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Introduction to Analysis

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$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
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Preface

Chapter 1

Set Theory

1.1 Relations

1.1.1 Orderings

Definition 1.1.1.1. Directed Set:
Let A be a set and $\leq \subset A \times A$ a binary relation on A. Then $(A, \leq)$ is said to be a **directed set** if ,

1. for each $\alpha \in A$, $\alpha \leq \alpha$
2. for each $\alpha, \beta, \gamma \in A$, $\alpha \leq \beta$ and $\beta \leq \gamma$ implies that $\alpha \leq \gamma$
3. for each $\alpha, \beta \in A$, there exists $\gamma \in A$ such that $\alpha, \beta \leq \gamma$
4. $A \neq \varnothing$

Definition 1.1.1.2. Poset:
Let A be a set and $\leq \subset A \times A$ a binary relation on A. Then

- $\leq$ is said to be a **partial ordering on** A if for each $a, b, c \in A$,
    1. $a \leq a$,
    2. $a \leq b$ and $b \leq a$ implies that $a = b$,
    3. $a \leq b$ and $b \leq c$ implies that $a \leq c$.
- $(A, \leq)$ is $(A, \leq)$ is said to be a **partially ordered set** or **poset** if $\leq$ is a partial ordering on A.

1.2 Functions

1.2.1 Introduction

Exercise 1.2.1.1. Let X, Y be sets, $f : X \to Y$ and $A \subset X$. Then $A \subset f^{-1}(f(A))$.

Proof. Let $x \in A$. Then $f(x) \in f(A)$. Set $B := f(A)$. Since $f(x) \in B$, we have that

$$\begin{aligned} x &\in f^{-1}(B) \\ &= f^{-1}(f(A)) \end{aligned}$$

Since $x \in A$ is arbitrary, we have that $A \subset f^{-1}(f(A))$. □

Exercise 1.2.1.2. Let X, Y be sets, $f : X \to Y$ and $B \subset Y$. Then $f(f^{-1}(B)) = B \cap f(X)$.

Proof. Let $y \in f(f^{-1}(B))$. Then there exists $x \in f^{-1}(B)$ such that $f(x) = y$. Thus

$$\begin{aligned} y &= f(x) \\ &\in B \end{aligned}$$

Since

$$\begin{aligned} f^{-1}(B) &= \{x \in X : f(x) \in B\} \\ &\subset X \end{aligned}$$

$$\begin{aligned} y &= f(x) \\ &\in f(X) \end{aligned}$$

Hence $y \in B \cap f(X)$. Since $y \in f(f^{-1}(B))$ is arbitrary, $f(f^{-1}(B)) \subset B \cap f(X)$.
Conversely, let $y \in B \cap f(X)$. Since $y \in f(X)$, there exists $x \in X$ such that $f(x) = y$. Since $y \in B$, $x \in f^{-1}(B)$. Hence

$$\begin{aligned} y &= f(x) \\ &\in f(f^{-1}(B)) \end{aligned}$$

Since $y \in B \cap f(X)$ is arbitrary, $B \cap f(X) \subset f(f^{-1}(B))$. Thus $f(f^{-1}(B)) = B \cap f(X)$. □

Exercise 1.2.1.3. Let X, Y be sets, $f : X \to Y$ and $\mathcal{A} \subset \mathcal{P}(X)$. Then

$$f\left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in \mathcal{A}} f(A)$$

Proof. Let $y \in f\left(\bigcup_{A \in \mathcal{A}} A\right)$ □

Exercise 1.2.1.4. Let X, Y be sets, $f : X \to Y$, $A \subset X$ and $B \subset Y$. Then

$$f(A \cap f^{-1}(B)) = f(A) \cap B$$

Proof. Let $y \in f(A \cap f^{-1}(B))$. Then there exists $a \in A \cap f^{-1}(B)$ such that $f(a) = y$. Then $a \in A$ and $f(a) \in B$. Hence

$$\begin{aligned} y &= f(a) \\ &\in f(A) \cap B. \end{aligned}$$

Since $y \in f(A \cap f^{-1}(B))$ is arbitrary, we have that $f(A \cap f^{-1}(B)) \subset f(A) \cap B$.
Conversely, let $y \in f(A) \cap B$. Then $y \in B$ and there exists $a \in A$ such that $f(a) = y$. Since $f(a) \in B$, we have that $a \in f^{-1}(B)$. Therefore

$$\begin{aligned} y &= f(a) \\ &\in f(A \cap f^{-1}(B)). \end{aligned}$$

Since $y \in f(A \cap f^{-1}(B))$ is arbitrary, we have that $f(A \cap f^{-1}(B)) \subset f(A \cap f^{-1}(B))$.
Hence $f(A \cap f^{-1}(B)) = f(A \cap f^{-1}(B))$. □

Exercise 1.2.1.5. Let X be a set, $K \subset X$, $(X_\alpha)_{\alpha \in A}$ a collection of sets and for each $\alpha \in A$, $f_\alpha : X \to X_\alpha$. For $\alpha \in A$, set $K_\alpha := f_\alpha(K)$. Then $K = \bigcap\limits_{\alpha \in A} f_\alpha^{-1}(K_\alpha)$.

Proof.

- Let $x \in K$ and $\alpha \in A$. By definition,
$$\begin{aligned} f_\alpha(x) &\in f_\alpha(K) \\ &= K_\alpha \end{aligned}$$
Thus $x \in f_\alpha^{-1}(K_\alpha)$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $x \in f_\alpha^{-1}(K_\alpha)$. Thus $x \in \bigcap_{\alpha \in A} f_\alpha^{-1}(K_\alpha)$. Since $x \in K$ is arbitrary, $K \subset \bigcap_{\alpha \in A} f_\alpha^{-1}(K_\alpha)$.
- Let $\alpha \in A$. Exercise 1.2.1.1 implies that
$$\begin{aligned} K &\subset f_\alpha^{-1}(f_\alpha(K)) \\ &= f_\alpha^{-1}(K_\alpha). \end{aligned}$$
Since $\alpha \in A$ is arbitrary, for each $\alpha \in A$, $K \subset f_\alpha^{-1}(K_\alpha)$. Thus $K \subset \bigcap_{\alpha \in A} f_\alpha^{-1}(K_\alpha)$.

Therefore $K = \bigcap_{\alpha \in A} f_\alpha^{-1}(K_\alpha)$. □

1.2.2 Bijections

Definition 1.2.2.1. Let X, Y be sets and $f : X \to Y$. Then f is said to be a **surjection** if for each $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Exercise 1.2.2.2. Let X, Y be sets and $f : X \to Y$. Suppose that f is a surjection. Then for each $B \subset Y$, $f(f^{-1}(B)) = B$.

Proof. Let $B \subset Y$. Since f is surjective, $f(X) = Y$. A previous exercise implies that
$$\begin{aligned} f(f^{-1}(B)) &= B \cap f(X) \\ &= B \cap Y \\ &= B \end{aligned}$$
□

Exercise 1.2.2.3. Let X, Y, Z be sets, $f : X \to Y$ and $g, h : Y \to Z$. Suppose that f is a surjection. If $g \circ f = h \circ f$, then $g = h$.

Proof. Suppose that $g \circ f = h \circ f$. Let $y \in Y$. Since f is a surjection, there exists $x \in X$ such that $y = f(x)$. Then
$$\begin{aligned} g(y) &= g \circ f(x) \\ &= h \circ f(x) \\ &= h(y). \end{aligned}$$
Since $y \in Y$ is arbitrary, we have that for each $y \in Y$, $g(y) = h(y)$. Hence $g = h$. □

1.2.3 Nets

Definition 1.2.3.1. Let X be a topological space, A a directed set and $x : A \to Y$. Then x is said to be a **net** in X. We typically write $(x_\alpha)_{\alpha \in A}$.

Definition 1.2.3.2. Let X be a set, A a directed set and $\alpha_0 in A$. We define the **α_0-tail operator**, denoted $L_{\alpha_0} : X^A \to X^{[\alpha_0, \infty)}$, by $L(x) := x|_{[\alpha_0, \infty)}$, i.e. $L_{\alpha_0}((x_\alpha)_{\alpha \in A}) := (x_\alpha)_{\alpha \in [\alpha_0, \infty)}$.

Definition 1.2.3.3. Let X be a topological space, $(x_\alpha)_{\alpha \in A}$, $(y_\beta)_{\beta \in B} \subset X$ nets and $\phi : B \to A$. Then $((y_\beta)_{\beta \in B}, \phi)$ is said to be a **subnet of** $(x_\alpha)_{\alpha \in A}$ if

1. for each $\beta \in B$, $y_\beta = x_{\phi(\beta)}$
2. for each $\alpha_0 \in A$, there exists $\beta_0 \in B$ such that for each $\beta \in B$, $\beta \geq \beta_0$ implies that $\phi(\beta) \geq \alpha_0$

Note 1.2.3.4. We usually supress ϕ and write α_β in place of $\phi(\beta)$.

1.2.4 Sequences

1.3 Products of Sets

Definition 1.3.0.1. Let $(X_\alpha)_{\alpha\in A}$ be a collection of sets. We define

- the **Cartesian product** of $(X_\alpha)_{\alpha\in A}$, denoted $\prod\limits_{\alpha\in A} X_\alpha$, by
$$\prod_{\alpha\in A} X_\alpha := \{f : A \to \bigcup_{\alpha\in A} X_\alpha : \text{ for each } \alpha\in A,\ f(\alpha)\in X_\alpha\}$$
- the α**-th projection map of** $\prod_{\alpha\in A} X_\alpha$ **onto** X_α, denoted $\pi_\alpha : \prod\limits_{\alpha\in A} X_\alpha \to X_\alpha$, by
$$\pi_\alpha(f) := f(\alpha)$$

Exercise 1.3.0.2. Let $(A_\lambda)_{\lambda\in\Lambda}$ be a collection of sets and B a set. Then
$$\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)\times B = \bigcup_{\lambda\in\Lambda}(A_\lambda\times B)$$

Proof. Let $(x,y)\in\left(\bigcup\limits_{\lambda\in\Lambda} A_\lambda\right)\times B$. Then $x\in\bigcup\limits_{\lambda\in\Lambda} A_\lambda$ and $y\in B$. Therefore, there exists $\lambda\in\Lambda$ such that $x\in A_\lambda$. Hence
$$\begin{aligned}(x,y) &\in A_\lambda\times B\\ &\subset \bigcup_{\lambda\in\Lambda}(A_\lambda\times B)\end{aligned}$$

Thus $\left(\bigcup\limits_{\lambda\in\Lambda} A_\lambda\right)\times B\subset \bigcup\limits_{\lambda\in\Lambda}(A_\lambda\times B)$.
Conversely, let $(x,y)\in\bigcup\limits_{\lambda\in\Lambda}(A_\lambda\times B)$. Then there exists $\lambda\in\Lambda$ such that $(x,y)\in A_\lambda\times B$. Then
$$\begin{aligned}x &\in A_\lambda\\ &\subset \bigcup_{\lambda\in\Lambda} A_\lambda\end{aligned}$$
and $y\in B$. Hence $(x,y)\in\left(\bigcup\limits_{\lambda\in\Lambda} A_\lambda\right)\times B$. So $\bigcup\limits_{\lambda\in\Lambda}(A_\lambda\times B)\subset\left(\bigcup\limits_{\lambda\in\Lambda} A_\lambda\right)\times B$. □

Definition 1.3.0.3. Let $(X_\alpha)_{\alpha\in A}$ and $(Y_\alpha)_{\alpha\in A}$ be collections of sets and $(f_\alpha)_{\alpha\in A}\in\prod\limits_{\alpha\in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha\in A$, $f_\alpha : X_\alpha\to Y_\alpha$. We define the **product of** $(f_\alpha)_{\alpha\in A}$, denoted $\prod_{\alpha\in A} f_\alpha : \prod\limits_{\alpha\in A} X_\alpha\to\prod\limits_{\alpha\in A} Y_\alpha$ by
$$\left(\left[\prod_{\alpha\in A} f_\alpha\right](x)\right)_\beta = f_\beta(x_\beta)$$

Exercise 1.3.0.4. Let $(X_\alpha)_{\alpha\in A}$ and $(Y_\alpha)_{\alpha\in A}$ be collections of sets and $(f_\alpha)_{\alpha\in A}\in\prod\limits_{\alpha\in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha\in A$, $f_\alpha : X_\alpha\to Y_\alpha$. Denote the α-th projection maps on $\prod\limits_{\alpha\in A} X_\alpha$ and $\prod\limits_{\alpha\in A} Y_\alpha$ by π_α^X and π_α^Y respectively. Then for each $\alpha\in A$, $\pi_\alpha^Y\circ\left[\prod\limits_{\alpha\in A} f_\alpha\right] = f_\alpha\circ\pi_\alpha^X$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\prod\limits_{\alpha\in A} X_\alpha \arrow[r, "\prod\limits_{\alpha\in A} f_\alpha"] \arrow[d, "\pi_\alpha^X"'] & \prod\limits_{\alpha\in A} Y_\alpha \arrow[d, "\pi_\alpha^Y"] \\
X_\alpha \arrow[r, "f_\alpha"'] & Y_\alpha
\end{tikzcd}$$

Proof. Set $X := \prod_{\alpha \in A} X_\alpha$, $Y := \prod_{\alpha \in A} Y_\alpha$ and define $f : X \to Y$ by $f := \prod_{\alpha \in A} f_\alpha$. Let $\alpha \in A$ and $x \in X$. Then

$$\begin{aligned} \pi_\alpha^Y \circ f(x) &= (f(x))_\alpha \\ &= f_\alpha(x_\alpha) \\ &= f_\alpha \circ \pi_\alpha^X(x) \end{aligned}$$

Since $\alpha \in A$ and $x \in X$ are arbitrary, for each $\alpha \in A$, $\pi_\alpha^Y \circ f = f_\alpha \circ \pi_\alpha^X$. □

Exercise 1.3.0.5. Let $(X_\alpha)_{\alpha \in A}$ and $(Y_\alpha)_{\alpha \in A}$ be collections of sets and $(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. Suppose that for each $\alpha \in A$, f_α is a bijection. Then $\prod_{\alpha \in A} f_\alpha$ is a bijection and $\left[\prod_{\alpha \in A} f_\alpha\right]^{-1} = \prod_{\alpha \in A} f_\alpha^{-1}$.

Proof. Set $X := \prod_{\alpha \in A} X_\alpha$, $Y := \prod_{\alpha \in A} Y_\alpha$ and define $f : X \to Y$ and $g : Y \to X$ by $f := \prod_{\alpha \in A} f_\alpha$ and $g := \prod_{\alpha \in A} f_\alpha^{-1}$. Denote the α-th projection maps on $\prod_{\alpha \in A} X_\alpha$ and $\prod_{\alpha \in A} Y_\alpha$ by π_α^X and π_α^Y respectively. Let $\alpha \in A$. Then

$$\begin{aligned} \pi_\alpha^X \circ g \circ f &= g_\alpha \circ \pi_\alpha^Y \circ f \\ &= g_\alpha \circ f_\alpha \circ \pi_\alpha^X \\ &= f_\alpha^{-1} \circ f_\alpha \circ \pi_\alpha^X \\ &= \pi_\alpha^X \end{aligned}$$

and

$$\begin{aligned} \pi_\alpha^Y \circ f \circ g &= f_\alpha \circ \pi_\alpha^X \circ g \\ &= f_\alpha \circ g_\alpha \circ \pi_\alpha^Y \\ &= f_\alpha \circ f_\alpha^{-1} \circ \pi_\alpha^Y \\ &= \pi_\alpha^Y \end{aligned}$$

Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $\pi_\alpha^X \circ g \circ f = \pi_\alpha^X$ and $\pi_\alpha^Y \circ f \circ g = \pi_\alpha^Y$. Hence $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Thus f is a bijection and $f^{-1} = g$. □

Definition 1.3.0.6. Let X be a a topological space. We define

- the **diagonal of** $X \times X$, denoted $\Delta_X \subset X \times X$, by $\Delta_X := \{(x, x) : x \in X\}$,
- the **diagonal of** $X^{\mathbb{N}}$, denoted $\Delta_{X^{\mathbb{N}}} \subset X^{\mathbb{N}}$, by $\Delta_{X^{\mathbb{N}}} := \{x \in X^{\mathbb{N}} : \text{ for each } m, n \in \mathbb{N}, \pi_m(x) = \pi_n(x)\}$.

Exercise 1.3.0.7. Let X be a set, $(B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$, $(Z_n)_{n \in \mathbb{N}}$ a collection of sets and $(f_n) \in \prod_{n \in \mathbb{N}} B_n^{Z_n}$. Suppose that for each $n \in \mathbb{N}$, f_n is a bijection. Set $Z_0 := \prod_{n \in \mathbb{N}} Z_n$. Define $f_0 : Z_0 \to X^{\mathbb{N}}$ and $Z \subset Z_0$ by $f_0 := \prod_{n \in \mathbb{N}} f_n$ and $Z := f_0^{-1}(\Delta_{X^{\mathbb{N}}})$. Then

1. for each $m, n \in \mathbb{N}$, $f_m \circ \pi_m|_Z = f_n \circ \pi_n|_Z$,
2. for each $n \in \mathbb{N}$, $f_n \circ \pi_n(Z) = \bigcap_{n \in \mathbb{N}} B_n$.

Proof.

1. Let $m, n \in \mathbb{N}$ and $z \in Z$. By construction, $f_0(z) \in \Delta_{X^{\mathbb{N}}}$. Therefore

$$\begin{aligned} f_n(\pi_n(z)) &= \pi_n(f_0(z)) \\ &= \pi_m(f_0(z)) \\ &= f_m(\pi_m(z)). \end{aligned}$$

   Since $z \in Z$ is arbitrary, we have that $f_m \circ \pi_m|_Z = f_n \circ \pi_n|_Z$. Since $m, n \in \mathbb{N}$ are arbitrary, we have that for each $m, n \in \mathbb{N}$, $f_m \circ \pi_m|_Z = f_n \circ \pi_n|_Z$.

2. Set $B := \bigcap_{n \in \mathbb{N}} B_n$. Let $n \in \mathbb{N}$.

   - Let $z \in Z$. The previous part implies that

$$\begin{aligned} f_n(\pi_n(z)) &= f_m(\pi_m(z)) \\ &\in B_m. \end{aligned}$$

     Since $m \in \mathbb{N}$ is arbitrary, we have that for each $m \in \mathbb{N}$, $f_n(\pi_n(z)) \in B_m$. Hence

$$\begin{aligned} f_n(\pi_n(z)) &\in \bigcap_{m \in \mathbb{N}} B_m \\ &= B. \end{aligned}$$

     Since $z \in Z$ is arbitrary, we have that $f_n \circ \pi_n(Z) \subset B$.
   - Let $x \in B$. Then for each $m \in \mathbb{N}$, $x \in B_m$. Define $z \in Z_0$ by $z_m := f_m^{-1}(x)$. Then for each $m \in \mathbb{N}$,

$$\begin{aligned} \pi_m \circ f_0(z) &= f_m \circ \pi_m(z) \\ &= f_m(z_m) \\ &= f_m(f_m^{-1}(x)) \\ &= x \end{aligned}$$

     In particular, for each $m \in \mathbb{N}$,

$$\begin{aligned} \pi_m \circ f_0(z) &= x \\ &= \pi_n \circ f_0(z). \end{aligned}$$

     Thus $f_0(z) \in \Delta_{X^{\mathbb{N}}}$ and $z \in Z$. Therefore

$$\begin{aligned} x &= \pi_n \circ f_0(z) \\ &= f_n \circ \pi_n(z) \\ &\in f_n \circ \pi_n(Z). \end{aligned}$$

     Since $x \in B$ is arbitrary, we have that $B \subset f_n \circ \pi_n(Z)$.

□

Definition 1.3.0.8. Let X be a set, $(Y_\alpha)_{\alpha \in A}$ a collection of sets and $(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} Y_\alpha^X$, i.e. for each $\alpha \in A$, $f_\alpha : X \to Y_\alpha$. Set $Y = \prod_{\alpha \in A} Y_\alpha$. We define the **tuple of** $(f_\alpha)_{\alpha \in A}$, denoted $(f_\alpha)_{\alpha \in A} : X \to Y$, by $(f_\alpha)_{\alpha \in A}(x) := (f_\alpha(x))_{\alpha \in A}$.

Exercise 1.3.0.9. Let X be a set, $(Y_\alpha)_{\alpha \in A}$ a collection of sets, $(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} Y_\alpha^X$ and $E \subset X$. Then $(f_\alpha)_{\alpha \in A}|_E = (f_\alpha|_E)_{\alpha \in A}$.

Proof. Clear. add details. □

Definition 1.3.0.10. Define slice maps here in terms of inclusion maps Let X, Y be sets and $U \subset X \times Y$. For each $(x_0, y_0) \in U$, we define $U_{x_0} = \{y \in Y : (x_0, y) \in U\}$ and $U^{y_0} = \{x \in X : (x, y_0) \in U\}$.

Definition 1.3.0.11. Let X, Y and Z be sets, $U \subset X \times Y$ and $f : U \to Z$. For each $(x_0, y_0) \in U$, we define $f_{x_0} : U_{x_0} \to Z$ and $f^{y_0} : U^{y_0} \to Z$ by $f_{x_0} = f(x_0, \cdot)$ and $f^{y_0} = f(\cdot, y_0)$.

Exercise 1.3.0.12. Let X, Y and Z be sets, $U \subset X \times Y$, $f : U \to Z$ and $(x_0, y_0) \in U$. Then for each $V \subset Z$, $(f_{x_0})^{-1}(V) = (f^{-1}(V))_{x_0}$ and $(f^{y_0})^{-1}(V) = (f^{-1}(V))^{y_0}$.

Proof. Let $V \subset Z$. Then for each $x \in U^{y_0}$,

$$\begin{aligned} x \in (f^{y_0})^{-1}(V) &\iff f^{y_0}(x) \in V \\ &\iff f(x, y_0) \in V \\ &\iff (x, y_0) \in f^{-1}(V) \\ &\iff x \in (f^{-1}(V))^{y_0} \end{aligned}$$

So $(f^{y_0})^{-1}(V) = (f^{-1}(V))^{y_0}$. Similarly, $(f_{x_0})^{-1}(V) = (f^{-1}(V))_{x_0}$. $\square$

Definition 1.3.0.13. Let X, Y, Z be sets. We define the **currying operator**, denoted $\mathrm{cur} : Z^{X \times Y} \to (Z^Y)^X$, by $\mathrm{cur}(f)(x)(y) = f(x, y)$.

1.4 Coproducts of Sets

Definition 1.4.0.1. Let $(X_\alpha)_{\alpha\in A}$ be a collection of sets. We define the **disjoint union** of $(X_\alpha)_{\alpha\in A}$, denoted $\coprod\limits_{\alpha\in A} X_\alpha$, by

$$\coprod_{\alpha\in A} X_\alpha = \{(\alpha, x) : x \in X_\alpha\}$$

Definition 1.4.0.2. Let $(X_\alpha)_{\alpha\in A}$ be a collection of sets. For $\alpha \in A$, we define the **α-th embedding map if X_α into $\coprod_{\alpha\in A} X_\alpha$**, denoted $\iota_\alpha : X_\alpha \to \coprod\limits_{\alpha\in A} X_\alpha$, by

$$\iota_\alpha(x) = (\alpha, x)$$

Definition 1.4.0.3. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in A}$ and $(Y_\alpha, \mathcal{S}_\alpha)_{\alpha\in A}$ be collections of topological spaces and $(f_\alpha)_{\alpha\in A} \in \prod\limits_{\alpha\in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. Set $X = \coprod\limits_{\alpha\in A} X_\alpha$ and $Y = \coprod\limits_{\alpha\in A} Y_\alpha$. We define the **coproduct of** $(f_\alpha)_{\alpha\in A}$, denoted $\coprod\limits_{\alpha\in A} f_\alpha : X \to Y$, by $\coprod\limits_{\alpha\in A} f_\alpha(\beta, x) = (\beta, f_\beta(x))$.

Exercise 1.4.0.4. Let $(X_\alpha)_{\alpha\in A}$ and $(Y_\alpha)_{\alpha\in A}$ be collections of sets and $(f_\alpha)_{\alpha\in A} \in \coprod\limits_{\alpha\in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. Set $f := \coprod\limits_{\alpha\in A} f_\alpha$. Then for each $\alpha \in A$, $f \circ \iota_\alpha^X = \iota_\alpha^Y \circ f_\alpha$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\coprod\limits_{\alpha\in A} X_\alpha \arrow[r, "f"] & \coprod\limits_{\alpha\in A} Y_\alpha \\
X_\alpha \arrow[u, "\iota_\alpha^X"] \arrow[r, "f_\alpha"'] & Y_\alpha \arrow[u, "\iota_\alpha^Y"']
\end{tikzcd}$$

Proof. Let $\alpha \in A$ and $x \in X_\alpha$. Then

$$\begin{aligned}
f \circ \iota_\alpha^X(x) &= f(\alpha, x) \\
&= (\alpha, f_\alpha(x)) \\
&= \iota_\alpha^Y \circ f_\alpha(x).
\end{aligned}$$

Since $x \in X$ are arbitrary, $f \circ \iota_\alpha^X = \iota_\alpha^Y \circ f_\alpha$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $f \circ \iota_\alpha^X = \iota_\alpha^Y \circ f_\alpha$. □

Exercise 1.4.0.5. Let $(X_\alpha)_{\alpha\in A}$ and $(Y_\alpha)_{\alpha\in A}$ be collections of sets and $(f_\alpha)_{\alpha\in A} \in \prod\limits_{\alpha\in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. Suppose that for each $\alpha \in A$, f_α is a bijection. Then $\coprod\limits_{\alpha\in A} f_\alpha$ is a bijection and $\left[\coprod\limits_{\alpha\in A} f_\alpha\right]^{-1} = \coprod\limits_{\alpha\in A} f_\alpha^{-1}$.

Proof. Set $X := \coprod\limits_{\alpha\in A} X_\alpha$, $Y := \coprod\limits_{\alpha\in A} Y_\alpha$ and define $f : X \to Y$ and $g : Y \to X$ by $f := \coprod\limits_{\alpha\in A} f_\alpha$ and $g := \coprod\limits_{\alpha\in A} f_\alpha^{-1}$. Denote the α-th embedding maps into X and Y by ι_α^X and ι_α^Y respectively. Let $\alpha \in A$. Then

$$\begin{aligned}
g \circ f \circ \iota_\alpha^X &= g \circ \iota_\alpha^Y \circ f_\alpha \\
&= \iota_\alpha^X \circ g_\alpha \circ f_\alpha \\
&= \iota_\alpha^X \circ f_\alpha^{-1} \circ f_\alpha \\
&= \iota_\alpha^X
\end{aligned}$$

and

$$\begin{aligned} f \circ g \circ \iota_\alpha^Y &= f \circ \iota_\alpha^X \circ g_\alpha \\ &= \iota_\alpha^Y \circ f_\alpha \circ g_\alpha \\ &= \iota_\alpha^Y \circ f_\alpha \circ f_\alpha^{-1} \\ &= \iota_\alpha^Y \end{aligned}$$

Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $g \circ f \circ \iota_\alpha^X = \iota_\alpha^X$ and $f \circ g \circ \iota_\alpha^Y = \iota_\alpha^Y$. Hence $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Thus f is a bijection and $f^{-1} = g$. $\square$

1.5 Quotients of Sets

Definition 1.5.0.1. Let X be a set and $\sim$ an equivalence relation on X. We define the **quotient set** of X by $\sim$, denoted $X/{\sim}$, by

$$X/{\sim} = \{\bar{x} : x \in X\}$$

1.6 Common Structures

1.6.1 Equalizers of Maps

Definition 1.6.1.1. Let X, Y be sets and $f, g : X \to Y$. We define the **equalizer of** f **and** g, denoted $\mathrm{Eq}(f, g)$, by

$$\mathrm{Eq}(f, g) := \{x \in X : f(x) = g(x)\}$$

Exercise 1.6.1.2.

1.6.2 Projective Limits of Sets

Note 1.6.2.1. Let $(X_j)_{j \in J}$ be a collection of sets. We denote the j-th projection map from $\prod_{j \in J} X_j$ onto X_j by $\mathrm{proj}_j : \prod_{j \in J} X_j \to X_j$.

Definition 1.6.2.2. Let $(J, \leq)$ be a directed poset, $(X_j)_{j \in J}$ a collection of sets and for each $(j, k) \in \leq$, $\pi_{j,k} : X_k \to X_j$. Suppose that for each $j, k, l \in J$,

1. $\pi_{j,j} = \mathrm{id}_{X_j}$,
2. $j \leq k$ and $k \leq l$ implies that $\pi_{j,k} \circ \pi_{k,l} = \pi_{j,l}$.

Then $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ is said to be a **projective system of sets**.

Definition 1.6.2.3. Let $(J, \leq)$ be a directed poset and $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a projective system of sets. We define

- the **inverse limit** of $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$, denoted $\varprojlim_{j \in J} X_j$, by

$$\varprojlim_{j \in J} X_j := \left\{x \in \prod_{j \in J} X_j : \text{ for each } (j, k) \in \leq, \, \pi_{j,k} \circ \mathrm{proj}_k(x) = \mathrm{proj}_j(x)\right\}$$

- the **j-th projection map of $\varprojlim_{j \in J} X_j$ onto** X_j, denoted $\pi_j : \varprojlim_{j \in J} X_j \to X_j$, by

$$\pi_j = \mathrm{proj}_j |_{\varprojlim_{j \in J} X_j}$$

Exercise 1.6.2.4. Let $(J, \leq)$ be a directed poset and $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a projective system of sets. Then for each $j, k \in J$, $j \leq k$ implies that $\pi_{j,k} \circ \pi_k = \pi_j$.

Proof. Let $x \in \varprojlim_{j \in J} X_j$. Let $j, k \in J$. Suppose that $j \leq k$. By definition,

$$\begin{aligned} \pi_{j,k} \circ \pi_k(x) &= \pi_{j,k} \circ \mathrm{proj}_k(x) \\ &= \mathrm{proj}_j(x) \\ &= \pi_j(x). \end{aligned}$$

Since $x \in \varprojlim_{j' \in J} X_{j'}$ is arbitrary, we have that $\pi_{j,k} \circ \pi_k = \pi_j$. □

Exercise 1.6.2.5. Let $(J, \leq)$ be a directed poset and $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a projective system of sets. Then

$$\varprojlim_{j \in J} X_j = \bigcap_{(j,k) \in \leq} \mathrm{Eq}(\pi_{j,k} \circ \mathrm{proj}_k, \mathrm{proj}_j)$$

Proof. Set $X :=$. By definition,

$$\begin{aligned}
\varprojlim_{j \in J} &= \bigcap_{(j,k) \in \leq} \{x \in X : \pi_{j,k} \circ \mathrm{proj}_k(x) = \mathrm{proj}_j(x)\} \\
&= \bigcap_{(j,k) \in \leq} \mathrm{Eq}(\pi_{j,k} \circ \mathrm{proj}_k, \mathrm{proj}_j).
\end{aligned}$$

□

Chapter 2

Real and Complex Numbers

2.1 Real Numbers

Note 2.1.0.1. As a starting point, we will take as fact the existence of the **natural numbers**

$$\mathbb{N} = \{1, 2, \cdots\}$$

the **integers**

$$\mathbb{Z} = \{\cdots, -2, -2, 0, 1, 2, \cdots\}$$

and the **rational numbers**

$$\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}\right\}$$

Definition 2.1.0.2. Let X be a set and $\leq$ a relation on X. Then $\leq$ is said to be a **total order** if for each $a, b, c \in X$,

1. $a \leq a$
2. $a \leq b$ and $b \leq c$ implies that $a \leq c$
3. $a \leq b$ and $b \leq a$ implies that $a = b$
4. $a \leq b$ or $b \leq a$

Exercise 2.1.0.3. We define the relation $\leq$ on $\mathbb{Q}$ defined by

$$\frac{a}{b} \leq \frac{c}{d} \text{ iff } ad \leq bc$$

Then $\leq$ is a total order of $\mathbb{Q}$.

Proof. Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Then

1. $\frac{a}{b} \leq \frac{a}{b}$ since $ab \leq ab$.
2. if $\frac{a}{b} \leq \frac{c}{d}$ and $\frac{c}{d} \leq \frac{e}{f}$, then $ad \leq bc$ and $cf \leq de$. Multiplying the first inequality by f and the second inequality by b, we obtain $adf \leq bcf \leq bde$. Dividing both sides by d yields $af \leq be$. Hence $\frac{a}{b} \leq \frac{e}{f}$.
3. if $\frac{a}{b} \leq \frac{c}{d}$ and $\frac{c}{d} \leq \frac{a}{b}$, then $ad \leq bc$ and $bc \leq ab$. This implies that $ad = bc$. Hence $\frac{a}{b} = \frac{c}{d}$.
4.

$\square$

Definition 2.1.0.4. Let A be a set

Exercise 2.1.0.5. Let A, B be sets and $f : A \times B \to \mathbb{R}$. Then

$$\sup_{(a,b)\in A\times B} f(a,b) = \sup_{a\in A}\left[\sup_{b\in B} f(a,b)\right] = \sup_{b\in B}\left[\sup_{a\in A} f(a,b)\right]$$

Proof. For $(a,b) \in A \times B$, set $s_a = \sup_{b\in B} f(a,b)$ and $t_b = \sup_{a\in A}$. Let $(a,b) \in A \times B$, $A \times \{b\}$. Then $\{a\} \times B \subset A \times B$. Therefore

$$\sup_{a\in A} s_a, \sup_{b\in B} t_b \leq \sup_{(x,y)\in A\times B} f(x,y)$$

Thus sup □

2.2 Extended Real Numbers

- define sup and inf.

Definition 2.2.0.1.

- We define the **extended real numbers**, denoted $\overline{\mathbb{R}}$, by

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$$

- For $a, b \in \overline{\mathbb{R}}$, we define

$$\begin{aligned}
a \pm \infty = \pm\infty + a &= \pm\infty, && a \neq \mp\infty \\
a \cdot (\pm\infty) = \pm\infty \cdot a &= \pm\infty, && a \in (0, +\infty] \\
a \cdot (\pm\infty) = \pm\infty \cdot a &= \mp\infty, && a \in [-\infty, 0) \\
\frac{a}{\pm\infty} &= 0, && a \in \mathbb{R} \\
\frac{\pm\infty}{a} &= \pm\infty, && a \in (0, +\infty) \\
\frac{\pm\infty}{a} &= \mp\infty, && a \in (-\infty, 0) \\
0 \cdot (\pm\infty) = \pm\infty \cdot 0 = 0
\end{aligned}$$

- We define $\leq_{\overline{\mathbb{R}}} \subset \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ by

$$\leq_{\overline{\mathbb{R}}} = \leq_{\mathbb{R}} \cup \{(a,b) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} : a = -\infty \text{ or } b = \infty\}$$

Definition 2.2.0.2. Let $A \subset \overline{\mathbb{R}}$. We define the **supremum of** A, denoted $\sup A :=$.

2.3 Complex Numbers

We define $\mathrm{Re}, \mathrm{Im} : \mathbb{C} \to \mathbb{R}$ by $\mathrm{Re}(a + ib) = a$ and $\mathrm{Im}(a + ib) = b$.

Chapter 3

Topology

3.1 Introduction

Definition 3.1.0.1. Let X be a set and $\mathcal{T} \subset \mathcal{P}(X)$. Then $\mathcal{T}$ is said to be a **topology on** X if

1. X, $\varnothing \in \mathcal{T}$
2. for each $(U_\alpha)_{\alpha \in A} \subset \mathcal{T}$,
$$\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$$
3. for each $(U_j)_{j=1}^n \subset \mathcal{T}$,
$$\bigcap_{j=1}^n U_j \in \mathcal{T}$$

Exercise 3.1.0.2. Let X be a set and $(\mathcal{T}_i)_{i \in I}$ a collection of topologies on X. Then $\bigcap\limits_{i \in I} \mathcal{T}_i$ is a topology on X.

Proof.

1. Since for each $i \in I$, $X, \varnothing \in \mathcal{T}_i$, we have that $X, \varnothing \in \bigcap\limits_{i \in I} \mathcal{T}_i$.
2. Let $(U_\alpha)_{\alpha \in A} \subset \bigcap\limits_{i \in I} \mathcal{T}_i$. Then for each $i \in I$, $(U_\alpha)_{\alpha \in A} \subset T_i$. So for each $i \in I$, $\bigcup\limits_{\alpha \in A} U_\alpha \in \mathcal{T}_i$. Thus $\bigcup\limits_{\alpha \in A} U_\alpha \in \bigcap\limits_{i \in I} \mathcal{T}_i$.
3. Let $(U_j)_{j=1}^n \subset \bigcap\limits_{i \in I} \mathcal{T}_i$. Then for each $i \in I$, $(U_j)_{j=1}^n \subset T_i$. So for each $i \in I$, $\bigcap\limits_{j=1}^n U_j \in \mathcal{T}_i$. Thus $\bigcap\limits_{j=1}^n U_j \in \bigcap\limits_{i \in I} \mathcal{T}_i$.

So $\bigcap\limits_{i \in I} \mathcal{T}_i$ is a topology on X. □

Definition 3.1.0.3. Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$. Set

$$\mathcal{S} = \{\mathcal{T} \subset \mathcal{P}(X) : \mathcal{T} \text{ is a topology on } X \text{ and } \mathcal{E} \subset \mathcal{T}\}$$

We define the **topology generated by** $\mathcal{E}$ on X, denoted $\tau(\mathcal{E})$, by

$$\tau(\mathcal{E}) = \bigcap_{\mathcal{T} \in \mathcal{S}} \mathcal{T}$$

Definition 3.1.0.4. Let X be a set and $\mathcal{T} \subset \mathcal{P}(X)$ a topology on X, $x \in X$ and $\mathcal{B}_x \subset \mathcal{T}$. Then $\mathcal{B}_x$ is said to be a **local basis for $\mathcal{T}$ at** x if

1. for each $U \in \mathcal{B}_x$, $x \in U$
2. for each $V \in \mathcal{T}$, if $x \in V$, then there exists $U \in \mathcal{B}_x$ such that $U \subset V$

Definition 3.1.0.5. Let X be a set and $\mathcal{T} \subset \mathcal{P}(X)$ a topology on X and $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is said to be a **basis for** $\mathcal{T}$ if for each $V \in \mathcal{T}$ and $x \in V$, there exists $U \in \mathcal{B}$ such that $x \subset U \subset V$.

Exercise 3.1.0.6. Let X be a set and $\mathcal{T} \subset \mathcal{P}(X)$ a topology on X and $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ iff for each $x \in X$, there exists $\mathcal{B}_x \subset \mathcal{B}$ such that $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x.

Proof. Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Let $x \in X$. Define $\mathcal{B}_x = \{U \in \mathcal{B} : x \in U\}$.

1. By definition, for each $U \in \mathcal{B}_x$, $x \in U$
2. Let $V \in \mathcal{T}$. Suppose that $x \in V$. Since $\mathcal{B}$ is a basis, there exists $U \in \mathcal{B}$ such that $x \in U \subset V$. By definition, $U \in \mathcal{B}_x$.

Hence $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x.
Conversely, suppose that for each $x \in X$, there exists $\mathcal{B}_x \subset \mathcal{B}$ such that $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x. Let $V \in \mathcal{T}$ and $x \in V$. By assumption, there exists $\mathcal{B}_x \subset \mathcal{B}$ such that $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x. Since $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x, there exists $U \in \mathcal{B}_x \subset \mathcal{B}$ such that $x \in U \subset V$. Hence $\mathcal{B}$ is a basis for $\mathcal{T}$. □

Exercise 3.1.0.7. Let X be a set and $\mathcal{T} \subset \mathcal{P}(X)$ a topology on X and $\mathcal{B} \subset \mathcal{T}$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ iff for each $V \in \mathcal{T}$, there exists a collection $\mathcal{C} \subset \mathcal{B}$ such that

$$V = \bigcup_{U \in \mathcal{C}} U$$

Proof. Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Let $V \in \mathcal{T}$. Since since $\mathcal{B}$ is a basis for $\mathcal{T}$, for each $x \in V$, there exists $U_x \in \mathcal{B}$ such that $x \in U_x \subset V$. Then $(U_x)_{x \in U} \subset \mathcal{B}$ satisfies $V = \bigcup\limits_{x \in U} U_x$.
Conversely, suppose that for each $V \in \mathcal{T}$, there exists a collection $\mathcal{C} \subset \mathcal{B}$ such that $V = \bigcup\limits_{U \in \mathcal{C}} U$. Let $V \in \mathcal{T}$ and $x \in V$. By assumption, there exists a collection $\mathcal{C} \subset \mathcal{B}$ such that $V = \bigcup\limits_{U \in \mathcal{C}} U$. Since $x \in V$, there exists $U \in \mathcal{C}$ such that $x \in U$. Hence there exists $U \in \mathcal{B}$ such that $x \in U \subset V$. Then $\mathcal{B}$ is a basis for $\mathcal{T}$. □

Exercise 3.1.0.8. Let X be a set and $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{P}(X)$ topologies on X and $\mathcal{B} \subset \mathcal{T}_1$. Suppose that $\mathcal{T}_1 \subset \mathcal{T}_2$. If $\mathcal{B}$ is a basis for $\mathcal{T}_2$, then $\mathcal{B}$ is a basis for $\mathcal{T}_1$.

Proof. Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}_2$. Let $V \in \mathcal{T}_1$. Then $V \in \mathcal{T}_2$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_2$, the previous exercise implies that there exists a collection $(U_\alpha)_{\alpha \in A} \subset \mathcal{B}$ such that $V = \bigcup\limits_{\alpha \in A} U_\alpha$. Thus the previous exercise implies that $\mathcal{B}$ is a basis for $\mathcal{T}_1$. □

Exercise 3.1.0.9. Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$. Define

$$\mathcal{T}_{\mathcal{B}} = \{U \subset X : \text{ for each } x \in U, \text{ there exists } V \in \mathcal{B} \text{ such that } x \in V \subset U\}$$

Then

1. $\mathcal{T}_{\mathcal{B}}$ is a topology on X iff
   (a) for each $x \in X$, there exists $V \in \mathcal{B}$ such that $x \in V$
   (b) for each $x \in X$ and $U, V \in \mathcal{B}$, if $x \in U \cap V$, then there exists $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$
2. if $\mathcal{T}_{\mathcal{B}}$ is a topology on X, then $\mathcal{B}$ is a basis for $\mathcal{T}_{\mathcal{B}}$
3. if $\mathcal{T}_{\mathcal{B}}$ is a topology on X, then $\mathcal{T}_{\mathcal{B}} = \tau(\mathcal{B})$

Proof.

1. • ($\Longrightarrow$) :
   Suppose that $\mathcal{T}_\mathcal{B}$ is a topology on X.
   (a) Let $x \in X$. Since $\mathcal{T}_\mathcal{B}$ is a topology on X, $X \in \mathcal{T}_\mathcal{B}$. Since $x \in X$, the definition of $\mathcal{T}_\mathcal{B}$ implies that there exists $V \in \mathcal{B}$ such that $x \in V \subset X$.
   (b) Let $x \in X$ and $U, V \in \mathcal{B}$. Suppose that $x \in U \cap V$. Since $\mathcal{B} \subset \mathcal{T}$, we have that $U, V \in \mathcal{T}_\mathcal{B}$. Since $\mathcal{T}_\mathcal{B}$ is a topology on X, $U \cap V \in \mathcal{T}_\mathcal{B}$. By definition of $\mathcal{T}_\mathcal{B}$, there exists $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.

   • ($\Longleftarrow$) :
   Conversely, suppose that (a) and (b) are satisfied.
   – Vacuously, $\varnothing \in \mathcal{T}_\mathcal{B}$. Condition (a) implies that $X \in \mathcal{T}_\mathcal{B}$.
   – Let $(U_\alpha)_{\alpha \in A} \subset \mathcal{T}_\mathcal{B}$ and $x \in \bigcup\limits_{\alpha \in A} U_\alpha$. Then there exists $\alpha \in A$ such that $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}_\mathcal{B}$, the definition of $\mathcal{T}_\mathcal{B}$ implies that there exists $V \in \mathcal{B}$ such that

$$\begin{aligned} x &\in V \\ &\subset U_\alpha \\ &\subset \bigcup_{\alpha \in A} U_\alpha \end{aligned}$$

   Since $x \in \bigcup\limits_{\alpha \in A} U_\alpha$ is arbitrary, $\bigcup\limits_{\alpha \in A} U_\alpha \in \mathcal{T}_\mathcal{B}$.
   – Let $U_1, U_2 \mathcal{T}_\mathcal{B}$ and $x \in U_1 \cap U_2$. The definition if $\mathcal{T}_\mathcal{B}$ implies that for $j \in \{1, 2\}$, there exists $V_j \in \mathcal{B}$ such that $x \in V_j \subset U_j$. This implies that $x \in V_1 \cap V_2$ and by condition (b), there exists $W \in \mathcal{B}$ such that

$$\begin{aligned} x &\in W \\ &\subset V_1 \cap V_2 \\ &\subset U_1 \cap U_2 \end{aligned}$$

   Since $x \in U_1 \cap U_2$ is arbitrary, $U_1 \cap U_2 \in \mathcal{T}_\mathcal{B}$.

   Thus $\mathcal{T}_\mathcal{B}$ is a topology on X.

2. Suppose that $\mathcal{T}_\mathcal{B}$ is a topology on X. Let $U \in \mathcal{T}_\mathcal{B}$ and $x \in U$. By definition of $\mathcal{T}_\mathcal{B}$, there exists $V \in \mathcal{B}$ such that $x \subset V \subset U$. Since $U \in \mathcal{T}_\mathcal{B}$ and $x \in U$ are arbitrary, $\mathcal{B}$ is a basis for $\mathcal{T}_\mathcal{B}$.

3. Suppose that $\mathcal{T}_\mathcal{B}$ is a topology on X. Since $\mathcal{B} \subset \mathcal{T}_\mathcal{B}$, we have that $\tau(\mathcal{B}) \subset \mathcal{T}_\mathcal{B}$. Let $U \in \tau(\mathcal{B})$.
   Conversely, let $U \in \mathcal{T}_\mathcal{B}$. Since $\mathcal{T}_\mathcal{B}$ is a topology on X, part (1) implies that $\mathcal{B}$ is a basis for $\mathcal{T}_\mathcal{B}$. Then there exists $\mathcal{C} \subset \mathcal{B}$ such that

$$\begin{aligned} U &= \bigcup_{V \in \mathcal{C}} V \\ &\in \tau(\mathcal{B}) \end{aligned}$$

   So $\mathcal{T}_\mathcal{B} \subset \tau(\mathcal{B})$. Hence $\mathcal{T}_\mathcal{B} = \tau(\mathcal{B})$.

□

Exercise 3.1.0.10. Let X be a set and $\mathcal{B} \subset \mathcal{P}(X)$. Then $\mathcal{B}$ is a basis for $\tau(\mathcal{B})$ iff

1. for each $x \in X$, there exists $V \in \mathcal{B}$ such that $x \in V$
2. for each $x \in X$ and $U, V \in \mathcal{B}$, if $x \in U \cap V$, then there exists $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$

Proof.

- ($\implies$) :
  Suppose that $\mathcal{B}$ is a basis for $\tau(\mathcal{B})$.

  1. Let $x \in X$. Since $X \in \tau(\mathcal{B})$ and $\mathcal{B}$ is a basis for $\tau(\mathcal{B})$, there exists $V \in \mathcal{B}$ such that $x \in V$.
  2. Let $x \in X$ and $U, V \in \mathcal{B}$. Suppose that $x \in U \cap V$. Since $\tau(\mathcal{B})$ is a topology, $U \cap V \in \tau(\mathcal{B})$. Since $\mathcal{B}$ is a basis for $\tau(\mathcal{B})$, there exists $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.

- ($\impliedby$) :
  Conversely, suppose that (1) and (2) are satisfied. The previous exercise implies that $\mathcal{B}$ is a basis for $\tau(\mathcal{B})$.

$\square$

Exercise 3.1.0.11. Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$. Define $\mathcal{B} \subset \mathcal{P}(X)$ by

$$\mathcal{B} = \{X, \varnothing\} \cup \left\{ \bigcap_{j=1}^{n} V_j : (V_j)_{j=1}^{n} \subset \mathcal{E} \right\}$$

Then

1. $\mathcal{B}$ is a basis for $\tau(\mathcal{E})$
2. $$\tau(\mathcal{E}) = \left\{ \bigcup_{\alpha \in A} V_\alpha : (V_\alpha)_{\alpha \in A} \subset \mathcal{B} \right\}$$

   That is, each element of $\tau(\mathcal{E})$ is either $X, \varnothing$ or a union of finite intersections of elements of $\mathcal{E}$.

Proof.

1. Referring to Exercise 3.1.0.9, since $X \in \mathcal{B}$, condition (1) is satisfied and since for each $U, V \in \mathcal{B}$, $U \cap V \in \mathcal{B}$, condition (2) is satisfied. Hence there exists a topology $\mathcal{T}$ on X such that $\mathcal{B}$ is a basis for $\mathcal{T}$. Since $\mathcal{B} \subset \mathcal{T}$ and $\tau(\mathcal{E}) = \tau(\mathcal{B})$, we have that $\tau(\mathcal{E}) \subset \mathcal{T}$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$ and $\mathcal{B} \subset \tau(\mathcal{E})$, Exercise 3.1.0.8 implies that $\mathcal{B}$ is a basis for $\tau(\mathcal{E})$.

2. Exercise 3.1.0.7 implies that
$$\tau(\mathcal{E}) = \left\{ \bigcup_{\alpha \in A} V_\alpha : (V_\alpha)_{\alpha \in A} \subset \mathcal{B} \right\}$$

$\square$

Definition 3.1.0.12. Let X be a set and $\mathcal{T}$ a topology on X. Then $(X, \mathcal{T})$ is said to be a **topological space**. Let $U \subset X$. Then U is said to be **open** if $U \in \mathcal{T}$ and U is said to be **closed** if U^c is open.

Definition 3.1.0.13. Let $(X, \mathcal{T})$ be a topological space and $S, N \subset X$. Then N is said to be a **neighborhood** of S if there exists $U \in \mathcal{T}$ such that $S \subset U \subset N$. For $S \in X$, we denote the set of neighborhoods of S by $\mathcal{N}_{\mathcal{T}}(S)$.

Note 3.1.0.14. We will typically write $\mathcal{N}(S)$ in place of $\mathcal{N}_{\mathcal{T}}(S)$ when the topology $\mathcal{T}$ is clear from the context.

Definition 3.1.0.15. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. We define

- the **collection of open subsets of** A, denoted $\mathcal{U}_A(X, \mathcal{T})$, by

$$\mathcal{U}_A(X, \mathcal{T}) := \{U \subset X : U \subset A \text{ and } U \text{ is open}\},$$

- the **collection of closed supersets of** A, denoted $\mathcal{C}_A(X, \mathcal{T})$, by

$$\mathcal{C}_A(X, \mathcal{T}) := \{C \subset X : A \subset C \text{ and } C \text{ is closed}\},$$

  the **interior of A**, denoted $\operatorname{Int} A$, by

$$\operatorname{Int} A = \bigcup_{U \in \mathcal{U}_A(X, \mathcal{T})} U$$

  the **closure of A**, denoted $\operatorname{cl} A$, by

$$\operatorname{cl} A = \bigcap_{C \in \mathcal{C}_A(X, \mathcal{T})} C$$

Note 3.1.0.16. When the context is clear, we write $\mathcal{U}_A$ and $\mathcal{C}_A$ in place of $\mathcal{U}_A(X, \mathcal{T})$ and $\mathcal{C}_A(X, \mathcal{T})$. For intuition, $\operatorname{Int} A$ is the largest open subset of A and $\operatorname{cl} A$ is the smallest closed superset of A.

Definition 3.1.0.17. Let X be a topological space and $A \subset X$. Then

1. A is open iff $A = \operatorname{Int} A$
2. A is closed iff $A = \operatorname{cl} A$

Proof. Clear. □

Exercise 3.1.0.18. Let X be a topological space and $A \subset X$. Then $(\operatorname{Int} A)^c = \operatorname{cl} A^c$.

Proof. Define $\mathcal{U}_A = \{U \subset X : U \subset A \text{ and } U \text{ is open}\}$ and $\mathcal{C}_{A^c} = \{C \subset X : A^c \subset C \text{ and } C \text{ is closed}\}$ as in Definition 3.1.0.15. We note that

1. for each $U \subset X$,

$$\begin{aligned} U \in \mathcal{U}_A &\iff U \subset A \text{ and } U \text{ is open} \\ &\iff A^c \subset U^c \text{ and } U^c \text{ is closed} \\ &\iff U^c \in \mathcal{C}_{A^c} \end{aligned}$$

2.

$$\begin{aligned} C \in \mathcal{C}_{A^c} &\iff A^c \subset C \text{ and } C \text{ is closed} \\ &\iff C^c \subset A \text{ and } C^c \text{ is open} \\ &\iff C^c \in \mathcal{U}_A \end{aligned}$$

Let $C \in \{U^c : U \in \mathcal{U}_A\}$. Then there exists $U \in \mathcal{U}_A$ such that $C = U^c$. By (1),

$$\begin{aligned} C &= U^c \\ &\in \mathcal{C}_{A^c} \end{aligned}$$

Since $C \in \{U^c : U \in \mathcal{U}_A\}$ is arbitrary, $\{U^c : U \in \mathcal{U}_A\} \subset \mathcal{C}_{A^c}$.
Conversely, let $C \in \mathcal{C}_{A^c}$. Then (2) implies that $C^c \in \mathcal{U}_A$ and therefore

$$\begin{aligned} C &= (C^c)^c \\ &\in \{U^c : U \in \mathcal{U}_A\} \end{aligned}$$

Since $C \in \mathcal{C}_{A^c}$ is arbitrary, $\mathcal{C}_{A^c} \subset \{U^c : U \in \mathcal{U}_A\}$. Hence $\mathcal{C}_{A^c} = \{U^c : U \in \mathcal{U}_A\}$ and therefore

$$\begin{aligned} (\operatorname{Int} A)^c &= \left(\bigcup_{U \in \mathcal{U}_A} U \right)^c \\ &= \bigcap_{U \in \mathcal{U}_A} U^c \\ &= \bigcap_{C \in \mathcal{C}_{A^c}} C \\ &= \operatorname{cl} A^c \end{aligned}$$

□

Exercise 3.1.0.19. Let X be a topological space and $A \subset X$. Then $(\operatorname{cl} A)^c = \operatorname{Int} A^c$.

Proof. Define $B = A^c$. The previous exercise implies that $(\operatorname{Int} B)^c = \operatorname{cl} B^c$. Therefore

$$\begin{aligned} \operatorname{Int} A^c &= \operatorname{Int} B \\ &= (\operatorname{cl} B^c)^c \\ &= (\operatorname{cl} A)^c \end{aligned}$$

□

Exercise 3.1.0.20. Let X be a topological space and $S, N \subset X$. Then $N \in \mathcal{N}(S)$ iff $S \subset \operatorname{Int} N$.

Proof. Suppose that $N \in \mathcal{N}(S)$. By definition, there exists $U \subset X$ such that U is open and $S \subset U \subset N$. Since U is open and $U \subset N$, Definition 3.1.0.15 implies that

$$\begin{aligned} S &\subset U \\ &\subset \operatorname{Int} N \end{aligned}$$

Conversely, suppose that $S \subset \operatorname{Int} N$. Since $\operatorname{Int} N \subset N$, we have that $S \subset \operatorname{Int} N \subset N$. So $N \in \mathcal{N}(S)$. □

Exercise 3.1.0.21. Let X be a topological space and $A \subset X$. Then A is open iff for each $x \in A$, there exists $U \in \mathcal{N}(x)$ such that U is open and $U \subset A$.

Proof. Suppose that A is open. Let $x \in A$. Then $A \in \mathcal{N}(x)$, A is open and $A \subset A$. Conversely, suppose that or each $x \in A$, there exists $U_x \in \mathcal{N}(x)$ such that U is open and $U_x \subset A$. Then

$$A = \bigcup_{x \in A} U_x$$

is open. □

Exercise 3.1.0.22. Let X be a topological space, $A \subset X$ and $x \in X$. Then $x \in \operatorname{cl} A$ iff for each $U \in \mathcal{N}(x)$, U is open implies that $A \cap U \neq \varnothing$.

Proof.

- $(\implies)$
  Suppose that $x \in \operatorname{cl} A$. Let $U \in \mathcal{N}(x)$. Suppose that U is open. For the sake of contradiction, suppose that $A \cap U = \varnothing$. Then $A \subset U^c$. Since U^c is closed, we have that

$$\begin{aligned} x &\in \operatorname{cl} A \\ &\subset U^c \end{aligned}$$

  This is a contradiction since $x \in U$. Hence $A \cap U \neq \varnothing$.

- $(\impliedby)$
  Suppose that for each $U \in \mathcal{N}(x)$, U is open implies that $A \cap U \neq \varnothing$. For the sake of contradiction, suppose that $x \notin \operatorname{cl} A$. By definition of closure, there exists $C \subset X$ such that C is closed, $A \subset X$ and $x \notin C$. Therefore C^c is open and $x \in C^c$. Thus $C^c \in \mathcal{N}(x)$. By assumption, $A \cap C^c \neq \varnothing$. This is a contradiction since $A \subset C$. So $x \in \operatorname{cl} A$.

□

Definition 3.1.0.23. Let X be a topological space, $A \subset X$ and $x \in X$. Then x is said to be a **limit point of** A if for each $U \in \mathcal{N}(x)$,

$$A \cap (U \setminus \{x\}) \neq \varnothing$$

We define $A' = \{x \in A : x \text{ is a limit point of } A\}$.

Exercise 3.1.0.24. Let X be a topological space and $A \subset X$. Then $\operatorname{cl} A = A \cup A'$.

Proof. Let $x \in A'$. For the sake of contradiction, suppose that $x \notin \operatorname{cl} A$. By definition of closure, there exists $C \subset X$ such thath C is closed, $A \subset C$ and $x \notin C$. Hence $x \in C^c \subset A^c$. Since C^c is open, $x \in \operatorname{Int} A^c$. Since $x \in A'$ and $\operatorname{Int} A^c \in \mathcal{N}(x)$, $[\operatorname{Int} A^c \setminus \{x\}] \cap A \neq \varnothing$. This is a contradiction since $\operatorname{Int} A^c \setminus \{x\} \subset A^c$. So $x \in \operatorname{cl} A$ and $A' \subset \operatorname{cl} A$. Since $A \subset \operatorname{cl} A$, we have that $A \cup A' \subset \operatorname{cl} A$.
Conversely, let $x \in \operatorname{cl} A$. For the sake of contradiction, suppose that $x \notin A \cup A'$. Then $x \in A^c \cap (A')^c$. Since $x \in (A')^c$, there exists $U \in \mathcal{N}(x)$ such that $(U \setminus \{x\}) \cap A = \varnothing$. Hence $U \setminus \{x\} \subset A^c$. Since $x \in A^c$,

$$\begin{aligned}\operatorname{Int} U &\subset U \\ &= (U \setminus \{x\}) \cup \{x\} \\ &\subset A^c\end{aligned}$$

Which implies that $A \subset (\operatorname{Int} U)^c$. Since $(\operatorname{Int} U)^c$ is closed,

$$\begin{aligned}x &\in \operatorname{cl} A \\ &\subset (\operatorname{Int} U)^c\end{aligned}$$

which is a contradiction since $x \in \operatorname{Int} U$. So $x \in A \cup A'$. Since $x \in \operatorname{cl} A$ is arbitrary, $\operatorname{cl} A \subset A \cup A'$. Therefore $\operatorname{cl} A = A \cup A'$. □

Definition 3.1.0.25. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then A is said to be **dense** in $(X, \mathcal{T})$ if $\operatorname{cl} A = X$.

Exercise 3.1.0.26. Let X be a topological space and $A \subset X$. Then $A = \varnothing$ iff $\operatorname{cl} A = \varnothing$.

Proof. Suppose $A = \varnothing$. Since A is closed,

$$\begin{aligned}\operatorname{cl} A &= A \\ &= \varnothing\end{aligned}$$

Conversely, suppose that $\operatorname{cl} A = \varnothing$. Since $A \subset \operatorname{cl} A$, $A = \varnothing$. □

Exercise 3.1.0.27. Let X be a topological space and $A \subset X$. Then A is dense in X iff for each $U \subset X$, U is open and $U \neq \varnothing$ implies that $A \cap U \neq \varnothing$.

Proof. Suppose that A is dense in X. Let $U \subset X$. Suppose that U is open. For the sake of contradiction, suppose that $A \cap U = \varnothing$. Then $U \subset A^c$. Thus $A \subset U^c$. Since U^c is closed, we have that

$$\begin{aligned}X &= \operatorname{cl} A \\ &\subset U^c\end{aligned}$$

Therefore, $X = U^c$ and hence $U = \varnothing$. This is a contradiction. So for each $U \subset X$, U is open and $U \neq \varnothing$ implies that $A \cap U \neq \varnothing$.
Conversely, suppose that for each $U \subset X$, if U is open and $U \neq \varnothing$, then $A \cap U \neq \varnothing$. Set $U = (\operatorname{cl} A)^c$. Then U is open. For the sake of contradiction, suppose that $U \neq \varnothing$. By assumption there exists $x \in X$ such that

$$\begin{aligned}x &\in A \cap U \\ &= A \cap (\operatorname{cl} A)^c \\ &\subset A \cap A^c \\ &= \varnothing\end{aligned}$$

which is a contradiction. Hence $U = \varnothing$. Then

$$\begin{aligned}X &= U^c \\ &= \operatorname{cl} A\end{aligned}$$

so that A is dense in X. □

Definition 3.1.0.28. Let X be a topological space and $A \subset X$. Then A is said to be **nowhere dense** in X if $\operatorname{Int}\operatorname{cl} A = \varnothing$.

Exercise 3.1.0.29. Let X be a topological space and $A \subset X$. If A is nowhere dense in X, then $\operatorname{cl} A$ is nowhere dense.

Proof. Suppose that A is nowhere dense in X. Then

$$\begin{aligned}\operatorname{Int}\operatorname{cl}\operatorname{cl} A &= \operatorname{Int}\operatorname{cl} A \\ &= \varnothing\end{aligned}$$

Hence $\operatorname{cl} A$ is nowhere dense. □

Exercise 3.1.0.30. Let X be a topological space and $A \subset X$. If A is nowhere dense in X, then A^c is dense.

Proof. Suppose that A is nowhere dense in X. Let $U \subset X$. Suppose that U is open and nonempty. For the sake of contradiction, suppose that $A^c \cap U = \varnothing$. Then

$$\begin{aligned}U &\subset (A^c)^c \\ &= A \\ &\subset \operatorname{cl} A\end{aligned}$$

Since U is open, we have that

$$\begin{aligned}U &\subset \operatorname{Int}\operatorname{cl} A \\ &= \varnothing\end{aligned}$$

Therefore, $U = \varnothing$. This is a contradiction since U is nonempty. Hence $A^c \cap U \neq \varnothing$. Since U is arbitrary open nonempty subset of X, we have that for each $U \subset X$, if U is open and nonempty, then $A^c \cap U \neq \varnothing$. Thus A^c is dense.

□

3.2 Continuous Maps

Definition 3.2.0.1. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then f is said to be **continuous** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$.

Definition 3.2.0.2. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces, $f : X \to Y$ and $x \in X$. Then f is said to be **continuous at** x if for each $V \in \mathcal{N}(f(x))$, there exists $U \in \mathcal{N}(x)$ such that $f(U) \subset V$.

Exercise 3.2.0.3. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces, $f : X \to Y$ and $x \in X$. Then f is continuous at x iff for each $V \in \mathcal{N}(f(x))$, $f^{-1}(V) \in \mathcal{N}(x)$.
Hint: for $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(f(x))$, consider $f^{-1}(f(U))$ and $f(f^{-1}(V))$

Proof. Suppose that f is continuous at x. Let $V \in \mathcal{N}(f(x))$. Then there exists $U \in \mathcal{N}(x)$ such that $f(U) \subset V$. Thus

$$\begin{aligned} x &\in \operatorname{Int} U \\ &\subset U \\ &\subset f^{-1}(f(U)) \\ &\subset f^{-1}(V) \end{aligned}$$

So $f^{-1}(V) \in \mathcal{N}(x)$.
Conversely, suppose that for each $V \in \mathcal{N}(f(x))$, $f^{-1}(V) \in \mathcal{N}(x)$. Let $V \in \mathcal{N}(f(x))$. Hence $f^{-1}(V) \in \mathcal{N}(x)$. Set $U = f^{-1}(V)$. Then

$$\begin{aligned} f(U) &= f(f^{-1}(V)) \\ &\subset V \end{aligned}$$

Thus f is continuous at x. □

Exercise 3.2.0.4. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then f is continuous iff for each $x \in X$, f is continuous at x.

Proof. Suppose that f is continuous. Let $x \in X$. Let $V \in \mathcal{N}(f(x))$. Then $\operatorname{Int} V \in \mathcal{B}$ and $f(x) \in \operatorname{Int} V$. Set $U = f^{-1}(\operatorname{Int} V)$. By continuity, $U \in \mathcal{A}$ and by construction, $x \in U$. Hence $U \in \mathcal{N}(x)$. Then

$$\begin{aligned} f(U) &= f(f^{-1}(\operatorname{Int} V)) \\ &\subset \operatorname{Int} V \\ &\subset V \end{aligned}$$

So f is continuous at x.
Conversely, suppose that for each $x \in X$, f is continuous at x. Let $B \in \mathcal{B}$. Let $x \in f^{-1}(B)$. Then $B \in \mathcal{N}(f(x))$. Continuity at x implies that $f^{-1}(B) \in \mathcal{N}(x)$. Then $x \in \operatorname{Int}(f^{-1}(B))$. Since $x \in f^{-1}(B)$ is arbitrary, $f^{-1}(B) \subset \operatorname{Int}(f^{-1}(B))$. Hence $f^{-1}(B) = \operatorname{Int}(f^{-1}(B))$ which implies that $f^{-1}(B) \in \mathcal{A}$. So f is continuous. □

Definition 3.2.0.5. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. We define the

1. **push-forward of** $\mathcal{A}$, denoted $f_*\mathcal{A}$, by

$$f_*\mathcal{A} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$$

2. **pull-back of** $\mathcal{B}$, denoted $f^*\mathcal{B}$, by

$$f^*\mathcal{B} = \{f^{-1}(B) : B \in \mathcal{B}\}$$

Exercise 3.2.0.6. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then

1. $f_*\mathcal{A}$ is a topology on Y

2. $f^*\mathcal{B}$ is a topology on X

Proof.

1.
   - Since $f^{-1}(Y) = X \in \mathcal{A}$ and $f^{-1}(\varnothing) = \varnothing \in \mathcal{A}$, $Y, \varnothing \in f_*\mathcal{A}$.
   - Let $(U_\alpha)_{\alpha \in A} \subset f_*\mathcal{A}$. Then for each $\alpha \in A$, $f^{-1}(U_\alpha) \in \mathcal{A}$. This implies that

$$\begin{aligned} f^{-1}\bigg(\bigcup_{\alpha \in A} U_\alpha\bigg) &= \bigcup_{\alpha \in A} f^{-1}(U_\alpha) \\ &\in \mathcal{A} \end{aligned}$$

     Hence $\bigcup\limits_{\alpha \in A} U_\alpha \in f_*\mathcal{A}$.
   - Let $(U_j)_{j=1}^n \subset f_*\mathcal{A}$. Then for each $j \in 1, \dots, n$, $f^{-1}(U_j) \in \mathcal{A}$. This implies that

$$\begin{aligned} f^{-1}\bigg(\bigcap_{j=1}^{n} U_j\bigg) &= \bigcap_{j=1}^{n} f^{-1}(U_j) \\ &\in \mathcal{A} \end{aligned}$$

     Hence $\bigcap\limits_{j=1}^{n} U_j \in f_*\mathcal{A}$.

   So $f_*\mathcal{A}$ is a topology on Y.
2. Similar to (1).

$\square$

Exercise 3.2.0.7. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces, $f : X \to Y$ and $\mathcal{E} \subset \mathcal{P}(Y)$. Suppose that $\mathcal{B} = \tau(\mathcal{E})$. Then f is continuous iff for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$.

Proof. Suppose that f is continuous. Since $\mathcal{E} \subset \mathcal{B}$, clearly for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$.
Conversely, suppose that for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. Then $\mathcal{E} \subset f_*\mathcal{A}$. Since $f_*\mathcal{A}$ is a topology on Y, we have that $\mathcal{B} = \tau(\mathcal{E}) \subset f_*\mathcal{A}$. So f is continuous. $\square$

Definition 3.2.0.8. Let X be a set, $(Y_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ a collection of topological spaces and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y_\alpha^X$ (i.e. $\mathcal{F} = (f_\alpha)_{\alpha \in A}$ where for each $\alpha \in A$, $f_\alpha : X \to Y_\alpha$). We define the **initial topology on X generated by $\mathcal{F}$**, denoted $\tau_X(\mathcal{F})$, by

$$\tau_X(\mathcal{F}) = \tau_X(\{f_\alpha^{-1}(B) : B \in \mathcal{T}_\alpha \text{ and } \alpha \in A\})$$

Note 3.2.0.9. The initial topology topology generated by $\mathcal{F}$ is also called the **weak topology generated by $\mathcal{F}$** and if $\mathcal{F} = \{f\}$, then $\tau_X(\mathcal{F}) = f^*\mathcal{B}$.

Exercise 3.2.0.10. Let X be a set, $(Y_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ a collection of topological spaces and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y_\alpha^X$ (i.e. $\mathcal{F} = (f_\alpha)_{\alpha \in A}$ where for each $\alpha \in A$, $f_\alpha : X \to Y_\alpha$). Then for each $\mathcal{T} \subset \mathcal{P}(X)$ if $\mathcal{T}$ is a topology on X and for each $\alpha \in A$, f_α is $(\mathcal{T}, \mathcal{T}_\alpha)$-continuous, then $\tau_X(\mathcal{F}) \subset \mathcal{T}$.

Proof. Let $\mathcal{T} \subset \mathcal{P}(X)$. Suppose that $\mathcal{T}$ is a topology on X and for each $\alpha \in A$, f_α is $(\mathcal{T}, \mathcal{T}_\alpha)$-continuous. Set $\mathcal{V} := \{f_\alpha^{-1}(V) : V \in \mathcal{T}_\alpha \text{ and } \alpha \in A\ \}$. By definition, $\tau_X(\mathcal{F}) = \tau_X(\mathcal{V})$. Since for each $\alpha \in A$, f_α is $(\mathcal{T}, \mathcal{T}_\alpha)$-continuous, we have that for each $\alpha \in A$ and $V \in \mathcal{T}_\alpha$, $f_\alpha^{-1}(V) \in \mathcal{T}$. Hence $\mathcal{V} \subset \mathcal{T}$. Therefore

$$\begin{aligned} \tau_X(\mathcal{F}) &= \tau_X(\mathcal{V}) \\ &\subset \mathcal{T}. \end{aligned}$$

$\square$

Note 3.2.0.11. Essentially, $\tau_X(\mathcal{F})$ is the smallest topology on X such that for each $\alpha \in A$, $f_\alpha : X \to Y_\alpha$ is continuous.

Exercise 3.2.0.12. Let $(Y_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a a collection of topological spaces, X a set, $(Z, \mathcal{C})$ a topological space, $\mathcal{F} = (f_\alpha)_{\alpha\in A} \in \prod\limits_{\alpha \in A} Y_\alpha^X$ and $g : Z \to X$. Then g is $(\mathcal{C}, \tau_X(\mathcal{F}))$-continuous iff for each $\alpha \in A$, $f_\alpha \circ g$ is $(\mathcal{C}, \mathcal{T}_\alpha)$-continuous:

$$\begin{tikzcd}
Y_\alpha & X \arrow[l, "f_\alpha"'] \\
 & Z \arrow[u, "g"'] \arrow[ul, "g\circ f_\alpha"]
\end{tikzcd}$$

Proof. If g is $(\mathcal{C}, \tau_X(\mathcal{F}))$-continuous, then clearly for each $\alpha \in A$, $f_\alpha \circ g$ is $(\mathcal{C}, \mathcal{T}_\alpha)$- continuous.
Conversely, suppose that for each $\alpha \in A$, $f_\alpha \circ g$ is $(\mathcal{C}, \mathcal{T}_\alpha)$-continuous. Let $\alpha \in A$ and $V \in \mathcal{T}_\alpha$. Continuity implies that,

$$\begin{aligned} g^{-1}(f_\alpha^{-1}(V)) &= (f_\alpha \circ g)^{-1}(V) \\ &\in \mathcal{C} \end{aligned}$$

Since $\alpha \in A$ and $V \in \mathcal{T}_\alpha$ are arbitrary, we have that for each $\alpha \in A$ and $V \in \mathcal{T}_\alpha$, $g^{-1}(f_\alpha^{-1}(V)) \in \mathcal{C}$. Since $\tau_X(\mathcal{F}) = \tau(\{f_\alpha^{-1}(V) : \alpha \in A \text{ and } V \in \mathcal{T}_\alpha)$, the previous exercise implies that g is $(\mathcal{C}, \tau_X(\mathcal{F}))$-continuous. $\square$

Exercise 3.2.0.13. Let $(X, \mathcal{T})$ be a topological space. Set $\mathcal{F} = \mathrm{Hom}_{\mathbf{Top}}((X, \mathcal{T}), (X, \mathcal{T}))$. Then $\tau_X(\mathcal{F}) = \mathcal{T}$.

Proof. Set $\mathcal{E} = \{f^{-1}(V) : V \in \mathcal{B} \text{ and } f \in \mathcal{F}\}$. Since for each $f \in \mathcal{F}$, f is $(\mathcal{T}, \mathcal{T})$-continuous, $\mathcal{E} \subset \mathcal{T}$.
Conversely, since $\mathrm{id}_X \in \mathcal{F}$, we have that for each $U \in \mathcal{T}$,

$$\begin{aligned} U &= \mathrm{id}_X^{-1}(U) \\ &\in \mathcal{E} \end{aligned}$$

So that $\mathcal{T} \subset \mathcal{E}$. Hence $\mathcal{E} = \mathcal{T}$ and

$$\begin{aligned} \tau_X(\mathcal{F}) &= \tau_X(\mathcal{E}) \\ &= \tau_X(\mathcal{T}) \\ &= \mathcal{T} \end{aligned}$$

$\square$

Definition 3.2.0.14. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a a collection of topological spaces, Y a set and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y^{X^\alpha}$ (i.e. $\mathcal{F} = (f_\alpha)_{\alpha \in A}$ where for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y$). We define the **final topology on** Y **generated by** $\mathcal{F}$, denoted $\tau_Y(\mathcal{F})$, by

$$\tau_Y(\mathcal{F}) := \tau_Y(\{V \subset Y : \text{ for each } \alpha \in A,\ f_\alpha^{-1}(V) \in \mathcal{A}_\alpha\})$$

Note 3.2.0.15. If $\mathcal{F} = \{f\}$, then $\tau_Y(\mathcal{F}) = f_*\mathcal{A}$.

Exercise 3.2.0.16. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a a collection of topological spaces, Y a set and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y^{X^\alpha}$ (i.e. $\mathcal{F} = (f_\alpha)_{\alpha \in A}$ where for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y$). Then for each $\mathcal{T} \subset \mathcal{P}(Y)$ if $\mathcal{T}$ is a topology on Y and for each $\alpha \in A$, f_α is $(\mathcal{T}_\alpha, \mathcal{T})$-continuous, then $\mathcal{T} \subset \tau_Y(\mathcal{F})$.

Proof. Let $\mathcal{T} \subset \mathcal{P}(Y)$. Suppose that $\mathcal{T}$ is a topology on Y and for each $\alpha \in A$, f_α is $(\mathcal{T}_\alpha, \mathcal{T})$-continuous. Set $\mathcal{V} := \{V \subset Y : \text{ for each } \alpha \in A,\ f_\alpha^{-1}(V) \in \mathcal{T}_\alpha\}$. By definition, $\tau_Y(\mathcal{F}) = \tau_Y(\mathcal{V})$. Let $V \in \mathcal{T}$. By assumption, for each $\alpha \in A$, f_α is $(\mathcal{T}_\alpha, \mathcal{T})$-measurable. Thus for each $\alpha \in A$, $f_\alpha^{-1}(V) \in \mathcal{T}_\alpha$. Therefore $V \in \mathcal{V}$. Since $V \in \mathcal{T}$ is arbitrary, we have that

$$\begin{aligned} \mathcal{T} &\subset \mathcal{V} \\ &\subset \tau_Y(\mathcal{V}) \\ &= \tau_Y(\mathcal{F}). \end{aligned}$$

$\square$

Note 3.2.0.17. Essentially, $\tau_X(\mathcal{F})$ is the largest topology on X such that for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y$ is continuous.

Exercise 3.2.0.18. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a a collection of topological spaces, Y a set, $(Z, \mathcal{C})$ a topological space, $\mathcal{F} = (f_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} Y^{X_\alpha}$ and $g : Y \to Z$. Then g is $(\tau_Y(\mathcal{F}), \mathcal{C})$-continuous iff for each $\alpha \in A$, $g \circ f_\alpha$ is $(\mathcal{T}_\alpha, \mathcal{C})$-continuous:

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[dr, "g \circ f_\alpha"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Proof. If g is $(\tau_Y(\mathcal{F}), \mathcal{C})$-continuous, then clearly for each $\alpha \in A$, $g \circ f_\alpha$ is $(\mathcal{T}_\alpha, \mathcal{C})$-continuous.
Conversely, suppose that for each $\alpha \in A$, $g \circ f_\alpha$ is $(\mathcal{T}_\alpha, \mathcal{C})$-continuous. Let $\alpha \in A$ and $V \in \mathcal{C}$. Continuity implies that

$$\begin{aligned} f_\alpha^{-1}(g^{-1}(V)) &= (g \circ f_\alpha)^{-1}(V) \\ &\in \mathcal{T}_\alpha \end{aligned}$$

Since $\alpha \in A$ is arbitrary, we have that by definition, $g^{-1}(V) \in \tau_Y(\mathcal{F})$. Since $V \in \mathcal{C}$ is arbitrary, g is $(\tau_Y(\mathcal{F}), \mathcal{C})$-continuous. □

Definition 3.2.0.19. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then

1. f is said to be **open** if for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$.

2. f is said to be **closed** if for each $A \subset X$, if $A^c \in \mathcal{A}$, then $f(A)^c \in \mathcal{B}$.

Exercise 3.2.0.20. Let $(X, \mathcal{T}), (Y, \mathcal{S})$ be topological spaces, $\mathcal{B} \subset \mathcal{T}$ a basis for $\mathcal{T}$ and $f : X \to Y$. Then f is open iff for each $U \in \mathcal{B}$, $f(U) \in \mathcal{S}$.
Hint: $f\left(\bigcup\limits_{\alpha \in A} A_\alpha \right) = \bigcup\limits_{\alpha \in A} f(A_\alpha)$.

Proof. Clearly if f is open, then for each $U \in \mathcal{B}$, $f(U) \in \mathcal{S}$.
Conversely, suppose that for each $U \in \mathcal{B}$, $f(U) \in \mathcal{S}$. Let $U \in \mathcal{T}$. Then there exists $(U_\alpha)_{\alpha \in A} \subset \mathcal{B}$ such that $U = \bigcup\limits_{\alpha \in A} U_\alpha$. Then

$$\begin{aligned} f(U) &= \bigcup_{\alpha \in A} f(U_\alpha) \\ &\in \mathcal{S} \end{aligned}$$

Since $U \in \mathcal{T}$ is arbitrary, f is open. □

Exercise 3.2.0.21. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $f : X \to Y$ and $\mathcal{B}_X$ a basis for $\mathcal{T}_X$. Suppose that f is surjective, continuous and open. Then $\{f(A) : A \in \mathcal{B}_X\}$ is a basis for $\mathcal{T}_Y$ is a basis for $\mathcal{T}_Y$.

Proof. Set $\mathcal{B}_Y = \{f(A) : A \in A \in \mathcal{T}_X\}$. Since f is open, $\mathcal{B}_Y \subset \mathcal{T}_Y$. Let $V \in \mathcal{T}_Y$. Set $U = f^{-1}(V)$. Since f is continuous, $U \in \mathcal{T}_X$. Since $\mathcal{B}_X$ is a basis for $\mathcal{T}_X$, there exist $\mathcal{B}'_X \subset \mathcal{B}_X$ such that $U = \bigcup\limits_{A \in \mathcal{B}'_X} A$. Define

$\mathcal{B}'_Y \subset \mathcal{B}_Y$ by $\mathcal{B}'_Y = \{f(A) : A \in \mathcal{B}'_X\}$. Since f is surjective, we have that $f(f^{-1}(V)) = f(V)$ and therefore

$$\begin{aligned} V &= f(f^{-1}(V)) \\ &= f(U) \\ &= f\bigg(\bigcup_{A \in \mathcal{B}'_X} B\bigg) \\ &= \bigcup_{A \in \mathcal{B}'_X} f(A) \\ &= \bigcup_{B \in \mathcal{B}'_Y} B \end{aligned}$$

Since $V \in \mathcal{T}_Y$ is arbitrary, we have that for each $V \in \mathcal{T}_Y$, there exists $\mathcal{B}'_Y \subset \mathcal{B}_Y$ such that $V = \bigcup_{B \in \mathcal{B}'_Y} B$. Thus $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$. $\square$

Exercise 3.2.0.22. Doob-Dynkin Lemma:
Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ and $(X_3, \mathcal{T}_3)$ be topological spaces and $f : X_1 \to X_2$ and $g : X_1 \to X_3$. Suppose that f is surjective and $\mathcal{T}_1$-$\mathcal{T}_2$ continuous and g is $\mathcal{T}_1$-$\mathcal{T}_3$ continuous and $(X_3, \mathcal{T}_3)$ is a T_1 space. Then g is $f^*\mathcal{T}_2$-$\mathcal{T}_3$ continuous iff there exists a unique $\phi : X_2 \to X_3$ such that ϕ is $\mathcal{T}_2$-$\mathcal{T}_3$ continuous and $g = \phi \circ f$.
Hint: For each $t \in X_3$, set $A_t = g^{-1}(\{t\}) \in \mathcal{F}_{(f^*\mathcal{T}_2)}$ and choose $B_t \in \mathcal{T}_2$ such that $A_t = f^{-1}(B_t)$. Set $\phi(y) = t$ for $y \in B_t \cap f(X_1)$ and $t \in g(X_1)$.

Proof. Suppose that there exists a unique $\phi : X_2 \to X_3$ such that ϕ is $\mathcal{T}_2$ - $\mathcal{T}_3$ measurable and $g = \phi \circ f$. Since f is $f^*\mathcal{T}_2$ - $\mathcal{T}_2$ continuous, we have that $g = \phi \circ f$ is $f^*\mathcal{T}_2$-$\mathcal{T}_3$ continuous.
Conversely, suppose that g is $f^*\mathcal{T}_2$-$\mathcal{T}_3$ continuous.

- **(Existence)**
  For each $t \in X_3$, set $A_t = g^{-1}(\{t\})$ Since $(X_3, \mathcal{T}_3)$ is a T_1 space, for each $t \in X_3$, $A_t \in \mathcal{F}_{f^*\mathcal{T}_2}$ and thus, there exists $B_t \in \mathcal{F}_{\mathcal{T}_2}$ such that $A_t = f^{-1}(B_t)$.
  Note that
  - for each $t \in g(X_1)$, there exists $x \in A_t$ such that $g(x) = t$. Hence $f(x) \in B_t$.
  - for $t_1, t_2 \in g(X_1)$, $t_1 \neq t_2$ implies that
  $$\begin{aligned} f^{-1}(B_{t_1} \cap B_{t_2}) &= A_{t_1} \cap A_{t_2} \\ &= g^{-1}(\{t_1\} \cap \{t_2\}) \\ &= \varnothing \end{aligned}$$
    and since f is surjective,
  $$\begin{aligned} B_{t_1} \cap B_{t_2} &= f(f^{-1}(B_{t_1} \cap B_{t_2})) \\ &= f(\varnothing) \\ &= \varnothing \end{aligned}$$
  - we have that
  $$\begin{aligned} f^{-1}\bigg(\bigcup_{t \in g(X_1)} B_t\bigg) &= \bigcup_{t \in g(X_1)} A_t \\ &= \bigcup_{t \in g(X_1)} g^{-1}(\{t\}) \\ &= g^{-1}(g(X_1)) \\ &= X_1 \end{aligned}$$

Since f is surjective, we have that

$$\begin{aligned} X_2 &= f(X_1) \\ &= f\bigg(f^{-1}\bigg(\bigcup_{t \in g(X_1)} B_t\bigg)\bigg) \\ &= \bigcup_{t \in g(X_1)} B_t \end{aligned}$$

Therefore,

- for each $t \in g(X_1)$, $B_t \neq \varnothing$
- $(A_t)_{t \in g(X_1)}$ is a partion of X_1
- $(B_t)_{t \in g(X_1)}$ is a partition of X_2

Define $\phi : X_2 \to X_3$ by $\phi(y) = t$ for $t \in g(X_1)$ and $y \in B_t$. Then the previous observations imply that ϕ is well defined and $\phi(X_2) = g(X_1)$. Since for each $t \in g(X_1)$ and $x \in A_t$, $f(x) \in B_t$ and $g(x) = t$, we have that $\phi \circ f(x) = t = g(x)$. So $\phi \circ f = g$.

To show that ϕ is continuous, let $C \in \mathcal{T}_3$. Choose $B \in \mathcal{T}_2$ such that $g^{-1}(C) = f^{-1}(B)$. Let $y \in \phi^{-1}(C) \subset X_2$. Set $t = \phi(y) \in C$ and choose $x \in X_1$ such that $y = f(x)$. Since

$$\begin{aligned} g(x) &= \phi \circ f(x) \\ &= \phi(y) \\ &= t \\ &\in C \end{aligned}$$

$x \in g^{-1}(C) = f^{-1}(B)$. Therefore, $y = f(x) \in B$. So $\phi^{-1}(C) \subset B$.
Let $y \in B$. Choose $x \in X_1$ such that $f(x) = y$. Then $x \in f^{-1}(B) = g^{-1}(C)$. So

$$\begin{aligned} \phi(y) &= \phi \circ f(x) \\ &= g(x) \\ &\in C \end{aligned}$$

and $y \in \phi^{-1}(C)$. So $B \subset \phi^{-1}(C)$. Hence $\phi^{-1}(C) = B \in \mathcal{T}_2$ and ϕ is $\mathcal{T}_2$ - $\mathcal{T}_3$ continuous.

- **(Uniqueness)**
  Let $\psi : X_2 \to X_3$. Suppose that ψ is $\mathcal{T}_2$-$\mathcal{T}_3$ continuous and $g = \psi \circ f$. Let $y \in X_2$. Then there exists $x \in X_1$ such that $y = f(x)$. Then

$$\begin{aligned} \psi(y) &= \psi \circ f(x) \\ &= g(x) \\ &= \phi \circ f(x) \\ &= \phi(y) \end{aligned}$$

  So $\psi = \phi$.

□

Exercise 3.2.0.23. Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ and $(X_3, \mathcal{T}_3)$ be topological spaces and $f : X_1 \to X_2$ and $g : X_1 \to X_3$. Suppose that f is $\mathcal{T}_1$-$\mathcal{T}_2$ continuous and g is $\mathcal{T}_1$-$\mathcal{T}_3$ continuous and $(X_3, \mathcal{T}_3)$ is a T_1 space. Then g is $f^*\mathcal{T}_2$-$\mathcal{T}_3$ continuous iff there exists a unique $\phi : f(X_1) \to X_3$ such that ϕ is $\mathcal{T}_2 \cap f(X_1)$ - $\mathcal{T}_3$ continuous and $g = \phi \circ f$.

Proof. A previous exercise implies that $f : X_1 \to f(X_1)$ is $\mathcal{T}_1$ - $\mathcal{T}_2 \cap f(X_1)$ continuous. Now apply the previous exercise. $\square$

Definition 3.2.0.24. Let X be a topological space, $x_0 \in X$ and $f : X \to \mathbb{R}$. We define the **limit inferior of f as $x \to x_0$ (resp. limit inferior of f as $x \to x_0$)**, denoted $\liminf\limits_{x \to x_0} f(x)$ (resp. $\liminf\limits_{x \to x_0} f(x)$), by

$$\liminf_{x \to x_0} f(x) = \sup_{V \in \mathcal{N}(x_0)} \inf_{x \in V \setminus \{x_0\}} f(x)$$

resp.

$$\limsup_{x \to x_0} f(x) = \inf_{V \in \mathcal{N}(x_0)} \sup_{x \in V \setminus \{x_0\}} f(x)$$

Exercise 3.2.0.25. Let X be a topological space, $x_0 \in X$ and $f : X \to \mathbb{R}$. Then f is continuous at x_0 iff $\liminf\limits_{x \to x_0} f(x) = \limsup\limits_{x \to x_0} f(x) = f(x_0)$

Proof. Suppose that
FINISH!!! $\square$

3.3 Nets

3.3.1 Common Directed Sets

Note 3.3.1.1. We recall the definition of a directed set from Definition 1.1.1.1.

Definition 3.3.1.2. Let X be a set. Define the **reverse inclusion ordering** on $\mathcal{N}(x)$, denoted $\leq$, by $U \leq V$ iff $V \subset U$.

Exercise 3.3.1.3. Let X be a topological space and $x \in X$. Then $\mathcal{N}(x)$ ordered by reverse inclusion is a directed set.

Proof.

1. Clearly, for each $U \in \mathcal{N}(x), U \leq U$.
2. Let $U, V, W \in \mathcal{N}(x)$. Suppose that $U \leq V$ and $V \leq W$. Then $W \subset V \subset U$ which implies that $W \subset U$ and hence $U \leq W$.
3. Let $U, V \in \mathcal{N}(x)$. Set $W = U \cap V$. Then $W \in \mathcal{N}(x)$ and $U, V \leq W$.

So $\mathcal{N}(x)$ is a directed set. □

Definition 3.3.1.4. Let $(A, \leq)$ be a directed set and $\alpha_0 \in A$. We define $[\alpha_0, \infty) \subset A$, by $[\alpha_0, \infty) := \{\alpha \in A : \alpha \geq \alpha_0\}$ and $\leq_{[\alpha_0,\infty)} := \leq \cap([\alpha_0, \infty) \times [\alpha_0, \infty))$.

Exercise 3.3.1.5. Let $(A, \leq)$ be a directed set and $\alpha_0 \in A$. Then $([\alpha_0, \infty), \leq_{[\alpha_0,\infty)})$ is a directed set.

Proof. Set $B := [\alpha_0, \infty)$.

1. Let $\alpha \in B$. Since $B \subset A$, $\alpha \in A$. Since $(A, \leq)$ is a directed set, $\alpha \leq \alpha$. Since $\alpha \in B$, we have that $\alpha \leq_B \alpha$. Since $\alpha \in B$ is arbitrary, we have that for each $\alpha \in B$, $\alpha \leq_B \alpha$.
2. Let $\alpha, \beta, \gamma \in B$. Suppose that $\alpha \leq_B \beta$ and $\beta \leq \gamma$. Since $\leq_B \subset \leq$, we have that $\alpha \leq \beta$ and $\beta \leq \gamma$. Since $(A, \leq)$ is a directed set, $\alpha \leq \gamma$. Since $\alpha, \gamma \in B$, we have that $\alpha \leq_B \gamma$. Since $\alpha, \beta, \gamma \in B$ are arbitrary, we have that for each $\alpha, \beta, \gamma \in B$, $\alpha \leq_B \beta$ and $\beta \leq_B \gamma$ implies that $\alpha \leq_B \gamma$.
3. Let $\alpha, \beta \in B$. Since $B \subset A$, there exists $\gamma \in A$ such that $\alpha, \beta \leq \gamma$. Since $\alpha \in B$, $\alpha \geq \alpha_0$. Since $\gamma \geq \alpha$, we have that $\gamma \geq \alpha_0$. Hence $\gamma \in B$. Since $\alpha, \beta, \gamma \in B$, we have that $\alpha, \beta \leq_B \gamma$. Since $\alpha, \beta \in B$ are arbitrary, we have that for each $\alpha, \beta \in B$, there exists $\gamma \in B$ such that $\alpha, \beta \leq_B \gamma$.
4. Since $\alpha_0 \in B$, $B \neq \varnothing$.

So $(B, \leq_B)$ is a directed set. □

Definition 3.3.1.6. Let X be a metric space and $x_0 \in X$. Define the **reverse distance from x_0 ordering** on $X \setminus \{x_0\}$, denoted $\leq_{x_0}$, by $x \leq_{x_0} y$ iff $d(x, x_0) \geq d(y, x_0)$.

Exercise 3.3.1.7. Let X be a metric space and $x_0 \in X$. Then $(X \setminus \{x_0\}, \leq_{x_0})$ is a directed set.

Proof.

1. Let $x \in X \setminus \{x_0\}$. Since $d(x, x_0) \geq d(x, x_0)$, $x \leq_{x_0} x$.
2. Let $x, y, z \in X \setminus \{x_0\}$. Suppose that $x \leq_{x_0} y$ and $y \leq_{x_0} z$. Then $d(x, x_0) \geq d(y, x_0)$ and $d(y, x_0) \geq d(z, x_0)$. Hence $d(x, x_0) \geq d(z, x_0)$ so that $x \leq z$.
3. Let $x, y \in X \setminus \{x_0\}$. Set
$$\begin{aligned} z &= \operatorname*{arg\,min}_{a \in \{x,y\}} d(a, x_0) \\ &\in X \setminus \{x_0\} \end{aligned}$$
Then $x, y \leq_{x_0} z$.

□

Definition 3.3.1.8. Let $(A, \leq_A)$ and $(B, \leq_B)$ be directed sets. We define the **product directed set of** $(A, \leq_A)$ **and** $(B, \leq_B)$, denoted $(A \times B, \leq)$, by

$$(a_1, b_1) \leq (a_2, b_2) \text{ iff } a_1 \leq a_2 \text{ and } b_1 \leq b_2$$

Exercise 3.3.1.9. Let $(A, \leq_A)$ and $(B, \leq_B)$ be directed sets. Then the product directed set of $(A, \leq_A)$ and $(B, \leq_B)$ is a directed set.

Proof.

1. Let $(a, b) \in A \times B$. Then $a \leq_A a$ and $b \leq_B b$. So $(a, b) \leq (a, b)$.
2. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A \times B$. Suppose that $(a_1, b_1) \leq (a_2, b_2)$ and $(a_2, b_2) \leq (a_3, b_3)$. Then $a_1 \leq_A a_2$, $a_2 \leq_A a_3$, $b_1 \leq_B b_2$ and $b_2 \leq_B b_3$. Therefore $a_1 \leq_A a_3$ and $b_1 \leq_B b_3$. Hence $(a_1, b_1) \leq (a_3, b_3)$.
3. Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then there exist $a \in A$ and $b \in B$ such that $a_1, a_2 \leq_A a$ and $b_1, b_2 \leq_B b$. Hence $(a_1, b_1), (a_2, b_2) \leq (a, b)$.

So $(A \times B, \leq)$ is directed. □

3.3.2 Nets in Topological Spaces

maybe move the basic definition of net to the set theory section and also introduce sequences there

Definition 3.3.2.1. Let X be a topological space, $(x_\alpha)_{\alpha \in A} \subset X$ a net and $U \subset X$. Then $(x_\alpha)_{\alpha \in A}$ is said to be

- **eventually in** U if there exists $\beta \in A$ such that for each $\alpha \in A$ $\alpha \geq \beta$ implies that $x_\alpha \in U$
- **frequently in** U if for each $\alpha \in A$, there exists $\beta \in A$ such that $\beta \geq \alpha$ and $x_\beta \in U$

Exercise 3.3.2.2. Let X be a topological space, $(x_\alpha)_{\alpha \in A} \subset X$ a net and $U \subset X$. Then $(x_\alpha)_{\alpha \in A}$ is eventually in U iff there exists $\alpha_0 \in A$ such that $(x_{\alpha \in [\alpha_0, \infty)})_{\alpha \in A} \subset U$.

Proof.

- $(\implies)$:
Suppose that $(x_\alpha)_{\alpha \in A}$ is eventually in U. Then there exists $\alpha_0 \in A$ such that for each $\alpha \in A$, $\alpha \geq \alpha_0$ implies that $x_\alpha \in U$. Then $(x_\alpha)_{\alpha \in [\alpha_0, \infty)} \subset U$.
- $(\impliedby)$:
Suppose that there exists $\alpha_0 \in A$ such that $(x_\alpha)_{\alpha \in [\alpha_0, \infty)} \subset U$. Then for each $\alpha \in A$, $\alpha \geq \alpha_0$ implies that $x_\alpha \in U$. Hence $(x_\alpha)_{\alpha \in A}$ is eventually in U.

□

Exercise 3.3.2.3. (make exercise about the tail net and being frequently in U)

Proof. □

Definition 3.3.2.4. Let X be a topological space, $(x_\alpha)_{\alpha \in A} \subset X$ a net and $x \in X$. Then $(x_\alpha)_{\alpha \in A}$ is said to **converge to** x, denoted $x_\alpha \to x$, if for each $U \in \mathcal{N}(x)$, $(x_\alpha)_{\alpha \in A}$ is eventually in U.

Definition 3.3.2.5. Let X be a topological space and $(x_\alpha)_{\alpha \in A} \subset X$ a net. Then $(x_\alpha)_{\alpha \in A}$ is said to **converge** if there exists $x \in X$ such that $x_\alpha \to x$.

Exercise 3.3.2.6. Let X be a metric space and $x_0 \in X$. Set $A = X \setminus \{x_0\}$. Order A by reverse distance from x_0. Define $(x_\alpha)_{\alpha \in A} \subset X$ by $x_\alpha = \alpha$. Then $x_\alpha \to x_0$.

Proof. Let $U \in \mathcal{N}(x_0)$. Since $x_0 \in \operatorname{Int} U$, there exists $\delta > 0$ such that $B(x_0, \delta) \subset \operatorname{Int} U$. Choose $\beta \in B^*(x_0, \delta)$. Let $\alpha \in A$. Suppose that $\alpha \geq \beta$. Then $d(\alpha, x_0) \leq d(\beta, x_0) < \delta$. Hence

$$\begin{aligned} x_\alpha &= \alpha \\ &\in B^*(x_0, \delta) \\ &\subset U \end{aligned}$$

Since $U \in \mathcal{N}(x_0)$ is arbitrary, $x_\alpha \to x_0$ $\square$

Exercise 3.3.2.7. Let X be a topological space, $S \subset X$ and $x \in X$. Then $x \in S'$ iff there exists a net $(x_\alpha)_{\alpha \in A} \subset S \setminus \{x\}$ such that $x_\alpha \to x$.

Proof. Suppose that $x \in S'$. Set $A = \mathcal{N}(x)$, ordered by reverse inclusion. Since $x \in S'$, for each $\alpha \in A$, there exists $x_\alpha \in (\alpha \setminus \{x\}) \cap S$. Then $(x_\alpha)_{\alpha \in A} \subset S \setminus \{x\}$. Let $V \in \mathcal{N}(x)$. Choose $\beta = V$. Let $\alpha \in \mathcal{N}(x)$. Suppose that $\alpha \geq \beta$. Then

$$\begin{aligned} x_\alpha &\in (\alpha \setminus \{x\}) \cap S \\ &\subset \alpha \\ &\subset \beta \\ &= V \end{aligned}$$

So $(x_\alpha)_{\alpha \in \mathcal{N}(x)}$ is eventually in V. Since $V \in \mathcal{N}(x)$ is arbitrary, $x_\alpha \to x$.
Conversely, suppose that there exists a net $(x_\alpha)_{\alpha \in A} \subset S \setminus \{x\}$ such that $x_\alpha \to x$. Let $U \in \mathcal{N}(x)$. Since $(x_\alpha)_{\alpha \in A}$ is eventually in U, there exits $\beta \in A$ such that $x_\beta \in U$. Then $x_\beta \in (U \setminus \{x\}) \cap S$ and $(U \setminus \{x\}) \cap S \neq \varnothing$. Since $U \in \mathcal{N}(x)$ is arbitrary, $x \in S'$. $\square$

Exercise 3.3.2.8. Let X be a topological space, $S \subset X$ and $x \in X$. Then $x \in \operatorname{cl} S$ iff there exists a net $(x_\alpha)_{\alpha \in A} \subset S$ such that $x_\alpha \to x$.

Proof. Suppose that $x \in \operatorname{cl} S$. Since $\operatorname{cl} S = S \cup S'$, $x \in S$ or $x \in S'$. If $x \in S$, define $(x_n)_{n \in \mathbb{N}} \subset S$ by $x_n = x$. Then $x_n \to x$. If $x \in S'$, the previous exercise implies that there exists a net $(x_\alpha)_{\alpha \in A} \subset S \setminus \{x\} \subset S$ such that $x_\alpha \to x$. $\square$

Definition 3.3.2.9. Let X be a topological space, $E \subset X$.

- Let $x \in X$. Then x is said to be a **boundary point of** E if for each $U \in \mathcal{N}(x)$, $U \cap E \neq \varnothing$ and $U \cap E^c \neq \varnothing$.
- We define the **boundary of** E, denoted ∂E, by

$$\partial E = \{x \in X : x \text{ is a boundary point } E\}$$

Exercise 3.3.2.10. Let X be a topological space and $E \subset X$. Then

1. $\partial E = \operatorname{cl} E \cap \operatorname{cl} E^c$
2. $\partial E = \operatorname{cl} E \setminus \operatorname{Int} E$

Proof.

1. Let $x \in \partial E$. Then for each $U \in \mathcal{N}(x)$, $U \cap E \neq \varnothing$ and $U \cap E^c \neq \varnothing$. The axiom of choice implies that there exist nets $(a_U)_{U \in \mathcal{N}(x)} \subset E$ $(b_U)_{U \in \mathcal{N}(x)} \subset E^c$ such that for each $U \in \mathcal{N}(x)$, $a_U, b_U \in U$. Then $a_U, b_U \to x$. Hence $x \in (\operatorname{cl} E) \cap (\operatorname{cl} E^c)$. Since $x \in \partial E$ is arbitrary, we have that $\partial E \subset (\operatorname{cl} E) \cap (\operatorname{cl} E^c)$.
   Conversely, let $x \in (\operatorname{cl} E) \cap (\operatorname{cl} E^c)$. Then there exists $(a_\alpha)_{\alpha \in A} \subset E$ and $(b_\beta)_{\beta \in B} \subset E^c$ such that $a_\alpha \to x$ and $b_\beta \to x$. Let $U \in \mathcal{N}(x)$. Then there exists $\alpha_0 \in A$ and $\beta_0 \in B$ such that for each $\alpha \in A$ and $\beta \in B$, $\alpha \geq \alpha_0$ implies that $a_\alpha \in A$ and $\beta \geq \beta_0$ implies that $b_\beta \in U$. In particular, $a_{\alpha_0} \in U \cap E$ and $b_{\beta_0} \in U \cap E^c$. Hence $U \cap E \neq \varnothing$ and $U \cap E^c \neq \varnothing$. Since $U \in \mathcal{N}(x)$ is arbitrary, we have that for each $U \in \mathcal{N}(x)$, $U \cap E \neq \varnothing$ and $U \cap E^c \neq \varnothing$. Thus $x \in \partial E$. Since $x \in (\operatorname{cl} E) \cap (\operatorname{cl} E^c)$ is arbitrary, $(\operatorname{cl} E) \cap (\operatorname{cl} E^c) \subset \partial E$.
   Therefore $\partial E = (\operatorname{cl} E) \cap (\operatorname{cl} E^c)$.

2. An exercise in introduction section and part (1) implies that

$$\begin{aligned}\partial E &= (\operatorname{cl} E) \cap (\operatorname{cl} E^c) \\ &= (\operatorname{cl} E) \cap (\operatorname{Int} E)^c \\ &= (\operatorname{cl} E) \setminus \operatorname{Int} E\end{aligned}$$

□

Exercise 3.3.2.11. Topology in Terms of Nets:
Let X be a topological space and $U \subset X$. Then U is open iff for each net $(x_\alpha)_{\alpha \in A} \subset X$ and $x \in U$, $x_\alpha \to x$ implies that $(x_\alpha)_{\alpha \in A}$ is eventually in U.

Proof. Suppose that U is open. Let $(x_\alpha)_{\alpha \in A} \subset X$ be a net and $x \in U$. Suppose that $x_\alpha \to x$. Since $U \in \mathcal{N}(x)$, $(x_\alpha)_{\alpha \in A}$ is eventually in U.
Conversely, suppose that for each net $(x_\alpha)_{\alpha \in A} \subset X$ and $x \in U$, $x_\alpha \to x$ implies that $(x_\alpha)_{\alpha \in A}$ is eventually in U. For the sake of contradiction, suppose that U^c is not closed. Then there exists $x \in \operatorname{cl} U^c$ such that $x \notin U^c$. Thus $x \in U$. Since $x \in \operatorname{cl} U^c$, a previous exercise implies that there exists a net $(x_\alpha)_{\alpha \in A} \subset U^c$ such that $x_\alpha \to x$. By assumption, $(x_\alpha)_{\alpha \in A}$ is eventually in U. This is a contradiction since $(x_\alpha)_{\alpha \in A} \subset U^c$. Hence U^c is closed and hence U is open. □

Exercise 3.3.2.12. Let X be a topological space, $U \in \mathcal{T}$ and $E \subset X$. If $U \cap \operatorname{cl} E \neq \varnothing$, then $U \cap E \neq \varnothing$.

Proof. Suppose that $U \cap \operatorname{cl} E \neq \varnothing$. Then there exists $x \in X$ such that $x \in U \cap \operatorname{cl} E$. Since $x \in \operatorname{cl} E$, there exists a net $(x_\alpha)_{\alpha \in A} \subset E$ such that $x_\alpha \to x$. Since $U \in \mathcal{N}(x)$, $(x_\alpha)_{\alpha \in A}$ is eventually in U. Thus there exists $\alpha_0 \in A$ such that for each $\alpha \geq \alpha_0$, $x_\alpha \in U$. In particular $x_{\alpha_0} \in U \cap E$. Hence $U \cap E \neq \varnothing$. □

Exercise 3.3.2.13. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces, $f : X \to Y$ and $x \in X$. Then f is continuous at x iff for each net $(x_\alpha)_{\alpha \in A} \subset X$, $x_\alpha \to x$ implies that $f(x_\alpha) \to f(x)$.

Proof. Suppose that f is continuous at x. Let $(x_\alpha)_{\alpha \in A} \subset X$ be a net. Suppose that $x_\alpha \to x$. Let $V \in \mathcal{N}(f(x))$. Continuity implies that $f^{-1}(V) \in \mathcal{N}(x)$. Since $x_\alpha \to x$, $(x_\alpha)_{\alpha \in A}$ is eventually in $f^{-1}(V)$. So there exists $\beta \in A$ such that for each $\alpha \in A$, $\alpha \geq \beta$ implies that $x_\alpha \in f^{-1}(V)$. Let $\alpha \in A$. Suppose that $\alpha \geq \beta$. Then $f(x_\alpha) \in V$. So $(f(x_\alpha))_{\alpha \in A}$ is eventually in V. Since $V \in \mathcal{N}(f(x))$ is arbitrary, $f(x_\alpha) \to f(x)$.
Conversely, suppose that f is not continuous at x. Then there exists $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$. Then $x \notin \operatorname{Int}(f^{-1}(V))$. So $x \in (\operatorname{Int}(f^{-1}(V)))^c = \operatorname{cl} f^{-1}(V^c)$. This implies that there exists a net $(x_\alpha)_{\alpha \in A} \subset f^{-1}(V^c)$ such that $x_\alpha \to x$. Since for each $\alpha \in A$, $f(x_\alpha) \in V^c$, $f(x_\alpha)$ is not eventually in V. So $f(x_\alpha) \not\to f(x)$. □

Exercise 3.3.2.14. Let $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$ be a collection of topological spaces, X a set and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y_\alpha^X$ with $\mathcal{F} = (f_\alpha)_{\alpha \in A}$. Equip X with $\tau_X(\mathcal{F})$. Let $(x_\gamma)_{\gamma \in \Gamma} \subset X$ be a net and $x \in X$. Then $x_\gamma \to x$ iff for each $\alpha \in A$, $f_\alpha(x_\gamma) \to f_\alpha(x)$. (maybe reword without $\mathcal{F}$ and similar instances elsewhere)

Proof. Suppose that $x_\gamma \to x$. Let $\alpha \in A$. Since f_α is continuous, the previous exercise implies that $f_\alpha(x_\gamma) \to f_\alpha(x)$.
Conversely, Suppose that for each $\alpha \in A$, $f_\alpha(x_\gamma) \to f_\alpha(x)$. Let $U \in \mathcal{N}(x)$. Since $\operatorname{Int} U \in \tau_X(\mathcal{F})$, Exercise 3.1.0.11 implies there exist $V_1 \in \mathcal{B}_{\alpha_1}, \dots, V_n \in \mathcal{B}_{\alpha_n}$ such that $\bigcap\limits_{j=1}^n f_{\alpha_j}^{-1}(V_j) \subset \operatorname{Int} U$ and $x \in \bigcap\limits_{j=1}^n f_{\alpha_j}^{-1}(V_j)$. Let $j \in \{1, \dots, n\}$. Since $f_{\alpha_j}^{-1}(V_j) \in \mathcal{N}(x)$, $V_j \in \mathcal{N}(f(x))$. By assumption, $f_{\alpha_j}(x_\gamma)$ is eventually in V_j. Thus there exist there exist $\gamma_j' \in \Gamma$ such that for each $\gamma \geq \gamma_j'$, $f_{\alpha_j}(x_\gamma) \in V_j$, or equivalently, $x_\gamma \in f_{\alpha_j}^{-1}(V_j)$. Since Γ is directed, there exists $\gamma' \in \Gamma$ such that for each $j \in \{1, \dots, n\}$, $\gamma' \geq \gamma_j'$. Let $\gamma \in \Gamma$. Suppose that $\gamma \geq \gamma'$. Then

$$\begin{aligned}x_\gamma &\in \bigcap_{j=1}^n f_{\alpha_j}^{-1}(V_j) \\ &\subset \operatorname{Int} U \\ &\subset U\end{aligned}$$

So $(x_\gamma)_{\gamma \in \Gamma}$ is eventually in U. Since $U \in \mathcal{N}(x)$ is arbitrary, $x_\gamma \to x$. $\square$

Exercise 3.3.2.15. reorganize to an iff
Let X be a set and $\mathcal{T}_1$, $\mathcal{T}_2$ topologies on X. Then the following are equivalent:

1. $\mathcal{T}_1 = \mathcal{T}_2$
2. for each net $(x_\alpha)_{\alpha \in A} \subset X$ and $x \in X$, $x_\alpha \to x$ in $\mathcal{T}_1$ iff $x_\alpha \to x$ in $\mathcal{T}_2$.

Proof.

- (1) $\implies$ (2):
  Clear.

- (2) $\implies$ (1):
  Let $U \in \mathcal{T}_1$ and $x \in U^c$. Since U^c is closed in $\mathcal{T}_1$, there exists a net $(x_\alpha)_{\alpha \in A} \subset U^c$ such that $x_\alpha \to x$ in $\mathcal{T}_1$. By assumption, $x_\alpha \to x$ in $\mathcal{T}_2$. So U^c is closed in $\mathcal{T}_2$ and $U \in \mathcal{T}_2$. Hence $\mathcal{T}_1 \subset \mathcal{T}_2$.
  Similarly, $\mathcal{T}_2 \subset \mathcal{T}_1$.

$\square$

Exercise 3.3.2.16. Let X, Y be topological spaces and $\phi : X \to Y$ a homeomorphism. Then for each $E \subset X$,

1. $\operatorname{cl} \phi(E) = \phi(\operatorname{cl} E)$
2. $\operatorname{Int} \phi(E) = \phi(\operatorname{Int} E)$

Proof.

1. Let $E \subset X$. Since $E \subset \operatorname{cl} E$, we have that $\phi(E) \subset \phi(\operatorname{cl} E)$. Since $\operatorname{cl} E$ is closed, $\phi(\operatorname{cl} E)$ is closed and thus $\operatorname{cl} \phi(E) \subset \phi(\operatorname{cl} E)$. Conversely, let $x \in \phi(\operatorname{cl} E)$. Then $\phi^{-1}(x) \in \operatorname{cl} E$. Then there exists a net $(y_\alpha)_{\alpha \in A} \subset E$ such that $y_\alpha \to \phi^{-1}(x)$. Then $(\phi(y_\alpha))_{\alpha \in A} \subset \phi(E)$ and $\phi(y_\alpha) \to x$. Thus $x \in \operatorname{cl} \phi(E)$ and $\phi(\operatorname{cl} E) \subset \operatorname{cl} \phi(E)$.
2. Similar

$\square$

Definition 3.3.2.17. Let X be a topological space, $(x_\alpha)_{\alpha \in A} \subset X$ a net and $x \in X$. Then x is said to be a **cluster point** or **accumulation point** of $(x_\alpha)_{\alpha \in A}$ if for each $U \in \mathcal{N}(x)$, $(x_\alpha)_{\alpha \in A}$ is frequently in U.

Note 3.3.2.18. We recall the definition of a subnet from Definition 1.2.3.3

Exercise 3.3.2.19. Let X be a topological space, $(x_\alpha)_{\alpha \in A} \subset X$ a net and $x \in X$. Then the following are equivalent:

1. x is a cluster point of $(x_\alpha)_{\alpha \in A}$
2. there exists a subnet $(x_{\alpha_\beta})_{\beta \in B}$ of $(x_\alpha)_{\alpha \in A}$ such that $x_{\alpha_\beta} \to x$
3. $x \in \bigcap_{\alpha \in A} \operatorname{cl}\{x_\beta : \beta \geq \alpha\}$

Hint: Order $\mathcal{N}(x)$ by reverse inclusion and consider the product directed set $B = A \times \mathcal{N}(x)$. If x is a cluster point of $(x_\alpha)_{\alpha \in A}$, then for each $\beta = (\gamma, U) \in B$, there exists $\alpha_\beta \in A$ such that $\alpha_\beta \geq \gamma$ and $\alpha_\beta \in U$.

Proof.

- (1) $\implies$ (2):
Suppose that x is a cluster point of $(x_\alpha)_{\alpha\in A}$. Set $B = A\times\mathcal{N}(x)$. Since x is a cluster point of $(x_\alpha)_{\alpha\in A}$, for each $(\gamma, U)\in B$, there exists $\alpha_{(\gamma,U)}\in A$ such that $\alpha_{(\gamma,U)}\geq\gamma$ and $x_{\alpha_{(\gamma,U)}}\in U$. Let $\alpha_0\in A$. Choose $\beta_0 = (\alpha_0, X)\in B$. Let $\beta = (\gamma, U)\in B$. Suppose that $\beta\geq\beta_0$. Then $\gamma\geq\alpha_0$ and
$$\begin{aligned}\alpha_\beta &= \alpha_{(\gamma,U)}\\ &\geq\gamma\\ &\geq\alpha_0\end{aligned}$$
So that $(x_{\alpha_\beta})_{\beta\in B}$ is a subnet of $(x_\alpha)_{\alpha\in A}$. Let $U_0\in\mathcal{N}(x)$. Choose $\alpha_0\in A$ and set $\beta_0 = (\alpha_0, U_0)$. Let $\beta = (\gamma, U)\in B$. Suppose that $\beta\geq\beta_0$. Then
$$\begin{aligned}x_{\alpha_\beta} &= x_{\alpha_{(\gamma,U)}}\\ &\in U\\ &\subset U_0\end{aligned}$$
Since $U_0\in\mathcal{N}(x)$ is arbitrary, $x_{\alpha_\beta}\to x$.

- (2) $\implies$ (3):
Suppose that that there exists a subnet $(x_{\alpha_\beta})_{\beta\in B}$ of $(x_\alpha)_{\alpha\in A}$ such that $x_{\alpha_\beta}\to x$. Let $\alpha\in A$. Then there exists $\beta_0\in B$ such that for each $\beta\in B$, $\beta\geq\beta_0$ implies that $\alpha_\beta\geq\alpha$. Therefore, for each $\beta\in B$, $\beta\geq\beta_0$ implies that $x_{\alpha_\beta}\in E_\alpha$. So $(x_{\alpha_\beta})_{\beta\in B}$ is eventually in E_α. Since $x_{\alpha_\beta}\to x$ and $(x_{\alpha_\beta})_{\beta\in B}$ is eventually in E_α, Exercise 3.3.2.8 implies that $x\in\operatorname{cl}E_\alpha$. Since $\alpha\in A$ is arbitrary, we have that $x\in\bigcap\limits_{\alpha\in A}\operatorname{cl}E_\alpha$.

- (3) $\implies$ (1):
Suppose that that $x\in\bigcap\limits_{\alpha\in A}\operatorname{cl}E_\alpha$. Let $U\in\mathcal{N}(x)$. Since
$$\begin{aligned}x &\in [\operatorname{Int}U]\cap\bigcap_{\alpha\in A}\operatorname{cl}E_\alpha\\ &= \bigcap_{\alpha\in A}([\operatorname{Int}U]\cap\operatorname{cl}E_\alpha)\end{aligned}$$
we have that for each $\alpha\in A$, $[\operatorname{Int}U]\cap\operatorname{cl}E_\alpha\neq\varnothing$. Exercise 3.3.2.12 implies that for each $\alpha\in A$,
$$\begin{aligned}\varnothing &\neq [\operatorname{Int}U]\cap E_\alpha\\ &\subset U\cap E_\alpha\end{aligned}$$
Let $\alpha\in A$. Since $U\cap E_\alpha\neq\varnothing$, there exists $x_0\in X$ such that $x_0\in U\cap E_\alpha$. Since $x_0\in E_\alpha$, there exists $\alpha_0\in A$ such that $\alpha_0\geq\alpha$ and
$$\begin{aligned}x_{\alpha_0} &= x_0\\ &\in U\end{aligned}$$
Thus $(x_\alpha)_{\alpha\in A}$ is frequently in U. Since $U\in\mathcal{N}(x)$ is arbitrary, we have that for each $U\in\mathcal{N}(x)$, $(x_\alpha)_{\alpha\in A}$ is frequently in U. Thus x is a cluster point of $(x_\alpha)_{\alpha\in A}$.

$\square$

Exercise 3.3.2.20. Let X be a topological space, $(x_\alpha)_{\alpha\in A}\subset X$ a net and $x\in X$. If $x_\alpha\to x$, then for each subnet $(x_{\alpha_\beta})_{\beta\in B}$ of $(x_\alpha)_{\alpha\in A}$, $x_{\alpha_\beta}\to x$.

Proof. Suppose that $x_\alpha\to x$. Let $(x_{\alpha_\beta})_{\beta\in B}$ be a subnet of $(x_\alpha)_{\alpha\in A}$ and $U\in\mathcal{N}(x)$. Since $x_\alpha\to x$, there exists $\alpha_0\in A$ such that for each $\alpha\geq\alpha_0$, $x_\alpha\in U$. Since $(x_{\alpha_\beta})_{\beta\in B}$ is a subnet of $(x_\alpha)_{\alpha\in A}$, there exists $\beta_0\in B$ such that for each $\beta\in B$, $\beta\geq\beta_0$ implies that $\alpha_{be}\geq\alpha_0$. Then for each $\beta\in B$, $\beta\geq\beta_0$ implies that $x_{\alpha_\beta}\in U$. Since $U\in\mathcal{N}(x)$ is arbitrary, $x_{\alpha_\beta}\to x$. $\square$

Exercise 3.3.2.21. Let X be a topological space, $(x_\alpha)_{\alpha\in A} \subset X$ a net and $x \in X$. Then $x_\alpha \to x$ iff for each subnet $(x_{\alpha_\beta})_{\beta\in B}$ of $(x_\alpha)_{\alpha\in A}$, there exists a subnet $(x_{\alpha_{\beta_\gamma}})_{\gamma\in\Gamma}$ of $(x_{\alpha_\beta})_{\beta\in B}$ such that $x_{\alpha_{\beta_\gamma}} \to x$.

Proof. FINISH!!! □

Definition 3.3.2.22. Let $(x_\alpha)_{\alpha\in A} \subset \overline{\mathbb{R}}$ a net.

- We define the **limit inferior** of $(x_\alpha)_{\alpha\in A}$, denoted $\liminf\limits_{\alpha\in A} x_\alpha \in [0,\infty]$, by

$$\liminf_{\alpha\in A} x_\alpha = \sup_{\beta\in A}\left[\inf_{\alpha\geq\beta} x_\alpha\right]$$

- We define the **limit superior** of $(x_\alpha)_{\alpha\in A}$, denoted $\limsup\limits_{\alpha\in A} x_\alpha \in [0,\infty]$, by

$$\limsup_{\alpha\in A} x_\alpha = \inf_{\beta\in A}\left[\sup_{\alpha\geq\beta} x_\alpha\right]$$

Exercise 3.3.2.23. Let $(x_\alpha)_{\alpha\in A} \subset \mathbb{R}$ a net. Then

$$\liminf_{\alpha\in A} x_\alpha \leq \limsup_{\alpha\in A} x_\alpha$$

Proof. Set $s := \liminf\limits_{\alpha\in A} x_\alpha$ and $S := \limsup\limits_{\alpha\in A} x_\alpha$. Let $\epsilon > 0$. Then there exists $\beta_1, \beta_2 \in A$ such that for each $\alpha \in A$, $\alpha \geq \beta_1$ implies that $\sup\limits_{\alpha\geq\beta_1} x_\alpha < S + \epsilon/2$ and $\inf\limits_{\alpha\geq\beta_2} x_\alpha > s - \epsilon/2$. Set $\beta_0 := \max(\beta_1, \beta_2)$. Then

$$\begin{aligned} s - \frac{\epsilon}{2} &< \inf_{\alpha\geq\beta_2} x_\alpha \\ &\leq \inf_{\alpha\geq\beta_0} x_\alpha \\ &\leq \sup_{\alpha\geq\beta_0} x_\alpha \\ &\leq \sup_{\alpha\geq\beta_1} x_\alpha \\ &< S + \frac{\epsilon}{2} \end{aligned}$$

Therefore $-\epsilon < S - s$. Since $\epsilon > 0$ is arbitrary, we have that $0 \leq S - s$. Hence

$$\begin{aligned} \liminf_{\alpha\in A} x_\alpha &= s \\ &\leq S \\ &= \limsup_{\alpha\in A} x_\alpha \end{aligned}$$

□

Exercise 3.3.2.24. Let $(x_\alpha)_{\alpha\in A} \subset \mathbb{R}$ a net and $x \in \mathbb{R}$. Then $x_\alpha \to x$ iff

$$\liminf x_\alpha = \limsup x_\alpha = x$$

Proof. Suppose that $x_\alpha \to x$. Let $\epsilon > 0$. Then there exist $\beta \in A$ such that for each $\alpha \in A$, $\alpha \geq \beta$ implies that $x_\alpha \in B(x,\epsilon)$. So $\inf\limits_{\alpha\geq\beta} x_\alpha \geq x - \epsilon$ and $\sup\limits_{\alpha\geq\beta} \leq x + \epsilon$. Therefore

$$\begin{aligned} \liminf x_\alpha &= \sup_{\beta\in A}\left[\inf_{\alpha\geq\beta} x_\alpha\right] \\ &\geq x - \epsilon \end{aligned}$$

and

$$\begin{aligned}\limsup x_\alpha &= \inf_{\beta\in A}\left[\sup_{\alpha\geq\beta} x_\alpha\right]\\ &\leq x+\epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\limsup x_\alpha \leq x \leq \liminf x_\alpha$$

Since $\liminf x_\alpha \leq \limsup x_\alpha$, we have that $\liminf x_\alpha = \limsup x_\alpha = x$. $\square$

Exercise 3.3.2.25. Let $(x_\alpha)_{\alpha\in A} \subset \mathbb{R}$ a net and $x \in \mathbb{R}$. Then

1. $\liminf\limits_{\alpha\in A} -x_\alpha = -\limsup\limits_{\alpha\in A} x_\alpha$
2. $(x_\alpha)_{\alpha\in A} \subset \mathbb{R}\setminus\{0\}$ implies that $\liminf\limits_{\alpha\in A} x_\alpha^{-1} = \left(\limsup\limits_{\alpha\in A} x_\alpha\right)^{-1}$.
3. generalize to any order reversing bijection of a totally ordered set (including $\overline{\mathbb{R}}$)

Proof.

