ORBIT SPACE METRICS AND MEASURES INDUCED BY ISOMETRIC GROUP ACTIONS

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Contents

1. Introduction

1.1. **Main Idea.** In these notes we do the following:

- for an isometric group action on metric spaces, we define an induced metric on the orbit space which metrizes the quotient topology
- for nice measures on metric spaces in the above case, we define nice induced measure on the orbit space
- give an application to Bayesian statistics

2. Group Actions on Metric Spaces

2.1. Introduction.

Note 2.1.1. For a set X, a group G and a (left) group action $\phi : G \times X \to X$, we will write $\phi(g, x)$ as $g \cdot x$. We denote the projection map by $\pi : X \to X/G$.

Definition 2.1.2. Let X be a set, G a group, $\phi : G \times X \to X$ a group action and $g \in G$. Define $l_g : X \to X$ by

$$l_g(x) = g \cdot x$$

Definition 2.1.3. Let X be a topological space, G a group and $\phi : G \times X \to X$ a group action. Then ϕ is said to be X-continuous if for each $g \in G$, l_g is continuous.

Exercise 2.1.4. Let X be a topological space, G a group and $\phi : G \times X \to X$ an X-continuous group action. Then for each $g \in G$, $l_g \in \text{Homeo}(X)$.

Proof. Let $g \in G$, then l_g and $l_g^{-1} = l_{g^{-1}}$ are continuous, so $l_g \in \text{Homeo}(G)$. □

Definition 2.1.5. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ a group action. Then ϕ is said to be an **isometric group action** if for each $g \in G$, $l_g : X \to X$ is an isometry.

Exercise 2.1.6. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Then ϕ is X-continuous.

Proof. Clear since isometries are continuous. □

Definition 2.1.7. Let X be a set, G a group and $\phi : G \times X \to X$ an X-continuous group action. Let $g \in G$. Define $L_g : \mathbb{C}^X \to \mathbb{C}^X$ by

$$\begin{aligned} L_g(f)(x) &= f \circ l_g^{-1} \\ &= f \circ l_{g^{-1}} \end{aligned}$$

Definition 2.1.8. Let X be a set, G a group, $\phi : G \times X \to X$ a group action and $f : X \to \mathbb{C}$. Then f is said to be G**-invariant** if for each $g \in G$, $L_g f = f$.

Exercise 2.1.9. Let X be a set, G a group, $\phi : G \times X \to X$ a group action and $f : X \to \mathbb{C}$. Then f is G-invariant iff for each $g \in G$ $x \in X$, $f(g \cdot x) = f(x)$.

Proof. Clear. □

Definition 2.1.10. Let X be a set, G a group, $\phi : G \times X \to X$ a group action and $f : X \to \mathbb{C}$. Suppose that f is G-invariant. Define $\bar{f} : X/G \to \mathbb{C}$ by $\bar{f}(\bar{x}) = f(x)$.

Exercise 2.1.11. Let X be a set, G a group, $\phi : G \times X \to X$ a group action and $f : X \to \mathbb{C}$. Suppose that f is G-invariant. Then $f = \bar{f} \circ \pi$.

Proof. Clear. □

2.2. **Induced Metrics on Orbit Spaces.**

Note 2.2.1. This section establishes the criteria for the existence of a metric on the orbit space of a metric space under a group action.

Definition 2.2.2. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ a group action. We define $\bar{d} : X/G \times X/G \to [0, \infty)$ by

$$\bar{d}(\bar{x}, \bar{y}) = \inf_{\substack{a \in \bar{x} \\ b \in \bar{y}}} d(a, b)$$

Exercise 2.2.3. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Then for each $x, y \in X$,

$$\bar{d}(\bar{x}, \bar{y}) = \inf_{g \in G} d(g \cdot x, y)$$

Proof. Let $x, y \in X$, $a \in \bar{x}$ and $b \in \bar{y}$. Then there exists there exists $g_a, g_b \in G$ such that $a = g_a \cdot x$ and $b = g_b \cdot y$. Set $g = g_b^{-1} g_a$. Since the map $z \mapsto g_b^{-1} \cdot z$ is an isometry,

$$\begin{aligned} d(a, b) &= d(g_a \cdot x, g_b \cdot y) \\ &= d(g_b^{-1} g_a \cdot x, y) \\ &= d(g \cdot x, y) \end{aligned}$$

