REAL ANALYSIS NOTES

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Contents

1. Measure

1.1. Product Measures.

Definition 1.1. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measurable spaces. Put $\mathcal{E} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$. Then $\mathcal{E}$ is an elementary family and thus $\mathcal{M}_0 = \{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n \subset \mathcal{E} \text{ are disjoint}\}$ is an algebra on $X \times Y$. We define $\pi_0 : \mathcal{M}_0 \to [0, \infty]$ by*

$$\pi_0(\bigcup_{i=1}^n A_i \times B_i) = \sum_{i=1}^n \mu(A_i)\nu(B_i)$$

Since $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{M}_0)$, we define a product measure $\mu \times \nu$ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ to be an extension of π_0 to $\mathcal{A} \otimes \mathcal{B}$. The existence of which is guaranteed by Caratheodory's theorem and on $\mathcal{A} \otimes \mathcal{B}$,

$$\mu \times \nu(E) = \inf\{\sum_{n\in\mathbb{N}} \pi_0(E_i) : (E_i)_{i\in\mathbb{N}} \subset \mathcal{M}_0 \ \text{and} \ E \subset \bigcup_{i\in\mathbb{N}} E_i\}$$

$$= \inf\{\sum_{n\in\mathbb{N}} \mu(A_i)\nu(B_i) : (A_i \times B_i)_{i\in\mathbb{N}} \subset \mathcal{E} \ \text{and} \ E \subset \bigcup_{i\in\mathbb{N}} A_i \times B_i\}$$

If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are both sigma finite, then so is π_0 and thus $\mu \times \nu$ is unique.

2. Integration

2.1. Measurable Functions.

Definition 2.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then f is said to be $\mathcal{A}$-$\mathcal{B}$ **measurable** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. When $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say that f is $\mathcal{A}$-**measurbale**. If $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}, \mathcal{L})$, then we say that f is **Borel measurable** or **Lebsgue measurbale** respectively.*

Lemma 2.2. *Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then*
  (1) $\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y
  (2) $\{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra on X

Lemma 2.3. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Suppose that there exists $\mathcal{E} \subset Y$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Let $f : X \to Y$. If for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$, then f is $\mathcal{A}$-$\mathcal{B}$ measurable.*

Corollary 2.4. *Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X \to Y$. If f is continuous, then f is $\mathcal{B}(X)$-$\mathcal{B}(Y)$ measurable.*

Definition 2.5. *Let X be a set and $f : X \to \mathbb{C}$. Then f is said to be **simple** if $f(X)$ is finite.*

Definition 2.6. *Let $(X, \mathcal{A})$ be a measurable space. We define $S^+(X, \mathcal{A}) = \{f : X \to [0, \infty) : f \text{ is simple, measurable}\}$ and $S(X, \mathcal{A}) = \{f : X \to \mathbb{C} : f \text{ is simple, measurable}\}$*

Theorem 2.7. *Let $(X, \mathcal{A})$ be a measurable space. Then*
  (1) If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.
  (2) If $f : X \to \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset S$ such that for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.

2.2. Integration of Nonnegative Functions.

Definition 2.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^+(X, \mathcal{A}, \mu) = \{f : X \to [0, \infty] : f \text{ is measurable}\}$ We will typically just write L^+.*

Theorem 2.9. *Monotone Convergence Theorem Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then*

$$\sup_{n\in\mathbb{N}} \int f_n = \int \sup_{n\in\mathbb{N}} f_n$$

.

Exercise 2.10. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$ and $f \in L^+$. Then*

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

.

Exercise 2.11. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Suppose that $\mu_1 \leq \mu_2$. Then for each $f \in L^+$,*

$$\int f d\mu_1 \leq \int f d\mu_2$$

Theorem 2.12. *Fatou's Lemma Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Then*

$$\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n.$$

Theorem 2.13. *Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Then*

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n.$$

Exercise 2.14. *Let $f \in L^+$ and suppose that $\int f < \infty$. Put $N = \{x \in X : f(x) = \infty\}$ and $S = \{x \in X : f(x) > 0\}$. Then $\mu(N) = 0$ and S is σ-finite.*

Exercise 2.15. *Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.*

Exercise 2.16. *Let $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that $f_n \xrightarrow{p.w.} f$, $\lim_{n \to \infty} \int f_n = \int f$ and $\int f < \infty$. Then for each $E \in \mathcal{A}$, $\lim_{n \to \infty} \int_E f_n = \int_E f$. This result may fail to be true if $\int f = \infty$*

Exercise 2.17. *Let $f \in L^+$. Define $\lambda : \mathcal{A} \to [0, \infty]$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{A}$ Then λ is a measure on $(X, \mathcal{A})$ and for each $g \in L^+$, $\int g d\lambda = \int g f d\mu$.*

Exercise 2.18. *Let $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$, $f_n \xrightarrow{p.w.} f$ and $\int f_1 < \infty$. Then $\lim_{n \to \infty} \int f_n = \int f$.*

2.3. **Integration of Complex Valued Functions.**

Definition 2.19. *Let $f : X \to \mathbb{C}$ be measurable. Then f is said to be **integrable** if*

$$\int |f| d\mu < \infty$$

Definition 2.20. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^1(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} : f$ is measurable and $\int |f| < \infty\}$*

Lemma 2.21. *Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff f^+ and f^- are integrable.*

Definition 2.22. *Let $f : X \to \mathbb{R}$ be measurable. Then f is said to be **extended integrable** if*

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

Lemma 2.23. *Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff $Re(f)$ and $Im(f)$ are integrable.*

Theorem 2.24. *Dominated Convergence Let $(f_n)_{n \in \mathbb{N}} \subset L^1$, f measurable and $g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. Then $f \in L^1$ and $\int f_n \to \int f$.*

Exercise 2.25. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Then*

(1) $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$

(2) for each $f \in L^1(\mu_1 + \mu_2)$, we have that

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

Theorem 2.26. *Let $(f_n)_{n \in \mathbb{N}} \subset L^1$. Suppose that*

$$\sum_{n \in \mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n \in \mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

Theorem 2.27. *Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$.*

Exercise 2.28. *Generalized Fatou's Lemma: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \geq -g$. Then $\int \liminf\limits_{n \to \infty} f_n \leq \liminf\limits_{n \to \infty} \int f_n$. What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?*

Exercise 2.29. *Let $(f_n)_{n \in \mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \to \mathbb{C}$. Suppose that $f_n \xrightarrow{uni} f$. Then*

(1) if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim\limits_{n \to \infty} \int f_n = \int f$

(2) if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on $\mathbb{R}$ with Lebesgue measure).

