

# INTRODUCTION TO ANALYSIS

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## PREFACE

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## 1. REAL AND COMPLEX NUMBERS

*Note 1.0.1.* As a starting point, we will take as fact the existence of the **natural numbers**

$$\mathbb{N} = \{1, 2, \dots\}$$

the **integers**

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

and the **rational numbers**

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\}$$

## 1.1. Real Numbers.

**Definition 1.1.1.** Let  $X$  be a set and  $\leq$  a relation on  $X$ . Then  $\leq$  is said to be a **total order** if for each  $a, b, c \in X$ ,

- (1)  $a \leq a$
- (2)  $a \leq b$  and  $b \leq c$  implies that  $a \leq c$
- (3)  $a \leq b$  and  $b \leq a$  implies that  $a = b$
- (4)  $a \leq b$  or  $b \leq a$

**Exercise 1.1.2.** We define the relation  $\leq$  on  $\mathbb{Q}$  defined by

$$\frac{a}{b} \leq \frac{c}{d} \text{ iff } ad \leq bc$$

Then  $\leq$  is a total order of  $\mathbb{Q}$ .

*Proof.* Let  $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$ . Then

- (1)  $\frac{a}{b} \leq \frac{a}{b}$  since  $ab \leq ab$ .
- (2) if  $\frac{a}{b} \leq \frac{c}{d}$  and  $\frac{c}{d} \leq \frac{e}{f}$ , then  $ad \leq bc$  and  $cf \leq de$ . Multiplying the first inequality by  $f$  and the second inequality by  $b$ , we obtain  $adf \leq bcf \leq bde$ . Dividing both sides by  $d$  yields  $af \leq be$ . Hence  $\frac{a}{b} \leq \frac{e}{f}$ .
- (3) if  $\frac{a}{b} \leq \frac{c}{d}$  and  $\frac{c}{d} \leq \frac{a}{b}$ , then  $ad \leq bc$  and  $bc \leq ad$ . This implies that  $ad = bc$ . Hence  $\frac{a}{b} = \frac{c}{d}$ .
- (4)

□

## 2. METRIC SPACES

## 2.1. Introduction.

## 3. TOPOLOGY

**Definition 3.0.1.** Let  $X$  be a topological space and  $S, N \subset X$ . Then  $N$  is said to be a **neighborhood** of  $S$  if there exists  $U \subset X$  such that  $U$  is open and  $S \subset U \subset N$ . For  $S \in X$ , we denote the set of neighborhoods of  $S$  by  $\mathcal{N}_S$

**Exercise 3.0.2.** Let  $X$  be a topological space and  $A \subset X$ . Then  $A$  is open iff for each  $a \in A$ , there exists  $U_a \in \mathcal{N}_a$  such that  $U_a$  is an open of  $a$  and  $U_a \subset A$ .

*Proof.* Suppose that  $A$  is open. Let  $a \in A$ . Then  $A \in \mathcal{N}_a$ ,  $A$  is an open and  $A \subset A$ . Conversely, suppose that for each  $a \in A$ , there exists  $U_a \in \mathcal{N}_a$  such that  $U_a$  is open and  $U_a \subset A$ . Then  $A = \bigcup_{a \in A} U_a$  is open.  $\square$

**Definition 3.0.3.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be topological spaces and  $f : X \rightarrow Y$ . Then

- (1)  $f$  is said to be **continuous** if for each  $B \in \mathcal{B}$ ,  $f^{-1}(B) \in \mathcal{A}$ .
- (2)  $f$  is said to be **open** if for each  $A \in \mathcal{A}$ ,  $f(A) \in \mathcal{B}$ .
- (3)  $f$  is said to be **closed** if for each  $A \subset X$ , if  $A^c \in \mathcal{A}$ , then  $f(A)^c \in \mathcal{B}$ .

**Exercise 3.0.4.** Let  $X, Y$  be topological spaces and  $\phi : X \rightarrow Y$  a homeomorphism. Then for each  $A \subset X$ ,

- (1)  $\overline{\phi(A)} = \phi(\overline{A})$
- (2)  $\phi(A)^\circ = \phi(A^\circ)$

*Proof.*

- (1) Let  $A \subset X$ . Since  $A \subset \overline{A}$ , we have that  $\phi(A) \subset \phi(\overline{A})$ . Since  $\overline{A}$  is closed,  $\phi(\overline{A})$  is closed and thus  $\overline{\phi(A)} \subset \phi(\overline{A})$ . Conversely, let  $x \in \phi(\overline{A})$ . Then  $\phi^{-1}(x) \in \overline{A}$ . Then there exists a net  $\langle y_\alpha \rangle \subset A$  such that  $y_\alpha \rightarrow \phi^{-1}(x)$ . Then  $\langle \phi(y_\alpha) \rangle \subset \phi(A)$  and  $\phi(y_\alpha) \rightarrow x$ . Thus  $x \in \overline{\phi(A)}$  and  $\phi(\overline{A}) \subset \overline{\phi(A)}$ .
- (2) Similar

$\square$

## 3.1. Semi-continuity.

**Definition 3.1.1.** Let  $X$  be a topological space,  $f : X \rightarrow (\infty, \infty]$  and  $x_0 \in X$ . Then  $f$  is said to be **lower semicontinuous (l.s.c.) at  $x_0$**  if

$$\liminf_{x \rightarrow x_0} f(x) \geq f(x_0)$$

and  $f$  is said to be **lower semicontinuous (l.s.c.)** if for each  $x_0 \in X$ ,  $f$  is lower semicontinuous at  $x_0$ .

**Exercise 3.1.2.** Let  $X$  be a topological space and  $f : X \rightarrow (\infty, \infty]$ . Then  $f$  is l.s.c. iff for each  $\alpha \in \mathbb{R}$ ,  $f^{-1}((\alpha, \infty])$  is open.

*Proof.* Suppose that  $f$  is l.s.c. Let  $\alpha \in \mathbb{R}$  and  $x_0 \in f^{-1}((\alpha, \infty])$ . Put  $\epsilon = f(x_0) - \alpha$ . By definition,

$$\sup_{V \in \mathcal{N}_{x_0}} \inf_{x \in V \setminus \{x_0\}} f(x) \geq f(x_0)$$

Choose  $V_\epsilon \in \mathcal{N}_{x_0}$  such that

$$\begin{aligned} \inf_{x \in V_\epsilon} f(x) &> f(x_0) - \epsilon \\ &= \alpha \end{aligned}$$

Then  $V_\epsilon^o \in \mathcal{N}_{x_0}$  is open and

$$\begin{aligned} V_\epsilon^o &\subset V_\epsilon \\ &\subset f^{-1}((\alpha, \infty]) \end{aligned}$$

So  $f^{-1}((\alpha, \infty])$  is open.

Conversely, suppose that for each  $\alpha \in \mathbb{R}$ ,  $f^{-1}((\alpha, \infty])$  is open. Let  $x_0 \in X$ . Put  $\alpha = f(x_0)$ . For  $n \in \mathbb{N}$ , define  $V_n = f^{-1}((f(x_0) - 1/n, \infty])$ . Then for each  $n \in \mathbb{N}$ ,  $V_n \in \mathcal{N}_{x_0}$  and

$$\begin{aligned} \liminf_{x \rightarrow x_0} f(x) &= \sup_{V \in \mathcal{N}_{x_0}} \inf_{x \in V \setminus \{x_0\}} f(x) \\ &\geq \sup_{n \in \mathbb{N}} \inf_{x \in V_n \setminus \{x_0\}} f(x) \\ &\geq \sup_{n \in \mathbb{N}} f(x_0) - 1/n \\ &= f(x_0) \end{aligned}$$

So  $f$  is l.s.c. □

## 4. BANACH SPACES

## 4.1. Introduction.

*Note 4.1.1.* In the following, we will consider vector spaces over  $\mathbb{C}$ . There are analogous results for real vector spaces as well, just replace every  $\mathbb{C}$  with  $\mathbb{R}$ .

**Definition 4.1.1.** Let  $X$  be a normed vector space. Then  $X$  is said to be a **Banach space** if  $X$  is complete.

**Definition 4.1.2.** Let  $X$  be a normed vector space and  $(x_i)_{i=1}^n \subset X$ . The series  $\sum_{i=1}^{\infty} x_i$  is said to **converge** if the sequence  $s_n := \sum_{i=1}^n x_i$  converges. The series  $\sum_{i=1}^{\infty} x_i$  is said to **converge absolutely** if  $\sum_{i \in \mathbb{N}} \|x_i\| < \infty$ .

**Exercise 4.1.3.** Let  $X$  be a normed vector space. Then  $X$  is complete iff for each  $(x_i)_{i \in \mathbb{N}} \subset X$ ,  $\sum_{i=1}^{\infty} x_i$  converges absolutely implies that  $\sum_{i=1}^{\infty} x_i$  converges.

**Hint:** Given a Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$ , obtain a subsequence  $(x_{n_j})_{j \in \mathbb{N}} \subset (x_n)_{n \in \mathbb{N}}$  such that for each  $j \in \mathbb{N}$ ,  $\|x_{n_{j+1}} - x_{n_j}\| < 2^{-j}$ . Define a new sequence  $(y_j)_{j \in \mathbb{N}} \subset X$  by

$$y_j = \begin{cases} x_{n_1} & j = 1 \\ x_{n_j} - x_{n_{j-1}} & j \geq 2 \end{cases}$$

*Proof.* Suppose that  $X$  is complete. Let  $(x_i)_{i \in \mathbb{N}} \subset X$ . Suppose that  $\sum_{i=1}^{\infty} x_i$  converges absolutely. Let  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $m, n \in \mathbb{N}$ , if  $m, n \geq N$  and  $m < n$ , then  $\sum_{i=m+1}^n \|x_i\| < \epsilon$ . Let  $m, n \in \mathbb{N}$ . Suppose that  $m < n$ . Then

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right\| \\ &= \left\| \sum_{i=m+1}^n x_i \right\| \\ &\leq \sum_{i=m+1}^n \|x_i\| \\ &< \epsilon \end{aligned}$$

Thus  $(s_n)_{n \in \mathbb{N}}$  is Cauchy. Since  $X$  is complete,  $\sum_{i=1}^{\infty} x_i$  converges. Conversely, Suppose that for each  $(x_i)_{i \in \mathbb{N}} \subset X$ ,  $\sum_{i=1}^{\infty} x_i$  converges absolutely implies that  $\sum_{i=1}^{\infty} x_i$  converges. Let  $(x_i)_{i \in \mathbb{N}} \subset X$  be Cauchy. Proceed inductively to create a strictly increasing sequence  $(n_i)_{i \in \mathbb{N}} \subset \mathbb{N}$  such that for each  $m, n \in \mathbb{N}$ , if  $m, n \geq n_i$ , then  $\|x_m - x_n\| < 2^{-i}$ . Define  $(y_i)_{i \in \mathbb{N}} \subset X$  by

$$y_i = \begin{cases} x_{n_1} & i = 1 \\ x_{n_i} - x_{n_{i-1}} & i \geq 2 \end{cases}$$

Then  $\sum_{i=1}^k y_i = x_{n_k}$  and

$$\begin{aligned} \sum_{i \in \mathbb{N}} \|y_i\| &= \|x_{n_1}\| + \sum_{i \in \mathbb{N}} \|x_{n_i} - x_{n_{i-1}}\| \\ &\leq \|x_{n_1}\| + 2 \sum_{i \in \mathbb{N}} 2^{-i} \\ &= \|x_{n_1}\| + 2 \end{aligned}$$

Hence  $(x_{n_k})_{k \in \mathbb{N}} = (\sum_{i=1}^k y_i)_{i \in \mathbb{N}}$  converges. Since  $(x_i)_{i \in \mathbb{N}}$  is cauchy and has a convergent subsequence, it converges. So  $X$  is complete.  $\square$

**Definition 4.1.4.** Let  $X, Y$  be a normed vector spaces. A linear map  $T : X \rightarrow Y$  is said to be **bounded** if there exists  $C \geq 0$  such that for each  $x \in X$ ,

$$\|Tx\| \leq C\|x\|$$

We define

$$L(X, Y) = \{T : X \rightarrow Y : T \text{ is bounded}\}$$

**Exercise 4.1.5.** Let  $X, Y$  be a normed vector spaces and  $T : X \rightarrow Y$  a linear map. Then  $T$  is bounded iff there exists  $r, s > 0$  such that  $T(B(0, r)) \subset B(0, s)$

*Proof.* Suppose that  $T$  is bounded. Then there exists  $C \geq 0$  such that for each  $x \in X$ ,  $\|Tx\| \leq C\|x\|$ . Thus  $T(B(0, 1)) \subset B(0, C + 1)$ . Conversely. Suppose that there exists  $r, s > 0$  such that  $T(B(0, r)) \subset B(0, s)$ . Define  $C = \frac{2s}{r}$ . Let  $x \in X$ . Put  $\alpha = \frac{r}{2\|x\|}$ . Then  $\alpha x \in B(0, r)$ . So  $T(\alpha x) = \alpha T(x) \in B(0, s)$ . Hence

$$\begin{aligned} \|T(\alpha x)\| &= \|\alpha T(x)\| \\ &= |\alpha| \|T(x)\| \\ &= \frac{r}{2\|x\|} \|T(x)\| \\ &< s. \end{aligned}$$

Thus

$$\|Tx\| < \frac{2s}{r} \|x\| = C\|x\|$$

So  $T$  is bounded.  $\square$

**Theorem 4.1.1.** Let  $X, Y$  be normed vector spaces and  $T : X \rightarrow Y$  a linear map. Then the following are equivalent:

- (1)  $T$  is continuous
- (2)  $T$  is continuous at  $x = 0$
- (3)  $T$  is bounded

*Proof.*

- (1)  $\implies$  (2):

Trivial

- (2)  $\implies$  (3):

Suppose that  $T$  is continuous at  $x = 0$ . Then there exists  $\delta > 0$  such that for each  $x \in X$ , if  $\|x\| < \delta$ , then  $\|Tx\| < 1$ . Choose  $C = \frac{2}{\delta}$ . If  $x = 0$ , then  $\|Tx\| \leq C\|x\|$ .

Suppose that  $\|x\| \neq 0$ . Define  $y = \frac{\delta}{2\|x\|}x$ . Then  $\|y\| < \delta$ . So

$$\begin{aligned} 1 &> \|Ty\| \\ &= \frac{\delta}{2\|x\|} \|Tx\| \end{aligned}$$

Thus

$$\begin{aligned} \|Tx\| &< \frac{2}{\delta} \|x\| \\ &= C\|x\| \end{aligned}$$

Hence  $T$  is bounded.

- (3)  $\implies$  (1)

Suppose that  $T$  is bounded. Then there exists  $C \geq 0$  such that for each  $x \in X$ ,  $\|Tx\| \leq C\|x\|$ . Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{C+1}$ . Let  $x, y \in X$ . Suppose that  $\|x - y\| < \delta$ . Then

$$\begin{aligned}\|Tx - Ty\| &= \|T(x - y)\| \\ &\leq C\|x - y\| \\ &< (C + 1)\delta \\ &= \epsilon\end{aligned}$$

So  $T$  is continuous. □

**Definition 4.1.6.** Let  $X, Y$  be normed vector spaces. Define  $\|\cdot\| : L(X, Y) \rightarrow [0, \infty)$  by

$$\|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

We call  $\|\cdot\|$  the **operator norm on  $L(X, Y)$**

**Exercise 4.1.7.** Let  $X, Y$  be normed vector spaces. If  $X \neq \{0\}$ , then the operator norm on  $L(X, Y)$  is given by:

$$(1) \|T\| = \sup_{\|x\|=1} \|Tx\|$$

$$(2) \|T\| = \sup_{x \neq 0} \|x\|^{-1} \|Tx\|$$

$$(3) \|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$$

*Proof.* Since  $X \neq \{0\}$ , the supremums in (1) and (2) are well defined. Let  $T \in L(X, Y)$ . By linearity of  $T$ , the sets over which the supremums are taken in (1) and (2) are the same. So (1) and (2) are equal.

Now, put  $M = \sup_{\|x\|=1} \|Tx\|$ ,  $m = \inf\{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$  and let  $x \in X$ . If  $\|x\| = 0$ , then  $\|Tx\| \leq M\|x\|$ . Suppose that  $\|x\| \neq 0$ . Then

$$\begin{aligned}\|Tx\| &= \left( \|T(x/\|x\|)\| \right) \|x\| \\ &\leq M\|x\|\end{aligned}$$

Hence  $M \in \{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$ . Therefore  $m \leq M$

Let  $C \in \{C \geq 0 : \text{for each } x \in X, \|Tx\| \leq C\|x\|\}$ . Suppose that  $\|x\| = 1$ . Then  $\|Tx\| \leq C\|x\| = C$ . So  $M \leq C$ . Therefore  $M \leq m$ . So  $M = m$  and the supremum in (1) is the same as the infimum in (3). □

*Note 4.1.2.* From here on, unless stated otherwise, we assume  $X \neq 0$ .

**Exercise 4.1.8.** Let  $X, Y$  be normed vector spaces and  $T \in L(X, Y)$ . Then for each  $x \in X$ ,  $\|Tx\| \leq \|T\|\|x\|$

*Proof.* This is just part of the previous exercise. Let  $x \in X$ . If  $x = 0$ , then  $\|Tx\| \leq \|T\|\|x\|$ . Suppose that  $x \neq 0$ . Then  $\|Tx\| = \|T(x/\|x\|)\| \|x\| \leq \|T\| \|x\|$   $\square$

**Exercise 4.1.9.** Let  $X, Y$  be normed vector spaces. Then the operator norm is a norm on  $L(X, Y)$ .

*Proof.* Let  $S, T \in L(X, Y)$  and  $\alpha \in \mathbb{C}$ . For each  $x \in X$ , we have that

$$\begin{aligned} \|(S + T)x\| &= \|Sx + Tx\| \\ &\leq \|Sx\| + \|Tx\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= (\|S\| + \|T\|)\|x\| \end{aligned}$$

So  $\|S + T\| \leq \|S\| + \|T\|$ .

Using the definition of  $\|T\|$ , we see that

$$\begin{aligned} \|\alpha T\| &= \sup_{\|x\|=1} \|(\alpha T)x\| \\ &= \sup_{\|x\|=1} |\alpha| \|Tx\| \\ &= |\alpha| \sup_{\|x\|=1} \|Tx\| \\ &= |\alpha| \|T\| \end{aligned}$$

So  $\|\alpha S\| = |\alpha| \|S\|$ .

Suppose that  $\|T\| = 0$ . Let  $x \in X$ . Then  $\|Tx\| \leq \|T\|\|x\| = 0$ . So  $Tx = 0$ . Since  $x \in X$  is arbitrary, we have that  $T = 0$ .  $\square$

**Exercise 4.1.10.** Let  $X$  be a normed vector space. Then addition and scalar multiplication are continuous on  $X \times X$  and  $\|\cdot\| : X \rightarrow [0, \infty)$  is continuous.

*Proof.* Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{2}$ . Let  $(x_1, y_1), (x_2, y_2) \in X \times X$ . Suppose that  $\|(x_1, y_1) - (x_2, y_2)\| = \max\{\|x_1 - x_2\|, \|y_1 - y_2\|\} < \delta$ . Then

$$\begin{aligned} \|(x_1 + y_1) - (x_2 + y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\ &\leq \|x_1 - x_2\| + \|y_1 - y_2\| \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

Hence addition is uniformly continuous.