1. We have that

$$\begin{aligned}\liminf_{\alpha\in A} -x_\alpha &= \sup_{\beta\in A}\left[\inf_{\alpha\geq\beta} -x_\alpha\right]\\ &= \sup_{\beta\in A}\left[-\sup_{\alpha\geq\beta} x_\alpha\right]\\ &= -\inf_{\alpha\in A}\left[\sup_{\alpha\geq\beta} x_\alpha\right]\\ &= -\limsup_{\alpha\in A} x_\alpha\end{aligned}$$

2. Suppose that $(x_\alpha)_{\alpha\in A} \subset \mathbb{R}\setminus\{0\}$. Then

$$\begin{aligned}\liminf_{\alpha\in A} x_\alpha^{-1} &= \sup_{\beta\in A}\left[\inf_{\alpha\geq\beta} x_\alpha^{-1}\right]\\ &= \sup_{\beta\in A}\left[\sup_{\alpha\geq\beta} x_\alpha\right]^{-1}\\ &= \left(\inf_{\alpha\in A}\left[\sup_{\alpha\geq\beta} x_\alpha\right]\right)^{-1}\\ &= \left(\limsup_{\alpha\in A} x_\alpha\right)^{-1}\end{aligned}$$

$\square$

Exercise 3.3.2.26. Let X be a topological space, $f : X \to \mathbb{R}$, $(x_\alpha)_{\alpha\in A} \subset X$ a net and $x \in X$. Suppose that $x_\alpha \to x$ and for each $\alpha \in A$, $x_\alpha \neq x$. Then

1. $\liminf\limits_{t\to x} f(t) \leq \liminf f(x_\alpha)$
2. $\limsup\limits_{t\to x} f(t) \geq \limsup f(x_\alpha)$

Proof.

1. Let $V \in \mathcal{N}(x)$. Then there exists $\beta_0 \in A$ such that for each $\alpha \in A$, $\alpha \geq \beta_0$ implies that $x_\alpha \in V \setminus \{x\}$. Thus

$$\begin{aligned} \inf_{t \in V \setminus \{x\}} f(t) &\leq \inf_{\alpha \geq \beta_0} f(x_\alpha) \\ &\leq \sup_{\beta \in A} \left[\inf_{\alpha \geq \beta} f(x_\alpha) \right] \\ &= \liminf_{\alpha \in A} f(x_\alpha) \end{aligned}$$

   and since $V \in \mathcal{N}(x)$ is arbitrary, we have that

$$\begin{aligned} \liminf_{t \to x} f(t) &= \sup_{V \in \mathcal{N}(x)} \left[\inf_{t \in V \setminus \{x\}} f(t) \right] \\ &\leq \liminf_{\alpha \in A} f(x_\alpha) \end{aligned}$$

2. Similar to (1).

$\square$

3.4 Subspace Topology

3.4.1 Introduction

Definition 3.4.1.1. Let X be a set and $A \subset X$. We define the **inclusion map from** A **to** B, denoted $\iota : A \to X$, by $\iota(x) = x$.

Definition 3.4.1.2. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. We define the **subspace topology on** A, denoted $\mathcal{T} \cap A$, by

$$\mathcal{T} \cap A = \iota^* \mathcal{T}$$

Exercise 3.4.1.3. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then

1. $\mathcal{T} \cap A = \{U \cap A : U \in \mathcal{T}\}$,
2. for each $E \subset A$, $E \in \mathcal{T} \cap A$ iff there exists $U \in \mathcal{T}$ such that $E = U \cap A$.

Proof.

1. Since for each $U \subset X$, $\iota^{-1}(U) = U \cap A$, we have that

$$\begin{aligned}\mathcal{T} \cap A &= \iota^* \mathcal{T} \\ &= \{\iota^{-1}(U) : U \in \mathcal{T}\} \\ &= \{U \cap A : U \in \mathcal{T}\}\end{aligned}$$

2. Clear

□

Exercise 3.4.1.4. Let $(X, \mathcal{T})$ be a topological space, $A \subset X$ and $B \subset A$. Then $\mathcal{T} \cap B = (\mathcal{T} \cap A) \cap B$.

Proof.

- Let $U \in (\mathcal{T} \cap A) \cap B$. Then there exists $U_A \in \mathcal{T} \cap A$ such that $U = U_A \cap B$. Similarly, there exists $U_X \in \mathcal{T}$ such that $U_A = U_X \cap A$. Therefore

$$\begin{aligned}U &= U_A \cap B \\ &= (U_X \cap A) \cap B \\ & U_X \cap (A \cap B) \\ &= U_X \cap B \\ &\in \mathcal{T} \cap B\end{aligned}$$

  Since $U \in (\mathcal{T} \cap A) \cap B$ is arbitrary, we have that $(\mathcal{T} \cap A) \cap B \subset \mathcal{T} \cap B$.

- Conversely, let $U \in \mathcal{T} \cap B$. Then there exists $U_X \in \mathcal{T}$ such that $U = U_X \cap B$. Then

$$\begin{aligned}U &= U_X \cap B \\ &= U_X \cap (A \cap B) \\ &= (U_X \cap A) \cap B \\ &\in (\mathcal{T} \cap A) \cap B\end{aligned}$$

  Since $U \in U_X \cap B$ is arbitrary, we have that $U_X \cap B \subset (\mathcal{T} \cap A) \cap B$.

Therefore $\mathcal{T} \cap B = (\mathcal{T} \cap A) \cap B$. □

Note 3.4.1.5. The previous exercise just indicates that the subspace topology on B does not depend on choice of the containing space as long as each containing space is a subspace of the original space maybe rephrase this in terms of projective system or something.

Exercise 3.4.1.6. Let $(X, \mathcal{T})$ be a topological space, $A \subset X$, $(x_\gamma)_{\gamma \in \Gamma} \subset A$ a net and $x \in A$. Then $x_\gamma \to x$ in $(A, \mathcal{T} \cap A)$ iff $x_\gamma \to x$ in $(X, \mathcal{T})$.

Proof.

- Suppose that $x_\gamma \to x$ in $(A, \mathcal{T} \cap A)$. Since $\iota : A \to X$ is $(\mathcal{T} \cap A, \mathcal{T})$-continuous,

$$\begin{aligned} x_\gamma &= \iota(x_\gamma) \\ &\to \iota(x) \\ &= x \end{aligned}$$

  So that $x_\gamma \to x$ in $(X, \mathcal{T})$.

- Conversely, suppose that $x_\gamma \to x$ in $(X, \mathcal{T})$. Let $V \in \mathcal{N}(x)$ in $(A, \mathcal{T} \cap A)$. Then $x \in \operatorname{Int} V$ in $(A, \mathcal{T} \cap A)$. Hence there exists $U \in \mathcal{T}$ such that $\operatorname{Int} V = U \cap A$. Thus $U \in \mathcal{N}(x)$ in $(X, \mathcal{T})$. Since $x_\gamma \to x$ in $(X, \mathcal{T})$ and $x \in U$, we have that that $(x_\gamma)_{\gamma \in \Gamma}$ is eventually in U. Then $(x_\gamma)_{\gamma \in \Gamma}$ is eventually in $U \cap A = \operatorname{Int} V \subset V$. So $x_\gamma \to x$ in $(A, \mathcal{T} \cap A)$.

□

Exercise 3.4.1.7. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then f is $(\mathcal{A}, \mathcal{B})$-continuous iff for each $x \in X$, there exists $U \in \mathcal{A}$ such that $x \in U$ and $f|_U$ is $(\mathcal{A} \cap U, \mathcal{B})$-continuous.

Proof.

- $(\implies)$:
  Suppose that f is continuous. Let $x \in X$. Define $U \in \mathcal{A}$ by $U := X$. Then $x \in U$ and $f|_U$ is $(\mathcal{A} \cap U, \mathcal{B})$-continuous.

- $(\impliedby)$:
  Suppose that for each $x \in X$, there exists $U \in \mathcal{A}$ such that $x \in U$ and $f|_U$ is $(\mathcal{A} \cap U, \mathcal{B})$-continuous. Let $x \in X$ and $V \in \mathcal{B}$. Suppose that $f(x) \in V$. By assumption, there exists $U_0 \in \mathcal{A}$ such that $x \in U_0$ and $f|_{U_0}$ is $(\mathcal{A} \cap U_0, \mathcal{B})$-continuous. Define $U \subset X$ by $U = f|_{U_0}^{-1}(V)$. Since $V \in \mathcal{B}$ and $f|_{U_0}$ is $(\mathcal{A} \cap U_0, \mathcal{B})$-continuous, $U \in \mathcal{A} \cap U_0$. Since $U_0 \in \mathcal{A}$, $\mathcal{A} \cap U_0 \subset \mathcal{A}$ and therefore $U \in \mathcal{A}$. We note that $x \in U$. Since $f|_{U_0}^{-1}(V) = U_0 \cap f^{-1}(V)$, we have that

$$\begin{aligned} f(U) &= f(f|_{U_0}^{-1}(V)) \\ &= f(U_0 \cap f^{-1}(V)) \\ &\subset V \end{aligned}$$

  Since $V \in \mathcal{B}$ with $f(x) \in V$ is arbitrary, we have that for each $V \in \mathcal{B}$, $f(x) \in V$ implies that there exists $U \in \mathcal{A}$ such that $x \in U$ and $f(U) \subset V$. Thus f is continuous at x. Since $x \in X$ is arbitrary, f is continuous.

□

Exercise 3.4.1.8. Let $(X, \mathcal{T})$ be a topological space, $A \subset X$ and $F \subset A$. Then F is closed in A iff there exists $C \subset X$ such that C is closed in X and $F = C \cap A$.

Proof. Suppose that F is closed in A. Then $A \setminus F$ is open in A. Hence there exists $U \in \mathcal{T}$ such that $A \setminus F = U \cap A$. Set $C = U^c$. Then C is closed in X and

$$\begin{aligned} F &= A \setminus (A \setminus F) \\ &= A \setminus (U \cap A) \\ &= A \cap [(U \cap A)^c] \\ &= A \cap (U^c \cup A^c) \\ &= (A \cap U^c) \cup (A \cap A^c) \\ &= A \cap U^c \\ &= A \cap C \end{aligned}$$

Conversely, suppose that there exists $C \subset X$ such that C is closed in X and $F = A \cap C$. Since $C^c \in \mathcal{T}$, we have that

$$\begin{aligned} A \setminus F &= A \cap F^c \\ &= A \cap [(A \cap C)^c] \\ &= A \cap (A^c \cup C^c) \\ &= (A \cap A^c) \cup (A \cap C^c) \\ &= A \cap C^c \\ &\in \mathcal{T} \cap A \end{aligned}$$

Thus $A \setminus F$ is open in A which implies that F is closed in A. □

Exercise 3.4.1.9. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then f is $(\mathcal{A}, \mathcal{B})$-continuous iff f is $(\mathcal{A}, \mathcal{B} \cap f(X))$-continuous.

Proof.

- $(\implies)$:
  Suppose that f is $(\mathcal{A}, \mathcal{B})$-continuous. Let $B \in \mathcal{B} \cap f(X)$. Then there exists $V \in \mathcal{B}$ such that $B = V \cap f(X)$. Then

  $$\begin{aligned} f^{-1}(B) &= f^{-1}(V \cap f(X)) \\ &= f^{-1}(V) \cap f^{-1}(f(X)) \\ &= f^{-1}(V) \cap X \\ &= f^{-1}(V) \\ &\in \mathcal{A} \end{aligned}$$

  Since $B \in \mathcal{B} \cap f(X)$ is arbitrary, f is $(\mathcal{A}, \mathcal{B} \cap f(X))$-continuous.

- $(\impliedby)$:
  Conversely, suppose that f is $(\mathcal{A}, \mathcal{B} \cap f(X))$-continuous. Let $V \in \mathcal{B}$. Then $V \cap f(X) \in \mathcal{B} \cap f(X)$ and

  $$\begin{aligned} f^{-1}(V) &= f^{-1}(V \cap f(X)) \\ &\in \mathcal{A} \end{aligned}$$

  Since $V \in \mathcal{B}$ is arbitrary, f is $(\mathcal{A}, \mathcal{B})$-continuous.

□

Exercise 3.4.1.10. Basis for Subspace Topology:
Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subset \mathcal{T}$ a basis for $\mathcal{T}$ on X and $A \subset X$. Then $\mathcal{B} \cap A$ is a basis for $\mathcal{T} \cap A$ on A.

Proof. Let $E \in \mathcal{T} \cap A$. Then there exists $E' \in \mathcal{T}$ such that $E = E' \cap A$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$ on X, there exists $\mathcal{U}' \subset \mathcal{B}$ such that $E' = \bigcup_{U' \in \mathcal{U}'} U'$. Define $\mathcal{U} \subset \mathcal{T} \cap A$ by $\mathcal{U} := \mathcal{U}' \cap A$. Then

$$\begin{aligned} \mathcal{U} &= \mathcal{U}' \cap A \\ &\subset \mathcal{B} \cap A \end{aligned}$$

and

$$\begin{aligned} E &= E' \cap E \\ &= \left(\bigcup_{U' \in \mathcal{U}} U' \right) \cap A \\ &= \bigcup_{U' \in \mathcal{U}'} (U' \cap A) \\ &= \bigcup_{U \in \mathcal{U}' \cap A} U \\ &= \bigcup_{U \in \mathcal{U}} U \end{aligned}$$

Since $E \in \mathcal{T} \cap A$ is arbitrary, we have that for each $E \in \mathcal{T} \cap A$, there exists $\mathcal{U} \subset \mathcal{B} \cap A$ such that $E = \bigcup_{U \in \mathcal{U}} U$. Hence $\mathcal{B} \cap A$ is a basis for $\mathcal{T} \cap A$ on A. $\square$

Exercise 3.4.1.11. Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces and $f : X \to Y$. If f is $(\mathcal{T}, \mathcal{S})$-open, then for each $U \in \mathcal{T}$, $f|_U$ is $(\mathcal{T} \cap U, \mathcal{S})$-open.

Proof. Suppose that f is $(\mathcal{T}, \mathcal{S})$-open. Let $U \in \mathcal{T}$ and $V \in \mathcal{T} \cap U$. Since $U \in \mathcal{T}$,

$$\begin{aligned} V &\in \mathcal{T} \cap U \\ &\subset \mathcal{T}. \end{aligned}$$

Since f is $(\mathcal{T}, \mathcal{S})$-open and $V \subset U$, we have that

$$\begin{aligned} f|_U(V) &= f(V) \\ &\in \mathcal{S}. \end{aligned}$$

Since $V \in \mathcal{T} \cap U$ is arbitrary, we have that for each $V \in \mathcal{T} \cap U$, $f|_U(V) \in \mathcal{S}$. Thus $f|_U$ is $(\mathcal{T} \cap U, \mathcal{S})$. Since $U \in \mathcal{T}$ is arbitrary, we have that for each $U \in \mathcal{T}$, $f|_U$ is $(\mathcal{T} \cap U, \mathcal{S})$. $\square$

Exercise 3.4.1.12. Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces and $f : X \to Y$. Then f is $(\mathcal{T}, \mathcal{S})$-open iff for each $x \in X$, there exists $U \in \mathcal{T}$ such that $x \in U$ and $f|_U$ is $(\mathcal{T} \cap U, \mathcal{S})$-open.

Proof.

- $(\implies)$:
  Suppose that f is $(\mathcal{T}, \mathcal{S})$-open. Let $x \in X$. Set $U := X$. Then $X \in \mathcal{T}$, $x \in U$ and $f|_U$ is $(\mathcal{T} \cap U, \mathcal{S})$-open.

- $(\impliedby)$:
  Suppose that for each $x \in X$, there exists $U \in \mathcal{T}$ such that $x \in U$ and $f|_U$ is $(\mathcal{T} \cap U, \mathcal{S})$-open. Let $V \in \mathcal{T}$. By assumption, for each $x \in V$, there exists $U_x \in \mathcal{T}$ such that $f|_{U_x}$ is $(\mathcal{T} \cap U_x, \mathcal{S})$-open. Then for each $x \in V$, $V \cap U_x \in \mathcal{T} \cap U_x$ and $V = \bigcup_{x \in V} (V \cap U_x)$. Since for each $x \in V$, $f|_{U_x}$ is $(\mathcal{T} \cap U_x, \mathcal{S})$-open,

we have that

$$\begin{aligned} f(V) &= f\bigg(\bigcup_{x \in V} (V \cap U_x)\bigg) \\ &= \bigcup_{x \in V} f(V \cap U_x) \\ &= \bigcup_{x \in V} f|_{U_x}(V \cap U_x) \\ &\in \mathcal{S} \end{aligned}$$

Since $V \in \mathcal{T}$ is arbitrary, we have that for each $V \in \mathcal{T}$, $f(V) \in \mathcal{S}$. Hence f is open.

□

Exercise 3.4.1.13. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $f : X \to Y$ and $E \subset X$. Suppose that f is $(\mathcal{T}_X, \mathcal{T}_Y)$-closed. If E is closed in $(X, \mathcal{T}_Y)$, then $f|_E$ is $(\mathcal{T}_X \cap E, \mathcal{T}_Y)$-closed.

Proof. Suppose that E is closed in $(X, \mathcal{T}_Y)$. Let $F \subset E$. Suppose that F is closed in $(E, \mathcal{T}_X \cap E)$. Exercise 3.4.1.8 implies that there exists $C \subset X$ such that C is closed in $(X, \mathcal{T}_X)$ and $F = C \cap E$. Since C, E are closed in $(X, \mathcal{T}_X)$, F is closed in $(X, \mathcal{T}_X)$. Since f is closed, $f(F)$ is closed in $(Y, \mathcal{T}_Y)$. Since $f|_E(F) = f(F)$, we have that $f|_E(F)$ is closed in $(Y, \mathcal{T}_Y)$. Since $F \subset E$ with F closed in $(E, \mathcal{T}_X \cap E)$ is arbitrary, we have that for each $F \subset E$, F is closed in $(E, \mathcal{T}_X \cap E)$ implies that $f|_E(F)$ is closed in $(Y, \mathcal{T}_Y)$. Hence $f|_E$ is $(\mathcal{T}_X \cap E, \mathcal{T}_Y)$-closed. □

Exercise 3.4.1.14. Let $(X, \mathcal{T}_X), (A, \mathcal{T}_A)$ be topological spaces. Suppose that $A \subset X$. If ι_A is a $(\mathcal{T}_A, \mathcal{T}_X \cap A)$-homeomorphism, then $\mathcal{T}_A = \mathcal{T}_X \cap A$.

Proof. Suppose that ι_A is a $(\mathcal{T}_A, \mathcal{T}_X \cap A)$-homeomorphism.

- Let $U \in \mathcal{T}_A$. Since $\iota_A(U) = U$ and ι_A is $(\mathcal{T}_A, \mathcal{T}_X \cap A)$-open, we have that

$$\begin{aligned} U &= \iota_A(U) \\ &\in \mathcal{T}_X \cap A. \end{aligned}$$

  Since $U \in \mathcal{T}_A$ is arbitrary, we have that $\mathcal{T}_A \subset \mathcal{T}_X \cap A$.

- Let $U \in \mathcal{T}_X \cap A$. Since ι_A is $(\mathcal{T}_A, \mathcal{T}_X \cap A)$-continuous and $U \subset A$, we have that we have that

$$\begin{aligned} U &= \iota_A^{-1}(U) \\ &\in \mathcal{T}_A. \end{aligned}$$

  Since $U \in \mathcal{T}_X \cap A$ is arbitrary, we have that $\mathcal{T}_X \cap A \subset \mathcal{T}_A$.

Hence $\mathcal{T}_A = \mathcal{T}_X \cap A$. □

Exercise 3.4.1.15. universal property

Proof. **FINISH!!!** □

3.4.2 Discrete Subsets

Definition 3.4.2.1. Let $(X, \mathcal{T})$ be a topological space, $A \subset X$ and $x \in A$. Then x is said to an **isolated point of** A if there exists $U \in \mathcal{T}$ such that $U \cap A = \{x\}$.

Exercise 3.4.2.2. Let $(X, \mathcal{T})$ be a topological space, $A \subset X$ and $x \in A$. Then x is an isolated point of A iff $\{x\} \in \mathcal{T} \cap A$.

Proof. Suppose that x is an isolated point of A. Then there exists $U \in \mathcal{T}$ such that

$$\begin{aligned}\{x\} &= U \cap A \\ &\in \mathcal{T} \cap A\end{aligned}$$

Conversely, suppose that $\{x\} \in \mathcal{T} \cap A$. Then there exists $U \in \mathcal{T}$ such that $\{x\} = U \cap A$. Hence x is an isolated point of A. $\square$

Definition 3.4.2.3. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then A is said to be **discrete** if for each $x \in A$, x is an isolated point of A.

Exercise 3.4.2.4. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then A is discrete iff $\mathcal{T} \cap A = \mathcal{P}(A)$.

Proof. Suppose that A is discrete. Then for each $x \in A$, $\{x\} \in \mathcal{T} \cap A$. Let $U \in \mathcal{P}(A)$. Then

$$\begin{aligned}U &= \bigcup_{x \in U} \{x\} \\ &\in \mathcal{T} \cap A\end{aligned}$$

Since $U \in \mathcal{P}(A)$ is arbitrary, we have that $\mathcal{P}(A) \subset \mathcal{T} \cap A$. Since $\mathcal{T} \cap A \subset \mathcal{P}(A)$, we have that $\mathcal{T} \cap A = \mathcal{P}(A)$. Conversely, suppose that $\mathcal{T} \cap A = \mathcal{P}(A)$. Let $x \in A$. Then

$$\begin{aligned}\{x\} &\in \mathcal{P}(A) \\ &= \mathcal{T} \cap A\end{aligned}$$

Hence x is an isolated point of A. Since $x \in A$ is arbitrary, A is discrete. $\square$

3.5 Product Topology

3.5.1 Bases of the product topology

Definition 3.5.1.1. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. We define the **product topology** on $\prod_{\alpha \in A} X_\alpha$, denoted $\bigotimes_{\alpha \in A} \mathcal{T}_\alpha$, by

$$\bigotimes_{\alpha \in A} \mathcal{T}_\alpha = \tau(\pi_\alpha : \alpha \in A)$$

i.e. $\bigotimes_{\alpha \in A} \mathcal{T}_\alpha$ is the initial (weak) topology on $\prod_{\alpha \in A} X_\alpha$ generated by the projection maps $(\pi_\alpha)_{\alpha \in A}$.

Exercise 3.5.1.2. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Define

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} B_\alpha : \text{ for each } \alpha \in A,\, B_\alpha \in \mathcal{T}_\alpha \text{ and } \#\{\alpha \in A : B_\alpha \neq X_\alpha\} < \infty \right\}$$

Then $\mathcal{B}$ is a basis for $\bigotimes_{\alpha \in A} \mathcal{T}_\alpha$.

Proof. Set $X = \prod_{\alpha \in A} X_\alpha$ and $\mathcal{T}_X = \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$. Set

$$\mathcal{E} = \{\pi_\alpha^{-1}(B_\alpha) : \alpha \in A, B_\alpha \in \mathcal{T}_\alpha\}$$

By definition, $\mathcal{T}_X = \tau_X(\mathcal{E})$. Let $\alpha \in A$ and $B_\alpha \in \mathcal{T}_\alpha$. For $\beta \in A$, set

$$C_\beta = \begin{cases} B_\beta & \beta = \alpha \\ X_\beta & \beta \neq \alpha \end{cases}$$

Then

$$\pi_\alpha^{-1}(B_\alpha) = \prod_{\beta \in A} C_\beta$$

Hence $\mathcal{B} = \left\{ \bigcap_{j=1}^n V_j : (V_j)_{j=1}^n \subset \mathcal{E} \right\} \subset \mathcal{T}_X$. A previous exercise implies that $\mathcal{B}$ is a basis for $\mathcal{T}_X$. □

Exercise 3.5.1.3. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Then for each $\alpha \in A$, $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ is open.

Proof. Let $\alpha \in A$. Define

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} B_\alpha : \text{ for each } \alpha \in A,\, B_\alpha \in \mathcal{T}_\alpha \text{ and } \#\{\alpha \in A : B_\alpha \neq X_\alpha\} < \infty \right\}$$

Then Exercise 3.5.1.2 implies that $\mathcal{B}$ is a basis for $\bigotimes_{\alpha \in A} \mathcal{T}_\alpha$. Let $U \in \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$. Then for each $\alpha' \in A$, there exist $B_{\alpha'} \in \mathcal{T}_{\alpha'}$ such that $U = \prod_{\alpha' \in A} B_{\alpha'}$ and $\#\{\alpha' \in A : B_{\alpha'} \neq X_{\alpha'}\} < \infty$. Then

$$\begin{aligned} \pi_\alpha(U) &= B_\alpha \\ &\in \mathcal{T}_\alpha \end{aligned}$$

Since $U \in \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$ is arbitrary, we have that for each $U \in \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$, $\pi_\alpha(U) \in \mathcal{T}_\alpha$. Exercise 3.2.0.20 implies that π_α is open. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, π_α is open. □

Exercise 3.5.1.4. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces, $x \in \prod_{\alpha \in A} X_\alpha$ and for each $\alpha \in A$, $\mathcal{B}_{x_\alpha} \subset \mathcal{T}_\alpha$. Suppose that for each $\alpha \in A$, $\mathcal{B}_{x_\alpha}$ is a local basis for $\mathcal{T}_\alpha$ at x_α. Define

$$\mathcal{B}_x = \left\{ \prod_{\alpha \in A} U_\alpha : \text{ [for each } \alpha \in A, U_\alpha \in \mathcal{T}_\alpha \text{ and } U_\alpha \neq X_\alpha \text{ implies that } U_\alpha \in \mathcal{B}_{x_\alpha}] \text{ and } \#\{\alpha \in A : U_\alpha \neq X_\alpha\} < \infty \right\}$$

Then $\mathcal{B}_x$ is a local basis for $\bigotimes_{\alpha \in A} \mathcal{T}_\alpha$ at x.

Proof. Set $X = \prod_{\alpha \in A} X_\alpha$ and $\mathcal{T} = \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$.

1. By construction, for each $V \in \mathcal{B}_x$, $x \in V$.
2. Let $U \in \mathcal{T}$. Suppose that $x \in U$. Set

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} B_\alpha : \text{ for each } \alpha \in A, B_\alpha \in \mathcal{T}_\alpha \text{ and } \#\{\alpha \in A : B_\alpha \neq X_\alpha\} < \infty \right\}$$

   The previous exercise implies that $\mathcal{B}$ is a basis for $\bigotimes_{\alpha \in A} \mathcal{T}_\alpha$. Thus for each $\alpha \in A$, there exists $B_\alpha \in \mathcal{T}_\alpha$ such that $\#\{\alpha \in A : B_\alpha \neq X_\alpha\} < \infty$ and $x \in \prod_{\alpha \in A} B_\alpha \subset U$. Set $J = \{\alpha \in A : B_\alpha \neq X_\alpha\}$. Since $x \in \prod_{\alpha \in A} B_\alpha$, for each $\alpha \in A$, $x_\alpha \in B_\alpha$. Since for each $\alpha \in A$, $\mathcal{B}_{x_\alpha}$ is a local basis for $\mathcal{T}_\alpha$ at x_α, the axiom of choice implies that there exists $(U_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} \mathcal{T}_\alpha$ such that for each if $\alpha \in J$, $U_\alpha \in \mathcal{B}_{x_\alpha}$ and $x_\alpha \in U_\alpha \subset B_\alpha$ and for each $\alpha \in J^c$, $U_\alpha = X_\alpha$. By definition, $\prod_{\alpha \in A} U_\alpha \in \mathcal{B}_x$. By construction,

$$\begin{aligned} x &\in \prod_{\alpha \in A} U_\alpha \\ &\subset \prod_{\alpha \in A} B_\alpha \\ &\subset U \end{aligned}$$

   Since $U \in \mathcal{T}$ such that $x \in U$ is arbitrary, we have that $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x.

□

Exercise 3.5.1.5. Let $(X_j, \mathcal{T}_j)_{j=1}^n$ be a collection of topological spaces. Set

$$\mathcal{B} = \left\{ \prod_{j=1}^n A_j : \text{for each } j \in \{1, \dots, n\}, A_j \in \mathcal{T}_j \right\}$$

Then $\mathcal{B}$ is a basis for the product topology on $\prod_{j=1}^n X_j$.

Proof. Clear by previous exercise. □

Exercise 3.5.1.6. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces and for each $\alpha \in A$, $\mathcal{B}_\alpha$ a basis for $\mathcal{T}_\alpha$. Set $X = \prod_{\alpha \in A} X_\alpha$ and denote the product topology on X by $\mathcal{T}_X$. Set

$$\begin{aligned} \mathcal{B} = \Bigg\{ \prod_{\alpha \in A} U_\alpha : &\text{there exists } J \subset A \text{ such that } \#J < \infty, \\ &\text{for each } \alpha \in J, U_\alpha \in \mathcal{B}_\alpha \text{ and for each } \alpha \in J^c, U_\alpha = X_\alpha \Bigg\} \end{aligned}$$

Then $\mathcal{B}$ is a basis for $\mathcal{T}_X$.

Proof. Set

$$\mathcal{B}' = \left\{ \prod_{\alpha \in A} V_\alpha : \text{ for each } \alpha \in A, V_\alpha \in \mathcal{T}_\alpha \text{ and } \#\{\alpha \in A : V_\alpha \neq X_\alpha\} < \infty \right\}$$

The previous exercise implies that $\mathcal{B}'$ is a basis for $\mathcal{T}_X$. Then $\mathcal{B} \subset \mathcal{B}' \subset \mathcal{T}_X$. Let $V \in \mathcal{T}$ and $x \in V$. Write $x = (x_\alpha)_{\alpha \in A}$. Since $\mathcal{B}'$ is a basis for $\mathcal{T}_X$, for each $\alpha \in A$, there exists $V_\alpha \in \mathcal{T}_\alpha$ such that for finitely many $\alpha \in A$, $V_\alpha \neq X_\alpha$ and $x \in \prod\limits_{\alpha \in A} V_\alpha \subset V$. Define $J \subset A$ by $J = \{\alpha \in A : V_\alpha \neq X_\alpha\}$. Then $\#J < \infty$. Let $\alpha \in J$. Then $x_\alpha \in V_\alpha$. Since $\mathcal{B}_\alpha$ is a basis for $\mathcal{T}_\alpha$, there exists $U'_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in U'_\alpha \subset V_\alpha$. For $\alpha \in A$, define $U_\alpha \in \mathcal{T}_\alpha$ by

$$U_\alpha = \begin{cases} U'_\alpha & \alpha \in J \\ X_\alpha & \alpha \in J^c \end{cases}$$

Set $U = \prod_{\alpha \in A} U_\alpha$. Then $U \in \mathcal{B}$ and

$$\begin{aligned} x &\in U \\ &= \prod_{\alpha \in A} U_\alpha \\ &\subset \prod_{\alpha \in A} V_\alpha \\ &\subset V \end{aligned}$$

Hence $\mathcal{B}$ is a basis for $\mathcal{T}_X$. □

3.5.2 Characteristics of the Product Topology

Exercise 3.5.2.1. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces and for each $\alpha \in A$, $E_\alpha \subset X_\alpha$. Then

$$\operatorname{cl}\left(\prod_{\alpha \in A} E_\alpha\right) = \prod_{\alpha \in A} \operatorname{cl} E_\alpha$$

Hint: Exercise 3.1.0.22

Proof. Since for each $\alpha \in A$, $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ is continuous and $\operatorname{cl} E_\alpha$ is closed, we have that for each $\alpha \in A$, $\pi_\alpha^{-1}(\operatorname{cl} E_\alpha)$ is closed and thus

$$\prod_{\alpha \in A} \operatorname{cl} E_\alpha = \bigcap_{\alpha \in A} \pi_\alpha^{-1}(\operatorname{cl} E_\alpha)$$

is closed. Since for each $\alpha \in A$, $E_\alpha \subset \operatorname{cl} E_\alpha$, we have that

$$\prod_{\alpha \in A} E_\alpha \subset \prod_{\alpha \in A} \operatorname{cl} E_\alpha$$

which implies that

$$\operatorname{cl}\left(\prod_{\alpha \in A} E_\alpha\right) \subset \prod_{\alpha \in A} \operatorname{cl} E_\alpha$$

Conversely, let $x \in \prod\limits_{\alpha \in A} \operatorname{cl} E_\alpha$ and $U \in \mathcal{N}(x)$. Suppose that U is open. Set

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} B_\alpha : \text{ for each } \alpha \in A, B_\alpha \in \mathcal{T}_\alpha \text{ and } \#\{\alpha \in A : B_\alpha \neq X_\alpha\} < \infty \right\}$$

A previous exercise implies that $\mathcal{B}$ is a basis for the product topology on $\prod_{\alpha \in A} X_\alpha$. So for each $\alpha \in A$, there exists $U_\alpha \in \mathcal{T}_\alpha$ such that $\#\{\alpha \in A : U_\alpha \neq X_\alpha\} < \infty$ and $x \in \prod\limits_{\alpha \in A} U_\alpha \subset U$. Then for each $\alpha \in A$,

$x_\alpha \in \operatorname{cl} E_\alpha \cap U_\alpha$. Let $\alpha \in A$. Since $x_\alpha \in \operatorname{cl} E_\alpha$ and $U_\alpha \in \mathcal{N}(x_\alpha)$ is an open neighborhood of x_α, Exercise 3.1.0.22 implies that $E_\alpha \cap U_\alpha \neq \varnothing$. Since $\alpha \in A$ is arbitrary, for each $\alpha \in A$, $E_\alpha \cap U_\alpha \neq \varnothing$. The axiom of choice implies that there exists

$$\begin{aligned} y &\in \prod_{\alpha \in A} (E_\alpha \cap U_\alpha) \\ &= \left(\prod_{\alpha \in A} E_\alpha \right) \cap \prod_{\alpha \in A} U_\alpha \\ &\subset \left(\prod_{\alpha \in A} E_\alpha \right) \cap U \end{aligned}$$

Hence $\left(\prod\limits_{\alpha \in A} E_\alpha \right) \cap U \neq \varnothing$. Since $U \in \mathcal{N}(x)$ is an arbirtary open neighborhood of x, Exercise 3.1.0.22 implies that $x \in \operatorname{cl} \prod\limits_{\alpha \in A} E_\alpha$. Since $x \in \prod\limits_{\alpha \in A} \operatorname{cl} E_\alpha$ is arbitrary, $\prod\limits_{\alpha \in A} \operatorname{cl} E_\alpha \subset \operatorname{cl} \prod\limits_{\alpha \in A} E_\alpha$.
Hence $\prod\limits_{\alpha \in A} \operatorname{cl} E_\alpha = \operatorname{cl} \prod\limits_{\alpha \in A} E_\alpha$ □

Exercise 3.5.2.2. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces and $(a_\gamma)_{\gamma \in \Gamma} \subset \prod_{\alpha \in A} X_\alpha$ a net and $a \in \prod_{\alpha \in A} X_\alpha$. Then $a_\gamma \to a$ in $(\prod_{\alpha \in A} X_\alpha, \bigotimes_{\alpha \in A} \mathcal{T}_\alpha)$ iff for each $\alpha \in A$, $\pi_\alpha(a_\gamma) \xrightarrow{\gamma} \pi_\alpha(a)$ in $(X_\alpha, \mathcal{T}_\alpha)$.

Proof. Clear by Exercise 3.3.2.14. □

3.5.3 Maps and the Product Topology

Exercise 3.5.3.1. Let X be a topological space, $(Y_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ a collection of topological spaces and $f : X \to \prod_{\alpha \in A} Y_\alpha$. Then f is continuous iff for each $\alpha \in A$, $\pi_\alpha \circ f$ is continuous.

Proof. Immediate by a Exercise 3.2.0.12. □

Exercise 3.5.3.2. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ and $(Y_\alpha, \mathcal{S}_\alpha)_{\alpha \in A}$ be collections of topological spaces and $(f_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. If for each $\alpha \in A$, f_α is continuous, then $\prod\limits_{\alpha \in A} f_\alpha$ is continous.

Proof. Set $X := \prod\limits_{\alpha \in A} X_\alpha$, $Y := \prod\limits_{\alpha \in A} Y_\alpha$ and define $f : X \to Y$ by $f := \prod\limits_{\alpha \in A} f_\alpha$. Denote the α-th projection maps on X and Y by π_α^X and π_α^Y respectively. Set $\mathcal{T} := \bigotimes\limits_{\alpha \in A} \mathcal{T}_\alpha$ and $\mathcal{S} := \bigotimes\limits_{\alpha \in A} \mathcal{S}_\alpha$. Suppose that for each $\alpha \in A$, f_α is continuous. Exercise 1.3.0.4 implies that for each $\alpha \in A$, $\pi_\alpha^Y \circ f = f_\alpha \circ \pi_\alpha^X$. Let $\alpha \in A$. Then $f_\alpha \circ \pi_\alpha^X$ is continuous. Hence $\pi_\alpha^Y \circ f$ is continuous. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $\pi_\alpha^Y \circ f$ is continuous. Exercise 3.5.3.1 implies that f is continuous. □

Exercise 3.5.3.3. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ and $(Y_\alpha, \mathcal{S}_\alpha)_{\alpha \in A}$ be collections of topological spaces and $(f_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. If $\#\{\alpha \in A : f_\alpha \text{ is not surjective}\} < \infty$ and for each $\alpha \in A$, f_α is open, then $\prod\limits_{\alpha \in A} f_\alpha$ is open.

Proof. Set $X := \prod\limits_{\alpha \in A} X_\alpha$, $Y := \prod\limits_{\alpha \in A} Y_\alpha$ and define $f : X \to Y$ by $f := \prod\limits_{\alpha \in A} f_\alpha$. Denote the α-th projection maps on X and Y by π_α^X and π_α^Y respectively. Set $\mathcal{T} := \bigotimes\limits_{\alpha \in A} \mathcal{T}_\alpha$ and $\mathcal{S} := \bigotimes\limits_{\alpha \in A} \mathcal{S}_\alpha$. Suppose that $\#\{\alpha \in A : f_\alpha \text{ is not surjective}\} < \infty$ and for each $\alpha \in A$, f_α is open. Set

$$\mathcal{B}_X := \left\{ \prod_{\alpha \in A} U_\alpha : \text{ for each } \alpha \in A,\, U_\alpha \in \mathcal{T}_\alpha \text{ and } \#\{\alpha \in A : U_\alpha \neq X_\alpha\} < \infty \right\}$$

$$\mathcal{B}_Y := \left\{ \prod_{\alpha \in A} V_\alpha : \text{ for each } \alpha \in A,\, V_\alpha \in \mathcal{S}_\alpha \text{ and } \#\{\alpha \in A : V_\alpha \neq Y_\alpha\} < \infty \right\}$$

A previous exercise implies that $\mathcal{B}_X$ is a basis for $\mathcal{T}$ and $\mathcal{B}_Y$ is a basis for $\mathcal{S}$. Let $U \in \mathcal{B}_X$. Then for each $\alpha \in A$ there exist $U_\alpha \in \mathcal{T}_\alpha$ such that $U = \prod_{\alpha \in A} U_\alpha$. Define

- $B_1 := \{\alpha \in A : f_\alpha \text{ is not surjective}\}$
- $B_2 := \{\alpha \in A : U_\alpha \neq X_\alpha\}$
- $B_3 := \{\alpha \in A : f_\alpha(U_\alpha) \neq Y_\alpha\}$

Let $\alpha \in A$. Suppose that $\alpha \in B_1^c \cap B_2^c$. Then f_α is surjective and $U_\alpha = X_\alpha$. Thus $f_\alpha(U_\alpha) = Y_\alpha$ and $\alpha \in B_3^c$. Therefore if $\alpha \in B_3$, then $\alpha \in B_1 \cup B_2$. Since $\alpha \in A$ is arbitrary, $B_3 \subset B_1 \cup B_2$. By assumption, $\#B_1 < \infty$ and $\#B_2 < \infty$. Thus

$$\begin{aligned} \#B_3 &\leq \#(B_1 \cup B_2) \\ &\leq \#B_1 + \#B_2 \\ &< \infty \end{aligned}$$

Since for each $\alpha \in A$, f_α is open, we have that for each $\alpha \in A$, $f_\alpha(U_\alpha) \in \mathcal{S}_\alpha$. Thus

$$\begin{aligned} f\bigg(\prod_{\alpha \in A} U_\alpha\bigg) &= \prod_{\alpha \in A} f_\alpha(U_\alpha) \\ &\in \mathcal{B}_Y \\ &\subset \mathcal{S} \end{aligned}$$

Since $U \in \mathcal{B}_X$ is arbitrary, we have that for each $U \in \mathcal{B}_X$, $f(U) \in \mathcal{S}$. Since $\mathcal{B}_X$ is a basis for $\mathcal{T}$, an exercise about open maps in the section on continuous maps implies that f is open. $\square$

Exercise 3.5.3.4. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ and $(Y_\alpha, \mathcal{S}_\alpha)_{\alpha \in A}$ be collections of topological spaces and for each $\alpha \in A$, $U_\alpha \subset X_\alpha$, $V_\alpha \subset Y_\alpha$ and $f_\alpha : U_\alpha \to V_\alpha$. If for each $\alpha \in A$, f_α is $(\mathcal{T}_\alpha \cap U_\alpha)$-continuous, then $\prod_{\alpha \in A} f_\alpha$ is $\bigg(\bigg[\bigotimes_{\alpha \in A} \mathcal{T}_\alpha\bigg] \cap \bigg[\prod_{\alpha \in A} U_\alpha\bigg], \bigg[\bigotimes_{\alpha \in A} \mathcal{S}_\alpha\bigg] \cap \bigg[\prod_{\alpha \in A} V_\alpha\bigg]\bigg)$-continuous.

Proof. Denote the α-th projection maps on X and Y by π_α^X and π_α^Y respectively. Let $(x_\gamma)_{\gamma \in \Gamma} \subset \prod_{\alpha \in A} U_\alpha$ and $x \in \prod_{\alpha \in A}$. Suppose that $x_\gamma \to x$ in $\bigg(\prod_{\alpha \in A} U_\alpha, \bigg[\bigotimes_{\alpha \in A} \mathcal{T}_\alpha\bigg] \cap \bigg[\prod_{\alpha \in A} U_\alpha\bigg]\bigg)$. An exercise in the section on the subspace topology implies that $x_\gamma \to x$ in $\bigg(\prod_{\alpha \in A} X_\alpha, \bigotimes_{\alpha \in A} \mathcal{T}_\alpha\bigg)$. Let $\alpha \in A$. Since π_α^X is $\bigg(\bigotimes_{\beta \in A} \mathcal{T}_\beta, \mathcal{T}_\alpha\bigg)$-continuous, we have that $\pi_\alpha^X(x_\gamma) \to \pi_\alpha^X(x)$ in $(X_\alpha, \mathcal{T}_\alpha)$. Another application of the same exercise implies that $\pi_\alpha^X(x_\gamma) \to \pi_\alpha^X(x)$ in $(U_\alpha, \mathcal{T}_\alpha \cap U_\alpha)$. Since f_α is $\bigg(\mathcal{T}_\alpha \cap U_\alpha, \mathcal{S}_\alpha \cap V_\alpha\bigg)$-continuous,

$$\begin{aligned} \pi_\alpha^Y \circ \bigg[\prod_{\alpha \in A} f_\alpha\bigg](x_\gamma) &= f_\alpha \circ \pi_\alpha^X(x_\gamma) \\ &\to f_\alpha \circ \pi_\alpha^X(x) \\ &= \pi_\alpha^Y \circ \bigg[\prod_{\alpha \in A} f_\alpha\bigg](x) \text{ in } (V_\alpha, \mathcal{S}_\alpha \cap V_\alpha) \end{aligned}$$

Another application of the exercise implies that $\pi_\alpha^Y \circ \bigg[\prod_{\alpha \in A} f_\alpha\bigg](x_\gamma) \to \pi_\alpha^Y \circ \bigg[\prod_{\alpha \in A} f_\alpha\bigg](x)$ in $(Y_\alpha, \mathcal{S}_\alpha)$. Since $(x_\gamma)_{\gamma \in \Gamma} \subset \prod_{\alpha \in A} U_\alpha$ and $x \in \prod_{\alpha \in A}$ with $x_\gamma \to x$ in $\bigg(\prod_{\alpha \in A} U_\alpha, \bigg[\bigotimes_{\alpha \in A} \mathcal{T}_\alpha\bigg] \cap \bigg[\prod_{\alpha \in A} U_\alpha\bigg]\bigg)$ is arbitary, we have that $\prod_{\alpha \in A} f_\alpha$ is $\left(\left[\bigotimes_{\alpha \in A} T_\alpha\right] \cap \left[\prod_{\alpha \in A} U_\alpha\right], \bigotimes_{\alpha \in A} S_\alpha\right)$-continuous. Another application of the exercise implies that $\prod_{\alpha \in A} f_\alpha$ is $\left(\left[\bigotimes_{\alpha \in A} T_\alpha\right] \cap \left[\prod_{\alpha \in A} U_\alpha\right], \left[\bigotimes_{\alpha \in A} S_\alpha\right] \cap \left[\prod_{\alpha \in A} V_\alpha\right]\right)$-continuous. □

Exercise 3.5.3.5. Let $(X, \mathcal{T}_X)$ be a topological space, $(Y_\alpha, \mathcal{T}_{Y_\alpha})_{\alpha \in A}$ a collection of topological spaces, $(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} Y_\alpha^X$. If for each $\alpha \in A$, f_α is $(\mathcal{T}_X, \mathcal{T}_{Y_\alpha})$-continuous, then $(f_\alpha)_{\alpha \in A}$ is $(\mathcal{T}_X, \bigotimes_{\alpha \in A} \mathcal{T}_{Y_\alpha})$-continuous.

Proof. Suppose that for each $\alpha \in A$, f_α is $(\mathcal{T}_X, \mathcal{T}_{Y_\alpha})$-continuous. Set $f := (f_\alpha)_{\alpha \in A}$. Since for each $\alpha \in A$, $\pi_\alpha \circ f = f_\alpha$, Exercise 3.5.3.1 implies that f is $(\mathcal{T}_X, \bigotimes_{\alpha \in A} \mathcal{T}_{Y_\alpha})$-continuous. □

Definition 3.5.3.6. define slice maps and then define the below sets in terms of them in the set theory section.

Exercise 3.5.3.7. Let X and Y be topological spaces and $U \subset X \times Y$ open. Then for each $(x_0, y_0) \in U$, U^{x_0} and U^{y_0} are open.

Proof. Let $(x_0, y_0) \in U$. Define $\phi : X \to X \times Y$ by $\phi(x) = (x, y_0)$. Since $\pi_X \circ \phi = \mathrm{id}_X$ and $\pi_Y \circ \phi$ is constant, $\pi_X \circ \phi$ and $\pi_Y \circ \phi$ are continous. Therefore, ϕ is continuous. Then U^{y_0} is open since U is open and $\phi^{-1}(U) = U^{y_0}$. Similarly, U_{x_0} is open. □

Exercise 3.5.3.8. Let X, Y and Z be topological spaces, $U \subset X \times Y$ open and $f : U \to Z$. Equip U with the subspace topology. Suppose that f is continuous. Let $(x_0, y_0) \in U$. Equip U_{x_0} and U^{y_0} with the subspace topology. Then $f_{x_0} : U_{x_0} \to Z$ and $f^{y_0} : U^{y_0} \to Z$ are continuous.

Proof. Let $(x_0, y_0) \in U$. Let $V \subset Z$. Suppose that V is open. Continuity of f implies that $f^{-1}(V)$ is open in U. Since U is open in $X \times Y$, $f^{-1}(V)$ is open in $X \times Y$. A previous exercise in the section on product sets implies that $(f^{y_0})^{-1}(V) = (f^{-1}(V))^{y_0}$. The previous exercise implies that $(f^{-1}(V))^{y_0}$ is open in X. So $(f^{y_0})^{-1}(V)$ is open in X. Since $(f^{y_0})^{-1}(V) \subset U^{y_0}$, $(f^{y_0})^{-1}(V)$ is open in U^{y^0}. Thus $f^{y_0} : U^{y_0} \to Z$ is continuous. Similarly, $f_{x_0} : U_{x_0} \to Z$ is continuous. □

3.6 Coproduct Topology

Definition 3.6.0.1. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. We define the **coproduct topology** on $\coprod_{\alpha \in A} X_\alpha$, denoted $\bigoplus_{\alpha \in A} \mathcal{A}_\alpha$, by

$$\bigoplus_{\alpha \in A} \mathcal{T}_\alpha = \tau(\iota_\alpha : \alpha \in A)$$

i.e. $\bigoplus_{\alpha \in A} \mathcal{T}_\alpha$ is the final topology on $\coprod_{\alpha \in A} X_\alpha$ generated by the embedding maps $(\iota_\alpha)_{\alpha \in A}$.

Exercise 3.6.0.2. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Then

$$\bigoplus_{\alpha \in A} \mathcal{T}_\alpha = \left\{ V \subset \coprod_{\alpha \in A} X_\alpha : \text{ for each } \alpha \in A,\ \iota_\alpha^{-1}(V) \in \mathcal{T}_\alpha \right\}$$

Proof. Set $X := \coprod_{\alpha \in A} X_\alpha$ and $\mathcal{T} := \left\{ V \subset \coprod_{\alpha \in A} X_\alpha : \text{ for each } \alpha \in A,\ \iota_\alpha^{-1}(V) \in \mathcal{T}_\alpha \right\}$.

- 1. Clearly $\varnothing, X \in \mathcal{T}$.
  2. Let $(U_\gamma)_{\gamma \in \Gamma} \subset \mathcal{T}$. Then by definition, for each $\gamma \in \Gamma$ and $\alpha \in A$, $\iota_\alpha^{-1}(U_\gamma) \in \mathcal{T}_\alpha$. Hence

$$\begin{aligned} \iota_\alpha^{-1}\left(\bigcup_{\gamma \in \Gamma} U_\gamma \right) &= \bigcup_{\gamma \in \Gamma} \iota_\alpha^{-1}(U_\gamma) \\ &\in \mathcal{T}_\alpha \end{aligned}$$

  3. Let $(U_j)_{j=1}^n \subset \mathcal{T}$. Then by definition, for each $\alpha \in A$ and $j \in [n]$, $\iota_\alpha^{-1}(U_j) \in \mathcal{T}_\alpha$. Hence

$$\begin{aligned} \iota_\alpha^{-1}\left(\bigcap_{j=1}^n U_j \right) &= \bigcap_{j=1}^n \iota_\alpha^{-1}(U_j) \\ &\in \mathcal{T}_\alpha \end{aligned}$$

  So $\mathcal{T}$ is a topology on X.

- Since $\mathcal{T}$ is a topology on X, we have that $\mathcal{T} = \tau_X(\mathcal{T})$. By Definition 3.2.0.14,

$$\begin{aligned} \mathcal{T} &= \tau_X(\mathcal{T}) \\ &= \tau_X(\iota_\alpha : \alpha \in A) \\ &= \bigoplus_{\alpha \in A} \mathcal{T}_\alpha. \end{aligned}$$

□

Exercise 3.6.0.3. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces. Then

$$\bigoplus_{\alpha \in A} \mathcal{T}_\alpha = \left\{ \coprod_{\alpha \in A} B_\alpha : B_\alpha \in \mathcal{T}_\alpha \right\}$$

Proof. Set

- $\mathcal{F} = \{V \subset \coprod_{\alpha \in A} X_\alpha : \text{ for each } \alpha \in A,\ \iota_\alpha^{-1}(V) \in \mathcal{T}_\alpha\}$
- $\mathcal{G} = \left\{ \coprod_{\alpha \in A} B_\alpha : \text{ for each } \alpha \in A,\ B_\alpha \in \mathcal{T}_\alpha \right\}$

Let $V \in \mathcal{G}$. Then for each $\alpha \in A$, there exists $B_\alpha \in \mathcal{T}_\alpha$ such that $V = \coprod_{\alpha \in A} B_\alpha$. Therefore, for each $\alpha \in A$,

$$\begin{aligned} \iota_\alpha^{-1}(V) &= \iota_\alpha^{-1}\bigg(\coprod_{\alpha \in A} B_\alpha\bigg) \\ &= B_\alpha \\ &\in \mathcal{T}_\alpha \end{aligned}$$

Hence $V \in \mathcal{F}$. Since $V \in \mathcal{G}$ is arbitrary, $\mathcal{G} \subset \mathcal{F}$.
Conversely, let $V \in \mathcal{F}$. Then for each $\alpha \in A$, $\iota_\alpha^{-1}(V) \in \mathcal{T}_\alpha$. For each $\alpha \in A$, define $B_\alpha \in \mathcal{T}_\alpha$ by $B_\alpha = \iota_\alpha^{-1}(V)$. Then

$$\begin{aligned} V &= \coprod_{\alpha \in A} B_\alpha \\ &\in \mathcal{G} \end{aligned}$$

Since $V \in \mathcal{F}$ is arbitrary, $\mathcal{F} \subset \mathcal{G}$. The previous exercise implies that

$$\begin{aligned} \mathcal{G} &= \mathcal{F} \\ &= \bigoplus_{\alpha \in A} \mathcal{T}_\alpha \end{aligned}$$

□

Exercise 3.6.0.4. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Then for each $\alpha \in A$, $\iota_\alpha : X_\alpha \to \coprod_{\alpha' \in A} X_{\alpha'}$ is $\bigg(\mathcal{T}_\alpha, \bigoplus_{\alpha' \in A} \mathcal{T}_{\alpha'}\bigg)$-open.

Proof. Let $\alpha, \beta \in A$ and $U \in \mathcal{T}_\alpha$. Then

$$\begin{aligned} \iota_\beta^{-1}(\iota_\alpha(U)) &= \iota_\beta^{-1}(\{\alpha\} \times U) \\ &= \begin{cases} U, & \beta = \alpha \\ \varnothing, & \beta \neq \alpha \end{cases} \\ &\in \mathcal{T}_\beta \end{aligned}$$

Since $\beta \in A$ is arbitrary, we have that for each $\beta \in A$, $\iota_\beta^{-1}(\iota_\alpha(U)) \in \mathcal{T}_\beta$. Exercise 3.6.0.2 implies that $\iota_\alpha(U) \in \bigoplus_{\beta \in A} \mathcal{T}_\beta$. Since $U \in \mathcal{T}_\alpha$ is arbitrary, we have that $\bigg(\mathcal{T}_\alpha, \bigoplus_{\alpha' \in A} \mathcal{T}_{\alpha'}\bigg)$-open. □

Exercise 3.6.0.5. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces and for each $\alpha \in A$, $\mathcal{B}_\alpha \subset \mathcal{T}_{a\langle}$. Suppose that for each $\alpha \in A$, B_α is a basis for $\mathcal{T}_\alpha$. Then $\{\iota_\alpha(U) : \alpha \in A \text{ and } U \in \mathcal{B}_\alpha\}$ is a basis for $\bigoplus_{\alpha \in A} \mathcal{T}_\alpha$.

Proof. Set $\mathcal{B} := \{\iota_\alpha(U) : \alpha \in A \text{ and } U \in \mathcal{B}_\alpha\}$ and $\mathcal{T} := \bigoplus_{\alpha \in A} \mathcal{T}_\alpha$. Exercise 3.6.0.4 implies that $\mathcal{B} \subset \mathcal{T}$. Let $W \in \mathcal{T}$ and $(\alpha_0, x_0) \in W$. Set $U := \iota_{\alpha_0}^{-1}(W)$. By assumption, $x_0 \in U$. Since ι_{α_0} is $(\mathcal{T}_{\alpha_0}, \mathcal{T})$-continuous, we have that $U \in \mathcal{T}_{\alpha_0}$. Exercise 3.6.0.4 implies that

$$\begin{aligned} \{\alpha_0\} \times U &= \iota_{\alpha_0}(U) \\ &\in \mathcal{T}) \end{aligned}$$

By construction,

$$\begin{aligned} (\alpha_0, x_0) &\in \{\alpha_0\} \times U \\ &= \iota_{\alpha_0}(U) \\ &= \iota_{\alpha_0}(\iota_{\alpha_0}^{-1}(W)) \\ &\subset W. \end{aligned}$$

Since $W \in \mathcal{T}$ and $(\alpha_0, x_0) \in W$ are arbitrary, we have that for each $W \in \mathcal{T}$ and $(\alpha_0, x_0) \in W$, there exists $B \in \mathcal{B}$ such that $(\alpha_0, x_0) \in B \subset W$. Therefore $\mathcal{B}$ is a basis for $\mathcal{T}$. □

Note 3.6.0.6. Let Γ be a directed set and $\gamma_0 \in \Gamma$. From Definition **??**, we recall the γ_0-tail operator $L_{\gamma_0} : X^\Gamma \to X^{[\gamma_0, \infty)}$.

Exercise 3.6.0.7. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces, $(\alpha_\gamma, x_\gamma)_{\gamma \in \Gamma} \subset \coprod_{\alpha \in A} X_\alpha$ a net and $(\alpha_0, x_0) \in \coprod_{\alpha \in A} X_\alpha$. Then $(\alpha_\gamma, x_\gamma) \to (\alpha_0, x_0)$ in $\left(\coprod_{\alpha \in A} X_\alpha, \bigoplus_{\alpha \in A} \mathcal{T}_\alpha\right)$ iff there exists $\gamma_0 \in \Gamma$ such that

1. for each $\gamma \in \Gamma$, $\gamma \geq \gamma_0$ implies that $\alpha_\gamma = \alpha_0$ and $x_\gamma \in X_{\alpha_0}$
2. $[L_{\gamma_0}(x)]_\gamma \to x_0$ in $(X_{\alpha_0}, \mathcal{T}_{\alpha_0})$.

Proof. Set $X := \coprod_{\alpha \in A} X_\alpha$ and $\mathcal{T} := \bigoplus_{\alpha \in A} \mathcal{T}_\alpha$.

- $(\implies)$:

  1. Suppose that $(\alpha_\gamma, x_\gamma) \to (\alpha_0, x_0)$ in Exercise 3.6.0.4 implies that $\{\alpha_0\} \times X_{\alpha_0} \in \mathcal{N}(\alpha_0, x_0)$. Since $(\alpha_\gamma, x_\gamma) \to (\alpha_0, x_0)$, $(\alpha_\gamma, x_\gamma)_{\gamma \in \Gamma}$ is eventually in $\{\alpha_0\} \times X_{\alpha_0}$. Exercise 3.3.2.2 implies that there exists $\gamma_0 \in \Gamma$ such that $(\alpha_\gamma, x_\gamma)_{\gamma \in [\gamma_0, \infty)} \subset \{\alpha_0\} \times X_{\alpha_0}$. Hence for each $\gamma \in \Gamma$, $\gamma \geq \gamma_0$ implies that $\alpha_\gamma = \alpha_0$ and $(x_\gamma)_{\gamma \in [\gamma_0, \infty)} \subset X_{\alpha_0}$.
  2. Define $(y_\gamma)_{\gamma \in [\gamma_0, \infty)} \subset X_{\alpha_0}$ by $y := L_{\gamma_0}(x)$. Let $U \in \mathcal{T}_{\alpha_0}$. Suppose that $x_0 \in U$. Exercise 3.6.0.4 implies that $\{\alpha_0\} \times U \in \mathcal{N}(\alpha_0, x_0)$. Since $(\alpha_\gamma, x_\gamma) \to (\alpha_0, x_0)$ in $(X, \mathcal{T})$, $(\alpha_\gamma, x_\gamma)_{\gamma \in \Gamma}$ is eventually in $\{\alpha_0\} \times U$. Exercise 3.3.2.2 implies that there exists $\gamma_1 \in \Gamma$ such that $(\alpha_\gamma, x_\gamma)_{\gamma \in [\gamma_1, \infty)} \subset \{\alpha_0\} \times U$. Since A is a directed set, there exists $\gamma_2 \in \Gamma$ such that $\gamma_0, \gamma_1 \leq \gamma_2$. Hence for each $\gamma \in [\gamma_2, \infty)$, $y_\gamma \in U$. So $(y_\gamma)_{\gamma \in [\gamma_0, \infty)}$ is eventually in U. Since $U \in \mathcal{T}_{\alpha_0}$ with $x_0 \in U$ is arbitrary, we have that $y_\gamma \to x_0$.

- $(\impliedby)$:
  Suppose that there exists $\gamma_0 \in \Gamma$ such that

  1. for each $\gamma \in \Gamma$, $\gamma \geq \gamma_0$ implies that $\alpha_\gamma = \alpha_0$ and $x_\gamma \in X_{\alpha_0}$
  2. $[L_{\gamma_0}(x)]_\gamma \to x_0$ in $(X_{\alpha_0}, \mathcal{T}_{\alpha_0})$.

  Let $W_0 \in \mathcal{N}(\alpha_0, x_0)$. Set $W := \operatorname{Int} W_0$. Then $W \in \mathcal{T}$ and $(\alpha_0, x_0) \in W$. Exercise 3.6.0.5 implies that there exists $U \in \mathcal{T}_{\alpha_0}$ such that $(\alpha_0, x_0) \in \{\alpha_0\} \times U \subset W$. By assumption, there exists $\gamma_1 \in \Gamma$ such that for each $\gamma \in \Gamma$, $\gamma \geq \gamma_1$ implies that $\alpha_\gamma = \alpha_0$ and $x_\gamma \in X_{\alpha_0}$. Since $[L_{\gamma_1}(x)]_\gamma \to x_0$ in $(X_{\alpha_0}, \mathcal{T}_{\alpha_0})$ and $U \in \mathcal{T}_{\alpha_0}$, there exists $\gamma_2 \in [\gamma_1, \infty)$ such that for each $\gamma \in [\gamma_1, \infty)$, $\gamma \geq \gamma_2$ implies that $x_\gamma \in U$. Since Γ is a directed set, there exists $\gamma_0 \in \Gamma$ such that $\gamma \geq \gamma_1, \gamma_2$. Let $\gamma \in \Gamma$. Suppose that $\gamma \geq \gamma_0$. Then

$$\begin{aligned}
(\alpha_\gamma, x_\gamma) &= (\alpha_0, x_\gamma) \\
&\in \{\alpha_0\} \times U \\
&\subset W \\
&\subset W_0
\end{aligned}$$

  Hence $(\alpha_\gamma, x_\gamma)$ is eventually in W_0. Since $W_0 \in \mathcal{N}(\alpha_0, x_0)$ is arbitrary, we have that for each $W_0 \in \mathcal{N}(\alpha_0, x_0)$, $(\alpha_\gamma, x_\gamma)$ is eventually in W_0. Thus $(\alpha_\gamma, x_\gamma) \to (\alpha_0, x_0)$ in $(X, \mathcal{T})$.

□

Exercise 3.6.0.8. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces, $(Y, \mathcal{S})$ a topological space and $f : \coprod_{\alpha \in A} X_\alpha \to Y$. Then f is $\left(\bigoplus_{\alpha \in A} \mathcal{T}_\alpha, \mathcal{S}\right)$-continuous iff for each $\alpha \in A$, $f \circ \iota_\alpha$ is $(\mathcal{T}_\alpha, \mathcal{S})$-continuous.

Proof. Clear by Exercise 3.2.0.18. add more details □

Exercise 3.6.0.9. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in A}$ and $(Y_\alpha, \mathcal{S}_\alpha)_{\alpha\in A}$ be collections of continuous spaces and $(f_\alpha)_{\alpha\in A} \in \coprod_{\alpha\in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. If for each $\alpha \in A$, f_α is $(\mathcal{T}_\alpha, \mathcal{S}_\alpha)$-continuous, then $\coprod_{\alpha\in A} f_\alpha$ is $(\bigoplus_{\alpha\in A} \mathcal{T}_\alpha, \bigoplus_{\alpha\in A} \mathcal{S}_\alpha)$-continuous.

Proof. Set $X := \coprod_{\alpha\in A} X_\alpha$, $Y := \coprod_{\alpha\in A} Y_\alpha$, $\mathcal{T} := \bigoplus_{\alpha\in A} \mathcal{T}_\alpha$ and $\mathcal{S} := \bigoplus_{\alpha\in A} \mathcal{S}_\alpha$. Suppose that for each $\alpha \in A$, f_α is $(\mathcal{T}_\alpha, \mathcal{S}_\alpha)$-continuous. Set $f = \coprod_{\alpha\in A} f_\alpha$. Denote the α-th embedding maps on X and Y by ι_α^X and ι_α^Y respectively. Let $\alpha \in A$. Exercise 1.4.0.4 implies that $f \circ \iota_\alpha^X = \iota_\alpha^Y \circ f_\alpha$. Since $\iota_\alpha^Y \circ f_\alpha$ is $(\mathcal{T}_\alpha, \mathcal{S})$-continuous, we have that $f \circ \iota_\alpha^X$ is $(\mathcal{T}_\alpha, \mathcal{S})$-continuous. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $f \circ \iota_\alpha^X$ is $(\mathcal{T}_\alpha, \mathcal{S})$-continuous. Exercise 3.6.0.8 implies that f is $(\mathcal{T}, \mathcal{S})$-continuous. □

Exercise 3.6.0.10. Let $(X, \mathcal{T})$ be a topological space and $(E_\alpha)_{\alpha\in A} \subset \mathcal{T}$. Define $\phi : \coprod_{\alpha\in A} E_\alpha \to \bigcup_{\alpha\in A} E_\alpha$ by $\phi(\alpha, x) := x$. If $(E_\alpha)_{\alpha\in A}$ is disjoint, then ϕ is a $\left(\bigoplus_{\alpha\in A} [\mathcal{T} \cap E_\alpha], \mathcal{T} \cap \left[\bigcup_{\alpha\in A} E_\alpha\right]\right)$-homeomorphism.

Proof. Suppose that $(E_\alpha)_{\alpha\in A}$ is disjoint.

- **(bijectivity) :**
  - **(injectivity) :**
    Let $(\alpha, x), (\beta, y) \in \coprod_{\alpha\in A} E_\alpha$. Suppose that $\phi(\alpha, x) = \phi(\beta, y)$. Then $x = y$. Thus $x \in E_\alpha \cap E_\beta$ and therefore $E_\alpha \cap E_\beta \neq \varnothing$. Since $(E_{\alpha'})_{\alpha'\in A}$ is disjoint, we have that $\alpha = \beta$. Hence $(\alpha, x) = (\beta, y)$. Since $(\alpha, x), (\beta, y) \in \coprod_{\alpha\in A} E_\alpha$ are arbitrary, we have that for each $(\alpha, x), (\beta, y) \in \coprod_{\alpha\in A} E_\alpha$, $\phi(\alpha, x) = \phi(\beta, y)$ implies that $(\alpha, x) = (\beta, y)$. Thus ϕ is injective.
  - **(surjectivity) :**
    Let $x \in \bigcup_{\alpha\in A} E_\alpha$. Then there exists $\alpha \in A$ such that $x \in E_\alpha$. Then $(\alpha, x) \in \coprod_{\alpha\in A} E_\alpha$ and $\phi(\alpha, x) = x$. Since $x \in \bigcup_{\alpha\in A} E_\alpha$ is arbitrary, we have that for each $x \in \bigcup_{\alpha\in A} E_\alpha$, there exists $a \in \coprod_{\alpha\in A} E_\alpha$ such that $\phi(a) = x$. Hence ϕ is surjective.

  So ϕ is a bijection.

- **(continuity) :**
  For each $\alpha \in A$, define $\iota_{E_\alpha} : E_\alpha \to \bigcup_{\alpha\in A} E_\alpha$ by $\iota_{E_\alpha}(x) = x$.
  - Let $\alpha \in A$. Since $\phi \circ \iota_\alpha = \iota_{E_\alpha}$ and ι_{E_α} is $\left(\mathcal{T} \cap E_\alpha, \mathcal{T} \cap \left[\bigcup_{\alpha\in A} E_\alpha\right]\right)$-continuous (maybe give more details), we have that $\phi \circ \iota_\alpha$ is $\left(\mathcal{T} \cap E_\alpha, \mathcal{T} \cap \left[\bigcup_{\alpha\in A} E_\alpha\right]\right)$-continuous. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $\phi \circ \iota_\alpha$ is $\left(\mathcal{T} \cap E_\alpha, \mathcal{T} \cap \left[\bigcup_{\alpha\in A} E_\alpha\right]\right)$-continuous. Exercise 3.6.0.8 implies that ϕ is $\left(\bigoplus_{\alpha\in A} (\mathcal{T} \cap E_\alpha), \mathcal{T} \cap \left[\bigcup_{\alpha\in A} E_\alpha\right]\right)$-continuous.
  - Let $B \in \bigoplus_{\alpha\in A} (\mathcal{T} \cap E_\alpha)$. Exercise 3.6.0.3 implies that for each $\alpha \in A$, there exist $B_\alpha \in \mathcal{T} \cap E_\alpha$ such that $B = \coprod_{\alpha\in A} B_\alpha$. Then for each $\alpha \in A$, there exists $C_\alpha \in \mathcal{T}$ such that $B_\alpha = C_\alpha \cap E_\alpha$. Since

$(E_\alpha)_{\alpha\in A} \subset \mathcal{T}$, we have that for each $\alpha \in A$, $C_\alpha \cap E_\alpha \in \mathcal{T}$. Hence for each $\alpha \in A$,

$$\begin{aligned} B_\alpha &= C_\alpha \cap E_\alpha \\ &= (C_\alpha \cap E_\alpha) \cap \left[\bigcup_{\alpha' \in \mathbb{N}} E_{\alpha'} \right] \\ &\in \mathcal{T} \cap \left[\bigcup_{\alpha' \in \mathbb{N}} E_{\alpha'} \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \phi(B) &= \phi\left(\coprod_{\alpha\in A} B_\alpha \right) \\ &= \phi\left(\bigcup_{\alpha\in A} \iota_\alpha(B_\alpha) \right) \\ &= \bigcup_{\alpha\in A} \phi \circ \iota_\alpha(B_\alpha) \\ &= \bigcup_{\alpha\in A} \iota_{E_\alpha}(B_\alpha) \\ &= \bigcup_{\alpha\in A} B_\alpha \\ &\in \mathcal{T} \cap \left[\bigcup_{\alpha\in A} E_\alpha \right]. \end{aligned}$$

Since $B \in \bigoplus_{\alpha\in A} (\mathcal{T} \cap E_\alpha)$ is arbitrary, we have that for each $B \in \bigoplus_{\alpha\in A} (\mathcal{T} \cap E_\alpha)$, $f(B) \in \mathcal{T}$.

Hence ϕ is a homeomorphism.

$\square$

3.7 Quotient Topology

3.7.1 Introduction

Definition 3.7.1.1. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be an $(\mathcal{T}_X, \mathcal{T}_Y)$**-quotient map** if

1. f is surjective
2. $\mathcal{T}_Y = f_* \mathcal{T}_X$

Note 3.7.1.2. We typically avoid specifying the topologies when they are clear from the context.

Exercise 3.7.1.3. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Suppose that f is surjective. Then

1. f is a quotient map iff
$$\text{for each } V \subset Y,\ V \in \mathcal{T}_Y \text{ iff } f^{-1}(V) \in \mathcal{T}_X$$
2. f is a quotient map iff
$$\text{for each } C \subset Y,\ C \text{ is closed in } Y \text{ iff } f^{-1}(C) \text{ is closed in } X$$

Proof.

1. • $(\implies)$
Suppose that f is a quotient map.
Let $V \subset Y$. Suppose that V is open, then
$$\begin{aligned} V &\in \mathcal{T}_Y \\ &= f_* \mathcal{T}_X \\ &= \{V' \subset Y : f^{-1}(V') \in \mathcal{T}_X\} \end{aligned}$$
Thus $f^{-1}(V) \in \mathcal{T}_X$. Hence $f^{-1}(V)$ is open.
Conversely, suppose that $f^{-1}(V)$ is open. Then $f^{-1}(V) \in \mathcal{T}_X$. Since
$$\begin{aligned} \mathcal{T}_Y &= f_* \mathcal{T}_X \\ &= \{V' \subset Y : f^{-1}(V') \in \mathcal{T}_X\}, \end{aligned}$$
we have that $V \in \mathcal{T}_Y$. Hence V is open.
Thus V is open iff $f^{-1}(V)$ is open. Since $V \subset Y$ is arbitrary, we have that for each $V \subset Y$, V is open iff $f^{-1}(V)$ is open.
   • $(\impliedby)$
Suppose that for each $V \subset Y$, V is open iff $f^{-1}(V)$ is open. Then
$$\begin{aligned} \mathcal{T}_Y &= \{V \subset Y : f^{-1}(V) \in \mathcal{T}_X\} \\ &= f_* \mathcal{T}_X \end{aligned}$$
So f is a quotient map.
2. • $(\implies)$
Suppose that f is a quotient map.
Let $C \subset Y$. Suppose that C is closed, then $C^c \in \mathcal{T}_Y$. Continuity implies that
$$\begin{aligned} f^{-1}(C)^c &= f^{-1}(C^c) \\ &\in \mathcal{T}_X \end{aligned}$$

Thus $f^{-1}(C)$ is closed.
Conversely, suppose that $f^{-1}(C)$ is closed. Then

$$\begin{aligned} f^{-1}(C^c) &= f^{-1}(C)^c \\ &\in \mathcal{T}_X \end{aligned}$$

Since f is a quotient map, $C^c \in \mathcal{T}_Y$. Hence C is closed.
Thus C is closed iff $f^{-1}(C)$ is closed. Since $C \subset Y$ is arbitrary, we have that for each $C \subset Y$, C is closed iff $f^{-1}(C)$ is closed.

- $(\Longleftarrow)$
  Suppose that for each $C \subset Y$, C is closed iff $f^{-1}(C)$ is closed.
  Let $V \subset Y$. Suppose that V is open. Then V^c is closed. By assumption $f^{-1}(V^c)$ is closed and therefore

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V^c)^c \\ &\in \mathcal{T}_X. \end{aligned}$$

  Thus $f^{-1}(V)$ is open.
  Conversely, suppose that $f^{-1}(V)$ is open. Then $f^{-1}(V)^c$ is closed. Since $f^{-1}(V^c) = f^{-1}(V)^c$, we have that $f^{-1}(V^c)$ is closed. By assumption, V^c is closed. Thus V is open.
  Thus V is open iff $f^{-1}(V)$ is open. Since $V \subset Y$ is arbitrary, we have that for each $V \subset Y$, V is open iff $f^{-1}(V)$ is open. Part (1) implies that f is a quotient map.

$\square$

Exercise 3.7.1.4. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. If f is a quotient map, then f is continuous.

Proof. Suppose that f is a quotient map. Let $V \subset Y$. Suppose that V is open. The previous exercise implies that $f^{-1}(V)$ is open. Since $V \subset Y$ with V open is arbitrary, we have that for each $V \subset Y$, V is open implies that $f^{-1}(V)$ is open. Hence f is continuous. $\square$

Exercise 3.7.1.5. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y), (Z, \mathcal{C})$ be topological spaces, $f : X \to Y$ and $g : Y \to Z$. If f is a quotient map, then g is continuous iff $g \circ f$ is continuous.

Proof. Suppose that f is a quotient map. Then $\mathcal{T}_Y = f_*\mathcal{T}_X$. An exercise in the section on continuous maps implies that g is continuous iff $g \circ f$ is continuous. $\square$

Exercise 3.7.1.6. Let $(X, \mathcal{T}_X)$, $(Y_1, \mathcal{T}_{Y_1})$, $(Y_2, \mathcal{T}_{Y_2})$ be topological spaces, $f_1 : X \to Y_1$, $f_2 : X \to Y_2$ and $\phi : Y_1 \to Y_2$. Suppose that f_1 and f_2 are quotient maps and ϕ is a bijection. If $\phi \circ f_1 = f_2$, then ϕ is a homeomorphism.

Proof. Since f_1 and f_2 are quotient maps, they are continous. Suppose that $\phi \circ f_1 = f_2$. Since f_2 is continuous, the previous exercise implies that ϕ is continuous. Since ϕ is a bijection, $f_1 = \phi^{-1} \circ f_2$. Similarly, since f_1 is continuous, the previous exercise implies that ϕ^{-1} is continuous. Hence ϕ is a homeomorphism. $\square$

Exercise 3.7.1.7. Restate the last exercise categorically: Let $U : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor. If $\phi \in \mathrm{Iso}_{U(X)/U(\mathbf{Top})}(U(f_1), U(f_2))$, then there exists $\phi' \in \mathrm{Iso}_{X/\mathbf{Top}}(f_1, f_2)$ such that $U(\phi') = \phi$, adjoint functor?...

Proof. FINISH!!! $\square$

Exercise 3.7.1.8. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Suppose that f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous and surjective. If f is $(\mathcal{T}_X, \mathcal{T}_Y)$-open or f is $(\mathcal{T}_X, \mathcal{T}_Y)$-closed, then f is a $(\mathcal{T}_X, \mathcal{T}_Y)$-quotient map.

Proof.

- Suppose that f is $(\mathcal{T}_X, \mathcal{T}_Y)$-open. Let $V \subset Y$.
  Suppose that $V \in \mathcal{T}_Y$. Continuity implies that $f^{-1}(V) \in \mathcal{T}_X$. Conversely, suppose that $f^{-1}(V) \in \mathcal{T}_X$. Since f is open $f(f^{-1}(V)) \in \mathcal{T}_Y$. Surjectivity implies that $V = f(f^{-1}(V))$. So $V \in \mathcal{T}_Y$. Exercise 3.7.1.3 then implies that f is a quotient map.

- Suppose that f is closed. Then similarly to above, f is a quotient map.

$\square$

Exercise 3.7.1.9. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Suppose that f is a quotient map. Then

1. f is open iff
$$\text{for each } U \subset X,\ U \text{ is open implies that } f^{-1}(f(U)) \text{ is open}$$
2. f is closed iff
$$\text{for each } C \subset X,\ C \text{ is closed implies that } f^{-1}(f(C)) \text{ is closed}$$

Proof.

1.
   - $(\implies)$
     Suppose that f is open.
     Let $U \subset X$. Suppose that U is open. Since f is open, $f(U)$ is open. Continuity implies that $f^{-1}(f(U))$ is open.

   - $(\impliedby)$
     Suppose that for each $U \subset X$, U is open implies that $f^{-1}(f(U))$ is open.
     Let $U \subset X$. Suppose that U is open. By assumption, $f^{-1}(f(U))$ is open. Since f is a quotient map, $f(U)$ is open. Since $U \subset X$ with U open is arbitrary, we have that for each $U \subset X$, U is open implies $f(U)$ is open. Thus f is open.
2.
   - $(\implies)$
     Suppose that f is closed.
     Let $C \subset X$. Suppose that C is closed. Since f is closed, $f(C)$ is closed. Continuity implies that $f^{-1}(f(C))$ is closed.

   - $(\impliedby)$
     Suppose that for each $X \subset X$, C is closed implies that $f^{-1}(f(C))$ is closed.
     Let $C \subset X$. Suppose that C is closed. By assumption, $f^{-1}(f(C))$ is closed. Since f is a quotient map, $f(C)$ is closed. Since $C \subset X$ with C closed is arbitrary, we have that for each $C \subset X$, C is closed implies $f(C)$ is closed. Thus f is closed.

$\square$

Exercise 3.7.1.10. Let $(X, \mathcal{T})$ be a topological space, Y a set and $f : X \to Y$. Suppose that f is surjective. Then $f : X \to Y$ is a $(\mathcal{T}, f_*\mathcal{T})$ quotient map.

Proof. Clear by definition. $\square$

Exercise 3.7.1.11. Let $(X, \mathcal{T})$ be a topological space, $\sim$ an eqivalence relation on X and $\pi : X \to X/\sim$ the projection map given by $x \mapsto \bar{x}$. Then π is a $(\mathcal{T}, \pi_*\mathcal{T})$-quotient map.

Proof. Since π is surjective, the previous exercise implies that π is a $(\mathcal{T}, \pi_*\mathcal{T})$-quotient map. $\square$

Definition 3.7.1.12. Let $(X, \mathcal{T})$ be a topological space, $\sim$ an eqivalence relation on X and $\pi : X \to X/\sim$ the projection map given by $x \mapsto \bar{x}$. We define the **quotient topology on** $X/\sim$ on $X/\sim$, denoted $\mathcal{T}_{X/\sim}$, by
$$\mathcal{T}_{X/\sim} = \pi_*\mathcal{T}$$

Definition 3.7.1.13. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $\sim_X$ an equivalence relation on X, $\sim_Y$ and equivalence relation on Y and $f : X \to Y$. Then f is said to be **$(\sim_X, \sim_Y)$-invariant** if for each $x, y \in X$, $x \sim_X y$ implies that $f(x) \sim_Y f(y)$.

Exercise 3.7.1.14. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $\sim_X, \sim_Y$ eqivalence relations on X and Y respectively, $\pi_X : X \to X/\sim_X$, $\pi_Y : Y \to Y/\sim_Y$ the respective projection maps and $f : X \to Y$ continuous. If f is $(\sim_X, \sim_Y)$-invariant, then there exists a unique $\bar{f} : X/{\sim_X} \to Y/{\sim_Y}$ such that $\bar{f}$ is continuous and $\bar{f} \circ \pi_X = \pi_Y \circ f$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ X/{\sim_X} \arrow[r, "\bar{f}"'] & Y/{\sim_Y} \end{tikzcd}$$

and $\bar{f}$ is $(\mathcal{T}_{X/\sim_X}, \mathcal{T}_{Y/\sim_Y})$-continuous.

Proof. Suppose that f is is $(\sim_X, \sim_Y)$-invariant.

- **Existence:**
  Define $\bar{f} : X/{\sim_Y} \to Y/{\sim_Y}$ by $\bar{f}(\bar{x}) = \overline{f(x)}$. Let $a, b \in X$. Then

$$\begin{aligned} \bar{a} = \bar{b} &\implies a \sim_X b \\ &\implies f(a) \sim_Y f(b) \\ &\implies \overline{f(a)} = \overline{f(b)} \\ &\implies \bar{f}(\bar{a}) = \bar{f}(\bar{b}) \end{aligned}$$

  So $\bar{f}$ is well defined. By construction $\bar{f} \circ \pi_X = \pi_Y \circ f$.

- **Uniqueness:**
  Let $g : X/{\sim_X} \to Y/{\sim_Y}$. Suppose that $g \circ \pi_X = \pi_Y \circ f$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ X/{\sim_X} \arrow[r, "g"'] & Y/{\sim_Y} \end{tikzcd}$$

  Then

$$\begin{aligned} g \circ \pi_X &= \pi_Y \circ f \\ &= \bar{f} \circ \pi_X \end{aligned}$$

  i.e. the following diagram commites:

$$\begin{tikzcd} X \arrow[r, "\pi_X"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\bar{f}"] \\ X/{\sim_X} \arrow[r, "g"'] & Y/{\sim_Y} \end{tikzcd}$$

  Since π_X is surjective, Exercise 1.2.2.3 implies that $\bar{f} = g$.

- **Continuity:**
  Let $V \in \mathcal{T}_{Y/\sim_Y}$. Continuity of f and π_Y implies that

$$\begin{aligned}\pi_X^{-1}(\bar{f}^{-1}(V)) &= (\bar{f} \circ \pi_X)^{-1}(V) \\ &= (\pi_Y \circ f)^{-1}(V) \\ &= f^{-1}(\pi_Y^{-1}(V)) \\ &\in \mathcal{T}_X\end{aligned}$$

  By definition of the quotient topology, $\bar{f}^{-1}(V) \in \mathcal{T}_{X/\sim_X}$. So $\bar{f}$ is $(\mathcal{T}_{X/\sim_X}, \mathcal{T}_{Y/\sim_Y})$-continuous.

□

Definition 3.7.1.15. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $\sim_X$ an equivalence relation on X and $f : X \to Y$. Then f is said to be **$\sim$-invariant** if f is $(\sim, =_Y)$-invariant.

Exercise 3.7.1.16. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $\sim \subset X \times X$ an eqivalence relation on X, $\pi : X \to X/\sim$ the projection map and $f : X \to Y$ continuous. If f is $\sim$-invariant, then there exists a unique $\bar{f} : X/\sim \to Y$ such that $\bar{f}$ is continuous and $\bar{f} \circ \pi = f$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X/\sim \arrow[r, "\bar{f}"'] & Y \end{tikzcd}$$

and $\bar{f}$ is $(\mathcal{T}_{X/\sim}, \mathcal{T}_Y)$-continuous.

Proof. Set $\sim_X := \sim$ and $\sim_Y := =_Y$ and use Exercise 3.7.1.14. add details □

Exercise 3.7.1.17. Let $(X, \mathcal{T})$, $(Y, \mathcal{T}_Y)$ be topological spaces, $\sim$ an eqivalence relation on X, $\pi : X \to X/\sim$ the projection map and $f : X \to Y$. Suppose f is a quotient map. If for each $a, b \in X$, $a \sim b$ iff $f(a) = f(b)$, then there exists a unique $\bar{f} : X/\sim \to Y$ such that $\bar{f}$ is a $(\mathcal{T}_{X/\sim}, \mathcal{T}_Y)$-homeomorphism and $\bar{f} \circ \pi = f$.

Proof. Suppose that for each $a, b \in X$, $a \sim b$ iff $f(a) = f(b)$. Then f is $\sim$-invariant. Since f is a quotient map, f is continuous. Exercise 3.7.1.16 implies that there exists a unique $\bar{f} : X/\sim \to Y$ such that $\bar{f}$ is continuous and $\bar{f} \circ \pi = f$. Let $y \in Y$. Since f is a quotient map, f is surjective. Therefore there exists $x \in X$ such that $f(x) = y$. Thus

$$\begin{aligned}\bar{f}(\bar{x}) &= f(x) \\ &= y\end{aligned}$$

Since $y \in Y$ is arbitrary, $\bar{f}$ is surjective. Let $\bar{a}, \bar{b} \in X/\sim$. Suppose that $\bar{f}(\bar{a}) = \bar{f}(\bar{b})$. Then

$$\begin{aligned}f(a) &= \bar{f}(\bar{a}) \\ &= \bar{f}(\bar{b}) \\ &= f(b)\end{aligned}$$

By assumption, $a \sim b$. Hence $\bar{a} = \bar{b}$. Since $\bar{a}, \bar{b} \in X/\sim$ are arbitrary, $\bar{f}$ is injective. Therefore, $\bar{f}$ is a bijection. Exercise 3.7.1.6 implies that $\bar{f}$ is a homeomorphism. □

Definition 3.7.1.18. Let $(X, \mathcal{T})$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. We define the relation $\sim_f \subset X \times X$ by $x \sim_f y$ iff $f(x) = f(y)$.

Exercise 3.7.1.19. Let $(X, \mathcal{T})$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then

1. $\sim_f$ is an equivlance relation on X,

2. if f is a quotient map, then there exists a unique $\bar{f} : X/\sim_f \to Y$ such that $\bar{f}$ is a $(\mathcal{T}_{X/\sim_f}, \mathcal{T}_Y)$-homeomorphism and $\bar{f} \circ \pi = f$.

Proof.

1. Clear.
2. Suppose that f is a quotient map. Exercise 3.7.1.17 implies that then there exists a unique $\bar{f} : X/\sim_f \to Y$ such that $\bar{f}$ is a $(\mathcal{T}_{X/\sim_f}, \mathcal{T}_Y)$-homeomorphism and $\bar{f} \circ \pi = f$.

$\square$

Exercise 3.7.1.20. Let X, X_1, X_2 be topological spaces and $f_1 : X \to X_1$, $f_2 : X \to X_2$ and $f_{1,2} : X_2 \to X_1$. Suppose that f_1 and f_2 are quotient maps. If $f_1 = f_{1,2} \circ f_2$, then there exists unique $\bar{f}_1 : X/\sim_{f_1} \to X_1$, $\bar{f}_2 : X/\sim_{f_2} \to X_2$, $\bar{f}_{1,2} : X/\sim_{f_2} \to X/\sim_{f_1}$ such that $\bar{f}_1$, $\bar{f}_2$ are homeomorphisms, $\bar{f}_{1,2}$ is continuous, $\bar{f}_1 \circ \pi_1 = f_1$, $\bar{f}_2 \circ \pi_2 = f_2$, $f_{1,2} \circ \bar{f}_1 = \bar{f}_2 \circ \bar{f}_{1,2}$, in which case the following diagram commutes:

$$
\begin{tikzcd}
X_1 \arrow[rr, "f_{1,2}"'] & & X_2 \\
X/\sim_{f_1} \arrow[u, "\bar{f}_1"] \arrow[rr, "\bar{f}_{1,2}"] & & X/\sim_{f_2} \arrow[u, "\bar{f}_2"'] \\
& X \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] \arrow[uul, bend left, "f_1"] \arrow[uur, bend right, "f_2"'] &
\end{tikzcd}
$$

Proof.

- **Existence and Uniqueness of $\bar{f}_k$:**
  Since f_1, f_2 are quotient maps, Exercise 3.7.1.19 implies that there exist unique $\bar{f}_1 : X/\sim_{f_1} \to X_1$, $\bar{f}_2 : X/\sim_{f_2} \to X_2$ such that $\bar{f}_1$, $\bar{f}_2$ are homeomorphisms, $\bar{f}_1 \circ \pi_1 = f_1$ and $\bar{f}_2 \circ \pi_2 = f_2$.

- **Existence of $\bar{f}_{1,2}$:**
  Define $\bar{f}_{1,2} : X/\sim_{f_1} \to X/\sim_{f_1}$ by $\bar{f}_{1,2} := \bar{f}_2^{-1} \circ f_{1,2} \circ \bar{f}_1$. Since $\bar{f}_2^{-1}$, $f_{1,2}$, $\bar{f}_1$ are continuous, $\bar{f}_{1,2}$ is continuous. By construction,

$$
\begin{aligned}
\bar{f}_2 \circ \bar{f}_{1,2} &= \bar{f}_2 \circ (\bar{f}_2^{-1} \circ f_{1,2} \circ \bar{f}_1) \\
&= (\bar{f}_2 \circ \bar{f}_2^{-1}) \circ (f_{1,2} \circ \bar{f}_1) \\
&= \mathrm{id}_{X_2} \circ (f_{1,2} \circ \bar{f}_1) \\
&= f_{1,2} \circ \bar{f}_1.
\end{aligned}
$$

- **Uniqueness of $\bar{f}_{1,2}$:**
  Let $g_{1,2} : X/\sim_{f_1} \to X/\sim_{f_2}$. Suppose that $\bar{f}_2 \circ g_{1,2} = f_{1,2} \circ \bar{f}_1$. Then

$$
\begin{aligned}
g_{1,2} &= \bar{f}_2^{-1} \circ f_{1,2} \circ \bar{f}_1 \\
&= \bar{f}_{1,2}.
\end{aligned}
$$

  Thus $\bar{f}_{1,2}$ is unique.

$\square$

3.7.2 Category of Topological Spaces with Equivalence Relations

Definition 3.7.2.1. We define the category **TopEq**

- $\mathrm{Obj}(\mathbf{TopEq}) = \{(X, \sim) :\ X \text{ is a topological space and } \sim \text{ is an equivalence relation on } X\}$

- $\text{Hom}_{\mathbf{TopEq}}((X, \sim_X), (Y, \sim_Y)) = \{f : X \to Y : f \text{ is continuous and } f \text{ is } (\sim_X, \sim_Y)\text{-invariant}\}$.

Definition 3.7.2.2. We define $F : \mathbf{TopEq} \to \mathbf{Top}$ by

- $F(X, \sim) = X/\sim$
- $F(f) = \bar{f}$

Exercise 3.7.2.3. We have that $F : \mathbf{TopEq} \to \mathbf{Top}$ is a functor

Proof.

1. Let $(X, \sim_X)$, $(Y, \sim_Y) \in \mathbf{TopEq}$ and $f \in \text{Hom}_{\mathbf{TopEq}}((X, \sim_X), (Y, \sim_Y))$. The previous exercise implies that $F(f) \in \text{Hom}_{\mathbf{Top}}(X/\sim_X, Y/\sim_Y)$.
2. Let $(X, \sim_X)$, $(Y, \sim_Y)$, $(Z, \sim_Z) \in \mathbf{TopEq}$, $f \in \text{Hom}_{\mathbf{TopEq}}((X, \sim_X), (Y, \sim_Y))$ and $g \in \text{Hom}_{\mathbf{TopEq}}((Y, \sim_Y), (Z, \sim_Z))$. Then

$$\begin{aligned}
\pi_Z \circ (g \circ f) &= (\pi_Z \circ g) \circ f \\
&= (\bar{g} \circ \pi_Y) \circ f \\
&= \bar{g} \circ (\pi_Y \circ f) \\
&= \bar{g} \circ (\bar{f} \circ \pi_X) \\
&= (\bar{g} \circ \bar{f}) \circ \pi_X
\end{aligned}$$

Uniqueness implies that $F(g \circ f) = F(g) \circ F(f)$.

□

Exercise 3.7.2.4. Let $(X, \sim_X)$, $(Y, \sim_Y)$, $(Z, =) \in \text{Obj}(\mathbf{TopEq})$, $\iota \in \text{Hom}_{\mathbf{TopEq}}((X, \sim_X), (Y, \sim_Y))$, $f \in \text{Hom}_{\mathbf{TopEq}}((X, \sim_X), (Z, =))$ and $g \in \text{Hom}_{\mathbf{TopEq}}((Y, \sim_Y), (Z, =))$. Then $f = g \circ \iota$ iff $\bar{f} = \bar{g} \circ \bar{\iota}$, in which case the following diagram commutes:

$$\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[rr, "\iota"] \arrow[d, "\pi_X"'] & & Y \arrow[ul, "g"'] \arrow[d, "\pi_Y"] \\
X/\sim_X \arrow[rr, "\bar{\iota}"'] \arrow[uur, bend left=60, "\bar{f}"] & & Y/\sim_Y \arrow[uul, bend right=60, "\bar{g}"']
\end{tikzcd}$$

Proof. Suppose that $f = g \circ \iota$. Functoriality implies that

$$\begin{aligned}
\bar{f} &= F(f) \\
&= F(g \circ \iota) \\
&= F(g) \circ F(\iota) \\
&= \bar{g} \circ \bar{\iota}
\end{aligned}$$

Conversely, suppose that $\bar{f} = \bar{g} \circ \bar{\iota}$. Then

$$\begin{aligned}
f &= \bar{f} \circ \pi_X \\
&= \bar{g} \circ \bar{\iota} \circ \pi_X \\
&= \bar{g} \circ \pi_Y \circ \iota \\
&= g \circ \iota
\end{aligned}$$

□

Exercise 3.7.2.5. Let G be a group, X a topological space and $\phi : G \times X \to X$ a group action. Suppose that for each $g \in G$, the map $\phi_g \in \mathrm{Sym}(X)$ defined by $\phi_g(x) = g \cdot x$ is continuous. Then $\pi : X \to X/G$ is open. move this before category theory stuff

Proof. Suppose that for each $g \in G$, ϕ_g is continuous. Let $g \in G$. Since $(\phi_g)^{-1} = \phi_{g^{-1}}$, ϕ_g is a homeomorphism. Hence for each $g \in G$ and $U \subset X$, U is open iff $g \cdot U$ is open. Let $U \subset X$. Suppose that U is open. Then $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$ is open. Since π is a quotient map, Exercise 3.7.1.9 implies that π is open. $\square$

3.8 Separation Axioms

Definition 3.8.0.1. Let X be a topological space. Then X is said to be

1. $\mathbf{T_0}$ of **Kolmogorov** if for each $x, y \in X$, if $x \neq y$, then there exists $U \subset X$ such that
   (a) U is open
   (b) $(x, y) \in U \times U^c$ or $(x, y) \in U^c \times U$
2. $\mathbf{T_1}$ if for each $x, y \in X$, if $x \neq y$, then there exists $U \in \mathcal{N}(x)$ such that U is open and $y \notin U$.
3. $\mathbf{T_2}$ or **Hausdorff** if for each $x, y \in X$, if $x \neq y$, then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that U and V are open and $U \cap V = \varnothing$.
4. $\mathbf{T_3}$ or **regular** if X is T_1 and for each $A \subset X$ and $x \in A^c$, if A is closed, then there exists $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(x)$ such that U and V are open and $U \cap V = \varnothing$.
5. $\mathbf{T_4}$ or **normal** if X is T_1 and for each $A, B \subset X$, if A and B are closed and $A \cap B = \varnothing$, then there exists $U \in \mathcal{N}(A)$ and $V \in \mathcal{N}(B)$ such that U and V are open and $U \cap V = \varnothing$.

Note 3.8.0.2. Some authors do not require the T_1 assumption for regularity or normality.

Exercise 3.8.0.3. Let X be a topological space. Then the following are equivalent:

1. X is T_1
2. for each $x \in X$, $\{x\}$ is closed
3. for each $A \subset X$, $A = \bigcap\limits_{U \in \mathcal{N}(A)} U$

Proof.

- (1) $\implies$ (2):
  Suppose that X is T_1. Let $x \in X$. Since X is T_1, for each $a \in \{x\}^c$, there exists $U_a \in \mathcal{N}(a)$ such that U_a is open and $U_a \subset \{x\}^c$. Therefore
  $$\{x\}^c = \bigcup_{a \in \{x\}^c} U_a$$
  which is open. Hence
  $$\{x\} = \bigcap_{a \in \{x\}^c} U_a^c$$
  which is closed.

- (2) $\implies$ (3):
  Suppose that for each $x \in X$, $\{x\}$ is closed. Clearly, $A \subset \bigcap\limits_{U \in \mathcal{N}(A)} U$. Since for each $x \in A^c$, $\{x\}^c \in \mathcal{N}(A)$, we have that
  $$\begin{aligned}\bigcap_{U \in \mathcal{N}(A)} U &\subset \bigcap_{x \in A^c} \{x\}^c \\ &= \bigg(\bigcup_{x \in A^c} \{x\}\bigg)^c \\ &= (A^c)^c \\ &= A\end{aligned}$$

- (3) $\implies$ (1):
  Suppose that for each $A \subset X$, $A = \bigcap\limits_{U \in \mathcal{N}(A)} U$. Let $x, y \in X$. Suppose that $x \neq y$. Since $\{x\} = \bigcap\limits_{V \in \mathcal{N}(x)} V$, $y \notin \bigcap\limits_{V \in \mathcal{N}(x)} V$. Thus there exists $V \in \mathcal{N}(x)$ such that $y \notin V$. Set $U = \operatorname{Int} V$. Then $U \in \mathcal{N}(x)$, U is open and $y \notin U$. Since $x, y \in X$ are arbitrary, X is T_1.

$\square$

Exercise 3.8.0.4. Let X be a topological space. Then

1. X is T_1 implies that X is T_0
2. X is T_2 implies that X is T_1
3. X is T_3 implies that X is T_2
4. X is T_4 implies that X is T_3

Proof. Clear by definition and the previous exercise. $\square$

Note 3.8.0.5. Let X be a set, we recall Definition 1.3.0.6 of Δ_X an $\Delta_{X^{\mathbb{N}}}$.

Exercise 3.8.0.6. Let X be a topological space. Then the following are equivalent:

1. X is Hausdorff
2. for each net $(x_\alpha)_{\alpha \in A} \subset X$ and $x, y \in X$, if $x_\alpha \to x$ and $x_\alpha \to y$, then $x = y$.
3. Δ_X is closed in $X \times X$.

Proof.

- (1) $\implies$ (2):
  Suppose that X is Hausdorff. Let $(x_\alpha)_{\alpha \in A} \subset X$ be a net and $x, y \in X$. Suppose that $x_\alpha \to x$ and $x_\alpha \to y$. For the sake of contradiction, suppose that $x \neq y$. Then there exist $U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that U and V are open and $U \cap V = \varnothing$. Since $x_\alpha \to x$, $(x_\alpha)_{\alpha \in A}$ is eventually in U and there exists $\beta_x \in A$ such that for each $\alpha \in A$, $\alpha \geq \beta_x$ implies that $x_\alpha \in U$. Since $x_\alpha \to y$, $(x_\alpha)_{\alpha \in A}$ is eventually in V and there exists $\beta_y \in A$ such that for each $\alpha \in A$, $\alpha \geq \beta_y$ implies that $x_\alpha \in V$. Since A is directed, there exists $\beta \in A$ such that $\beta \geq \beta_x, \beta_y$. Hence
  $$\begin{aligned} x_\beta &\in U \cap V \\ &= \varnothing \end{aligned}$$
  which is a contradiction. So $x = y$.

- (2) $\implies$ (3):
  Let $(x_\alpha, y_\alpha)_{\alpha \in A} \subset \Delta_X$ be a net and $(x, y) \in X \times X$. Then for each $\alpha \in A$, $x_\alpha = y_\alpha$. Suppose that $(x_\alpha, y_\alpha) \to (x, y)$. So $x_\alpha \to x$ and $x_\alpha \to y$. Hence $x = y$ and $(x, y) \in \Delta_X$. Thus Δ_X is closed.

- (3) $\implies$ (1):
  Suppose that Δ_X is closed. Let $x, y \in X$. Suppose that $x \neq y$. Then $(x, y) \in \Delta_X^c$. Recall that $\mathcal{B} = \{A \times B : A, B \subset X \text{ and } A, B \text{ are open}\}$ is a basis for the product topology on $X \times X$. Since Δ_X^c is open and $(x, y) \in \Delta_X^c$, there exist $A \times B \in \mathcal{B}$ such that $(x, y) \in A \times B \subset \Delta_X^c$. Suppose that $A \cap B \neq \varnothing$. Then there exists $z \in A \cap B$. Hence $(z, z) \in A \times B$. This is a contradiction since $A \times B \subset \Delta_X^c$. Thus $x \in A$, $y \in B$ and $A \cap B = \varnothing$ and A, B are open. Since $x, y \in X$ are arbitrary, X is Hausdorff.

$\square$

Exercise 3.8.0.7. Let $(X, \mathcal{T})$ be a topological space. If X is Hausdorff, then $\Delta_{X^{\mathbb{N}}}$ is closed in $(X^{\mathbb{N}}, \mathcal{T}^{\otimes \mathbb{N}})$.

Proof. Suppose that X is Hausdorff. Let $(x_\alpha)_{\alpha\in A} \subset \Delta_{X^{\mathbb{N}}}$ and $x \in X^{\mathbb{N}}$. Suppose that $x_\alpha \to x$. Then for each $n \in \mathbb{N}$, $\pi_n(x_\alpha) \xrightarrow{\alpha} \pi_n(x)$. Since $(x_\alpha)_{\alpha\in A} \subset \Delta_{X^{\mathbb{N}}}$, we have that for each $\alpha \in A$ and $m, n \in \mathbb{N}$, $\pi_m(x_\alpha) = \pi_m(x_\alpha)$. Let $n \in \mathbb{N}$. Then

$$\begin{aligned}\pi_n(x_\alpha) &= \pi_1(x_\alpha) \\ &\xrightarrow{\alpha} \pi_1(x).\end{aligned}$$

Since $\pi_n(x_\alpha) \xrightarrow{\alpha} \pi_n(x)$, $\pi_n(x_\alpha) \xrightarrow{\alpha} \pi_1(x)$ and X is Hausdorff, Exercise 3.8.0.6 implies that $\pi_1(x) = \pi_n(x)$. Since $n \in \mathbb{N}$ is arbitrary, we have that for each $m, n \in \mathbb{N}$,

$$\begin{aligned}\pi_m(x) &= \pi_1(x) \\ &= \pi_n(x).\end{aligned}$$

Hence $x \in \Delta_{X^{\mathbb{N}}}$. Since $(x_\alpha)_{\alpha\in A} \subset \Delta_{X^{\mathbb{N}}}$ and $x \in X^{\mathbb{N}}$ with $x_\alpha \to x$ are arbitrary, we have that for each $(x_\alpha)_{\alpha\in A} \subset \Delta_{X^{\mathbb{N}}}$ and $x \in X^{\mathbb{N}}$, if $x_\alpha \to x$, then $x \in \Delta_{X^{\mathbb{N}}}$. Hence $\Delta_{X^{\mathbb{N}}}$ is closed in $(X^{\mathbb{N}}, \mathcal{T}^{\otimes\mathbb{N}})$. $\square$

Exercise 3.8.0.8. Let X be a topological space. Suppose that X is T_1. Then X is regular iff for each $x \in X$ and $U \in \mathcal{N}(x)$, U is open implies that there exists $V \in \mathcal{N}(x)$ such that $\operatorname{cl} V \subset U$.

Proof.

- ($\Longrightarrow$):
  Suppose that X is regular. Let $x \in X$ and $U \in \mathcal{N}(x)$. Suppose that U is open. Then U^c is closed. Since $x \notin U^c$, there exists $V_x \in \mathcal{N}(x)$ and $V_{U^c} \in \mathcal{N}(U^c)$ such that V_x and V_{U^c} are open and $V_x \cap V_{U^c} = \varnothing$. Therefore, $V_{U^c}^c$ is closed and $V_x \subset V_{U^c}^c \subset U$. Hence

$$\begin{aligned}x &\in V_x \\ &\subset \operatorname{cl} V_x \\ &\subset \operatorname{cl} V_{U^c}^c \\ &= V_{U^c}^c \\ &\subset U\end{aligned}$$

- ($\Longleftarrow$):
  Suppose that for each $x \in X$ and $U \in \mathcal{N}(x)$, U is open implies that there exists $V \in \mathcal{N}(x)$ such that $\operatorname{cl} V \subset U$. Let $x \in X$ and $A \subset X$. Suppose that A is closed and $x \notin A$. Then A^c is open and $x \in A^c$. By assumption, there exists $V \in \mathcal{N}(x)$ such that $\operatorname{cl} V \subset A^c$. Set $U_x = \operatorname{Int} V$ and $U_A = \operatorname{Int} V^c$. Then

$$\begin{aligned}A &\subset (\operatorname{cl} V)^c \\ &= \operatorname{Int} V^c \\ &= U_A\end{aligned}$$

  so that U_x and U_A are open, $U_x \in \mathcal{N}(x)$, $U_A \in \mathcal{N}(A)$ and $U_x \cap U_A = \varnothing$. Hence X is regular.

$\square$

Exercise 3.8.0.9. lemma for Uryshohns lemma

Proof. FINISH!!! $\square$

Exercise 3.8.0.10. Urysohn's Lemma for Normal Spaces:
Let X be a topological space. Suppose that X is normal. Let $A, B \subset X$. Suppose that A and B are closed and $A \cap B = \varnothing$. Then there exists $f \in C(X, [0, 1])$ such that $f|_A = 0$ and $f|_B = 1$.

Proof. FINISH!!! $\square$

Exercise 3.8.0.11. Tietze Extension Theorem for Normal Spaces:
Let X be a topological space. Suppose that X is normal. Let $A, B \subset X$. Suppose that A and B are closed and $A \cap B = \varnothing$. Then there exists $f \in C(X, [0, 1])$ such that $f|_A = 0$ and $f|_B = 1$.

Proof. FINISH!!! $\square$

3.8.1 Separation and Subspaces

Exercise 3.8.1.1. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. If $(X, \mathcal{T})$ is T_1, then $(A, \mathcal{T} \cap A)$ is T_1.

Proof. Suppose that $(X, \mathcal{T})$ is T_1. Let $x \in A$. Since $(X, \mathcal{T})$ is T_1, $\{x\}$ is closed in X. Thus $\{x\} = \{x\} \cap A$ is closed in $(A, \mathcal{T} \cap A)$. Since $x \in A$ is arbitrary, $(A, \mathcal{T} \cap A)$ is T_1. $\square$

Exercise 3.8.1.2. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. If $(X, \mathcal{T})$ is Hausdorff, then $(A, \mathcal{T} \cap A)$ is Hausdorff.

Proof. Suppose that $(X, \mathcal{T})$ is Hausdorff. Let $x, y \in A$. Since $(X, \mathcal{T})$ is Hausdorff, there exist $U' \in \mathcal{N}(x)$, $V' \in \mathcal{N}(y)$ such that $U, V \in \mathcal{T}$ and $U' \cap V' = \varnothing$. Set $U = U' \cap A$ and $V = V' \cap A$. Then $U, V \in \mathcal{T} \cap A$, $x \in U$, $y \in V$ and

$$\begin{aligned} U \cap V &= (U' \cap A) \cap (V' \cap A) \\ &= (U' \cap V') \cap A \\ &= \varnothing \end{aligned}$$

Since $x, y \in A$ are arbitary, $(A, \mathcal{T} \cap A)$ is Hausdorff. $\square$

Exercise 3.8.1.3. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. If $(X, \mathcal{T})$ is regular, then $(A, \mathcal{T} \cap A)$ is regular.

Proof. Suppose that $(X, \mathcal{T})$ is regular. Let $x \in A$ and $U \in \mathcal{N}(x)(\mathcal{T} \cap A)$. Suppose that $U \in \mathcal{T} \cap A$. Then there exists $U' \in \mathcal{T}$ such that $U = U' \cap A$. Since $(X, \mathcal{T})$ is regular, there exist $V' \in \mathcal{N}(x)(\mathcal{T})$ such that, $\mathrm{cl}_{\mathcal{T}} V' \subset U'$. Set $V = V' \cap A$. Then $V \in \mathcal{N}(x)(\mathcal{T} \cap A)$

$$\begin{aligned} \mathrm{cl}_{\mathcal{T} \cap A} V &= \mathrm{cl}_{\mathcal{T}} V' \cap A \\ &\subset U' \cap A \end{aligned}$$

Since $x \in A$ are arbitary, $(A, \mathcal{T} \cap A)$ is regular.
FINISH!!! $\square$

3.8.2 Separation and Product Spaces

Exercise 3.8.2.1. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Set $X = \prod_{\alpha \in A} X_\alpha$ and denote the product topology on X by $\mathcal{T}_X$. If for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is T_1, then $(X, \mathcal{T}_X)$ is T_1.

Proof. Suppose that for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is T_1. Let $(x_\alpha)_{\alpha \in A}, (y_\alpha)_{\alpha \in A} \in X$. Suppose that $(x_\alpha)_{\alpha \in A} \neq (y_\alpha)_{\alpha \in A}$. Then there exists $\alpha_0 \in A$ such that $x_{\alpha_0} \neq y_{\alpha_0}$. Then there exists $U_{\alpha_0} \in \mathcal{T}_{\alpha_0}$ such that $x_{\alpha_0} \in U_{\alpha_0}$ and $y_{\alpha_0} \notin U_{\alpha_0}$. Set $U = \pi_{\alpha_0}^{-1}(U_{\alpha_0})$. Then $U \in \mathcal{T}_X$, $(x_\alpha)_{\alpha \in A} \in U$ and $(y_\alpha)_{\alpha \in A} \notin U$. Since $(x_\alpha)_{\alpha \in A}, (y_\alpha)_{\alpha \in A} \in X$ are arbitrary, $(X, \mathcal{T}_X)$ is T_1. $\square$

Exercise 3.8.2.2. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Set $X = \prod_{\alpha \in A} X_\alpha$ and $\mathcal{T} := \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$. If for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is Hausdorff, then $(X, \mathcal{T})$ is Hausdorff.

Proof. Suppose that for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is Hausdorff. Let $(x_\alpha)_{\alpha \in A}, (y_\alpha)_{\alpha \in A} \in X$. Suppose that $(x_\alpha)_{\alpha \in A} \neq (y_\alpha)_{\alpha \in A}$. Then there exists $\alpha_0 \in A$ such that $x_{\alpha_0} \neq y_{\alpha_0}$. Then there exists $U_{\alpha_0}, V_{\alpha_0} \in \mathcal{T}_{\alpha_0}$ such that $x_{\alpha_0} \in U_{\alpha_0}$, $y_{\alpha_0} \in V_{\alpha_0}$ and $U_{\alpha_0} \cap V_{\alpha_0} = \varnothing$. Set $U = \pi_{\alpha_0}^{-1}(U_{\alpha_0})$ and $V = \pi_{\alpha_0}^{-1}(V_{\alpha_0})$. Then $U, V \in \mathcal{T}$, $(x_\alpha)_{\alpha \in A} \in U$, $(y_\alpha)_{\alpha \in A} \in V$ and

$$\begin{aligned} U \cap V &= \pi_{\alpha_0}^{-1}(U_{\alpha_0}) \cap \pi_{\alpha_0}^{-1}(V_{\alpha_0}) \\ &= \pi_{\alpha_0}^{-1}(U_{\alpha_0} \cap V_{\alpha_0}) \\ &= \pi_{\alpha_0}^{-1}(\varnothing) \\ &= \varnothing \end{aligned}$$

Since $(x_\alpha)_{\alpha \in A}, (y_\alpha)_{\alpha \in A} \in X$ are arbitrary, $(X, \mathcal{T})$ is Hausdorff. $\square$

Exercise 3.8.2.3. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Set $X = \prod_{\alpha \in A} X_\alpha$ and denote the product topology on X by $\mathcal{T}_X$. If for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is regular, then $(X, \mathcal{T}_X)$ is regular.

Proof. Let $x \in X$ and $U \in \mathcal{N}(x)$. Suppose that U is open. Set

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} B_\alpha : \text{ for each } \alpha \in A,\, B_\alpha \in \mathcal{T}_\alpha \text{ and } \#\{\alpha \in A : B_\alpha \neq X_\alpha\} < \infty \right\}$$

Then $\mathcal{B}$ is a basis for $\mathcal{T}_X$. So for each $\alpha \in A$, there exist $U_\alpha \in \mathcal{T}_\alpha$ such that $\#\{\alpha \in A : B_\alpha \neq X_\alpha\} < \infty$ and $x \in \prod_{\alpha \in A} U_\alpha \subset U$. Set $J = \{\alpha \in A : B_\alpha \neq X_\alpha\}$. Let $\alpha \in A$. Suppose that $\alpha \in J$. Then $x_\alpha \in U_\alpha$. Since $U_\alpha \in \mathcal{N}(x)$ is an open neighborhood of x_α and X_α is regular, the previous exercise implies that there exists $V_\alpha \in \mathcal{N}(x_\alpha)$ such that $\operatorname{cl} V_\alpha \subset U_\alpha$. If $\alpha \in J^c$, set $V_\alpha = X_\alpha$. Define $V = \prod_{\alpha \in A} V_\alpha$. Then $V \in \mathcal{N}(x)$ and an exercise in the section on the product topology implies that

$$\begin{aligned} \operatorname{cl} V &= \operatorname{cl} \prod_{\alpha \in A} V_\alpha \\ &= \prod_{\alpha \in A} \operatorname{cl} V_\alpha \\ &\subset \prod_{\alpha \in A} U_\alpha \\ &\subset U \end{aligned}$$

□

3.8.3 Separation and Quotient Spaces

Definition 3.8.3.1. Let $(X, \mathcal{T})$ be a topological space. Define $\sim_0 \subset X \times X$ by

$$\sim_0 = \{(x, y) \in X \times X : \text{for each } U \in \mathcal{T},\, x \in U \text{ iff } y \in U\}.$$

Exercise 3.8.3.2. Let $(X, \mathcal{T})$ be a topological space. Then

1. $\sim_0$ is an equivalence relation on X,
2. for each $x, y \in X$, $x \not\sim_0 y$ iff there exists $U \in \mathcal{T}$ such that $(x, y) \in U \times U^c$ or $(x, y) \in U^c \times U$,
3. for each $x, y \in X$, $x \sim_0 y$ iff $\operatorname{cl}\{x\} = \operatorname{cl}\{y\}$,
4. X is T_0 iff for each $x, y \in X$, $x \neq y$ iff $x \not\sim_0 y$,
5. (a) for each $U \in \mathcal{T}$, $\pi^{-1}(\pi(U)) = U$
   (b) $\pi : X \to X/\sim_0$ is open,
6. $X/\sim_0$ is T_0,
7. X is T_0 iff $X/\sim_0 \cong X$.

Proof.

1. Let $x, y, z \in X$.
   - Clearly $x \sim_0 x$.
   - Clearly $x \sim_0 y$ implies that $y \sim_0 x$.
   - Suppose that $x \sim_0 y$ and $y \sim_0 z$. Let $U \in \mathcal{T}$.
     - Suppose that $x \in U$. Since $x \sim_0 y$, we have that $y \in U$. Since $y \sim_0 z$, we have that $z \in U$. Thus $x \in U$ implies that $z \in U$.

– Similarly, $z \in U$ implies that $x \in U$.

Thus $x \in U$ iff $z \in U$. Since $U \in \mathcal{T}$ is arbitrary, we have that for each $U \in \mathcal{T}$, $x \in U$ iff $z \in U$. Therefore $x \sim_0 z$.

Since $x, y, z \in X$ are arbitrary, we have that $\sim_0$ is an equivalence relation on X.

2. Let $x, y \in X$. By definition,

$$\begin{aligned} x \not\sim_0 y &\iff \neg[\forall U \in \mathcal{T}, (x \in U \iff y \in U)] \\ &\iff \exists U \in \mathcal{T} \text{ s.t. } \neg[(x \in U \implies y \in U) \text{ and } (y \in U \implies x \in U)] \\ &\iff \exists U \in \mathcal{T} \text{ s.t. } \neg(x \in U \implies y \in U) \text{ or } \neg(y \in U \implies x \in U) \\ &\iff \exists U \in \mathcal{T} \text{ s.t. } (x \in U \wedge y \notin U) \text{ or } (y \in U \wedge x \notin U) \\ &\iff \exists U \in \mathcal{T} \text{ s.t. } (x, y) \in U \times U^c \text{ or } (x, y) \in U^c \times U \end{aligned}$$

3. Let $x, y \in X$.

- ($\implies$) :
Suppose that $x \sim_0 y$. Define $\mathcal{C}_x := \{C \subset X : C \text{ is closed and } x \in C\}$ and $\mathcal{C}_y := \{C \subset X : C \text{ is closed and } y \in C\}$. Let $C \in \mathcal{C}_x$. For the sake of contradiction, suppose that $y \notin C$. Then $y \in C^c$. Since $x \sim_0 y$, $C^c \in \mathcal{T}$ and $y \in C^c$, we have that $x \in C^c$. This is a contradiction since $x \in C$. Thus $y \in C$ and therefore $C \in \mathcal{C}_y$. Since $C \in \mathcal{C}_x$ is arbitrary, we have that $\mathcal{C}_x \subset \mathcal{C}_y$. Similarly, $\mathcal{C}_y \subset \mathcal{C}_x$. Thus $\mathcal{C}_y = \mathcal{C}_x$. Therefore

$$\begin{aligned} \operatorname{cl}\{x\} &= \bigcap_{C \in \mathcal{C}_x} C \\ &= \bigcap_{C \in \mathcal{C}_y} C \\ &= \operatorname{cl}\{y\} \end{aligned}$$

- ($\impliedby$) :
Suppose that $x \not\sim_0 y$. Then part (2) implies that there exists $U \in \mathcal{T}$ such that $(x, y) \in U \times U^c$ or $(x, y) \in U^c \times U$.
  - Suppose that $(x, y) \in U \times U^c$. Then $U^c \in \mathcal{C}_y$ and therefore

$$\begin{aligned} \operatorname{cl}\{y\} &= \bigcap_{C \in \mathcal{C}_y} C \\ &\subset U^c \end{aligned}$$

    Since $x \in U$, $\{x\} \cap \operatorname{cl}\{y\} = \varnothing$. Since $x \in \operatorname{cl}\{x\}$, we have that $\operatorname{cl}\{x\} \not\subset \operatorname{cl}\{y\}$. Hence $\operatorname{cl}\{x\} \neq \operatorname{cl}\{y\}$.
  - Similarly, if $(x, y) \in U^c \times U$, then $\operatorname{cl}\{x\} \neq \operatorname{cl}\{y\}$.

4. - ($\implies$) :
Suppose that X is T_0. Let $x, y \in X$. Since X is T_0, we have that

$$\begin{aligned} x \neq y &\iff \exists U \in \mathcal{T} \text{ s.t. } (x, y) \in U \times U^c \text{ or } (x, y) \in U^c \times U \\ &\iff x \not\sim_0 y \end{aligned}$$

- ($\impliedby$) :
Suppose that for each $x, y \in X$, $x \neq y$ iff $x \not\sim_0 y$. Let $x, y \in X$. Suppose that $x \neq y$. By assumption, $x \not\sim_0 y$. Thus there exists $U \in \mathcal{T}$ such that $(x, y) \in U \times U^c$ or $(x, y) \in U^c \times U$. Since $x, y \in X$ are arbitrary, we have that X is T_0.

5. (a) Let $U \in \mathcal{T}$ and $x \in \pi^{-1}(\pi(U))$. Then $\pi(x) \in \pi(U)$. Thus there exists $y \in U$ such that $\pi(x) = \pi(y)$. By definition of π, $x \sim_0 y$. By definition of $\sim_0$, $x \in U$. Since $x \in \pi^{-1}(\pi(U))$ is arbitrary, we have that $\pi^{-1}(\pi(U)) \subset U$. Exercise 1.2.1.1 implies that $U \subset \pi^{-1}(\pi(U))$. Thus $\pi^{-1}(\pi(U)) = U$. Since $U \in \mathcal{T}$ is arbitrary, we have that for each $U \in \mathcal{T}$, $\pi^{-1}(\pi(U)) = U$.

   (b) In particular, for each $U \in \mathcal{T}$,

$$\begin{aligned} \pi^{-1}(\pi(U)) &= U \\ &\in \mathcal{T} \end{aligned}$$

   and since π is a quotient map, Exercise 3.7.1.9 implies that π is open.

6. Let $a, b \in X/\sim_0$. For the sake of contradiction, suppose that $a \sim_0 b$ and $a \neq b$. Since π is surjective, there exist $x, y \in X$ such that $\pi(x) = a$ and $\pi(y) = b$. Since $\pi(x) \neq \pi(y)$, $x \not\sim_0 y$. Therefore there exists $U \in \mathcal{T}$ such that $(x, y) \in U \times U^c$ or $(x, y) \in U^c \times U$.

   - Suppose that $(x, y) \in U \times U^c$. Since $x \in U$ and π is open, we have that

$$\begin{aligned} a &= \pi(x) \\ &\in \pi(U) \end{aligned}$$

     Since $a \sim_0 b$, $a \in \pi(U)$ and $\pi(U) \in \mathcal{T}_{X/\sim_0}$, we have that $b \in \pi(U)$. Therefore

$$\begin{aligned} y &\in \pi^{-1}(\pi(U)) \\ &= U \end{aligned}$$

     This is a contradiction since $y \in U^c$.
   - Similarly, if $(x, y) \in U^c \times U$, then $x \in U$ which is a contradiction.

   Hence $a \sim_0 b$ implies that $a = b$.
   Conversely, since $\sim_0$ is an equivalence relation, $a = b$ implies that $a \sim_0 b$.
   Thus $a = b$ iff $a \sim_0 b$. Since $a, b \in X/\sim_0$ are arbitrary, we have that for each $a, b \in X/\sim_0$, $a = b$ iff $a \sim_0 b$. Part (4) implies that $X/\sim_0$ is T_0.

7. FINISH!!!

□

Exercise 3.8.3.3. Let X be a topological space and $\sim$ an equivalence relation on X. If $\pi : X \to X/\sim$ is open, then $X/\sim$ is Hausdorff iff $\sim$ is closed in $X \times X$.

Proof. Suppose that $\pi : X \to X/\sim$ is open.

- ($\Longrightarrow$):
  Suppose that $X/\sim$ is Hausdorff. Let $(x_\alpha, y_\alpha)_{\alpha \in A} \subset \sim$ be a net and $(x, y) \in X \times X$. Suppose that $x_\alpha, y_\alpha \to (x, y)$. Then $x_\alpha \to x$ and $y_\alpha \to y$. Since $\pi : X \to X/\sim$ is continuous, $\pi(x_\alpha) \to \pi(x)$ and $\pi(y_\alpha) \to \pi(y)$. Since for each $\alpha \in A$, $x_\alpha \sim y_\alpha$, we have that

$$\begin{aligned} \pi(x_\alpha) &= \pi(y_\alpha) \\ &\to \pi(y) \end{aligned}$$

  Since $X/\sim$ is Hausdorff, $\pi(x) = \pi(y)$. Hence $x \sim y$ and $(x, y) \in \sim$. Thus $\sim$ is closed in $X \times X$.