Exercise 2.30. *Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$, $g_n \xrightarrow{a.e.} g$, $|f_n| \leq g_n$ and $\int g_n \to \int g$. Then $\int f_n \to \int f$.*

Exercise 2.31. *Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.*

Exercise 2.32. *Let $(r_n)_{n \in \mathbb{N}}$ be an enummeration of the rationals. Define $f : \mathbb{R} \to [0, \infty)$ by*

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}$$

and define $g : X \to [0, \infty]$ by

$$g(x) = \sum_{n \in \mathbb{N}} 2^{-n} f(x - r_n).$$

Then

(1) $g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.

(2) $g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of $\mathbb{R}$

(3) *Taking $g \in L^1$, g is unbounded on each subinterval of $\mathbb{R}$ and discontinuous everywhere and remains so after redefinition on a null set*

Exercise 2.33. *Let $f \in L^1$.*

(1) *If f is bounded, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\int_E |f| < \epsilon$.*

(2) *The same conclusion holds for f unbounded.*

Exercise 2.34. *Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Define $F : \mathbb{R} \to \mathbb{R}$ by*

$$F(x) = \int_{(-\infty, x]} f \, dm.$$

Then F is continuous.

Exercise 2.35. *Denote by δ_x the point mass measure at $x \in X$ on measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$. Then*

$$\int f \, d\delta_x = f(x)$$

Exercise 2.36. *Denote by $\#$ the counting measure on the measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$ and suppose that $f \in L^1$. Then*

$$\int f \, d\# = \sum_{x \in X} f(x).$$

In particular, if f is integrable, then $\{x \in X : f(x) \neq 0\}$ is countable.

Exercise 2.37. *Let $f, g : X \to \mathbb{R}$. Suppose that $f, g \in L^1$. Then $f \leq g$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$.*

Definition 2.38. *Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is said to be **uniformly integrable** if for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$. (i.e. $\lim_{k \to \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$).*

Exercise 2.39. *Suppose that μ is finite. Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is uniformly integrable iff*

(1) *there exists $M > 0$ such that $\sup_{f \in \mathcal{F}} \int |f| \leq M$*

(2) *for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\sup_{f \in \mathcal{F}} \int_E |f| < \epsilon$.*

2.4. Integration on Product Spaces.

Definition 2.40. *Let X, Y, and Z be sets, $E \subset X \times Y$ and $f : X \times Y \to Z$. For each $x \in X$, define $E_x = \{y \in Y : (x, y) \in E\}$ and $f_x : Y \to Z$ by $f_x(y) = f(x, y)$. For each $y \in Y$, define $E^y = \{x \in X : (x, y) \in E\}$ and $f^y : X \to Z$ by $f^y(x) = f(x, y)$.*

Note 2.41. *It is often helpful to observe that $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.*

Lemma 2.42. *Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces, $Z = [0, \infty]$ or $\mathbb{C}$ and $f : X \times Y \to Z$.*

(1) *For each $E \in \mathcal{A} \otimes \mathcal{B}$, $x \in X$, $y \in Y$, we have that $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$*

(2) If f is $\mathcal{A}\otimes\mathcal{B}$-measurable, then for each $x \in X$, $y \in Y$, we have that f_x is $\mathcal{B}$-measurable and f^y is $\mathcal{A}$-measurable.

Theorem 2.43. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then for each $E \in \mathcal{A}\otimes\mathcal{B}$, the maps $\phi : X \to [0, \infty]$ and $\psi : Y \to [0, \infty]$ defined by $\phi(x) = \nu(E_x)$ and $\psi(y) = \mu(E^y)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, respectively and*

$$\mu \times \nu(E) = \int_X \nu(E_x)d\mu(x) = \int_Y \mu(E^y)d\nu(y)$$

Theorem 2.44. *Fubini, Tonelli: Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.*

(1) *(Tonelli) For each $f \in L^+(X \times Y)$, the functions $g : X \to [0, \infty]$, $h : Y \to [0, \infty]$ defined by $g(x) = \int_Y f_x(y)d\nu(y)$ and $h(y) = \int_X f^y(x)d\mu(x)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable respectively and*

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

(2) *(Fubini) For each $f \in L^1(X \times Y)$, $f_x \in L^1(\nu)$ for μ-a.e. $x \in X$ and $f^y \in L^1(\mu)$ for ν-a.e. $y \in Y$, respectively and the functions (after redefinition of f on a null set) $g : X \to \mathbb{C}$, $h : Y \to \mathbb{C}$ defined by $g(x) = \int_Y f_x(y)d\nu(y)$ and $h(y) = \int_X f^y(x)d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Furthermore*

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

Note 2.45. *We usually just write $\int\int f d\mu d\nu$ and $\int\int f d\nu d\mu$ instead of $\int h d\nu$ and $\int g d\mu$ respectively. We have a similar result for complete product measure spaces. See*

Exercise 2.46. *Take $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1]), \mathcal{B} = \mathcal{P}([0, 1])$ and μ, ν to be Lebesgue measure and counting measure respectively. Define $D = \{(x, y) \in [0, 1]^2 : x = y\}$ Show that*

$$\int \chi_D d\mu \times \nu, \int\int \chi_D d\mu d\nu \text{ and } \int\int \chi_D d\nu d\mu$$

are all different. (Hint: for the first integral, use the definition of $\mu \times \nu$)

Exercise 2.47. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $f : X \to [0, \infty) \in L^+$. Show that $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ and $\mu \times m(G) = \int_X f d\mu$. The same is true if we replace "$\geq$" with "$>$". (Hint: to show that G is measurable, split up $(x, y) \mapsto f(x) - y$ into the composition of measurable functions.*

Exercise 2.48. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$. Define $h : X \times Y \to \mathbb{C}$ by $h(x, y) = f(x)g(y)$.*

(1) *If f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable, then h is $\mathcal{A} \otimes \mathcal{B}$-measurable.*
(2) *If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and*

$$\int_{X \times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu$$

Note 2.49. *In the above exercise part (2), we can replace L^1 with L^+ and get the same result by the same method.*

Exercise 2.50. *Let $f : \mathbb{R} \to [0, \infty) \in L^+$. Show that*

$$\int_\mathbb{R} f dm = \int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\})dm(t)$$

