

PORTFOLIO THEORY NOTES

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Note 0.1. In these notes we will mostly consider a probability space (Ω, \mathcal{F}, P) and a random variable $X : \Omega \rightarrow \mathbb{R}$. We assume that $X \in L^1(P)$ and $F_X : \mathbb{R} \rightarrow (0, 1)$ is strictly increasing and continuous. We will call such a random variable "nice". The random variable X will usually refer to the return on some portfolio. As such, we will define the loss of X to be $L_X = -X$.

1. RISK MEASURES

1.1. Value at Risk.

Definition 1.1. Let X be a nice random variable and $\alpha \in (0, 1)$. We define the **value at risk of X at confidence level α** , denoted by $VaR_\alpha(X)$, to be

$$\begin{aligned} VaR_\alpha(X) &= F_{-X}^{-1}(\alpha) \\ &= F_{L_X}^{-1}(\alpha) \end{aligned}$$

Thus

$$P(L_X > VaR_\alpha(X)) = 1 - \alpha$$

Note 1.2. In practice, we take $\alpha = .95$ or $\alpha = .99$.

1.2. Estimating the Value at Risk.

1.3. Average Value at Risk.

Definition 1.3. Let X be a nice random variable and $\alpha \in (0, 1)$. We define the **average value at risk of X with tail probability α** , denoted by $AVaR_\alpha(X)$, to be

$$AVaR_\alpha(X) = \frac{1}{1 - \alpha} \int_{[\alpha, 1)} VaR_p(X) dm(p)$$

Note 1.4. If X represents the return on a portfolio, then $AVaR_\alpha(X)$ is just the average of the $VaR_p(X)$ on the interval $(\alpha, 1]$.

Exercise 1.5. Let X be a nice random variable and $\alpha \in (0, 1)$. Then

$$AVaR_\alpha(X) = E[L_X | L_X \geq VaR_\alpha(X)]$$

Proof. Recall that for measurable spaces (X, \mathcal{A}) and (Y, \mathcal{B}) , a measurable function $f : X \rightarrow Y$ and a measure $\mu : \mathcal{A} \rightarrow [0, \infty]$, we may form the push-forward measure of μ by f , $f_*\mu : \mathcal{B} \rightarrow [0, \infty]$ with the following property: for each $g : Y \rightarrow \mathbb{C}$, $g \in L^1(f_*\mu)$ iff $g \circ f \in L^1(\mu)$ and for each $B \in \mathcal{B}$,

$$\int_{f^{-1}(B)} g \circ f d\mu = \int_B g df_*\mu$$

Also recall that for an increasing continuous, bijective $F : \mathbb{R} \rightarrow (0, 1)$, we may form the Borel measure μ_F with $\mu_F((a, b]) = F(b) - F(a)$. Then observe that $F_*\mu_F = m$ because

$$\begin{aligned} F_*\mu_F((a, b]) &= \mu_F(F^{-1}((a, b])) \\ &= \mu_F((F^{-1}(a), F^{-1}(b)]) \\ &= F(F^{-1}(b)) - F(F^{-1}(a)) \\ &= b - a \end{aligned}$$

Then

$$\begin{aligned} E[L_X | L_X \geq VaR_\alpha(X)] &= E[L_X | L_X \geq F_{L_X}^{-1}(\alpha)] \\ &= \frac{1}{1 - \alpha} E[L_X \mathbf{1}_{\{L_X \geq F_{L_X}^{-1}(\alpha)\}}] \\ &= \frac{1}{1 - \alpha} \int_{\{L_X \geq F_{L_X}^{-1}(\alpha)\}} L_X dP \\ &= \frac{1}{1 - \alpha} \int_{[F_{L_X}^{-1}(\alpha), \infty)} x dF_{L_X}(x) \end{aligned}$$

Using the facts recalled earlier, we have

$$\begin{aligned} \int_{[F_{L_X}^{-1}(\alpha), \infty)} x dF_{L_X}(x) &= \int_{[F_{L_X}^{-1}(\alpha), \infty)} (F_{L_X}^{-1} \circ F_{L_X})(x) dF_{L_X}(x) \\ &= \int_{[\alpha, 1)} F_{L_X}^{-1}(x) dm(x) \\ &= \int_{[\alpha, 1)} VaR_\alpha(X) dm(x) \end{aligned}$$

