1

Introduction to Group Theory

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Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Prelimiaries

1.1 Category Theory

- **Hilb**:
  - $\mathrm{Obj}(\mathbf{Hilb}) = \{H : H \text{ is a Hilbert space}\}$
  - $\mathrm{Hom}_{\mathbf{Hilb}}(H_1, H_2) = \{T \in \mathbf{Vect}_{\mathbb{C}}(H_1, H_2) : T \text{ is continuous}\}$
- **Mon**

1.1.1 The Unitary Group

Definition 1.1.1.1. Let $H_1, H_2 \in \mathrm{Obj}(\mathbf{Hilb})$. We define the **unitary group from** H_1 **to** H_2, denoted $U(H_1, H_2)$, by

$$U(H_1, H_2) = \{T \in \mathrm{Iso}_{\mathbf{Hilb}}(H_1, H_2) : T^* = T^{-1}\}$$

We write $U(H)$ in place of $U(H, H)$. We equip $U(H_1, H_2)$ with the strong operator topology.

Exercise 1.1.1.2. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$. Then $\mathcal{T}^s_{U(H)} = \mathcal{T}^w_{U(H)}$. strong weak operator topologies coincide

Exercise 1.1.1.3. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$. Then $U(H)$ is a topological group.

Proof. content... □

Chapter 2

Representation Theory

2.1 Group Representations

2.1.1 Unitary representations

Definition 2.1.1.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $H \in \mathrm{Obj}(\mathbf{Hilb})$ and $\pi \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(H))$. Then (H, π) is said to be a **unitary representation of** G. We define the **dimension of** (H, π), denoted $\dim(H, \pi)$, by $\dim(H, \pi) := \dim V$.

Definition 2.1.1.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, (H_π, π), (H_ρ, ρ) unitary representations of G and $T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho)$. Then T is said to be (π, ρ)**-equivariant** if for each $g \in G$, $T \circ \pi(g) = \rho(g) \circ T$, i.e. the following diagram commutes:

$$\begin{tikzcd} H_\pi \arrow[r, "T"] \arrow[d, "\pi(g)"'] & H_\rho \arrow[d, "\rho(g)"] \\ H_\pi \arrow[r, "T"'] & H_\rho \end{tikzcd}$$

Definition 2.1.1.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. We define $\mathbf{URep}(G)$ by

- $\mathrm{Obj}(\mathbf{URep}(G)) = \{(H, \pi) : (H, \pi) \text{ is a unitary representation of } G\}$.
- for $(H_\pi, \pi), (H_\rho, \rho) \in \mathrm{Obj}(\mathbf{URep}(G))$,
$$\mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) = \{T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho) : T \text{ is } (\pi, \rho)\text{-equivariant}\}$$
- for $(H_\pi, \pi), (H_\rho, \rho), (H_\mu, \mu) \in \mathrm{Obj}(\mathbf{URep}(G))$, $T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$ and $S \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\rho, \rho), (H_\mu, \mu))$,
$$S \circ_{\mathbf{URep}(G)} T = S \circ T$$

Exercise 2.1.1.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. Then $\mathbf{URep}(G)$ is a category.

Proof. FINISH!!! □

Definition 2.1.1.5. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H_\pi, \pi), (H_\rho, \rho) \in \mathbf{URep}(G)$. Then (H_π, π) is said to be **unitarily equivalent** to (H_ρ, ρ), denoted $(H_\pi, \pi) \equiv (H_\rho, \rho)$, if $\mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) \cap U(H_\pi, H_\rho) \neq \varnothing$.

Note 2.1.1.6. Let $\pi \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(H))$. Since $U(H)$ is equipped with the strong operator topology, we have that for each $u \in H$, the map $g \mapsto \pi(g)u$ is continuous.

Definition 2.1.1.7. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. We define the **induced group action of** G **on** H, denoted $\phi_{(H,\pi)} : G \times H \to H$, by

$$\phi_{(H,\pi)}(g, v) = \pi(g)v$$

Note 2.1.1.8. When the context is clear, we write $g \cdot v$ in place of $\phi_{(H,\pi)}(g,v)$.

Exercise 2.1.1.9. Let $G \in \text{Obj}(\mathbf{TopGrp})$ and $(H,\pi) \in \text{Obj}(\mathbf{URep}(G))$. Then

1. $\phi_{(H,\pi)}$ is a linear group action.
2. G is locally compact implies that $\phi_{(H,\pi)}$ is continuous

Proof.

