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# Introduction to Category Theory

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# Notation

| | |
|---|---|
| $\mathcal{M}_+(X, \mathcal{A})$ | finite measures on $(X, \mathcal{A})$ |
| $v$ | velocity |

# Preface

# Chapter 1

# Basic Concepts

## 1.1 von Neumann–Bernays–Gödel Set Theory

**Definition 1.1.0.1.** Let $x$ be a class. Then $x$ is said to be a set iff there exists a class $A$ such that $x \in A$.

**Definition 1.1.0.2.** product of two classes

**Definition 1.1.0.3.** Let $A$, $B$ be classes and $R \subset A \times B$. elation from $A$ to $B$.

**Note 1.1.0.4.** We can define cartesion products, relations, and functions for classes just like for sets.

**Axiom 1.1.0.5. Axiom of Extensionality:**
Let $x$ and $y$ be classes. If for each set $a$, $a \in x$ iff $a \in y$, then $x = y$.

**Axiom 1.1.0.6. Axiom of Pairing:**
Let $a$, $b$ be sets. Then there exists a set $p$ such that for each for each set $x$, $x \in p$ iff $x = a$ or $x = b$.

**Exercise 1.1.0.7.** Let $a$, $b$ be sets. Then there exists a unique set $p$ such that for each for each set $x$, $x \in p$ iff $x = a$ or $x = b$.

*Proof.* By Axiom **??** implies that there exists a set $p$ such that for each for each set $x$, $x \in p$ iff $x = a$ or $x = b$. Let $q$ be a set. Suppose that for each for each set $x$, $x \in q$ iff $x = a$ or $x = b$. Then □

**Definition 1.1.0.8.** Let $x$ and $y$ be sets. We define $(x, y) = \{\}$, denoted

**Axiom 1.1.0.9. Axiom of Replacement:**
Let $A$, $B$ be classes and $f : A \to B$. If $A$ is a set, then $f(A)$ is a set.

**Axiom 1.1.0.10. Schema of Specification:**
Let $\phi$ a propositional function on sets. Then there exists a class $A$ such that for each set $x$, $x \in A$ iff $\phi(x)$.

**Exercise 1.1.0.11.** There exists a class $A$ such that for each class $x$, $x \in A$ iff $x$ is a set.

*Proof.* Define $\phi$ by

$$\phi(x) : x = x$$

Axiom 1.1.0.7 implies that there exists a class $A$ such that for each set $x$, $x \in A$ iff $x = x$. Let $x$ be a class. If $x \in A$, then by definition, $x$ is a set.
Conversely, if $x$ is a set, then by construction, $x \in A$. □

**Exercise 1.1.0.12.** There exists a class $A$ such that for each class $G$ and $* : G \times G \to G$, $(G, *) \in A$ iff $(G, *)$ is a group.

*Proof.* Define $\phi_1$, $\phi_2$ and $\phi_3$ by

- $\phi_1(G, *) : * : G \times G \to G$ is associative

- $\phi_2(G, *)$ : there exists $e \in G$ such that for each $g \in G$, $e * g = g * e = g$
- $\phi_3(G, *)$ : for each $g \in G$ there exists $h \in G$ such that $g * h = h * g = e$

Define $\phi$ by

$$\phi(G, *) : \phi_1(G, *) \text{ and } \phi_2(G, *) \text{ and } \phi_3(G, *)$$

Then there exists a class $A$ such that for each set $G$ and $* : G \times G \to G$, $(G, *) \in A$ iff $\phi(G, *)$ $(G, *)$ "is a group". Therefore, for each group $(G, *)$, $(G, *) \in A$. **FINISH!!!** $\square$

### 1.1.1 TO DO

1. cover existence of subclasses, products of classes to be able to define class relations and subsequently class functions

2.

## 1.2 Categories

### 1.2.1 Introduction

**Definition 1.2.1.1.** Let $\mathcal{C}_0, \mathcal{C}_1$ be classes and $\mathrm{dom}, \mathrm{cod} : \mathcal{C}_1 \to \mathcal{C}_0$ class functions. Set $\mathcal{C} = (C_0, C_1, \mathrm{dom}, \mathrm{cod})$. Then $\mathcal{C}$ is said to be a **category** if

1. (composition): for each $f, g \in C_1$, if $\mathrm{cod}(f) = \mathrm{dom}(g)$, then there exists $g \circ f \in C_1$ such that $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$
2. (associativity): for each $f, g, h \in C_1$, if $\mathrm{cod}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(g) = \mathrm{dom}(h)$, then
$$(h \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} f = h \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} f)$$
3. (identity): for each $X \in \mathcal{C}_0$, there exists $\mathrm{id}_X^{\mathcal{C}} \in C_1$ such that $\mathrm{dom}(\mathrm{id}_X^{\mathcal{C}}) = \mathrm{cod}(\mathrm{id}_X^{\mathcal{C}}) = X$ and for each $f, g \in C_1$, if $\mathrm{dom}(f) = X$ and $\mathrm{cod}(g) = X$, then
$$f \circ_{\mathcal{C}} \mathrm{id}_X^{\mathcal{C}} = f \text{ and } \mathrm{id}_X^{\mathcal{C}} \circ_{\mathcal{C}} g = g$$

We define the

- **objects of** $\mathcal{C}$, denoted $\mathrm{Obj}(\mathcal{C})$, by $\mathrm{Obj}(\mathcal{C}) = C_0$
- **morphisms of** $\mathcal{C}$, denoted $\mathrm{Hom}_{\mathcal{C}}$, by $\mathrm{Hom}_{\mathcal{C}} = C_1$

For $X, Y \in \mathrm{Obj}(\mathcal{C})$, we define the **morphisms of** $\mathcal{C}$ **from** $X$ **to** $Y$, denoted $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, by $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{f \in \mathrm{Hom}(\mathcal{C}) : \mathrm{dom}(f) = X \text{ and } \mathrm{cod}(f) = Y\}$.

**Note 1.2.1.2.** When the context is clear, we write $g \circ f$ and $\mathrm{id}_X$ in place of $g \circ_{\mathcal{C}} f$ and $\mathrm{id}_X^{\mathcal{C}}$ respectively.

**Definition 1.2.1.3.** Let $\mathcal{C}$ be a category. We define $\mathrm{Hom}_{\mathcal{C}}^{(2)} = \{(g, f) \in \mathrm{Hom}_{\mathcal{C}} \times \mathrm{Hom}_{\mathcal{C}} : \mathrm{cod}(f) = \mathrm{dom}(g)\}$.

**Exercise 1.2.1.4.** Let $\mathcal{C}$ be a category. Then

1. $\circ \in \mathcal{R}$
2. $\circ : \mathrm{Hom}_{\mathcal{C}}^{(2)} \to \mathrm{Hom}_{\mathcal{C}}$

*Proof.* Let $(g, f) \in \mathrm{Hom}_{\mathcal{C}}^{(2)}$. Since $\mathcal{C}$ is a category, there exists $g$

$\square$

**Note 1.2.1.5.** We typically define a category $\mathcal{C}$ by specifying

- $\mathrm{Obj}(\mathcal{C})$
- for $X, Y \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$
- for $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, the composite morphism $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$.

and then show

- well-definedness of composition
- associativity of composition
- existence of identities

**Definition 1.2.1.6.** We define the **empty category**, denoted $\mathbf{0}$, by

- $\mathrm{Obj}(\mathbf{0}) = \varnothing$

- $\mathrm{Hom}_{\mathbf{0}} = \varnothing$

**Exercise 1.2.1.7.** We have that **0** is a category.

*Proof.* Vacuously true. □

**Definition 1.2.1.8.** We define the **trivial category**, denoted **1**, by

- $\mathrm{Obj}(\mathbf{1}) = \{*\}$
- $\mathrm{Hom}_{\mathbf{1}} = \{\mathrm{id}_*\}$

**Exercise 1.2.1.9.** We have that **1** is a category.

*Proof.* Clear. □

**Definition 1.2.1.10.** We define **Set** by

- $\mathrm{Obj}(\mathbf{Set}) = \{A : A \text{ is a set}\}$
- for each $A, B \in \mathrm{Obj}(\mathbf{Set})$, $\mathrm{Hom}_{\mathbf{Set}}(A, B) = \{f : f : A \to B\}$
- for $A, B, C \in \mathbf{Set}$, $f \in \mathrm{Hom}_{\mathbf{Set}}(A, B)$ and $g \in \mathrm{Hom}_{\mathbf{Set}}(B, C)$, $g \circ_{\mathbf{Set}} f = g \circ f$.

**Exercise 1.2.1.11.** We have that **Set** is a category.

*Proof.*

- **well-definedness of composition:**
- **associativity of composition:**
- **existence of identities:**

**FINISH!!!** □

**Definition 1.2.1.12.** Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is said to be

- **small** if $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}$ are sets
- **locally small** if for each $A, B \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set

**Exercise 1.2.1.13.** Let $\mathcal{C}$ be a category. If $\mathcal{C}$ is small, then $\mathcal{C}$ is a set.

*Proof.* Suppose that $\mathcal{C}$ is small. Then $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}$ are sets. Then $\mathcal{P}(\mathrm{Obj}(\mathcal{C}))$, $\mathcal{P}(\mathrm{Hom}_{\mathcal{C}})$ and $\mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}}$ are sets. Hence $\mathrm{Obj}(\mathcal{C}) \times \mathrm{Hom}_{\mathcal{C}} \times \mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}} \times \mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}}$ is a set. By definition, $\mathcal{C} = (\mathrm{Obj}(\mathcal{C}), \mathrm{Hom}_{\mathcal{C}}, \mathrm{dom}, \mathrm{cod}) \in \mathrm{Obj}(\mathcal{C}) \times \mathrm{Hom}_{\mathcal{C}} \times \mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}} \times \mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}}$. By definition, $\mathcal{C}$ is a set. □

**Exercise 1.2.1.14.** There exists a class $A$ such that $\mathcal{C} \in A$ iff $\mathcal{C}$ is a small category.

*Proof.* Exercise 1.2.1.13 implies that for each category $\mathcal{C}$, $\mathcal{C}$ is small implies that $\mathcal{C}$ is a set. Define $\phi$ by

$$\phi(\mathcal{C}) : \mathcal{C} \text{ is a small category}$$

Then Axiom 1.1.0.7 implies that there exists a class $A$ such that $\mathcal{C} \in A$ iff $\mathcal{C}$ is a small category. □

### 1.2.2 Opposite Category

**Definition 1.2.2.1.** Let $\mathcal{C}$ be a category, we define the dual of $\mathcal{C}$ or the **opposite of** $\mathcal{C}$, denoted $\mathcal{C}^{\text{op}}$, by

- $\text{Obj}(\mathcal{C}^{\text{op}}) = \text{Obj}(\mathcal{C})$
- for $X, Y \in \text{Obj}(\mathcal{C}^{\text{op}})$, $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$
- for $X, Y, Z \in \text{Obj}(\mathcal{C}^{\text{op}})$ and $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y), g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$, $g \circ_{\mathcal{C}^{\text{op}}} f = f \circ_{\mathcal{C}} g$

**Exercise 1.2.2.2.** Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\text{op}}$ is a category.

*Proof.*

- for $W, X, Y, Z \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(W, X)$ and $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$ and $h \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$. Then

$$\begin{aligned}(h \circ_{\mathcal{C}^{\text{op}}} g) \circ_{\mathcal{C}^{\text{op}}} f &= f \circ_{\mathcal{C}} (h \circ_{\mathcal{C}^{\text{op}}} g)\\ &= f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h)\\ &= (f \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} h\\ &= h \circ_{\mathcal{C}^{\text{op}}} (f \circ_{\mathcal{C}} g)\\ &= h \circ_{\mathcal{C}^{\text{op}}} (g \circ_{\mathcal{C}^{\text{op}}} f)\end{aligned}$$

  So composition is associative.