□

Lemma 1.6. Let $\alpha \in (0, 1)$. Define $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_\alpha(\theta) = \theta + \frac{1}{1 - \alpha} E[(L_X - \theta)^+]$$

Then f_α is convex and

$$\frac{df_\alpha}{d\theta}(\theta) = \frac{F_{L_X}(\theta) - \alpha}{1 - \alpha}$$

Proof. Recall that given $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, if for each $\omega \in \Omega$, $g(\omega, \theta)$ is convex in θ , then $E[g(\cdot, \theta)]$ is convex in θ . For $\omega \in \Omega, \theta \in \mathbb{R}$, put

$$g(\omega, \theta) = (L_X(\omega) - \theta)^+$$

So

$$f_\alpha(\theta) = \theta + \frac{1}{1-\alpha} E[g(\cdot, \theta)]$$

Then for each $\omega \in \Omega$, $g(\omega, \cdot)$ is convex. This implies that for each $\alpha \in (0, 1)$, f_α is convex and therefore continuous.

Now Let $\theta, \theta' \in \mathbb{R}$. Suppose that $\theta < \theta'$. Then

$$\frac{f_\alpha(\theta') - f_\alpha(\theta)}{\theta' - \theta} = 1 + \frac{1}{1-\alpha} E \left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta} \right]$$

For $\omega \in \Omega$, we have that

$$\frac{(L_X(\omega) - \theta')^+ - (L_X(\omega) - \theta)^+}{\theta' - \theta} = \begin{cases} -1 & \theta' \leq L_X(\omega) \\ 0 & L_X(\omega) \leq \theta \\ \epsilon \in (-1, 0) & \theta < L_X(\omega) < \theta' \end{cases}$$

This implies that

$$\begin{aligned} -(F_{L_X}(\theta') - F_{L_X}(\theta)) &= -P(\theta < L_X < \theta') \\ &\leq E \left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta} \mathbf{1}_{L_X \in (\theta, \theta')} \right] \\ &< 0 \end{aligned}$$

Thus there exists $c \in (0, 1)$ such that

$$E \left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta} \mathbf{1}_{L_X \in (\theta, \theta')} \right] = -c(F_{L_X}(\theta') - F_{L_X}(\theta))$$

In addition, $P(\theta' \leq L_X) = 1 - F_{L_X}(\theta')$. Therefore

$$E \left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta} \right] = -(1 - F_{L_X}(\theta')) - c[F_{L_X}(\theta') - F_{L_X}(\theta)]$$

This implies that

$$\lim_{\theta' \rightarrow \theta^+} E \left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta} \right] = F_{L_X}(\theta) - 1$$

Finally we have that

$$\begin{aligned} \lim_{\theta' \rightarrow \theta^+} \frac{f_\alpha(\theta') - f_\alpha(\theta)}{\theta' - \theta} &= 1 + \frac{1}{1 - \alpha} \lim_{\theta' \rightarrow \theta^+} E \left[\frac{(L_X - \theta')^+ - (L_X - \theta)^+}{\theta' - \theta} \right] \\ &= 1 + \frac{F_{L_X}(\theta) - 1}{1 - \alpha} \\ &= \frac{F_{L_X}(\theta) - \alpha}{1 - \alpha} \end{aligned}$$

The case is similar for the lefthand limit. □

Theorem 1.7. *Let X be a nice random variable and $\alpha \in (0, 1)$. Then*

$$VaR_\alpha(X) = \arg \min_{\theta \in \mathbb{R}} \left(\theta + \frac{1}{1 - \alpha} E[(L_X - \theta)^+] \right)$$

and

$$AVaR_\alpha(X) = \min_{\theta \in \mathbb{R}} \left(\theta + \frac{1}{1 - \alpha} E[(L_X - \theta)^+] \right)$$

Proof. Define f_α as before. The previous lemma tells us that

$$\frac{df_\alpha}{d\theta}(\theta) = \frac{F_{L_X}(\theta) - \alpha}{1 - \alpha}$$