2.5. **Convergence.**

Definition 2.51. *Let $(X, \mathcal{A})$ be a measurable space. For convencience we will define $L^0 = \{f : X \to \mathbb{C} : f \text{ is measurable}\}$.*

Definition 2.52. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **in measure** if for each $\epsilon > 0$, $\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$. This is written $f_n \xrightarrow{\mu} f$.*

Definition 2.53. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then f_n converges to f **almost uniformly** if for each $\epsilon > 0$, there exists $N \in \mathcal{A}$ such that $\mu(N) < \epsilon$ and $f_n \xrightarrow{uni} f$ on N^c. This is written $f_n \xrightarrow{a.u.} f$.*

Theorem 2.54. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $f_{n_k} \xrightarrow{a.e.} f$.*

Theorem 2.55. *(Egoroff): Suppose that $\mu(X) < \infty$. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.e.}$. Then $f_n \xrightarrow{a.u.} f$.*

Exercise 2.56. *Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.*

Exercise 2.57. *Suppose $\mu(X) < \infty$. Define $d : L^0 \times L^0 \to [0, \infty)$ by*

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \quad f, g \in L^0$$

Then d is a metric on L^0 if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each $f, g \in L^0$, $d(f, g) \leq \mu(X)$.

Exercise 2.58. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \xrightarrow{\mu} f$. Then $f \geq 0$ a.e. and $\int f \leq \liminf\limits_{n \to \infty} \int f_n$.*

Exercise 2.59. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \xrightarrow{\mu} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.*

Exercise 2.60. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$.*

(1) If ϕ is continuous, and $f_n \xrightarrow{a.e.} f$ then $\phi \circ f_n \xrightarrow{a.e.} \phi \circ f$.
(2) If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly or in measure, respectively.
(3) Find a counter example to (2) if we drop the word "uniform".

Exercise 2.61. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.u} f$. Then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{a.e.} f$.*

Exercise 2.62. *Let $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then*

(1) $f_n + g_n \xrightarrow{\mu} f + g$
(2) if $\mu(X) < \infty$, then $f_n g_n \xrightarrow{\mu} fg$

Exercise 2.63. *Let $(f_n)_{n \in \mathbb{N}} L^0$ and $f \in L^0$. Suppose that for each $\epsilon > 0$,*

$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then $f_n \xrightarrow{a.e.} f$.

3. Differentiation

3.1. Signed Measures.

Definition 3.1. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$. Then ν is said to be a **signed measure** if*

(1) for each $E \in \mathcal{A}$, $\nu(E) < \infty$ or for each $E \in \mathcal{A}$, $\nu(E) > -\infty$.
(2) $\nu(\varnothing) = 0$
(3) for each $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ if $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is disjoint, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$ and if $|\sum_{n \in \mathbb{N}} \nu(E_n)| < \infty$, then $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely.

Exercise 3.2. *Let $\nu : \mathcal{A} \to [0, \infty]$ be a signed measure and $(E_n)_{n \in \mathbb{N}}$, $(F_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. If $(E_n)_{n \in \mathbb{N}}$ is increasing, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \to \infty} \nu(E_n)$. If $(F_n)_{n \in \mathbb{N}}$ is decreasing and $|\nu(E_1)| < \infty$, then $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \to \infty} \nu(F_n)$.*

Definition 3.3. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$ a signed measure and $E \in \mathcal{A}$. Then E is said to be ν-**positive**, ν-**negative** and ν-**null** if for each $F \in \mathcal{A}$, $F \subset E$ implies that $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ respectively.*

Exercise 3.4. *Let $E \subset \mathcal{A}$. If E is positive, negative or null, then for each $F \in \mathcal{A}$, if $F \subset E$, then F is positive, negative or null respectively.*

Exercise 3.5. *Let $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ be positive, negative or null. Then $\bigcup_{n \in \mathbb{N}} E_n$ is positive, negative or null respectively.*

Theorem 3.6. *Hahn Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist $P, N \in \mathcal{A}$ such that P is positive, N is negative, $X = N \cup P$ and $N \cap P = \varnothing$. Furthermore, these two sets are unique in the following sense: For any $P', N' \in \mathcal{A}$, if N, P satisfy the properties above, $P' \Delta P = N' \Delta N$ is null.*

Definition 3.7. *Let ν be a signed measure on $(X, \mathcal{A})$ and $P, N \in \mathcal{A}$. Then P and N are said to form a **Hahn decomposition** of X with respect to ν if P, N satisfy the results in the above theorem.*

Definition 3.8. *Let μ, ν be signed measures on $(X, \mathcal{A})$. Then μ and ν are said to be **mutually singular** if there exist $E, F \in \mathcal{A}$ such that $X = E \cup F$, $E \cap F = \varnothing$ and E is μ-null and F is ν-null. We will denote this by $\mu \perp \nu$.*

Theorem 3.9. *Jordan Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{A})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Definition 3.10. *Let ν be a signed measure on $(X, \mathcal{A})$. Then ν^+ and ν^- from the last theorem are called the **positive** and **negative variations** of ν respectively. We define the **total variation** measure $|\nu|$ on $(X, \mathcal{A})$ by $|\nu| = \nu^+ + \nu^-$.*

Definition 3.11. *Let ν be a signed measure on $(X, \mathcal{A})$. Then ν is said to be σ-finite if $|\nu|$ is σ-finite.*

Exercise 3.12. *Let ν be a signed measure and λ, μ positive measures on $(X, \mathcal{A})$. Suppose that $\nu = \lambda - \mu$. Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.*

Exercise 3.13. *Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Hint: use the last exercise)*

Note 3.14. *Recall that a previous exercise from the section on complex valued functions tells us that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.*

Definition 3.15. *Let ν be a signed measure on $(X, \mathcal{A})$. Then we define $L^1(\nu) = L^1(|\nu|)$. For $f \in L^1(\nu)$, we define*

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

Exercise 3.16. *Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$*

Exercise 3.17. *Let ν, μ be signed measures on $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then*

   (1) E is ν-null iff $|\nu|(E) = 0$
   (2) $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Exercise 3.18. *Let ν be a signed measure on $(X, \mathcal{A})$. Then*

   (1) for $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$
   (2) if ν is finite, then for each $E \in \mathcal{A}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : f$ is measurable and $|f| \leq 1\}$

Exercise 3.19. *Let μ be a positive measure on $(X, \mathcal{A})$ and $f \in L^0(X, \mathcal{A})$ extended μ-integrable. Define ν on $(X, \mathcal{A})$ by $\nu(E) = \int_E f d\mu$. Then*

   (1) ν is a signed measure
   (2) for each $E \in \mathcal{A}$, $|\nu|(E) = \int_E |f| d\mu$.

