INTRODUCTION TO FOURIER ANALYSIS

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Contents

1. Fourier Analysis on $\mathbb{R}^n$

1.1. **Schwartz Space.**

Definition 1.1.1. Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

1. $\langle x, y\rangle = \sum_j x_j y_j$
2. $|x| = \langle x, x\rangle^{1/2}$
3. $|\alpha| = \alpha_1 + \cdots + \alpha_n$
4. $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
5. $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

Definition 1.1.2. Let $f \in C^\infty(\mathbb{R}^n)$,$\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}^n} (1 + |x|^N)|\partial^\alpha f(x)|$$

We define Schwartz space, denoted $\mathcal{S}$, by

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n,\, N \in \mathbb{N}_0,\, \|f\|_{\alpha,N} < \infty\}$$

Exercise 1.1.3. For each $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}$, $\alpha \in \mathbb{N}_0^n$. Then there exists $C \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^{-1}$$

Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = (1 + |x|^2)^{-1}$. Then $g \in L^1(\mathbb{R}^n)$ which implies that $\partial^\alpha f \in L^1(\mathbb{R}^n)$. □

Definition 1.1.4.

1.2. **The Convolution.**

Definition 1.2.1. Let $f, g \in L^0(\mathbb{R}^n)$. If for a.e. $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) < \infty$$

we define the **convolution of f with g**, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$, by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

Exercise 1.2.2. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = f(x-y)g(y)$. Tonelli's theorem implies that,

$$\begin{aligned}
\int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}^n} |g(y)| \left[\int_{\mathbb{R}^n} |f(x-y)|dm(y) \right] dm(x) \\
&= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dm(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

Then $h \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Fubini's theorem implies that $f * g \in L^1(\mathbb{R}^n)$. Clearly

$$\begin{aligned}
\|f * g\|_1 &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 \\
&\leq \|f\|_1 \|g\|_1
\end{aligned}$$

□

Exercise 1.2.3. Let $f, g, h \in L^1(\mathbb{R}^n)$. Then $(f * g) * h = f * (g * h)$.
Hint: use the substitution $z \mapsto z - y$

Proof. Let $x \in \mathbb{R}^n$. Then using the substitution $z \mapsto z - y$ and Fubini's theorem, we obtain

$$
\begin{aligned}
(f * g) * h(x) &= \int f * g(x-y)h(y)dm(y) \\
&= \int \bigg[\int f(x-y-z)g(z)dm(z)\bigg] h(y)dm(y) \\
&= \int \bigg[\int f(x-z)g(z-y)dm(z)\bigg] h(y)dm(y) \\
&= \int \bigg[\int f(x-z)g(z-y)h(y)dm(z)\bigg] dm(y) \\
&= \int \bigg[\int f(x-z)g(z-y)h(y)dm(y)\bigg] dm(z) \\
&= \int f(x-z)\bigg[\int g(z-y)h(y)dm(y)\bigg] dm(z) \\
&= \int f(x-z)g * h(z)dm(z) \\
&= f * (g * h)(z)
\end{aligned}
$$

So $(f * g) * h = f * (g * h)$. $\square$

Exercise 1.2.4. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g = g * f$.

Proof. Let $x \in \mathbb{R}^n$. Using the transformation $y \mapsto x - y$, we obtain that

$$
\begin{aligned}
f * g(x) &= \int f(x-y)g(y)dm(y) \\
&= \int f(y)g(x-y)dm(y) \\
&= \int g(x-y)f(y)dm(y) \\
&= g * f(x)
\end{aligned}
$$

So $f * g = g * f$. $\square$

Note 1.2.5. To summarize, $(L^1(\mathbb{R}^n), *)$ is a commutative Banach algebra.

Exercise 1.2.6. Young's Inequality:
Let $p \in [1, \infty]$, $f \in L^1$ and $g \in L^p$. Then $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof. Define $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $K(x, y) = f(x - y)$. Since for each $x, y \in \mathbb{R}^n$,

$$
\begin{aligned}
\int |K(x,y)|dm(x) &= \int |K(x,y)|dm(y) \\
&= \|f\|_p
\end{aligned}
$$

an exercise in section 5.1 of Introduction to Measure and Integration implies that $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. $\square$

Exercise 1.2.7. Let $p, q \in [1, \infty]$ be conjugate, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then

(1) for each $x \in \mathbb{R}^n$, $f * g(x)$ exists.

(2) $\|f * g\|_u \leq \|f\|_p \|g\|_q$
(3)

Proof. (1) Let $x \in \mathbb{R}^n$. Holder's inequality implies that

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \leq \|f\|_p \|g\|_q$$

Then $f * g(x)$ exists.

(2) Let $x \in \mathbb{R}^n$. Then in part (1) we showed that

$$\begin{aligned} |f * g(x)| &= \left| \int_{\mathbb{R}^n} f(x-y)g(y)dm(y) \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \\ &\leq \|f\|_p \|g\|_q \end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, $\|f * g\|_u \leq \|f\|_p \|g\|_q$.

(3)

$\square$

Exercise 1.2.8. Let $f \in L^1(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $g \in C^k(\mathbb{R}^n)$. Suppose that for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $\partial^\alpha g \in L^\infty$. Then for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $f * g \in C^k$ and

$$\partial^\alpha (f * g) = f * \partial^\alpha g$$

Proof. Let $\alpha \in \mathbb{N}_0^n$. Suppose that $|\alpha| = 1$. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x,y) = g(x-y)f(y)$. Young's inequality imples that for a.e. $x \in \mathbb{R}^n$, $h(x, \cdot) \in L^1(m)$. For each $y \in \mathbb{R}^n$, $\partial^\alpha h(\cdot, y) = \partial^\alpha g(\cdot - y)f(y)$ and for each $x, y \in \mathbb{R}^n$, $|\partial^\alpha h(x,y)| \leq \|\partial^\alpha g\|_\infty |f(y)| \in L^1(\mathbb{R}^n)$. An exercise in section 3.3 of Introduction to Measure and Integration implies that for a.e. $x \in \mathbb{R}^n$, $\partial^\alpha (g * f)(x)$ exists and

$$\begin{aligned} \partial^\alpha (f * g)(x) &= \partial^\alpha (g * f)(x) \\ &= \partial^\alpha \int_{\mathbb{R}^n} h(x,y)dm(y) \\ &= \int_{\mathbb{R}^n} \partial^\alpha g(x-y)f(y)dm(y) \\ &= (\partial^\alpha g) * f(x) \\ &= f * (\partial^\alpha g)(x) \end{aligned}$$

Now proceed by induction on $|\alpha|$. $\square$

1.3. **The Fourier Transform on $L^1(\mathbb{R}^n)$.**

Definition 1.3.1.

Exercise 1.3.2. Let $\phi : \mathbb{R} \to S^1$ be a measurable homomorphism.

(1) Then $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ and there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
(2) Define
$$c = \left[\int_{(0,a]} \phi dm \right]^{-1}$$
Then For each $x \in \mathbb{R}$,
$$\phi(x) = c \int_{(x,x+a]} \phi dm$$
(3) $\phi \in C^\infty(\mathbb{R})$ and $\phi' = c(\phi(a) - 1)\phi$

(4) Define $b = c(\phi(a) - 1)$ and $g \in C^\infty(\mathbb{R})$ by $g(x) = e^{-bx}\phi(x)$. Then g is constant and there exists $\xi \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i \xi x}$.

Proof.

(1) Let $K \subset \mathbb{R}$ be compact. Then
$$\int_K |\phi| dm = m(K) < \infty$$
So $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$. For the sake of contradiction, suppose that for each $a > 0$,
$$\int_{(0,a]} \phi dm = 0$$
Then the FTC implies that $\phi = 0$ a.e. on $[0, \infty)$, which is a contradiction. So there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
(2) For $x \in \mathbb{R}$,
$$\begin{aligned}
\phi(x) &= c \int_{(0,a]} \phi(x)\phi(t) dm(t) \\
&= c \int_{(0,a]} \phi(x + t) dm(t) \\
&= c \int_{(x,x+a]} \phi dm
\end{aligned}$$
(3) Part (2) and the FTC imply that ϕ is continuous. Let $d \in \mathbb{R}$. Define $f_d \in C((d, \infty))$ by
$$f_d(x) = \int_{(d,x]} \phi dm$$

Since ϕ is continuous, the FTC implies that f_d is differentiable and for each $x > d$ $f_d'(x) = \phi(x)$. Part (2) implies that for each $x > d$,

$$\begin{aligned}\phi(x) &= c\int_{(x,x+a]} \phi dm \\ &= c(f_d(x+a) - f_d(x))\end{aligned}$$

So for each $x > d$, ϕ is differentiable at x and

$$\begin{aligned}\phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x)\end{aligned}$$

Since $d \in \mathbb{R}$ is arbitrary, ϕ is differentiable and $\phi' = c(\phi(a) - 1)\phi$. This implies that $\phi \in C^\infty(\mathbb{R})$.

(4) Let $x \in \mathbb{R}$. Then

$$\begin{aligned}g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0\end{aligned}$$

So $g' = 0$ and g is constant. Hence there exists $k \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = ke^{bx}$. Since $\phi(0) = 1$, $k = 1$. Since $|\phi| = 1$, there exists $\xi \in \mathbb{R}$ such that $b = 2\pi i\xi$.

□

Note 1.3.3. To summarize, for each measurable homomorphism $\phi : \mathbb{R} \to S^1$, there exists $\xi \in \mathbb{R}$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

Exercise 1.3.4. Let $\phi : \mathbb{R}^n \to S^1$ be a measurable homomorphism. Then there $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi(x) = e^{2\pi i\langle\xi,x\rangle}$.

Proof. When done in the category of measurable groups, an exercise in the section on direct products of groups of Introduction to Group Theory implies that there exist measurable homomorphism $(\phi_j)_{j=1}^n \subset (S^1)^{\mathbb{R}}$ such that $\phi = \bigotimes_{j=1}^n \phi_j$. The previous exercise imples that there exist $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi_j(x_j) = e^{2\pi i\xi_j x_j}$. Then for each $x \in \mathbb{R}^n$,

$$\begin{aligned}\phi(x) &= \prod_{j=1}^n \phi_j(x_j) \\ &= \prod e^{2\pi i\xi_j x_j} \\ &= e^{2\pi i \sum\limits_{j=1}^n \xi_j x_j} \\ &= e^{2\pi i\langle\xi,x\rangle}\end{aligned}$$

□

Definition 1.3.5. Let $f \in L^1(\mathbb{R}^n)$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle\xi,x\rangle}dm(x)$$

2. Fourier Analysis on LCA Groups