- ($\Longleftarrow$):
  Conversely, suppose that $\sim$ is closed in $X \times X$ is closed. Let $\bar{x}, \bar{y} \in X/\sim$. Suppose that $\bar{x} \neq \bar{y}$. Then $(x, y) \in \sim^c$. Recall that $\mathcal{B} = \{A \times B : A, B \subset X \text{ and } A, B \text{ are open}\}$ is a basis for $X \times X$. Since $\sim^c$ is open and $(x, y) \in \sim^c$, there exist $A, B \subset X$ such that A, B are open and $(x, y) \in A \times B \subset \sim^c$.

Thus $x \in A$ and $y \in B$. Since π is open, $\pi(A) = \bar{A}$ and $\pi(B) = \bar{B}$ are open. Suppose for the sake of contradiction that $\pi(A) \cap \pi(B) \neq \varnothing$. Then there exists $z \in X$ such that $\bar{z} \in \pi(A) \cap \pi(B)$. Therefore there exist $z_A \in A$ and $z_B \in B$ such that $z_A \sim z$ and $\sim z_B$. Then $(z_A, z_B) \in A \times B$ and $(z_A, z_B) \in \sim$. This is a contradiction since $A \times B \subset \sim^c$. So $\pi(A) \cap \pi(B) = \varnothing$. Thus $\bar{x} \in \pi(A)$, $\bar{y} \in \pi(B)$, $\pi(A)$, $\pi(B)$ are open and $\pi(A) \cap \pi(B) = \varnothing$. Since $\bar{x}, \bar{y} \in X/\sim$ are arbitrary, $X/\sim$ is Hausdorff.

$\square$

3.9 Countability Axioms

3.9.1 First-Countability

Definition 3.9.1.1. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is said to be **first-countable** if for each $x \in X$, there exists $\mathcal{B}_x \subset \mathcal{T}$ such that

1. $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x
2. $\mathcal{B}_x$ is countable

Exercise 3.9.1.2. Let $(X, \mathcal{T})$ be a topological space. Suppose that $(X, \mathcal{T})$ is first-countable. Then for each $x \in X$, there exists $(U_{x,n})_{n\in\mathbb{N}} \subset \mathcal{T}$ such that

1. $(U_{x,n})_{n\in\mathbb{N}}$ is a local basis for $\mathcal{T}$ at X
2. for each $n \in \mathbb{N}$, $U_{x,n+1} \subset U_{x,n}$

Proof.

1. Let $x \in X$. Since $(X, \mathcal{T})$ is first-countable, there exists $(E_{x,j})_{j\in\mathbb{N}} \subset \mathcal{T}$ such that $(E_{x,j})_{j\in\mathbb{N}}$ is a local basis for $\mathcal{T}$ at x. Define $(U_{x,n})_{n\in\mathbb{N}} \subset \mathcal{T}$ by
$$U_{x,n} = \bigcap_{j=1}^{n} E_{x,j}$$
   - Since $(E_{x,j})_{j\in\mathbb{N}}$ is a local basis for $\mathcal{T}$ at x, for each $j \in \mathbb{N}$, $x \in E_{x,j}$. Therefore for each $n \in \mathbb{N}$,
$$\begin{aligned} x &\in \bigcap_{j=1}^{n} E_{x,j} \\ &= U_{x,n} \end{aligned}$$
   - Let $V \in \mathcal{T}$. Suppose that $x \in V$. Since $(E_{x,j})_{j\in\mathbb{N}}$ is a local basis for $\mathcal{T}$ at x, there exists $n \in \mathbb{N}$ such that $E_{x,n} \subset V$. Then
$$\begin{aligned} U_{x,n} &= \bigcap_{j=1}^{n} E_{x,j} \\ &\subset E_{x,n} \\ &\subset V \end{aligned}$$

   Thus $(U_{x,n})_{n\in\mathbb{N}}$ is a local basis for $\mathcal{T}$ at x.
2. By construction, for each $n \in \mathbb{N}$,
$$\begin{aligned} U_{x,n+1} &= \bigcap_{j=1}^{n+1} E_{x,j} \\ &\subset \bigcap_{j=1}^{n} E_{x,j} \\ &= U_{x,n} \end{aligned}$$

□

Exercise 3.9.1.3. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Suppose that $(X, \mathcal{T})$ is first-countable. Then f is continous iff for each sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$, $x_n \to x$ implies that $f(x_n) \to x$.

Proof.

- ($\implies$):
  Suppose that f is continuous. Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence and $x \in X$. Suppose that $x_n \to x$. Since $(x_n)_{n\in\mathbb{N}}$ is a net, a previous exercise implies that $f(x_n) \to x$.

- ($\impliedby$):
  Conversely, suppose that for each sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$, $x_n \to x$ implies that $f(x_n) \to x$. Since $(X, \mathcal{T})$ is first-countable, the previous exercise implies that there exists $(U_{x,n})_{n\in\mathbb{N}} \subset \mathcal{T}$ such that

  1. $(U_{x,n})_{n\in\mathbb{N}}$ is a local basis for $\mathcal{T}$ at X
  2. for each $n \in \mathbb{N}$, $U_{x,n+1} \subset U_{x,n}$

  For the sake of contradiction, suppose that f is not continuous. Then there exists $x \in X$ such that f is not continuous at x. Thus there exists $V \in \mathcal{N}(f(x))$ such that for each $U \in \mathcal{N}(x)$, $f(U) \not\subset V$. In particular, for each $n \in \mathbb{N}$, $f(U_{x,n}) \cap V^c \neq \varnothing$ and therefore $U_{x,n} \cap f^{-1}(V^c) \neq \varnothing$. The axiom of choice implies that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that for each $n \in \mathbb{N}$, $x_n \in U_{x,n}$ and $f(x_n) \in V^c$. Let $U \in \mathcal{N}(x)$. Since $(U_{x,n})_{n\in\mathbb{N}}$ is a local basis for $\mathcal{T}$ at x, there exists $N \in \mathbb{N}$ such that $U_{x,N} \subset \operatorname{Int} U$. Then for each $n \in \mathbb{N}$, $n \geq N$ implies that

  $$\begin{aligned} x_n &\in U_{x,n} \\ &\subset U_{x,N} \\ &\subset \operatorname{Int} U \\ &\subset U \end{aligned}$$

  Hence $(x_n)_{n\in\mathbb{N}}$ is eventually in U. Since $U \in \mathcal{N}(x)$ is arbitrary, $x_n \to x$. By assumption, $f(x_n) \to f(x)$. This is a contradiction since for each $n \in \mathbb{N}$, $f(x_n) \in V^c$ and therefore it is not the case that $(f(x_n))_{n\in\mathbb{N}}$ is eventually in V. Hence f is continuous.

□

Exercise 3.9.1.4. Let $(X_n, \mathcal{T}_n)_{n\in\mathbb{N}} \subset \operatorname{Obj}(\mathbf{Top})$. If for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is first-countable, then $\left(\prod\limits_{n\in\mathbb{N}} X_n, \bigotimes_{n\in\mathbb{N}} \mathcal{T}_n\right)$ is first-countable.

Proof. Set $X = \prod\limits_{n\in\mathbb{N}} X_n$ and $\mathcal{T} = \bigotimes_{n\in\mathbb{N}} \mathcal{T}_n$. Let $x \in X$. Since for each $n \in \mathbb{N}$, X_n is first-countable, we have that for each $n \in \mathbb{N}$, there exists $\mathcal{B}_{x_n} \subset \mathcal{T}_n$ such that

1. $\mathcal{B}_{x_n}$ is a local basis for $\mathcal{T}_n$ at x_n
2. $\mathcal{B}_{x_n}$ is countable

Set

$$\mathcal{B}_x = \left\{ \prod_{n\in\mathbb{N}} U_n : \text{ [for each } n \in \mathbb{N}, U_n \in \mathcal{T}_n \text{ and } U_n \neq X_n \text{ implies that } U_n \in \mathcal{B}_{x_n}] \text{ and } \#\{n \in \mathbb{N} : U_n \neq X_n\} < \infty \right\}$$

Then $\mathcal{B}_x$ is countable and an exercise in the section on the product topology implies that $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x. Since $x \in X$ is arbitrary, we have that for each $x \in X$, there exists $\mathcal{B}_x \subset \mathcal{T}$ such that

1. $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x
2. $\mathcal{B}_x$ is countable

Hence $(X, \mathcal{T})$ is first-countable. □

Exercise 3.9.1.5. Let $(X, \mathcal{T})$ be a topological space, $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$. Suppose that $(X, \mathcal{T})$ is first countable and x is a cluster point of $(x_n)_{n\in\mathbb{N}}$. Then there exists $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \to x$.

Proof.

- Since $(X, \mathcal{T})$ is first-countable, there exists $(U_n)_{n \in \mathbb{N}} \subset \mathcal{T}$ such that $(U_n)_{n \in \mathbb{N}}$ is a local basis for $\mathcal{T}$ at x. Define $(V_n)_{n \in \mathbb{N}} \subset \mathcal{T}$ by $V_n := \bigcap_{j=1}^{n} U_j$. Then for each $n \in \mathbb{N}$, $V_n \in \mathcal{N}(x)$.

- We define $(A_j)_{j,k \in \mathbb{N}} \subset \mathcal{P}(\mathbb{N})$ by $A_{k,j} := \{n \in \mathbb{N} : n \geq j \text{ and } x_n \in V_k\}$. Since x is a cluster point of $(x_n)_{n \in \mathbb{N}}$, for each $j, k \in \mathbb{N}$, $A_{k,j} \neq \varnothing$.

- We define $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ by

$$n_k := \begin{cases} \min A_{1,1}, & k = 1 \\ \min A_{k, n_{k-1}+1}, & k \geq 2 \end{cases}$$

  By construction for each $k \in \mathbb{N}$, $n_k > n_{k-1}$. Thus $(x_{n_k})_{k \in \mathbb{N}} \subset (x_n)_{n \in \mathbb{N}}$.

- Let $U \in \mathcal{N}(x)$. Since $(U_n)_{n \in \mathbb{N}}$ is a local basis for $\mathcal{T}$ at x, there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} x &\in V_{k_0} \\ &\subset U_{k_0} \\ &\subset \operatorname{Int} U \\ &\subset U \end{aligned}$$

  Let $k \in \mathbb{N}$. Suppose that $k \geq k_0$. Since

$$\begin{aligned} n_k &> n_{k-1} \\ &\geq n_{k_0}, \end{aligned}$$

  we have that $V_{n_k} \subset V_{n_{k_0}}$ and

$$\begin{aligned} x_{n_k} &\in V_{n_k} \\ &\subset V_{k_0} \\ &\subset U. \end{aligned}$$

  Hence there exists $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq k_0$ implies that $x_{n_k} \in U$. Thus (x_{n_k}) is eventually in U. Since $U \in \mathcal{N}(x)$ is arbitrary, we have that for each $U \in \mathcal{N}(x)$, (x_{n_k}) is eventually in U. Therefore $x_{n_k} \to x$ in $(X, \mathcal{T})$.

□

3.9.2 Second-Countability

Definition 3.9.2.1. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is said to be **second-countable** if there exists $\mathcal{B} \subset \mathcal{T}$ such that

1. $\mathcal{B}$ is a basis for $\mathcal{T}$
2. $\mathcal{B}$ is countable

Exercise 3.9.2.2. Let $(X, \mathcal{T})$ be a topological space. Suppose that there exist $(U_n)_{n \in \mathbb{N}} \subset \mathcal{T}$ such that

1. $X = \bigcup_{n \in \mathbb{N}} U_n$
2. for each $n \in \mathbb{N}$, $(U_n, \mathcal{T} \cap U_n)$ is second-countable

Then $(X, \mathcal{T})$ is second-countable

Proof. Since for each $n \in \mathbb{N}$, $(U_n, \mathcal{T}_\cap U_n)$ is second-countable, we have that for each $n \in \mathbb{N}$, there exists $\mathcal{B}_n \subset \mathcal{T}_\cap U_n$ such that $\mathcal{B}_n$ is a basis for $\mathcal{T}_\cap U_n$ and $\mathcal{B}_n$ is countable. Since $(U_n)_{n\in\mathbb{N}} \subset \mathcal{T}$, we have that for each $n \in \mathbb{N}$,

$$\begin{aligned}\mathcal{B}_n &\subset \mathcal{T} \cap U_n \\ &\subset \mathcal{T}\end{aligned}$$

Define $\mathcal{B} \subset \mathcal{T}$ by $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$. Then $\mathcal{B}$ is countable. Let $V \in \mathcal{T}$ and $n \in \mathbb{N}$. Then $V \cap U_n \in \mathcal{T} \cap U_n$. Since $V \cap U_n \in \mathcal{T} \cap U_n$, there exist $(B_{n,j})_{j\in\mathbb{N}} \subset \mathcal{B}_n$ such that $V \cap U_n = \bigcup_{j\in\mathbb{N}} B_{n,j}$. Then $(B_{n,j})_{n,j\in\mathbb{N}} \subset \mathcal{B}$ and

$$\begin{aligned}V &= V \cap X \\ &= V \cap \left(\bigcup_{n\in\mathbb{N}} U_n\right) \\ &= \bigcup_{n\in\mathbb{N}} V \cap U_n \\ &= \bigcup_{n\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} B_{n,j}\end{aligned}$$

Since $V \in \mathcal{T}$ is arbitrary, we have that for each $V \in \mathcal{T}$, there exits $\mathcal{B}' \subset \mathcal{B}$ such that $V = \bigcup_{B\in\mathcal{B}'} B$. Hence $\mathcal{B}$ is a basis for $\mathcal{T}$. Since $\mathcal{B}$ is countable, $(X, \mathcal{T})$ is second-countable. □

Exercise 3.9.2.3. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Suppose that f is surjective, continuous and open. If X is second countable, then Y is second-countable.

Proof. Suppose that X is second-countable. Then there exists $\mathcal{B}_X \subset \mathcal{T}_X$ such that $\mathcal{B}_X$ is a basis for $\mathcal{T}_X$ and $\mathcal{B}_X$ is countable. Set $\mathcal{B}_Y = \{f(A) : A \in A \in \mathcal{T}_X\}$. Since $\mathcal{B}_X$ is countable, $\mathcal{B}_Y$ is countable. Since f is surjective, continuous and open, a previous exercise implies that $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$. Hence $(Y, \mathcal{T}_Y)$ is second countable. □

Exercise 3.9.2.4. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Suppose that f is a homoemorphism. Then $(X, \mathcal{T}_X)$ is second countable iff $(Y, \mathcal{T}_Y)$ is second countable.

Proof.

- $(\implies)$:
  Suppose that $(X, \mathcal{T}_X)$ is second-countable. Since f is surjective, continuous and open, the previous exercise implies that $(Y, \mathcal{T}_Y)$ is second countable.

- $(\impliedby)$:
  Conversely, suppose that $(Y, \mathcal{T}_Y)$ is second-countable. Since $f^{-1} : Y \to X$ is surjective, continuous and open, the previous exercise implies that $(X, \mathcal{T}_X)$ is second countable.

□

Definition 3.9.2.5. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is said to be **separable** if there exists $S \subset X$ such that S is dense in X and S is countable.

Exercise 3.9.2.6. Let $(X, \mathcal{T})$ be a topological space. If $(X, \mathcal{T})$ is second-countable, then $(X, \mathcal{T})$ is separable.

Proof. Suppose that $(X, \mathcal{T})$ is second-countable. Then there exists $\mathcal{B} \subset \mathcal{T}$ such that $\mathcal{B}$ is a basis for $\mathcal{T}$ and $\mathcal{B}$ is countable. The axiom of choice implies that there exists $(x_U)_{U\in\mathcal{B}} \subset X$ such that each $U \in \mathcal{B}$, $x_U \in U$. Let $V \in \mathcal{T}$. Suppose that $V \neq \varnothing$. Then there exists $x \in V$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, there exists $U \in \mathcal{B}$ such that $x \in U \subset V$. Hence $x_U \in (x_U)_{U\in\mathcal{B}} \cap V$ which implies that $(x_U)_{U\in\mathcal{B}} \cap V \neq \varnothing$. Since $V \in \mathcal{T}$ such that $V \neq \varnothing$ is arbitrary, we have that for each $V \in \mathcal{T}$, $V \neq \varnothing$ implies that $(x_U)_{U\in\mathcal{B}} \cap V \neq \varnothing$. Exercise 3.1.0.27 implies that $(x_U)_{U\in\mathcal{B}}$ is dense in X. Since $\mathcal{B}$ is countable, $(X, \mathcal{T})$ is separable. □

Exercise 3.9.2.7. Let $(X, \mathcal{T}), (Y, \mathcal{T}_Y)$ be a topological spaces and $f : X \to Y$ a $(\mathcal{T}_X, \mathcal{T}_Y)$-homeomorphism. Then $(X, \mathcal{T})$ is second-countable iff $(Y, \mathcal{T}_Y)$ is second-countable.

Proof.
- $(\implies)$:
  Suppose that $(X, \mathcal{T})$ is second-countable. Then there exists $\mathcal{B}_X \subset \mathcal{T}_X$ such that $\mathcal{B}_X$ is a basis for $\mathcal{T}_X$ and $\mathcal{B}_X$ is countable. Define $\mathcal{B}_Y := \{f(U) : U \in \mathcal{B}_X\}$. Since f is a $(\mathcal{T}_X, \mathcal{T}_Y)$-homeomorphism, we have that f is open and $\mathcal{B}_Y \subset \mathcal{T}_Y$. Since $\mathcal{B}_X$ is countable, $\mathcal{B}_Y$ is countable. Let $V \in \mathcal{T}_Y$ and $y \in V$. Set $U := f^{-1}(V)$ and $x := f^{-1}(y)$. Since f is continuous, $U \in \mathcal{T}_X$. By construction, $x \in U$. Since $\mathcal{B}_X$ is a basis for $\mathcal{T}_X$, there exists $B_X \in \mathcal{B}_X$ such that $x \in B_X$ and $B_X \subset U$. Set $B_Y := f(B_X)$. By definition, $B_Y \in \mathcal{B}_Y$ and

$$\begin{aligned} y &= f(x) \\ &\in f(B_X) \\ &\subset f(U) \\ &= V \end{aligned}$$

  and

$$\begin{aligned} y &\in f(B_X) \\ &= B_Y. \end{aligned}$$

  Since $V \in \mathcal{T}_Y$ and $y \in V$ are arbitrary, we have that for each $V \in \mathcal{T}_Y$ and $y \in V$, there exists $B_Y \in \mathcal{B}_Y$ such that $y \in B_Y \subset V$. Hence $\mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$.
- $(\impliedby)$:
  Similar to $(\implies)$.

$\square$

Definition 3.9.2.8. Let X be a topological space. Then X is said to be **Lindelöf** if for each open cover $\mathcal{U}$ of X, there exists a subcover $\mathcal{U}' \subset \mathcal{U}$ of X such that $\mathcal{U}'$ is countable.
NEED TO DEFINE COVER AND SUBCOVER
FINISH!!!

Exercise 3.9.2.9. Let $(X, \mathcal{T})$ be a topological space. If $(X, \mathcal{T})$ is second countable, then $(X, \mathcal{T})$ is Lindelöf.

Proof. Suppose that X is second countable. Then there exists $\mathcal{B} \subset \mathcal{T}$ such that $\mathcal{B}$ is a basis for $\mathcal{T}$ and $\mathcal{B}$ is countable. Let $\mathcal{U}$ be an open cover of X. For $B \in \mathcal{B}$, define $\mathcal{U}_B \subset \mathcal{U}$ by $\mathcal{U}_B = \{U \in \mathcal{U} : B \subset U\}$. Set $\Gamma = \{B \in \mathcal{B} : \mathcal{U}_B \neq \varnothing\}$. The axiom of choice implies that there exists $(V_B)_{B\in\Gamma} \subset \mathcal{U}$ such that for each $B \in \Gamma$, $V_B \in \mathcal{U}_B$. Set $\mathcal{U}' = (V_B)_{B\in\Gamma}$. Let $x \in X$. Since $\mathcal{U}$ is an open cover of X, there exists $U \in \mathcal{U}$ such that $x \in U$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Thus $B \in \Gamma$ and $x \in V_B$. So $\mathcal{U}'$ is an subcover of $\mathcal{U}$. Since $\mathcal{B}$ is countable, $\mathcal{U}'$ is countable. Since $\mathcal{U}$ is an arbitrary open cover of X, we have that for each open cover $\mathcal{U}$ of X, there exists a countable subscover $\mathcal{U}' \subset \mathcal{U}$. Hence $(X, \mathcal{T})$ is Lindelöf. $\square$

Second-Countability and Subspaces

Exercise 3.9.2.10. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. If $(X, \mathcal{T})$ is second-countable, then $(A, \mathcal{T} \cap A)$ is second countable.

Proof. Suppose that $(X, \mathcal{T})$ is second-countable. Then there exists $\mathcal{B} \subset \mathcal{T}_d$ such that $\mathcal{B}$ is a basis for $\mathcal{T}_d$ and $\mathcal{B}$ is countable. Exercise 3.4.1.10 implies that $\mathcal{B} \cap A$ is a basis for $\mathcal{T} \cap A$. Since $\mathcal{T} \cap A$ is countable, $(A, \mathcal{T} \cap A)$ is second countable. $\square$

Second-Countability and Product Spaces

Exercise 3.9.2.11. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in A}$ be a collection of topological spaces. Set $X = \prod_{\alpha\in A} X_\alpha$ and $\mathcal{T}_X := \bigotimes_{\alpha\in A} \mathcal{T}_\alpha$. Suppose that A is countable. If for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is second-countable, then $(X, \mathcal{T}_X)$ is second-countable.

Proof. Suppose that for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is second-countable. Then for each $\alpha \in A$, there exists $\mathcal{B}_\alpha \subset \mathcal{T}_\alpha$ such that $\mathcal{B}_\alpha$ is a basis for $\mathcal{T}_\alpha$ and $\mathcal{B}_\alpha$ is countable. Set

$$\mathcal{B} = \Bigg\{ \prod_{\alpha \in A} U_\alpha : \text{there exists } J \subset A \text{ such that } \#J < \infty, \\ \text{for each } \alpha \in J, U_\alpha \in \mathcal{B}_\alpha \text{ and for each } \alpha \in J^c, U_\alpha = X_\alpha \Bigg\}$$

An exercise in the section on the product topology implies that $\mathcal{B}$ is a basis for $\mathcal{T}_X$. Since A is countable, $\mathcal{B}$ is countable. Hence $\mathcal{T}_X$ is second-countable. □

Second-Countability and Coproduct Spaces

Exercise 3.9.2.12. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Set $X = \coprod_{\alpha \in A} X_\alpha$ and $\mathcal{T} := \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$. Suppose that A is countable. If for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is second-countable, then $(X, \mathcal{T})$ is second-countable.

Proof. Suppose that for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is second-countable. Then for each $\alpha \in A$, there exists $\mathcal{B}_\alpha \subset \mathcal{T}_\alpha$ such that $\mathcal{B}_\alpha$ is a basis for $\mathcal{T}_\alpha$ and $\mathcal{B}_\alpha$ is countable. Define $\mathcal{B} := \bigcup_{\alpha \in A} \mathcal{B}_\alpha$. Since A is countable and for each $\alpha \in A$, $\mathcal{B}_\alpha$ is countable, we have that $\mathcal{B}$ is countable. Exercise 3.6.0.5 implies that $\mathcal{B}$ is a basis for $\mathcal{T}$. Hence $(X, \mathcal{T})$ is second-countable. □

Second-Countability and Quotient Spaces

3.10 Compactness

3.10.1 Basic Properties

Definition 3.10.1.1. Let $(X, \mathcal{T})$ be a topological space $E \subset X$ and $\mathcal{U} \subset \mathcal{P}(X)$. Then $\mathcal{U}$ is said to be an **open cover** of E in $(X, \mathcal{T})$ if

1. $\mathcal{U} \subset \mathcal{T}$
2. $E \subset \bigcup_{U \in \mathcal{U}} U$

Definition 3.10.1.2. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is said to be **compact** if for each $\mathcal{U} \subset \mathcal{P}(X), \mathcal{U}$ is an open cover of X in $(X, \mathcal{T})$ implies that there exists $\mathcal{U}_0 \subset \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of X in $(X, \mathcal{T})$ and $\mathcal{U}_0$ is finite.

Definition 3.10.1.3. Let $(X, \mathcal{T})$ be topological space and $E \subset X$. Then E is said to be compact in $(X, \mathcal{T})$ if $(E, \mathcal{T} \cap E)$ is compact.

Exercise 3.10.1.4. Let $(X, \mathcal{T})$ be a topological space, $E \subset X$ and $A \subset E$. Then A is compact in $(E, \mathcal{T} \cap E)$ iff A is compact in $(X, \mathcal{T})$.

Proof. We note that since $A \subset E$, $(\mathcal{T} \cap E) \cap A = \mathcal{T} \cap A$. Suppose that A is compact in $(E, \mathcal{T} \cap E)$. By definition, $(A, (\mathcal{T} \cap E) \cap A)$ is compact. Since $(A, (\mathcal{T} \cap E) \cap A) = (A, \mathcal{T} \cap A)$, by definition, A is compact in $(X, \mathcal{T})$.
Coversely, suppose that A is compact in $(X, \mathcal{T})$. By definition, $(A, \mathcal{T} \cap A)$ is compact. Similarly, since $(A, \mathcal{T} \cap A) = (A, (\mathcal{T} \cap E) \cap A)$, A is compact in $(E, \mathcal{T} \cap E)$. □

Exercise 3.10.1.5. Let $(X, \mathcal{T})$ be a topological space and $K \subset X$. Then K is compact in $(X, \mathcal{T})$ iff for each $\mathcal{U} \subset \mathcal{P}(X), \mathcal{U}$ is an open cover of K in $(X, \mathcal{T})$ implies that there exists $\mathcal{U}_0 \subset \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover K in $(X, \mathcal{T})$ and $\mathcal{U}_0$ is finite.

Proof.

- $(\implies)$:
  Suppose that K is compact in $(X, \mathcal{T})$. Let $\mathcal{U} \subset \mathcal{P}(X)$. Suppose that $\mathcal{U}$ is an open cover of K in $(X, \mathcal{T})$. Then $\mathcal{U} \subset \mathcal{T}$ and $K \subset \bigcup_{U \in \mathcal{U}} U$. Therefore $\mathcal{U} \cap K \subset \mathcal{T} \cap K$ and

$$\begin{aligned} K &\subset \left[\bigcup_{U \in \mathcal{U}} U\right] \cap K \\ &= \bigcup_{U \in \mathcal{U}} U \cap K \end{aligned}$$

  Hence $\mathcal{U} \cap K$ is an open cover of K in $(K, \mathcal{T} \cap K)$. Since K is compact in $(X, \mathcal{T})$, Exercise 3.10.1.4 implies that $(K, \mathcal{T} \cap K)$ is compact. By Definition 3.10.1.2, there exists $\mathcal{V}_0 \subset \mathcal{U} \cap K$ such that $\mathcal{V}_0$ is an open cover of K in $(K, \mathcal{T} \cap K)$ and $\mathcal{V}_0$ is finite. Since $\mathcal{V}_0 \subset \mathcal{U} \cap K$, for each $V \in \mathcal{V}_0$, there exists $U_V \in \mathcal{U}$ such that $V = U_V \cap K$. Set $\mathcal{U}_0 = \{U_V : V \in \mathcal{V}_0\}$. Then $\mathcal{U}_0 \subset \mathcal{U}$.

  1. Since $\mathcal{U} \subset \mathcal{T}$

$$\begin{aligned} \mathcal{U}_0 &\subset \mathcal{U} \\ &\subset \mathcal{T} \end{aligned}$$

2. Since $\mathcal{V}_0$ is an open cover of K in $(K, \mathcal{T} \cap K)$, we have that

$$\begin{aligned} K &\subset \bigcup_{V \in \mathcal{V}_0} V \\ &= \bigcup_{U \in \mathcal{U}_0} U \cap K \\ &\subset \bigcup_{U \in \mathcal{U}_0} U \end{aligned}$$

By Definition 3.10.1.1, $\mathcal{U}_0$ is an open cover of K in $(X, \mathcal{T})$. By construction, $\mathcal{U}_0$ is finite.

- ($\Longleftarrow$):
Suppose that for each $\mathcal{U} \subset \mathcal{P}(X)$, if $\mathcal{U}$ is an open cover of K in $(X, \mathcal{T})$, then there exists $\mathcal{U}_0 \subset \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover K in $(X, \mathcal{T})$ and $\mathcal{U}_0$ is finite. Let $\mathcal{V} \subset \mathcal{P}(K)$. Suppose that $\mathcal{V}$ is an open cover of K in $(K, \mathcal{T} \cap K)$. Then $\mathcal{V} \subset \mathcal{T} \cap K$ and $K \subset \bigcup_{V \in \mathcal{V}} V$. By definition of $\mathcal{T} \cap K$, for each $V \in \mathcal{V}$, there exists $U \in \mathcal{T}$ such that $V = U \cap K$. The axiom of choice implies that there exists $(U_V)_{V \in \mathcal{V}} \subset \mathcal{T}$ such that for each $V \in \mathcal{V}$, $V = U_V \cap K$. Therefore

$$\begin{aligned} K &\subset \bigcup_{V \in \mathcal{V}} V \\ &= \bigcup_{V \in \mathcal{V}} U_V \cap K \\ &\subset \bigcup_{V \in \mathcal{V}} U_V \end{aligned}$$

Hence $(U_V)_{V \in \mathcal{V}}$ is an open cover of K in $(X, \mathcal{T})$. By assumption, there exists $\mathcal{V}_0 \subset \mathcal{V}$ such that $(U_V)_{V \in \mathcal{V}_0}$ is an open cover of K in $(X, \mathcal{T})$ and $\mathcal{V}_0$ is finite.

1. Since $\mathcal{V} \subset \mathcal{T} \cap K$, we have that

$$\begin{aligned} \mathcal{V}_0 &\subset V \\ &\subset \mathcal{T} \cap K \end{aligned}$$

2. Since $(U_V)_{V \in \mathcal{V}_0}$ is an open cover of K in $(X, \mathcal{T})$, we have that

$$\begin{aligned} K &= K \cap K \\ &\subset \left[\bigcup_{V \in \mathcal{V}_0} U_V \right] \cap K \\ &= \bigcup_{V \in \mathcal{V}_0} U_V \cap K \\ &= \bigcup_{V \in \mathcal{V}_0} V \end{aligned}$$

Therefore $\mathcal{V}_0$ is an open cover of K in $(K, \mathcal{T} \cap K)$. Since $\mathcal{V} \subset \mathcal{P}(K)$ such that $\mathcal{V}$ is an open cover of K in $(K, \mathcal{T} \cap K)$ is arbitrary, we have that for each $\mathcal{V} \subset \mathcal{P}(K)$, if $\mathcal{V}$ is an open cover of K in $(K, \mathcal{T} \cap K)$, then there exists $\mathcal{V}_0 \subset \mathcal{V}$ such that $\mathcal{V}_0$ is an open cover of K in $(K, \mathcal{T} \cap K)$ and $\mathcal{V}_0$ is finite. Hence $(K, \mathcal{T} \cap K)$ is compact. By definition, K is compact in $(X, \mathcal{T})$.

$\square$

Exercise 3.10.1.6. Let $(X, \mathcal{T})$ be a topological space and $K, L \subset X$. If K and L are compact, then $K \cup L$ is compact.

Proof. Suppose that K and L are compact. Let $\mathcal{U} \subset \mathcal{P}(X)$. Suppose that $\mathcal{U}$ is an open cover of $K \cup L$ in $(X, \mathcal{T})$. Since $K, L \subset K \cup L$, $\mathcal{U}$ is an open cover of K and L. Since K, L are compact, there exist $\mathcal{U}_K, \mathcal{U}_L \subset \mathcal{U}$ such that $\mathcal{U}_K$ is an open cover of K, $\mathcal{U}_L$ is an open cover of L and $\mathcal{U}_K$, $\mathcal{U}_L$ are finite. Define $\mathcal{U}_0 \subset \mathcal{U}$ by $\mathcal{U}_0 := \mathcal{U}_K \cup \mathcal{U}_L$. Then $\mathcal{U}_0$ is an open cover of $K \cup L$ and $\mathcal{U}_0$ is finite. Since $\mathcal{U} \subset \mathcal{P}(X)$ with $\mathcal{U}$ an open cover of $K \cup L$ arbitrary, we have that for each $\mathcal{U} \subset \mathcal{P}(X)$, if $\mathcal{U}$ is an open cover of $K \cup L$, there exists $\mathcal{U}_0 \subset \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of $K \cup L$ and $\mathcal{U}_0$ is finite. Thus $K \cup L$ is compact. □

Exercise 3.10.1.7. Let $(X, \mathcal{T})$ be a topological space and $K \subset X$. Suppose that $(X, \mathcal{T})$ is Hausdorff. If K is compact in X, then K is closed in X.

Proof. Suppose that K is compact. Let $y \in K^c$. Since $(X, \mathcal{T})$ is Hausdorff, for each $x \in K$, there exists $U_x, V_x \in \mathcal{T}$, such that $x \in U_x$, $y \in V_x$ and $U_x \cap V_x = \varnothing$. Thus $(U_x)_{x \in K}$ is an open cover of K in $(X, \mathcal{T})$. Since K is compact, there exist $x_1, \ldots, x_n \in K$ such that $(U_{x_j})_{j=1}^n$ is an open cover of K in $(X, \mathcal{T})$. Set $V = \bigcap_{j=1}^{n} V_{x_j}$. Then $V \in \mathcal{T}$ and $y \in V$. Since for each $j \in \{1, \ldots, n\}$, $V \subset V_{x_j}$, we have that

$$\begin{aligned}
V \cap K &\subset V \cap \left[\bigcup_{j=1}^{n} U_{x_j}\right] \\
&= \bigcup_{j=1}^{n} (V \cap U_{x_j}) \\
&\subset \bigcup_{j=1}^{n} (V_{x_j} \cap U_{x_j}) \\
&= \bigcup_{j=1}^{n} \varnothing \\
&= \varnothing
\end{aligned}$$

Thus $V \subset K^c$. Since $y \in K^c$ is arbitrary, we have that for each $y \in K^c$, there exists $V \in \mathcal{T}$ such that $y \in V$ and $V \subset K^c$. Hence K^c is open. Thus K is closed. □

Exercise 3.10.1.8. Let $(X, \mathcal{T})$ be a topological space and $E \subset X$. If $(X, \mathcal{T})$ is compact and E is closed in $(X, \mathcal{T})$, then E is compact in $(X, \mathcal{T})$.

Proof. Suppose that $(X, \mathcal{T})$ is compact and E is closed in $(X, \mathcal{T})$. Let $\mathcal{U} \subset \mathcal{P}(X)$. Suppose that $\mathcal{U}$ is an open cover of E in $(X, \mathcal{T})$. Since E is closed in $(X, \mathcal{T})$, $E^c \in \mathcal{T}$. Set $\mathcal{U}' = \mathcal{U} \cup \{E^c\}$. Then $\mathcal{U}' \subset \mathcal{T}$ and $\mathcal{U}'$ is an open cover of X in $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is compact, there exists $\mathcal{U}_0' \subset \mathcal{U}'$ such that $\mathcal{U}_0'$ is an open cover of X in $(X, \mathcal{T})$ and $\mathcal{U}_0'$ is finite. Set $\mathcal{U}_0 = \mathcal{U}_0' \setminus \{E^c\}$. Then $\mathcal{U}_0$ is an open cover for E in $(X, \mathcal{T})$. Since $\mathcal{U}$ such that $\mathcal{U}$ is an open cover of E in $(X, \mathcal{T})$ is arbitrary, Exercise 3.10.1.5 implies that E is compact in $(X, \mathcal{T})$.
GIVE MORE DETAILS
FINISH!!! □

Definition 3.10.1.9. Let X be a topological space and $E \subset X$. Then E is said to be **precompact** if $\operatorname{cl} E$ is compact.

Exercise 3.10.1.10. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ continuous. Then for each $K \subset X$, if K is compact in $(X, \mathcal{T}_X)$, then $f(K)$ is compact in $(Y, \mathcal{T}_Y)$.

Proof. Let $K \subset X$. Suppose that K is compact in $(X, \mathcal{T}_X)$. Let $\mathcal{V} \subset \mathcal{P}(Y)$. Suppose that $\mathcal{V}$ is an open cover of $f(K)$ in $(Y, \mathcal{T}_Y)$. By definition, $\mathcal{V} \subset \mathcal{T}_Y$ and $f(K) \subset \bigcup_{V \in \mathcal{V}} V$. Define $\mathcal{U} \subset \mathcal{P}(X)$ by $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$.

Since f is continuous and $\mathcal{V} \subset \mathcal{T}_Y$, $\mathcal{U} \subset \mathcal{T}_X$ and by construction

$$\begin{aligned}
K &\subset f^{-1}(f(K)) \\
&\subset f^{-1}\bigg(\bigcup_{V \in \mathcal{V}} V\bigg) \\
&= \bigcup_{V \in \mathcal{V}} f^{-1}(V) \\
&= \bigcup_{U \in \mathcal{U}} U
\end{aligned}$$

Hence $\mathcal{U}$ is an open cover of K in $(X, \mathcal{T})$. Since K is compact, there exists $\mathcal{U}_0 \subset \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of K in $(X, \mathcal{T}_X)$ and $\mathcal{U}_0$ is finite. For $U \in \mathcal{U}_0$, set $A_U = \{V \in \mathcal{V} : f^{-1}(V) = U\}$. By construction, for each $U \in \mathcal{U}_0$, $A_U \neq \varnothing$. Since $\mathcal{U}_0$ is finite, $\prod\limits_{U \in \mathcal{U}_0} A_U \neq \varnothing$. Thus there exists $\alpha \in \prod\limits_{U \in \mathcal{U}_0} A_U$. Set $\mathcal{V}_0 = \{\alpha_U : U \in \mathcal{U}_0\}$. By construction $\mathcal{V}_0 \subset \mathcal{V}$ and for each $U \in \mathcal{U}_0$, $f^{-1}(\alpha_U) = U$. Therefore

$$\begin{aligned}
f(K) &\subset f\bigg(\bigcup_{U \in \mathcal{U}_0} U\bigg) \\
&= \bigcup_{U \in \mathcal{U}_0} f(U) \\
&= \bigcup_{U \in \mathcal{U}_0} f(f^{-1}(\alpha_U)) \\
&= \bigcup_{U \in \mathcal{U}_0} \alpha_U \cap f(X) \\
&\subset \bigcup_{U \in \mathcal{U}_0} \alpha_U \\
&= \bigcup_{V \in \mathcal{V}_0} V
\end{aligned}$$

Hence $\mathcal{V}_0$ is an open cover of $f(K)$. Since $\mathcal{V} \subset \mathcal{P}(X)$ with $\mathcal{V}$ an open cover of $f(K)$ is arbitrary, we have that $f(K)$ is compact. $\square$

Exercise 3.10.1.11. Let $(X, \mathcal{T})$ be a topological space, $K \subset X$ and $x \in K^c$. Suppose that $(X, \mathcal{T})$ is Hausdorff. If K is compact in $(X, \mathcal{T})$, then there exist $U, V \in \mathcal{T}$ such that $U \cap V = \varnothing$, $x \in U$ and $K \subset V$.

Proof. Suppose that K is compact in $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is Hausdorff, we have that for each $y \in K$, there exist $U_y, V_y \in \mathcal{T}$ such that $U_y \cap V_y = \varnothing$, $x \in U_y$ and $y \in V_y$. Define $\mathcal{U}, \mathcal{V} \subset \mathcal{P}(X)$ by $\mathcal{U} := \{U_y : y \in K\}$ and $\mathcal{V} := \{V_y : y \in K\}$. Then $\mathcal{V}$ is an open cover of K in $(X, \mathcal{T})$. Since K is compact in $(X, \mathcal{T})$, there exists $\mathcal{V}_0 \subset \mathcal{V}$ such that $\mathcal{V}_0$ is finite and $\mathcal{V}_0$ is an open cover of K in $(X, \mathcal{T})$. By definition of $\mathcal{V}$, there exist $y_1, \ldots, y_n \in K$ such that $\mathcal{V}_0 = \{V_{y_j}\}_{j=1}^n$. Define $\mathcal{U}_0 \subset \mathcal{U}$ by $\mathcal{U}_0 = \{U_{y_j}\}_{j=1}^n$. Define $U, V \subset \mathcal{T}$ by $U := \bigcap\limits_{j=1}^n U_{y_j}$ and $V := \bigcup\limits_{j=1}^n V_{y_j}$. Since $\mathcal{V}_0$ is an open cover of K in $(X, \mathcal{T})$, we have that $K \subset V$. Since for each $y \in K$,

$x \in U_y$, we have that $x \in U$. Since for each $y \in K$, $U_y \cap V_y = \varnothing$, we have that

$$\begin{aligned} U \cap V &= \left(\bigcap_{j=1}^{n} U_{y_j} \right) \cap \left(\bigcup_{k=1}^{n} V_{y_k} \right) \\ &= \bigcup_{k=1}^{n} \left[\left(\bigcap_{j=1}^{n} U_{y_j} \right) \cap V_{y_k} \right] \\ &\subset \bigcup_{k=1}^{n} U_{y_k} \cap V_{y_k} \\ &= \bigcup_{k=1}^{n} \varnothing \\ &= \varnothing \end{aligned}$$

□

Exercise 3.10.1.12. Let $(X, \mathcal{T})$ be a topological space and $U \in \mathcal{T}$. Suppose that $(X, \mathcal{T})$ is Hausdorff. If $\operatorname{cl} U$ is compact, then for each $x \in U$, there exists $K \in \mathcal{N}(x)$ such that $K \subset U$ and K is compact.

Proof. Suppose that $\operatorname{cl} U$ is compact. Since $U \in \mathcal{T}$, $U \cap \partial U = \varnothing$. Thus

$$\begin{aligned} x &\in U \\ &\subset (\partial U)^c \end{aligned}$$

$x \notin \partial U$. Since $\operatorname{cl} U$ is compact, ∂U is closed and $\partial U \subset \operatorname{cl} U$, we have that ∂U is compact. The previous exercise implies that there exist $V, W \in \mathcal{T}$ such that $V \cap W = \varnothing$, $x \in V$ and $\partial U \subset W$. Since $V \subset W^c$ and W^c is closed, we have that $\operatorname{cl} V \subset W^c$. Since $\partial U \subset W$, we have that $(\operatorname{cl} V) \cap \partial U = \varnothing$. Hence $\operatorname{cl} V \subset U$. Set $K = \operatorname{cl} V$. Since K is closed, $K \subset \operatorname{cl} U$ and $\operatorname{cl} U$ is compact, we have that K is compact. By construction,

$$\begin{aligned} x &\in V \\ &= \operatorname{Int} K \end{aligned}$$

so $K \in \mathcal{N}(x)$. □

Exercise 3.10.1.13. Let $(X, \mathcal{T})$ be a topological space. If $(X, \mathcal{T})$ is compact and Hausdorff, then $(X, \mathcal{T})$ is normal.

Proof. Suppose that $(X, \mathcal{T})$ is compact and Hausdorff. Since $(X, \mathcal{T})$ is Hausdorff, $(X, \mathcal{T})$ is $\mathbf{T_1}$. Let $E, F \subset X$. Suppose that E, F are closed and $E \cap F = \varnothing$. Since $(X, \mathcal{T})$ is compact and E, F are closed, E, F are compact. A previous exercise implies that for each $x \in E$, there exists $U_x, V_x \in \mathcal{T}$ such that $U_x \cap V_x = \varnothing$, $x \in U_x$ and $F \subset V_x$. The axiom of choice implies that there exist $(U_x)_{x \in E}, (V_x)_{x \in E} \subset \mathcal{T}$ such that for each $x \in X$, $U_x \cap V_x = \varnothing$, $x \in U_x$ and $F \subset V_x$. Then $(U_x)_{x \in E}$ is an open cover of E. Since E is compact, there exist $x_1, \ldots, x_n \in E$ such that $(U_{x_j})_{j=1}^n$ is an open cover of E. Define $U, V \in \mathcal{T}$ by $U := \bigcup\limits_{j=1}^{n} U_{x_j}$ and

$V := \bigcap_{k=1}^{n} V_{x_k}$. Then

$$
\begin{aligned}
U \cap V &= \left(\bigcup_{j=1}^{n} U_{x_j}\right) \cap \left(\bigcap_{k=1}^{n} V_{x_k}\right) \\
&= \bigcup_{j=1}^{n} \left[U_{x_j} \cap \left(\bigcap_{k=1}^{n} V_{x_k}\right)\right] \\
&\subset \bigcup_{j=1}^{n} (U_{x_j} \cap V_{x_j}) \\
&= \bigcup_{j=1}^{n} \varnothing \\
&= \varnothing
\end{aligned}
$$

By construction,

$$
\begin{aligned}
E &\subset \bigcup_{j=1}^{n} U_{x_j} \\
&= U
\end{aligned}
$$

and

$$
\begin{aligned}
F &\subset \bigcap_{k=1}^{n} V_{x_j} \\
&= V
\end{aligned}
$$

Since $E, F \subset X$ with E, F closed and $E \cap F = \varnothing$ are arbitrary, we have that for each $E, F \subset X$, E, F are closed and $E \cap F = \varnothing$ implies that there exist $U, V \in \mathcal{T}$ such that $U \cap V = \varnothing$, $E \subset U$, $F \subset V$. So $(X, \mathcal{T})$ is normal. □

3.10.2 The Finite Intersection Property

Definition 3.10.2.1. Let $(X, \mathcal{T})$ be a set and $\mathcal{A} \subset \mathcal{P}(X)$. Then $\mathcal{A}$ is said to have the **finite intersection property** if for each $\mathcal{B} \subset \mathcal{A}$, $\mathcal{B}$ is finite implies that $\bigcap_{B \in \mathcal{B}} B \neq \varnothing$. We define

$$\text{FIP}(X, \mathcal{T}) = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \text{ has the finite intersection property}\}$$

and order $\text{FIP}(X, \mathcal{T})$ by inclusion.

Note 3.10.2.2. When the context is clear, we write $\text{FIP}(X)$ in place of $\text{FIP}(X, \mathcal{T})$.

Exercise 3.10.2.3. Let X be a set. Then $\text{FIP}(X)$ ordered by inclusion is a poset.

Proof. Clear. □

Exercise 3.10.2.4. Let X be a set and $\mathcal{A}_0 \in \text{FIP}(X)$. Then there exists $\mathcal{A} \in \text{FIP}(X)$ such that $\mathcal{A}$ is maximal in $[\mathcal{A}_0, \infty)$.

Proof. Let $\mathcal{C} \subset [\mathcal{A}_0, \infty)$. Suppose that $\mathcal{C}$ is a chain.

- Suppose that $\mathcal{C} = \varnothing$. Set $S := \mathcal{A}_0$. Then $\mathcal{S} \in [\mathcal{A}_0, \infty)$ and it is vacuously true that for each $\mathcal{E} \in \mathcal{C}$, $\mathcal{E} \subset \mathcal{S}$. Since $\mathcal{C} \subset [\mathcal{A}_0, \infty)$ with $\mathcal{C}$ a chain is arbitrary, we have that for each $\mathcal{C} \subset [\mathcal{A}_0, \infty)$, if $\mathcal{C}$ is a chain, then there exists $\mathcal{S} \in [\mathcal{A}_0, \infty)$ such that $\mathcal{S}$ is an upper bound for $\mathcal{C}$.

- Suppose that $\mathcal{C} \neq \varnothing$. Define $\mathcal{S} \in \mathcal{P}(X)$ by $\mathcal{S} := \bigcup_{\mathcal{E} \in \mathcal{C}} \mathcal{E}$. Since $\mathcal{C} \neq \varnothing$, there exists $\mathcal{E}_0 \in [\mathcal{A}_0, \infty)$ such that $\mathcal{E}_0 \in \mathcal{C}$. Since $\mathcal{A}_0 \subset \mathcal{E}_0$, $\mathcal{A}_0 \subset \mathcal{S}$. Let $\mathcal{B} \subset \mathcal{S}$. Suppose that $\mathcal{B}$ is finite. Since $\mathcal{B} \subset \mathcal{S}$ and $\mathcal{S} = \bigcup_{\mathcal{E} \in \mathcal{C}} \mathcal{E}$, we have that for each $B \in \mathcal{B}$, there exists $\mathcal{E}_B \in \mathcal{C}$ such that $B \in \mathcal{E}_B$. Since $\mathcal{B}$ is finite and $\mathcal{C}$ is totally ordered, there exists $B_0 \in \mathcal{B}$ such that $\mathcal{E}_{B_0} = \max_{B \in \mathcal{B}} \mathcal{E}_B$. Therefore for each $B \in \mathcal{B}$,

$$\begin{aligned} B &\in \mathcal{E}_B \\ &\subset \mathcal{E}_{B_0} \end{aligned}$$

Hence $\mathcal{B} \subset \mathcal{E}_{B_0}$ which implies that

$$\bigcap_{B \in \mathcal{E}_{B_0}} B \subset \bigcap_{B \in \mathcal{B}} B.$$

Since $\mathcal{E}_{B_0} \in \mathcal{C}$, and $\mathcal{C} \subset \text{FIP}(X)$, we have that $\mathcal{E}_{B_0} \in \text{FIP}(X)$. Since $\mathcal{E}_{B_0} \in \text{FIP}(X)$, $\mathcal{B} \subset \mathcal{E}_{B_0}$ and $\mathcal{B}$ is finite, we have that $\bigcap_{B \in \mathcal{E}_{B_0}} B \neq \varnothing$. Thus there exists $x \in X$ such that $x \in \bigcap_{B \in \mathcal{E}_{B_0}} B$. Since $\bigcap_{B \in \mathcal{E}_{B_0}} B \subset \bigcap_{B \in \mathcal{B}} B$, we have that $x \in \bigcap_{B \in \mathcal{B}} B$. Hence $\bigcap_{B \in \mathcal{B}} B \neq \varnothing$. Since $\mathcal{B} \subset \mathcal{S}$ with $\mathcal{B}$ finite is arbitrary, we have that for each $\mathcal{B} \subset \mathcal{S}$, $\mathcal{B}$ is finite implies that $\bigcap_{B \in \mathcal{B}} B \neq \varnothing$. Thus $\mathcal{S} \in \text{FIP}(X)$. Since $\mathcal{A}_0 \subset \mathcal{S}$ and $\mathcal{S} \in \text{FIP}(X)$, we have that $\mathcal{S} \in [\mathcal{A}_0, \infty)$. By construction, for each $\mathcal{E} \in \mathcal{C}$, $\mathcal{E} \subset \mathcal{S}$ so that $\mathcal{S}$ is an upper bound for $\mathcal{C}$. Since $\mathcal{C} \subset [\mathcal{A}_0, \infty)$ with $\mathcal{C}$ a chain is arbitrary, we have that for each $\mathcal{C} \subset [\mathcal{A}_0, \infty)$, if $\mathcal{C}$ is a chain, then there exists $\mathcal{S} \in [\mathcal{A}_0, \infty)$ such that $\mathcal{S}$ is an upper bound for $\mathcal{C}$. Zorn's lemma implies that there exists $\mathcal{A} \in [\mathcal{A}_0, \infty]$ such that $\mathcal{A}$ is maximal.

□

Exercise 3.10.2.5. Let X be a set and $\mathcal{A} \in \text{FIP}(X)$. Suppose that $\mathcal{A}$ is maximal. Then

1. for each $\mathcal{B} \subset \mathcal{A}$, if $\mathcal{B}$ is finite, then $\bigcap_{B \in \mathcal{B}} B \in \mathcal{A}$,
2. for each $B \subset X$, if for each $A \in \mathcal{A}$, $B \cap A \neq \varnothing$, then $B \in \mathcal{A}$.
   Hint: use part (1)

Proof.

1. Let $\mathcal{B} \subset \mathcal{A}$. Suppose that $\mathcal{B}$ is finite. Set $B_0 := \bigcap_{B \in \mathcal{B}} B$ and $\mathcal{A}_0 = \mathcal{A} \cup \{B_0\}$. Let $\mathcal{C} \subset \mathcal{A}_0$. Suppose that $\mathcal{C}$ is finite.

   - Suppose that $B_0 \notin \mathcal{C}$. Since $\mathcal{A} \in \text{FIP}(X)$, $\mathcal{C} \subset \mathcal{A}$, $\mathcal{C}$ is finite,

   $$\begin{aligned} \bigcap_{C \in \mathcal{C}} C &= \bigcap_{C \in \mathcal{C} \cap \mathcal{A}} C \\ &\neq \varnothing \end{aligned}$$

   - Now suppose that $B_0 \in \mathcal{C}$. Since $\mathcal{C}$ is finite and $\mathcal{B}$ is finite, we have that $(\mathcal{C} \cap \mathcal{A}) \cup \mathcal{B}$ is finite. Since $\mathcal{A} \in \text{FIP}(X)$ and $(\mathcal{C} \cap \mathcal{A}) \cup \mathcal{B} \subset \mathcal{A}$,

   $$\begin{aligned} \bigcap_{C \in \mathcal{C}} C &= \left(\bigcap_{C \in \mathcal{C} \cap \mathcal{A}} C \right) \cap B_0 \\ &= \left(\bigcap_{C \in \mathcal{C} \cap \mathcal{A}} C \right) \cap \left(\bigcap_{B \in \mathcal{B}} B \right) \\ &= \bigcap_{C \in (\mathcal{C} \cap \mathcal{A}) \cup \mathcal{B}} C \\ &\neq \varnothing \end{aligned}$$

Therefore $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Since $\mathcal{C} \subset \mathcal{A}_0$ with $\mathcal{C}$ finite is arbitrary, we have that for each $\mathcal{C} \subset \mathcal{A}_0$, $\mathcal{C}$ is finite implies that $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Hence $\mathcal{A}_0 \in \mathrm{FIP}(X)$. Since $\mathcal{A} \in \mathrm{FIP}(X)$ is maximal and $\mathcal{A} \subset \mathcal{A}_0$, we have that $\mathcal{A} = \mathcal{A}_0$ and therefore $B_0 \in \mathcal{A}$.

2. Let $B \subset X$. Suppose that for each $A \in \mathcal{A}$, $B \cap A \neq \varnothing$. Define $\mathcal{A}_0 = \mathcal{A} \cup \{B\}$. Let $\mathcal{C} \subset \mathcal{A}_0$. Suppose that $\mathcal{C}$ is finite.

   - If $B \notin \mathcal{C}$, then

$$\begin{aligned} \bigcap_{C \in \mathcal{C}} C &= \bigcap_{C \in \mathcal{C} \cap \mathcal{A}} C \\ &\neq \varnothing \end{aligned}$$

   - Suppose that $B \in \mathcal{C}$. Since $\mathcal{C} \cap \mathcal{A}$ is finite, part (1) implies that $\bigcap\limits_{C \in \mathcal{C} \cap \mathcal{A}} C \in \mathcal{A}$. Then by assumption, $\left(\bigcap\limits_{C \in \mathcal{C} \cap \mathcal{A}} C\right) \cap B \neq \varnothing$. Therefore

$$\begin{aligned} \bigcap_{C \in \mathcal{C}} C &= \bigcap_{C \in (\mathcal{C} \cap \mathcal{A}) \cup \{B\}} C \\ &= \left(\bigcap_{C \in \mathcal{C} \cap \mathcal{A}} C\right) \cap B \\ &\neq \varnothing \end{aligned}$$

Therefore $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Since $\mathcal{C} \subset \mathcal{A}_0$ with $\mathcal{C}$ finite is arbitrary, we have that for each $\mathcal{C} \subset \mathcal{A}_0$, $\mathcal{C}$ is finite implies that $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Hence $\mathcal{A}_0 \in \mathrm{FIP}(X)$. Since $\mathcal{A} \in \mathrm{FIP}(X)$ is maximal and $\mathcal{A} \subset \mathcal{A}_0$, we have that $\mathcal{A} = \mathcal{A}_0$ and therefore $B \in \mathcal{A}$.

$\square$

Note 3.10.2.6. Recall the definition of $\mathcal{C}_A(X, \mathcal{T})$ in Definition 3.1.0.15.

Exercise 3.10.2.7. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is compact iff for each $\mathcal{C} \subset \mathcal{C}_\varnothing(X, \mathcal{T})$, $\mathcal{C} \in \mathrm{FIP}(X, \mathcal{T})$ implies that $\bigcap\limits_{C \in \mathcal{C}} C \neq \varnothing$.
Hint: consider $\{C^c : C \in \mathcal{C}\}$ and whether it is an open cover

Proof.

- ($\Longrightarrow$):
  Suppose that $(X, \mathcal{T})$ is compact. Let $\mathcal{C} \subset \mathcal{C}_\varnothing$. Suppose that $\mathcal{C} \in \mathrm{FIP}(X, \mathcal{T})$. For the sake of contradiction, suppose that $\bigcap\limits_{C \in \mathcal{C}} C = \varnothing$. Define $\mathcal{U} \subset \mathcal{T}$ by $\mathcal{U} := \{C^c : C \in \mathcal{C}\}$. Then

$$\begin{aligned} \bigcup_{U \in \mathcal{U}} U &= \bigcup_{C \in \mathcal{C}} C^c \\ &= \left(\bigcap_{C \in \mathcal{C}} C\right)^c \\ &= \varnothing^c \\ &= X \end{aligned}$$

  Thus $\mathcal{U}$ is an open cover of X. Since $(X, \mathcal{T})$ is compact, there exists $\mathcal{U}' \subset \mathcal{U}$ such that $\mathcal{U}'$ is finite and

$\mathcal{U}'$ is an open cover of X. Define $\mathcal{C}' \subset \mathcal{C}$ by $\mathcal{C}' := \{U^c : U \in \mathcal{U}'\}$. Then

$$\begin{aligned}\bigcap_{C \in \mathcal{C}'} C &= \bigcap_{U \in U'} U^c \\ &= \bigg(\bigcup_{U \in U'} U\bigg)^c \\ &= X^c \\ &= \varnothing\end{aligned}$$

However, since $\mathcal{C} \in \text{FIP}(X, \mathcal{T})$, $\mathcal{C}' \subset \mathcal{C}$ and $\mathcal{C}'$ is finite, we have that $\bigcap_{C \in \mathcal{C}'} C \neq \varnothing$. This is a contradiction. Hence $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Since $\mathcal{C} \subset \mathcal{C}_\varnothing$ such that $\mathcal{C} \in \text{FIP}(X, \mathcal{T})$ is arbitrary, we have have that for each $\mathcal{C} \subset \mathcal{C}_\varnothing$, $\mathcal{C} \in \text{FIP}(X, \mathcal{T})$ implies that $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$.

- ($\Longleftarrow$):
Suppose that for each $\mathcal{C} \subset \mathcal{C}_\varnothing$, $\mathcal{C} \in \text{FIP}(X, \mathcal{T})$ implies that $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. Let $\mathcal{U} \subset \mathcal{P}(X)$. Suppose that $\mathcal{U}$ is an open cover of X in $(X, \mathcal{T})$. Then $\mathcal{U} \subset \mathcal{T}$ and $X = \bigcup_{U \in \mathcal{U}} U$. Define $\mathcal{C} \subset \mathcal{C}_\varnothing$ by $\mathcal{C} = \{U^c : U \in \mathcal{U}\}$. Then

$$\begin{aligned}\bigcap_{C \in \mathcal{C}} C &= \bigcap_{U \in \mathcal{U}} U^c \\ &= \bigg(\bigcup_{U \in \mathcal{U}} U\bigg)^c \\ &= X^c \\ &= \varnothing\end{aligned}$$

By assumption, $\mathcal{C} \notin \text{FIP}(X, \mathcal{T})$. Thus there exists $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{C}'$ is finite and $\bigcap_{C \in \mathcal{C}'} C = \varnothing$. Define $\mathcal{U}' \subset \mathcal{U}$ by $\mathcal{U}' := \{C^c : C \in \mathcal{C}'\}$. Then $\mathcal{U}'$ is finite and

$$\begin{aligned}X &= \varnothing^c \\ &= \bigg(\bigcap_{C \in \mathcal{C}'} C\bigg)^c \\ &= \bigcup_{C \in \mathcal{C}'} C^c \\ &= \bigcup_{U \in \mathcal{U}'} U\end{aligned}$$

Hence $\mathcal{U}'$ is an open cover of X in $(X, \mathcal{T})$. Since $\mathcal{U} \subset \mathcal{P}(X)$ such that $\mathcal{U}$ is an open cover of X in $(X, \mathcal{T})$, we have that for each $\mathcal{U} \subset \mathcal{P}(X)$, $\mathcal{U}$ is an open cover of X in $(X, \mathcal{T})$ implies that there exists $\mathcal{U}' \subset \mathcal{U}$ such that $\mathcal{U}'$ is finite and $\mathcal{U}'$ is an open cover of X in $(X, \mathcal{T})$. Thus $(X, \mathcal{T})$ is compact.

$\square$

Exercise 3.10.2.8. Let $(X, \mathcal{T})$ be a topological space. Then the following are equivalent:

1. $(X, \mathcal{T})$ is compact
2. for each net $(x_\alpha)_{\alpha \in A} \subset X$, there exists $x \in X$ such that x is a cluster point of $(x_\alpha)_{\alpha \in A}$.
3. for each net $(x_\alpha)_{\alpha \in A} \subset X$, there exists a subnet $(x_{\alpha_\beta})_{\beta \in B}$ of $(x_\alpha)_{\alpha \in A}$ and $x \in X$ such that $x_{\alpha_\beta} \to x$.

Hint:

- (1) $\implies$ (2):
For $\alpha \in A$, set $E_\alpha := \{x_{\alpha'} : \alpha' \geq \alpha\}$. Then $\{\operatorname{cl} E_\alpha : \alpha \in A\} \in \operatorname{FIP}(X, \mathcal{T})$.

- (3) $\implies$ (1):
If $(X, \mathcal{T})$ is not compact, choose open cover $\mathcal{U}$ of X such that for each $\mathcal{U}_0 \subset \mathcal{U}$, $\mathcal{U}_0$ is finite implies that $\mathcal{U}_0$ is not an open cover of X. Consider $\mathcal{F}_\mathcal{U} = \{\mathcal{U}' \subset \mathcal{U} : \mathcal{U}' \text{ is finite}\}$ ordered by inclusion. Then there exists a net $(x_{\mathcal{U}'})_{\mathcal{U}' \in \mathcal{F}_\mathcal{U}} \subset X$ such that for each $x_{\mathcal{U}'} \notin \bigcup\limits_{U \in \mathcal{U}'} U$.

Proof.

- (1) $\implies$ (2):
Suppose that $(X, \mathcal{T})$ is compact. Let $(x_\alpha)_{\alpha \in A} \subset X$ be a net. For $\alpha_0 \in A$, define $E_\alpha = \{x_{\alpha'} : \alpha' \geq \alpha\}$. Then for each $\alpha_1, \alpha_2 \in A$, $\alpha_1 \leq \alpha_2$ implies that $E_{\alpha_2} \subset E_{\alpha_1}$. Since A is directed, for each $\alpha \in A$, $E_\alpha \neq \varnothing$ and for each $A_0 \subset A$, A_0 is finite implies that there exists $\alpha_0 \in A$ such that for each $\alpha \in A_0$, $\alpha_0 \geq \alpha$.

  Define $\mathcal{E} \subset \mathcal{P}(X)$ by $\mathcal{E} := \{\operatorname{cl} E_\alpha : \alpha \in A\}$. Let $A_0 \subset A$. Suppose that A_0 is finite. Then there exists $\alpha_0 \in A$ such that for each $\alpha \in A_0$, $\alpha_0 \geq \alpha$. Then for each $\alpha \in A$, $E_{\alpha_0} \subset E_\alpha$. Thus

$$\begin{aligned} \varnothing &\neq E_{\alpha_0} \\ &\subset \bigcap_{\alpha \in A_0} E_\alpha \\ &\subset \bigcap_{\alpha \in A_0} \operatorname{cl} E_\alpha \end{aligned}$$

  Since $A_0 \subset A$ with A_0 finite is arbitrary, we have that for each $A_0 \subset A$, A_0 is finite implies that $\bigcap_{\alpha \in A_0} \operatorname{cl} E_\alpha \neq \varnothing$. Thus $\mathcal{E} \in \operatorname{FIP}(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is compact, the previous exercise implies that $\bigcap\limits_{\alpha \in A} \operatorname{cl} E_\alpha \neq \varnothing$. Thus there exists $x \in X$ such that $x \in \bigcap\limits_{\alpha \in A} \operatorname{cl} E_\alpha$. Exercise 3.3.2.19 implies that x is a cluster point of $(x_\alpha)_{\alpha \in A}$.

- (2) $\implies$ (3):
Immediate by Exercise 3.3.2.19.

- (3) $\implies$ (1):
Suppose that $(X, \mathcal{T})$ is not compact. Then there exists $\mathcal{U} \subset \mathcal{P}(X)$ such that $\mathcal{U}$ is an open cover of X in $(X, \mathcal{T})$ and for each $\mathcal{U}_0 \subset \mathcal{U}$, $\mathcal{U}_0$ is finite implies that $\mathcal{U}_0$ is not an open cover of X in $(X, \mathcal{T})$. Define $\mathcal{F}_\mathcal{U} \subset \mathcal{P}(X)$ by $\mathcal{F}_\mathcal{U} := \{\mathcal{U}_0 \subset \mathcal{U} : \mathcal{U}_0 \text{ is finite}\}$. We define $\leq \subset \mathcal{F}_\mathcal{U} \times \mathcal{F}_\mathcal{U}$ by inclusion so that $\mathcal{U}_1 \leq \mathcal{U}_2$ iff $\mathcal{U}_1 \subset \mathcal{U}_2$. Then $(\mathcal{F}_U, \leq)$ is a directed set. By construction for each $\mathcal{U}' \in \mathcal{F}_\mathcal{U}$, $\left(\bigcup\limits_{U \in \mathcal{U}'} U\right)^c \neq \varnothing$. The axiom of choice implies that there exists a net $(x_{\mathcal{U}'})_{\mathcal{U}' \in \mathcal{F}_\mathcal{U}} \subset X$ such that for each $\mathcal{U}' \in \mathcal{F}_\mathcal{U}$, $x_{\mathcal{U}'} \in \left(\bigcup\limits_{U \in \mathcal{U}'} U\right)^c$.

  For the sake of contradiction suppose that there exists a subnet $(x_{\mathcal{U}'_\beta})_{\beta \in B}$ of $(x_{\mathcal{U}'})_{\mathcal{U}' \in \mathcal{F}_\mathcal{U}}$ and $x \in X$ such that $x_{\mathcal{U}'_\beta} \to x$. Since $\mathcal{U}$ is an open cover of X in $(X, \mathcal{T})$, there exists $U_0 \in \mathcal{U}$ such that $x \in U_0$. Since $x_{\mathcal{U}'_\beta} \to x$ and $U_0 \in \mathcal{N}(x)$, there exists $\beta_0 \in B$ such that for each $\beta \in B$, $\beta \geq \beta_0$ implies that $x_{\mathcal{U}'_\beta} \in U_0$. Define $\mathcal{U}_0 \in \mathcal{F}_U$ by $\mathcal{U}_0 := \{U_0\}$. Since $(x_{\mathcal{U}'_\beta})_{\beta \in B}$ is a subnet of $(x_{\mathcal{U}'})_{\mathcal{U}' \in \mathcal{F}_\mathcal{U}}$, there exists $\beta_1 \in B$ such that for each $\beta \in B$, $\beta \geq \beta_1$ implies that $\mathcal{U}'_\beta \geq \mathcal{U}_0$. Since B is a directed set, there exists $\beta_2 \in B$ such that $\beta_2 \geq \beta_0, \beta_1$.

  Since $\beta_2 \geq \beta_0$, we have that $x_{\mathcal{U}'_{\beta_2}} \in U_0$. Since $\beta_2 \geq \beta_1$, we have that $\mathcal{U}'_{\beta_2} \geq \mathcal{U}_0$. Hence $\mathcal{U}_0 \subset \mathcal{U}'_{\beta_2}$ and therefore

$$\begin{aligned} U_0 &= \bigcup_{U \in \mathcal{U}_0} U \\ &\subset \bigcup_{U \in \mathcal{U}'_{\beta_2}} U \end{aligned}$$

By construction,

$$\begin{aligned} x_{\mathcal{U}'_{\beta_2}} &\in \bigg(\bigcup_{U \in \mathcal{U}'_{\beta_2}} U \bigg)^c \\ &\subset \bigg(\bigcup_{U \in \mathcal{U}_0} U \bigg)^c \\ &= U_0^c \end{aligned}$$

This is a contradiction. Thus for each subnet $(x_{\mathcal{U}'_\beta})_{\beta \in B}$ of $(x_{\mathcal{U}'})_{\mathcal{U}' \in \mathcal{F}_\mathcal{U}}$ and $x \in X$, we have that $x_{\mathcal{U}'_\beta} \not\to x$. Therefore there exists a net $(x_\alpha)_{\alpha \in A} \subset X$ such that for each subnet $(x_{\alpha_\beta})_{\beta \in B}$ of $(x_\alpha)_{\alpha \in A}$ and $x \in X$, $x_{\alpha_\beta} \not\to x$. By contrapositive, we have that (3) $\implies$ (1).

□

Exercise 3.10.2.9. Tychonoff's Theorem:
Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Suppose that for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is compact. Then $\left(\prod_{\alpha \in A} X_\alpha, \bigotimes_{\alpha \in A} \mathcal{T}_\alpha \right)$ is compact.
Hint:

Proof. Set $X := \prod_{\alpha \in A} X_\alpha$ and $\mathcal{T} := \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$. Let $\mathcal{C} \subset \mathcal{C}_\varnothing$. Suppose that $\mathcal{C} \in \mathrm{FIP}(X, \mathcal{T})$. A previous exercise implies that there exists $\mathcal{D} \in \mathrm{FIP}(X, \mathcal{T})$ such that $\mathcal{D}$ is maximal in $[\mathcal{C}, \infty)$. Let $\alpha \in A$. Set $\mathcal{D}_\alpha := \{\pi_\alpha(D) : D \in \mathcal{D}\}$.

Let $\mathcal{D}_{\alpha,0} \subset \mathcal{D}_\alpha$. Suppose that $\mathcal{D}_{\alpha,0}$ is finite. Then there exist $\mathcal{D}_0 \subset \mathcal{D}$ such that $\mathcal{D}_0$ is finite and $\mathcal{D}_{\alpha,0} = \{\pi_\alpha(D) : D \in \mathcal{D}_0\}$. Since $\mathcal{D} \in \mathrm{FIP}(X, \mathcal{T})$, $\bigcap_{D \in \mathcal{D}_0} D \neq \varnothing$. Therefore

$$\begin{aligned} \varnothing \neq \pi_\alpha \bigg(\bigcap_{D \in \mathcal{D}_0} D \bigg) &\\ \subset \bigcap_{D \in \mathcal{D}_0} \pi_\alpha(D) &\\ = \bigcap_{D \in \mathcal{D}_{\alpha,0}} D & \end{aligned}$$

Since $\mathcal{D}_{\alpha,0} \subset \mathcal{D}_\alpha$ with $\mathcal{D}_{\alpha,0}$ finite is arbitrary, we have that for each $\mathcal{D}_{\alpha,0} \subset \mathcal{D}_\alpha$, $\mathcal{D}_{\alpha,0}$ is finite implies that $\bigcap_{D \in \mathcal{D}_{\alpha,0}} D \neq \varnothing$. Hence $\mathcal{D}_\alpha \in \mathrm{FIP}(X, \mathcal{T})$.

For the sake of contradiction, suppose that $\bigcap_{D \in \mathcal{D}} \operatorname{cl} \pi_\alpha(D) = \varnothing$. Then $X = \bigcup_{D \in \mathcal{D}} [\operatorname{cl} \pi_\alpha(D)]^c$. Since $\{[\operatorname{cl} \pi_\alpha(D)]^c : D \in \mathcal{D}\} \subset \mathcal{T}_\alpha$ and $(X_\alpha, \mathcal{T}_\alpha)$ is compact, there exists $\mathcal{D}_0 \subset \mathcal{D}$ such that $\mathcal{D}_0$ is finite and $X = \bigcup_{D \in \mathcal{D}_0} [\operatorname{cl} \pi_\alpha(D)]^c$. Therefore $\bigcap_{D \in \mathcal{D}_0} \operatorname{cl} \pi_\alpha(D) = \varnothing$. Since $\mathcal{D}_\alpha \in \mathrm{FIP}(X, \mathcal{T})$ and $\{\operatorname{cl} \pi_\alpha(D) : D \in \mathcal{D}_0\} \subset \mathcal{D}_\alpha$ is finite, we have that $\bigcap_{D \in \mathcal{D}_0} \operatorname{cl} \pi_\alpha(D) \neq \varnothing$. This is a contradiction. Hence $\bigcap_{D \in \mathcal{D}} \operatorname{cl} \pi_\alpha(D) \neq \varnothing$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $\bigcap_{D \in \mathcal{D}} \operatorname{cl} \pi_\alpha(D) \neq \varnothing$.

The axiom of choice implies that there exists $x \in X$ such that for each $\alpha \in A$, $x_\alpha \in \bigcap_{D \in \mathcal{D}} \operatorname{cl} \pi_\alpha(D)$. Set

$$\mathcal{E}_x := \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in \mathcal{T}_\alpha, x_\alpha \in E_\alpha\}$$

Let $\alpha \in A$ and $E_\alpha \in \mathcal{T}_\alpha$. Suppose that $x \in \pi_\alpha^{-1}(E_\alpha)$. Then $x_\alpha \in E_\alpha$. Let $D \in \mathcal{D}$. Since $x_\alpha \in E_\alpha \cap \operatorname{cl} \pi_\alpha(D)$, $E_\alpha \cap \operatorname{cl} \pi_\alpha(D) \neq \varnothing$. Exercise 3.3.2.12 implies that $E_\alpha \cap \pi_\alpha(D) \neq \varnothing$. Therefore $\pi_\alpha^{-1}(E_\alpha) \cap D \neq \varnothing$. Since $D \in \mathcal{D}$ is arbitrary, we have that for each $D \in \mathcal{D}$, $\pi_\alpha^{-1}(E_\alpha) \cap D \neq \varnothing$. Since $\mathcal{D} \in \operatorname{FIP}(X, \mathcal{T})$, a previous exercise implies that $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{D}$. Since $\alpha \in A$ and $E_\alpha \in \mathcal{T}_\alpha$ are arbitrary, we have that $\mathcal{E}_x \subset \mathcal{D}$. Set

$$\mathcal{B}_x := \left\{ \bigcap_{j=1}^n V_j : (V_j)_{j=1}^n \subset \mathcal{E}_x \right\}$$

Then an exercise in the section on the product topology implies that $\mathcal{B}_x \subset \mathcal{T}_X$ and $\mathcal{B}_x$ is a local basis for $\mathcal{T}$ at x. Since $\mathcal{D} \in \operatorname{FIP}(X, \mathcal{T})$, a previous exercise implies that $\mathcal{B}_x \subset \mathcal{D}$.

Let $D \in \mathcal{D}$ and $E \in B_x$. Since $\mathcal{D} \in \operatorname{FIP}(X, \mathcal{T})$ and $D, E \in \mathcal{D}$, we have that $D \cap E \neq \varnothing$. Since $E \in \mathcal{B}_x$ is arbitary we have that for each $E \in \mathcal{B}_x$, $D \cap E \neq \varnothing$. An exercise in the introduction to topology section implies that $x \in \operatorname{cl} D$. Since $D \in \mathcal{D}$ is arbitrary, we have that for each $D \in \mathcal{D}$, $x \in \operatorname{cl} D$. Thus $x \in \bigcap\limits_{D \in \mathcal{D}} \operatorname{cl} D$ and therefore $\bigcap\limits_{D \in \mathcal{D}} \operatorname{cl} D \neq \varnothing$. Since $\mathcal{C} \subset \mathcal{C}_\varnothing$, for each $C \in \mathcal{C}$, $C = \operatorname{cl} C$. By construction, $\mathcal{C} \subset \mathcal{D}$, which implies that

$$\begin{aligned}
\varnothing &\neq \bigcap_{D \in \mathcal{D}} \operatorname{cl} D \\
&\subset \bigcap_{C \in \mathcal{C}} \operatorname{cl} C \\
&= \bigcap_{C \in \mathcal{C}} C
\end{aligned}$$

Since $\mathcal{C} \subset \mathcal{C}_\varnothing$ with $\mathcal{C} \in \operatorname{FIP}(X, \mathcal{T})$ is arbitrary, we have that for each $\mathcal{C} \subset \mathcal{C}_\varnothing$, $\mathcal{C} \in \operatorname{FIP}(X, \mathcal{T})$ implies that $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$. The previous exercise implies that $(X, \mathcal{T})$ is compact. $\square$

Exercise 3.10.2.10. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Suppose that for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is compact. Set $X := \prod\limits_{\alpha \in A} X_\alpha$ and $\mathcal{T}_X := \bigotimes\limits_{\alpha \in A} \mathcal{T}_\alpha$. Let $\alpha_0 \in A$. Set $A' := A \setminus \{\alpha_0\}$, $Y := \prod\limits_{\alpha \in A'} X_\alpha$ and $\mathcal{T}_Y := \bigotimes\limits_{\alpha \in A'} \mathcal{T}_\alpha$. Let $E \subset X$ and $a \in \pi_{\alpha_0}(E)^c$. Suppose that E is closed in $(X, \mathcal{T}_X)$.

1. For each $y \in Y$, there exists $x^y \in X$ such that $\pi_{\alpha_0}(x^y) = a$, for each $\alpha \in A'$, $\pi_\alpha(x^y) = y_\alpha$ and $x^y \in E^c$.
2. For each $y \in Y$, there exists $U_{\alpha_0}^y \in \mathcal{T}_{\alpha_0}$, $V^y \in \mathcal{T}_Y$ and $W^y \in \mathcal{T}_X$ such that $U_{\alpha_0}^y = \pi_{\alpha_0}(W^y)$, $V^y = \prod\limits_{\alpha \in A'} \pi_\alpha(W^y)$, $a \in U_{\alpha_0}^y$, $y \in V^y$ and $x^y \in W^y \subset E^c$.
   Hint: E is closed
3. There exist $U_{\alpha_0} \in \mathcal{T}_{\alpha_0}$ such that
   (a) $a \in U_{\alpha_0}$,
   Hint: consider the intersection of a finite open cover of Y.
   (b) $U_{\alpha_0} \subset \pi_{\alpha_0}(E)^c$
   proof by contradiction

Proof.

1. Let $y \in Y$. Define $x^y \in X$ by $x_\alpha^y := \begin{cases} a, & \alpha = \alpha_0 \\ y_\alpha, & \alpha \neq \alpha_0. \end{cases}$ For the sake of contradiction, suppose that $x^y \in E$. Then

$$\begin{aligned}
a &= x_{\alpha_0}^y \\
&= \pi_{\alpha_0}(x^y) \\
&\in \pi_{\alpha_0}(E).
\end{aligned}$$

This is a contradiction since $a \in (\pi_{\alpha_0}(E))^c$. Hence $x^y \in E^c$.

2. Since E is closed in $(X,\mathcal{T}_X)$, $E^c \in \mathcal{T}_X$. Exercise 3.5.1.2 then implies that there exist $(U^y_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} \mathcal{T}_\alpha$ such that $\prod_{\alpha\in A} U^y_\alpha \in \mathcal{T}_X$, $x^y \in \prod_{\alpha\in A} U^y_\alpha \subset E^c$ and $\#\{\alpha \in A : U^y_\alpha \neq X_\alpha\} < \infty$. Set $W^y := \prod_{\alpha\in A} U^y_\alpha$ and $V^y := \prod_{\alpha\in A'} U^y_\alpha$. Since $\#\{\alpha \in A : U^y_\alpha \neq X_\alpha\} < \infty$, we have that $\#\{\alpha \in A' : U^y_\alpha \neq X_\alpha\} < \infty$ and therefore $V^y \in \mathcal{T}_Y$. Then

$$\begin{aligned} a &= x^y_{\alpha_0} \\ &\in U^y_{\alpha_0} \end{aligned}$$

and $y \in V^y$. By construction, $U^y_{\alpha_0} = \pi_{\alpha_0}(W^y)$ and $V^y = \prod_{\alpha\in A'} \pi_\alpha(W^y)$. Since $y \in Y$ is arbitrary, we have that for each $y \in Y$, there exists $U^y_{\alpha_0} \in \mathcal{T}_{\alpha_0}$, $V^y \in \mathcal{T}_Y$ and $W^y \in \mathcal{T}_X$ such that $U^y_{\alpha_0} = \pi_{\alpha_0}(W^y)$, $V^y = \prod_{\alpha\in A'} \pi_\alpha(W^y)$, $a \in U^y_{\alpha_0}$, $y \in V^y$ and $x^y \in W^y \subset E^c$.

3. (a) Exercise 3.10.2.9 implies that $(Y,\mathcal{T}_Y)$ is compact. Since $Y \subset \bigcup_{y\in Y} V^y$ and $(Y,\mathcal{T}_Y)$ is compact, there exists $y_1,\dots,y_n \in Y$ such that $Y \subset \bigcup_{j=1}^n V^{y_j}$. Set $U_{\alpha_0} := \bigcap_{j=1}^n U^{y_j}_{\alpha_0}$. Then $U_{\alpha_0} \in \mathcal{T}_{\alpha_0}$ and $a \in U_{\alpha_0}$.

   (b) For the sake of contradiction, suppose that $U_{\alpha_0} \not\subset \pi_{\alpha_0}(E)^c$. Then there exists $b \in U_{\alpha_0}$ such that $b \in \pi_{\alpha_0}(E)$. Since $b \in \pi_{\alpha_0}(E)$, there exists $z \in E$ such that $\pi_{\alpha_0}(z) = b$. Since $Y \subset \bigcup_{j=1}^n V^{y_j}$, there exists $j_0 \in [n]$ such that $(z_\alpha)_{\alpha\in A'} \in V^{y_{j_0}}$. Since

$$\begin{aligned} z_{\alpha_0} &= b \\ &\in U_{\alpha_0} \\ &= \bigcap_{j=1}^n U^{y_j}_{\alpha_0} \\ &\subset U^{y_{j_0}}_{\alpha_0}, \end{aligned}$$

$$\begin{aligned} (z_\alpha)_{\alpha\in A'} &\in V^{y_{j_0}} \\ &= \prod_{\alpha\in A'} U^{y_{j_0}}_\alpha \end{aligned}$$

   and $W^{y_{j_0}} = \prod_{\alpha\in A} U^{y_{j_0}}_\alpha$, we have that $z \in W^{y_{j_0}}$. Since $z \in E$,

$$\begin{aligned} z &\in W^{y_{j_0}} \cap E \\ &= \varnothing, \end{aligned}$$

   which is a contradiction. Hence $U_{\alpha_0} \subset \pi_{\alpha_0}(E)^c$.

□

Exercise 3.10.2.11. Let $(X_\alpha,\mathcal{T}_\alpha)_{\alpha\in A}$ be a collection of topological spaces. Suppose that for each $\alpha \in A$, $(X_\alpha,\mathcal{T}_\alpha)$ is compact. Then for each $\alpha \in A$, $\pi_\alpha : \prod_{\alpha\in A} X_\alpha \to X_\alpha$ is closed.

Hint: Exercise 3.10.2.10

Proof. Set $X := \prod_{\alpha\in A} X_\alpha$ and $\mathcal{T}_X := \bigotimes_{\alpha\in A} \mathcal{T}_\alpha$. Let $\alpha_0 \in A$ and $E \subset X$. Suppose that E is closed in $(X,\mathcal{T}_X)$. Let $a \in \pi_{\alpha_0}(E)^c$. Exercise 3.10.2.10 implies that there exists $U_{\alpha_0} \in \mathcal{T}_{\alpha_0}$ such that $a \in U_{\alpha_0}$ and $U_{\alpha_0} \subset \pi_{\alpha_0}(E)^c$. Since $a \in \pi_{\alpha_0}(E)^c$ is arbitrary, we have that for each $a \in \pi_{\alpha_0}(E)^c$, there exists $U_{\alpha_0} \in \mathcal{T}_{\alpha_0}$ such that $a \in U_{\alpha_0}$ and $U_{\alpha_0} \subset \pi_{\alpha_0}(E)^c$. Therefore $\pi_{\alpha_0}(E)^c \in \mathcal{T}_{\alpha_0}$ and $\pi_{\alpha_0}(E)$ is closed in $(X_{\alpha_0},\mathcal{T}_{\alpha_0})$. Since $E \subset X$ with E closed in $(X,\mathcal{T}_X)$ is arbitrary, we have that for each $E \subset X$, E closed in $(X,\mathcal{T}_X)$ implies that $\pi_{\alpha_0}(E)$ is closed in $(X_{\alpha_0},\mathcal{T}_{\alpha_0})$. Hence π_{α_0} is closed. Since $\alpha_0 \in A$ is arbitrary, we have that for each $\alpha \in A$, $\pi_\alpha : \prod_{\alpha\in A} X_\alpha \to X_\alpha$ is closed. □

3.10.3 Sequential Compactness

Definition 3.10.3.1. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is said to be **sequentially compact** if for each $(x_n)_{n\in\mathbb{N}} \subset X$, there exists $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ and $x\ inX$ such that $x_{n_k} \to x$.

Exercise 3.10.3.2. Let $(X, \mathcal{T})$ be a topological space and $(x_n)_{n\in\mathbb{N}} \subset X$. Suppose that $(X, \mathcal{T})$ is first-countable and $(X, \mathcal{T})$ is not sequentially compact. Then

1. for each $x \in X$, there exists $U_x \in \mathcal{T}$ and $n_x \in \mathbb{N}$ such that $x \in U_x$ and for each $n \in \mathbb{N}$, $n \geq n_x$ implies that $x_n \in U^c$,
   Hint: Exercise 3.9.1.5

2. $(X, \mathcal{T})$ is not compact

Proof.

1. Since $(X, \mathcal{T})$ is not sequentially compact, there exists $(x_n)_{n\in\mathbb{N}} \subset X$ such that for each $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ and $x \in X$, $x_{n_k} \not\to x$. Exercise 3.9.1.5 implies that for each $x \in X$, x is not a cluster point of $(x_n)_{n\in\mathbb{N}}$. Thus for each $x \in X$, there exists $U_x \in \mathcal{T}$ and $n_x \in \mathbb{N}$ such that $x \in U_x$ and for each $n \in \mathbb{N}$, $n \geq n_x$ implies that $x_n \in U_x^c$.

2. For the sake of contradiction suppose that $(X, \mathcal{T})$ is compact. Define $\mathcal{U} \subset \mathcal{T}$ by $\mathcal{U} := \{U_x : x \in X\}$. Then $\mathcal{U}$ is an open cover of X in $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is compact, there exists $J \in \mathbb{N}$ and $(a_j)_{j=1}^J \subset X$ such that $(U_{a_j})_{j=1}^J$ is an open cover of X in $(X, \mathcal{T})$. Set $n_0 := \max(n_{a_j} : j \in [J])$. Let $n \in \mathbb{N}$. Suppose that $n \geq n_0$. Then for each $j \in [J]$,

$$\begin{aligned} n &\geq n_0 \\ &\geq n_{a_j}. \end{aligned}$$

   Thus for each $j \in [J]$, $x_n \in U_{a_j}^c$. Hence

$$\begin{aligned} x_n &\in \bigcap_{j=1}^J U_{a_j}^c \\ &= \left(\bigcup_{j=1}^J U_{a_j} \right)^c \\ &= X^c \\ &= \varnothing. \end{aligned}$$

   This is a contradiction. Therefore $(X, \mathcal{T})$ is not compact.

□

Exercise 3.10.3.3. Let $(X, \mathcal{T})$ be a topological space and $(x_n)_{n\in\mathbb{N}} \subset X$. Suppose that $(X, \mathcal{T})$ is first-countable. If $(X, \mathcal{T})$ is compact, then $(X, \mathcal{T})$ is sequentially compact.

Proof. Clear by the previious exercise. (add detail?) □

3.10.4 Compactness and Continuity

Exercise 3.10.4.1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces, $K \subset X$ and $f : X \to Y$. Suppose that f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous. If K is compact in $(X, \mathcal{T}_X)$, then $f(K)$ is compact in $(Y, \mathcal{T}_Y)$.

Proof. Suppose that K is compact in $(X, \mathcal{T}_X)$. Let $\mathcal{V} \subset \mathcal{P}(Y)$. Suppose that $\mathcal{V}$ is an open cover of $f(K)$ in $(Y, \mathcal{T}_Y)$. Then $\mathcal{V} \subset \mathcal{T}_Y$ and $f(K) \subset \bigcup_{V\in\mathcal{V}} V$. Define $\mathcal{U} \subset \mathcal{P}(X)$ by $\mathcal{U} := \{f^{-1}(V) : V \in \mathcal{V}\}$. Since f is continous, $\mathcal{U} \subset \mathcal{T}_X$ and we have that

$$\begin{aligned} K &\subset f^{-1}(f(K)) \\ &\subset f^{-1}\left(\bigcup_{V \in \mathcal{V}} V\right) \\ &= \bigcup_{V \in \mathcal{V}} f^{-1}(V) \\ &= \bigcup_{U \in \mathcal{U}} U. \end{aligned}$$

Hence $\mathcal{U}$ is an open cover of K in $(X, \mathcal{T}_X)$. Since K is compact, there exists $\mathcal{U}_0 \subset \mathcal{U}$ such that $\mathcal{U}_0$ is an open cover of K in $(X, \mathcal{T}_X)$ and $\mathcal{U}_0$ is finite. By choice (maybe make set theory exercise about how finite choice is given by ZFC), there exists $(\mathcal{V}_U)_{U \in \mathcal{U}_0} \subset \mathcal{V}$ such that for each $U \in \mathcal{U}_0$, $U = f^{-1}(V_U)$. Set $\mathcal{V}_0 := \{V_U : U \in \mathcal{U}_0\}$. Then $\mathcal{V}_0$ is finite and

$$\begin{aligned} f(K) &\subset f\left(\bigcup_{U \in \mathcal{U}_0} U\right) \\ &= \bigcup_{U \in \mathcal{U}_0} f(U) \\ &= \bigcup_{U \in \mathcal{U}_0} f(f^{-1}(V_U)) \\ &= \bigcup_{V \in \mathcal{V}_0} f(f^{-1}(V)) \\ &\subset \bigcup_{V \in \mathcal{V}_0} V \end{aligned}$$

Hence $\mathcal{V}_0$ is an open cover of $f(K)$. Since $\mathcal{V} \subset \mathcal{P}(Y)$ such that $\mathcal{V}$ is an open cover of $f(K)$ is arbitrary, we have that for each $\mathcal{V} \subset \mathcal{T}_Y$, if $\mathcal{V}$ is an open cover of $f(K)$, then there exists $\mathcal{V}_0 \subset \mathcal{V}$ such that $\mathcal{V}_0$ is an open cover of $f(K)$ and $\mathcal{V}_0$ is finite. Hence $f(K)$ is compact in $(Y, \mathcal{T}_Y)$. $\square$

Definition 3.10.4.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be **proper** if for each $K \subset X$, K is compact in $(Y, \mathcal{T}_Y)$ implies that $f^{-1}(K)$ is compact in $(X, \mathcal{T})$.

3.11 Locally Compact Hausdorff Spaces

3.11.1 Introduction

Definition 3.11.1.1. Let X be a topological space. Then

- X is said to be **locally compact** if for each $x \in X$, there exists $K \in \mathcal{N}(x)$ such that K is compact
- X is said to be a locally compact Hausdorff (LCH) space if X is locally compact and X is Hausdorff.

Exercise 3.11.1.2. Let X be a LCH space and $U \subset X$. Suppose that U is open. Then for each $x \in U$, there exists $K \in \mathcal{N}(x)$ such that $K \subset U$ and K is compact.

Proof. Let $x \in U$. Since X is locally compact, there exists $K_0 \in \mathcal{N}(x)$ such that K_0 is compact. Set $U_0 = (\operatorname{Int} K_0) \cap U$. Since $\operatorname{cl} U_0 \subset K_0$, $\operatorname{cl} U_0$ is closed and K_0 is compact, we have that $\operatorname{cl} U_0$ is compact. Since U_0 is open and $x \in U_0$, an exercise in the section on compactness implies that there exists $K \in \mathcal{N}(x)$ such that $K \subset U$ and K is compact. □

Exercise 3.11.1.3. Let X be a LCH space and $U \subset X$ and $K \subset U$. If U is open and K is compact, then there exists $V \subset X$ such that V is open, $K \subset V$, $\operatorname{cl} V \subset U$ and V is precompact.

Proof. Suppose that U is open and K is compact. The previous exercise implies that for each $x \in K$, there exist $N \in \mathcal{N}(x)$ such that $N \subset U$ and N is compact. The axiom of choice implies that there exists $(N_x)_{x \in K} \subset \mathcal{P}(X)$ such that for each $x \in K$, $N_x \in \mathcal{N}(x)$, $N_x \subset U$ and N_x is compact. Then $(\operatorname{Int} N_x)_{x \in K}$ is an open cover of K. Since K is compact, there exist $x_1, \ldots, x_n \in K$ such that $(\operatorname{Int} N_{x_j})_{j=1}^n$ is an open cover of K. Set $V = \bigcup_{j=1}^n \operatorname{Int} N_{x_j}$. Then V is open and since $(\operatorname{Int} N_{x_j})_{j=1}^n$ is an open cover of K, we have that

$$\begin{aligned} K &\subset \bigcup_{j=1}^n \operatorname{Int} N_{x_j} \\ &= V \end{aligned}$$

By construction, $\operatorname{cl} V = \bigcap_{j=1}^n N_{x_j}$ which is compact, so V is precompact. Finally

$$\begin{aligned} \operatorname{cl} V &= \bigcap_{j=1}^n N_{x_j} \\ &\subset U \end{aligned}$$

□

Exercise 3.11.1.4. Urysohn's Lemma for LCH Spaces:
Let X be a LCH space, $U \subset X$ and $K \subset U$. If U is open and K is compact, then there exists $f \in C_c(X, [0,1])$ such that $f|_K = 1$ and $\operatorname{supp} f \subset U$.

Proof. Suppose that U is open and K is compact. The previous exercise implies that there exists $V \subset X$ such that V is open, $K \subset V$, $\operatorname{cl} V \subset U$ and V is precompact. □

Definition 3.11.1.5. Let X be a LCH space and $f \in C(X)$. Then f is said to **vanish at infinity** if for each $\epsilon > 0$, $|f|^{-1}([\epsilon, \infty))$ is compact.

Exercise 3.11.1.6. Let X be a LCH space. Then $\operatorname{cl} C_c(X) = C_0(X)$.

Proof. FINISH!!! □

Exercise 3.11.1.7. Let X, Y be topological spaces and $f : X \to Y$. Suppose that Y is a LCH space and f is continuous. If f is proper, then f is closed.
Hint: Let $(y_\alpha)_{\alpha \in A} \subset f(C)$ be a net and $y \in Y$. Suppose that $y_\alpha \to y$ and consider $K_y \in \mathcal{N}(X)$, then $f(C \cap f^{-1}(K_y)) = f(C) \cap K_y$.

Proof. Suppose that f is proper. Let $C \subset X$. Suppose that C is closed in X. Let $(y_\alpha)_{\alpha \in A} \subset f(C)$ be a net and $y \in Y$. Suppose that $y_\alpha \to y$.

Since Y is LCH, there exists $K_y \in \mathcal{N}(y)$ such that K_y is compact. Since Y is Hausdorff, Exercise 3.10.1.7 implies that K_y is closed. Since f is continuous, $f^{-1}(K_y)$ is closed. Hence $C \cap f^{-1}(K_y)$ is closed. Since f is proper, $f^{-1}(K_y)$ is compact. Exercise 3.10.1.8 implies that $C \cap f^{-1}(K_y)$ is compact. Since f is continous, $f(C \cap f^{-1}(K_y))$ is compact. Exercise 1.2.1.4 implies that $f(C \cap f^{-1}(K_y)) = f(C) \cap K_y$. Since $f(C) \cap K_y$ is compact and Y is Hausdorff, Exercise 3.10.1.7 implies that $f(C) \cap K_y$ is closed in Y.

Since $K_y \in \mathcal{N}(y)$ and $y_\alpha \to y$, there exists $\alpha_0 \in A$ such that for each $\alpha \in A$, $\alpha \geq \alpha_0$ implies that $y_\alpha \in K_y$. Hence for each $\alpha \in A$, $\alpha \geq \alpha_0$ implies that $y_\alpha \in f(C) \cap K_y$. Since $f(C) \cap K_y$ is closed in Y, we have that

$$\begin{aligned} y &\in f(C) \cap K_y \\ &\subset f(C). \end{aligned}$$

Since $(y_\alpha)_{\alpha \in A} \subset f(C)$ a net and $y \in Y$ such that $y_\alpha \to y$ are arbitrary, we have that for each net $(y_\alpha)_{\alpha \in A} \subset f(C)$ and $y \in Y$, $y_\alpha \to y$ implies that $y \in f(C)$. Hence $f(C)$ is closed in Y. Since $C \subset X$ such that C is closed in X is arbitrary, we have that for each $C \subset X$, C is closed in X implies that $f(C)$ is closed in Y. Therefore f is closed. □

3.11.2 Arzela-Ascoli Theorem

3.11.3 Stone-Weierstrass Theorem

Definition 3.11.3.1. Let X be a compact Hausdorff space and $\mathcal{A} \subset C(X, \mathbb{C})$ a subalgebra. Then $\mathcal{A}$ is said to **separate the points of** X if for each $x, y \in X$, $x \neq y$ implies that there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Exercise 3.11.3.2. Equip $\mathbb{R}^2$ with pointwise addition and multiplication.

1. For each $\mathcal{A} \in \{\{(0,0)\}, \operatorname{span}\{(1,0)\}, \operatorname{span}\{(0,1)\}, \operatorname{span}\{(1,1), \mathbb{R}^2\}\}$, $\mathcal{A}$ is a subalgebra of $\mathbb{R}^2$.
2. For each $\mathcal{A} \subset \mathbb{R}^2$, $\mathcal{A}$ is a subalgebra implies that $\mathcal{A} \in \{\{(0,0)\}, \operatorname{span}\{(1,0)\}, \operatorname{span}\{(0,1)\}, \operatorname{span}\{(1,1), \mathbb{R}^2\}\}$.
   Hint: If $(s,t) \in \mathcal{A}$, then $(s^2, t^2) \in \mathcal{A}$.

Proof.

1. Clear.
2. Since $\dim \mathbb{R}^2 = 2$, we have that $\dim \mathcal{A} \in \{0,1,2\}$. If $\dim \mathcal{A} = 0$, then $\mathcal{A} = \{(0,0)\}$. If $\dim \mathcal{A} = 2$, then $\mathcal{A} = \mathbb{R}^2$. Suppose that $\dim \mathcal{A} = 1$. Then there exists $(s,t) \in \mathcal{A}$ such that $\mathcal{A} = \operatorname{span}\{(s,t)\}$. Thus $(s,t) \neq (0,0)$. Since $(s,t) \in \mathcal{A}$, $(s^2,t^2) \in \mathcal{A}$. Since $(s,t) \neq (0,0)$, $(s^2,t^2) \neq (0,0)$. Since $\mathcal{A} = \operatorname{span}\{(s,t)\}$, there exists $\lambda \in \mathbb{R}$ such that $(s^2,t^2) = \lambda(s,t)$. Thus $s(s-\lambda) = 0$ and $t(t-\lambda) = 0$. Since $(s^2,t^2) \neq (0,0)$, $\lambda \neq 0$. Hence $(s,t) \in \{(\lambda,0),(0,\lambda),(\lambda,\lambda)\}$. Therefore

$$\begin{aligned} \mathcal{A} &= \operatorname{span}\{(s,t)\} \\ &\in \{\operatorname{span}\{(1,0)\}, \operatorname{span}\{(0,1)\}, \operatorname{span}\{(1,1)\}\}. \end{aligned}$$

□

Exercise 3.11.3.3. For each $\epsilon > 0$, there exists $p \in \mathbb{R}[x]$ such that $p(0) = 0$ and for each $x \in [-1,1]$, $\Big||x| - p(x)\Big| < \epsilon$.
Hint: Define $f : (-1,1) \to \mathbb{R}$ by $f(t) := (1-t)^{1/2}$. Consider the Taylor series for f at $t = 0$.

Proof. FINISH!!! □

Exercise 3.11.3.4. Let X be a compact Hausdorff space and $\mathcal{A} \subset C(X, \mathbb{C})$ a closed subalgebra.

Exercise 3.11.3.5. Real Stone-Weierstrass Theorem Let X be a compact Hausdorff space and $\mathcal{A} \subset C(X, \mathbb{C})$ a closed subalgebra. Then $\mathcal{A} = C(X, \mathbb{R})$ or there exists $x_0 \in X$ such that $\mathcal{A} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$.

Proof. content... □

FINISH!!!

3.12 Tychonoff Spaces

3.13 Compactification

Definition 3.13.0.1. Let X and Y be topological spaces and $\phi : X \to Y$. Then (Y, ϕ) is said to be a **compactification of** X if

1. Y is compact
2. $\phi(X)$ is dense in Y
3. $\phi : X \to \phi(X)$ is a homeomorphism

Definition 3.13.0.2. Let $X, X^* \in \text{Obj}(\textbf{Top})$ and $\iota_X \in \text{Hom}_{\textbf{Top}}(X, Y)$. Then (X', ι_X) is said to be a **Stone-Čech compactification of** X if

1. (X', ι_X) is a compactification of X
2. for each compactification (Y, ϕ) of X, there exists a unique $\phi' \in \text{Hom}_{\textbf{Top}}(X', Y)$ such that $\phi' \circ \iota_X = \phi$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, "\phi"'] & X' \arrow[d, "\phi'"] \\ & Y \end{tikzcd}$$

3.14 Common Structures

3.14.1 Equalizers of Continuous Maps

Note 3.14.1.1. Let X, Y be sets and $f, g : X \to Y$. We denote the equalizer of f and g by $\mathrm{Eq}(f, g)$ as in Definition 1.6.1.1.

Exercise 3.14.1.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f, g : X \to Y$ $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous. If $(Y, \mathcal{T}_Y)$ is Hausdorff, then $\mathrm{Eq}(f, g)$ is closed in $(X, \mathcal{T}_X)$.

Proof. Suppose that $(Y, \mathcal{T}_Y)$ is Hausdorff. Exercise 3.8.0.6 implies that Δ_Y is closed in $(Y \times Y, \mathcal{T}_Y \otimes \mathcal{T}_Y)$. Since f, g are $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous, Exercise 3.5.3.5 implies that $(f, g) : X \to Y \times Y$ is $(\mathcal{T}_X, \mathcal{T}_Y \otimes \mathcal{T}_Y)$-continuous. Since $\mathrm{Eq}(f, g) = (f, g)^{-1}(\Delta_Y)$, we have that $\mathrm{Eq}(f, g)$ is closed in $(X, \mathcal{T}_X)$. □

3.14.2 Projective Limits of Topological Spaces

Note 3.14.2.1. Let $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a projective system of sets.

- We denote the j-th projection map from $\prod_{j \in J} X_j$ onto X_j by $\mathrm{proj}_j : \prod_{j \in J} X_j \to X_j$.
- We denote the j-th projection map from $\varprojlim_{j \in J} X_j$ into X_j by $\pi_j : \varprojlim_{j \in J} X_j \to X_j$ as in Definition 1.6.2.3

Definition 3.14.2.2. Let $(J, \leq)$ be a directed poset, $(X_j, \mathcal{T}_j)_{j \in J}$ a collection of topological spaces and for each $(j, k) \in \leq$, $\pi_{j,k} : X_k \to X_j$ a $(\mathcal{T}_k, \mathcal{T}_j)$-continuous map. Suppose that for each $j, k, l \in J$,

1. $\pi_{j,j} = \mathrm{id}_{X_j}$,
2. $j \leq k$ and $k \leq l$ implies that $\pi_{j,k} \circ \pi_{k,l} = \pi_{j,l}$.

Then $((X_j, \mathcal{T}_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ is said to be a **projective system of topological spaces**.

Definition 3.14.2.3. Let $(J, \leq)$ be a directed poset, $((X_j, \mathcal{T}_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a projective system of topological spaces. We define the **projective limit topology** on $\varprojlim_{j \in J} X_j$, denoted $\varprojlim_{j \in J} \mathcal{T}_j$, by

$$\varprojlim_{j \in J} \mathcal{T}_j := \tau(\pi_j : j \in J).$$

Exercise 3.14.2.4. $\varprojlim_{j \in J} \mathcal{T}_j = \left[\bigotimes_{j \in J} \mathcal{T}_j \right] \cap \varprojlim_{j \in J} X_j$. FINISH!!!

Exercise 3.14.2.5. Let $(J, \leq)$ be a directed poset, $((X_j, \mathcal{T}_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a projective system of topological spaces. If for each $j \in J$, $(X_j, \mathcal{T}_j)$ is Hausdorff, then $\varprojlim_{j \in J} X_j$ is closed in $(\prod_{j \in J} X_j, \bigotimes_{j \in J} \mathcal{T}_j)$.

Proof. Suppose that for each $j \in J$, $(X_j, \mathcal{T}_j)$ is Hausdorff. Set $X := \varprojlim_{j \in J} X_j$ and $\mathcal{T} := \varprojlim_{j \in J} \mathcal{T}_j$. Exercise 1.6.2.5 implies that $X = \bigcap_{(j,k) \in \leq} \mathrm{Eq}(\pi_{j,k} \circ \mathrm{proj}_k, \mathrm{proj}_j)$. Let $(j, k) \in \leq$. Since $\pi_{j,k}$ is $(\mathcal{T}_k, \mathcal{T}_j)$-continuous and π_k is $(\mathcal{T}, \mathcal{T}_k)$-continuous, we have that $\pi_{j,k} \circ \pi_k$ is $(\mathcal{T}, \mathcal{T}_j)$-continuous. Since $\pi_{j,k} \circ \pi_k$ and π_j are $(\mathcal{T}, \mathcal{T}_j)$-continuous and $(X_j, \mathcal{T}_j)$ is Hausdorff, Exercise 3.14.1.2 implies that $\mathrm{Eq}(\pi_{j,k} \circ \mathrm{proj}_k, \mathrm{proj}_j)$ is closed in $(X, \mathcal{T})$. Since $(j, k) \in \leq$ is arbitrary, we have that for each $(j, k) \in \leq$, $\mathrm{Eq}(\pi_{j,k} \circ \mathrm{proj}_k, \mathrm{proj}_j)$ is closed in $(X, \mathcal{T})$. Since $X = \bigcap_{(j,k) \in \leq} \mathrm{Eq}(\pi_{j,k} \circ \mathrm{proj}_k, \mathrm{proj}_j)$, we have that X is closed in $(\prod_{j \in J} X_j, \bigotimes_{j \in J} \mathcal{T}_j)$. □

Exercise 3.14.2.6. Let $(J, \leq)$ be a directed poset, $((X_j, \mathcal{T}_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a projective system of topological spaces. Set $X := \varprojlim_{j \in J} X_j$ and $\mathcal{T} := \varprojlim_{j \in J} \mathcal{T}_j$. If for each $j \in J$, $(X_j, \mathcal{T}_j)$ is compact Hausdorff, then for each $j \in J$, $\pi_j : X \to X_j$ is $(\mathcal{T}, \mathcal{T}_j)$-closed.

Proof. Suppose that for each $j \in J$, $(X_j, \mathcal{T}_j)$ is compact Hausdorff. Let $j_0 \in J$. Set $X_0 := \prod_{j \in J} X_j$ and $\mathcal{T}_0 := \bigotimes_{j \in J} T_j$. Since for each $j \in J$, $(X_j, \mathcal{T}_j)$ is compact, Exercise 3.10.2.11 implies that for each $j \in J$, $\text{proj}_j : X_0 \to X_j$ is $(\mathcal{T}_0, \mathcal{T}_j)$-closed. Since for each $j \in J$, $(X_j, \mathcal{T}_j)$ is Hausdorff, Exercise 3.14.2.5 implies that X is closed in $(X_0, \mathcal{T}_0)$. Since $\pi_{j_0} = \text{proj}_{j_0}|_X$, Exercise 3.4.1.13 then implies that $\pi_{j_0} : X \to X_j$ is $(\mathcal{T}, \mathcal{T}_j)$-closed. Since $j_0 \in J$ is arbitrary, we have that for each $j \in J$, $\pi_j : X \to X_j$ is $(\mathcal{T}, \mathcal{T}_j)$-closed. $\square$

Exercise 3.14.2.7. Let $(J, \leq)$ be a directed poset, $((X_j, \mathcal{T}_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a projective system of topological spaces. If for each $j \in J$, $(X_j, \mathcal{T}_j)$ is compact Hausdorff, then $(\varprojlim_{j \in J} X_j, \varprojlim_{j \in J} \mathcal{T}_j)$ is compact Hausdorff.

Proof. Suppose that for each $j \in J$, $(X_j, \mathcal{T}_j)$ is compact Hausdorff. Set $X_0 := \prod_{j \in J} X_j$, $X := \varprojlim_{j \in J} X_j$, $\mathcal{T}_0 := \bigotimes_{j \in J} \mathcal{T}_j$ and $\mathcal{T} := \varprojlim_{j \in J} \mathcal{T}_j$. Since for each $j \in J$, $(X_j, \mathcal{T}_j)$ is Hausdorff, Exercise 3.8.2.1 implies that $(X_0, \mathcal{T}_0)$ is Hausdorff. Since $\mathcal{T} = \mathcal{T}_0 \cap X$, Exercise 3.8.1.2 implies that $(X, \mathcal{T})$ is Hausdorff. Since for each $j \in J$, $(X_j, \mathcal{T}_j)$ is compact, Exercise 3.10.2.9 implies that $(X_0, \mathcal{T}_0)$ is compact. Exercise 3.14.2.5 implies that X is closed in $(X_0, \mathcal{T}_0)$. Since X is closed in $(X_0, \mathcal{T}_0)$, Exercise 3.10.1.8 implies that $(X, \mathcal{T})$ is compact. $\square$

Definition 3.14.2.8. Let $(J, \leq)$ be a directed poset $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ be a projective system of topological spaces. Set $X := \varprojlim_{j \in J} X_j$. For each $j \in J$, we define the **j-th projective equivalence relation on** X, denoted $\sim_j \subset X \times X$, by $x \sim_j y$ iff $\pi_j(x) = \pi_j(y)$ and we denote the projection of X onto $X/\sim_j$ by $\pi_j^Q : X \to X/\sim_j$.

Exercise 3.14.2.9. Let $(J, \leq)$ be a directed poset $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ be a projective system of topological spaces. Suppose that for each $j \in J$, X_j is a compact Hausdorff space and π_j is surjective. Then

1. for each $j \in J$, there exists a unique $\bar{\pi} : X/\sim_j \to X_j$ such that $\bar{\pi}_j \circ \pi_j^Q = \pi_j$ and $\bar{\pi}_j$ is a homeomorphism,

2. for each $(j,k) \in \leq$, there exists a unique $\pi_{j,k}^Q : X/\sim_k \to X/\sim_j$ such that $\pi_{j,k}^Q$ is continuous and $\pi_j^Q = \pi_{j,k}^Q \circ \pi_k^Q$, i.e. the following diagram commutes:

$$\begin{tikzcd}
X_2 \arrow[rr, "f_{1,2}"] & & X_1 \\
X/\sim_{f_2} \arrow[u, "\bar{f}_2"] \arrow[rr, "\bar{f}_{1,2}"] & & X/\sim_{f_1} \arrow[u, "\bar{f}_1"'] \\
& X \arrow[ul, "\pi_2"] \arrow[ur, "\pi_1"'] \arrow[uul, bend left=60, "f_2"] \arrow[uur, bend right=60, "f_1"'] &
\end{tikzcd}$$

3. $((X/\sim_j)_{j \in J}, (\pi_{j,k}^Q)_{(j,k) \in \leq})$ is a projective system of topological spaces.

Proof.

1. Let $j \in J$. By construction, π_j is continuous. By assumption, π_j is surjective. Since for each $j' \in J$, $X_{j'}$ is compact Hausdorff, Exercise 3.14.2.6 implies that π_j is closed. Exercise 3.7.1.8 implies that π_j is a quotient map. Exercise 3.7.1.19 then implies that there exists a unique $\bar{\pi} : X/\sim_j \to X_j$ such that $\pi_j = \bar{\pi}_j \circ \pi_j^Q$ and $\bar{\pi}_j$ is a homeomorphism.

2. Let $(j,k) \in \leq$. looke here Exercise 3.7.1.20 Define $\pi_{j,k}^Q : X/\sim_k \to X/\sim_j$ by $\pi_{j,k}^Q := \bar{\pi}_j^{-1} \circ \pi_{j,k} \circ \bar{\pi}_k$. Since $\bar{\pi}_j^{-1}$, $\pi_{j,k}$, $\bar{\pi}_k$ are continuous, we have that $\pi_{j,k}^Q$ is continuous. By construction,

$$\pi_{j,k}^Q \circ \pi_k^Q = (\bar{\pi}_j^{-1} \circ \pi_{j,k} \circ \bar{\pi}_k) \circ \pi_k^Q$$

□

Exercise 3.14.2.10. For each $j \in J$, define $C_j \subset C(X)$ by

$$C_j := \{f \in C(X) : \text{ for each } x, y \in X,\ \pi_j(x) = \pi_j(y) \text{ implies that } f(x) = f(y)\}$$

Define $C \subset C(X)$ by $C := \bigcup_{j \in J} C_j$. Then

1. $C_j = \{f \in C(X) : f \text{ is } \sim_j\text{-invariant}\}$
2. C is a subalgebra of $C(X)$,
3. C is dense in $C(X)$
4. for each $f \in C_j$, there exists a unique $\bar{f} : X/{\sim_j} \to \mathbb{C}$ such that $\bar{f} \circ \bar{\pi}_j^{-1} \circ \pi_j = f$

Proof. 1.

2. Let $f_1, f_2 \in C$ and $\lambda \in \mathbb{C}$. Then there exist $j_1, j_2 \in J$ such that $f_1 \in C_{j_1}$ and $f_2 \in C_{j_2}$. Since J is directed, there exists $j_0 \in J$ such that $j_0 \geq j_1, j_2$. Let $x, y \in X$. Suppose that $\pi_{j_0}(x) = \pi_{j_0}(y)$. Then

$$\begin{aligned} \pi_{j_1}(x) &= \pi_{j_1,j_0} \circ \pi_{j_0}(x) \\ &= \pi_{j_1,j_0} \circ \pi_{j_0}(y) \\ &= \pi_{j_1}(y) \end{aligned}$$

and similarly, $\pi_{j_2}(x) = \pi_{j_2}(y)$. Therefore $f_1(x) = f_1(y)$ and $f_2(x) = f_2(y)$. Hence

$$\begin{aligned} (f_1 + \lambda f_2)(x) &= f_1(x) + \lambda f_2(x) \\ &= f_1(y) + \lambda f_2(y) \\ &= (f_1 + \lambda f_2)(y) \end{aligned}$$

and

$$\begin{aligned} (f_1 \cdot f_2)(x) &= f_1(x) f_2(x) \\ &= f_1(y) f_2(y) \\ &= (f_1 \cdot f_2)(y). \end{aligned}$$

Since $x, y \in X$ with $\pi_{j_0}(x) = \pi_{j_0}(y)$ are arbitrary, we have that for each $x, y \in X$, $\pi_{j_0}(x) = \pi_{j_0}(y)$ implies that $(f_1 + \lambda f_2)(x) = (f_1 + \lambda f_2)(y)$ and $(f_1 \cdot f_2)(x) = (f_1 \cdot f_2)(y)$. Thus

$$\begin{aligned} f_1 + \lambda f_2 &\in C_{j_0} \\ &\subset C \end{aligned}$$

and

$$\begin{aligned} f_1 \cdot f_2 &\in C_{j_0} \\ &\subset C \end{aligned}$$

Since $f_1, f_2 \in C$ and $\lambda \in \mathbb{C}$ are arbitrary, for each $f_1, f_2 \in C$ and $\lambda \in \mathbb{C}$, $f_1 + \lambda f_2 \in C$ and $f_1 \cdot f_2 \in C$. Hence C is a subalgebra of $C(X)$.

3.

□

3.15 Semi-continuity

Definition 3.15.0.1. Let X be a topological space, $f : X \to (\infty, \infty]$ and $x_0 \in X$. Then f is said to be **lower semicontinuous at** x_0 if

$$\liminf_{x \to x_0} f(x) \geq f(x_0)$$

and f is said to be **lower semicontinuous** if for each $x_0 \in X$, f is lower semicontinuous at x_0.

Exercise 3.15.0.2. Let X be a topological space and $f : X \to (\infty, \infty]$. Then f is lower semicontinuous iff for each $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty])$ is open.

Proof.

- $(\implies)$:
  Suppose that f is lower semicontinuous. Let $\alpha \in \mathbb{R}$ and $x_0 \in f^{-1}(\alpha, \infty]$.
  - Suppose that $f(x_0) = \infty$. Since f is lower semicontinuous,

$$\begin{aligned} \sup_{V \in \mathcal{N}(x_0)} \left[\inf_{x \in V \setminus \{x_0\}} f(x) \right] &= \liminf_{x \to x_0} f(x) \\ &\geq f(x_0) \\ &= \infty \end{aligned}$$

    Thus there exists $V_\alpha \in \mathcal{N}(x)$ such that

$$\begin{aligned} \inf_{x \in V_\alpha \setminus \{x_0\}} f(x) &\geq \alpha + 1 \\ &> \alpha \end{aligned}$$

    Hence for each $x \in V_\alpha \setminus \{x_0\}$,

$$\begin{aligned} f(x) &\geq \inf_{t \in V_\alpha \setminus \{x_0\}} f(t) \\ &> \alpha \end{aligned}$$

    Since $f(x_0) = \infty$, $f(x_0) > \alpha$. Hence

$$\begin{aligned} \operatorname{Int} V_\alpha &\subset V_\alpha \\ &\subset f^{-1}((\alpha, \infty]) \end{aligned}$$

    Thus there exists $V \in \mathcal{N}(x_0)$ such that V is open and $V \subset f^{-1}((\alpha, \infty])$.
  - Suppose that $f(x_0) < \infty$. Set $\epsilon := f(x_0) - \alpha$. By definition,

$$\begin{aligned} \sup_{V \in \mathcal{N}(x_0)} \left[\inf_{x \in V \setminus \{x_0\}} f(x) \right] &\geq f(x_0) \\ &> f(x_0) - \epsilon \end{aligned}$$

    Choose $V_\epsilon \in \mathcal{N}(x_0)$ such that

$$\begin{aligned} \inf_{x \in V_\epsilon \setminus \{x_0\}} f(x) &> f(x_0) - \epsilon \\ &= \alpha \end{aligned}$$

    Then

$$\begin{aligned} \operatorname{Int} V_\epsilon &\subset V_\epsilon \\ &\subset f^{-1}((\alpha, \infty]) \end{aligned}$$

    Thus there exists $V \in \mathcal{N}(x_0)$ such that V is open and $V \subset f^{-1}((\alpha, \infty])$.

Since $x_0 \in f^{-1}((\alpha, \infty])$ is arbitrary, we have that for each $x_0 \in f^{-1}((\alpha, \infty])$, there exists $V \subset f^{-1}((\alpha, \infty])$ such that V is open and $x_0 \in V$. Hence $f^{-1}((\alpha, \infty])$ is open. Since $\alpha \in \mathbb{R}$ is arbitrary, we have that for each $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty])$ is open.

- $(\Longleftarrow)$:
  Suppose that for each $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty])$ is open. Let $x_0 \in X$.
  - Suppose that $f(x_0) = \infty$. For $M > 0$, define $V_M = f^{-1}((M, \infty])$. Then for each $M > 0$, $V_M \in \mathcal{N}(x_0)$ and

$$\begin{aligned} \liminf_{x \to x_0} f(x) &= \sup_{V \in \mathcal{N}(x_0)} \left[\inf_{x \in V \setminus \{x_0\}} f(x) \right] \\ &\geq \sup_{M > 0} \left[\inf_{x \in V_M \setminus \{x_0\}} f(x) \right] \\ &\geq \sup_{n \in \mathbb{N}} M \\ &= \infty \\ &= f(x_0) \end{aligned}$$

  - Suppose that $f(x_0) < \infty$. Set $\alpha := f(x_0)$. For $n \in \mathbb{N}$, define $V_n = f^{-1}((f(x_0) - 1/n, \infty])$. Then for each $n \in \mathbb{N}$, $V_n \in \mathcal{N}(x_0)$ and

$$\begin{aligned} \liminf_{x \to x_0} f(x) &= \sup_{V \in \mathcal{N}(x_0)} \left[\inf_{x \in V \setminus \{x_0\}} f(x) \right] \\ &\geq \sup_{n \in \mathbb{N}} \left[\inf_{x \in V_n \setminus \{x_0\}} f(x) \right] \\ &\geq \sup_{n \in \mathbb{N}} f(x_0) - 1/n \\ &= f(x_0) \end{aligned}$$

  Hence So f is lower semicontinuous at x_0. Since $x_0 \in X$ is arbitrary, we have that for each $x_0 \in X$, f is lower semicontinuous at x_0. Therefore f is lower semicontinuous.

□

Definition 3.15.0.3. Let X be a topological space and $f : X \to \mathbb{R}$. We define the **epigraph of** f, denoted $\operatorname{epi} f$, by

$$\operatorname{epi} f = \{(x, y) \in X \times \mathbb{R} : f(x) \leq y\}$$

Exercise 3.15.0.4. Let X be a topological space and $f : X \to \mathbb{R}$. Then f is lower semicontinuous iff $\operatorname{epi} f$ is closed.

Proof. Suppose that f is lower semicontinuous. Let $(x_\alpha, y_\alpha)_{\alpha \in A} \subset \operatorname{epi} f$ be a net and $(x, y) \in X \times \mathbb{R}$. Then for each $\alpha \in A$, $f(x_\alpha) \leq y_\alpha$. Suppose that $(x_\alpha, y_\alpha) \to (x, y)$. Then $x_\alpha \to x$ and $y_\alpha \to y$. Therefore

$$\begin{aligned} f(x) &\leq \liminf_{t \to x} f(t) \\ &\leq \liminf_{\alpha \in A} f(x_\alpha) \\ &\leq \liminf_{\alpha \in A} y_\alpha \\ &= y \end{aligned}$$

So $(x, y) \in \operatorname{epi} f$ and $\operatorname{epi} f$ is closed.
Conversely, suppose that $\operatorname{epi} f$ is closed. □

Exercise 3.15.0.5. Let X be a topological space and $(f_\lambda)_{\lambda \in \Lambda} \subset (-\infty, \infty]^X$. Suppose that for each $\lambda \in \Lambda$, f_λ is lower semicontinuous. Set $f = \sup\limits_{\lambda \in \Lambda} f_\lambda$. Then f is lower semicontinuous.

Proof. Let $\alpha \in \mathbb{R}$ and $x \in X$. Then

$$\begin{aligned} x \in f^{-1}((\alpha, \infty]) &\iff \sup_{\lambda \in \Lambda} f_\lambda(x) > \alpha \\ &\iff \text{there exists } \lambda \in \Lambda \text{ such that } f_\lambda(x) > \alpha \\ &\iff \text{there exists } \lambda \in \Lambda \text{ such that } x \in f_\lambda^{-1}((\alpha, \infty]) \\ &\iff x \in \bigcup_{\lambda \in \Lambda} f_\lambda^{-1}((\alpha, \infty]) \end{aligned}$$

Since for each $\lambda \in \Lambda$, $f_\lambda^{-1}((\alpha, \infty])$ is open, $f^{-1}((\alpha, \infty]) = \bigcup\limits_{\lambda \in \Lambda} f_\lambda^{-1}((\alpha, \infty])$ is open. So f is lower semicontinuous. $\square$

Chapter 4

Metric Spaces

4.1 Introduction

Metrics

Definition 4.1.0.1. Let X be a set and $d : X \times X \to \mathbb{R}$. Then d is said to be a **metric on** X if for each $x, y, z \in X$,

1. $d(x, y) = d(y, x)$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) \leq d(x, z) + d(z, y)$

Exercise 4.1.0.2. Let X be a set and $d : X \times X \to \mathbb{R}$ a metric on X. Then for each $x, y \in X$, $d(x, y) \geq 0$.