Let $\epsilon > 0$. Then there exist $a^* \in \bar{x}$ and $b^* \in \bar{y}$ such that $d(a^*, b^*) < \bar{d}(\bar{x}, \bar{y}) + \epsilon$. The above argument implies that that there exists $g^* \in G$ such that

$$\begin{aligned} \inf_{g \in G} d(g \cdot x, y) &\leq d(g^* \cdot x, y) \\ &= d(a^*, b^*) \\ &< \bar{d}(\bar{x}, \bar{y}) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\inf_{g \in G} d(g \cdot x, y) \leq \bar{d}(\bar{x}, \bar{y})$$

Conversely, since $\{(g \cdot x, y) : g \in G\} \subset \{(a, b) : a \in \bar{x}, b \in \bar{y}\}$, we have that

$$\inf_{g \in G} d(g \cdot x, y) \geq \bar{d}(\bar{x}, \bar{y})$$

$\square$

Exercise 2.2.4. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Then for each $x, y, z \in X$,

$$\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y})$$

Proof. Let $x, y, z \in X$. An exercise in section (2.1) implies that $d(\bar{x}, \bar{y}) \leq d(\bar{x}, z) + d(z, \bar{y})$. The previous exercise implies that

$$\begin{aligned} d(\bar{x}, z) &= \inf_{a \in \bar{x}} d(a, z) \\ &= \inf_{g \in G} d(g \cdot x, z) \\ &= \bar{d}(\bar{x}, \bar{z}) \end{aligned}$$

Similarly, $d(z,\bar{y}) = \bar{d}(\bar{z},\bar{y})$. Then

$$\begin{aligned} d(\bar{x},\bar{y}) &\leq d(\bar{x},z) + d(z,\bar{y}) \\ &= \bar{d}(\bar{x},\bar{z}) + \bar{d}(\bar{z},\bar{y}) \end{aligned}$$

$\square$

Exercise 2.2.5. Let (X,d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. If for each $x \in X$, $\bar{x}$ is closed, then for each $x,y \in X$, $\bar{d}(\bar{x},\bar{y}) = 0$ implies that $\bar{x} = \bar{y}$.

Proof. Suppose that for each $x \in X$, $\bar{x}$ is closed. Let $x,y \in X$. Suppose that $\bar{d}(\bar{x},\bar{y}) = 0$. Then $\inf\limits_{g \in G} d(g \cdot x, y) = 0$. Hence there exists $(g_n)_{n\in N} \subset G$ such that $g_n \cdot x \to y$. Since $(g_n \cdot x)_{n\in\mathbb{N}} \subset \bar{x}$ and $\bar{x}$ is closed, $y \in \bar{x}$. Thus $\bar{x} = \bar{y}$. $\square$

Exercise 2.2.6. Let (X,d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. If for each $x \in X$, $\bar{x}$ is closed, then $\bar{d}$ is a metric on X/G.

Proof. Clear by preceeding exercises. $\square$

Exercise 2.2.7. Let (X,d) be a metric space, (G,τ) a topological group, and $\phi : G\times X \to X$ an isometric group action. Suppose that G is compact and for each $x \in X$, the map $g \mapsto g\cdot x$ is continuous. Then $\bar{d}$ is a metric on X/G.

Proof. Let $x \in X$. Since G is compact and the map $g \mapsto g \cdot x$ is continuous, $\bar{x} = G \cdot x$ is compact and therefore closed. The previous exercise implies that $\bar{d}$ is a metric. $\square$

Exercise 2.2.8. Let (X,d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric on X/G. Then the projection map $\pi : X \to X/G$ is Lipschitz and therefore continuous.

Proof. Let $x,y \in X$. Then

$$\begin{aligned} \bar{d}(\pi(x),\pi(y)) &= \bar{d}(\bar{x},\bar{y}) \\ &= \inf_{g\in G} d(g\cdot x, y) \\ &\leq d(x,y) \end{aligned}$$

$\square$

Exercise 2.2.9. Let (X,d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric on X/G. Let $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$. Then $\bar{x}_n \xrightarrow{\bar{d}} \bar{x}$ iff there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $g_n \cdot x_n \xrightarrow{d} x$.