3.2. **The Lebesgue-Radon-Nikodym Theorem.**

Definition 3.20. *Let $(X, \mathcal{A})$ be a measureable space, ν be a signed measure on $(X, \mathcal{A})$ and μ a measure on $(X, \mathcal{A})$. Then ν is said to be **absolutely continuous** with respect to μ, denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.*

Note 3.21. *If there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that for each $E \in \mathcal{A}$, $\nu(E) = \int_E f d\mu$, then we write $d\nu = f d\mu$.*

Theorem 3.22. *Let $(X, \mathcal{A})$ be a measureable space, ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that $\rho = f d\mu$ and f is unique μ-a.e.*

Definition 3.23. *The decomposition $\nu = \lambda + \rho$ is referred to as the **Lebesgue decomposition of ν with respect to** μ. In the case $\nu \ll \mu$, we have $\lambda = 0$ and $\rho = \nu$ and we define the **Radon-Nikodym derivative of ν with respect to** μ, denoted by $d\nu/d\mu$, to be $d\nu/d\mu = f$ where $d\nu = f d\mu$.*

Theorem 3.24. *Let ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then*

   (1) for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2) $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-}a.e.$$

Exercise 3.25. *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures and μ a measure.*

   (1) *If for each $n \in \mathbb{N}$, $\nu_n \ll \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.*
   (2) *If for each $n \in \mathbb{N}$, $\nu_n \perp \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \perp \mu$.*

Exercise 3.26. *Choose $X = [0,1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$. Let m be Lebesgue measure and μ the counting measure.*
Then

   (1) *$m \ll \mu$ but for each $f \in L^+$, $dm \neq f d\mu$*
   (2) *There is no Lebesgue decomposition of μ with respect to m.*

Exercise 3.27. *Let $(X, \mathcal{F}, \mu)$ be a measure space and $\mathcal{E}$ a sub σ-alg of $\mathcal{F}$ and $f \in L^1(\mu)$. Define $\nu : \mathcal{E} \to [0, \infty]$ by $\nu(E) = \int_E f d\mu$. Let $\overline{\mu}$ be the restriction of μ to $\mathcal{E}$. Define the* **expectation of f given $\mathcal{E}$** *to be $E[f|\mathcal{E}] = d\nu/d\overline{\mu}$. Then for each $E \in \mathcal{E}$,*

$$\int_E E[f|\mathcal{E}]d\mu = \int_E f d\mu$$

3.3. Complex Measures.

Definition 3.28. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to \mathbb{C}$. Then ν is said to be a* **complex measure** *if*

   (1) *$\nu(\varnothing) = 0$*
   (2) *for each sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if $(E_n)_{n\in\mathbb{N}}$ is disjoint, then $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \nu(E_n)$ and $\sum_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely.*

Note 3.29. *We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.*

Definition 3.30. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. We define $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$. For $f \in L^1(\nu)$, we define*

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

Theorem 3.31. *Let $(X, \mathcal{A})$ be a measurable space, ν a complex measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exists a complex measure λ on $(X, \mathcal{A})$ and $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$ and such that for each complex measure λ' on $(X, \mathcal{A})$, $f' \in L^1(\mu)$, if $\nu = d\lambda' + f'd\mu$, then $\lambda = \lambda'$ and $f = f'$ μ-a.e.*

Theorem 3.32. *Let ν be a complex measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then*

   (1) *for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and*

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

   (2) *$\nu \ll \lambda$ and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-}a.e.$$

Definition 3.33. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. Define $\mu = |\nu_1| + |\nu_2|$. Then $\nu \ll \mu$ and thus There exists $f \in L^1(\mu)$ such that $d\nu = fd\mu$. Define $|\nu| : \mathcal{A} \to [0, \infty)$ by $|\nu|(E) = \int_E |f| d\mu$ for each $E \in \mathcal{A}$. We call $|\nu|$ the **total variation of** ν.*

Exercise 3.34. *Let ν be a complex measure on $(X, \mathcal{A})$ and μ a σ-finite measures on $(X, \mathcal{A})$. If $\nu \ll \mu$, then $\{x \in X : d\nu/d\mu(x) = 0\}$ is ν-null.*

Exercise 3.35. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. Then $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu2|$.*

Exercise 3.36. *Let $(X, \mathcal{A})$ be a measurable space, ν a complex measure on $(X, \mathcal{A})$ and $c \in \mathbb{C}$. Then $|c\nu| = |c||\nu|$.*

Exercise 3.37. *Let $(X, \mathcal{A})$ be a measurable space and ν a complex measure on $(X, \mathcal{A})$. Then*

(1) for each $E \in \mathcal{A}$, $|\nu(E)| \leq |\nu|(E)$.
(2) $\nu \ll |\nu|$ and $\left|d\nu/d|\nu|\right| = 1$ $|\nu|$-a.e.
(3) $L^1(\nu) = L^1(|\nu|)$ and for each $g \in L^1(\nu)$, $|\int g d\nu| \leq \int |g| d|\nu|$

3.4. **Differentiation.**

Definition 3.38. *Let $f : \mathbb{R}^n \to \mathbb{C}$. Then f is said to be **locally integrable** (with respect to Lebesgue measure) if f is measurable and for each $K \subset \mathbb{R}$, K is compact implies $\int_K |f| dm < \infty$. We define $L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is locally integrable}\}$*

Definition 3.39. *For $f \in L^1_{loc}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$, we define the **average of** f **over** $B(x, r)$, denoted by $Af(x, r)$, to be*

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x,r)} f dm$$

Exercise 3.40. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Define*

$$H^*f(x) = \sup\{\frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B\} \quad (x \in \mathbb{R}^n)$$

Then $Hf \leq H^*f \leq 2^n Hf$.