- Let $X \in \text{Obj}(\mathcal{C})$ and $f, g \in \text{Hom}_{\mathcal{C}^{\text{op}}}$. Suppose that $\text{dom}(f) = X$ and $\text{cod}(g) = X$ Then

$$\begin{aligned}f \circ_{\mathcal{C}^{\text{op}}} \text{id}_X &= \text{id}_X \circ_{\mathcal{C}} f\\ &= f\end{aligned}$$

  and

$$\begin{aligned}\text{id}_X \circ_{\mathcal{C}^{\text{op}}} g &= g \circ_{\mathcal{C}} \text{id}_X\\ &= g\end{aligned}$$

  So $(\text{id}_X)_{\mathcal{C}^{\text{op}}} = (\text{id}_X)_{\mathcal{C}}$.

□

### 1.2.3 Slice Category

**Definition 1.2.3.1.** Let $\mathcal{C}$ be a category and $X \in \text{Obj}(\mathcal{C})$. We define the **slice category of** $\mathcal{C}$ **over** $X$, denoted $\mathcal{C}/X$, by

- $\text{Obj}(\mathcal{C}/X) = \{f \in \text{Hom}_{\mathcal{C}} : \text{cod}(f) = X\}$
- for $f, g \in \text{Obj}(\mathcal{C}/X)$,

$$\text{Hom}_{\mathcal{C}/X}(f, g) = \{\alpha \in \text{Hom}_{\mathcal{C}} : \text{dom}(\alpha) = \text{dom}(f),\ \text{cod}(\alpha) = \text{dom}(g) \text{ and } f = g \circ \alpha\}$$

  i.e. for $f \in \text{Hom}_{\mathcal{C}}(A, X)$ and $g \in \text{Hom}_{\mathcal{C}}(B, X)$, $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$ iff the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

- for $f, g, h \in \text{Obj}(\mathcal{C}/X)$, $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \text{Hom}_{\mathcal{C}/X}(g, h)$,

$$\beta \circ_{\mathcal{C}/X} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

**Exercise 1.2.3.2.** Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. Then $\mathcal{C}/X$ is a category.

*Proof.*

- $f, g, h \in \mathrm{Obj}(\mathcal{C}/X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f,g)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}/X}(g,h)$. Then $f = g \circ_{\mathcal{C}} \alpha$ and $g = h \circ_{\mathcal{C}} \beta$, i.e. the following diagrams commute:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & \mathrm{dom}(g) \arrow[dl, "g"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{dom}(g) \arrow[rr, "\beta"] \arrow[dr, "g"'] & & \mathrm{dom}(h) \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

  Therefore, we have that

$$\begin{aligned} f &= g \circ_{\mathcal{C}} \alpha \\ &= (h \circ_{\mathcal{C}} \beta) \circ_{\mathcal{C}} \alpha \\ &= h \circ_{\mathcal{C}} (\beta \circ_{\mathcal{C}} \alpha) \end{aligned}$$

  i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\beta \circ_{\mathcal{C}} \alpha"] \arrow[dr, "f"'] & & \mathrm{dom}(h) \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

  which implies that

$$\begin{aligned} \beta \circ_{\mathcal{C}/X} \alpha &= \beta \circ_{\mathcal{C}} \alpha \\ &\in \mathrm{Hom}_{\mathcal{C}/X}(f,h) \end{aligned}$$

  and composition is well defined.

- Associativity of $\circ_{\mathcal{C}/X}$ follows from associativity of $\circ_{\mathcal{C}}$.

- Let $f \in \mathrm{Obj}(\mathcal{C}/X)$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}/X}$. Since $f \circ \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} = f$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{dom}_{\mathcal{C}}(f) \arrow[rr, "\mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)}"] \arrow[dr, "f"'] & & \mathrm{dom}_{\mathcal{C}}(f) \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

  we have that $\mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} \in \mathrm{Hom}_{\mathcal{C}/X}(f,f)$. Suppose that $\mathrm{dom}_{\mathcal{C}/X}(\alpha) = f$ and $\mathrm{cod}_{\mathcal{C}/X}(\beta) = f$. Then

$$\begin{aligned} \alpha \circ_{\mathcal{C}/X} \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} &= \alpha \circ_{C} \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} \\ &= \alpha \end{aligned}$$

  and

$$\begin{aligned} \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} \circ_{\mathcal{C}/X} \beta &= \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} \circ_{\mathcal{C}} \beta \\ &= \beta \end{aligned}$$

  So $\mathrm{id}_f = \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)}$.

$\square$

### 1.2.4 Subcategories

**Definition 1.2.4.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $\mathcal{D}$ is said to be a **subcategory of** $\mathcal{C}$, denoted $\mathcal{D} \subset \mathcal{C}$, if

1. $\mathrm{Obj}(\mathcal{D}) \subset \mathrm{Obj}(\mathcal{C})$
2. for each $A, B \in \mathrm{Obj}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{D}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B)$
3. for each $A, B, C \in \mathrm{Obj}(\mathcal{D})$, $d \in \mathrm{Hom}_{\mathcal{D}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(B, C)$, $g \circ_{\mathcal{D}} f = g \circ_{\mathcal{C}} f$
4. for each $A \in \mathrm{Obj}(\mathcal{D})$, $\mathrm{id}_A$

### 1.2.5 Product Categories

**Definition 1.2.5.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We define the **product category of** $\mathcal{C}$ **and** $\mathcal{D}$, denoted $\mathcal{C} \times \mathcal{D}$ by

- $\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) = \{(A, B) : A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D})\}$
- for each $(A, A'), (B, B') \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B')) = \{(f, g) : f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \text{ and } g \in \mathrm{Hom}_{\mathcal{C}}(A', B')\}$
- for each $(A, A'), (B, B'), (C, C') \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $(f, f') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$ and $(g, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'))$,
$$(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') = (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f')$$

**Exercise 1.2.5.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $\mathcal{C} \times \mathcal{D}$ is a category.

*Proof.*

- **well-definedness of composition:**
Let $(A, A'), (B, B'), (C, C') \in \mathrm{Obj}(\mathcal{C}\times\mathcal{D})$, $(f, f') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$ and $(g, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'))$. Then $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $f' \in \mathrm{Hom}_{\mathcal{D}}(A', B')$, and $g' \in \mathrm{Hom}_{\mathcal{D}}(B', C')$. Hence $g \circ_{\mathcal{C}} f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ and $g' \circ_{\mathcal{D}} f' \in \mathrm{Hom}_{\mathcal{D}}(A', C')$. Thus
$$\begin{aligned}(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') &= (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f') \\ &\in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (C, C'))\end{aligned}$$
Thus, composition is well defined.

- **associativity of composition:**
Let $(A, A'), (B, B'), (C, C'), (D, D') \in \mathrm{Obj}(\mathcal{C}\times\mathcal{D})$, $(f, f') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$, $(g, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'$ and $(h, h') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C, C'), (D, D'))$. Then
$$\begin{aligned}[(h, h') \circ_{\mathcal{C}\times\mathcal{D}} (g, g')] \circ_{\mathcal{C}\times\mathcal{D}} (f, f') &= (h \circ_{\mathcal{C}} g, h' \circ_{\mathcal{D}} g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') \\ &= ((h \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} f, (h' \circ_{\mathcal{D}} g') \circ_{\mathcal{D}} f') \\ &= (h \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} f), h' \circ_{\mathcal{D}} (g' \circ_{\mathcal{D}} f')) \\ &= (h, h') \circ_{\mathcal{C}\times\mathcal{D}} (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f') \\ &= (h, h') \circ_{\mathcal{C}\times\mathcal{D}} [(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f')]\end{aligned}$$
Thus composition is associative.

- **existence of identities:**
  Let $(A, B) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $(f, f'), (g, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}$. Suppose that $\mathrm{dom}_{\mathcal{C}\times\mathcal{D}}(f, f') = (A, B)$ and $\mathrm{cod}_{\mathcal{C}\times\mathcal{D}}(g, g') = (A, B)$. Then $\mathrm{dom}_{\mathcal{C}}(f) = A$, $\mathrm{dom}_{\mathcal{D}}(f') = B$, $\mathrm{cod}_{\mathcal{C}}(g) = A$ and $\mathrm{cod}_{\mathcal{D}}(g') = B$. Hence

$$\begin{aligned}(f, f') \circ_{\mathcal{C}\times\mathcal{D}} (\mathrm{id}_A, \mathrm{id}_B) &= (f \circ_{\mathcal{C}} \mathrm{id}_A, f' \circ_{\mathcal{D}} \mathrm{id}_B)\\ &= (f, f)\end{aligned}$$

  and

$$\begin{aligned}(\mathrm{id}_A, \mathrm{id}_B) \circ_{\mathcal{C}\times\mathcal{D}} (g, g') &= (\mathrm{id}_A \circ_{\mathcal{C}} g, \mathrm{id}_B \circ g')\\ &= (g, g')\end{aligned}$$

  Therefore $(\mathrm{id}_{(A,B)})_{\mathcal{C}\times\mathcal{D}} = (\mathrm{id}_A, \mathrm{id}_B)$.

$\square$

# 1.3 Functors

## 1.3.1 Introduction

**Definition 1.3.1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, $F_1 : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}$ class functions. Set $F = (F_0, F_1)$. Then $F$ is said to be a functor from $\mathcal{C}$ to $\mathcal{D}$, denoted $F : \mathcal{C} \to \mathcal{D}$, if

1. for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F_1(f) \in \mathrm{Hom}_{\mathcal{D}}(F_0(A), F_0(B))$
2. for each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$
3. for each $A \in \mathrm{Obj}(\mathcal{C})$, $F_1(\mathrm{id}_A) = \mathrm{id}_{F_0(A)}$

**Note 1.3.1.2.** For $A \in \mathrm{Obj}(C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}$, we typically write $F(A)$ and $F(f)$ instead of $F_0(A)$ and $F_1(f)$ respectively.

**Definition 1.3.1.3.** Let $\mathcal{C}$ be a category. We define the **empty functor** from $\mathbf{0}$ to $\mathcal{C}$, denoted $E_{\mathcal{C}} : \mathbf{0} \to \mathcal{C}$ by $(E_{\mathcal{C}})_0 = (E_{\mathcal{C}})_1 = \varnothing$.

**Exercise 1.3.1.4.** Let $\mathcal{C}$ be a category. Then $E_{\mathcal{C}} : \mathbf{0} \to \mathcal{C}$ is a functor.

*Proof.* Since $\mathrm{Obj}(\mathbf{0}) = \varnothing$ and $\mathrm{Hom}_{\mathbf{0}} = \varnothing$, this is vacuously true. □

**Definition 1.3.1.5.** Let $\mathcal{C}, \mathcal{D}$ be categories and $X \in \mathrm{Obj}(\mathcal{D})$. We define the **constant functor** from $\mathcal{C}$ onto $X$, denoted $\Delta_X^{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ by

- $\Delta_X^{\mathcal{C}}(A) = X$
- $\Delta_X^{\mathcal{C}}(f) = \mathrm{id}_X$

**Exercise 1.3.1.6.** Let $\mathcal{C}, \mathcal{D}$ be categories and $X \in \mathrm{Obj}(\mathcal{D})$. Then $\Delta_X^{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ is a functor.