Proof. Suppose that $\bar{x}_n \xrightarrow{\bar{d}} \bar{x}$. For $n \in \mathbb{N}$, choose $g_n \in G$ such that $d(g_n \cdot x_n, x) < \bar{d}(\bar{x}_n,\bar{x}) + 2^{-n}$. Then $d(g_n \cdot x_n, x) \to 0$ and $g_n \cdot x_n \xrightarrow{d} x$.

Conversely, suppose that that there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $g_n \cdot x_n \xrightarrow{d} x$. Since $\pi : X \to X/G$ is continuous, we have that

$$\begin{aligned} g_n \cdot x_n \xrightarrow{d} x &\implies \pi(g_n \cdot x_n) \xrightarrow{\bar{d}} \pi(x) \\ &\implies \bar{x}_n \xrightarrow{\bar{d}} \bar{x} \end{aligned}$$

$\square$

Exercise 2.2.10. Let X be a set, $d_1, d_2 : X^2 \to [0, \infty)$ metrics, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that d_1 and d_2 are topologically equivalent.

(1) Then $\bar{d}_1$ is a metric on X/G iff $\bar{d}_2$ is a metric on X/G
(2) If $\bar{d}_1$ and $\bar{d}_2$ are metrics, then $\bar{d}_1$ and $\bar{d}_2$ are topologically equivalent.

Proof.

(1)
- $\implies$ Suppose that $\bar{d}_1$ is a metric. Let $x, y \in X$. Suppose that $\bar{d}_2(\bar{x}, \bar{y}) = 0$. Then there exist $(g_n)_{n\in\mathbb{N}} \subset G$ such that $d_2(g_n \cdot x, y) \to 0$. Since d_1 and d_2 are topologically equivalent, $d_1(g_n \cdot x, y) \to 0$. Thus $\bar{d}_1(\bar{x}, \bar{y}) = 0$. Since $\bar{d}_1$ is a metric, $\bar{x} = \bar{y}$. Hence $\bar{d}_2$ is a metric.
- $\impliedby$ Similar.

(2) Suppose that $\bar{d}_1$ and $\bar{d}_2$ are metrics. Let $(\bar{x}_n)_{n\in\mathbb{N}} \subset X/G$ and $\bar{x} \in X/G$.
- Suppose that $\bar{x}_n \xrightarrow{\bar{d}_1} \bar{x}$. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $g_n \cdot x_n \xrightarrow{d_1} x$. Since d_1 and d_2 are topologically equivalent, $g_n \cdot x_n \xrightarrow{d_2} x$. This implies that $\bar{x}_n \xrightarrow{\bar{d}_2} \bar{x}$.
- Suppose that $\bar{x}_n \xrightarrow{\bar{d}_2} \bar{x}$. Then similarly to above, $\bar{x}_n \xrightarrow{\bar{d}_1} \bar{x}$.

□

Exercise 2.2.11. Let X be a set, $d_1, d_2 : X^2 \to [0, \infty)$ metrics on X, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that d_1 and d_2 are equivalent. If $\bar{d}_1$ and $\bar{d}_2$ are metrics, then $\bar{d}_1$ and $\bar{d}_2$ are equivalent.

Proof. Suppose that $\bar{d}_1$ and $\bar{d}_2$ are metrics. Since d_1 d_2 are equivalent, there exist $C_1, C_2 > 0$ such that for each $x, y \in X$, $C_1 d_1(x, y) \leq d_2(x, y) \leq C_2 d_1(x, y)$. Let $x, y \in X$. Then

$$\begin{aligned}
C_1 \bar{d}_1(\bar{x}, \bar{y}) &= C_1 \inf_{g\in G} d_1(g \cdot x, y) \\
&= \inf_{g\in G} C_1 d_1(g \cdot x, y) \\
&\leq \inf_{g\in G} d_2(g \cdot x, y) \\
&= \bar{d}_2(\bar{x}, \bar{y})
\end{aligned}$$

and

$$\begin{aligned}
\bar{d}_2(\bar{x}, \bar{y}) &= \inf_{g\in G} d_2(g \cdot x, y) \\
&\leq \inf_{g\in G} C_2 d_1(g \cdot x, y) \\
&= C_2 \inf_{g\in G} d_1(g \cdot x, y) \\
&= C_2 \bar{d}_1(\bar{x}, \bar{y})
\end{aligned}$$

So that $C_1 \bar{d}_1 \leq \bar{d}_2 \leq C_2 \bar{d}_1$ □

Exercise 2.2.12. Let (X, d) be a metric space, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Then $\pi : X \to X/G$ is a quotient map.

