E |\phi| < \epsilon/2$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then

$$\begin{aligned}\int_E |f| &\leq \int_E |f-\phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon\end{aligned}$$

□

Exercise 1.20. *Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Define $F : \mathbb{R} \to \mathbb{R}$ by*

$$F(x) = \int_{(-\infty,x]} f dm.$$

Then F is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Since $f \in L^1$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| dm < \epsilon.$$

Let $x \in \mathbb{R}$. Suppose that $|x - x_0| < \delta$. Then

$$\begin{aligned}|F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f| dm \\ &< \epsilon\end{aligned}$$

So F is continuous.

□

Exercise 1.21. *Denote by δ_x the point mass measure at $x \in X$ on measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$. Then*

$$\int f d\delta_x = f(x)$$

Proof. First assume that f is simple. Then there exist $a_1, a_2, \cdots, a_n \in \mathbb{C}$ and $E_1, E_2, \cdots, E_n \in \mathcal{P}(X)$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$ Thus $\int f d\delta_x = f(x)$. Now assume that f, which is measurable by choice of σ-algebra, satisfies $f(X) \subset [0, \infty)$. Choose a sequence $(\phi_n)_{n\in\mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1}$ and $\phi_n \xrightarrow{\text{p.w}} f$. From before, we see that for each $n \in \mathbb{N}$, $\int \phi_n d\delta_x = \phi_n(x)$. Monotone convergence tells us that $\int f d\delta_x = f(x)$. Now just extend to complex valued functions.

□

Exercise 1.22. *Denote by $\#$ the counting measure on the measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$ and suppose that $f \in L^1$. Then*

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if f is integrable, then $\{x \in X : f(x) \neq 0\}$ is countable.

Proof. Please refer to the definition of the sum in the appendix. First suppose that $f(X) \subset [0, \infty)$. For $n \in \mathbb{N}$, put $X_n = \{x \in X : f(x) > 1/n\}$ and define $X^* = \{x \in X : f(x) > 0\}$, $X_0 = \{x \in X : f(x) = 0\}$ Then $X^* = \bigcup_{n \in \mathbb{N}} X_n$. Since $f \in L^1$, we have that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\infty &> \int f d\# \\
&\geq \int_{X_n} f d\# \\
&\geq \frac{1}{n} \#(X_n).
\end{aligned}$$

Thus for each $n \in \mathbb{N}$, X_n is finite and X^* is countable. Thus there exists $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that $X^* = \{x_n\}_{n \in \mathbb{N}}$. For $n \in \mathbb{N}$, define $E_n = \{x_1, x_2, \cdots, x_n\}$ and

$$\begin{aligned}
f_n &= f \chi_{E_n} \\
&= \sum_{i=1}^{n} f(x_i) \chi_{\{x_i\}}
\end{aligned}$$

Then $f_n \xrightarrow{\text{p.w.}} f \chi_{X^*} = f$ and for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. So

$$\begin{aligned}
\int f &= \sup_{n \in \mathbb{N}} \int f_n \\
&= \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} f(x_i) \\
&= \sum_{x \in X^*} f(x) \\
&= \sum_{x \in X} f(x).
\end{aligned}$$

For $f : X \to \mathbb{C}$, our L^1 assumption and the result above tell us that

$$\sum_{x \in X} |f(x)| < \infty.$$

Thus writing $f = g + ih$, we see that the same is true for f^+, f^-, g^+, g^-. Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f d\# = \sum_{x \in X} f(x).$$

$\square$

Exercise 1.23. *Let $f, g : X \to \mathbb{R}$. Suppose that $f, g \in L^1$. Then $f \leq g$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$.*

Proof. Suppose $f \leq g$ a.e. Put $N = \{x \in X : f(x) > g(x)\} \subset N$. Then $\mu(N) = 0$ and $g - f \geq 0$ on N^c. So for each $E \in \mathcal{A}$,

$$\begin{aligned}\int_E g - \int_E f &= \int_E (g-f) \\ &= \int_{E \cap N^c} (g-f) \\ &\geq 0\end{aligned}$$

Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$. Put $N_n = \{x \in X : f(x) - g(x) > 1/n\}$ and $N = \{x \in X : f(x) > g(x)\}$. Then $N = \bigcup\limits_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned}0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n}\mu(N_n) \\ &\geq 0.\end{aligned}$$

So that $\mu(N_n) = 0$. Thus for each $n \in \mathbb{N}$, $\mu(N_n) = 0$ which implies $\mu(N) = 0$. Therefore $f \leq g$ a.e. as required. □

Exercise 1.24.