Lemma 3.41. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then $Af : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ is continuous.*

Definition 3.42. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define its **Hardy Littlewood maximal function**, denoted by Hf to be*

$$Hf(x) = \sup_{r>0} A|f|(x, r) \quad x \in \mathbb{R}^n$$

Theorem 3.43. *There exists $C > 0$ such that for each $f \in L^1(m)$ and $\alpha > 0$,*

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{a} \int |f| dm$$

Exercise 3.44. *Let $f \in L^1(\mathbb{R}^n)$. Suppose that $||f||_1 > 0$. Then there exist $C, R > 0$ such that for each $x \in \mathbb{R}^n$, if $|x| > R$, then $Hf(x) \geq C|x|^{-n}$. Hence there exists $C' > 0$ such that for each $\alpha > 0$, $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$ when α is small.*

Theorem 3.45. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{r \to 0} Af(x, r) = f(x)$$

. Equivalently, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} [f(y) - f(x)] dm(y) \right] = 0$$

Note 3.46. *We can a stronger result of the same flavor.*

Definition 3.47. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define the **Lebesgue set of** f, denoted by L_f, to be*

$$L_f = \{x \in \mathbb{R}^n : \lim_{r \to 0} A|f - f(x)|(x, r) = 0\}$$

$$= \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) \right] = 0 \right\}$$

Exercise 3.48. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. If f is continuous at x, then $x \in L_f$.*

Theorem 3.49. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then $m((L_f)^c) = 0$*

Definition 3.50. *Let $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then $(E_r)_{r>0}$ is said to **shrink nicely to** x if*

  (1) for each $r > 0$, $E_r \subset B(x, r)$
  (2) there exists $\alpha > 0$ such that for each $r > 0$, $m(E_r) > \alpha m(B(x, r))$

Theorem 3.51. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then for each $x \in L_f$,*

$$\lim_{r \to 0} \left[\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \right] = 0$$

  and

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

Definition 3.52. *Let $\mu : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$ be a Borel measure. Then μ is said to be **regular** if*

  (1) for each $K \subset \mathbb{R}^n$, if K is compact, then $\mu(K) < \infty$
  (2) for each $E \in \mathcal{B}(\mathbb{R}^n)$, $\mu(E) = \inf\{\mu(U) : U$ is open and $E \subset U\}$

  Let ν be a signed or complex Borel measure on $\mathbb{R}^n$. Then ν is said to be regular if $|\nu|$ is regular.

Theorem 3.53. *Let ν be a regular signed or complex measure on $\mathbb{R}^n$. Let $d\nu = d\lambda + f dm$ be the Lebesgue decomposition of ν with respect to m. Then for m-a.e. $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(R^n)$, if $(E_r)_{r>0}$ shrinks nicely to x, then*

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

3.5. **Functions of Bounded Variation.**

Definition 3.54. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Define $F_+ : \mathbb{R} \to \mathbb{R}$ by*

$$F_+(x) = \lim_{t \to x^+} F(t) = \inf\{F(t) : t > x\}$$

Note 3.55. *Observe that $F \leq F_+$ and F_+ is increasing.*

Exercise 3.56. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then for each $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in (x, x + \delta)$, $0 \leq F_+(y) - F(y) \leq \epsilon$.*

Exercise 3.57. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then F_+ is right continuous.*

Theorem 3.58. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then*
   (1) $\{x \in \mathbb{R} : F$ is not continuous at $x\}$ is countable
   (2) F and F_+ are differentiable a.e. and $F' = F'_+$ a.e.

Definition 3.59. *Let $F : \mathbb{R} \to \mathbb{C}$. Define $T_F : \mathbb{R} \to \mathbb{R}$ by*

$$T_F(x) = \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^{n} \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \quad (x \in \mathbb{R})$$

   *T_F is called the **total variation function of** F.*

Exercise 3.60. *Let $F : \mathbb{R} \to \mathbb{C}$. Then T_F is increasing.*

Lemma 3.61. *Let $F : \mathbb{R} \to \mathbb{R}$. Then $T_F + F$ and $T_F - F$ are increasing.*

Exercise 3.62. *For each $F : \mathbb{R} \to \mathbb{C}$, $T_{|F|} \leq T_F$.*

Definition 3.63. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to have **bounded variation** if $\lim_{x \to \infty} T_F(x) < \infty$. The **total variation of** F, denoted by $TV(F)$, is defined to be $TV(F) = \lim_{x \to \infty} T_F(x)$. We define $BV = \{F : \mathbb{R} \to \mathbb{C} : TV(F) < \infty\}$.*

Definition 3.64. *Let $a, b \in \mathbb{R}$ and $F : [a, b] \to \mathbb{C}$. Define $G_F : \mathbb{R} \to \mathbb{C}$ by $G_F = F(a)\chi_{(-\infty,a)} + F\chi_{[a,b]} + F(b)\chi_{(b,\infty)}$. Then F is said to have **bounded variation on** $[a, b]$ if $G_F \in BV$. The **total variation of** F **on** $[a, b]$, denoted by $TV(F, [a, b])$, is defined to be $TV(F, [a, b]) = TV(G_F)$ We define $BV([a, b]) = \{F : [a, b] \to \mathbb{C} : TV(F, [a, b]) < \infty\}$.*

Note 3.65. *Equivalently, $TV(F, [a, b]) = \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^{n} \subset [a, b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$ and $F \in BV([a, b])$ iff $TV(F, [a, b]) < \infty$. In general,*

Exercise 3.66. *Let $F \in BV$. Then F is bounded.*

Exercise 3.67. *Let $F : \mathbb{R} \to \mathbb{R}$. If F is bounded and increasing, then $F \in BV$.*

Exercise 3.68. *Let $F : \mathbb{R} \to \mathbb{C}$. If F is differentiable and F' is bounded on $[a, b]$, then, $F \in BV([a, b])$.*

Exercise 3.69. *Define $F, G : \mathbb{R} \to \mathbb{R}$ by*

$$F(x) = \begin{cases} x^2 \sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 \sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Then F and G are differentiable, $F \in BV([-1, 1])$ and $G \notin BV([-1, 1])$.

