PORTFOLIO THEORY NOTES

Contents

Note 0.1. *In these notes we will mostly consider random variables X that model returns. As such we may assume that $X \in L^1(\mathbb{P})$ and $F_X : \mathbb{R} \to (0,1)$ is bijective and continuous. We will call such random variables "nice". The random variable X will usually refer to the return on some portfolio. As such, we will define the loss of X to be $L_X = -X$.*

1. Risk Measures

1.1. Value at Risk.

Definition 1.1. *Let X be a nice random variable and $\alpha \in (0,1)$. We define the* ***value at risk of*** X ***at confidence level*** α*, denoted by $VaR_\alpha(X)$, to be*

$$\begin{aligned} VaR_\alpha(X) &= F_{-X}^{-1}(\alpha) \\ &= F_{L_X}^{-1}(\alpha) \end{aligned}$$

Thus

$$\mathbb{P}(L_X > VaR_\alpha(X)) = 1 - \alpha$$

Note 1.2. *In practice, we take $\alpha = .95$ or $\alpha = .99$.*

1.2. Estimating the Value at Risk.

1.3. Average Value at Risk.

Definition 1.3. *Let X be a nice random variable and $\alpha \in (0,1)$. We define the* ***average value at risk of*** X ***with tail probability*** α*, denoted by $AVaR_\alpha(X)$, to be*

$$AVaR_\alpha(X) = \frac{1}{1-\alpha} \int_{[\alpha,1)} VaR_p(X)dm(p)$$

Note 1.4. *If X represents the return on a portfolio, then $AVaR_\alpha(X)$ is just the average of the $VaR_p(X)$ on the interval $(\alpha, 1]$.*

Exercise 1.5. *Let X be a nice random variable and $\alpha \in (0,1)$. Then*

$$AVaR_\alpha(X) = \mathbb{E}[L_X | L_X \geq Var_\alpha(X)]$$

Proof. Recall that for measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a measurable function $f : X \to Y$ and a measure $\mu : \mathcal{A} \to [0, \infty]$, we may form the push-foreward measure of μ by f, $f_*\mu : \mathcal{B} \to [0, \infty]$ with the following property: for each $g : Y \to \mathbb{C}$, $g \in L^1(f_*\mu)$ iff $g \circ f \in L^1(\mu)$ and for each $B \in \mathcal{B}$,

$$\int_{f^{-1}(B)} g \circ f d\mu = \int_B g df_*\mu$$

Also recall that for an increasing continuous, bijective $F : \mathbb{R} \to (0,1)$, we may form the Borel measure μ_F with $\mu_F((a,b]) = F(b) - F(a)$. Then observe that $F_*\mu_F = m$ because

$$\begin{aligned}
F_*\mu_F((a,b]) &= \mu_F(F^{-1}((a,b])) \\
&= \mu_F((F^{-1}(a), F^{-1}(b)]) \\
&= F(F^{-1}(a)) - F(F^{-1}(b)) \\
&= b - a
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}[L_X | L_X \geq VaR_\alpha(X)] &= \mathbb{E}[L_X | L_X \geq F_{L_X}^{-1}(\alpha)] \\
&= \frac{1}{1-\alpha} \mathbb{E}[L_X \mathbf{1}_{\{L_X \geq F_{L_X}^{-1}(\alpha)\}}] \\
&= \frac{1}{1-\alpha} \int_{\{L_X \geq F_{L_X}^{-1}(\alpha)\}} L_X dP \\
&= \frac{1}{1-\alpha} \int_{[F_{L_X}^{-1}(\alpha), \infty)} x dF_{L_X}(x)
\end{aligned}$$

Using the facts recalled earilier, we have

$$\begin{aligned}
\int_{[F_{L_X}^{-1}(\alpha), \infty)} x dF_{L_X}(x) &= \int_{[F_{L_X}^{-1}(\alpha), \infty)} (F_{L_X}^{-1} \circ F_{L_X})(x) dF_{L_X}(x) \\
&= \int_{[\alpha, 1)} F_{L_X}^{-1}(x) dm(x) \\
&= \int_{[\alpha, 1)} VaR_\alpha(X) dm(x)
\end{aligned}$$