Definition 1.25. *Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is said to be* ***uniformly integrable*** *if for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup\limits_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$. (i.e. $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| = 0$).*

Suppose that μ is finite. Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is uniformly integrable iff

(1) *there exists $M > 0$ such that $\sup\limits_{f \in \mathcal{F}} \int |f| \leq M$*
(2) *for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\sup\limits_{f \in \mathcal{F}} \int_E |f| < \epsilon$.*

Proof. ($\Rightarrow$): (1) Suppose that $\mathcal{F}$ is uniformly integrable. Then there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| < 1$. Choose $M = \mu(X)K + 1$. Then for each $f \in \mathcal{F}$,

$$\begin{aligned}\int |f| &= \int_{\{|f|>K\}} |f| + \int_{\{|f| \leq K|\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M\end{aligned}$$

(2) Let $\epsilon > 0$. Then choose $K \in \mathbb{N}$ such that $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>K\}} |f| < \epsilon/2$ and choose $\delta = \epsilon/2K$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then for $f \in \mathcal{F}$,

$$\begin{aligned}\int_E |f| &= \int_{E \cap \{|f|>K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon\end{aligned}$$

($\Leftarrow$): Choose $M > 0$ as in (1). Suppose that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $f \in \mathcal{F}$ such that $\mu(\{|f| > K\}) \geq \epsilon$. Choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $f_K \in \mathcal{F}$ such that $\mu(\{|f_K| > K\}) \geq \epsilon$. Then

$$\begin{aligned}\int |f_K| &\geq \int_{\{|f_K|>K\}} |f| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M,\end{aligned}$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $f \in \mathcal{F}$, $\mu(\{|f| > K\}) < \epsilon$. Since $\mu(\{|f| > k\})$ is a decreasing sequence in k, we have that $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \mu(\{|f| > k\}) = 0$. Now, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Choose $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$, $\mu(\{|f| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$,

$$\int_{\{|f|>k\}} |f| < \epsilon.$$

Thus $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| = 0$ as required.

$\square$

1.3. **Convergence.**

Definition 1.26. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f* ***in measure*** *if for each $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$. This is written $f_n \xrightarrow{\mu} f$.*

Definition 1.27. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f* ***almost uniformly*** *if for each $\epsilon > 0$, there exists $N \in \mathcal{A}$ such that $\mu(N) < \epsilon$ and $f_n \xrightarrow{uni} f$ on N^c. This is written $f_n \xrightarrow{a.u.} f$.*

Theorem 1.28. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $f_{n_k} \xrightarrow{a.e.} f$.*

Theorem 1.29. *(Egoroff): Suppose that $\mu(X) < \infty$. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.e.}$. Then $f_n \xrightarrow{a.u.} f$.*

Exercise 1.30. *Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.*

Proof. Let $\epsilon > 0$. for $n \in \mathbb{N}$, define $E_{e,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$. Then for $n \in \mathbb{N}$,

$$\begin{aligned} \int |f - f_n| &\geq \int_{E_{\epsilon,n}} |f - f_n| \\ &\geq \epsilon\mu(E_{\epsilon,n}). \end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$. Since $\int |f - f_n| \to 0$, we have that $\mu(E_{\epsilon,n}) \to 0$. Since $\epsilon > 0$ is arbitrary, $f_n \xrightarrow{\mu} f$ as required. □

Exercise 1.31. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \xrightarrow{\mu} f$. Then $f \geq 0$ a.e. and $\int f \leq \liminf\limits_{n\to\infty} \int f_n$.*

Proof. First choose a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $\int f_{n_k} \to \liminf\limits_{n\to\infty} \int f_n$. Since $f_n \xrightarrow{\mu} f$ so does $(f_{n_k})_{k\in N}$. Therefore there exists a subsequence $(f_{n_{k_j}})_{k\in\mathbb{N}}$ of $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Thus $f \geq 0$ a.e. and Fatou's lemma tells us that

$$\begin{aligned} \int f &\leq \liminf_{j\in\mathbb{N}} \int f_{n_{k_j}} \\ &= \liminf_{n\to\infty} \int f_n. \end{aligned}$$

□

Exercise 1.32. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \xrightarrow{\mu} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.*

Proof. Clearly $(f_n)_{n\in\mathbb{N}} \subset L^1$. Since $f_n \xrightarrow{\mu} f$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$. This implies that $|f| \leq g$ a.e. and so $f \in L^1$. For $n \in \mathbb{N}$, put $h_n = 2g - |f_n - f|$. Then for each $n \in \mathbb{N}$, $h_n \geq 0$ and $h_n \xrightarrow{\mu} 2g$. By the previous exercise

$$\begin{aligned} \int 2g &\leq \liminf_{n\to\infty} \int (2g - |f_n - f|) \\ &= \int 2g - \limsup_{n\to\infty} \int |f_n - f|. \end{aligned}$$

So $\limsup\limits_{n\to\infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $f_n \xrightarrow{L^1} f$ as required. □

2. Appendix

2.1. Summation.

Definition 2.1. *Let $f : X \to [0, \infty)$, Then we define*

$$\sum_{x\in X} f(x) := \sup_{\substack{F\subset X \\ F \text{ finite}}} \sum_{x\in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x\in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. *In this case, we may define*

$$\sum_{x \in X} f(x)$$

in the obvious way.