Exercise 3.70. *The following is stated for BV, but is also true for $BV([a,b])$.*

(1) *For each $F, G \in BV$, $T_{F+G} \leq T_F + T_G$ and therefore BV is a vector space.*
(2) *For each $F : \mathbb{R} \to \mathbb{C}$, $F \in BV$ iff $Re(f) \in BV$ and $Im(F) \in BV$.*
(3) *For each $F : \mathbb{R} \to \mathbb{R}$, $F \in BV$ iff there exist functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$*
(4) *For each $F \in BV$ and $x \in \mathbb{R}$, $\lim_{t \to x^+} F(t)$ and $\lim_{t \to x^-} F(t)$ exist.*
(5) *For each $F \in BV$, $\{x \in R : F$ is not continuous at $x\}$ is countable.*
(6) *For each $F \in BV$, F and F_+ are differentiable a.e. and $F' = (F_+)'$ a.e.*
(7) *For each $F \in BV, c \in \mathbb{R}$, $F - c \in BV$*

Lemma 3.71. *Let $F \in BV$. Then $\lim_{x \to -\infty} T_F(x) = 0$ and if F is right continuous, then T_F is right continuous.*

Definition 3.72. *Define $NBV = \{F \in BV : F$ is right continuous and $\lim_{x \to -\infty} F(x) = 0\}$.*

Theorem 3.73. *Let $M(\mathbb{R})$ be the set of complex Borel measures on $\mathbb{R}$. For $F \in NBV$, define $\mu_F \in M(\mathbb{R})$ by $\mu_F((-\infty, x]) = F(x)$. Then $F \mapsto \mu_F$ defines a bijection $NBV \to M(\mathbb{R})$. In addition, $|\mu_F| = \mu_{T_F}$*

Theorem 3.74. *Let $F \in NBV$. Then $F' \in L^1(m)$, $\mu_F \perp m$ iff $F' = 0$ a.e. and $\mu_F \ll m$ iff for each $x \in \mathbb{R}$, $\int_{(-\infty, x]} F' dm = F(x)$*

Definition 3.75. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to be **absolutely continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Definition 3.76. *Let $F : [a,b] \to \mathbb{C}$. Then F is said to be **absolutely continuous on** $[a,b]$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a,b])$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Proposition 3.77. *Let $F : [a,b] \to \mathbb{C}$. If F is absolutely continuous on $[a,b]$, then $F \in BV[a,b]$.*

Exercise 3.78. *Let $F : \mathbb{R} \to \mathbb{C}$. Suppose that there exists $f \in L^1(m)$ such that $F(x) = \int_{(-\infty, x)} f dm$. Then $F \in NBV$.*

Lemma 3.79. *Let $F \in NBV$. Then F is absolutely continuous iff $\mu_F \ll m$.*

Exercise 3.80. *Fundamental Theorem of Calculus: Let $F : [a,b] \to \mathbb{C}$. The following are equivalent:*

(1) *F is absolutely continuous on $[a,b]$.*
(2) *there exists $f \in L^1([a,b], m)$ such that for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} f dm$*
(3) *F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b], m)$ and for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} F' dm$*

Exercise 3.81. *Let $F : \mathbb{R} \to \mathbb{C}$. If F is absolutely continuous. Then F is differentiable a.e.*

Exercise 3.82. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is Lipschitz continuous iff F is absolutely continuous and F' is bounded a.e.*

Exercise 3.83. *Construct an increasing function $F : \mathbb{R} \to \mathbb{R}$ whose discontinuities is $\mathbb{Q}$.*

Exercise 3.84. *Let $(F_n)_{n \in \mathbb{N}} \in NBV$ be a sequence of nonnegative, increasing functions. If for each $x \in \mathbb{R}$, $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$, then for a.e. $x \in \mathbb{R}$, F is differentiable at x and $F'(x) = \sum_{n \in \mathbb{N}} F_n'(x)$.*

Exercise 3.85. *Let $F : [0, 1] \to [0, 1]$ be the Cantor function. Extend F to $\mathbb{R}$ by setting $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $([a_n, b_n])_{n \in \mathbb{N}}$ be an ennumeration of the closed subintervals of $[0, 1]$ with rational endpoints. For $n \in \mathbb{N}$, define $F_n : \mathbb{R} \to [0, 1]$ by $F_n(x) = F(\frac{x - a_n}{b_n - a_n})$. Define $G : \mathbb{R} \to \mathbb{R}$ by $G = \sum_{n \in \mathbb{N}} 2^{-n} F_n$. Then G is continuous, strictly increasing on $[0, 1]$ and $G' = 0$ a.e.*

4. Topology

Definition 4.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then*
  (1) *f is said to be **continuous** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$.*
  (2) *f is said to be open if for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$.*
  (3) *f is said to be **closed** if for each $A \subset X$, if $A^c \in \mathcal{A}$, then $f(A)^c \in \mathcal{B}$.*

5. L^p Spaces

6. Functional Analysis

6.1. **Normed Vector Spaces.**

Note 6.1. *In the following, we will consider vector spaces over $\mathbb{C}$. There are analogous results for real vector spaces as well, just replace every $\mathbb{C}$ with $\mathbb{R}$.*

Definition 6.2. *Let X be a normed vector space. Then X is said to be a **Banach space** if X is complete.*

Definition 6.3. *Let X be a normed vector space and $(x_i)_{i=1}^n \subset X$. The series $\sum_{i=1}^\infty x_i$ is said to **converge** if the sequence $s_n := \sum_{i=1}^n x_i$ converges. The series $\sum_{i=1}^\infty x_i$ is said to **converge absolutely** if $\sum_{i \in \mathbb{N}} \|x_i\| < \infty$.*

Theorem 6.4. *Let X be a normed vector space. Then X is complete iff for each $(x_i)_{i \in \mathbb{N}} \subset X$, $\sum_{i=1}^\infty x_i$ converges absolutely implies that $\sum_{i=1}^\infty x_i$ converges.*

Definition 6.5. *Let X, Y be a normed vector spaces. A linear map $T : X \to Y$ is said to be **bounded** if there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$.*

Exercise 6.6. *Let X, Y be a normed vector spaces and $T : X \to Y$ a linear map. Then T is bounded iff there exists $r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$*

Theorem 6.7. *Let X, Y be normed vector spaces and $T : X \to Y$ a linear map. Then the following are equivalent:*
  (1) *T is continuous*
  (2) *T is continuous at $x = 0$*
  (3) *T is bounded*

Definition 6.8. *Let X, Y be normed vector spaces. Define $L(X, Y) = \{T : X \to Y : T$ is bounded$\}$. Define $\|\cdot\| : L(X, Y) \to [0, \infty)$ by*

$$\|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

*We call $\|\cdot\|$ the **operator norm on** $L(X, Y)$*

Exercise 6.9. *Let X, Y be normed vector spaces. If $X \neq \{0\}$, then the operater norm on $L(X, Y)$ is given by:*