Let  $(\lambda_1, x_1) \in \mathbb{C} \times X$  and  $\epsilon > 0$ . Choose  $\delta = \min\{\frac{\epsilon}{2(|\lambda_1| + \|x_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$ . Let  $(\lambda_2, x_2) \in \mathbb{C} \times X$ .



Suppose that  $\|(\lambda_1, x_1) - (\lambda_2, x_2)\| = \max\{|\lambda_1 - \lambda_2|, \|x_1 - x_2\|\} < \delta$ . Then

$$\begin{aligned}
 \|\lambda_1 x_1 - \lambda_2 x_2\| &= \|\lambda_1 x_1 - \lambda_1 x_2 + \lambda_1 x_2 - \lambda_2 x_2\| \\
 &= \|\lambda_1(x_1 - x_2) + (\lambda_1 - \lambda_2)x_2\| \\
 &\leq |\lambda_1|\|x_1 - x_2\| + |\lambda_1 - \lambda_2|\|x_2\| \\
 &\leq |\lambda_1|\|x_1 - x_2\| + |\lambda_1 - \lambda_2|(\|x_1 - x_2\| + \|x_1\|) \\
 &< |\lambda_1|\delta + \delta(\delta + \|x_1\|) \\
 &= (|\lambda_1| + \|x_1\|)\delta + \delta^2 \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon
 \end{aligned}$$

Since  $(\lambda_1, x_1) \in \mathbb{C} \times X$  is arbitrary, scalar multiplication is continuous.

Let  $\epsilon > 0$ . Choose  $\delta = \epsilon$ . Let  $x, y \in X$ . Suppose that  $\|x - y\| < \delta$ . Then

$$\begin{aligned}
 \left| \|x\| - \|y\| \right| &\leq \|x - y\| \\
 &< \delta \\
 &= \epsilon
 \end{aligned}$$

So  $\|\cdot\| : X \rightarrow [0, \infty)$  is uniformly continuous.  $\square$

**Exercise 4.1.11.** Let  $X, Y$  be normed vector spaces. If  $Y$  is complete, then so is  $L(X, Y)$ .

*Proof.* Suppose that  $Y$  is complete. Let  $(T_n)_{n \in \mathbb{N}} \subset L(X, Y)$ . Suppose that  $(T_n)_{n \in \mathbb{N}}$  is Cauchy. Since for each  $m, n \in \mathbb{N}$ ,  $|\|T_m\| - \|T_n\|| \leq \|T_m - T_n\|$ , we have that  $(\|T_n\|)_{n \in \mathbb{N}} \subset [0, \infty)$  is Cauchy. Hence  $\lim_{n \rightarrow \infty} \|T_n\|$  exists.

Let  $x \in X$  and  $m, n \in \mathbb{N}$ . Then

$$\begin{aligned}
 \|T_m x - T_n x\| &= \|(T_m - T_n)x\| \\
 &\leq \|T_m - T_n\|\|x\|
 \end{aligned}$$

So  $(T_n x)_{n \in \mathbb{N}} \subset Y$  is Cauchy and hence converges. Define  $T : X \rightarrow Y$  by  $Tx = \lim_{n \rightarrow \infty} T_n x$ .

Since addition and scalar multiplication are continuous,  $T$  is linear. Let  $x \in X$  and  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then  $\|Tx - T_n x\| < \epsilon$ . Then for each  $n \in \mathbb{N}$ , if  $n \geq N$  we have that

$$\begin{aligned}
 \|Tx\| &\leq \|Tx - T_n x\| + \|T_n x\| \\
 &< \epsilon + \|T_n x\| \\
 &\leq \epsilon + \|T_n\|\|x\|
 \end{aligned}$$

Thus  $\|Tx\| \leq \epsilon + (\lim_{n \rightarrow \infty} \|T_n\|)\|x\|$ . Since  $\epsilon > 0$  is arbitrary,  $\|Tx\| \leq (\lim_{n \rightarrow \infty} \|T_n\|)\|x\|$ . Thus  $T \in L(X, Y)$  and  $\|T\| \leq \lim_{n \rightarrow \infty} \|T_n\|$ .

Note that since addition, scalar multiplication and  $\|\cdot\|$  are continuous, we have that for each  $n \in \mathbb{N}$  and  $x \in X$ ,  $\|(T_n - T_m)x\|$  converges to  $\|(T_n - T)x\|$  because

$$\begin{aligned} \lim_{m \rightarrow \infty} \|(T_n - T_m)x\| &= \lim_{m \rightarrow \infty} \|T_n x - T_m x\| \\ &= \|T_n x - \lim_{m \rightarrow \infty} T_m x\| \\ &= \|T_n x - T x\| \\ &= \|(T_n - T)x\| \end{aligned}$$

Let  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $m, n \in \mathbb{N}$  if  $n, m \geq N$ , then  $\|T_n - T_m\| < \epsilon$ . Then for each  $n \in \mathbb{N}$  if  $n \geq N$ , then for each  $x \in X$ ,

$$\|(T_n - T_m)x\| \leq \|T_n - T_m\| \|x\| < \epsilon \|x\|$$

Combining this with the previous fact, we see that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then for each  $x \in X$ ,

$$\|(T_n - T)x\| \leq \epsilon \|x\|$$

In particular, for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then

$$\|T_n - T\| = \sup_{\|x\|=1} \|(T_n - T)x\| \leq \epsilon$$

This implies that  $T_n$  converges to  $T$  in  $L(X, Y)$ . Since

$$|\|T_n\| - \|T\|| \leq \|T_n - T\|$$

it is clear that  $\lim_{n \rightarrow \infty} \|T_n\| = \|T\|$  □

**Definition 4.1.12.** Let  $X$  be a normed vector space and  $M \subset X$  a closed subspace. Define  $\|\cdot\| : X/M \rightarrow [0, \infty)$  by

$$\|x + M\| := \inf_{y \in M} \|x + y\|$$

We call  $\|\cdot\|$  the **subspace norm on  $X/M$**

**Exercise 4.1.13.** Let  $X$  be a normed vector space and  $M \subsetneq X$  a proper, closed subspace of  $X$ . Then

- (1) The previously defined subspace norm on  $X/M$  is well defined and is a norm.
- (2) For each  $\epsilon > 0$ , there exists  $x \in X$  such that  $\|x\| = 1$  and  $\|x + M\| \geq 1 - \epsilon$ .
- (3) The projection map  $\pi : X \rightarrow X/M$  defined by  $\pi(x) = x + M$  is continuous and  $\|\pi\| = 1$ .
- (4) If  $X$  is complete, then  $X/M$  is complete.

*Proof.* (1) Let  $x, y \in X$  and  $\alpha \in \mathbb{C}$ . Suppose that  $x + M = y + M$ . Then there exists  $m \in M$  such that  $x = y + m$ . Since  $M$  is a subspace, the map  $T : M \rightarrow M$  given by  $Tx = x + m$  is a bijection. So

$$\inf_{z \in M} \|y + m + z\| = \inf_{z \in M} \|y + z\|$$

which implies that

$$\begin{aligned}
 \|x + M\| &= \inf_{z \in M} \|x + z\| \\
 &= \inf_{z \in M} \|y + m + z\| \\
 &= \inf_{z \in M} \|y + z\| \\
 &= \|y + M\|
 \end{aligned}$$

So  $\|\cdot\| : X/M \rightarrow [0, \infty)$  is well defined.

We observe that for each  $z, w \in M$ ,

$$\|x + y + z\| \leq \|x + w\| + \|y + w + z\|$$

Taking infimums over  $M$  with respect to  $z$  in this inequality implies that for each  $w \in M$ ,

$$\begin{aligned}
 \inf_{z \in M} \|x + y + z\| &\leq \inf_{z \in M} \left( \|x + w\| + \|y + w + z\| \right) \\
 &= \|x + w\| + \inf_{z \in M} \|y + w + z\|
 \end{aligned}$$

Again we use the fact that for each  $w \in M$ ,

$$\inf_{z \in M} \|y + w + z\| = \inf_{z \in M} \|y + z\|$$

This implies that for each  $w \in M$ ,

$$\inf_{z \in M} \|x + y + z\| \leq \|x + w\| + \inf_{z \in M} \|y + z\|$$

Therefore, taking infimums over  $M$  with respect to  $w$  in this inequality yields

$$\begin{aligned}
 \|x + y + M\| &= \inf_{z \in M} \|x + y + z\| \\
 &\leq \inf_{w \in M} \left( \|x + w\| + \inf_{z \in M} \|y + z\| \right) \\
 &= \inf_{w \in M} \|x + w\| + \inf_{z \in M} \|y + z\| \\
 &= \|x + M\| + \|y + M\|
 \end{aligned}$$

If  $\alpha = 0$ , then  $\alpha x = 0$ . Choosing  $z = 0 \in M$  gives  $\|\alpha x + M\| = 0 = |\alpha| \|x + M\|$ . Suppose that  $\alpha \neq 0$ . Then the map  $T : M \rightarrow M$  given by  $Tx = \alpha^{-1}x$  is a bijection and thus  $\inf_{z \in M} \|x + \alpha^{-1}z\| = \inf_{z \in M} \|x + z\|$ . Hence we have that

$$\begin{aligned}
 \|\alpha x + M\| &= \inf_{z \in M} \|\alpha x + z\| \\
 &= \inf_{z \in M} |\alpha| \|x + \alpha^{-1}z\| \\
 &= |\alpha| \inf_{z \in M} \|x + \alpha^{-1}z\| \\
 &= |\alpha| \inf_{z \in M} \|x + z\| \\
 &= |\alpha| \|x + M\|
 \end{aligned}$$

Suppose that  $\|x\| = 0$ . Choose a sequence  $(z_n)_{n \in \mathbb{N}} \subset M$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x - z_n\| &= \inf_{z \in M} \|x + z\| \\ &= 0 \end{aligned}$$

Then  $\lim_{n \rightarrow \infty} z_n = x$ . Since  $M$  is closed,  $x \in M$ . Hence  $x + M = 0 + M$ .

- (2) Since  $M$  is a proper subspace, there exists  $v \in X$  such that  $v \notin M$ . Then  $\|v + M\| \neq 0$ . Let  $\epsilon > 0$ . Then  $(1 - \epsilon)^{-1}\|v + M\| > \|v + M\|$ . So there exists  $z \in M$  such that

$$0 < \|v + M\| \leq \|v + z\| < (1 - \epsilon)^{-1}\|v + M\|$$

Choose  $x = \|v + z\|^{-1}(v + z)$ . Then  $\|x\| = 1$  and

$$\begin{aligned} \|x + M\| &= \|v + z\|^{-1}\|v + z + M\| \\ &= \|v + z\|^{-1}\|v + M\| \\ &> 1 - \epsilon \end{aligned}$$

- (3) Let  $x \in X$ . Taking  $z = 0$ , we see that  $\|\pi(x)\| = \|x + M\| \leq \|x + z\| = \|x\|$ . So  $\pi$  is bounded and in particular,

$$\sup_{\|x\|=1} \|\pi(x)\| \leq 1$$

From (2) we see that

$$\sup_{\|x\|=1} \|\pi(x)\| \geq 1$$

Hence  $\|\pi\| = 1$ .

- (4) Suppose that  $X$  is complete. Let  $(x_i + M)_{i \in \mathbb{N}} \subset X/M$ . Suppose that  $\sum_{i \in \mathbb{N}} \|x_i + M\| < \infty$ . Let  $\epsilon > 0$ . Then for each  $i \in \mathbb{N}$ , there exists  $z_i \in M$  such that  $\|x_i + z_i\| < \|x_i + M\| + \epsilon 2^{-i}$ . Define the sequence  $(a_i)_{i \in \mathbb{N}} \subset X$  by  $a_i = x_i + z_i$ . Then we have

$$\begin{aligned} \sum_{i \in \mathbb{N}} \|a_i\| &= \sum_{i \in \mathbb{N}} \|x_i + z_i\| \\ &\leq \sum_{i \in \mathbb{N}} \left( \|x_i + M\| + \epsilon 2^{-i} \right) \\ &= \sum_{i \in \mathbb{N}} \|x_i + M\| + \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, it follows that

$$\sum_{i \in \mathbb{N}} \|a_i\| \leq \sum_{i \in \mathbb{N}} \|x_i + M\| < \infty$$

Since  $X$  is complete,  $\sum_{i=1}^{\infty} a_i$  converges in  $X$ . Define  $(s_n)_{n \in \mathbb{N}} \subset X$  and  $s \in X$  by  $s_n = \sum_{i=1}^n a_i$  and  $s = \sum_{i=1}^{\infty} a_i$ . Since  $\lim_{n \rightarrow \infty} s_n = s$ , and  $\pi : X \rightarrow X/M$  is continuous, it follows that  $\lim_{n \rightarrow \infty} \pi(s_n) = \pi(s)$ . Since

$$\begin{aligned} \pi(s_n) &= \sum_{i=1}^n a_i + M \\ &= \sum_{i=1}^n x_i + M \end{aligned}$$

We have that  $\sum_{i=1}^{\infty} x_i + M$  converges which implies that  $X/M$  is complete. □

**Exercise 4.1.14.** Let  $X, Y$  be normed vector spaces and  $T \in L(X, Y)$ . Then

- (1)  $\ker T$  is closed
- (2) there exists a unique map  $S : X/\ker T \rightarrow T(X)$  such that  $T = S \circ \pi$ . Furthermore  $S$  is a bounded linear bijection and  $\|S\| = \|T\|$ .

*Proof.* (1) Since  $T$  is continuous and  $\ker T = T^{-1}(\{0\})$ , we have that  $\ker T$  is closed.

- (2) Suppose that there exists  $S_1, S_2 \in L(X/\ker T, T(X))$  such that  $T = S_1 \circ \pi$  and  $T = S_2 \circ \pi$ . Let  $x \in X$ . Then

$$S_1(x + \ker T) = S_1(\pi(x)) = T(x) = S_2(\pi(x)) = S_2(x + \ker T)$$

So  $S_1 = S_2$ . Therefore such a map is unique.

Define  $S : X/\ker T \rightarrow T(X)$  by  $S(x + \ker T) = T(x)$ . Then  $S$  is clearly a linear bijection that satisfies  $T = S \circ \pi$ . Let  $x \in X$  and  $z \in \ker T$ . Then

$$\begin{aligned} \|S(x + \ker T)\| &= \|T(x)\| \\ &= \|T(x + z)\| \\ &\leq \|T\| \|x + z\| \end{aligned}$$

Thus

$$\|S(x + \ker T)\| \leq \|T\| \inf_{z \in \ker T} \|x + z\| = \|T\| \|x + \ker T\|$$

So  $S$  is bounded and  $\|S\| \leq \|T\|$ . This implies that

$$\|T\| = \|S \circ \pi\| \leq \|S\| \|\pi\| = \|S\|$$

Thus  $\|S\| = \|T\|$ . □

**Exercise 4.1.15.** Let  $X, Y$  be normed vector spaces. Define  $\phi : L(X, Y) \times X \rightarrow Y$  by  $\phi(T, x) = Tx$ . Then  $\phi$  is continuous.

*Proof.* Let  $(T_1, x_1) \in L(X, Y) \times X$  and  $\epsilon > 0$ . Choose  $\delta = \min\{\frac{\epsilon}{2(\|x_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$ . Let  $(t_2, x_2) \in L(X, Y) \times X$ . Suppose that

$$\|(T_1, x_1) - (T_2, x_2)\| = \max\{\|T_1 - T_2\|, \|x_1 - x_2\|\} < \delta$$

Then

$$\begin{aligned}
\|\phi(T_1, x_1) - \phi(T_2, x_2)\| &= \|T_1x_1 - T_2x_2\| \\
&= \|T_1x_1 - T_2x_1 + T_2x_1 - T_2x_2\| \\
&\leq \|(T_1 - T_2)x_1\| + \|T_2(x_1 - x_2)\| \\
&\leq \|T_1 - T_2\|\|x_1\| + \|T_2\|\|x_1 - x_2\| \\
&\leq \|T_1 - T_2\|\|x_1\| + (\|T_1 - T_2\| + \|T_1\|)\|x_1 - x_2\| \\
&< \delta\|x_1\| + (\delta + \|T_1\|)\delta \\
&= \delta(\|T_1\| + \|x_1\|) + \delta^2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So  $\phi$  is continuous.  $\square$

**Exercise 4.1.16.** Let  $X$  be a normed vector space and  $M \subset X$  a subspace. Then  $\overline{M}$  is a subspace.

*Proof.* Let  $x, y \in \overline{M}$  and  $\alpha \in \mathbb{C}$ . Then there exist sequences  $(x_n)_{n \in \mathbb{N}} \subset M$  and  $(y_n)_{n \in \mathbb{N}} \subset M$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . Since  $M$  is a subspace,  $(x_n + y_n)_{n \in \mathbb{N}} \subset M$  and  $(\alpha x_n)_{n \in \mathbb{N}} \subset M$ . Since addition and scalar multiplication are continuous, we have that  $x_n + y_n \rightarrow x + y$  and  $\alpha x_n \rightarrow \alpha x$ . Thus  $x + y \in \overline{M}$  and  $\alpha x \in \overline{M}$  and hence  $\overline{M}$  is a subspace.  $\square$

**Exercise 4.1.17.** Let  $X, Y, Z$  be normed vector spaces,  $T \in L(X, Y)$  and  $S \in L(Y, Z)$ . Define  $ST : X \rightarrow Z$  by  $STx = S(Tx)$ . Then  $ST \in L(X, Z)$  and  $\|ST\| \leq \|S\|\|T\|$ .

*Proof.* Clearly  $ST$  is linear. Let  $x \in X$ . Then

$$\begin{aligned}
\|STx\| &= \|S(Tx)\| \\
&\leq \|S\|\|Tx\| \\
&\leq \|S\|\|T\|\|x\|
\end{aligned}$$

So  $\|ST\| \leq \|S\|\|T\|$ .  $\square$

**Definition 4.1.18.** Let  $X, Y$  be normed vector spaces and  $T \in L(X, Y)$ . Then  $T$  is said to be **invertible** or an **isomorphism** if  $T$  is a bijection and  $T^{-1} \in L(Y, X)$ .

**Definition 4.1.19.** Let  $X$  be a Banach space. Define  $GL(X) := \{T \in L(X, X) : T \text{ is invertible}\}$ .

**Exercise 4.1.20.** Let  $X$  be a Banach space. Then

- (1) For each  $T \in L(X, X)$ , if  $\|I - T\| < 1$ , then  $T$  is invertible and

$$T^{-1} = \sum_{n=0}^{\infty} (I - T)^n$$

- (2) For each  $S, T \in L(X, X)$ , if  $S$  is invertible and  $\|S - T\| < \|S^{-1}\|^{-1}$ , then  $T$  is invertible.  
(3)  $GL(X)$  is open.

*Proof.*

(1) Let  $T \in L(X, X)$ . Suppose that  $\|I - T\| < 1$ . Then

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n < \infty$$

Since  $X$  is a complete, so is  $L(X, X)$  and thus  $\sum_{n=0}^{\infty} (I - T)^n$  converges in  $L(X, X)$ .