Proof. Let $x, y, z \in X$. Then $d(x, z) \leq d(x, y) + d(y, z)$. This implies that $d(x, z) - d(x, y) \leq d(y, z)$. Since z is arbitrary, taking $z = x$, we obtain

$$\begin{aligned} d(x, x) - d(x, y) \leq d(y, x) &\implies -d(x, y) \leq d(x, y) \\ &\implies 0 \leq 2d(x, y) \\ &\implies d(x, y) \geq 0 \end{aligned}$$

□

Definition 4.1.0.3. Let X be a set and $d : X \times X \to [0, \infty)$ a metric. Then (X, d) is called a **metric space**.

Note 4.1.0.4. We usually suppress the metric and write X in place of (X, d).

Exercise 4.1.0.5. Reverse Triangle Inequality:
Let (X, d) be a metric space. Then for each $x, y, z \in X$, $|d(x, y) - d(y, z)| \leq d(x, z)$.

Proof. Let $x, y, z \in X$. The triangle inequality implies that $d(x, y) \leq d(x, z) + d(z, y)$. Hence $d(x, y) - d(z, y) \leq d(x, z)$. Similarly, the triangle inequality implies that $d(y, z) \leq d(y, x) + d(x, z)$. Then $-d(y, z) \geq -d(y, x) - d(x, z)$. Therefore

$$-d(x, z) \leq d(x, y) - d(z, y) \leq d(x, z)$$

and $|d(x, y) - d(z, y)| \leq d(x, z)$. □

Definition 4.1.0.6. Let (X, d) be a metric space, $A \subset X$ and $x \in X$. We define **the distance between** A **and** x, denoted $d(x, A)$, by

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

Exercise 4.1.0.7. Let (X, d) be a metric space, $A \subset X$. Then for each $x, y \in X$, $|d(x, A) - d(y, A)| \leq d(x, y)$.

Proof. Let $x, y \in X$. Let $\epsilon > 0$. Then there exists $a \in A$ such that $d(y, a) < d(y, A) + \epsilon$. The triangle inequality implies that

$$\begin{aligned} d(x, A) &\leq d(x, a) \\ &\leq d(x, y) + d(y, a) \\ &< d(x, y) + d(y, A) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $d(x, A) \leq d(x, y) + d(y, A)$. Hence $d(x, A) - d(y, A) \leq d(x, y)$. Similarly, we have that $d(y, A) - d(x, A) \leq d(x, y)$. Therefore $|d(y, A) - d(x, A)| \leq d(x, y)$. □

Definition 4.1.0.8. Let (X, d) be a metric space and $A, B \subset X$. We define the **distance between** A **and** B, denoted $d(A, B)$, by

$$d(A, B) = \inf_{\substack{a \in A \\ b \in B}} d(a, b)$$

Exercise 4.1.0.9. Let (X, d) be a metric space. Then for each $A, B \subset X$ and $c \in X$,

$$d(A, B) \leq d(A, c) + d(c, B)$$

Proof. Let $A, B \subset X$, $c \in X$ and $\epsilon > 0$. Choose $a \in A$ and $b \in B$ such that $d(a, c) < d(A, c) + \epsilon/2$ and $d(c, b) < d(c, B) + \epsilon/2$. Then

$$\begin{aligned} d(A, B) &\leq d(a, b) \\ &\leq d(a, c) + d(c, b) \\ &< d(A, c) + \frac{\epsilon}{2} + d(c, B) + \frac{\epsilon}{2} \\ &= d(A, c) + d(c, B) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $d(A, B) \leq d(A, c) + d(c, B)$. □

Metric Topology

Definition 4.1.0.10. Let X be a metric space, $x \in X$ and $r > 0$. We define the

- **open ball of radius** r **at** x, denoted $B(x, r)$, by

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

- **closed ball of radius** r **at** x, denoted $\bar{B}(x, r)$, by

$$\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$$

Definition 4.1.0.11. Let (X, d) be a metric space. We define the **metric topology on X**, denoted $\mathcal{T}_d$, by

$$\mathcal{T}_d = \{U \subset X : \text{ for each } x \in U, \text{ there exists } \delta > 0 \text{ such that } B(x, \delta) \subset U\}$$

Exercise 4.1.0.12. Let (X, d) be a metric space. Then

1. $\mathcal{T}_d$ is a topology on X,
2. $\{B(x, \delta) : x \in X, \delta > 0\}$ is a basis for $\mathcal{T}_d$.

Proof.

1. (a) Clearly $X \in \mathcal{T}_d$ and it is vacuously true that $\varnothing \in \mathcal{T}_d$.

(b) Let $(U_\alpha)_{\alpha\in A} \subset \mathcal{T}_d$. Let $x \in \bigcup_{\alpha\in A} U_\alpha$. Then there exists $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. By definition, there exists $\delta > 0$ such that $B(x,\delta) \subset U_{\alpha_0}$. Then

$$\begin{aligned} B(x,\delta) &\subset U_{\alpha_0} \\ &\subset \bigcup_{\alpha\in A} U_\alpha \end{aligned}$$

Hence $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}_d$.

(c) Let $(U_j)_{j=1}^n \subset \mathcal{T}_d$ and $x \in \bigcap_{j=1}^n U_j$. Then for each $j \in [n]$, $x \in U_j$. By definition, for each $j \in [n]$, there exists $\delta_j > 0$ such that $B(x,\delta_j) \subset U_j$. Set $\delta := \min(\delta_1,\dots,\delta_n)$. Then for each $j \in [n]$,

$$\begin{aligned} B(x,\delta) &\subset B(x,\delta_j) \\ &\subset U_j. \end{aligned}$$

Therefore $B(x,\delta) \subset \bigcap_{j=1}^n U_j$. Hence $\bigcap_{j=1}^n U_j \in \mathcal{T}_d$.

Thus $\mathcal{T}_d$ is a topology on X.

2. Set $\mathcal{B} := \{B(x,\delta) : x \in X, \delta > 0\}$. Let $U \in \mathcal{T}_d$. Let $x \in U$. By definition, there exists $\delta > 0$ such that $B(x,\delta) \subset U$. Set $B := B(x,\delta)$. Then $B \in \mathcal{B}$ and $x \in B \subset U$. Since $x \in U$ is arbitrary, we have that for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. Therefore $\mathcal{B}$ is a basis for $\mathcal{T}_d$.

□

Exercise 4.1.0.13. Let X be a metric space and $U \subset X$. Then $U \in \mathcal{T}_d$ iff for each $x \in A$, there exists $\delta > 0$ such that $B(x,\delta) \subset U$.

Proof. Clear by Exercise 4.1.0.12. □

Exercise 4.1.0.14. Let (X,d) be a metric space and $x \in X$. Set $\mathcal{B}_x = \{B(x,\delta) : \delta > 0\}$. Then $\mathcal{B}_x$ is a local basis for $\mathcal{T}_d$ at x.

Proof.

1. Clearly for each $U \in \mathcal{B}_x$, $x \in U$.

2. Let $V \in \mathcal{T}_d$. Suppose that $x \in V$. By definition of $\mathcal{T}_d$, there exists $\delta > 0$ such that $B(x,\delta) \in V$. Set $U := B(x,\delta)$. Then $U \in \mathcal{B}_x$ and $U \subset V$.

Hence $\mathcal{B}_x$ is a local basis for $\mathcal{T}_d$ at x. □

Exercise 4.1.0.15. Let (X,d_X), (Y,d_Y) be a metric spaces, $f : X \to Y$ and $x_0 \in X$. Then f is continuous at x_0 iff for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x \in X$, $d_X(x_0,x) < \delta$ implies that $d_Y(f(x_0,f(x))) < \epsilon$.

Proof.

- $(\implies)$:
  Suppose that f is continuous at x_0. Let $\epsilon > 0$. Set $V := B_Y(f(x_0),\epsilon)$. Since $V \in \mathcal{N}(f(x_0))$, continuity at x_0 and Exercise 3.2.0.3 imply that $f^{-1}(V) \in \mathcal{N}(x_0)$. Exercise 4.1.0.13 implies that there exists $\delta > 0$ such that $x \in B_X(x_0,\delta) \subset f^{-1}(V)$. Set $U := B_X(x_0,\delta)$. Let $x \in X$. Suppose that $d_X(x_0,x) < \delta$. Then $x \in U$. Since $U \subset f^{-1}(V)$, we have that

$$\begin{aligned} f(x) &\in f(U) \\ &\subset V \\ &= B_Y(f(x_0),\epsilon) \end{aligned}$$

and $d_Y(f(x_0), f(x)) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x \in X$, $d_X(x_0, x) < \delta$ implies that $d_Y(f(x_0), f(x)) < \epsilon$.

- $(\Longleftarrow)$:
  Suppose that for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x \in X$, $d_X(x_0, x) < \delta$ implies that $d_Y(f(x_0, f(x))) < \epsilon$. FINISH!!!

$\square$

Exercise 4.1.0.16. Let (X, d) be a metric space, $(x_\gamma)_{\gamma\in\Gamma}$ a net and $x \in X$. Then $x_\gamma \to x$ in $(X, \mathcal{T}_d)$ iff $d(x_\gamma, x) \to 0$ in $(\mathbb{R}, \mathcal{T}_\mathbb{R})$.

Proof.

- $(\Longrightarrow)$:
  Suppose that $x_\gamma \to x$ in $(X, \mathcal{T}_d)$. Let $U \in \mathcal{N}_\mathbb{R}(0)$. Then there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset U$. Since $B(x, \epsilon) \in \mathcal{N}(x)$ and $x_\gamma \to x$ in $(X, \mathcal{T}_d)$, we have that $(x_\gamma)_{\gamma\in\Gamma}$ is eventually in $B(x, \epsilon)$. Thus there exists $\gamma_0 \in \Gamma$ such that for each $\gamma \in \Gamma$, $\gamma \geq \gamma_0$ implies that $x_\gamma \in B(x, \epsilon)$. Let $\gamma \in \Gamma$. Suppose that $\gamma \geq \gamma_0$. Then

$$\begin{aligned} d(x_\gamma, x) &\in (-\epsilon, \epsilon) \\ &\subset U. \end{aligned}$$

  Thus $(d(x_\gamma, x))_{\gamma\in\Gamma}$ is eventually in U. Since $U \in \mathcal{N}_\mathbb{R}(0)$ is arbitrary, we have that for each $U \in \mathcal{N}_\mathbb{R}(0)$, $(d(x_\gamma, x))_{\gamma\in\Gamma}$ is eventually in U. Hence $d(x_\gamma, x) \to 0$.

- $(\Longleftarrow)$:
  Conversely, suppose that $d(x_\gamma, x) \to 0$ in $(\mathbb{R}, \mathcal{T}_\mathbb{R})$. Let $U \in \mathcal{N}_X(x)$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Since $(-\epsilon, \epsilon) \in \mathcal{N}_\mathbb{R}(0)$ and $d(x_\gamma, x) \to 0$ in $(\mathbb{R}, \mathcal{T}_\mathbb{R})$, we have that $(d(x_\gamma, x))_{\gamma\in\Gamma}$ is eventually in $(-\epsilon, \epsilon)$. Thus there exists $\gamma_0 \in \Gamma$ such that for each $\gamma \in \Gamma$, $\gamma \geq \gamma_0$ implies that $d(x_\gamma, x) \in (-\epsilon, \epsilon)$. Let $\gamma \in \Gamma$. Suppose that $\gamma \geq \gamma_0$. Then $d(x_\gamma, x) < \epsilon$ and

$$\begin{aligned} x_\gamma &\in B(x, \epsilon) \\ &\subset U. \end{aligned}$$

  Thus (x_γ) is eventually in U. Since $U \in \mathcal{N}(x)$ is arbitrary, we have that for each $U \in \mathcal{N}(x)$, $(x_\gamma)_{\gamma\in\Gamma}$ is eventually in U. Therefore $x_\gamma \to x$.

$\square$

Definition 4.1.0.17. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Then f is said to be **uniformly continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x_1, x_2 \in X$, $d_X(x_1, x_2) < \delta$ implies that $d_Y(f(x_1), f(x_2)) < \epsilon$.

Exercise 4.1.0.18. Let (X, d) be a metric space, $A \subset X$. Then $d(\cdot, A)$ is uniformly continuous.

Proof. Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x, y \in X$. Suppose that $d(x, y) < \delta$. Then

$$\begin{aligned} |d(y, A) - d(x, A)| &\leq d(x, y) \\ &< \delta \\ &= \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, y \in X$, $d(x, y) < \delta$ implies that $|d(y, A) - d(x, A)| < \epsilon$. Hence $d(\cdot, A)$ is uniformly continuous. $\square$

Definition 4.1.0.19. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Then f is said to be **Lipchitz** if there exists $K > 0$ such that for each $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$.

Exercise 4.1.0.20. Define $\phi : [0, \infty) \to [0, 1)$ by $\phi(t) := t/(1 + t)$. Then ϕ is Lipschitz.

Proof. Let $t_1, t_2 \in [0, \infty)$. Then

$$\begin{aligned}\phi(t_1) - \phi(t_2) &= \frac{t_1}{1+t_1} - \frac{t_2}{1+t_2} \\ &= \frac{t_1(1+t_2) - t_2(1+t_1)}{(1+t_1)(1+t_2)} \\ &= \frac{t_1 - t_2}{(1+t_1)(1+t_2)}.\end{aligned}$$

Therefore

$$\begin{aligned}|\phi(t_1) - \phi(t_2)| &= \left|\frac{t_1 - t_2}{(1+t_1)(1+t_2)}\right| \\ &= \frac{|t_1 - t_2|}{|1+t_1||1+t_2|} \\ &\leq |t_1 - t_2|.\end{aligned}$$

Since $t_1, t_2 \in [0, \infty)$ is arbitrary, we have that ϕ is Lipschitz. □

Exercise 4.1.0.21. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. If f is Lipschitz, then f is uniformly continuous.

Proof. Suppose that f is Lipschitz. Let $\epsilon > 0$. Since f is Lipschitz, there exists $K > 0$ such that for each $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$. Set $\delta := \epsilon/K$. Then $\delta > 0$. Let $x_1, x_2 \in X$. Suppose that $d_X(x_1, x_2) < \delta$. Then

$$\begin{aligned}d_Y(f(x_1), f(x_2)) &\leq K d_X(x_1, x_2) \\ &< K\delta \\ &= \frac{K\epsilon}{K} \\ &= \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x_1, x_2 \in X$, $d_X(x_1, x_2) < \delta$ implies that $d_Y(f(x_1), f(x_2)) < \epsilon$. Hence f is uniformly continuous. □

Exercise 4.1.0.22. Let (X, d) be a metric space, $C \subset X$ and $x \in X$. Suppose that C is closed. Then $d(x, C) = 0$ iff $x \in C$.

Proof. Suppose that $d(x, C) = 0$. Then for each $n \in \mathbb{N}$, there exists $c_n \in C$ such that $d(x, c_n) < 1/n$. Then $c_n \to x$. Since C is closed, $x \in C$.
Conversely, suppose that $x \in C$. Then

$$\begin{aligned}d(x, C) &= \inf\{d(x, c) : c \in C\} \\ &\leq d(x, x) \\ &= 0\end{aligned}$$

Hence $d(x, C) = 0$. □

Definition 4.1.0.23. Let (X, d) be a metric space, $A \subset X$ and $\epsilon > 0$. We define the **ϵ-enlargement of** A, denoted A_ϵ, by

$$A_\epsilon = \{x \in X : d(x, A) < \epsilon\}$$

Exercise 4.1.0.24. Let (X, d) be a metric space, $A \subset X$ and $\epsilon > 0$. Then A_ϵ is open.

Proof. Let $x \in A_\epsilon$. By definition, $d(x, A) < \epsilon$. Set $\delta = (\epsilon - d(x, A))/2$. Then $\delta > 0$ and thus there exists $a \in A$ such that $d(x, a) < d(x, A) + \delta$. Let $y \in B(x, \delta)$. Therefore

$$\begin{aligned}
d(y, A) &= \inf\{d(y, b) : b \in A\} \\
&\leq d(y, a) \\
&\leq d(y, x) + d(x, a) \\
&< \delta + d(x, A) + \delta \\
&= d(x, A) + 2\delta \\
&= d(x, A) + \epsilon - d(x, A) \\
&= \epsilon
\end{aligned}$$

Hence $y \in A_\epsilon$. Since $y \in B(x, \delta)$ is arbitrary, $B(x, \delta) \subset A_\epsilon$. Since $x \in A_\epsilon$ is arbitrary, we have that for each $x \in A_\epsilon$, there exists $\delta > 0$ such that $B(x, \delta) \subset A_\epsilon$. Hence A_ϵ is open. □

Exercise 4.1.0.25. Let (X, d) be a metric space and $C, U \subset X$.

1. If C is closed, then $C = \bigcap_{n \in \mathbb{N}} C_{1/n}$.
2. If U is open, then $U = \bigcup_{n \in \mathbb{N}} [(U^c)_{1/n}]^c$.

Proof.

1. Suppose that C is closed. Since for each $n \in \mathbb{N}$, $C \subset C_{1/n}$, we have that $C \subset \bigcap_{n \in \mathbb{N}} C_{1/n}$.

   For the sake of contradiction, suppose that $\bigcap_{n \in \mathbb{N}} C_{1/n} \not\subset C$. Then there exists $x \in \bigcap_{n \in \mathbb{N}} C_{1/n}$ such that $x \notin C$. Since C is closed, a previous exercise implies that $d(x, C) > 0$. Set $\epsilon = d(x, C)$. Since $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Since $x \in \bigcap_{n \in \mathbb{N}}$, $x \in C_{1/N}$. Thus

   $$\begin{aligned}
   d(x, C) &< 1/N \\
   &< \epsilon \\
   &= d(x, C)
   \end{aligned}$$

   which is a contradiction. Hence $\bigcap_{n \in \mathbb{N}} C_{1/n} \subset C$. Thus $C = \bigcap_{n \in \mathbb{N}} C_{1/n}$.

2. Suppose that U is open. Then U^c is closed. The previous part implies that

   $$\begin{aligned}
   U &= (U^c)^c \\
   &= (\bigcap_{n \in \mathbb{N}} (U^c)_{1/n})^c \\
   &= \bigcup_{n \in \mathbb{N}} [(U^c)_{1/n}]^c
   \end{aligned}$$

□

Definition 4.1.0.26. Let X be a topological space and $A \subset X$. Then

- A is said to be an F_σ**-set** if there exist $(C_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ such that for each $n \in \mathbb{N}$, C_n is closed and $A = \bigcup_{n \in \mathbb{N}} C_n$
- A is said to be an G_δ**-set** if there exist $(U_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ such that for each $n \in \mathbb{N}$, U_n is open and $A = \bigcap_{n \in \mathbb{N}} U_n$

Exercise 4.1.0.27. Let (X, d) be a metric space. Then

1. for each $U \subset X$, if U is open, then U is an F_σ set,

2. for each $C \subset X$, if C is closed, then C is a G_δ set.

Proof. Clear by the previous exercise. $\square$

Exercise 4.1.0.28. Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}} \subset X$, $x \in X$ and $r > 0$. Suppose that $x_n \to x$. Then

1. for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $B(x_n, r) \subset B(x, r+\epsilon)$.

2. for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $\bar{B}(x_n, r) \subset \bar{B}(x, r+\epsilon)$.

Proof.

1. Let $\epsilon > 0$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $d(x_n, x) < \epsilon$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Let $y \in B(x_n, r)$. Then

$$\begin{aligned} d(y, x) &\leq d(y, x_n) + d(x_n, x) \\ &< r + \epsilon \end{aligned}$$

   Thus $y \in B(x, r+\epsilon)$. Since $y \in B(x_n, r)$ is arbitrary, $B(x_n, r) \subset B(x, r+\epsilon)$.

2. Similar to (1).

$\square$

Exercise 4.1.0.29. Let (X, d) be a metric space and $E \subset X$. Then E is dense in $(X, \mathcal{T})$ iff for each $x \in X$ and $\epsilon > 0$, there exists $x' \in E$ such that $d(x, x') < \epsilon$.

Proof.

- $(\implies)$:
  Suppose that E is dense in $(X, \mathcal{T})$. Let $x \in X$ and $\epsilon > 0$. Since $B(x, \epsilon) \in \mathcal{T}_d$ and $B(x, \epsilon) \neq \varnothing$, Exercise 3.1.0.27 implies that there exists $x' \in E$ such that $x' \in B(x, \epsilon)$. Then $d(x, x') < \epsilon$. Since $x \in X$ and $\epsilon > 0$ are arbitrary, we have that for each $x \in X$ and $\epsilon > 0$, there exists $x' \in E$ such that $d(x, x') < \epsilon$.

- $(\impliedby)$:
  Conversely, suppose that for each $x \in X$ and $\epsilon > 0$, there exists $x' \in E$ such that $d(x, x') < \epsilon$. Let $U \in \mathcal{T}_d$. Suppose that $U \neq \varnothing$. Then there exists $x \in X$ such that $x \in U$. Exercise 4.1.0.13 implies that there exists $\delta > 0$ such that

$\square$

Exercise 4.1.0.30. Let (X, d) be a complete separable metric space and $U \subset X$. Suppose that U is open. Set $C := \{(x, d(x, U^c)^{-1}) : x \in U\}$ and define $f : U \to C$ by $f(x) := (x, d(\cdot, U^c)^{-1})$. Then

1. f is a homeomorphism,

2. C is closed in $X \times \mathbb{R}$.

Proof.

1. - We note that $\mathrm{proj}_1|_C \circ f = \mathrm{id}_X$ and $f \circ \mathrm{proj}_1|_C = \mathrm{id}_C$. Thus f is a bijection with $f^{-1} = \mathrm{proj}_1|_C$.
   - Exercise 4.1.0.18 and Exercise 4.1.0.22 imply that f is continous. Since $\mathrm{proj}_1 : X \times \mathbb{R} \to X$ is continuous, $f^{-1} = \mathrm{proj}_1|_C$ is continuous. Hence f is a homeomorphism.

2. Let $(y_n)_{n\in\mathbb{N}} \subset C$ and $y \in X\times\mathbb{R}$. Suppose that $y_n \to y$. Since $C = f(U)$, there exist $(x_n)_{n\in\mathbb{N}} \subset U$ such that for each $n \in \mathbb{N}$, $y_n = f(x_n)$. Since $y \in X \times \mathbb{R}$, there exists $x \in X$ and $a \in \mathbb{R}$ such that $y = (x, a)$. Since $y_n \to y$, we have that

$$\begin{aligned} (x_n, d(x_n, U^c)^{-1}) &= y_n \\ &\to y \\ &= (x, a). \end{aligned}$$

Hence $x_n \to x$ and $d(x_n, U^c)^{-1} \to a$. For the sake of contradiction, suppose that $x \in U^c$. Since $x_n \to x$,

$$\begin{aligned} d(x_n, U^c) &\leq d(x_n, x) \\ &\to 0. \end{aligned}$$

Therefore

$$\begin{aligned} d(x_n, U^c)^{-1} &\to \infty \\ &\neq a. \end{aligned}$$

This is a contradiction. Hence $x \in U$. By continuity,

$$\begin{aligned} (x_n, d(x_n, U^c)^{-1}) &= f(x_n) \\ &\to f(x) \\ &= (x, d(x, U^c)^{-1}). \end{aligned}$$

In particular, $d(x_n, U^c)^{-1} \to d(x, U^c)^{-1}$. Since $d(x_n, U^c)^{-1} \to a$, we have that $a = d(x, U^c)^{-1}$. Therefore

$$\begin{aligned} y &= (x, a) \\ &= (x, d(x, U^c)^{-1}) \\ &= f(x) \\ &\in f(U) \\ &= C. \end{aligned}$$

Since $(y_n)_{n\in\mathbb{N}} \subset C$ and $y \in X \times \mathbb{R}$ with $y_n \to y$ are arbitrary, we have that for each $(y_n)_{n\in\mathbb{N}} \subset C$ and $y \in X \times \mathbb{R}$, $y_n \to y$ implies that $y \in C$. Thus C is closed in $X \times \mathbb{R}$.

□

Exercise 4.1.0.31. Let (X, d) be a complete separable metric space and $(U_n)_{n\in\mathbb{N}} \subset \mathcal{T}_d$. Set $E := \bigcap\limits_{n\in\mathbb{N}} U_n$ and define $C \subset X \times \mathbb{R}^{\mathbb{N}}$ and $f : E \to C$ by

$$C := \{(x, d(x, U_1^c)^{-1}, d(x, U_2^c)^{-1}, \dots) : x \in X\} \quad \text{and} \quad f(x) := (x, d(x, U_1^c)^{-1}, d(x, U_2^c)^{-1}, \dots).$$

Then

1. f is a homeomorphism,
2. C is closed in $X \times \mathbb{R}^{\mathbb{N}}$.

Proof.

1.
   - We note that $\mathrm{proj}_1|_C \circ f = \mathrm{id}_X$ and $f \circ \mathrm{proj}_1|_C = \mathrm{id}_C$. Thus f is a bijection with $f^{-1} = \mathrm{proj}_1|_C$.
   - Exercise 4.1.0.18 and Exercise 4.1.0.22 imply that f is continous. Since $\mathrm{proj}_1 : X \times \mathbb{R} \to X$ is continuous, $f^{-1} = \mathrm{proj}_1|_C$ is continuous. Hence f is a homeomorphism.

2. Let $(y_n)_{n\in\mathbb{N}} \subset C$ and $y \in X \times \mathbb{R}^{\mathbb{N}}$. Suppose that $y_n \to y$. Since $C = f(U)$, there exist $(x_n)_{n\in\mathbb{N}} \subset U$ such that for each $n \in \mathbb{N}$, $y_n = f(x_n)$. Since $y \in X \times \mathbb{R}^{\mathbb{N}}$, there exists $x \in X$ and $a \in \mathbb{R}^{\mathbb{N}}$ such that $y = (x, a)$. Since $y_n \to y$, we have that

$$\begin{aligned}(x_n, d(x_n, U_1^c)^{-1}, d(x_n, U_2^c)^{-1}, \dots) &= y_n \\ &\to y \\ &= (x, a_1, a_2, \dots).\end{aligned}$$

Hence $x_n \to x$ and for each $k \in \mathbb{N}$ $d(x_n, U_k^c)^{-1} \xrightarrow{n} a_k$. For the sake of contradiction, suppose that $x \in E^c$. Since

$$\begin{aligned}E^c &= \left(\bigcap_{k\in\mathbb{N}} U_k\right)^c \\ &= \bigcup_{k\in\mathbb{N}} U_k^c\end{aligned}$$

there exists $k_0 \in \mathbb{N}$ such that $x \in U_{k_0}^c$. Since $x_n \to x$,

$$\begin{aligned}d(x_n, U_{k_0}^c) &\le d(x_n, x) \\ &\to 0.\end{aligned}$$

Therefore

$$\begin{aligned}d(x_n, U_{k_0}^c)^{-1} &\to \infty \\ &\neq a_{k_0}.\end{aligned}$$

This is a contradiction. Hence $x \in E$. By continuity,

$$\begin{aligned}(x_n, d(x_n, U_1^c)^{-1}, d(x_n, U_2^c)^{-1}, \dots) &= f(x_n) \\ &\to f(x) \\ &= (x, d(x, U_1^c)^{-1}, d(x, U_2^c)^{-1}, \dots).\end{aligned}$$

In particular, for each $k \in \mathbb{N}$, $d(x_n, U_k^c)^{-1} \xrightarrow{n} d(x, U_k^c)^{-1}$. Since $d(x_n, U_k^c)^{-1} \xrightarrow{n} a_k$, we have that $a = d(x, U^c)^{-1}$. Therefore

$$\begin{aligned}y &= (x, a_1, a_2, \dots) \\ &= (x, d(x, U_1^c)^{-1}, d(x, U_2^c)^{-1}, \dots) \\ &= f(x) \\ &\in f(U) \\ &= C.\end{aligned}$$

Since $(y_n)_{n\in\mathbb{N}} \subset C$ and $y \in X \times \mathbb{R}^{\mathbb{N}}$ with $y_n \to y$ are arbitrary, we have that for each $(y_n)_{n\in\mathbb{N}} \subset C$ and $y \in X \times \mathbb{R}$, $y_n \to y$ implies that $y \in C$. Thus C is closed in $X \times \mathbb{R}$.

□

Isometries

Definition 4.1.0.32. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Then f is said to be a (d_X, d_Y)**-isometry** if for each $x_1, x_2 \in X$, $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$.

Exercise 4.1.0.33. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. If f is a (d_X, d_Y)-isometry, then f is injective.

Proof. Suppose that f is an (d_X, d_Y)-isometry. Let $x_1.x_2 \in X$. Suppose that $f(x_1) = f(x_2)$. Then

$$\begin{aligned} d_X(x_1, x_2) &= d_Y(f(x_1), f(x_2)) \\ &= 0. \end{aligned}$$

Hence $x_1 = x_2$. Since $x_1.x_2 \in X$ are arbitrary, we have that for each $x_1.x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. Thus f is injective. □

Definition 4.1.0.34. Let X a set, (Y, d) be a metric space and $f : X \to Y$ an injection. We define the **pullback of d by f**, denoted $f^*d : X \times X \to [0, \infty)$, by

$$f^*d(x_1, x_2) := d(f(x_1), f(x_2))$$

Exercise 4.1.0.35. Let X a set, (Y, d) be a metric space and $f : X \to Y$ an injection. Then f^*d is a metric on X.

Proof. Let $x_1, x_2, x_3 \in X$. Then

1. $$\begin{aligned} f^*d(x_1, x_2) &= d(f(x_1), f(x_2)) \\ &= d(f(x_2), f(x_1)) \\ &= f^*d(x_2, x_1) \end{aligned}$$

2. $$\begin{aligned} f^*d(x_1, x_2) = 0 &\iff d(f(x_1), f(x_2)) = 0 \\ &\iff f(x_1) = f(x_2) \\ &\iff x_1 = x_2 \end{aligned}$$

3. $$\begin{aligned} &f^*d(x_1, x_3) \\ &\quad = d(f(x_1), f(x_3)) \\ &\quad \leq d(f(x_1), f(x_2)) + d(f(x_2), f(x_3)) \\ &\quad = f^*d(x_1, x_2) + f^*d(x_2, x_3) \end{aligned}$$

   So f^*d is a metric on X.

□

Exercise 4.1.0.36. Let X a set, (Y, d) be a metric space and $f : X \to Y$ an injection. Then f is an (f^*d, d)-isometry.

Proof. Let $x_1, x_2 \in X$. Then

$$d(f(x_1), f(x_2)) = f^*d(x_1, x_2).$$

Since $x_1, x_2 \in X$ are arbitrary, we have that f is (f^*d, d)-isometry. □

Exercise 4.1.0.37. Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$. Then f is a (d_X, d_Y)-isometry iff f is injective and $d_X = f^*d_Y$.

Proof.
- $(\implies)$:
  Suppose that f is a (d_X, d_Y)-isometry. Exercise 4.1.0.33 implies that f is injective. Then for each $x_1, x_2 \in X$, we have that

  $$\begin{aligned} d_X(x_1, x_2) &= d_Y(f(x_1), f(x_2)) \\ &= f^*d_Y(x_1, x_2) \end{aligned}$$

  Since $x_1, x_2 \in X$ are arbitrary, we have that $d_X = f^*d_Y$.

- ($\Longleftarrow$) :
  Conversely, suppose that f is injective and $d_X = f^*d_Y$. Exercise 4.1.0.37 implies that f is a (f^*d_Y, d_Y)-isometry. Since $d_X = f^*d_Y$, we have that f is a (d_X, d_Y)-isometry.

$\square$

Exercise 4.1.0.38. Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$. Suppose that f is a (d_X, d_Y)-isometry and f is a bijection. Then f^{-1} is a (d_Y, d_X)-isometry.

Proof. Since f^{-1} is a bijection, f^{-1} is injective. Let $y_1, y_2 \in Y$. Exercise 4.1.0.37 implies that $f^*d_Y = d_X$. Set $x_1 := f^{-1}(y_1)$ and $x_2 := f^{-1}(y_2)$. Then

$$\begin{aligned}(f^{-1})^*d_X(y_1, y_2) &= d_X(f^{-1}(y_1), f^{-1}(y_2))\\ &= d_X(x_1, x_2)\\ &= f^*d_Y(x_1, x_2)\\ &= d_Y(f(x_1), f(x_2))\\ &= d_Y(y_1, y_2).\end{aligned}$$

Since $y_1, y_2 \in Y$ are arbitrary, we have that for each $y_1, y_2 \in Y$, $(f^{-1})^*d_X(y_1, y_2) = d_Y(y_1, y_2)$. Hence $d_Y = (f^{-1})^*d_X$. Exercise 4.1.0.37 then implies that f^{-1} is a (d_Y, d_X)-isometry. $\square$

Exercise 4.1.0.39. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, $d_X : X \times X \to [0, \infty)$ a metric on X and $f : X \to Y$ a $(\mathcal{T}_X, \mathcal{T}_Y)$-homeomorphism. Then $\mathcal{T}_Y = \mathcal{T}_{(f^{-1})^*d_X}$.

Proof. Set $d_Y := (f^{-1})^*d_X$. Let $(y_n)_{n\in\mathbb{N}} \subset Y$ and $y \in Y$. Define $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$ by $x_n := f^{-1}(y_n)$ and $x := f^{-1}(y)$.

- Suppose that $y_n \to y$ in $(Y, \mathcal{T}_Y)$. Since f^{-1} is a $(\mathcal{T}_Y, \mathcal{T}_X)$-homeomorphism,

$$\begin{aligned}x_n &= f^{-1}(y_n)\\ &\to f^{-1}(y)\\ &= x\end{aligned}$$

  in $(X, \mathcal{T}_X)$. Since $\mathcal{T}_X = \mathcal{T}_{d_X}$, we have that

$$\begin{aligned}d_Y(y_n, y) &= (f^{-1})^*d_X(y_n, y)\\ &= d_X(f^{-1}(y_n), f^{-1}(y))\\ &= d_X(x_n, x)\\ &\to 0.\end{aligned}$$

  Thus $y_n \to y$ in $(X, \mathcal{T}_{d_Y})$.

- Conversely, suppose that $y_n \to y$ in $(Y, \mathcal{T}_{d_Y})$. Then

$$\begin{aligned}d_X(x_n, x) &= d_X(f^{-1}(y_n), f^{-1}(y))\\ &= (f^{-1})^*d_X(y_n, y)\\ &= d_Y(y_n, y)\\ &\to 0.\end{aligned}$$

  Hence $x_n \to x$ in $(X, \mathcal{T}_{d_X})$. Since $\mathcal{T}_{d_X} = \mathcal{T}_X$ and f is a $(\mathcal{T}_X, \mathcal{T}_Y)$-homeomorphism, we have that

$$\begin{aligned}y_n &= f(x_n)\\ &\to f(x)\\ &= y\end{aligned}$$

  in $(Y, \mathcal{T}_Y)$.

Since $(y_n)_{n\in\mathbb{N}}$ and $y \in Y$ are arbitrary, we have that for each $(y_n)_{n\in\mathbb{N}}$ and $y \in Y$, $y_n \to y$ in $(Y, \mathcal{T}_Y)$ iff $y_n \to y$ in $(Y, \mathcal{T}_{d_Y})$. Exercise 3.3.2.15 implies that $\mathcal{T}_Y = \mathcal{T}_{d_Y}$. □

Countability

Exercise 4.1.0.40. Let (X, d) be a metric space. Then $(X, \mathcal{T}_d)$ is second countable iff $(X, \mathcal{T}_d)$ is separable.

Proof.

- $(\implies)$:
  Suppose that $(X, \mathcal{T}_d)$ is second countable. Exercise 3.9.2.6 implies that $(X, \mathcal{T}_d)$ is separable.
- $(\impliedby)$:
  Conversely, suppose that $(X, \mathcal{T}_d)$ is separable. Then there exists $(x_n)_{n\in\mathbb{N}} \subset X$ such that $(x_n)_{n\in\mathbb{N}}$ is dense in X. Define $\mathcal{B} := \{B(x_n, q) : (n, q) \in \mathbb{N} \times \mathbb{Q} \cap (0, 1)\}$. Then $\mathcal{B}$ is countable.

  Let $U \in \mathcal{T}_d$ and $x \in U$. Since U is open, Exercise 4.1.0.13 implies that there exists $\delta_0 > 0$ such that $B(x, \delta_0) \subset U$. Set $\delta := \delta_0/2$. Since $B(x, \delta)$ is open, $B(x, \delta) \neq \varnothing$ and $(x_n)_{n\in\mathbb{N}}$ is dense in X, Exercise 3.1.0.27 implies that there exists $n \in \mathbb{N}$ such that $x_n \in B(x, \delta)$. Set $B := B(x_n, \delta)$. By construction $B \in \mathcal{B}$ and $x \in B$. Let $y \in B$. Then

  $$\begin{aligned} d(x, y) &\leq d(x, x_n) + d(x_n, y) \\ &< \delta + \delta \\ &= 2\delta \\ &= \delta_0 \end{aligned}$$

  Thus

  $$\begin{aligned} y &\in B(x, \delta_0) \\ &\subset U \end{aligned}$$

  Since $y \in B$ is arbitrary, we have that $B \subset U$. Therefore, there exists $B \in \mathcal{B}$ such that

  $$\begin{aligned} x &\in B \\ &\subset U. \end{aligned}$$

  Hence $\mathcal{B}$ is a basis for $\mathcal{T}_d$. Since $\mathcal{B}$ is countable, $(X, \mathcal{T}_d)$ is second-countable.

□

Exercise 4.1.0.41. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a $(\mathcal{T}_{d_X}, \mathcal{T}_{d_Y})$-homeomorphism. Then $(X, \mathcal{T}_{d_X})$ is separable iff $(Y, \mathcal{T}_{d_Y})$ is separable.

Proof.

- $(\implies)$:
  Suppose that $(X, \mathcal{T}_{d_X})$ is separable. Exercise 4.1.0.40 implies that $(X, \mathcal{T}_{d_X})$ is second-countable. Exercise 3.9.2.7 implies that $(Y, \mathcal{T}_{d_Y})$ is second-countable. Another application of Exercise 4.1.0.40 implies that $(Y, \mathcal{T}_{d_Y})$ is separable.
- $(\impliedby)$:
  Similar to $(\implies)$.

□

4.2 Top-Equivalent Metrics

Definition 4.2.0.1. Let X be a set, $d_1, d_2 : X \times X \to [0,\infty)$ metrics on X. Then d_1 and d_2 are said to be **Top-equivalent**, denoted $d_1 \sim_{\mathbf{Top}} d_2$, if $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.

Exercise 4.2.0.2. Let X be a set, $d_1, d_2 : X \times X \to [0,\infty)$ metrics on X. Then d_1 and d_2 are **Top**-equivalent iff for each $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$, $x_n \xrightarrow{d_1} x$ iff $x_n \xrightarrow{d_2} x$.

Proof. FINISH!!! $\square$

Definition 4.2.0.3. Let $\phi : [0,\infty) \to [0,\infty)$. Then ϕ is said to be **Top metric-preserving** if for each set X and metric d on X,

1. $\phi \circ d$ is a metric on X
2. $\phi \circ d \sim_{\mathbf{Top}} d$

Definition 4.2.0.4. Let (X, d) be a metric space and $\phi : [0,\infty) \to [0,\infty)$. Suppose that ϕ is said to be **Top** metric-preserving. We define the **ϕ-iterate of** d, denoted d^ϕ, by $d^\phi = \phi \circ d$.

Exercise 4.2.0.5. Let $\phi : [0,\infty) \to [0,\infty)$. Suppose that

1. ϕ is continuous
2. ϕ is increasing
3. $\phi^{-1}(\{0\}) = \{0\}$

Then for each $(s_n)_{n\in\mathbb{N}} \subset [0,\infty)$, $s_n \to 0$ iff $\phi(s_n) \to 0$.

Proof. Let $(s_n)_{n\in\mathbb{N}} \subset [0,\infty)$. Suppose that $s_n \to 0$. Since ϕ is continuous,

$$\begin{aligned} \phi(s_n) &\to \phi(0) \\ &= 0 \end{aligned}$$

Conversely, suppose that $\phi(s_n) \to 0$. For the sake of contradiction, suppose that $s_n \not\to 0$. Then there exists $\epsilon > 0$ and a subsequence $(s_{n_k})_{k\in\mathbb{N}} \subset (s_n)_{n\in\mathbb{N}}$ such that $(s_{n_k})_{k\in\mathbb{N}} \subset B(0,\epsilon)^c$. Since $\phi^{-1}(\{0\}) = \{0\}$, for each $k \in \mathbb{N}$, $\phi(s_{n_k}) > 0$. Since $\phi(s_{n_k}) \to 0$, there exists a subsequence $(s_{n_{k_j}})_{j\in\mathbb{N}} \subset (s_{n_k})_{k\in\mathbb{N}}$ such that for each $j \in \mathbb{N}$, $\phi(s_{n_{k_{j+1}}}) < \phi(s_{n_{k_j}})$. Define $(t_j)_{j\in\mathbb{N}} \subset B(0,\epsilon)^c$ by $t_j = s_{n_{k_j}}$. For the sake of contradiction, suppose that there exists $j \in \mathbb{N}$ such that $t_j \leq t_{j+1}$. Since ϕ is increasing, $\phi(t_j) \leq \phi(t_{j+1})$. This is a contradiction since by construction, $\phi(t_{j+1}) < \phi(t_j)$. Therefore for each $j \in \mathbb{N}$, $t_{j+1} < t_j$. Hence $(t_j)_{j\in\mathbb{N}}$ is decreasing and $t_j \to \inf_{j\in\mathbb{N}} t_j$. Set $t = \inf_{j\in\mathbb{N}} t_j$. Since $(t_j)_{j\in\mathbb{N}} \subset B(0,\epsilon)^c$, $t \in B(0,\epsilon)^c$. Since $t \neq 0$ and $\phi^{-1}(\{0\}) = \{0\}$, we have that $\phi(t) \neq 0$. Since ϕ is continuous, $\phi(t_j) \to \phi(t)$. By construction $\phi(t_j) \to 0$. Hence $\phi(t) = 0$. This is a contradiction. Hence $s_n \to 0$. $\square$

Exercise 4.2.0.6. Let $\phi : [0,\infty) \to [0,\infty)$. Suppose that

1. ϕ is continuous
2. ϕ is increasing
3. ϕ is subadditive
4. $\phi^{-1}(\{0\}) = \{0\}$

Then ϕ is **Top** metric-preserving.

Proof. Let (X, d) be a metric space.

1. (a) Let $x, y \in X$. Suppose that $x = y$. Then $d(x,y) = 0$. Since $0 \in \phi^{-1}(\{0\})$, we have that

$$\begin{aligned} d^{\phi}(x,y) &= \phi(d(x,y)) \\ &= \phi(0) \\ &= 0 \end{aligned}$$

Conversely, suppose that $d^{\phi}(x,y) = 0$. Then $\phi(d(x,y)) = 0$ and therefore

$$\begin{aligned} d(x,y) &\in \phi^{-1}(\{0\}) \\ &= \{0\} \end{aligned}$$

Thus $d(x,y) = 0$. Since d is a metric on X, $x = y$. Hence $d^{\phi}(x,y) = 0$ iff $x = y$.

(b) Let $x, y, z \in X$. Since ϕ is increasing and subadditive, we have that

$$\begin{aligned} d_{\phi(x,z)} &= \phi(d(x,z)) \\ &\leq \phi(d(x,y) + d(y,z)) \\ &\leq \phi(d(x,y)) + \phi(d(y,z)) \\ &= d^{\phi}(x,y) + d^{\phi}(y,z) \end{aligned}$$

Therefore d^{ϕ} is a metric on X.

2. Let $(x_n)_{n \in \mathbb{N}} \subset X$ and $x \in X$. Suppose that $x_n \xrightarrow{d} x$. Then $d(x_n, x) \to 0$. Since ϕ is continuous,

$$\begin{aligned} d^{\phi}(x_n,x) &= \phi(d(x_n,x)) \\ &\to 0 \end{aligned}$$

So $x_n \xrightarrow{d^{\phi}} x$.
Conversely, suppose that $x_n \xrightarrow{d^{\phi}} x$. Then

$$\begin{aligned} \phi(d(x_n,x)) &= d^{\phi}(x_n,x) \\ &\to 0 \end{aligned}$$

The previous exercise implies that $d(x_n, x) \to 0$. Hence $x_n \xrightarrow{d} x$. Since $(x_n)_{n \in \mathbb{N}} \subset X$ and $x \in X$ are arbitrary, we have that $(x_n)_{n \in \mathbb{N}} \subset X$ and $x \in X$, $x_n \xrightarrow{d} x$ iff $x_n \xrightarrow{d^{\phi}} x$. Therefore $d^{\phi} \sim_{\mathbf{Top}} d$.

Since (X, d) is arbitrary, ϕ is **Top** metric-preserving. □

Exercise 4.2.0.7. Define $\phi : [0, \infty) \to [0, 1)$ by

$$\phi(t) = \frac{t}{1+t}$$

Then ϕ is **Top** metric-preserving.

Proof.

1. We note that $\phi \in C^{\infty}([0,\infty))$ and for each $t \in [0, \infty)$,

$$\phi'(t) = \frac{1}{(1+t)^2} \quad \text{and} \quad \phi''(t) = -\frac{2}{(1+t)^3}$$

In particular, ϕ is continuous.

2. Since $\phi' > 0$, ϕ is strictly increasing.

3. Since $\phi'' < 0$, ϕ is strictly concave. Since $\phi(0) = 0$, an exercise in the section on convex functions implies that ϕ is subadditive. reference section on convex functions

4. Clearly $\phi^{-1}(\{0\}) = \{0\}$.

So ϕ is **Top** metric-preserving. □

Exercise 4.2.0.8. Let $a \in (0, \infty)$. Define $\phi_a : [0, \infty) \to [0, \infty)$ by

$$\phi_a(t) = t \wedge a$$

Then ϕ_a is **Top** metric-preserving.

Proof.

1. Clearly ϕ is continuous.

2. Clearly, ϕ is increasing.

3. Since the minimum of two concave functions is concave, ϕ is concave. Since $\phi(0) = 0$, an exercise in the section on convex functions implies that ϕ is subadditive. reference section on convex functions

4. Clearly $\phi^{-1}(\{0\}) = \{0\}$.

So ϕ_a is **Top** metric-preserving. □

4.3 Subspaces

4.3.1 Introduction

Exercise 4.3.1.1. Let (X, d) be a metric space. Then $\mathcal{T}_{d|_{E\times E}} = \mathcal{T}_d \cap E$.

Proof. Set $d_E := d|_{E\times E}$. Let $(x_n)_{n\in\mathbb{N}} \subset E$ and $x \in E$.

- Suppose that $x_n \to x$ in $(E, \mathcal{T}_{d_E})$. Then $d_E(x_n, x) \to 0$. Hence

$$\begin{aligned} d(x_n, x) &= d_E(x_n, x) \\ &\to 0. \end{aligned}$$

  Therefore $x_n \to x$ in $(X, \mathcal{T}_d)$. Exercise 3.4.1.6 implies that $x_n \to x$ in $(E, \mathcal{T}_d \cap E)$.

- Conversely, suppose that $x_n \to x$ in $(E, \mathcal{T}_d \cap E)$. Exercise 3.4.1.6 implies that $x_n \to x$ in $(X, \mathcal{T}_d)$. Then $d(x_n, x) \to 0$. Hence

$$\begin{aligned} d_E(x_n, x) &= d(x_n, x) \\ &\to 0. \end{aligned}$$

  Therefore $x_n \to x$ in $(E, \mathcal{T}_{d_E})$.

Hence $x_n \to x$ in $(E, \mathcal{T}_{d_E})$ iff $x_n \to x$ in $(E, \mathcal{T}_d \cap E)$. Exercise 3.3.2.15 implies that $\mathcal{T}_d \cap E = \mathcal{T}_{d_E}$. $\square$

Exercise 4.3.1.2. Let (X, d) be a metric space and $E \subset X$. If $(X, \mathcal{T}_d)$ is separable, then $(E, \mathcal{T}_d \cap E)$ is separable.

Proof. Suppose that $(X, \mathcal{T}_d)$ is separable. Exercise 4.1.0.40 implies that $(X, \mathcal{T}_d)$ is second countable. Exercise 3.9.2.10 implies that $(E, \mathcal{T}_d \cap E)$ is second-countable. Another application of Exercise 4.1.0.40 implies that $(E, \mathcal{T}_d \cap E)$ is separable. $\square$

4.3.2 Discrete Subsets

Definition 4.3.2.1. Let (X, d) be a metric space and $S \subset X$. Then S is said to be **discrete** if for each $x \in X$, there exists $r > 0$ such that $B(0, r) \cap X = \{x\}$.

Exercise 4.3.2.2. Let (X, d) be a metric space and $S \subset X$. Then S is discrete iff $\mathcal{T}_{d|_S} = \mathcal{T}_{dscrt(X)}$.

Proof. FINISH!!! $\square$

Exercise 4.3.2.3. compare with discrete sets in topology section Let (X, d) be a metric space, $A \subset X$ and $x \in A$. Then x is an isolated point of A iff there exists $r > 0$ such that $B(x, r) \cap A = \{x\}$.

Proof. Suppose that x is an isolated point of A. Then there exists $U \subset X$ such that U is open in X and $U \cap A = \{x\}$. Since U is open, $x \in U$ and $\{B(x, r) : r > 0\}$ is a local basis for the topology on X at x, there exists $r > 0$ such that $B(x, r) \subset U$. Hence

$$\begin{aligned} B(x, r) \cap A &\subset U \cap A \\ &= \{x\} \end{aligned}$$

Since $x \in B(x,r) \cap A$, we have that $\{x\} \subset B(x,r) \cap A$. Hence $B(x,r) \cap A = \{x\}$.
Conversely, suppose that there exists $r > 0$ such that $B(x,r) \cap A = \{x\}$. Since $B(x,r)$ is open in X, x is an isolated point of A. $\square$

Exercise 4.3.2.4. Let (X,d) be a metric space and $A \subset X$. Suppose that A is discrete. If X is separable, then A is countable.
Hint: If $E \subset X$ is countable and dense in X, then for each $x \in A$, there exists $y \in E$ and $q \in \mathbb{Q} \cap (0,\infty)$ such that $x \in B(y,q)$.

Proof. Suppose that X is separable. Let $x \in A$. Since X is separable, there exists $(x_n)_{n\in\mathbb{N}} \subset X$ such that $(x_n)_{n\in\mathbb{N}}$ is dense in X. Since A is discrete, x is an isolated point of A and the previous exercise implies that there exists $r > 0$ such that $B(x,r) \cap A = \{x\}$. Choose $q \in \mathbb{Q} \cap (0,r)$. Set $\epsilon = \min(r-q, q)$. Then $\epsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is dense in X, there exists $N \in \mathbb{N}$ such that $d(x_N, x) < \epsilon$. Let $y \in B(x_N, q)$. Then

$$\begin{aligned} d(y,x) &\leq d(y,x_N) + d(x_N,x) \\ &< q + \epsilon \\ &\leq q + (r-q) \\ &= r \end{aligned}$$

Thus $y \in B(x,r)$. Since $y \in B(x_N,q)$ is arbitrary, we have that $B(x_N,q) \subset B(x,r)$. In addition,

$$\begin{aligned} d(x_N,x) &< \epsilon \\ &\leq q \end{aligned}$$

which implies that $x \in B(x_N,q)$ and therefore $\{x\} \subset B(x_N,q) \cap A$. Conversely,

$$\begin{aligned} B(x_N,q) \cap A &\subset B(x,r) \cap A \\ &= \{x\} \end{aligned}$$

Hence $B(x_N,q) \cap A = \{x\}$. Since $x \in A$ is arbitrary, we have that for each $x \in A$,

$$\{B(x_n,q) : (n,q) \in \mathbb{N} \times (\mathbb{Q} \cap (0,\infty)) \text{ and } B(x_n,q) \cap A = \{x\}\} \neq \varnothing$$

For each $x \in A$, define

$$V(x) := \{B(x_n,q) : (n,q) \in \mathbb{N} \times (\mathbb{Q} \cap (0,\infty)) \text{ and } B(x_n,q) \cap A = \{x\}\}$$

Define

$$V := \{B(x_n,q) : (n,q) \in \mathbb{N} \times (\mathbb{Q} \cap (0,\infty))\}$$

Since $\bigcup\limits_{x\in A} V(x) \subset V$ and V is countable, we have that $\bigcup\limits_{x\in A} V(x)$ is countable. The axiom of choice implies that there exists $\phi : A \to \bigcup\limits_{x\in A} V(x)$ such that for each $x \in A$, $\phi(x) \in V(x)$. Let $x, y \in A$. Suppose that $\phi(x) = \phi(y)$. By construction,

$$\begin{aligned} \{x\} &= \phi(x) \cap A \\ &= \phi(y) \cap A \\ &= \{y\} \end{aligned}$$

Hence $x = y$. Since $x, y \in A$ are arbitrary, ϕ is injective. Since $\phi : A \to \bigcup\limits_{x\in A} V(x)$ is injective, and $\bigcup\limits_{x\in A} V(x)$ is countable, we have that A is countable. $\square$

4.4 Product Spaces

Definition 4.4.0.1. Let $(X_n, d_n)_{n\in\mathbb{N}}$ be a collection of metric spaces. Set $X := \prod_{n\in\mathbb{N}} X_n$. Define $\phi : [0,\infty) \to [0,1)$ by $\phi(t) := t/(1+t)$. We define the **product metric on** X, denoted $d_X : X \times X \to [0,\infty)$ by

$$d_X((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} 2^{-n} d_n^{\phi}(x_n, y_n)$$

Exercise 4.4.0.2. Let $(X_n, d_n)_{n\in\mathbb{N}}$ be a collection of metric spaces. Set $X := \prod_{n\in\mathbb{N}} X_n$ and $\mathcal{T} := \bigotimes_{n\in\mathbb{N}} \mathcal{T}_{d_n}$. Then

1. d_X is a metric on X,
2. $\mathcal{T}_{d_X} = \mathcal{T}$.

Proof.

1. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in X$.

   (a)

$$\begin{aligned} d_X((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) &= \sum_{n\in\mathbb{N}} 2^{-n} d_n^{\phi}(x_n, y_n) \\ &= \sum_{n\in\mathbb{N}} 2^{-n} d_n^{\phi}(y_n, x_n) \\ &= d_X((y_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}) \end{aligned}$$

   (b) • Suppose that $(x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$. Then for each $n \in \mathbb{N}$, $x_n = y_n$. Thus for each $n \in \mathbb{N}$, $d_n^{\phi}(x_n, y_n) = 0$. Hence

$$\begin{aligned} d_X((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) &= \sum_{n\in\mathbb{N}} 2^{-n} d_n^{\phi}(x_n, y_n) \\ &= 0 \end{aligned}$$

   • Suppose that $d_X((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = 0$. Then for each $n \in \mathbb{N}$, $d_n^{\phi}(x_n, y_n) = 0$. Therefore for each $n \in \mathbb{N}$, $x_n = y_n$. Hence $(x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$.

   Therefore $d_X((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = 0$ iff $(x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$.

   (c)

$$\begin{aligned} d_X((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) &= \sum_{n\in\mathbb{N}} 2^{-n} d_n^{\phi}(x_n, y_n) \\ &\leq \sum_{n\in\mathbb{N}} 2^{-n} [d_n^{\phi}(x_n, z_n) + d_n^{\phi}(z_n, y_n)] \\ &= \sum_{n\in\mathbb{N}} 2^{-n} d_n^{\phi}(x_n, z_n) + \sum_{n\in\mathbb{N}} 2^{-n} d_n^{\phi}(z_n, y_n) \\ &= d_X((x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}) + d_X((z_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \end{aligned}$$

   So d_X is a metric.

2. Let $(a_m)_{m\in\mathbb{N}} \subset X$ and $a \in X$. Then for each $m \in \mathbb{N}$, there exist $(x_{m,n})_{n\in\mathbb{N}} \in X$ such that $a_m = (x_{m,n})$ and there exists $(x_n)_{n\in\mathbb{N}} \in X$ such that $a = (x_n)_{n\in\mathbb{N}}$.

- Suppose that $a_m \to a$ in $(X, \mathcal{T})$. Let $\epsilon > 0$. Choose $N_0 \in \mathbb{N}$ such that $\sum\limits_{n \geq N_0+1} 2^{-n} < \epsilon/2$. Since $a_m \to a$ in $(X, \mathcal{T})$, we have that for each $n \in \mathbb{N}$, $x_{m,n} \xrightarrow{m} x_n$ in $(X_n, \mathcal{T}_{d_n})$. Hence for each $n \in \mathbb{N}$, $d_n(x_{m,n}, x_n) \xrightarrow{m} 0$. Exercise 4.2.0.7 implies that for each $n \in \mathbb{N}$, $d_n^\phi(x_{m,n}, x_n) \xrightarrow{m} 0$. Let $n \in [N_0]$. Since $d_n^\phi(x_{m,n}, x_n) \xrightarrow{m} 0$, there exists $M_n \in \mathbb{N}$ such that for each $m \in \mathbb{N}$, $m \geq M_n$ implies that $d_n^\phi(x_{m,n}, x_n) < \epsilon/2$. Set $M := \max(M_1, \dots, M_{N_0})$. Let $m \in \mathbb{N}$. Suppose that $m \geq M$. Then

$$\begin{aligned}
d_X(a_m, a) &= \sum_{n \in \mathbb{N}} 2^{-n} d_n^\phi(x_{m,n}, x_n) \\
&= \sum_{n=1}^{N_0} 2^{-n} d_n^\phi(x_{m,n}, x_n) + \sum_{n \geq N_0+1} 2^{-n} d_n^\phi(x_{m,n}, x_n) \\
&< \sum_{n=1}^{N_0} 2^{-n} \left(\frac{\epsilon}{2}\right) + \sum_{n \geq N_0+1} 2^{-n} \\
&= \frac{\epsilon}{2} \sum_{n=1}^{N_0} 2^{-n} + \sum_{n \geq N_0+1} 2^{-n} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

  Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for each $m \in \mathbb{N}$, $m \geq M$ implies that $d_X(a_m, a) < \epsilon$. Hence $d_X(a_m, a) \to 0$ and therefore $a_m \to a$ in $(X, \mathcal{T}_{d_X})$.

- Conversely, suppose that $a_m \to a$ in $(X, \mathcal{T}_{d_X})$. Then $d_X(a_m, a) \to 0$. Let $n \in \mathbb{N}$. Then

$$\begin{aligned}
2^{-n} d_n^\phi(x_{m,n}, x_n) &\leq d_X(a_m, a) \\
&\to 0
\end{aligned}$$

  Hence $d_n^\phi(x_{m,n}, x_n) \to 0$. Thus $x_{m,n} \xrightarrow{m} x_n$ in $(X, \mathcal{T}_{d_n^\phi})$. Since $\mathcal{T}_{d_n^\phi} = \mathcal{T}_{d_n}$, we have that $x_{m,n} \xrightarrow{m} x_n$ in $(X, \mathcal{T}_{d_n})$. Since $n \in \mathbb{N}$ is arbitrary, we have that $a_m \to a$ in $(X, \mathcal{T})$.

Since $(a_m)_{m \in \mathbb{N}} \subset X$ and $a \in X$ are arbitrary, we have that for each $(a_m)_{m \in \mathbb{N}} \subset X$ and $a \in X$, $a_m \to a$ in $(X, \mathcal{T})$ iff $a_m \to a$ in $(X, \mathcal{T}_{d_X})$. Exercise 3.3.2.15 implies that $\mathcal{T}_{d_X} = \mathcal{T}$.

$\square$

Note 4.4.0.3. Thus the product topolgy is basically the topology of waning importance. More importance is given to earlier entries than later entries in a point of the product space. should have another example of a metric compatible with the product, maybe the supremum/n one

4.5 Coproduct Spaces

Definition 4.5.0.1. Let $(X_\alpha, d_\alpha)_{\alpha \in A}$ be a collection of metric space. Set $X := \coprod_{n \in \mathbb{N}} X_n$. Define $\phi : [0, \infty) \to [0, 1)$ by $\phi(t) := t/(1+t)$. We define the **coproduct metric on** X, denoted $d_X : X \times X \to [0, \infty)$, by

$$d_X((\alpha, x), (\beta, y)) := \begin{cases} d_\alpha^\phi(x, y), & \alpha = \beta \\ 1, & \alpha \neq \beta \end{cases}$$

Exercise 4.5.0.2. Let $(X_\alpha, d_\alpha)_{\alpha \in A}$ be a collection of metric space. Set $X := \coprod_{\alpha \in A} X_\alpha$ and $\mathcal{T} := \bigotimes_{\alpha \in A} \mathcal{T}_{d_\alpha}$. Then

1. d_X is a metric on X,
2. $\mathcal{T}_{d_X} = \mathcal{T}$.

Proof.

1. Let $(\alpha_1, x_1), (\alpha_2, x_2), (\alpha_3, x_3) \in X$.

   (a) • Suppose that $\alpha_1 = \alpha_2$. Then

   $$\begin{aligned} d_X((\alpha_1, x_1), (\alpha_2, x_2)) &= d_{\alpha_1}^\phi(x_1, x_2) \\ &= d_{\alpha_1}^\phi(x_2, x_1) \\ &= d_X((\alpha_2, x_2), (\alpha_1, x_1)). \end{aligned}$$

   • Suppose that $\alpha_1 \neq \alpha_2$. Then

   $$\begin{aligned} d_X((\alpha_1, x_1), (\alpha_2, x_2)) &= 1 \\ &= d_X((\alpha_2, x_2), (\alpha_1, x_1)). \end{aligned}$$

   (b) • If $(\alpha_1, x_1) = (\alpha_2, x_2)$, then $\alpha_1 = \alpha_2$ and $x_1 = x_2$. Thus

   $$\begin{aligned} d_X((\alpha_1, x_1), (\alpha_2, x_2)) &= d_{\alpha_1}^\phi(x_1, x_1) \\ &= 0. \end{aligned}$$

   • Suppose that $d_X((\alpha_1, x_1), (\alpha_2, x_2)) = 0$. For the sake of contradiction, suppose that $\alpha_1 \neq \alpha_2$. Then

   $$\begin{aligned} 0 &= d_X((\alpha_1, x_1), (\alpha_2, x_2)) \\ &= 1 \end{aligned}$$

   which is a contradiction. Thus $\alpha_1 = \alpha_2$. Then

   $$\begin{aligned} 0 &= d_X((\alpha_1, x_1), (\alpha_2, x_2)) \\ &= d_{\alpha_1}^\phi(x_1, x_2). \end{aligned}$$

   Hence $x_1 = x_2$ and $(\alpha_1, x_1) = (\alpha_2, x_2)$.

   (c) • Suppose that $\alpha_1 = \alpha_2$ and $\alpha_1 = \alpha_3$. Then $\alpha_2 = \alpha_3$ and

   $$\begin{aligned} d_X((\alpha_1, x_1), (\alpha_2, x_2)) &= d_{\alpha_1}^\phi(x_1, x_2) \\ &\leq d_{\alpha_1}^\phi(x_1, x_3) + d_{\alpha_1}^\phi(x_3, x_2) \\ &= d_X((\alpha_1, x_1), (\alpha_3, x_3)) + d_X((\alpha_3, x_3), (\alpha_2, x_2)). \end{aligned}$$

- Suppose that $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_3$. Then $\alpha_2 \neq \alpha_3$ and

$$\begin{aligned} d_X((\alpha_1, x_1), (\alpha_2, x_2)) &= d^\phi_{\alpha_1}(x_1, x_2) \\ &\leq 2 \\ &= d_X((\alpha_1, x_1), (\alpha_3, x_3)) + d_X((\alpha_3, x_3), (\alpha_2, x_2)). \end{aligned}$$

- Suppose that $\alpha_1 \neq \alpha_2$ and $\alpha_1 = \alpha_3$. Then $\alpha_2 \neq \alpha_3$ and

$$\begin{aligned} d_X((\alpha_1, x_1), (\alpha_2, x_2)) &= 1 \\ &\leq d_X((\alpha_1, x_1), (\alpha_3, x_3)) + 1 \\ &= d_X((\alpha_1, x_1), (\alpha_3, x_3)) + d_X((\alpha_3, x_3), (\alpha_2, x_2)). \end{aligned}$$

- Suppose that $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq \alpha_3$. Then

$$\begin{aligned} d_X((\alpha_1, x_1), (\alpha_2, x_2)) &= 1 \\ &\leq 1 + d_X((\alpha_3, x_3), (\alpha_2, x_2)) \\ &= d_X((\alpha_1, x_1), (\alpha_3, x_3)) + d_X((\alpha_3, x_3), (\alpha_2, x_2)). \end{aligned}$$

2. Let $(\alpha_j, x_j)_{j \in \mathbb{N}} \subset X$ and $(\alpha_0, x_0) \in X$.

   - Suppose that $(\alpha_j, x_j) \to (\alpha_0, x_0)$ in $(X, \mathcal{T})$. Exercise 3.6.0.7 implies that there exists $j_0 \in \mathbb{N}$ such that

     (a) for each $j \in \mathbb{N}$, $j \geq j_0$ implies that $\alpha_j = \alpha_0$,

     (b) $[L_{j_0}(x)]_j \to x_0 \in (X_{\alpha_0}, \mathcal{T}_{d_{\alpha_0}})$.

     Hence $d_{\alpha_0}(x_{j+j_0}, x_0) \to 0$. Therefore

$$\begin{aligned} d_X((\alpha_{j+j_0}, x_{j+j_0}), (\alpha_0, x_0)) &= d^\phi(x_j, x_0) \\ &\to 0. \end{aligned}$$

     Hence $(\alpha_j, x_j) \to (\alpha_0, x_0)$ in $(X, \mathcal{T}_{d_X})$.

   - Conversely, suppose that $(\alpha_j, x_j) \to (\alpha_0, x_0)$ in $(X, \mathcal{T}_{d_X})$. Then $d_X((\alpha_j, x_j), (\alpha_0, x_0)) \to 0$. Therefore there exists $j_0 \in \mathbb{N}$ such that for each $j \in \mathbb{N}$, $j \geq j_0$ implies that $\alpha_j = \alpha_0$ and $d^\phi_{\alpha_0}(x_{j+j_0}, x_0) \to 0$. Therefore $d_{\alpha_0}(L_{j_0}(x), x_0) \to 0$. Exercise 3.6.0.7 implies that $(\alpha_j, x_j) \to (\alpha_0, x_0)$ in $(X, \mathcal{T})$.

   Since $(\alpha_j, x_j) \to (\alpha_0, x_0)$ in $(X, \mathcal{T}_{d_X})$ iff $(\alpha_j, x_j) \to (\alpha_0, x_0)$ in $(X, \mathcal{T})$, Exercise 3.3.2.15 implies that $\mathcal{T} = \mathcal{T}_{d_X}$.

$\square$

Exercise 4.5.0.3. Let $(X_\alpha, d_\alpha)_{\alpha \in A}$ be a collection of metric space. Set $X := \coprod_{\alpha \in A} X_\alpha$ and $\mathcal{T} := \bigotimes_{\alpha \in A} \mathcal{T}_{d_\alpha}$. Then for each $\alpha \in A$, $\iota_\alpha : X_\alpha \to X$ is Lipschitz.

Proof. Define $\phi : [0, \infty) \to [0, 1)$ by $\phi(t) := t/(1 + t)$. Exercise 4.1.0.20 implies that ϕ is Lipschitz. Hence there exists $K > 0$ such that for each $a, b \in [0, \infty)$, $|\phi(a) - \phi(b)| \leq K|a - b|$. Let $\alpha \in A$ and $x_1, x_2 \in X_\alpha$. Then

$$\begin{aligned} d_X(\iota_\alpha(x_1)\iota_\alpha(x_2)) &= d_X((\alpha, x_1), (\alpha, x_2)) \\ &= d^\phi_\alpha(x_1, x_2) \\ &= \phi(d_\alpha(x_1, x_2)) \\ &= \phi(|d_\alpha(x_1, x_2) - 0|) \qquad \leq K|d_\alpha(x_1, x_2) - \phi(0)| \\ &= K d_\alpha(x_1, x_2) \end{aligned}$$

Since $x_1, x_2 \in X_\alpha$ are arbitrary, we have that ι_α is Lipschitz. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, ι_α is Lipschitz. $\square$

4.6 Completeness

Definition 4.6.0.1. Let (X, d) be a metric space and $(a_n)_{n\in\mathbb{N}} \subset X$. Then $(a_n)_{n\in\mathbb{N}}$ is said to be **Cauchy** in (X, d) if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d(a_m, a_n) < \epsilon$.

Exercise 4.6.0.2. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. If f is uniformly continuous, then for each $(a_n)_{n\in\mathbb{N}} \subset X$, $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X) implies that $f(a_n)_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y).

Proof. Suppose that f is uniformly continuous. Let $(a_n)_{n\in\mathbb{N}} \subset X$. Suppose that $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X). Let $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that for each $x_1, x_2 \in X$, $d_X(x_1, x_2) < \delta$ implies that $d_Y(f(x_1), f(x_2)) < \epsilon$. Since $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X), there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d_X(a_m, a_n) < del$. Let $m, n \in \mathbb{N}$. Suppose that $m, n \geq N$. Then $d_Y(f(a_m), f(a_n)) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d_Y(f(a_m), f(a_n)) < \epsilon$. Hence $(f(a_n))_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y). FINISH and then update proof of preserving cauchy sequences in completeness section □

Exercise 4.6.0.3. Let (X, d) be a metric space and $\phi : [0, \infty) \to [0, \infty)$. Suppose that ϕ is **Top** metric-preserving and

1. ϕ is homeomorphism, (relax this to homeo near 0)
2. $\phi(0) = 0$.

Then for each $(x_n)_{n\in\mathbb{N}} \subset X$, $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d) iff $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d^ϕ). (try showing that a homoemorphism with $\phi(0) = 0$ is strictly increasing)

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset X$. Since ϕ is a homoemorphism and $\phi(0) = 0$, we have that ϕ is injective and strictly increasing.

- $(\implies)$:
  Suppose that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d). Let $\epsilon > 0$. Since ϕ is injective and $\phi(0 = 0)$, $\phi^{-1}(\epsilon) > 0$ and there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d(x_m, x_n) < \phi^{-1}(\epsilon)$. Let $m, n \in \mathbb{N}$. Suppose that $m, n \geq N$. Since ϕ is increasing, we have that

$$\begin{aligned} d^\phi(x_m, x_n) &= \phi \circ d(x_m, x_n) \\ &\leq \phi(\phi^{-1}(\epsilon)) \\ &= \epsilon \end{aligned}$$

  Thus $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d^ϕ).

- $(\impliedby)$:
  Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d^ϕ). Let $\epsilon > 0$. Since ϕ is injective and $\phi(0 = 0)$, $\phi(\epsilon) > 0$ and there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d^\phi(x_m, x_n) < \phi(\epsilon)$. Let $m, n \in \mathbb{N}$. Suppose that $m, n \geq N$. Since ϕ^{-1} is increasing, we have that

$$\begin{aligned} d(x_m, x_n) &= \phi^{-1} \circ \phi \circ d(x_m, x_n) \\ &= \phi^{-1} \circ d^\phi(x_m, x_n) \\ &\leq \phi^{-1}(\phi(\epsilon)) \\ &= \epsilon \end{aligned}$$

  Thus $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d^ϕ).

□

Definition 4.6.0.4. Let (X, d) be a metric space. Then (X, d) is said to be **complete** if for each $(a_n)_{n\in\mathbb{N}} \subset X$, $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d) implies that there exists $a \in X$ such that $a_n \to a$ in $(X, \mathcal{T}_d)$.

Exercise 4.6.0.5. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : X \to Y$ a (d_X, d_Y)-isometry and $(x_n)_{n\in\mathbb{N}}$. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X) iff $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y).

Proof. Since f is a (d_X, d_Y)-isometry, Exercise 4.1.0.37 implies that f is injective and $d_X = f^* d_Y$.

- $(\implies)$:
  Suppose that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X). Let $\epsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X), there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d_X(x_m, x_n) < \epsilon$. Let $m, n \in \mathbb{N}$. Suppose that $m, n \geq N$. Then

  $$\begin{aligned} d_Y(f(x_m), f(x_n)) &= f^* d_Y(x_m, x_n) \\ &= d_X(x_m, x_n) \\ &< \epsilon. \end{aligned}$$

  Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d_Y(f(x_m), f(x_n)) < \epsilon$. Hence $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y).

- $(\impliedby)$:
  Conversely, suppose that $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y). Let $\epsilon > 0$. Since $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y), there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d_Y(f(x_m), f(x_n)) < \epsilon$. Let $m, n \in \mathbb{N}$. Suppose that $m, n \geq N$. Then

  $$\begin{aligned} d_X(x_m, x_n) &= f^* d_Y(x_m, x_n) \\ &= d_Y(f(x_m), f(x_n)) \\ &< \epsilon. \end{aligned}$$

  Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $d_X(x_m, x_n) < \epsilon$. Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X).

$\square$

Exercise 4.6.0.6. Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ a surjective (d_X, d_Y)-isometry. Then (X, d_X) is complete iff (Y, d) is complete.

Proof. Exercise 4.1.0.37 implies that f is injective and $d_X = f^* d_Y$. Since f is surjective, f is a bijection. Exercise 4.1.0.38 implies that f^{-1} is a (d_Y, d_X)-isometry and Exercise 4.1.0.37 implies that $d_Y = (f^{-1})^* d_X$.

- $(\implies)$:
  Suppose that (X, d_X) is complete. Let $(y_n)_{n\in\mathbb{N}} \subset Y$. Suppose that $(y_n)_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y). Define $(x_n)_{n\in\mathbb{N}} \subset X$ by $x_n := f^{-1}(y_n)$. Since f^{-1} is a (d_Y, d_X)-isometry, Exercise 4.6.0.5 then implies that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X). Since (X, d_X) is complete, there exists $x \in X$ such that $x_n \to x$ in (X, d_X). Set $y := f(x)$. Then

  $$\begin{aligned} d_Y(y_n, y) &= (f^{-1})^* d_X(y_n, y) \\ &= d_X(f^{-1}(y_n), f^{-1}(y)) \\ &= d_X(x_n, x) \\ &\to 0 \end{aligned}$$

  Exercise 4.1.0.16 implies that $y_n \to y$. Since $(y_n)_{n\in\mathbb{N}} \subset Y$ with $(y_n)_{n\in\mathbb{N}}$ Cauchy in (Y, d_Y) is arbitrary, we have that for each $(y_n)_{n\in\mathbb{N}} \subset Y$, $(y_n)_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y) implies that there exists $y \in Y$ such that $y_n \to y$. Hence (Y, d_Y) is complete.

- $(\impliedby)$:
  Similar to $(\implies)$.

$\square$

4.6.1 Completeness and Subspaces

Exercise 4.6.1.1. Let (X,d) be a metric space, $E \subset X$ and $(a_n)_{n\in\mathbb{N}} \subset E$. Then $(a_n)_{n\in\mathbb{N}}$ is Cauchy in $(E, d|_{E\times E})$ iff $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d).

Proof. Set $d_E := d|_{E\times E}$.

- $(\implies)$:
  Suppose that $(a_n)_{n\in\mathbb{N}}$ is Cauchy in $(E, d|_{E\times E})$. Let $\epsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (E, d_E), there exists $N \in \mathbb{N}$ such that for each $m,n \in \mathbb{N}$, $m,n \geq N$ implies that $d_E(a_m,a_n) < \epsilon$. Thus for each $m,n \in \mathbb{N}$, $m,n \geq N$ implies that
  $$\begin{aligned} d(a_m,a_n) &= d_E(a_m,a_n) \\ &< \epsilon \end{aligned}$$
  Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m,n \in \mathbb{N}$, $m,n \geq N$ implies that $d(a_m,a_n) < \epsilon$. Hence $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d).

- $(\impliedby)$:
  Suppose that $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d). Let $\epsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d), there exists $N \in \mathbb{N}$ such that for each $m,n \in \mathbb{N}$, $m,n \geq N$ implies that $d(a_m,a_n) < \epsilon$. Thus for each $m,n \in \mathbb{N}$, $m,n \geq N$ implies that
  $$\begin{aligned} d_E(a_m,a_n) &= d(a_m,a_n) \\ &< \epsilon \end{aligned}$$
  Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m,n \in \mathbb{N}$, $m,n \geq N$ implies that $d_E(a_m,a_n) < \epsilon$. Hence $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (E,d_E).

$\square$

Exercise 4.6.1.2. Let (X,d) be a metric space and $C \subset X$. Suppose that (X,d) is complete. If C is closed, then $(C, d|_{C\times C})$ is complete.

Proof. Suppose that C is closed. Set $d_C := d|_{C\times C}$. Let $(a_n)_{n\in\mathbb{N}} \subset C$. Suppose that $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (C,d_C). Exercise 4.6.1.1 implies that $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d). Since (X,d) is complete, there exists $a \in X$ such that $a_n \to a$ in $(X,\mathcal{T}_X)$. Since C is closed, $a \in C$. Exercise 3.4.1.6 implies that $a_n \to a$ in $(C, \mathcal{T}_X \cap C)$. Since $(a_n)_{n\in\mathbb{N}} \subset C$ is arbitrary, we have that for each $(a_n)_{n\in\mathbb{N}} \subset C$, $(a_n)_{n\in\mathbb{N}}$ is Cauchy in (C,d_C) implies that there exists $a \in C$ such that $a_n \to a$ in $(C, \mathcal{T}_X \cap C)$. Hence (C,d_C) is complete. $\square$

4.6.2 Completeness and Product Spaces

Exercise 4.6.2.1. Let $(X_n,d_n)_{n\in\mathbb{N}}$ be a collection of metric spaces. Set $X := \prod_{n\in\mathbb{N}} X_n$ and define $d_X : X \times X \to [0,\infty)$ as in Definition 4.4.0.1. Then for each $(a_m)_{m\in\mathbb{N}} \subset X$, $(a_m)_{m\in\mathbb{N}}$ is Cauchy in (X,d_X) iff for each $n \in \mathbb{N}$, $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n,d_n).

Proof. Define $\phi : [0,\infty) \to [0,\infty)$ as in Definition 4.4.0.1. Let $(a_m)_{m\in\mathbb{N}} \subset X$.

- Suppose that $(a_m)_{m\in\mathbb{N}}$ is Cauchy in (X,d_X). Let $n \in \mathbb{N}$ and $\epsilon > 0$. Since $(a_m)_{m\in\mathbb{N}}$ is Cauchy in (X,d_X), there exists $M \in \mathbb{N}$ such that for each $m_1,m_2 \in \mathbb{N}$, $m_1,m_2 \geq M$ implies that $d_X(a_{m_1},a_{m_2}) < 2^{-n}\epsilon$. Let $m_1,m_2 \in \mathbb{N}$. Suppose that $m_1,m_2 \geq M$. Then
  $$\begin{aligned} d_n^\phi(\pi_n(a_{m_1}),\pi_n(a_{m_2})) &\leq 2^n d_X(a_{m_1},a_{m_2}) \\ &< \frac{2^n\epsilon}{2^n} \\ &= \epsilon. \end{aligned}$$
  Thus $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n,d_n^ϕ). Exercise 4.6.0.3 and Exercise 4.2.0.7 imply that $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n,d_n).

- Conversely, suppose that for each $n \in \mathbb{N}$, $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n). Exercise 4.6.0.3 and Exercise 4.2.0.7 imply that for each $n \in \mathbb{N}$, $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n^{ϕ}). Let $\epsilon > 0$. Choose $N_0 \in \mathbb{N}$ such that $\sum\limits_{n\geq N_0+1} 2^{-n} < \epsilon/2$. Since for each $n \in \mathbb{N}$, $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n^{ϕ}), we have that for each $n \in [N_0]$, there exists $M_n \in \mathbb{N}$ such that for each $m_1, m_2 \in \mathbb{N}$, $m_1, m_2 \geq M_n$ implies that $d_n^{\phi}(\pi_n(a_{m_1}), \pi_n(a_{m_2})) < \epsilon/2$. Set $M := \max(M_1, \dots, M_{N_0})$. Let $m_1, m_2 \in \mathbb{N}$. Suppose that $m_1, m_2 \geq M$. Then

$$\begin{aligned}
d_X(a_{m_1}, a_{m_1}) &= \sum_{n\in\mathbb{N}} 2^{-n} d_n^{\phi}(\pi_n(a_{m_1}), \pi_n(a_{m_2})) \\
&= \sum_{n=1}^{N_0} 2^{-n} d_n^{\phi}(\pi_n(a_{m_1}), \pi_n(a_{m_2})) + \sum_{n\geq N_0+1} 2^{-n} d_n^{\phi}(\pi_n(a_{m_1}), \pi_n(a_{m_2})) \\
&< \sum_{n=1}^{N_0} 2^{-n} \left(\frac{\epsilon}{2}\right) + \sum_{n\geq N_0+1} 2^{-n} \\
&= \frac{\epsilon}{2} \sum_{n=1}^{N_0} 2^{-n} + \sum_{n\geq N_0+1} 2^{-n} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

  Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for each $m_1, m_2 \in \mathbb{N}$, $m_1, m_2 \geq M$ implies that $d_X(a_{m_1}, a_{m_1}) < \epsilon$. Hence $(a_m)_{m\in\mathbb{N}}$ is Cauchy in (X, d_X).

Thus (a_m) is Cauchy in $(X < d_X)$ iff for each $n \in \mathbb{N}$, $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n). Since $(a_m)_{m\in\mathbb{N}} \subset X$ is arbitrary, we have that for each $(a_m)_{m\in\mathbb{N}} \subset X$, $(a_m)_{m\in\mathbb{N}}$ is Cauchy in (X, d_X) iff for each $n \in \mathbb{N}$, $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n). □

Exercise 4.6.2.2. Let $(X_n, d_n)_{n\in\mathbb{N}}$ be a collection of metric spaces. Set $X := \prod\limits_{n\in\mathbb{N}} X_n$ and define $d_X : X \times X \to [0, \infty)$ as in Definition 4.4.0.1. Then (X, d_X) is complete iff for each $n \in \mathbb{N}$, (X_n, d_n) is complete.

Proof. Define $\phi : [0, \infty) \to [0, 1)$ and $\mathcal{T}$ as in Definition 4.4.0.1.

- Suppose that (X, d_X) is complete. Let $n \in \mathbb{N}$ and $(x_m)_{m\in\mathbb{N}} \subset X_n$. Suppose that $(x_m)_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n). Choose $a_0 \in X$. Define $(a_m)_{m\in\mathbb{N}} \subset X$ by

$$\pi_k(a_m) = \begin{cases} \pi_k(a_0), & k \neq n \\ x_m, & k = n \end{cases}$$

  Let $\epsilon > 0$. Since $(x_m)_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n), Exercise 4.6.0.3 implies that $(x_m)_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n^{ϕ}). Thus there exists $M \in \mathbb{N}$ such that for each $m_1, m_2 \in \mathbb{N}$, $m_1, m_2 \geq M$ implies that $d_n^{\phi}(x_{m_1}, x_{m_2}) < 2^n\epsilon$. Let $m_1, m_2 \in \mathbb{N}$. Suppose that $m_1, m_2 \geq M$. Then

$$\begin{aligned}
d_X(a_{m_1}, a_{m_2}) &= \sum_{k\in\mathbb{N}} 2^{-k} d_k^{\phi}(\pi_k(a_{m_1}), \pi_k(a_{m_2})) \\
&= 2^{-n} d_n^{\phi}(\pi_n(a_{m_1}), \pi_n(a_{m_2})) + \sum_{k\neq n} 2^{-k} d_k^{\phi}(\pi_k(a_{m_1}), \pi_k(a_{m_2})) \\
&= 2^{-n} d_n^{\phi}(x_{m_1}, x_{m_2}) + \sum_{k\neq n} 2^{-k} d_k^{\phi}(\pi_k(a_0), \pi_k(a_0)) \\
&= 2^{-n} d_n^{\phi}(x_{m_1}, x_{m_2}) \\
&< 2^{-n}(2^n\epsilon) \\
&= \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that for each $m_1, m_2 \in \mathbb{N}$, $m_1, m_2 \geq M$ implies that $d_X(a_{m_1}, a_{m_2}) < \epsilon$. Therefore $(a_m)_{m\in\mathbb{N}}$ is Cauchy in (X, d_X). Since (X, d_X) is complete, there exists $a \in X$ such that $a_m \to a$ in $(X, \mathcal{T}_{d_X})$. Exercise 4.4.0.2 implies that $a_m \to a$ in $(X, \mathcal{T})$. Define $x \in X_n$ by $x := \pi_n(a)$. Exercise 3.5.2.2 implies that

$$\begin{aligned} x_m &= \pi_n(a_m) \\ &\to \pi_n(a) \\ &= x \end{aligned}$$

in $(X_n, \mathcal{T}_{d_n})$. Since $(x_m)_{m\in\mathbb{N}} \subset X_n$ with $(x_m)_{m\in\mathbb{N}}$ Cauchy is arbitrary, we have that for each $(x_m)_{m\in\mathbb{N}} \subset X_n$, if $(x_m)_{m\in\mathbb{N}}$ Cauchy, then there exists $x \in X_n$ such that $x_m \to x$ in $(X, \mathcal{T}_{d_n})$. Hence (X_n, d_n) is Complete. Since $n \in \mathbb{N}$ is arbitrary, we have that for each $n \in \mathbb{N}$, (X_n, d_n) is complete.

- Converesly, suppose that for each $n \in \mathbb{N}$, (X_n, d_n) is complete. Let $(a_m)_{m\in\mathbb{N}} \subset X$. Suppose that $(a_m)_{m\in\mathbb{N}}$ is Cauchy in (X, d_X). Exercise 4.6.2.1 implies that for each $n \in \mathbb{N}$, $(\pi_n(a_m))_{m\in\mathbb{N}}$ is Cauchy in (X_n, d_n). Since for each $n \in \mathbb{N}$, (X_n, d_n) is complete, we have that for each $n \in \mathbb{N}$, there exists $x_n \in X_n$ such that $\pi_n(a_m) \xrightarrow{m} x_n$ in $(X_n, \mathcal{T}_{d_n})$. Define $a \in X$ by $a := (x_n)_{n\in\mathbb{N}}$. Exercise 3.5.2.2 implies that $a_m \to a$ in $(X, \mathcal{T})$. Exercise 4.4.0.2 implies that $a_m \to a$ in $(X, \mathcal{T}_{d_X})$. Since $(a_m)_{m\in\mathbb{N}} \subset X$ is arbitrary, we have that for each $(a_m)_{m\in\mathbb{N}} \subset X$, if $(a_m)_{m\in\mathbb{N}}$ is Cauchy in (X, d_X), then there exists $a \in X$ such that $a_m \to a$ in $(X, \mathcal{T}_{d_X})$. Hence (X, d_X) is complete.

□

Exercise 4.6.2.3. Let $(X_n, d_n)_{n\in\mathbb{N}}$ be a collection of metric spaces. Suppose that for each $n \in \mathbb{N}$, (X_n, d_n) is complete. Set $X := \prod_{n\in\mathbb{N}} X_n$ and define $d_X : (\prod_{n\in\mathbb{N}} X_n)^2 \to [0, \infty)$ as in Definition 4.4.0.1. Then (X, d_X) is complete.

Proof. Set $X := \prod_{n\in\mathbb{N}} X_n$ and $\mathcal{T} := \bigotimes_{n\in\mathbb{N}} \mathcal{T}_n$. Let $(a_m)_{m\in\mathbb{N}} \subset X$. Then for each $m \in \mathbb{N}$, there exists $(x_{m,n})_{n\in\mathbb{N}} \in X$ such that $a_m = (x_{m,n})_{n\in\mathbb{N}}$. Suppose that $(a_m)_{m\in\mathbb{N}}$ is Cauchy. Let $n \in \mathbb{N}$ and $\epsilon > 0$. Then $2^{-n}\epsilon > 0$. Since $(a_m)_{m\in\mathbb{N}}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for each $m_1, m_2 \in \mathbb{N}$, $m_1, m_2 \geq N$ implies that $d(a_{m_1}, a_{m_2}) < 2^{-n}\epsilon$. Let $m_1, m_2 \in \mathbb{N}$. Suppose that $m_1, m_2 \geq N$. Then

$$\begin{aligned} 2^{-n} d_n(x_{m_1,n}, x_{m_2,n}) &\leq d(a_{m_1}, a_{m_2}) \\ &< 2^{-n}\epsilon \end{aligned}$$

and thus $d_n(x_{m_1,n}, x_{m_2,n}) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m_1, m_2 \in \mathbb{N}$, $m_1, m_2 \geq N$ implies that $d_n(x_{m_1,n}, x_{m_2,n}) < \epsilon$. Therefore $(x_{m,n})_{m\in\mathbb{N}}$ is Cauchy. Since (X_n, d_n) is complete, there exists $x_n \in X_n$ such that $x_{m,n} \xrightarrow{m} x_n$. Since $n \in \mathbb{N}$ is arbitrary, we have that for each $n \in \mathbb{N}$, there exists $x_n \in X_n$ such that $\pi_n(a_m) \xrightarrow{m} x_n$. Define $a \in X$ by $a_n := x_n$. An exercise or definition in product topology section implies that $a_m \to a$. (maybe clean up and cite product metric space exercise). Since $(a_m)_{m\in\mathbb{N}} \subset X$ with $(a_m)_{m\in\mathbb{N}}$ Cauchy is arbitrary, we have that for each $(a_m)_{m\in\mathbb{N}} \subset X$, if $(a_m)_{m\in\mathbb{N}}$ Cauchy, then there exists $a \in X$ such that $a_m \to a$. Thus (X, d) is complete. □

4.6.3 Completeness and Coproduct Spaces

Exercise 4.6.3.1. Let $(X_\alpha, d_\alpha)_{\alpha\in A}$ be a collection of metric spaces. Set $X := \coprod_{\alpha\in A} X_\alpha$ and define $\phi : [0, \infty) \to [0, 1)$ and $d_X : X \times X \to [0, \infty)$ as in Definition 4.5.0.1. Then for each $(\alpha_j, x_j)_{j\in\mathbb{N}} \subset X$, $(\alpha_j, x_j)_{j\in\mathbb{N}}$ is Cauchy in (X, d_X) iff there exists $\alpha_0 \in A$ and $j_0 \in \mathbb{N}$ such that

1. for each $j \in \mathbb{N}$, $j \geq j_0$ implies that $\alpha_j = \alpha_0$ and $x_j \in X_{\alpha_0}$,
2. $(x_{j'+j_0})_{j'\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d_{\alpha_0})$.

Proof. Let $(\alpha_j, x_j)_{j\in\mathbb{N}} \subset X$.

- ($\Longrightarrow$) :
Suppose that $(\alpha_j, x_j)_{j\in\mathbb{N}}$ is Cauchy in (X, d_X). Set $\epsilon_0 := 1/2$. Since $(\alpha_j, x_j)_{j\in\mathbb{N}}$ is Cauchy in (X, d_X), there exists $j_0 \in \mathbb{N}$ such that for each $j, k \in \mathbb{N}$, $j, k \geq j_0$ implies that $d_X((\alpha_j, x_j), (\alpha_k, x_k)) < \epsilon_0$. Set $\alpha_0 := \alpha_{j_0}$.

  1. Since $\epsilon_0 < 1$, we have that for each $j \in \mathbb{N}$, $j \geq j_0$ implies that $\alpha_j = \alpha_0$ and therefore $x_j \in X_{\alpha_0}$.
  2. Define $(y_j)_{j\in\mathbb{N}} \subset X_{\alpha_0}$ by $y_j := x_{j+j_0}$. Let $\epsilon > 0$. Since $(\alpha_j, x_j)_{j\in\mathbb{N}}$ is Cauchy in $(X, \mathcal{T}_X)$, there exists $N_0 \in \mathbb{N}$ such that for each $j, k \in \mathbb{N}$, $j, k \geq N_0$ implies that $d_X((\alpha_j, x_j), (\alpha_k, x_k)) < \epsilon$. Set $N := \max(j_0 + 1, N_0) - j_0$. Then for each $j, k \in \mathbb{N}$, $j, k \geq N$ implies that $j + j_0, k + j_0 \geq N_0$ and
$$\begin{aligned} d^{\phi}_{\alpha_0}(y_j, y_k) &= d^{\phi}_{\alpha_0}(x_{j+j_0}, y_{k+j_0}) \\ &= d_X((\alpha_j, x_{j+j_0}), (\alpha_k, y_{k+j_0})) \\ &< \epsilon. \end{aligned}$$
Thus $(y_j)_{j\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d^{\phi}_{\alpha_0})$. Exercise 4.6.0.3 implies that $(y_j)_{j\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d_{\alpha_0})$.

- ($\Longleftarrow$) :
Conversely, suppose that there exists $\alpha_0 \in A$ and $j_0 \in \mathbb{N}$ such that

  1. for each $j \in \mathbb{N}$, $j \geq j_0$ implies that $\alpha_j = \alpha_0$ and $x_j \in X_{\alpha_0}$,
  2. $(x_{j'+j_0})_{j'\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d_{\alpha_0})$.

  Define $(y_j)_{j\in\mathbb{N}} \subset X_{\alpha_0}$ by $y_j := x_{j+j_0}$. Since $(y_j)_{j\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d_{\alpha_0})$, we have that $(y_j)_{j\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d^{\phi}_{\alpha_0})$. Let $\epsilon > 0$. Since $(y_j)_{j\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d^{\phi}_{\alpha_0})$, there exists $N_0 \in \mathbb{N}$ such that for each $j, k \in \mathbb{N}$, $j, k \geq N_0$ implies that $d^{\phi}(y_j, j_k) < \epsilon$. Set $N := \max(j_0, N_0) + j_0$. Then for each $j, k \in \mathbb{N}$, $j, k \geq N$ implies that $j - j_0, k - j_0 \geq N_0, j_0$. Thus
$$\begin{aligned} d_X((\alpha_j, x_j), (\alpha_k, x_k)) &= d^{\phi}(x_j, x_k) \\ &= d^{\phi}(y_{j-j_0}, y_{k-j_0}) \\ &< \epsilon. \end{aligned}$$
Hence $(\alpha_j, x_j)_{j\in\mathbb{N}}$ is Cauchy in (X, d_X).

□

Exercise 4.6.3.2. Let $(X_\alpha, d_\alpha)_{\alpha\in A}$ be a collection of metric spaces. Set $X := \coprod_{\alpha\in A} X_\alpha$ and define $\phi : [0, \infty) \to [0, 1)$ and $d_X : X \times X \to [0, \infty)$ as in Definition 4.5.0.1. Then (X, d_X) is complete iff for each $\alpha \in A$, (X_α, d_α) is complete.

Proof.

- ($\Longrightarrow$) :
Suppose that (X, d_X) is complete. Let $\alpha \in A$ and $(x_n)_{n\in\mathbb{N}} \subset X_\alpha$. Suppose that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X_α, d_α). Exercise 4.5.0.3 implies that $\iota_\alpha : X_\alpha \to X$ is Lipschitz. Therefore ι_α is uniformly continuous and Exercise 4.6.0.2 implies that $(\iota_\alpha(x_n))_{n\in\mathbb{N}}$ is Cauchy in (X, d_X). Since (X, d_X) is complete, there exists $(\alpha_0, x_0) \in X$ such that $(\iota_\alpha(x_n)) \to (\alpha_0, x_0)$ in (X, d_X). Exercise 4.5.0.2 and Exercise 3.6.0.7 imply that $x_n \to x_0$. Since $(x_n)_{n\in\mathbb{N}} \subset X_\alpha$ with $(x_n)_{n\in\mathbb{N}}$ Cauchy in (X_α, d_α) is arbitrary, we have that for each $(x_n)_{n\in\mathbb{N}} \subset X_\alpha$, if $(x_n)_{n\in\mathbb{N}}$ is Cauchy, then there exists $x_0 \in X_\alpha$ such that $x_n \to x_0$. Hence (X_α, d_α) is complete. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, (X_α, d_α) is complete.

- ($\Longleftarrow$) :
Conversely, suppose that for each $\alpha \in A$, (X_α, d_α) is complete. Let $(\alpha_n, x_n)_{n\in\mathbb{N}} \subset X$. Suppose that $(\alpha_n, x_n)_{n\in\mathbb{N}}$ is Cauchy in (X, d_X). Exercise 4.6.3.1 implies that there exists $\alpha_0 \in A$ and $j_0 \in \mathbb{N}$ such that

  1. for each $j \in \mathbb{N}$, $j \geq j_0$ implies that $\alpha_j = \alpha_0$ and $x_j \in X_{\alpha_0}$,

2. $(x_{j'+j_0})_{j'\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d_{\alpha_0})$.

Define $(y_j)_{j\in\mathbb{N}} \subset X_{\alpha_0}$ by $y_j := x_{j+j_0}$. Since $(X_{\alpha_0}, d_{\alpha_0})$ is complete, and $(y_j)_{j\in\mathbb{N}}$ is Cauchy in $(X_{\alpha_0}, d_{\alpha_0})$, there exists $x_0 \in X_{\alpha_0}$ such that $y_n \to x_0$ in $(X, \mathcal{T}_{d_{\alpha_0}})$. Thus $y_n \to x_0$ in $(X, \mathcal{T}_{d^\phi_{\alpha_0}})$. Let $\epsilon > 0$. Since $y_n \to x_0$ in $(X, \mathcal{T}_{d^\phi_{\alpha_0}})$, there exists $N_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N_0$ implies that $d^\phi_{\alpha_0}(y_n, x_0) < \epsilon$. Set $N := \max(j_0, N_0) + j_0$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Then $n - j_0 \geq j_0, N_0$ and

$$\begin{aligned}
d_X((\alpha_n, x_n), (\alpha_0, x_0)) &= d_X((\alpha_0, x_n), (\alpha_0, x_0)) \\
&= d^\phi(x_n, x_0) \\
&= d^\phi(y_{n-j_0}, x_0) \\
&< \epsilon.
\end{aligned}$$

Hence $(\alpha_n, x_n) \to (\alpha_0, x_0)$ in $(X, \mathcal{T}_{d_X})$. Since $(\alpha_n, x_n)_{n\in\mathbb{N}} \subset X$ with $(\alpha_n, x_n)_{n\in\mathbb{N}}$ Cauchy is arbitrary, we have that for each $(\alpha_n, x_n)_{n\in\mathbb{N}} \subset X$, $(\alpha_n, x_n)_{n\in\mathbb{N}}$ is Cauchy implies that there exists $(\alpha_0, x_0) \in X$ such that $(\alpha_n, x_n) \to (\alpha_0, x_0)$ in $(X, \mathcal{T}_{d_X})$. Thus $(X, \mathcal{T}_{d_X})$ is complete.

$\square$

4.7 The Baire Category Theorem

Exercise 4.7.0.1. Let X be a complete metric space and $(U_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$. Suppose that for each $n \in \mathbb{N}$, U_n is open and dense in X. Then $\bigcap\limits_{n\in\mathbb{N}} U_n$ is dense in X.

Hint: Let $W \subset X$. Suppose that W is open. Since U_1 is open and dense in X, Exercise 3.1.0.27 implies that $U_1 \cap W$ is open and nonempty. Hence there exists $x_1 \in U_1 \cap W$ and $r_1 \in (0, 2^{-1})$ such that $\operatorname{cl} B(x_1, r_1) \subset U_1 \cap W$. Inductively define $(x_n)_{n\in\mathbb{N}} \subset X$ and $(r_n)_{n\in\mathbb{N}} \subset (0, 1)$.

Proof. Set $U = \bigcap\limits_{n\in\mathbb{N}} U_n$.Let $W \subset X$. Suppose that W is open and nonempty. Since U_1 is open and dense in X, Exercise 3.1.0.27 implies that $U_1 \cap W$ is open and nonempty. Hence there exists $x_1 \in U_1 \cap W$ and $r_1 \in (0, 2^{-1})$ such that $\operatorname{cl} B(x_1, r_1) \subset U_1 \cap W$. For $n \geq 2$, Exercise 3.1.0.27 implies that $U_n \cap B(x_{n-1}, r_{n-1})$ is open and nonempy. Hence there exists $x_n \in U_n \cap B(x_{n-1}, r_{n-1})$ and $r_n \in (0, 2^{-n})$ such that $\operatorname{cl} B(x_n, r_n) \subset U_n \cap B(x_{n-1}, r_{n-1})$. Note that for each $N, n \in \mathbb{N}$, if $n \geq N$, then by definition,

$$\begin{aligned} x_n &\in B(x_n, r_n) \\ &\subset U_n \cap B(x_{n-1}, r_{n-1}) \\ &\subset \left(\bigcap_{j=N+1}^{n} U_j \right) \cap B(x_N, r_N) \end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $2^{1-N} < \epsilon$. Let $n, m \in \mathbb{N}$. Suppose that $n, m \geq N$. Then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_N) + d(x_N, x_m) \\ &\leq 2^{-N} + 2^{-N} \\ &= 2^{1-N} \\ &< \epsilon \end{aligned}$$

Thus $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since X is complete, there exists $x \in X$ such that $x_n \to x$. Let $n \in \mathbb{N}$. Since $(x_n)_{n\geq N} \subset \operatorname{cl} B(x_1, r_1)$, we have that

$$\begin{aligned} x &\in \operatorname{cl} B(x_1, r_1) \\ &\subset W \end{aligned}$$

Similarly, for each $n \in \mathbb{N}$,

$$\begin{aligned} x_n &\in U_n \cap \operatorname{cl} B(x_n, r_n) \\ &\subset U_n \end{aligned}$$

which implies that $x \in U$. Hence $\bigcap\limits_{n\in\mathbb{N}} U_n \cap W \neq \varnothing$. Since W is an arbitrary open nonempty subset of X, we have that for each $W \subset X$, if W is open and nonempty, then $U \cap W \neq \varnothing$. By definition, W is dense in X. $\square$

Exercise 4.7.0.2. Let X be a complete metric space and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$. If for each $n \in \mathbb{N}$, A_n is nowhere dense, then $X \neq \bigcup\limits_{n\in\mathbb{N}} A_n$.

Proof. Suppose that for each $n \in \mathbb{N}$, A_n is nowhere dense. Exercise 3.1.0.29 and Exercise 3.1.0.30 imply that for each $n \in \mathbb{N}$, $(\operatorname{cl} A_n)^c$ is dense and open. For the sake of contradiction, suppose that $X = \bigcup\limits_{n\in\mathbb{N}} A_n$. Then $X = \bigcup\limits_{n\in\mathbb{N}} \operatorname{cl} A_n$. Exercise 4.7.0.1 implies that $\varnothing = \bigcap\limits_{n\in\mathbb{N}} (\operatorname{cl} A_n)^c$ is dense in X. This is a contradiction. Hence $X \neq \bigcup\limits_{n\in\mathbb{N}} A_n$. $\square$

Definition 4.7.0.3. Let X be a topological space. Set $\mathcal{D}_{\mathcal{O}}(X) = \{U \subset X : U \text{ is open and dense in } X\}$. Then X is said to be a **Baire space** if for each $(U_n)_{n\in\mathbb{N}} \subset \mathcal{D}_{\mathcal{O}}(X)$, $\bigcap\limits_{n\in\mathbb{N}} U_n$ is dense in X.

Definition 4.7.0.4. Let X be a topological space. Set $\mathcal{D}_{\mathcal{N}}(X) = \{U \subset X : U \text{ is nowhere dense in } X\}$. Let $E \subset X$. Then E is said to be **meager** in X if there exist $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}_{\mathcal{N}}(X)$ such that $E = \bigcup\limits_{n\in\mathbb{N}} A_n$.

Theorem 4.7.0.5. Baire Category Theorem:
Let X be a complete metric space. Then

1. X is a Baire space
2. X is not meager

Proof. Immediate by Exercise 4.7.0.1 and Exercise 4.7.0.2. □

Definition 4.7.0.6. content...

4.8 Metrizable Spaces

4.8.1 Introduction

Definition 4.8.1.1. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is said to be **metrizable** if there exists a metric $d : X \times X \to [0, \infty)$ such that $\mathcal{T} = \mathcal{T}_d$.

Definition 4.8.1.2. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is said to be **completely metrizable** if there exists a metric $d : X \times X \to [0, \infty)$ such that $\mathcal{T} = \mathcal{T}_d$ and (X, d) is complete.

Exercise 4.8.1.3. Let $(X, \mathcal{T})$ be a topological space and $C \subset X$. Suppose that $(X, \mathcal{T})$ is completely metrizable. If C is closed, then $(C, \mathcal{T} \cap C)$ is completely metrizable. Then $(X, \mathcal{T})$ is said to be **completely metrizable** if there exists a metric $d : X \times X \to [0, \infty)$ such that $\mathcal{T} = \mathcal{T}_d$ and (X, d) is complete.

Proof. Suppose that C is closed. Since $(X, \mathcal{T})$ is completely metrizable, there exists a metric $d : X \times X \to [0, \infty)$ such that $\mathcal{T} = \mathcal{T}_d$ and (X, d) is complete. Exercise 4.6.1.2 implies that $(C, d|_{C \times C})$ is complete. Exercise 4.3.1.1 implies that $\mathcal{T} \cap C = \mathcal{T}_{d|_{C \times C}}$. Thus $(C, \mathcal{T} \cap C)$ is completely metrizable. □

Exercise 4.8.1.4. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ a $(\mathcal{T}_X, \mathcal{T}_Y)$-homeomorphism. Then $(X, \mathcal{T}_X)$ is completely metrizable iff $(Y, \mathcal{T}_Y)$ is completely metrizable.

Proof.

- $(\implies)$:
  Suppose that $(X, \mathcal{T}_X)$ is completely metrizable. Then there exists a metric $d_X : X \times X \to [0, \infty)$ such that $\mathcal{T}_X = \mathcal{T}_{d_X}$ and (X, d_X) is complete. Set $d_Y := (f^{-1})^* d_X$.
  - Exercise 4.1.0.39 implies that $\mathcal{T}_Y = \mathcal{T}_{d_Y}$. So $(Y, \mathcal{T}_Y)$ is metrizable.
  - Exercise 4.1.0.37 implies that f^{-1} is a (d_Y, d_X)-isometry. Since (X, d_X) is complete, Exercise 4.6.0.6 implies that (Y, d_Y) is complete.

  Hence $(Y, \mathcal{T}_Y)$ is completely metrizable.
- $(\impliedby)$:
  Similar to $(\implies)$.

□

4.8.2 Metrizability of Subspaces

Exercise 4.8.2.1. Let $(X, \mathcal{T})$ be a topological space and $C \subset X$. Suppose that $(X, \mathcal{T})$ is completely metrizable. If C is closed, then $(C, \mathcal{T} \cap C)$ is completely metrizable.

Proof. Suppose that C is closed. Since $(X, \mathcal{T})$ is completely metrizable, there exists a metric $d : X \times X \to [0, \infty)$ such that $\mathcal{T} = \mathcal{T}_d$ and (X, d) is complete. Exercise 4.6.1.2 implies that $(C, d|_{C \times C})$ is complete. Exercise 4.3.1.1 implies that $\mathcal{T}_{d|_{C \times C}} = \mathcal{T}_d \cap C$. Hence $(C, \mathcal{T}_d \cap C)$ is completely metrizable. □

4.8.3 Metrizability of Product Spaces

Exercise 4.8.3.1.

Exercise 4.8.3.2. Let $(X_n, \mathcal{T}_n)_{n \in \mathbb{N}}$ be a collection of topological spaces. Suppose that for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is completely metrizable. Then $(\prod\limits_{n \in \mathbb{N}} X_n, \bigotimes\limits_{n \in \mathbb{N}} \mathcal{T}_n)$ is completely metrizable.

Proof. Define $\phi : [0, \infty) \to [0, \infty)$, X and $\mathcal{T}$ as in Definition 4.4.0.1. Since for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is completely metrizable, for each $n \in \mathbb{N}$, there exists a metric $d_n : X_n \times X_n \to [0, \infty)$ such that (X_n, d_n) is a complete metric space. Exercise 4.6.2.2 implies that (X, d_X) is complete. Since Exercise 4.4.0.2 implies that $\mathcal{T} = \mathcal{T}_{d_X}$, we have that $(X, \mathcal{T})$ is completely metrizable. □

4.8.4 Metrizability of Coproduct Spaces

Exercise 4.8.4.1. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Suppose that for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is completely metrizable. Then $(\coprod_{\alpha \in A} X_\alpha, \bigoplus_{\alpha \in A} \mathcal{T}_\alpha)$ is completely metrizable.

Proof. Define $\phi : [0, \infty) \to [0, \infty)$, X and $\mathcal{T}$ as in Definition 4.5.0.1. Since for each $\alpha \in A$, $(X_\alpha, \mathcal{T}_\alpha)$ is completely metrizable, for each $\alpha \in A$, there exists a metric $d_\alpha : X_\alpha \times X_\alpha \to [0, \infty)$ such that (X_α, d_α) is a complete metric space. Define $d_X : X \times X \to [0, \infty)$ as in Definition 4.5.0.1. Exercise 4.6.3.2 implies that (X, d_X) is complete. Since Exercise 4.5.0.2 implies that $\mathcal{T} = \mathcal{T}_{d_X}$, we have that $(X, \mathcal{T})$ is completely metrizable. $\square$

4.9 Polish Spaces

Definition 4.9.0.1. Let $(X, \mathcal{T})$ be a topological space. Then $(X, \mathcal{T})$ is said to be a **Polish** space if

1. $(X, \mathcal{T})$ is completely metrizable
2. $(X, \mathcal{T})$ is separable

Exercise 4.9.0.2. Let $(X, \mathcal{T})$ be a Polish space and $C \subset X$. If C is closed in $(X, \mathcal{T})$, then $(C, \mathcal{T} \cap C)$ is a Polish space.

Proof. Suppose that C is closed.

1. Since $(X, \mathcal{T})$ is completely metrizable and C is closed, Exercise 4.8.2.1 implies that $(C, \mathcal{T} \cap C)$ is completely metriable.
2. Since $(X, \mathcal{T})$ is separable, Exercise 4.3.1.2 implies that $(C, \mathcal{T} \cap C)$ is separable.

Therefore $(C, \mathcal{T} \cap C)$ is a Polish space. □

Exercise 4.9.0.3. Let $(X_n, \mathcal{T}_n)_{n\in\mathbb{N}}$ be a collection of Polish spaces. Then $(\prod_{n\in\mathbb{N}} X_n, \bigotimes_{n\in\mathbb{N}} \mathcal{T}_n)$ is a Polish space.

Proof. Since for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is a Polish space, we have that for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is completely metrizable and $(X_n, \mathcal{T}_n)$ is separable. Set $X := \prod_{n\in\mathbb{N}} X_n$ and $\mathcal{T} := \bigotimes_{n\in\mathbb{N}} \mathcal{T}_n$.

1. Exercise 4.8.3.2 implies that $(X, \mathcal{T})$ is completely metrizable.
2. Since for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is separable, Exercise 4.1.0.40 implies that for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is second-countable. Exercise 3.9.2.11 then implies that $(X, \mathcal{T})$ is second-countable. Another application of Exercise 4.1.0.40 implies that $(X, \mathcal{T})$ is separable.

Thus $(X, \mathcal{T})$ is a Polish space. □

Exercise 4.9.0.4. Let $(X_n, \mathcal{T}_n)_{n\in\mathbb{N}}$ be a collection of Polish spaces. Then $(\coprod_{n\in\mathbb{N}} X_n, \bigoplus_{n\in\mathbb{N}} \mathcal{T}_n)$ is Polish.

Proof. Set $X := \coprod_{n\in\mathbb{N}} X_n$ and $\mathcal{T} := \bigoplus_{n\in\mathbb{N}} \mathcal{T}_n$. Since for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is a Polish space, for each $n \in \mathbb{N}$, $(X_n, \mathcal{T}_n)$ is completely metrizable and second-countable. Exercise 4.8.4.1 implies that $(X, \mathcal{T})$ is completely metrizable and Exercise 3.9.2.12 implies that $(X, \mathcal{T})$ is second-countable. Hence $(X, \mathcal{T})$ is a Polish space. □

Exercise 4.9.0.5. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ a $(\mathcal{T}_X, \mathcal{T}_Y)$-homeomorphism. Then $(X, \mathcal{T}_X)$ is a Polish space iff $(Y, \mathcal{T}_Y)$ is a Polish space.

Proof.

- $(\implies)$:
  Suppose that $(X, \mathcal{T}_X)$ is a Polish space. Then $(X, \mathcal{T}_X)$ is completely metrizable and $(X, \mathcal{T}_X)$ is separable.
  1. Exercise 4.8.1.4 implies that $(Y, \mathcal{T}_Y)$ is completely metrizable.
  2. Since $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are metrizable, the exist metrics $d_X : X \times X \to [0, \infty)$ and $d_Y : Y \times Y \to [0, \infty)$ on X and Y respectively such that $\mathcal{T}_X = \mathcal{T}_{d_X}$ and $\mathcal{T}_Y = \mathcal{T}_{d_Y}$. Since $(X, \mathcal{T}_X)$ is separable and f is a $(\mathcal{T}_X, \mathcal{T}_Y)$-homeomorphism, we have that Exercise 4.1.0.41 implies that $(Y, \mathcal{T}_Y)$ is separable.

  Thus $(Y, \mathcal{T}_Y)$ is a Polish space.
- $(\impliedby)$:
  Similar to $(\implies)$.

□

Exercise 4.9.0.6. Let $(X, \mathcal{T})$ be a Polish space and $U \in \mathcal{T}$. Then $(U, \mathcal{T} \cap U)$ is a Polish space.
Hint: Exercise 4.1.0.30

Proof. Define $C \subset X \times \mathbb{R}$ and $f : U \to C$ as in Exercise 4.1.0.30. Exercise 4.1.0.30 implies that f is a $(\mathcal{T} \cap U, (\mathcal{T} \otimes \mathcal{T}_{\mathbb{R}}) \cap C)$-homeomorphism and C is closed. Exercise 4.9.0.3 implies that $(X \times \mathbb{R}, \mathcal{T} \otimes \mathcal{T}_{\mathbb{R}})$ is a Polish space. Exercise 4.9.0.5 implies that $(C, (\mathcal{T} \otimes \mathcal{T}_{\mathbb{R}}) \cap C)$ is a Polish space. $\square$

Exercise 4.9.0.7. Let $(X, \mathcal{T})$ be a Polish space and $E \subset X$. Then E is a G_δ-set iff $(E, \mathcal{T} \cap E)$ is a Polish space.
Hint: Exercise 4.1.0.31

Proof.

- $(\implies)$:
  Suppose that E is a G_δ-set. Set $\mathcal{S} := \mathcal{T} \otimes (\mathcal{T}_{\mathbb{R}})^{\otimes \mathbb{N}}$. Since E is a G_δ-set, there exists $(U_n)_{n \in \mathbb{N}} \subset \mathcal{T}$ such that $E = \bigcap\limits_{n \in \mathbb{N}} U_n$. Exercise 4.1.0.31 implies that there exists $C \subset X \times \mathbb{R}^{\mathbb{N}}$ and $f : E \to C$ such that f is a $(\mathcal{T} \cap E, \mathcal{S} \cap C)$-homeomorphism and C is closed in $X \times \mathbb{R}^{\mathbb{N}}$. Exercise 4.9.0.3 implies that $X \times \mathbb{R}^{\mathbb{N}}$ is a Polish space. Since C is closed in $X \times \mathbb{R}^{\mathbb{N}}$, Exercise 4.9.0.2 implies that $(C, \mathcal{S} \cap C)$ is a Polish space. Exercise 4.9.0.5 implies that $(E, \mathcal{T} \cap E)$ is a Polish space.

- $(\impliedby)$:
  Conversely, suppose that $(E, \mathcal{T} \cap E)$ is a Polish space.

$\square$

Exercise 4.9.0.8. Let $(X, \mathcal{T})$ be a Polish space. Set $\Delta_{X^{\mathbb{N}}} := \{x \in X : \text{ for each } m, n \in \mathbb{N}, \pi_m(x) = \pi_n(x)\}$. Then $(\Delta_{X^{\mathbb{N}}}, \mathcal{T}^{\otimes \mathbb{N}} \cap \Delta_{X^{\mathbb{N}}})$ is a Polish space. **Hint:** Exercise 3.8.0.7

Proof. Since $(X, \mathcal{T})$ is Polish, $(X, \mathcal{T})$ is Hausdorff. Exercise 4.9.0.3 implies that $(X^{\mathbb{N}}, \mathcal{T}^{\otimes \mathbb{N}})$ is a Polish space. and Exercise 3.8.0.7 implies that $\Delta_{X^{\mathbb{N}}}$ is closed in $(X^{\mathbb{N}}, \mathcal{T}^{\otimes \mathbb{N}})$. Exercise 4.9.0.2 then implies that $(\Delta_{X^{\mathbb{N}}}, \mathcal{T}^{\otimes \mathbb{N}} \cap \Delta_{X^{\mathbb{N}}})$ is a Polish space. $\square$

4.10 Ultrametric Spaces

Ultrametric spaces are given by sequences of partitions of X, $(\mathcal{P}_j)_{j\in\mathbb{N}}$, where for each j and $E \in \mathcal{P}_j$, there exists $\mathcal{E} \subset \mathcal{P}_{j+1}$ such that $E = \bigcup_{F\in\mathcal{R}} F$. Then set $d(x,y) = 2^{-n}$ if $n = \max(j \in \mathbb{N} :$ there exists $E \in \mathcal{P}_j$: such that $x, y \in E)$.

Definition 4.10.0.1. Let X be a set and $d : X \times X \to [0, \infty)$. Then d is said to be and **ultrametric on** X if for each $x, y, z \in X$,

1. **(symmetry):** $d(x,y) = d(y,x)$
2. **(definiteness):** $d(x,y) = 0$ iff $x = y$
3. **(strong triangle inequality):** $d(x,z) \leq \max(d(x,y), d(y,z))$

Exercise 4.10.0.2. Let X be a set and d an ultrametric on X. Then d is a metric on X.

Proof. Let $x, y, z \in X$. Since $(d(x,y), d(y,z) \geq 0$, we have that $d(x,y), d(y,z) \leq d(x,y) + d(y,z)$. Therefore

$$\begin{aligned} d(x,y) &\leq \max(d(x,y), d(y,z)) \\ &\leq d(x,y) + d(y,z) \end{aligned}$$

□

Definition 4.10.0.3. Let X be a set and d an ultrametric on X. Then (X,d) is said to be an **ultrametric space**.

Exercise 4.10.0.4. Isosceles Triangle Property:
Let (X,d) be an ultrametric space and $x, y, z \in X$.

1. If $d(x,y) < d(y,z)$, then

$$\begin{aligned} d(y,z) &= d(x,z) \\ &= \max(d(x,y), d(y,z)) \\ &= \max(d(x,y), d(x,z)) \end{aligned}$$

2. If $d(x,y) \neq d(y,z)$, then $d(x,y) = \max(d(x,y), d(y,z))$.

Proof.

1. $d(x,y) < d(y,z)$. Then

$$\begin{aligned} d(x,z) &\leq \max(d(x,y), d(y,z)) \\ &= d(y,z) \end{aligned}$$

For the sake of contradiction, suppose that $d(x,z) \leq d(x,y)$. Since $d(x,y) < d(y,z)$, we have that

$$\begin{aligned} d(y,z) \leq \max(d(x,y), d(x,z)) \\ = d(x,y) \\ < d(y,z) \end{aligned}$$

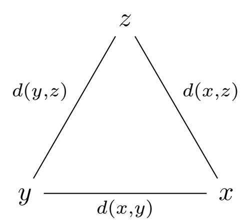
which is a contradictiton. Hence $d(x,y) < d(x,z)$. Therefore, we have that

$$\begin{aligned} d(x,z) &\leq \max(d(x,y), d(y,z)) \\ &= d(y,z) \\ &\leq \max(d(y,x), d(x,z)) \\ &= d(x,z) \end{aligned}$$

Hence

$$\begin{aligned} d(y,z) &= d(x,z) \\ &= \max(d(x,y), d(y,z)) \\ &= \max(d(x,y), d(x,z)) \end{aligned}$$

and we have the following isosceles triangle:



2. Suppose that $d(x,y) \neq d(y,z)$.
   If $d(x,y) < d(y,z)$, then part (1) implies that $d(x,z) = \max(d(x,y), d(y,z))$.
   Suppose that $d(y,z) < d(x,y)$. Then $d(z,y) < d(y,x)$. If we permute x and z in part (1), we see that

$$\begin{aligned} d(x,z) &= \max(d(z,y), d(y,x)) \\ &= \max(d(x,y), d(y,z)) \end{aligned}$$

□

Definition 4.10.0.5. Let (X,d) be an ultrametric space and $r > 0$. We define the

- **open r-ball relation** on X, denoted $\sim_r \subset X \times X$ by $x \sim_r y$ iff $d(x,y) < r$
- **closed r-ball relation** on X, denoted $\simeq_r \subset X \times X$ by $x \simeq_r y$ iff $d(x,y) \leq r$

Exercise 4.10.0.6. Let (X,d) be an ultrametric space and $r > 0$. Then

1. $\sim_r$ is an equivalence relation on X
2. $\simeq_r$ is an equivalence relation on X.

Proof.

1. (a) Let $x \in X$. Since

$$\begin{aligned} d(x,x) &= 0 \\ &< r \end{aligned}$$

   we have that $x \sim_r x$.

   (b) Let $x, y \in X$. Suppose that $x \sim_r y$. Then $d(x,y) < r$. This implies that

$$\begin{aligned} d(y,x) &= d(x,y) \\ &< r \end{aligned}$$

   So $y \sim_r x$.

   (c) Let $x, y, z \in X$. Suppose that $x \sim_r y$ and $y \sim_r z$. Then $d(x,y) < r$ and $d(y,z) < r$. The strong triangle inequality implies that

$$\begin{aligned} d(x,z) &\leq \max(d(x,y), d(y,z)) \\ &< r \end{aligned}$$

   Hence $x \sim_r z$.

2. (a) Let $x \in X$. Since

$$\begin{aligned} d(x,x) &= 0 \\ &\leq r \end{aligned}$$

we have that $x \simeq_r x$.

(b) Let $x, y \in X$. Suppose that $x \simeq_r y$. Then $d(x,y) \leq r$. This implies that

$$\begin{aligned} d(y,x) &= d(x,y) \\ &\leq r \end{aligned}$$

So $y \simeq_r x$.

(c) Let $x, y, z \in X$. Suppose that $x \simeq_r y$ and $y \simeq_r z$. Then $d(x,y) \leq r$ and $d(y,z) \leq r$. The strong triangle inequality implies that

$$\begin{aligned} d(x,z) &\leq \max(d(x,y), d(y,z)) \\ &\leq r \end{aligned}$$

Hence $x \simeq_r z$.

□

Definition 4.10.0.7. Let (X,d) be an ultrametric space and $r > 0$. We denote

- the projection of X onto $X/\sim_r$ by $\pi^d_r : X \to X/\sim_r$
- the projection of X onto $X/\simeq_r$ by $\bar{\pi}^d_r : X \to X/\simeq_r$

Exercise 4.10.0.8. Let (X,d) be an ultrametric space, $x \in X$ and $r > 0$. Then

1. $\pi^d_r(x) = B(x,r)$
2. $\bar{\pi}^d_r(x) = \bar{B}(x,r)$.

Proof.

1. For each $y \in X$,

$$\begin{aligned} y \sim_r x &\iff d(x,y) < r \\ &\iff y \in B(x,r) \end{aligned}$$

so that $\pi^d_r(x) = \bar{B}(x,r)$.

2. For each $y \in X$,

$$\begin{aligned} y \sim_r x &\iff d(x,y) \leq r \\ &\iff y \in \bar{B}(x,r) \end{aligned}$$

so that $\bar{\pi}^d_r(x) = \bar{B}(x,r)$.

□

Exercise 4.10.0.9. Let (X,d) be an ultrametric space, $x, y \in X$ and $s \in [0,\infty)$. If $y \in \bar{B}(x,s)$, then for each $r \in [0,\infty)$, $r \leq s$ implies that $\bar{B}(y,r) \subset \bar{B}(x,s)$.