Proof.

- Clearly π is surjective.

- Let $C \subset X/G$. Suppose that C is closed. Since π is continuous, if $\pi^{-1}(C)$ is closed. Conversely, suppose that $\pi^{-1}(C)$ is closed. Let $(\bar{x}_\alpha)_\alpha \subset C$ be a net and $\bar{x} \in X/G$. Suppose that $\bar{x}_\alpha \to \bar{x}$. Then there exists $(g_\alpha)_{\alpha\in A} \subset G$ such that $g_\alpha \cdot x_\alpha \to x$. Since $(g_\alpha \cdot x_\alpha)_{\alpha\in A} \subset \pi^{-1}(C)$, $x \in \pi^{-1}(C)$. Hence $\bar{x} \in C$ and C is closed. Then Exercise 4.1.4 implies that π is a quotient map.

$\square$

Exercise 2.2.13. Let (X, d) be a metric space, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Then $\pi : X \to X/G$ is open.

Proof. Let $U \subset X$. Suppose that U is open. Then

$$\pi^{-1}(\pi(U)) = \bigcup_{g\in G} g \cdot U$$

Since for each $g \in G$, $l_g \in \text{Homeo}(X)$, we have that for each $g \in G$, $g \cdot U$ is open. Therefore $\bigcup_{g\in G} g \cdot U$ is open. Hence $\pi^{-1}(\pi(U))$ is open. Then Exercise 4.1.6 implies that π is open. $\square$

Exercise 2.2.14. Let (X, d) be a metric space, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Then $\bar{d}$ metrizes the quotient topology $\pi_*\tau(d)$ on X/G.

Proof. Immediate by the previous exercise and Exercise 4.1.9. $\square$

Exercise 2.2.15. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Let $f : X \to \mathbb{C}$. Suppose that f is G-invariant. Suppose that $\bar{d}$ is a metric. If $f \in C(X)$, then $\bar{f} \in C(X/G)$.
Hint: Doob-Dynkin Lemma

Proof. Suppose that $f \in C(X)$. Let $(x_\alpha)_{\alpha\in A}$ be a net in X and $x \in X$. Suppose that $x_\alpha \to x$ in the $\tau(\pi)$ topology. Then $\bar{x}_\alpha \to \bar{x}$. This implies that there exists $(g_\alpha)_{\alpha\in A} \subset G$ such that $g_\alpha \cdot x_\alpha \xrightarrow{d} x$. Since f is G-invariant and continuous, we have that

$$\begin{aligned} f(x_\alpha) &= f(g_\alpha \cdot x_\alpha) \\ &\to f(x) \end{aligned}$$

So f is $\tau(\pi)$-$\tau(\mathbb{C})$ continuous. The Doob-Dynkin lemma for continuous functions implies that there exists a continuous unique $g : X/G \to \mathbb{C}$ such that $f = g \circ \pi$. Since $f = \bar{f} \circ \pi$, we have that $\bar{f} = g$ and $\bar{f}$ is continuous. $\square$

Note 2.2.16. I would have liked to show that f is $\sigma(\pi)$-$\mathcal{B}(\mathbb{C})$ measurable and used the Doob-Dynkin lemma for measurable functions to show that $\bar{f}$ is measurable, but was unable to do this.

2.3. **Induced Measures on Isometric Orbit Spaces.**

Note 2.3.1. This section assumes familiarity with induced metrics on orbit spaces of metric spaces under isometric group actions. See section 9.1 of [2] for details.

Note 2.3.2.

Definition 2.3.3. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $(X/G, \bar{d})$ is a metric space. Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a measure on X. We define $\bar{\mu} : \mathcal{B}(X/G) \to [0, \infty]$ by $\bar{\mu} = \pi_* \mu$.

Note 2.3.4. If $\mu \ll H_p^X$, where X has Hausdorff dimension p, I want to be able to define $\bar{\mu}$ in terms of $H_q^{X/G}$ where X/G has Hausdorff dimension q. I was unable to do this. It might be possible with some manifold theory, for instance $O(2)$ acting on $\mathbb{R}^2$.