*Proof.*

1. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned}\Delta_X^{\mathcal{C}}(f) &= \mathrm{id}_X \\ &\in \mathrm{Hom}_{\mathcal{D}}(X, X) \\ &= \mathrm{Hom}_{\mathcal{D}}(\Delta_X^{\mathcal{C}}(A), \Delta_X^{\mathcal{C}}(B))\end{aligned}$$

2. Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

$$\begin{aligned}\Delta_X^{\mathcal{C}}(g \circ f) &= \mathrm{id}_X \\ &= \mathrm{id}_X \circ \mathrm{id}_X \\ &= \Delta_X^{\mathcal{C}}(g) \circ \Delta_X^{\mathcal{C}}(f)\end{aligned}$$

3. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}\Delta_X^{\mathcal{C}}(\mathrm{id}_A) &= \mathrm{id}_X \\ &= \mathrm{id}_{\Delta_X^{\mathcal{C}}(A)}\end{aligned}$$

So $\Delta_X^{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ is a functor. □

### 1.3.2 Category of Small Categories

**Definition 1.3.2.1.** Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ functors. We define the **composition of $G$ with $F$**, denoted $G \circ F : \mathcal{C} \to \mathcal{E}$, by

- $G \circ F(A) = G(F(A))$
- $G \circ F(f) = G(F(f))$

**Exercise 1.3.2.2.** Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ functors. Then $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor.

*Proof.*

1. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$, we have that $G(F(f)) \in \mathrm{Hom}_{\mathcal{E}}(G(F(A)), G(F(B)))$. Then

$$\begin{aligned} G \circ F(f) &= G(F(f)) \\ &\in \mathrm{Hom}_{\mathcal{E}}(G(F(A)), G(F(B))) \\ &= \mathrm{Hom}_{\mathcal{E}}(G \circ F(A), G \circ F(B)) \end{aligned}$$

2. Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

$$\begin{aligned} G \circ F(g \circ f) &= G(F(g \circ f)) \\ &= G(F(g) \circ F(f)) \\ &= G(F(g)) \circ G(F(f)) \\ &= G \circ F(g) \circ G \circ F(f) \end{aligned}$$

3. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned} G \circ F(\mathrm{id}_A) &= G(F(\mathrm{id}_A)) \\ &= G(\mathrm{id}_{F(A)}) \\ &= \mathrm{id}_{G(F(A))} \\ &= \mathrm{id}_{G \circ F(A)} \end{aligned}$$

So $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor. □

**Exercise 1.3.2.3.** Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, $H : \mathcal{E} \to \mathcal{F}$ functors. Then $(H \circ G) \circ F = H \circ (G \circ F)$.

*Proof.* Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

- $$\begin{aligned} (H \circ G) \circ F(A) &= H \circ G(F(A)) \\ &= H(G(F(A))) \\ &= H(G \circ F(A)) \\ &= H \circ (G \circ F)(A) \end{aligned}$$

- $$\begin{aligned} (H \circ G) \circ F(f) &= H \circ G(F(f)) \\ &= H(G(F(f))) \\ &= H(G \circ F(f)) \\ &= H \circ (G \circ F)(f) \end{aligned}$$

Hence $(H \circ G) \circ F = H \circ (G \circ F)$. □

**Definition 1.3.2.4.** Let $\mathcal{C}$ be a category. We define the **identity functor from $\mathcal{C}$ to $\mathcal{C}$**, denoted $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, by

- $\mathrm{id}_{\mathcal{C}}(A) = A$, $(A \in \mathrm{Obj}(\mathcal{C}))$
- $\mathrm{id}_{\mathcal{C}}(f) = f$, $(f \in \mathrm{Hom}_{\mathcal{C}})$

**Exercise 1.3.2.5.** Let $\mathcal{C}$ be a category. Then $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a functor.

*Proof.*

1. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then
$$\begin{aligned} \mathrm{id}_{\mathcal{C}}(f) &= f \\ &\in \mathrm{Hom}_{\mathcal{C}}(A, B) \\ &= \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B)) \end{aligned}$$

2. Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then
$$\begin{aligned} \mathrm{id}_{\mathcal{C}}(g \circ f) &= g \circ f \\ &= \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f) \end{aligned}$$

3. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then
$$\begin{aligned} \mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) &= \mathrm{id}_A \\ &= \mathrm{id}_{\mathrm{id}_{\mathcal{C}}(A)} \end{aligned}$$

□

**Exercise 1.3.2.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. Then

1. $\mathrm{id}_{\mathcal{D}} \circ F = F$
2. $F \circ \mathrm{id}_{\mathcal{C}} = F$

*Proof.*

1. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then
$$\begin{aligned} \mathrm{id}_{\mathcal{D}} \circ F(A) &= \mathrm{id}_{\mathcal{D}}(F(A)) \\ &= F(A) \end{aligned}$$
and
$$\begin{aligned} \mathrm{id}_{\mathcal{D}} \circ F(f) &= \mathrm{id}_{\mathcal{D}}(F(f)) \\ &= F(f) \end{aligned}$$
Since $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ are arbitrary, $\mathrm{id}_{\mathcal{D}} \circ F = F$.

2. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then
$$\begin{aligned} F \circ \mathrm{id}_{\mathcal{C}}(A) &= F(\mathrm{id}_{\mathcal{C}}(A)) \\ &= F(A) \end{aligned}$$
and
$$\begin{aligned} F \circ \mathrm{id}_{\mathcal{C}}(f) &= F(\mathrm{id}_{\mathcal{C}}(f)) \\ &= F(f) \end{aligned}$$
Since $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ are arbitrary, $F \circ \mathrm{id}_{\mathcal{C}} = F$.

□

**Exercise 1.3.2.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. If $\mathcal{C}$ is small, then $F$ is a set.

*Proof.* Suppose that $\mathcal{C}$ is small. Then $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}$ are sets. By definition, there exist $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $F_1 : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}$ such that $F = (F_0, F_1)$. Axiom 1.1.0.6 implies that $F_0(\mathrm{Obj}(\mathcal{C}))$ and $F_1(\mathrm{Hom}_{\mathcal{C}})$ are sets. Therefore, $\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C))$ and $\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}})$ are sets. Hence $\mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C)))$ and $\mathcal{P}(\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}}))$ are sets. Since $F_0 \subset \mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C))$ and $F_1 \subset \mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}})$, we have that $F_0 \in \mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C)))$ and $F_1 \in \mathcal{P}(\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}}))$. Hence $F_0$ and $F_1$ are sets. Thus $F = (F_0, F_1)$ is a set. □

**Exercise 1.3.2.8.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Then there exists a class $A$ such that for each class $F$, $F \in A$ iff $F : \mathcal{C} \to \mathcal{D}$.

*Proof.* Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Define $\phi$ by

$$\phi(F) : F : \mathcal{C} \to \mathcal{D}$$

Then there exists a class $A$ such that for each set $F$, $F \in A$ iff $\phi(F)$. Let $F$ be a class. Suppose that $F \in A$. By Definition 1.1.0.1, $F$ is a set. Since $F$ is a set and $F \in A$, we have that $\phi(F)$. Hence $F : \mathcal{C} \to \mathcal{D}$. Conversely, suppose that $F : \mathcal{C} \to \mathcal{D}$. Exercise 1.3.2.7 implies that $F$ is a set. Since $F$ is a set and $\phi(F)$ is true, we have that $F \in A$. □

**Definition 1.3.2.9.** We define **Cat** by

- $\mathrm{Obj}(\mathbf{Cat}) = \{\mathcal{C} : \mathcal{C} \text{ is a small category}\}$.
- for $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathbf{Cat})$,
$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) = \{F : F : \mathcal{C} \to \mathcal{D}\}$$
- for $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathbf{Cat})$, $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ and $G \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{D}, \mathcal{E})$,
$$G \circ_{\mathbf{Cat}} F = G \circ F$$

**Exercise 1.3.2.10.** We have that **Cat** is

1. a category
2. locally small

*Proof.*

1. Exercise 1.3.2.2 implies that composition is well defined. Exercise 1.3.2.3 implies that composition is associative. Exercise 1.3.2.5 and Exercise 1.3.2.6 imply the existence of identities.

2. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathbf{Cat})$ and $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$. Definition 1.2.1.12 implies that $\mathrm{Obj}(\mathcal{C})$, $\mathrm{Obj}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{C}}$ and $\mathrm{Hom}_{\mathcal{D}}$ are sets. Then $\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}$ and $\mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ are sets. Hence $\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})} \times \mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ is a set. Let $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$. Then there exist $F_0 \in \mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}$ and $F_1 \in \mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ such that $F = (F_0, F_1)$. Therefore $F \in \mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})} \times \mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$. Since $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ is arbitrary,
$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})} \times \mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$$
which implies that $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ is a set. Therefore, **Cat** is locally small.

□

### 1.3.3 Comma Categories

**Definition 1.3.3.1.** Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be a categories and $S : \mathcal{A} \to \mathcal{C}$, $T : \mathcal{B} \to \mathcal{C}$ functors. We define the **comma category of** $S$ **to** $T$, denoted $(S \downarrow T)$, by

- $\mathrm{Obj}(S \downarrow T) = \{(A, B, h) : A \in \mathrm{Obj}(\mathcal{A}), B \in \mathrm{Obj}(\mathcal{B}), \text{ and } h \in \mathrm{Hom}_{\mathcal{C}}(S(A), T(B))\}$
- For $(A_1, B_1, h_1), (A_2, B_2, h_2) \in \mathrm{Obj}(S \downarrow T)$,
$$\begin{aligned} &\mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2)) = \\ &\quad \{(\alpha, \beta) : \alpha \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2),\ \beta \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_2) \text{ and } T(\beta) \circ_{\mathcal{C}} h_1 = h_2 \circ_{\mathcal{C}} S(\alpha)\} \end{aligned}$$
i.e. for $(A_1, B_1, h_1), (A_2, B_2, h_2) \in \mathrm{Obj}(S \downarrow T)$, $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ and $\beta \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_2)$, $(\alpha, \beta) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$ iff the following diagram commutes:
$$\begin{tikzcd} S(A_1) \arrow[r, "S(\alpha)"] \arrow[d, "h_1"'] & S(A_2) \arrow[d, "h_2"] \\ T(B_1) \arrow[r, "T(\beta)"'] & T(B_2) \end{tikzcd}$$
- For
  - $(A_1, B_1, h_1), (A_2, B_2, h_2), (A_3, B_3, h_3) \in \mathrm{Obj}(S \downarrow T)$
  - $(\alpha_{12}, \beta_{12}) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$
  - $(\alpha_{23}, \beta_{23}) \in \mathrm{Hom}_{(S\downarrow T)}((A_2, B_2, h_2), (A_3, B_3, h_3))$

  we define
$$(\alpha_{23}, \beta_{23}) \circ_{(S\downarrow T)} (\alpha_{12}, \beta_{12}) = (\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}, \beta_{23} \circ_{\mathcal{B}} \beta_{12})$$

**Exercise 1.3.3.2.** Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be a categories and $S : \mathcal{A} \to \mathcal{C}$, $T : \mathcal{B} \to \mathcal{C}$ functors. Then $(S \downarrow T)$ is a category.