Define  $(S_k)_{k=0}^{\infty} \subset L(X, X)$  and  $S \in L(X, X)$  by  $S_k = \sum_{n=0}^k (I - T)^n$  and

$S = \sum_{n=0}^{\infty} (I - T)^n$ . Then for each  $k \in \mathbb{N}$ ,

$$\begin{aligned} S_k T &= S_k - S_k(I - T) \\ &= (I - T)^0 - (I - T)^{k+1} \\ &= I - (I - T)^{k+1} \end{aligned}$$

and  $\|S_k T - I\| \leq \|I - T\|^{k+1}$ . Since multiplication on Banach algebras is continuous, we have that

$$ST = \left( \lim_{k \rightarrow \infty} S_k \right) T = \lim_{k \rightarrow \infty} S_k T = I$$

Similarly  $TS = I$ . Thus  $T$  is invertible and  $T^{-1} = S \in L(X, X)$ .

(2) Let  $S, T \in L(X, X)$ . Suppose that  $S$  is invertible and  $\|S - T\| < \|S^{-1}\|^{-1}$ . Then

$$\begin{aligned} \|I - S^{-1}T\| &= \|S^{-1}(S - T)\| \\ &\leq \|S^{-1}\| \|S - T\| \\ &< 1 \end{aligned}$$

So  $S^{-1}T$  is invertible. Thus  $T = S(S^{-1}T)$  is invertible.

(3) Let  $T \in GL(X)$ . Choose  $\delta = \|T^{-1}\|^{-1}$ . By (2),  $B(T, \delta) \subset GL(X)$ .

□

**Definition 4.1.21.** Let  $(X_n)_{n \in \mathbb{N}}$  be a collection of normed vector spaces. Put  $X = \bigoplus_{n \in \mathbb{N}} X_n$ .

Let  $p \in [1, \infty]$  and define  $\|\cdot\|_p : X \rightarrow [0, \infty)$  by

$$\|(x_n)_{n \in \mathbb{N}}\|_p = \begin{cases} \left( \sum_{n \in \mathbb{N}} \|x_n\|^p \right)^{1/p} & p < \infty \\ \sup_{n \in \mathbb{N}} \|x_n\| & p = \infty \end{cases}$$

We define

$$\bigoplus_{n \in \mathbb{N}}^p X_n = \{x \in X : \|x\|_p < \infty\}$$

and

$$\bigoplus_{n \in \mathbb{N}}^0 X_n = \left\{ x \in \bigoplus_{n \in \mathbb{N}}^{\infty} X_n : \lim_{n \rightarrow \infty} \|x_n\| = 0 \right\}$$

**Exercise 4.1.22.** Let  $(X_n)_{n \in \mathbb{N}}$  be a collection of Banach spaces. Then for each  $p \in [1, \infty] \cup \{0\}$ ,  $\bigoplus_{n \in \mathbb{N}}^p X_n$  is a Banach space.

**Definition 4.1.23.** Let  $X_1, \dots, X_n, Y$  be vector spaces and  $T : \bigoplus_{i=1}^n X_i \rightarrow Y$ . Then  $T$  is said to be **multilinear** if for each  $x_1 \in X_1, \dots, x_n \in X_n$ , and  $i \in \{1, \dots, n\}$  the maps  $T_i : X_i \rightarrow Y$  defined by

$$T_i(x) = T(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$$

are linear.

**Definition 4.1.24.** Let  $X_1, \dots, X_n, Y$  be normed vector spaces and  $T : \bigoplus_{i=1}^n X_i \rightarrow Y$  multilinear. Then  $T$  is said to be **bounded** if there exists  $C \geq 0$  such that for each  $x_1, \dots, x_n \in X$ ,

$$\|T(x_1, \dots, x_n)\| \leq C\|x_1\| \cdots \|x_n\|$$

**Exercise 4.1.25.** Let  $X_1, \dots, X_n, Y$  be normed vector spaces and  $T : \bigoplus_{i=1}^n X_i \rightarrow Y$  multilinear. Then the following are equivalent:

- (1)
- (2)
- (3)



## 4.2. Linear and Sublinear Functionals.

### Definition 4.2.1.

- (1) Let  $X$  be a  $\mathbb{C}$ -vector space and  $T : X \rightarrow \mathbb{C}$ . Then  $T$  is said to be a **linear functional on  $X$**  if  $T$  is linear. We define the **dual space of  $X$** , denoted  $X^*$ , by  $X^* = \{T : X \rightarrow \mathbb{C} : T \text{ is linear}\}$
- (2) If  $X$  is a normed  $\mathbb{C}$ -vector space, then  $T$  is said to be a **bounded linear functional on  $X$**  if  $T \in L(X, \mathbb{C})$ . We define the **dual space of  $X$** , denoted  $X^*$ , by  $X^* = L(X, \mathbb{C})$ .

*Note 4.2.1.* We define  $X^*$  similarly when  $X$  is an  $\mathbb{R}$ -vector space or normed  $\mathbb{R}$ -vector space.

**Definition 4.2.2.** Let  $X$  be a normed vector space and  $p : X \rightarrow \mathbb{R}$ . Then  $p$  is said to be a **sublinear functional** if for each  $x, y \in X$ ,  $\lambda \geq 0$ ,

- (1)  $p(x + y) \leq p(x) + p(y)$
- (2)  $p(\lambda x) = \lambda p(x)$

**Exercise 4.2.3.** Let  $X$  be a vector space and  $\|\cdot\| : X \rightarrow [0, \infty)$  be a seminorm, then  $\|\cdot\|$  is a sublinear functional.

*Proof.* Clear □

**Exercise 4.2.4.** Let  $X$  be a vector space and  $p : X \rightarrow \mathbb{R}$  a sublinear functional. Then for each  $x, y \in X$

- (1)  $-p(-x) \leq p(x)$
- (2)  $-p(y - x) \leq p(x) - p(y) \leq p(x - y)$

*Proof.* Let  $x, y \in X$ .

- (1) We have

$$\begin{aligned} 0 &= p(0) \\ &= p(x - x) \\ &\leq p(x) + p(-x) \end{aligned}$$

So  $-p(-x) \leq p(x)$ .

- (2) We have

$$\begin{aligned} p(x) &= p(x - y + y) \\ &\leq p(x - y) + p(y) \end{aligned}$$

So  $p(x) - p(y) \leq p(x - y)$ . Switching  $x$  and  $y$  gives us  $p(y) - p(x) \leq p(y - x)$  and multiplying both sides by  $-1$  yields  $-p(y - x) \leq p(x) - p(y)$

Putting these two together, we see that

$$-p(y - x) \leq p(x) - p(y) \leq p(x - y)$$

□

**Definition 4.2.5.** Let  $X$  be a normed vector space and  $p : X \rightarrow \mathbb{R}$  a sublinear functional. Then  $p$  is said to be **bounded** if there exists  $M > 0$  such that for each  $x \in X$ ,  $p(x) \leq M\|x\|$ .

**Exercise 4.2.6.** Let  $X$  be a normed vector space and  $p : X \rightarrow \mathbb{R}$  a sublinear functional. Then  $p$  is bounded iff  $p$  is Lipschitz.

*Proof.* Suppose that  $p$  is bounded. Then there exists  $M > 0$  such that for each  $x \in X$ ,  $p(x) \leq M\|x\|$ . Let  $x, y \in X$ . Then the previous exercise implies that

$$\begin{aligned} -M\|x - y\| &= -M\|y - x\| \\ &\leq -p(y - x) \\ &\leq p(x) - p(y) \\ &\leq p(x - y) \\ &\leq M\|x - y\| \end{aligned}$$

So that

$$|p(x) - p(y)| \leq M\|x - y\|$$

and  $p$  is Lipschitz. Conversely, suppose that  $p$  is Lipschitz. Then there exists  $M > 0$  such that for each  $x, y \in X$ ,  $|p(x) - p(y)| \leq M\|x - y\|$ . Let  $x \in X$ . Then

$$\begin{aligned} p(x) &\leq |p(x)| \\ &= |p(x) - p(0)| \\ &\leq M\|x - 0\| \\ &\leq M\|x\| \end{aligned}$$

So  $p$  is bounded. □

**Theorem 4.2.1. Hahn-Banach Theorem:** Let  $X$  be a vector space,  $p : X \rightarrow \mathbb{R}$  a sublinear functional,  $M \subset X$  a subspace and  $f : M \rightarrow \mathbb{R}$  a linear functional. If for each  $x \in M$ ,  $f(x) \leq p(x)$ , then there exists a linear functional  $F : X \rightarrow \mathbb{R}$  such that for each  $x \in X$ ,  $F(x) \leq p(x)$  and  $F|_M = f$ .

**Exercise 4.2.7.** Let  $X$  be a vector space and  $p : X \rightarrow \mathbb{R}$  a sublinear functional. Then there exists  $F : X \rightarrow \mathbb{R}$  such that for each  $x \in X$ ,  $F(x) \leq p(x)$ .

*Proof.* Take  $M = \{0\}$  and  $f \equiv 0$  and apply the Hahn-Banach theorem. □

**Exercise 4.2.8. Equivalency of linearity (General Case)** Let  $X$  be a vector space and  $p : X \rightarrow \mathbb{R}$  a sublinear functional. Then the following are equivalent:

- (1) there exists a unique  $F \in X^*$  such that  $F \leq p$
- (2) for each  $x \in X$ ,  $-p(-x) = p(x)$
- (3)  $p$  is linear

**Hint:** If there exists  $x \in X$  such that  $-p(-x) \neq p(x)$ , define  $f_1, f_2 : \text{span}(x) \rightarrow \mathbb{R}$  by  $f_1(tx) = tp(x)$  and  $f_2(tx) = -tp(-x)$

*Proof.*

- (1)  $\Rightarrow$  (2):

Suppose that there exists a unique  $F \in X^*$  such that  $F \leq p$ . For the sake of contradiction, suppose that there exists  $x \in X$  such that  $-p(-x) \neq p(x)$ . Define  $f_1, f_2 : \text{span}(x) \rightarrow \mathbb{R}$  by

$$f_1(tx) = tp(x)$$

and

$$f_2(tx) = -tp(-x)$$

Let  $y \in \text{span}(x)$ . Then there exists  $t \in \mathbb{R}$  such that  $y = tx$ . Then for each  $k \in \mathbb{R}$ ,

$$\begin{aligned} f_1(ky) &= f_1(ktx) \\ &= ktp(x) \\ &= kf_1(tx) \\ &= kf_1(y) \end{aligned}$$

Similarly,  $f_2(ky) = kf_2(y)$  and so  $f_1, f_2 \in \text{span}(x)^*$ . If  $t \geq 0$ , then

$$\begin{aligned} f_1(y) &= f_1(tx) \\ &= tp(x) \\ &= p(tx) \\ &= p(y) \end{aligned}$$

If  $t < 0$ , then

$$\begin{aligned} f_1(y) &= f_1(tx) \\ &= tp(x) \\ &= -|t|p(x) \\ &= -p(|t|x) \\ &= -p(-tx) \\ &\leq p(tx) \\ &= p(y) \end{aligned}$$

So  $f_1 \leq p$  on  $\text{span}(x)$ . Similarly,  $f_2 \leq p$  on  $\text{span}(x)$ . The Hahn-Banach theorem implies that there exist  $F_1, F_2 \in X^*$  such that  $F_1, F_2 \leq p$  and  $F_1 = f_1, F_2 = f_2$  on  $\text{span}(x)$ . By the assumption of uniqueness,  $F_1 = F_2$ . This is a contradiction since

$$\begin{aligned} F_1(x) &= p(x) \\ &\neq -p(-x) \\ &= F_2(x) \end{aligned}$$

So for each  $x \in X$ ,  $-p(-x) = p(x)$ .

- (2)  $\Rightarrow$  (3):

Suppose that for each  $x \in X$ ,  $-p(-x) = p(x)$ . The previous exercise implies that there exists  $F \in X^*$  such that  $F \leq p$ . Let  $x \in X$ . Then

$$\begin{aligned} -F(x) &= F(-x) \\ &\leq p(-x) \\ &= -p(x) \end{aligned}$$

So  $p(x) \leq F(x)$  and  $p \leq F$ . Therefore  $p = F$  and  $p$  is linear.

- (3)  $\Rightarrow$  (1):

Suppose that  $p$  is linear. Let  $F \in X^*$ . Suppose that  $F \leq p$ . Let  $x \in X$ . Then as in

the case for (2)  $\Rightarrow$  (3), we have that

$$\begin{aligned} -F(x) &= F(-x) \\ &\leq p(-x) \\ &= -p(x) \end{aligned}$$

which implies that  $p = F$ . So  $p$  is the unique linear function  $F \in X^*$  such that  $F \leq p$ .  $\square$

**Exercise 4.2.9.** Let  $X$  be a normed vector space,  $p : X \rightarrow \mathbb{R}$  a bounded sublinear functional and  $\phi : X \rightarrow \mathbb{R}$  a linear functional. If  $\phi \leq p$ , then  $\phi \in X^*$ .

*Proof.* Since  $p$  is Lipschitz, there exists  $M > 0$  such that for each  $x \in X$ ,  $|p(x)| \leq M\|x\|$ . Let  $x \in X$ . Then

$$\begin{aligned} \phi(x) &\leq p(x) \\ &\leq |p(x)| \\ &\leq M\|x\| \end{aligned}$$

and therefore

$$\begin{aligned} -M\|x\| &= -M\|-x\| \\ &\leq -p(-x) \\ &\leq -\phi(-x) \\ &= \phi(x) \end{aligned}$$

So that  $|\phi(x)| \leq M\|x\|$  and  $\phi \in X^*$ .  $\square$

**Exercise 4.2.10.** Let  $X$  be a normed vector space and  $p : X \rightarrow \mathbb{R}$  a bounded sublinear functional. Then there exists  $\phi \in X^*$  such that for each  $x \in X$ ,  $\phi(x) \leq p(x)$ .

*Proof.* A previous exercise implies there exists  $\phi : X \rightarrow \mathbb{R}$  such that  $\phi$  is linear and  $\phi \leq p$ . The previous exercise implies that  $\phi \in X^*$ .  $\square$

**Exercise 4.2.11. Equivalency of linearity (Bounded Case)** Let  $X$  be a normed vector space and  $p : X \rightarrow \mathbb{R}$  a bounded sublinear functional. Then the following are equivalent:

- (1) there exists a unique  $\phi \in X^*$  such that  $\phi \leq p$
- (2) for each  $x \in X$ ,  $-p(-x) = p(x)$
- (3)  $p$  is linear

*Proof.* Basically the same as last time.  $\square$

**Theorem 4.2.2. Complex Hahn-Banach Theorem:** Let  $X$  be a vector space,  $p : X \rightarrow \mathbb{R}$  a seminorm,  $M \subset X$  a subspace and  $f : M \rightarrow \mathbb{C}$  a linear functional. If for each  $x \in M$ ,  $|f(x)| \leq p(x)$ , then there exists a linear functional  $F : X \rightarrow \mathbb{C}$  such that for each  $x \in X$ ,  $|F(x)| \leq p(x)$  and  $F|_M = f$ .

**Exercise 4.2.12.** Let  $X$  be a normed vector space,  $M \subset X$  a subspace and  $f \in M^*$ . Then there exists  $F \in X^*$  such that  $\|F\| = \|f\|$  and  $F|_M = f$ .

*Proof.* If  $f = 0$ , Choose  $F = 0$ . Suppose  $f \neq 0$ . Then  $\|f\| \neq 0$  and there exists  $x_0 \in M$  such that  $x_0 \neq 0$ . Thus  $\|f\| = \sup\{|f(x)| : x \in M \text{ and } \|x\| = 1\}$ . Define  $p : X \rightarrow [0, \infty)$  by  $p(x) = \|f\|\|x\|$ . Then  $p$  is a sublinear functional on  $X$  and for each  $x \in M$ ,  $|f(x)| \leq p(x)$ . So there exists a linear functional  $F : X \rightarrow \mathbb{C}$  such that for each  $x \in X$ ,  $|F(x)| \leq p(x) = \|f\|\|x\|$  and  $F|_M = f$ . Thus  $F \in X^*$  with  $\|F\| \leq \|f\|$ . Also

$$\|F\| = \sup_{\substack{x \in X \\ \|x\|=1}} |F(x)| \geq \sup_{\substack{x \in M \\ \|x\|=1}} |F(x)| = \sup_{\substack{x \in M \\ \|x\|=1}} |f(x)| = \|f\|$$

So  $\|F\| = \|f\|$ . □

**Exercise 4.2.13.** Let  $X$  be a normed vector space,  $M \subsetneq X$  a proper closed subspace and  $x \in X \setminus M$ . Then there exists  $F \in X^*$  such that  $F|_M = 0$ ,  $\|F\| = 1$  and  $F(x) = \|x + M\| \neq 0$ . (**Hint:** Consider  $f : M + \mathbb{C}x \rightarrow \mathbb{C}$  defined by  $f(m + \lambda x) = \lambda\|x + M\|$ .)

*Proof.* Define  $f : M + \mathbb{C}x \rightarrow \mathbb{C}$  as above. Clearly  $f$  is linear and  $f|_M = 0$ . Let  $m \in M$  and  $\lambda \in \mathbb{C}$ . If  $\lambda = 0$ , then  $|f(m + \lambda x)| = 0 \leq \|m + \lambda x\|$ . Suppose that  $\lambda \neq 0$ . Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \inf_{z \in M} \|z + \lambda x\| \\ &\leq \|m + \lambda x\| \end{aligned}$$

So  $f \in (M + \mathbb{C}x)^*$  and  $\|f\| \leq 1$ . Let  $\epsilon > 0$ . A previous exercise tells us that there exist  $m \in M, \lambda \in \mathbb{C}$  such that  $\|m + \lambda x\| = 1$  and  $\|m + \lambda x + M\| > 1 - \epsilon$ . Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \|m + \lambda x + M\| \\ &> 1 - \epsilon \end{aligned}$$

So

$$\|f\| = \sup_{\substack{z \in M + \mathbb{C}x \\ \|z\|=1}} |f(z)| \geq 1$$

Hence  $\|f\| = 1$ . The same exercise also tells us that  $f(x) = \|x + M\| \neq 0$ . Using the previous exercise, there exists  $F \in X^*$  such that  $\|F\| = \|f\| = 1$  and  $F|_{M + \mathbb{C}x} = f$ . □

**Exercise 4.2.14.** Let  $X$  be a normed vector space and  $x \in X$ . If  $x \neq 0$ , then there exists  $F \in X^*$  such that  $\|F\| = 1$  and  $F(x) = \|x\|$ .