1.
   - Let $g, h \in G$ and $v \in H$.
     (a) Since $\pi \in \text{Hom}_{\mathbf{TopGrp}}(G, U(H))$,
     $$\begin{aligned} e \cdot v &= \pi(e)v \\ &= \text{id}_H v \\ &= v \end{aligned}$$
     (b) Since $\pi \in \text{Hom}_{\mathbf{TopGrp}}(G, U(H))$,
     $$\begin{aligned} g \cdot (h \cdot v) &= \pi(g)[\pi(h)v] \\ &= [\pi(g)\pi(h)]v \\ &= \pi(gh)v \\ &= (gh) \cdot v \end{aligned}$$
     Since $g, h \in G$ and $v \in H$ are arbitrary, $\phi_{(H,\pi)}$ is a group action of G on H.
   - Let $g \in G$, $\lambda \in \mathbb{C}$ and $v, w \in H$. Then
     $$\begin{aligned} &g \cdot (\lambda v + w) \\ &\quad = \pi(g)(\lambda v + w) \\ &\quad = \lambda \pi(g)v + \pi(g)w \\ &\quad = \lambda g \cdot v + g \cdot w \end{aligned}$$
     Since $g \in G$, $\lambda \in \mathbb{C}$ and $v, w \in H$ are arbitrary, $\phi_{(H,\pi)}$ is a linear action.
2. Suppose that G is locally compact. Let $(g_0, v_0) \in G \times H$ and $\epsilon > 0$. Since G is locally compact, there exists $K \subset G$ such that $g_0 \in \text{Int}\, K$ and K is compact. Let $v \in H$. Define $f_v : G \to H$ by $f_v(g) = g \cdot v$. Since $\pi : G \to U(H)$ is continuous, f_v is continuous. Thus $\|f_v\|$ is continuous. Since K is compact, $\|f_v\|(K)$ is compact. Thus
   $$\begin{aligned} \sup_{g \in K} \|g \cdot v\| &= \sup_{g \in K} \|f_v(g)\| \\ &< \infty \end{aligned}$$
   Since $v \in H$ is arbitrary, we have that for each $v \in H$, $\sup_{g \in K} \|g \cdot v\| < \infty$. The uniform boundedness principle implies that there exists $M > 0$ such that $\sup_{g \in K} \|\pi(g)\| \leq M$. Since f_{v_0} is continuous, there exists $U \subset K$ such that U is open, $g_0 \in U$, and $f_{v_0}(U) \subset B(f_{v_0}(g_0), \epsilon/2)$. Let $(g_1, v_1) \in U \times B(v_0, (2M)^{-1}\epsilon)$. Then
   $$\begin{aligned} \|\phi_{(H,\pi)}(g_0, v_0) - \phi_{(H,\pi)}(g_1, v_1)\| &= \|g_0 \cdot v_0 - g_1 \cdot v_1\| \\ &\leq \|g_0 \cdot v_0 - g_1 \cdot v_0\| + \|g_1 \cdot v_0 - g_1 \cdot v_1\| \\ &= \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + \|\pi(g_1)(v_0 - v_1)\| \\ &\leq \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + \|\pi(g_1)\|\|v_0 - v_1\| \\ &\leq \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + M\|v_0 - v_1\| \\ &\leq \frac{\epsilon}{2} + M\frac{\epsilon}{2M} \\ &= \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\phi_{(H,\pi)}$ is continuous at (g_0, v_0). Since $(g_0, v_0) \in G \times H$ is arbitrary, we have that $\phi_{(H,\pi)} : G \times H \to H$ is continuous.

□

2.1.2 Subrepresentations

Definition 2.1.2.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. Then E is said to be

- **nontrivial** if $E \neq H, \varnothing$
- (H, π)**-invariant** if for each $g \in G$, $\pi(g)(E) = E$

Definition 2.1.2.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $\mathbb{K} \in \mathrm{Obj}(\mathbf{Field})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then

- (H, π) is said to be **reducible** if there exists a closed subspace $E \subset H$ such that E is not trivial and E is (H, π)-invariant
- (H, π) is said to be **irreducible** if (H, π) is not reducible.

Exercise 2.1.2.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. Suppose that E is (H, π)-invariant. Then for each $g \in G$, $\pi(g)|_E \in U(E)$.

Proof. Let $g \in G$. Since E is (H, π)-invariant, for each $g \in G$, $\pi(g)(E) = E$. Since $\pi(g) \in U(H)$, $\pi(g)|_E \in U(E)$. □

Definition 2.1.2.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. Suppose that E is (H, π)-invariant.