*Proof.*

- **well-definedness of composition:**
Let
  - $(A_1, B_1, h_1), (A_2, B_2, h_2), (A_3, B_3, h_3) \in \mathrm{Obj}(S \downarrow T)$
  - $(\alpha_{12}, \beta_{12}) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$
  - $(\alpha_{23}, \beta_{23}) \in \mathrm{Hom}_{(S\downarrow T)}((A_2, B_2, h_2), (A_3, B_3, h_3))$

  By definition, $\alpha_{12} \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$, $\alpha_{23} \in \mathrm{Hom}_{\mathcal{A}}(A_2, A_3)$, $\beta_{12} \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_2)$, $\beta_{23} \in \mathrm{Hom}_{\mathcal{B}}(B_2, B_3)$, $T(\beta_{12}) \circ_{\mathcal{C}} h_1 = h_2 \circ S(\alpha_{12})$ and $T(\beta_{23}) \circ_{\mathcal{C}} h_2 = h_3 \circ_{\mathcal{C}} S(\alpha_{23})$,
i.e. the following diagram commutes:
$$\begin{tikzcd} S(A_1) \arrow[r, "S(\alpha_{12})"] \arrow[d, "h_1"'] & S(A_2) \arrow[r, "S(\alpha_{23})"] \arrow[d, "h_2"] & S(A_3) \arrow[d, "h_3"] \\ T(B_1) \arrow[r, "T(\beta_{12})"'] & T(B_2) \arrow[r, "T(\beta_{23})"'] & T(B_3) \end{tikzcd}$$
Then $\alpha_{23} \circ_{\mathcal{A}} \alpha_{12} \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_3)$, $\beta_{23} \circ_{\mathcal{B}} \beta_{12} \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_3)$ and
$$\begin{aligned} T(\beta_{23} \circ_{\mathcal{B}} \beta_{12}) \circ_{\mathcal{C}} h_1 &= (T(\beta_{23}) \circ_{\mathcal{C}} T(\beta_{12})) \circ_{\mathcal{C}} h_1 \\ &= T(\beta_{23}) \circ_{\mathcal{C}} (T(\beta_{12}) \circ_{\mathcal{C}} h_1) \\ &= T(\beta_{23}) \circ_{\mathcal{C}} (h_2 \circ_{\mathcal{C}} S(\alpha_{12})) \\ &= (T(\beta_{23}) \circ_{\mathcal{C}} h_2) \circ_{\mathcal{C}} S(\alpha_{12}) \\ &= (h_3 \circ_{\mathcal{C}} S(\alpha_{23})) \circ_{\mathcal{C}} S(\alpha_{12}) \\ &= h_3 \circ_{\mathcal{C}} (S(\alpha_{23}) \circ_{\mathcal{C}} S(\alpha_{12})) \\ &= h_3 \circ_{\mathcal{C}} S(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} S(A_1) & \xrightarrow{S(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12})} & S(A_3) \\ {\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle h_3} \\ T(B_1) & \xrightarrow[T(\beta_{23} \circ_{\mathcal{B}} \beta_{12})]{} & T(B_3) \end{array}$$

Hence $(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}, \beta_{23} \circ_{\mathcal{B}} \beta_{12}) \in \mathrm{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_3, B_3, h_3))$ and composition is well defined.

- **associativity of composition:**
  Let
  - $(A_1, B_1, h_1), (A_2, B_2, h_2), (A_3, B_3, h_3), (A_4, B_4, h_4) \in \mathrm{Obj}(S \downarrow T)$
  - $(\alpha_{12}, \beta_{12}) \in \mathrm{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$
  - $(\alpha_{23}, \beta_{23}) \in \mathrm{Hom}_{(S \downarrow T)}((A_2, B_2, h_2), (A_3, B_3, h_3))$
  - $(\alpha_{34}, \beta_{34}) \in \mathrm{Hom}_{(S \downarrow T)}((A_3, B_3, h_3), (A_4, B_4, h_4))$

  Then

$$\begin{aligned} [(\alpha_{34}, \beta_{34}) \circ_{(S\downarrow T)} (\alpha_{23}, \beta_{23})] \circ_{(S\downarrow T)} (\alpha_{12}, \beta_{12}) &= (\alpha_{34} \circ_{\mathcal{A}} \alpha_{23}, \beta_{34} \circ_{\mathcal{B}} \beta_{23}) \circ_{(S\downarrow T)} (\alpha_{12}, \beta_{12}) \\ &= ([\alpha_{34} \circ_{\mathcal{A}} \alpha_{23}] \circ_{\mathcal{A}} \alpha_{12}, [\beta_{34} \circ_{\mathcal{B}} \beta_{23}] \circ_{\mathcal{B}} \beta_{12}) \\ &= (\alpha_{34} \circ_{\mathcal{A}} [\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}], \beta_{34} \circ_{\mathcal{B}} [\beta_{23} \circ_{\mathcal{B}} \beta_{12}]) \\ &= (\alpha_{34}, \beta_{34}) \circ_{(S\downarrow T)} (\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}, \beta_{23} \circ_{\mathcal{B}} \beta_{12}) \\ &= (\alpha_{34}, \beta_{34}) \circ_{(S\downarrow T)} [(\alpha_{23}, \beta_{23}) \circ_{(S\downarrow T)} (\alpha_{12}, \beta_{12})] \end{aligned}$$

  So composition is associative.

- **existence of identities:**
  Let
  - $(A_1, B_1, h_1), (A_2, B_2, h_2), \in \mathrm{Obj}(S \downarrow T)$
  - $(\alpha, \beta) \in \mathrm{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$

  By definition,
  - $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$, $\beta \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_2)$
  - $h_1 \in \mathrm{Hom}_{\mathcal{C}}(S(A_1), T(B_1))$, $h_2 \in \mathrm{Hom}_{\mathcal{C}}(S(A_2), T(B_2))$
  - $T(\beta) \circ h_1 = h_2 \circ S(\alpha)$

  Since $\mathrm{id}_{A_1} \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_1)$, $\mathrm{id}_{B_1} \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_1)$, and

$$\begin{aligned} T(\mathrm{id}_{B_1}) \circ_{\mathcal{C}} h_1 &= \mathrm{id}_{T(B_1)} \circ_{\mathcal{C}} h_1 \\ &= h_1 \\ &= h_1 \circ_{\mathcal{C}} \mathrm{id}_{S(A_1)} \\ &= h_1 \circ_{\mathcal{C}} S(\mathrm{id}_{A_1}) \end{aligned}$$

  i.e. the following diagram commutes:

$$\begin{array}{ccc} S(A_1) & \xrightarrow{S(\mathrm{id}_{A_1})} & S(A_1) \\ {\scriptstyle h_1}\big\downarrow & & \big\downarrow{\scriptstyle h_1} \\ T(B_1) & \xrightarrow[T(\mathrm{id}_{B_1})]{} & T(B_1) \end{array}$$

we have that $(\mathrm{id}_{A_1}, \mathrm{id}_{B_1}) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_1, B_1, h_1))$. Similarly $(\mathrm{id}_{A_2}, \mathrm{id}_{B_2}) \in \mathrm{Hom}_{(S\downarrow T)}((A_2, B_2, h_2), (A_2, B$
Therefore

$$\begin{aligned}(\alpha, \beta) \circ_{(S\downarrow T)} (\mathrm{id}_{A_1}, \mathrm{id}_{B_1}) &= (\alpha \circ_{\mathcal{A}} \mathrm{id}_{A_1}, \beta \circ_{\mathcal{B}} \mathrm{id}_{B_1}) \\ &= (\alpha, \beta)\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id}_{A_2}, \mathrm{id}_{B_2}) \circ_{(S\downarrow T)} (\alpha, \beta) &= (\mathrm{id}_{A_2} \circ_{\mathcal{A}} \alpha, \mathrm{id}_{B_2} \circ_{\mathcal{B}} \beta) \\ &= (\alpha, \beta)\end{aligned}$$

Since $(A_1, B_1, h_1), (A_2, B_2, h_2), \in \mathrm{Obj}(S \downarrow T)$ and
$(\alpha, \beta) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$ are arbitrary, we have that for each $(A, B, h) \in \mathrm{Obj}(S \downarrow T)$, $\mathrm{id}_{(A,B,h)} = (\mathrm{id}_A, \mathrm{id}_B)$.

□

**Definition 1.3.3.3.** Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. We define the **comma category from $X$ to $F$**, denoted $(X \downarrow F)$, by $(X \downarrow F) = (\Delta^{\mathbf{1}}_X \downarrow F)$.
We may make the following identification:

- $\mathrm{Obj}(X \downarrow F) = \{(A, f) : A \in \mathrm{Obj}(\mathcal{C}) \text{ and } f \in \mathrm{Hom}_{\mathcal{D}}(X, F(A))\}$
- For $(A_1, f_1), (A_2, f_2) \in \mathrm{Obj}(X \downarrow F)$,

$$\mathrm{Hom}_{(X\downarrow F)}((A_1, f_1), (A_2, f_2)) = \{\alpha \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2) \text{ and } F(\alpha) \circ f_1 = f_2\}$$

  i.e. for $(A_1, f_1), (A_2, f_2) \in \mathrm{Obj}(X \downarrow F)$ and $\alpha \in \mathrm{Hom}_{A_1,A_2}$, $\alpha \in \mathrm{Hom}_{(X\downarrow F)}((A_1, f_1), (A_2, f_2))$ iff the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ F(A_1) \arrow[rr, "F(\alpha)"'] & & F(A_2) \end{tikzcd}$$

- For
  - $(A_1, f_1), (A_2, f_2), (A_3, f_3) \in \mathrm{Obj}(X \downarrow F)$
  - $\alpha \in \mathrm{Hom}_{(X\downarrow F)}((A_1, f_1), (A_2, f_2))$
  - $\beta \in \mathrm{Hom}_{(X\downarrow F)}((A_2, f_2), (A_3, f_3))$

  we define

$$\beta \circ_{(X\downarrow F)} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

**Definition 1.3.3.4.** Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. We define the **comma category from $F$ to $X$**, denoted $(F \downarrow X)$, by $(F \downarrow X) = (F \downarrow \Delta^{\mathbf{1}}_X)$.
We may make the following identification:

- $\mathrm{Obj}(F \downarrow X) = \{(A, f) : A \in \mathrm{Obj}(\mathcal{C}) \text{ and } f \in \mathrm{Hom}_{\mathcal{D}}(F(A), X)\}$
- For $(A_1, f_1), (A_2, f_2) \in \mathrm{Obj}(F \downarrow X)$,

$$\mathrm{Hom}_{(X\downarrow F)}((A_1, f_1), (A_2, f_2)) = \{\alpha \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2) \text{ and } f_2 \circ F(\alpha) = f_1\}$$

  i.e. for $(A_1, f_1), (A_2, f_2) \in \mathrm{Obj}(F \downarrow X)$ and $\alpha \in \mathrm{Hom}_{A_1,A_2}$, $\alpha \in \mathrm{Hom}_{(F\downarrow X)}((A_1, f_1), (A_2, f_2))$ iff the following diagram commutes:

$$\begin{tikzcd} F(A_1) \arrow[rr, "F(\alpha)"] \arrow[dr, "f_1"'] & & F(A_2) \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

- For
  - $(A_1, f_1), (A_2, f_2), (A_3, f_3) \in \mathrm{Obj}(F \downarrow X)$
  - $\alpha \in \mathrm{Hom}_{(F \downarrow X)}((A_1, f_1), (A_2, f_2))$
  - $\beta \in \mathrm{Hom}_{(F \downarrow X)}((A_2, f_2), (A_3, f_3))$

  we define
  $$\beta \circ_{(F \downarrow X)} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

## 1.4 Natural Transformations

### 1.4.1 Introduction

**Definition 1.4.1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : \text{Obj}(\mathcal{C}) \to \text{Hom}_{\mathcal{D}}$. Then $\alpha$ is said to be a **natural transformation from** $F$ **to** $G$, denoted $\alpha : F \Rightarrow G$, if

1. for each $A \in \text{Obj}(\mathcal{C})$, $\alpha_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$
2. for each $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$, i.e. the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$

### 1.4.2 Category of Functors

**Definition 1.4.2.1.** Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ natural transformations. We define the **composition of** $\beta$ **with** $\alpha$, denoted $\beta \circ \alpha : F \Rightarrow H$, by

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$$

**Exercise 1.4.2.2.** Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ natural transformations. Then $\beta \circ \alpha : F \Rightarrow H$ is a natural transformation.