*Proof.* Define  $f : \mathbb{C}x \rightarrow \mathbb{C}$  by  $f(\lambda x) = \lambda\|x\|$ . Then  $f$  is linear and  $f(x) = \|x\|$ . Clearly

$$\sup_{\substack{z \in \mathbb{C}x \\ \|z\|=1}} |f(z)| = 1$$

So  $f \in (\mathbb{C}x)^*$  and  $\|f\| = 1$ . By a previous exercise, there exists  $F \in X^*$  such that  $\|F\| = \|f\| = 1$  and  $F|_{\mathbb{C}x} = f$ . □

**Exercise 4.2.15.** Let  $X$  be a normed vector space. Then  $X^*$  separates the points of  $X$ .

*Proof.* Let  $x, y \in X$ . Suppose that  $x \neq y$ . Then  $x - y \neq 0$ . The previous exercise implies that there exists  $F \in X^*$  such that  $\|F\| = 1$  and

$$F(x) - F(y) = F(x - y) = \|x - y\| \neq 0$$

Thus  $F(x) \neq F(y)$  and  $X^*$  separates the points of  $X$ . □

**Definition 4.2.16.** Let  $X, Y$  be metric spaces and  $T : X \rightarrow Y$ . Then  $T$  is said to be an **isometry** if for each  $x_1, x_2 \in X$ ,  $d(Tx_1, Tx_2) = d(x_1, x_2)$ .

**Exercise 4.2.17.** Let  $X, Y$  be metric spaces and  $T : X \rightarrow Y$  an isometry. Then  $T$  is injective.

*Proof.* Let  $x_1, x_2 \in X$ . Suppose that  $Tx_1 = Tx_2$ . Then  $0 = d(Tx_1, Tx_2) = d(x_1, x_2)$ . So  $x_1 = x_2$ . Hence  $T$  is injective. □

*Note 4.2.2.* Let  $X, Y$  be metric spaces and  $T : X \rightarrow Y$  an isometry. Then  $T$  is clearly continuous. If  $T$  is surjective, then  $T^{-1}$  is an isometry and therefore continuous. Hence  $T$  is a homeomorphism.

**Exercise 4.2.18.** Let  $X$  be a normed vector space and  $x \in X$ . Define  $\hat{x} : X^* \rightarrow \mathbb{C}$  by  $\hat{x}(f) = f(x)$ . Then  $\hat{x} \in X^{**}$  and  $\|\hat{x}\| = \|x\|$ .

*Proof.* Let  $f, g \in X^*$  and  $\lambda \in \mathbb{C}$ . Then

$$\hat{x}(f + \lambda g) = (f + \lambda g)(x) = f(x) + \lambda g(x) = \hat{x}(f) + \lambda \hat{x}(g)$$

So  $\hat{x}$  is linear. For each  $f \in X^*$ ,

$$|\hat{x}(f)| = |f(x)| \leq \|x\| \|f\|$$

Hence  $\hat{x} \in X^{**}$  with  $\|\hat{x}\| \leq \|x\|$ . If  $x = 0$ , then  $\hat{x} = 0$  and  $\|\hat{x}\| = \|x\|$ . Suppose that  $x \neq 0$ . Then a previous exercise implies that there exists  $F \in X^*$  such that  $\|F\| = 1$  and  $F(x) = \|x\|$ . Then we have that

$$\sup_{\substack{f \in X^* \\ \|f\|=1}} |\hat{x}(f)| = \sup_{\substack{f \in X^* \\ \|f\|=1}} |f(x)| \geq |F(x)| = \|x\|$$

Hence  $\|\hat{x}\| = \|x\|$ . □

**Exercise 4.2.19.** Let  $X$  be a normed vector space. Define  $\phi : X \rightarrow X^{**}$  by  $\phi(x) = \hat{x}$ . Then  $\phi$  is a linear isometry.

*Proof.* Let  $x, y \in X$  and  $\lambda \in \mathbb{C}$ . Then for each  $f \in X^*$ , we have that

$$\begin{aligned} \phi(x + \lambda y)(f) &= \widehat{x + \lambda y}(f) \\ &= f(x + \lambda y) \\ &= f(x) + \lambda f(y) \\ &= \hat{x}(f) + \lambda \hat{y}(f) \\ &= \phi(x)(f) + \lambda \phi(y)(f) \end{aligned}$$

So  $\phi(x + \lambda y) = \phi(x) + \lambda \phi(y)$  and  $\phi$  is linear. The previous exercise tells us that

$$\begin{aligned} \|\phi(x) - \phi(y)\| &= \|\phi(x - y)\| \\ &= \|\widehat{x - y}\| = \|x - y\| \end{aligned}$$

So  $\phi$  is an isometry. □

**Definition 4.2.20.** Let  $X$  be a normed vector space and define  $\phi : X \rightarrow X^{**}$  as above. We define  $\widehat{X} = \phi(X) \subset X^{**}$ . Since  $\widehat{X}$  and  $X$  are isomorphic, we may identify  $X$  as a subset of  $X^{**}$ .

**Definition 4.2.21.** Let  $X$  be a normed vector space and define  $\phi : X \rightarrow X^{**}$  as above. Then  $X$  is said to be reflexive if  $\phi$  is surjective. In this case  $\phi$  is then an isomorphism

**Exercise 4.2.22.** Let  $X$  be a normed vector space and  $f : X \rightarrow \mathbb{C}$  a linear functional on  $X$ . Then  $f$  is bounded iff  $\ker f$  is closed.

*Proof.* Suppose that  $f$  is continuous. Since  $\{0\}$  is closed, we have that  $\ker f = f^{-1}(\{0\})$  is closed. Conversely, suppose that  $\ker f$  is closed. If  $\ker f = X$ , then  $f = 0$  and  $f$  is continuous. Suppose that  $\ker f \neq X$ . Then  $\ker f$  is a proper, closed subspace of  $X$ . A previous exercise tells us that there exists  $x \in X$  such that  $\|x\| = 1$  and  $\|x + \ker f\| > \frac{1}{2}$ . Let  $y \in X$ . Suppose that  $\|y\| < \frac{1}{2}$ . Then for each  $z \in \ker f$ ,

$$\begin{aligned} \|z - (x + y)\| &= \|(z - x) - y\| \\ &\geq \|z - x\| - \|y\| \\ &> \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

So  $x + y \notin \ker f$ . Therefore  $f(B(x, \frac{1}{2})) \cap \{0\} = \emptyset$ . If  $f(B(x, \frac{1}{2}))$  is unbounded, then  $f(B(x, \frac{1}{2})) = \mathbb{C}$  by linearity. This is a contradiction since  $0 \notin f(B(x, \frac{1}{2}))$ . So There exists  $s > 0$  such that  $f(B(x, \frac{1}{2})) \subset B(0, s)$  and thus  $f$  is bounded.  $\square$

**Exercise 4.2.23.** Let  $X$  be a normed vector space.

- (1) Let  $M \subsetneq X$  be a proper closed subspace of  $X$  and  $x \in X \setminus M$ . Then  $M + \mathbb{C}x$  is closed.
- (2) Let  $M \subset X$  be a finite dimensional subspace of  $X$ . Then  $M$  is closed.

*Proof.* (1) Let  $y \in X$  and  $(y_n)_{n \in \mathbb{N}} \subset M + \mathbb{C}x$ . Suppose that  $y_n \rightarrow y$ . If  $y \in M$ , then  $y \in M + \mathbb{C}x$ . Suppose that  $y \notin M$ . For each  $n \in \mathbb{N}$ , there exists  $m_n \in M$  and  $\lambda_n \in \mathbb{C}$  such that  $y_n = m_n + \lambda_n x$ . A previous exercise tells us that there exists  $F \in X^*$  such that  $\|F\| = 1$ ,  $F|_M = 0$  and  $F(x) = \|x + M\| \neq 0$ . Since  $F$  is continuous,  $F(y_n) \rightarrow F(y)$ . Since for each  $n \in \mathbb{N}$ ,

$$F(y_n) = F(m_n + \lambda_n x) = F(m_n) + \lambda_n(Fx) = \lambda_n F(x)$$

we have that  $\lambda_n F(x) \rightarrow F(y)$ . Since  $F(x) \neq 0$ , this implies that  $\lambda_n \rightarrow F(x)^{-1}F(y)$ . It follows that  $\lambda_n x \rightarrow F(x)^{-1}F(y)x$ . Since for each  $n \in \mathbb{N}$ ,  $m_n = y_n - \lambda_n x$ , we know that  $m_n \rightarrow y - F(x)^{-1}F(y)x$ . Since  $(m_n)_{n \in \mathbb{N}} \subset M$  and  $M$  is closed, we have that  $y - F(x)^{-1}F(y)x \in M$  and therefore  $y \in M + \mathbb{C}x$ . Hence  $M + \mathbb{C}x$  is closed.

- (2) If  $M = X$ , then  $M$  is closed. Suppose that  $M \neq X$ . Let  $(x_i)_{i=1}^n$  be a basis for  $M$ . Define  $N_0 = \{0\}$  and for each  $i = 1, 2, \dots, n$ , define  $N_i = N_{i-1} + \mathbb{C}x_i$ . Since  $N_0$  is a proper closed subspace of  $X$  and  $x_1 \in X \setminus N_0$ , (1) implies that  $N_1$  is closed. Proceed inductively to obtain that  $M = N_n$  is closed.

$\square$

**Exercise 4.2.24.** Let  $X$  be an infinite-dimensional normed vector space.

- (1) There exists a sequence  $(x_n)_{n \in \mathbb{N}} \subset X$  such that for each  $m, n \in \mathbb{N}$ ,  $\|x_n\| = 1$  and if  $m \neq n$ , then  $\|x_m - x_n\| > \frac{1}{2}$ .
- (2)  $X$  is not locally compact.

*Proof.* (1) Define  $N_0 = \{0\}$ . Then  $N_0$  is a closed proper subspace of  $X$ . Choose  $x_1 \in X$  such that  $\|x_1\| = 1$ . Using the results of previous exercises, we proceed inductively. For each  $n \geq 2$  we define  $N_{n-1} = \text{span}(x_1, x_2, \dots, x_{n-1})$ . Then  $N_{n-1}$  is a closed proper subspace of  $X$ . Thus we may choose  $x_n \in X$  such that  $\|x_n\| = 1$  and  $\|x_n + N_{n-1}\| > \frac{1}{2}$ . Let  $m, n \in \mathbb{N}$ . Suppose that  $m < n$ . Then  $x_m \in N_{n-1}$ . Thus  $\|x_n - x_m\| \geq \|x_n + N_{n-1}\| > \frac{1}{2}$ .

- (2) Suppose that  $X$  is locally compact. Then  $\overline{B(0, 1)}$  is compact and therefore sequentially compact. Using  $(x_n)_{n \in \mathbb{N}} \subset \overline{B(0, 1)}$  defined in (1), we see that there exists a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$ ,  $x \in \overline{B(0, 1)}$  such that  $x_{n_k} \rightarrow x$ . Then  $(x_{n_k})_{k \in \mathbb{N}}$  is Cauchy. So there exists  $N \in \mathbb{N}$  such that for each  $j, k \in \mathbb{N}$ , if  $j, k \geq N$ , then  $\|x_{n_j} - x_{n_k}\| < \frac{1}{2}$ . Then  $\|x_{n_N} - x_{n_{N+1}}\| < \frac{1}{2}$ . This is a contradiction since by construction,  $\|x_{n_N} - x_{n_{N+1}}\| > \frac{1}{2}$ . Thus  $X$  is not locally compact. □

**Exercise 4.2.25.** Let  $X, Y$  be normed vector spaces and  $T \in L(X, Y)$ .

- (1) Define the **adjoint of  $T$** , denoted  $T^* : Y^* \rightarrow X^*$  by  $T^*(f) = f \circ T$ . Then  $T^* \in L(Y^*, X^*)$ .
- (2) Applying the result from (1) twice, we have that  $T^{**} \in L(X^{**}, Y^{**})$ . We have that for each  $x \in X$ ,  $T^{**}(\hat{x}) = \widehat{T(x)}$ .
- (3)  $T^*$  is injective iff  $T(X)$  is dense in  $Y$ .
- (4) If  $T^*(Y^*)$  is dense in  $X^*$ , then  $T$  is injective. The converse is true if  $X$  is reflexive.

*Proof.* (1) Let  $f \in Y^*$ . Then  $\|T^*(f)\| = \|f \circ T\| \leq \|T\|\|f\|$ . So  $T^* \in L(Y^*, X^*)$  with  $\|T^*\| \leq \|T\|$ .

- (2) Let  $x \in X$ . Let  $f \in Y^*$ . Then

$$\begin{aligned}
 T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\
 &= \hat{x}(T^*(f)) \\
 &= \hat{x}(f \circ T) \\
 &= f \circ T(x) \\
 &= f(T(x)) \\
 &= \widehat{T(x)}(f)
 \end{aligned}$$

Hence  $T^{**}(\hat{x}) = \widehat{T(x)}$ .



- (3) Suppose that  $T(X)$  is not dense in  $Y$ . Then  $\overline{T(X)} \neq Y$ . So  $T(X)$  is a proper closed subspace of  $Y$  and there exists  $y \in Y$  such that  $y \notin \overline{T(X)}$ . By a previous exercise, there exists  $f \in Y^*$  such that  $f(y) = \|y + \overline{T(X)}\| \neq 0$ ,  $\|f\| = 1$  and  $f|_{\overline{T(X)}} = 0$ . Let  $x \in X$ . Then  $T^*(f)(x) = f \circ T(x) = 0$ . Hence  $T^*(f) = 0 = T^*(0)$ . Since  $f \neq 0$ ,  $T^*$  is not injective.

Now suppose that  $T(X)$  is dense in  $Y$ . Let  $f, g \in Y^*$ . Define  $h \in Y^*$  by  $h = f - g$ . Suppose that  $T^*(f) = T^*(g)$ . Then  $T^*(h) = 0$ . So for each  $x \in X$ ,  $h(T(x)) = 0$ . Let  $y \in Y$  and  $\epsilon > 0$ . By continuity, there exists  $\delta > 0$  such that for each  $y' \in Y$ , if  $\|y - y'\| < \delta$ , then  $\|h(y) - h(y')\| < \epsilon$ . Since  $T(X)$  is dense in  $Y$ , there exists  $x \in X$  such that  $\|y - T(x)\| < \delta$ . Thus

$$\begin{aligned} \|h(y)\| &\leq \|h(y) - h(T(x))\| + \|h(T(x))\| \\ &= \|h(y) - h(T(x))\| \\ &< \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary,  $\|h(y)\| = 0$ . This implies that  $h(y) = 0$  and therefore  $f(y) = g(y)$ . Since  $y \in Y$  is arbitrary,  $f = g$  and  $T^*$  is injective.

- (4) For the sake of contradiction, suppose that  $T^*(Y^*)$  is dense in  $X^*$  and  $T$  is not injective. Then there exist  $x_1, x_2 \in X$  such that  $x_1 \neq x_2$  and  $T(x_1) = T(x_2)$ . Define  $x = x_1 - x_2$ . Then  $x \neq 0$  and  $T(x) = 0$ . A previous exercise implies that there exists  $F \in X^*$  such that  $F(x) = \|x\| \neq 0$  and  $\|F\| = 1$ . Let  $\epsilon > 0$ . Choose  $g \in Y^*$  such that  $\|F - T^*(g)\| < \epsilon$ . Then

$$\begin{aligned} \|x\| &= |F(x)| \\ &\leq |F(x) - T^*(g)(x)| + |T^*(g)(x)| \\ &< \epsilon\|x\| + |g(T(x))| \\ &= \epsilon\|x\| \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we have that  $\|x\| = 0$  which is a contradiction. Hence if  $T^*(Y^*)$  is dense in  $X^*$ , then  $T$  is injective.

Now, suppose that  $X$  is reflexive and  $T$  is injective. Let  $\phi_1, \phi_2 \in X^{**}$ . Suppose that  $T^{**}(\phi_1) = T^{**}(\phi_2)$ . Then  $T^{**}(\phi_1 - \phi_2) = 0$ . Since  $X$  is reflexive, there exist  $x_1, x_2 \in X$  such that  $\phi_1 = \hat{x}_1$  and  $\phi_2 = \hat{x}_2$ . Define  $x = x_1 - x_2$ . Then  $T^{**}(\hat{x}) = 0$ . So for each  $f \in Y^*$ ,

$$\begin{aligned} T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= 0 \end{aligned}$$

Suppose that  $T(x) \neq 0$ . Then a previous exercise implies that there exists  $g \in Y^*$  such that  $g(T(x)) = \|T(x)\| \neq 0$  and  $\|g\| = 1$ . This is a contradiction since  $g(T(x)) = 0$ .

So  $T(x) = 0$ . Since  $T$  is injective, this implies that  $x = 0$ . Hence  $\hat{x} = 0$  and thus  $\phi_1 = \phi_2$ . Thus  $T^{**}$  is injective. By (3), we have that  $T^*(Y^*)$  is dense in  $X^*$ .  $\square$

**Exercise 4.2.26.** Let  $X$  be a normed vector space. Then  $X$  is reflexive iff  $X^*$  is reflexive.

*Proof.* Suppose that  $X$  is reflexive. Let  $\alpha \in X^{***}$ . Define  $f : X \rightarrow \mathbb{C}$  by  $f(x) = \alpha(\hat{x})$ . Clearly  $f$  is linear and a previous exercise tells us that for each  $x \in X$ ,

$$\begin{aligned} |f(x)| &\leq \|\alpha\| \|\hat{x}\| \\ &= \|\alpha\| \|x\| \end{aligned}$$

So  $f \in X^*$ . Let  $\phi \in X^{**}$ . Since  $X$  is reflexive, there exists  $x \in X$  such that  $\phi = \hat{x}$ . Then

$$\begin{aligned} \alpha(\phi) &= \alpha(\hat{x}) \\ &= f(x) \\ &= \hat{x}(f) \\ &= \hat{f}(\hat{x}) \\ &= \hat{f}(\phi) \end{aligned}$$

Hence  $\alpha = \hat{f}$ . Thus the map  $X^* \rightarrow X^{***}$  given by  $f \mapsto \hat{f}$  is surjective and so  $X^*$  is reflexive.

Conversely, suppose that  $X^*$  is reflexive. Since  $\phi : X \rightarrow X^{**}$  given by  $\phi(x) = \hat{x}$  is an isometry,  $\hat{X} \subset X^{**}$  is closed. For the sake of contradiction, suppose that  $\hat{X} \neq X^{**}$ . Then there exists  $\alpha \in X^{**}$  such that  $\alpha \notin \hat{X}$ . Thus there exists  $F \in X^{***}$  such that  $\|F\| = 1$ ,  $F(\alpha) = \|\alpha + \hat{X}\| \neq 0$  and  $F|_{\hat{X}} = 0$ . Since  $X^*$  is reflexive, there exists  $f \in X^*$  such that  $F = \hat{f}$ . A previous exercise tells us that  $\|f\| = \|\hat{f}\| = \|F\| = 1$ . Since for each  $x \in X$ ,  $f(x) = \hat{x}(f) = \hat{f}(\hat{x}) = F(\hat{x}) = 0$ , we have that  $f = 0$ . Thus  $\|f\| = 0$ , a contradiction. So  $\hat{X} = X^{**}$  and  $X$  is reflexive.  $\square$

### 4.3. The Baire Category and Closed Graph Theorems.

**Theorem 4.3.1.** *Let  $X, Y$  be Banach spaces and  $T \in L(X, Y)$ . If  $T$  is surjective, then  $T$  is open.*

**Corollary 4.3.2.** *Let  $X, Y$  be Banach spaces and  $T \in L(X, Y)$ . If  $T$  is a bijection, then  $T^{-1} \in L(X, Y)$ .*

**Definition 4.3.1.** Let  $X, Y$  be sets and  $f : X \rightarrow Y$ . We define the **graph of  $f$** ,  $\Gamma(f)$ , by  $\Gamma(f) = \{(x, y) \in X \times Y : f(x) = y\}$ .