Definition 2.3.5. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $(X/G, \bar{d})$ is a metric space. Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a measure on X. Then μ is said to be G-invariant if for each $g \in G$, $U \in \mathcal{B}(X)$,

$$\mu(g \cdot U) = \mu(U)$$

Exercise 2.3.6. Let X be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Then for each $p \geq 0$, H_p is G-invariant.

Proof. Clear. $\square$

Exercise 2.3.7. Let X be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a measure on X. Suppose that $\mu \ll H_p$. Then μ is G-invariant iff $d\mu/dH_p$ is G-invariant.

Proof. Suppose that μ is G-invariant. Let $g \in G$ and $U \in \mathcal{B}(X)$. Then

$$\begin{aligned}
\int_U L_g \frac{d\mu}{dH_p}(x)\, dH_p(x) &= \int_U \frac{d\mu}{dH_p} \circ l_g^{-1}(x)\, dH_p(x) \\
&= \int_{l_g^{-1}(U)} \frac{d\mu}{dH_p}(x)\, d(l_g^{-1})_* H_p(x) \\
&= \int_{g^{-1} \cdot U} \frac{d\mu}{dH_p}(x)\, dH_p(x) \\
&= \mu(g^{-1} \cdot U) \\
&= \mu(U)
\end{aligned}$$

So that

$$L_g \frac{d\mu}{dH_p} = \frac{d\mu}{dH_p}$$

The Converse is similar. $\square$

Exercise 2.3.8. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a measure on X. Suppose that μ is G-invariant, $\mu \ll H_p^X$ and $d\mu/dH_p^X$ is continuous. Then $\bar{\mu} \ll \bar{H}_p^X$, $d\bar{\mu}/d\bar{H}_p^X$ is G-invariant, $d\bar{\mu}/d\bar{H}_p^X$ is continuous and

$$\frac{d\bar{\mu}}{d\bar{H}_p^X} = \overline{\frac{d\mu}{dH_p^X}}$$

Proof. A previous exercise implies that $\bar{\mu} \ll \bar{H}_p^X$. Set $f = d\mu/dH_p^X$. Since μ is G-invariant, f is G-invariant. Since f is continuous, an exercise in section 9.2 of [2] implies that $\bar{f}$ is continuous and $f = \bar{f} \circ \pi$. Let $E \in \mathcal{B}(X/G)$. Then

$$\begin{aligned}\int_E \bar{f} d\bar{H}_p^X &= \int_{\pi^{-1}(E)} \bar{f} \circ \pi dH_p^X \\ &= \int_{\pi^{-1}(E)} f dH_p^X \\ &= \mu(\pi^{-1}(E)) \\ &= \bar{\mu}(E)\end{aligned}$$

Therefore, by definition, we have that

$$\frac{d\bar{\mu}}{d\bar{H}_p^X} = \bar{f} = \overline{\frac{d\mu}{dH_p^X}}$$

□

3. Applications

3.1. Applications to Bayesian Statistics.

Exercise 3.1.1. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \to \Theta$ an isometric group action. Suppose that $\bar{d}$ is a metric on Θ/G. Let

- H_p^Θ be the Hausdorff measure on Θ, $\mu_\mathcal{X}$ a measure on $\mathcal{X}$,
- p a denisty on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on $\mathcal{X}$.
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose that p is G-invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G-invariant and continuous on Θ. For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \ldots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \to [0, 1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)})\, dH_p^\Theta(\theta)$$

Then there exists a continuous density $\bar{p}(\cdot|X^{(n)})$ on Θ/G such that

$$d\bar{P}_{\Theta|X^{(n)}}(\theta) = \bar{p}(\theta|X^{(n)})\, d\bar{H}_p^\Theta(\theta)$$

Proof. Clear from previous work. □

Exercise 3.1.2. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \to \Theta$ an isometric group action. Suppose that $\bar{d}$ is a metric on Θ/G. Let

- H_p^Θ be the Hausdorff measure on Θ, $\mu_\mathcal{X}$ a measure on $\mathcal{X}$,
- p a denisty on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on $\mathcal{X}$.
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose that p is G-invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G-invariant and continuous on Θ. For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \ldots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \to [0, 1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)})\, dH_p^\Theta(\theta)$$