*Proof.*

1. Let $A \in \text{Obj}(\mathcal{C})$. Since $\alpha_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$ and $\beta_A \in \text{Hom}_{\mathcal{D}}(G(A), H(A))$, we have that

$$\begin{aligned} (\beta \circ \alpha)_A &= \beta_A \circ \alpha_A \\ &\in \text{Hom}_{\mathcal{D}}(F(A), H(A)) \end{aligned}$$

2. Let $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$. Since $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ and $H(f) \circ \beta_A = \beta_B \circ G(f)$. Therefore

$$\begin{aligned} H(f) \circ (\beta \circ \alpha)_A &= H(f) \circ (\beta_A \circ \alpha_A) \\ &= (H(f) \circ \beta_A) \circ \alpha_A \\ &= (\beta_B \circ G(f)) \circ \alpha_A \\ &= \beta_B \circ (G(f) \circ \alpha_A) \\ &= \beta_B \circ (\alpha_B \circ F(f)) \\ &= (\beta_B \circ \alpha_B) \circ F(f) \\ &= (\beta \circ \alpha)_B \circ F(f) \end{aligned}$$

So $\beta \circ \alpha : F \Rightarrow H$ is a natural transformation. $\square$

**Exercise 1.4.2.3.** Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H, I : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ and $\gamma : H \Rightarrow I$ natural transformations. Then

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

*Proof.* Let $A \in \text{Obj}(\mathcal{C})$. By definition,

$$\begin{aligned}[(\gamma \circ \beta) \circ \alpha]_A &= (\gamma \circ \beta)_A \circ \alpha_A \\ &= (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &= \gamma_A \circ (\beta \circ \alpha)_A \\ &= [\gamma \circ (\beta \circ \alpha)]_A\end{aligned}$$

Since $A \in \text{Obj}(\mathcal{C})$ is arbitrary,

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

□

**Definition 1.4.2.4.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. We define the **identity natural transformation from $F$ to $F$**, denoted $\text{id}_F : F \Rightarrow F$, by

$$(\text{id}_F)_A = \text{id}_{F(A)}$$

**Exercise 1.4.2.5.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. Then $\text{id}_F : F \Rightarrow F$ is a natural transformation from $F$ to $F$.

*Proof.*

1. Let $A \in \text{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\text{id}_F)_A &= \text{id}_{F(A)} \\ &\in \text{Hom}_{\mathcal{D}}(F(A), F(A))\end{aligned}$$

2. Let $A, B \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned}F(f) \circ (\text{id}_F)_A &= F(f) \circ \text{id}_{F(A)} \\ &= F(f) \\ &= \text{id}_{F(B)} \circ F(f) \\ &= (\text{id}_F)_B \circ F(f)\end{aligned}$$

□

**Exercise 1.4.2.6.** Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Then

1. $\text{id}_G \circ \alpha = \alpha$

2. $\alpha \circ \text{id}_F = \alpha$

*Proof.*

1. Let $A \in \text{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\text{id}_G \circ \alpha)_A &= (\text{id}_G)_A \circ \alpha_A \\ &= \text{id}_{G(A)} \circ \alpha_A \\ &= \alpha_A\end{aligned}$$

   Since $A \in \text{Obj}(C)$ is arbitrary, $\text{id}_G \circ \alpha = \alpha$

2. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ (\mathrm{id}_F)_A \\ &= \alpha_A \circ \mathrm{id}_{F(A)} \\ &= \alpha_A\end{aligned}$$

   Since $A \in \mathrm{Obj}(C)$ is arbitrary, $\alpha \circ \mathrm{id}_F = \alpha$.

□

**Exercise 1.4.2.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. If $\mathcal{C}$ is small, then $\alpha$ is a set.

*Proof.* Suppose that $\mathcal{C}$ is small. Then $\mathrm{Obj}(\mathcal{C})$ is a set. Since $\alpha : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Hom}_{\mathcal{D}}$, Axiom 1.1.0.6 implies that $\alpha(\mathrm{Obj}(\mathcal{C}))$ is a set. Then $\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C}))$ is a set. Therefore $\mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C})))$ is a set. Since $\alpha \subset \mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C}))$, we have that $\alpha \in \mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C})))$ which implies that $\alpha$ is a set. □

**Exercise 1.4.2.8.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$. If $\mathcal{C}$ is small, then there exists a class $A$ such that for each class $\alpha$, $\alpha \in A$ iff $\alpha : F \Rightarrow G$.

*Proof.* Suppose that $\mathcal{C}$ is small. Define $\phi$ by

$$\phi(\alpha) : \alpha : F \Rightarrow G$$

Axiom 1.1.0.7 implies that there exists a class $A$ such that for each set $\alpha$, $\alpha \in A$ iff $\phi(\alpha)$. Let $\alpha$ be a class. Suppose that $\alpha \in A$. By Definition 1.1.0.1, $\alpha$ is a set. Since $\alpha$ is a set and $\alpha \in A$, we have that $\phi(\alpha)$. Hence $\alpha : F \Rightarrow G$.
Conversely, suppose that $\alpha : F \Rightarrow G$. Since $\mathcal{C}$ is small, Exercise 1.4.2.7 implies that $\alpha$ is a set. Since $\phi(\alpha)$, we have that $\alpha \in A$. □

**Definition 1.4.2.9.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. We define the **functor category from $\mathcal{C}$ to $\mathcal{D}$**, denoted $\mathcal{D}^{\mathcal{C}}$, by

- $\mathrm{Obj}(\mathcal{D}^{\mathcal{C}}) = \{F : F : \mathcal{C} \to \mathcal{D}\}$
- For $F, G \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G) = \{\alpha : \alpha : F \Rightarrow G\}$
- For $F, G, H \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$ and $\beta \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(G, H)$, $\beta \circ_{\mathcal{D}^{\mathcal{C}}} \alpha = \beta \circ \alpha$

**Exercise 1.4.2.10.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Then $\mathcal{D}^{\mathcal{C}}$ is a category.

*Proof.* Exercise 1.4.2.2 implies that composition is well-defined. Exercise 1.4.2.3 implies that composition is associative. Exercise 1.4.2.5 and Exercise 1.4.2.6 imply the existence of identities. □

### 1.4.3 Diagonal Functor

**Definition 1.4.3.1.** Let $\mathcal{C}$, $\mathcal{D}$ be categories, $X, Y \in \mathrm{Obj}(\mathcal{D})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$. We define the **constant natural transformation on $\mathcal{C}$ at $f$**, denoted $\delta^{\mathcal{C}}_f : \Delta^{\mathcal{C}}_X \Rightarrow \Delta^{\mathcal{C}}_Y$, by

$$(\delta^{\mathcal{C}}_f)_A = f$$

**Exercise 1.4.3.2.** Let $\mathcal{C}$, $\mathcal{D}$ be categories, $X, Y \in \mathrm{Obj}(\mathcal{D})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$. Then $\delta^{\mathcal{C}}_f : \Delta^{\mathcal{C}}_X \Rightarrow \Delta^{\mathcal{C}}_Y$ is a natural transformation.

*Proof.*

1. By definition, for each $A \in \mathrm{Obj}(\mathcal{C})$ $(\delta^{\mathcal{C}}_f)_A \in \mathrm{Hom}_{\mathcal{D}}(\Delta^{\mathcal{C}}_X(A), \Delta^{\mathcal{C}}_Y(A))$.

2. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned}\Delta^{\mathcal{C}}_Y(g) \circ (\delta^{\mathcal{C}}_f)_A &= \mathrm{id}_Y \circ f \\ &= f \\ &= f \circ \mathrm{id}_X \\ &= (\delta^{\mathcal{C}}_f)_B \circ \Delta^{\mathcal{C}}_X(g)\end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} \Delta^{\mathcal{C}}_X(A) \arrow[r, "(\delta^{\mathcal{C}}_f)_A"] \arrow[d, "\Delta^{\mathcal{C}}_X(g)"'] & \Delta^{\mathcal{C}}_Y(A) \arrow[d, "\Delta^{\mathcal{C}}_Y(g)"] \\ \Delta^{\mathcal{C}}_X(B) \arrow[r, "(\delta^{\mathcal{C}}_f)_B"] & \Delta^{\mathcal{C}}_Y(B) \end{tikzcd} = \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "\mathrm{id}_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

So $\delta^{\mathcal{C}}_f : \Delta^{\mathcal{C}}_X \Rightarrow \Delta^{\mathcal{C}}_Y$ is a natural transformation. □

**Exercise 1.4.3.3.** Let $\mathcal{C}, \mathcal{D}$ be categories, $X, Y, Z \in \mathrm{Obj}(\mathcal{D})$, $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$. Then $\delta^{\mathcal{C}}_{g\circ f} = \delta^{\mathcal{C}}_g \circ \delta^{\mathcal{C}}_f$.

*Proof.* Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\delta^{\mathcal{C}}_{g\circ f})_A &= g \circ f \\ &= (\delta^{\mathcal{C}}_g)_A \circ (\delta^{\mathcal{C}}_f)_A \\ &= (\delta^{\mathcal{C}}_g \circ \delta^{\mathcal{C}}_f)_A\end{aligned}$$

Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\delta^{\mathcal{C}}_{g\circ f} = \delta^{\mathcal{C}}_g \circ \delta^{\mathcal{C}}_f$. □

**Exercise 1.4.3.4.** Let $\mathcal{C}, \mathcal{D}$ be categories and $X \in \mathrm{Obj}(\mathcal{D})$. Then $\delta^{\mathcal{C}}_{\mathrm{id}_X} = \mathrm{id}_{\Delta^{\mathcal{C}}_X}$.

*Proof.* Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\delta^{\mathcal{C}}_{\mathrm{id}_X})_A &= \mathrm{id}_X \\ &= \mathrm{id}_{\Delta^{\mathcal{C}}_X(A)} \\ &= (\mathrm{id}_{\Delta^{\mathcal{C}}_X})_A\end{aligned}$$

Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\delta^{\mathcal{C}}_{\mathrm{id}_X} = \mathrm{id}_{\Delta^{\mathcal{C}}_X}$ □

**Definition 1.4.3.5.** Let $\mathcal{C}$, $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. We define the **$\mathcal{C}$-ary diagonal functor** on $\mathcal{D}$, denoted by $\Delta^{\mathcal{C}} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$, by

- $\Delta^{\mathcal{C}}(X) = \Delta^{\mathcal{C}}_X$
- $\Delta^{\mathcal{C}}(f) = \delta^{\mathcal{C}}_f$

**Exercise 1.4.3.6.** Let $\mathcal{C}$, $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Then $\Delta^{\mathcal{C}} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$ is a functor.