- We define $\pi^E \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(E))$ by $\pi^E(g) := \pi(g)|_E$
- We define the **restriction** (H, π) **to** E, denoted $(H, \pi)|_E$, by $(H, \pi)|_E := (E, \pi^E)$

Exercise 2.1.2.5. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace.

1. If E is nontrivial, then $E^\perp$ is nontrivial.
2. If E is (H, π)-invariant, then $E^\perp$ is (H, π)-invariant.

Proof.

1. Suppose that E is nontrivial. Then $E \neq \{0\}, H$. Then $E^\perp \neq \{0\}, H$. Thus $E^\perp$ is nontrivial.
2. Suppose that E is (H, π)-invariant. Let $g \in G$. Since $\pi(g) \in U(H)$ and $\pi(g)(E) = E$, An exercise in the analysis notes section on Hilbert spaces implies that $\pi(g)(E^\perp) = E^\perp$. Since $g \in G$ is arbitrary, $E^\perp$ is (H, π)-invariant.

□

Definition 2.1.2.6. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $u \in H$. We define the **cyclic subspace of** H **generated by** u **under** (H, π), denoted $\mathrm{cyc}_{(H,\pi)}(u)$, by

$$\mathrm{cyc}_{(H,\pi)}(u) := \mathrm{cl}\,\mathrm{span}(\phi_{(H,\pi)}(G, u))$$

Note 2.1.2.7. When the context is clear, we write $\mathrm{cyc}(u)$ in place of $\mathrm{cyc}_{(H,\pi)}(u)$.

Exercise 2.1.2.8. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $u \in H$. Then $\mathrm{cyc}(u)$ is (H, π)-invariant. this should largely be a result about linear group actions.

Proof. Let $g \in G$. Since G acts linearly and homeomorphically on H,

$$\begin{aligned} g \cdot \operatorname{cyc}(u) &= g \cdot \operatorname{cl}\operatorname{span}(G \cdot u) \\ &= \operatorname{cl} g \cdot \operatorname{span}(G \cdot u) \\ &= \operatorname{cl}\operatorname{span}[g \cdot (G \cdot u)] \\ &= \operatorname{cl}\operatorname{span}(G \cdot u) \\ &= \operatorname{cyc}(u) \end{aligned}$$

Since $g \in G$ is arbitrary, $\operatorname{cyc}(u)$ is G-invariant. □

Definition 2.1.2.9. Let $G \in \operatorname{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \operatorname{Obj}(\mathbf{URep}(G))$.

- Let $u \in H$. Then u is said to be (H, π)**-cyclic** if $\operatorname{cyc}(u) = H$.
- Then (H, π) is said to be **cyclic** if there exists $u \in H$ such that u is (H, π)-cyclic.

2.1.3 Direct Sum of Representations

Definition 2.1.3.1. Let $G \in \operatorname{Obj}(\mathbf{TopGrp})$ and $(H_\alpha, \pi_\alpha)_{\alpha \in A} \subset \operatorname{Obj}(\mathbf{URep}(G))$.

- We define $\bigoplus\limits_{\alpha \in A} \pi_\alpha \in \operatorname{Hom}_{\mathbf{TopGrp}}(G, U(\bigoplus\limits_{\alpha \in A} H_\alpha))$ by

$$\left[\bigoplus_{\alpha \in A} \pi_\alpha\right](g) = \bigoplus_{\alpha \in A} \pi_\alpha(g)$$

- We define the **direct sum** of $(H_\alpha, \pi_\alpha)_{\alpha \in A}$, denoted $\bigoplus\limits_{\alpha \in A}(H_\alpha, \pi_\alpha)$, by

$$\bigoplus_{\alpha \in A}(H_\alpha, \pi_\alpha) = \left(\bigoplus_{\alpha \in A} H_\alpha, \bigoplus_{\alpha \in A} \pi_\alpha\right)$$

Note 2.1.3.2. FINISH!!! the last definition works for internal or external direct sum, just need to define inner or external sum of H_α and π_α in either case.

Exercise 2.1.3.3. Let $G \in \operatorname{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \operatorname{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. If E is nontrivial and (H, π)-invariant, then $(H, \pi) = (E \oplus E^\perp, \pi^E \oplus \pi^{E^\perp})$.