Proof. Suppose that $y \in \bar{B}(x,s)$. Let $r \in [0,\infty)$. Suppose that $r \leq s$. Let

$$\begin{aligned} z &\in \bar{B}(y,r) \\ &\subset \bar{B}(y,s) \end{aligned}$$

Then $z \simeq_s y$. Since $y \simeq_s x$, the previous exercise implies that $z \simeq_s x$. Hence $z \in \bar{B}(x,s)$. Since $z \in \bar{B}(y,r)$ is arbitrary, $\bar{B}(y,r) \subset \bar{B}(x,s)$. □

Exercise 4.10.0.10. Let (X, d) be an ultrametric space, $x \in X$ and $r > 0$. Then

1. $B(x, r)$ is closed and open
2. $\bar{B}(x, r)$ is closed and open

Proof.

1. Since d is a metric, for each $y \in X$, $B(y, r)$ is open. In particular, $B(x, r)$ is open. Since $\sim_r$ is an equivalence relation, we have that

$$B(x, r)^c = \bigcup_{y \in B(x,r)^c} B(y, r)$$

which is open. Hence $B(x, r)$ is closed.

2. Since d is a metric, $\bar{B}(x, r)$ is closed. Let $y \in \partial\bar{B}(x, r)$. By definition, $d(x, y) = r$. Let $z \in B(y, r)$. Then

$$\begin{aligned} d(y, z) &< r \\ &= d(x, y) \end{aligned}$$

A previous exercise implies that

$$\begin{aligned} d(x, z) &= \max(d(x, y), d(y, z)) \\ &= d(x, y) \\ &= r \end{aligned}$$

Hence $z \in \partial\bar{B}(x, r)$. Since $z \in B(y, r)$ is arbitrary, $B(y, r) \subset \partial\bar{B}(x, r)$. Since $B(y, r)$ is open, and $y \in \partial\bar{B}(x, r)$ is arbitrary, we have that for each $y \in \partial\bar{B}(x, r)$, there exists $U \subset \partial\bar{B}(x, r)$ such that U is open and $y \in U$. Thus $\partial\bar{B}(x, r)$ is open. Therefore

$$\bar{B}(x, r) = B(x, r) \cup \partial\bar{B}(x, r)$$

which is open.

□

Definition 4.10.0.11. Let X be a set. We define the **collection of partitions of** X, denoted $\text{Part}(X)$, by

$$\text{Part}(X) = \{\mathcal{P} \subset \mathcal{P}(X) : \mathcal{P} \text{ is a partition of } X\}$$

Let Γ be a totally ordered set and $\mathcal{P} : \Gamma \to \text{Part}(X)$.

- For each $r \in \Gamma$, we define the **r-th partition relation** on X, denoted $\sim_{\mathcal{P}_r}$, by $x \sim_{\mathcal{P}_r} y$ iff there exists $E \in \mathcal{P}_r$ such that $x \in E$ and $y \in E$.
- For $r \in \Gamma$, we denote the projection of X onto $X/\sim_{\mathcal{P}_r}$ by $\pi^{\mathcal{P}}_r : X \to X/\sim_{\mathcal{P}_r}$ so that

$$\mathcal{P}_r = \{\pi^{\mathcal{P}}_r(x) : x \in X\}$$

Then

- $\mathcal{P}$ is said to **separate points** if for each $x, y \in X$, $x \neq y$ implies that there exists $r \in \Gamma$ such that $\pi^{\mathcal{P}}_r(x) \neq \pi^{\mathcal{P}}_r(y)$
- $\mathcal{P}$ is said to **collect points** if for each $x, y \in X$, there exists $r \in \Gamma$ such that $\pi^{\mathcal{P}}_r(x) = \pi^{\mathcal{P}}_r(y)$.
- $\mathcal{P}$ is said to be **decreasing** if for each $r, s \in \Gamma$, $r \leq s$ implies that for each $x \in X$, there exists $\mathcal{F}_x \subset X$ such that

$$\pi^{\mathcal{P}}_r(x) = \bigcup_{y \in \mathcal{F}_x} \pi^{\mathcal{P}}_s(y)$$

Exercise 4.10.0.12. Let X be a set, Γ a totally ordered set and $\mathcal{P} : \Gamma \to \mathrm{Part}(X)$. Suppose that $\mathcal{P}$ is decreasing. Let $x \in X$ and $r, s \in \Gamma$. If $r \leq s$, then $\pi_s^{\mathcal{P}}(x) \subset \pi_r^{\mathcal{P}}(x)$.

Proof. Suppose that $r \leq s$. Since $\mathcal{P}$ is decreasing, there exists $\mathcal{F}_x \subset X$ such that

$$\pi_r^{\mathcal{P}}(x) = \bigcup_{y \in \mathcal{F}_x} \pi_s^{\mathcal{P}}(y)$$

Since $x \in \pi_r^{\mathcal{P}}(x)$, there exists $y_0 \in \mathcal{F}_x$ such that $x \in \pi_s^{\mathcal{P}}(y_0)$. Then $\pi_s^{\mathcal{P}}(y_0) = \pi_s^{\mathcal{P}}(x)$ and

$$\begin{aligned}\pi_s^{\mathcal{P}}(x) &= \pi_s^{\mathcal{P}}(y_0) \\ &\subset \bigcup_{y \in \mathcal{F}_x} \pi_s^{\mathcal{P}}(y) \\ &= \pi_r^{\mathcal{P}}(x)\end{aligned}$$

□

Exercise 4.10.0.13. Let X be a set, Γ a totally ordered set and $\mathcal{P} : \Gamma \to \mathrm{Part}(X)$. Suppose that $\mathcal{P}$ is decreasing. Then for each $x, y \in X$ and $s \in \Gamma$, if $\pi_s^{\mathcal{P}}(x) = \pi_s^{\mathcal{P}}(y)$, then for each $r \in \Gamma$, $r \leq s$ implies that $\pi_r^{\mathcal{P}}(x) = \pi_r^{\mathcal{P}}(y)$.

Proof. Let $x, y \in X$ and $s \in \Gamma$. Suppose that $\pi_s^{\mathcal{P}}(x) = \pi_s^{\mathcal{P}}(y)$. Let $r \in \Gamma$. Suppose that $r \leq s$. Since $\mathcal{P}$ is decreasing, there exists $\mathcal{F}_x \subset X$ such that $\pi_r^{\mathcal{P}}(x) = \bigcup\limits_{z \in \mathcal{F}_x} \pi_s^{\mathcal{P}}(z)$. Since $\mathcal{P}_s$ is a partition of X, there exists $x' \in \mathcal{F}_x$ such that $\pi_s^{\mathcal{P}}(x') = \pi_s^{\mathcal{P}}(x)$. Since $\pi_s^{\mathcal{P}}(x) = \pi_s^{\mathcal{P}}(y)$, we have that

$$\begin{aligned}y &\in \bigcup_{z \in \mathcal{F}_x} \pi_s^{\mathcal{P}}(z) \\ &= \pi_r^{\mathcal{P}}(x)\end{aligned}$$

Since $\mathcal{P}_r$ is a partition of X, $\pi_r^{\mathcal{P}}(y) = \pi_r^{\mathcal{P}}(x)$. □

Exercise 4.10.0.14. Let X be a set, Γ a totally ordered set and $\mathcal{P} : \Gamma \to \mathrm{Part}(X)$. Suppose that $\mathcal{P}$ is decreasing. Let $x, y \in X$ and $r, s \in \Gamma$. Suppose $r \leq s$. If $\pi_r^{\mathcal{P}}(x) \cap \pi_s^{\mathcal{P}}(y) \neq \varnothing$, then $\pi_s^{\mathcal{P}}(y) \subset \pi_r^{\mathcal{P}}(x)$.

Proof. Suppose that $\pi_r^{\mathcal{P}}(x) \cap \pi_s^{\mathcal{P}}(y) \neq \varnothing$. Then there exists $z \in X$ such that $z \in \pi_r^{\mathcal{P}}(x) \cap \pi_s^{\mathcal{P}}(y)$. Therefore $\pi_r^{\mathcal{P}}(z) = \pi_r^{\mathcal{P}}(x)$ and $\pi_s^{\mathcal{P}}(z) = \pi_s^{\mathcal{P}}(y)$. Since $r \leq s$, the previous exercise implies that $\pi_r^{\mathcal{P}}(z) = \pi_r^{\mathcal{P}}(y)$. Since $\mathcal{P}$ is decreasing, we have that

$$\begin{aligned}\pi_s^{\mathcal{P}}(y) &\subset \pi_r^{\mathcal{P}}(y) \\ &= \pi_r^{\mathcal{P}}(z) \\ &= \pi_r^{\mathcal{P}}(x)\end{aligned}$$

□

Exercise 4.10.0.15. Let X be a set, Γ a totally ordered set and $\mathcal{P} : \Gamma \to \mathrm{Part}(X)$. Suppose that $\mathcal{P}$ is decreasing. Let $x, y \in X$. Suppose that there exists $r \in \Gamma$ such that $\pi_r^{\mathcal{P}}(x) \neq \pi_r^{\mathcal{P}}(y)$. Then for each $s \geq r$, $\pi_s^{\mathcal{P}}(x) \neq \pi_s^{\mathcal{P}}(y)$.

Proof. Let $x, y \in X$. Let $s \geq r$. Since $\mathcal{P}$ is decreasing, there exist $\mathcal{F}_x, \mathcal{F}_y \subset X$ such that $\pi_r^{\mathcal{P}}(x) = \bigcup\limits_{z \in \mathcal{F}_x} \pi_s^{\mathcal{P}}(z)$ and $\pi_r^{\mathcal{P}}(y) = \bigcup\limits_{w \in \mathcal{F}_y} \pi_s^{\mathcal{P}}(w)$. Since $\pi_r^{\mathcal{P}}(x) \cap \pi_r^{\mathcal{P}}(y) = \varnothing$, we have that for each $z \in \mathcal{F}_x$ and $w \in \mathcal{F}_y$, $\pi_s^{\mathcal{P}}(z) \cap \pi_s^{\mathcal{P}}(w) = \varnothing$. Since $\mathcal{P}_s$ is a partition of X, there exist $x' \in \mathcal{F}_x$ and $y' \in \mathcal{F}_y$ such that $\pi_s^{\mathcal{P}}(x') = \pi_s^{\mathcal{P}}(x)$ and $\pi_s^{\mathcal{P}}(y') = \pi_s^{\mathcal{P}}(y)$. Therefore $\pi_s^{\mathcal{P}}(x) \cap \pi_s^{\mathcal{P}}(y) = \varnothing$ and thus $\pi_s^{\mathcal{P}}(x) \neq \pi_s^{\mathcal{P}}(y)$. □

Definition 4.10.0.16. Let X be a set and $\mathcal{P} : (0, \infty) \to \mathrm{Part}(X)$. For $x, y \in X$, we define

$$A^{\mathcal{P}}(x,y) = \{r \in (0,\infty) : \pi_r^{\mathcal{P}}(x) = \pi_r^{\mathcal{P}}(y)\} \quad \text{and} \quad \alpha^{\mathcal{P}}(x,y) = \sup A^{\mathcal{P}}(x,y)$$

Then $\mathcal{P}$ is said to be **left-continuous** if for each $x, y \in X$, $A^{\mathcal{P}}(x,y) \neq \varnothing$ and $\alpha^{\mathcal{P}}(x,y) < \infty$ implies that $\alpha^{\mathcal{P}}(x,y) \in A^{\mathcal{P}}(x,y)$

Definition 4.10.0.17. Let X be a set and $\mathcal{P} : \mathbb{N} \to \mathrm{Part}(X)$. For $x, y \in X$, we define the **left-continuous extension of** $\mathcal{P}$ denoted $\bar{\mathcal{P}} : (0, \infty) \to \mathrm{Part}(X)$, by

$$\bar{\mathcal{P}}_r = \mathcal{P}_{\lceil r \rceil}$$

Exercise 4.10.0.18. Let X be a set and $\mathcal{P} : \mathbb{N} \to \mathrm{Part}(X)$. Then $\bar{\mathcal{P}}$ is left-continuous.

Proof. Let $x, y \in X$. Suppose that $A^{\bar{\mathcal{P}}}(x,y) \neq \varnothing$ and $\alpha^{\bar{\mathcal{P}}}(x,y) < \infty$. Set $s = \alpha^{\bar{\mathcal{P}}}(x,y)$.

- For the sake of contradiction, suppose that $s \neq \lceil s \rceil$. Then $\lfloor s \rfloor < s < \lceil s \rceil$. Set $\epsilon = 2^{-1} \min(s - \lfloor s \rfloor, \lceil s \rceil - s)$. Then $\epsilon > 0$, $s - \epsilon \in (\lfloor s \rfloor, s)$ and $s + \epsilon \in (s, \lceil s \rceil)$. Since $s - \epsilon \in (\lfloor s \rfloor, s)$, there exists $r \in A^{\bar{\mathcal{P}}}(x,y)$ such that $r \in (s - \epsilon, s]$. Set $t = s + \epsilon$. Since $r, t \in (\lfloor s \rfloor, \lceil s \rceil)$, we have that

$$\lceil r \rceil, \lceil t \rceil = \lceil s \rceil$$

  Therefore, the definition of $\bar{\mathcal{P}}$ implies that

$$\begin{aligned} \pi_t^{\bar{\mathcal{P}}}(x) &= \pi_{\lceil t \rceil}^{\mathcal{P}}(x) \\ &= \pi_{\lceil s \rceil}^{\mathcal{P}}(x) \end{aligned}$$

  Similarly, $\pi_t^{\bar{\mathcal{P}}}(y) = \pi_{\lceil s \rceil}^{\mathcal{P}}(y)$, $\pi_r^{\bar{\mathcal{P}}}(x) = \pi_{\lceil s \rceil}^{\mathcal{P}}(x)$ and $\pi_r^{\bar{\mathcal{P}}}(y) = \pi_{\lceil s \rceil}^{\mathcal{P}}(y)$.
  Since $r \in A^{\bar{\mathcal{P}}}(x,y)$, we have that $\pi_r^{\bar{\mathcal{P}}}(x) = \pi_r^{\bar{\mathcal{P}}}(y)$. By definition of $\bar{\mathcal{P}}$, we have that

$$\begin{aligned} \pi_t^{\bar{\mathcal{P}}}(x) &= \pi_{\lceil s \rceil}^{\mathcal{P}}(x) \\ &= \pi_r^{\bar{\mathcal{P}}}(x) \\ &= \pi_r^{\bar{\mathcal{P}}}(y) \\ &= \pi_{\lceil s \rceil}^{\mathcal{P}}(y) \\ &= \pi_t^{\bar{\mathcal{P}}}(y) \end{aligned}$$

  Hence $t \in A^{\bar{\mathcal{P}}}(x,y)$. This is a contradiction since

$$\begin{aligned} \sup A^{\bar{\mathcal{P}}}(x,y) &= s \\ &< t \\ &\leq \sup A^{\bar{\mathcal{P}}}(x,y) \end{aligned}$$

  Hence $s = \lceil s \rceil$ and $s \in \mathbb{N}$.

- Choose $r \in A^{\bar{\mathcal{P}}}(x,y)$ such that $r \in (s-1, s]$. Then $\lceil r \rceil = s$, and

$$\begin{aligned} \pi_s^{\bar{\mathcal{P}}}(x) &= \pi_{\lceil s \rceil}^{\mathcal{P}}(x) \\ &= \pi_s^{\mathcal{P}}(x) \\ &= \pi_{\lceil r \rceil}^{\mathcal{P}}(x) \\ &= \pi_r^{\bar{\mathcal{P}}}(x) \end{aligned}$$

Similarly, $\pi_s^{\bar{\mathcal{P}}}(y) = \pi_r^{\bar{\mathcal{P}}}(y)$. Since $r \in A^{\bar{\mathcal{P}}}(x, y)$, $\pi_r^{\bar{\mathcal{P}}}(x) = \pi_r^{\bar{\mathcal{P}}}(y)$. Hence

$$\begin{aligned} \pi_s^{\bar{\mathcal{P}}}(x) &= \pi_r^{\bar{\mathcal{P}}}(x) \\ &= \pi_r^{\bar{\mathcal{P}}}(y) \\ &= \pi_s^{\bar{\mathcal{P}}}(y) \end{aligned}$$

Hence

$$\begin{aligned} \alpha^{\bar{\mathcal{P}}}(x, y) &= s \\ &\in A^{\bar{\mathcal{P}}}(x, y) \end{aligned}$$

Since $x, y \in X$ with $A^{\bar{\mathcal{P}}}(x, y) \neq \varnothing$ and $\alpha^{\bar{\mathcal{P}}}(x, y) < \infty$ are arbitrary, $\bar{\mathcal{P}}$ is left-continuous. $\square$

Definition 4.10.0.19. Let X be a set and $\mathcal{P} : (0, \infty) \to \mathrm{Part}(X)$. Then $\mathcal{P}$ is said to be **ultrametric-equivalent** if

1. $\mathcal{P}$ separates points
2. $\mathcal{P}$ collects points
3. $\mathcal{P}$ is decreasing
4. $\mathcal{P}$ is left-continuous

Exercise 4.10.0.20. Let X be a set and $\mathcal{P} : \mathbb{N} \to \mathrm{Part}(X)$. Suppose that $\mathcal{P}$ separates points, collects points and is decreasing. Then $\bar{\mathcal{P}}$ is ultrametric-equivalent.

Proof.

1. Let $x, y \in X$. Suppose that $x \neq y$. Since $\mathcal{P}$ separates points, there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} \pi_n^{\bar{\mathcal{P}}}(x) &= \pi_n^{\mathcal{P}}(x) \\ &\neq \pi_n^{\mathcal{P}}(y) \\ &= \pi_n^{\bar{\mathcal{P}}}(y) \end{aligned}$$

   Since $x, y \in X$ with $x \neq y$ are arbitrary, $\bar{\mathcal{P}}$ separates points.

2. Let $x, y \in X$. Since $\mathcal{P}$ collects points, there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} \pi_n^{\bar{\mathcal{P}}}(x) &= \pi_n^{\mathcal{P}}(x) \\ &= \pi_n^{\mathcal{P}}(y) \\ &= \pi_n^{\bar{\mathcal{P}}}(y) \end{aligned}$$

   Since $x, y \in X$ are arbitrary, $\bar{\mathcal{P}}$ collects points.

3. Let $r, s \in (0, \infty)$. Suppose that $r \leq s$. Let $x \in X$. Since $r \leq s$, we have that $\lceil r \rceil \leq \lceil s \rceil$. Since $\mathcal{P}$ is decreasing, there exists $\mathcal{F}_x \subset X$ such that

$$\begin{aligned} \pi_r^{\bar{\mathcal{P}}}(x) &= \pi_{\lceil r \rceil}^{\mathcal{P}}(x) \\ &= \bigcup_{y \in \mathcal{F}_x} \pi_{\lceil s \rceil}^{\mathcal{P}}(y) \\ &= \bigcup_{y \in \mathcal{F}_x} \pi_s^{\bar{\mathcal{P}}}(y) \end{aligned}$$

   Since $r, s \in (0, \infty)$ with $r \leq s$ and $x \in X$ are arbitrary, we have that $\bar{\mathcal{P}}$ is decreasing.

4. The previous exercise implies that $\bar{\mathcal{P}}$ is left continuous.

Since $\bar{\mathcal{P}}$ separates points, collects points, is decreasing and is left continuous, $\bar{\mathcal{P}}$ is ultrametric-equivalent. $\square$

Exercise 4.10.0.21. Let X be a set and $\mathcal{P} : (0, \infty) \to \text{Part}(X)$. Suppose that $\mathcal{P}$ is ultrametric-equivalent. Then for each $x, y \in X$, if $x \neq y$, then $\alpha^{\mathcal{P}}(x, y)$ exists and $A^{\mathcal{P}}(x, y) = [0, \alpha^{\mathcal{P}}(x, y)]$

Proof. Let $x, y \in X$. Suppose that $x \neq y$. Since $\mathcal{P}$ collects points, there exists $r > 0$ such that $\pi_r^{\mathcal{P}}(x) = \pi_r^{\mathcal{P}}(y)$. Hence $A^{\mathcal{P}}(x, y) \neq \varnothing$. Since $\mathcal{P}$ separates points, there exists $r > 0$ such that $\pi_r^{\mathcal{P}}(x) \neq \pi_r^{\mathcal{P}}(y)$. The previous exercise implies that $A^{\mathcal{P}}(x, y) \subset [0, r)$. Since $A^{\mathcal{P}}(x, y)$ is nonempty and bounded above, $\alpha^{\mathcal{P}}(x, y)$ exists and $\alpha^{\mathcal{P}}(x, y) < \infty$. By definition of the supremum, $A^{\mathcal{P}}(x, y) \subset [0, \alpha^{\mathcal{P}}(x, y)]$. Since $\mathcal{P}$ is left-continuous, $\alpha^{\mathcal{P}}(x, y) \in A^{\mathcal{P}}(x, y)$. A previous exercise implies that $[0, \alpha^{\mathcal{P}}(x, y)] \subset A^{\mathcal{P}}(x, y)$. Hence $A^{\mathcal{P}}(x, y) = [0, \alpha^{\mathcal{P}}(x, y)]$. $\square$

Exercise 4.10.0.22. Fundamental Example:
Let X be a set and $\mathcal{P} : (0, \infty) \to \text{Part}(X)$. Suppose that $\mathcal{P}$ is ultrametric-equivalent. Define $d^{\mathcal{P}} : X \times X \to (0, \infty)$ by

$$d^{\mathcal{P}}(x, y) = \begin{cases} e^{-a(x,y)} & x \neq y \\ 0 & x = y \end{cases}$$

Then $d^{\mathcal{P}}$ is an ultrametric on X.

Proof. Let $x, y \in X$.

1. Suppose that $x \neq y$. Since $\sim_{\mathcal{P}_r}$ is symmetric, $\alpha^{\mathcal{P}}(x, y) = \alpha^{\mathcal{P}}(y, x)$. Hence $d^{\mathcal{P}}(x, y) = d^{\mathcal{P}}(y, x)$. If $x = y$, then $d^{\mathcal{P}}(x, y) = d^{\mathcal{P}}(y, x)$.

2. By definition, $d^{\mathcal{P}}(x, y) = 0$ iff $x = y$.

3. If $x = z$, $x = y$ or $y = z$, then $d^{\mathcal{P}}(x, z) \leq \max(d^{\mathcal{P}}(x, y), d^{\mathcal{P}}(y, z))$. Suppose that $x \neq z$, $x \neq y$ and $y \neq z$. Then $d^{\mathcal{P}}(x, z) \neq 0$, $d^{\mathcal{P}}(x, y) \neq 0$ and $d^{\mathcal{P}}(y, z) \neq 0$.
   Suppose that $d^{\mathcal{P}}(x, y) \leq d^{\mathcal{P}}(y, z)$. Then $\alpha^{\mathcal{P}}(y, z) \leq \alpha^{\mathcal{P}}(x, y)$. The previous exercises imply that $\alpha^{\mathcal{P}}(y, z) \in A^{\mathcal{P}}(x, y) \cap A^{\mathcal{P}}(y, z)$. Hence

$$\begin{aligned} \pi^{\mathcal{P}}_{\alpha^{\mathcal{P}}(y,z)}(x) &= \pi^{\mathcal{P}}_{\alpha^{\mathcal{P}}(y,z)}(y) \\ &= \pi^{\mathcal{P}}_{\alpha^{\mathcal{P}}(y,z)}(z) \end{aligned}$$

   Hence $\alpha^{\mathcal{P}}(y, z) \in A^{\mathcal{P}}(x, z)$. Therefore $\alpha^{\mathcal{P}}(y, z) \leq \alpha^{\mathcal{P}}(x, z)$ which implies that

$$\begin{aligned} d^{\mathcal{P}}(x, z) &\leq d^{\mathcal{P}}(y, z) \\ &= \max(d^{\mathcal{P}}(x, y), d^{\mathcal{P}}(y, z)) \end{aligned}$$

   Simlilarly, if $d^{\mathcal{P}}(y, z) \leq d^{\mathcal{P}}(x, y)$, then

$$\begin{aligned} d^{\mathcal{P}}(x, z) &\leq d^{\mathcal{P}}(x, y) \\ &= \max(d^{\mathcal{P}}(x, y), d^{\mathcal{P}}(y, z)) \end{aligned}$$

Hence $d^{\mathcal{P}}$ satisfies the strong triangle inequality. Therefore $d^{\mathcal{P}}$ is an ultrametric on X. $\square$

Definition 4.10.0.23. Let (X, d) be an ultrametric space. We define $\mathcal{P}^d : (0, \infty) \to \text{Part}(X)$ by

$$\mathcal{P}_r^d = \{\bar{\pi}^d_{r^{-1}}(x) : x \in X\}$$

Exercise 4.10.0.24. Let (X, d) be an ultrametric space. Then $\mathcal{P}^d$ is ultrametric-equivalent.

Proof.

1. Let $x, y \in X$. Suppose that $x \neq y$. Then $d(x,y) > 0$. Set $r = 2d(x,y)^{-1}$. Then

$$\begin{aligned} d(x,y) &> 2^{-1}d(x,y) \\ &= r^{-1} \end{aligned}$$

Hence $x \not\simeq_{r^{-1}} y$ and therefore

$$\begin{aligned} \pi_r^{\mathcal{P}^d}(x) \cap \pi_r^{\mathcal{P}^d}(y) &= \bar{B}(x, r^{-1}) \cap \bar{B}(y, r^{-1}) \\ &= \varnothing \end{aligned}$$

Since $x, y \in X$ with $x \neq y$ are arbitrary, $\mathcal{P}^d$ separates points.

2. Let $x, y \in X$. Set $r = (d(x,y)+1)^{-1}$. Then $d(x,y) \leq r^{-1}$. So $x \simeq_{r^{-1}} y$ and therefore

$$\begin{aligned} \pi_r^{\mathcal{P}^d}(x) &= \bar{B}(x, r^{-1}) \\ &= \bar{B}(y, r^{-1}) \\ &= \pi_r^{\mathcal{P}^d}(y) \end{aligned}$$

Since $x, y \in X$ are arbitrary, $\mathcal{P}^d$ collects points.

3. Let $r, s \in (0, \infty)$. Suppose that $r \leq s$. Then $s^{-1} \leq r^{-1}$. Let $x \in X$. Choose $\mathcal{F}_x = \pi_r^{\mathcal{P}^d}(x)$. Let $y \in \mathcal{F}_x$. By definition of $\mathcal{P}^d$ and a previous exercise,

$$\begin{aligned} \pi_r^{\mathcal{P}^d}(x) &= \pi_{r^{-1}}^d(x) \\ &= \bar{B}(x, r^{-1}) \end{aligned}$$

and

$$\begin{aligned} \pi_s^{\mathcal{P}^d}(y) &= \pi_{s^{-1}}^d(y) \\ &= \bar{B}(y, s^{-1}) \end{aligned}$$

Since $s^{-1} \leq r^{-1}$, the previous exercise implies that

$$\begin{aligned} \pi_s^{\mathcal{P}^d}(y) &= \bar{B}(y, s^{-1}) \\ &\subset \bar{B}(x, r^{-1}) \\ &= \pi_r^{\mathcal{P}^d}(x) \end{aligned}$$

Since $y \in \mathcal{F}_x$ is arbitrary,

$$\pi_r^{\mathcal{P}^d}(x) = \bigcup_{y \in \mathcal{F}_x} \pi_s^{\mathcal{P}^d}(y)$$

Therefore $\mathcal{P}^d$ is decreasing.

4. Let $x, y \in X$. Suppose that $A^{\mathcal{P}^d}(x,y) \neq \varnothing$ and $\alpha^{\mathcal{P}^d}(x,y) < \infty$. For the sake of contradiction, suppose that $\alpha^{\mathcal{P}^d}(x,y) \notin A^{\mathcal{P}^d}(x,y)$. Then there exists $(\alpha_n)_{n\in\mathbb{N}} \subset A^{\mathcal{P}^d}(x,y)$ such that for each $n \in \mathbb{N}$, $\alpha^{\mathcal{P}^d}(x,y) = \sup\limits_{n\in\mathbb{N}} \alpha_n$ and $\alpha_n \neq \alpha^{\mathcal{P}^d}(x,y)$. Let $n \in \mathbb{N}$. Since $\alpha_n \in A^{\mathcal{P}^d}(x,y)$,

$$\begin{aligned} \pi_{\alpha_n}^{\mathcal{P}^d}(x) = \pi_{\alpha_n}^{\mathcal{P}^d}(y) &\implies \bar{B}(x, \alpha_n^{-1}) = \bar{B}(y, \alpha_n^{-1}) \\ &\implies d(x,y) \leq \alpha_n^{-1} \end{aligned}$$

Since $n \in \mathbb{N}$ is arbitrary,

$$\begin{aligned} d(x,y) &\leq \inf_{n \in \mathbb{N}} \alpha_n^{-1} \\ &= (\sup_{n \in \mathbb{N}} \alpha_n)^{-1} \\ &= \alpha^{\mathcal{P}^d}(x,y)^{-1} \end{aligned}$$

Hence

$$\begin{aligned} \pi^{\mathcal{P}^d}_{\alpha^{\mathcal{P}^d}(x,y)}(x) &= \bar{B}(x, \alpha^{\mathcal{P}^d}(x,y)^{-1}) \\ &= \bar{B}(y, \alpha^{\mathcal{P}^d}(x,y)^{-1}) \\ &= \pi^{\mathcal{P}^d}_{\alpha^{\mathcal{P}^d}(x,y)}(y) \end{aligned}$$

Therefore $\alpha^{\mathcal{P}^d}(x,y) \in A^{\mathcal{P}^d}(x,y)$ and $\mathcal{P}^d$ is left-continuous

Hence $\mathcal{P}^d$ is ultrametric-equivalent. □

Exercise 4.10.0.25. Let (X,d) be an ultrametric space. Then $d^{\mathcal{P}^d} \sim_{\mathbf{Top}} d$.
FINISH!!!

Exercise 4.10.0.26. Conjecture:
Let (X,d) be an ultrametric space. Then there exists $\mathcal{P} : \mathbb{N} \to \text{Part}(X)$ such that $\bar{\mathcal{P}} = \mathcal{P}^d$ iff d is bounded above and $d(X \times X) \setminus \{0\}$ is discrete.

Proof.

- $(\implies)$:
  Suppose that there exists $\mathcal{P} : \mathbb{N} \to \text{Part}(X)$ such that $\bar{\mathcal{P}} = \mathcal{P}^d$
- $(\impliedby)$:

□

want to categorize when a discrete valued metric is basically a tree.

4.11 LEFTOVERS

Definition 4.11.0.1. Let X be a metric space. Then X is said to be **separable** if there exists $D \subset X$ such that D is countable and for each $x \in X$ and $\epsilon > 0$, there exists $y \in D$ such that $d(x, y) < \epsilon$.

Exercise 4.11.0.2. Let X be a metric space. If X is separable, then for each $A \subset X$, if A is open, then

1. there there exist $(x_n)_{n\in\mathbb{N}} \subset X$ and $(r_n)_{n\in\mathbb{N}} \subset (0, \infty)$ such that

$$A = \bigcup_{n\in\mathbb{N}} B(x_n, r_n)$$

   i.e. A is a countable union of open balls

2. there there exist $(x_n)_{n\in\mathbb{N}} \subset X$ and $(r_n)_{n\in\mathbb{N}} \subset (0, \infty)$ such that

$$A = \bigcup_{n\in\mathbb{N}} \bar{B}(x_n, r_n)$$

   i.e. A is a countable union of closed balls.

Proof. Suppose that X is separable. Then there exists $(x_n)_{n\in\mathbb{N}} \subset X$ such that for each $x \in X$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x, x_N) < \epsilon$. Let $A \subset X$. Suppose that A is open.

1. Set

$$\mathcal{B} = \{B(x_n, r) : r \in \mathbb{Q} \text{ and } B(x_n, r) \subset A\}$$

   Note that $\mathcal{B}$ is countable. Let $x \in A$. Since A is open, there exists $s \in \mathbb{R}$ such that $B(x, s) \subset A$. Then there exists $r \in \mathbb{Q} \cap (0, r)$. Choose $N \in \mathbb{N}$ such that $d(x, x_N) < r/2$. Let $y \in B(x_N, r/2)$, then

$$\begin{aligned} d(x, y) &\leq d(x, x_N) + d(x_N, y) \\ &< r/2 + r/2 \\ &= r \end{aligned}$$

   Therefore

$$\begin{aligned} x &\in B(x_N, r/2) \\ &\subset B(x, r) \\ &\subset A \end{aligned}$$

   Hence $B(x_N, r/2) \in \mathcal{B}$ and $x \in \bigcup_{B\in\mathcal{B}} B$. Since $x \in A$ is arbitrary, $A \subset \bigcup_{B\in\mathcal{B}} B$.

2. Similar, but take $r/4$ instead of $r/2$.

□

Definition 4.11.0.3. Let X be a set, $d_1, d_2 : X \times X \to [0, \infty)$ metrics on X. Then d_1 and d_2 are said to be **equivalent** if there exist $A, B > 0$ such that

$$Ad_1 \leq d_2 \leq Bd_1$$

Definition 4.11.0.4. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Then f is said to be **Lipchitz** if there exists $K \geq 0$ such that for each $a, b \in X$,

$$d_Y(f(a), f(b)) \leq K d_X(a, b)$$

Exercise 4.11.0.5. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. If f is Lipchitz, then f is uniformly continuous.

Proof. By definition, there exists $K \geq 0$ such that for each $a, b \in X$,

$$d_Y(f(a), f(b)) \leq K d_X(a, b)$$

Let $\epsilon > 0$. Choose $\delta = \epsilon/(K+1)$. Let $a, b \in X$. Suppose that $d_X(a, b) < \delta$. Then

$$\begin{aligned} d_Y(f(a), f(b)) &\leq K d_X(a, b) \\ &< K\delta \\ &= K\frac{\epsilon}{K+1} \\ &< \epsilon \end{aligned}$$

□

Definition 4.11.0.6. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ and $x_0 \in X$. Then f is said to be **locally Lipchitz at** x_0 if there exists $U \in \mathcal{N}(x_0)$ such that f is Lipschitz on U.

Definition 4.11.0.7. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Then f is said to be **locally Lipchitz** if for each $x_0 \in X$, f is locally Lipschitz at x_0.

Definition 4.11.0.8. Let X, Y be metric spaces and $T : X \to Y$. Then T is said to be an **isometry** if for each $x_1, x_2 \in X$, $d(Tx_1, Tx_2) = d(x_1, x_2)$.

Exercise 4.11.0.9. Let X, Y be metric spaces and $T : X \to Y$ and isometry. Then T is injective.

Proof. Let $x_1, x_2 \in X$. Suppose that $Tx_1 = Tx_2$. Then $0 = d(Tx_1, Tx_2) = d(x_1, x_2)$. So $x_1 = x_2$. Hence T is injective. □

Note 4.11.0.10. Let X, Y be metric spaces and $T : X \to Y$ an isometry. Then T is clearly continuous. If T is surjective, then T^{-1} is an isometry and therefore continuous. Hence T is a homeomorphism.

Definition 4.11.0.11. Let (X, d) be a metric space. Then (X, d) is said to be a **Polish space** if (X, d) is complete and separable.

Exercise 4.11.0.12. Let (X, d) be a compact metric space, $E \subset X$ closed, $U \subset X$ open. Suppose that $E \subset U$. Then there exists $\delta > 0$ such that for each $x \in E$, $B(x, \delta) \subset U$.

Proof. Since X is compact, E and U^c are compact. Then there exist $x_0 \in E$ and $y_0 \in U^c$ such that $d(E, U^c) = d(x_0, y_0)$. Since $E \cap U^c = \varnothing$, $x_0 \neq y_0$ and $d(E, U^c) > 0$. Put $\epsilon = d(E, U^c)$ and $\delta = \frac{\epsilon}{2}$. Let $x \in E$, $w \in B(x, \delta)$ and $y \in U^c$. Then

$$\begin{aligned} d(y, w) &\geq d(y, x) - d(x, w) \\ &> \epsilon - \delta \\ &= \epsilon - \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2} \\ &> 0 \end{aligned}$$

So $y \neq w$. Since and $y \in U^c$ and $w \in B(x, \delta)$ are arbitrary, $B(x, \delta) \subset U$. □

Definition 4.11.0.13. Let S be a set, (X, d) a metric space and $B(S, X) = \{f : S \to X : f \text{ is bounded}\}$. We define the **supremum metric**, denoted $d_u : B(S, X) \times B(S, X) \to [0, \infty)$, by

$$d_u(f, g) = \sup_{x \in X} d(f(x), g(x))$$

Exercise 4.11.0.14. Let X be a set, (Y, d_Y), (Z, d_Z) metric spaces, $(f_n)_{n \in \mathbb{N}} \subset B(X, Y)$, $f \in B(X, Y)$ and $g \in C(Y, Z)$. Suppose that g is uniformly continuous. If $f_n \xrightarrow{u} f$, then $g \circ f_n \xrightarrow{u} g \circ f$.

Proof. Suppose that $f_n \xrightarrow{\text{u}} f$. Let $\epsilon > 0$. Uniform continuity of g implies that there exists $\delta > 0$ such that for each $y_1, y_2 \in Y$, $d_Y(y_1, y_2) < \delta$ implies that $d_Z(g(y_1), g(y_2)) < \epsilon/2$. Uniform convergence implies that there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq \mathbb{N}$ implies that $d_u(f_n, f) < \delta/2$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Let $x \in X$. Then $d_Y(f_n(x), f(x)) < \delta$. This implies that $d_Z(g(f_n(x)), g(f(x))) < \epsilon/2$. Hence $\sup\limits_{x \in X} d_Z(g \circ f_n(x), g \circ f(x)) \leq \epsilon/2$. Thus $d_u(g \circ f_n, g \circ f) < \epsilon$. So $g \circ f_n \xrightarrow{\text{u}} g \circ f$. □

Definition 4.11.0.15. Let (X, d) be a metric space. Define

1. $\text{Aut}(X) = \{\sigma : X \to X : \sigma \text{ is a homeomorphism}\}$
2. $\text{Aut}(X, d) = \{\sigma : X \to X : \sigma \text{ is an isometric isomorphism}\}$

Exercise 4.11.0.16. Let (X, d) be a compact metric space, $E \subset X$ closed, $U \subset X$ open. Suppose that $E \subset U$. Let $(f_n)_{n \in \mathbb{N}} \in \text{Aut}(X)$, $f \in \text{Aut}(X)$. Suppose that $f_n \xrightarrow{\text{u}} f$. Then there exists $N \in \mathbb{N}$ such that for each $n \geq N$, $f(E) \subset f_n(U)$.

Proof. Since f is a homeomorphism, E is closed and U is open, $f(E)$ is compact and $f(U)$ is open and $f(E) \subset f(U)$. Then $d(f(E), f(U^c)) > 0$. Put $\epsilon = d(f(E), f(U^c))$. Choose $\delta = \epsilon/2$. Then there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $\sup\limits_{z \in X} d(f(z), f_n(z)) < \delta$. Let $n \geq N$, $x \in E$ and $w \in B(f(x), \delta)$. For the sake of contradiction, suppose that $w \in f_n(U^c)$. Then there exist $p \in U^c$ such that $w = f_n(p)$. Put $z = f(p) \in f(U^c)$. Then

$$\begin{aligned}
\epsilon &\leq d(f(x), z) \\
&\leq d(f(x), w) + d(w, z) \\
&= d(f(x), w) + d(f_n(p), f(p)) \\
&< \delta + \delta \\
&= \epsilon
\end{aligned}$$

which is a contradiction. So $w \in f_n(U)$. Hence $B(f(x), \delta) \subset f_n(U)$ □

Chapter 5

Topological Vector Spaces

5.1 Introduction

Definition 5.1.0.1. Let X be a vector space and $\mathcal{T}$ a topology on X. Then X is said to be a **topological vector space** if

1. addition $X \times X \to X$ is continuous
2. scalar multiplication $\mathbb{C} \times X \to X$ is continuous

Note 5.1.0.2. We usually suppress the topology $\mathcal{T}$.

Exercise 5.1.0.3. Let X be a topological vector space, $(\lambda_\alpha)_{\alpha \in A} \subset \mathbb{C}$, $(x_\alpha)_{\alpha \in A}$, $(y_\alpha)_{\alpha \in A} \subset X$ nets and $\lambda \in \mathbb{C}$, $x, y \in X$. If $\lambda_\alpha \to \lambda$, $x_\alpha \to x$ and $y_\alpha \to y$, then $x_\alpha + \lambda_\alpha y_\alpha \to x + \lambda y$.

Proof. Clear since addition and scalar multiplication are continuous. □

Exercise 5.1.0.4. Let X be a topological vector space, $y \in X$ and $\lambda \in \mathbb{C}^\times$. Define $f, g : X \to X$ by $f(x) = x + y$ and $g(x) = \lambda x$. Then f and g are homeomorphisms.

Proof. Since X is a topological vector space, f and g are continuous. Clearly f and g are bijections with $f^{-1}(x) = x - y$ and $g^{-1}(x) = \lambda^{-1}x$. Again, since X is a topological vector space, f^{-1} and g^{-1} are continuous. □

Exercise 5.1.0.5. Let X be a topological vector space. Then X is Hausdorff iff $\{0\}$ is closed.

Proof. An exercise in a previous section implies that X is Hausdorff iff for each $x \in X$, $\{x\}$ is closed. Thus, if X is Hausdorff, then $\{0\}$. Conversely, if $\{0\}$ is closed, then the previous exercise implies that for each $x \in X$, $\{x\}$ is closed. Hence X is Hausdorff. □

Exercise 5.1.0.6. Let $(\mathbb{C}, \mathcal{T})$ be a topological vector space.

Exercise 5.1.0.7. Let X be a topological vector space, $x, y \in X$ and $U \in \mathcal{N}(x)$. If U is open, then there exists $r > 0$ such that for each $t \in \mathbb{R}$, $|t| \leq r$ implies that $x + ty \in U$.

Proof. Suppose that U is open. For the sake of contradiction, suppose that for each $r > 0$, there exists $t \in \mathbb{R}$ such that $t \leq r$ and $x + ty \notin U$. Then for each $n \in \mathbb{N}$, there exists $t_n \in \mathbb{R}$ such that $|t_n| \leq 1/n$ and $x + t_n y \in U^c$. Since $t_n \to 0$,

$$\begin{aligned} x + t_n y &\to x + 0y \\ &= x \end{aligned}$$

Since U^c is closed, $x \in U^c$. This is a contradiction. Hence there exists $r > 0$ such that for each $t \in \mathbb{R}$, $|t| \leq r$ implies that $x + ty \in U$. □

Exercise 5.1.0.8. Let X be a topological vector space and A, $B \subset X$. If A is open, then $A+B$ is open.

Proof. Suppose that A is open. Then for each $b \in B$, $A+b$ is open. Since

$$A + B = \bigcup_{b \in B} A + b$$

we have that $A+B$ is open. $\square$

Exercise 5.1.0.9. Let X be a topological vector space and $A, B \subset X$. Suppose that A is compact, B is closed and $A \cap B = \varnothing$. Then there exists $U \in \mathcal{N}(0)$ such that U is open and $(A+U) \cap B = \varnothing$.

Proof. Set $\Gamma = \{U \in \mathcal{N}(0) : U \text{ is open}\}$ and order Γ by reverse inclusion, so that Γ is a directed set. For the sake of contradiction, suppose that for each $U \in \Gamma$, $(A+U) \cap B \neq \varnothing$. Then for each $\gamma \in \Gamma$, there exist $a_\gamma \in A$ and $u_\gamma \in \gamma$ such that $a_\gamma + u_\gamma \in B$. Let $V \in \mathcal{N}(0)$. Since $\operatorname{Int} V \in \Gamma$

$$\begin{aligned} u_{\operatorname{Int} V} &\in \operatorname{Int} V \\ &\subset V \end{aligned}$$

Since $V \in \mathcal{N}(0)$ is arbitrary, $u_\gamma \to 0$. Since A is compact, there exists $a \in A$ and a subnet $(a_{\gamma_\zeta})_{\zeta \in Z}$ of $(a_\gamma)_{\gamma \in \Gamma}$ such that $a_{\gamma_\zeta} \to a$. Then $a_{\gamma_\zeta} + u_{\gamma_\zeta} \to a$. Since $(a_{\gamma_\zeta} + u_{\gamma_\zeta})_{\zeta \in Z} \subset B$ and B is closed, we have that $a \in B$. This is a contradiction since $A \cap B = \varnothing$. So there exists $U \in \mathcal{N}(0)$ such that U is open and $(A+U) \cap B = \varnothing$. $\square$

Exercise 5.1.0.10. Let X be a topological vector space and $U \in \mathcal{N}(0)$. If U is open, then there exists $V \in \mathcal{N}(0)$ such that V is open and $V + V \subset U$.

Proof. Suppose that U is open. Set $\Gamma = \{V \in \mathcal{N}(0) : V \text{ is open}\}$ and order Γ by reverse inclusion, so that Γ is a directed set. For the sake of contradiction, suppose that for each $V \in \mathcal{N}(0)$, if V is open, then $V + V \not\subset U$. Then for each $\gamma \in \Gamma$, there exists $x_\gamma, y_\gamma \in \gamma$ such that $x_\gamma + y_\gamma \in U^c$. Let $W \in \mathcal{N}(0)$. Set $\beta = \operatorname{Int} V$. Then $\beta \in \Gamma$. Then for each $\gamma \geq \beta$,

$$\begin{aligned} x_\gamma, y_\gamma &\in \gamma \\ &\subset \beta \\ &\subset W \end{aligned}$$

So that $(x_\gamma)_{\gamma \in \Gamma}$ and $(y_\gamma)_{\gamma \in \Gamma}$ are eventually in W. Since $W \in \mathcal{N}(0)$ is arbitrary, $x_\gamma \to 0$ and $y_\gamma \to 0$. Therefore $x_\gamma + y_\gamma \to 0$. Since for each $\gamma \in \Gamma$, $x_\gamma + y_\gamma \in U^c$ and U^c is closed, $0 \in U^c$. This is a contradiction, so there exists $V \in \mathcal{N}(0)$ such that V is open and $V + V \subset U$. $\square$

Definition 5.1.0.11. Let X, Y be topological vector spaces. We define $L(X;Y) := \{T : X \to Y : T \text{ is linear and continuous}\}$.

Definition 5.1.0.12. Let X be a topological vector space over $\mathbb{K}$. We define the **dual space of** X, denoted X^*, by $X^* = \{T : X \to \mathbb{K} : T \text{ is linear and continuous}\}$

Exercise 5.1.0.13. Let X be a topological vector space. Then X^* is a vector space.

Proof. Clear. $\square$

Exercise 5.1.0.14. Let X, Y be topological vector spaces and $\phi : X \to Y$. Suppose that ϕ is linear. Then ϕ is continuous iff ϕ is continuous at 0.

Proof. If ϕ is continuous, then ϕ is continuous at 0.
Conversely, suppose that ϕ is continous at 0. Let $(x_\alpha)_{\alpha \in A} \subset X$ be a net and $x \in X$. Suppose that $x_\alpha \to x$. Then $x_\alpha - x \to 0$. Hence

$$\begin{aligned} \phi(x_\alpha) - \phi(x) &= \phi(x_\alpha - x) \\ &\to \phi(0) \\ &= 0 \end{aligned}$$

Therefore $\phi(x_\alpha) \to \phi(x)$ and ϕ is continuous at x. Since $x \in X$ is arbitrary, ϕ is continuous. $\square$

Exercise 5.1.0.15. Let X be a topological vector space and $\phi : X \to \mathbb{C}$ linear. Then $\phi \in X^*$ iff $|\phi|$ is continuous.

Proof. Suppose that ϕ is continuous. Since $|\cdot| : \mathbb{C} \to [0, \infty)$ is continuous, $|\phi|$ is continuous.
Conversely, suppose that $|\phi|$ is continuous. Let $(x_\alpha)_{\alpha \in A} \subset X$ be a net and $x \in X$. Suppose that $x_\alpha \to x$. Then $x_\alpha - x \to 0$. Therefore

$$\begin{aligned} |\phi(x_\alpha) - \phi(x)| &= |\phi(x_\alpha - x)| \\ &\to |\phi(0)| \\ &= 0 \end{aligned}$$

So $\phi(x_\alpha) \to \phi(x)$ and ϕ is continuous. □

Exercise 5.1.0.16. Let X be a real topological vector space and $\phi \in X^*$. If ϕ is not constant, then ϕ is open.
Hint: There exists $x_* \in X$ such that $\phi(x_*) = 1$ and for each $U \subset X$ open and $x \in U$, there exists $r > 0$ such that for each $t \in \mathbb{R}$, $|t| \leq r$ implies that $x + tx_* \in U$.

Proof. Suppose that ϕ is not constant. Then there exists $x_0 \in X$ such that $\phi(x_0) \neq 0$. Set $x_* = \phi(x_0)^{-1}x_0$. Then

$$\begin{aligned} \phi(x_*) &= \phi(\phi(x_0)^{-1}x_0) \\ &= \phi(x_0)^{-1}\phi(x_0) \\ &= 1 \end{aligned}$$

Let $U \subset X$ be open and $y \in \phi(U)$. Then there exists $x \in U$ such that $\phi(x) = y$. Sine U is open, a previous exercise implies that there exists $r > 0$ such that for each $t \in \mathbb{R}$, $\|t\| \leq r$ implies that $x + tx_* \in U$. Let $t \in (-r, r)$. Then $\phi(x + tx_*) \in \phi(U)$. Since

$$\begin{aligned} \phi(x + tx_*) &= \phi(x) + t\phi(x_*) \\ &= y + t \end{aligned}$$

we have that $(y - r, y + r) \subset \phi(U)$. Since $y \in U$ is arbitrary, $\phi(U)$ is open thus ϕ is open. □

Definition 5.1.0.17. Let X be a vector space and $\phi : X \to \mathbb{C}$. Then ϕ is said to be **real-linear** if for each $x, y \in X$ and $\lambda \in \mathbb{R}$, $\phi(x + \lambda y) = \phi(x) + \lambda\phi(y)$.

Exercise 5.1.0.18. Let X be a topological vector space and $\phi \in X^*$. Then $\operatorname{Re}\phi$ is continuous and real-linear.

Proof. Clear. □

Exercise 5.1.0.19. Let X be a topological vector space and $f : X \to \mathbb{R}$. If f is continuous and real-linear, then there exists a unique $\phi \in X^*$ such that $\operatorname{Re}\phi = f$.
Hint: For each $z \in \mathbb{C}$, $z = \pi_{\mathbb{R}}(z) - i\pi_{\mathbb{R}}(iz)$

Proof. Suppose that f is continuous and real-linear. Define $\phi : X \to \mathbb{C}$ by $\phi(x) = f(x) - if(ix)$. Then ϕ is continuous. Let $x, y \in X$ and $\lambda \in C$. Write $\lambda = a + bi$. Then

$$\begin{aligned} \phi(x + \lambda y) &= f(x + \lambda y) - if(i(x + \lambda y)) \\ &= f(x + ay + iby) - if(ix + iay - by) \\ &= f(x) + af(y) + bf(iy) - if(ix) - iaf(iy) + ibf(y) \\ &= [f(x) - if(ix)] + a[f(y) - if(iy)] + ib[f(y) - if(iy)] \\ &= \phi(x) + a\phi(y) + ibf(y) \\ &= \phi(x) + \lambda\phi(y) \end{aligned}$$

So ϕ is linear and $\phi \in X^*$. Let $\psi \in X^*$. Suppose that $f = \operatorname{Re}\psi$. Then for each $x \in X$,

$$\begin{aligned}
\phi(x) &= f(x) - if(ix) \\
&= \operatorname{Re}[\psi(x)] - i\operatorname{Re}[\psi(ix)] \\
&= \operatorname{Re}[\psi(x)] - \operatorname{Re}[i\psi(x)] \\
&= \operatorname{Re}[\psi(x)] + \operatorname{Im}[\psi(x)] \\
&= \psi(x)
\end{aligned}$$

So $\psi = \phi$ and ϕ is unique. □

5.2 Sublinear Functionals

Definition 5.2.0.1. Let X be a real vector space and $p : X \to \mathbb{R}$. Then p is said to be a **sublinear functional** if for each $x, y \in X$, $\lambda \geq 0$,

1. $p(x + y) \leq p(x) + p(y)$
2. $p(\lambda x) = \lambda p(x)$

Exercise 5.2.0.2. Let X be a vector space and $p : X \to \mathbb{R}$ be a sublinear functional. Then $p(0) = 0$.

Proof. Set $\lambda = 0$. Then

$$\begin{aligned} 0 &= \lambda p(0) \\ &= p(\lambda 0) \\ &= p(0) \end{aligned}$$

□

Proof. Clear □

Exercise 5.2.0.3. Let X be a vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then for each $x, y \in X$

1. $-p(-x) \leq p(x)$
2. $-p(y - x) \leq p(x) - p(y) \leq p(x - y)$

Proof. Let $x, y \in X$.

1. We have

$$\begin{aligned} 0 &= p(0) \\ &= p(x - x) \\ &\leq p(x) + p(-x) \end{aligned}$$

   So $-p(-x) \leq p(x)$.

2. We have

$$\begin{aligned} p(x) &= p(x - y + y) \\ &\leq p(x - y) + p(y) \end{aligned}$$

   So $p(x) - p(y) \leq p(x - y)$. Switching x and y gives us $p(y) - p(x) \leq p(y - x)$ and multiplying both sides by -1 yields $-p(y - x) \leq p(x) - p(y)$
   Putting these two together, we see that

$$-p(y - x) \leq p(x) - p(y) \leq p(x - y)$$

□

Theorem 5.2.0.4. Hahn-Banach Theorem for Sublinear Functionals
Let X be a vector space, $p : X \to \mathbb{R}$ a sublinear functional, $M \subset X$ a subspace and $f : M \to \mathbb{R}$ a linear functional. If for each $x \in M$, $f(x) \leq p(x)$, then there exists a linear functional $F : X \to \mathbb{R}$ such that for each $x \in X$, $F(x) \leq p(x)$ and $F|_M = f$.

Exercise 5.2.0.5. Let X be a vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then there exists a linear functional $F : X \to \mathbb{R}$ such that for each $x \in X$, $F(x) \leq p(x)$.

Proof. Take $M = \{0\}$ and $f \equiv 0$ and apply the Hahn-Banach theorem. □

Exercise 5.2.0.6. Equivalency of linearity (General Case) Let X be a vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then the following are equivalent:

1. there exists a unique $F \in X^*$ such that $F \leq p$
2. for each $x \in X$, $-p(-x) = p(x)$
3. p is linear

Hint: If there exists $x \in X$ such that $-p(-x) \neq p(x)$, define $f_1, f_2 : \text{span}(x) \to \mathbb{R}$ by $f_1(tx) = tp(x)$ and $f_2(tx) = -tp(-x)$

Proof.

- (1) $\implies$ (2):
  Suppose that there exists a unique $F \in X^*$ such that $F \leq p$. For the sake of contradiction, suppose that there exists $x \in X$ such that $-p(-x) \neq p(x)$. Define $f_1, f_2 : \text{span}(x) \to \mathbb{R}$ by
  $$f_1(tx) = tp(x)$$
  and
  $$f_2(tx) = -tp(-x)$$
  Let $y \in \text{span}(x)$. Then there exists $t \in \mathbb{R}$ such that $y = tx$. Then for each $k \in \mathbb{R}$,
  $$\begin{aligned} f_1(ky) &= f_1(ktx) \\ &= ktp(x) \\ &= kf_1(tx) \\ &= kf_1(y) \end{aligned}$$
  Similarly, $f_2(ky) = kf_2(y)$ and so $f_1, f_2 \in \text{span}(x)^*$. If $t \geq 0$, then
  $$\begin{aligned} f_1(y) &= f_1(tx) \\ &= tp(x) \\ &= p(tx) \\ &= p(y) \end{aligned}$$
  If $t < 0$, then
  $$\begin{aligned} f_1(y) &= f_1(tx) \\ &= tp(x) \\ &= -|t|p(x) \\ &= -p(|t|x) \\ &= -p(-tx) \\ &\leq p(tx) \\ &= p(y) \end{aligned}$$
  So $f_1 \leq p$ on $\text{span}(x)$. Similarly, $f_2 \leq p$ on $\text{span}(x)$. The Hahn-Banach theorem implies that there exist $F_1, F_2 \in X^*$ such that $F_1, F_2 \leq p$ and $F_1 = f_1, F_2 = f_2$ on $\text{span}(x)$. By the assumption of uniqueness, $F_1 = F_2$. This is a contradiction since
  $$\begin{aligned} F_1(x) &= p(x) \\ &\neq -p(-x) \\ &= F_2(x) \end{aligned}$$
  So for each $x \in X$, $-p(-x) = p(x)$.

- (2) $\Rightarrow$ (3):
  Suppose that for each $x \in X$, $-p(-x) = p(x)$. The previous exercise implies that there exists $F \in X^*$ such that $F \leq p$. Let $x \in X$. Then

$$\begin{aligned} -F(x) &= F(-x) \\ &\leq p(-x) \\ &= -p(x) \end{aligned}$$

  So $p(x) \leq F(x)$ and $p \leq F$. Therefore $p = F$ and p is linear.

- (3) $\implies$ (1):
  Suppose that p is linear. Let $F \in X^*$. Suppose that $F \leq p$. Let $x \in X$. Then as in the case for (2) $\implies$ (3), we have that

$$\begin{aligned} -F(x) &= F(-x) \\ &\leq p(-x) \\ &= -p(x) \end{aligned}$$

  which implies that $p = F$. So p is the unique linear function $F \in X^*$ such that $F \leq p$.

$\square$

5.3 Seminorms

Definition 5.3.0.1. Let X be a vector space and $p : X \to \mathbb{R}$. Then p is said to be a **seminorm** if for each $x, y \in X$, $\lambda \in \mathbb{R}$,

1. $p(x+y) \leq p(x) + p(y)$
2. $p(\lambda x) = |\lambda| p(x)$

Exercise 5.3.0.2. Let X be a vector space and $p : X \to \mathbb{R}$ be a seminorm, then p is a sublinear functional.

Proof. Clear □

Exercise 5.3.0.3. Let X be a vector space and $\phi \in X^*$. Then $|\phi|$ is a seminorm on X.

Proof. Clear. □

Exercise 5.3.0.4. Let X, Y be a vector spaces, $T \in L(X;Y)$ and p a seminorm on Y. Then $p \circ T$ is a seminorm on X.

Proof. Clear. □

Exercise 5.3.0.5. Let X be a vector space and $p : X \to \mathbb{R}$ be a seminorm. Then $p \geq 0$.

Proof. Let $x \in X$. Then

$$\begin{aligned} 0 &= p(0) \\ &= p(x-x) \\ &\leq p(x) + p(-x) \\ &= p(x) + p(x) \\ &= 2p(x) \end{aligned}$$

So $p(x) \geq 0$. □

Exercise 5.3.0.6. Reverse Triangle Inequality:
Let X be a vector space and $p : X \to [0, \infty)$ be a seminorm on X. Then for each $x, y \in X$, $|p(x) - p(y)| \leq p(x-y)$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} p(x) &= p(x-y+y) \\ &\leq p(x-y) + p(y) \end{aligned}$$

So $p(x) - p(y) \leq p(x-y)$. Similarly, $p(y) \leq p(y-x) + p(y)$ and so $p(x) - p(y) \leq p(x-y)$. Therefore $|p(x) - p(y)| \leq p(x-y)$. □

Exercise 5.3.0.7. Let X be a vector space, $p : X \to [0, \infty)$ a seminorm and $\phi \in X^*$. Then $\phi \leq p$ iff $|\phi| \leq p$.

Proof. Suppose that $\phi \leq p$. Let $x \in X$. Then

$$\begin{aligned} -\phi(x) &= \phi(-x) \\ &\leq p(-x) \\ &= p(x) \end{aligned}$$

So $-p(x) \leq \phi(x)$. Hence $-p \leq \phi \leq p$. Thus $|\phi| \leq p$.
Conversely, if $|\phi| \leq p$, then clearly $\phi \leq p$. □

Definition 5.3.0.8. Let X be a vector space and $p : X \to [0, \infty)$ be a seminorm on X. We define the **kernel of** p, denoted $\ker p$, by $\ker p = p^{-1}(\{0\})$.

Exercise 5.3.0.9. Let X be a vector space and $p : X \to [0, \infty)$ a seminorm. Then $\ker p$ is a subspace of X.

Proof. Let $x, y \in \ker p$ and $\lambda \in \mathbb{C}$. Then $p(x) = p(y) = 0$. Thus

$$\begin{aligned} p(x + \lambda y) &\leq p(x) + p(\lambda y) \\ &= p(x) + |\lambda| p(y) \\ &= 0 \end{aligned}$$

So $x + \lambda y \in N$ and N is a subspace. □

Definition 5.3.0.10. Let X be a vector space and $p : X \to [0, \infty)$ a seminorm on X. We define the **norm induced by** p, denoted $\bar{p} : X/\ker p \to [0, \infty)$, by

$$\bar{p}(\bar{x}) = p(x)$$

Exercise 5.3.0.11. Let X be a vector space and $p : X \to [0, \infty)$ a seminorm on X. Then $\bar{p} : X/\ker p \to [0, \infty)$ is well defined and a norm.

Proof. Let $x, y \in X$. Suppose that $\bar{x} = \bar{y}$. Then there exists $n \in \ker p$ such that $x = y + n$. Therefore,

$$\begin{aligned} \bar{p}(\bar{x}) &= p(x) \\ &= p(y + n) \\ &\leq p(y) + p(n) \\ &= p(y) \\ &= \bar{p}(\bar{y}) \end{aligned}$$

and

$$\begin{aligned} \bar{p}(\bar{y}) &= p(y) \\ &= p(x - n) \\ &\leq p(x) + p(n) \\ &= p(x) \\ &= \bar{p}(\bar{x}) \end{aligned}$$

So $\bar{p}(\bar{x}) = \bar{p}(\bar{y})$ and $\bar{p} : X/\ker p \to [0, \infty)$ is well defined. Let $x \in X$. Suppose that $\bar{x} = \bar{0}$. Then there exists $n \in \ker p$ such that $x = n$. Therefore

$$\begin{aligned} \bar{p}(\bar{x}) &= p(x) \\ &= p(n) \\ &= 0 \end{aligned}$$

So $\bar{p}$ is a norm. □

Definition 5.3.0.12. Let X be a vector space, $p : X \to [0, \infty)$ a seminorm on X, $x \in X$ and $r > 0$. We define the

- **open semiball of** p **at** x **of radius** r, denoted $B_p(x, r)$, by

$$B_p(x, r) = \{y \in X : p(x - y) < r\}$$

- **closed semiball of** p **at** x **of radius** r, denoted $\bar{B}_p(x, r)$, by

$$\bar{B}_p(x, r) = \{y \in X : p(x - y) \leq r\}$$

Exercise 5.3.0.13. Let X be a vector space, $p : X \to [0, \infty)$ a seminorm on X, $x \in X$ and $r > 0$. Then $B_p(x, r) = x + rB_p(0, 1)$.

Proof. Let $y \in B_p(x, r)$. Then

$$\begin{aligned} p(r^{-1}(y - x)) &= r^{-1}p(y - x) \\ &< r^{-1}r \\ &= 1 \end{aligned}$$

So $r^{-1}(y - x) \in B_p(0, 1)$. By definition, there exists $u \in B_p(0, 1)$ such that $r^{-1}(y - x) = u$, which implies that

$$\begin{aligned} y &= x + ru \\ &\in x + rB_p(0, 1) \end{aligned}$$

Conversely, let $y \in x + rB_p(0, 1)$. By definition, there exists $u \in B_p(0, 1)$ such that $y = x + ru$. Then

$$\begin{aligned} p(y - x) &= p(ru) \\ &= rp(u) \\ &< r \end{aligned}$$

So $y \in B_p(x, r)$ □

Exercise 5.3.0.14. Let X be a vector space and $p, q : X \to [0, \infty)$ seminorms on X. Then $p \leq q$ iff $B_q(0, 1) \subset B_p(0, 1)$.

Proof. Suppose that $p \leq q$. Let $x \in B_q(0, 1)$. Then

$$\begin{aligned} p(x) &\leq q(x) \\ &< 1 \end{aligned}$$

So $x \in B_p(0, 1)$.
Conversely, suppose that $B_q(0, 1) \subset B_p(0, 1)$. Let $x \in X$. If $p(x) = 0$, then $p(x) \leq q(x)$. Suppose that $p(x) > 0$. For the sake of contradiction, suppose that $p(x) > q(x)$. Then

$$\begin{aligned} q\left(\frac{x}{p(x)}\right) &= \frac{q(x)}{p(x)} \\ &< 1 \end{aligned}$$

Therefore, $x/p(x) \in B_q(0, 1) \subset B_p(0, 1)$. By definition,

$$\begin{aligned} \frac{p(x)}{p(x)} &= p\left(\frac{x}{p(x)}\right) \\ &< 1 \end{aligned}$$

which is a contradiction. So $p(x) \leq q(x)$. Since $x \in X$ is arbitrary, $p \leq q$. □

Exercise 5.3.0.15. Let X be a topological vector space and $p : X \to [0, \infty)$ a continuous seminorm. Then

1. $B_p(0, 1)$ is open
2. $\bar{B}_p(0, 1)$ is closed

Proof.

1. Let $(x_\alpha)_{\alpha \in A}$ be a net in $B_p(0, 1)^c$ and $x \in X$. Suppose that $x_\alpha \to x$. Then $p(x_\alpha) \to p(x)$. Since for each $\alpha \in A$, $p(x_\alpha) \geq 1$, $p(x) \geq 1$. Hence $x \in B_p(0, 1)^c$. So $B_p(0, 1)^c$ is closed which implies that $B_p(0, 1)$ is open.

2. Let $(x_\alpha)_{\alpha\in A}$ be a net in $\bar{B}_p(0,1)$ and $x \in X$. Suppose that $x_\alpha \to x$. Then $p(x_\alpha) \to p(x)$. Since for each $\alpha \in A$, $p(x_\alpha) \le 1$, $p(x) \le 1$. Hence $x \in \bar{B}_p(0,1)$. So $\bar{B}_p(0,1)$ is closed.

□

Exercise 5.3.0.16. Let X be a topological vector space and $p : X \to [0,\infty)$ a seminorm. Then the following are quivalent:

1. p is continuous
2. $B_p(0,1)$ is open
3. $\bar{B}_p(0,1) \in \mathcal{N}(0)$
4. p is continuous at 0.

Proof.

- (1) $\implies$ (2):
  Clear from previous exercise.

- (2) $\implies$ (3):
  Clear since $B_p(0,1) \subset \bar{B}_p(0,1)$.

- (3) $\implies$ (4):
  Let $(x_\alpha)_{\alpha\in A} \subset X$ be a net. Suppose that $x_\alpha \to 0$. Let $U \subset \mathbb{R}$. Suppose that $U \in \mathcal{N}(0)$. Then there exists $\epsilon > 0$ such that $\bar{B}(0,\epsilon) \subset U$. Since the map $f_\epsilon : X \to X$ defined by $f_\epsilon(x) = \epsilon x$ is a homeomorphism, $\bar{B}_p(0,\epsilon) = \epsilon\bar{B}_p(0,1) \in \mathcal{N}(0)$. Hence there exists $\beta \in A$ such that for each $\alpha \ge \beta$, $x_\alpha \in \bar{B}_p(0,\epsilon)$. Let $\alpha \in A$. Suppose that $\alpha \ge \beta$. By definition, $p(x_\alpha) \le \epsilon$. So $p(x_\alpha) \in \bar{B}(0,\epsilon) \subset U$. Hence $(p(x_\alpha))_{\alpha\in A}$ is eventually in U. Since $U \in \mathcal{N}(0)$ is arbitrary, $p(x_\alpha) \to 0$. So p is continuous at 0.

- (4) $\implies$ (1):
  Let $(x_\alpha)_{\alpha\in A} \subset X$ be a net and $x \in X$. Suppose that $x_\alpha \to x$. Then $x_\alpha - x \to 0$. Therefore $p(x_\alpha - x) \to 0$. The reverse triangle inequality implies that $p(x_\alpha) \to p(x)$. So p is continuous.

□

Exercise 5.3.0.17. Let X be a topological vector space and $p : X \to [0,\infty)$ a seminorm. Then p is continuous iff there exists a continuous seminorm $q : X \to [0,\infty)$ such that $p \le q$.

Proof. Suppose that p is continuous. Set $q = p$. Then q is a continuous and $p \le q$
Conversely, suppose that there exists a continuous seminorm $q : X \to [0,\infty)$ such that $p \le q$. Then $\bar{B}_q(0,1) \subset \bar{B}_p(0,1)$. The previous exercise tells us that

$$\begin{aligned} q \text{ is continuous} &\iff \bar{B}_q(0,1) \in \mathcal{N}(0) \\ &\implies \bar{B}_p(0,1) \in \mathcal{N}(0) \\ &\iff p \text{ is continuous} \end{aligned}$$

□

Theorem 5.3.0.18. Hahn-Banach Theorem for Seminorms
Let X be a vector space, $p : X \to \mathbb{R}$ a seminorm, $M \subset X$ a subspace and $f : M \to \mathbb{C}$ a linear functional. If for each $x \in M$, $|f(x)| \le p(x)$, then there exists a linear functional $F : X \to \mathbb{C}$ such that for each $x \in X$, $|F(x)| \le p(x)$ and $F|_M = f$.

5.4 Minkowski Functionals

Definition 5.4.0.1. Let X be a vector space and $A \subset X$. Then A is said to be **convex** if for each $x, y \in A$, $t \in [0,1]$, $tx + (1-t)y \in A$.

Exercise 5.4.0.2. Let X be a vector space and $\mathcal{A} \subset \mathcal{P}(X)$, Suppose that for each $A \in \mathcal{A}$, A is convex. Then

$$\bigcap_{A \in \mathcal{A}} A$$

is convex.

Proof. Let $x, y \in \bigcap_{A \in \mathcal{A}} A$ and $t \in [0,1]$. Then for each $A \in \mathcal{A}$, $x, y \in A$. Let $A \in \mathcal{A}$. Since A is convex, $tx + (1-t)y \in A$. Since $A \in \mathcal{A}$ is arbitrary, $tx + (1-t)y \in \bigcap_{A \in \mathcal{A}} A$. So $\bigcap_{A \in \mathcal{A}} A$ is convex. □

Definition 5.4.0.3. Let X be a vector space and $A \subset X$. Set

$$\mathcal{S} = \{S \subset X : S \text{ is convex and } A \subset S\}$$

We define the **convex hull of** A, denoted $\operatorname{conv} A$, by

$$\operatorname{conv} A = \bigcap_{S \in \mathcal{S}} S$$

Note 5.4.0.4. We may think of $\operatorname{conv} A$ as the smallest convex set containing A.

Definition 5.4.0.5. Let X be a vector space, $A \subset X$ and $x \in X$. Then x is said to be a **convex combinations of elements of** A if there exist $(a_j)_{j=1}^n \subset A$ and $(t_j)_{j=1}^n \subset [0,1]$ such that $x = \sum_{j=1}^n t_j a_j$ and $\sum_{j=1}^n t_j = 1$. We define $C_A \subset X$ by

$$C_A = \{x \in X : x \text{ is a convex combination of elements of } A\}$$

Exercise 5.4.0.6. Let X be a vector space and $A \subset X$. Then

1. $A \subset C_A$
2. C_A is convex

Proof.

1. Let $x \in A$, then

$$\begin{aligned} x &= 1x \\ &\in C_A \end{aligned}$$

So $A \subset C_A$.

2. Let $x, y \in C_A$. and $\lambda \in [0,1]$. Then there exist $(a_i)_{i=1}^n$, $(b_j)_{j=1}^m \subset A$ and $(s_i)_{i=1}^n$, $(t_j)_{j=1}^m \subset [0,1]$ such that $x = \sum_{i=1}^n s_i a_i$ and $y = \sum_{j=1}^m t_j b_j$. Then

$$\begin{aligned} \lambda x + (1-\lambda) y &= \lambda[\sum_{i=1}^n s_i a_i] + (1-\lambda)[\sum_{j=1}^m t_j b_j] \\ &= \sum_{i=1}^n \lambda s_i a_i + \sum_{j=1}^m (1-\lambda) t_j b_j \end{aligned}$$

Since

(a) for each $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, we have that $\lambda s_i \in [0,1]$ and $(1-\lambda)t_j \in [0,1]$

(b)

$$\begin{aligned}\sum_{i=1}^{n} \lambda s_i + \sum_{j=1}^{m}(1-\lambda)t_j &= \lambda \sum_{i=1}^{n} s_i + (1-\lambda)\sum_{j=1}^{m} t_j \\ &= \lambda + (1-\lambda) \\ &= 1\end{aligned}$$

we have that $\lambda x + (1-\lambda)y \in C_A$. So C_A is convex.

□

Exercise 5.4.0.7. Let X be a vector space and $A \subset X$. Let $(a_j)_{j=1}^{n} \subset A$ and $(t_j)_{j=1}^{n} \subset [0,1]$. Suppose that $\sum_{j=1}^{n} t_j = 1$. If A is convex, then $\sum_{j=1}^{n} t_j a_j \in A$.
Hint: proceed by induction on n

Proof. Suppose that A is convex. If $n = 2$, then by definition, $\sum_{j=1}^{n} t_j a_j \in A$.

Suppose that the claim is true for $n-1$. Since $\sum_{j=1}^{n} t_j = 1$, then there $k \in \{1, \dots, n\}$ such that $t_k > 0$. Choose Choose $l \in \{1, \dots, n\}$ such that $l \neq k$. Set $S = \{1, \dots, n\} \setminus \{t_l\}$. Then $1 - t_l > 0$ and

$$\begin{aligned}x &= \sum_{j=1}^{n} t_j a_j \\ &= t_l a_l + \sum_{j \in S} t_j a_j \\ &= t_l a_l + (1-t_l)\sum_{j \in S} \frac{t_j}{1-t_l} a_j\end{aligned}$$

Since

$$\begin{aligned}\sum_{j \in S} \frac{t_j}{1-t_l} &= \frac{1-t_l}{1-t_l} \\ &= 1\end{aligned}$$

our induction hypothesis implies that

$$\sum_{j \in S} \frac{t_j}{1-t_l} a_j \in A$$

Since A is convex, by definition we have that

$$\begin{aligned}x &= t_l a_l + (1-t_l)\left[\sum_{j \in S} \frac{t_j}{1-t_l} a_j\right] \\ &\in A\end{aligned}$$

□

Exercise 5.4.0.8. Let X be a vector space and $A \subset X$. Then

$$\operatorname{conv} A = C_A$$

Proof. Since $A \subset C_A$ and C_A is convex, $\operatorname{conv} A \subset C_A$.
Conversely, Let $x \in C_A$. Then there exist $(a_j)_{j=1}^n \subset A$ and $(t_j)_{j=1}^n \subset [0,1]$ such that $x = \sum\limits_{j=1}^n t_j a_j$ and $\sum\limits_{j=1}^n t_j = 1$. Since $A \subset \operatorname{conv} A$ and $\operatorname{conv} A$ is convex, the previous exercise implies that $x \in \operatorname{conv} A$. So $C_A \subset \operatorname{conv} A$. Hence $\operatorname{conv} A = C_A$. □

Exercise 5.4.0.9. Let X be a vector space and $A, B \subset X$ convex and $\lambda \in \mathbb{C}$. Then

1. $A + B$ is convex
2. λA is convex

Proof.

1. Let $x, y \in A + B$ and $t \in [0,1]$. Then there exist $a_x, a_y \in A$, $b_x, b_y \in B$ such that $x = a_x + b_x$ and $y = a_y + b_y$. Since A and B are convex, $ta_x + (1-t)a_y \in A$ and $tb_x + (1-t)b_y \in B$. Hence

$$\begin{aligned} tx + (1-t)y &= ta_x + tb_x + (1-t)a_y + (1-t)b_y \\ &= [ta_x + (1-t)a_y] + [tb_x + (1-t)b_y] \\ &\in A + B \end{aligned}$$

So $A + B$ is convex.

2. Let $x, y \in \lambda A$ and $t \in [0,1]$. Then there exist $a_x, a_y \in A$ such that $x = \lambda a_x$ and $y = \lambda a_y$. Since A is convex, $ta_x + (1-t)a_y \in A$. Therefore

$$\begin{aligned} tx + (1-t)y &= t\lambda a_x + (1-t)\lambda a_y \\ &= \lambda[ta_x + (1-t)a_y] \\ &\in \lambda A \end{aligned}$$

So λA is convex.

□

Definition 5.4.0.10. Let X be a vector space and $A \subset X$. Then A is said to be **balanced** if for each $x \in A$, $c \in \mathbb{C}$, $|c| \leq 1$ implies that $cx \in A$.

Exercise 5.4.0.11. Let X be a vector space and $\mathcal{A} \subset \mathcal{P}(X)$, Suppose that for each $A \in \mathcal{A}$, A is balanced. Then

$$\bigcup_{A \in \mathcal{A}} A$$

is balanced.

Proof. Let $x \in \bigcap\limits_{A \in \mathcal{A}} A$ and $r \in \mathbb{C}$. Suppose that $|r| \leq 1$. Then there exists $B \in \mathcal{A}$ such that $x \in B$. Since A is balanced,

$$\begin{aligned} rx &\in B \\ &\subset \bigcap_{A \in \mathcal{A}} A \end{aligned}$$

So $\bigcap\limits_{A \in \mathcal{A}} A$ is balanced. □

Definition 5.4.0.12. Let X be a vector space and $A \subset X$. We define the **balanced hull of** A, denoted $\operatorname{bal} A$, by

$$\operatorname{bal} A = \bigcup_{\substack{r \in \mathbb{C} \\ |r| \leq 1}} rA$$

Exercise 5.4.0.13. Let X be a vector space and $A \subset X$. Then $\operatorname{bal} A$ is balanced.

Proof. Let $x \in \operatorname{bal} A$ and $r \in \mathbb{C}$. Suppose that $|r| \leq 1$. By definition, there exists $s \in \mathbb{C}$ and $a \in A$ such that $|s| \leq 1$ and $x = sa$. Then

$$\begin{aligned} |rs| &= |r||s| \\ &\leq 1 \end{aligned}$$

which implies that

$$\begin{aligned} rx &= rsa \\ &\in rsA \\ &\subset \bigcup_{\substack{q \in \mathbb{C} \\ |q| \leq 1}} qA \\ &= \operatorname{bal} A \end{aligned}$$

So $\operatorname{bal} A$ is balanced. □

Note 5.4.0.14. We may think of $\operatorname{bal} A$ as the smallest balanced set containing A.

Exercise 5.4.0.15. Let X be a vector space and $A \subset X$. Suppose that $A \neq \varnothing$. If A is balanced, then $0 \in A$.

Proof. Clear by definition. □

Exercise 5.4.0.16. Let X be a vector space, $A \subset X$, $x \in X$ and $\lambda \in \mathbb{C}$. Suppose that A is balanced. Then $\lambda x \in A$ iff $|\lambda| x \in A$.

Proof. If $\lambda = 0$, then the claim is clearly true. Suppose that $\lambda \neq 0$. Set $s = \operatorname{sgn}(\lambda)$. Suppose that $\lambda x \in A$. Since A is balanced and $|s| = |s^{-1}| = 1$,

$$\begin{aligned} |\lambda| x &= s^{-1} \lambda x \\ &\in A \end{aligned}$$

Conversely, suppose that $|\lambda| x \in A$. Then

$$\begin{aligned} \lambda x &= s |\lambda| x \\ &\in A \end{aligned}$$

□

Exercise 5.4.0.17. Let X be a vector space and $A \subset X$. If A is balanced, then $\operatorname{conv} A$ is balanced.

Proof. Suppose that A is balanced. Let $x \in \operatorname{conv} A$ and $r \in \mathbb{C}$. Suppose that $|r| \leq 1$. Then there exist $(a_j)_{j=1}^n \subset A$ and $(t_j)_{j=1}^n \subset [0,1]$ such that $x = \sum_{j=1}^n t_j a_j$ and $\sum_{j=1}^n t_j = 1$. Since A is balanced, for each $j \in \{1, \ldots, n\}$,

$$\begin{aligned} r a_j &\in A \\ &\subset \operatorname{conv} A \end{aligned}$$

Since $\operatorname{conv} A$ is convex, we have that

$$\begin{aligned} rx &= r \sum_{j=1}^n t_j a_j \\ &= \sum_{j=1}^n t_j r a_j \\ &\in \operatorname{conv} A \end{aligned}$$

Hence $\operatorname{conv} A$ is balanced.. □

Definition 5.4.0.18. Let X be a vector space and $A \subset X$. Then A is said to be **absorbing** if for each $x \in X$, there exists $r > 0$ such that for each $c \in \mathbb{R}$, $|c| \geq r$ implies that $x \in cA$.

Exercise 5.4.0.19. Let X be a topological vector space and $A \in \mathcal{N}(0)$. Then A is absorbing.

Proof. Let $x \in A$. For the sake of contradiction, suppose that for each $r > 0$, there exists $c \in \mathbb{R}$ such that $|c| \geq r$ and $c^{-1}x \in A^c$. Then there exists a sequence $(c_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that for each $n \in \mathbb{N}$, $c_n \geq n$ and $c_n^{-1}x \in A^c$. Since $c_n^{-1} \to 0$, $c_n^{-1}x \to 0$. Since $A \in \mathcal{N}(0)$, $(c_n^{-1}x)_{n\in\mathbb{N}}$ is eventually in A. This is a contradiction. So there exists $r > 0$ such that for each $c \in \mathbb{R}$, $|c| \geq r$ implies that $x \in cA$. Hence A is absorbing. □

Exercise 5.4.0.20.

Proof. □

Definition 5.4.0.21. Let X be a vector space and $A \subset X$. For $x \in X$, set

$$T_x^A = \{t > 0 : x \in tA\}$$

We define the **Minkowski functional**, denoted $p_A : X \to [0, \infty]$, by

$$p_A(x) = \inf T_x^A$$

Exercise 5.4.0.22. Let X be a vector space and $A \subset X$. Suppose that A is convex, absorbing and $0 \in A$. Then

1. $p_A : X \to [0, \infty)$
2. $p(0) = 0$
3. p_A is a sublinear functional on X

Proof.

1. Since A is absorbing, there exists $r > 0$ such that for each $c \in \mathbb{R}$, $|c| \geq r$ implies that $x \in cA$. Therefore $p_A(x) \leq |c|$ and $p_A : X \to [0, \infty)$.
2. Since $0 \in A$,
$$\begin{aligned} p_A(0) &= \inf T_0^A \\ &= 0 \end{aligned}$$
3.
   - Let $\epsilon > 0$. Choose $t_x \in T_x^A$ and $t_y \in T_y^A$ such that $t_x < p_A(x) + \epsilon/2$ and $t_y < p_A(y) + \epsilon/2$. By definition, $t_x^{-1}x$, $t_y^{-1}y \in A$. Set $\theta = t_x(t_x + t_y)^{-1} \in (0, 1)$. Since A is convex,
   $$\begin{aligned} (t_x + t_y)^{-1}(x + y) &= (t_x + t_y)^{-1}x + (t_x + t_y)^{-1}y \\ &= \theta t_x^{-1}x + (1 - \theta)t_y^{-1}y \\ &\in A \end{aligned}$$
   Therefore, $t_x + t_y \in T_{x+y}^A$ and
   $$\begin{aligned} p_A(x + y) &\leq t_x + t_y \\ &< p_A(x) + \frac{\epsilon}{2} + p_A(y) + \frac{\epsilon}{2} \\ &= p_A(x) + p_A(y) + \epsilon. \end{aligned}$$
   Since $\epsilon > 0$ is arbitrary, $p_A(x + y) \leq p_A(x) + p_A(y)$.

- If $\lambda = 0$, then

$$\begin{aligned} p_A(\lambda x) &= p_A(0) \\ &= 0 \\ &= |\lambda| p_A(x) \end{aligned}$$

Suppose that $\lambda > 0$. Let $t > 0$. Then

$$\begin{aligned} p_A(\lambda x) &= \inf\{t > 0 : \lambda x \in tA\} \\ &= \inf\{t > 0 : x \in \lambda^{-1} tA\} \\ &= \inf\{\lambda s > 0 : x \in sA\} \\ &= \lambda \inf\{s > 0 : x \in sA\} \\ &= \lambda p_A(x) \end{aligned}$$

So p is a sublinear functional on X. $\square$

Exercise 5.4.0.23. Let X be a vector space and $A \subset X$. Suppose that A is convex, absorbing and $0 \in A$. Then $p_A^{-1}[0,1) \subset A$.

Proof. Let $x \in p_A^{-1}[0,1)$. Then $p_A(x) < 1$. By definition, there exists $t \in (0,1)$ such that $x \in tA$. Thus $t^{-1}x \in A$. Since $0 \in A$ and A is convex, we have that

$$\begin{aligned} x &= t(t^{-1}x) + (1-t)0 \\ &\in A \end{aligned}$$

Since $x \in p_A^{-1}[0,1)$ is arbitrary, $p_A^{-1}[0,1) \subset A$. $\square$

Exercise 5.4.0.24. Let X be a topological vector space and $A \subset X$. Suppose that A is open, convex, and $0 \in A$. Then $p_A^{-1}[0,1) = A$.
Hint: for $x \in A$, consider the sequence $(1 + 1/n)x$

Proof. Since A is open and $0 \in A$, $A \in \mathcal{N}(0)$ which implies that A is absorbing. The previous exercise implies that $p_A^{-1}[0,1) \subset A$.
Conversely, let $x \in A$. Since A is open, $A \in \mathcal{N}(x)$. Since $1 + 1/n \to 1$, $(1 + 1/n)x \to x$. Therefore, there exits $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $(1 + 1/n)x \in A$. In particular, $x \in (1 + 1/N)^{-1}A$. Hence $(1 + 1/N)^{-1} \in T_x^A$ and

$$\begin{aligned} p_A(x) &= \leq (1 + 1/N)^{-1} \\ &< 1 \end{aligned}$$

So $x \in p_A^{-1}[0,1)$ and $A \subset B_{p_A}(0,1)$.

$\square$

Exercise 5.4.0.25. Let X be a topological vector space, $A \subset X$ and $x_0 \in A^c$. Suppose that A is convex, $A \in \mathcal{N}(0)$ and A is open. Then there exists $F \in X^*$ such that $\operatorname{Re} F(x_0) = 1$ and $\operatorname{Re} F|_A < 1$.
Hint: Assume X is real.

1. **Existence:** Consider a special $f \in (\mathbb{R}x_0)^*$ and use p_A to apply the Hahn-Banach theorem.

2. **Continuity:** for $\epsilon > 0$, consider the neighborhood $U_\epsilon = \epsilon A \cap -\epsilon A$

Proof. Assume that X is real.

1. Define $f \in (\mathbb{R}x_0)^*$ by $f(tx_0) = t$. Then $f(x_0) = 1$. Since $A \in \mathcal{N}(0)$, $0 \in A$ and a previous exercise implies that A is absorbing. Since A is convex, absorbing and $0 \in A$, $p_A : X \to [0, \infty)$ is a sublinear functional on X. Since $x_0 \in A^c$, the previous exercise implies that $1 \leq p_A(x_0)$. Let $x \in \mathbb{R}x_0$. Then there exists $t \in \mathbb{R}$ such that $x = tx_0$.

- If $t \geq 0$, then

$$\begin{aligned} f(x) &= t \\ &\leq t p_A(x_0) \\ &= p_A(t x_0) \\ &= p_A(x) \end{aligned}$$

- If $t < 0$, then $-t > 0$ and an exercise from the section on sublinear functionals implies that

$$\begin{aligned} f(x) &= t \\ &=< 0 \\ &\leq p_A(x) \end{aligned}$$

So $f \leq p_A$ on $\mathbb{R}x_0$. The Hahn-Banach theorem implies that there exists $F : X \to \mathbb{R}$ such that F is linear, $F|_{\mathbb{R}x_0} = f$ and $F \leq p_A$. The previous exercise implies that $p_A|_A < 1$. Hence $F|_A < 1$.

2. Let $V \in \mathcal{N}(0_\mathbb{R})$. Choose $\epsilon > 0$ such that $B(0, \epsilon) \subset V$. Set $U_\epsilon = \epsilon A \cap -\epsilon A$. Then $U_\epsilon \in \mathcal{N}(0)$. Let $u \in U_\epsilon$. Then $\epsilon^{-1} u, -\epsilon^{-1} u \in A$. A previous exercise implies that $p_A^{-1}([0, 1)) = A$. Hence

$$\begin{aligned} \epsilon^{-1} F(u) &= F(\epsilon^{-1} u) \\ &\leq p_A(\epsilon^{-1} u) \\ &< 1 \end{aligned}$$

So $F(u) < \epsilon$. Similarly, $F(-u) < \epsilon$. So $-\epsilon < F(u) < \epsilon$ and

$$\begin{aligned} F(U_\epsilon) &\subset B(0, \epsilon) \\ &\subset V \end{aligned}$$

Since $V \in \mathcal{N}(0_\mathbb{R})$ is arbitrary, F is continuous at 0. Since F is linear and F is continuous at 0, F is continuous. Hence $F \in X^*$.

If X is complex, then the previous part implies that there exists $G : X \to \mathbb{R}$ such that G is continuous, real-linear, $G(x_0) = 1$ and $G|_A < 1$. A previous exercise implies that there exists a unique $F \in X^*$ such that $\operatorname{Re} F = G$. □

Exercise 5.4.0.26. Hahn-Banach Separation Theorem 1:
Let X be a topological vector space and $A, B \subset X$. Suppose that A, B are nonempty, convex and disjoint. If A is open, then there exists $\phi \in X^*$ and $c \in \mathbb{R}$ such that for each $x \in A$, $y \in B$,

$$\operatorname{Re} \phi(x) < c \leq \operatorname{Re} \phi(y)$$

Hint: Assume X is real.

1. Choose $a_0 \in A$ and $b_0 \in B$ and set $x_0 = b_0 - a_0$ and $C = A - B + x_0$. Then there exists $\phi \in X^*$ such that $\phi(x_0) = 1$ and $\phi|_C < 1$.

2. For each $a \in A$, $b \in B$, $\phi(a) < \phi(b)$. Set $c = \sup\limits_{a \in A} \phi(a)$. Since ϕ is not constant, ϕ is open.

Proof. Assume X is real.

1. Since A, B are nonempty, there exist $a_0 \in A$ and $b_0 \in B$. Set $x_0 = b_0 - a_0$. Previous exercises imply that $A - B$ is open and convex. Set $C = A - B + x_0$. Then C is open and convex. Since

$$\begin{aligned} 0 &= a_0 - b_0 + x_0 \\ &\in C \end{aligned}$$

$C \in \mathcal{N}(0)$. For the sake of contradiction, suppose that $x_0 \in C$. Then there exist $a \in A$, $b \in B$ such that $x_0 = a - b + x_0$. This implies that $a = b$. This is a contradiction since $A \cap B = \varnothing$. Hence $x_0 \notin C$. The previous exercise implies that there exists a $\phi \in X^*$ such that $\phi(x_0) = 1$ and $\phi|_C < 1$.

2. Let $x \in A$ and $y \in B$. Then

$$\begin{aligned}\phi(a) - \phi(b) + 1 &= \phi(a) - \phi(b) + \phi(x_0) \\ &= \phi(a - b + x_0) \\ &< 1\end{aligned}$$

So $\phi(a) < \phi(b)$. Set $c = \sup_{a \in A} \phi(a)$. Since A is open and $\phi \in X^*$ is open. Thus for each $x \in A$, $y \in B$,

$$\phi(x) < c \leq \phi(y)$$

If X is complex, then the previous part implies that there exists $f : X \to \mathbb{R}$ and $c \in \mathbb{R}$ such that f is continuous, real-linear and for each $x \in A$ and $y \in B$,

$$f(x) < c \leq f(y)$$

A previous exercise implies that there exists a unique $\phi \in X^*$ such that $\operatorname{Re} \phi = f$. $\square$

Definition 5.4.0.27. Let X be a vector space and $A \subset X$. Then A is said to be an **absorbing disk** if A is convex, absorbing and balanced.

Exercise 5.4.0.28. Let X be a vector space, $p : X \to [0, \infty)$ a seminorm on X and $r > 0$. Then $B_p(0, r)$ is an absorbing disk.

Proof.

1. Let $a, b \in B_p(0, r)$ and $t \in [0, 1]$. Then $p(a - x) < r$ and $p(b) < r$. So

$$\begin{aligned}p([ta + (1-t)b]) &\leq p(ta + p((1-t)b) \\ &= tp(a) + (1-t)p(b) \\ &< tr + (1-t)r \\ &= r\end{aligned}$$

So $ta + (1-t)b \in B_p(0, r)$ and $B_p(0, r)$ is convex.

2. Let $a \in X$. Set $s = (p(a) + 1)/r$. Then for each $t \geq s$, $tr \geq p(a) + 1$ so that

$$\begin{aligned}a &\in B_p(0, p(a) + 1) \\ &\subset B_p(0, tr) \\ &= tB_p(0, r)\end{aligned}$$

So $B_p(0, r)$ is absorbing.

3. Let $a \in B_p(0, r)$ and $u \in \mathbb{C}$. Uppose that $|u| \leq 1$. Then

$$\begin{aligned}p(ua) &= |u|p(a) \\ &< |u|r \\ &\leq r\end{aligned}$$

So $ua \in B_p(0, r)$ and $B_p(0, r)$ is balanced.

Since $B_p(0, r)$ is convex, absorbing and balanced, it is an absorbing disk. $\square$

Exercise 5.4.0.29. Let X be a vector space and $A \subset X$. Suppose that A is an absorbing disk. Then $p_A : X \to [0, \infty)$ is a seminorm on X.

Proof. Since A is an absorbing disk, A is convex, absorbing and balanced. So $0 \in A$ and the previous exercise tells us that p is a sublinear functional on X. Let $x \in X$ and $\lambda \in \mathbb{C}$.
If $\lambda = 0$, then

$$\begin{aligned} p_A(\lambda x) &= p_A(0) \\ &= 0 \\ &= |\lambda| p_A(x) \end{aligned}$$

Suppose that $\lambda \neq 0$. Since A is balanced, for $t > 0$, $\lambda t^{-1} x \in A$ iff $|\lambda| t^{-1} x \in A$. So

$$\begin{aligned} p_A(\lambda x) &= \inf\{t > 0 : \lambda x \in tA\} \\ &= \inf\{t > 0 : x \in |\lambda|^{-1} tA\} \\ &= \inf\{|\lambda| s > 0 : x \in sA\} \\ &= |\lambda| \inf\{s > 0 : x \in sA\} \\ &= |\lambda| p_A(x) \end{aligned}$$

So p is a seminorm on X. □

Exercise 5.4.0.30. Let X be a topological vector space and $A \subset X$. Suppose that A is an absorbing disk and A is open. Then $B_{p_A}(0,1) = A$.

Proof. Clear by previous exercise. □

Exercise 5.4.0.31. Let X be a topological vector space and $A \subset X$. Suppose that A is an absorbing disk. Then $p_A : X \to [0, \infty)$ is continuous iff A is open.

Proof. If A is open, then

$$\begin{aligned} A &= B_{p_A}(0,1) \\ &\subset \bar{B}_{p_A}(0,1) \end{aligned}$$

which implies that $\bar{B}_{p_A}(0,1) \in \mathcal{N}(0)$. An exercise in the previous section implies that p_A is continuous. Conversely, if p_A is continuous, then an exercise in the previous section implies that $B_{p_A}(0,1)$ is open. □

5.5 Locally Convex Spaces

Definition 5.5.0.1. Let X be a vector space and $p : X \to [0, \infty)$ a seminorm on X. We equip $X/\ker p$ with the topology induced by the norm $\bar{p} : X/p \to [0, \infty)$. We define the projection $\pi_p : X \to X/\ker p$ by $\pi_p(x) = \bar{x} = x + \ker p$.

Definition 5.5.0.2. Let X be a vector space and $\mathcal{P}$ a family of seminorms on X. Then $\mathcal{P}$ is said to **separate points of** X if for each $x \in X$, if $x \neq 0$, then there exists $p \in \mathcal{P}$ such that $p(x) \neq 0$.

Definition 5.5.0.3. Let X be a vector space, $\mathcal{T}$ a topology on X and $\mathcal{P}$ a family of seminorms. Then $(X, \mathcal{T})$ is said to be a **locally convex space with associated family of seminorms** $\mathcal{P}$ if

- $\mathcal{P}$ separates points of X
- $\mathcal{T} = \tau_X(\pi_p : p \in \mathcal{P})$

Note 5.5.0.4. We will generally suppress the family $\mathcal{P}$ of seminorms and the induced topology $\mathcal{T}$.

Exercise 5.5.0.5. Let X be a locally convex space and $(x_\alpha)_{\alpha \in A} \subset X$ a net and $x \in X$. Then $x_\alpha \to x$ iff for each $p \in \mathcal{P}$, $p(x_\alpha - x) \to 0$.

Proof. Suppose that $x_\alpha \to x$. Let $p \in \mathcal{P}$. By assumption,

$$\begin{aligned} \bar{x}_\alpha &= \pi_p(x_\alpha) \\ &\to \pi_p(x) \\ &= \bar{x} \end{aligned}$$

So

$$\begin{aligned} p(x_\alpha - x) &= \bar{p}(\bar{x}_\alpha - \bar{x}) \\ &\to 0 \end{aligned}$$

Conversely, suppose that for each $p \in \mathcal{P}$, $p(x_\alpha - x) \to 0$. Let $p \in \mathcal{P}$. Then

$$\begin{aligned} \bar{p}(\bar{x}_\alpha - \bar{x}) &= p(x_\alpha - x) \\ &\to 0 \end{aligned}$$

So $\pi_p(x_\alpha) \to \pi_p(x)$. Since $p \in \mathcal{P}$ is arbitrary, $x_\alpha \to x$. □

Exercise 5.5.0.6. Let X be a locally convex space. Then for each $p \in \mathcal{P}$, p is continuous.

Proof. Let $(x_\alpha)_{\alpha \in A} \subset X$ be a net and $x \in X$. Suppose that $x_\alpha \to x$. Let $p \in \mathcal{P}$. Then $p(x_\alpha - x) \to 0$. The reverse triangle inequality implies that

$$\begin{aligned} |p(x_\alpha) - p(x)| &\leq p(x_\alpha - x) \\ &\to 0 \end{aligned}$$

So $p(x_\alpha) \to p(x)$ and p is continuous. □

Exercise 5.5.0.7. Let X be a locally convex space. Then X is a Hausdorff topological vector space.

Proof.

1. Let $(x_\alpha)_{\alpha \in A}$, $(x_\alpha)_{\alpha \in A} \subset X$ and $(\lambda_\alpha)_{\alpha \in A} \subset \mathbb{C}$ be nets and x,$y \in X$, $\lambda \in \mathbb{C}$. Suppose that $x_\alpha \to x$, $y_\alpha \to y$ and $\lambda_\alpha \to \lambda$. Let $P \in \mathcal{P}$. Then

$$\begin{aligned} p([x_\alpha + y_\alpha] - [x + y]) &= p([x_\alpha - x] + [y_\alpha - y]) \\ &\leq p(x_\alpha - x) + p(y_\alpha - y) \\ &\to 0 \end{aligned}$$

Since $p \in \mathcal{P}$ is arbitrary, $x_\alpha + y_\alpha \to x + y$ and addition $X \times X \to X$ is continuous.

2. Similiarly,

$$\begin{aligned}
p(\lambda_\alpha x_\alpha - \lambda x) &= p([\lambda_\alpha x_\alpha - \lambda x_\alpha] + [\lambda x_\alpha - \lambda x]) \\
&\leq p(\lambda_\alpha x_\alpha - \lambda x_\alpha) + p(\lambda x_\alpha - \lambda x) \\
&= p([\lambda_\alpha - \lambda] x_\alpha) + p(\lambda [x_\alpha - x]) \\
&= |\lambda_\alpha - \lambda| p(x_\alpha) + |\lambda| p(x_\alpha - x) \\
&\to 0
\end{aligned}$$

So scalar multiplication $\mathbb{C} \times X \to X$ is continuous.

3. Let $x, y \in X$. Suppose that $x \neq y$. Since $\mathcal{P}$ separates points of X, there exists $p \in \mathcal{P}$ such that $p(x-y) \neq 0$. Thus $\bar{p}(\bar{x} - \bar{y}) \neq 0$. Thus $\bar{x} \neq \bar{y}$. Since $X/\ker p$ is Hausdorff, there exists $U' \in \mathcal{N}(\bar{x})$ and $V' \in \mathcal{N}(\bar{y})$ such that $U' \cap V' = \varnothing$. Set $U = \pi_p^{-1}(U')$ and $V = \pi_p^{-1}(V')$. Then $U \in \mathcal{N}(x)$, $V \in \mathcal{N}(y)$ and

$$\begin{aligned}
U \cap V &= \pi_p^{-1}(U') \cap \pi_p^{-1}(V') \\
&= \pi_p^{-1}(U' \cap V') \\
&= \pi_p^{-1}(\varnothing) \\
&= \varnothing
\end{aligned}$$

So X is Hausdorff.

□

Exercise 5.5.0.8. Let X be a locally convex space and $U \in \mathcal{N}(0)$ open. Then there exist $p \in \mathcal{P}$ and $r > 0$ such that $B_p(0, r) \subset U$.

Proof. For the sake of contradiction, suppose that for each $p \in \mathcal{P}$ and $r > 0$, $B_p(0, r) \not\subset U$. Then there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset U^c$ such that for each $p \in \mathcal{P}$ and $n \in \mathbb{N}$, $p(x_n) < 1/n$. So $x_n \to 0$. Since U^c is closed, $0 \in U^c$ which is a contradiction. Hence there exist $p \in \mathcal{P}$ and $r > 0$ such that $B_p(0, r) \subset U$. □

Exercise 5.5.0.9. Let X be a locally convex space. Then for each $U \in \mathcal{N}(0)$, if U is open, then there exists $V \subset U$ such that V is an open absorbing disk.

Proof. Let $U \in \mathcal{N}(0)$. Suppose that U is open. The previous exercise implies that there exists $p \in \mathcal{P}$ and $r > 0$ such that $B_p(0, 1) \subset U$. A previous exercise tells us that $B_p(0, 1)$ is an open absorbing disk. □

Exercise 5.5.0.10. Let $(X, \mathcal{T})$ be a locally convex space with associated family of seminorms $\mathcal{P}$ and $M \subset X$ a subspace. Define $\mathcal{P}_M = \{p|_M : p \in \mathcal{P}\}$. Then $(M, \mathcal{T} \cap M)$ is a locally convex space with associated family of seminorms $\mathcal{P}_M$.

Proof. Let $(x_\alpha)_{\alpha \in A} \subset M$ be a net and $x \in M$. Suppose that $x_\alpha \to x$ in $\mathcal{T} \cap M$. Then an exercise in the section on the subspace topology implies that $x_\alpha \to x$ in $\mathcal{T}$. Let $q \in \mathcal{P}_M$. Then there exists $p \in \mathcal{P}$ such that $q = p|_M$. Therefore

$$\begin{aligned}
q(x_\alpha - x) &= p|_M (x_\alpha - x) \\
&= p(x_\alpha - x) \\
&\to 0
\end{aligned}$$

Hence $x_\alpha \to x$ in $\tau_X(\pi_q : q \in \mathcal{P}_M)$.
Conversely, suppose that $x_\alpha \to x$ in $\tau_X(\pi_q : q \in \mathcal{P}_M)$. Let $p \in \mathcal{P}$. Then

$$\begin{aligned}
p(x_\alpha - x) &= p|_M (x_\alpha - x) \\
&\to 0
\end{aligned}$$

Hence $x_\alpha \to x$ in $\mathcal{T}$. So $x_\alpha \to x$ in $\mathcal{T} \cap M$. Therefore $\mathcal{T} \cap M = \tau_X(\pi_q : q \in \mathcal{P}_M)$. □

Exercise 5.5.0.11. Let X be a locally convex space, $M \subset X$ a subspace and $f \in M^*$. Then there exists $F \in X^*$ such that $F|_M = f$.

Proof. Set $p_f = |f|$. Since p_f is a continuous seminorm, $B_{p_f}(0,1)$ is open in M. Therefore, there exists $U \subset X$ open such that $B_{p_f}(0,1) = U \cap M$. A previous exercise implies that there exists $p \in \mathcal{P}$ and $r > 0$ such that $B_p(0,r) \subset U$. Set $A = B_p(0,r)$. Since A is open, $p_A : X \to [0,\infty)$ is continuous and $A = B_{p_A}(0,1)$. Hence

$$\begin{aligned} B_{p_A|_M}(0,1) &= A \cap M \subset U \cap M \\ &= B_{p_f}(0,1) \end{aligned}$$

Therefore $p_f \leq p_A|_M$ and $|f| \leq p_A$ on M. The Hahn-Banach theorem implies that there exists $F : X \to \mathbb{C}$ such that F is linear, $F|_M = f$ and $|F| \leq p_A$. Since p_A is continuous, $|F|$ is continuous, which implies that F is continuous. So $F \in X^*$. □

Exercise 5.5.0.12. Hahn-Banach Separation Theorem 2:
Let X be a locally convex space and A, $B \subset X$. Suppose that A, B are nonempty, convex and disjoint. If A is compact and B is closed, then there exists $\phi \in X^*$ and $c_1, c_2 \in \mathbb{R}$ such that for each $x \in A$, $y \in B$,

$$\operatorname{Re}\phi(x) < c_1 < c_2 \leq \operatorname{Re}\phi(y)$$

Hint: Assume X is real. Since X is locally convex, there exists $V \subset U$ such that V is an open absorbing disk and $(A + V) \cap B = \varnothing$. Then apply the first Hahn-Banach separation theorem to $A + V$ and B.

Proof. Assume X is real. Suppose that A is compact and B is closed. A previous exercise implies that there exists $U \in \mathcal{N}(0)$ such that U is open and $(A + U) \cap B = \varnothing$. Since X is locally convex, there exists $V \subset U$ such that V is an open absorbing disk. Then $(A + V)$ is open and convex. By the first Hahn-Banach separation theorem, there exist $\phi \in X^*$ and $c_2 \in \mathbb{R}$ such that for each $x \in A + V$, $y \in B$,

$$\phi(x) < c_2 \leq \phi(y)$$

Specifically, $c_2 = \sup\limits_{x \in A+V} \phi(x)$. Since $\phi \in X^*$ is not constant, ϕ is open and thus $\phi(A+V)$ is open. Continuity of ϕ implies that $\phi(A)$ is compact. Therefore, $\sup \phi(A) < \sup \phi(A + V)$. So there exists $c_1 \in \phi(A + V)$ such that $\sup \phi(A) < c_1$. Hence there exists $x_1 \in A + V$ such that $\phi(x_1) = c_1$. Then for each $x \in A$ and $y \in B$,

$$\begin{aligned} \phi(x) &\leq \sup \phi(A) \\ &< c_1 \\ &= \phi(x_1) \\ &< c_2 \\ &\leq \phi(y) \end{aligned}$$

If X is complex, then the previous part implies that there exists $f : X \to \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$ such that f is continuous, real-linear and for each $x \in A$ and $y \in B$,

$$f(x) < c_1 < c_2 \leq f(y)$$

A previous exercise implies that there exists a unique $\phi \in X^*$ such that $\operatorname{Re}\phi = f$. □

Exercise 5.5.0.13. Let X be a locally convex space and $M \subset X$ a closed subspace. If $M \neq X$, then there exists $\phi \in X^*$ such that $\phi \neq 0$ and $\phi|_M = 0$.

Proof. Assume that X is real. Suppose that $M \neq X$. Then there exists $x_0 \in X$ such that $x_0 \notin M$. Since $\{x_0\}$ is compact and convex, M is closed and convex and $\{x_0\} \cap M = \varnothing$, the second Hahn-Banach separation theorem implies that there exists $\phi \in X^*$ such that for each $x \in M$,

$$\phi(x_0) < \phi(x)$$

Since $0 \in M$,

$$\begin{aligned} \phi(x_0) &< \phi(0) \\ &= 0 \end{aligned}$$

so that $\phi \neq 0$. For the sake of contradiction, suppose that $\phi|_M \neq 0$. Then there exists $x_1 \in M$ such that $\phi(x_1) \neq 0$. Then for each $t \in \mathbb{R}$,

$$\begin{aligned} \phi(x_0) &< \phi(tx_1) \\ &= t\phi(x_1) \end{aligned}$$

Set $t = \frac{\phi(x_0)}{\phi(x_1)}$. Then

$$\begin{aligned} \phi(x_0) &< t\phi(x_1) \\ &= \phi(x_0) \end{aligned}$$

which is a contradiction. So $\phi|_M = 0$. $\square$

Exercise 5.5.0.14. Let X be a locally convex space. Then X^* separates the points of X.

Proof. Let $x, y \in X$. The second Hahn-Banach separation theorem implies that there exists $\phi \in X^*$ such that $\phi(x) \neq \phi(y)$. $\square$

5.6 Direct Sums

5.7 Quotient Spaces

Exercise 5.7.0.1. Let X be a topological vector space and $M \subset X$ a subspace. Then $\pi : X \to X/M$ is open.

Proof. Define the action $\phi : M \times X \to X$ by $m \cdot x = x + m$. Then $M \cdot x = x + M$. Since for each $m \in M$, the map $x \mapsto x + m$ is continuous, Exercise 3.7.2.5 implies that $\pi : X \to X/M$ is open. $\square$

Exercise 5.7.0.2. Let $(X, \mathcal{T})$ be a topological vector space and $M \subset X$ a subspace. Then $(X/M, \mathcal{T}_{X/M})$ is a topological vector space.

Proof. Denote addition on X and X/M by $A : X^2 \to X$ and $\bar{A} : (X/M)^2 \to X/M$ respectively. Similarly, denote scalar multiplication on X and X/M by $\Lambda : \mathbb{C} \times X \to X$ and $\bar{\Lambda} : \mathbb{C} \times (X/M) \to X/M$ respectively.

- Let $\bar{x}, \bar{y} \in X/M$. Let $U \in \mathcal{N}(\bar{x} + \bar{y})$. Since $\pi : X \to X/M$ is continuous, we have that $\pi^{-1}(U) \in \mathcal{N}(x + y)$. Since addition $A : X^2 \to X$ is continuous,

$$\begin{aligned}(\pi \circ A)^{-1}(U) &= A^{-1}(\pi^{-1}(U)) \\ &\in \mathcal{N}(x, y)\end{aligned}$$

  Since $\mathcal{B} = \{A \times B : A, B \subset X \text{ and } A, B \text{ are open}\}$ is a basis for the product topology on X^2, there exist $V_x \times V_y \in \mathcal{B}$ such that $(x, y) \in V_x \times V_y \subset (\pi \circ A)^{-1}(U)$. Thus $V_x \in \mathcal{N}(x), V_y \in \mathcal{N}(y)$ and $V_x \times V_y \in \mathcal{N}(x, y)$. Recall that $\pi \times \pi : X^2 \to (X/M)^2$ is defined by $\pi \times \pi(x, y) = (\pi(x), \pi(y))$. For $x, y \in X$, we have that

$$\begin{aligned}\bar{A} \circ (\pi \times \pi)(x, y) &= \bar{A}(\bar{x}, \bar{y}) \\ &= \bar{x} + \bar{y} \\ &= \pi(x) + \pi(y) \\ &= \pi(x + y) \\ &= \pi \circ A(x, y)\end{aligned}$$

  So $\bar{A} \circ (\pi \times \pi) = \pi \circ A$. Since π is open, an exercise in the section on the product topology implies that $\pi \times \pi$ is open and therefore $\pi \times \pi(V_x \times V_y) \in \mathcal{N}(\bar{x}, \bar{y})$. Hence

$$\begin{aligned}\bar{A} \circ (\pi \times \pi)(V_x \times V_y) &\subset \bar{A} \circ (\pi \times \pi)((\pi \circ A)^{-1}(U)) \\ &= \bar{A} \circ (\pi \times \pi)((\bar{A} \circ (\pi \times \pi))^{-1}(U)) \\ &\subset U\end{aligned}$$

  So for each $U \in \mathcal{N}(\bar{x} + \bar{y})$, there exists $\pi \times \pi(V_x \times V_y) \in \mathcal{N}(\bar{x}, \bar{y})$ such that $\bar{A}(\pi \times \pi(V_x \times V_y)) \subset U$. Hence $\bar{A}$ is continuous at $(\bar{x}, \bar{y})$. Since $\bar{x}, \bar{y} \in X/M$ are arbitrary, $\bar{A}$ is continuous.

- Let $\lambda \in \mathbb{C}$ and $\bar{x} \in X/M$. Let $U \in \mathcal{N}(\lambda \bar{x})$. Since π is continuous, $\pi^{-1}(U) \in \mathcal{N}(\lambda x)$. Since scalar multiplication $\Lambda : \mathbb{C} \times X \to X$ is continuous,

$$\begin{aligned}\Lambda^{-1}(\pi^{-1}(U)) &= (\pi \circ \Lambda)^{-1}(U) \\ &\in \mathcal{N}(\lambda, x)\end{aligned}$$

  Since $\mathcal{B} = \{A \times B : A \subset \mathbb{C},\ B \subset X \text{ and } A, B \text{ are open}\}$ is a basis for the product topology on $\mathbb{C} \times X$, there exist $V_\lambda \times V_x \in \mathcal{B}$ such that $(\lambda, x) \in V_x \times V_y \subset (\pi \circ \Lambda)^{-1}(U)$. Thus $V_\lambda \in \mathcal{N}(\lambda), V_x \in \mathcal{N}(x)$ and $V_\lambda \times V_x \in \mathcal{N}(\lambda, x)$. As in the previous part, $\pi \circ \Lambda = \bar{\Lambda} \circ (\mathrm{id}_{\mathbb{C}} \times \pi)$ and $\mathrm{id}_{\mathbb{C}}$ is open. Hence $\mathrm{id}_{\mathbb{C}} \times \pi$ is open and $\mathrm{id}_{\mathbb{C}} \times \pi(V_\lambda \times V_x) \in \mathcal{N}(\lambda, \bar{x})$. As in the previous part we have that

$$\begin{aligned}\bar{\Lambda} \circ (\mathrm{id}_{\mathbb{C}} \times \pi)(V_\lambda \times V_x) &\subset \bar{\Lambda} \circ (\mathrm{id}_{\mathbb{C}} \times \pi)((\pi \circ \Lambda)^{-1}(U)) \\ &= \bar{\Lambda} \circ (\mathrm{id}_{\mathbb{C}} \times \pi)((\bar{\Lambda} \circ (\mathrm{id}_{\mathbb{C}} \times \pi))^{-1}(U)) \\ &\subset U\end{aligned}$$

  So for each $U \in \mathcal{N}(\lambda \bar{x})$, there exists $\mathrm{id}_{\mathbb{C}} \times \pi(V_\lambda \times V_x) \in \mathcal{N}(\lambda, \bar{x})$ such that $\bar{\Lambda}(\mathrm{id}_{\mathbb{C}} \times \pi(V_\lambda \times V_x)) \subset U$. Hence $\bar{\Lambda}$ is continuous at $(\lambda, \bar{x})$. Since $\lambda \in \mathbb{C}$ and $\bar{x} \in X/M$ are arbitrary, $\bar{\Lambda}$ is continuous.

$\square$

Exercise 5.7.0.3. Let X be a topological vector space and $M \subset X$ a subspace. Then M is closed iff X/M is Hausdorff.

Proof. Suppose that M is closed. Define the action $\phi : M \times X \to X$ by $m \cdot x = m + x$. Denote by $\sim$, the equivalence relation induced by ϕ (i.e. $x \sim y$ iff $x - y \in M$). A previous exercise implies that $\pi : X \to X/M$ is open. Let $(x_\alpha, y_\alpha)_{\alpha \in A} \subset \sim$ be a net and $(x, y) \in X \times X$. Suppose that $(x_\alpha, y_\alpha) \to (x, y)$. Then $x_\alpha \to x$ and $y_\alpha \to y$. Therefore $x_\alpha - y_\alpha \to x - y$. Since for each $\alpha \in A$, $x_\alpha - y_\alpha \in M$ and M is closed, we have that $x - y \in M$. Hence $(x, y) \in \sim$ and $\sim$ is closed. Since π is open, a previous exercise in the section on separation and countability implies that X/M is Hausdorff.
Conversely, suppose that X/M is Hausdorff. Then $\{0 + M\}$ is closed in X/M. Since $\pi : X \to X/M$ is continuous, we have that $M = \pi^{-1}(0 + M)$ is closed in X. $\square$

Exercise 5.7.0.4. Let X be a topological vector spaces and $\phi : X \to \mathbb{C}$ linear. Then $\ker \phi$ is closed iff ϕ is continuous.
Note: need to show that if $T : X \to Y$ is linear, then T is continuous iff $\bar{T} : X/\ker T \to Y$ is continuous

Proof. Suppose that ϕ is continuous. Since $\{0\} \subset \mathbb{C}$ is closed, $\ker \phi = \phi^{-1}(\{0\})$ is closed.
Conversely, suppose that $\ker \phi$ is closed. Then $X/\ker \phi$ is Hausdorff. Hence FINISH!!! $\square$

Exercise 5.7.0.5. Let X be a topological vector space and $\phi, \psi \in X^*$. If $\ker \phi \subset \ker \psi$, then there exists $\lambda \in \mathbb{C}$ such that $\psi = \lambda \phi$.
Hint: This is just a fact about vector spaces. The isomorphism theorems imply that there exists $g : \operatorname{Im} \phi \to \operatorname{Im} \psi$ such that $\psi = g \circ \phi$.

Proof. Suppose that $\ker \phi \subset \ker \psi$. If $\phi = 0$, then

$$\begin{aligned} X &= \ker \phi \\ &\subset \ker \psi \end{aligned}$$

So

$$\begin{aligned} \psi &= 0 \\ &= \phi \end{aligned}$$

Suppose that $\phi \neq 0$. Then $\operatorname{Im} \phi = \mathbb{C}$. Let $\pi_\phi : X \to X/\ker \phi$ and $\pi_\psi : X \to X/\ker`\psi$ be the canonical projection maps and let $\tilde{\phi} : X/\ker \phi \to \operatorname{Im} \phi$ and $\tilde{\psi} : X/\ker \psi \to \operatorname{Im} \psi$ be the unique maps such that $\tilde{\phi} \circ \pi_\phi = \phi$ and $\tilde{\psi} \circ \pi_\psi = \psi$. Note that $\tilde{\phi}$ and $\tilde{\psi}$ are vector space isomorphisms. Define the linear map $\iota : X/\ker \phi \to X/\ker \psi$ by $\iota(x + \ker \phi) = x + \ker \psi$. Let $x, y \in X$. If $x + \ker \phi = y + \ker \phi$, then

$$\begin{aligned} x - y &\in \ker \phi \\ &\subset \ker \psi \end{aligned}$$

So

$$\begin{aligned} \iota(x) &= x + \ker \psi \\ &= y + \ker \psi \\ &= \iota(y) \end{aligned}$$

and ι is well defined. Define $g : \operatorname{Im} \phi \to \operatorname{Im} \psi$ by $g(y) = \tilde{\psi} \circ \iota \circ \tilde{\phi}^{-1}$. Set $\lambda = g(1)$. Since $g : \mathbb{C} \to \mathbb{C}$ is linear, $g = \lambda \operatorname{id}_{\mathbb{C}}$. Thus the following diagram commutates:

$$\begin{tikzcd} & X_\alpha \arrow[dl, "\pi_\phi"'] \arrow[dr, "\pi_\psi"] & \\ X/\ker \phi \arrow[rr, "\iota"] \arrow[d, "\tilde{\phi}"] & & X/\ker \psi \arrow[d, "\tilde{\psi}"] \\ \operatorname{Im} \phi \arrow[rr, "g = \lambda \operatorname{id}_{\mathbb{C}}"'] & & \operatorname{Im} \psi \end{tikzcd}$$

Hence

$$\begin{aligned}\psi &= g \circ \phi \\ &= \lambda \operatorname{id}_{\mathbb{C}} \circ \phi \\ &= \lambda \phi\end{aligned}$$

□

5.8 Duality

Definition 5.8.0.1. Let X, Y and Z be topological vector spaces (over the same field) and $b : X \times Y \to Z$. Then b is said to be a **pairing of X with Y over Z** if b is bilinear.

Definition 5.8.0.2. Let X, Y and Z be topological vector spaces and $b : X \times Y \to Z$ a pairing. We define the **dual pairing** of b, denoted $b^* : Y \times X \to Z$, by $b^*(y, x) = b(x, y)$. Then b is a pairing.

Exercise 5.8.0.3. Let X, Y and Z be topological vector spaces and $b : X \times Y \to Z$ a pairing. Then b^* is a pairing.

Proof. Clear. □

Definition 5.8.0.4. Let X, Y and Z be topological vector spaces and $b : X \times Y \to Z$ a pairing. We define the **weak topology on X induced by** b, denoted $\sigma_b(X, Y)$ by

$$\sigma_b(X, Y) = \tau_X(b(\cdot, y) : X \to Z : y \in Y)$$

We define the **weak topology on Y induced by** b, denoted $\sigma_b(Y, X)$, by $\sigma_b(Y, X) = \sigma_{b^*}(Y, X)$.

Exercise 5.8.0.5. Let X, Y and Z be topological vector spaces and $b : X \times Y \to Z$ a pairing. Then

1. $(X, \sigma_b(X, Y))$ is a topological vector space
2. $(Y, \sigma_b(Y, X))$ is a topological vector space

Proof.

1. Let $(u_\alpha)_{\alpha\in A}$, $(v_\alpha)_{\alpha\in A} \subset X$ and $(\lambda_\alpha)_{\alpha\in A} \subset \mathbb{C}$ be nets and $u, v \in X$ and $\lambda \in \mathbb{C}$. Suppose that $u_\alpha \to u$, $v_\alpha \to v$ and $\lambda_\alpha \to \lambda$. Let $y \in Y$. Since Z is a topological vector space,

$$\begin{aligned} b(u_\alpha + v_\alpha, y) &= b(u_\alpha, y) + b(v_\alpha, y) \\ &\to b(u, y) + b(v, y) \\ &= b(u + v, y) \end{aligned}$$

   and

$$\begin{aligned} b(\lambda_\alpha u_\alpha, y) &= \lambda_\alpha b(u_\alpha, y) \\ &\to \lambda b(u, y) \\ &= b(\lambda u, y) \end{aligned}$$

   Since $y \in Y$ is arbitrary, $u_\alpha + v_\alpha \to u + v$ and $\lambda_\alpha u_\alpha \to \lambda u$. Hence addition $X \times X \to X$ and scalar multiplication $\mathbb{C} \times X \to X$ are continuous.

2. Since $\sigma_b(X, Y) = \sigma_{b^*}(Y, X)$, (1) implies (2).

□

Definition 5.8.0.6. Let X, Y and Z be topological vector spaces and $b : X \times Y \to Z$ a pairing. Then

- Y is said to **separate the points of X via** b if for each $x \in X$, $x \neq 0$ implies that there exists $y \in Y$ such that $b(x, y) \neq 0$
- X is said to **separate the points of Y via** b if X separates the points of Y via b^*

Exercise 5.8.0.7. Let X, Y and Z be topological vector spaces and $b : X \times Y \to Z$ a pairing. Suppose that Z is Hausdorff.

1. if Y separates the points of X via b, then $(X, \sigma_b(X, Y))$ is Hausdorff
2.

Proof.

1. Suppose that Y separates the points of X via b. Let $x_1, x_2 \in X$. Suppose that $x_1 \neq x_2$. Then $x_1 - x_2 \neq 0$. Hence there exists $y \in Y$ such that

$$\begin{aligned} b(x_1, y) - b(x_2, y) &= b(x_1 - x_2, y) \\ &\neq 0 \end{aligned}$$

Since Z is Hausdorff, there exist $V_1 \in \mathcal{N}(b(x_1, y)), V_2 \in \mathcal{N}(b(x_2, y))$ such that V_1 and V_2 are open and $V_1 \cap V_2 = \varnothing$. Set $U_1 = b(\cdot, y)^{-1}(V_1)$ and $U_2 = b(\cdot, y)^{-1}(V_2)$. By definition of $\sigma_b(X, Y)$, $b(\cdot, y) : X \to Z$ is continuous. Thus U_1, $U_2 \in \sigma_b(X, Y)$, $x_1 \in U_1$, $x_2 \in U_2$ and

$$\begin{aligned} U_1 \cap U_2 &= b(\cdot, y)^{-1}(V_1) \cap b(\cdot, y)^{-1}(V_2) \\ &= b(\cdot, y)^{-1}(V_1 \cap V_2) \\ &= b(\cdot, y)^{-1}(\varnothing) \\ &= \varnothing \end{aligned}$$

Therefore $(X, \sigma_b(X, Y))$ is Hausdorff.