Thus

$$\lim_{\theta \rightarrow \infty} \frac{df_\alpha}{d\theta}(\theta) = 1$$

and

$$\lim_{\theta \rightarrow -\infty} \frac{df_\alpha}{d\theta}(\theta) = -\frac{\alpha}{1 - \alpha} < 0$$

Thus $\lim_{\theta \rightarrow \infty} f_\alpha(\theta) = \lim_{\theta \rightarrow -\infty} f_\alpha(\theta) = \infty$. The convexity of f_α implies that there exists a unique $\theta^* \in \mathbb{R}$ such that $f_\alpha(\theta^*) = \inf_{\theta \in \mathbb{R}} f_\alpha(\theta)$. Thus

$$\frac{df_\alpha}{d\theta}(\theta^*) = 0$$

which implies that

$$F_{L_X}(\theta^*) = \alpha$$

By definition, $\theta^* = VaR_\alpha(X)$. Finally, evaluating f_α at θ^* shows us that

$$\begin{aligned}
f_\alpha(\theta^*) &= \theta^* + \frac{1}{1-\alpha} E[(L_X - \theta^*)^+] \\
&= \theta^* + \frac{1}{P(L_X > \theta^*)} E[(L_X - \theta^*) \mathbf{1}_{\{L_X > \theta^*\}}] \\
&= \theta^* + \frac{1}{P(L_X > \theta^*)} E[L_X \mathbf{1}_{\{L_X > \theta^*\}}] - \frac{1}{P(L_X > \theta^*)} E[\theta^* \mathbf{1}_{\{L_X > \theta^*\}}] \\
&= \theta^* + \frac{1}{P(L_X > \theta^*)} E[L_X \mathbf{1}_{\{L_X > \theta^*\}}] - \theta^* \\
&= E[L_X | L_X > \theta^*] \\
&= E[L_X | L_X > VaR_\alpha(X)] \\
&= AVaR_\alpha(X)
\end{aligned}$$

□

1.4. Estimating the Value at Risk.

Definition 1.8. Let X be a random nice random variable, $X_1, \dots, X_n \stackrel{iid}{\sim} X$ and $\alpha \in (0, 1)$. Define

$$\widehat{VaR}_\alpha(X) =$$

1.5. Estimating the Average Value at Risk.

Definition 1.9. Let X be a random nice random variable, $X_1, \dots, X_n \stackrel{iid}{\sim} X$ and $\alpha \in (0, 1)$. Define

$$\widehat{AVaR}_\alpha(X) =$$

Lemma 1.10. Let X be a random nice random variable, $X_1, \dots, X_n \stackrel{iid}{\sim} X$ and $\alpha \in (0, 1)$. Then $\widehat{AVaR}_\alpha(X)$ is an unbiased estimator for $AVaR_\alpha(X)$.

Proof. For each $\alpha \in (0, 1)$, $\omega \in \Omega$ and $\theta \in \mathbb{R}$, define

$$f_\alpha(\omega)(\theta) = \theta + \frac{1}{n(1-\alpha)} \sum_{i=1}^n \max(-X_i(\omega) - \theta, 0)$$

Note that for each $\alpha \in (0, 1)$ and $\omega \in \Omega$, $f_\alpha(\omega)$ is convex and continuous. In this case with no expectation, it is easy to show that

$$\lim_{\theta \rightarrow \infty} \frac{\partial f_\alpha(\omega)}{\partial \theta}(\theta) = 1$$

and

$$\lim_{\theta \rightarrow -\infty} \frac{\partial f_\alpha(\omega)}{\partial \theta}(\theta) = -\frac{\alpha}{1-\alpha} < 0$$

So for each $\alpha \in (0, 1)$ and $\omega \in \Omega$, $f_\alpha(\omega)$ achieves its minimum at . Then $\{\theta \in \mathbb{R} : f_\alpha(\omega)(\theta) \leq m + 1\}$ is bounded

Since f_α is continuous, we have that

$$\inf_{\theta \in \mathbb{R}} f_\alpha(\theta) = \inf_{\theta \in \mathbb{Q}} f_\alpha(\theta)$$

which is measurable.

□

REFERENCES