Proof. Suppose that E is nontrivial and (H, π)-invariant. A previous exercise implies that $E^\perp$ is nontrivial and (H, π)-invariant. It is clear that $H = E \oplus E^\perp$. Let $g \in G$ and $u \in H$. Since $H = E \oplus E^\perp$, there exists $v \in E$ and $w \in E^\perp$ such that $u = v + w$. Then

$$\begin{aligned} \pi(g)(u) &= \pi(g)(v + w) \\ &= \pi(g)(v) + \pi(g)(w) \\ &= \pi(g)|_E(v) + \pi(g)|_{E^\perp}(w) \\ &= \pi^E(g)(v) + \pi^{E^\perp}(g)(w) \\ &= [\pi^E(g) \oplus \pi^E(g)](v + w) \\ &= [\pi^E \oplus \pi^E](g)(v + w) \\ &= [\pi^E \oplus \pi^E](g)(u) \end{aligned}$$

Since $u \in H$ is arbitrary, $\pi(g) = [\pi^E \oplus \pi^E](g)$. Since $g \in G$ is arbitrary, $\pi = \pi^E \oplus \pi^E$. □

Definition 2.1.3.4. Let $G \in \operatorname{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \operatorname{Obj}(\mathbf{URep}(G))$ and $\mathcal{E} \subset \mathcal{P}(H)$. Then $\mathcal{E}$ is said to be an (H, π)**-orthocyclic system** if for each $E, F \in \mathcal{E}$,

1. E is a closed subspace of H
2. $(H, \pi)|_E$ is cyclic
3. if $E \neq F$, then $E \perp F$

Exercise 2.1.3.5. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then there exists $(H_\alpha, \pi_\alpha)_{\alpha \in A} \subset \mathrm{Obj}(\mathbf{URep}(G))$ such that for each $\alpha \in A$, (H_α, π_α) is cyclic and $(H, \pi) = \bigoplus\limits_{\alpha \in A} (H_\alpha, \pi_\alpha)$.

Hint: Zorn's lemma

Proof. Define $\mathcal{P} = \{\mathcal{E} : \mathcal{E}$ is an (H, π)-orthocyclic system$\}$. We partially order $\mathcal{P}$ by inclusion. Let $\mathcal{C} \subset \mathcal{P}$ be a chain. Set $\mathcal{E}_0 = \bigcup\limits_{\mathcal{E} \in \mathcal{C}} \mathcal{E}$. Let $E_1, E_2 \in \mathcal{E}_0$. Then there exist $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}$ such that $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$. Since $\mathcal{C}$ is a chain, $\mathcal{E}_1 \subset \mathcal{E}_2$ or $\mathcal{E}_2 \subset \mathcal{E}_1$.

Suppose that $\mathcal{E}_1 \subset \mathcal{E}_2$. Then $E_1 \in \mathcal{E}_2$. Since $\mathcal{E}_2$ is an (H, π)-orthocyclic system, we have that E_1 is a closed subspaces of H, $(H, \pi)|_{E_1}$ is cyclic and if $E_1 \neq E_2$, then $E_1 \perp E_2$. Similarly, $\mathcal{E}_2 \subset \mathcal{E}_1$ implies the same conclusion. Since $E_1, E_2 \in \mathcal{E}_0$ are arbitrary, we have that for each $E_1, E_2 \in \mathcal{E}_0$

1. E_1 is a closed subspaces of H and E_1 is (H, π)-invariant
2. $(H, \pi)|_{E_1}$ is cyclic
3. if $E_1 \neq E_2$, then $E_1 \perp E_2$

Thus $\mathcal{E}_0$ is an (H, π)-orthocyclic system. Hence $\mathcal{E}_0 \in \mathcal{P}$. By construction, for each $\mathcal{E} \in \mathcal{C}$, $\mathcal{E} \subset \mathcal{E}_0$. So $\mathcal{E}_0$ is an upper bound of $\mathcal{C}$. Since $\mathcal{C} \subset \mathcal{P}$ such that $\mathcal{C}$ is a chain is arbitrary, we have that for each $\mathcal{C} \subset \mathcal{P}$, if $\mathcal{C}$ is a chain, then there exists $\mathcal{E}_0 \in \mathcal{P}$ such that $\mathcal{E}_0$ is an upper bound of $\mathcal{C}$. Zorn's lemma implies that there exists $\mathcal{E} \in \mathcal{P}$ such that $\mathcal{E}$ is maximal. Set $E = \bigoplus\limits_{E_0 \in \mathcal{E}} E_0$. For the sake of contradiction, suppose that $H \neq E$. Then $E^\perp \neq \{0\}$. Thus there exists $u \in E^\perp$ such that $u \neq 0$. Therefore $\mathrm{cyc}(u) \neq 0$ and $\mathrm{cyc}(u) \subset E^\perp$. Let $E_0 \in \mathcal{E}$. By construction, $E_0 \subset E$. Thus

$$\begin{aligned} \mathrm{cyc}(u) &\subset E^\perp \\ &\subset E_0^\perp \end{aligned}$$