**Theorem 4.3.3.** *Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  a linear map. If  $\Gamma(T)$  is closed, then  $T \in L(X, Y)$ .*

*Note 4.3.1.* We recall that  $\Gamma(T)$  is closed iff for each  $(x_n)_{n \in \mathbb{N}} \subset X$ ,  $x \in X$  and  $y \in Y$ ,  $x_n \rightarrow x$  and  $T(x_n) \rightarrow y$  implies that  $T(x) = y$ .

**Theorem 4.3.4.** *Let  $X, Y$  be Banach spaces and  $S \subset L(X, Y)$ . If for each  $x \in X$ ,*

$$\sup_{T \in S} \|Tx\| < \infty$$

*then*

$$\sup_{T \in S} \|T\| < \infty$$

**Exercise 4.3.2.** Let  $\mu$  be counting measure on  $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ . Define  $h : \mathbb{N} \rightarrow \mathbb{N}$  and  $\nu$  on  $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$  by  $h(n) = n$  and  $d\nu = h d\mu$ . Define  $X = L^1(\nu)$  and  $Y = L^1(\mu)$ . Equip both  $X$  and  $Y$  with the  $L^1$  norm with respect to  $\mu$ .

- (1) We have that  $X$  is a proper subspace of  $Y$  and therefore  $X$  is not complete.
- (2) Define  $T : X \rightarrow Y$  by  $Tf(n) = nf(n)$ . Then  $T$  is linear,  $\Gamma(T)$  is closed, and  $T$  is unbounded.
- (3) Define  $S : Y \rightarrow X$  by  $Sg(n) = \frac{1}{n}g(n)$ . Then  $S \in L(Y, X)$ ,  $S$  is surjective and  $S$  is not open.

*Proof.*

- (1) Note that for each  $f : \mathbb{N} \rightarrow \mathbb{C}$ ,

$$\begin{aligned} \|f\|_{\mu,1} &= \sum_{n=1}^{\infty} |f(n)| \\ &\leq \sum_{n=1}^{\infty} n |f(n)| \\ &= \|f\|_{\nu,1} \end{aligned}$$

Hence  $X$  is a subspace of  $Y$ . Define  $f : \mathbb{N} \rightarrow \mathbb{C}$  by  $f(n) = \frac{1}{n^2}$ . Then

$$\|f\|_{\mu,1} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

So  $f \in Y$ . However

$$\|f\|_{\nu,1} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

So  $f \notin X$ . Thus  $X$  is a proper subspace of  $Y$ . Let  $g \in Y$  and  $\epsilon > 0$ . Since the simple functions are dense in  $L^1(\mu)$ , there exists  $\phi \in L^1(\mu)$  such that  $\phi$  is simple and  $\|g - \phi\|_{\mu,1} < \epsilon$ . Then there exist  $(c_i)_{i=1}^k \subset \mathbb{C}$  and  $(E_i)_{i=1}^k \subset \mathcal{P}(\mathbb{N})$  such that for each  $i, j \in \{1, 2, \dots, k\}$ ,  $E_i$  is finite,  $i \neq j$  implies that  $E_i \cap E_j = \emptyset$  and

$$\phi = \sum_{i=1}^k c_i \chi_{E_i}$$

Define  $c = \max\{|c_i| : i = 1, 2, \dots, k\}$  and  $m = \max \left[ \bigcup_{i=1}^k E_i \right]$ . Then

$$\begin{aligned} \|\phi\|_{\nu,1} &= \sum_{n=1}^m n |\phi(n)| \\ &\leq \sum_{n=1}^m mc \\ &= cm^2 \\ &< \infty \end{aligned}$$

Hence  $\phi \in X$  and  $X$  is dense in  $Y$ . Since  $X$  is a dense, proper subspace, it is not closed. Since  $Y$  is complete and  $X \subset Y$  is not closed, we have that  $X$  is not complete.

- (2) Clearly  $T$  is linear. Let  $(f_j)_{j \in \mathbb{N}} \subset X$ ,  $f \in X$  and  $g \in Y$ . Suppose that  $f_j \xrightarrow{L^1(\mu)} f$  and  $Tf_j \xrightarrow{L^1(\mu)} g$ .

Note that for each  $j \in \mathbb{N}$  and  $n \in \mathbb{N}$ ,

$$|f_j(n) - f(n)| \leq \sum_{n=1}^{\infty} |f_j(n) - f(n)| = \|f_j - f\|_{\mu,1}$$

and

$$|nf_j(n) - g(n)| \leq \sum_{n=1}^{\infty} |nf_j(n) - g(n)| = \|Tf_j - g\|_{\mu,1}$$

Thus for each  $n \in \mathbb{N}$ ,  $f_j(n) \xrightarrow{j} f(n)$  and  $nf_j(n) \xrightarrow{j} g(n)$ . This implies that for each  $n \in \mathbb{N}$ ,  $nf(n) = g(n)$ . Thus  $Tf = g$  which implies that  $\Gamma(T)$  is closed. Suppose, for the sake of contradiction, that  $T$  is bounded. Then there exists  $C \geq 0$  such that for each  $f \in X$ ,  $\|Tf\|_{\mu,1} \leq C\|f\|_{\mu,1}$ . Choose  $n \in \mathbb{N}$  such that  $n > C$ . Define  $f : \mathbb{N} \rightarrow \mathbb{C}$  by  $f = \chi_{\{n\}}$ . As established above,  $S^+ \subset L^1(\mu)$ . Then  $\|f\|_{\mu,1} = 1$  and

$$\begin{aligned} \|Tf\|_{\mu,1} &= n \\ &> C \\ &= C\|f\|_{\mu,1} \end{aligned}$$

which is a contradiction. So  $T$  is unbounded.

(3) Clearly  $S$  is linear. Let  $g \in Y$ . Then

$$\begin{aligned}\|Sg\|_{\mu,1} &= \sum_{n=1}^{\infty} \frac{1}{n} |g(n)| \\ &\leq \sum_{n=1}^{\infty} |g(n)| \\ &= \|g\|_{\mu,1}\end{aligned}$$

So  $S$  is bounded and  $\|S\| \leq 1$ . Thus  $S \in L(Y, X)$ . Let  $f \in X$ . Define  $g : \mathbb{N} \rightarrow \mathbb{C}$  by  $g(n) = nf(n)$ . By definition,  $g \in Y$  and we have that

$$\begin{aligned}Sg(n) &= \frac{1}{n} g(n) \\ &= f(n)\end{aligned}$$

Hence  $Sg = f$  and thus  $S$  is surjective. Let  $g \in Y$ . Suppose that  $Sg = 0$ . Then

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n)| = \|Sg\| = 0$$

Thus for each  $n \in \mathbb{N}$ ,  $g(n) = 0$ . Hence  $\ker S = \{0\}$  and  $S$  is injective. Note that for each  $A \subset Y$ ,  $S(A) = T^{-1}(A)$ . If  $S$  is open, then  $T$  is continuous which as shown above is a contradiction. So  $g$  is not open. □

**Exercise 4.3.3.** Let  $X = C^1([0, 1])$  and  $Y = C([0, 1])$ . Equip both  $X$  and  $Y$  with the uniform norm.

- (1) Then  $X$  is not complete
- (2) Define  $T : X \rightarrow Y$  by  $Tf = f'$ . Then  $\Gamma(T)$  is closed and  $T$  is not bounded.

*Proof.* (1) Recall that for each  $a, b \geq 0$  and  $p \in \mathbb{N}$ ,

$$(a^{\frac{1}{p}} + b^{\frac{1}{p}})^p = \sum_{n=0}^p \binom{p}{n} a^{\frac{n}{p}} b^{\frac{p-n}{p}} \geq a + b$$

Thus  $(a + b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$ .

For each  $n \in \mathbb{N}$ , define  $f_n : [0, 1] \rightarrow \mathbb{C}$  by  $f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n^2}}$ . Then  $(f_n)_{n \in \mathbb{N}} \subset X$ . Define  $f : [0, 1] \rightarrow \mathbb{C}$  by  $f(x) = |x - \frac{1}{2}|$ . Then  $f \in Y \cap X^c$ . Note that for each  $n \in \mathbb{N}$ ,  $f \leq f_n$ . Our observation above implies that for each  $x \in X$ ,

$$\begin{aligned}f_n(x) &= \left[ (x - \frac{1}{2})^2 + \frac{1}{n^2} \right]^{\frac{1}{2}} \\ &\leq |x - \frac{1}{2}| + \frac{1}{n}\end{aligned}$$

Thus  $0 \leq f_n - f \leq \frac{1}{n}$ . This implies that  $f_n \xrightarrow{u} f$ . Since  $f \notin X$ ,  $X$  is not complete.

- (2) Let  $(f_n)_{n \in \mathbb{N}} \subset X$ ,  $f \in X$  and  $g \in Y$ . Suppose that  $f_n \xrightarrow{u} f$  and  $Tf_n \xrightarrow{u} g$ . Let  $x \in [0, 1]$ . Then  $f_n(x) \rightarrow f(x)$  and  $f_n(0) \rightarrow f(0)$  and  $f'_n \xrightarrow{u} g$ . Applying the DCT to this sequence of integrable functions that converges uniformly to an integrable function on a finite measure space (a previous exercise) we have that

$$\begin{aligned} f_n(x) - f_n(0) &= \int_{[0,x]} f'_n dm \\ &\rightarrow \int_{[0,x]} g dm \end{aligned}$$

Since  $f_n(x) - f_n(0) \rightarrow f(x) - f(0)$ , we know that

$$f(x) - f(0) = \int_{[0,x]} g dm$$

. Thus  $Tf = g$  and  $\Gamma(T)$  is closed.

Suppose for the sake of contradiction that  $T$  is bounded. Then there exists  $C \geq 0$  such that for each  $f \in X$ ,  $\|Tf\| \leq C\|f\|$ . Choose  $n \in \mathbb{N}$  such that  $n > C$ . Define  $f \in X$  by  $f(x) = x^n$ . Then  $\|f\| = 1$  and

$$\begin{aligned} \|Tf\| &= \|f'\| \\ &= n \\ &> C \\ &= C\|f\| \end{aligned}$$

which is a contradiction. So  $T$  is not bounded. □

**Exercise 4.3.4.** Let  $X, Y$  be Banach spaces and  $T \in L(X, Y)$ . Then  $X/\ker T \cong T(X)$  iff  $T(X)$  is closed.

*Proof.* Since  $X$  is a Banach space and  $T$  is continuous, we have that  $\ker T$  is closed and  $X/\ker T$  is a Banach space. Suppose that  $X/\ker T \cong T(X)$ . Then  $T(X)$  is complete. Since  $Y$  is complete, this implies that  $T(X)$  is closed.

Conversely Suppose that  $T(X)$  is closed. Then  $T(X)$  is complete. Define  $S : X/\ker T \rightarrow T(X)$  by  $S(x + \ker T) = T(x)$ . A previous exercise tells us that the map  $S : X/\ker T \rightarrow T(X)$  defined by  $S(x + \ker T) = T(x)$  is a bounded linear bijection. Since  $T(X)$  is complete and  $S$  is surjective,  $S^{-1}$  is bounded and thus  $S$  is an isomorphism. □

**Exercise 4.3.5.** Let  $X$  be a separable Banach space. Define  $B_X = \{x \in X : \|x\| < 1\}$ . Let  $(x_n)_{n \in \mathbb{N}} \subset B_X$  a dense subset of the unit ball and  $\mu$  the counting measure on  $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ . Define  $T : L^1(\mu) \rightarrow X$  by

$$Tf = \sum_{n=1}^{\infty} f(n)x_n$$

Then

- (1)  $T$  is well defined and  $T \in L(L^1(\mu), X)$
- (2)  $T$  is surjective
- (3) There exists a closed subspace  $K \subset L^1(\mu)$  such that  $L^1(\mu)/K \cong X$

*Proof.* (1) Let  $f \in L^1(\mu)$ . Since  $X$  is complete and

$$\begin{aligned} \sum_{n=1}^{\infty} \|f(n)x_n\| &= \sum_{n=1}^{\infty} |f(n)| \|x_n\| \\ &\leq \sum_{n=1}^{\infty} |f(n)| \\ &< \infty \end{aligned}$$

we have that  $\sum_{n=1}^{\infty} f(n)x_n$  converges and thus  $Tf \in X$ . Hence  $T$  is well defined.

Clearly  $T$  is linear. Let  $f \in L^1(\mu)$ . Then

$$\begin{aligned} \|Tf\| &= \left\| \sum_{n=1}^{\infty} f(n)x_n \right\| \\ &\leq \sum_{n=1}^{\infty} \|f(n)x_n\| \\ &\leq \sum_{n=1}^{\infty} |f(n)| \\ &= \|f\|_1 \end{aligned}$$

So  $T$  is bounded with  $\|T\| \leq 1$ .

- (2) Let  $x \in X$ . Suppose that  $\|x\| < 1$ . Then  $x \in B_X$ . So there exists  $n_1 \in \mathbb{N}$  such that  $\|x - x_{n_1}\| < \frac{1}{2}$ . Then  $2(x - x_{n_1}) \in B_X$ . Since for each  $j \in \mathbb{N}$ ,  $B_X \setminus (x_n)_{n=1}^j$  is dense in  $B_X$ , there exists  $n_2 \in \mathbb{N}$  such that  $x_{n_2} \notin (x_n)_{n=1}^{n_1}$  and  $\|2(x - x_{n_1}) - x_{n_2}\| < \frac{1}{2}$  which implies that  $\|x - (x_{n_1} - \frac{1}{2}x_{n_2})\| < \frac{1}{4}$ .

Proceed inductively to obtain a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  such that for each  $k \geq 2$ ,  $x_{n_k} \notin (x_n)_{n=1}^{n_{k-1}}$  and  $\|x - \sum_{j=1}^k 2^{1-j}x_{n_j}\| < \frac{1}{2^k}$ . Then  $x = \sum_{k=1}^{\infty} 2^{1-k}x_{n_k}$ .

Define  $f : \mathbb{N} \rightarrow \mathbb{C}$  by  $f = \sum_{k=1}^{\infty} 2^{1-k}\chi_{\{n_k\}}$ . Then  $\|f\|_1 = \sum_{k=1}^{\infty} 2^{1-k} < \infty$ , so  $f \in L^1(\mu)$  and  $Tf = \sum_{k=1}^{\infty} 2^{1-k}x_{n_k} = x$ . Now, suppose that  $\|x\| \geq 1$ , then  $\frac{1}{2\|x\|}x \in B_X$ . The above argument shows that there exists  $f \in L^1(\mu)$  such that  $Tf = \frac{1}{2\|x\|}x$ . Then  $2\|x\|f \in L^1(\mu)$  and  $T(2\|x\|f) = 2\|x\|Tf = x$ .

So for each  $x \in X$ , there exists  $f \in L^1(\mu)$  such that  $Tf = x$  and thus  $T$  is surjective.

- (3) Since  $X$  is a Banach space and  $T$  is surjective, the previous exercise implies that  $L^1(\mu)/\ker T \cong X$ . □

**Exercise 4.3.6.** Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  a linear map. If for each  $f \in Y^*$ ,  $f \circ T \in X^*$ , then  $T \in L(X, Y)$ .

*Proof.* Suppose that for each  $f \in Y^*$ ,  $f \circ T \in X^*$ . Let  $x \in X$ , □

#### 4.4. Banach Algebras.

**Definition 4.4.1.** Let  $X$  be a Banach space and an associative algebra. Then  $X$  is said to be a **Banach algebra** if for each  $S, T \in X$ ,  $\|ST\| \leq \|S\|\|T\|$ . If there exists  $I \in X$  such that  $I \neq 0$  and for each  $T \in X$ ,  $IT = TI = T$ , then  $X$  is said to be **unital** with identity  $I$ . An element  $T \in X$  is said to be **invertible** if there exists  $S \in X$  such that  $TS = ST = I$ .

**Exercise 4.4.2.** Let  $X$  be a unital Banach algebra. Then  $\|I\| \leq 1$ .

*Proof.* Since  $I \neq 0$ ,  $\|I\| \neq 0$ . By definition,

$$\|I\| = \|II\| \leq \|I\|\|I\|$$

Hence  $1 \leq \|I\|$ . □

*Note 4.4.1.* If  $X$  is a Banach space, then a previous exercise implies that  $L(X, X)$  equipped with composition is a unital Banach algebra where  $I$  is the identity operator. It is easy to see that  $\|I\| = 1$ .

*Note 4.4.2.* Let  $X$  be a Banach algebra. Then the set of invertible elements in  $X$  is a group.

**Exercise 4.4.3.** Let  $X$  be a Banach algebra. Then multiplication is continuous.

*Proof.* Let  $(S_1, T_1) \in X \times X$  and  $\epsilon > 0$ . Choose  $\delta = \min\{\frac{\epsilon}{2(\|S_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$ . Let  $(S_2, T_2) \in X \times X$ . Suppose that

$$\|(S_1, T_1) - (S_2, T_2)\| = \max\{\|S_1 - S_2\|, \|T_1 - T_2\|\} < \delta$$

Then

$$\begin{aligned} \|S_1T_1 - S_2T_2\| &= \|S_1T_1 - S_2T_1 + S_2T_1 - S_2T_2\| \\ &\leq \|S_1 - S_2\|\|T_1\| + \|S_2\|\|T_1 - T_2\| \\ &\leq \|S_1 - S_2\|\|T_1\| + (\|S_1 - S_2\| + \|S_1\|)\|T_1 - T_2\| \\ &\leq \delta\|T_1\| + (\delta + \|S_1\|)\delta \\ &= \delta(\|S_1\| + \|T_1\|) + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□



#### 4.5. Differentiation.

*Note 4.5.1.* In this section, we assume all Banach spaces to be over  $\mathbb{R}$ .

**Definition 4.5.1.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open,  $f : A \rightarrow Y$ ,  $x_0 \in A$  and  $x \in X$ . Then  $f$  is said to be

- (1) **right-hand-differentiable at  $x_0$  in the direction  $x$**  if the limit

$$\lim_{t \rightarrow 0^+} \frac{f(x_0 + tx) - f(x_0)}{t}$$

exists. If  $f$  is right-hand-differentiable at  $x_0$  in the direction  $x$ , we define the **right-hand derivative** of  $f$  at  $x_0$  in the direction  $x$ , denoted by  $d^+f(x_0; x)$ , to be the above limit.

- (2) **left-hand-differentiable at  $x_0$  in the direction  $x$**  if the limit

$$\lim_{t \rightarrow 0^-} \frac{f(x_0 + tx) - f(x_0)}{t}$$

exists. If  $f$  is left-hand-differentiable at  $x_0$  in the direction  $x$ , we define the **left-hand derivative** of  $f$  at  $x_0$  in the direction  $x$ , denoted by  $d^-f(x_0; x)$ , to be the above limit.

- (3) **differentiable at  $x_0$  in the direction  $x$**  if the limit

$$\lim_{t \rightarrow 0} \frac{f(x_0 + tx) - f(x_0)}{t}$$

exists. If  $f$  is differentiable at  $x_0$  in the direction  $x$ , we define the **derivative** of  $f$  at  $x_0$  in the direction  $x$ , denoted by  $df(x_0; x)$ , to be the above limit.