2.

□

Definition 5.8.0.8. Let X be a topological vector space.

- We define the **canonical pairing of X with X^***, denoted $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{C}$, by $\langle x, \phi \rangle = \phi(x)$.
- For each $x \in X$, we define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x} = \langle x, \cdot \rangle$.
- We define $\hat{X} \subset \mathbb{C}^{X^*}$ by $\hat{X} = \{\hat{x} : x \in X\}$.

Definition 5.8.0.9. Let X be a topological vector space. We define the **weak topology on** X, denoted $\mathcal{T}_w$, by $\mathcal{T}_w = \tau_X(X^*)$.

Note 5.8.0.10. The weak topology on X is the initial topology on X generated by X^*, i.e. the weak topology on X induced by the canonical pairing $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{C}$.

Definition 5.8.0.11. Let X be a topological vector space, $(x_\alpha)_{\alpha \in A} \subset X$ and $x \in X$. Then $(x_\alpha)_{\alpha \in A}$ is said to **converge weakly to** x, denoted $x_\alpha \xrightarrow{w} x$ if $(x_\alpha)_{\alpha \in A}$ converges to x in the weak topology.

Exercise 5.8.0.12. Let X be a topological vector, $(x_\alpha)_{\alpha \in A} \subset X$ a net and $x \in X$. Then $x_\alpha \xrightarrow{w} x$ iff for each $\lambda \in X^*$, $\lambda(x_\alpha) \to \lambda(x)$.

Proof. Immediate by Exercise 3.3.2.14. □

Definition 5.8.0.13. Let X be a topological vector space. We define the **weak-* topology on** X^*, denoted $\mathcal{T}_{w*}$, by $\mathcal{T}_{w*} = \tau_X(\hat{X})$.

Note 5.8.0.14. The weak-* topology on X^* is the initial topology on X^* generated by $\hat{X}$, i.e. the weak topology on X^* induced by the canonical pairing $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{C}$.

Definition 5.8.0.15. Let X be a topological vector space, $(\lambda_\alpha)_{\alpha \in A} \subset X^*$ and $\lambda \in X^*$. Then $(\lambda_\alpha)_{\alpha \in A}$ is said to **converge in weak-* to** λ, denoted $\lambda_\alpha \xrightarrow{w^*} \lambda$ if $(\lambda_\alpha)_{\alpha \in A}$ converges to λ in the weak-* topology.

Exercise 5.8.0.16. Let X be a topological vector, $(\lambda_\alpha)_{\alpha \in A} \subset X^*$ a net and $\lambda \in X^*$. Then $\lambda_\alpha \xrightarrow{w^*} \lambda$ iff for each $x \in X$, $\lambda_\alpha(x) \to \lambda(x)$.

Proof. Immediate by Exercise 3.3.2.14. □

Exercise 5.8.0.17. Let X be a topological vector space.

1. If X^* separates the points of X, then $(X, \mathcal{T}_w)$ is a locally convex space
2. $(X^*, \mathcal{T}_{w^*})$ is a locally convex space

Proof.

1. Suppose that X^* separates the points of X. For $\lambda \in X^*$, define $p_\lambda : X \to [0, \infty)$ by $p_\lambda = |\lambda|$. Set $\mathcal{P}_w = \{p_\lambda : \lambda \in X^*\}$. Then $\mathcal{P}_w$ separates the points of X. Let $(x_\alpha)_{\alpha \in A} \subset X$ be a net and $x \in X$. Suppose that $x_\alpha \xrightarrow{w} x$. Let $\lambda \in X^*$. Then

$$\begin{aligned} p_\lambda(x_\alpha - x) &= |\lambda(x_\alpha - x)| \\ &= |\lambda(x_\alpha) - \lambda(x)| \\ &\to 0 \end{aligned}$$

So $x_\alpha \to x$ in $\tau_X(\pi_p : p \in \mathcal{P}_w)$.
Conversely, suppose that $x_\alpha \to x$ in $\tau_X(\pi_p : p \in \mathcal{P}_w)$. Then for each $x \in X$,

$$\begin{aligned} |\lambda(x_\alpha) - \lambda(x)| &= p_\lambda(x_\alpha - x) \\ &\to 0 \end{aligned}$$

So that $\lambda(x_\alpha) \to \lambda(x)$ and $x_\alpha \xrightarrow{w} x$. Hence $\mathcal{T}_w = \tau_X(\pi_p : p \in \mathcal{P}_w)$ and $(X, \mathcal{T}_w)$ is a locally convex space.

2. For $x \in X$, define $p_x : X^* \to [0, \infty)$ by $p_x = |\hat{x}|$. Set $\mathcal{P}_{w^*} = \{p_x : x \in X\}$. Let $\phi \in X^*$. Suppose that $\phi \neq 0$. Then there exists $x \in X$ such that

$$\begin{aligned} \hat{x}(\phi) &= \phi(x) \\ &\neq 0 \end{aligned}$$

So $\mathcal{P}_{w^*}$ separates the points of X^*. Let $(\lambda_\alpha)_{\alpha \in A} \subset X^*$ be a net and $\lambda \in X^*$. Suppose that $\lambda_\alpha \xrightarrow{w^*} \lambda$. Let $x \in X$. Then

$$\begin{aligned} p_x(\lambda_\alpha - \lambda) &= |\hat{x}(\lambda_\alpha - \lambda)| \\ &= |\hat{x}(\lambda_\alpha) - \hat{x}(\lambda)| \\ &\to 0 \end{aligned}$$

So $\lambda_\alpha \to \lambda$ in $\tau_{X^*}(\pi_p : p \in \mathcal{P}_{w^*})$.
Conversely, suppose that $\lambda_\alpha \to \lambda$ in $\tau_{X^*}(\pi_p : p \in \mathcal{P}_{w^*})$. Then for each $x \in X$,

$$\begin{aligned} |\hat{x}(\lambda_\alpha) - \hat{x}(\lambda)| &= p_x(\lambda_\alpha - \lambda) \\ &\to 0 \end{aligned}$$

So that $\hat{x}(\lambda_\alpha) \to \hat{x}(\lambda)$ and $\lambda_\alpha \xrightarrow{w^*} \lambda$. Hence $\mathcal{T}_{w^*} = \tau_{X^*}(\pi_p : p \in \mathcal{P}_{w^*})$ and $(X^*, \mathcal{T}_{w^*})$ is a locally convex space.

□

Note 5.8.0.18. Let X be a topological vector space. When we equip X^* with the weak-$*$ topology, we write X^{**} in place of $(X^*)^*$.

Exercise 5.8.0.19. Let X be a topological vector space. Then $X^{**} = \hat{X}$.
Hint: Hahn-Banach theorem

Proof. Let $f \in X^{**}$. Define $p_f = |f|$. Then p_f is a continuous seminorm on X^*. Therefore $B_{p_f}(0,1)$ is open. A previous exercise implies that there exists $p \in \mathcal{P}_{w^*}$ and $r > 0$ such that

$$\begin{aligned} B_{r^{-1}p}(0,1) &= B_p(0,r) \\ &\subset B_{p_f}(0,1) \end{aligned}$$

A previous exercise implies that $p_f \leq r^{-1}p$. By definition of $\mathcal{P}_{w^*}$, there exists $x \in X$ such that $p = |\hat{x}|$. Thus

$$\begin{aligned} p_f &= |f| \\ &\leq r^{-1}p \\ &= |r^{-1}\hat{x}| \end{aligned}$$

Therefore $\ker \hat{x} \subset \ker f$. An exercise in the section on quotient spaces of locally convex spaces implies that there exists $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} f &= \lambda r^{-1}\hat{x} \\ &\in \hat{X} \end{aligned}$$

So $X^{**} = \hat{X}$. $\square$

5.9 Continous Linear Maps

redo in terms of "boundedness", need to define bounded subsets, then continuous maps should send bounded sets to bounded sets

Definition 5.9.0.1. Let X, Y be topological vector spaces. We define

$$L(X;Y) = \{T : X \to Y : T \text{ is linear and continuous}\}$$

Definition 5.9.0.2. Let X, Y be locally convex spaces with respective associated families of seminorms $\mathcal{P}$ and $\mathcal{Q}$ and $p \in \mathcal{P}$, $q \in \mathcal{Q}$. We define $\|\cdot\|_{p,q} : L(X;Y) \to [0,\infty)$ by

$$\|T\|_{p,q} = \inf\{C \geq 0 : \text{ for each } x \in X,\, q(Tx) \leq Cp(x)\}$$

Exercise 5.9.0.3. Let X, Y be locally convex spaces with respective associated families of seminorms $\mathcal{P}$ and $\mathcal{Q}$, $p \in \mathcal{P}$, $q \in \mathcal{Q}$ and $T \in L(X;Y)$. Then for each $x \in X$, $q(Tx) \leq \|T\|_{p,q}p(x)$.

Proof. Set $A = \{C \geq 0 : \text{ for each } x \in X,\, q(Tx) \leq Cp(x)\}$. Let $C \in A$ and $x \in X$. Let $\epsilon > 0$. Then $\epsilon/[1 + p(x)] > 0$. Hence there exists $C \in A$ such that

$$C < \|T\|_{p,q} + \frac{\epsilon}{1 + p(x)}$$

Therefore,

$$\begin{aligned} q(Tx) &\leq Cp(x) \\ &\leq \left[\|T\|_{p,q} + \frac{\epsilon}{1 + p(x)}\right]p(x) \\ &< \|T\|_{p,q}p(x) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $q(Tx) \leq \|T\|_{p,q}p(x)$. Since $x \in X$ is arbitrary, $\|T\|_{p,q} \in A$. □

Exercise 5.9.0.4. Let X, Y be locally convex spaces with respective associated families of seminorms $\mathcal{P}$ and $\mathcal{Q}$, $p \in \mathcal{P}$, $q \in \mathcal{Q}$ and $T \in L(X;Y)$. Then

$$\|T\|_{p,q} = \sup\{q(Tx) : p(x) = 1\}$$

Proof. Let □

Exercise 5.9.0.5. Let X, Y be locally convex spaces with respective associated families of seminorms $\mathcal{P}$ and $\mathcal{Q}$ and $p \in \mathcal{P}$, $q \in \mathcal{Q}$. Then $\|\cdot\|_{p,q}$ is a seminorm on $L(X;Y)$.

Proof. Let $S, T \in L(X;Y)$ and $\lambda \in \mathbb{C}$.

1. Let $x \in X$. Then

$$\begin{aligned} q((S+T)(x)) &= q(Sx + Tx) \\ &\leq q(Sx) + q(Tx) \\ &\leq \|S\|_{p,q}p(x) + \|T\|_{p,q}p(x) \\ &= (\|S\|_{p,q} + \|T\|_{p,q})p(x) \end{aligned}$$

   Since $x \in X$ is arbitrary, $\|S + T\|_{p,q} \leq \|S + T\|_{p,q}$

2. Let $x \in X$. Then

$$\begin{aligned} q((\lambda T)(x)) &= q(\lambda Tx) \\ &= |\lambda|q(Tx) \\ &\leq |\lambda|\|T\|_{p,q}p(x) \end{aligned}$$

   Since $x \in X$ is arbitrary, $\|\lambda T\|_{p,q} \leq$

□

Chapter 6

Banach Spaces

6.1 Introduction

Note 6.1.0.1. In the following, we will consider vector spaces over $\mathbb{C}$. There are analogous results for real vector spaces as well, just replace every $\mathbb{C}$ with $\mathbb{R}$.

Definition 6.1.0.2. Let X be a normed vector space. Then X is said to be a **Banach space** if X is complete.

Definition 6.1.0.3. Let X be a normed vector space and $(x_i)_{i=1}^n \subset X$. The series $\sum_{i=1}^\infty x_i$ is said to **converge** if the sequence $s_n := \sum_{i=1}^n x_i$ converges. The series $\sum_{i=1}^\infty x_i$ is said to **converge absolutely** if $\sum_{i\in\mathbb{N}} \|x_i\| < \infty$.

Exercise 6.1.0.4. Let X be a normed vector space. Then X is complete iff for each $(x_i)_{i\in\mathbb{N}} \subset X$, $\sum_{i=1}^\infty x_i$ converges absolutely implies that $\sum_{i=1}^\infty x_i$ converges.
Hint: Given a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$, obtain a subsequence $(x_{n_j})_{j\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ such that for each $j \in \mathbb{N}$, $\|x_{n_{j+1}} - x_{n_j}\| < 2^{-j}$. Define a new sequence $(y_j)_{j\in\mathbb{N}} \subset X$ by

$$y_j = \begin{cases} x_{n_1} & j = 1 \\ x_{n_j} - x_{n_{j-1}} & j \geq 2 \end{cases}$$

Proof. Suppose that X is complete. Let $(x_i)_{i\in\mathbb{N}} \subset X$. Suppose that $\sum_{i=1}^\infty x_i$ converges absolutely. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq N$ and $m < n$, then $\sum_{m+1}^n \|x_i\| < \epsilon$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then

$$\begin{aligned} \|s_n - s_m\| &= \left\| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right\| \\ &= \left\| \sum_{i=m+1}^n x_i \right\| \\ &\leq \sum_{i=m+1}^n \|x_i\| \\ &< \epsilon \end{aligned}$$

Thus $(s_n)_{n\in N}$ is Cauchy. Since X is complete, $\sum_{i=1}^\infty x_i$ converges.
Conversely, Suppose that for each $(x_i)_{i\in\mathbb{N}} \subset X$, $\sum_{i=1}^\infty x_i$ converges absolutely implies that $\sum_{i=1}^\infty x_i$ converges. Let $(x_i)_{i\in\mathbb{N}} \subset X$ be Cauchy. Proceed inductively to create a strictly increasing sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq n_i$, then $\|x_m - x_n\| < 2^{-i}$. Define $(y_i)_{i\in\mathbb{N}} \subset X$ by

$$y_i = \begin{cases} x_{n_1} & i = 1 \\ x_{n_i} - x_{n_{i-1}} & i \geq 2 \end{cases}$$

Then $\sum_{i=1}^k y_i = x_{n_k}$ and

$$\begin{aligned}\sum_{i\in\mathbb{N}}\|y_i\| &= \|x_{n_1}\| + \sum_{i\in\mathbb{N}}\|x_{n_i} - x_{n_{i-1}}\| \\ &\leq \|x_{n_1}\| + 2\sum_{i\in\mathbb{N}} 2^{-i} \\ &= \|x_{n_1}\| + 2\end{aligned}$$

Hence $(x_{n_k})_{k\in\mathbb{N}} = (\sum_{i=1}^k y_i)_{i\in\mathbb{N}}$ converges. Since $(x_i)_{i\in\mathbb{N}}$ is cauchy and has a convergent subsequence, it converges. So X is complete. □

Exercise 6.1.0.5. Let X be a normed vector space. Then addition $X\times X\to X$ and scalar multiplication $\mathbb{C}\times X\to X$ are continuous and $\|\cdot\| : X\to[0,\infty)$ is continuous.

Proof. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Let $(x_1, y_1), (x_2, y_2) \in X\times X$. Suppose that

$$\max\{\|x_1 - x_2\|, \|y_1 - y_2\|\} < \delta$$

Then

$$\begin{aligned}\|(x_1 + y_1) - (x_2 + y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\ &\leq \|x_1 - x_2\| + \|y_1 - y_2\| \\ &< 2\delta \\ &= \epsilon\end{aligned}$$

Hence addition is uniformly continuous.

Let $(\lambda_1, x_1) \in \mathbb{C}\times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(|\lambda_1|+\|x_1\|+1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(\lambda_2, x_2) \in \mathbb{C}\times X$. Suppose that

$$\max\{|\lambda_1 - \lambda_2|, \|x_1 - x_2\|\} < \delta$$

Then

$$\begin{aligned}\|\lambda_1 x_1 - \lambda_2 x_2\| &= \|\lambda_1 x_1 - \lambda_1 x_2 + \lambda_1 x_2 - \lambda_2 x_2\| \\ &= \|\lambda_1(x_1 - x_2) + (\lambda_1 - \lambda_2)x_2\| \\ &\leq |\lambda_1|\|x_1 - x_2\| + |\lambda_1 - \lambda_2|\|x_2\| \\ &\leq |\lambda_1|\|x_1 - x_2\| + |\lambda_1 - \lambda_2|(\|x_1 - x_2\| + \|x_1\|) \\ &< |\lambda_1|\delta + \delta(\delta + \|x_1\|) \\ &= (|\lambda_1| + \|x_1\|)\delta + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

Since $(\lambda_1, x_1) \in \mathbb{C}\times X$ is arbitrary, scalar multiplication is continuous.

Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x, y \in X$. Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned}\big|\|x\| - \|y\|\big| &\leq \|x - y\| \\ &< \delta \\ &= \epsilon\end{aligned}$$

So $\|\cdot\| : X\to[0,\infty)$ is uniformly continuous. □

6.2 Bounded Operators

Definition 6.2.0.1. Let X, Y be a normed vector spaces and $T : X \to Y$ linear. Then T is said to be **bounded** if $T(\operatorname{cl} B(0,1))$ is bounded. We define

$$L(X;Y) = \{T : X \to Y : T \text{ is linear and bounded}\}$$

When $X = Y$, we write $L(X)$.

Exercise 6.2.0.2. Let X, Y be normed vector spaces and $T : X \to Y$ linear. Then T is bounded iff there exists $C \geq 0$ such that for each $x \in X$,

$$\|Tx\| \leq C\|x\|$$

Proof. Suppose that T is bounded. If $T = 0$, choose $C = 0$. Suppose that $T \neq 0$. Set $A = \{\|Tx\| : \|x\| = 1\}$. Since $T \neq 0$, there exists $x_0 \in X$ such that $\|x_0\| = 1$ so that $A \neq \varnothing$. Boundedness of T implies that A is bounded. Set $C = \sup A$. Let $x \in X$. If $x = 0$, then $Tx = 0$ and $\|Tx\| \leq C\|x\|$. Suppose that $x \neq 0$. Then $Tx = \|x\|T(\|x\|^{-1}x)$. Since $\|\|x\|^{-1}x\| = 1$, we have that

$$\begin{aligned} \|Tx\| &= \|T(\|x\|^{-1}x)\|\|x\| \\ &\leq C\|x\| \end{aligned}$$

Conversely, suppose that there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Let $x \in \operatorname{cl} B(0,1)$. Then

$$\begin{aligned} \|Tx\| &\leq C\|x\| \\ &\leq C \end{aligned}$$

So that $T(\operatorname{cl} B(0,1))$ is bounded. □

Exercise 6.2.0.3. Set $X = C^1([0,1])$ and $Y = C([0,1])$. Equip both X and Y with the sup norm. Define $T : X \to Y$ by $Tf = f'$. Then T is not bounded.

Proof. For the sake of contradiction, suppose that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\| \leq C\|f\|$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f \in X$ by $f(x) = x^n$. Then

$$\begin{aligned} n &= \|Tf\| \\ &\leq C\|f\| \\ &= C \end{aligned}$$

which is a contradiction. Hence T is not bounded. □

Exercise 6.2.0.4. Let X, Y be a normed vector spaces and $T : X \to Y$ a linear map. Then T is bounded iff there exists $r, s > 0$ such that $T(B(0,r)) \subset B(0,s)$

Proof. Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Thus $T(B(0,1)) \subset B(0, C+1)$. Conversely. Suppose that there exists $r, s > 0$ such that $T(B(0,r)) \subset B(0,s)$. Define $C = \frac{2s}{r}$. Let $x \in X$. Put $\alpha = \frac{r}{2\|x\|}$ Then $\alpha x \in B(0,r)$. So $T(\alpha x) = \alpha T(x) \in B(0,s)$. Hence

$$\begin{aligned} \|T(\alpha x)\| &= \|\alpha T(x)\| \\ &= |\alpha|\|T(x)\| \\ &= \frac{r}{2\|x\|}\|T(x)\| \\ &< s. \end{aligned}$$

Thus

$$\|Tx\| < \frac{2s}{r}\|x\| = C\|x\|$$

So T is bounded. □

Exercise 6.2.0.5. Let X, Y be normed vector spaces and $T : X \to Y$. Suppose that T is linear. Then there exists $x_0 \in X$ such that T is continuous at x_0 iff T is continuous at 0.

Proof. Suppose that there exists $x_0 \in X$ such that T is continuous at x_0. Since T is linear, $T(0) = 0$. Let $(x_n)_{n\in\mathbb{N}} \subset X$. Suppose that $x_n \to 0$. Then $x_n + x_0 \to x_0$. Hence

$$\begin{aligned} T(x_n) + T(x_0) &= T(x_n + x_0) \\ &\to T(x_0) \end{aligned}$$

This implies that

$$\begin{aligned} T(x_n) &\to 0 \\ &= T(0) \end{aligned}$$

Therefore T is continuous at 0.
Conversely, if T is continuous at 0, then trivially, there exists $x_0 \in X$ such that T is continuous at x_0. $\square$

Exercise 6.2.0.6. Let X, Y be normed vector spaces and $T : X \to Y$ a linear map. Then the following are equivalent:

1. T is continuous
2. T is continuous at $x = 0$
3. T is bounded

Proof.

- (1) $\implies$ (2):
  Trivial
- (2) $\implies$ (3):
  Suppose that T is continuous at $x = 0$. Then there exists $\delta > 0$ such that for each $x \in X$, if $\|x\| < \delta$, then $\|Tx\| < 1$. Choose $C = \frac{2}{\delta}$. If $x = 0$, then $\|Tx\| \leq C\|x\|$. Suppose that $\|x\| \neq 0$. Define $y = \frac{\delta}{2\|x\|}x$. Then $\|y\| < \delta$. So

  $$\begin{aligned} 1 &> \|Ty\| \\ &= \frac{\delta}{2\|x\|}\|Tx\| \end{aligned}$$

  Thus

  $$\begin{aligned} \|Tx\| &< \frac{2}{\delta}\|x\| \\ &= C\|x\| \end{aligned}$$

  Hence T is bounded.
- (3) $\implies$ (1)
  Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{C+1}$. Let $x, y \in X$ Suppose that $\|x - y\| < \delta$. Then

  $$\begin{aligned} \|Tx - Ty\| &= \|T(x - y)\| \\ &\leq C\|x - y\| \\ &< (C + 1)\delta \\ &= \epsilon \end{aligned}$$

  So T is continuous.

□

Definition 6.2.0.7. Let X, Y be normed vector spaces. Define $\|\cdot\| : L(X;Y) \to [0,\infty)$ by

$$\|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \ \|Tx\| \leq C\|x\|\}$$

We call $\|\cdot\|$ the **operator norm on** $L(X;Y)$

Exercise 6.2.0.8. Let X, Y be normed vector spaces. If $X \neq \{0\}$, then the operater norm on $L(X;Y)$ is given by:

1. $\|T\| = \sup\limits_{\|x\|=1} \|Tx\|$
2. $\|T\| = \sup\limits_{x \neq 0} \|x\|^{-1}\|Tx\|$
3. $\|T\| = \inf\{C \geq 0 : \text{for each } x \in X, \ \|Tx\| \leq C\|x\|\}$

Proof. Since $X \neq \{0\}$, the supremums in (1) and (2) are well defined. Let $T \in L(X;Y)$. By linearity of T, the sets over which the supremums are taken in (1) and (2) are the same. So (1) and (2) are equal.
Now, set $M = \sup\limits_{\|x\|=1} \|Tx\|$ and $m = \inf\{C \geq 0 : \text{ for each } x \in X, \ \|Tx\| \leq C\|x\|\}$. Let $x \in X$. If $\|x\| = 0$, then $\|Tx\| \leq M\|x\|$. Suppose that $\|x\| \neq 0$. Then

$$\begin{aligned} \|Tx\| &= \bigg(\big\|T(\|x\|^{-1}x)\big\|\bigg)\|x\| \\ &\leq M\|x\| \end{aligned}$$

Hence $M \in \{C \geq 0 : \text{ for each } x \in X, \ \|Tx\| \leq C\|x\|\}$ and $m \leq M$. Let $C \in \{C \geq 0 : \text{ for each } x \in X, \ \|Tx\| \leq C\|x\|\}$. Suppose that $\|x\| = 1$. Then $\|Tx\| \leq C\|x\| = C$. So $M \leq C$. Therefore $M \leq m$. So $M = m$ and the supremum in (1) is the same as the infimum in (3). □

Note 6.2.0.9. From here on, unless stated otherwise, we assume $X \neq 0$.

Exercise 6.2.0.10. Let X, Y be normed vector spaces and $T \in L(X;Y)$. Then for each $x \in X$, $\|Tx\| \leq \|T\|\|x\|$

Proof. Let $x \in X$. If $x = 0$, then $\|Tx\| \leq \|T\|\|x\|$. Suppose that $x \neq 0$. The previous exercise implies that

$$\begin{aligned} \|Tx\| &= \|T(\|x\|^{-1}x)\|\|x\| \\ &\leq \bigg(\sup_{\|x\|=1} \|Tx\|\bigg)\|x\| \\ &= \|T\|\|x\| \end{aligned}$$

□

Exercise 6.2.0.11. Let X, Y be normed vector spaces. Then $L(X;Y)$ is a vector space and the operator norm is a norm on $L(X;Y)$.

Proof. Let $S, T \in L(X;Y)$ and $\alpha \in \mathbb{C}$.

- It is clear that $S + T$ is linear. For each $x \in X$,

$$\begin{aligned} \|(S+T)x\| &= \|Sx + Tx\| \\ &\leq \|Sx\| + \|Tx\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= \big(\|S\| + \|T\|\big)\|x\| \end{aligned}$$

  So $S + T \in L(X;Y)$ and $\|S + T\| \leq \|S\| + \|T\|$

- It is clear that αT is linear. For each $x \in X$,

$$\begin{aligned}\|\alpha T\| &= \sup_{\|x\|=1} \|(\alpha T)x\| \\ &= \sup_{\|x\|=1} |\alpha|\|Tx\| \\ &= |\alpha| \sup_{\|x\|=1} \|Tx\| \\ &= |\alpha|\|T\|\end{aligned}$$

  So $\alpha T \in L(X;Y)$ and $\|\alpha T\| \leq |\alpha|\|T\|$.

- Suppose that $\|T\| = 0$. Let $x \in X$. Then

$$\begin{aligned}\|Tx\| &\leq \|T\|\|x\| \\ &= 0\end{aligned}$$

  So $Tx = 0$. Since $x \in X$ is arbitrary, we have that $T = 0$.

Therefore $L(X;Y)$ is a vector space and $\|\cdot\| : L(X;Y) \to [0, \infty)$ is a norm. □

Exercise 6.2.0.12. Let X, Y, Z be normed vector spaces, $T \in L(X;Y)$ and $S \in L(Y, Z)$. Define $ST : X \to Z$ by $STx = S(Tx)$. Then $ST \in L(X, Z)$ and $\|ST\| \leq \|S\|\|T\|$.

Proof. Clearly ST is linear. Let $x \in X$. Then

$$\begin{aligned}\|STx\| &= \|S(Tx)\| \\ &\leq \|S\|\|Tx\| \\ &\leq \|S\|\|T\|\|x\|\end{aligned}$$

So $\|ST\| \leq \|S\|\|T\|$. □

Definition 6.2.0.13. Let X, Y be a normed vector spaces and $T \in L(X;Y)$. Then T is said to be **invertible** or an **isomorphism** if T is a bijection and $T^{-1} \in L(Y, X)$.

Definition 6.2.0.14. Let X be a normed vector space. Define $GL(X) := \{T \in L(X, X) : T \text{ is invertible}\}$.

Exercise 6.2.0.15. Let X, Y be normed vector spaces. If Y is complete, then $L(X;Y)$ is complete.

Proof. Suppose that Y is complete. Let $(T_n)_{n\in\mathbb{N}} \subset L(X;Y)$. Suppose that $(T_n)_{n\in\mathbb{N}}$ is Cauchy.

- Since for each $m, n \in \mathbb{N}$, $\big|\|T_m\| - \|T_n\|\big| \leq \|T_m - T_n\|$, we have that $(\|T_n\|)_{n\in\mathbb{N}} \subset [0, \infty)$ is Cauchy. Hence $\lim\limits_{n\to\infty} \|T_n\|$ exists.

  Let $x \in X$ and $m, n \in \mathbb{N}$. Then

$$\begin{aligned}\|T_m x - T_n x\| &= \|(T_m - T_n)x\| \\ &\leq \|T_m - T_n\|\|x\|\end{aligned}$$

  So $(T_n x)_{n\in\mathbb{N}} \subset Y$ is Cauchy and hence converges. Define $T : X \to Y$ by $Tx = \lim\limits_{n\to\infty} T_n x$. Since addition and scalar multiplication are continuous, for each $x_1, x_2 \in X$ and $\lambda \in \mathbb{K}$,

$$\begin{aligned}T(x_1 + \lambda x_2) &= \lim_{n\to\infty} T_n(x_1 + \lambda x_2) \\ &= \lim_{n\to\infty} [T_n(x_1) + \lambda T_n(x_2)] \\ &= \lim_{n\to\infty} T_n(x_1) + \lambda \lim_{n\to\infty} T_n(x_2) \\ &= T(x_1) + \lambda T(x_2).\end{aligned}$$

Hence T is linear. Let $x \in X$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in N$, if $n \geq N$, then $\|Tx - T_n x\| < \epsilon$. Then for each $n \in \mathbb{N}$, if $n \geq N$ we have that

$$\begin{aligned}\|Tx\| &\leq \|Tx - T_n x\| + \|T_n x\| \\ &< \epsilon + \|T_n x\| \\ &\leq \epsilon + \|T_n\|\|x\|\end{aligned}$$

Thus $\|Tx\| \leq \epsilon + (\lim_{n\to\infty} \|T_n\|)\|x\|$. Since $\epsilon > 0$ is arbitrary, $\|Tx\| \leq (\lim_{n\to\infty} \|T_n\|)\|x\|$. Since $x \in X$ is arbitrary, $T \in L(X;Y)$ and $\|T\| \leq \lim_{n\to\infty} \|T_n\|$.

- Let $\epsilon > 0$. Since $(T_n)_{n\in\mathbb{N}}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $n, m \geq N$ implies that $\|T_n - T_m\| < \epsilon$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Since addition $A : X \times X \to X$, scalar multiplication $M : \mathbb{K} \times X \to X$ and the norm $\|\cdot\| : X \to [0, \infty)$ are continuous, we have that for each $x \in X$,

$$\begin{aligned}\|(T_n - T)(x)\| &\|T_n(x) - T(x)\| \\ &\|T_n(x) - \lim_{m\to\infty} T_m(x)\| \\ &= \lim_{m\to\infty} \|T_n(x) - T_m(x)\| \\ &= \liminf_{m\to\infty} \|T_n(x) - T_m(x)\| \\ &\leq \liminf_{m\to\infty} \|T_n - T_m\|\|x\| \\ &\leq \epsilon\|x\|.\end{aligned}$$

  Hence $\|T_n - T\| \leq \epsilon$. Since $n \in \mathbb{N}$ with $n \geq N$ is arbitrary, we have that for each $n \in \mathbb{N}$, $n \geq N$ implies that $\|T_n - T\| \leq \epsilon$. Therefore $T_n \to T$.

Since $(T_n)_{n\in\mathbb{N}} \subset L(X;Y)$ with $(T_n)_{n\in\mathbb{N}}$ Cauchy is arbitrary, we have that for each $(T_n)_{n\in\mathbb{N}} \subset L(X;Y)$, $(T_n)_{n\in\mathbb{N}}$ is Cauchy implies that there exists $T \in L(X;Y)$ such that $T_n \to T$. Hence $L(X;Y)$ is complete. $\square$

6.3 Direct Sums

Definition 6.3.0.1. Let X, Y be normed vector spaces and $p \in [1, \infty]$. Let $\| \cdot \|'_p : \mathbb{R}^2 \to [0, \infty)$ denote the usual l^p norm. We define $\| \cdot \|_p : X \oplus Y \to [0, \infty)$ by

$$\|(x, y)\|_p = \|(\|x\|, \|y\|)\|'_p$$

Exercise 6.3.0.2. Let X, Y be normed vector spaces. Then

1. for each $p \in [1, \infty]$, $\| \cdot \|_p : X \oplus Y \to [0, \infty)$ is a norm on $X \oplus Y$
2. $\{\| \cdot \|_p : p \in [1, \infty]\}$ are equivalent.

Proof.

1. Let $p \in [1, \infty]$, $(x_1, y_1), (x_2, y_2) \in X \oplus Y$ and $\lambda \in \mathbb{C}$.
   - Clearly if $(x_1, y_1) = (0, 0)$, then $\|S\|_p = 0$. Conversely, suppose that $\|(x_1, y_1)\|_p = 0$. Then $\|x_1\| = 0$ and $\|y_1\| = 0$. So $x_1 = 0$ and $y_1 = 0$. Therefore $S = 0$.
   - $$\begin{aligned} \|\lambda(x_1, y_1)\|_p &= \|(\|\lambda x_1\|, \|\lambda y_1\|)\|'_p \\ &= \|(|\lambda|\|x_1\|, |\lambda|\|y_1\|)\|'_p \\ &= \||\lambda|(\|x_1\|, \|y_1\|)\|'_p \\ &= |\lambda|\|(\|x_1\|, \|y_1\|)\|'_p \\ &= |\lambda|\|(x_1, y_1)\|_p \end{aligned}$$
   - $$\begin{aligned} \|(x_1, y_1) + (x_2, y_2)\|_p &= \|(\|x_1 + x_2\|, \|y_1 + y_2\|)\|'_p \\ &\le \|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|'_p \\ &= \|(\|x_1\|, \|y_1\|) + (\|x_2\|, \|y_2\|)\|'_p \\ &\le \|(\|x_1\|, \|y_1\|)\|'_p + \|(\|x_2\|, \|y_2\|)\|'_p \\ &= \|(x_1, y_1)\|_p + \|(x_2, y_2)\|_p \end{aligned}$$
2. All norms on $\mathbb{R}^2$ are equivalent.

□

Exercise 6.3.0.3. Let X, Y be Banach spaces. Then $X \oplus Y$ equipped with $\| \cdot \|_p : X \oplus Y \to [0, \infty)$ is a Banach space.

Proof. □

Exercise 6.3.0.4. Let X, Y and Z be Banach spaces and $p \in [0, \infty]$. Equip $Y \oplus Z$ with $\| \cdot \|_p$. Let $T \in L(X, Y \oplus Z)$ with $T = (T_Y, T_Z)$. Then $T_Y \in L(X; Y)$ and $T_Z \in L(X, Z)$.

Proof. Let $x \in X$. Then $\|T_Y(x)\|, \|T_Z(x)\| \le$
FINISH!!! □

Definition 6.3.0.5. Let X, Y and Z be Banach spaces and $p \in [0, \infty]$. Let $\| \cdot \|'_p : \mathbb{R}^2 \to [0, \infty)$ denote the usual l^p norm. Equip $Y \oplus Z$ with $\| \cdot \|_p$. Let $T \in L(X, Y \oplus Z)$ with $T = (T_Y, T_Z)$. Define $\| \cdot \|_p : L(X, Y \oplus Z) \to [0, \infty)$ by

$$\|T\|_p = \|(\|T_Y\|, \|T_Z\|)\|'_p$$

Exercise 6.3.0.6. Let X, Y and Z be Banach spaces and $p \in [0, \infty]$. Then $\|\cdot\|_p : L(X, Y \oplus Z) \to [0, \infty)$ is a norm on $L(X, Y \oplus Z)$.

Proof. Let $\lambda \in \mathbb{C}$ and $S, T \in L(X, Y \oplus Z)$ with $S = (S_Y, S_Z)$ and $T = (T_Y, T_Z)$.

- Clearly if $S = 0$, then $\|S\|_p = 0$. Conversely, suppose that $\|S\|_p = 0$. Then $\|S_Y\| = 0$ and $\|S_Z\| = 0$. So $S_Y = 0$ and $S_Z = 0$. Therefore $S = 0$.
-
$$\begin{aligned}
\|\lambda S\|_p &= \|(\|\lambda S_Y\|, \|\lambda S_Z\|)\|_p' \\
&= \|(|\lambda|\|S_Y\|, |\lambda|\|S_Z\|)\|_p' \\
&= \||\lambda|(\|S_Y\|, \|S_Z\|)\|_p' \\
&= |\lambda|\|(\|S_Y\|, \|S_Z\|)\|_p' \\
&= |\lambda|\|S\|_p
\end{aligned}$$
-
$$\begin{aligned}
\|S + T\|_p &= \|(\|S_Y + T_Y\|, \|S_Z + T_Z\|)\|_p' \\
&\le \|(\|S_Y\| + \|T_Y\|, \|S_Z\| + \|T_Z\|)\|_p' \\
&= \|(\|S_Y\|, \|S_Z\|) + (\|T_Y\|, \|T_Z\|)\|_p' \\
&\le \|(\|S_Y\|, \|S_Z\|)\|_p' + \|(\|T_Y\|, \|T_Y\|)\|_p' \\
&= \|S\|_p + \|T\|_p
\end{aligned}$$

So $\|\cdot\|_p : L(X, Y \oplus Z) \to [0, \infty)$ is a norm on $L(X, Y \oplus Z)$. $\square$

Exercise 6.3.0.7. Let X, Y and Z be Banach spaces and $p \in [0, \infty]$. Equip $Y \oplus Z$ with $\|\cdot\|_p$. Let $T \in L(X, Y \oplus Z)$ with $T = (T_Y, T_Z)$. Then $\|T\| \le 2^{1/p}\|T\|_p$.

Proof. Let $x \in X$. If $p < \infty$, then

$$\begin{aligned}
\|T(x)\|_p &= \|(T_Y(x), T_Z(x))\|_p \\
&\|(\|T_Y(x)\|, \|T_Z(x)\|)\|_p' \\
&= \left(\|T_Y(x)\|^p + \|T_Z(x)\|^p\right)^{1/p} \\
&\le \left(\|T_Y\|^p\|x\|^p + \|T_Z\|^p\|x\|^p\right)^{1/p} \\
&\le \left[(\|T_Y\|^p + \|T_Z\|^p)\|x\|^p + (\|T_Y\|^p + \|T_Z\|^p)\|x\|^p\right]^{1/p} \\
&= \left[2(\|T_Y\|^p + \|T_Z\|^p)\|x\|^p\right]^{1/p} \\
&= 2^{1/p}\|T\|_p\|x\|
\end{aligned}$$

Hence $\|T\| \le 2^{1/p}\|T\|_p$ If $p = \infty$, then

$$\begin{aligned}
\|T(x)\|_\infty &= \max(\|T_Y(x)\|, \|T_Z(x)\|) \\
&\le \max(\|T_Y\|\|x\|, \|T_Z\|\|x\|) \\
&\le \max\left[\max(\|T_Y\|, \|T_Z\|)\|x\|, \max(\|T_Y\|, \|T_Z\|)\|x\|\right] \\
&= \max(\|T_Y\|, \|T_Z\|)\|x\| \\
&= \|T\|_\infty\|x\|
\end{aligned}$$

Hence

$$\begin{aligned}\|T\| &\leq \|T\|_\infty \\ &= 2^{1/\infty}\|T\|_\infty\end{aligned}$$

□

Exercise 6.3.0.8. Let X and $X_1, \cdots, X_n$ be Banach spaces and $p \in [0, \infty]$. Equip $\bigoplus_{j=1}^{n} X_j$ with $\|\cdot\|_p$. Let $T \in L(X, \bigoplus_{j=1}^{n} X_j)$. Then $\|T\| \leq n^{1/p}\|T\|_p$.

Proof. Similar to the previous exercise. □

6.4 Quotient Spaces

Definition 6.4.0.1. Let X be a normed vector space and $M \subset X$ a closed subspace. Define $\|\cdot\| : X/M \to [0, \infty)$ by

$$\|x + M\| := \inf_{y \in M} \|x + y\|$$

We call $\|\cdot\|$ the **subspace norm on** X/M

Exercise 6.4.0.2. Let X be a normed vector space and $M \subsetneq X$ a proper, closed subspace of M. Then

1. The previously defined subspace norm on X/M is well defined and is a norm.
2. For each $\epsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.
3. The projection map $\pi : X \to X/M$ defined by $\pi(x) = x + M$ is continuous and $\|\pi\| = 1$.
4. If X is complete, then X/M is complete.

Proof.

1. Let $x, y \in X$ and $\alpha \in \mathbb{C}$. Suppose that $x + M = y + M$. Then there exists $m \in M$ such that $x = y + m$. Since M is a subspace, the map $T : M \to M$ given by $Tx = x + m$ is a bijection. So

$$\inf_{z \in M} \|y + m + z\| = \inf_{z \in M} \|y + z\|$$

which implies that

$$\begin{aligned} \|x + M\| &= \inf_{z \in M} \|x + z\| \\ &= \inf_{z \in M} \|y + m + z\| \\ &= \inf_{z \in M} \|y + z\| \\ &= \|y + M\| \end{aligned}$$

So $\|\cdot\| : X/M \to [0, \infty)$ is well defined.

We observe that for each $z, w \in M$,

$$\|x + y + z\| \leq \|x + w\| + \|y + w + z\|$$

Taking infimums over M with respect to z in this inequality implies that for each $w \in M$,

$$\begin{aligned} \inf_{z \in M} \|x + y + z\| &\leq \inf_{z \in M} \left(\|x + w\| + \|y + w + z\| \right) \\ &= \|x + w\| + \inf_{z \in M} \|y + w + z\| \end{aligned}$$

Again we use the fact that for each $w \in M$,

$$\inf_{z \in M} \|y + w + z\| = \inf_{z \in M} \|y + z\|$$

This implies that for each $w \in M$,

$$\inf_{z \in M} \|x + y + z\| \leq \|x + w\| + \inf_{z \in M} \|y + z\|$$

Therefore, taking infimums over M with respect to w in this inequality yields

$$\begin{aligned} \|x + y + M\| &= \inf_{z \in M} \|x + y + z\| \\ &\leq \inf_{w \in M} \left(\|x + w\| + \inf_{z \in M} \|y + z\| \right) \\ &= \inf_{w \in M} \|x + w\| + \inf_{z \in M} \|y + z\| \\ &= \|x + M\| + \|y + M\| \end{aligned}$$

If $\alpha = 0$, then $\alpha x = 0$. Choosing $z = 0 \in M$ gives $\|\alpha x + M\| = 0 = |\alpha|\|x + M\|$. Suppose that $\alpha \neq 0$. Then the map $T : M \to M$ given by $Tx = \alpha^{-1}x$ is a bijection and thus $\inf\limits_{z\in M} \|x+\alpha^{-1}z\| = \inf\limits_{z\in M} \|x+z\|$. Hence we have that

$$\begin{aligned}
\|\alpha x + M\| &= \inf_{z\in M} \|\alpha x + z\| \\
&= \inf_{z\in M} |\alpha|\|x + \alpha^{-1}z\| \\
&= |\alpha| \inf_{z\in M} \|x + \alpha^{-1}z\| \\
&= |\alpha| \inf_{z\in M} \|x + z\| \\
&= |\alpha|\|x + M\|
\end{aligned}$$

Suppose that $\|x\| = 0$. Choose a sequence $(z_n)_{n\in N} \subset M$ such that

$$\begin{aligned}
\lim_{n\to\infty} \|x - z_n\| &= \inf_{z\in M} \|x + z\| \\
&= 0
\end{aligned}$$

Then $\lim\limits_{n\to\infty} z_n = x$. Since M is closed, $x \in M$. Hence $x + M = 0 + M$.

2. Since M is a proper subspace, there exists $v \in X$ such that $v \notin M$. Then $\|v + M\| \neq 0$. Let $\epsilon > 0$. Then

$$\begin{aligned}
(1-\epsilon)^{-1}\|v + M\| &> \|v + M\| \\
&= \inf_{y\in M} \|x + y\|
\end{aligned}$$

So there exists $z \in M$ such that $\|v + z\| < (1-\epsilon)^{-1}\|v + M\|$. Since $v + M \neq 0 + M$, we have that $v + z \neq 0$. Choose $x = \|v + z\|^{-1}(v + z)$. Then $\|x\| = 1$ and

$$\begin{aligned}
\|x + M\| &= \|v + z\|^{-1}\|v + z + M\| \\
&= \|v + z\|^{-1}\|v + M\| \\
&> 1 - \epsilon
\end{aligned}$$

3. Let $x \in X$. Taking $z = 0$, we we see that $\|\pi(x)\| = \|x + M\| \leq \|x + z\| = \|x\|$. So π is bounded and in particular,

$$\sup_{\|x\|=1} \|\pi(x)\| \leq 1$$

From (2) we see that

$$\sup_{\|x\|=1} \|\pi(x)\| \geq 1$$

Hence $\|\pi\| = 1$.

4. Suppose that X is complete. Let $(x_i + M)_{i\in\mathbb{N}} \subset X/M$. Suppose that $\sum\limits_{i\in\mathbb{N}} \|x_i + M\| < \infty$. Let $\epsilon > 0$. Then for each $i \in \mathbb{N}$, there exists $z_i \in M$ such that $\|x_i + z_i\| < \|x_i + M\| + \epsilon 2^{-i}$. Define the sequence

$(a_i)_{i\in\mathbb{N}} \subset X$ by $a_i = x_i + z_i$. Then we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|a_i\| &= \sum_{i\in N} \|x_i + z_i\| \\ &\leq \sum_{i\in N} \left(\|x_i + M\| + \epsilon 2^{-i}\right) \\ &= \sum_{i\in\mathbb{N}} \|x_i + M\| + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\sum_{i\in\mathbb{N}} \|a_i\| \leq \sum_{i\in\mathbb{N}} \|x_i + M\| < \infty$$

Since X is complete, $\sum\limits_{i=1}^{\infty} a_i$ converges in X. Define $(s_n)_{n\in\mathbb{N}} \subset X$ and $s \in X$ by $s_n = \sum\limits_{i=1}^{n} a_i$ and $s = \sum\limits_{i=1}^{\infty} a_i$. Since $\lim\limits_{n\to\infty} s_n = s$, and $\pi : X \to X/M$ is continuous, it follows that $\lim\limits_{n\to\infty} \pi(s_n) = \pi(s)$. Since

$$\begin{aligned}\pi(s_n) &= \sum_{i=1}^{n} a_i + M \\ &= \sum_{i=1}^{n} x_i + M\end{aligned}$$

We have that $\sum\limits_{i=1}^{\infty} x_i + M$ converges which implies that X/M is complete.

□

Exercise 6.4.0.3. Let X, Y be normed vector spaces and $T \in L(X;Y)$. Then

1. $\ker T$ is closed
2. there exists a unique map $S : X/\ker T \to T(X)$ such that $T = S \circ \pi$. Furthermore S is a bounded linear bijection and $\|S\| = \|T\|$.

Proof.

1. Since T is continuous and $\ker T = T^{-1}(\{0\})$, we have that $\ker T$ is closed.
2. Suppose that there exists $S_1, S_2 \in L(X/\ker T, T(X))$ such that $T = S_1 \circ \pi$ and $T = S_2 \circ \pi$. Let $x \in X$. Then
$$S_1(x + \ker T) = S_1(\pi(x)) = T(x) = S_2(\pi(x)) = S_2(x + \ker T)$$
So $S_1 = S_2$. Therefore such a map is unique.
Define $S : X/\ker T \to T(X)$ by $S(x + \ker T) = T(x)$. Then S is clearly a linear bijection that satisfies $T = S \circ \pi$. Let $x \in X$ and $z \in \ker T$. Then
$$\begin{aligned}\|S(x + \ker T)\| &= \|T(x)\| \\ &= \|T(x + z)\| \\ &\leq \|T\|\|x + z\|\end{aligned}$$
Thus
$$\|S(x + \ker T)\| \leq \|T\| \inf_{z\in\ker T} \|x + z\| = \|T\|\|x + \ker T\|$$
So S is bounded and $\|S\| \leq \|T\|$. This implies that
$$\|T\| = \|S \circ \pi\| \leq \|S\|\|\pi\| = \|S\|$$
Thus $\|S\| = \|T\|$.

□

Exercise 6.4.0.4. Let X, Y be normed vector spaces. Define $\phi : L(X;Y) \times X \to Y$ by $\phi(T, x) = Tx$. Then ϕ is continuous.

Proof. Let $(T_1, x_1) \in L(X;Y) \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|x_1\|+\|T_1\|+1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(t_2, x_2) \in L(X;Y) \times X$. Suppose that

$$\|(T_1, x_1) - (T_2, x_2)\| = \max\{\|T_1 - T_2\|, \|x_1 - x_2\|\} < \delta$$

Then

$$\begin{aligned}
\|\phi(T_1, x_1) - \phi(T_2 - x_2)\| &= \|T_1 x_- T_2 x_2\| \\
&= \|T_1 x_1 - T_2 x_1 + T_2 x_1 - T_2 x_2\| \\
&\leq \|(T_1 - T_2)x_1\| + \|T_2(x_1 - x_2)\| \\
&\leq \|T_1 - T_2\|\|x_1\| + \|T_2\|\|x_1 - x_2\| \\
&\leq \|T_1 - T_2\|\|x_1\| + \big(\|T_1 - T_2\| + \|T_1\|\big)\|x_1 - x_2\| \\
&< \delta\|x_1\| + (\delta + \|T_1\|)\delta \\
&= \delta(\|T_1\| + \|x_1\|) + \delta^2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So ϕ is continuous. □

Exercise 6.4.0.5. Let X be a normed vector space and $M \subset X$ a subspace. Then $\overline{M}$ is a subspace.

Proof. Let $x, y \in \overline{M}$ and $\alpha \in \mathbb{C}$. Then there exist sequences $(x_n)_{n\in\mathbb{N}} \subset M$ and $(y_n)_{n\in\mathbb{N}} \subset M$ such that $x_n \to x$ and $y_n \to y$. Since M is a subspace, $(x_n + y_n)_{n\in\mathbb{N}} \subset M$ and $(\alpha x_n)_{n\in\mathbb{N}} \subset M$. Since addition and scalar multiplication are continuous, we have that $x_n + y_n \to x + y$ and $\alpha x_n \to \alpha x$. Thus $x + y \in \overline{M}$ and $\alpha x \in \overline{M}$ and hence $\overline{M}$ is a subspace. □

6.5 Applications of the Hahn-Banach Theorem

Definition 6.5.0.1. Let X be a normed vector space over $\mathbb{C}$, and $T : X \to \mathbb{C}$. Then T is said to be a **bounded linear functional on** X if $T \in L(X, \mathbb{C})$. We define the **dual space of** X, denoted X^*, by $X^* = L(X, \mathbb{C})$.

Note 6.5.0.2. We define X^* similarly when X is a normed vector space over $\mathbb{R}$.

Definition 6.5.0.3. Let X be a normed vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then p is said to be **bounded** if there exists $M > 0$ such that for each $x \in X$, $p(x) \leq M\|x\|$.

Exercise 6.5.0.4. Let X be a normed vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then p is bounded iff p is Lipschitz.

Proof. Suppose that p is bounded. Then there exists $M > 0$ such that for each $x \in X$, $p(x) \leq M\|x\|$. Let $x, y \in X$. Then the previous exercise implies that

$$\begin{aligned} -M\|x-y\| &= -M\|y-x\| \\ &\leq -p(y-x) \\ &\leq p(x) - p(y) \\ &\leq p(x-y) \\ &\leq M\|x-y\| \end{aligned}$$

So that

$$|p(x) - p(y)| \leq M\|x-y\|$$

and p is Lipschitz. Conversely, suppose that p is Lipschitz. Then there exists $M > 0$ such that for each $x, y \in X$, $|p(x) - p(y)| \leq M\|x-y\|$. Let $x \in X$. Then

$$\begin{aligned} p(x) &\leq |p(x)| \\ &= |p(x) - p(0)| \\ &\leq M\|x-0\| \\ &\leq M\|x\| \end{aligned}$$

So p is bounded. $\square$

Exercise 6.5.0.5. Let X be a normed vector space, $p : X \to \mathbb{R}$ a bounded sublinear functional and $\phi : X \to \mathbb{R}$ a linear functional. If $\phi \leq p$, then $\phi \in X^*$.

Proof. Since p is Lipschitz, there exists $M > 0$ such that for each $x \in X$,

$$\begin{aligned} p(x) &\leq |p(x)| \\ &\leq M\|x\| \end{aligned}$$

Let $x \in X$. Then

$$\begin{aligned} \phi(x) &\leq p(x) \\ &\leq |p(x)| \\ &\leq M\|x\| \end{aligned}$$

and therefore

$$\begin{aligned} -M\|x\| &= -M\|-x\| \\ &\leq -p(-x) \\ &\leq -\phi(-x) \\ &= \phi(x) \end{aligned}$$

So that $|\phi(x)| \leq M\|x\|$ and $\phi \in X^*$. $\square$

Exercise 6.5.0.6. Let X be a normed vector space and $p : X \to \mathbb{R}$ a bounded sublinear functional. Then there exists $\phi \in X^*$ such that for each $x \in X$, $\phi(x) \leq p(x)$.

Proof. A previous exercise in the section on sublinear functionals in the topologoical vector space chapter implies there exists $\phi : X \to \mathbb{R}$ such that ϕ is linear and $\phi \leq p$. The previous exercise implies that $\phi \in X^*$. $\square$

Exercise 6.5.0.7. Equivalency of linearity (Bounded Case)
Let X be a normed vector space and $p : X \to \mathbb{R}$ a bounded sublinear functional. Then the following are equivalent:

1. there exists a unique $\phi \in X^*$ such that $\phi \leq p$
2. for each $x \in X$, $-p(-x) = p(x)$
3. p is linear

Proof. Basically the same as last time. $\square$

Exercise 6.5.0.8. Let X be a normed vector space, $M \subset X$ a subspace and $f \in M^*$. Then there exists $F \in X^*$ such that $\|F\| = \|f\|$ and $F|_M = f$.

Proof. If $f = 0$, Choose $F = 0$. Suppose $f \neq 0$. Then $\|f\| \neq 0$ and there exists $x_0 \in M$ such that $x_0 \neq 0$. Thus $\|f\| \neq 0$. Define $p : X \to [0, \infty)$ by $p(x) = \|f\|\|x\|$. Then p is a sublinear functional on X and for each $x \in M$, $|f(x)| \leq p(x)$. So there exists a linear functional $F : X \to \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x) = \|f\|\|x\|$ and $F|_M = f$. Thus $F \in X^*$ with $\|F\| \leq \|f\|$. Also

$$\|F\| = \sup_{\substack{x \in X \\ \|x\|=1}} |F(x)| \geq \sup_{\substack{x \in M \\ \|x\|=1}} |F(x)| = \sup_{\substack{x \in M \\ \|x\|=1}} |f(x)| = \|f\|$$

So $\|F\| = \|f\|$. $\square$

Exercise 6.5.0.9. Let X be a normed vector space, $M \subsetneq X$ a proper closed subspace and $x \in X \setminus M$. Then there exists $F \in X^*$ such that $F|_M = 0$, $\|F\| = 1$ and $F(x) = \|x + M\| \neq 0$.
Hint: Consider $f : M + \mathbb{C}x \to \mathbb{C}$ defined by $f(m + \lambda x) = \lambda \|x + M\|$.

Proof. Define $f : M + \mathbb{C}x \to \mathbb{C}$ as above. Clearly f is linear and $f|_M = 0$. Let $m \in M$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $|f(m + \lambda x)| = 0 \leq \|m + \lambda x\|$. Suppose that $\lambda \neq 0$. Then

$$\begin{aligned}
|f(m + \lambda x)| &= |\lambda|\|x + M\| \\
&= \|\lambda x + M\| \\
&= \inf_{z \in M} \|z + \lambda x\| \\
&\leq \|m + \lambda x\|
\end{aligned}$$

So $f \in (M + \mathbb{C}x)^*$ and $\|f\| \leq 1$. Let $\epsilon > 0$. A previous exercise tells us that there exist $m \in M, \lambda \in \mathbb{C}$ such that $\|m + \lambda x\| = 1$ and $\|m + \lambda x + M\| > 1 - \epsilon$. Then

$$\begin{aligned}
|f(m + \lambda x)| &= |\lambda|\|x + M\| \\
&= \|\lambda x + M\| \\
&= \|m + \lambda x + M\| \\
&> 1 - \epsilon
\end{aligned}$$

So

$$\|f\| = \sup_{\substack{z \in M + \mathbb{C}x \\ \|z\|=1}} |f(z)| \geq 1$$

Hence $\|f\| = 1$. The same exercise also tells us that $f(x) = \|x + M\| \neq 0$. Using the previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{M + \mathbb{C}x} = f$. $\square$

Exercise 6.5.0.10. Let X be a normed vector space and $x \in X$. If $x \neq 0$, then there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$.

Proof. Define $f : \mathbb{C}x \to \mathbb{C}$ by $f(\lambda x) = \lambda \|x\|$. Then f is linear and $f(x) = \|x\|$. Clearly

$$\sup_{\substack{z \in \mathbb{C}x \\ \|z\|=1}} |f(z)| = 1$$

So $f \in (\mathbb{C}x)^*$ and $\|f\| = 1$. By a previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{\mathbb{C}x} = f$. □

Exercise 6.5.0.11. Let X be a normed vector space and $x \in X$. Then $x = 0$ iff for each $\phi \in X^*$, $\phi(x) = 0$.

Proof. Clear by previous exercise. □

Exercise 6.5.0.12. Let X be a normed vector space. Then X^* separates the points of X.

Proof. Let $x, y \in X$. Suppose that $x \neq y$. Then $x - y \neq 0$. The previous exercies implies that there exists $F \in X^*$ such that $\|F\| = 1$ and

$$F(x) - F(y) = F(x - y) = \|x - y\| \neq 0$$

Thus $F(x) \neq F(y)$ and X^* separates the points of X. □

Exercise 6.5.0.13. Let X be a normed vector space and $f : X \to \mathbb{C}$ a linear functional on X. Then f is bounded iff $\ker f$ is closed.

Proof. Suppose that f is continuous. Since $\{0\}$ is closed, we have that $\ker f = f^{-1}(\{0\})$ is closed. Conversely, suppose that $\ker f$ is closed. If $\ker f = X$, then $f = 0$ and f is continuous. Suppose that $\ker f \neq X$. Then $\ker f$ is a proper, closed subspace of X. A previous exercise tells us that there exists $x \in X$ such that $\|x\| = 1$ and $\|x + \ker f\| > \frac{1}{2}$. Let $y \in X$. Suppose that $\|y\| < \frac{1}{2}$. Then for each $z \in \ker f$,

$$\begin{aligned}
\|z - (x + y)\| &= \|(z - x) - y\| \\
&\geq \|z - x\| - \|y\| \\
&> \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}$$

So $x + y \notin \ker f$. Therefore $f(B(x, \frac{1}{2})) \cap \{0\} = \varnothing$. If $f(B(x, \frac{1}{2}))$ is unbounded, then $f(B(x, \frac{1}{2})) = \mathbb{C}$ by linearity. This is a contradiction since $0 \notin f(B(x, \frac{1}{2}))$. So There exists $s > 0$ such that $f(B(x, \frac{1}{2})) \subset B(0, s)$ and thus f is bounded. □

Exercise 6.5.0.14. Let X be a normed vector space.

1. Let $M \subset X$ be a proper closed subspace of X and $x \in X \setminus M$. Then $M + \mathbb{C}x$ is closed.
2. Let $M \subset X$ be a finite dimensional subspace of X. Then M is closed.

Proof. 1. Let $y \in X$ and $(y_n)_{n \in \mathbb{N}} \subset M + \mathbb{C}x$. Suppose that $y_n \to y$. If $y \in M$, then $y \in M + \mathbb{C}x$. Suppose that $y \notin M$. For each $n \in \mathbb{N}$, there exists $m_n \in M$ and $\lambda_n \in \mathbb{C}$ such that $y_n = m_n + \lambda_n x$. A previous exercise tells us that there exists $F \in X^*$ such that $\|F\| = 1$, $F|_M = 0$ and $F(x) = \|x + M\| \neq 0$. Since F is continuous, $F(y_n) \to F(y)$. Since for each $n \in \mathbb{N}$,

$$F(y_n) = F(m_n + \lambda_n x) = F(m_n) + \lambda_n (F_x) = \lambda_n F(x)$$

we have that $\lambda_n F(x) \to F(y)$. Since $F(x) \neq 0$, this implies that $\lambda_n \to F(x)^{-1} F(y)$. It follows that $\lambda_n x \to F(x)^{-1} F(y) x$. Since for each $n \in \mathbb{N}$, $m_n = y_n - \lambda_n x$, we know that $m_n \to y - F(x)^{-1} F(y) x$. Since $(m_n)_{n \in \mathbb{N}} \subset M$ and M is closed, we have that $y - F(x)^{-1} F(y) x \in M$ and therefore $y \in M + \mathbb{C}x$. Hence $M + \mathbb{C}x$ is closed.

2. If $M = X$, then M is closed. Suppose that $M \neq X$. Let $(x_i)_{i=1}^n$ be a basis for M. Define $N_0 = \{0\}$ and for each $i = 1, 2, \cdots, n$, define $N_i = N_{i-1} + \mathbb{C}x_i$. Since N_0 is a proper closed subpace of X and $x_1 \in X \setminus N_0$, (1) implies that N_1 is closed. Proceed inductively to obtain that $M = N_n$ is closed.

□

Exercise 6.5.0.15. Let X be an infinite-dimensional normed vector space.

1. There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that for each $m, n \in \mathbb{N}$, $\|x_n\| = 1$ and if $m \neq n$, then $\|x_m - x_n\| > \frac{1}{2}$.
2. X is not locally compact.

Proof.

1. Define $N_0 = \{0\}$. Then N_0 is a closed proper subspace of X. Choose $x_1 \in X$ such that $\|x_1\| = 1$. Using the results of previous exercises, we proceed inductively. For each $n \geq 2$ we define $N_{n-1} = \text{span}(x_1, x_2, \cdots, x_{n-1})$. Then N_{n-1} is a closed proper subspace of X. Thus we may choose $x_n \in X$ such that $\|x_n\| = 1$ and $\|x_n + N_{n-1}\| > \frac{1}{2}$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then $x_m \in N_{n-1}$. Thus $\|x_n - x_m\| \geq \|x_n + N_{n-1}\| > \frac{1}{2}$

2. Suppose that X is locally compact. Then $\operatorname{cl} B(0,1)$ is compact and therefore sequentially compact. Using $(x_n)_{n\in\mathbb{N}} \subset \operatorname{cl} B(0,1)$ defined in (1), we see that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, $x \in \operatorname{cl} B(0,1)$ such that $x_{n_k} \to x$. Then $(x_{n_k})_{k\in\mathbb{N}}$ is Cauchy. So there exists $N \in N$ such that for each $j, k \in \mathbb{N}$, if $j, k \geq N$, then $\|x_{n_j} - x_{n_k}\| < \frac{1}{2}$. Then $\|x_{n_N} - x_{n_{N+1}}\| < \frac{1}{2}$. This is a contradiction since by construction, $\|x_{n_N} - x_{n_{N+1}}\| > \frac{1}{2}$. Thus X is not locally compact.

□

6.6 Applications of the Baire Category Theorem

Theorem 6.6.0.1. Open Mapping Theorem:
Let X, Y be Banach spaces and $T \in L(X;Y)$. If T is surjective, then T is open.

Corollary 6.6.0.2. Let X, Y be Banach spaces and $T \in L(X;Y)$. If T is a bijection, then $T^{-1} \in L(X;Y)$.

Definition 6.6.0.3. Let X, Y be sets and $f : X \to Y$. We define the **graph of f**, $\Gamma(f)$, by $\Gamma(f) = \{(x, y) \in X \times Y : f(x) = y\}$.

Theorem 6.6.0.4. Closed Graph Theorem:
Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If $\Gamma(T)$ is closed, then $T \in L(X;Y)$.

Note 6.6.0.5. We recall that $\Gamma(T)$ is closed iff for each $(x_n)_{n\in\mathbb{N}} \subset X$, $x \in X$ and $y \in Y$, $x_n \to x$ and $T(x_n) \to y$ implies that $T(x) = y$.

Exercise 6.6.0.6. Uniform Boundedness Principle:
Let X, Y be Banach spaces and $S \subset L(X;Y)$. If for each $x \in X$,

$$\sup_{T\in S} \|Tx\| < \infty$$

then

$$\sup_{T\in S} \|T\| < \infty$$

Proof. Finish!!! □

Exercise 6.6.0.7. Let μ be counting measure on $(N, \mathcal{P}(\mathbb{N}))$. Define $h : \mathbb{N} \to \mathbb{N}$ and ν on $(N, \mathcal{P}(\mathbb{N}))$ by $h(n) = n$ and $d\nu = hd\mu$. Define $X = L^1(\nu)$ and $Y = L^1(\mu)$. Equip both X and Y with the L^1 norm with respect to μ.

1. We have that X is a proper subspace of Y and therefore X is not complete.
2. Define $T : X \to Y$ by $Tf(n) = nf(n)$. Then T is linear, $\Gamma(T)$ is closed, and T is unbounded.
3. Define $S : Y \to X$ by $Sg(n) = \frac{1}{n}g(n)$. Then $S \in L(Y, X)$, S is surjective and S is not open.

Proof.

1. Note that for each $f : \mathbb{N} \to \mathbb{C}$,

$$\begin{aligned}
\|f\|_{\mu,1} &= \sum_{n=1}^{\infty} |f(n)| \\
&\leq \sum_{n=1}^{\infty} n|f(n)| \\
&= \|f\|_{\nu,1}
\end{aligned}$$

Hence X is a subspace of Y. Define $f : \mathbb{N} \to \mathbb{C}$ by $f(n) = \frac{1}{n^2}$. Then

$$\|f\|_{\mu,1} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

So $f \in Y$. However

$$\|f\|_{\nu,1} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

So $f \notin X$. Thus X is a proper subspace of Y. Let $g \in Y$ and $\epsilon > 0$. Since the simple functions are dense in $L^1(\mu)$, there exists $\phi \in L^1(\mu)$ such that ϕ is simple and $\|g - \phi\|_{\mu,1} < \epsilon$. Then there exist

$(c_i)_{i=1}^k \subset \mathbb{C}$ and $(E_i)_{i=1}^k \subset \mathcal{P}(\mathbb{N})$ such that for each $i, j \in \{1, 2, \cdots, k\}$, E_i is finite, $i \neq j$ implies that $E_i \cap E_j = \varnothing$ and

$$\phi = \sum_{i=1}^k c_i \chi_{E_i}$$

Define $c = \max\{|c_i| : i = 1, 2, \cdots k\}$ and $m = \max\left[\bigcup_{i=1}^k E_i\right]$. Then

$$\begin{aligned}
\|\phi\|_{\nu,1} &= \sum_{n=1}^m n|\phi(n)| \\
&\leq \sum_{n=1}^m mc \\
&= cm^2 \\
&< \infty
\end{aligned}$$

Hence $\phi \in X$ and X is dense in Y. Since X is a dense, proper subspace, it is not closed. Since Y is complete and $X \subset Y$ is not closed, we have that X is not complete.

2. Clearly T is linear. Let $(f_j)_{j\in\mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_j \xrightarrow{L^1(\mu)} f$ and $Tf_j \xrightarrow{L^1(\mu)} g$. Note that for each $j \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$|f_j(n) - f(n)| \leq \sum_{n=1}^\infty |f_j(n) - f(n)| = \|f_j - f\|_{\mu,1}$$

and

$$|nf_j(n) - g(n)| \leq \sum_{n=1}^\infty |nf_j(n) - g(n)| = \|Tf_j - g\|_{\mu,1}$$

Thus for each $n \in \mathbb{N}$, $f_j(n) \xrightarrow{j} f(n)$ and $nf_j(n) \xrightarrow{j} g(n)$. This implies that for each $n \in \mathbb{N}$, $nf(n) = g(n)$. Thus $Tf = g$ which implies that $\Gamma(T)$ is closed. Suppose, for the sake of contradiction, that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\|_{\mu,1} \leq C\|f\|_{\mu,1}$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f : \mathbb{N} \to \mathbb{C}$ by $f = \chi_{\{n\}}$. As established above, $S^+ \subset L^1(\mu)$. Then $\|f\|_{\mu,1} = 1$ and

$$\begin{aligned}
\|Tf\|_{\mu,1} &= n \\
&> C \\
&= C\|f\|_{\mu,1}
\end{aligned}$$

which is a contradiction. So T is unbounded.

3. Clearly S is linear. Let $g \in Y$. Then

$$\begin{aligned}
\|Sg\|_{\mu,1} &= \sum_{n=1}^\infty \frac{1}{n}|g(n)| \\
&\leq \sum_{n=1}^\infty |g(n)| \\
&= \|g\|_{\mu,1}
\end{aligned}$$

So S is bounded and $\|S\| \leq 1$. Thus $S \in L(Y, X)$. Let $f \in X$. Define $g : \mathbb{N} \to \mathbb{C}$ by $g(n) = nf(n)$. By defnition, $g \in Y$ and we have that

$$\begin{aligned}
Sg(n) &= \frac{1}{n}g(n) \\
&= f(n)
\end{aligned}$$

Hence $Sg = f$ and thus S is surjective. Let $g \in Y$. Suppsose that $Sg = 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n}|g(n)| = \|Sg\| = 0$$

Thus for each $n \in \mathbb{N}$, $g(n) = 0$. Hence $\ker S = \{0\}$ and S is injective. Note that for each $A \subset Y$, $S(A) = T^{-1}(A)$. If S is open, then T is continuous which as shown above is a contradiction. So g is not open.

□

Exercise 6.6.0.8. Let $X = C^1([0,1])$ and $Y = C([0,1])$. Equip both X and Y with the uniform norm.

1. Then X is not complete
2. Define $T : X \to Y$ by $Tf = f'$. Then $\Gamma(T)$ is closed and T is not bounded.

Proof. 1. Recall that for each $a, b \geq 0$ and $p \in \mathbb{N}$,

$$(a^{\frac{1}{p}} + b^{\frac{1}{p}})^p = \sum_{n=0}^{p} \binom{p}{n} a^{\frac{n}{p}} b^{\frac{p-n}{p}} \geq a + b$$

Thus $(a+b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$.
For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{C}$ by $f_n(x) = \sqrt{(x-\frac{1}{2})^2 + \frac{1}{n^2}}$. Then $(f_n)_{n\in\mathbb{N}} \subset X$. Define $f : [0,1] \to \mathbb{C}$ by $f(x) = |x - \frac{1}{2}|$. Then $f \in Y \cap X^c$. Note that for each $n \in \mathbb{N}$, $f \leq f_n$. Our observation above implies that for each $x \in X$,

$$\begin{aligned} f_n(x) &= \left[(x - \frac{1}{2})^2 + \frac{1}{n^2}\right]^{\frac{1}{2}} \\ &\leq |x - \frac{1}{2}| + \frac{1}{n} \end{aligned}$$

Thus $0 \leq f_n - f \leq \frac{1}{n}$. This implies that $f_n \xrightarrow{\text{u}} f$. Since $f \notin X$, X is not complete.

2. Let $(f_n)_{n\in\mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_n \xrightarrow{\text{u}} f$ and $Tf_n \xrightarrow{\text{u}} g$. Let $x \in [0,1]$. Then $f_n(x) \to f(x)$ and $f_n(0) \to f(0)$ and $f_n' \xrightarrow{\text{u}} g$. Applying the DCT to this sequence of integrable functions that converges uniformly to an integrable function on a finite measure space (a previous exercise) we have that

$$\begin{aligned} f_n(x) - f_n(0) &= \int_{[0,x]} f_n' dm \\ &\to \int_{[0,x]} g dm \end{aligned}$$

Since $f_n(x) - f_n(0) \to f(x) - f(0)$, we know that

$$f(x) - f(0) = \int_{[0,x]} g dm$$

Thus $Tf = g$ and $\Gamma(T)$ is closed.
By Exercise 6.2.0.3, T is not bounded.

□

Exercise 6.6.0.9. Let X, Y be Banach spaces and $T \in L(X;Y)$. Then $X/\ker T \cong T(X)$ iff $T(X)$ is closed.

Proof. Since X is a banach space and T is continuous, we have that $\ker T$ is closed and $X/\ker T$ is a Banach space. Suppose that $X/\ker T \cong T(X)$. Then $T(X)$ is complete. Since Y is complete, this implies that $T(X)$ is closed.
Conversely Suppose that $T(X)$ is closed. Then $T(X)$ is complete. Define $S : X/\ker T \to T(X)$ by $S(x + \ker T) = T(x)$. A previous exercise tells us that the map $S : X/\ker T \to T(X)$ defined by $S(x+\ker T) = T(x)$ is a bounded linear bijection. Since $T(X)$ is complete and S is surjective, S^{-1} is bounded and thus S is an isomorphism. □

Exercise 6.6.0.10. Let X be a separable Banach space. Define $B_X = \{x \in X : \|x\| < 1\}$. Let $(x_n)_{n\in\mathbb{N}} \subset B_X$ a dense subset of the unit ball and μ the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Define $T : L^1(\mu) \to X$ by

$$Tf = \sum_{n=1}^{\infty} f(n)x_n$$

Then

1. T is well defined and $T \in L(L^1(\mu), X)$
2. T is surjective
3. There exists a closed subspace $K \subset L^1(\mu)$ such that $L^1(\mu)/K \cong X$

Proof. 1. Let $f \in L^1(\mu)$. Since X is complete and

$$\begin{aligned}\sum_{n=1}^{\infty} \|f(n)x_n\| &= \sum_{n=1}^{\infty} |f(n)|\|x_n\| \\ &\leq \sum_{n=1}^{\infty} |f(n)| \\ &< \infty\end{aligned}$$

we have that $\sum_{n=1}^{\infty} f(n)x_n$ converges and thus $Tf \in X$. Hence T is well defined.

Clearly T is linear. Let $f \in L^1(\mu)$. Then

$$\begin{aligned}\|Tf\| &= \|\sum_{n=1}^{\infty} f(n)x_n\| \\ &\leq \sum_{n=1}^{\infty} \|f(n)x_n\| \\ &\leq \sum_{n=1}^{\infty} |f(n)| \\ &= \|f\|_1\end{aligned}$$

So T is bounded with $\|T\| \leq 1$.

2. Let $x \in X$. Suppose that $\|x\| < 1$. Then $x \in B_X$. So there exists $n_1 \in \mathbb{N}$ such that $\|x - x_{n_1}\| < \frac{1}{2}$. Then $2(x - x_{n_1}) \in B_X$. Since for each $j \in \mathbb{N}$, $B_X \setminus (x_n)_{n=1}^{j}$ is dense in B_X, there exists $n_2 \in \mathbb{N}$ such that $x_{n_2} \notin (x_n)_{n=1}^{n_1}$ and $\|2(x - x_{n_1}) - x_{n_2}\| < \frac{1}{2}$ which implies that $\|x - (x_{n_1} - \frac{1}{2}x_{n_2})\| < \frac{1}{4}$.

Proceed inductively to obtain a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that for each $k \geq 2$, $x_{n_k} \notin (x_n)_{n=1}^{n_{k-1}}$ and

$\|x - \sum_{j=1}^{k} 2^{1-j} x_{n_j}\| < \frac{1}{2^k}$. Then $x = \sum_{k=1}^{\infty} 2^{1-k} x_{n_k}$.

Define $f : \mathbb{N} \to \mathbb{C}$ by $f = \sum_{k=1}^{\infty} 2^{1-k} \chi_{\{n_k\}}$. Then $\|f\|_1 = \sum_{k=1}^{\infty} 2^{1-k} < \infty$, so $f \in L^1(\mu)$ and $Tf = \sum_{k=1}^{\infty} 2^{1-k} x_{n_k} = x$. Now, suppose that $\|x\| \geq 1$, then $\frac{1}{2\|x\|} x \in B_X$. The above argument shows that there exists $f \in L^1(\mu)$ such that $Tf = \frac{1}{2\|x\|} x$. Then $2\|x\| f \in L^1(\mu)$ and $T(2\|x\| f) = 2\|x\| Tf = x$. So for each $x \in X$, there exists $f \in L^1(\mu)$ such that $Tf = x$ and thus T is surjective.

3. Since X is a Banach space and T is surjective, the previous exercise implies that $L^1(\mu)/\ker T \cong X$.

□

6.7 Duality

Note 6.7.0.1. Let X be a normed vector space. Then X^* is a normed vector space. In general the weak-$*$ topology on X^* is not necessarily the same as the norm topology on X^*. In the context of normed vector spaces, we will write X^{**} to denote $(X^*)^*$ when X^* is equipped with the norm topology and $\hat{X}$ to denote $(X^*)^*$ when X^* is equipped with the weak-$*$ topology.

Exercise 6.7.0.2. Let X be a normed vector space and $x \in X$. Define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| = \|x\|$.
Hint: Hahn-Banach theorem

Proof. Let $f, g \in X^*$ and $\lambda \in \mathbb{C}$. Then

$$\hat{x}(f + \lambda g) = (f + \lambda g)(x) = f(x) + \lambda g(x) = \hat{x}(f) + \lambda \hat{x}(g)$$

So $\hat{x}$ is linear. For each $f \in X^*$,

$$|\hat{x}(f)| = |f(x)| \leq \|x\|\|f\|$$

Hence $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leq \|x\|$. If $x = 0$, then $\hat{x} = 0$ and $\|\hat{x}\| = \|x\|$. Suppose that $x \neq 0$. Then a previous exercise implies that there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$. Then we have that

$$\sup_{\substack{f \in X^* \\ \|f\|=1}} |\hat{x}(f)| = \sup_{\substack{f \in X^* \\ \|f\|=1}} |f(x)| \geq |F(x)| = \|x\|$$

Hence $\|\hat{x}\| = \|x\|$. □

Exercise 6.7.0.3. Let X be a topological vector space. Then

1. $\mathcal{T}_{w^*} \subset \mathcal{T}_{X^*}$
2. For each $E \subset X^*$, if E is weak-* closed, then E is norm closed

Proof.

1. Since $\hat{X} \subset X^{**}$, we have that

$$\begin{aligned} \mathcal{T}_{w^*} &= \tau_{X^*}(\hat{X}) \\ &\subset \tau_{X^*}(X^{**}) \\ &= \mathcal{T}_{X^*} \end{aligned}$$

2. Let $E \subset X^*$. Suppose that E is weak-* closed. Then

$$\begin{aligned} E^c &\in \mathcal{T}_{w^*} \\ &\subset \mathcal{T}_{X^*} \end{aligned}$$

So E is norm closed.

□

Exercise 6.7.0.4. Let X be a normed vector space. If X is separable, then there exist $(\phi_n)_{n \in \mathbb{N}} \subset X^*$ such that for each $n \in \mathbb{N}$, $\|\phi_n\| = 1$ and for each $x \in X$,

$$\|x\| = \sup_{n \in \mathbb{N}} |\phi_n(x)|$$

Hint: Let $(x_n)_{n \in \mathbb{N}} \subset X$ be a dense subset. A previous exercise on the Hahn-Banach theorem implies that for each n, there exists $\phi_n \in X^*$ such that $\|\phi_n\| = 1$ and $\phi_n(x_n) = \|x_n\|$. Then for each $x \in X$,

$$\|x\| = \sup_{n \in \mathbb{N}} |\phi_n(x)|$$

.

Proof. Suppose that X is separable. Then there exists $(x_n)_{n\in\mathbb{N}} \subset X$ such that $(x_n)_{n\in\mathbb{N}}$ is dense in X. A previous exercise on the Hahn-Banach theorem implies that for each n, there exists $\phi_n \in X^*$ such that $\|\phi_n\| = 1$ and $\phi_n(x_n) = \|x_n\|$. Let $x \in X$. Then

$$\begin{aligned}
\|x\| &= \|\hat{x}\| \\
&= \sup_{\substack{\phi\in X^* \\ \|\phi\|=1}} \|\hat{x}(\phi)\| \\
&= \sup_{\substack{\phi\in X^* \\ \|\phi\|=1}} \|\phi(x)\| \\
&\geq \sup_{n\in\mathbb{N}} \|\phi_n(x)\|
\end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|x - x_N\| < \epsilon/2$. Then

$$\begin{aligned}
\|x\| &\leq \|x - x_N\| + \|x_N\| \\
&= \|x - x_N\| + |\phi_N(x_N)| \\
&\leq \|x - x_N\| + |\phi_N(x_N - x)| + |\phi_N(x)| \\
&\leq \|x - x_N\| + \|\phi_N\|\|x_N - x\| + |\phi_N(x)| \\
&\leq 2\|x - x_N\| + |\phi_N(x)| \\
&< 2\frac{\epsilon}{2} + |\phi_N(x)| \\
&\leq \epsilon + \sup_{n\in\mathbb{N}} |\phi_n(x)|
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|x\| \leq \sup_{n\in\mathbb{N}} |\phi_n(x)|$. So $\|x\| = \sup_{n\in\mathbb{N}} |\phi_n(x)|$. □

Exercise 6.7.0.5. Let X be a normed vector space. Define $\phi : X \to X^{**}$ by $\phi(x) = \hat{x}$. Then ϕ is a linear isometry.

Proof. Let $x, y \in X$ and $\lambda \in \mathbb{C}$. Then for each $f \in X^*$, we have that

$$\begin{aligned}
\phi(x + \lambda y)(f) &= \widehat{x + \lambda y}(f) \\
&= f(x + \lambda y) \\
&= f(x) + \lambda f(y) \\
&= \hat{x}(f) + \lambda\hat{y}(f) \\
&= \phi(x)(f) + \lambda\phi(y)(f)
\end{aligned}$$

So $\phi(x + \lambda y) = \phi(x) + \lambda\phi(y)$ and ϕ is linear. The previous exercise tells us that

$$\begin{aligned}
\|\phi(x) - \phi(y)\| &= \|\phi(x - y)\| \\
&= \|\widehat{x - y}\| = \|x - y\|
\end{aligned}$$

So ϕ is an isometry. □

Definition 6.7.0.6. Let X be a normed vector space and define $\phi : X \to X^{**}$ as above. We define $\widehat{X} = \phi(X) \subset X^{**}$. Since $\widehat{X}$ and X are isomorphic, we may identify X as a subset of X^{**}.

Definition 6.7.0.7. Let X be a normed vector space and define $\phi : X \to X^{**}$ as above. Then X is said to be **reflexive** if ϕ is surjective. In this case ϕ is then an isomorphism

Definition 6.7.0.8. Let X, Y be normed vector spaces and $T \in L(X; Y)$. Define the **adjoint of** T, denoted $T^* : Y^* \to X^*$, by $T^*(f) = f \circ T$.

Exercise 6.7.0.9. Let X, Y be normed vector spaces and $T \in L(X;Y)$.

1. Then $T^* \in L(Y^*, X^*)$.
2. Applying the result from (1) twice, we have that $T^{**} \in L(X^{**}, Y^{**})$. We have that for each $x \in X$, $T^{**}(\hat{x}) = \widehat{T(x)}$.
3. T^* is injective iff $T(X)$ is dense in Y.
4. If $T^*(Y^*)$ is dense in X^*, then T is injective. The converse is true if X is reflexive.

Proof.

1. Let $f \in Y^*$. Then $\|T^*(f)\| = \|f \circ T\| \leq \|T\|\|f\|$. So $T^* \in L(Y^*, X^*)$ with $\|T^*\| \leq \|T\|$.

2. Let $x \in X$. Let $f \in Y^*$. Then
$$\begin{aligned} T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= \widehat{T(x)}(f) \end{aligned}$$
Hence $T^{**}(\hat{x}) = \widehat{T(x)}$.

3. Suppose that $T(X)$ is not dense in Y. Then $\operatorname{cl} T(X) \neq Y$. So $T(X)$ is a proper closed subspace of Y and there exists $y \in Y$ such that $y \notin \operatorname{cl} T(X)$. By a previous exercise, there exists $f \in Y^*$ such that $f(y) = \|y + \operatorname{cl} T(X)\| \neq 0$, $\|f\| = 1$ and $f|_{\operatorname{cl} T(X)} = 0$. Let $x \in X$. Then $T^*(f)(x) = f \circ T(x) = 0$. Hence $T^*(f) = 0 = T^*(0)$. Since $f \neq 0$, T^* is not injective.
Now suppose that $T(X)$ is dense in Y. Let $f, g \in Y^*$. Define $h \in Y^*$ by $h = f - g$ Suppose that $T*(f) = T^*(g)$ Then $T^*(h) = 0$. So for each $x \in X$, $h(T(x)) = 0$. Let $y \in Y$ and $\epsilon > 0$. By continuity, there exists $\delta > 0$ such that for each $y' \in Y$, if $\|y - y'\| < \delta$, then $\|h(y) - h(y')\| < \epsilon$. Since $T(X)$ is dense in Y, there exists $x \in X$ such that $\|y - T(x)\| < \delta$. Thus
$$\begin{aligned} \|h(y)\| &\leq \|h(y) - h(T(x))\| + \|h(T(x))\| \\ &= \|h(y) - h(T(x))\| \\ &< \epsilon \end{aligned}$$
Since $\epsilon > 0$ is arbitrary, $\|h(y)\| = 0$. This implies that $h(y) = 0$ and therefore $f(y) = g(y)$. Since $y \in Y$ is arbitrary, $f = g$ and T^* is injective.

4. For the sake of contradiction, suppose that $T^*(Y^*)$ is dense in X^* and T is not injective. Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $T(x_1) = T(x_2)$. Define $x = x_1 - x_2$. Then $x \neq 0$ and $T(x) = 0$. A previous exercise implies that there exists $F \in X^*$ such that $F(x) = \|x\| \neq 0$ and $\|F\| = 1$. Let $\epsilon > 0$. Choose $g \in Y^*$ such that $\|F - T^*(g)\| < \epsilon$. Then
$$\begin{aligned} \|x\| &= |F(x)| \\ &\leq |F(x) - T^*(g)(x)| + |T^*(g)(x)| \\ &< \epsilon\|x\| + |g(T(x))| \\ &= \epsilon\|x\| \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\|x\| = 0$ which is a contradiction. Hence if $T^*(Y^*)$ is dense in X^*, then T is injective.

Now, suppose that X is reflexive and T is injective. Let $\phi_1, \phi_2 \in X^{**}$. Suppose that $T^{**}(\phi_1) = T^{**}(\phi_2)$. Then $T^{**}(\phi_1 - \phi_2) = 0$. Since X is reflexive, there exist $x_1, x_2 \in X$ such that $\phi_1 = \hat{x_1}$ and $\phi_2 = \hat{x_2}$. Define $x = x_1 - x_2$. Then $T^{**}(\hat{x}) = 0$. So for each $f \in Y^*$,

$$\begin{aligned} T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= 0 \end{aligned}$$

Suppose that $T(x) \neq 0$. Then a previous exercise implies that there exists $g \in Y^*$ such that $g(T(x)) = \|T(x)\| \neq 0$ and $\|g\| = 1$. This is a contradiction since $g(T(x)) = 0$. So $T(x) = 0$. Since T is injective, this implies that $x = 0$. Hence $\hat{x} = 0$ and thus $\phi_1 = \phi_2$. Thus T^{**} is injective. By (3), we have that $T^*(Y^*)$ is dense in X^*.

□

Exercise 6.7.0.10. Let X be a normed vector space. Then X is reflexive iff X^* is reflexive.

Proof. Suppose that X is reflexive. Let $\alpha \in X^{***}$. Define $f : X \to \mathbb{C}$ by $f(x) = \alpha(\hat{x})$. Clearly f is linear and a previous exercise tells us that for each $x \in X$,

$$\begin{aligned} |f(x)| &\leq \|\alpha\|\|\hat{x}\| \\ &= \|\alpha\|\|x\| \end{aligned}$$

So $f \in X^*$. Let $\phi \in X^{**}$. Since X is reflexive, there exists $x \in X$ such that $\phi = \hat{x}$. Then

$$\begin{aligned} \alpha(\phi) &= \alpha(\hat{x}) \\ &= f(x) \\ &= \hat{x}(f) \\ &= \hat{f}(\hat{x}) \\ &= \hat{f}(\phi) \end{aligned}$$

Hence $\alpha = \hat{f}$. Thus the map $X^* \to X^{***}$ given by $f \mapsto \hat{f}$ is surjective and so X^* is reflexive.

Conversely, suppose that X^* is reflexive. Since $\phi : X \to X^{**}$ given by $\phi(x) = \hat{x}$ is an isometry, $\widehat{X} \subset X^{**}$ is closed. For the sake of contradiction, suppose that $\widehat{X} \neq X^{**}$. Then there exists $\alpha \in X^{**}$ such that $\alpha \notin \widehat{X}$. Thus there exists $F \in X^{***}$ such that $\|F\| = 1$, $F(\alpha) = \|\alpha + \widehat{X}\| \neq 0$ and $F|_{\widehat{X}} = 0$. Since X^* is reflexive, there exists $f \in X^*$ such that $F = \hat{f}$. A previous exercise tells us that $\|f\| = \|\hat{f}\| = \|F\| = 1$. Since for each $x \in X$, $f(x) = \hat{x}(f) = \hat{f}(\hat{x}) = F(\hat{x}) = 0$, we have that $f = 0$. Thus $\|f\| = 0$, a contradiction. So $\widehat{X} = X^{**}$ and X is reflexive.

□

Definition 6.7.0.11. Let X be a normed vector space, $M \subset X$ and $N \subset X^*$. We define the **annihilator** of M and the annihilator of N, denoted by $M^\perp \subset X^*$ and ${}^\perp N \subset X$ respectively, by

$$\begin{aligned} M^\perp &= \{\phi \in X^* : \text{for each } x \in M, \, \phi(x) = 0\} \\ {}^\perp N &= \{x \in X : \text{for each } \phi \in N, \, \phi(x) = 0\} \end{aligned}$$

Exercise 6.7.0.12. Let X be a normed vector space, $M \subset X$ and $N \subset X^*$. Then

1. $$M^{\perp} = \bigcap_{x \in M} \ker \hat{x}$$

2. $$^{\perp}N = \bigcap_{\phi \in N} \ker \phi$$

Proof.

1. $$\begin{aligned} M^{\perp} &= \{\phi \in X^* : \text{for each } x \in M,\, \phi(x) = 0\} \\ &= \bigcap_{x \in M} \{\phi \in X^* : \phi(x) = 0\} \\ &= \bigcap_{x \in M} \{\phi \in X^* : \hat{x}(\phi) = 0\} \\ &= \bigcap_{x \in M} \ker \hat{x} \end{aligned}$$

2. $$\begin{aligned} {}^{\perp}N &= \{x \in X : \text{for each } \phi \in N,\, \phi(x) = 0\} \\ &= \bigcap_{\phi \in N} \{x \in X : \phi(x) = 0\} \\ &= \bigcap_{\phi \in N} \ker \phi \end{aligned}$$

□

Exercise 6.7.0.13. Let X be a normed vector space, $M \subset X$ and $N \subset X^*$. Then

1. $M^{\perp}$ is weak-* closed
2. ${}^{\perp}N$ is closed

Proof.

1. Let $(\phi_n)_{n \in \mathbb{N}} \subset M^{\perp}$ and $\phi \in X^*$. Suppose that $\phi_n \xrightarrow{w^*} \phi$. Then for each $x \in X$, $\phi_n(x) \to \phi(x)$. Let $x \in M$. By definition, for each $n \in \mathbb{N}$, $\phi_n(x) = 0$. Thus $\phi_n(x) \to 0$ which implies that $\phi(x) = 0$ and $\phi \in \ker \hat{x}$. Since $x \in M$ is arbitrary,
$$\begin{aligned} \phi &\in \bigcap_{x \in M} \ker \hat{x} \\ &= M^{\perp} \end{aligned}$$

2. Let $(x_n)_{n \in \mathbb{N}} \subset {}^{\perp}N$ and $x \in X$. Suppose that $x_n \to x$. Let $\phi \in N$. Continuity implies that $\phi(x_n) \to \phi(x)$. By definition, for each $n \in \mathbb{N}$, $\phi(x_n) = 0$. Thus $\phi(x_n) \to 0$ which implies that $\phi(x) = 0$. So $x \in \ker \phi$. Since $\phi \in N$ is arbitrary,
$$\begin{aligned} x &\in \bigcap_{\phi \in N} \ker \phi \\ &= {}^{\perp}N \end{aligned}$$

□

Exercise 6.7.0.14. Let X be a normed vector space, $M \subset X$ and $N \subset X^*$. Then

1. ${}^{\perp}(M^{\perp}) = \operatorname{cl} M$, i.e. the norm closure of M
2. $({}^{\perp}N)^{\perp} = \operatorname{cl}_{w^*}(N)$, i.e. the weak-* closure of N.

Proof.

1. Let $x \in M$, then by definition, for each $\phi \in M^{\perp}$, $\phi(x) = 0$. Again by definition, $x \in {}^{\perp}(M^{\perp})$. So $M \subset {}^{\perp}(M^{\perp})$. Since ${}^{\perp}(M^{\perp})$ is closed, $\operatorname{cl} M \subset {}^{\perp}(M^{\perp})$. For the sake of contradiction, suppose that ${}^{\perp}(M^{\perp}) \not\subset \operatorname{cl} M$. Then there exists $x \in {}^{\perp}(M^{\perp})$ such that $x \notin \operatorname{cl} M$. Exercise 6.5.0.9 implies that there exists $\phi \in X^*$ such that $\phi|_{\operatorname{cl} M} = 0$, $\|\phi\| = 1$ and $\phi(x) = \|x + \operatorname{cl} M\| > 0$. By definition, $\phi \in M^{\perp}$. Since $\phi(x) \neq 0$, we have that $x \notin {}^{\perp}(M^{\perp})$. This is a contradiction and so ${}^{\perp}(M^{\perp}) \subset \operatorname{cl} M$.

2.

□

Exercise 6.7.0.15. Banach-Alaoglu Theorem:
Let X be a normed vector space. Then $\operatorname{cl} B(0,1)$ is w^*-compact.

Proof. For $x \in X$, define $D_x \subset \mathbb{C}$ by $D_x := \operatorname{cl} B_{\mathbb{C}}(0, \|x\|)$. Then for each $x \in X$, D_x is compact. Define $D \subset \mathbb{C}^X$ by $D := \prod_{x \in X} D_x$. Tychonoff's theorem implies that D is compact. Let $\phi \in \bar{B}_{X^*}(0,1)$. Then for each $x \in X$, $|\phi(x)| \leq \|x\|$. Hence $\phi \in D$. Since $\phi \in \bar{B}_{X^*}(0,1)$ is arbitrary, we have that $\bar{B}_{X^*}(0,1) \subset D$. Let $(\phi_n)_{n \in \mathbb{N}} \subset \bar{B}_{X^*}(0,1)$ and $\phi \in \bar{B}_{X^*}(0,1)$. Since $\phi_n \to \phi$ in weak-$*$ iff $\phi_n \to \phi$ in D, we have that $\bar{B}_{X^*}(0,1)$ in weak-$*$ is a closed subspace of D. Since D is compact, $\bar{B}_{X^*}(0,1)$ in weak-$*$ is compact.

FINISH!!! or clean up □

6.8 Compact Operators

Definition 6.8.0.1.

6.9 Multilinear Maps

Definition 6.9.0.1. Let $X_1, \cdots, X_n, Y$ be normed vector spaces and $T : \prod_{j=1}^{n} X_j \to Y$. Suppose that T is multilinear. Then T is said to be **bounded** if there exists $C \geq 0$ such that for each $x_1, \cdots, x_n \in X$,

$$\|T(x_1, \cdots, x_n)\| \leq C\|x_1\| \cdots \|x_n\|$$

We define

$$L^n(X_1, \ldots, X_n; Y) = \left\{T : \prod_{j=1}^{n} X_j \to Y : T \text{ is multilinear and bounded}\right\}$$

If $X_1 = \cdots = X_n = X$, we write $L^n(X; Y)$ in place of $L^n(X, \ldots, X; Y)$.

Note 6.9.0.2. For the remainder of this section we will primarily consider $L^2(X_1, X_2; Y)$ to avoid notational clutter, but all results immediately generalize to $L^n(X_1, \ldots, X_n; Y)$

Exercise 6.9.0.3. Let X_1, X_2 and Y be normed vector spaces and $T : X_1 \times X_2 \to Y$ bilinear. Then the following are equivalent:

1. T is continuous
2. T is continuous at $(0, 0)$
3. T is bounded

Proof.

- (1) $\implies$ (2):
  Trivial
- (2) $\implies$ (3):
  Suppose that T is continuous at $(0, 0)$. For the sake of contradiction, suppose that T is not bounded. Then for each $C \geq 0$, there exist $(x_1, x_2) \in X_1 \times X_2$ such that $\|T(x_1, x_2)\| > C\|x_1\|\|x_2\|$. Hence there exist $(a_n)_{n\in\mathbb{N}} \subset X_1$ and $(b_n)_{n\in\mathbb{N}} \subset X_2$ such that for each $n \in \mathbb{N}$, $\|T(a_n, b_n)\| > n^2\|a_n\|\|b_n\|$. Hence for each $n \in \mathbb{N}$, $\|a_n\|$, $\|b_n\| > 0$. Define
  $$(a'_n)_{n\in\mathbb{N}} \subset X_1$$
  and $(b'_n)_{n\in\mathbb{N}} \subset X_2$ by $a'_n = \frac{a_n}{n\|a_n\|}$ and $b'_n = \frac{b_n}{n\|b_n\|}$. Then $(a'_n, b'_n) \to (0, 0)$. Continuiuty implies that $T(a'_n, b'_n) \to 0$. By construction, for each $n \in \mathbb{N}$,
  $$\begin{aligned} \|T(a'_n, b'_n)\| &= \frac{1}{n^2\|a_n\|\|b_n\|} T(a_n, b_n) \\ &> \frac{n^2\|a_n\|\|b_n\|}{n^2\|a_n\|\|b_n\|} \\ &= 1 \end{aligned}$$
  which is a contradiction. So T is bounded.
- (3) $\implies$ (1):
  Suppose that T is bounded. Then there exists $C > 0$ such that for each $(x_1, x_2) \in X_1 \times X_2$, $\|T(x_1, x_2)\| \leq C\|x_1\|\|x_2\|$. Let $(a, b) \in X_1 \times X_2$ and $(a_n, b_n)_{n\in\mathbb{N}} \subset X_1 \times X_2$. Suppose that $(a_n, b_n) \to (a, b)$. Then $a_n \to a$, $b_n \to b$ and $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are bounded. So there exists $B \geq 0$ such that for each $n \in \mathbb{N}$ $\|b_n\| \leq B$. Hence
  $$\begin{aligned} \|T(a_n, b_n) - T(a, b)\| &= \|T(a_n, b_n) - T(a, b_n) + T(a, b_n) - T(a, b)\| \\ &\leq \|T(a_n, b_n) - T(a, b_n)\| + \|T(a, b_n) - T(a, b)\| \\ &= \|T(a_n - a, b_n)\| + \|T(a, b_n - b)\| \\ &\leq C(\|a_n - a\|\|b_n\| + \|a\|\|b_n - b\|) \\ &\leq C(\|a_n - a\|B + \|a\|\|b_n - b\|) \\ &\to 0 \end{aligned}$$

Thus T is continuous.

$\square$

Definition 6.9.0.4. Let X_1, X_2 and Y be normed vector spaces and $T \in L^2(X_1, X_2; Y)$. We define the **operator norm** on $L^2(X_1, X_2; Y)$, denoted $\|\cdot\| : L^2(X_1, X_2; Y) \to [0, \infty)$, by

$$\|T\| = \inf\{C \geq 0 : \text{for each } (x_1, x_2) \in X_1 \times X_2,\ \|T(x_1, x_2)\| \leq C\|x_1\|\|x_2\|\}$$

Exercise 6.9.0.5. Let X_1, X_2 and Y be normed vector spaces. If $X_1 \neq \{0\}$ and $X_2 \neq \{0\}$, then the operater norm on $L^2(X_1, X_2; Y)$ is given by:

1. $\|T\| = \sup\limits_{\|x_1\|=1, \|x_2\|=1} \|T(x_1, x_2)\|$
2. $\|T\| = \sup\limits_{x_1 \neq 0, x_2 \neq 0} \|x_1\|^{-1}\|x_2\|^{-1}\|T(x_1, x_2)\|$
3. $\|T\| = \inf\{C \geq 0 : \text{for each } (x_1, x_2) \in X_1 \times X_2,\ \|T(x_1, x_2)\| \leq C\|x_1\|\|x_2\|\}$

Proof. Since $X_1 \neq \{0\}$ and $X_2 \neq \{0\}$, the supremums in (1) and (2) are well defined. Let $T \in L^2(X_1, X_2; Y)$. Bilinearity of T implies that the sets over which the supremums are taken in (1) and (2) are the same. So (1) and (2) are equal.
Now, set

$$M = \sup_{\|x_1\|=1, \|x_2\|=1} \|T(x_1, x_2)\|$$

and

$$m = \inf\{C \geq 0 : \text{for each } (x_1, x_2) \in X_1 \times X_2,\ \|T(x_1, x_2)\| \leq C\|x_1\|\|x_2\|\}$$

Let $(x_1, x_2) \in X_1 \times X_2$. If $\|x_1\| = 0$ or $\|x_2\| = 0$, then $T(x_1, x_2) = 0$ and $\|T(x_1, x_2)\| \leq M\|x_1\|\|x_2\|$. Suppose that $\|x_1\| \neq 0$ and $\|x_2\| \neq 0$. Then

$$\begin{aligned}\|T(x_1, x_2)\| &= \left(\left\|T(\|x_1\|^{-1}x_1, \|x_2\|^{-1}x_2)\right\|\right)\|x_1\|\|x_2\| \\ &\leq M\|x_1\|\|x_2\|\end{aligned}$$

Hence $M \in \{C \geq 0 : \text{ for each } x \in X,\ \|Tx\| \leq C\|x\|\}$ and $m \leq M$. Let $C \in \{C \geq 0 : \text{for each } (x_1, x_2) \in X_1 \times X_2,\ \|T(x_1, x_2)\| \leq C\|x_1\|\|x_2\|\}$. Suppose that $\|x_1\| = 1$ and $\|x_2\| = 1$. Then $\|T(x_1, x_2)\| \leq C\|x_1\|\|x_2\| = C$. So $M \leq C$. Therefore $M \leq m$. So $M = m$ and the supremum in (1) is the same as the infimum in (3). $\square$

Exercise 6.9.0.6. Let X_1, X_2 and Y be normed vector spaces and $T \in L(X_1, X_2; Y)$. Then for each $(x_1, x_2) \in X_1 \times X_2$, $\|T(x_1, x_2)\| \leq \|T\|\|x_1\|\|x_2\|$.

Proof. Let $(x_1, x_2) \in X_1 \times X_2$. If $x_1 = 0$ or $x_2 = 0$, then

$$\begin{aligned}\|T(x_1, x_2)\| &= \|0\| \\ &= 0 \\ &= \|T\|\|x_1\|\|x_2\|\end{aligned}$$

Suppose that $x_1 \neq 0$ and $x_2 \neq 0$. The previous exercise implies that

$$\begin{aligned}\|T(x_1, x_2)\| &= \|T(\|x_1\|^{-1}\|x_1\|, \|x_2\|^{-1}\|x_2\|)\|\|x_1\|\|x_2\| \\ &\leq \left(\sup_{\|x_1\|=1, \|x_2\|=1} \|T(x_1, x_2)\|\right)\|x_1\|\|x_2\| \\ &= \|T\|\|x_1\|\|x_2\|\end{aligned}$$

$\square$

Exercise 6.9.0.7. Let X_1, X_2 and Y be normed vector spaces. Then $L(X_1, X_2; Y)$ is a vector space and $\|\cdot\| : L^2(X_1, X_2; Y) \to [0, \infty)$ is a norm.

Proof. Let $S, T \in L(X_1, X_2; Y)$ and $\lambda \in \mathbb{C}$.

- It is clear that $S + T : X_1 \times X_2 \to Y$ is multilinear. For each $(x_1, x_2) \in X_1 \times X_2$,

$$\begin{aligned}\|(S+T)(x_1,x_2)\| &= \|S(x_1,x_2) + T(x_1,x_2)\| \\ &\leq \|S(x_1,x_2)\| + \|T(x_1,x_2)\| \\ &\leq \|S\|\|x_1\|\|x_2\| + \|T\|\|x_1\|\|x_2\| \\ &= (\|S\| + \|T\|)\|x_1\|\|x_2\|\end{aligned}$$

  So $S + T \in L(X_1, X_2; Y)$ and $\|S + T\| \leq \|S\| + \|T\|$.

- It is clear that $\lambda T : X_1 \times X_2 \to Y$ is multilinear. For each $(x_1, x_2) \in X_1 \times X_2$,

$$\begin{aligned}\|(\lambda T)(x_1,x_2)\| &= \|\lambda T(x_1,x_2)\| \\ &= |\lambda|\|T(x_1,x_2)\| \\ &\leq |\lambda|\|T\|\|x_1\|\|x_2\|\end{aligned}$$

  So $\lambda T \in L(X_1, X_2; Y)$ and $\|\lambda T\| \leq |\lambda|\|T\|$.

- Suppose that $\|T\| = 0$. Let $(x_1, x_2) \in X_1 \times X_2$. If $x_1 = 0$ or $x_2 = 0$, then $T(x_1, x_2) = 0$. Suppose that $x_1 \neq 0$ and $x_2 \neq 0$. Then

$$\begin{aligned}\|T(x_1,x_2)\| &\leq \|T\|\|x_1\|\|x_2\| \\ &= 0\end{aligned}$$

  So $T(x_1, x_2) = 0$. Since $(x_1, x_2) \in X_1 \times X_2$ is arbitrary, $T = 0$.

Therefore $L(X_1, X_2; Y)$ is a vector space and $\|\cdot\| : L(X_1, X_2; Y) \to [0, \infty)$ is a norm. □

Note 6.9.0.8. Let X, Y, Z be sets. We recall the definition of $\mathrm{cur} : Y^{X \times Y} \to (Z^Y)^X$ from Definition 1.3.0.13.

Exercise 6.9.0.9. Let X_1, X_2, Y be normed vector spaces. Then

1. $\mathrm{cur} : (X_1 \times X_2)^Y \to (Y^{X_2})^{X_1}$ is linear
2. $\mathrm{cur}|_{L(X_1,X_2;Y)} : L(X_1, X_2; Y) \to L(X_1; L(X_2; Y))$
3. $\mathrm{cur}|_{L(X_1,X_2;Y)}$ is an isometry
4. $\mathrm{cur}|_{L(X_1,X_2;Y)}$ is surjective
5. $\mathrm{cur}|_{L(X_1,X_2;Y)} : L(X_1, X_2; Y) \to L(X_1; L(X_2; Y))$ is an isometric isomorphism.

Proof.

1. Let $S, T \in (X_1 \times X_2)^Y$, $\lambda \in \mathbb{C}$ and $(x_1, x_2) \in X_1 \times X_2$. Then

$$\begin{aligned}\mathrm{cur}(S + \lambda T)(x_1)(x_2) &= (S + \lambda T)(x_1, x_2) \\ &= S(x_1, x_2) + \lambda T(x_1, x_2) \\ &= \mathrm{cur}(S)(x_1)(x_2) + \lambda\,\mathrm{cur}(T)(x_1)(x_2) \\ &\mathrm{cur}(S)(x_1)(x_2) + \lambda\,\mathrm{cur}(T)(x_1)(x_2) \\ &= [\mathrm{cur}(S) + \lambda\,\mathrm{cur}(T)](x_1)(x_2)\end{aligned}$$

   Since $(x_1, x_2) \in X_1 \times X_2$ is arbitrary, $\mathrm{cur}(S + \lambda T) = \mathrm{cur}(S) + \lambda\,\mathrm{cur}(T)$. Since $S, T \in (X_1 \times X_2)^Y$ and $\lambda \in \mathbb{C}$ are arbitrary, $\mathrm{cur} : (X_1 \times X_2)^Y \to (Y^{X_2})^{X_1}$ is linear.

2. Let $T \in L(X_1, X_2; Y)$ and $x_1 \in X_1$. Since T is bilinear, for each $u, v \in X_2$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}\operatorname{cur}(T)(x_1)(u + \lambda v) &= T(x_1, u + \lambda b) \\ &= T(x_1, u) + \lambda T(x_1, v) \\ &= \operatorname{cur}(T)(x_1)(u) + \lambda \operatorname{cur}(T)(x_1)(v)\end{aligned}$$

   So for each $x_1 \in X_1$, $\operatorname{cur}(T)(x_1)$ is linear. Let $a, b \in X_1$, $\alpha \in \mathbb{C}$ and $x_2 \in X_2$. Then

$$\begin{aligned}\operatorname{cur}(T)(a + \alpha b)(x_2) &= T(a + \alpha b, x_2) \\ &= T(a, x_2) + \alpha T(b, x_2) \\ &= \operatorname{cur}(T)(a)(x_2) + \alpha \operatorname{cur}(T)(b)(x_2) \\ &= [\operatorname{cur}(T)(a) + \alpha \operatorname{cur}(T)(b)](x_2)\end{aligned}$$

   Since $x_2 \in X_2$ is arbitrary, $\operatorname{cur}(T)(a + \alpha b) = \operatorname{cur}(T)(a) + \alpha \operatorname{cur}(T)(b)$. Since $a, b \in X_1$ and $\alpha \in \mathbb{C}$ are arbitrary, $\operatorname{cur}(T)$ is linear. Let $(x_1, x_2) \in X_1 \times X_2$. Then

$$\begin{aligned}\|\operatorname{cur}(T)(x_1)(x_2)\| &= \|T(x_1, x_2)\| \\ &\leq (\|T\|\|x_1\|)\|x_2\|\end{aligned}$$

   So $\operatorname{cur}(T)(x_1) \in L(X_2, Y)$ and $\|\operatorname{cur}(T)(x_1)\| \leq \|T\|\|x_1\|$. Since $x_1 \in X_1$ is arbitrary, $\operatorname{cur}(T) \in L(X_1; L(X_2; Y))$ and $\|\operatorname{cur} T\| \leq \|T\|$. Since $T \in L(X_1, X_2; Y)$ is arbitrary, $\operatorname{cur}(L(X_1, X_2; Y)) \subset L(X_1; L(X_2; Y))$. Therefore $\operatorname{cur}|_{L(X_1, X_2; Y)} : L(X_1, X_2; Y) \to L(X_1; L(X_2; Y))$.

3. Let $T \in L(X_1, X_2; Y)$. A previous exercise and an exercise in the section on real numbers imply that

$$\begin{aligned}\|\operatorname{cur}(T)\| &= \sup_{\|x_1\|=1} \|T(x_1)\| \\ &= \sup_{\|x_1\|=1} \left[\sup_{\|x_2\|=1} \|T(x_1)(x_2)\|\right] \\ &= \sup_{\|x_1\|=1, \|x_2\|=2} \|T(x_1)(x_2)\| \\ &= \|T\|\end{aligned}$$

   So $\operatorname{cur}|_{L(X_1, X_2; Y)} : L(X_1, X_2; Y) \to L(X_1; L(X_2; Y))$ is an isometry.

4. Let $T \in L(X_1; L(X_2; Y))$. Define $S : X_1 \times X_2 \to Y$ by $S(x_1, x_2) = T(x_1)(x_2)$. It is straightforward to show that S is bilinear and for each $(x_1, x_2) \in X_1 \times X_2$,

$$\begin{aligned}\|S(x_1, x_2)\| &= \|T(x_1)(x_2)\| \\ &\leq \|T(x_1)\|\|x_2\| \\ &\leq \|T\|\|x_1\|\|x_2\|\end{aligned}$$

   So $S \in L(X_1, X_2; Y)$ and $\|S\| \leq \|T\|$. By construction $\operatorname{cur}(S) = T$. Since $T \in L(X_1; L(X_2; Y))$ is arbitrary, we have that for each $T \in L(X_1; L(X_2; Y))$, there exists $S \in L(X_1, X_2; Y)$ such that $T = \operatorname{cur}(S)$. Hence $\operatorname{cur}|_{L(X_1, X_2; Y)}$ is surjective.

5. Since $\operatorname{cur}|_{L(X_1, X_2; Y)}$ is an isometry, $\operatorname{cur}|_{L(X_1, X_2; Y)}$ is injective. From the previous part, we know that $\operatorname{cur}|_{L(X_1, X_2; Y)}$ is surjective. Hence $\operatorname{cur}|_{L(X_1, X_2; Y)}$ is a bijection. The first part implies that $\operatorname{cur}|_{L(X_1, X_2; Y)}$ is linear. Hence $\operatorname{cur}|_{L(X_1, X_2; Y)} : L(X_1, X_2; Y) \to L(X_1; L(X_2; Y))$ is an isometric isomorphism.

□

Exercise 6.9.0.10. Let X_1, X_2, Y be normed vector spaces. If Y is complete, then $L(X_1, X_2; Y)$ is complete.

Proof. Suppose that Y is complete. Then $L(X_1; L(X_2; Y))$ is complete. Since $L(X_1; L(X_2; Y))$ is isometrically isomorphic to $L(X_1, X_2; Y)$, we have that $L(X_1, X_2; Y)$ is complete. □

Definition 6.9.0.11. Let X_1, X_2 be normed vector spaces, $\phi_1 \in X_1^*$ and $\phi_2 \in X_2^*$. Define $\phi_1 \otimes \phi_2 : X_1 \times X_2$ by $\phi_1 \otimes \phi_2(x_1, x_2) = \phi_1(x_1)\phi_2(x_2)$.

Exercise 6.9.0.12. Let X_1, X_2 be normed vector spaces, $\phi_1 \in X_1^*$ and $\phi_2 \in X_2^*$. Then $\phi_1 \otimes \phi_2 \in L^2(X_1, X_2; \mathbb{C})$.

Proof. Clear. □

Exercise 6.9.0.13. Let X_1, X_2 be normed vector spaces and $(x_1, x_2) \in X_1 \times X_2$. If for each $\phi_1 \in X_1^*$ and $\phi_2 \in X_2^*$, $\phi_1 \otimes \phi_2(x_1, x_2) = 0$, then $x_1 = 0$ or $x_2 = 0$.

Proof. Suppose that $x_1 \neq 0$ and $x_2 \neq 0$. The previous section implies that there exist $\phi_1 \in X_1^*$ and $\phi_2 \in X_2^*$ such that $\phi_1(x_1) = \|x_1\| \neq 0$ and $\phi_2(x_2) = \|x_2\| \neq 0$. Then

$$\begin{aligned}\phi_1 \otimes \phi_2(x_1, x_2) &= \phi_1(x_1)\phi_2(x_2) \\ &\neq 0\end{aligned}$$

□

6.10 Tensor Products of Banach Spaces

(cite "intro to tensor products of banach spaces by Ryan")

6.11 Injective Tensor Product

Definition 6.11.0.1. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. We define $x \otimes_\epsilon y : X^* \times Y^* \to \mathbb{K}$ by $x \otimes_\epsilon y(\phi, \psi) := \phi(x)\psi(y)$.

Exercise 6.11.0.2. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. Then $x \otimes_\epsilon y \in L^2(X^*, Y^*; \mathbb{K})$.

Proof. Let $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} x \otimes_\epsilon y(\phi_1 + \lambda\phi_2, \psi) &= [\phi_1 + \lambda\phi_2(x)]\psi(y) \\ &= \phi_1(x)\psi(y) + \lambda\phi_2(x)\psi(y) \\ &= x \otimes_\epsilon y(\phi_1, \psi) + \lambda x \otimes_\epsilon y(\phi_2, \psi) \end{aligned}$$

Since $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$ are arbitrary, we have that for each $\psi \in Y^*$, $x \otimes_\epsilon y(\cdot, \psi)$ is linear. Similarly for each $\phi \in X^*$, $x \otimes_\epsilon y(\phi, \cdot)$ is linear. Hence $x \otimes_\epsilon y$ is bilinear and $x \otimes_\epsilon y \in L^2(X^*, Y^*; \mathbb{K})$. □

Definition 6.11.0.3. Let X, Y be vector spaces. We define

- the **injective tensor product of X and Y**, denoted $X \otimes Y \subset L^2(X^*, Y^*; \mathbb{K})$, by

$$X \otimes_\epsilon Y := \operatorname{span}(x \otimes_\epsilon y : x \in X \text{ and } y \in Y),$$

- the **injective tensor map**, denoted $\otimes_\epsilon : X \times Y \to X \otimes Y$, by $\otimes_\epsilon(x, y) := x \otimes_\epsilon y$.

Exercise 6.11.0.4. Let X, Y be vector spaces, $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$. The following are equivalent:

1. $\sum_{j=1}^n x_j \otimes y_j = 0$
2. for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum_{j=1}^n \phi(x_j)\psi(y_j) = 0$
3. for each $\phi \in X^*$, $\sum_{j=1}^n \phi(x_j)y_j = 0$
4. for each $\psi \in Y^*$, $\sum_{j=1}^n \psi(y_j)x_j = 0$

Proof.

1. (1) $\implies$ (2) :
   Suppose that $\sum_{j=1}^n x_j \otimes y_j = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Then

$$\begin{aligned} \sum_{j=1}^n \phi(x_j)\psi(y_j) &= \phi\bigg(\sum_{j=1}^n \psi(y_j)x_j\bigg) \\ &= \end{aligned}$$

2.
3.

□

6.11.1 Projective Tensor Product

Definition 6.11.1.1. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. We define $x \otimes_\pi y : L^2(X, Y; \mathbb{K}) \to \mathbb{K}$ by $x \otimes_\pi y(\phi, \psi) := \phi(x)\psi(y)$.

Exercise 6.11.1.2. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. Then $x \otimes_\pi y \in L^2(X, Y; \mathbb{K})^*$.

Proof. Let $A, B \in L^2(X, Y; \mathbb{K})$ and $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} x \otimes_\pi y(A + \lambda B) &= (A + \lambda B)(x, y) \\ &= A(x, y) + \lambda B(x, y) \\ &= x \otimes_\pi y(A) + \lambda x \otimes_\pi y(B) \end{aligned}$$

and

$$\begin{aligned} |x \otimes_\pi y(A)| &= |A(x, y)| \\ &\leq \|x\|\|y\|\|A\| \end{aligned}$$

Since $A, B \in L^2(X, Y; \mathbb{K})$ and $\lambda \in \mathbb{K}$ are arbitrary, we have that $x \otimes_\pi y$ is linear and $\|x \otimes_\pi y\| \leq \|x\|\|y\|$. Hence $x \otimes_\pi y \in L^2(X, Y; \mathbb{K})^*$ □

Definition 6.11.1.3. Let X, Y be vector spaces. We define

- the **projective tensor product of** X **and** Y, denoted $X \otimes Y \subset L^2(X^*, Y^*; \mathbb{K})$, by
$$X \otimes_\pi Y := \operatorname{span}(x \otimes_\pi y : x \in X \text{ and } y \in Y),$$
- the **projective tensor map**, denoted $\otimes_\pi : X \times Y \to X \otimes Y$, by $\otimes(x, y) := x \otimes_\pi y$.

Exercise 6.11.1.4. Let X, Y be vector spaces, $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$. The following are equivalent:

1. $\sum\limits_{j=1}^n x_j \otimes y_j = 0$
2. for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum\limits_{j=1}^n \phi(x_j)\psi(y_j) = 0$
3. for each $\phi \in X^*$, $\sum\limits_{j=1}^n \phi(x_j) y_j = 0$
4. for each $\psi \in Y^*$, $\sum\limits_{j=1}^n \psi(y_j) x_j = 0$

Hint: For (4) $\implies$ (1), set $E := \operatorname{span}(x_j)_{j=1}^n$, $F := \operatorname{span}(y_j)_{j=1}^n$ and define $B \in L^2(E, F; \mathbb{K})^*$ by $B := A|_{E \times F}$. Use the fact that $L^2(E, F; \mathbb{K}) = E^* \otimes_\epsilon F^*$ make this an exercise in the section on multilinear maps.

Proof.

1. (1) $\implies$ (2) :
Suppose that $\sum\limits_{j=1}^n x_j \otimes_\pi y_j = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Define $B_{\phi,\psi} \in L^2(X, Y; \mathbb{K})$ by $B_{\phi,\psi}(x, y) := \phi(x)\psi(y)$. Then

$$\begin{aligned} \sum_{j=1}^n \phi(x_j)\psi(y_j) &= \sum_{j=1}^n B_{\phi,\psi}(x_j, y_j) \\ &= \sum_{j=1}^n x_j \otimes_\pi y_j(B_{\phi,\psi}) \\ &= \left[\sum_{j=1}^n x_j \otimes_\pi y_j\right](B_{\phi,\psi}) \\ &= u(B_{\phi,\psi}) \\ &= 0 \end{aligned}$$

Since $\phi \in X^*$ and $\psi \in Y^*$ are arbitrary, for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum\limits_{j=1}^n \phi(x_j)\psi(y_j) = 0$.

2. (2) $\implies$ (3) :
   Suppose that for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum\limits_{j=1}^n \phi(x_j)\psi(y_j) = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Then

$$\begin{aligned}\psi\bigg(\sum_{j=1}^n \phi(x_j)y_j\bigg) &= \sum_{j=1}^n \phi(x_j)\psi(y_j)\\ &= 0.\end{aligned}$$

   Since $\psi \in Y^*$ is arbitrary, Exercise 6.5.0.11 implies that $\sum\limits_{j=1}^n \phi(x_j)y_j = 0$. Since $\phi \in X^*$ is arbitrary, we have that for each $\phi \in X^*$, $\sum\limits_{j=1}^n \phi(x_j)y_j = 0$.

3. (3) $\implies$ (4):
   Suppose that for each $\phi \in X^*$, $\sum\limits_{j=1}^n \phi(x_j)y_j = 0$. Let $\psi \in Y^*$. Then

$$\begin{aligned}\phi\bigg(\sum_{j=1}^n \psi(y_j)x_j\bigg) &= \sum_{j=1}^n \phi(x_j)\psi(y_j)\\ &= \psi\bigg(\sum_{j=1}^n \phi(x_j)y_j\bigg)\\ &= \psi(0)\\ &= 0.\end{aligned}$$

   Since $\phi \in X^*$ is arbitrary, we have that $\sum\limits_{j=1}^n \psi(y_j)x_j = 0$. Since $\psi \in Y^*$ is arbitrary, we have that for each $\psi \in Y^*$, $\sum\limits_{j=1}^n \psi(y_j)x_j = 0$.

4. (4) $\implies$ (1):
   Suppose that for each $\psi \in Y^*$, $\sum\limits_{j=1}^n \psi(y_j)x_j = 0$. Set $E := \operatorname{span}(x_j : j \in [n])$, $F := \operatorname{span}(y_j : j \in [n])$ and define $B \in L^2(E, F; \mathbb{K})$ by $B := A|_{E\times F}$. make exercise about $\phi_{0,k} \otimes_\epsilon \psi_{0,l} : k, l$ is a basis when $(\phi_{0,k})_{k=1}^n$ and $(\psi_{0,l})_{l=1}^n$ are bases for E^*, which can be obtained from a basis for E and F, there exist $(\phi_{0,j})_{j=1}^n \subset E^*$ and $(\psi_{0,j})_{j=1}^n \subset F^*$ such that $B = \sum_{k=1}^n \phi_{0,k} \otimes_\epsilon \psi_{0,k}$. Let $j \in [n]$. Exercise 6.5.0.8 implies that there exist $(\phi_j)_{j=1}^n \subset X^*$ and $(\psi_j)_{j=1}^n \subset Y^*$ such that for each $j \in [n]$, $\|\phi_j\| = \|\phi_{0,j}\|$,

$\|\psi_j\| = \|\psi_{0,j}\|$, $\phi_j|_E = \phi_{0,j}$ and $\psi_j|_F = \psi_{0,j}$. Then

$$\begin{aligned}
u(A) &= \sum_{j=1}^n x_j \otimes y_j(A) \\
&= \sum_{j=1}^n A(x_j, y_j) \\
&= \sum_{j=1}^n B(x_j, y_j) \\
&= \sum_{j=1}^n \left[\sum_{k=1}^n \phi_{0,k} \otimes_\epsilon \psi_{0,k}(x_j, y_j)\right] \\
&= \sum_{j=1}^n \left[\sum_{k=1}^n \phi_{0,k}(x_j)\psi_{0,k}(y_j)\right] \\
&= \sum_{j=1}^n \left[\sum_{k=1}^n \phi_k(x_j)\psi_k(y_j)\right] \\
&= \sum_{k=1}^n \left[\sum_{j=1}^n \phi_k(x_j)\psi_k(y_j)\right] \\
&= \sum_{k=1}^n \left[\phi_k\left(\sum_{j=1}^n \psi_k(y_j)x_j\right)\right] \\
&= \sum_{k=1}^n \phi_k(0) \\
&= 0
\end{aligned}$$

Since $A \in L^2(X, Y; \mathbb{K})$ is arbitrary, we have that for each $A \in L^2(X, Y; \mathbb{K})$, $u(A) = 0$. Hence $u = 0$.