□

Lemma 1.6. *Let $\alpha \in (0,1)$. Define $f_\alpha : \mathbb{R} \to \mathbb{R}$ by*

$$f_\alpha(\theta) = \theta + \frac{1}{1-\alpha} \mathbb{E}[(L_X - \theta)^+]$$

Then f_α is convex and

$$\frac{df_\alpha}{d\theta}(\theta) = \frac{F_{L_X} - \alpha}{1-\alpha}$$

Proof. Recall that given $g : \Omega \times \mathbb{R} \to \mathbb{R}$, if for each $\omega \in \Omega$, $g(\omega, \theta)$ is convex in θ, then $\mathbb{E}[g(\cdot, \theta)]$ is convex in theta. For $\omega \in \Omega, \theta \in \mathbb{R}$, put

$$g(\omega, \theta) = (L_X(\omega) - \theta)^+$$

So

$$f_\alpha(\theta) = \theta + \frac{1}{1-\alpha}\mathbb{E}[g(\theta)]$$

Then for each $\omega \in \Omega$, $g(\omega, \cdot)$ is convex. This implies that for each $\alpha \in (0,1)$, f_α is convex and therefore continuous.

Now Let $\theta, \theta' \in \mathbb{R}$. Suppose that $\theta < \theta'$. Then

$$\frac{f_\alpha(\theta') - f_\alpha(\theta)}{\theta' - \theta} = 1 + \frac{1}{1-\alpha}\mathbb{E}\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\right]$$

For $\omega \in \Omega$, we have that

$$\frac{(L_X(\omega) - \theta')^+ - (L_X(\omega) - \theta)^+}{\theta' - \theta} = \begin{cases} -1 & \theta' \leq L_X(\omega) \\ 0 & L_X(\omega) \leq \theta \\ \epsilon \in (-1, 0) & \theta < L_X(\omega) < \theta' \end{cases}$$

This implies that

$$\begin{aligned} -(F_{L_X}(\theta') - F_{L_X}(\theta)) &= -\mathbb{P}(\theta < L_X < \theta') \\ &\leq \mathbb{E}\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\mathbf{1}_{L_X \in (\theta, \theta')}\right] \\ &< 0 \end{aligned}$$

Thus there exists $c \in (0,1)$ such that

$$\mathbb{E}\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\mathbf{1}_{L_X \in (\theta, \theta')}\right] = -c(F_{L_X}(\theta') - F_{L_X}(\theta))$$

In addition, $\mathbb{P}(\theta' \leq L_X(\omega)) = 1 - F_{L_X}(\theta')$. Therefore

$$\mathbb{E}\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\right] = -(1 - F_{L_X}(\theta')) - c(F_{L_X}(\theta') - F_{L_X}(\theta))$$

This implies that

$$\lim_{\theta' \to \theta^+} \mathbb{E}\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\right] = F_{L_X}(\theta) - 1$$

Finally we have that

$$\begin{aligned} \lim_{\theta' \to \theta^+} \frac{f_\alpha(\theta') - f_\alpha(\theta)}{\theta' - \theta} &= 1 + \frac{1}{1-\alpha} \lim_{\theta' \to \theta^+} \mathbb{E}\left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta}\right] \\ &= 1 + \frac{F_{L_X}(\theta) - 1}{1-\alpha} \\ &= \frac{F_{L_X}(\theta) - \alpha}{1-\alpha} \end{aligned}$$

The case is similar for the lefthand limit.