**Exercise 4.5.2.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open,  $f : A \rightarrow \mathbb{R}$  and  $x_0 \in A$ . Then  $df(x_0; 0) = 0$ .

*Proof.* Clear. □

**Definition 4.5.3.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open,  $f : A \rightarrow Y$  and  $x_0 \in A$ . Then  $f$  is said to be

- (1) **right-hand Gateaux differentiable at  $x_0$**  if for each  $x \in X$ ,  $d^+f(x_0; x)$  exists. We define the **right-hand Gateaux derivative** of  $f$  at  $x_0$ , denoted  $d^+f(x_0) : X \rightarrow \mathbb{R}$ , to be

$$d^+f(x_0)(x) = d^+f(x_0; x)$$

- (2) **left-hand Gateaux differentiable at  $x_0$**  if for each  $x \in X$ ,  $d^-f(x_0; x)$  exists. We define the **left-hand Gateaux derivative** of  $f$  at  $x_0$ , denoted  $d^-f(x_0) : X \rightarrow \mathbb{R}$ , to be

$$d^-f(x_0)(x) = d^-f(x_0; x)$$

- (3) **Gateaux differentiable at  $x_0$**  if for each  $x \in X$ ,  $df(x_0; x)$  exists. We define the **Gateaux derivative** of  $f$  at  $x_0$ , denoted  $df(x_0) : X \rightarrow \mathbb{R}$ , to be

$$df(x_0)(x) = df(x_0; x)$$

**Exercise 4.5.4.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open,  $f, g : A \rightarrow Y$ ,  $\lambda \in \mathbb{R}$  and  $x_0 \in A$ . If  $f, g$  are Gateaux differentiable at  $x_0$ , then  $f + \lambda g$  Gateaux differentiable at  $x_0$  and  $d[f + \lambda g](x_0) = df(x_0) + \lambda dg(x_0)$ .

*Proof.* Similar to the case of the derivative from Calc I.  $\square$

**Exercise 4.5.5.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open,  $f : A \rightarrow Y$  and  $x_0 \in A$ . Suppose that  $f$  is Gateaux differentiable at  $x_0$ . Then for each  $\lambda \in \mathbb{R}$  and  $x \in X$ ,

$$df(x_0)(\lambda x) = \lambda df(x_0)(x) \in X^*$$

*Proof.* Let  $\lambda \in \mathbb{R}$  and  $x \in X$ . Then

$$\begin{aligned} df(x_0)(\lambda x) &= \lim_{t \rightarrow 0} \frac{f(x_0 + t\lambda x) - f(x_0)}{t} \\ &= \lim_{t \rightarrow 0} \lambda \frac{f(x_0 + t\lambda x) - f(x_0)}{\lambda t} \\ &= \lambda \lim_{t \rightarrow 0} \frac{f(x_0 + t\lambda x) - f(x_0)}{\lambda t} \\ &= \lambda \lim_{t \rightarrow 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= \lambda df(x_0)(x) \end{aligned}$$

$\square$

**Exercise 4.5.6.** Let  $X$  be a Banach space,  $A \subset X$  open,  $f : A \rightarrow \mathbb{R}$  and  $x_0 \in A$ . If  $f$  is Gateaux differentiable at  $x_0$  and  $f$  has a local extremum at  $x_0$ , then  $df(x_0) = 0$ .

*Proof.* Suppose that  $f$  is Gateaux differentiable at  $x_0$  and  $f$  has a local minimum at  $x_0$ . Then there exists  $\delta > 0$  such that  $B(x_0, \delta) \subset A$  and for each  $y \in B(x_0, \delta)$ ,  $f(x_0) \leq f(y)$ . For the sake of contradiction, suppose that  $df(x_0) \neq 0$ . Then there exists  $x \in X$  such that  $x \neq 0$  and  $df(x_0)(x) \neq 0$ .

First, suppose that  $df(x_0)(x) < 0$ . Choose  $\epsilon = -df(x_0)(x) > 0$ . Then there exists  $t_0 > 0$  such that for each  $t \in B^*(0, t_0)$ ,  $x_0 + tx \in B(x_0, \delta)$  and

$$\left| \frac{f(x_0 + tx) - f(x_0)}{t} - df(x_0)(x) \right| < \epsilon$$

This implies that for each  $t \in B^*(0, t_0)$ ,

$$\begin{aligned} \frac{f(x_0 + tx) - f(x_0)}{t} &< \epsilon + df(x_0)(x) \\ &= 0 \end{aligned}$$

and hence  $f(x_0 + tx) < f(x_0)$ , which is a contradiction.

Now, suppose that  $df(x_0)(x) > 0$ . Then

$$\begin{aligned} df(x_0)(-x) &= -df(x_0)(x) \\ &< 0 \end{aligned}$$

Similarly to above, this implies that there exists  $t_0 > 0$  such that for each  $t \in B^*(0, t_0)$ ,  $x_0 - tx \in B(x_0, \delta)$  and  $f(x_0 - tx) < f(x_0)$  which is a contradiction. So  $df(x_0)(x) = 0$  and  $df(x_0) = 0$ .

If  $f$  has a local maximum at  $x_0$ , then  $-f$  has a local minimum at  $x_0$ . Then

$$\begin{aligned} df(x_0) &= -d[-f](x_0) \\ &= -0 \\ &= 0 \end{aligned}$$

□

**Exercise 4.5.7.** Let  $X, Y$  be a normed vector spaces and  $\phi : X \rightarrow Y$  linear. If  $\phi(h) = o(\|h\|)$  as  $h \rightarrow 0$ , then  $\phi = 0$ .

*Proof.* Let  $h_0 \in X$ . If  $h_0 = 0$ , then  $\phi(h_0) = 0$ . Suppose that  $h_0 \neq 0$ . Define  $(h_n)_{n \in \mathbb{N}} \subset X$  by

$$h_n = \frac{h_0}{n}$$

Then  $h_n \rightarrow 0$ . By continuity of  $\phi$  and our initial assumption we have that

$$\begin{aligned} \|h_0\|^{-1}\phi(h_0) &= \phi\left(\frac{h_0}{\|h_0\|}\right) \\ &= \phi\left(\frac{h_n}{\|h_n\|}\right) \\ &= \frac{\phi(h_n)}{\|h_n\|} \\ &\rightarrow 0 \end{aligned}$$

which implies that  $\|h_0\|^{-1}\phi(h_0) = 0$ . So  $\phi(h_0) = 0$  and hence  $\phi = 0$ . □

**Exercise 4.5.8.** Let  $X, Y$  be a normed vector spaces,  $A \subset X$  open,  $f : A \rightarrow Y$  and  $x_0 \in A$ . Suppose that there exists  $\phi : X \rightarrow Y$  such that  $\phi$  is linear and

$$f(x_0 + h) = f(x_0) + \phi(h) + o(\|h\|) \quad \text{as } h \rightarrow 0$$

then  $\phi$  is unique.

*Proof.* Suppose that there exists  $\psi : X \rightarrow Y$  such that  $\psi$  is linear and such that

$$f(x_0 + h) = f(x_0) + \psi(h) + o(\|h\|) \quad \text{as } h \rightarrow 0$$

Then  $\phi(h) - \psi(h) = o(h)$ . Since  $\phi - \psi$  is linear, the previous exercise implies that  $\phi = \psi$ . □

**Definition 4.5.9.** Let  $X, Y$  be a Banach spaces,  $A \subset X$  open,  $f : A \rightarrow Y$  and  $x_0 \in A$ . Then  $f$  is said to be **Frechet differentiable at  $x_0$**  if there exists  $Df(x_0) \in L(X, Y)$  such that,

$$f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|) \quad \text{as } h \rightarrow 0$$

If  $f$  is Frechet differentiable at  $x_0$ , we define the **Frechet derivative of  $f$  at  $x_0$**  to be  $Df(x_0)$ .

**Exercise 4.5.10.** Let  $X, Y$  be a Banach spaces,  $A \subset X$  open,  $f, g : A \rightarrow Y$ ,  $\lambda \in \mathbb{R}$  and  $x_0 \in A$ . If  $f$  and  $g$  are Frechet differentiable at  $x_0$ , then  $f + \lambda g$  is Frechet differentiable at  $x_0$  and  $D(f + \lambda g)(x_0) = Df(x_0) + \lambda Dg(x_0)$ .

*Proof.* Suppose that  $f$  and  $g$  are Frechet differentiable at  $x_0$ . Then

$$f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|) \quad \text{as } h \rightarrow 0$$

and

$$g(x_0 + h) = g(x_0) + Dg(x_0)(h) + o(\|h\|) \quad \text{as } h \rightarrow 0$$

This implies that

$$\begin{aligned} (f + \lambda g)(x_0 + h) &= f(x_0 + h) + \lambda g(x_0 + h) \\ &= f(x_0) + Df(x_0)(h) + o(\|h\|) + \lambda g(x_0) + \lambda Dg(x_0)(h) + o(\|h\|) \\ &= (f + \lambda g)(x_0) + [Df(x_0) + \lambda Dg(x_0)](h) + o(\|h\|) \quad \text{as } h \rightarrow 0 \end{aligned}$$

Since  $Df(x_0) + \lambda Dg(x_0) \in L(X, Y)$ ,  $f + \lambda g$  is Frechet differentiable at  $x_0$  and  $D(f + \lambda g)(x_0) = Df(x_0) + \lambda Dg(x_0)$ .  $\square$

**Exercise 4.5.11.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open,  $f : A \rightarrow Y$  and  $x_0 \in A$ . If  $f$  is Frechet differentiable at  $x_0$ , then  $f$  is Gateaux differentiable at  $x_0$  and  $df(x_0) = Df(x_0)$ .

*Proof.* Suppose that  $f$  is Frechet differentiable at  $x_0$ . Then  $f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|)$  as  $h \rightarrow 0$ . Let  $x \in X$ . Then  $f(x_0 + tx) - f(x_0) = tDf(x_0)(x) + o(t)$  as  $t \rightarrow 0$ . This implies that  $f$  is differentiable at  $x_0$  in the direction  $x$  and

$$\begin{aligned} df(x_0)(x) &= \lim_{t \rightarrow 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= Df(x_0)(x) \end{aligned}$$

Since  $x \in X$  is arbitrary,  $f$  is Gateaux differentiable at  $x_0$  and  $df(x_0) = Df(x_0)$ .  $\square$

**Exercise 4.5.12.** Let  $X$  be a Banach space,  $A \subset X$  open,  $f : A \rightarrow \mathbb{R}$  and  $x_0 \in A$ . If  $f$  is Frechet differentiable at  $x_0$  and  $f$  has a local extremum at  $x_0$ , then  $df(x_0) = 0$ .

*Proof.* Suppose that  $f$  is Frechet differentiable at  $x_0$  and  $f$  has a local extremum at  $x_0$ , then  $df(x_0) = 0$ . Two previous exercises imply that  $f$  is Gateaux differentiable at  $x_0$  and

$$\begin{aligned} Df(x_0) &= df(x_0) \\ &= 0 \end{aligned}$$

$\square$

**Exercise 4.5.13.** Let  $A \subset \mathbb{R}$  be open and  $f : A \rightarrow \mathbb{R}$ . Then  $f$  is differentiable iff  $f$  is Frechet differentiable and for each  $x_0 \in A$  and  $h \in \mathbb{R}$

$$Df(x_0)(h) = hf'(x_0)$$

*Proof.* Clear.  $\square$

*Note 4.5.2.* Recall that for Banach spaces  $X$  and  $Y$ , there is an isomorphic isometry  $L(X, L(X, \dots, L(X, Y)) \dots) \rightarrow L^n(X, Y)$  given by  $\phi \mapsto \psi_\phi$  where

$$\psi_\phi(x_1, x_2, \dots, x_n) = \phi(x_1)(x_2), \dots, (x_n)$$

**Definition 4.5.14.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open and  $f : A \rightarrow Y$ . Then  $f$  is said to be **Frechet differentiable** (or **1-st order Frechet differentiable**) if for each  $x \in A$ ,  $f$  is Frechet differentiable at  $x$ .

If  $f$  is Frechet differentiable, we define the **(first order) Frechet derivative of  $f$** , denoted  $D^{(1)}f : A \rightarrow L(X, Y)$ , by  $x \mapsto D^{(1)}f(x)$ . We define higher order Frechet derivatives inductively:

Let  $x_0 \in A$  and  $n \geq 2$ . Then  $f$  is said to be  **$n$ -th order Frechet differentiable at  $x_0$**  if  $f$  is  $(n - 1)$ -th order Frechet differentiable and  $D^{n-1}f$  is Frechet differentiable at  $x_0$ . If  $f$  is  $n$ -th order Frechet differentiable at  $x_0$ , we define  $D^n f(x_0) \in L^n(X, Y)$  by

$$D^n f(x_0) = D[D^{n-1}f](x_0)$$

Finally,  $f$  is said to be  **$n$ -th order Frechet differentiable** if  $f$  is  $(n-1)$ -th order Frechet differentiable and for each  $x \in A$ ,  $D^{n-1}f$  is Frechet differentiable at  $x$ . If  $f$  is  $n$ -th order Frechet differentiable, we define the  **$n$ -th order Frechet derivative of  $f$** , denoted  $D^n f : A \rightarrow L^n(X, Y)$  by

$$D^n f = D[D^{n-1}f]$$

If  $f$  is  $n$ -th order differentiable, then  $f$  is said to be **continuously  $n$ -th order differentiable** if  $D^n f$  is continuous. We define

$$C_Y^n(A) = \{f : A \rightarrow Y : f \text{ is continuously } n\text{-th order differentiable}\}$$

**Exercise 4.5.15. Mean Value Theorem:**

Let  $X, Y$  be Banach spaces,  $A \subset X$  open and convex and  $f : A \rightarrow Y$ . If  $f$  is Frechet differentiable, then for each  $x, y \in A$ , there exists  $t \in (0, 1)$  such that

$$\|f(x) - f(y)\| \leq \|Df(tx + (1-t)y)\| \|x - y\|$$

**Hint:** For  $x, y \in A$  with  $f(x) \neq f(y)$ , using a Hahn-Banach argument, find  $\lambda \in Y^*$  such that  $\|\lambda\| = 1$  and  $\lambda(f(x) - f(y)) = \|f(x) - f(y)\|$ .

*Proof.* Suppose that  $f$  is Frechet differentiable. Let  $x, y \in A$ . The claim is clearly true when  $f(x) = f(y)$ . Suppose that  $f(x) \neq f(y)$ . An exercise in the section on linear functionals implies that there exists  $\lambda \in Y^*$  such that  $\lambda(f(x) - f(y)) = \|f(x) - f(y)\|$  and  $\|\lambda\| = 1$ . Define  $g : [0, 1] \rightarrow \mathbb{R}$  by

$$g(t) = \lambda(f(tx + (1-t)y))$$

Then  $g$  is continuous and (Frechet) differentiable on  $(0, 1)$  with

$$Dg(t)(h) = \lambda \circ Df(tx + (1-t)y)((x-y)h)$$

which implies that

$$\begin{aligned} g'(t) &= Dg(t)(1) \\ &= \lambda \circ Df(tx + (1-t)y)((x-y)) \end{aligned}$$

The mean value theorem implies that there exists  $t \in (0, 1)$  such that

$$\begin{aligned} \|f(x) - f(y)\| &= \lambda(f(x) - f(y)) \\ &= g(1) - g(0) \\ &= g'(t) \\ &= \lambda \circ Df(tx + (1-t)y)((x-y)) \end{aligned}$$

Taking absolute values, we see that

$$\begin{aligned} \|f(x) - f(y)\| &= |\lambda \circ Df(tx + (1-t)y)((x-y))| \\ &\leq \|\lambda\| \|Df(tx + (1-t)y)\| \|x - y\| \\ &\leq \|Df(tx + (1-t)y)\| \|x - y\| \end{aligned}$$

□

**Exercise 4.5.16.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open and convex and  $f : A \rightarrow Y$ . Suppose that  $f$  is Frechet differentiable. If for each  $x \in A$ ,  $Df(x) = 0$ , then  $f$  is constant.

*Proof.* Suppose that for each  $x \in A$ ,  $Df(x) = 0$ . Let  $x, y \in A$ . Then the mean value theorem implies that there exists  $t \in (0, 1)$  such that

$$\begin{aligned} \|f(x) - f(y)\| &\leq \|Df(tx + (1-t)y)\| \|x - y\| \\ &= 0 \end{aligned}$$

So  $f(x) = f(y)$ . □

**Exercise 4.5.17.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open and convex and  $f, g : A \rightarrow Y$ . Suppose that  $f$  and  $g$  are Frechet differentiable. If  $Df = Dg$ , then there exists  $c \in Y$  such that  $f = g + c$ .

*Proof.* Suppose that  $Df = Dg$ . Then  $D(f - g) = 0$  and the previous exercise implies that  $f - g$  is constant. □

**Exercise 4.5.18.** Let  $Y$  be a separable Banach space,  $f : [a, b] \rightarrow Y$  continuous so that  $f$  is Bochner-integrable. Define  $F : (a, b) \rightarrow Y$  by

$$F(x) = \int_{(a, x]} f dm$$

Then  $F \in C_Y^1((a, b))$  and for each  $x_0 \in (a, b)$  and  $h \in \mathbb{R}$ ,  $DF(x_0)(h) = hf(x_0)$

*Proof.* Let  $x_0 \in (a, b)$  and  $h \in (0, b - x_0)$ . Then continuity implies that

$$\begin{aligned} \frac{1}{\|h\|} \left\| \int_{(x_0, x_0+h]} f - f(x_0) dm \right\| &\leq \frac{1}{\|h\|} \max_{x \in (x_0, x_0+h]} \|f(x) - f(x_0)\| \|h\| \\ &= \max_{x \in [x_0, x_0+h]} \|f(x) - f(x_0)\| \\ &\rightarrow 0 \text{ as } h \rightarrow 0 \end{aligned}$$

So

$$\int_{(x_0, x_0+h]} f - f(x_0) dm = o(\|h\|) \quad \text{as } h \rightarrow 0$$

Therefore

$$\begin{aligned} F(x_0 + h) &= \int_{(a, x_0+h]} f dm \\ &= \int_{(a, x_0]} f dm + \int_{(x_0, x_0+h]} f dm \\ &= \int_{(a, x_0]} f dm + hf(x_0) + \int_{(x_0, x_0+h]} f - f(x_0) dm \\ &= F(x_0) + hf(x_0) + o(\|h\|) \quad \text{as } h \rightarrow 0 \end{aligned}$$

The case is similar for  $h \in (x_0 - b, 0)$ . Since the map  $h \mapsto f(x_0)h$  is bounded,  $F$  is Frechet differentiable at  $x_0$  and  $DF(x_0)(h) = f(x_0)h$ . Let  $x, y \in (a, b)$  and  $h \in \mathbb{R}$ . Then

$$\begin{aligned} \|(DF(x) - DF(y))(h)\| &= \|DF(x)(h) - DF(y)(h)\| \\ &= \|hf(x) - hf(y)\| \\ &= \|h(f(x) - f(y))\| \\ &= \|f(x) - f(y)\| \|h\| \end{aligned}$$

So  $\|DF(x) - DF(y)\| \leq \|f(x) - f(y)\|$ . Then continuity of  $f$  implies continuity of  $DF$ . Hence  $F \in C_Y^1(a, b)$ .  $\square$

**Exercise 4.5.19. Fundamental Theorem of Calculus:** Let  $Y$  be a separable Banach space and  $f \in C_Y^1(a, b)$ . Then for each  $x, x_0 \in (a, b)$ ,  $x_0 < x$  implies that

- (1)  $Df(\cdot)(1)$  is Bochner integrable on  $(x_0, x]$
- (2)

$$f(x) - f(x_0) = \int_{(x_0, x]} Df(t)(1) dm(t)$$

*Proof.* (1) Since  $Df : (a, b) \rightarrow Y$  is continuous and for each  $x, y \in (a, b)$ ,

$$\begin{aligned} \|Df(x)(1) - Df(y)(1)\| &= \|(Df(x) - Df(y))(1)\| \\ &\leq \|Df(x) - Df(y)\| \end{aligned}$$

and we have that  $Df(\cdot)(1) : (a, b) \rightarrow Y$  is continuous. Let  $x, x_0 \in (a, b)$ . Suppose that  $x_0 < x$ . Choose  $c, d \in (a, b)$  such that  $a < c < x_0 < x < d < b$ . Then  $Df(\cdot)(1)$  is continuous on  $[c, d]$  and hence Bochner-integrable on  $(c, d]$  and  $(x_0, x]$ .