$\square$

Definition 6.11.1.5. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. We define the **projective norm on** $X \otimes Y$, denoted $\|\cdot\|_\pi : X \otimes Y \to [0, \infty)$, by

$$\|u\|_\pi := \inf\left\{\sum_{j=1}^n \|x_j\|_X \|y_j\|_Y : (x_j)_{j=1}^n \subset X, (y_j)_{j=1}^n \subset Y \text{ and } u = \sum_{j=1}^n x_j \otimes y_j \right\}$$

Exercise 6.11.1.6. Let X, Y be Banach spaces. Then

1. $\|\cdot\|_\pi$ is a norm on $X \otimes Y$
2. for each $x \in X$ and $y \in Y$, $\|x \otimes_\pi y\|_\pi = \|x\|\|y\|$.

Proof.

1. (a) Let $u \in X \otimes Y$. Suppose that $\|u\|_\pi = 0$. Set

$$V_u := \left\{\sum_{j=1}^n \|x_j\|\|y_j\| : (x_j)_{j=1}^n \subset X, (y_j)_{j=1}^n \subset Y \text{ and } u = \sum_{j=1}^n x_j \otimes y_j \right\}$$

Let $\phi \in X^*$, $\psi \in Y^*$ and $\epsilon > 0$. Then there exist $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$ such that

$u = \sum\limits_{j=1}^n x_j \otimes y_j$ and

$$\begin{aligned}\sum_{j=1}^n \|x_j\|\|y_j\| &< \inf V_u + \epsilon/(\|\phi\|\|\psi\|+1) \\ &= \|u\|_\pi + \epsilon/(\|\phi\|\|\psi\|+1) \\ &= \epsilon/(\|\phi\|\|\psi\|+1).\end{aligned}$$

$$\begin{aligned}|u(\phi,\psi)| &= \sum_{j=1}^n |\phi(x_j)\psi(y_j)| \\ &\leq \sum_{j=1}^n \|\phi\|\|\psi\|\|x_j\|\|y_j\| \\ &= \|\phi\|\|\psi\| \sum_{j=1}^n \|x_j\|\|y_j\| \\ &< \|\phi\|\|\psi\| \frac{\epsilon}{\|\phi\|\|\psi\|+1} \\ &\leq \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $u(\phi,\psi) = 0$. Since $\phi \in X^*$ and $\psi \in Y^*$ are arbitrary, we have that for each $\phi \in X^*$ and $\psi \in Y^*$, $u(\phi,\psi) = 0$. Therefore $u = 0$.

(b) Let $u \in X \otimes Y$ and $\lambda \in \mathbb{K}$. Set

$$V_u := \left\{ \sum_{j=1}^n \|x_j\|\|y_j\| : (x_j)_{j=1}^n \subset X, (y_j)_{j=1}^n \subset Y \text{ and } u = \sum_{j=1}^n x_j \otimes y_j \right\}$$

and

$$V_{\lambda u} := \left\{ \sum_{j=1}^n \|x_j\|\|y_j\| : (x_j)_{j=1}^n \subset X, (y_j)_{j=1}^n \subset Y \text{ and } \lambda u = \sum_{j=1}^n x_j \otimes y_j \right\}.$$

Let $\epsilon > 0$. Since $\|u\|_\pi = \inf V_u$, there exists $a \in V_u$ such that $a < \|u\|_\pi + \epsilon/(|\lambda|+1)$. Therefore, there exists $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$ such that $u = \sum\limits_{j=1}^n x_j \otimes y_j$ and $a = \sum_{j=1}^n \|x_j\|\|y_j\|$. Then $\lambda u = \sum\limits_{j=1}^n (\lambda x_j) \otimes y_j$ and

$$\begin{aligned}\|\lambda u\|_\pi &= \inf V_{\lambda u} \\ &\leq \sum_{j=1}^n \|\lambda x_j\|\|y_j\| \\ &= |\lambda| \sum_{j=1}^n \|x_j\|\|y_j\| \\ &= |\lambda| a \\ &< |\lambda| \left(\|u\|_\pi + \frac{\epsilon}{|\lambda|+1} \right) \\ &= |\lambda|\|u\|_\pi + \epsilon \frac{|\lambda|}{|\lambda|+1} \\ &< |\lambda|\|u\|_\pi + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\|\lambda u\|_\pi \leq |\lambda|\|u\|_\pi$.

- Suppose that $\lambda = 0$. Then $\lambda u = 0 = 0 \otimes 0$. Hence

$$\begin{aligned} 0 \leq & \|\lambda u\|_\pi \\ &= \inf V_{\lambda u} \\ &\leq \|0\|\|0\| \\ &= 0 \end{aligned}$$

  Thus

$$\begin{aligned} \|\lambda u\|_\pi &= 0 \\ &= |\lambda|\|u\|_\pi \end{aligned}$$

- Suppose that $\lambda \neq 0$. Then

$$\begin{aligned} \|u\|_\pi &= \|\lambda^{-1}(\lambda u)\|_\pi \\ &\leq |\lambda^{-1}|\|\lambda u\|_\pi \\ &= |\lambda|^{-1}\|\lambda u\|_\pi \end{aligned}$$

  Hence $|\lambda|\|u\|_\pi \leq \|\lambda u\|_\pi$. Thus $\|\lambda u\|_\pi = |\lambda|\|u\|_\pi$.

(c)

2.

$\square$

Chapter 7

Hilbert Spaces

7.1 TODO

- Express $V^* \cong \overline{V}$ where $\overline{V}$ is just V, but with $\lambda * v = \lambda^* v$. so Rieze rep theorem reads $V \cong \overline{V^*}$ or $V \cong \overline{V}^*$
- discuss projection maps
- show internal direct sum isomorphic to external
- discuss quotient hilbert space?
- discus subspaces

7.2 Introduction

Definition 7.2.0.1. Let H be a vector space and $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$. Then

- $\langle \cdot, \cdot \rangle$ is said to be an **inner product** on H if for each $x, y, z \in H$and $c \in \mathbb{C}$
  1. $\langle x, y + cz \rangle = \langle x, y \rangle + c\langle x, z \rangle$
  2. $\langle x, y \rangle = \langle y, x \rangle^*$
  3. $\langle x, x \rangle \geq 0$
  4. if $\langle x, x \rangle = 0$, then $x = 0$.
- $(H, \langle \cdot, \cdot \rangle)$ is said to be a **inner product space** if $\langle \cdot, \cdot \rangle$ is an inner product on H.

Note 7.2.0.2. When the context is clear, we supress the inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$.

Note 7.2.0.3. In mathematics, inner products are conventionally linear in the first argument. The convention in physics is linearity in the second argument. The physics convention notationally generalizes the dot product as matrix multiplication when identifying $\mathbb{C}^n$ with $\mathbb{C}^{n \times 1}$ as is done in an introductory linear algebra class. For example, for $x, y \in \mathbb{C}^n$ $\langle x, y \rangle = \bar{x}^\top y$.

Exercise 7.2.0.4. Let H be an inner product space, $(x_j)_{j=1}^n, (y_j)_{j=1}^n \subset H$ and $(\alpha_j)_{j=1}^n, (\beta_j)_{j=1}^n \subset \mathbb{C}$. Then

$$\left\langle \sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^n \beta_j y_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^n \alpha_i^* \beta_j \langle x_i, y_j \rangle$$

Proof. Clear. □

Definition 7.2.0.5. Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Define the **induced norm**, denoted $\|\cdot\| :\to \mathbb{C}$, by

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

Note 7.2.0.6. Unless otherwise specified, we only consider the induced norm on any given Hilbert space.

Exercise 7.2.0.7. Cauchy-Schwarz Inequality
Let H be an inner product space. Then for each $x, y \in H$, $|\langle x, y \rangle| \leq \|x\|\|y\|$ and $|\langle x, y \rangle| = \|x\|\|y\|$ iff $x \in \text{span}(y)$.
Hint: For $x, y \in H$, put $z = \text{sgn}\langle x, y \rangle^* y$ and Consider $f : \mathbb{R} \to [0, \infty)$ defined by $f(t) = \|x - tz\|^2$

Proof. Let $x, y \in H$. If $y = 0$, then the claim holds trivially. Suppose that $y \neq 0$. Put $z = \text{sgn}\langle x, y \rangle^* y$. So $\langle x, z \rangle = |\langle x, y \rangle|$ and $\|z\| = \|y\|$. Define $f : \mathbb{R} \to [0, \infty)$ by

$$f(t) = \|x - tz\|^2$$

Then for each $t \in \mathbb{R}$,

$$\begin{aligned} 0 &\leq f(t) \\ &= \|x - tz\|^2 \\ &= \|x\|^2 + |t|^2\|z\|^2 - 2\text{Re}\,(t\langle x, z \rangle) \\ &= \|x\|^2 + t^2\|y\|^2 - 2t|\langle x, y \rangle| \end{aligned}$$

Thus f is a quadratic with a minimum at $t_0 = \frac{|\langle x, y \rangle|}{\|y\|^2}$. Hence

$$\begin{aligned} 0 &\leq f(t_0) \\ &= \|x\|^2 + \frac{|\langle x, y \rangle|}{\|y\|^2} - 2\frac{|\langle x, y \rangle|}{\|y\|^2} \\ &= \|x\|^2 - \frac{|\langle x, y \rangle|}{\|y\|^2} \end{aligned}$$

Which implies that

$$|\langle x, y \rangle|^2 \leq \|x\|^2\|y\|^2$$

and hence the claim holds. Clearly if $x \in \text{span}(y)$, then equality holds. Conversely, if equality holds, then $x - z = 0$ which implies that $x \in \text{span}(y)$. $\square$

Exercise 7.2.0.8. Let H be an inner product space. Then the induced norm, $\|\cdot\| : H \to \mathbb{C}$, is a norm.

Proof. Let $x, y \in H$ and $\lambda \in \mathbb{C}$. Then

1. By definition, if $\|x\| = 0$, then $\langle x, x \rangle = 0$, which implies that $x = 0$.
2. Note that

$$\begin{aligned} \|\lambda x\|^2 &= \langle \lambda x, \lambda x \rangle \\ &= \lambda * \lambda \langle x, x \rangle \\ &= |\lambda|^2\|x\|^2 \end{aligned}$$

   So $\|\lambda x\| = |\lambda|\|x\|$
3. The Cauchy-Schwarz inequality implies that

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + \|y\|^2 + 2\text{Re}\,(\langle x, y \rangle) \\ &\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y \rangle| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

   Hence $\|x + y\| \leq \|x\| + \|y\|$.

□

Exercise 7.2.0.9. Let H be an inner-product space and $y \in H$. Then $y = 0$ iff for each $x \in H$, $\langle y, x\rangle = 0$.

Proof.

- ($\implies$):
  Suppose that $y = 0$. Let $x \in H$. Then

$$\begin{aligned}\langle y, x\rangle &= \langle 0, x\rangle \\ &= \langle 0 + 0, x\rangle \\ &= \langle 0, x\rangle + \langle 0, x\rangle \\ &= \langle y, x\rangle + \langle y, x\rangle\end{aligned}$$

  Hence $\langle y, x\rangle = 0$. Since $x \in H$ is arbitrary, we have that for each $x \in H$, $\langle y, x\rangle = 0$.

- ($\impliedby$):
  Suppose that $y \neq 0$. Then

$$\begin{aligned}0 &\neq \|y\| \\ &= \langle y, y\rangle\end{aligned}$$

  Define $x \in H$ by $x := y$. Then $\langle y, x\rangle \neq 0$. Hence $y \neq 0$ implies that there exists $x \in H$ such that $\langle y, x\rangle \neq 0$. By contrapositive, if for each $x \in H$, $\langle y, x\rangle = 0$, then $y = 0$.

□

Exercise 7.2.0.10. Parallelogram Law:
Let H be an inner product space. Then for each $x, y \in H$,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y^2\|)$$

Proof. Let $x, y \in H$. Then

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= (\|x\|^2 + \|y^2\| + 2\operatorname{Re}(\langle x, y\rangle)) + (\|x\|^2 + \|y^2\| - 2\operatorname{Re}(\langle x, y\rangle)) \\ &= 2(\|x\|^2 + \|y^2\|)\end{aligned}$$

□

Definition 7.2.0.11. Let H be an inner product space, $x, y \in H$ and $S \subset H$. Then

1. x and y are said to be **orthogonal**, denoted $x \perp y$, if $\langle x, y\rangle = 0$.
2. S is said to be **orthogonal** if for each $x, y \in S$, $x \perp y$.

Definition 7.2.0.12. Let H be an inner product space and $E \subset H$ a closed subspace. We define the **orthogonal complement of** E, denoted $E^\perp$, by

$$E^\perp = \{x \in H : \text{ for each } y \in E,\ x \perp y\}$$

Exercise 7.2.0.13. Let H be an inner product space and $E \subset H$. Then $E^\perp$ is a closed subspace of H.

Proof.

- Let $x, y \in E^\perp$ and $\lambda \in \mathbb{C}$. Then for each $z \in E$,

$$\begin{aligned}\langle x + \lambda y, z\rangle &= \langle x, z\rangle + \lambda\langle y, z\rangle \\ &= 0\end{aligned}$$

  Hence $x + \lambda y \in E^\perp$. Thus $E^\perp$ is a subspace of H.

- Let $(x_n)_{n\in\mathbb{N}} \subset E^\perp$ and $x \in H$. Suppose that $x_n \to x$. Then for each $z \in E$, continuity implies that

$$\begin{aligned}\langle x, z\rangle &= \lim_{n\to\infty} \langle x_n, z\rangle \\ &= \lim_{n\to\infty} 0 \\ &= 0\end{aligned}$$

Hence $x \in E^\perp$. Since $(x_n)_{n\in\mathbb{N}} \subset E^\perp$ and $x \in H$ with $x_n \to x$ are arbitrary, we have that $E^\perp$ is closed.

□

Exercise 7.2.0.14. Let H be a Hilbert space. Then

1. $H^\perp = \{0\}$.
2. $\{0\}^\perp = H$

Proof.

1. Let $z \in H^\perp$. By definition, for each $x \in H$, $\langle z, x\rangle = 0$. A previous exercise implies that $z = 0$. Since $z \in H^\perp$ is arbitrary, $H^\perp = \{0\}$.
2. Let $z \in H$. Trivially, for each $x \in \{0\}$, $\langle z, x\rangle = 0$. Thus $z \in \{0\}^\perp$. Since $z \in H$ is arbitrary, $H \subset \{0\}^\perp$. Since trivially, $\{0\}^\perp \subset H$, we have that $\{0\}^\perp = H$.

□

Exercise 7.2.0.15. Let H be an inner product space and $E, F \subset H$. If $E \subset F$, then $F^\perp \subset E^\perp$.

Proof. Suppose that $E \subset F$. Let $x \in F^\perp$ and $z \in E$. By definition, for each $y \in F$, $\langle x, y\rangle = 0$. Since $E \subset F$, $z \in F$. Hence $\langle x, z\rangle = 0$. Since $x \in E$ is arbitrary, we have that for each $z \in E$, $\langle x, z, \rangle = 0$. Hence $x \in E^\perp$. Since $x \in F^\perp$ is arbitrary, we have that $F^\perp \subset E^\perp$. □

Exercise 7.2.0.16. Pythagorean theorem:
Let H be an inner product space and $(x_j)_{j=1}^n \subset H$ an orthogonal set. Then

$$\left\|\sum_{j=1}^n x_j\right\|^2 = \sum_{j=1}^n \|x_j\|^2$$

Proof. We have that

$$\begin{aligned}\left\|\sum_{j=1}^n x_j\right\|^2 &= \left\langle \sum_{i=1}^n x_i, \sum_{j=1}^n x_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \langle x_j, x_j\rangle \\ &= \sum_{j=1}^n \langle x_j, x_j\rangle \\ &= \sum_{j=1}^n \|x_j\|^2\end{aligned}$$

□

Exercise 7.2.0.17. Let H be an inner product space and $S \subset H$. Suppose that $0 \notin S$. If S is orthogonal, then S is linearly independent.

Proof. Let $x_1, \cdots, x_n \in S$ and $c_1, \cdots, c_n \in \mathbb{C}$. Suppose that $\sum_{j=1}^{n} c_j x_j = 0$. Since $(c_j x_j)_{j=1}^n$ is orthogonal, the Pythagorean theorem implies that

$$\begin{aligned} 0 &= \left\| \sum_{i=1}^{n} c_i x_i \right\| \\ &= \sum_{j=1}^{n} |c_j|^2 \|x_j\| \end{aligned}$$

So for each $j \in \{1, \cdots, n\}$, $c_j = 0$ and S is linearly independent. □

Definition 7.2.0.18. Let H be an inner product space and $S \subset H$. Then S is said to be **orthonormal** if S is orthogonal and for each $x \in S$, $\|x\| = 1$.

Exercise 7.2.0.19. Bessel's Inequality:
Let H be an inner product space and $S \subset H$. If S is orthonormal, then for each $x \in H$,

1. $\sum_{u \in S} |\langle u, x \rangle|^2 \leq \|x\|$
2. $\{u \in S : \langle u, x \rangle \neq 0\}$ is countable.

Proof.

1. Suppose that S is orthonormal. Let $x \in H$. We consider the measure space $(S, \mathcal{P}(S), \#)$. Define $f_x : S \to [0, \infty)$ by $f_x(u) = \langle u, x \rangle$. Basic results about counting measure imply that

$$\begin{aligned} \sum_{u \in S} |\langle u, x \rangle|^2 &= \int |f_x|^2 \, d\# \\ &= \sup \left\{ \sum_{u \in F} |f_x(u)|^2 : F \subset S \text{ and } \#(F) < \infty \right\} \end{aligned}$$

Let $F \subset S$ finite. Then the Pythagorean theorem implies that

$$\begin{aligned} 0 &\leq \left\| x - \sum_{u \in F} \langle u, x \rangle u \right\|^2 \\ &= \|x\|^2 + \left\| \sum_{u \in F} \langle u, x \rangle u \right\|^2 - 2 \operatorname{Re} \left\langle x, \sum_{u \in F} \langle u, x \rangle u \right\rangle \\ &= \|x\|^2 + \sum_{u \in F} |\langle u, x \rangle|^2 \|u\|^2 - 2 \sum_{u \in F} |\langle u, x \rangle|^2 \\ &= \|x\|^2 - \sum_{u \in F} |\langle u, x \rangle|^2 \end{aligned}$$

Thus

$$\sum_{u \in F} |\langle u, x \rangle|^2 \leq \|x\|$$

Since $F \subset X$ such that $\#(F) < \infty$ was arbitrary, we have that

$$\sum_{u \in S} |\langle u, x \rangle|^2 \leq \|x\|$$

2. Since

$$\int |f_x|^2 \, d\# < \infty$$

basic results about counting measure imply that $\{u \in S : \langle u, x \rangle \neq 0\}$ is countable.

□

Definition 7.2.0.20. Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. Then $(H, \langle \cdot, \cdot \rangle_H)$ is said to be a **Hilbert space** if $(H, \|\cdot\|)$ is a Banach space.

Exercise 7.2.0.21. Let H be a Hilbert space, $E \subset H$ a closed subspace of H, $x \in H$, $y \in E$ and $z \in E^\perp$. If $x = y + z$, then $\|x + E\| = \|z\|$.

Proof. Suppose that $x = y + z$. Let $y' \in E$. The Pythagorean theorem implies that

$$\begin{aligned} \|x - y'\|^2 &= \|y + z - y'\|^2 \\ &= \|y - y'\|^2 + \|z\|^2 \\ &\geq \|z\|^2 \end{aligned}$$

Thus

$$\begin{aligned} \|x + E\| &= \inf_{y' \in E} \|x - y'\| \\ &\geq \|z\| \end{aligned}$$

Also,

$$\begin{aligned} \|x + E\| &\leq \|x - y\| \\ &= \|z\| \end{aligned}$$

Hence $\|x + E\| = \|z\|$. □

Exercise 7.2.0.22. Let H be a Hilbert space, $E \subset H$ a closed subspace of H and $x \in H$. Then

1. there exists a unique $y_0 \in E$ such that $\|x - y_0\| = \|x + E\|$
   Hint: Suppose $(y_n)_{n\in\mathbb{N}} \subset E$ satisfies $\|x - y_n\| \to \|x + E\|$. Show that $(y_n)_{n\in\mathbb{N}}$ is Cauchy using the parallelogram law.

2. there exist unique $y_0 \in E$ and $z_0 \in E^\perp$ such that $x = y_0 + z_0$ and $\|z_0\| = \|x + E\|$
   Hint: Set $z_0 := x - y_0$ and for $u \in E$, choose $\lambda \in \mathbb{C}$ such that $\langle x, \lambda u \rangle \in \mathbb{R}$. Consider $f(t) = \|z - t\lambda u\|^2$.

Proof. Set $s := \|x + E\|$.

1. • **(Existence):**
   Choose $(y_n)_{n\in\mathbb{N}} \subset E$ such that for each $n \in \mathbb{N}$, $\|x - y_n\| < s + 1/n$. Define $(a_n)_{n\in\mathbb{N}} \subset H$ by $a_n = x - y_n$. Let $m, n \in \mathbb{N}$. The parallelogram law implies that

$$\begin{aligned} 2(\|x - y_n\|^2 + \|x - y_m\|^2) &= 2(\|a_n\|^2 + \|a_m\|^2) \\ &= \|a_n + a_m\|^2 + \|a_n - a_m\|^2 \\ &= \|2x - (y_n + y_m)\|^2 + \|y_m - y_n\|^2 \\ &= 4\|x - 2^{-1}(y_n + y_m)\|^2 + \|y_m - y_n\|^2 \end{aligned}$$

   Since $y_n, y_m \in E$, $2^{-1}(y_n + y_m) \in E$ and therefore $\|x - 2^{-1}(y_n + y_m)\| \geq s$. Sine $m, n \in \mathbb{N}$ are arbitrary, we have that for each $m, n \in \mathbb{N}$,

$$\begin{aligned} \|y_m - y_n\|^2 &= 2(\|x - y_n\|^2 + \|x - y_m\|^2) - 4\|x - 2^{-1}(y_n + y_m)\|^2 \\ &\leq 2(\|x - y_n\|^2 + \|x - y_m\|^2) - 4s^2 \end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $1/N < (s^2 + \epsilon^2/4)^{1/2} - s$. Let $m, n \in \mathbb{N}$. Suppose that $m, n \geq N$. Then

$$\begin{aligned}
\|y_m - y_n\|^2 &= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - 4s^2 \\
&< 2(s + 1/n)^2 + 2(s + 1/m)^2 - 4s^2 \\
&< 2(s + 1/N)^2 + 2(s + 1/N)^2 - 4s^2 \\
&= 4(s + 1/N)^2 - 4s^2 \\
&< 4(s^2 + \epsilon^2/4) - 4s^2 \\
&= \epsilon^2
\end{aligned}$$

Thus $\|y_m - y_n\| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $\|y_m - y_n\| < \epsilon$. So $(y_n)_{n \in \mathbb{N}}$ is Cauchy and since H is complete, there exists $y_0 \in H$ such that $y_n \to y$. Coninutiy of the inner product, addition and scalar multiplication implies that

$$\begin{aligned}
\|x - y_0\| &= \lim_{n \to \infty} \|x - y_n\| \\
&= s \\
&= \|x + E\|
\end{aligned}$$

- **(Uniqueness):**
  Let $y_1 \in E$. Suppose that $\|x - y_1\| = \|x + E\|$. Similarly to part (1), the parallelogram law implies that

$$\begin{aligned}
\|y_1 - y_0\|^2 &\leq 2(\|x - y_1\|^2 + \|x - y_0\|^2) - 4s^2 \\
&= 2(s^2 + s^2) - 4s^2 \\
&= 0
\end{aligned}$$

  Hence $\|y_1 - y_0\| = 0$ and $y_1 = y_0$.

2. - **(Existence):**
  Set $z_0 := x - y_0$. Let $u \in E$. Set

$$\lambda := \begin{cases} (\operatorname{sgn}\langle z_0, u \rangle)^{-1} & \langle z_0, u \rangle \neq 0 \\ 1 & \langle z_0, u \rangle = 0 \end{cases}$$

  and $v := \lambda u$. So $\langle z_0, v \rangle \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \|z_0 - tv\|^2$. By construction, since $y_0, v \in E$, we have that for each $t \in \mathbb{R}$, $y_0 - tv \in E$ and therefore

$$\begin{aligned}
f(t) &= \|z_0 - tv\|^2 \\
&= \|x - (y_0 - tv)\|^2 \\
&\geq \inf_{y \in E} \|x - y\|^2 \\
&= \|x - y_0\|^2 \\
&= f(0)
\end{aligned}$$

  Since f is smooth and has a local minimum at $t = 0$, $f'(0) = 0$. Furthermore, for each $t \in \mathbb{R}$,

$$\begin{aligned}
f(t) &= \|z_0\|^2 - 2t \operatorname{Re}(\langle z_0, v \rangle) + t^2 \|v\|^2 \\
&= \|z_0\|^2 - 2t \langle z_0, v \rangle + t^2 \|v\|^2
\end{aligned}$$

  so that $f'(0) = 2\langle z_0, v \rangle$. Thus

$$\begin{aligned}
\langle z_0, u \rangle &= \lambda^{-1} \langle z_0, \lambda u \rangle \\
&= \lambda^{-1} \langle z_0, v \rangle \\
&= 0
\end{aligned}$$

Since $u \in E$ is arbitrary, we have that $z_0 \in E^\perp$. **(Uniqueness):**
Suppose that there exist $y_1 \in E$ and $z_1 \in E^\perp$ such that $x = y_1 + z_1$ and $\|z_1\| = \|x + E\|$. Since $z_1 = x - y_1$, by assmuption,

$$\begin{aligned} \|x - y_1\| &= \|z_1\| \\ &= \|x + E\| \end{aligned}$$

Since $y_1 \in E$, uniqueness in part (1) implies that $y_1 = y_0$. Hence

$$\begin{aligned} z_1 &= x - y_1 \\ &= x - y_0 \\ &= z_0 \end{aligned}$$

□

Exercise 7.2.0.23. Let H be a Hilbert space and $E \subset H$ a closed subspace of H. Then $(E^\perp)^\perp = E$.

Proof.

- Let $x \in (E^\perp)^\perp$ and $x_0 \in H$. The previous exercise implies that there exist unique $y, y_0 \in E$ and $z, z_0 \in E^\perp$ such that $x = y + z$, $x_0 = y_0 + z_0$, $\|x\| = \|y + E\|$ and $\|z_0\| = \|x + E\|$. Since $x \in (E^\perp)^\perp$, $z_0 \in E^\perp$ and $y_0 \in E$, we have that

$$\begin{aligned} \langle x, x_0 \rangle &= \langle x, y_0 + z_0 \rangle \\ &= \langle x, y_0 \rangle + \langle x, z_0 \rangle \\ &= \langle x, y_0 \rangle \\ &= \langle y + z, y_0 \\ &= \langle y, y_0 \rangle + \langle z, y_0 \rangle \\ &= \langle y, y_0 \rangle \end{aligned}$$

  Similarly, since $y \in E$ and $z_0 \in E^\perp$, we have that

$$\begin{aligned} \langle y, x_0 \rangle &= \langle y, y_0 + z_0 \rangle \\ &= \langle y, y_0 \rangle + \langle y, z_0 \rangle \\ &= \langle y, y_0 \rangle \end{aligned}$$

  Therefore

$$\begin{aligned} \langle z, x_0 \rangle &= \langle x - y, x_0 \rangle \\ &= \langle x, x_0 \rangle - \langle y, x_0 \rangle \\ &= \langle y, y_0 \rangle - \langle y, y_0 \rangle \\ &= 0 \end{aligned}$$

  Since $x_0 \in H$ is arbitrary, a previous exercise implies that $z = 0$. Hence

$$\begin{aligned} x &= y + z \\ &= y \\ &\in E \end{aligned}$$

  Since $x \in (E^\perp)^\perp$ is arbitrary, we have that $(E^\perp)^\perp \subset E$.

- Let $y \in E$ and $z \in E^\perp$. By definition of $E^\perp$, $\langle y, z \rangle = 0$. Since $z \in E^\perp$ is arbitrary, we have that for each $z \in E^\perp$, $\langle y, z \rangle = 0$. Hence $y \in (E^\perp)^\perp$. Since $y \in E$ is arbitrary, we have that $E \subset (E^\perp)^\perp$.

Since $(E^\perp)^\perp \subset E$ and $E \subset (E^\perp)^\perp$, we have that $(E^\perp)^\perp = E$. $\square$

Exercise 7.2.0.24. Let H be a Hilbert space and $E, F \subset H$ closed subspaces of H. Then $E = F$ iff $E^\perp = F^\perp$.

Proof. If $E = F$, then clearly $E^\perp = F^\perp$.
Conversely, suppose that $E^\perp = F^\perp$. Then the previous exercise and part (1) imply that

$$\begin{aligned} E &= (E^\perp)^\perp \\ &= (F^\perp)^\perp \\ &= F \end{aligned}$$

$\square$

Exercise 7.2.0.25. Riesz Representation Theorem:
Let H be a Hilbert space. For each $\phi \in H^*$, there exists a unique $y \in H$ such that for each $x \in H$, $\phi(x) = \langle y, x \rangle$.
Hint: If $x \notin \ker \phi$, then there exists $z \in E^\perp$ such that $\|z\| = 1$. Consider $u := \phi(x)z - \phi(z)x$. Then $u \in E$ and consider $\langle z, u \rangle$.

Proof. Let $\phi \in H^*$.

- **(Existence):**

  - Suppose that $\phi = 0$. Set $y := 0$. A previous exercise implies that for each $x \in H$, $\phi(x) = \langle y, x \rangle$.
  - Suppose that $\phi \neq 0$. Set $E = \ker \phi$. Since ϕ is continuous, E is a closed subspace of H. Since $\phi \neq 0$, $E \neq H$. The previous exercise then implies that $E^\perp \neq \{0\}$. Thus there exists $z \in E^\perp$ such that $\|z\| = 1$. Define $y \in H$ by $y := \phi(z)^* z$. Let $x \in H$. Define $u \in H$ by $u := \phi(x)z - \phi(z)x$. Then

    $$\begin{aligned} \phi(u) &= \phi(x)\phi(z) - \phi(z)\phi(x) \\ &= 0 \end{aligned}$$

    Therefore $u \in E$. Since $z \in E^\perp$, we have that

    $$\begin{aligned} 0 &= \langle z, u \rangle \\ &= \langle z, \phi(x)z - \phi(z)x \rangle \\ &= \langle z, \phi(x)z \rangle - \langle z, \phi(z)x \rangle \\ &= \phi(x)\langle z, z \rangle - \phi(z)\langle z, x \rangle \\ &= \phi(x)\|z\|^2 - \phi(z)\langle z, x \rangle \\ &= \phi(x) - \langle \phi(z)^* z, x \rangle \\ &= \phi(x) - \langle y, x \rangle \end{aligned}$$

    Since $x \in H$ is arbitrary, we have that for each $x \in H$, $\phi(x) = \langle y, x \rangle$.

- **(Uniqueness):**
  Let $y' \in H$. Suppose that for each $x \in H$, $\phi(x) = \langle y', x \rangle$. Then for each $x \in H$, $\langle y - y', x \rangle = 0$. A previous exercise implies that $y - y' = 0$. Thus $y' = y$.

$\square$

Definition 7.2.0.26. Let H be a hilbert space. We

Exercise 7.2.0.27. Let H be a Hilbert space and $S \subset H$. Suppose that S is orthonormal. Then the following are equivalent:

1. For each $x \in H$, if for each $u \in S$, $\langle u, x\rangle = 0$, then $x = 0$.
2. For each $x \in H$, there exist $(u_j)_{j\in\mathbb{N}} \subset S$ such that $x = \sum\limits_{j\in\mathbb{N}} \langle u_j, x\rangle u_j$ and for each $u \notin (u_j)_{j\in\mathbb{N}}$, $\langle u, x\rangle = 0$.
3. For each $x \in H$, $\|x\|^2 = \sum\limits_{u\in S} |\langle u, x\rangle|^2$.

Proof.

- (1) $\implies$ (2):
Suppose that for each $x \in H$, if for each $u \in S$, $\langle u, x\rangle = 0$, then $x = 0$. Let $x \in H$. Set $S_* := \{u \in S : \langle u, x\rangle \neq 0\}$. The previous exercise implies that S_* is countable. Write $S_* = (u_j)_{j\in\mathbb{N}}$. The previous exercise tells us that $\sum\limits_{j\in\mathbb{N}} |\langle u_j, x\rangle|^2 \leq \|x\|^2$ and hence converges. Thus for $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that if $m < n$, then

$$\sum_{m+1}^{n} |\langle u_j, x\rangle|^2 < \epsilon$$

Define $(y_n)_{n\in\mathbb{N}} \subset H$ by

$$y_n = \sum_{j=1}^{n} \langle u_j, x\rangle u_j$$

Then for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that if $m < n$, then

$$\begin{aligned}\|y_n - y_m\|^2 &= \left\| \sum_{1}^{n} \langle u_j, x\rangle u_j - \sum_{1}^{m} \langle u_j, x\rangle u_j \right\|^2 \\ &= \left\| \sum_{m+1}^{n} \langle u_j, x\rangle u_j \right\|^2 \\ &= \sum_{m+1}^{n} |\langle u_j, x\rangle|^2 \\ &< \epsilon\end{aligned}$$

So $(y_n)_{n\in\mathbb{N}}$ is Cauchy. Since H is complete, there exists $y \in H$ such that $y_n \to y$. By definition,

$$y = \sum_{j\in\mathbb{N}} \langle u_j, x\rangle u_j$$

Continuity of $\langle \cdot, \cdot\rangle : H \times H \to \mathbb{C}$ implies that

  1. for each $u \in S \setminus S_*$,

$$\begin{aligned}\langle u, x - y\rangle &= \langle u, x\rangle - \langle u, y\rangle \\ &= \langle u, x\rangle - \lim_{n\to\infty} \langle u, y_n\rangle \\ &= \langle u, x\rangle - \lim_{n\to\infty} \sum_{j=1}^{n} \langle u_j, x\rangle \langle u, u_j\rangle \\ &= 0 - 0 \\ &= 0\end{aligned}$$

2. for each $k \in \mathbb{N}$,

$$\begin{aligned}\langle u_k, x - y\rangle &= \langle u_k, x\rangle - \langle u_k, y\rangle \\ &= \langle u_k, x\rangle - \lim_{n\to\infty} \langle u_k, y_n\rangle \\ &= \langle u_k, x\rangle - \lim_{n\to\infty} \sum_{j=1}^{n} \langle u_j, x\rangle\langle u_k, u_j\rangle \\ &= \langle u_k, x\rangle - \langle u_k, x\rangle \\ &= 0\end{aligned}$$

So for each $u \in S$, $\langle u, x - y\rangle = 0$. By assumption, $x - y = 0$ and hence

$$x = \sum_{j\in\mathbb{N}} \langle u_j, x\rangle u_j$$

- (2) $\implies$ (3):
Suppose that for each $x \in H$, there exist $(u_j)_{j\in\mathbb{N}} \subset S$ such that $x = \sum_{j\in\mathbb{N}} \langle u_j, x\rangle u_j$ and for each $u \notin (u_j)_{j\in\mathbb{N}}$, $\langle u, x\rangle = 0$. Then continuity of $\|\cdot\| : H \to [0, \infty)$ implies that

$$\begin{aligned}\|x\|^2 &= \left\| \lim_{n\to\infty} \sum_{j=1}^{n} \langle u_j, x\rangle u_j \right\|^2 \\ &= \lim_{n\to\infty} \left\| \sum_{j=1}^{n} \langle u_j, x\rangle u_j \right\|^2 \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} |\langle u_j, x\rangle|^2 \\ &= \sum_{j\in\mathbb{N}} |\langle u_j, x\rangle|^2 \\ &= \sum_{u\in S} |\langle u, x\rangle|^2\end{aligned}$$

- (3) $\implies$ (1):
Suppose that for each $x \in H$, $\|x\|^2 = \sum_{u\in S} |\langle u, x\rangle|^2$. Let $x \in H$. Suppose that for each $u \in S$, $\langle u, x\rangle = 0$. Then

$$\begin{aligned}\|x\|^2 &= \sum_{u\in S} |\langle u, x\rangle|^2 \\ &= 0\end{aligned}$$

So $x = 0$

□

Definition 7.2.0.28. Let H be a Hilbert space and $S \subset H$. Then S is said to be an **orthonormal basis of** H if

1. S is orthonormal
2. for each $x \in H$, if for each $u \in S$, $\langle u, x\rangle = 0$, then $x = 0$

7.3 Operators and Functionals on Hilbert Spaces

Exercise 7.3.0.1. Let H_1, H_2 be Hilbert spaces and $A \in L(H_1, H_2)$. Then there exists a unique $B \in L(H_2, H_1)$ such that for each $x_1 \in H_1$ and $x_2 \in H_2$,

$$\langle x_1, Bx_2 \rangle = \langle Ax_1, x_2 \rangle$$

Proof. □

Definition 7.3.0.2. Adjoint of an Operator:
Let H_1, H_2 be a Hilbert space and $A \in L(H_1, H_2)$. We define the **adjoint of** A, denoted A^*, to be the unique $B \in L(H_2, H_1)$ such that for each $x_1 \in H_1$ and $x_2 \in H_2$,

$$\langle x_1, Bx_2 \rangle = \langle Ax_1, x_2 \rangle$$

Note 7.3.0.3. In physics, the adjoint of A is typically denoted by $A^\dagger$.

Exercise 7.3.0.4. Let H be a Hilbert space, $A, B \in L(H)$ and $\lambda \in \mathbb{C}$, then

1. $(A^*)^* = A$
2. $(A + B)^* = A^* + B^*$
3. $(AB)^* = B^* A^*$
4. $(\lambda A)^* = \lambda^* A^*$
5. A and B commute iff A^* and B^* commute.

Proof. Let x_1, $x_2 \in H$. Then

1.
$$\begin{aligned} \langle Ax_1, x_2 \rangle &= \langle x_2, Ax_1 \rangle^* \\ &= \langle A^* x_2, x_1 \rangle^* \quad \text{(by definition)} \\ &= \langle x_1, A^* x_2 \rangle \end{aligned}$$

2.
$$\begin{aligned} \langle x_1, (A + B)x_2 \rangle &= \langle x_1, Ax_2 \rangle + \langle x_1, Bx_2 \rangle \\ &= \langle A^* x_1, x_2 \rangle + \langle B^* x_1, x_2 \rangle \\ &= \langle (A^* + B^*)x_1, x_2 \rangle \end{aligned}$$

3.
$$\begin{aligned} \langle x_1, ABx_2 \rangle &= \langle A^* x_1, Bx_2 \rangle \\ &= \langle B^* A^* x_1, x_2 \rangle \end{aligned}$$

4.
$$\begin{aligned} \langle x_1, \lambda Ax_2 \rangle &= \lambda \langle x_1, Ax_2 \rangle \\ &= \lambda \langle A^* x_1, x_2 \rangle \\ &= \langle \lambda^* A^* x_1, x_2 \rangle \end{aligned}$$

5. If A and B commute, then

$$\begin{aligned} A^*B^* &= (BA)^* \\ &= (AB)^* \\ &= B^*A^* \end{aligned}$$

Conversely, if A^* and B^* commute then

$$\begin{aligned} AB &= (B^*A^*)^* \\ &= (A^*B^*)^* \\ &= BA \end{aligned}$$

□

Definition 7.3.0.5. Let H be a Hilbert space and $Q \in L(H)$. Then Q is said to be **self-adjoint** if

$$Q = Q^*$$

Exercise 7.3.0.6. Let H be a Hilbert space and $Q \in L(H)$. If Q is a self-adjoint then

1. the eigenvalues of Q are real.
2. the eigenvectors of Q corresponding to distinct eigenvalues are orthogonal.

Proof. Suppose that Q is self-adjoint.

1. Let λ be an eigenvalue of Q with corresponding eigenvector x. Then

$$\begin{aligned} \lambda\langle x, x\rangle &= \langle x, Qx\rangle \\ &= \langle Qx, x\rangle \\ &= \lambda^*\langle x, x\rangle \end{aligned}$$

Thus $\lambda = \lambda^*$ and is real

2. Let λ_1 and λ_2 be eigenvalues of Q with corresponding eigenvectors x_1 and x_2. Suppose that $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned} \lambda_2\langle x_1, x_2\rangle &= \langle x_1, Qx_2\rangle \\ &= \langle Qx_1, x_2\rangle \\ &= \lambda_1\langle x_1, x_2\rangle \end{aligned}$$

So $(\lambda_2 - \lambda_1)\langle x_1, x_2\rangle = 0$. Which implies that $\langle x_1, x_2\rangle = 0$

□

Exercise 7.3.0.7. Let H be a Hilbert space, $A, B \in L(H)$ and $\lambda \in \mathbb{R}$. Suppose that A, B are self-adjoint. If A and B commute and then λAB is self-adjoint.

Proof.

$$\begin{aligned} (\lambda AB)^* &= \lambda^*(AB)^* \\ &= \lambda B^*A^* \\ &= \lambda BA \\ &= \lambda AB \end{aligned}$$

□

Definition 7.3.0.8. Adjoint of a Vector:
Let H be a Hilbert space and $x \in H$. We define the **adjoint** of x, denoted $x^* \in H^*$, by $x^*y = \langle x, y\rangle$.

Note 7.3.0.9. In mathematics, where linearity of the inner product is in the first argument, x^* is typically referred to by $u_x \in H^*$ where $u_x(y) = \langle y, x\rangle$. In physics, where the inner product with linearity in the second argument, $x^*\phi$ is usually written in the so-called "bra-ket" notation as $\langle x|\phi\rangle$ which works smoothly since it aligns with the linearity of $u_x(\phi_1 + \lambda\phi_2)$ and the conjugate-linearity of $u_{x_1+\lambda x_2}(\phi)$. In this way, it generalizes the notation for $\langle x, y\rangle = x^Ty$ for $\mathbb{R}^n$ to $\langle x, y\rangle = x^*y$ for $\mathbb{C}^n$.

Exercise 7.3.0.10. Let H be a Hilbert space, $x, y \in H$ and $\lambda \in \mathbb{C}$. Then

1. $(x + y)^* = x^* + y^*$
2. $(\lambda x)^* = \lambda^* x^*$

Proof. Clear. □

Definition 7.3.0.11. Let H be a Hilbert space, $x, y \in H$ and $A \in L(H)$. We define

1. $x^*A \in H^*$ by $(x^*A)y = x^*(Ay)$
2. $xy^* \in L(H)$ by $(xy^*)z = (y^*z)x$

Exercise 7.3.0.12. Let H be a Hilbert space, $A \in L(H)$ and $x \in H$. Then

$$(Ax)^* = x^*A^*$$

Proof. Let $y \in H$. Then

$$\begin{aligned}(Ax)^*y &= \langle Ax, y\rangle \\ &= \langle x, A^*y\rangle \\ &= x^*A^*y\end{aligned}$$

□

Definition 7.3.0.13. Commutator:
Let H be a Hilbert space and $A, B \in L(H)$. The **commutator** of A and B, denoted $[A, B]$, is defined by

$$[A, B] = AB - BA$$

Exercise 7.3.0.14. Let H be a Hilbert space and $A, B, C \in L(H)$. Then

1. $[AB, C] = A[B, C] + [A, C]B$
2. $[A, BC] = B[A, C] + [A, B]C$

Proof.

1. $$\begin{aligned}[AB, C] &= ABC - CAB \\ &= ABC - ACB + ACB - CAB \\ &= A(BC - CB) + (AC - CA)B \\ &= A[B, C] + [A, C]B\end{aligned}$$

2. Similar to (1).

□

7.4 Subspaces of Hilbert Spaces

Exercise 7.4.0.1. Let $(H, \langle \cdot, \cdot \rangle) \in \text{Obj}(\textbf{Hilb})$ and $E \subset H$ a closed subspace of H. Then $(E, \langle \cdot, \cdot \rangle|_{E \times E})$ is a Hilbert space.

Proof. Clearly $(E, \langle \cdot, \cdot \rangle|_{E \times E})$ is an inner product space. Since E is closed and H is complete, E is complete. Hence $(E, \langle \cdot, \cdot \rangle|_{E \times E})$ is a Hilbert space. $\square$

Definition 7.4.0.2. Let $H \in \text{Obj}(\textbf{Hilb})$ and $P \in L(H)$. Then P is said to be **idemptoent** if $P^2 = P$.

Exercise 7.4.0.3. Let $H \in \text{Obj}(\textbf{Hilb})$ and $P \in L(H)$. Then P is idempotent iff $I - P$ is idempotent.

Proof.

- $(\implies)$:
  Suppose that P is idempotent. Then

$$\begin{aligned}(I - P)^2 &= (I - P)(I - P) \\ &= I^2 - IP - PI + P^2 \\ &= I - 2P + P \\ &= I - P\end{aligned}$$

  So $I - P$ is idempotent.

- $(\impliedby)$:
  Suppose that $I - P$ is idempotent. Part (1) implies that $I - (I - P) = P$ is idempotent.

$\square$

Exercise 7.4.0.4. Let $H \in \text{Obj}(\textbf{Hilb})$, $E \subset H$ a closed subspace of H and $P \in L(H)$. If $P|_E = I_E$ and $\operatorname{Im} P \subset E$, then P is idempotent

Proof. Suppose that $P|_E = I_E$ and $\operatorname{Im} P \subset E$. Let $x \in H$ and define $y \in E$ by $y := P(x)$. Then

$$\begin{aligned}P^2(x) &= P(y) \\ &= P|_E(y) \\ &= I_E(y) \\ &= y \\ &= P(x)\end{aligned}$$

Since $x \in H$ is arbitrary, we have that $P^2 = P$ and P is idempotent. $\square$

Definition 7.4.0.5. Let $H \in \text{Obj}(\textbf{Hilb})$ and $E \subset H$ a closed subspace of H. We define the **orthogonal projection onto** E, denoted $P_E : H \to E$ by

$$P_E(x) = \underset{y \in E}{\arg\min} \|x - y\|$$

Note 7.4.0.6. An exercise in introduction section implies that P_E is well-defined.

Exercise 7.4.0.7. Let $H \in \text{Obj}(\textbf{Hilb})$ and $E \subset H$ a closed subspace of H. Then

1. $P_E|_E = I_E$
2. $\operatorname{Im} P_E = E$
3. P_E is linear
4. $P_E \in L(H, E)$ and $\|P_E\| = \begin{cases} 0 & E = \{0\} \\ 1 & E \neq \{0\} \end{cases}$ 1

5. P_E is self-adjoint
6. P_E is idempotent

Proof.

1. Let $x \in E$. Then

$$\begin{aligned} P_E(x) &= \arg\min_{y \in E} \|x - y\| \\ &= x \end{aligned}$$

Since $x \in E$ is arbitrary, $P_E|_E = I_E$.

2. By definition, $\operatorname{Im} P_E \subset E$. The previous part implies that $E \subset \operatorname{Im} P_E$. Thus $\operatorname{Im} P_E = E$.

3. Let $x_1, x_2 \in H$ and $\lambda \in \mathbb{C}$. A previous exercise in the introduction section implies that there exist unique $y_1, y_2 \in E$ and $z_1, z_2 \in E^\perp$ such that $x_1 = y_1 + z_1$, $x_2 = y_2 + y_2$, $\|x_1 + E\| = \|z_1\|$ and $\|x_2 + E\| = \|z_2\|$. Then $y_1 + \lambda y_2 \in E$, $z_1 + \lambda z_2 \in E^\perp$ and $x_1 + \lambda x_2 = (y_1 + \lambda y_2) + (z_1 + \lambda z_2)$. An exercise in the introduction section implies that $\|x_1 + \lambda x_2 + E\| = \|z_1 + \lambda z_2\|$. Uniqueness implies that

$$\begin{aligned} P_E(x_1 + \lambda x_2) &= y_1 + \lambda y_2 \\ &= P_E(x_1) + \lambda P_E(x_2) \end{aligned}$$

Since $x_1, x_2 \in H$ and $\lambda \in \mathbb{C}$ are arbitrary, P_E is linear.

4. Let $x \in H$. Then there exist unique $y \in E$ and $z \in E^\perp$ such that $x = y + z$ and $\|x + E\| = \|z\|$. The Pythagorean theorem implies that

$$\begin{aligned} \|P_E(x)\|^2 &= \|y\|^2 \\ &\leq \|y\|^2 + \|z\|^2 \\ &= \|y + z\|^2 \\ &= \|x\|^2 \end{aligned}$$

So $\|P_E(x)\| \leq \|x\|$. Since $x \in H$ is arbitrary, $P_E \in L(H, E)$ and $\|P_E\| \leq 1$. If $E = \{0\}$, then $P_E = 0$ and therefore $\|P_E\| = 0$. Suppose that $E \neq \{0\}$. Then there exists $y \in E$ such that $\|y\| = 1$. Hence

$$\begin{aligned} \|P_E\| &= \sup_{y' \neq 0} \|y'\|^{-1} \|P_E(y')\| \\ &\geq \|P_E(y)\| \\ &= \|y\| \\ &= 1 \end{aligned}$$

So $\|P_E\| = 1$.

5. FINISH!!!

6. Since $P \in L(H)$, $P|_E = I_E$ and $\operatorname{Im} P \subset E$, a previous exercise implies that P_E is idempotent.

$\square$

Exercise 7.4.0.8. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$, $E \subset H$ a closed subspace of H. Then there exists a unique $P \in L(H)$ such that

1. P is self-adjoint
2. P is idempotent
3. $\operatorname{Im} P = E$

Hint: for uniqueness, if P and Q satsify $(1)-(3)$, consider $(P-Q)^*(P-Q)$

Proof.

- **(Existence):**

- **(Uniqueness):**
  Let $Q \in L(H,E)$. Suppose that Q is self-adjoint, Q is idempotent and $\operatorname{Im} Q = E$. The previous exercise implies that $P|_E, Q|_E = I_E$. Then

$$\begin{aligned}(P-Q)^*(P-Q) &= P^*P - P^*Q - Q^*P + Q^*Q \\ &= P^2 - PQ - QP + Q^2 \\ &= P^2 - P|_E Q - Q|_E P + Q^2 \\ &= P - Q - P + Q \\ &= 0\end{aligned}$$

□

Definition 7.4.0.9. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$, $E \subset H$ a closed subspace of H and $P \in L(H,E)$. Then P is said to be an **orthogonal projection onto** E if

1. P is self-adjoint
2. $P^2 = P$
3. $\operatorname{Im} P = E$

Exercise 7.4.0.10. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$, $E \subset H$ a closed subspace of H. Then there exists a unique $P \in L(H,E)$ such that P is an othogonal projection onto E.

Definition 7.4.0.11. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$, $E \subset H$ a closed subspace of H and $P \in L(H,E)$. Then P is said to be an **orthogonal projection onto** E, denoted $P_E \in L(H,E)$, by

$$P_E(x) = \operatorname*{arg\,min}_{y \in E} \|x - y\|$$

7.5 Direct Sums of Hilbert spaces

Definition 7.5.0.1. Let $(H_\alpha, \langle\cdot,\cdot\rangle_\alpha)_{\alpha\in A} \subset \text{Obj}(\textbf{Hilb})$. We define $s_A : \prod_{\alpha\in A} H_\alpha \to [0,\infty)^A$ by $s_A(x) = (\|x_\alpha\|_\alpha)_{\alpha\in A}$.

Exercise 7.5.0.2. Let $(H_\alpha, \langle\cdot,\cdot\rangle_\alpha)_{\alpha\in A} \subset \text{Obj}(\textbf{Hilb})$. Then

1. for each $x, y \in s_A^{-1}(l^2(A))$, $(\langle x_\alpha, y_\alpha\rangle_\alpha)_{\alpha\in A} \in l^1(A)$
2. for each $x, y \in \prod_{\alpha\in A} H_\alpha$, $\|s_A(x) - s_A(y)\|_2 \leq \|s_A(x-y)\|_2$

Proof.

1. Let $x, y \in s_A^{-1}(l^2(A))$. Then $s_A(x), s_A(y) \in l^2(A)$. Therefore $s_A(x)s_A(y) \in l^1(A)$ and
$$\begin{aligned}\|\langle x_\alpha, y_\alpha\rangle_\alpha\|_1 &= \sum_{\alpha\in A} |\langle x_\alpha, y_\alpha\rangle_\alpha| \\ &\leq \sum_{\alpha\in A} \|x_\alpha\|_\alpha \|y_\alpha\|_\alpha \\ &= \|s_A(x)s_A(y)\|_1 \\ &< \infty\end{aligned}$$
2. Let $x, y \in \prod_{\alpha\in A} H_\alpha$. The reverse triangle inequality implies that
$$\begin{aligned}\|s_A(x) - s_A(y)\|_2^2 &= \|(\|x_\alpha\|_\alpha)_{\alpha\in A} - (\|y_\alpha\|_\alpha)_{\alpha\in A}\|_2^2 \\ &= \|(\|x_\alpha\|_\alpha - \|y_\alpha\|_\alpha)_{\alpha\in A}\|_2^2 \\ &= \sum_{\alpha\in A} |\|x_\alpha\|_\alpha - \|y_\alpha\|_\alpha|^2 \\ &\leq \sum_{\alpha\in A} \|x_\alpha - y_\alpha\|_\alpha^2 \\ &= \|(\|(x-y)_\alpha\|_\alpha)_{\alpha\in A}\|_2^2 \\ &= \|s_A(x-y)\|_2^2\end{aligned}$$
Thus $\|s_A(x) - s_A(y)\|_2 \leq \|s_A(x-y)\|_2^2$.

□

Definition 7.5.0.3. Let $(H_\alpha, \langle\cdot,\cdot\rangle_\alpha)_{\alpha\in A} \subset \text{Obj}(\textbf{Hilb})$. We define $\bigoplus_{\alpha\in A} H_\alpha \subset \prod_{\alpha\in A} H_\alpha$ and $\langle\cdot,\cdot\rangle : \left[\bigoplus_{\alpha\in A} H_\alpha\right]^2 \to \mathbb{C}$ by
$$\bigoplus_{\alpha\in A} H_\alpha = s_A^{-1}(l^2(A))$$
and
$$\langle x, y\rangle = \sum_{\alpha\in A} \langle x_\alpha, y_\alpha\rangle_\alpha$$
We define the **direct sum of** $(H_\alpha, \langle\cdot,\cdot\rangle_\alpha)_{\alpha\in A}$ to be $(\bigoplus_{\alpha\in A} H_\alpha, \langle\cdot,\cdot\rangle)$.

Exercise 7.5.0.4. Let $(H_\alpha, \langle\cdot,\cdot\rangle_\alpha)_{\alpha\in A} \subset \text{Obj}(\textbf{Hilb})$. Then

1. for each $x \in \prod_{\alpha\in A} H_\alpha$, $x \in \bigoplus_{\alpha\in A} H_\alpha$ iff $s_A(x) \in l^2(A)$

2. $s_A|_{\bigoplus\limits_{\alpha \in A} H_\alpha} : \bigoplus\limits_{\alpha \in A} H_\alpha \to l^2(A)$

Proof. Immediate by definition ☐

Exercise 7.5.0.5. Let $(H_\alpha, \langle \cdot, \cdot \rangle_\alpha)_{\alpha \in A} \subset \mathrm{Obj}(\mathbf{Hilb})$. Set $H := \bigoplus\limits_{\alpha \in A} H_\alpha$. Then

1. H is a vector space
2. $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ is an inner product
3. $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space

Proof.

1. Clearly $\prod\limits_{\alpha \in A} H_\alpha$ is a vector space. Let $x, y \in H$ and $\lambda \in \mathbb{C}$. The previous exercise implies that $s_A(x), s_A(y) \in l^2(A)$. Therefore $s_A(x)s_A(y) \in L^1(A)$ and

$$\begin{aligned}
\sum_{\alpha \in A} \|(x + \lambda y)_\alpha\|_\alpha^2 &= \sum_{\alpha \in A} \|x_\alpha + \lambda y_\alpha\|_\alpha^2 \\
&= \sum_{\alpha \in A} \left[\|x_\alpha\|_\alpha^2 + |\lambda|^2 \|y_\alpha\|_\alpha^2 + 2\,\mathrm{Re}(\langle x_\alpha, y_\alpha \rangle_\alpha) \right] \\
&\leq \sum_{\alpha \in A} \|x_\alpha\|_\alpha^2 + |\lambda|^2 \sum_{\alpha \in A} \|y_\alpha\|_\alpha^2 + 2 \sum_{\alpha \in A} \|x_\alpha\|_\alpha \|y\|_\alpha \\
&= \|s_A(x)\|_2^2 + |\lambda|^2 \|s_A(y)\|_2^2 + 2\|s_A(x)s_A(y)\|_1 \\
&< \infty
\end{aligned}$$

Thus H is a vector space.

2. Let $x, y, z \in H$ and $\lambda \in \mathbb{C}$. Then

(a)

$$\begin{aligned}
\langle x, y + \lambda z \rangle &= \sum_{\alpha \in A} \langle x_\alpha, (y + \lambda z)_\alpha \rangle_\alpha \\
&= \sum_{\alpha \in A} \langle x_\alpha, y_\alpha + \lambda z_\alpha \rangle_\alpha \\
&= \sum_{\alpha \in A} \left[\langle x_\alpha, y_\alpha \rangle_\alpha + \lambda \langle x_\alpha, z_\alpha \rangle_\alpha \right] \\
&= \sum_{\alpha \in A} \langle x_\alpha, y_\alpha \rangle_\alpha + \lambda \sum_{\alpha \in A} \langle x_\alpha, z_\alpha \rangle_\alpha \\
&= \langle x, y \rangle + \lambda \langle x, z \rangle
\end{aligned}$$

(b)

$$\begin{aligned}
\langle x, y \rangle &= \sum_{\alpha \in A} \langle x_\alpha, y_\alpha \rangle_\alpha \\
&= \sum_{\alpha \in A} \langle y_\alpha, x_\alpha \rangle_\alpha^* \\
&= \left(\sum_{\alpha \in A} \langle y_\alpha, x_\alpha \rangle_\alpha \right)^* \\
&= \langle y, x \rangle
\end{aligned}$$

(c)

$$\begin{aligned}\langle x, x\rangle &= \sum_{\alpha \in A} \langle x_\alpha, x_\alpha\rangle_\alpha \\ &\geq 0\end{aligned}$$

(d) Suppose that $\langle x, x\rangle = 0$. Then

$$\begin{aligned}0 &= \langle x, x\rangle \\ &= \sum_{\alpha \in A} \langle x_\alpha, x_\alpha\rangle_\alpha \\ &= \sum_{\alpha \in A} \|x_\alpha\|_\alpha^2\end{aligned}$$

Thus for each $\alpha \in A$, $\|x_\alpha\|_\alpha = 0$. Therefore for each $\alpha \in A$, $x_\alpha = 0$. Hence $x = 0$.

So $\langle \cdot, \cdot\rangle : H \times H \to \mathbb{C}$ is an inner product on H.

3. Let $(x_j)_{j\in\mathbb{N}} \subset H$. Suppose that $(x_j)_{j\in\mathbb{N}}$ Cauchy. Let $\alpha \in A$ and $\epsilon > 0$. Since $(x_j)_{j\in\mathbb{N}}$ Cauchy, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $\|x_m - x_n\| < \epsilon$. Let $m, n \in \mathbb{N}$. Suppose that $m, n \geq N$. Then

$$\begin{aligned}\|x_{m,\alpha} - x_{n,\alpha}\|_\alpha^2 &\leq \sum_{\beta \in A} \|x_{m,\beta} - x_{n,\beta}\|_\beta^2 \\ &= \|x_m - x_n\|^2 \\ &< \epsilon^2\end{aligned}$$

Thus $\|x_{m,\alpha} - x_{n,\alpha}\|_\alpha \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $\|x_{m,\alpha} - x_{n,\alpha}\|_\alpha$. Hence $(x_{j,\alpha})_{j\in\mathbb{N}}$ is Cauchy. Since H_α is complete, there exists $x_\alpha \in H_\alpha$ such that $x_{j,\alpha} \to x_\alpha$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, there exists $x_\alpha \in H_\alpha$ such that $x_{j,\alpha} \to x_\alpha$. Define $x \in \prod_{\alpha \in A} H_\alpha$ by $x = (x_\alpha)_{\alpha \in A}$.

Let $\epsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $\|x_m - x_n\| \leq \epsilon$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Fatou's lemma imply that

$$\begin{aligned}\|s_A(x - x_n)\|_2^2 &= \sum_{\alpha \in A} \|x_\alpha - x_{n,\alpha}\|_\alpha^2 \\ &= \sum_{\alpha \in A} \lim_{m\to\infty} \|x_{m,\alpha} - x_{n,\alpha}\|_\alpha^2 \\ &\leq \liminf_{m\to\infty} \sum_{\alpha \in A} \|x_{m,\alpha} - x_{n,\alpha}\|_\alpha^2 \\ &= \liminf_{m\to\infty} \|x_m - x_n\|^2 \\ &\leq \epsilon^2\end{aligned}$$

Thus $\|s_A(x - x_n)\|_2 \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N_\epsilon$ implies that $\|s_A(x - x_n)\|_2 \leq \epsilon$.

In particular, setting $\epsilon = 1$ and $n = N_1$, A previous exercise implies that

$$\begin{aligned}\|s_A(x)\|_2 &\leq \|s_A(x) - s_A(x_{N_1})\|_2 + \|s_A(x_{N_1})\|_2 \\ &= \|s_A(x) - s_A(x_{N_1})\|_2 + \|x_{N_1}\| \\ &\leq \|s_A(x - x_\epsilon)\|_2 + \|x_{N_1}\| \\ &\leq 1 + \|x_{N_1}\| \\ &< \infty\end{aligned}$$

so that $s_A(x) \in l^2(A)$ and therefore $x \in \bigoplus\limits_{\alpha \in A} H_\alpha$.

Since $x \in H$, we have that for each $n \in \mathbb{N}$, $\|x - x_n\| = \|s_A(x - x_n)\|_2$. Thus from before, for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N_\epsilon$ implies that $\|x - x_n\| \leq \epsilon$. Hence $x_n \to x$. Since $(x_n)_{n\in\mathbb{N}} \subset H$ with $(x_n)_{n\in\mathbb{N}}$ Cauchy is arbitrary, we have that for each $(x_n)_{n\in\mathbb{N}} \subset H$, $(x_n)_{n\in\mathbb{N}}$ is Cauchy implies that there exists $x \in H$ such that $x_n \to x$. Hence H is complete. Thus $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

$\square$

Note 7.5.0.6. This construction might work for Banach spaces with norms satisfying $\|x_\alpha + y_\alpha\|_\alpha^2 \leq \|x_\alpha\|_\alpha^2 + \|y_\alpha\|_\alpha^2 + 2\|x_\alpha\|_\alpha \|y_\alpha\|_\alpha$.

Exercise 7.5.0.7. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$, A an index set and for each $\alpha \in A$, E_α a closed subspace of H. Then $H \cong \bigoplus_{\alpha \in A} E_\alpha$ iff

1. for each $\alpha, \beta \in A$, $\alpha \neq \beta$ implies that $E_\alpha \cap E_\beta = \{0\}$
2. for each $x \in H$, there exist $(x_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} E_\alpha$ such that $x = \sum\limits_{\alpha \in A}$

Proof.

- ($\implies$):
  Suppose that $H \cong \bigoplus_{\alpha \in A} E_\alpha$. Let $x \in H$.
- ($\impliedby$):

$\square$

7.6 Tensor Products

Note 7.6.0.1. This section assumes familiarity with the algebraic tensor product of two vector spaces. See section ??? of [1] for details.

Definition 7.6.0.2. Let X, Y and Z be Banach spaces and $\phi \in L^2(X, Y; Z)$. Then (Z, ϕ) is said to be a **tensor product** of X with Y if

1. $\operatorname{span}\phi(X \times Y)$ is dense in Z
2. for each Banach space W and $\psi \in L^2(X, Y; W)$, there exists a unique $\psi' \in L(Z, W)$ such that $\psi' \circ \phi = \psi$, i.e. such that the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\phi"] \arrow[dr, "\psi"'] & Z \arrow[d, "\psi'"] \\ & W \end{tikzcd}$$

If (Z, ϕ) is a tensor product of X with Y. We often write $Z = X \otimes Y$ and for each $x \in X$, $y \in Y$, we often write $\phi(x, y) = x \otimes y$.

Exercise 7.6.0.3. Let X and Y be Banach spaces, $U \subset X$ and $V \subset Y$. Set $W = \{u \otimes v : u \in U \text{ and } v \in V\} \subset X \otimes Y$. If U and V are linearly independent, then W is linearly independent.
Hint: For $\phi \in X^*$, $\psi \in Y^*$, define $T \in L^2(X, Y; \mathbb{C})$ by $T(x, y) = \phi(x)\psi(y)$.

Proof. Let $w = \sum\limits_{u \in U} \sum\limits_{v \in V} \lambda_{u,v} u \otimes v$. Suppose that $w = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Define $T \in L^2(X, Y; \mathbb{C})$ by $T(x, y) = \phi(x)\psi(y)$. By definition of the tensor product, there exists a unique $T' \in L(X \otimes Y, \mathbb{C})$ such that for each $x \in X$ and $y \in Y$, $T'(x \otimes y) = T(x, y)$. Then

$$\begin{aligned} 0 &= T'(w) \\ &= T'\Big(\sum_{u \in U} \sum_{v \in V} \lambda_{u,v} u \otimes v\Big) \\ &= \sum_{u \in U} \sum_{v \in V} \lambda_{u,v} T'(u \otimes v) \\ &= \sum_{u \in U} \sum_{v \in V} \lambda_{u,v} T(u, v) \\ &= \sum_{u \in U} \sum_{v \in V} \lambda_{u,v} \phi(u)\psi(v) \\ &= \phi\bigg(\sum_{u \in U} \sum_{v \in V} \lambda_{u,v} \psi(v) u\bigg) \end{aligned}$$

Since $\phi \in X^*$ is arbitary, a previous exercise in the section on linear functionals implies that

$$\begin{aligned} 0 &= \sum_{u \in U} \sum_{v \in V} \lambda_{u,v} \psi(v) u \\ &= \sum_{u \in U} \bigg(\sum_{v \in V} \lambda_{u,v} \psi(v)\bigg) u \end{aligned}$$

Linear independence of U implies that for each $u \in U$,

$$\begin{aligned} 0 &= \sum_{v \in V} \lambda_{u,v} \psi(v) \\ &= \psi\bigg(\sum_{v \in V} \lambda_{u,v} v\bigg) \end{aligned}$$

Since $\psi \in Y^*$ is arbitary, for each $u \in U$,

$$\sum_{v \in V} \lambda_{u,v} v = 0$$

Linear independence of V implies that for each $u \in U, v \in V$, $\lambda_{u,v} = 0$. Hence W is linearly independent. $\square$

Exercise 7.6.0.4. Uniqueness:
Let X, Y and Z be Banach spaces and $\phi \in L^2(X, Y; Z)$. Suppose that (Z, ϕ) is a tensor product of X with Y. Then (Z, ϕ) is unique up to isomorphism.

Proof. Let W be a Banach space and $\psi \in L^2(X, Y; W)$. Suppose that (W, ψ) is a tensor product of X with Y. Since (Z, ϕ) is a tensor product of X with Y, there exists a unique $\psi' \in L(Z, W)$ such that $\psi' \circ \phi = \psi$. Since (W, ψ) is a tensor product of X with Y, there exists a unique $\phi' \in L(W, Z)$ such that $\phi' \circ \psi = \phi$. Thus the following diagram commutes:

$$\begin{tikzcd}
 & W \arrow[d, "\phi'"] \\
X \times Y \arrow[ur, "\psi"] \arrow[r, "\phi"] \arrow[dr, "\psi"'] & Z \arrow[d, "\psi'"] \\
 & W
\end{tikzcd}$$

On the other hand, since (W, ψ) is a tensor product of X with Y, there exists a unique $\Psi \in L(W)$ such that $\Psi \circ \psi = \psi$. Thus the following diagram commutes:

$$\begin{tikzcd}
X \times Y \arrow[r, "\psi"] \arrow[dr, "\psi"'] & W \arrow[d, "\Psi"] \\
 & W
\end{tikzcd}$$

Since $I_W \in L(W)$ and $I_W \circ \psi = \psi$, uniqueness of Ψ implies that $\Psi = I_W$. From the first diagram, we see that $\psi' \circ \phi'$ satisfies $(\psi' \circ \phi') \circ \psi = \psi$. Since $\psi' \circ \phi' \in L(W)$, uniqueness of Ψ implies that $\Psi = \psi' \circ \phi'$. Thus $\psi' \circ \phi' = I_W$.
Similarly, we could have initially considered the following diagram:

$$\begin{tikzcd}
 & Z \arrow[d, "\psi'"] \\
X \times Y \arrow[ur, "\phi"] \arrow[r, "\psi"] \arrow[dr, "\phi"'] & W \arrow[d, "\phi'"] \\
 & Z
\end{tikzcd}$$

Playing a similar game, we could use the fact that there exists a unique $\Phi \in L(Z)$ such that $\Phi \circ \phi = \phi$ to obtain the following diagram:

$$\begin{tikzcd}
X \times Y \arrow[r, "\phi"] \arrow[dr, "\phi"'] & Z \arrow[d, "\Phi"] \\
 & Z
\end{tikzcd}$$

As before, uniqueness enables us to conclude that $\phi' \circ \psi' = I_Z$. Thus ψ' and ϕ' are isomorphisms and $Z \cong W$. $\square$

Note 7.6.0.5. The following definitions and exercises will cover the explicit construction of a tensor product of Banach spaces.

Definition 7.6.0.6. Let X and Y be Banach spaces. Define $X \otimes^{\text{alg}} Y = \operatorname{span}\{x \otimes y : x \in X \text{ and } y \in Y\}$ to be the algebraic tensor product of X with Y (see section ??? of [1] for details).

Exercise 7.6.0.7. Let X and Y be Banach spaces and $x \otimes y \in X \otimes^{\text{alg}} Y$. If for each $\phi \in X^*$ and $\psi \in Y^*$, $\phi \otimes \psi(x, y) = 0$, then $x \otimes y = 0$.

Proof. The previous section tells us that for each $\phi \in X^*$ and $\psi \in Y^*$, $\phi \otimes psi(x, y) = 0$, then $x = 0$ or $y = 0$. This implies that $x \otimes y = 0$. ☐

Definition 7.6.0.8. The Projective Norm:
Define $\| \cdot \|_\pi : X \otimes^{\text{alg}} Y \to [0, \infty)$ by

$$\|u\|_\pi = \inf \left\{ \sum_{j=1}^{n} \|x_j\| \|y_j\| : (x_j)_{j=1}^n \subset X, \, (y_j)_{j=1}^n \subset Y \text{ and } u = \sum_{j=1}^{n} x_j \otimes y_j \right\}$$

Exercise 7.6.0.9. Let X and Y be Banach spaces. Then $\| \cdot \|_\pi : X \otimes^{\text{alg}} Y \to [0, \infty)$ is a norm on $X \otimes^{\text{alg}} Y$.

Proof.

- Let $\lambda \in \mathbb{C}$, $u \in X \otimes^{\text{alg}} Y$. If $\lambda = 0$, then $\lambda u = 0u = 0 \otimes 0$ and clearly $\|\lambda u\|_\pi = 0 = |\lambda| \|u\|_\pi$. Suppose that $\lambda \neq 0$. Let $\epsilon > 0$. Then there exist $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$ such that $u = \sum\limits_{j=1}^{n} x_j \otimes y_j$ and $\sum\limits_{j=1}^{n} \|x_j\| \|y_j\| < \|u\|_\pi + \epsilon / |\lambda|$. Then $\lambda u = \sum\limits_{j=1}^{n} (\lambda x_j) \otimes y_j$. Therefore

$$\begin{aligned} \|\lambda u\|_\pi &\leq \sum_{j=1}^{n} \|\lambda x_j\| \|y_j\| \\ &\leq |\lambda| \sum_{j=1}^{n} \|x_j\| \|y_j\| \\ &< |\lambda| \left(\|u\|_\pi + \frac{\epsilon}{|\lambda|} \right) \\ &= |\lambda| \|u\|_\pi + \epsilon \end{aligned}$$

  Since $\epsilon > 0$ is arbitrary, $\|\lambda u\|_\pi \leq |\lambda| \|u\|_\pi$. For the sake of contradiction, suppose that $\|\lambda u\|_\pi < |\lambda| \|u\|_\pi$. Then there exists $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$ such that $\lambda u = \sum\limits_{j=1}^{n} x_j \otimes y_j$ and $\sum\limits_{j=1}^{n} \|x_j\| \|y_j\| < |\lambda| \|u\|_\pi$. Hence $u = \sum\limits_{j=1}^{n} (\lambda^{-1} x_j) \otimes y_j$. This implies that

$$\begin{aligned} \|u\|_\pi &\leq \sum_{j=1}^{n} \|\lambda^{-1} x_j\| \|y_j\| \\ &= |\lambda|^{-1} \sum_{j=1}^{n} \|x_j\| \|y_j\| \\ &< |\lambda|^{-1} |\lambda| \|u\|_\pi \\ &= \|u\|_\pi \end{aligned}$$

  which is a contradiction. Therefore $\|\lambda u\|_\pi \geq |\lambda| \|u\|_\pi$ which implies that $\|\lambda u\|_\pi = |\lambda| \|u\|_\pi$

- Let $u, v \in X \otimes^{\text{alg}} Y$ and $\epsilon > 0$. Then there exist $(x_j)_{j=1}^n$, $(a_k)_{k=1}^m \subset X$ and $(y_j)_{j=1}^n$, $(b_k)_{k=1}^m \subset Y$ such that $u = \sum\limits_{j=1}^{n} x_j \otimes y_j$, $v = \sum\limits_{k=1}^{m} a_k \otimes b_k$, $\sum\limits_{j=1}^{n} \|x_j\| \|y_j\| < \|u\|_\pi + \epsilon/2$ and $\sum\limits_{k=1}^{m} \|a_k\| \|b_k\| < \|u\|_\pi + \epsilon/2$. Then

$u + v = \sum_{j=1}^n x_j \otimes y_j + \sum_{k=1}^m a_k \otimes b_k$ which implies that

$$\begin{aligned} \|u+v\|_\pi &\leq \sum_{j=1}^n \|x_j\|\|y_j\| + \sum_{k=1}^m \|a_k\|\|b_k\| \\ &< \|u\|_\pi + \epsilon/2 + \|v\|_\pi + \epsilon/2 \\ &= \|u\|_\pi + \|v\|_\pi + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|u+v\|_\pi \leq \|u\|_\pi + \|v\|_\pi$.

- Let $u \in X \otimes^{\text{alg}} Y$. Suppose that $\|u\| = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$ and $\epsilon > 0$. Then there exist $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$ such that $u = \sum_{j=1}^n x_j \otimes y_j$ and

$$\sum_{j=1}^n \|x_j\|\|y_j\| < \frac{\epsilon}{\|\phi\|\|\psi\| + 1}$$

Then

$$\begin{aligned} \sum_{j=1}^n |\phi \otimes \psi(x_j, y_j)| &= \sum_{j=1}^n |\phi(x_j)\psi(y_j)| \\ &\leq \sum_{j=1}^n \|\phi\|\|x_j\|\|\psi\|\|y_j\| \\ &= \|\phi\|\|\psi\| \sum_{j=1}^n \|x_j\|\|y_j\| \\ &< \|\phi\|\|\psi\| \frac{\epsilon}{\|\phi\|\|\psi\| + 1} \\ &< \epsilon \end{aligned}$$

Then for each $j \in \{1, \ldots, n\}$, $|\phi \otimes \psi(x_j, y_j)| < \epsilon$. **FINISH!!!** Try using sequences and continuity and a common refinement of representation and averaging

□

Exercise 7.6.0.10. Existence:

Proof. □

7.7 MISC, unitary transformations

Definition 7.7.0.1. Let H be a Hilbert space, $T \in GL(H)$ and $E \subset H$ a closed subspace. Then E is said to be **invariant under** T if $T(E) = E$.

Exercise 7.7.0.2. Let H be a Hilbert space, $U \in U(H)$ and $E \subset H$ a closed subspace. If E is invariant under U, then $U(E^\perp) = U(E)^\perp$.

Proof. Suppose that E is invariant under U. Let $y \in E$ and $x_0 \in E^\perp$. Since E is invariant under U, $U(E) = E$. Hence there exists $x \in E$ such that $Ux = y$. Since $x_0 \in E^\perp$ and $x \in E$, $\langle x_0, x\rangle = 0$. Since $U \in U(H)$,

$$\begin{aligned} \langle U(x_0), y\rangle &= \langle U(x_0), y\rangle \\ &= \langle U(x_0), U(x)\rangle \\ &= \langle x_0, x\rangle \\ &= 0 \end{aligned}$$

Since $y \in E$ is arbitrary, we have that for each $y \in E$, $\langle U(x_0), y \rangle = 0$. Therefore $Ux_0 \in E^\perp$. Since $x_0 \in E^\perp$ is arbitrary, we have that for each $x_0 \in E^\perp$, $Ux_0 \in E^\perp$. Thus $U(E^\perp) \subset E^\perp$. For the sake of contradiction, suppose that $E^\perp \not\subset U(E^\perp)$. Then there exists $y \in E^\perp$ such that $y \notin U(E^\perp)$. Since Since $H = E \oplus E^\perp$ and FINISH!!! show $U(E \oplus E^\perp) = U(E) \oplus U(E^\perp)$. $\square$

Chapter 8

Differentiation

8.1 TODO

- Finish implicit and inverse function theorems

Note 8.1.0.1. Much of the material in this chapter discusses maps $f : X \to Y$ where X and Y are Banach spaces. It is often the case that a discussion requires the base fields of X and Y to agree or to both be real. We note that in these cases, every complex vector space is also a real vector space. In particular, if X is a finite dimensional complex vector space with dimension n, then X is a finite dimensional real vector space of dimension $2n$.

8.2 The Gateaux Derivative

Definition 8.2.0.1. Let X, Y be a Banach spaces, $A \subset X$ open, $f : A \to Y$, $x_0 \in A$ and $x \in X$. Then f is said to be

1. **right-hand-differentiable at** x_0 **in the direction** x if the limit
$$\lim_{t\to 0^+} \frac{f(x_0 + tx) - f(x_0)}{t}$$
exists. If f is right-hand-differentiable at x_0 in the direction x, we define the **right-hand derivative** of f at x_0 in the direction x, denoted by $d^+f(x_0; x)$, to be the above limit.

2. **left-hand-differentiable at** x_0 **in the direction** x if the limit
$$\lim_{t\to 0^-} \frac{f(x_0 + tx) - f(x_0)}{t}$$
exists. If f is right-hand-differentiable at x_0 in the direction x, we define the **left-hand derivative** of f at x_0 in the direction x, denoted by $d^-f(x_0; x)$, to be the above limit.

3. **differentiable at** x_0 **in the direction** x if the limit
$$\lim_{t\to 0} \frac{f(x_0 + tx) - f(x_0)}{t}$$
exists. If f is differentiable at x_0 in the direction x, we define the **derivative** of f at x_0 in the direction x, denoted by $df(x_0; x)$, to be the above limit.

Exercise 8.2.0.2. Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to \mathbb{R}$ and $x_0 \in A$. Then $df(x_0; 0) = 0$.

Proof. Clear. □

Definition 8.2.0.3. The Gateaux Derivative:
Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Then f is said to be

1. **right-hand Gateaux differentiable** at x_0 if for each $x \in X$, $d^+f(x_0; x)$ exits. We define the **right-hand Gateaux derivative** of f at x_0, denoted $d^+f(x_0) : X \to \mathbb{R}$, to be
$$d^+f(x_0)(x) = d^+f(x_0; x)$$

2. **left-hand Gateaux differentiable** at x_0 if for each $x \in X$, $d^-f(x_0; x)$ exits. We define the **left-hand Gateaux derivative** of f at x_0, denoted $d^-f(x_0) : X \to \mathbb{R}$, to be
$$d^-f(x_0)(x) = d^-f(x_0; x)$$

3. **Gateaux differentiable** at x_0 if for each $x \in X$, $df(x_0; x)$ exits. We define the **Gateaux derivative** of f at x_0, denoted $df(x_0) : X \to \mathbb{R}$, to be
$$df(x_0)(x) = df(x_0; x)$$

Definition 8.2.0.4. Let Y be a Banach space, $A \subset \mathbb{R}$ open and $f : A \to Y$. Then f is said to be **Gateaux differentiable** if for each $x \in A$, f is Gateaux differentiable at x. If f is Gateaux differentiable, we define $df : A \to Y^X$ by $x_0 \mapsto df(x_0)$.

Exercise 8.2.0.5. Let X, Y be Banach spaces, $A \subset X$ open, $f, g : A \to Y$, $\lambda \in \mathbb{R}$ and $x_0 \in A$. If f, g are Gateaux differentiable at x_0, then $f + \lambda g$ is Gateaux differentiable at x_0 and $d[f+\lambda g](x_0) = df(x_0) + \lambda dg(x_0)$.

Proof. Similar to the case of the derivative from Calc I. $\square$

Exercise 8.2.0.6. Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Suppose that f is Gateaux differentiable at x_0. Then for each $\lambda \in \mathbb{R}$ and $x \in X$,

$$df(x_0)(\lambda x) = \lambda df(x_0)(x)$$

Proof. Let $\lambda \in \mathbb{R}$ and $x \in X$. Then

$$\begin{aligned} df(x_0)(\lambda x) &= \lim_{t \to 0} \frac{f(x_0 + t\lambda x) - f(x_0)}{t} \\ &= \lim_{t \to 0} \lambda \frac{f(x_0 + t\lambda x) - f(x_0)}{\lambda t} \\ &= \lambda \lim_{t \to 0} \frac{f(x_0 + t\lambda x) - f(x_0)}{\lambda t} \\ &= \lambda \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= \lambda df(x_0)(x) \end{aligned}$$

$\square$

Exercise 8.2.0.7. Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to Y$. If f is constant, then f is Gateaux differentiable and for each $x_0 \in A, x \in X$,

$$df(x_0)(x) = 0$$

Proof. Suppose that f is constant. Then there exists $c \in Y$ such that for each $x \in A$, $f(x) = c$. Let $x_0 \in A, x \in X$. Then

$$\begin{aligned} df(x_0)(x) &= \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= \lim_{t \to 0} \frac{c - c}{t} \\ &= 0 \end{aligned}$$

$\square$

Exercise 8.2.0.8. Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to Y$. If f is linear, then f is Gateaux differentiable and for each $x_0 \in A, x \in X$,

$$df(x_0)(x) = f(x)$$

Proof. Suppose that f is linear. Let $x_0 \in A, x \in X$. Then

$$\begin{aligned} df(x_0)(x) &= \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= \lim_{t \to 0} \frac{f(x_0) + tf(x) - f(x_0)}{t} \\ &= f(x) \end{aligned}$$

$\square$

Exercise 8.2.0.9. There exist Banach spaces X, Y, and $f : X \to Y$ such that f is Gateaux differentiable and f is nowhere continuous.
Hint: use Exercise 8.2.0.8

Proof. Set $X = C^1([0,1])$ and $Y = C([0,1])$. Equip both X and Y with the sup norm. Define $T : X \to Y$ by $Tf = f'$. Then Exercise 6.2.0.3 implies that T is not bounded. Since T is linear, Exercise 8.2.0.8 implies that T is Gateaux differentiable. Since T is not bounded, Exercise 6.2.0.6 implies that T is not continuous at 0. Then Exercise 6.2.0.5 tells us that T is nowhere continuous. □

Exercise 8.2.0.10. Set $A = \{(x,y) \in \mathbb{R}^2 : y = -x^2 \text{ and } x \neq 0\}$. Define $f : \mathbb{R}^2 \setminus A \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{x^4 y}{x^6 + y^3} & \text{otherwise} \end{cases}$$

Then f is Gateaux differentiable at $(0,0)$ and f is not continuous at $(0,0)$.
Hint: Consider the set $B = \{(x, x^2 : x \in \mathbb{R})\} \subset \mathbb{R}^2 \setminus A$.

Proof. □

Exercise 8.2.0.11. Let Y be a Banach space, $A \subset \mathbb{R}$ open, $f : A \to Y$ and $x_0 \in A$. Suppose that f is Gateaux differentiable at x_0. Then $df(x_0) \in L(\mathbb{R}, Y)$.

Proof. Let $x, y, \lambda \in \mathbb{R}$.

1. The previous exercise implies

$$\begin{aligned} df(x_0)(x + \lambda y) &= df(x_0)((x + \lambda y)1) \\ &= (x + \lambda y) df(x_0)(1) \\ &= x df(x_0)(1) + \lambda y df(x_0)(1) \\ &= df(x_0)(x) + \lambda df(x_0)(y) \end{aligned}$$

   So $df(x_0) : \mathbb{R} \to Y$ is linear.

2. Since

$$\begin{aligned} \|df(x_0)(x)\| &= \|x df(x_0)(1)\| \\ &= |x| \|df(x_0)(1)\| \end{aligned}$$

   We have that $df(x_0) : \mathbb{R} \to Y$ is bounded with $\|df(x_0)\| \leq \|df(x_0)(1)\|$.

□

Exercise 8.2.0.12. Let X be a Banach space, $A \subset X$ open, $f : A \to \mathbb{R}$ and $x_0 \in A$. If f is Gateaux differentiable at x_0 and f has a local extremum at x_0, then $df(x_0) = 0$.

Proof. Suppose that f is Gateaux differentiable at x_0 and f has a local minimum point at x_0. Then there exists $\delta > 0$ such that $B(x_0, \delta) \subset A$ and for each $y \in B(x_0, \delta)$, $f(x_0) \leq f(y)$.
For the sake of contradiction, suppose that $df(x_0) \neq 0$. Then there exists $x \in X$ such that $x \neq 0$ and $df(x_0)(x) \neq 0$.
First, suppose that $df(x_0)(x) < 0$. Choose $\epsilon = -df(x_0)(x) > 0$. Then there exists $t_0 > 0$ such that for each $t \in B^*(0, t_0)$, $x_0 + tx \in B(x_0, \delta)$ and

$$\left| \frac{f(x_0 + tx) - f(x_0)}{t} - df(x_0)(x) \right| < \epsilon$$

This implies that for each $t \in B^*(0, t_0)$,

$$\begin{aligned} \frac{f(x_0 + tx) - f(x_0)}{t} &< \epsilon + df(x_0)(x) \\ &= 0 \end{aligned}$$

and hence $f(x_0+tx) < f(x_0)$, which is a contradiction.
Now, suppose that $df(x_0)(x) > 0$. Then

$$\begin{aligned} df(x_0)(-x) &= -df(x_0)(x) \\ &< 0 \end{aligned}$$

Similarly to above, this implies that there exists $t_0 > 0$ such that for each $t \in B^*(0,t_0)$, $x_0 - tx \in B(x_0,\delta)$ and $f(x_0-tx) < f(x_0)$ which is a contradiction. So $df(x_0)(x) = 0$ and $df(x_0) = 0$.
If f has a local maximum at x_0, then $-f$ has a local minimum point at x_0. Then

$$\begin{aligned} df(x_0) &= -d[-f](x_0) \\ &= -0 \\ &= 0 \end{aligned}$$

□

Exercise 8.2.0.13. Let X, Y, Z be a Banach spaces, $A \subset X$ open, $B \subset Y$ open, $f : A \to Y$, $g : B \to Z$ and $x_0 \in A$. Suppose that f is affine. If g is Gateaux differentiable at $f(x_0)$, then $g \circ f$ is Gateaux differentiable at x_0 and

$$d(g \circ f)(x_0)(x) = dg(f(x_0))(df(x_0)(x))$$

Proof. Suppose that g is Gateaux differentiable at $f(x_0)$. Since f is affine, there exists $h : A \to Y$ and $c \in Y$ such that h is linear and $f = h + c$. Then

$$\begin{aligned} df(x_0) &= dh(x_0) \\ &= h \end{aligned}$$

Let $x \in X$. Choose $\delta > 0$ such that for each $t \in B(0,\delta) \subset \mathbb{R}$, $f(x_0) + th(x) \in B$. Then for each $t \in B^*(0,\delta)$,

$$\begin{aligned} g \circ f(x_0+tx) &= g\bigg(f(x_0) + t\frac{f(x_0+tx) - f(x_0)}{t}\bigg) \\ &= g(f(x_0) + th(x)) \end{aligned}$$

This implies that

$$\begin{aligned} d(g \circ f)(x_0) &= \lim_{t \to 0} \frac{g \circ f(x_0+tx) - g(f(x_0))}{t} \\ &= \lim_{t \to 0} \frac{g(f(x_0) + th(x)) - g(x_0)}{t} \\ &= dg(f(x_0))(h(x)) \\ &= dg(f(x_0))(df(x_0)(x)) \end{aligned}$$

□

8.3 The Frechet Derivative

Exercise 8.3.0.1. Let X, Y be a normed vector spaces and $\phi : X \to Y$ linear. If $\phi(h) = o(\|h\|)$ as $h \to 0$, then $\phi = 0$.

Proof. Let $h_0 \in X$. If $h_0 = 0$, then $\phi(h_0) = 0$. Suppose that $h_0 \neq 0$. Define $(h_n)_{n\in\mathbb{N}} \subset X$ by

$$h_n = \frac{h_0}{n}$$

Then $h_n \to 0$. By continuity of ϕ and our initial assumption we have that

$$\begin{aligned}\|h_0\|^{-1}\phi(h_0) &= \phi\bigg(\frac{h_0}{\|h_0\|}\bigg) \\ &= \phi\bigg(\frac{h_n}{\|h_n\|}\bigg) \\ &= \frac{\phi(h_n)}{\|h_n\|} \\ &\to 0\end{aligned}$$

which implies that $\|h_0\|^{-1}\phi(h_0) = 0$. So $\phi(h_0) = 0$ and hence $\phi = 0$. □

Exercise 8.3.0.2. Let X, Y be a normed vector spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Suppose that there exists $\phi : X \to Y$ such that ϕ is linear and

$$f(x_0 + h) = f(x_0) + \phi(h) + o(\|h\|) \qquad \text{as } h \to 0$$

then ϕ is unique.

Proof. Suppose that there exists $\psi : X \to Y$ such that ψ is linear and such that

$$f(x_0 + h) = f(x_0) + \psi(h) + o(\|h\|) \qquad \text{as } h \to 0$$

Then $\phi(h) - \psi(h) = o(h)$. Since $\phi - \psi$ is linear, the previous exercise implies that $\phi = \psi$. □

Note 8.3.0.3. Recall that for Banach spaces X and Y,

$$\text{cur} : L^n(X; Y) \to L(X; L(X; \cdots; L(X; Y)) \cdots)$$

is an isometric isomorphism and we may identify $L(X; L(X; \cdots; L(X; Y)) \cdots)$ as $L^n(X; Y)$.

Definition 8.3.0.4. Frechet Derivative:
Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$.

1. - Then f is said to be **Frechet differentiable at** x_0 if there exists $Df(x_0) \in L(X; Y)$ such that,

   $$f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|) \qquad \text{as } h \to 0$$

   - If f is Frechet differentiable at x_0, we define the **Frechet derivative of f at** x_0 to be $Df(x_0)$.
   - We say that f is **Frechet differentiable** if for each $x \in A$, f is Frechet differentiable at x.
   - If f is Frechet differentiable, we define the **Frechet derivative of** f, denoted $Df : A \to L(X; Y)$, by $x \mapsto D^{(1)}f(x)$.

2. Continuing inductively, we set $D^0 f = f$ and for $n \geq 2$,

   - f is said to be **n-th order Frechet differentiable at** x_0 if f is $(n-1)$-th order Frechet differentiable and $D^{n-1}f$ is Frechet differentiable at x_0.

- If f is n-th order Frechet differentiable at x_0, we define $D^n f(x_0) \in L^n(X;Y)$ by

$$D^n f(x_0) = D[D^{n-1} f](x_0)$$

- We say that f is **n-th order Frechet differentiable** if f is $(n-1)$-th order Frechet differentiable and for each $x \in A$, $D^{n-1} f$ is Frechet differentiable at x.
- If f is n-th order Frechet differentiable, we define the **n-th order Frechet derivative of f**, denoted $D^n f : A \to L^n(X;Y)$, by $x \mapsto D^n f(x)$

3. If f is n-th order differentiable, then f is said to be **continuously n-th order differentiable** if $D^n f$ is continuous. We define

$$C^n(A, Y) = \{f : A \to Y : f \text{ is continuously } n\text{-th order differentiable}\}$$

Exercise 8.3.0.5. Let X, Y be a banach spaces, $A \subset X$ open, $f, g : A \to Y$, $\lambda \in \mathbb{R}$ and $x_0 \in A$. If f and g are Frechet differentiable at x_0, then $f + \lambda g$ is Frechet differentiable at x_0 and $D(f+\lambda g)(x_0) = Df(x_0) + \lambda Dg(x_0)$.

Proof. Suppose that f and g are Frechet differentiable at x_0. Then

$$f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|) \qquad \text{as } h \to 0$$

and

$$g(x_0 + h) = g(x_0) + Dg(x_0)(h) + o(\|h\|) \qquad \text{as } h \to 0$$

This implies that

$$\begin{aligned}(f + \lambda g)(x_0 + h) &= f(x_0 + h) + \lambda g(x_0 + h) \\ &= f(x_0) + Df(x_0)(h) + o(\|h\|) + \lambda g(x_0) + \lambda Dg(x_0)(h) + o(\|h\|) \\ &= (f + \lambda g)(x_0) + [Df(x_0) + \lambda Dg(x_0)](h) + o(\|h\|) \qquad \text{as } h \to 0\end{aligned}$$

Since $Df(x_0) + \lambda Dg(x_0) \in L(X;Y)$, $f + \lambda g$ is Frechet differentiable at x_0 and $D(f + \lambda g)(x_0) = Df(x_0) + \lambda Dg(x_0)$. □

Exercise 8.3.0.6. Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. If f is Frechet differentiable at x_0, then f is continuous at x_0.

Proof. Suppose that f is Frechet differentiable at x_0. Then $f(x) - f(x_0) = Df(x_0)(x - x_0) + o(\|x - x_0\|)$ as $x \to x_0$. Hence $\|f(x) - f(x_0)\| \leq \|Df(x_0)\|\|x - x_0\| + o(\|x - x_0\|)$ as $x \to x_0$. This implies that $f(x) \to f(x_0)$ as $x \to x_0$ and therefore f is continuous at x_0. □

Exercise 8.3.0.7. Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. If f is Frechet differentiable at x_0, then f is Gateaux differentiable at x_0 and $df(x_0) = Df(x_0)$.

Proof. Suppose that f is Frechet differentiable at x_0. Then $f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|)$ as $h \to 0$. Let $x \in X$. Then $f(x_0 + tx) - f(x_0) = tDf(x_0)(x) + o(t)$ as $t \to 0$. This implies that f is differentiable at x_0 in the direction x and

$$\begin{aligned}df(x_0)(x) &= \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= Df(x_0)(x)\end{aligned}$$

Since $x \in X$ is arbitrary, f is Gateaux differentiable at x_0 and $df(x_0) = Df(x_0)$. □

Exercise 8.3.0.8. Let X be a Banach space, $A \subset X$ open, $f : A \to \mathbb{R}$ and $x_0 \in A$. If f is Frechet differentiable at x_0 and f has a local extremum at x_0, then $Df(x_0) = 0$.

Proof. Suppose that f is Frechet differentiable at x_0 and f has a local extremum at x_0. Two previous exercises imply that f is Gateaux differentiable at x_0 and

$$\begin{aligned} Df(x_0) &= df(x_0) \\ &= 0 \end{aligned}$$

□

Definition 8.3.0.9. Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Suppose that f is Frechet differentiable at x_0. Define $R_f(x_0) : A - x_0 \to Y$ by

$$R_f(x_0)(h) = f(x_0 + h) - f(x_0) - Df(x_0)(h)$$

Exercise 8.3.0.10. Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. If f is Frechet differentiable at x_0, then

$$f(x_0 + h) - f(x_0) = O(\|h\|) \text{ as } h \to 0$$

Proof. Suppose that f is Frechet differentiable at x_0. Then $R_f(h) = o(\|h\|)$ as $h \to 0$. Hence there exists $\delta > 0$ such that $B(0, \delta) \subset A - x_0$ and for each $h \in B(0, \delta)$, $\|R_f(h)\| \leq \|h\|$. Hence for each $h \in B(0, \delta)$

$$\begin{aligned} \|f(x_0 + h) - f(x_0)\| &= \|Df(x_0)(h) + R_f(x_0)(h)\| \\ &\leq \|Df(x_0)(h)\| + \|R_f(x_0)(h)\| \\ &\leq \|Df(x_0)\|\|(h)\| + \|h\| \\ &= (\|Df(x_0)\| + 1)\|h\| \end{aligned}$$

□

Exercise 8.3.0.11. Chain Rule:
Let X, Y, Z be a Banach spaces, $A \subset X$ open, $B \subset Y$ open, $f : A \to Y$, $g : B \to Z$ and $x_0 \in A$. Suppose that $f(x_0) \in B$. If f is Frechet differentiable at x_0 and g is Frechet differentiable at $f(x_0)$, then $g \circ f$ is Frechet differentiable at x_0 and

$$D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$$

Proof. Suppose that f is Frechet differentiable at x_0 and g is Frechet differentiable at $f(x_0)$.

- The previous exercise implies that there exists $\delta^* > 0$ and $K > 0$ such that for each $h \in B(0, \delta^*)$, $\|f(x_0 + h) - f(x_0)\| \leq K\|h\|$. Let $\epsilon > 0$. Since $R_g(f(x_0))(h') = o(\|h'\|)$ as $h' \to 0$, there exists $\delta' > 0$ such that for each $h' \in B(0, \delta')$, $\|R_g(f(x_0))(h')\| \leq \frac{\epsilon}{K}\|h'\|$.
  Choose $\delta = \min(\delta'/K, \delta^*)$. Let $h \in B(0, \delta)$. Then

  $$\begin{aligned} \|f(x_0 + h) - f(x_0)\| &\leq K\|h\| \\ &< \delta' \end{aligned}$$

  This implies that

  $$\begin{aligned} \|R_g(f(x_0))(f(x_0 + h) - f(x_0))\| &\leq \frac{\epsilon}{K}\|f(x_0 + h) - f(x_0)\| \\ &\leq \frac{\epsilon}{K}K\|h\| \\ &\leq \epsilon\|h\| \end{aligned}$$

  So $R_g(f(x_0))(f(x_0 + h) - f(x_0)) = o(\|h\|)$ as $h \to 0$.

- Since $\|Dg(f(x_0))(R_f(x_0)(h))\| \leq \|Dg(f(x_0))\|\|R_f(x_0)(h)\|$ and $R_f(x_0)(h) = o(h)$ as $h \to 0$, we have that $Dg(f(x_0))(R_f(x_0)(h)) = o(h)$ as $h \to 0$.

- Combining the previous two observations, we have that $Dg(f(x_0))(R_f(x_0)(h))+R_g(f(x_0))(f(x_0+h)-f(x_0)) = o(\|h\|)$ as $h \to 0$.

- All together, we obtain

$$\begin{aligned}
g \circ f(x_0+h) &= g(f(x_0)) + f(x_0+h) - f(x_0)) \\
&= g(f(x_0)) + Dg(f(x_0))(f(x_0+h) - f(x_0)) + R_g(f(x_0))(f(x_0+h) - f(x_0)) \\
&= g(f(x_0)) + Dg(f(x_0))(Df(x_0)(h) + R_f(x_0)(h)) \\
&\qquad + R_g(f(x_0))(f(x_0+h) - f(x_0)) \\
&= g(f(x_0)) + Dg(f(x_0))(Df(x_0)(h)) + Dg(f(x_0))(R_f(x_0)(h)) \\
&\qquad + R_g(f(x_0))(f(x_0+h) - f(x_0)) \\
&= g \circ f(x_0) + Dg(f(x_0)) \circ Df(x_0)(h) + o(\|h\|) \text{ as } h \to 0
\end{aligned}$$

So $g \circ f$ is Frechet differentiable at x_0 and $D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$.

□

Exercise 8.3.0.12. Let Y be a Banach space, $A \subset \mathbb{R}$ open and $f : A \to Y$. Then f is Gateaux differentiable iff f is Frechet differentiable.

Proof. Suppose that f is Gateaux differentiable. Let $x_0 \in A$. A previous exercise implies that $df(x_0) \in L(\mathbb{R}, Y)$. By defintion,

$$\lim_{h \to 0} \left\| \frac{f(x_0+h) - f(x_0)}{h} - df(x_0)(1) \right\| = 0$$

This is equivalent to saying that

$$f(x_0+h) = f(x_0) + df(x_0)(h) + o(|h|) \qquad \text{as } h \to 0$$

So f is Frechet differentiable at x_0 and $Df(x_0) = df(x_0)$. □

8.4 The Calc I Derivative

Definition 8.4.0.1. Calc I Derivative:
Let Y be a Banach space, $A \subset \mathbb{R}$ or $\mathbb{C}$ open, $f : A \to Y$ and $x_0 \in A$.

1. - If f is Frechet differentiable at x_0, we define the **calc I derivative of f at x_0**, denoted

     $$f'(x_0) \text{ or } \frac{df}{dt}(x_0)$$

     by

     $$\begin{aligned} f'(x_0) &= \lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t} \\ &= df(x_0)(1) \\ &= Df(x_0)(1) \end{aligned}$$

   - If f is Frechet differentiable, we define $f' : A \to Y$ by $x \mapsto f'(x)$.

2. Continuing inductively, we set $f^{(0)} = f$ and for $n \geq 1$,
   - if $f^{(n-1)}$ is Frechet differentiable at x_0, we define the **(n)-th order calc I derivative of f at** x_0, denoted $f^{(n)}(x_0)$, by
     $$f^{(n)}(x_0) = [f^{(n-1)}]'(x_0)$$
   - if $f^{(n-1)}$ is Frechet differentiable, we define $f^{(n)} : A \to Y$ by
     $$f^{(n)} = [f^{(n-1)}]'$$

Exercise 8.4.0.2. Let Y be a Banach space, $A \subset \mathbb{R}$ open and $f : A \to Y$. If f is n-th order Frechet differentiable, then for each $x_0 \in A$ and $k \in \{1, \cdots, n\}$,

$$f^{(k)}(x_0) = D^k f(x_0)(1^{\oplus k})$$

Proof. Let $x_0 \in A$. We proceed by induction. The base case is true by definition. Let $k \in \{1, \cdots, n\}$. Suppose the claim is true for $k - 1$. Then

$$f^{(k-1)}(x_0) = D^{k-1} f(x_0)(1^{\oplus(k-1)})$$

Since f is n-th order Frechet differentiable,

$$D^{k-1} f(x_0 + h) = D^{k-1} f(x_0) + D^k f(x_0)(h) + o(\|h\|) \qquad \text{as } h \to 0$$

This implies that

$$\begin{aligned} f^{(k-1)}(x_0 + h) &= D^{k-1} f(x_0 + h)(1^{\oplus(k-1)}) \\ &= D^{k-1} f(x_0)(1^{\oplus(k-1)}) + D^k f(x_0)(h)(1^{\oplus(k-1)}) + o(\|h\|) \qquad \text{as } h \to 0 \end{aligned}$$

Therefore for each $h \in \mathbb{R}$,

$$Df^{(k-1)}(x_0)(h) = D^k f(x_0)(h)(1^{\oplus(k-1)})$$

and by definition,

$$\begin{aligned} f^{(k)}(x_0) &= [f^{(k-1)}]'(x_0) \\ &= Df^{(k-1)}(x_0)(1) \\ &= D^k f(x_0)(1^{\oplus k}) \end{aligned}$$

□

Exercise 8.4.0.3. Let X, Y be Banach spaces, $A \subset X$ open, $f \in C^n(A, Y), x_0 \in A$, and $h \in X$. Suppose that $\{x_0 + th : t \in [0,1]\} \subset A$. Define and $g : (0,1) \to Y$ by

$$g(t) = f(x_0 + th)$$

Then for each $k \in \{1 \ldots, n\}$ and $t \in (0,1)$,

$$g^{(k)}(t) = D^k f(x_0 + th)(h^{\oplus k})$$

Proof. We proceed by induction. It is straightforward to show that the claim is true for $k = 1$. Let $k \in \{1 \ldots, n\}$. Suppose that $g^{(k-1)}(t) = D^{k-1} f(x_0 + th)(h^{\oplus(k-1)})$. Since $f \in C^k(A, Y)$,

$$D^{k-1} f(x_0 + s_0 h + th) = D^{k-1} f(x_0 + s_0 h) + D^k f(x_0 + s_0 h)(th) + o(\|t\|) \quad \text{as } t \to 0$$

The previous exercise implies that

$$\begin{aligned} g^{(k-1)}(s_0 + t) &= D^{k-1} g(s_0 + t)(1^{\oplus(k-1)}) \\ &= D^{k-1} f(x_0 + s_0 h + th)(h^{\oplus(k-1)}) \\ &= D^{k-1} f(x_0 + s_0 h)(h^{\oplus(k-1)}) + D^k f(x_0 + s_0 h)(th)(h^{\oplus(k-1)}) + o(\|t\|) \quad \text{as } t \to 0 \end{aligned}$$

Hence

$$Dg^{(k-1)}(s_0)(t) = D^k f(x_0 + s_0 h)(th)(h^{\oplus(k-1)})$$

and

$$\begin{aligned} g^{(k)}(t) &= Dg^{(k-1)}(t)(1) \\ &= D^k f(x_0 + th)(h^{\oplus k}) \end{aligned}$$

$\square$

8.5 Mean Value Theorem

Exercise 8.5.0.1. Let X be a Banach space, $A \subset X$ open and convex, and $f : A \to \mathbb{R}$. If f is continuous and Gateaux differentiable, then for each $x, y \in A$, there exists $t^* \in (0,1)$ such that $f(x) - f(y) = df(t^*x + (1-t^*)y)(x-y)$.

Proof. Suppose that f is continuous and Gateaux differentiable. Let $x, y \in A$. Define $h : [0,1] \to X$ by $h(t) = tx + (1-t)y$. Set $g = f \circ h : [0,1] \to \mathbb{R}$. Then g is continuous on $[0,1]$ and Exercise 8.2.0.13 implies that g is Gateaux differentiable on $(0,1)$. Then Exercise 8.3.0.12 Exercise 8.2.0.13 and the mean value theorem implies that there exists $t^* \in (0,1)$ such that

$$\begin{aligned}
f(x) - f(y) &= g(1) - g(0) \\
&= g'(t^*) \\
&= dg(t^*)(1) \\
&= df(h(t^*))(dh(t^*)(1)) \\
&= df(h(t^*))(h'(t^*)) \\
&= df(t^*x + (1-t^*)y)(x-y)
\end{aligned}$$

□

Exercise 8.5.0.2. Let X be a Banach space, $A \subset X$ open and convex, and $f : A \to \mathbb{R}$. If f is Frechet differentiable, then for each $x, y \in A$, there exists $t^* \in (0,1)$ such that $f(x) - f(y) = Df(t^*x + (1-t^*)y)(x-y)$.

Proof. Suppose that f is Frechet differentiable. Then f is continuous and Gateaux differentiable. Now apply the previous exercise. □

Exercise 8.5.0.3. Mean Value Theorem:
Let X, Y be a Banach spaces, $A \subset X$ open and convex and $f : A \to Y$. If f is Frechet differentiable, then for each $x, y \in A$, there exists $t^* \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \|Df(t^*x + (1-t^*)y)\| \|x-y\|$$

Hint: For $x, y \in A$ with $f(x) \neq f(y)$, using a Hahn-Banach argument, find $\lambda \in Y^*$ such that $\|\lambda\| = 1$ and $\lambda(f(x) - f(y)) = \|f(x) - f(y)\|$.

Proof. Suppose that f is Frechet differentiable. Let $x, y \in A$. The claim is clearly true when $f(x) = f(y)$. Suppose that $f(x) \neq f(y)$. An exercise in the section on linear functionals implies that there exists $\lambda \in Y^*$ such that $\lambda(f(x) - f(y)) = \|f(x) - f(y)\|$ and $\|\lambda\| = 1$ Define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = \lambda(f(tx + (1-t)y))$$

Then g is continuous and (Frechet) differentiable on $(0,1)$ with

$$Dg(t)(h) = \lambda \circ Df(tx + (1-t)y)((x-y)h)$$

which implies that

$$\begin{aligned}
g'(t) &= Dg(t)(1) \\
&= \lambda \circ Df(tx + (1-t)y)((x-y))
\end{aligned}$$

The mean value theorem implies that there exists $t^* \in (0,1)$ such that

$$\begin{aligned}
\|f(x) - f(y)\| &= \lambda(f(x) - f(y)) \\
&= g(1) - g(0) \\
&= g'(t^*) \\
&= \lambda \circ Df(t^*x + (1-t^*)y)((x-y))
\end{aligned}$$

Taking absolute values, we see that

$$\begin{aligned}\|f(x)-f(y)\| &= |\lambda \circ Df(t^*x+(1-t^*)y)((x-y))| \\ &\leq \|\lambda\|\|Df(t^*x+(1-t^*)y)\|\|x-y\| \\ &\leq \|Df(t^*x+(1-t^*)y)\|\|x-y\|\end{aligned}$$

□

Exercise 8.5.0.4. Let X, Y be Banach spaces, $A \subset X$ open and convex and $f : A \to Y$. Suppose that f is Frechet differentiable. If for each $x \in A$, $Df(x) = 0$, then f is constant.

Proof. Suppose that for each $x \in A$, $Df(x) = 0$. Let $x, y \in A$. Then the mean value theorem implies that there exists $t \in (0,1)$ such that

$$\begin{aligned}\|f(x)-f(y)\| &\leq \|Df(tx+(1-t)y)\|\|x-y\| \\ &= 0\end{aligned}$$

So $f(x) = f(y)$. □

Exercise 8.5.0.5. Let X, Y be Banach spaces, $A \subset X$ open and convex and $f, g : A \to Y$. Suppose that f and g are Frechet differentiable. If $Df = Dg$, then there exists $c \in Y$ such that $f = g + c$.

Proof. Suppose that $Df = Dg$. Then $D(f-g) = 0$ and the previous exercise implies that $f-g$ is constant. □

Exercise 8.5.0.6. Let X, Y be a Banach spaces, $A \subset \mathbb{R}$ open and $f : A \to Y$. Suppose that f is Frechet differentiable. Then $f' \in C(A, Y)$ iff $f \in C^1(A, Y)$.

Proof. Suppose that $f' \in C(A, Y)$. Let $x, y \in A$ and $h \in \mathbb{R}$. Then

$$\begin{aligned}\|(Df(x)-Df(y))(h)\| &= \|Df(x)(h)-Df(y)(h)\| \\ &= \|hf'(x)-hf'(y)\| \\ &= \|h(f'(x)-f'(y))\| \\ &= \|f'(x)-f'(y)\||h|\end{aligned}$$

So $\|Df(x)-Df(y)\| \leq \|f'(x)-f'(y)\|$. Hence continuity of f' implies continuity of Df and $f \in C^1(A, Y)$. Conversely, suppose that $f \in C^1(A, Y)$. Let $x, y \in A$. Then

$$\begin{aligned}\|f'(x)-f'(y)\| &= \|Df(x)(1)-Df(y)(1)\| \\ &= \|(Df(x)-Df(y))(1)\| \\ &\leq \|Df(x)-Df(y)\|\end{aligned}$$

Hence continuity of Df implies continuity of f' and $f' \in C(A, Y)$. □

Exercise 8.5.0.7. Let X, Y be Banach spaces, $A \subset X$ open and convex and $f : A \to Y$. Suppose that f is Frechet differentiable. Then f is Lipschitz iff Df is bounded.

Proof. Suppose that f is Lipschitz. Then there exists $M > 0$ such that for each $x, y \in A$, $\|f(y)-f(x)\| \leq M\|y-x\|$. Let $x \in A$ and $h \in X$. Suppose that $\|h\| = 1$. Since $f(x+th) = f(x) + Df(x)(th) + o(|t|)$ as $t \to 0$, we have that

$$\begin{aligned}|t|\|Df(x)(h)\| &= \|Df(x)(th)\| \\ &\leq \|f(x+th)-f(x)\| + o(|t|) \text{ as } t \to 0 \\ &\leq M\|th\| + o(|t|) \text{ as } t \to 0 \\ &= M|t| + o(|t|) \text{ as } t \to 0\end{aligned}$$

Hence $\|Df(x)(h)\| \leq M + o(1)$ as $t \to 0$ which implies that $\|Df(x)(h)\| \leq M$. Thus

$$\begin{aligned}\|Df(x)\| &= \sup\{\|Df(x)(h)\| : h \in X \text{ and } \|h\| = 1\} \\ &\leq M\end{aligned}$$

Since $x \in A$ is arbitrary, Df is bounded.
Conversely, suppose that Df is bounded. Then there exists $M > 0$ such that for each $x \in A$, $\|Df(x)\| \leq M$. Let $x, y \in A$. The mean value theorem implies that there exists $t^* \in (0,1)$ such that

$$\begin{aligned}\|f(x) - f(y)\| &\leq \|Df(t^*x + (1-t^*)y)\|\|x - y\| \\ &\leq M\|x - y\|\end{aligned}$$

Therefore f is Lipschitz. $\square$

8.6 Taylor's Theorem

Exercise 8.6.0.1. Let Y be a separable Banach space, $f : [a, b] \to Y$ continuous so that f is Bochner-integrable. Define $F : (a, b) \to Y$ by

$$F(x) = \int_{(a,x]} f dm$$

Then $F \in C^1((a, b), Y)$ and for each $x_0 \in (a, b)$ and $F'(x_0) = f(x_0)$.

Proof. Let $x_0 \in (a, b)$ and $h \in (0, b - x_0)$. Then continuity implies that

$$\begin{aligned}
\frac{1}{\|h\|}\left| \int_{(x_0,x_0+h]} f - f(x_0)dm \right| &\leq \frac{1}{\|h\|} \max_{x\in(x_0,x_0+h]} |f(x) - f(x_0)|\|h\| \\
&= \max_{x\in[x_0,x_0+h]} |f(x) - f(x_0)| \\
&\to 0 \text{ as } h \to 0
\end{aligned}$$

So

$$\int_{(x_0,x_0+h]} f - f(x_0)dm = o(\|h\|) \qquad \text{as } h \to 0$$

Therefore

$$\begin{aligned}
F(x_0 + h) &= \int_{(a,x_0+h]} f dm \\
&= \int_{(a,x_0]} f dm + \int_{(x_0,x_0+h]} f dm \\
&= \int_{(a,x_0]} f dm + hf(x_0) + \int_{(x_0,x_0+h]} f - f(x_0)dm \\
&= F(x_0) + hf(x_0) + o(\|h\|) \qquad \text{as } h \to 0
\end{aligned}$$

The case is similar for $h \in (x_0 - b, 0)$. Since the map $h \mapsto f(x_0)h$ is bounded, F is Frechet differentiable at x_0 and $DF(x_0)(h) = f(x_0)h$. This implies that $F'(x_0) = f(x_0)$ and a previous exercise implies tells us that continuity of f implies continuity of DF. So $F \in C^1(A, Y)$. □

Exercise 8.6.0.2. Fundamental Theorem of Calculus: Let Y be a separable Banach space and $f \in C^1((a, b), Y)$. Then for each $x, x_0 \in (a, b)$, $x_0 < x$ implies that

1. f' is Bochner integrable on $(x_0, x]$

2. $$f(x) - f(x_0) = \int_{(x_0,x]} f' dm$$

Proof.

1. Since $f \in C^1((a, b), Y)$, a previous exercise tells us that $f' \in C_Y(a, b)$. Let $x, x_0 \in (a, b)$. Suppose that $x_0 < x$. Choose $c, d \in (a, b)$ such that $a < c < x_0 < x < d < b$. Then f' is continuous on $[c, d]$ and hence Bochner-integrable on $(c, d]$ and $(x_0, x]$.

2. Define $g : (c, d) \to Y$ by
$$g(\xi) = \int_{(c,\xi]} f' dm$$

Then the previous exercise implies that $g \in C^1_Y(c,d)$ and for each $t \in (c,d)$, $g'(t) = f'(t)$. Let $t \in (c,d)$ and $h \in \mathbb{R}$. Then

$$\begin{aligned} Dg(t)(h) &= hg'(t) \\ &= hf'(t) \\ &= Df(t)(h) \end{aligned}$$

So $Dg = Df$ on (c,d). A previous exercise implies that there exists $c \in Y$ such that $f = g + c$ on (c,d). Then

$$\begin{aligned} f(x) - f(x_0) &= g(x) + c - (g(x_0) + c) \\ &= g(x) - g(x_0) \\ &= \int_{(c,x]} f' dm - \int_{(c,x_0]} f' dm \\ &= \int_{(x_0,x]} f' dm \end{aligned}$$

□

Exercise 8.6.0.3. Let Y be a Banach space, $A \subset \mathbb{R}$ open and $g : A \to Y$. If g is n-th order Frechet differentiable, then

$$\frac{d}{dt} \sum_{k=0}^{n-1} \frac{(1-t)^k}{k!} g^{(k)}(t) = \frac{(1-t)^{n-1}}{(n-1)!} g^{(n)}(t)$$

Proof. Taking the derivative yields a telescoping series. □

Exercise 8.6.0.4. Taylor's Theorem I:
Let X be a Banach space, Y a separable Banach space, $A \subset X$ open and convex, $f \in C^{n+1}(A, Y)$, $x_0 \in A$, and $h \in A - x_0$. Then

$$f(x_0 + h) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(x_0)(h^{\oplus k}) + R(x_0)(h)$$

where $R(x_0) : A - x_0 \to Y$ is defined by

$$R(x_0)(h) = \frac{1}{n!} \int_{(0,1)} (1-t)^n D^{n+1} f(x_0 + th)(h^{\oplus (n+1)}) dm(t)$$

and $R(x_0)(h) = o(\|h\|^n)$ as $h \to 0$.
Hint: Define $g : (0,1) \to Y$ by

$$g(t) = f(x_0 + th)$$

Then use the previous exercise and the fundamental theorem of calculus.

Proof. Let $h \in X$. Suppose that $x_0 + h \in A$. Define $g : (0,1) \to Y$ by

$$g(t) = f(x_0 + th)$$

For each $k \in \{1, \ldots, n+1\}$, a previous exercise implies that $g^{(k)}(t) = D^k f(x_0 + th)(h^{\oplus k})$, so $g^{(k)}(0) =$

$D^k f(x_0)(h^{\oplus k})$. The previous exercise and the fundamental theorem of calculus tell us that

$$\begin{aligned}
f(x_0+h) - \sum_{k=0}^{n} \frac{1}{k!} D^k f(x_0)(h^{\oplus k}) &= g(1) - \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(0) \\
&= \int_{(0,1)} \left[\frac{d}{dt} \sum_{k=0}^{n} \frac{(1-t)^k}{k!} g^{(k)}(t) \right] dm(t) \\
&= \int_{(0,1)} \frac{(1-t)^n}{n!} g^{(n+1)}(t) dm(t) \\
&= \frac{1}{n!} \int_{(0,1)} (1-t)^n D^{n+1} f(x_0+th)(h^{\oplus(n+1)}) dm(t) \\
&= R(x_0)(h)
\end{aligned}$$

Note that

$$\frac{1}{n+1} = \frac{1}{n!} \int_{(0,1)} (1-t)^n dm(t)$$

Since $D^{n+1} f$ is continuous at x_0, there exists $\delta_1 > 0$ such that for each $h \in B(0, \delta_1)$, $x_0 + h \in A$ and

$$\|D^{n+1} f(x_0+h) - D^{n+1} f(x_0)\| < 1$$

Let $\epsilon > 0$. Choose $\delta_2 > 0$ such that

$$\frac{1}{n+1} \bigg(\|D^{n+1} f(x_0)\| + 1 \bigg) \delta_2 < \epsilon$$

Set $\delta = \min(\delta_1, \delta_2)$. Let $h \in B(0, \delta)$. Then

$$\begin{aligned}
\|R(x_0)(h)\| &= \bigg\| \int_{(0,1)} \frac{1}{n!} \int_{(0,1)} (1-t)^n D^{n+1} f(x_0+th)(h^{\oplus(n+1)}) dm(t) \bigg\| \\
&\leq \frac{1}{n!} \int_{(0,1)} \|(1-t)^n D^{n+1} f(x_0+th)(h^{\oplus(n+1)})\| dm(t) \\
&\leq \max_{t \in [0,1]} \|D^{n+1} f(x_0+th)\| \|h\|^{n+1} \frac{1}{n!} \int_{(0,1)} (1-t)^n dm(t) \\
&\leq \frac{1}{n+1} \bigg(\|D^{n+1} f(x_0)\| + \max_{t \in [0,1]} \|D^{n+1} f(x_0+th) - D^{n+1} f(x_0)\| \bigg) \|h\|^{n+1} \\
&< \frac{1}{n+1} \bigg(\|D^{n+1} f(x_0)\| + 1 \bigg) \|h\|^{n+1} \\
&< \epsilon \|h\|^n
\end{aligned}$$

So $R(x_0)(h) = o(\|h\|^n)$ as $h \to 0$. □

Exercise 8.6.0.5. Taylor's Theorem II:
Let X be a Banach space, Y a separable Banach space, $A \subset X$ open and convex, $f \in C^n(A, Y)$, $x_0 \in A$, and $h \in A - x_0$. Then there exists $R(x_0) : A - x_0 \to Y$ such that

$$f(x_0+h) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(x_0)(h^{\oplus k}) + R(x_0)(h)$$

and $R(x_0)(h) = o(\|h\|^n)$ as $h \to 0$.
Hint: use Taylor's theorem and expand the derivative inside the integral.

Proof. This is clear by definition for $n = 1$. Suppose that $n \geq 2$. Taylor's theorem implies that

$$f(x_0 + h) = \sum_{k=0}^{n-2} \frac{1}{k!} D^k f(x_0)(h^{\oplus k}) + S(x_0)(h)$$

where $S(x_0) : A - x_0 \to Y$ is defined by

$$S(x_0)(h) = \frac{1}{(n-2)!} \int_{(0,1)} (1-t)^{n-2} D^{n-1} f(x_0 + th)(h^{\oplus(n-1)}) dm(t)$$

and $S(x_0; h) = o(\|h\|^{n-2})$ as $h \to 0$. Define $T^{n-1}(x_0) : A - x_0 \to L^{n-1}(X; Y)$ by

$$T^{n-1}(x_0)(h) = D^{n-1} f(x_0 + h) - D^{n-1} f(x_0) - D^n f(x_0)(h)$$

so that

$$D^{n-1} f(x_0 + h) = D^{n-1} f(x_0) + D^n f(x_0)(h) + T^{n-1}(x_0)(h)$$

and $T^{n-1}(x_0)(h) = o(\|h\|)$ as $h \to 0$.
Define $R(x_0) : A - x_0 \to Y$ by

$$R(x_0)(h) = \frac{1}{(n-2)!} \int_{(0,1)} (1-t)^{n-2} T^{n-1}(x_0)(th)(h^{\oplus(n-1)}) dm(t)$$

Note that

- $$\int_0^1 (1-t)^{n-2} dt = \frac{1}{n-1}$$

- $$\int_0^1 (1-t)^{n-2} t dt = \frac{1}{n(n-1)}$$

Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $h \in B(0, \delta)$, $h \in A - x_0$ and

$$\|T^{n-1}(x_0)(h)\| \leq \epsilon n! \|h\|$$

Let $h \in B(0, \delta)$. Then

$$\begin{aligned}
\|R(x_0)(h)\| &= \left\| \frac{1}{(n-2)!} \int_{(0,1)} (1-t)^{n-2} T^{n-1}(x_0)(th)(h^{\oplus(n-1)}) dm(t) \right\| \\
&\leq \frac{1}{(n-2)!} \int_{(0,1)} (1-t)^{n-2} \|T^{n-1}(x_0)(th)(h^{\oplus(n-1)})\| dm(t) \\
&\leq \frac{1}{(n-2)!} \int_{(0,1)} (1-t)^{n-2} \|T^{n-1}(x_0)(th)\| \|h\|^{n-1} dm(t) \\
&\leq \frac{\epsilon}{(n-2)!} n! \|h\|^n \int_{(0,1)} (1-t)^{n-2} t dm(t) \\
&= \epsilon \|h\|^n
\end{aligned}$$

So that $R(x_0)(h) = o(\|h\|^n)$ as $h \to 0$.