(1) $\|T\| = \sup\limits_{\|x\|=1} \|Tx\|$

(2) $\|T\| = \sup\limits_{x \neq 0} \|x\|^{-1}\|Tx\|$

(3) $\|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$

Note 6.10. *From here on, unless stated otherwise, we assume $X \neq 0$.*

Exercise 6.11. *Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then for each $x \in X$, $\|Tx\| \leq \|T\|\|x\|$*

Exercise 6.12. *Let X, Y be normed vector spaces. Then the operator norm is a norm on $L(X, Y)$.*

Exercise 6.13. *Let X be a normed vector space. Then addition and scalar multiplication are continuous on $X \times X$ and $\|\cdot\| : X \to [0, \infty)$ is continuous.*

Exercise 6.14. *Let X, Y be normed vector spaces. If Y is complete, then so is $L(X, Y)$.*

Definition 6.15. *Let X be a normed vector space and $M \subset X$ a closed subspace. Define $\|\cdot\| : X/M \to [0, \infty)$ by*

$$\|x + M\| := \inf_{y \in M} \|x + y\|$$

*We call $\|\cdot\|$ the **subspace norm on** X/M*

Exercise 6.16. *Let X be a normed vector space and $M \subsetneq X$ a proper, closed subspace of M. Then*

(1) *The previously defined subspace norm on X/M is well defined and is a norm.*

(2) *For each $\epsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.*

(3) *The projection map $\pi : X \to X/M$ defined by $\pi(x) = x + M$ is continuous and $\|\pi\| = 1$.*

(4) *If X is complete, then X/M is complete.*

Exercise 6.17. *Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then*

(1) $\ker T$ *is closed*

(2) *there exists a unique map $S : X/\ker T \to T(X)$ such that $T = S \circ \pi$. Furthermore S is a bounded linear bijection and $\|S\| = \|T\|$.*

Exercise 6.18. *Let X, Y be normed vector spaces. Define $\phi : L(X, Y) \times X \to Y$ by $\phi(T, x) = Tx$. Then ϕ is continuous.*

Exercise 6.19. *Let X be a normed vector space and $M \subset X$ a subspace. Then $\overline{M}$ is a subspace.*

Exercise 6.20. *Let X, Y, Z be normed vector spaces, $T \in L(X, Y)$ and $S \in L(Y, Z)$. Define $ST : X \to Z$ by $STx = S(Tx)$. Then $ST \in L(X, Z)$ and $\|ST\| \leq \|S\|\|T\|$.*

Definition 6.21. *Let X be a Banach space and an associative algebra. Then X is said to be a Banach algebra if for each $S, T \in X$, $\|ST\| \leq \|S\|\|T\|$. If there exists $I \in X$ such that $I \neq 0$ and for each $T \in X$, $IT = TI = T$, then X is said to be **unital** with identity I. An element $T \in X$ is said to be **invertible** if there exists $S \in X$ such that $TS = ST = I$.*

Exercise 6.22. *Let X be a unital Banach algebra. Then $\|I\| \leq 1$.*

Note 6.23. *If X is a Banach space, then a previous exercise implies that $L(X, X)$ equipped with composition is a unital Banach algebra where I is the identity operator. It is easy to see that $\|I\| = 1$.*

Note 6.24. *Let X be a Banach algebra. Then the set of invertible elements in X is a group.*

Exercise 6.25. *Let X be a Banach algebra. Then mulitplication is continuous.*

Definition 6.26. *Let X, Y be a normed vector spaces and $T \in L(X, Y)$. Then T is said to be **invertible** or an **isomorphism** if T is a bijection and $T^{-1} \in L(Y, X)$.*

Definition 6.27. *Let X be a Banach space. Define $GL(X) := \{T \in L(X, X) : T \text{ is invertible}\}$.*

Exercise 6.28. *Let X be a Banach space. Then*
  (1) For each $T \in L(X, X)$, if $\|I - T\| < 1$, then T is invertible and

$$T^{-1} = \sum_{n=0}^{\infty} (I - T)^n$$

  (2) For each $S, T \in L(X, X)$, if S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$, then T is invertible.
  (3) $GL(X)$ is open.

Exercise 6.29. *Let $M(X, \mathcal{A})$ denote the set of complex measures on the measurable space $(X, \mathcal{A})$. Define $\|\cdot\| : M(X, \mathcal{A}) \to [0, \infty)$ by $\|\mu\| = |\mu|(X)$. Then $\|\cdot\|$ is a norm on $M(X, \mathcal{A})$.*

6.2. **Linear Functionals.**

Definition 6.30. *Let X be a normed vector space and $T : X \to \mathbb{C}$. Then T is said to be a **linear functional on** X if T is linear and T is said to be a **bounded linear functional on** X if $T \in L(X, \mathbb{C})$. We define the **dual space of** X, denoted X^*, by $X^* = L(X, \mathbb{C})$.*

Definition 6.31. *Let X be a normed vector space and $p : X \to \mathbb{R}$. Then p is said to be a **sublinear functional** if for each $x, y \in X$, $\lambda \geq 0$,*
  (1) $p(x + y) \leq p(x) + p(y)$
  (2) $p(\lambda x) = \lambda p(x)$

Note 6.32. *Let X be a vector space and $\|\cdot\| : X \to [0, \infty)$ be a seminorm, then $\|\cdot\|$ is a sublinear functional.*

Theorem 6.33. *Hahn-Banach Theorem: Let X be a vector space, $p : X \to \mathbb{R}$ a sublinear functional, $M \subset X$ a subspace and $f : M \to C$ a linear functional. If for each $x \in M$, $|f(x)| \leq p(x)$, then there exists a linear functional $F : X \to \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x)$ and $F|_M = f$.*

Exercise 6.34. *Let X be a normed vector space, $M \subset X$ a subspace and $f \in M^*$. Then there exists $F \in X^*$ such that $\|F\| = \|f\|$ and $F|_M = f$.*

Exercise 6.35. *Let X be a normed vector space, $M \subsetneq X$ a proper closed subspace and $x \in X \setminus M$. Then there exists $F \in X^*$ such that $F|_M = 0$, $\|F\| = 1$ and $F(x) = \|x + M\| \neq 0$. (Hint: Consider $f : M + \mathbb{C}x \to \mathbb{C}$ defined by $f(m + \lambda x) = \lambda \|x + M\|$.)*