*Proof.*

1. Exercise 1.4.3.2 implies that for each $X, Y \in \mathrm{Obj}(\mathcal{D})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$, $\Delta^{\mathcal{C}}(f) \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta^{\mathcal{C}}(X), \Delta^{\mathcal{C}}(Y))$
2. Exercise 1.4.3.3 implies that for each $X, Y, Z \in \mathrm{Obj}(\mathcal{D})$, $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$, $\Delta^{\mathcal{C}}(g \circ f) = \Delta^{\mathcal{C}}(g) \circ \Delta^{\mathcal{C}}(f)$
3. Exercise 1.4.3.4 implies that for each $X \in \mathrm{Obj}(\mathcal{D})$, $\Delta^{\mathcal{C}}(\mathrm{id}_X) = \mathrm{id}_{\Delta^{\mathcal{C}}(X)}$

So $\Delta^{\mathcal{C}} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$ is a functor. □

## 1.5 Algebra of Morphisms

### 1.5.1 Isomorphisms

**Exercise 1.5.1.1. Uniqueness of Identities:**
Let $\mathcal{C}$ be a category. Then for each $A \in \text{Obj}(\mathcal{C})$, there exists a unique $e_A \in \text{Hom}_{\mathcal{C}}(A, A)$ such that for each $B \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, A)$, $f \circ e_A = f$ and $e_A \circ g = g$.

*Proof.* Let $A \in \text{Obj}(\mathcal{C})$.

- **Existence:**
  Since $\mathcal{C}$ is a category, by definition there exists $\text{id}_A \in \text{Hom}_{\mathcal{C}}(A, A)$ such that for each $B \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, A)$, $f \circ \text{id}_A = f$ and $\text{id}_A \circ g = g$.

- **Uniqueness:**
  Let $e_A \in \text{Hom}_{\mathcal{C}}(A, A)$. Suppose that for each $B \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, A)$, $f \circ e_A = f$ and $e_A \circ g = g$. Then

$$\begin{aligned} e_A &= e_A \circ \text{id}_A \\ &= \text{id}_A \end{aligned}$$

□

**Definition 1.5.1.2.** Let $\mathcal{C}$ be a category, $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$. Then $f$ is said to be an **isomorphism** if there exists $g \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

**Exercise 1.5.1.3. Uniqueness of Inverses:**
Let $\mathcal{C}$ be a category, $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$. If $f$ is an isomorphism, then there exists a unique $g \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

*Proof.* Suppose that $f$ is an isomorphism.

- **Existence:**
  By definition, there exists $g \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

- **Uniqueness:**
  Let $g' \in \text{Hom}_{\mathcal{C}}(B, A)$. Suppose that $g' \circ f = \text{id}_A$, $f \circ g' = \text{id}_B$. Then

$$\begin{aligned} g' &= g' \circ \text{id}_B \\ &= g' \circ (f \circ g) \\ &= (g' \circ f) \circ g \\ &= \text{id}_A \circ g \\ &= g \end{aligned}$$

□

**Definition 1.5.1.4.** Let $\mathcal{C}$ be a category, $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$. Suppose that $f$ is an isomorphism. We define the **inverse of** $f$, denoted $f^{-1}$, to be the unique $g \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

**Exercise 1.5.1.5.** Let $\mathcal{C}$ be a category and $A \in \text{Obj}(\mathcal{C})$. Then $\text{id}_A$ is an isomorphism and $(\text{id}_A)^{-1} = \text{id}_A$.

*Proof.* Since $\text{id}_A \circ \text{id}_A = \text{id}_A$, we have that $\text{id}_A$ is an isomorphism and $(\text{id}_A)^{-1} = \text{id}_A$. □

**Exercise 1.5.1.6.** Let $\mathcal{C}$ be a category and $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$. If $f$ is an isomorphism, then $f^{-1}$ is an isomorphism and $(f^{-1})^{-1} = f$.

*Proof.* Suppose that $f$ is an isomorphism. By definition, $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$. Hence $f^{-1}$ is an isomorphism and $(f^{-1})^{-1} = f$. □

**Exercise 1.5.1.7.** Let $\mathcal{C}$ be a category, $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. If $f$ and $g$ are isomorphisms, then $g \circ f$ is an isomorphism and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

*Proof.* Suppose that $f$ and $g$ are isomorphisms. Then

$$\begin{aligned}
(f^{-1} \circ g^{-1}) \circ (g \circ f) &= ((f^{-1} \circ g^{-1}) \circ g) \circ f \\
&= (f^{-1} \circ (g^{-1} \circ g)) \circ f \\
&= (f^{-1} \circ \mathrm{id}_B) \circ f \\
&= f^{-1} \circ f \\
&= \mathrm{id}_A
\end{aligned}$$

and

$$\begin{aligned}
(g \circ f) \circ (f^{-1} \circ g^{-1}) &= ((g \circ f) \circ f^{-1}) \circ g^{-1} \\
&= (g \circ (f \circ f^{-1})) \circ g^{-1} \\
&= (g \circ \mathrm{id}_B) \circ g^{-1} \\
&= g \circ g^{-1} \\
&= \mathrm{id}_C
\end{aligned}$$

So $g \circ f$ is an isomorphism and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

**Definition 1.5.1.8.** Let $\mathcal{C}$ be a category and $A, B \in \mathrm{Obj}(\mathcal{C})$. Then $A$ is said to be **isomorphic** to $B$ if there exists $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that $f$ is an isomorphism.

**Exercise 1.5.1.9.** Let $\mathcal{C}$ be a category. We define the relation $\cong$ on $\mathrm{Obj}(\mathcal{C})$ by $A \cong B$ iff $A$ is isomorphic to $B$. Then $\cong$ is an equivalence relation on $\mathrm{Obj}(\mathcal{C})$.

*Proof.*

1. **reflexivity:**
   Let $A \in \mathrm{Obj}(\mathcal{C})$. Exercise 1.5.1.5 implies that $\mathrm{id}_A$ is an isomorphism. So $A \cong A$. Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, we have that for each $A \in \mathrm{Obj}(\mathcal{C})$, $A \cong A$ and thus $\cong$ is reflexive.

2. **symmetry:**
   Let $A, B \in \mathrm{Obj}(\mathcal{C})$. Suppose that $A \cong B$. Then there exists $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that $f$ is an isomorphism. Exercise 1.5.1.6 implies that $f^{-1}$ is an isomorphism. Since $f^{-1} \in \mathrm{Hom}_{\mathcal{C}}(B, A)$, $B \cong A$. Since $A, B \in \mathrm{Obj}(\mathcal{C})$ are arbitrary, we have that for each $A, B \in \mathrm{Obj}(\mathcal{C})$, $A \cong B$ implies that $B \cong A$ and thus $\cong$ is reflexive.

3. **transitivity:** Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$. Suppose that $A \cong B$ and $B \cong C$. Then there exist $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ such that that $f$ and $g$ are isomorphisms. Exercise 1.5.1.7 implies that $g \circ f$ is an isomorphism. Since $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$, $A \cong C$. Since $A, B, C \in \mathrm{Obj}(\mathcal{C})$ are arbitrary, we have that for each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $A \cong B$ and $B \cong C$ implies that $A \cong C$ and thus $\cong$ is transitive.

Since $\cong$ is reflexive, symmetric and transitive, $\cong$ is an equivalence relation on $\mathrm{Obj}(\mathcal{C})$. □

**Definition 1.5.1.10.** Let $\mathcal{C}$ be a category, $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f : A \to B$. Then

- $f$ is said to be a **monomorphism** if for each $C \in \mathrm{Obj}(C)$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(C, A)$, $f \circ g = f \circ h$ implies that $g = h$, i.e. we have the following implication of commutative diagrams:

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "h"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\implies
\begin{tikzcd}
C \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & A
\end{tikzcd}$$

- $f$ is said to be an **epimorphism** if for each $C \in \text{Obj}(C)$ and $g, h \in \text{Hom}_{\mathcal{C}}(B, C)$, $g \circ f = h \circ f$ implies that $g = h$, i.e. we have the following implication of commutative diagrams:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ B \arrow[r, "h"'] & C \end{tikzcd} \implies \begin{tikzcd} B \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & C \end{tikzcd}$$

**Exercise 1.5.1.11.** Let $A, B \in \text{Obj}(\mathbf{Set})$ and $f \in \text{Hom}_{\mathbf{Set}}(A, B)$. Then

1. $f$ is a monomorphism iff $f$ is injective
2. $f$ is an epimorphism iff $f$ is surjective

**Hint:** consider $C = \{0\}$ and $C = \{0, 1\}$.

*Proof.*

1. Suppose that $f$ is injective. Let $C \in \text{Obj}(\mathbf{Set})$ and $g, h \in \text{Hom}_{\mathbf{Set}}(C, A)$. Suppose that $f \circ g = f \circ h$. Let $x \in C$. Then $f(g(x)) = f(h(x))$. Injectivity of $f$ implies that $g(x) = h(x)$. Since $x \in C$ is arbitrary, $g = h$. Hence $f$ is a monomorphism.
   Conversely, suppose that $f$ is a monomorphism. Let $a, b \in A$. Suppose that $f(a) = f(b)$. Set $C = \{0\}$ and define $g, h : C \to A$ by $g(0) = a$ and $h(0) = b$. Then
$$\begin{aligned} f \circ g(0) &= f(g(0)) \\ &= f(a) \\ &= f(b) \\ &= f(h(0)) \\ &= f \circ h(0) \end{aligned}$$
   Therefore $f \circ g = f \circ h$. Since $f$ is a monomorphism, we have that $g = h$. Hence
$$\begin{aligned} a &= g(0) \\ &= h(0) \\ &= b \end{aligned}$$
2. Suppose that $f$ is surjective. Let $C \in \text{Obj}(\mathcal{C})$ and $g, h \in \text{Hom}_{\mathbf{Set}}(B, C)$. Suppose that $g \circ f = h \circ f$. Let $y \in B$. Surjective of $f$ implies that there exists $x \in A$ such that $y = f(x)$. Then
$$\begin{aligned} g(y) &= g(f(x)) \\ &= g \circ f(x) \\ &= h \circ f(x) \\ &= h(f(x)) \\ &= h(y) \end{aligned}$$
   Since $y \in B$ is arbitrary, $g = h$. Hence $f$ is an epimorphism.
   Conversely, suppose that $f$ is an epimorphism. Set $C = \{0, 1\}$ and define $g, h : B \to C$ by $g = \chi_{f(A)}$ and $h = \chi_B$. Then $g \circ f = h \circ f$. Since $f$ is an epimorphism, $g = h$ and $f(A) = B$. Hence $f$ is surjective.

$\square$

**Exercise 1.5.1.12.** Let $\mathcal{C}$ be a category, $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$. If $f$ is an isomorphism, then $f$ is a monomorphism and $f$ is an epimorphism.

*Proof.* Suppose that $f$ is an isomorphism.

- (monomorphism)
  Let $C \in \mathrm{Obj}(\mathcal{C})$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(C, A)$. Suppose that $f \circ g = f \circ h$. Then

$$\begin{aligned} g &= \mathrm{id}_A \circ g \\ &= (f^{-1} \circ f) \circ g \\ &= f^{-1} \circ (f \circ g) \\ &= f^{-1} \circ (f \circ h) \\ &= (f^{-1} \circ f) \circ h \\ &= \mathrm{id}_A \circ h \\ &= h \end{aligned}$$

  So $f$ is a monomorphism.