Suppose that $(P_{\Theta|X^{(n)}})_{n\in\mathbb{N}}$ concentrates on $\bar{\theta}_0 \subset \Theta$ a.s. or in probability. Then $(\bar{P}_{\Theta|X^{(n)}})_{n\in\mathbb{N}}$ concentrates a.s. or in probability on $\{\bar{\theta}_0\} \subset \Theta/G$ (i.e. is consistent a.s. or in probability)

Proof. Let $V \in \mathcal{N}_{\bar{\theta}_0}$. Then $\pi^{-1}(V) \in \mathcal{N}_{\bar{\theta}_0}$. By definition,

$$\begin{aligned}\bar{P}_{\Theta|X^{(n)}}(V^c) &= P_{\Theta|X^{(n)}}(\pi^{-1}(V^c)) \\ &= P_{\Theta|X^{(n)}}(\pi^{-1}(V)^c) \\ &\xrightarrow{\text{a.s.}/P} 0\end{aligned}$$

□

Note 3.1.3. Some examples of G-invariant priors would be the uniform distribution, or $N_n(0, \sigma^2 I)$ on $\mathbb{R}^n$ when acted on by $O(n)$. An example of a G-invariant likelihood would be $f(A|Z) \sim \mathrm{Ber}(ZZ^T)$ as in a latent position random graph model where $Z \in \mathbb{R}^{n\times d}$ is the parameter is invariant under right multiplication by $U \in O(d)$.

Note 3.1.4. Next steps are to come up with a model that is computationally expensive, but on the oprbit space, computationally viable, get an estimate for the orbit of the parameter, map back.

3.2. **Applications to Network Statistics.**

Note 3.2.1. We will define $L(d,n) = \{L \in \mathbb{R}^{d\times n} : L \text{ is lower-triangular}\}$.

Note 3.2.2. Consider the isometric group action $O(d)\times\mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ given by $(Q,\theta) \mapsto Q\theta$. Using the QL decomposition, for each $\theta \in \mathbb{R}^{d\times n}$, there exists $Q \in O(d)$ and $L \in L(d,n)$ such that $\theta = QL$. So that for each $\theta \in \mathbb{R}^{d\times n}$, there exists $L \in L(d,n)$ such that $\bar{\theta} = \bar{L}$ in $\mathbb{R}^{d\times n}/O(d)$. We may embed $\mathbb{R}^{\binom{d+1}{2}}$ into $L(d,n)$ via the map $\mathbb{R}^{\binom{d+1}{2}} \to L(d,n)$ given by $\eta \mapsto \theta_\eta$ where θ_η is lower triangular whose entries are the entries of η.

Define $\sigma : \mathbb{R} \to (0,1)$ and $g : \mathbb{R}^{d\times n} \to \mathbb{R}^{n\times n}$ by

$$\sigma(t) = \frac{1}{1+e^{-t}}$$

and

$$g(\theta)_{i,j} = \sigma((\theta^T\theta)_{i,j})$$

.

Consider the model

- $A_{ij}^{(t)} \overset{iid}{\sim} \text{Bern}(g(\theta_0)_{i,j})$ for $i,j \in \{1,\dots,n\}$ and $t \in \{1,\dots,T\}$
- $\theta_0 \in \mathbb{R}^{d\times n}$

Write $\theta = QL$, let $\eta \in \mathbb{R}^{\binom{d+1}{2}}$ such that $\theta_\eta = L$ and write $\theta_\eta = (U_\eta, 0)$ where $U_\eta \in \mathbb{R}^{d\times d}$ is lower-triangular. Then $g(\theta) = g(\theta_\eta)$ and

$$\theta_\eta^T\theta_\eta = \begin{pmatrix} U_\eta^T U_\eta & 0 \\ 0 & 0 \end{pmatrix}$$