□

Theorem 1.7. *Let X be a nice random variable and $\alpha \in (0,1)$. Then*

$$AVaR_\alpha(X) = \min_{\theta\in\mathbb{R}} \left(\theta + \frac{1}{1-\alpha}\mathbb{E}[(L_X - \theta)^+]\right)$$

Proof. The previous lemma tells us that

$$\frac{df_\alpha}{d\theta}(\theta) = \frac{F_{L_X}(\theta) - \alpha}{1-\alpha}$$

For further details, see [**?**]. Thus

$$\lim_{\theta\to\infty} \frac{df_\alpha}{d\theta}(\theta) = 1$$

and

$$\lim_{\theta\to-\infty} \frac{df_\alpha}{d\theta}(\theta) = -\frac{\alpha}{1-\alpha} < 0$$

The convexity of f_α implies that there exists $\theta^* \in \mathbb{R}$ such that $f_\alpha(\theta^*) = \inf_{\theta\in\mathbb{R}} f_\alpha(\theta)$ Thus

$$\frac{df_\alpha}{d\theta}(\theta^*) = 0$$

which implies that

$$F_{L_X}(\theta^*) = \alpha$$

By definition, $\theta^* = VaR_\alpha(X)$. Finally, evaluating f_α at θ^* shows us that

$$\begin{aligned}
f_\alpha(\theta^*) &= \theta^* + \frac{1}{1-\alpha}\mathbb{E}[(L_X - \theta^*)^+] \\
&= \theta^* + \frac{1}{\mathbb{P}(L_X > \theta^*)}\mathbb{E}[(L_X - \theta^*)\mathbf{1}_{\{L_X>\theta^*\}}] \\
&= \theta^* + \frac{1}{\mathbb{P}(L_X > \theta^*)}\mathbb{E}[L_X\mathbf{1}_{\{L_X>\theta^*\}}] - \frac{1}{\mathbb{P}(L_X > \theta^*)}\mathbb{E}[\theta^*\mathbf{1}_{\{L_X>\theta^*\}}] \\
&= \theta^* + \frac{1}{\mathbb{P}(L_X > \theta^*)}\mathbb{E}[L_X\mathbf{1}_{\{L_X>\theta^*\}}] - \theta^* \\
&= \mathbb{E}[L_X | L_X > \theta^*] \\
&= \mathbb{E}[L_X | L_X > VaR_\alpha(X)] \\
&= AVaR_\alpha(X)
\end{aligned}$$

□

1.4. **Estimating the Value at Risk.**

Definition 1.8. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\alpha \in (0,1)$. Define*

$$\widehat{VaR_\alpha}(X) =$$

1.5. **Estimating the Average Value at Risk.**

Definition 1.9. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\alpha \in (0,1)$. Define*

$$\widehat{AVaR_\alpha}(X) =$$

Lemma 1.10. *Let X be a random nice random variable, $X_1, \cdots, X_n \overset{iid}{\sim} X$ and $\alpha \in (0,1)$. Then $\widehat{AVarR_\alpha}(X)$ is an unbiased estimator for $AVaR_\alpha(X)$.*

Proof. For each $\alpha \in (0,1), \omega \in \Omega$ and $\theta \in \mathbb{R}$, define

$$f_\alpha(\omega)(\theta) = \theta + \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \max(-X_i(\omega) - \theta, 0)$$

Note that for each $\alpha \in (0,1)$ and $\omega \in \Omega$, $f_\alpha(\omega)$ is convex and continuous. In this case with no expectation, it is easy to show that

$$\lim_{\theta \to \infty} \frac{\partial f_\alpha(\omega)}{\partial \theta}(\theta) = 1$$

and

$$\lim_{\theta \to -\infty} \frac{\partial f_\alpha(\omega)}{\partial \theta}(\theta) = -\frac{\alpha}{1-\alpha} < 0$$

So for each $\alpha \in (0,1)$ and $\omega \in \Omega$, $f_\alpha(\omega)$ achieves its minimum at . Then $\{\theta \in \mathbb{R} : f_\alpha(\omega)(\theta) \leq m+1\}$ is bounded

Since f_α is continuous, we have that

$$\inf_{\theta \in \mathbb{R}} f_\alpha(\theta) = \inf_{\theta \in \mathbb{Q}} f_\alpha(\theta)$$

which is measurable. □

References