- (epimorphism)
  Let $C \in \mathrm{Obj}(\mathcal{C})$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Suppose that $g \circ f = h \circ f$. Then

$$\begin{aligned} g &= g \circ \mathrm{id}_B \\ &= g \circ (f \circ f^{-1}) \\ &= (g \circ f) \circ f^{-1} \\ &= (h \circ f) \circ f^{-1} \\ &= h \circ (f \circ f^{-1}) \\ &= h \circ \mathrm{id}_B \\ &= h \end{aligned}$$

  So $f$ is an epimorphism.

□

**Definition 1.5.1.13.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Then $\alpha$ is said to be a **natural isomorphism** if for each $A \in \mathrm{Obj}(\mathcal{C})$, $\alpha_A$ is an isomorphism.

**Definition 1.5.1.14.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Suppose that $\alpha$ is a natural isomorphism. We define $\alpha^{-1} : G \Rightarrow F$ by $(\alpha^{-1})_A = \alpha_A^{-1}$.

**Exercise 1.5.1.15.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Suppose that $\alpha$ is a natural isomorphism. Then $\alpha^{-1} : G \Rightarrow F$ is a natural transformation.

*Proof.*

1. Let $A \in \mathrm{Obj}(\mathcal{C})$. Since $\alpha_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$, we have that

$$\begin{aligned} (\alpha^{-1})_A &= \alpha_A^{-1} \\ &\in \mathrm{Hom}_{\mathcal{D}}(G(A), F(A)) \end{aligned}$$

2. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since $G(f) \circ \alpha_A = \alpha_B \circ F(f)$, i.e. the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$

we have that

$$\begin{aligned}
F(f) \circ (\alpha^{-1})_A &= F(f) \circ \alpha_A^{-1} \\
&= \mathrm{id}_{F(B)} \circ (F(f) \circ \alpha_A^{-1}) \\
&= (\alpha_B^{-1} \circ \alpha_B) \circ (F(f) \circ \alpha_A^{-1}) \\
&= \alpha_B^{-1} \circ (\alpha_B \circ (F(f) \circ \alpha_A^{-1})) \\
&= \alpha_B^{-1} \circ ((\alpha_B \circ F(f)) \circ \alpha_A^{-1}) \\
&= \alpha_B^{-1} \circ ((G(f) \circ \alpha_A) \circ \alpha_A^{-1}) \\
&= \alpha_B^{-1} \circ (G(f) \circ (\alpha_A \circ \alpha_A^{-1})) \\
&= \alpha_B^{-1} \circ (G(f) \circ \mathrm{id}_{G(A)}) \\
&= \alpha_B^{-1} \circ G(f) \\
&= (\alpha^{-1})_B \circ G(f)
\end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
G(A) \arrow[r, "(\alpha^{-1})_A"] \arrow[d, "G(f)"'] & F(A) \arrow[d, "F(f)"] \\
G(B) \arrow[r, "(\alpha^{-1})_B"] & F(B)
\end{tikzcd}$$

So $\alpha^{-1} : G \Rightarrow F$.

□

**Exercise 1.5.1.16.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Suppose that $\alpha$ is a natural isomorphism. Then $\alpha^{-1} \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \alpha^{-1} = \mathrm{id}_G$.

*Proof.* Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}
(\alpha^{-1} \circ \alpha)_A &= (\alpha^{-1})_A \circ \alpha_A \\
&= \alpha_A^{-1} \circ \alpha_A \\
&= \mathrm{id}_{F(A)} \\
&= (\mathrm{id}_F)_A
\end{aligned}$$

and

$$\begin{aligned}
(\alpha \circ \alpha^{-1})_A &= \alpha_A \circ (\alpha^{-1})_A \\
&= \alpha_A \circ \alpha_A^{-1} \\
&= \mathrm{id}_{G(A)} \\
&= (\mathrm{id}_G)_A
\end{aligned}$$

Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\alpha^{-1} \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \alpha^{-1} = \mathrm{id}_G$. □

**Exercise 1.5.1.17.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Let $F, G \in \mathcal{D}^{\mathcal{C}}$ and $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$. Then $\alpha$ is an isomorphism iff $\alpha$ is a natural isomorphism.

*Proof.* Suppose that $\alpha$ is an isomorphism. Then there exists $\beta \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(G, F)$ such that $\beta \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \beta - \mathrm{id}_G$. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}
\beta_A \circ \alpha_A &= (\beta \circ \alpha)_A \\
&= (\mathrm{id}_F)_A \\
&= \mathrm{id}_{F(A)}
\end{aligned}$$

and

$$\begin{aligned}\alpha_A \circ \beta_A &= (\alpha \circ \beta)_A \\ &= (\mathrm{id}_G)_A \\ &= \mathrm{id}_{G(A)}\end{aligned}$$

Hence $\alpha_A$ is an isomorphism. Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\alpha$ is a natural isomorphism.
Conversely, suppose that $\alpha$ is a natural isomorphism. Exercise 1.5.1.15 and Exercise 1.5.1.16 imply that $\alpha$ is an isomorphism. $\square$

## 1.5.2 Initial and Final Objects

**Definition 1.5.2.1.** Let $\mathcal{C}$ be a category and $0 \in \mathrm{Obj}(\mathcal{C})$. Then 0 is said to be **initial** if for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $f \in \mathrm{Hom}_{\mathcal{C}}(0, A)$ such that $\mathrm{Hom}_{\mathcal{C}}(0, A) = \{f\}$.

**Definition 1.5.2.2.** Let $\mathcal{C}$ be a category and $1 \in \mathrm{Obj}(\mathcal{C})$. Then 1 is said to be **final** if for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $f \in \mathrm{Hom}_{\mathcal{C}}(A, 1)$ such that $\mathrm{Hom}_{\mathcal{C}}(A, 1) = \{f\}$.

**Exercise 1.5.2.3.** Let $\mathcal{C}$ be a category and $0 \in \mathrm{Obj}(\mathcal{C})$. If 0 is initial, then $\mathrm{Hom}_{\mathcal{C}}(0, 0) = \{\mathrm{id}_0\}$.

*Proof.* Suppose that 0 is initial. Then there exists a $f \in \mathrm{Hom}_{\mathcal{C}}(0, 0)$ such that $\mathrm{Hom}_{\mathcal{C}}(0, 0) = \{f\}$. Since $\mathrm{id}_0 \in \mathrm{Hom}_{\mathcal{C}}(0, 0)$, $f = \mathrm{id}_0$ and therefore $\mathrm{Hom}_{\mathcal{C}}(0, 0) = \{\mathrm{id}_0\}$. $\square$

**Exercise 1.5.2.4.** Let $\mathcal{C}$ be a category and $1 \in \mathrm{Obj}(\mathcal{C})$. If 1 is final, then $\mathrm{Hom}_{\mathcal{C}}(1, 1) = \{\mathrm{id}_1\}$.

*Proof.* Similar to Exercise 1.5.2.3 $\square$

**Exercise 1.5.2.5.** Let $\mathcal{C}$ be a category and $0, 0' \in \mathrm{Obj}(\mathcal{C})$. If 0 and $0'$ are initial, then 0 and $0'$ are isomorphic.

*Proof.* Suppose that 0 and $0'$ are initial. By definition, there exist $f \in \mathrm{Hom}_{\mathcal{C}}(0, 0')$ and $f' \in \mathrm{Hom}_{\mathcal{C}}(0', 0)$ such that $\mathrm{Hom}_{\mathcal{C}}(0, 0') = \{f\}$ and $\mathrm{Hom}_{\mathcal{C}}(0', 0) = \{f'\}$, i.e. we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[loop left, "f' \circ f"] \arrow[r, bend right, "f"'] & 0' \arrow[l, bend right, "f'"'] \arrow[loop right, "f \circ f'"] \end{tikzcd}$$

Exercise 1.5.2.3 implies that $f' \circ f = \mathrm{id}_0$ and $f \circ f' = \mathrm{id}_{0'}$. Hence $f$ is an isomorphism. Since $f \in \mathrm{Hom}_{\mathcal{C}}(0, 0')$, we have that $0 \cong 0'$. $\square$

**Exercise 1.5.2.6.** Let $\mathcal{C}$ be a category and $1, 1' \in \mathrm{Obj}(\mathcal{C})$. If 1 and $1'$ are final, then 1 and $1'$ are isomorphic.

*Proof.* Similar to Exercise 1.5.2.5 $\square$

**Exercise 1.5.2.7.** We have that $\varnothing$ is initial in **Set**.

*Proof.* Let $A \in \mathrm{Obj}(\mathbf{Set})$. Define $f \in \mathrm{Hom}_{\mathbf{Set}}(\varnothing, A)$ by $f = \varnothing$. Let $g \in \mathrm{Hom}_{\mathbf{Set}}(\varnothing, A)$. Then $g = f$. Since $g \in \mathrm{Hom}_{\mathbf{Set}}(\varnothing, A)$ is arbitrary, $\mathrm{Hom}_{\mathbf{Set}}(\varnothing, A) = \{f\}$. Hence $\varnothing$ is initial. $\square$

**Exercise 1.5.2.8.** We have that $\{\varnothing\}$ is terminal in **Set**.

*Proof.* Let $A \in \mathrm{Obj}(\mathbf{Set})$. Define $f \in \mathrm{Hom}_{\mathbf{Set}}(A, \{\varnothing\})$ by $f(x) = \varnothing$. Let $g \in \mathrm{Hom}_{\mathbf{Set}}(A, \{\varnothing\})$. Then $g = f$. Since $g \in \mathrm{Hom}_{\mathbf{Set}}(A, \{\varnothing\})$ is arbitrary, $\mathrm{Hom}_{\mathbf{Set}}(A, \{\varnothing\}) = \{f\}$. Hence $\{\varnothing\}$ is final. $\square$

**Exercise 1.5.2.9.** We have that **0** is initial in **Cat**.

*Proof.* Let $\mathcal{C} \in \mathrm{Obj}(\mathbf{Cat})$. It is clear that $\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{0}, \mathcal{C}) = \{E_{\mathcal{C}}\}$. Hence **0** is initial in **Cat**. $\square$

**Exercise 1.5.2.10.** We have that **1** is final in **Cat**.

*Proof.* Let $\mathcal{C} \in \mathrm{Obj}(\mathbf{Cat})$. It is clear that $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathbf{1}) = \{\Delta_*^{\mathcal{C}}\}$. Hence $\mathbf{1}$ is final in $\mathbf{Cat}$. $\square$

**Definition 1.5.2.11.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $0 \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. Suppose that 0 is initial in $\mathcal{D}$. Then for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $f_A \in \mathrm{Hom}_{\mathcal{D}}(0, F(A))$ such that $\mathrm{Hom}_{\mathcal{D}}(0, F(A)) = \{f_A\}$. We define the **initial natural transformation induced by** 0 from $\Delta_0^{\mathcal{C}}$ to $F$, denoted $\zeta_0 : \Delta_0^{\mathcal{C}} \Rightarrow F$, by $(\eta_0)_A = f_A$.

**Definition 1.5.2.12.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $1 \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. Suppose that 1 is final in $\mathcal{D}$. Then for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $f_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), 1)$ such that $\mathrm{Hom}_{\mathcal{D}}(F(A), 1) = \{f_A\}$. We define the **final natural transformation induced by** 1 from $F$ to $\Delta_1^{\mathcal{C}}$, denoted $\phi_1 : F \Rightarrow \Delta_1^{\mathcal{C}}$, by $(\phi_1)_A = f_A$.

**Exercise 1.5.2.13.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $0 \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. Suppose that 0 is initial in $\mathcal{D}$. Then $\eta_0 : \Delta_0^{\mathcal{C}} \Rightarrow F$ is a natural transformation.

*Proof.*

1. By definition, for each $A \in \mathrm{Obj}(\mathcal{C})$, $(\eta_0)_A \in \mathrm{Hom}_{\mathcal{D}}(\Delta_0^{\mathcal{C}}(A), F(A))$
2. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since

$$\begin{aligned} F(f) \circ (\eta_0)_A &\in \mathrm{Hom}_{\mathcal{D}}(0, F(B)) \\ &= \{(\eta_0)_B\} \end{aligned}$$

we have that

$$\begin{aligned} F(f) \circ (\eta_0)_A &= (\eta_0)_B \\ &= (\eta_0)_B \circ \mathrm{id}_0 \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} \Delta_0^{\mathcal{C}}(A) & \xrightarrow{(\eta_0)_A} & F(A) \\ {\scriptstyle \Delta_0^{\mathcal{C}}(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ \Delta_0^{\mathcal{C}}(B) & \xrightarrow{(\eta_0)_B} & F(B) \end{array} = \begin{array}{ccc} 0 & \xrightarrow{(\eta_0)_A} & F(A) \\ {\scriptstyle \mathrm{id}_0}\downarrow & & \downarrow{\scriptstyle F(f)} \\ 0 & \xrightarrow{(\eta_0)_B} & F(B) \end{array}$$

So $\eta_0 : \Delta_0^{\mathcal{C}} \Rightarrow F$ is a natural transformation. $\square$

**Exercise 1.5.2.14.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $1 \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. Suppose that 1 is final in $\mathcal{D}$. Then $\phi_1 : F \Rightarrow \Delta_0^{\mathcal{C}}$ is a natural transformation.

*Proof.* Similar to Exercise 1.5.2.13 $\square$

**Exercise 1.5.2.15.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $0 \in \mathrm{Obj}(\mathcal{D})$. Suppose that $\mathcal{C}$ is small. If 0 is initial in $\mathcal{D}$, then $\Delta_0^{\mathcal{C}}$ is initial in $\mathcal{D}^{\mathcal{C}}$.

*Proof.* Suppose that 0 is initial in $\mathcal{D}$. Let $F \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F)$ and $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned} \alpha_A &\in \mathrm{Hom}_{\mathcal{D}}(\Delta_0^{\mathcal{C}}(A), F(A)) \\ &= \mathrm{Hom}_{\mathcal{D}}(0, F(A)) \\ &= \{(\eta_0)_A\} \end{aligned}$$

Hence $\alpha_A = (\eta_0)_A$. Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\alpha = \eta_0$. Since $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F)$ is arbitrary, $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F) = \{\eta_0\}$. Therefore $\Delta_0^{\mathcal{C}}$ is initial in $\mathcal{D}^{\mathcal{C}}$. $\square$

**Exercise 1.5.2.16.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $1 \in \mathrm{Obj}(\mathcal{D})$. Suppose that $\mathcal{C}$ is small. If 1 is final in $\mathcal{D}$, then $\Delta_1^{\mathcal{C}}$ is final in $\mathcal{D}^{\mathcal{C}}$.

*Proof.* Similar to Exercise 1.5.2.15. $\square$

# Chapter 2

# Universal Morphisms and Limits

### 2.0.1 Universal Morphisms

**Definition 2.0.1.1.** Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$, $F : \mathcal{C} \to \mathcal{D}$, $A \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, F(A))$. Then $(A, f)$ is said to be a **universal morphism** from $X$ to $F$ if for each $A' \in \mathrm{Obj}(\mathcal{C})$ $f' \in \mathrm{Hom}_{\mathcal{D}}(X, F(A'))$, there exists a unique $\alpha \in \mathrm{Hom}_{\mathcal{C}}(A, A')$ such that $f' = F(\alpha) \circ f$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & F(A) \arrow[d, dashed, "F(\alpha)"] & A \arrow[d, dashed, "\alpha"] \\ & F(A') & A' \end{tikzcd}$$

**Definition 2.0.1.2.** Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$, $F : \mathcal{C} \to \mathcal{D}$, $A \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(F(A), X)$. Then $(A, f)$ is said to be a **universal morphism** from $F$ to $X$ if for each $A' \in \mathrm{Obj}(\mathcal{C})$ $f' \in \mathrm{Hom}_{\mathcal{D}}(F(A'), X)$, there exists a unique $\alpha \in \mathrm{Hom}_{\mathcal{C}}(A', A)$ such that $f' = f \circ F(\alpha)$, i.e. the following diagram commutes:

$$\begin{tikzcd} X & F(A) \arrow[l, "f"'] & A \\ & F(A') \arrow[ul, "f'"] \arrow[u, dashed, "F(\alpha)"'] & A' \arrow[u, dashed, "\alpha"'] \end{tikzcd}$$

**Exercise 2.0.1.3.** Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$, $F : \mathcal{C} \to \mathcal{D}$, $A \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, F(A))$. Then $(A, f)$ is a universal morphism from $X$ to $F$ iff $(A, f)$ is initial in $(X \downarrow F)$.

*Proof.* $\square$

**Exercise 2.0.1.4.** Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$, $F : \mathcal{C} \to \mathcal{D}$ $A \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(F(A), X)$. Then $(A, f)$ is a universal morphism from $F$ to $X$ iff $(A, f)$ is terminal in $(F \downarrow X)$.

*Proof.* $\square$

## 2.1 Limits

**Definition 2.1.0.1.** Let $\mathcal{J}$, $\mathcal{C}$ be categories and $D : \mathcal{J} \to \mathcal{C}$. Then $D$ is said to be a **diagram of type $\mathcal{J}$ in $\mathcal{C}$**.

**Note 2.1.0.2.** We are usually interested in the case that $\mathcal{J}$ is small. We will identify a diagram $D$ with its image.

**Example 2.1.0.3.** Define $\mathcal{J}$ by

- $\mathrm{Obj}(\mathcal{J}) = \{1, 2, 3\}$ and for $i, j \in \mathrm{Obj}(\mathcal{J})$, $\mathrm{Hom}_{\mathcal{J}}(i, j) = \{a_{i,j}\}$,
- for $i, j \in \mathrm{Obj}(\mathcal{J})$, $\mathrm{Hom}_{\mathcal{J}}(i, j) = \{a_{ij}\}$.

Let $\mathcal{C}$ be a category and $D : \mathcal{J} \to \mathcal{C}$. Without including the identity morphisms or compositions, we can visualize $D$ as follows:

$$\begin{tikzcd}
& 1 \arrow[r, "b"] \arrow[dl, "a"'] & 2 \arrow[dl, "c"] & & & D_1 \arrow[r] \arrow[dl] & D2 \arrow[dl] \\
3 \arrow[r, "d"'] & 4 & & \xrightarrow{D} & D_3 \arrow[r] & D_4 &
\end{tikzcd}$$

**Definition 2.1.0.4.** Let $\mathcal{J}$, $\mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \mathrm{Obj}(\mathcal{C}^{\mathcal{J}})$. We define the **category of cones to** $D$, denoted $\mathbf{Cone}(D)$, by $\mathbf{Cone}(D) = (\Delta^{\mathcal{J}} \downarrow D)$.

**Example 2.1.0.5.** Let $\mathcal{J}$

$$\begin{tikzcd}
& X, \arrow[d, "\phi_1"] \arrow[drr, "\phi_2"] \arrow[ddl, "\phi_3"'] \arrow[ddr, "\phi_4"] & & \\
& D_1 \arrow[rr] \arrow[dl] & & D2 \arrow[dl] \\
D_3 \arrow[rr] & & D_4 &
\end{tikzcd}$$

**Definition 2.1.0.6.** Let $\mathcal{J}$, $\mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \mathrm{Obj}(\mathcal{C}^{\mathcal{J}})$. We define the **category of cocones from** $D$, denoted $\mathbf{Cocone}(D)$, by $\mathbf{Cocone}(D) = (D \downarrow \Delta^{\mathcal{J}})$.

**Definition 2.1.0.7.** Let $\mathcal{J}$, $\mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \mathrm{Obj}(\mathcal{C}^{\mathcal{J}})$ and $(X, \phi) \in \mathbf{Cone}(D)$. Then $(X, \phi)$ is said to be a **limit of** $D$ if $(X, \phi)$ is a universal morphism from $\Delta^{\mathcal{J}}$ to $D$.

**Note 2.1.0.8.** Let $\mathcal{J}$, $\mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \mathrm{Obj}(\mathcal{C}^{\mathcal{J}})$ and $(X, \phi) \in \mathbf{Cone}(D)$. Then

$$\begin{aligned}
(X, \phi) \text{ is a limit of } D &\iff (X, \phi) \text{ is terminal in } \mathbf{Cone}(D) \\
&\iff \text{for each } (Y, \psi) \in \mathbf{Cone}(D), \text{ there exists a unique} \\
&\qquad f \in \mathrm{Hom}_{\mathcal{C}}(Y, X) \text{ such that for each } j \in \mathcal{J},\ \psi_j = \phi_j \circ f
\end{aligned}$$

**Definition 2.1.0.9.** Let $\mathcal{J}$, $\mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \mathrm{Obj}(\mathcal{C}^{\mathcal{J}})$ and $(X, \phi) \in \mathbf{Cocone}(D)$. Then $(X, \phi)$ is said to be a **colimit of** $D$ if $(X, \phi)$ is a universal morphism from $D$ to $\Delta^{\mathcal{J}}$.

**Note 2.1.0.10.** Let $\mathcal{J}$, $\mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \mathrm{Obj}(\mathcal{C}^{\mathcal{J}})$ and $(X, \phi) \in \mathbf{Cone}(D)$. Then

$$\begin{aligned}
(X, \phi) \text{ is a colimit of } D &\iff (X, \phi) \text{ is initial in } \mathbf{Cocone}(D) \\
&\iff \text{for each } (Y, \psi) \in \mathbf{Cocone}(D), \text{ there exists a unique} \\
&\qquad f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \text{ such that for each } j \in \mathcal{J},\ \psi_j = f \circ \phi_j
\end{aligned}$$

### 2.1.1 Products and Coproducts

### 2.1.2 Equalizers and Coequalizers

## 2.2 TO DO

- Define subcategories and full subcategories and show that if $\mathrm{Obj}(D) \subset \mathrm{Obj}(C)$ and for each $X, Y \in \mathrm{Obj}(D)$, $\mathrm{Hom}_D(X,Y) = \mathrm{Hom}_C(X,Y)$, then $D$ is a full subcategory of $C$. I used this in differential

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# Chapter 3

# Monoidal Categories

**Definition 3.0.0.1.**

# Appendix A

# App

## A.1 Reading Diagrams and associated digraphs of diagrams

**Definition A.1.0.1.** Let

$$
\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "h"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\implies
\begin{tikzcd}
C \arrow[rr, bend left, "g"] \arrow[rr, bend right, "h"'] & & A
\end{tikzcd}
$$

see an intro to the language of category theory by roman for description

**Definition A.1.0.2.** A diagram is said to be **commutative** if for each path of length $\geq 2$, in the associated digraph gives the same morphism.