We can therefore consider the function $g' : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ given by

$$g'(U)_{i,j} = \sigma((U^TU)_{i,j})$$

The likelihood $L(\theta)$ is given by

$$\begin{aligned}
L(\theta) &= L(\theta_\eta) \\
&= \prod_{t=1}^{T}\prod_{i<j} g(\theta_\eta)_{i,j}^{A_{i,j}^{(t)}}(1-g(\theta_\eta)_{i,j})^{1-A_{i,j}^{(t)}} \\
&= \prod_{t=1}^{T}\prod_{i<j\le d} g(\theta_\eta)_{i,j}^{A_{i,j}^{(t)}}(1-g(\theta_\eta)_{i,j})^{1-A_{i,j}^{(t)}}(1/2)^{n^2-d^2} \\
&= \prod_{t=1}^{T}\prod_{i<j\le d} g'(U_\eta)_{i,j}^{A_{i,j}^{(t)}}(1-g'(U_\eta)_{i,j})^{1-A_{i,j}^{(t)}}(1/2)^{n^2-d^2}
\end{aligned}$$

Since $L(\theta) = L(\theta_\eta)$, to optimize, we may drop the constant multiple (the only dependence on n) and maximize the log-likelihood $l'(\eta)$, which is given by

$$\begin{aligned} l'(\eta) &= l(\theta_\eta) \\ &= \sum_{t=1}^{T} \sum_{i<j\leq d} A_{i,j}^{(t)} \log[g'(U_\eta)_{i,j}] + (1 - A_{i,j}^{(t)}) \log[1 - g'(U_\eta)_{i,j}] \end{aligned}$$

4. Appendix

4.1. Quotient Topology.

Definition 4.1.1. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Suppose that f is surjective. Then f is said to be a **$\mathcal{A}$-$\mathcal{B}$ quotient map** if

(1) f is surjective
(2) for each $V \subset Y$, $V \in \mathcal{B}$ iff $f^{-1}(V) \in \mathcal{A}$.

Note 4.1.2. We typically avoid specifying the topologies when they are clear from the context.

Exercise 4.1.3. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. If f is a quotient map, then f is continuous.

Proof. Suppose that f is a quotient map. Let $V \subset Y$. Suppose that V is open. By definition, $f^{-1}(V)$ is open. Hence f is continuous. $\square$

Exercise 4.1.4. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Suppose that f is continuous and surjective. Then f is a quotient map iff

$$\text{for each } C \subset Y,\ C \text{ is closed iff } f^{-1}(C) \text{ is closed}$$

Proof.

- $(\implies)$
  Suppose that f is a quotient map.
  Let $C \subset Y$. If C is closed, then continuity implies that $f^{-1}(C)$ is closed.
  Conversely, suppose that $f^{-1}(C)$ is closed. Then $f^{-1}(C^c) = (f^{-1}(C))^c$ is open. Since f is a quotient map, $f(f^{-1}(C^c))$ is open. Surjectivity implies that $f(f^{-1}(C^c)) = C^c$. So C is closed.
- $(\impliedby)$
  Suppose that for each $C \subset Y$, C is closed iff $f^{-1}(C)$ is closed.
  Let $V \subset Y$. If V is open. Continuity implies that $f^{-1}(V)$ is open.
  Conversely, suppose that $f^{-1}(V)$ is open. Then $f^{-1}(V^c) = (f^{-1}(V))^c$ is closed. Therefore, $f(f^{-1}(V^c))$ is closed. Surjectivity implies that $V^c = f(f^{-1}(V^c))$. So U is open.

$\square$

Exercise 4.1.5. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Suppose that f is continuous and surjective. If f is open or closed, then f is a quotient map.

Proof.

- Suppose that f is open. Let $V \subset Y$.
  Suppose that V is open. Then continuity implies that $f^{-1}(V)$ is open. Conversely, suppose that $f^{-1}(V)$ is open. Since f is open $f(f^{-1}(V))$ is open. Surjectivity implies that $V = f(f^{-1}(V))$. So V is open. By definition, f is a quotient map.

- Suppose that f is open. Then similarly to above, f is a quotient map.

$\square$

Exercise 4.1.6. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Suppose that f is a quotient map. Then f is open iff

$$\text{for each } U \subset X,\, U \text{ is open implies that } f^{-1}(f(U)) \text{ is open}$$

Proof.

- $(\implies)$
  Suppose that f is open.
  Let $U \subset X$. Suppose that U is open. Since f is open, $f(U)$ is open. Continuity implies that $f^{-1}(f(U))$ is open.