Since $E_0 \in \mathcal{E}$ is arbitrary, we have that for each $E_0 \in \mathcal{E}$, $\mathrm{cyc}(u) \subset E_0^\perp$. Set $\mathcal{E}' = \mathcal{E} \cup \{\mathrm{cyc}(u)\}$. Then for each $E, F \in \mathcal{E}'$,

1. E is a closed subspaces of H and E is (H, π)-invariant
2. $(H, \pi)|_E$ is cyclic
3. if $E \neq F$, then $E \perp F$

Hence $\mathcal{E}' \in \mathcal{P}$. Since $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{E}$ $\square$

2.2 Tannaka Duality

Definition 2.2.0.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. We define the **forgetful functor from $\mathbf{URep}(G)$ to Hilb**, denoted $U : \mathbf{URep}(G) \to \mathbf{Hilb}$, by

- $U(H, \pi) = H, \quad (H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$
- $U(T) = T, \quad T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$.

Need to find out if quotienting by equivalence of isomorphism makes $\mathbf{URep}(G)$ a small category so that we can talk about the functor category $\mathbf{Hilb}^{\mathbf{URep}(G)}$ containing the forgetful functor as an object.

Definition 2.2.0.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $g \in G$. We define $\hat{g} : U \Rightarrow U$ by

$$\hat{g}_{(H,\pi)} = \pi(g)$$

Exercise 2.2.0.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $g \in G$. Then

1. $\hat{g} : U \Rightarrow U$ is a natural transformation.
2. $\hat{g} \in \mathrm{Aut}_{\mathbf{Hilb}^{\mathbf{URep}(G)}}(U)$

Proof.

1. (a) Let $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. By definition,

$$\begin{aligned} \hat{g}_{(H,\pi)} &= \pi(g) \\ &\in U(H) \\ &\subset \mathrm{Aut}_{\mathbf{Hilb}}(U(H, \pi)) \end{aligned}$$

   (b) Let $(H_\pi, \pi), (H_\rho, \rho) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$. By definition, $T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho)$ and T is (π, ρ)-equivariant. Therefore

$$\begin{aligned} U(T) \circ \hat{g}_{(H_\pi,\pi)} &= T \circ \pi(g) \\ &= \rho(g) \circ T \\ &= \hat{g}_{(H_\rho,\rho)} \circ U(T) \end{aligned}$$

   i.e. the following diagram commutes:

$$\begin{tikzcd} U(H_\pi, \pi) \arrow[r, "\hat{g}_{(H_\pi,\pi)}"] \arrow[d, "U(T)"'] & U(H_\pi, \pi) \arrow[d, "U(T)"] \\ U(H_\rho, \rho) \arrow[r, "\hat{g}_{(H_\rho,\rho)}"] & U(H_\rho, \rho) \end{tikzcd} = \begin{tikzcd} H_\pi \arrow[r, "\pi(g)"] \arrow[d, "T"'] & H_\pi \arrow[d, "T"] \\ H_\rho \arrow[r, "\rho(g)"] & H_\rho \end{tikzcd}$$

   Thus $\hat{g} : U \Rightarrow U$ is a natural transformation.

2. Set $h = g^{-1}$. Part (1) implies that $\hat{g}, \hat{h} \in \mathrm{End}_{\mathbf{Hilb}^{\mathbf{URep}(G)}}(U)$. Let $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then

$$(\hat{g} \circ \hat{h})_{(H,\pi)} = \hat{g}_{(H,\pi)}$$

   The previous part implies that

$$\begin{aligned} \hat{g} &\in \mathrm{Hom}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U, U) \\ &= \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U) \end{aligned}$$

∎

Definition 2.2.0.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. We define the (H, π)**-projection**, denoted $\pi_{(H,\pi)} : \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U) \to \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}}(V)$, by $\pi_{(H,\pi)}(\alpha) = \alpha_{(H,\pi)}$. We define the **topology of endomorphisms of** U, denoted $\mathcal{T}_{\mathcal{E}(U)}$, by

$$\mathcal{T}_{\mathcal{E}(U)} = \tau(\pi_{(H,\pi)} : (H, \pi) \in \mathbf{URep}(G))$$

Definition 2.2.0.5. define addition of endomorphisms of U pointwise

Exercise 2.2.0.6. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. Then $(\mathrm{Aut}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U), \mathcal{T}_{\mathcal{E}(U)})$ is a topological unital algebra.

Proof. $\square$

Chapter 3

Groupoids

Definition 3.0.0.1.

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration