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Introduction to Group Theory

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Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Representation Theory

1.1 Tannaka-Krein Duality

Definition 1.1.0.1. Let $G \in \text{Obj}(\textbf{TopMon})$, $V \in \text{Obj}(\textbf{Top Vect}_{\mathbb{C}})$ and $\pi \in \text{Hom}_{\textbf{TopMon}}(G, \text{End}_{\textbf{TopVect}_{\mathbb{C}}}(V))$. Then (V, π) is said to be a **$\mathbb{C}$-representation of** G. We denote the set of $\mathbb{C}$-representations of G by $\mathcal{R}(G, \mathbb{C})$.

Definition 1.1.0.2. Let $(V, \pi) \in \mathcal{R}(G < \mathbb{C})$. We define the **dimension of** (V, π), denoted $\dim(V, \pi)$, by $\dim(V, \pi) = \dim V$. Then (V, π) is said to be **finite dimensional** if $\dim(V, \pi) < \infty$.

Definition 1.1.0.3. Let $G \in \text{Obj}(\textbf{TopMon})$, $(V, \pi), (W, \rho) \in \mathcal{R}(G, \mathbb{C})$ and $T \in \text{Hom}_{\textbf{TopVect}_{\mathbb{C}}}(V, W)$ Then T is said to be (π, ρ)**-equivariant** if for each $g \in G$, $T \circ \pi(g) = \rho(g) \circ T$, i.e. the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi(g)"'] & W \arrow[d, "\rho(g)"] \\ V \arrow[r, "T"'] & W \end{tikzcd}$$

Definition 1.1.0.4. Let $G \in \text{Obj}(\textbf{TopMon})$. We define $\textbf{Rep}(G, \mathbb{C})$ by

- $\text{Obj}(\textbf{Rep}(G, \mathbb{C})) = \mathcal{R}(G, \mathbb{C})$.
- for $(V, \pi), (W, \rho) \in \text{Obj}(\textbf{Rep}(G, \mathbb{C}))$,

$$\text{Hom}_{\textbf{Rep}(G,\mathbb{C})}((V, \pi), (W, \rho)) = \{T \in \text{Hom}_{\textbf{TopVect}_{\mathbb{C}}}(V, W) : T \text{ is } (\pi, \rho)\text{-equivariant}\}$$

- for $(V, \pi), (W, \rho), (Z, \mu) \in \text{Obj}(\textbf{Rep}(G, \mathbb{C}))$, $T \in \text{Hom}_{\textbf{Rep}(G,\mathbb{C})}((V, \pi), (W, \rho))$ and $S \in \text{Hom}_{\textbf{Rep}(G,\mathbb{C})}((W, \rho), (Z, \mu))$,

$$S \circ_{\textbf{Rep}(G,\mathbb{C})} T = S \circ T$$

Exercise 1.1.0.5. Let $G \in \text{Obj}(\textbf{TopMon})$. Then $\textbf{Rep}(G, \mathbb{C})$ is a category.

Proof. □

Definition 1.1.0.6. Let $G \in \text{Obj}(\textbf{TopMon})$. We define the **forgetful functor from $\textbf{Rep}(G, \mathbb{C})$ to $\textbf{TopVect}_{\mathbb{C}}$**, denoted $U : \textbf{Rep}(G, \mathbb{C}) \to \textbf{TopVect}_{\mathbb{C}}$, by

- $U(V, \pi) = V, \quad (V, \pi) \in \text{Obj}(\textbf{Rep}(G, \mathbb{C}))$
- $U(T) = T, \quad T \in \text{Hom}_{\textbf{Rep}(G,\mathbb{C})}((V, \pi), (W, \rho))$.

Definition 1.1.0.7. Let $G \in \text{Obj}(\textbf{TopMon})$ and $g \in G$. We define $\hat{g} : U \Rightarrow U$ by $\hat{g}_{(V,\pi)} = \pi(g)$.

Exercise 1.1.0.8. Let $G \in \text{Obj}(\textbf{TopMon})$ and $g \in G$. Then

1. $\hat{g} : U \Rightarrow U$ is a natural transformation.

2. $\hat{g} \in \operatorname{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U)$

Proof.

1. (a) Let $(V, \pi) \in \operatorname{Obj}(\mathbf{Rep}(G, \mathbb{C}))$. By definition,

$$\begin{aligned} \hat{g}_{(V,\pi)} &= \pi(g) \\ &\in \operatorname{End}_{\mathbf{TopVect}_{\mathbb{C}}}(V) \\ &= \operatorname{Hom}_{\mathbf{TopVect}_{\mathbb{C}}}(U(V,\pi), U(V,\pi)) \end{aligned}$$

(b) Let $(V, \pi), (W, \rho) \in \operatorname{Obj}(\mathbf{Rep}(G, \mathbb{C}))$ and $T \in \operatorname{Hom}_{\mathbf{Rep}(G,\mathbb{C})}((V, \pi), (W, \rho))$. By definition, $T \in \operatorname{Hom}_{\mathbf{TopVect}_{\mathbb{C}}}(V, W)$ and T is (π, ρ)-equivariant. Therefore

$$\begin{aligned} U(T) \circ \hat{g}_{(V,\pi)} &= T \circ \pi(g) \\ &= \rho(g) \circ T \\ &= \hat{g}_{(W,\rho)} \circ U(T) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} U(V,\pi) \arrow[r, "\hat{g}_{(V,\pi)}"] \arrow[d, "U(T)"'] & U(V,\pi) \arrow[d, "U(T)"] \\ U(W,\rho) \arrow[r, "\hat{g}_{(W,\rho)}"] & U(W,\rho) \end{tikzcd}$$

Thus $\hat{g} : U \Rightarrow U$ is a natural transformation.

2. The previous part implies that

$$\begin{aligned} \hat{g} &\in \operatorname{Hom}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U, U) \\ &= \operatorname{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U) \end{aligned}$$

□

Definition 1.1.0.9. Let $G \in \operatorname{Obj}(\mathbf{TopMon})$ and $(V, \pi) \in \operatorname{Obj}(\mathbf{Rep}(G, \mathbb{C}))$. We define the (V, π)**-projection**, denoted $\pi_{(V,\pi)} : \operatorname{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U) \to \operatorname{End}_{\mathbf{TopVect}_{\mathbb{C}}}(V)$, by $\pi_{(V,\pi)}(\alpha) = \alpha_{(V,\pi)}$. We define the **topology of endomorphisms of** U, denoted $\mathcal{T}_{\mathcal{E}(U)}$, by

$$\mathcal{T}_{\mathcal{E}(U)} = \tau(\pi_{(V,\pi)} : (V, \pi) \in \mathbf{Rep}(G, \mathbb{C}))$$

Definition 1.1.0.10. define addition of endomorphisms of U pointwise

Exercise 1.1.0.11. Let $G \in \operatorname{Obj}(\mathbf{TopMon})$. Then $(\operatorname{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U), \mathcal{T}_{\mathcal{E}(U)})$ is a topological unital algebra.

Proof. □

Bibliography

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