- $(\impliedby)$
  Suppose that for each $U \subset X$, U is open implies that $f^{-1}(f(U))$ is open.
  Since f is a quotient map, $f(U)$ is open. So f is open.

$\square$

Definition 4.1.7. Let $(X, \mathcal{T})$ be a topological space, Y a set and $f : X \to Y$. Suppose that f is surjective. We call $f_*\mathcal{T}$ the **quotient topology** on Y.

Exercise 4.1.8. Let $(X, \mathcal{T})$ be a topological space, Y a set and $f : X \to Y$. Suppose that f is surjective. Then $f : X \to Y$ is a $\mathcal{T}$-$f_*\mathcal{T}$ quotient map.

Proof. Clear. $\square$

Exercise 4.1.9. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be topological spaces, and $f : X \to Y$. Suppose that f is surjective and continuous. If f is open or closed, then $f_*\mathcal{A} = \mathcal{B}$.

Proof. Continuity, $\mathcal{B} \subset f_*\mathcal{A}$.

- Suppose that f is open. Let $V \in f_*\mathcal{A}$. By definiiton, $f^{-1}(V) \in \mathcal{A}$. Since f is open, $f(f^{-1}(V)) \in \mathcal{B}$. Surjectivity implies that $V = f(f^{-1}(V))$.
- The case is similar if f is closed.

$\square$

4.2. **Hausdorff Measure.**

Definition 4.2.1. Let X be a metric space and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ an outer measure on X. Then μ^* is said to be a **metric outer measure on** X if for each $A, B \subset X$, $d(A, B) > 0$ implies that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

Exercise 4.2.2. Let X be a metric space and $\mu^* : \mathcal{P}(X) \to [0, \infty]$ a metric outer measure on X. Then for each $A \in \mathcal{B}(X)$, A is μ^*-outer measurable.

Proof. □

Definition 4.2.3. Let X be a metric space, $E \subset X$ and $\delta > 0$. Define $\mathcal{A}_{E,\delta} \subset \mathcal{P}(X)^{\mathbb{N}}$ by

$$\mathcal{A}_{E,\delta} = \inf \left\{ (A_j)_{j\in\mathbb{N}} \subset \mathcal{P}(X) : E \subset \bigcup_{j\in\mathbb{N}} A_j \text{ and for each } j \in \mathbb{N},\ \operatorname{diam}(A_j) < \delta \right\}$$

Exercise 4.2.4. Let X be a metric space, $E \subset X$ and $\delta_1, \delta_2 > 0$. If $\delta_1 \leq \delta_2$, then $\mathcal{A}_{E,\delta_1} \subset \mathcal{A}_{E,\delta_2}$.

Proof. Clear. □

Definition 4.2.5. Let X be a metric space, $d \geq 0$ and $\delta > 0$. Define $H_{d,\delta} : \mathcal{P}(X) \to [0, \infty]$ by

$$H_{d,\delta}(E) = \inf \left\{ \sum_{j\in\mathbb{N}} \operatorname{diam}(A_j)^d : (A_j)_{j\in\mathbb{N}} \in \mathcal{A}_{E,\delta} \right\}$$

Exercise 4.2.6. Let X be a metric space, $d \geq 0$ and $\delta_1, \delta_2 > 0$. If $\delta_1 \leq \delta_2$, then $H_{d,\delta_2} \leq H_{d,\delta_1}$.

Proof. Clear. □

Definition 4.2.7. Let X be a metric space and $d \geq 0$. We define the **d-dimensional Hausdorff outer measure**, denoted $H_d : \mathcal{P}(X) \to [0, \infty]$, by

$$\begin{aligned} H_d(E) &= \sup_{\delta > 0} H_{d,\delta}(E) \\ &= \lim_{\delta \to 0^+} H_{d,\delta}(E) \end{aligned}$$

Exercise 4.2.8. Let X be a metric space and $d \geq 0$. Then $H_d : \mathcal{P}(X) \to [0, \infty]$ is an outer measure on X.

Proof. □

Exercise 4.2.9. Let X be a metric space and $d \geq 0$. Then $H_d : \mathcal{P}(X) \to [0, \infty]$ is a metric outer measure on X.

Proof. □

References

[1] Introduction to Algebra
[2] Introduction to Analysis
[3] Introduction to Fourier Analysis
[4] Introduction to Measure and Integration