- (2) Define  $g : (c, d) \rightarrow Y$  by

$$g(\xi) = \int_{(c, \xi]} Df(t)(1) dm(t)$$

Then the previous exercise implies that  $g \in C_Y^1$  and for each  $t \in (c, d)$  and  $h \in \mathbb{R}$ ,

$$\begin{aligned} Dg(t)(h) &= hDf(t)(1) \\ &= Df(t)(h) \end{aligned}$$

A previous exercise implies that there exists  $c \in Y$  such that  $f = g + c$  on  $(c, d)$ . Then

$$\begin{aligned} f(x) - f(x_0) &= g(x) + c - (g(x_0) + c) \\ &= g(x) - g(x_0) \\ &= \int_{(c, x]} Df(t)(1) dm(t) - \int_{(c, x_0]} Df(t)(1) dm(t) \\ &= \int_{(x_0, x]} Df(t)(1) dm(t) \end{aligned}$$

$\square$

**Exercise 4.5.20.** Let  $X, Y$  be Banach spaces,  $A \subset X$  open,  $f : A \rightarrow Y$ ,  $x_0 \in A$  and  $n \in \mathbb{N}$ . If  $Y$  is separable, then  $f$  is  $n$ -th order Frechet differentiable at  $x_0$  iff for each  $j \in \{1, \dots, n\}$ , there exists  $\phi_j \in L^j(X, Y)$  such that

$$f(x + h) = f(x) + \sum_{j=1}^n \phi_j(h, \dots, h) + o(\|h\|^n) \quad \text{as } h \rightarrow 0$$

*Proof.*

$\square$

## 5. HILBERT SPACES

**Definition 5.0.1.** Let  $H$  be a vector space and  $\langle \cdot, \cdot \rangle : H \rightarrow \mathbb{C}$ . Then  $\langle \cdot, \cdot \rangle$  is said to be an **inner product** on  $H$  if for each  $x, y, z \in H$  and  $c \in \mathbb{C}$

- (1)  $\langle x, y + cz \rangle = \langle x, y \rangle + c\langle x, z \rangle$
- (2)  $\langle x, y \rangle = \langle y, x \rangle^*$
- (3)  $\langle x, x \rangle \geq 0$
- (4) if  $\langle x, x \rangle = 0$ , then  $x = 0$ .

**Exercise 5.0.2.** Let  $H$  be an inner product space,  $(x_j)_{j=1}^n, (y_j)_{j=1}^n \subset H$  and  $(\alpha_j)_{j=1}^n, (\beta_j)_{j=1}^n \subset \mathbb{C}$ . Then

$$\left\langle \sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^n \beta_j y_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^n \alpha_i^* \beta_j \langle x_i, y_j \rangle$$

*Proof.* Clear. □

**Definition 5.0.3.** Let  $H$  be an inner product space. Define the **induced norm**, denoted  $\| \cdot \| : H \rightarrow \mathbb{C}$ , by

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

**Exercise 5.0.4.** Let  $H$  be an inner product space. Then the induced norm,  $\| \cdot \| : H \rightarrow \mathbb{C}$ , is a norm.

*Proof.* Let  $x, y \in H$  and  $c \in \mathbb{C}$ . Then

- (1)  $\|x + y\|$
- (2) Note that

$$\begin{aligned} \|cx\|^2 &= \langle cx, cx \rangle \\ &= c * c \langle x, x \rangle \\ &= |c|^2 \|x\|^2 \end{aligned}$$

$$\text{So } \|cx\| = |c| \|x\|$$

□

**Definition 5.0.5.** Let  $x_1, x_2 \in H$  and  $S \subset H$ . Then  $x_1$  and  $x_2$  are said to be **orthogonal** if  $\langle x_1, x_2 \rangle = 0$  and  $S$  is said to be **orthogonal** if for each  $x_1, x_2 \in S$ ,  $x_1, x_2$  are orthogonal.



## 6. CONVEXITY

## 6.1. Introduction.

*Note 6.1.1.* In this section, we assume all vector spaces are real.

**Definition 6.1.1.** Let  $X$  be a vector space and  $A \subset X$ . Then  $A$  is said to be **convex** if for each  $x, y \in A$ , and  $t \in [0, 1]$ ,  $tx + (1 - t)y \in A$ .

**Definition 6.1.2.** Let  $X$  be a vector space and  $f : A \rightarrow \mathbb{R}$ . Then  $f$  is said to be **convex** if for each  $x, y \in A$ ,  $t \in [0, 1]$ ,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

**Exercise 6.1.3.** Let  $X$  be a vector space,  $f \in X^*$  and  $g : X \rightarrow \mathbb{R}$  constant. Then  $f$  and  $g$  are convex.

*Proof.* Let  $x, y \in X$  and  $t \in [0, 1]$ . Put  $c = g(0)$ . Then

$$f(tx + (1 - t)y) = tf(x) + (1 - t)f(y)$$

and

$$\begin{aligned} g(tx + (1 - t)y) &= c \\ &= tc + (1 - t)c \\ &= tg(x) + (1 - t)g(y) \end{aligned}$$

So  $f$  and  $g$  are convex. □

**Exercise 6.1.4.** Let  $X$  be a vector space,  $A \subset X$  convex,  $f, g : A \rightarrow \mathbb{R}$  and  $\lambda \geq 0$ . If  $f, g$  are convex, then

- (1)  $f + g$  is convex
- (2)  $\lambda f$  is convex

*Proof.* Suppose that  $f$  and  $g$  are convex. Let  $x, y \in A$  and  $t \in [0, 1]$ . Then

$$\begin{aligned} (f + \lambda g)(tx + (1 - t)y) &= f(tx + (1 - t)y) + \lambda g(tx + (1 - t)y) \\ &\leq tf(x) + (1 - t)f(y) + t\lambda g(x) + (1 - t)\lambda g(y) \\ &= t(f(x) + \lambda g(x)) + (1 - t)(f(y) + \lambda g(y)) \\ &= t(f + \lambda g)(x) + (1 - t)(f + \lambda g)(y) \end{aligned}$$

□

**Definition 6.1.5.** Let  $X$  be a vector space and  $f : X \rightarrow \mathbb{R}$ . Then  $f$  is said to be **affine** if there exists  $\phi \in X^*$ ,  $a \in \mathbb{R}$  constant such that  $f = \phi + a$ .

**Exercise 6.1.6.** Let  $X$  be a vector space and  $f : X \rightarrow \mathbb{R}$ . If  $f$  is affine, then  $f$  is convex.

*Proof.* Suppose that  $f$  is affine. Then there exists  $\phi \in X^*$ ,  $a \in \mathbb{R}$  constant such that  $f = \phi + a$ . Then  $\phi$  is convex and  $g : X \rightarrow \mathbb{R}$  defined by  $g(x) = a$  is convex. So  $f = \phi + g$  is convex. □

**Exercise 6.1.7.** Let  $X$  be a vector space,  $A \subset X$  convex,  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : A \rightarrow \mathbb{R}$ . If  $f$  is convex and increasing and  $g$  is convex, then  $f \circ g$  is convex.

*Proof.* Let  $t \in [0, 1]$  and  $x, y \in A$ . Then convexity of  $g$  implies that

$$g(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y)$$

and we have

$$\begin{aligned} f \circ g(tx + (1 - t)y) &= f(g(tx + (1 - t)y)) \\ &\leq f(tg(x) + (1 - t)g(y)) && (f \text{ increasing}) \\ &\leq tf(g(x)) + (1 - t)f(g(y)) && (f \text{ convex}) \\ &= tf \circ g(x) + (1 - t)f \circ g(y) \end{aligned}$$

So  $f \circ g$  is convex. □

**Exercise 6.1.8.** Let  $X$  be a vector space,  $A \subset X$  convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . Then  $f$  has a local minimum at  $x_0$  iff  $f$  has a global minimum at  $x_0$ .

*Proof.* If  $f$  has a global minimum at  $x_0$ , then  $f$  has a local minimum at  $x_0$ . Conversely, suppose that  $f$  has a local minimum at  $x_0$ . Then there exists  $\delta > 0$  such that for each  $x \in B(x_0, \delta)$ ,  $f(x_0) \leq f(x)$ . For the sake of contradiction, suppose that  $f$  does not have a global minimum at  $x_0$ . Then there exists  $x' \in A$  such that  $f(x') < f(x_0)$ . Put  $t_0 = \min(\frac{\delta}{\|x' - x_0\| + 1}, 1) > 0$ . Let  $t \in (0, t_0)$ , then

$$\begin{aligned} \|(tx' + (1 - t)x_0) - x_0\| &= t\|x' - x_0\| \\ &< \frac{\|x' - x_0\|\delta}{\|x' - x_0\| + 1} \\ &< \delta \end{aligned}$$

so that  $tx' + (1 - t)x_0 \in B(x_0, \delta)$  and hence  $f(x_0) \leq f(tx' + (1 - t)x_0)$ . Therefore

$$\begin{aligned} f(x_0) &\leq f(tx' + (1 - t)x_0) \\ &\leq tf(x') + (1 - t)f(x_0) \quad (\text{convexity of } f) \\ &< tf(x_0) + (1 - t)f(x_0) \\ &= f(x_0) \end{aligned}$$

which is a contradiction. Hence  $f$  has a global minimum at  $x_0$ . □

**Definition 6.1.9.** Let  $X, Y$  be vector spaces,  $A \subset X \oplus Y$ . For  $y \in Y$ , define

$$A^y = \{x \in X : (x, y) \in A\}$$

and  $f^y : A^y \rightarrow \mathbb{R}$  by

$$f^y(x) = f(x, y)$$

**Exercise 6.1.10.** Let  $X, Y$  be vector spaces,  $A \subset X \oplus Y$  convex and  $f : A \rightarrow \mathbb{R}$  convex. Then for each  $y \in \pi_2(A)$ ,

- (1)  $A^y$  is convex
- (2)  $f^y$  is convex

where  $\pi_2 : X \times Y \rightarrow Y$ , the canonical projection of  $X \times Y$  onto  $Y$  given by  $\pi_2(x, y) = y$ .

*Proof.* Let  $y \in \pi_2(A)$ ,  $x_1, x_2 \in A^y$  and  $t \in [0, 1]$ . Then by definition,  $(x_1, y), (x_2, y) \in A$ .

- (1) Convexity of  $A$  implies that  $(tx_1 + (1 - t)x_2, y) \in A$ . Hence  $tx_1 + (1 - t)x_2 \in A^y$  and  $A^y$  is convex.

(2) Convexity of  $f$  implies that

$$\begin{aligned} f^y(tx_1 + (1-t)x_2) &= f(tx_1 + (1-t)x_2, y) \\ &= f(t(x_1, y) + (1-t)(x_2, y)) \\ &\leq tf(x_1, y) + (1-t)f(x_2, y) \\ &= tf^y(x_1) + (1-t)f^y(x_2) \end{aligned}$$

and so  $f^y$  is convex. □

**Exercise 6.1.11.** Let  $X, Y$  be vector spaces and  $A \subset X, B \subset Y$ . If  $A$  and  $B$  are convex, then  $A \times B \subset X \oplus Y$  is convex.

*Proof.* Suppose that  $A$  and  $B$  are convex. Let  $(x_1, y_1), (x_2, y_2) \in A \times B$  and  $t \in [0, 1]$ . Convexity of  $A$  and  $B$  implies that  $tx_1 + (1-t)x_2 \in A$  and  $ty_1 + (1-t)y_2 \in B$ . Therefore

$$\begin{aligned} t(x_1, y_1) + (1-t)(x_2, y_2) &= (tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \\ &\in A \times B \end{aligned}$$
□

**Exercise 6.1.12.** Let  $X, Y$  be vector spaces and  $A \subset X, B \subset Y$  convex (implying that  $A \times B$  is convex) and  $f : A \times B \rightarrow \mathbb{R}$  convex. Suppose that for each  $y \in B$ ,  $\{f(x, y) : x \in A\}$  is bounded below. Then  $\inf_{y \in B} f^y$  is convex

*Proof.* Put  $g = \inf_{y \in B} f^y$ . Let  $x_1, x_2 \in A, y_1, y_2 \in B$  and  $t \in [0, 1]$ . Put  $y' = ty_1 + (1-t)y_2$ . Then convexity of  $f$  implies that

$$\begin{aligned} g(tx_1 + (1-t)x_2) &\leq f^{y'}(tx_1 + (1-t)x_2) \\ &= f(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \\ &= f(t(x_1, y_1) + (1-t)(x_2, y_2)) \\ &\leq tf(x_1, y_1) + (1-t)f(x_2, y_2) \\ &= tf^{y_1}(x_1) + (1-t)f^{y_2}(x_2) \end{aligned}$$

Since  $y_1 \in B$  is arbitrary, we have that

$$g(tx_1 + (1-t)x_2) \leq tg(x_1) + (1-t)f^{y_2}(x_2)$$

Similarly, since  $y_2 \in B$  is arbitrary, we have that

$$g(tx_1 + (1-t)x_2) \leq tg(x_1) + (1-t)g(x_2)$$

and  $f$  is convex. □

**Exercise 6.1.13.** Let  $X$  be a vector space,  $A \subset X$  convex and  $(f_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}^A$ . Suppose that for each  $\lambda \in \Lambda$ ,  $f_\lambda$  is convex. Then  $\sup_{\lambda \in \Lambda} f_\lambda$  is convex.

*Proof.* Define  $f = \sup_{\lambda \in \Lambda} f_\lambda$ . Let  $x, y \in A, t \in [0, 1]$  and  $\lambda \in \Lambda$ . Then

$$\begin{aligned} f_\lambda(tx + (1-t)y) &\leq tf_\lambda(x) + (1-t)f_\lambda(y) \\ &\leq tf(x) + (1-t)f(y) \end{aligned}$$

Since  $\lambda \in \Lambda$  is arbitrary,  $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ .  $\square$

**Exercise 6.1.14.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . If  $f$  is continuous at  $x_0$ , then  $f$  is locally Lipschitz at  $x_0$ . (**Hint:** Given  $x_1, x_2$  near  $x_0$  Choose a  $z$  near  $x_0$  s.t.  $x_1$  is a convex combination of  $x_2$  and  $z$ . Then repeat but with  $x_2$  as a convex combination of  $x_1$  and  $z$ )

*Proof.* By continuity,  $f$  is locally bounded at  $x_0$ . So there exist  $M, \delta > 0$  such that  $B(x_0, \delta) \subset A$  and for each  $x \in B(x_0, \delta)$ ,  $|f(x)| \leq M$ . Put  $\delta' = \frac{\delta}{2}$  and choose  $U = B(x_0, \delta')$ . Then  $U \subset A$ ,  $U$  is open and  $U \in N_{x_0}$ .

Let  $x_1, x_2 \in U$ . Suppose that  $x_1 \neq x_2$ . Define  $\alpha = \|x_1 - x_2\| > 0$ ,  $p = \frac{\alpha}{\alpha + \delta'}$ ,  $q = 1 - p$  and  $z = p^{-1}(x_1 - qx_2)$ . Then  $x_1 = pz + qx_2$  and

$$\begin{aligned} \|z - x_1\| &= \|(p^{-1} - 1)x_1 - p^{-1}qx_2\| \\ &= \frac{1-p}{p}\alpha \\ &= \frac{\delta'}{\alpha}\alpha \\ &= \delta' \end{aligned}$$

Therefore

$$\begin{aligned} \|z - x_0\| &\leq \|z - x_1\| + \|x_1 - x_0\| \\ &< \delta' + \delta' \\ &= \delta \end{aligned}$$

So  $z \in B(x_0, \delta)$ , which implies that

$$\begin{aligned} f(z) - f(x_2) &\leq |f(z) - f(x_2)| \\ &\leq |f(z)| + |f(x_2)| \\ &\leq 2M \end{aligned}$$

Since  $x_1 = pz + qx_2$ , convexity of  $f$  implies that  $f(x_1) \leq pf(z) + qf(x_2)$ . Hence

$$\begin{aligned} f(x_1) - f(x_2) &\leq pf(z) - pf(x_2) \\ &= p(f(z) - f(x_2)) \\ &\leq p2M \\ &= \frac{\alpha}{\alpha + \delta'}2M \\ &\leq \alpha 2M \\ &= 2M\|x_1 - x_2\| \end{aligned}$$

Similarly, choosing  $z = p^{-1}(x_2 - qx_1)$ , yields  $f(x_2) - f(x_1) \leq 2M\|x_1 - x_2\|$  which implies that

$$|f(x_1) - f(x_2)| \leq 2M\|x_1 - x_2\|$$

and  $f$  is Lipschitz on  $U$ .  $\square$

## 6.2. Differentiation.

**Exercise 6.2.1.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex,  $x_0 \in A$  and  $x \in X$ . Define  $T = \{t \in \mathbb{R} : x_0 + tx \in A\}$ . Then there exist  $a, b \in (0, \infty]$  such that  $T = (-a, b)$ .

*Proof.* Continuity of scalar multiplication and addition implies that  $T$  is an open neighborhood of 0. Let  $t > 0$  and  $s \in [0, t]$ . Then  $\frac{s}{t} \in [0, 1]$  and by convexity of  $A$ ,  $x_0 + tx \in A$  implies that

$$\begin{aligned} x_0 + sx &= \frac{s}{t}(x_0 + tx) + \left(1 - \frac{s}{t}\right)x_0 \\ &\in A \end{aligned}$$

Thus  $[0, t] \subset T$ . Similarly,  $x_0 - tx \in A$  implies that  $[-t, 0] \subset T$ .

Define  $a, b \in (0, \infty]$  by  $a = \sup\{t > 0 : x_0 - tx \in A\}$  and  $b = \sup\{t > 0 : x_0 + tx \in A\}$ . Then  $(-a, b) = T$ .  $\square$

**Definition 6.2.2.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex,  $x_0 \in A$  and  $x \in X$ . Define  $T$  as in the previous exercise and choose  $t_0 > 0$  such that  $(-t_0, t_0) \subset T$ . For  $t \in (0, t_0)$ , define the difference quotient  $q : (-t_0, t_0) \setminus \{0\} \rightarrow \mathbb{R}$  by

$$q(t) = \frac{f(x_0 + tx) - f(x_0)}{t}$$

**Exercise 6.2.3.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex,  $x_0 \in A$  and  $x \in X$ . Define  $t_0$  as above. Then

- (1)  $q(t)$  is increasing on  $(0, t_0)$
- (2)  $q(-t)$  decreasing on  $(0, t_0)$

(**Hint:** As an example, look at the graph of  $f(x) = x^2$ . For the algebra, start at the desired end inequality and work backwards)

*Proof.* Let  $s, t \in (0, t_0)$  and suppose that  $s \leq t$ . Then  $x_0 + sx, x_0 + tx \in A$ . Note that since  $0 < s \leq t$ ,  $\frac{s}{t} \in (0, 1]$  and  $1 - \frac{s}{t} = \frac{t-s}{t} \in (0, 1]$ . Also, since  $A$  is convex, we have that

$$\left(\frac{t-s}{t}\right)x_0 + \left(\frac{s}{t}\right)(x_0 + tx) \in A$$

Convexity of  $f$  implies that

$$\begin{aligned} f(x_0 + sx) &= f\left(\left(\frac{t-s}{t}\right)x_0 + \left(\frac{s}{t}\right)(x_0 + tx)\right) \\ &\leq \left(\frac{t-s}{t}\right)f(x_0) + \left(\frac{s}{t}\right)f(x_0 + tx) \end{aligned}$$

This implies that

$$tf(x_0 + sx) \leq (t-s)f(x_0) + sf(x_0 + tx)$$

and after rearranging, we get

$$tf(x_0 + sx) - tf(x_0) \leq sf(x_0 + tx) - sf(x_0)$$

and so finally, dividing both sides by  $st$ , we obtain

$$\begin{aligned} q(s) &= \frac{f(x_0 + sx) - f(x_0)}{s} \\ &\leq \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= q(t) \end{aligned}$$

as desired.

Similar to (1). □

**Exercise 6.2.4.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex,  $x_0 \in A$  and  $x \in X$ . Define  $t_0$  as before. Then for each  $t \in (0, t_0)$ ,

$$q(-t) \leq q(t)$$

(**Hint:** for sufficiently small  $t$ , convexity of  $f$  implies that  $f(x_0) \leq \frac{1}{2}f(x_0 - 2tx) + \frac{1}{2}f(x_0 + 2tx)$ )

- (1) *Proof.* Choose  $t_0$  as in the previous exercise. Since convexity of  $f$  implies that for each  $t \in (0, t_0/2)$ ,

$$f(x_0) \leq \frac{1}{2}f(x_0 - 2tx) + \frac{1}{2}f(x_0 + 2tx)$$

we have that for each  $t \in (0, t_0/2)$ ,

$$\begin{aligned} q(-2t) &= \frac{f(x_0 - 2tx) - f(x_0)}{-2t} \\ &\leq \frac{f(x_0 + 2tx) - f(x_0)}{2t} \\ &= q(2t) \end{aligned}$$

So for each  $t \in (0, t_0)$ ,  $q(-t) \leq q(t)$ . □

**Exercise 6.2.5.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . Then

- (1)  $f$  is left-hand and right-hand Gateaux differentiable at  $x_0$  with  $d^-f(x_0) \leq d^+f(x_0)$
- (2) for each  $x \in X$ ,  $d^-f(x_0)(x) = -d^+f(x_0)(-x)$

*Proof.*

- (1) Let  $x \in X$ . Choose  $t_0 > 0$  as in the previous two exercises. Let  $t, u \in (0, t_0)$ . Choose  $s \in (0, \min(u, t))$ . The previous two exercises imply that

$$\begin{aligned} q(-u) &\leq q(-s) \\ &\leq q(s) \\ &\leq q(t) \end{aligned}$$

and therefore  $q(t)$  is an upper bound for  $\{q(-u) : u \in (0, t_0)\}$  and  $d^-f(x_0)(x) = \sup_{u \in (0, t_0)} q(-u)$  exists with  $d^-f(x_0)(x) \leq q(t)$ .

Since  $t \in (0, t_0)$  is arbitrary,  $d^-f(x_0)(x)$  is a lower bound for  $\{q(t) : t \in (0, t_0)\}$ . Therefore

$$d^+f(x_0)(x) = \inf_{t \in (0, t_0)} q(t)$$

exists with  $d^+f(x_0)(x) \geq d^-f(x_0)(x)$ .

(2) By definition, we have

$$\begin{aligned} d^-f(x_0)(x) &= \lim_{t \rightarrow 0^+} \frac{f(x_0 + -tx) - f(x_0)}{-t} \\ &= - \lim_{t \rightarrow 0^+} \frac{f(x_0 + -tx) - f(x_0)}{t} \\ &= -d^+f(x_0)(-x) \end{aligned}$$

□

**Exercise 6.2.6.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . Then  $d^+f(x_0) : X \rightarrow \mathbb{R}$  is a sublinear functional.

*Proof.* Let  $x, y \in X$  and  $k \geq 0$ . If  $k = 0$ , then clearly

$$d^+f(x_0)(kx) = kd^+(x_0)(x)$$

If  $k > 0$ . Then

$$\begin{aligned} d^+f(x_0)(kx) &= \lim_{t \rightarrow 0^+} \frac{f(x_0 + tkx) - f(x_0)}{t} \\ &= k \lim_{t \rightarrow 0^+} \frac{f(x_0 + tkx) - f(x_0)}{tk} \\ &= kd^+f(x_0)(x) \end{aligned}$$

Define  $t_0 > 0$  as before and let  $t \in (0, \frac{t_0}{2})$ . Note that

$$x_0 + tx + ty = \frac{1}{2}(x_0 + 2tx) + \frac{1}{2}(x_0 + 2ty)$$

Convexity of  $f$  implies that

$$f(x_0 + tx + ty) \leq \frac{1}{2}f(x_0 + 2tx) + \frac{1}{2}f(x_0 + 2ty)$$

which implies that

$$\frac{f(x_0 + tx + ty) - f(x_0)}{t} \leq \frac{f(x_0 + 2tx) - f(x_0)}{2t} + \frac{f(x_0 + 2ty) - f(x_0)}{2t}$$

Therefore

$$\begin{aligned} d^+f(x_0)(x + y) &= \lim_{t \rightarrow 0^+} \frac{f(x_0 + t(x + y)) - f(x_0)}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{f(x_0 + tx + ty) - f(x_0)}{t} \\ &\leq \lim_{t \rightarrow 0^+} \left[ \frac{f(x_0 + 2tx) - f(x_0)}{2t} + \frac{f(x_0 + 2ty) - f(x_0)}{2t} \right] \\ &= \lim_{t \rightarrow 0^+} \frac{f(x_0 + 2tx) - f(x_0)}{2t} + \lim_{t \rightarrow 0^+} \frac{f(x_0 + 2ty) - f(x_0)}{2t} \\ &= d^+f(x_0)(x) + d^+f(x_0)(y) \end{aligned}$$

□

**Exercise 6.2.7.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . Then for each  $x \in A$ ,

$$d^+f(x_0)(x - x_0) \leq f(x) - f(x_0)$$

*Proof.* Let  $x \in A$ . Define  $T = \{t \in \mathbb{R} : x_0 + t(x - x_0) \in A\}$  similarly to earlier. Clearly  $1 \in T$  and

$$\begin{aligned} d^+f(x_0)(x - x_0) &= \inf_{t \in (0,1]} \frac{f(x_0 + t(x - x_0)) - f(x_0)}{t} \\ &\leq f(x) - f(x_0) \end{aligned}$$

□

**Exercise 6.2.8.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . If  $f$  is continuous at  $x_0$ , then  $d^+f(x_0)$  is Lipschitz (equivalently bounded).

*Proof.* Suppose that  $f$  is continuous at  $x_0$ . A previous exercise about convex functions tells us that  $f$  is locally Lipschitz at  $x_0$ , so there exists  $\delta, M > 0$  such that for each  $x_1, x_2 \in B(x_0, \delta)$ ,  $|f(x_1) - f(x_2)| \leq M\|x_1 - x_2\|$ . Let  $x \in X$  and define  $t_0 = \frac{\delta}{\|x\|+1}$  so that for each  $t \in (0, t_0)$ ,

$$\begin{aligned} \|(x_0 + tx) - x_0\| &= t\|x\| \\ &\leq t_0\|x\| \\ &= \frac{\delta\|x\|}{\|x\|+1} \\ &< \delta \end{aligned}$$

and  $x_0 + tx \in B(x_0, \delta)$ . Then for each  $t \in (0, t_0)$ ,

$$\begin{aligned} d^+f(x_0)(x) &\leq \frac{f(x_0 + tx) - f(x_0)}{t} \\ &\leq \frac{|f(x_0 + tx) - f(x_0)|}{t} \\ &\leq t^{-1}M\|(x_0 + tx) - x_0\| \\ &= M\|x\| \end{aligned}$$

Thus  $d^+f(x_0)$  is a bounded sublinear functional and a previous exercise in the section on sublinear functionals implies this is equivalent to  $d^+f(x_0)$  being Lipschitz. □

**Exercise 6.2.9.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . If  $f$  is continuous at  $x_0$ , then there exists  $\phi \in X^*$  such that  $\phi \leq d^+f(x_0)$ .

*Proof.* Suppose that  $f$  is continuous at  $x_0$ . The previous exercise implies that  $d^+f(x_0)$  is Lipschitz (equivalently bounded). A previous exercise in the section discussing sublinear functionals tells us that boundedness of  $d^+f(x_0)$  implies that there exists  $\phi \in X^*$  such that  $\phi \leq d^+f(x_0)$ . □

**Definition 6.2.10.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . We define the **subdifferential of  $f$  at  $x_0$** , denoted  $\partial f(x_0)$ , to be

$$\partial f(x_0) = \{\phi \in X^* : \text{for each } x \in A, f(x_0) + \phi(x - x_0) \leq f(x)\}$$

**Exercise 6.2.11.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . If  $f$  is continuous at  $x_0$ , then  $\partial f(x_0) \neq \emptyset$ .



*Proof.* Suppose that  $f$  is continuous at  $x_0$ . The previous exercise tells us that there exists  $\phi \in X^*$  such that  $\phi \leq d^+f(x_0)$ . Let  $x \in A$ . A previous exercise implies that

$$\begin{aligned}\phi(x - x_0) &\leq d^+f(x_0)(x - x_0) \\ &\leq f(x) - f(x_0)\end{aligned}$$

Then  $f(x_0) + \phi(x - x_0) \leq f(x)$ . □

**Exercise 6.2.12.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex,  $\phi \in X^*$  and  $x_0 \in A$ . Then

(1) for each  $x \in A$ ,

$$\phi(x - x_0) \leq f(x) - f(x_0)$$

iff

$$\phi \leq d^+f(x_0)$$

(2)  $\partial f(x_0) = \{\phi \in X^* : \phi \leq d^+f(x_0)\}$

*Proof.*

(1) Suppose that for each  $x \in A$ ,  $\phi(x - x_0) \leq f(x) - f(x_0)$ . Let  $x \in X$ . Define  $t_0$  as before. Then for each  $t \in (0, t_0)$ ,

$$\begin{aligned}t\phi(x) &= \phi((x_0 + tx) - x_0) \\ &\leq f(x_0 + tx) - f(x_0)\end{aligned}$$

This implies that  $\phi(x) \leq d^+f(x_0)(x)$ .

Conversely, suppose that  $\phi \leq d^+f(x_0)$ . Let  $x \in A$ . A previous exercise implies that,

$$\begin{aligned}\phi(x - x_0) &\leq d^+f(x_0)(x - x_0) \\ &\leq f(x) - f(x_0)\end{aligned}$$

(2) Clear. □

**Exercise 6.2.13.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . If  $f$  is continuous at  $x_0$ , then the following are equivalent:

- (1)  $f$  is Gateaux differentiable at  $x_0$
- (2)  $d^+f(x_0)$  is linear
- (3)  $\#\partial f(x_0) = 1$

*Proof.* Suppose that  $f$  is continuous at  $x_0$ . Then  $d^+f(x_0)$  is Lipschitz and bounded.

- (1)  $\Rightarrow$  (2):

Suppose that  $f$  is Gateaux differentiable at  $x_0$ . Let  $x \in X$ . Then a previous exercise implies that

$$\begin{aligned}-df^+(x_0)(-x) &= df^-f(x_0)(x) \\ &= df^+f(x_0)(x)\end{aligned}$$

An exercise in the section on sublinear functionals implies that  $df^+f(x_0)$  is linear.

- (2)  $\Rightarrow$  (3):

Suppose that  $df^+f(x_0)$  is linear. Let  $\phi \in \partial f(x_0)$ . The previous exercise implies that  $\phi \leq df^+f(x_0)$ . Equivalence of linearity in the section on sublinear functionals implies that  $d^+f(x_0) = \phi$ .

- (3)  $\Rightarrow$  (1):

Suppose that  $\#\partial f(x_0) = 1$ . Since  $\partial f(x_0) = \{\phi \in X^* : \phi \leq d^+f(x_0)\}$ , equivalence of linearity in the section on sublinear functionals implies that  $d^+f(x_0)$  is linear. This implies that  $d^+f(x_0) = d^-f(x_0)$  and which implies that  $f$  is Gateaux differentiable at  $x_0$ . □

**Exercise 6.2.14.** Let  $X$  be a Banach space,  $A \subset X$  open and convex,  $f : A \rightarrow \mathbb{R}$  convex and  $x_0 \in A$ . If  $f$  is continuous at  $x_0$ , then  $f$  has a global minimum at  $x_0$  iff  $0 \in \partial f(x_0)$ .

*Proof.* Suppose that  $f$  has a global minimum at  $x_0$  iff  $0 \in \partial f(x_0)$  Let  $x \in X$ . Then

$$\begin{aligned} d^+f(x_0)(x) &= \lim_{t \rightarrow 0^+} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &\geq 0 \end{aligned}$$

So  $0 \leq df^+(x_0)$  and  $0 \in \partial f(x_0)$ .

Conversely, suppose that  $0 \in \partial f(x_0)$ . Let  $x \in A$ . Then

$$\begin{aligned} 0 &= 0(x - x_0) \\ &\leq f(x) - f(x_0) \end{aligned}$$

So that  $f(x_0) \leq f(x)$  which implies that  $f$  has a global minimum at  $x_0$ . □

### 6.3. Conjugacy.

**Definition 6.3.1.** Let  $X$  be a Banach space,  $A \subset X$  and  $f : A \rightarrow \mathbb{R}$ . Define  $A^* \subset X^*$  and  $f^* : A^* \rightarrow \mathbb{R}$  by

$$A^* = \left\{ \phi \in X^* : \sup_{x \in A} [\phi(x) - f(x)] < \infty \right\}$$

and

$$f^*(\phi) = \sup_{x \in A} [\phi(x) - f(x)]$$

If  $X$  is a Hilbert space, we may define  $A^* \subset X$  and  $f^* : A^* \rightarrow \mathbb{R}$  via the Riesz representation theorem by

$$A^* = \left\{ y \in X : \sup_{x \in A} [\langle y, x \rangle - f(x)] < \infty \right\}$$

and  $f^* : A^* \rightarrow \mathbb{R}$  and

$$f^*(y) = \sup_{x \in A} [\langle y, x \rangle - f(x)]$$

**Exercise 6.3.2.** Let  $X$  be a Banach space,  $A \subset X$  and  $f : A \rightarrow \mathbb{R}$ . Then  $f^*$  is convex.

*Proof.* For  $x \in A$ , define  $g_x : X^* \rightarrow [\infty, \infty)$  by  $g_x(\phi) = \phi(x) - f(x)$ . Then for each  $x \in A$ ,  $g_x$  is convex since it is affine. Thus  $f^* = \sup_{x \in A} g_x$  is convex.  $\square$

**Exercise 6.3.3.** Let  $X$  be a Banach space,  $A \subset X$  and  $f : A \rightarrow \mathbb{R}$ . Then for each  $x \in X$  and  $\phi \in X^*$ ,  $f(x) \geq \phi(x) - f^*(\phi)$ .

*Proof.* Clear  $\square$

**Exercise 6.3.4.**

**Definition 6.3.5.** Let

**Definition 6.3.6.**  $\partial f$

**Exercise 6.3.7.**

#### 6.4. Functional Optimization.

**Exercise 6.4.1.** Let  $X$  be a Banach space,  $(S, \mathcal{S}, \mu)$  a measure space,  $A \subset X$ ,  $K \in L^0(A, \mathbb{R})$  and  $\Lambda \subset L^0(S, A) \cap \{f : S \rightarrow A : K \circ f \in L^1(\mu)\}$ . Suppose that  $A$  and  $\Lambda$  are convex. Define  $\phi : \Lambda \rightarrow \mathbb{R}$  by

$$\phi f = \int K \circ f d\mu$$

Then  $K$  is convex implies that  $\phi$  is convex.

*Proof.* Suppose that  $K$  is convex. Let  $t \in [0, 1]$  and  $f, g \in \Lambda$ . Convexity of  $K$  implies that for each  $s \in S$ ,

$$K[tf(s) + (1-t)g(s)] \leq tK[f(s)] + (1-t)K[g(s)]$$

So

$$K \circ [tf + (1-t)g] \leq tK \circ f + (1-t)K \circ g$$

Therefore

$$\begin{aligned} \phi[tf + (1-t)g] &= \int K \circ [tf + (1-t)g] d\mu \\ &\leq \int tK \circ f + (1-t)K \circ g d\mu \\ &= t \int K \circ f d\mu + (1-t) \int K \circ g d\mu \\ &= t\phi f + (1-t)\phi g \end{aligned}$$

and  $\phi$  is convex. □

## 7. APPENDIX

### 7.1. Asymptotic Notation.

**Definition 7.1.1.** Let  $X$  be a topological space,  $Y, Z$  be normed vector spaces,  $f : X \rightarrow Y$ ,  $g : X \rightarrow Z$  and  $x_0 \in X \cup \{\infty\}$ . Then we write

$$f = o(g) \quad \text{as } x \rightarrow x_0$$

if for each  $\epsilon > 0$ , there exists  $U \in \mathcal{N}_{x_0}$  such that  $U$  is open and for each  $x \in U$ ,

$$\|f(x)\| \leq \epsilon \|g(x)\|$$

**Exercise 7.1.2.** Let  $X$  be a topological space,  $Y, Z$  be normed vector spaces,  $f : X \rightarrow Y$ ,  $g : X \rightarrow Z$  and  $x_0 \in X \cup \{\infty\}$ . If there exists  $U \in \mathcal{N}_{x_0}$  such that  $U$  is open and for each  $x \in U \setminus \{x_0\}$ ,  $g(x) > 0$ , then

$$f = o(g) \text{ as } x \rightarrow x_0 \quad \text{iff} \quad \lim_{x \rightarrow x_0} \frac{\|f(x)\|}{\|g(x)\|} = 0$$