Exercise 6.36. *Let X be a normed vector space and $x \in X$. If $x \neq 0$, then there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$.*

Exercise 6.37. *Let X be a normed vector space. Then X^* separates the points of X.*

Definition 6.38. *Let X, Y be metric spaces and $T : X \rightarrow Y$. Then T is said to be an **isometry** if for each $x_1, x_2 \in X$, $d(Tx_1, Tx_2) = d(x_1, x_2)$.*

Exercise 6.39. *Let X, Y be metric spaces and $T : X \rightarrow Y$ and isometry. Then T is injective.*

Note 6.40. *Let X, Y be metric spaces and $T : X \rightarrow Y$ an isometry. Then T is clearly continuous. If T is surjective, then T^{-1} is an isometry and therefore continuous. Hence T is a homeomorphism.*

Exercise 6.41. *Let X be a normed vector space and $x \in X$. Define $\hat{x} : X^* \rightarrow \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| = \|x\|$.*

Exercise 6.42. *Let X be a normed vector space. Define $\phi : X \rightarrow X^{**}$ by $\phi(x) = \hat{x}$. Then ϕ is a linear isometry.*

Definition 6.43. *Let X be a normed vector space and define $\phi : X \rightarrow X^{**}$ as above. We define $\widehat{X} = \phi(X) \subset X^{**}$. Since $\widehat{X}$ and X are isomorphic, we may identify X as a subset of X^{**}.*

Definition 6.44. *Let X be a normed vector space and define $\phi : X \rightarrow X^{**}$ as above. Then X is said to be reflexive if ϕ is surjective. In this case ϕ is then an isomorphism*

Exercise 6.45. *Let X be a normed vector space and $f : X \rightarrow \mathbb{C}$ a linear functional on X. Then f is bounded iff $\ker f$ is closed.*

Exercise 6.46. *Let X be a normed vector space.*
  (1) Let $M \subsetneq X$ be a proper closed subspace of X and $x \in X \setminus M$. Then $M + \mathbb{C}x$ is closed.
  (2) Let $M \subset X$ be a finite dimensional subspace of X. Then M is closed.

Exercise 6.47. *Let X be an infinite-dimensional normed vector space.*
  (1) There exists a sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that for each $m, n \in \mathbb{N}$, $\|x_n\| = 1$ and if $m \neq n$, then $\|x_m - x_n\| > \frac{1}{2}$.
  (2) X is not locally compact.

Exercise 6.48. *Let X, Y be normed vector spaces and $T \in L(X, Y)$.*
  *(1) Define the **adjoint of** T, $T^* : Y^* \rightarrow X^*$ by $T^*(f) = f \circ T$. Then $T^* \in L(Y^*, X^*)$.*
  (2) Applying the result from (1) twice, we have that $T^{**} \in L(X^{**}, Y^{**})$. We have that for each $x \in X$, $T^{**}(\hat{x}) = \widehat{T(x)}$.
  (3) T^* is injective iff $T(X)$ is dense in Y.
  (4) If $T^*(Y^*)$ is dense in X^*, then T is injective. The converse is true if X is reflexive.

Exercise 6.49. *Let X be a normed vector space. Then X is reflexive iff X^* is reflexive.*

6.3. **The Baire Category Theorem and Consequences.**

Theorem 6.50. *Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is surjective, then T is open.*

Corollary 6.51. *Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is a bijection, then $T^{-1} \in L(X, Y)$.*

Definition 6.52. *Let X, Y be sets and $f : X \to Y$. We define the **graph of f**, $\Gamma(f)$, by $\Gamma(f) = \{(x, y) \in X \times Y : f(x) = y\}$.*

Theorem 6.53. *Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If $\Gamma(T)$ is closed, then $T \in L(X, Y)$.*

Note 6.54. *We recall that $\Gamma(T)$ is closed iff for each $(x_n)_{n \in \mathbb{N}} \subset X$, $x \in X$ and $y \in Y$ if $x_n \to x$ and $T(x_n) \to y$, then $T(x) = y$.*

Theorem 6.55. *Let X, Y be Banach spaces and $S \subset L(X, Y)$. If for each $x \in X$,*

$$\sup_{T \in S} \|Tx\| < \infty$$

then

$$\sup_{T \in S} \|T\| < \infty$$

Exercise 6.56. *Let μ be counting measure on $(N, \mathcal{P}(\mathbb{N}))$. Define $h : \mathbb{N} \to \mathbb{N}$ and ν on $(N, \mathcal{P}(\mathbb{N}))$ by $h(n) = n$ and $d\nu = h d\mu$. Define $X = L^1(\nu)$ and $Y = L^1(\mu)$. Equip both X and Y with the L^1 norm with respect to μ.*

    (1) We have that X is a proper subspace of Y and therefore X is not complete.
    (2) Define $T : X \to Y$ by $Tf(n) = nf(n)$. Then T is linear, $\Gamma(T)$ is closed, and T is unbounded.
    (3) Define $S : Y \to X$ by $Sg(n) = \frac{1}{n}g(n)$. Then $S \in L(Y, X)$, S is surjective and S is not open.

Exercise 6.57. *Let $X = C^1([0, 1])$ and $Y = C([0, 1])$. Equip both X and Y with the uniform norm.*

    (1) Then X is not complete
    (2) Define $T : X \to Y$ by $Tf = f'$. Then $\Gamma(T)$ is closed and T is not bounded.

Exercise 6.58. *Let X, Y be Banach spaces and $T \in L(X, Y)$. Then $X/\ker T \cong T(X)$ iff $T(X)$ is closed.*

Exercise 6.59. *Let X be a separable Banach space. Define $B_X = \{x \in X : \|x\| < 1\}$. Let $(x_n)_{n \in \mathbb{N}} \subset B_X$ a dense subset of the unit ball and μ the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Define $T : L^1(\mu) \to X$ by*

$$Tf = \sum_{n=1}^{\infty} f(n)x_n$$

Then

    (1) T is well defined and $T \in L(L^1(\mu), X)$
    (2) T is surjective
    (3) There exists a closed subspace $K \subset L^1(\mu)$ such that $L^1(\mu)/K \cong X$

Exercise 6.60. *Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If for each $f \in Y^*$, $f \circ T \in X^*$, then $T \in L(X, Y)$.*

7. Appendix

7.1. Summation.

Definition 7.1. *Let $f : X \to [0, \infty)$, Then we define*

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subseteq X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note 7.2. *Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.*