Then

$$\begin{aligned}
S(x_0)(h) &= \frac{1}{(n-2)!}\int_{(0,1)}(1-t)^{n-2}D^{n-1}f(x_0+th)(h^{\oplus(n-1)})dm(t) \\
&= \frac{1}{(n-2)!}\int_{(0,1)}(1-t)^{n-2}D^{n-1}f(x_0)(h^{\oplus(n-1)})dm(t) \\
&\quad + \frac{1}{(n-2)!}\int_{(0,1)}(1-t)^{n-2}tD^{n}f(x_0)(h)(h^{\oplus(n-1)})dm(t) \\
&\quad + \frac{1}{(n-2)!}\int_{(0,1)}(1-t)^{n-2}T^{n-1}(x_0)(th)(h^{\oplus(n-1)})dm(t) \\
&= \frac{1}{(n-1)!}D^{n-1}f(x_0)(h^{\oplus(n-1)}) + \frac{1}{n!}D^nf(x_0)(h^{\oplus n}) + R_f(x_0)(h)
\end{aligned}$$

Hence

$$\begin{aligned}
f(x_0+h) &= \sum_{k=0}^{n-2}\frac{1}{k!}D^kf(x_0)(h^{\oplus k}) + S(x_0)(h) \\
&= \sum_{k=0}^{n}\frac{1}{k!}D^kf(x_0)(h^{\oplus k}) + R(x_0)(h)
\end{aligned}$$

□

Exercise 8.6.0.6. Taylor's Theorem III:
Let X be a Banach space, $A \subset X$ open and convex, $f \in C^n(A)$, $x_0 \in A$, and $h \in A - x_0$. Then there exists $t^* \in (0,1)$ such that

$$f(x_0+h) = \sum_{k=0}^{n-1}\frac{1}{k!}D^kf(x_0)(h^{\oplus k}) + \frac{1}{(n-1)!}(1-t^*)^{n-1}D^nf(x_0+t^*h)(h^{\oplus n})$$

Hint: use Taylor's theorem and the mean value theorem.

Proof. Taylors Theorem implies that

$$f(x_0+h) = \sum_{k=0}^{n-1}\frac{1}{k!}D^kf(x_0)(h^{\oplus k}) + R(x_0)(h)$$

where

$$R(x_0)(h) = \frac{1}{(n-1)!}\int_{(0,1)}(1-t)^{n-1}D^nf(x_0+th)(h^{\oplus n})dm(t)$$

Define $F \in C^1([0,1])$ by

$$F(t) = \int_{(0,t]}\frac{1}{(n-1)!}(1-s)^{n-1}D^nf(x_0+sh)(h^{\oplus n})dm(s)$$

Then the fundamental theorem of calculus implies that

$$F'(t) = \frac{1}{(n-1)!}(1-t)^{n-1}D^nf(x_0+th)(h^{\oplus n})$$

The mean value theorem implies that there exists $t^* \in (0,1)$ such that

$$\begin{aligned}
R(x_0)(h) &= F(1) - F(0) \\
&= F'(t^*) \\
&= \frac{1}{(n-1)!}(1-t^*)^{n-1}D^nf(x_0+t^*h)(h^{\oplus n})
\end{aligned}$$

□

Exercise 8.6.0.7. Let X be a Banach space, $A \subset X$ open and convex and $f \in C^2(A)$, $x_0 \in A$. If f has a local minimum at x_0, then $D^2 f(x_0)$ is positive semidefinite.

Proof. Suppose that f has a local minimum at x_0, then $Df(x_0) = 0$. Let $x \in X$. Then

$$\begin{aligned} 0 &\leq f(x+h) - f(x_0) \\ &= \frac{1}{2} D^2 f(x_0)(h,h) + o(\|h\|^2) \quad \text{as } h \to 0 \end{aligned}$$

Let $h \in X$. Then

$$0 \leq \frac{1}{2} t^2 D^2 f(x_0)(h,h) + o(t^2) \quad \text{as } t \to 0$$

This implies that $D^2 f(x_0)(h,h) \geq 0$. So $D^2 f(x_0)$ is positive semidefinite. □

8.7 Implicit and Inverse Function Theorems

Definition 8.7.0.1. Let $(x_0, y_0) \in U$. Then f is said to be **partial Frechet differentiable with respect to X at** (x_0, y_0) if f^{y_0} is Frechet differentiable at x_0.
Suppose that f is partial Frechet differentiable with respect to X at (x_0, y_0). We define the **partial Frechet derivative of f with respect to X at** (x_0, y_0), denoted $D_X f(x_0, y_0) \in L(X, Z)$, by

$$D_X f(x_0, y_0) = Df^{y_0}(x_0)$$

Suppose that for each $y \in Y$, f^y is Frechet differentiable. We define the **partial Frechet derivative of f with respect to** X, denoted $D_X f : X \times Y \to L(X, Z)$, by

$$D_X f(x, y) = Df^y(x)$$

We define partial Frechet differentiability with respect to Y similarly.

Exercise 8.7.0.2. Let X, Y and Z be Banach spaces, $f : X \times Y \to Z$ and $(x_0, y_0) \in X \times Y$. If f is Frechet differentiable at (x_0, y_0), then f is partial Frechet differentiable at (x_0, y_0) with respect to X and Y and for each $h_X \in X$, $h_Y \in Y$,

$$Df(x_0, y_0)(h_X, h_Y) = D_X f(x_0, y_0)(h_X) + D_Y f(x_0, y_0)(h_Y)$$

Proof. Suppose that f is Frechet differentiable at (x_0, y_0). Then

$$f[(x_0, y_0) + (h_X, h_Y)] = f(x_0, y_0) + Df(x_0, y_0)(h_X, h_Y) + o(\|(h_X, h_Y)\|_{X \oplus Y}) \text{ as } (h_X, h_Y) \to (0, 0)$$

Since there exist $C_1, C_2 > 0$ such that for each $h_X \in X$ and $h_Y \in Y$,

$$C_1(\|x\| + \|y\|) \leq \|(h_x, h_y)\|_{X \oplus Y} \leq C_2(\|x\| + \|y\|)$$

we have that

$$f^{y_0}(x_0 + h_X) = f^{y_0}(x_0) + Df(x_0, y_0)(h_X, 0) + o(\|h_X\|) \text{ as } h_X \to 0$$

Therefore $f^{y_0} : X \to Z$ is Frechet differentiable at x_0 and $Df^{y_0}(x_0) = Df(x_0, y_0)(h_X, 0)$. Hence f is partial Frechet differentiable at (x_0, y_0) with respect to X and for each $h_X \in X$, $D_X f(x_0, y_0)(h_x) = Df(x_0, y_0)(h_X, 0)$. Similarly, f is partial Frechet differentiable at (x_0, y_0) with respect to Y and for each $h_Y \in Y$, $D_Y f(x_0, y_0)(h_Y) = Df(x_0, y_0)(0, h_Y)$. Let $h_X \in X$ and $h_Y \in Y$. Then

$$\begin{aligned}
Df(x_0, y_0)(h_X, h_Y) &= Df(x_0, y_0)[(h_X, 0) + (0, h_Y)] \\
&= Df(x_0, y_0)(h_X, 0) + Df(x_0, y_0)(0, h_Y) \\
&= D_X f(x_0, y_0)(h_x) + D_Y f(x_0, y_0)(h_Y)
\end{aligned}$$

□

Exercise 8.7.0.3. Let X, Y and Z be Banach spaces, $U \subset X \times Y$ open, $f : U \to Z$ and $n \in \mathbb{N}$. If f is $C^1(U, Z)$, then $D_X f$,$D_Y f \in C(U, Z)$.

Proof. Suppose that f is $C^1(U, Z)$. Then $Df \in C(U, Z)$. Define $\phi_X : X \to X \times Y$ and $\phi_Y : Y \to X \times Y$ by $\phi_X(x) = (x, 0)$ and $\phi_Y(y) = (0, y)$. Then $\phi_X \in L(X, X \times Y)$ and $\phi_Y \in L(Y, X \times Y)$. The previous exercise implies that for each $(x, y) \in U$, $D_X f(x, y) = Df(x, y) \circ \phi_X$. Let (x, y), $(x_0, y_0) \in U$. Then

$$\begin{aligned}
\|D_X f(x, y) - D_X f(x_0, y_0)\| &= \|Df(x, y) \circ \phi_X - Df(x_0, y_0) \circ \phi_X\| \\
&= \|(Df(x, y) - Df(x_0, y_0)) \circ \phi_X\| \\
&\leq \|Df(x, y) - Df(x_0, y_0)\| \|\phi_X\|
\end{aligned}$$

□

Exercise 8.7.0.4. Let X, Y and Z be Banach spaces, $U \subset X \times Y$ open, $F : U \to Z$, $(x_0, y_0) \in U$. Suppose that F is partial Frechet differentiable with respect to Y on U and F and $D_Y F$ continuous at (x_0, y_0). Then there

Proof. Set $L = D_Y F(x_0, y_0)$. Define $G : U \to Z$ by $G(x, y) = y - L^{-1}F(x, y)$. Then $G(x_0, y_0) = y_0$ and since $F \in C^1(U, Z)$, $G \in C^1(U, Z)$. The previous exercise implies that $D_Y G \in C(U, Z)$. Note that for each $(x, y) \in U$,

$$\begin{aligned} D_Y G(x, y) &= \mathrm{id}_Y - L^{-1} D_Y F(x, y) \\ &= L^{-1}(L - D_Y F(x, y)) \end{aligned}$$

which implies that $D_Y G(x_0, y_0) = 0$. Set $\epsilon = 1/2$. Since U is open and $D_Y G$ is continuous at (x_0, y_0) there exist δ_X, $\delta_Y > 0$ such that for each $x \in B(x_0, \delta_X)$ and $y \in B(y_0, \delta_Y)$, $(x, y) \in U$ and

$$\begin{aligned} \|D_Y G(x, y)\| &= \|D_Y G(x, y) - D_Y G(x_0, y_0)\| \\ &< \epsilon \end{aligned}$$

Set $A = B(x_0, \delta_X)$ and $B = B(y_0, \delta_Y)$. Let $x \in A$ and $y_1, y_2 \in B$. Define $l : [0, 1] \to B$ by $l(t) = ty_1 + (1-t)y_2$. The mean value theorem implies that

$$\begin{aligned} \|G(x, y_1) - G(x, y_2)\| &\leq \sup_{t \in [0,1]} \|D_Y G(x, l(t))\| \|y_1 - y_2\| \\ &\leq \epsilon \|y_1 - y_2\| \\ &= \frac{1}{2} \|y_1 - y_2\| \end{aligned}$$

Hence, for each $x \in X$ and $y \in Y$, $\|G(x, y)\| \leq \frac{1}{2}\|y_1 - y_2\|$ For $x \in A$, define $T_x : B \to B$ by $T_x(y) = G(x, y)$. $\square$

8.8 The Gradient

Definition 8.8.0.1. Let H be a Hilbert space, $f : H \to \mathbb{C}$ and $x_0 \in H$. Suppose that f is Frechet differentiable at x_0. Then $Df(x_0) \in H^*$. We define the **gradient of** f **at** x_0, denoted $\nabla f(x_0) \in H$, via the Riesz representation theorem to be the unique element of H satisfying

$$\langle \nabla f(x_0), y \rangle = Df(x_0)(y) \quad \text{for each } y \in H$$

Chapter 9

Banach Algebras

9.1 Introduction

Definition 9.1.0.1. Let X be a Banach space and $\mu : X \times X \to X$. Then (X, μ) is said to be a **Banach algebra** if

1. (X, μ) is an associative algebra
2. $\mu \in L^2(X)$ and $\|\mu\| \leq 1$

Note 9.1.0.2. By definition in the section on multilinear maps, condition (2) is equivalent to the assumption that for each $x, y \in X$, $\|xy\| \leq \|x\|\|y\|$.

Definition 9.1.0.3. Let X be a Banach algebra and $e \in X$. Then e is said to be an **identity** if for each $x \in X$, $ex = xe = x$.

Definition 9.1.0.4. Let X be a Banach algebra. Then X is said to be **unital** if there exists $e \in X$ such that e is an identity.

Exercise 9.1.0.5. Let X be a unital Banach algebra. Then there exists a unique $e \in X$ such that e is an identity.

Proof.

- **Existence:**
  By definition, there exists $e \in X$ such that e is an identity.
- **Uniqueness:**
  Let $e' \in X$. Suppose that e' is an identity. Then

$$\begin{aligned} e' &= e'e \\ &= e' \end{aligned}$$

□

Exercise 9.1.0.6. Let X be a unital Banach algebra. If $X \neq \{0\}$, then $1 \leq \|e\|$.

Proof. Suppose that $X \neq \{0\}$. Then $e \neq 0$ which implies that $\|e\| > 0$. Since

$$\|e\| = \|ee\| \leq \|e\|\|e\|$$

we have that $1 \leq \|e\|$. □

Exercise 9.1.0.7. Fundamental Example:
Let X be a Banach space. Then $GL(X)$ is a unital Banach algebra.

Proof. Clear. □

Definition 9.1.0.8. Let X be a unital Banach algebra and $x, y \in X$. Then y is said to be an **inverse** of x if $xy = yx = e$.

Definition 9.1.0.9. Let X be a unital Banach algebra and $x \in X$. Then y is said to be **invertible** if there exists $y \in X$ such that y is an inverse of x.

Exercise 9.1.0.10. Let X be a unital Banach algebra and $x \in X$. If x is invertible, then there exists a unique $y \in X$ such that y is an inverse of x.

Proof. Suppose that x is invertible.

- **Existence:**
  By definition, there exists $y \in X$ such that y is an inverse of x.
- **Uniqueness:**
  Let $y' \in X$. Suppose that y' is an inverse of x. Then

$$\begin{aligned} y' &= y'e \\ &= y'(xy) \\ &= (y'x)y \\ &= ey \\ &= y \end{aligned}$$

□

Definition 9.1.0.11. Let X be a unital Banach algebra. We define $G(X) = \{x \in X : x \text{ is invertible}\}$.

Exercise 9.1.0.12. Let X be a unital Banach algebra. Then $G(X)$ is a group.

Proof. Clear. □

Definition 9.1.0.13. Let X be a unital Banach algebra and $x \in G(X)$. We define the **inverse of** x, denoted x^{-1}, to be the unique $y \in X$ such that $yx = xy = e$.

Exercise 9.1.0.14. Let X be a unital Banach algebra, $x \in G(X)$ and $\lambda \in \mathbb{C}^\times$. Then $\lambda x \in G(X)$ is and $(\lambda x)^{-1} = \lambda^{-1}x^{-1}$.

Proof. We have that

$$\begin{aligned} (\lambda^{-1}x^{-1})(\lambda x) &= ((\lambda^{-1}\lambda)x^{-1})x \\ &= (1x^{-1})x \\ &= x^{-1}x \\ &= e \end{aligned}$$

Similarly, $(\lambda x)(\lambda^{-1}x^{-1}) = e$. Hence $\lambda x \in G(X)$ and $(\lambda x)^{-1} = \lambda^{-1}x^{-1}$. □

Exercise 9.1.0.15. Let X be a unital Banach algebra and $x, y \in G(X)$. Then $xy \in G(X)$ is and $(xy)^{-1} = y^{-1}x^{-1}$.

Proof. We have that

$$\begin{aligned} (y^{-1}x^{-1})(xy) &= y^{-1}(x^{-1}(xy)) \\ &= y^{-1}((x^{-1}x)y) \\ &= y^{-1}(ey) \\ &= y^{-1}y \\ &= e \end{aligned}$$

Similarly, $(xy)(y^{-1}x^{-1}) = e$. Hence $xy \in G(X)$ and $(xy)^{-1} = y^{-1}x^{-1}$. □

Exercise 9.1.0.16. Let X be a unital Banach algebra and $x, y \in X$.

1. If $xy \in G(X)$ and $y \in G(X)$, then $x \in G(X)$.
2. If $xy \in G(X)$ and $yx \in G(X)$, then $x \in G(X)$ and $y \in G(X)$.
3. If $xy = yx$ and $x \notin G(X)$, then $xy \notin G(X)$.

Proof.

1. Suppose that $xy \in G(X)$ and $y \in G(X)$. Since $xy \in G(X)$, there exists $z \in G(X)$ such that $z(xy) = (xy)z = e$. Since $z, y \in G(X)$, we have that $yz \in G(X)$ and and $(yz)^{-1} = z^{-1}y^{-1}$. Therefore

$$\begin{aligned} z(xy) = e &\implies xy = z^{-1} \\ &\implies x = z^{-1}y^{-1} \\ &\implies x = (yz)^{-1} \end{aligned}$$

   Hence $x \in G(X)$.

2. Suppose that $xy, yx \in G(X)$. Then there exists $z \in G(X)$ such that $z(xy) = (xy)z = e$. Then $x(yz) = e$ and since $yx \in G(X)$, we have that

$$\begin{aligned} z(xy) = e &\implies (zx)y = e \\ &\implies (zx)yx = x \\ &\implies zx = x(yx)^{-1} \\ &\implies y(zx) = y(x(yx)^{-1}) \\ &\implies (yz)x = (yx)(yx)^{-1} \\ &\implies (yz)x = e \end{aligned}$$

   Since $(yz)x = x(yz) = e$, we have that $x \in G(X)$. Similarly, $y \in G(X)$.

3. Suppose that $xy = yx$ and $x \notin G(X)$. Part (2) implies that $xy \notin G(X)$ or $yx \notin G(X)$. Since $xy = yx$, we have that $xy \notin G(X)$.

$\square$

Exercise 9.1.0.17. Let X be a unital Banach algebra.

1. For each $x \in X$, if $\|x\| < 1$, then $e - x \in G(X)$ and

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n$$

2. For each $x \in X$ and $\lambda \in \mathbb{C}^\times$, if $\|x\| < |\lambda|$, then $\lambda e - x \in G(X)$ and

$$(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} x^n$$

3. For each $x, y \in X$, if $x \in G(X)$ and $\|y\| < \|x^{-1}\|^{-1}$, then $x - y \in G(X)$ and

$$(x - y)^{-1} = x^{-1} \sum_{n=0}^{\infty} (yx^{-1})^n$$

4. For each $x, y \in X$, if $x \in G(X)$ and $\|x - y\| < \|x^{-1}\|^{-1}$, then $y \in G(X)$ and

$$y^{-1} = x^{-1} \sum_{n=0}^{\infty} (e - yx^{-1})^n$$

5. $G(X)$ is open

Proof.

1. Let $x \in X$. Suppose that $\|x\| < 1$. Then

$$\sum_{n=0}^{\infty} \|x^n\| \leq \sum_{n=0}^{\infty} \|x\|^n < \infty$$

Since X is a complete, $\sum_{n=0}^{\infty} x^n$ converges in X.

Define $(s_k)_{k=0}^{\infty} \subset X$ and $s \in X$ by $s_k = \sum_{n=0}^{k} x^n$ and $s = \sum_{n=0}^{\infty} x^n$. Then for each $k \in \mathbb{N}$,

$$\begin{aligned}(e - x)s_k &= s_k - xs_k \\ &= e - x^{k+1}\end{aligned}$$

Since $x^k \to 0$ as $k \to \infty$, we have that $(e - x)s_k \to e$ as $k \to \infty$. Since multiplication on Banach algebras is continuous, we have that $(e - x)s_k \to (e - x)s$ as $k \to \infty$. Uniqueness of limits implies that $(e - x)s = e$. A similar argument implies that $s(e - x) = e$. Thus $e - x \in G(X)$ and $(e - x)^{-1} = \sum_{n=0}^{\infty} x^n$.

2. Let $x \in X$ and $\lambda \in \mathbb{C}^{\times}$. Suppose that $\|x\| < |\lambda|$. Then

$$\begin{aligned}&\|\lambda^{-1}x\| \\ &= |\lambda^{-1}|\|x\| \\ &= |\lambda|^{-1}\|x\| \\ &< |\lambda|^{-1}|\lambda| \\ &= 1\end{aligned}$$

By (1), we have that $e - \lambda^{-1}x \in G(X)$ and

$$\begin{aligned}(e - \lambda^{-1}x)^{-1} &= \sum_{n=0}^{\infty} (\lambda^{-1}x)^n \\ &= \sum_{n=0}^{\infty} \lambda^{-n}x^n\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda e - x &= \lambda(e - \lambda^{-1}x) \\ &\in G(X)\end{aligned}$$

and

$$\begin{aligned}(\lambda e - x)^{-1} &= (\lambda(e - \lambda^{-1}x))^{-1} \\ &= \lambda^{-1}(e - \lambda^{-1}x)^{-1} \\ &= \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n}x^n \\ &= \sum_{n=0}^{\infty} \lambda^{-(n+1)}x^n\end{aligned}$$

3. Let $x, y \in X$. Suppose that $x \in G(X)$ and $\|y\| < \|x^{-1}\|^{-1}$. Then
$$\begin{aligned}\|yx^{-1}\| &\leq \|y\|\|x^{-1}\| \\ &< \|x^{-1}\|^{-1}\|x^{-1}\| \\ &= 1\end{aligned}$$
Hence $e - yx^{-1} \in G(X)$ and
$$(e - yx^{-1}) = \sum_{n=0}^{\infty}(yx^{-1})^n$$
This implies that
$$\begin{aligned}x - y &= (e - yx^{-1})x \\ &\in G(X)\end{aligned}$$
and
$$\begin{aligned}(x - y)^{-1} &= ((e - yx^{-1})x)^{-1} \\ &= x^{-1}(e - yx^{-1})^{-1} \\ &= x^{-1}\sum_{n=0}^{\infty}(yx^{-1})^n\end{aligned}$$

4. Let $x, y \in X$. Suppose that $x \in G(X)$ and $\|x - y\| < \|x^{-1}\|^{-1}$. Then (2) implies that
$$\begin{aligned}y &= x - (x - y) \\ &\in G(X)\end{aligned}$$
and
$$\begin{aligned}y^{-1} &= (x - (x - y))^{-1} \\ &= x^{-1}\sum_{n=0}^{\infty}((x - y)x^{-1})^n \\ &= x^{-1}\sum_{n=0}^{\infty}(e - yx^{-1})^n\end{aligned}$$

5. Let $x \in G(X)$. Choose $\delta = \|x^{-1}\|^{-1}$. By (3), $B(x, \delta) \subset G(X)$. Since $x \in G(X)$ is arbitrary, $G(X)$ is open.

□

Definition 9.1.0.18. Let X be a unital Banach algebra. We define $\iota_\mu : G(X) \to G(X)$ by $\iota_\mu(x) = x^{-1}$.

Exercise 9.1.0.19. Let X be a unital Banach algebra. Then

1. for each $x, y \in X$, if $x \in G(X)$ and $\|y\| \leq \frac{1}{2}\|x^{-1}\|^{-1}$ so that $x - y \in G(X)$, then
$$\|(x - y)^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2\|y\|$$
2. $\iota_\mu : G(X) \to G(X)$ is continuous
3. $G(X)$ is a topological group

Proof.

1. Let $x, y \in X$. Suppose that $x \in G(X)$ and $\|y\| \leq 2^{-1}\|x^{-1}\|^{-1}$. The previous exercise implies that

$$\begin{aligned}
\|(x-y)^{-1} - x^{-1}\| &= \left\| x^{-1} \sum_{n=0}^{\infty} (yx^{-1})^n - x^{-1} \right\| \\
&= \left\| x^{-1} \sum_{n=1}^{\infty} (yx^{-1})^n \right\| \\
&\leq \|x^{-1}\| \sum_{n=1}^{\infty} (\|y\|\|x^{-1}\|)^n \\
&= \|x^{-1}\|^2 \|y\| \sum_{n=0}^{\infty} (\|y\|\|x^{-1}\|)^n \\
&= \|x^{-1}\|^2 \|y\| \sum_{n=0}^{\infty} \frac{1}{2^n} \\
&= 2\|x^{-1}\|^2 \|y\|
\end{aligned}$$

2. Let $(x_n)_{n\in\mathbb{N}} \subset G(X)$ and $x \in G(X)$. Suppose that $x_n \to x$ in $G(X)$. Then $x_n \to x$ in X. Define $(y_n)_{n\in\mathbb{N}} \subset X$ by $y_n = x - x_n$. Then $y_n \to 0$ in X. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that

$$\|y_n\| < \max\left(\frac{\epsilon}{2\|x^{-1}\|^2}, \frac{1}{2}\|x^{-1}\|^{-1} \right)$$

Let $n \in \mathbb{N}$. Suppose that $n \geq N$. By (1), we have that

$$\begin{aligned}
\|\iota_\mu(x_n) - \iota_\mu(x)\| &= \|x_n^{-1} - x^{-1}\| \\
&= \|(x-y_n)^{-1} - x^{-1}\| \\
&\leq 2\|x^{-1}\|^2 \|y_n\| \\
&< \epsilon
\end{aligned}$$

Hence $\iota_\mu(x_n) \to \iota_\mu(x)$ in X. Thus $\iota_\mu(x_n) \to \iota_\mu(x)$ in $G(X)$. So $\iota_\mu : G(X) \to G(X)$ is continuous.

3. Since multiplication $G(X) \times G(X) \to G(X)$ and multiplicative inversion $\iota_\mu : G(X) \to G(X)$ are continuous, $G(X)$ is a topological group.

□

Exercise 9.1.0.20. Let X be a unital Banach algebra. Then $\iota_\mu : G(X) \to G(X)$ is Frechet differentiable and for each $x \in G(X)$, $h \in X$, $D\iota_\mu(x)(h) = x^{-1}hx^{-1}$.

Proof. Let $x \in G(X)$ and $h \in B(x, \|x^{-1}\|^{-1})$. A previous exercise implies that $x + h \in G(X)$ and

$$\begin{aligned}
\iota_\mu(x+h) &= (x+h)^{-1} \\
&= x^{1-} \sum_{n=0}^{\infty} [(-h)x^{-1}]^n \\
&= x^{-1} - x^{-1}hx^{-1} + x^{-1} \sum_{n=2}^{\infty} [(-h)x^{-1}]^n \\
&= x^{-1} - x^{-1}hx^{-1} + o(\|h\|) \text{ as } h \to 0
\end{aligned}$$

Since the map $X \to X$ given by $h \mapsto -x^{-1}hx^{-1}$ is a bounded linear operator and $x^{-1} \sum\limits_{n=2}^{\infty} [(-h)x^{-1}]^n = o(\|h\|)$ as $h \to 0$, we have that ι_μ is differentiable at x and for each $h \in X$, $D\iota_\mu(x)(h) = -x^{-1}hx^{-1}$. Since $x \in G(X)$ is arbitrary, we have that $\iota_\mu(x)$ is differentiable. □

do all the other derivatives like power rule, product rule, etc

Exercise 9.1.0.21.

Proof. content... □

Exercise 9.1.0.22.

Proof. content... □

Exercise 9.1.0.23.

Proof. content... □

9.2 Spectral Theory

Definition 9.2.0.1. Let X be a unital Banach algebra and $x \in X$. We define the

- **resolvent** of x, denoted $\rho(x)$, by
$$\rho(x) = \{\lambda \in \mathbb{C} : \lambda e - x \in G(X)\}$$
- **spectrum** of x, denoted $\sigma(x)$, by
$$\sigma(x) = \rho(x)^c$$

Exercise 9.2.0.2. Let X be a unital Banach algebra and $x \in X$. Then

1. $\rho(x)$ is open
2. $\sigma(x)$ is closed
3. $\sigma(x) \subset \operatorname{cl} B(0, \|x\|)$
4. $\sigma(x)$ is compact

Proof.

1. Let $\lambda \in \rho(x)$. Set $\delta = (\|(\lambda e - x)^{-1}\|\|e\|)^{-1} > 0$. Let $\lambda' \in B(\lambda, \delta)$. Then
$$\begin{aligned} \|(\lambda e - x) - (\lambda' e - x)\| &= |\lambda - \lambda'|\|e\| \\ &< \delta\|e\| \\ &= \|(\lambda e - x)^{-1}\|^{-1} \end{aligned}$$
A previous exercise implies that $\lambda' e - x \in G(X)$. Hence $\lambda' \in \rho(x)$. Since $\lambda' \in B(\lambda, \delta)$ is arbitrary, we have that $B(\lambda, \delta) \subset \rho(x)$. So for each $\lambda \in \rho(x)$, there exists $\delta > 0$ such that $B(\lambda, \delta) \subset \rho(x)$. Hence $\rho(x)$ is open.
2. Since $\sigma(x) = \rho(x)^c$ and $\rho(x)$ is open, we have that $\sigma(x)$ is closed.
3. Let $\lambda \in \sigma(x)$. For the sake of contradiction, suppose that $\|x\| < |\lambda|$. A previous exercise implies that $\lambda e - x \in G(X)$. Hence
$$\begin{aligned} \lambda &\in \rho(x) \\ &= \sigma(x)^c \end{aligned}$$
which is a contradiction. So $|\lambda| \leq \|x\|$ and thus $\lambda \in \operatorname{cl} B(0, \|x\|)$. Since $\lambda \in \sigma(x)$ is arbitrary, $\sigma(x) \subset \operatorname{cl} B(0, \|x\|)$.
4. Since $\sigma(x) \subset \mathbb{C}$ is closed and bounded, $\sigma(x)$ is compact.

$\square$

Exercise 9.2.0.3. Let X be a unital Banach algebra and $x \in X$ and $p \in \mathbb{C}[t]$. Suppose that $\deg p \geq 1$. Then $\sigma(p(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}$.
Hint: Consider the roots of $p(x) - p(\lambda)e$.

Proof. Let $\lambda \in \sigma(x)$. Then λe is a root of $p(t) - p(\lambda)e$. Therefore, there exists $q \in \mathbb{C}[t]$ such that $\deg q = \deg p - 1$ and $p(x) - p(\lambda)e = (x - \lambda e)q(x)$. Since $q(x)(x - \lambda e) = (x - \lambda e)q(x)$ and $(x - \lambda e) \notin G(X)$, a previous exercise implies that $p(x) - p(\lambda)e \notin G(X)$. Thus $p(\lambda) \in \sigma(p(x))$. Since $\lambda \in \sigma(x)$ is arbitrary, we have that

$\{p(\lambda) : \lambda \in \sigma(x)\} \subset \sigma(p(x))$.
Conversely, let $\mu \in \sigma(p(x))$. Set $n = \deg p$. Then there exist $(a_j)_{j=1}^n \subset \mathbb{C}$ and $a \in \mathbb{C}$ such that

$$p(x) - \mu e = a \prod_{j=1}^{n} (x - a_j e)$$

Since $p(x) - \mu e \notin G(X)$, there exists $j \in \{1, \dots, n\}$ such that $(x - a_j e) \notin G(X)$. Thus $a_j \in \sigma(x)$. By construction

$$\begin{aligned}(p(a_j) - \mu)e &= p(a_j)e - \mu e \\ &= p(a_j e) - \mu e \\ &= 0\end{aligned}$$

Thus

$$\begin{aligned}\mu &= p(a_j) \\ &\in \{p(\lambda) : \lambda \in \sigma(x)\}\end{aligned}$$

Since $\mu \in \sigma(p(x))$ is arbitrary, we have that $\sigma(p(x)) \subset \{p(\lambda) : \lambda \in \sigma(x)\}$. Hence $\sigma(p(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}$. $\square$

Definition 9.2.0.4. Let X be a unital Banach algebra and $x \in X$. We define the **resolvent function** of x, denoted $R_x : \rho(x) \to G(X)$, by

$$R_x(\lambda) = (\lambda e - x)^{-1}$$

Exercise 9.2.0.5. Let X be a unital Banach algebra and $x \in X$. Then

1. $R_x : \rho(x) \to G(X)$ is Frechet differentiable and for each $\lambda \in \rho(x)$,

$$R_x' = -R_x^2$$

2. $R_x \in C^\infty(\rho(x))$ and for each $n \in \mathbb{N}$, $R_x^{(n)} = (-1)^n n! R_x^{n+1}$

Proof.

1. Define $S_x : \rho(x) \to G(X)$ by $S_x(\lambda) = \lambda e - x$. Then $R_x = \iota_\mu \circ S_x$. Since S_x and ι_μ are differentiable, $R_x = \iota_\mu \circ S_x$ is differentiable. Previous exercises imply that for each $\lambda \in \rho(x)$, we have that

$$\begin{aligned}R_x'(\lambda) &= DR_x(\lambda)(1) \\ &= [D\iota_\mu(S_x(\lambda)) \circ DS_x(\lambda)](1) \\ &= D\iota_\mu(S_x(\lambda))(DS_x(\lambda)(1)) \\ &= D\iota_\mu(S_x(\lambda))(e) \\ &= -S_x(\lambda)^{-1} e S_x(\lambda)^{-1} \\ &= -S_x(\lambda)^{-2} \\ &= -R_x(\lambda)^2\end{aligned}$$

2. Let $n \in \mathbb{N}$. Suppose that $R_x \in C^{n-1}(\rho(x))$ and $R_x^{(n-1)} = (-1)^{n-1}(n-1)! R_x^n$. Then

$$\begin{aligned}R_x^{(n)} &= (R_x^{(n-1)})' \\ &= [(-1)^{n-1}(n-1)! R_x^n]' \\ &= (-1)^{n-1}(n-1)!(nR_x^{n-1})(-R_x^2) \\ &= (-1)^n n! R_x^{n+1}\end{aligned}$$

By induction, for each $n \in \mathbb{N}$, $R_x \in C^n(\rho(x))$ and $R_x^{(n)} = (-1)^n n! R_x^{n+1}$. A previous exercise in the section of differentiability implies that for each $n \in \mathbb{N}$, $R_x \in C^n(\rho(x))$ iff $R_x^{(n)} \in C^0(\rho(x))$. Hence for each $n \in \mathbb{N}$, $R_x \in C^n(\rho(x))$ and therefore $R_x \in C^\infty(\rho(x))$.

□

Exercise 9.2.0.6. Let X be a unital Banach algebra and $x \in X$. Then $\sigma(x) \neq \varnothing$.
Hint: R_x is bounded and apply Louiville's theorem

Proof. Suppose that $\sigma(x) = \varnothing$. Then $\rho(x) = \mathbb{C}$ and the previous exercise implies that $R_x : \mathbb{C} \to G(X)$ is differentiable. We observe that for each $\lambda \in \mathbb{C}^\times$,

$$\begin{aligned} R_x(\lambda) &= (\lambda e - x)^{-1} \\ &= \lambda^{-1}(e - \lambda^{-1}x)^{-1} \end{aligned}$$

Since $\lambda^{-1} \to 0$ as $\lambda \to \infty$ and $\iota_\mu : G(X) \to G(X)$ is continuous, we have that

$$\begin{aligned} (e - \lambda^{-1}x)^{-1} &\to e^{-1} \\ &= e \end{aligned}$$

as $\lambda \to \infty$. Hence $R_x(\lambda) \to 0$ as $\lambda \to \infty$. Thus $R_x : \mathbb{C} \to G(X)$ is bounded. Louiville's theorem implies that $R_x = 0$. This is a contradiction since $0 \notin G(X)$. □

Definition 9.2.0.7. Let X be a unital Banach algebra and $x \in X$. We define the **spectral radius of** x, denoted by $r(x)$, by

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

Exercise 9.2.0.8. Let X be a unital Banach algebra and $x \in X$. Then

1. $r(x) \leq \liminf \|x^n\|^{1/n}$
2. for each $\phi \in X^*$, $\phi \circ X$ is bounded and

Proof.

1. Let $\lambda \in \sigma(x)$ and $n \in \mathbb{N}$. The previous exercise implies that $\lambda^n \in \sigma(x^n)$. Since $\lambda^n e - x^n \notin G(X)$, we have that

$$\begin{aligned} |\lambda|^n &= |\lambda^n| \\ &\leq \|x^n\| \end{aligned}$$

Therefore $|\lambda| \leq \|x^n\|^{1/n}$. Since $n \in \mathbb{N}$ is arbitrary, $|\lambda| \leq \liminf \|x^n\|^{1/n}$. Since $\lambda \in \sigma(x)$ is arbitrary, we have that

$$\begin{aligned} r(x) &= \sup_{\lambda \in \sigma(x)} |\lambda| \\ &\leq \liminf \|x^n\|^{1/n} \end{aligned}$$

2.

□

Exercise 9.2.0.9. Let X be a unital Banach algebra and $x \in X$. Then

$$r(x) = \lim_{n \to \infty} \|x^n\|^{1/n}$$

Chapter 10

Semigroup Theory

Chapter 11

Banach Modules

11.1 Introduction

Definition 11.1.0.1. Let A be a Banach algebra and X a Banach space and $\mu : A \times X \to X$. Then (X, A, μ) is said to be a **left Banach A-module** if

1. (X, A, μ) is a left A-module
2. $\mu \in L(A, X; X)$ and $\|\mu\| \leq 1$

Note 11.1.0.2. Condition (2) is equivalent to the assumption that for each $a \in A$ and $x \in X$, $\|ax\|_X \leq \|a\|_A \|x\|_X$.

Chapter 12

Convexity

12.1 Introduction

Note 12.1.0.1. In this section, we assume all vector spaces are real.

Definition 12.1.0.2. Let X be a vector space and $A \subset X$. Then A is said to be **convex** if for each $x, y \in A$, and $t \in [0,1]$, $tx + (1-t)y \in A$.

Definition 12.1.0.3. Let X be a vector space and $f : A \to R$. Then f is said to be **convex** if for each $x, y \in A$, $t \in [0,1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Definition 12.1.0.4. Let X be a vector space and $f : A \to R$. Then f is said to be **strictly convex** if for each $x, y \in A$, $t \in (0,1)$, $x \neq y$ implies that

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y)$$

Exercise 12.1.0.5. Let X be a vector space, $f \in X^*$ and $g : X \to \mathbb{R}$ constant. Then f and g are convex.

Proof. Let $x, y \in X$ and $t \in [0,1]$. Put $c = g(0)$. Then

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

and

$$\begin{aligned} g(tx + (1-t)y) &= c \\ &= tc + (1-t)c \\ &= tg(x) + (1-t)g(y) \end{aligned}$$

So f and g are convex. □

Exercise 12.1.0.6. Star-shapedness: Let $f : [0,\infty) \to \mathbb{R}$ be convex. If $f(0) \leq 0$, then for each $x \in [0,\infty)$, $t \in [0,1]$, $f(tx) \leq tf(x)$.

Proof. Suppose that $f(0) \leq 0$. Let $x \in [0,\infty)$ and $t \in [0,1]$. Then

$$\begin{aligned} f(tx) &= f(tx + (1-t)0) \\ &\leq tf(x) + (1-t)f(0) \\ &\leq tf(x) \end{aligned}$$

□

Exercise 12.1.0.7. Superadditivity:
Let $f : [0, \infty) \to \mathbb{R}$ be convex. If $f(0) = 0$, then for each $x, y \in [0, \infty)$,

$$f(x) + f(y) \leq f(x + y)$$

Hint: $f(x) = f\left(\frac{x}{x+y}(x + y)\right)$

Proof. Suppose that $f(0) = 0$. Let $x, y \in [0, \infty)$. If $x+y = 0$, then $x = y = 0$ and $f(x)+f(y) = 0 = f(x+y)$. Suppose that $x + y \neq 0$. Then the previous exercise implies that

$$\begin{aligned} f(x) + f(y) &= f\left(\frac{x}{x+y}(x + y)\right) + f\left(\frac{y}{x+y}(x + y)\right) \\ &\leq \frac{x}{x+y} f(x + y) + \frac{y}{x+y} f(x + y) \\ &= f(x + y) \end{aligned}$$

□

Exercise 12.1.0.8. Let X be a vector space, $A \subset X$ convex, $f, g : A \to \mathbb{R}$ and $\lambda \geq 0$. If f, g are convex, then

1. $f + g$ is convex
2. λf is convex

Proof. Suppose that f and g are convex. Let $x, y \in A$ and $t \in [0, 1]$. Then

$$\begin{aligned} (f + \lambda g)(tx + (1 - t)y) &= f(tx + (1 - t)y) + \lambda g(tx + (1 - t)y) \\ &\leq tf(x) + (1 - t)f(y) + t\lambda g(x) + (1 - t)\lambda g(y) \\ &= t(f(x) + \lambda g(x)) + (1 - t)(f(y) + \lambda g(y)) \\ &= t(f + \lambda g)(x) + (1 - t)(f + \lambda g)(y) \end{aligned}$$

□

Definition 12.1.0.9. Let X be a vector space and $f : X \to \mathbb{R}$. Then f is said to be **affine** if there exists $\phi \in X^*$, $a \in \mathbb{R}$ constant such that $f = \phi + a$.

Exercise 12.1.0.10. Let X be a vector space and $f : X \to \mathbb{R}$. If f is affine, then f is convex.

Proof. Suppose that f is affine. Then there exists $\phi \in X^*$, $a \in R$ constant such that $f = \phi + a$. Then ϕ is convex and $g : X \to \mathbb{R}$ defined by $g(x) = a$ is convex. So $f = \phi + g$ is convex. □

Exercise 12.1.0.11. Let X be a vector space, $A \subset X$ convex, $f : \mathbb{R} \to \mathbb{R}$ and $g : A \to \mathbb{R}$. If f is convex and increasing and g is convex, then $f \circ g$ is convex.

Proof. Let $t \in [0, 1]$ and $x, y \in A$. Then convexity of g implies that

$$g(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y)$$

and we have

$$\begin{aligned} f \circ g(tx + (1 - t)y) &= f(g(tx + (1 - t)y)) \\ &\leq f(tg(x) + (1 - t)g(y)) && (f \text{ increasing}) \\ &\leq tf(g(x)) + (1 - t)f(g(y)) && (f \text{ convex}) \\ &= tf \circ g(x) + (1 - t)f \circ g(y) \end{aligned}$$

So $f \circ g$ is convex. □

Exercise 12.1.0.12. Let X be a vector space, $A \subset X$ convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Then f has a local minimum point at x_0 iff f has a global minimum point at x_0.

Proof. If f has a global minimum point at x_0, then f has a local minimum point at x_0. Conversely, suppose that f has a local minimum point at x_0. Then there exists $\delta > 0$ such that for each $x \in B(x_0, \delta) \cap A$, $f(x_0) \leq f(x)$. For the sake of contradiction, suppose that f does not have a global minimum point at x_0. Then there exits $x' \in A$ such that $f(x') < f(x_0)$. Put $t_0 = \min(\frac{\delta}{\|x'-x_0\|+1}, 1) > 0$. Let $t \in (0, t_0)$, then

$$\begin{aligned}\|(tx' + (1-t)x_0) - x_0\| &= t\|x' - x_0\| \\ &< \frac{\|x' - x_0\|\delta}{\|x' - x_0\| + 1} \\ &< \delta\end{aligned}$$

so that $tx' + (1-t)x_0 \in B(x_0, \delta) \cap A$ and hence $f(x_0) \leq f(tx' + (1-t)x_0)$. Therefore

$$\begin{aligned}f(x_0) &\leq f(tx' + (1-t)x_0) \\ &\leq tf(x') + (1-t)f(x_0) \qquad \text{(convexity of } f\text{)} \\ &< tf(x_0) + (1-t)f(x_0) \\ &= f(x_0)\end{aligned}$$

which is a contradiction. Hence f has a global minimum point at x_0. □

Exercise 12.1.0.13. Let X be a vector space, $A \subset X$ convex, $f : A \to \mathbb{R}$ strictly convex and $x_0 \in X$. If f has a local minimum point at x_0, then f has a unique global minimum point at x_0.

Proof. Suppose that f has a local minimum point at x_0. The previous exercise implies that f has a global minimum point at x_0. For the sake of contradiction suppose that there exists $x_1 \in X$ such that f has a global minimum point at x_1 and $x_0 \neq x_1$. This implies $f(x_0) = f(x_1)$. Set $t = 1/2$. Strict convexity implies that

$$\begin{aligned}f(tx_0 + (1-t)x_1) &< tf(x_0) + (1-t)f(x_1) \\ &= f(x_0)\end{aligned}$$

which is a contradiction since f has a global minimum point at x_0. □

Definition 12.1.0.14. Let X, Y be vector spaces, $A \subset X \oplus Y$. For $y \in Y$, define

$$A^y = \{x \in X : (x, y) \in A\}$$

and $f^y : A^y \to \mathbb{R}$ by

$$f^y(x) = f(x, y)$$

Exercise 12.1.0.15. Let X, Y be vector spaces, $A \subset X \oplus Y$ convex and $f : A \to \mathbb{R}$ convex. Then for each $y \in \pi_2(A)$,

1. A^y is convex
2. f^y is convex

where $\pi_2 : X \times Y \to Y$, the canonical projection of $X \times Y$ onto Y given by $\pi_2(x, y) = y$.

Proof. Let $y \in \pi_2(A)$, $x_1, x_2 \in A^y$ and $t \in [0, 1]$. Then by definition, (x_1, y), $(x_2, y) \in A$.

1. Convexity of A implies that $(tx_1 + (1-t)x_2, y) \in A$. Hence $tx_1 + (1-t)x_2 \in A^y$ and A^y is convex.

2. Convexity of f implies that

$$\begin{aligned} f^y(tx_1+(1-t)x_2) &= f(tx_1+(1-t)x_2,y) \\ &= f(t(x_1,y)+(1-t)(x_2,y)) \\ &\leq tf(x_1,y)+(1-t)f(x_2,y) \\ &= tf^y(x_1)+(t-t)f^y(x_2) \end{aligned}$$

and so f^y is convex.

$\square$

Exercise 12.1.0.16. Let X, Y be vector spaces and $A \subset X, B \subset Y$. If A and B are convex, then $A \times B \subset X \oplus Y$ is convex.

Proof. Suppose that A and B are convex. Let $(x_1,y_1),(x_2,y_2) \in A \times B$ and $t \in [0,1]$. Convexity of A and B implies that $tx_1+(1-t)x_2 \in A$ and $ty_1+(1-t)y_2 \in B$. Therefore

$$\begin{aligned} t(x_1,y_1)+(1-t)(x_2,y_2) &= (tx_1+(1-t)x_2, ty_1+(1-t)y_2) \\ &\in A \times B \end{aligned}$$

$\square$

Exercise 12.1.0.17. Let X, Y be vector spaces and $A \subset X$, $B \subset Y$ convex (implying that $A \times B$ is convex) and $f : A \times B \to \mathbb{R}$ convex. Suppose that for each $y \in B$, $\{f(x,y) : x \in A\}$ is bounded below. Then $\inf\limits_{y \in B} f^y$ is convex

Proof. Put $g = \inf\limits_{y \in B} f^y$. Let $x_1, x_2 \in A$, $y_1, y_2 \in B$ and $t \in [0,1]$. Put $y' = ty_1+(1-t)y_2$. Then convexity of f implies that

$$\begin{aligned} g(tx_1+(1-t)x_2) &\leq f^{y'}(tx_1+(1-t)x_2) \\ &= f(tx_1+(1-t)x_2, ty_1+(1-t)y_2) \\ &= f(t(x_1,y_1)+(1-t)(x_2,y_2)) \\ &\leq tf(x_1,y_1)+(1-t)f(x_2,y_2) \\ &= tf^{y_1}(x_1)+(1-t)f^{y_2}(x_2) \end{aligned}$$

Since $y_1 \in B$ is arbitrary, we have that

$$g(tx_1+(1-t)x_2) \leq tg(x_1)+(1-t)f^{y_2}(x_2)$$

Similarly, since $y_2 \in B$ is arbitrary, we have that

$$g(tx_1+(1-t)x_2) \leq tg(x_1)+(1-t)g(x_2)$$

and f is convex. $\square$

Exercise 12.1.0.18. Let X be a vector space, $A \subset X$ convex and $(f_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}^A$. Suppose that for each $\lambda \in \Lambda$, f_λ is convex. Define

1. $A^* = \{x \in A : \sup\limits_{\lambda \in \Lambda} f_\lambda(x) < \infty\}$
2. $f^* : A^* \to \mathbb{R}$ by $f^*(x) = \sup\limits_{\lambda \in \Lambda} f_\lambda(x)$

Then

1. A^* is convex

2. f^* is convex

Proof. 1. Let $x, y \in A$ and $t \in [0,1]$. By definition, $\sup_{\lambda \in \Lambda} f_\lambda(x), \sup_{\lambda \in \Lambda} f_\lambda(y) < \infty$. Therefore

$$\begin{aligned}\sup_{\lambda \in \Lambda} f_\lambda(tx + (1-t)y) &\leq \sup_{\lambda \in \Lambda}[t f_\lambda(x) + (1-t) f_\lambda(y)] \\ &\leq t \sup_{\lambda \in \Lambda} f_\lambda(x) + (1-t) \sup_{\lambda \in \Lambda} f_\lambda(y) \\ &< \infty\end{aligned}$$

So $tx + (1-t)y \in A$.

2. By definition, the previous part implies that for each $x, y \in A^*$, $f^*(tx+(1-t)y) \leq t f^*(x) + (1-t) f^*(y)$. So $f^* : A^* \to \mathbb{R}$ is convex.

□

Exercise 12.1.0.19. Let X be a normed vector space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then f is locally Lipschitz at x_0.
Hint: Given x_1, x_2 near x_0 Choose a z near x_0 s.t. x_1 is a convex combination of x_2 and z. Then repeat but with x_2 as a convex combination of x_1 and z

Proof. By continuity, f is locally bounded at x_0. So there exist $M, \delta > 0$ such that $B(x_0, \delta) \subset A$ and for each $x \in B(x_0, \delta)$, $|f(x)| \leq M$. Put $\delta' = \frac{\delta}{2}$ and choose $U = B(x_0, \delta')$. Then $U \subset A$ and $U \in \mathcal{N}(x_0)$.
Let $x_1, x_2 \in U$. Suppose that $x_1 \neq x_2$. Define $\alpha = \|x_1 - x_2\| > 0$, $p = \frac{\alpha}{\alpha + \delta'}$, $q = 1 - p$ and $z = p^{-1}(x_1 - q x_2)$. Then $x_1 = pz + q x_2$ and

$$\begin{aligned}\|z - x_1\| &= \|(p^{-1} - 1)x_1 - p^{-1} q x_2\| \\ &= \frac{1-p}{p}\alpha \\ &= \frac{\delta'}{\alpha}\alpha \\ &= \delta'\end{aligned}$$

Therefore

$$\begin{aligned}\|z - x_0\| &\leq \|z - x_1\| + \|x_1 - x_0\| \\ &< \delta' + \delta' \\ &= \delta\end{aligned}$$

So $z \in B(x_0, \delta)$, which implies that

$$\begin{aligned}f(z) - f(x_2) &\leq |f(z) - f(x_2)| \\ &\leq |f(z)| + |f(x_2)| \\ &\leq 2M\end{aligned}$$

Since $x_1 = pz + q x_2$, convexity of f implies that $f(x_1) \leq p f(z) + q f(x_2)$. Hence

$$\begin{aligned}f(x_1) - f(x_2) &\leq p f(z) - p f(x_2) \\ &= p(f(z) - f(x_2)) \\ &\leq p 2M \\ &= \frac{\alpha}{\alpha + \delta'} 2M \\ &\leq \alpha 2M \\ &= 2M\|x_1 - x_2\|\end{aligned}$$

Similarly, choosing $z = p^{-1}(x_2 - q x_1)$, yields $f(x_2) - f(x_1) \leq 2M\|x_1 - x_2\|$ which implies that

$$|f(x_1) - f(x_2)| \leq 2M\|x_1 - x_2\|$$

and f is Lipschitz on U. □

12.2 The Subdifferential

Exercise 12.2.0.1. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $x_0 \in A$ and $x \in X$. Define $T = \{t \in \mathbb{R} : x_0 + tx \in A\}$. Then there exist $a, b \in (0, \infty]$ such that $T = (-a, b)$.

Proof. Continuity of scalar multiplication and addition implies that T is an open neighborhood of 0. Let $t > 0$ and $s \in [0, t]$. Then $\frac{s}{t} \in [0, 1]$ and by convexity of A, $x_0 + tx \in A$ implies that

$$\begin{aligned} x_0 + sx &= \frac{s}{t}(x_0 + tx) + \left(1 - \frac{s}{t}\right)x_0 \\ &\in A \end{aligned}$$

Thus $[0, t] \subset T$. Similarly, $x_0 - tx \in A$ implies that $[-t, 0] \subset T$.
Define $a, b \in (0, \infty]$ by $a = \sup\{t > 0 : x_0 - tx \in A\}$ and $b = \sup\{t > 0 : x_0 + tx \in A\}$. Then $(-a, b) = T$. $\square$

Definition 12.2.0.2. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $x_0 \in A$ and $x \in X$. Define T as in the previous exercise and choose $t_0 > 0$ such that $(-t_0, t_0) \subset T$. For $t \in (0, t_0)$, define the difference quotient $q : (-t_0, t_0) \setminus \{0\} \to \mathbb{R}$ by

$$q(t) = \frac{f(x_0 + tx) - f(x_0)}{t}$$

Exercise 12.2.0.3. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $x_0 \in A$ and $x \in X$. Define t_0 as above. Then

1. $q(t)$ is increasing on $(0, t_0)$
2. $q(-t)$ decreasing on $(0, t_0)$

Hint: As an example, look at the graph of $f(x) = x^2$. For the algebra, start at the desired end inequality and work backwards

Proof.

1. Let $s, t \in (0, t_0)$ and suppose that $s \leq t$. Then $x_0 + sx$, $x_0 + tx \in A$. Note that since $0 < s \leq t$, $\frac{s}{t} \in (0, 1]$ and $1 - \frac{s}{t} = \frac{t-s}{t} \in (0, 1]$. Also, since A is convex, we have that

$$\left(\frac{t-s}{t}\right)x_0 + \left(\frac{s}{t}\right)(x_0 + tx) \in A$$

Convexity of f implies that

$$\begin{aligned} f(x_0 + sx) &= f\left(\left(\frac{t-s}{t}\right)x_0 + \left(\frac{s}{t}\right)(x_0 + tx)\right) \\ &\leq \left(\frac{t-s}{t}\right)f(x_0) + \left(\frac{s}{t}\right)f(x_0 + tx) \end{aligned}$$

This implies that

$$tf(x_0 + sx) \leq (t - s)f(x_0) + sf(x_0 + tx)$$

and after rearranging, we get

$$tf(x_0 + sx) - tf(x_0) \leq sf(x_0 + tx) - sf(x_0)$$

and so finally, dividing both sides by st, we obtain

$$\begin{aligned} q(s) &= \frac{f(x_0 + sx) - f(x_0)}{s} \\ &\leq \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= q(t) \end{aligned}$$

as desired.

2. Similar to (1).

□

Exercise 12.2.0.4. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $x_0 \in A$ and $x \in X$. Define t_0 as before. Then for each $t \in (0, t_0)$,

$$q(-t) \leq q(t)$$

Hint: for sufficiently small t, convexity of f implies that $f(x_0) \leq \frac{1}{2}f(x_0 - 2tx) + \frac{1}{2}f(x_0 + 2tx)$

Proof. Choose t_0 as in the previous exercise. Since convexity of f implies that for each $t \in (0, t_0/2)$,

$$f(x_0) \leq \frac{1}{2}f(x_0 - 2tx) + \frac{1}{2}f(x_0 + 2tx)$$

we have that for each $t \in (0, t_0/2)$,

$$\begin{aligned} q(-2t) &= \frac{f(x_0 - 2tx) - f(x_0)}{-2t} \\ &\leq \frac{f(x_0 + 2tx) - f(x_0)}{2t} \\ &= q(2t) \end{aligned}$$

So for each $t \in (0, t_0)$, $q(-t) \leq q(t)$. □

Exercise 12.2.0.5. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Then

1. f is left-hand and right-hand Gateaux differentiable at x_0 with $d^- f(x_0) \leq d^+ f(x_0)$
2. for each $x \in X$, $d^- f(x_0)(x) = -d^+ f(x_0)(-x)$

Proof.

1. Let $x \in X$. Choose $t_0 > 0$ as in the previous two exercises. Let $t, u \in (0, t_0)$. Choose $s \in (0, \min(u, t))$. The previous two exercises imply that

   $$\begin{aligned} q(-u) &\leq q(-s) \\ &\leq q(s) \\ &\leq q(t) \end{aligned}$$

   and therefore $q(t)$ is an upper bound for $\{q(-u) : u \in (0, t_0)\}$ and $d^- f(x_0)(x) = \sup_{u \in (0,t_0)} q(-u)$ exists with $d^- f(x_0)(x) \leq q(t)$.
   Since $t \in (0, t_0)$ is arbitrary, $d^- f(x_0)(x)$ is a lower bound for $\{q(t) : t \in (0, t_0)\}$. Therefore

   $$d^+ f(x_0)(x) = \inf_{t \in (0,t_0)} q(t)$$

   exists with $d^+ f(x_0)(x) \geq d^- f(x_0)(x)$.

2. By definition, we have

   $$\begin{aligned} d^- f(x_0)(x) &= \lim_{t \to 0^+} \frac{f(x_0 + -tx) - f(x_0)}{-t} \\ &= -\lim_{t \to 0^+} \frac{f(x_0 + -tx) - f(x_0)}{t} \\ &= -d^+ f(x_0)(-x) \end{aligned}$$

□

Exercise 12.2.0.6. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Then $d^+f(x_0) : X \to \mathbb{R}$ is a sublinear functional.

Proof. Let $x, y \in X$ and $k \geq 0$. If $k = 0$, then clearly

$$d^+f(x_0)(kx) = kd^+(x_0)(x)$$

If $k > 0$. Then

$$\begin{aligned} d^+f(x_0)(kx) &= \lim_{t \to 0^+} \frac{f(x_0 + tkx) - f(x_0)}{t} \\ &= k \lim_{t \to 0^+} \frac{f(x_0 + tkx) - f(x_0)}{tk} \\ &= kd^+f(x_0)(x) \end{aligned}$$

Define $t_0 > 0$ as before and let $t \in (0, \frac{t_0}{2})$. Note that

$$x_0 + tx + ty = \frac{1}{2}(x_0 + 2tx) + \frac{1}{2}(x_0 + 2ty)$$

Convexity of f implies that

$$f(x_0 + tx + ty) \leq \frac{1}{2}f(x_0 + 2tx) + \frac{1}{2}f(x_0 + 2ty)$$

which implies that

$$\frac{f(x_0 + tx + ty) - f(x_0)}{t} \leq \frac{f(x_0 + 2tx) - f(x_0)}{2t} + \frac{f(x_0 + 2ty) - f(x_0)}{2t}$$

Therefore

$$\begin{aligned} d^+f(x_0)(x + y) &= \lim_{t \to 0^+} \frac{f(x_0 + t(x + y)) - f(x_0)}{t} \\ &= \lim_{t \to 0^+} \frac{f(x_0 + tx + ty) - f(x_0)}{t} \\ &\leq \lim_{t \to 0^+} \left[\frac{f(x_0 + 2tx) - f(x_0)}{2t} + \frac{f(x_0 + 2ty) - f(x_0)}{2t} \right] \\ &= \lim_{t \to 0^+} \frac{f(x_0 + 2tx) - f(x_0)}{2t} + \lim_{t \to 0^+} \frac{f(x_0 + 2ty) - f(x_0)}{2t} \\ &= d^+f(x_0)(x) + d^+f(x_0)(y) \end{aligned}$$

□

Exercise 12.2.0.7. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Then for each $x \in A$,

$$d^+f(x_0)(x - x_0) \leq f(x) - f(x_0)$$

Proof. Let $x \in A$. Define $T = \{t \in \mathbb{R} : x_0 + t(x - x_0) \in A\}$ similarly to earlier. Clearly $1 \in T$ and

$$\begin{aligned} d^+f(x_0)(x - x_0) &= \inf_{t \in (0,1]} \frac{f(x_0 + t(x - x_0)) - f(x_0)}{t} \\ &\leq f(x) - f(x_0) \end{aligned}$$

□

Exercise 12.2.0.8. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then $d^+ f(x_0)$ is Lipschitz (equivalently bounded).

Proof. Suppose that f is continuous at x_0. A previous exercise about convex functions tells us that f is locally Lipschitz at x_0, so there exists $\delta, M > 0$ such that for each $x_1, x_2 \in B(x_0, \delta)$, $|f(x_1) - f(x_2)| \leq M\|x_1 - x_2\|$. Let $x \in X$ and define $t_0 = \frac{\delta}{\|x\|+1}$ so that for each $t \in (0, t_0)$,

$$\begin{aligned} \|(x_0 + tx) - x_0\| &= t\|x\| \\ &\leq t_0\|x\| \\ &= \frac{\delta\|x\|}{\|x\| + 1} \\ &< \delta \end{aligned}$$

and $x_0 + tx \in B(x_0, \delta)$. Then for each $t \in (0, t_0)$,

$$\begin{aligned} d^+ f(x_0)(x) &\leq \frac{f(x_0 + tx) - f(x_0)}{t} \\ &\leq \frac{|f(x_0 + tx) - f(x_0)|}{t} \\ &\leq t^{-1} M\|(x_0 + tx) - x_0\| \\ &= M\|x\| \end{aligned}$$

Thus $d^+ f(x_0)$ is a bounded sublinear functional and a previous exercise in the section on sublinear functionals implies this is eqivalent to $d^+ f(x_0)$ being Lipschitz. $\square$

Exercise 12.2.0.9. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then there exists $\phi \in X^*$ such that $\phi \leq d^+ f(x_0)$.

Proof. Suppose that f is continuous at x_0. The previous exercise implies that $d^+ f(x_0)$ is Lipschitz (equivalently bounded). A previous exercise in the section discussing sublinear functionals tells us that boundedness of $d^+ f(x_0)$ implies that there exists $\phi \in X^*$ such that $\phi \leq d^+ f(x_0)$. $\square$

Definition 12.2.0.10. Subdifferential:
Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. We define the **subdifferential of** f **at** x_0, denoted $\partial f(x_0)$, to be

$$\partial f(x_0) = \{\phi \in X^* : \text{for each } x \in A, f(x_0) + \phi(x - x_0) \leq f(x)\}$$

Exercise 12.2.0.11. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then $\partial f(x_0) \neq \varnothing$.

Proof. Suppose that f is continuous at x_0. The previous exercise tells us that there exists $\phi \in X^*$ such that $\phi \leq d^+ f(x_0)$. Let $x \in A$. A previous exercise implies that

$$\begin{aligned} \phi(x - x_0) &\leq d^+ f(x_0)(x - x_0) \\ &\leq f(x) - f(x_0) \end{aligned}$$

Then $f(x_0) + \phi(x - x_0) \leq f(x)$. $\square$

Exercise 12.2.0.12. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $\phi \in X^*$ and $x_0 \in A$. Then

1. for each $x \in A$,
$$\phi(x - x_0) \leq f(x) - f(x_0)$$
iff
$$\phi \leq d^+ f(x_0)$$

2. $\partial f(x_0) = \{\phi \in X^* : \phi \leq d^+ f(x_0)\}$

Proof.

1. Suppose that for each $x \in A$, $\phi(x - x_0) \leq f(x) - f(x_0)$. Let $x \in X$. Define t_0 as before. Then for each $t \in (0, t_0)$,

$$\begin{aligned} t\phi(x) &= \phi((x_0 + tx) - x_0) \\ &\leq f(x_0 + tx) - f(x_0) \end{aligned}$$

This implies that $\phi(x) \leq d^+ f(x_0)(x)$.
Conversely, suppose that $\phi \leq d^+ f(x_0)$. Let $x \in A$. A previous exercise implies that,

$$\begin{aligned} \phi(x - x_0) &\leq d^+ f(x_0)(x - x_0) \\ &\leq f(x) - f(x_0) \end{aligned}$$

2. Clear.

□

Exercise 12.2.0.13. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then the following are equivalent:

1. f is Gateaux differentiable at x_0
2. $d^+ f(x_0)$ is linear
3. $\#\partial f(x_0) = 1$

Proof. Suppose that f is continuous at x_0. Then $d^+ f(x_0)$ is Lipschitz and bounded.

- (1) $\implies$ (2):
Suppose that f is Gateaux differentiable at x_0. Let $x \in X$. Then a previous exercise implies that

$$\begin{aligned} -df^+(x_0)(-x) &= df^- f(x_0)(x) \\ &= df^+ f(x_0)(x) \end{aligned}$$

An exercise in the section on sublinear functionals implies that $df^+ f(x_0)$ is linear.

- (2) $\implies$ (3):
Suppose that $df^+ f(x_0)$ is linear. Let $\phi \in \partial f(x_0)$. The previous exercise implies that $\phi \leq df^+ f(x_0)$. Equivalence of linearity in the section on sublinear functionals implies that $d^+ f(x_0) = \phi$.

- (3) $\implies$ (1):
Suppose that $\#\partial f(x_0) = 1$. Since $\partial f(x_0) = \{\phi \in X^* : \phi \leq d^+ f(x_0)\}$, equivalence of linearity in the section on sublinear functionals implies that $d^+ f(x_0)$ is linear. This implies that $d^+ f(x_0) = d^- f(x_0)$ and which implies that f is Gateaux differentiable at x_0.

□

Exercise 12.2.0.14. Let X be a Banach space, $A \subset X$ open and convex, $f, g : A \to \mathbb{R}$ convex, $\lambda \geq 0$ and $x_0 \in A$. Then

$$\partial f(x_0) + \lambda \partial g(x_0) \subset \partial [f + \lambda g](x_0)$$

Proof. Let $\zeta \in \partial f(x_0) + \lambda \partial g(x_0)$. Then there exist $\phi \in \partial f(x_0)$ and $\psi \in \partial g(x_0)$ such that $\zeta = \phi + \lambda \psi$. A previous exercise implies that $\phi \leq d^+ f(x_0)$ and $\lambda \psi \leq \lambda d^+ g(x_0) = d^+[\lambda g](x_0)$. Hence

$$\begin{aligned} \zeta &= \phi + \lambda \psi \\ &\leq d^+ f(x_0) + d^+[\lambda g](x_0) \\ &= d^+[f + \lambda g](x_0) \end{aligned}$$

So $\zeta \in \partial [f + \lambda g](x_0)$ □

Exercise 12.2.0.15. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then f has a global minimum point at x_0 iff $0 \in \partial f(x_0)$.

Proof. Suppose that f has a global minimum point at x_0. Let $x \in X$. Then

$$\begin{aligned} d^+ f(x_0)(x) &= \lim_{t \to 0^+} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &\geq 0 \end{aligned}$$

So $0 \leq df^+(x_0)$ and $0 \in \partial f(x_0)$.
Conversely, suppose that $0 \in \partial f(x_0)$. Let $x \in A$. Then

$$\begin{aligned} 0 &= 0(x - x_0) \\ &\leq f(x) - f(x_0) \end{aligned}$$

So that $f(x_0) \leq f(x)$ which implies that f has a global minimum point at x_0. □

Exercise 12.2.0.16. et X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is Frechet differentiable at x_0, then $\partial f(x_0) = \{Df(x_0)\}$.

Proof. Clear. □

Exercise 12.2.0.17. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Suppose that f is Frechet differentiable at x_0. If $Df(x_0) = 0$, then f has a global minimum point at x_0.

Proof. Suppose that $Df(x_0) = 0$. Since $\partial f(x_0) = \{Df(x_0)\}$, a previous exercise implies that f has a global minimum point at x_0. □

Exercise 12.2.0.18. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Suppose that f is Frechet differentiable at x_0. Then for each $x \in A$, $f(x) \geq f(x_0) + Df(x_0)(x - x_0)$

Proof. Since $Df(x_0) \in \partial f(x_0)$, for each $x \in A$, $Df(x_0)(x - x_0) \leq f(x) - f(x_0)$. □

Exercise 12.2.0.19. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$. Suppose that f is Frechet differentiable. Then f is convex iff for each $x_0, x \in A$, $f(x) \geq f(x_0) + Df(x_0)(x - x_0)$.

Proof. Suppose that f is convex. Then the previous exercise implies that for each $x_0, x \in A$, $f(x) \geq f(x_0) + Df(x_0)(x - x_0)$. Conversely, suppose that for each $x_0, x \in A$, $f(x) \geq f(x_0) + Df(x_0)(x - x_0)$. Let $x_0, x, y \in A$. Then $f(x) \geq f(x_0) + Df(x_0)(x - x_0)$ and $f(y) \geq f(x_0) + Df(x_0)(y - x_0)$.
FINISH!!! □

Exercise 12.2.0.20. Let X be a Banach space, $A \subset X$ open and convex, and $f \in C^2(A)$. Then f is convex iff for each $x_0 \in A$, $D^2 f(x_0)$ is positive semidefinite.
Hint: Define $g : A \to \mathbb{R}$ by $g(x) = f(x) - Df(x_0)(x - x_0)$ and show g is convex and use Taylor's Theorem

Proof. Suppose that f is convex. Let $x_0 \in X$. Define $g : A \to \mathbb{R}$ by $g(x) = f(x) - Df(x_0)(x - x_0)$. Since g is the sum of a convex function and an affine function, g is convex. Since $f \in C^2(A)$, we have that $g \in C^2(A)$ and it is straightforward to show that for each $x \in A$, $Dg(x) = Df(x) - Df(x_0)$ and $D^2 g(x) = D^2 f(x)$. In particular, $Dg(x_0) = 0$. Hence g has a global minimum point at x_0. This implies that $D^2 f(x_0)$ is positive semidefinite. Conversely, suppose that for each $x_0 \in A$, $D^2 f(x_0)$ is positive semidefinite. Let
FINISH!!! □

12.3 Conjugacy

Definition 12.3.0.1. Let X be a Banach space, $A \subset X$ and $f : A \to \mathbb{R}$. Define

1. $A^* \subset X^*$ and $f^* : A^* \to \mathbb{R}$
2. $A^{**} \subset X$ and $f^{**} : A^{**} \to \mathbb{R}$

by

1. $$A^* = \left\{\phi \in X^* : \sup_{x \in A} \Big[\phi(x) - f(x)\Big] < \infty\right\}$$
   and
   $$f^*(\phi) = \sup_{x \in A} \Big[\phi(x) - f(x)\Big]$$
2. $$A^{**} = \left\{x \in X : \sup_{\phi \in A^*} \Big[\hat{x}(\phi) - f^*(\phi)\Big] < \infty\right\}$$
   and
   $$f^{**}(x) = \sup_{\phi \in A^*} \Big[\hat{x}(\phi) - f^*(\phi)\Big]$$

Note 12.3.0.2. If X is a Hilbert space, we may define $A^* \subset X$ and $f^* : A^* \to \mathbb{R}$ via the Riesz representation theorem by

$$A^* = \left\{y \in X : \sup_{x \in A} \Big[\langle y, x\rangle - f(x)\Big] < \infty\right\}$$

and $f^* : A^* \to \mathbb{R}$ and

$$f^*(y) = \sup_{x \in A} \Big[\langle y, x\rangle - f(x)\Big]$$

Exercise 12.3.0.3. Let X be a Banach space, $A \subset X$ and $f : A \to \mathbb{R}$. Then

1. A^* is convex and $f^* : A^* \to \mathbb{R}$ is convex and weak* lower semicontinuous.
2. A^{**} is convex and $f^{**} : A^{**} \to \mathbb{R}$ is convex and weakly lower semicontinuous.

Proof.

1. For $x \in A$, define $g_x : X^* \to \mathbb{R}$ by $g_x(\phi) = \hat{x}(\phi) - f(x)$. Then for each $x \in A$, g_x is convex and weak* lower semicontinuous since it is affine and weak* continuous. Exercise 12.1.0.18 implies that $A^* = \{\phi \in X^* : \sup\limits_{x \in A} g_x(\phi) < \infty\}$ is convex and $f^* = \sup\limits_{x \in A} g_x$ is convex.
2. For $\phi \in A^*$, define $h_\phi : X \to \mathbb{R}$ by $h_\phi(x) = \phi(x) - f^*(\phi)$. Then for each $\phi \in A^*$, h_ϕ is convex and weakly lower semicontinuous since it is affine and weakly continuous. Exercise 12.1.0.18 implies that $A^{**} = \{x \in X : \sup\limits_{\phi \in A^*} h_\phi(x) < \infty\}$ is convex and $f^{**} = \sup\limits_{\phi \in A^*} h_\phi$ is convex.

$\square$

Exercise 12.3.0.4. Let X be a Banach space, $A \subset X$ and $f : A \to \mathbb{R}$. Then for each $x \in A$ and $\phi \in A^*$, $f^*(\phi) \geq \phi(x) - f(x)$.

Proof. Clear by definition. $\square$

Exercise 12.3.0.5. Let X be a Banach space, $A \subset X$ and $f : A \to \mathbb{R}$. Then $A \subset A^{**}$.

Proof. Let $x \in A$. Then the previous exercise implies that

$$\begin{aligned}\sup_{\phi \in A^*} [\phi(x) - f^*(\phi)] &\leq f(x) \\ &< \infty\end{aligned}$$

So $x \in A^{**}$. □

Exercise 12.3.0.6. Let X be a Banach space, $A \subset X$ convex, $f : A \to \mathbb{R}$ convex and lower semicontinuous and $x_0 \in A$.

1. if $x_0 \in A$, then for each $\epsilon > 0$, there exists $\phi \in A^*$ such that for each $x \in A$, $f(x) > f(x_0) + \phi(x - x_0) - \epsilon$
2. if $x_0 \notin A$, then for each $M \in \mathbb{R}$, there exists $\phi \in A^*$ such that for each $x \in A$, $f(x) > M + \phi(x - x_0)$

Hint: Apply second Hahn-Banach separation theorem to $\{(x_0, f(x_0) - \epsilon)\}$ and epi f.

Proof.

1. Suppose that $x_0 \in A$. Let $\epsilon > 0$. Since f is convex and lower semicontinuous, epi $f \subset X \times \mathbb{R}$ is convex and closed, $\{(x_0, f(x_0) - \epsilon)\} \subset X \times \mathbb{R}$ is convex and compact and $\{(x_0, f(x_0) - \epsilon)\} \cap \operatorname{epi} f = \varnothing$. Thus, there exists $\lambda \in \mathbb{R}$, $\psi \in X^*$ and $k \in \mathbb{R}$ such that for each $x \in A$ and $r \geq f(x)$,

$$\psi(x) + \lambda r < k < \psi(x_0) + \lambda(f(x_0) - \epsilon)$$

Taking $(x, r) = (x_0, f(x_0))$ implies that $0 < -\lambda\epsilon$ and hence that $\lambda < 0$. Set $\phi = |\lambda|^{-1}\psi$. For $x \in A$, set $r = f(x)$. Then

$$\begin{aligned} &\psi(x) - |\lambda| f(x) < \psi(x_0) - |\lambda|(f(x_0) - \epsilon) \\ \iff &|\lambda|^{-1}\psi(x) - f(x) < |\lambda|^{-1}\psi(x_0) - (f(x_0) - \epsilon) \\ \iff &\phi(x) - f(x) < \phi(x_0) - (f(x_0) - \epsilon) \\ \iff &f(x) > f(x_0) + \phi(x - x_0) - \epsilon \end{aligned}$$

Since for each $x \in A$, $\phi(x) - f(x) < \phi(x_0) - f(x_0) + \epsilon$, we have that

$$\begin{aligned}\sup_{a \in A}[\phi(x) - f(x)] &\leq \phi(x_0) - f(x_0) + \epsilon \\ &< \infty\end{aligned}$$

So $\phi \in A^*$.

2. Suppose that $x_0 \notin A$. Let $M \in \mathbb{R}$. Repeat the previous argument for (x_0, M) and epi f.

□

Exercise 12.3.0.7. Let X be a Banach space, $A \subset X$ convex and $f : A \to \mathbb{R}$ convex and lower semicontinuous. Then

1. $A = A^{**}$
2. $f = f^{**}$

Proof.

1. A previous exercise implies that $A \subset A^{**}$. Let $x_0 \in X$. Suppose that $x_0 \notin A$. Let $M \in \mathbb{R}$. The previous exercise implies that there exists $\phi_0 \in A^*$ such that for each $x \in A$, $f(x) > M + \phi_0(x - x_0)$. Then

$$\begin{aligned}\phi_0(x_0) - f^*(\phi_0) &= \phi_0(x_0) - \sup_{x \in A}[\phi_0(x) - f(x)] \\ &= \phi_0(x_0) + \inf_{x \in A}[f(x) - \phi_0(x)] \\ &\geq \phi_0(x_0) + (M - \phi_0(x_0)) \\ &= M\end{aligned}$$

Therefore

$$\begin{aligned}\sup_{\phi \in A^*} [\phi(x_0) - f^*(\phi)] &\geq \phi_0(x_0) - f^*(\phi_0) \\ &\geq M\end{aligned}$$

Since $M \in \mathbb{R}$ is arbitrary,

$$\sup_{\phi \in A^*} [\phi(x_0) - f^*(\phi)] = \infty$$

and $x_0 \notin A^{**}$. So $A^c \subset (A^{**})^c$, which implies that $A^{**} \subset A$. Thus $A^{**} = A$.

2. Part (1) and a previous exercise imply that $f^{**} \leq f$. Suppose that $f \not\leq f^{**}$. Then there exists $x_0 \in A$ such that $f(x_0) > f^{**}(x_0)$. Choose $\epsilon > 0$ such that $f(x_0) > f^{**}(x_0) + 2\epsilon$. A previous exercise implies that there exists $\phi \in A^*$ such that for each $x \in A$, $f(x) > f(x_0) + \phi(x - x_0) - \epsilon$. Choose $a \in A$ such that $f^*(\phi) - \epsilon < \phi(a) - f(a)$. Then

$$\begin{aligned}f(x_0) &> f^{**}(x_0) + 2\epsilon \\ &\geq \phi(x_0) - f^*(\phi) + 2\epsilon \\ &> \phi(x_0 - a) + f(a) + \epsilon \\ &> \phi(x_0 - a) + f(x_0) + \phi(a - x_0) - \epsilon + \epsilon \\ &= f(x_0)\end{aligned}$$

which is a contradiction. So $f \leq f^{**}$ and hence $f = f^{**}$.

$\square$

Definition 12.3.0.8. Let

Definition 12.3.0.9. ∂f

Exercise 12.3.0.10.

Chapter 13

Topological Groups

13.1 Introduction

Definition 13.1.0.1. Let G be a group, we define $\text{mult} : G \times G \to G$ and $\text{inv} : G \to G$ by $\text{mult}(g, h) = gh$ and $\text{inv}(g) = g^{-1}$ respectively.

Definition 13.1.0.2. Let G be a group and $\mathcal{T}$ a topology on G. Then $(G, \mathcal{T})$ is said to be a **topological group** if $\text{mult} : G \times G \to G$ and $\text{inv} : G \to G$ are continuous.

Note 13.1.0.3. For the remainder of this chapter, measurablility is in reference to $(G, \mathcal{B}(\mathcal{T}))$. That is, the measurable sets are the Borel sets.

Definition 13.1.0.4. Let G be a topological group. We define

$$\text{Homeo}(G) = \{\phi : G \to G : \phi \text{ is a homeomorphism}\}$$

Note 13.1.0.5. Let G be a topological group. Then $\text{Homeo}(G)$ is a group.

Exercise 13.1.0.6. Let G be a topological group. Then $\text{inv} \in \text{Homeo}(G)$.

Proof. By assumption inv is continuous. We know from basic group theory that inv is a bijection with $\text{inv}^{-1} = \text{inv}$. $\square$

Definition 13.1.0.7. Let G be a group and $S \subset G$, then S is said to be **symmetric** if $\text{inv}(S) = S$, (i.e. $S^{-1} = S$).

Definition 13.1.0.8. Let G be a topological group and $\phi : G \to G$. Then ϕ is said to be an **automorphism** of G if ϕ is a homomorphism and a homeomorphism. We define

$$\text{Aut}(G) = \{\phi : G \to G : \phi \text{ is an automorphism}\}$$

Exercise 13.1.0.9. Let G be a topological group. Then $\text{inv} \in \text{Aut}(G)$ iff G is abelian.

Proof. Basic group theory tells us that inv is a homomorphism iff G is abelian. $\square$

Definition 13.1.0.10. Let G be a group and $g \in G$. Define $l_g : G \to G$ and $r_g : G \to G$ by $l_g(x) = gx$ and $r_g(x) = xg^{-1}$.

Exercise 13.1.0.11. Let G be a topological group and $g \in G$. Then $l_g, r_g \in \text{Homeo}(G)$.

Proof. By assumption l_g and r_g are continuous. We know from basic group theory that l_g and r_g are bijections with $l_g^{-1} = l_{g^{-1}}$ and $r_g^{-1} = r_{g^{-1}}$ so l_g and r_g. are homeomorphisms. $\square$

Exercise 13.1.0.12. Let G be a toplogical group. Define $\phi, \psi : G \to \text{Homeo}(G)$ by $\phi(g) = l_g$ and $\psi(g) = r_g$. Then ϕ, ψ are homomorphisms.

Proof. Let $g_1, g_2 \in G$. Then
$$l_{g_1} \circ l_{g_2}(x) = l_{g_1}(g_2 x) = g_1 g_2 x = l_{g_1 g_2}(x)$$
and
$$r_{g_1} \circ r_{g_2}(x) = r_{g_1}(x g_2^{-1}) = x g_2^{-1} g_1^{-1} = x(g_1 g_2)^{-1} = r_{g_1 g_2}(x)$$
□

Exercise 13.1.0.13. Let G be a topological group. Then for each $U \subset G$ and $g \in G$, if U is open, then gU, Ug and U^{-1} are open.

Proof. Let $U \subset G$ and $g \in G$. Suppose that U is open. Since l_g, r_g and inv are homeomorphisms, $l_g(U) = gU$, $r_g(U) = Ug$ and $\mathrm{inv}(U) = U^{-1}$ are open. □

Definition 13.1.0.14. Let G be a topological group, $y \in G$ and $f \in L^0$. Define $L_y, R_y : L^0(G) \to L^0(G)$ by $L_y f = f \circ l_y^{-1}$ and $R_y f = f \circ r_y^{-1}$, that is, $L_y f(x) = f(y^{-1}x)$ and $R_y f(x) = f(xy)$.

Exercise 13.1.0.15. Let G be a topological group and $y \in G$. Then $L_y, R_y \in \mathrm{Sym}(L^0(G))$.

Proof. It is straight forward to show that $L_y^{-1} = L_{y^{-1}}$ and $R_y^{-1} = R_{y^{-1}}$. □

Exercise 13.1.0.16. Let G be a topological group. Define $\phi, \psi : G \to \mathrm{Sym}(L^0(G))$ by $\phi(y) = L_y$ and $\psi(y) = R_y$. Then ϕ and ψ are homomorphisms.

Proof. Let $y, z \in G$ and $f \in L^0(G)$. Then
$$\begin{aligned} L_y \circ L_z(f) &= L_y(L_z(f)) \\ &= L_y(f \circ l_z^{-1}) \\ &= (f \circ l_z^{-1}) \circ l_y^{-1} \\ &= f \circ (l_z^{-1} \circ l_y^{-1}) \\ &= f \circ (l_y \circ l_z)^{-1} \\ &= f \circ l_{yz}^{-1} \\ &= L_{yz}(f) \end{aligned}$$

The case is similar for R_y and R_z. □

Exercise 13.1.0.17. Let G be a topological group, $U \in \mathcal{B}(G)$ and $y \in G$. Then $L_y \chi_U = \chi_{yU}$ and $R_y \chi_U = \chi_{Uy^{-1}}$.

Proof. Let $x \in G$. Then
$$\begin{aligned} L_y \chi_U(x) = 1 &\iff y^{-1}x \in U \\ &\iff x \in yU \\ &\iff \chi_{yU}(x) = 1 \end{aligned}$$
The case is similar for R_y □

Exercise 13.1.0.18. Let G be a topological group, $y \in G$ and $f \in L^0(G)$. Then $\mathrm{supp}(L_y f) = y\,\mathrm{supp}(f)$ and $\mathrm{supp}(R_y f) = \mathrm{supp}(f) y^{-1}$

Proof. Put $A = \{x \in G : L_y f(x) \neq 0\}$ and $B = \{x \in G : f(x) \neq 0\}$. Then
$$\begin{aligned} x \in A &\iff L_y f(x) \neq 0 \\ &\iff f(y^{-1}x) \neq 0 \\ &\iff y^{-1}x \in B \\ &\iff x \in yB \end{aligned}$$
Thus $A = yB$ which implies that $\mathrm{cl}\, A = y\, \mathrm{cl}\, B$. Therefore $\mathrm{supp}(L_y f) = y\,\mathrm{supp}(f)$. □

Exercise 13.1.0.19. Let G be a topological group and $y \in G$. Then L_y, R_y are linear and if we restrict to the bounded measurable functions, then $L_y, R_y \in L(B(G))$ and $\|L_y\|, \|R_y\| = 1$.

Proof. Let $f, g \in L^0(G)$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} L_y(\lambda f + g)(x) &= (\lambda f + g)(y^{-1}x) \\ &= \lambda f(y^{-1}x) + g(y^{-1}x) \\ &= \lambda L_y f(x) + L_y g(x) \end{aligned}$$

So L_y is linear. Next, we restrict to $B(G) \cap L^0$. We note that

$$\{|f(y^{-1}x)| : x \in y\,\mathrm{supp}(f)\} = \{|f(x)| : x \in \mathrm{supp}(f)\}$$

This implies that

$$\begin{aligned} \|L_y f\|_u &= \sup_{x \in \mathrm{supp}(L_y f)} |L_y f(x)| \\ &= \sup_{x \in y\,\mathrm{supp}(f)} |f(y^{-1}x)| \\ &= \sup_{x \in \mathrm{supp}(f)} |f(x)| \\ &= \|f\|_u \end{aligned}$$

So L_y is bounded. Hence $L_y \in L(L^0)$. The case is similar for R_y. $\square$

Definition 13.1.0.20. Let G be a topological group. We say that G is a **locally compact group** if G is locally compact and Hausdorff.

13.2 Group Actions

13.2.1 Introduction

Note 13.2.1.1. Let X, Y, X be sets. We recall that for $f : X \times Y \to Z$, $a \in X$ and $b \in Y$, the maps $f_a : Y \to Z$ and $f^b : X \to Z$ are defined by

$$f_a(y) = f(a, y) \quad f^b(x) = f(x, b)$$

Definition 13.2.1.2. Let $\mathcal{C}$ a concrete category with products, $G, X \in \mathrm{Obj}(C)$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(G \times X, X)$. Suppose that G is a group. Then ϕ is said to be a **group action** of G on X if

1. for each $x \in X$, $\phi_e = \mathrm{id}_X$
2. for each $g, h \in G$, $\phi_{gh} = \phi_g \circ \phi_h$

Note 13.2.1.3. When the context is clear, we will write $g \cdot x$ in place of $\phi(g, x)$.

Exercise 13.2.1.4. Let $\mathcal{C}$ a category with products, $G, X \in \mathrm{Obj}(C)$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(G \times X, X)$. Suppose that G is a group and ϕ group action. Then for each $g \in G$, $\phi_g \in \mathrm{Aut}(X)$.

Proof. Let $g \in G$. Then

$$\begin{aligned}
\phi_g \circ \phi_{g^{-1}}(x) &= \phi_g(\phi_{g^{-1}}(x)) \\
&= g \cdot (g^{-1} \cdot x) \\
&= (gg^{-1}) \cdot x \\
&= e \cdot x \\
&= x
\end{aligned}$$

Since $x \in X$ is arbitrary, $\phi_g \circ \phi_{g^{-1}} = \mathrm{id}_X$. Similarly, $\phi_{g^{-1}} \circ \phi_g = \mathrm{id}_X$. Hence $\phi_g \in \mathrm{Aut}(X)$. □

Definition 13.2.1.5. Let $\mathcal{C}$ a category with products, $G, X \in \mathrm{Obj}(C)$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(G \times X, X)$. Suppose that G is a group and ϕ group action. We define $\hat{\phi} : G \to \mathrm{Aut}(X)$ by $\hat{\phi}(g) = \phi_g$.

Exercise 13.2.1.6. Let $\mathcal{C}$ a category with products, $G, X \in \mathrm{Obj}(C)$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(G \times X, X)$. Suppose that G is a group and ϕ group action. Then $\hat{\phi} : G \to \mathrm{Aut}(X)$ is a group homomorphism.

Proof. Clear by definition. □

13.2.2 Homogeneous Spaces

Definition 13.2.2.1. Let G be a topological group, X a topological space and $\phi : G \times X \to X$ a continous group action. Then (X, ϕ) is said to be a **homogeneous G-space** if

- ϕ is transitive
- for each $x \in X$, $\phi_x : G \to X$ is open

Definition 13.2.2.2. Let G be a topological group, $H < G$. We define $\phi_H : H \times G \to G$ by $\phi(h, g) = gh^{-1}$.

Exercise 13.2.2.3.

Exercise 13.2.2.4. Let G be a topological group, $H < G$ a closed subgroup of G. Then $(G/H, \phi_H)$

Exercise 13.2.2.5. Let G be a topological group, $H < G$ a closed subgroup of G and (X, ϕ) a homogeneous G-space.

13.2.3 Common Examples

Exercise 13.2.3.1. Let H be a Hilbert space and $x, y \in H$. Then $\|x\| = \|y\|$ iff there exists $U \in U(H)$ such that $x = Uy$.

Proof.

- $(\implies)$: Suppose that $\|x\| = \|y\|$. An exercise
- $(\impliedby)$:

□

Exercise 13.2.3.2. Let H be a Hilbert space. Then

1. $\|\cdot\| : H \to [0, \infty)$ is a quotient map
2. $H/U(H)$ is homeomorphic to $[0, \infty)$

Proof. content... □

Definition 13.2.3.3. Let $n, k \in \mathbb{N}$. Suppose that $n \geq k$. We define the **Stiefel manifold**, denoted $V_k(\mathbb{R}^n)$, by

$$V_k(\mathbb{R}^n) = \{A \in \mathbb{R}^{n \times k} : A^*A = I\}$$

We define the **orthogonal matrices**, denoted by $O(n)$, by

$$O(n) = V_n(\mathbb{R}^n)$$

Note 13.2.3.4. We note that for each $X \in V_k(\mathbb{R}^n)$, $\operatorname{rank} X = k$ and for each $U \in O(n)$, $UU^* = I$.

Exercise 13.2.3.5. Let $X, Y \in \mathbb{R}^{n \times k}$. Suppose that $\operatorname{rank} X = k$ and $\operatorname{rank} Y = k$. Then $XX^* = YY^*$ iff there exists $U \in O(k)$ such that $X = YU$.
Hint: $\operatorname{rank} X = \operatorname{rank} X^*X$.

Proof.

- $(\implies)$:
  Suppose that $XX^* = YY^*$. Since $\operatorname{rank} X = k$, we have that

$$\begin{aligned} \operatorname{rank} XX^* &= \operatorname{rank} X \\ &= k \end{aligned}$$

  Since $X^*X \in \mathbb{R}^{k \times k}$, X^*X is invertible. Hence

$$\begin{aligned} X &= XI \\ &= X(X^*X)(X^*X)^{-1} \\ &= (XX^*)X(X^*X)^{-1} \\ &= (YY^*)X(X^*X)^{-1} \\ &= Y(Y^*X)(X^*X)^{-1} \end{aligned}$$

  Set $U = (Y^*X)(X^*X)^{-1}$. Then $X = YU$ and

$$\begin{aligned} U^*U &= \left((Y^*X)(X^*X)^{-1}\right)^*(Y^*X)(X^*X)^{-1} \\ &= (X^*X)^{-1}(X^*Y)(Y^*X)(X^*X)^{-1} \\ &= (X^*X)^{-1}X^*(YY^*)X(X^*X)^{-1} \\ &= (X^*X)^{-1}X^*(XX^*)X(X^*X)^{-1} \\ &= (X^*X)^{-1}(X^*X)(X^*X)(X^*X)^{-1} \\ &= I \end{aligned}$$

  Thus $U \in O(k)$.

- ($\Longleftarrow$):
  Suppose that there exists $U \in O(k)$ such that $X = YU$. Then

$$\begin{aligned} XX^* &= (YU)(YU)^* \\ &= (YU)(U^*Y^*) \\ &= Y(UU^*)Y^* \\ &= YIY^* \\ &= YY^* \end{aligned}$$

□

Exercise 13.2.3.6. Define $f : V$

13.3 Quotient Groups

Definition 13.3.0.1. Let

13.4 Automorphism Groups of Metric Spaces

Definition 13.4.0.1. Let (X, τ) be a topological space. Define

$$\text{Aut}(X) = \{\sigma : X \to X : \sigma \text{ is a homeomorphism}\}$$

Exercise 13.4.0.2. Let (X, d) be a compact metric space. Then $(\text{Aut}(X), d_u)$ is a topological group.

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}} \subset \text{Aut}(X)$ and $\sigma, \tau \in \text{Aut}(X)$. Suppose that $\sigma_n \xrightarrow{\text{u}} \sigma$ and $\tau_n \xrightarrow{\text{u}} \tau$.

1. Let $\epsilon > 0$. Since X is compact and σ is continuous, σ is uniformly continuous. Then there exists $\delta > 0$ such that for each $x, y \in X$, $d(x, y) < \delta$ implies that $d(\sigma(x), \sigma(y)) \leq \epsilon/2$. Choose $N_\sigma \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq \mathbb{N}$ implies that $d_u(\sigma_n, \sigma) < \epsilon/2$. Choose $N_\tau \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq \mathbb{N}$ implies that $d_u(\tau_n, \tau) < \delta$. Put $N = \max(N_\sigma, N_\tau)$. Let $n \in \mathbb{N}$ and $x \in X$. Suppose that $n \geq N$. Then

$$\begin{aligned} d(\sigma_n \circ \tau_n(x), \sigma \circ \tau(x)) &\leq d(\sigma_n(\tau_n(x)), \sigma(\tau_n(x))) + d(\sigma(\tau_n(x)), \sigma(\tau(x))) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

   So $d_u(\sigma_n \circ \tau_n, \sigma \circ \tau) \leq \epsilon$ and $\circ : \text{Aut}(X)^2 \to \text{Aut}(X)$ is continuous.

2. Suppose that $\sigma = \text{id}_X$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $d_u(\sigma_n, \text{id}_X) < \epsilon$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Then

$$\begin{aligned} \sup_{x\in X} d(\sigma_n^{-1}(x), x) &= \sup_{x\in\sigma_n(X)} d(\sigma_n^{-1}(x), x) \\ &= \sup_{x\in X} d(\sigma_n^{-1}(\sigma_n(x)), \sigma_n(x)) \\ &= \sup_{x\in X} d(x, \sigma_n(x)) \\ &< \epsilon \end{aligned}$$

   So $\sigma_n^{-1} \xrightarrow{\text{u}} \text{id}_X$. Now suppose that $\sigma \neq \text{id}_X$. Since $\sigma_n \xrightarrow{\text{u}} \sigma$, part (1) implies that $\sigma^{-1} \circ \sigma_n \xrightarrow{\text{u}} \text{id}_X$. Applying the result from above, we get that $\sigma_n^{-1} \circ \sigma \xrightarrow{\text{u}} \text{id}_X$. Applying part (1) again implies that $\sigma_n^{-1} \xrightarrow{\text{u}} \sigma^{-1}$. So the map $\sigma \mapsto \sigma^{-1}$ is continuous.

Hence $\text{Aut}(X)$ is a topological group. □

Definition 13.4.0.3. Let (X, d) be a metric space. Define

$$\text{Aut}(X, d) = \{\sigma : X \to X : \sigma \text{ is an isometric isomorphism}\}$$

Exercise 13.4.0.4. Let (X, d) be a compact metric space. Then $(\text{Aut}(X, d), d_u)$ is a compact subgroup of $(\text{Aut}(X), d_u)$.

Proof. Clearly, $(\text{Aut}(X, d), d_u)$ is a topological subgroup. To show compactness, use the Arzela Ascoli theorem. □

Definition 13.4.0.5. Let (X, τ) be a topological space and $\mu : \mathcal{B}(X) \to \mathbb{R}$ a Borel measure. Define

$$\text{Aut}(X, \mu) = \{\sigma \in \text{Aut}(X) : \sigma_*\mu = \mu\}$$

Exercise 13.4.0.6. Let (X, d) be a compact metric space and $\mu : \mathcal{B}(X) \to \mathbb{R}$ an outer-regular Borel measure. Then $\text{Aut}(X, \mu)$ is a closed subgroup of $\text{Aut}(X)$.

Proof. It is clear that $\text{Aut}(X, \mu)$ is a subgroup of $\text{Aut}(X)$. Let $(\sigma_n)_{n\in\mathbb{N}} \subset \text{Aut}(X, \mathcal{B}(X), \mu)$ and $\sigma \in \text{Aut}(X)$. Suppose that $\sigma_n \xrightarrow{\text{u}} \sigma$. Let $E \subset X$ be closed, $U \subset X$ open and suppose that $E \subset U$. An exercise in the section on metric spaces tells us that there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $\sigma(E) \subset \sigma_n(U)$. Then

$$\begin{aligned} \mu(\sigma(E)) &\leq \mu(\sigma_N(U)) \\ &= \mu(U) \end{aligned}$$

Therefore, since μ is outer regular, $\mu(\sigma(E)) \leq \mu(E)$. Since $\sigma_n^{-1} \xrightarrow{\mathrm{u}} \sigma^{-1}$, we may apply the above argument to obtain that

$$\begin{aligned} \mu(E) &= \mu(\sigma^{-1}(\sigma(E))) \\ &\leq \mu(\sigma(E)) \end{aligned}$$

Hence $\mu(E) = \mu(\sigma(E))$. Applying the whole argument above thus far to σ^{-1}, we see that $\mu(E) = \mu(\sigma^{-1}(E))$. Since $E \subset X$ is an arbitrary closed set and $\mathcal{B}(X) = \sigma(E \subset X : E \text{ is closed})$, we have that $\mu = \sigma_* \mu$. Thus $\sigma \in \operatorname{Aut}(X, \mu)$ which implies that $\operatorname{Aut}(X, \mu)$ is closed. □

Definition 13.4.0.7. Let (X, d) be a compact metric space and $\mu : \mathcal{B}(X) \to \mathbb{R}$ an outer-regular Borel measure. Define $\operatorname{Aut}(X, d, \mu) = \operatorname{Aut}(X, d) \cap \operatorname{Aut}(X, \mu)$.

Exercise 13.4.0.8. Let (X, d) be a compact metric space and $\mu : \mathcal{B}(X) \to \mathbb{R}$ an outer-regular Borel measure. Then $\operatorname{Aut}(X, d, \mu)$ is compact.

Proof. Since $\operatorname{Aut}(X, d)$ is compact and $\operatorname{Aut}(X, \mu)$ is closed, $\operatorname{Aut}(X, d, \mu)$ is compact. □

Chapter 14

Group Actions

14.1 Introduction

Note 14.1.0.1. For a set X, a group G and a (left) group action $\phi : G \times X \to X$, we will write $\phi(g, x)$ as $g \cdot x$.

Definition 14.1.0.2. Let X be a set, G a group, $\phi : G \times X \to X$ a group action and $g \in G$. Define $l_g : X \to X$ by

$$l_g(x) = g \cdot x$$

Definition 14.1.0.3. Let X be a topological space, G a group and $\phi : G \times X \to X$ a group action. Then ϕ is said to be X-continuous if for each $g \in G$, ϕ_g is continuous.

Exercise 14.1.0.4. Let X be a topological space, G a group and $\phi : G \times X \to X$ an X-continuous group action. Then for each $g \in G$, $\phi_g \in \text{Homeo}(X)$.

Proof. Let $g \in G$, then ϕ_g and $\phi_g^{-1} = \phi_{g^{-1}}$ are continuous, so $\phi_g \in \text{Homeo}(G)$. $\square$

Definition 14.1.0.5. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ a group action. Then ϕ is said to be an **isometric group action** if for each $g \in G$, $\phi_g : X \to X$ is an isometry.

Exercise 14.1.0.6. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Then ϕ is X-continuous.

Proof. Clear since isometries are continuous. $\square$

Definition 14.1.0.7. Let X be a set, G a group, $\phi : G \times X \to X$ a group action. We define the relation $\sim_\phi \subset X \times X$ by

$$\sim_\phi = \{(a, b) \in X \times X : \text{ there exists } g \in G : a = g \cdot b\}$$

Exercise 14.1.0.8. Let X be a set, G a group, $\phi : G \times X \to X$ a group action. Then $\sim_\phi$ is an equivalence relation on X.

Proof. Let $a, b, c \in X$.

- **(reflexivity):** Then $a = e \cdot a$. Hence $a \sim_\phi a$.
- **(symmetry):** Suppose that $a \sim_\phi b$. Then there exists $g \in G$ such that $a = g \cdot b$. Hence $b = g^{-1} \cdot a$. Thus $b \sim_\phi a$.
- **(transitivity):** Suppose that $a \sim_\phi b$ and $b \sim_\phi c$. Then there exist $g, h \in G$ such that $a = g \cdot b$ and $b = h \cdot c$. Then

$$\begin{aligned} a &= g \cdot b \\ &= g \cdot (h \cdot c) \\ &= (gh) \cdot c \end{aligned}$$

Hence $a \sim_\phi c$.

□

Definition 14.1.0.9. Let X be a set, G a group and $\phi : G \times X \to X$ a group action. We define the **quotient of** X **by** G, denoted X/G, by

$$X/G = X/ \sim_\phi$$

We denote the projection from X onto X/G by $\pi : X \to X/G$.

Definition 14.1.0.10. Let X be a set, G a group, $\phi : G \times X \to X$ a group action and $f : X \to \mathbb{C}$. Then f is said to be ϕ**-invariant** if for each $g \in G$ and $x \in X$, $f(g \cdot x) = f(x)$.

Exercise 14.1.0.11. Let X be a set, G a group, $\phi : G \times X \to X$ a group action and $f : X \to \mathbb{C}$. Then f is ϕ-invariant iff f is $\sim_\phi$-invariant.

Proof.

- $(\implies)$:
  Suppose that f is ϕ-invariant. Let $a, b \in X$. Suppose that $a \sim_\phi b$. Then there exists $g \in G$ such that $a = g \cdot b$. Since f is ϕ-invariant,

  $$\begin{aligned} f(a) &= f(g \cdot b) \\ &= f(b) \end{aligned}$$

  Since $a, b \in X$ such that $a \sim_\phi b$ are arbitrary, we have that f is $\sim_\phi$-invariant.

- $(\impliedby)$:
  Suppose that f is $\sim_\phi$-invariant. Let $g \in G$ and $x \in X$. By definition, $x \sim_\phi g \cdot x$. Since f is $\sim_\phi$-invariant, $f(g \cdot x) = f(x)$. Since $g \in G$ and $x \in X$ are arbitrary, f is ϕ-invariant.

□

Exercise 14.1.0.12. Let X, Y be a topological spaces, G a topological group, $\phi : G \times X \to X$ a continuous group action and $f : X \to Y$ a homeomorphism.

14.2 Group Actions on Metric Spaces

Note 14.2.0.1. This section establishes the criteria for the existence of a metric on the orbit space of a metric space under a group action.

Definition 14.2.0.2. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ a group action. We define $\bar{d} : X/G \times X/G \to [0, \infty)$ by

$$\bar{d}(\bar{x}, \bar{y}) = \inf_{\substack{a \in \bar{x} \\ b \in \bar{y}}} d(a, b)$$

Exercise 14.2.0.3. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Then for each $x, y \in X$,

$$\bar{d}(\bar{x}, \bar{y}) = \inf_{g \in G} d(g \cdot x, y)$$

Proof. Let $x, y \in X$, $a \in \bar{x}$ and $b \in \bar{y}$. Then there exists there exists $g_a, g_b \in G$ such that $a = g_a \cdot x$ and $b = g_b \cdot y$. Set $g = g_b^{-1} g_a$. Since the map $z \mapsto g_b^{-1} \cdot z$ is an isometry,

$$\begin{aligned} d(a, b) &= d(g_a \cdot x, g_b \cdot y) \\ &= d(g_b^{-1} g_a \cdot x, y) \\ &= d(g \cdot x, y) \end{aligned}$$

Let $\epsilon > 0$. Then there exist $a^* \in \bar{x}$ and $b^* \in \bar{y}$ such that $d(a^*, b^*) < \bar{d}(\bar{x}, \bar{y}) + \epsilon$. The above argument implies that that there exists $g^* \in G$ such that

$$\begin{aligned} \inf_{g \in G} d(g \cdot x, y) &\leq d(g^* \cdot x, y) \\ &= d(a^*, b^*) \\ &< \bar{d}(\bar{x}, \bar{y}) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\inf_{g \in G} d(g \cdot x, y) \leq \bar{d}(\bar{x}, \bar{y})$$

Conversely, since $\{(g \cdot x, y) : g \in G\} \subset \{(a, b) : a \in \bar{x}, b \in \bar{y}\}$, we have that

$$\inf_{g \in G} d(g \cdot x, y) \geq \bar{d}(\bar{x}, \bar{y})$$

□

Exercise 14.2.0.4. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Then for each $x, y, z \in X$,

$$\bar{d}(\bar{x}, \bar{y}) \leq \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y})$$

Proof. Let $x, y, z \in X$. An exercise in section (2.1) implies that $d(\bar{x}, \bar{y}) \leq d(\bar{x}, z) + d(z, \bar{y})$. The previous exercise implies that

$$\begin{aligned} d(\bar{x}, z) &= \inf_{a \in \bar{x}} d(a, z) \\ &= \inf_{g \in G} d(g \cdot x, z) \\ &= \bar{d}(\bar{x}, \bar{z}) \end{aligned}$$

Similarly, $d(z, \bar{y}) = \bar{d}(\bar{z}, \bar{y})$. Then

$$\begin{aligned} d(\bar{x}, \bar{y}) &\leq d(\bar{x}, z) + d(z, \bar{y}) \\ &= \bar{d}(\bar{x}, \bar{z}) + \bar{d}(\bar{z}, \bar{y}) \end{aligned}$$

□

Exercise 14.2.0.5. Let (X,d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. If for each $x \in X$, $\bar{x}$ is closed, then for each $x, y \in X$, $\bar{d}(\bar{x}, \bar{y}) = 0$ implies that $\bar{x} = \bar{y}$.

Proof. Suppose that for each $x \in X$, $\bar{x}$ is closed. Let $x, y \in X$. Suppose that $\bar{d}(\bar{x}, \bar{y}) = 0$. Then $\inf_{g \in G} d(g \cdot x, y) = 0$. Hence there exists $(g_n)_{n \in N} \subset G$ such that $g_n \cdot x \to y$. Since $(g_n \cdot x)_{n \in \mathbb{N}} \subset \bar{x}$ and $\bar{x}$ is closed, $y \in \bar{x}$. Thus $\bar{x} = \bar{y}$. □

Exercise 14.2.0.6. Let (X,d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. If for each $x \in X$, $\bar{x}$ is closed, then $\bar{d}$ is a metric on X/G.

Proof. Clear by preceeding exercises. □

Exercise 14.2.0.7. Let (X,d) be a metric space, $(G, \mathcal{T}_G)$ a topological group, and $\phi : G \times X \to X$ an isometric group action. Suppose that G is compact and for each $x \in X$, the map $g \mapsto g \cdot x$ is $(\mathcal{T}_G, \mathcal{T}_d)$-continuous. Then $\bar{d}$ is a metric on X/G.

Proof. Let $x \in X$. Since G is compact and the map $g \mapsto g \cdot x$ is $(\mathcal{T}_G, \mathcal{T}_d)$-continuous, $\bar{x} = G \cdot x$ is compact and therefore closed. The previous exercise implies that $\bar{d}$ is a metric. □

Exercise 14.2.0.8. Let (X,d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric on X/G. Then the projection map $\pi : X \to X/G$, is $(d, \bar{d})$-Lipschitz and therefore $(\mathcal{T}_d, \mathcal{T}_{\bar{d}})$-continuous.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \bar{d}(\pi(x), \pi(y)) &= \bar{d}(\bar{x}, \bar{y}) \\ &= \inf_{g \in G} d(g \cdot x, y) \\ &\leq d(x, y) \end{aligned}$$

□

Exercise 14.2.0.9. Let (X,d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric on X/G. Let $(x_n)_{n \in \mathbb{N}} \subset X$ and $x \in X$. Then $\bar{x}_n \xrightarrow{\bar{d}} \bar{x}$ iff there exists a sequence $(g_n)_{n \in \mathbb{N}}$ such that $g_n \cdot x_n \xrightarrow{d} x$.

Proof. Suppose that $\bar{x}_n \xrightarrow{\bar{d}} \bar{x}$. For $n \in \mathbb{N}$, choose $g_n \in G$ such that $d(g_n \cdot x_n, x) < \bar{d}(\bar{x}_n, \bar{x}) + 2^{-n}$. Then $d(g_n \cdot x_n, x) \to 0$ and $g_n \cdot x_n \xrightarrow{d} x$.
Conversely, suppose that that there exists a sequence $(g_n)_{n \in \mathbb{N}}$ such that $g_n \cdot x_n \xrightarrow{d} x$. Since $\pi : X \to X/G$ is $(\mathcal{T}_d, \mathcal{T}_{\bar{d}})$-continuous, we have that

$$\begin{aligned} g_n \cdot x_n \xrightarrow{d} x &\implies \pi(g_n \cdot x_n) \xrightarrow{\bar{d}} \pi(x) \\ &\implies \bar{x}_n \xrightarrow{\bar{d}} \bar{x} \end{aligned}$$

□

Exercise 14.2.0.10. Let X be a set, $d_1, d_2 : X^2 \to [0, \infty)$ metrics, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that d_1 and d_2 are **Top**-equivalent.

1. Then $\bar{d}_1$ is a metric on X/G iff $\bar{d}_2$ is a metric on X/G
2. If $\bar{d}_1$ and $\bar{d}_2$ are metrics, then $\bar{d}_1$ and $\bar{d}_2$ are **Top**-equivalent.

Proof.

1. - $\implies$ Suppose that $\bar{d}_1$ is a metric. Let $x, y \in X$. Suppose that $\bar{d}_2(\bar{x}, \bar{y}) = 0$. Then there exist $(g_n)_{n\in\mathbb{N}} \subset G$ such that $d_2(g_n \cdot x, y) \to 0$. Since d_1 and d_2 are **Top**-equivalent, $d_1(g_n \cdot x, y) \to 0$. Thus $\bar{d}_1(\bar{x}, \bar{y}) = 0$. Since $\bar{d}_1$ is a metric, $\bar{x} = \bar{y}$. Hence $\bar{d}_2$ is a metric.
   - $\impliedby$ Similar.
2. Suppose that $\bar{d}_1$ and $\bar{d}_2$ are metrics. Let $(\bar{x}_n)_{n\in\mathbb{N}} \subset X/G$ and $\bar{x} \in X/G$.
   - Suppose that $\bar{x}_n \xrightarrow{\bar{d}_1} \bar{x}$. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $g_n \cdot x_n \xrightarrow{d_1} x$. Since d_1 and d_2 are **Top**-equivalent, $g_n \cdot x_n \xrightarrow{d_2} x$. This implies that $\bar{x}_n \xrightarrow{\bar{d}_2} \bar{x}$.
   - Suppose that $\bar{x}_n \xrightarrow{\bar{d}_2} \bar{x}$. Then similarly to above, $\bar{x}_n \xrightarrow{\bar{d}_1} \bar{x}$.

□

Exercise 14.2.0.11. Let X be a set, $d_1, d_2 : X^2 \to [0, \infty)$ metrics on X, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that d_1 and d_2 are equivalent. If $\bar{d}_1$ and $\bar{d}_2$ are metrics, then $\bar{d}_1$ and $\bar{d}_2$ are equivalent.

Proof. Suppose that $\bar{d}_1$ and $\bar{d}_2$ are metrics. Since d_1 d_2 are equivalent, there exist $C_1, C_2 > 0$ such that for each $x, y \in X$, $C_1 d_1(x, y) \le d_2(x, y) \le C_2 d_1(x, y)$. Let $x, y \in X$. Then

$$\begin{aligned} C_1\bar{d}_1(\bar{x}, \bar{y}) &= C_1 \inf_{g\in G} d_1(g \cdot x, y) \\ &= \inf_{g\in G} C_1 d_1(g \cdot x, y) \\ &\le \inf_{g\in G} d_2(g \cdot x, y) \\ &= \bar{d}_2(\bar{x}, \bar{y}) \end{aligned}$$

and

$$\begin{aligned} \bar{d}_2(\bar{x}, \bar{y}) &= \inf_{g\in G} d_2(g \cdot x, y) \\ &\le \inf_{g\in G} C_2 d_1(g \cdot x, y) \\ &= C_2 \inf_{g\in G} d_1(g \cdot x, y) \\ &= C_2\bar{d}_1(\bar{x}, \bar{y}) \end{aligned}$$

So that $C_1\bar{d}_1 \le \bar{d}_2 \le C_2\bar{d}_1$ □

Exercise 14.2.0.12. Let (X, d) be a metric space, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Then $\pi : X \to X/G$ is a $(\mathcal{T}_d, \mathcal{T}_{\bar{d}})$-quotient map.

Proof.

- Clearly π is surjective.
- Let $C \subset X/G$. Suppose that C is closed. Since π is $(\mathcal{T}_d, \mathcal{T}_{\bar{d}})$-continuous, if $\pi^{-1}(C)$ is closed. Conversely, suppose that $\pi^{-1}(C)$ is closed. Let $(\bar{x}_n)_{n\in\mathbb{N}} \subset C$ and $\bar{x} \in X/G$. Suppose that $\bar{x}_n \xrightarrow{\bar{d}} \bar{x}$. Then there exists $(g_n)_{\alpha\in A} \subset G$ such that $g_n \cdot x_n \xrightarrow{d} x$. Since $(g_n \cdot x_n)_{n\in\mathbb{N}} \subset \pi^{-1}(C)$, $x \in \pi^{-1}(C)$. Hence $\bar{x} \in C$ and C is closed. Then Exercise 3.7.1.3 implies that π is a $(\mathcal{T}_d, \mathcal{T}_{\bar{d}})$-quotient map.

□

Exercise 14.2.0.13. Let (X, d) be a metric space, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Then $\pi : X \to X/G$ is $(\mathcal{T}_d, \mathcal{T}_{\bar{d}})$-open.

Proof. Let $U \subset X$. Suppose that U is open. Then

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

Since for each $g \in G$, ϕ_g is an isometry and thus a homeomorphism, we have that for each $g \in G$, $g \cdot U$ is open. Therefore

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g \cdot U$$

is open. Exercise 3.7.1.9 implies that π is open. $\square$

Exercise 14.2.0.14. Let (X, d) be a metric space, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Then $\bar{\pi} : X/G \to X/G$ is a $(\mathcal{T}_{X/G}, \mathcal{T}_{\bar{d}})$-homeomorphism.

Proof. The previous exercises imply that $\pi : X \to X/G$ is a $(\mathcal{T}_d, \mathcal{T}_{\bar{d}})$-quotient map and $(\mathcal{T}_d, \mathcal{T}_{\bar{d}})$-open. Since for each $a, b \in X$, $a \sim b$ iff $\pi(a) = \pi(b)$, Exercise **??** implies that $\bar{\pi} : X/G \to X/G$ is a $(\mathcal{T}_{X/G}, \mathcal{T}_{\bar{d}})$-homeomorphism. $\square$

Exercise 14.2.0.15. Let (X, d) be a metric space, G a group and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Then $\bar{d}$ metrizes the quotient topology $\pi_* \mathcal{T}_d$ on X/G.

Proof. Immediate by the previous exercise. $\square$

14.3 Fundamental Examples

Note 14.3.0.1. This section uses results from the previous two sections to establish metrics on some fundamental orbit spaces of metric spaces under a group action.

Exercise 14.3.0.2. Procrustes Distance:
Consider the metric space $(\mathbb{C}^{n\times d}, \|\cdot\|_F)$, topological group $(U_d, \|\cdot\|_F)$ and the (right) action $\phi : X \times U_d \to X$ by $X \cdot U = XU$. Then

1. ϕ is a continuous isometric group action
2. U_d is compact
3. $\bar{d}$ is a metric on $\mathbb{C}^{n\times d}/U_d$

Proof. Clear. □

Exercise 14.3.0.3. Let X be a compact metric space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Borel measure. Define the (right) group action $\phi : L^1(\mu) \times \mathrm{Aut}(X, \mu) \to L^1(\mu)$ by

$$f \cdot \sigma = f \circ \sigma$$

Then ϕ is an isometric group action.

Proof. Let $\sigma \in \mathrm{Aut}(X, \mu)$ and $f \in L^1(\mu)$. Then

$$\begin{aligned}
\|f \cdot \sigma\|_1 &= \int_X |f \circ \sigma| d\mu \\
&= \int_X |f| \circ \sigma d\mu \\
&= \int_{\sigma(X)} |f| d\sigma_* \mu \\
&= \int_{\sigma(X)} |f| d\mu \\
&= \int_X |f| d\mu \\
&= \|f\|_1
\end{aligned}$$

□

Exercise 14.3.0.4. Let X be a compact metric space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure. Define the (right) group action $\phi : L^1(\mu) \times \mathrm{Aut}(X, \mu) \to L^1(\mu)$ by

$$f \cdot \sigma = f \circ \sigma$$

Then for each $f \in L^1(\mu)$, the map $\sigma \mapsto f \cdot \sigma$ is continuous.

Proof. Let $f \in L^1(\mu)$, $(\sigma_n)_{n\in\mathbb{N}} \subset \mathrm{Aut}(X, \mu)$ and $\sigma \in \mathrm{Aut}(X, \mu)$. Suppose that $\sigma_n \xrightarrow{\mathrm{u}} \sigma$. Since μ is Radon, $C_c(X)$ is dense in $L^1(\mu)$ and therefore, there exists $\phi \in C_c(X)$ such that $\|\phi - f\| < \epsilon/3$. Since X is compact and μ is Radon, $\mu(X) < \infty$. Since ϕ is uniformly continuous, Exercise 4.11.0.14 implies that $\phi \circ \sigma_n \xrightarrow{\mathrm{u}} \phi \circ \sigma$. So there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $\|\phi \circ \sigma_n - \phi \circ \sigma\|_u < \frac{\epsilon}{3(\mu(X)+1)}$. Let $n \in \mathbb{N}$. Suppose that $n \geq \mathbb{N}$. Then

$$\begin{aligned}
\|f \circ \sigma_n - f \circ \sigma\|_1 &\leq \|f \circ \sigma_n - \phi \circ \sigma_n\|_1 + \|\phi \circ \sigma_n - \phi \circ \sigma\|_1 + \|\phi \circ \sigma - f \circ \sigma\|_1 \\
&= \|(f - \phi) \circ \sigma_n\|_1 + \|\phi \circ \sigma_n - \phi \circ \sigma\|_1 + \|(\phi - f) \circ \sigma\|_1 \\
&= \|f - \phi\|_1 + \|\phi \circ \sigma_n - \phi \circ \sigma\|_1 + \|\phi - f\|_1 \\
&= \|f - \phi\|_1 + \|\phi \circ \sigma_n - \phi \circ \sigma\|_u \mu(X) + \|\phi - f\|_1 \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

So that $f \circ \sigma_n \xrightarrow{\text{u}} f \circ \sigma$ which implies that the map $\sigma \mapsto f \cdot \sigma$ is continuous. □

Exercise 14.3.0.5. Cut Distance:
Let X be a compact metric space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure. Define the (right) group action $\phi : L^1(\mu) \times \operatorname{Aut}(X, \mu) \to L^1(\mu)$ by

$$f \cdot \sigma = f \circ \sigma$$

Then

1. ϕ is an isometric group action
2. $\operatorname{Aut}(X, d, \mu)$ is compact
3. for each $f \in L^1(\mu)$, the map $\sigma \mapsto f \cdot \sigma$ is continuous.
4. $\bar{d}$ is a metric on $L^1(\mu)/\operatorname{Aut}(X, d, \mu)$

Proof. Clear by the preceeding exercises. □

Note 14.3.0.6. The preceeding distance is not quite the Cut distance, as the Cut norm only considers a subset of measurable sets for a function of two variables, but with some work, maybe I can show it is a distance.

Appendix A

Summation

Definition A.0.0.1. Let $f : X \to [0, \infty)$, Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note A.0.0.2. Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.

Appendix B

Asymptotic Notation

Definition B.0.0.1. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = o(g) \qquad \text{as } x \to x_0$$

if for each $\epsilon > 0$, there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U$,

$$\|f(x)\| \leq \epsilon \|g(x)\|$$

Exercise B.0.0.2. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. If there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U \setminus \{x_0\}$, $g(x) > 0$, then

$$f = o(g) \text{ as } x \to x_0 \quad \text{iff} \quad \lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = 0$$

Exercise B.0.0.3. Let X and Y a be normed vector spaces, $A \subset X$ open and $f : A \to Y$. Suppose that $0 \in A$. If $f(h) = o(\|h\|)$ as $h \to 0$, then for each $h \in X$, $f(th) = o(|t|)$ as $t \to 0$.

Proof. Suppose that $f(h) = o(\|h\|)$ as $h \to 0$. Let $h \in X$ and $\epsilon > 0$. Choose $\delta' > 0$ such that for each $h' \in B(0, \delta')$, $h' \in A$ and

$$\|f(h')\| \leq \frac{\epsilon}{\|h\| + 1} \|h'\|$$

Choose $\delta > 0$ such that for each $t \in B(0, \delta)$, $th \in B(0, \delta')$. Let $t \in B(0, \delta)$. Then

$$\begin{aligned} \|f(th)\| &\leq \frac{\epsilon}{\|h\| + 1} |t| \|h\| \\ &< \epsilon |t| \end{aligned}$$

So $f(th) = o(|t|)$ as $t \to 0$. $\square$

Definition B.0.0.4. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = O(g) \qquad \text{as } x \to x_0$$

if there exists $U \in \mathcal{N}(x_0)$ and $M \geq 0$ such that for each $x \in U$,

$$\|f(x)\| \leq M \|g(x)\|$$

Appendix C

Vector Spaces

it might be better to cover some category theory and write everything in terms of $\text{Hom}_{\mathbf{Vect}_{\mathbb{K}}}$ and $\text{Obj}(\mathbf{Vect}_{\mathbb{K}})$

C.1 Introduction

Definition C.1.0.1. Let X be a set, $\mathbb{K}$ a field, $+ : X \times X \to X$ and $\cdot : \mathbb{K} \times X \to X$. Then $(X, +, \cdot)$ is said to be a $\mathbb{K}$**-vector space** if

1. $(X, +)$ is an abelian group
2.

Definition C.1.0.2. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$. Then $(E, +_E, \cdot_E)$ is said to be a subspace of X if

1. $+_E = +_X|_{E \times E}$
2. $\cdot_E = \cdot_X|_{\mathbb{K} \times E}$

Exercise C.1.0.3. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$.

Exercise C.1.0.4. Let $(X, +, \cdot)$ be a vector space and $E \subset X$. Then E is a subspace of X

Definition C.1.0.5. Let X be a vector space and $(E_j)_{j \in J}$ a collection of subspaces of X. Then $\bigcap\limits_{j \in J} E_j$ is a subspace of X.

Proof. Set $E := \bigcap\limits_{j \in J} E_j$. Let $x, y \in E$ and $\lambda \in \mathbb{K}$. Then for each $j \in J$, $x, y \in E_j$. Since for each $j \in J$, E_j is a subspace of X, we have that for each $j \in J$, $x + \lambda y \in E_j$. Thus $x + \lambda y \in E$. Since $x, y \in E$ and $\lambda \in \mathbb{K}$ are arbitrary, (cite exercise here) we have that E is a subspace of X. $\square$

Definition C.1.0.6. Let X, Y be vector spaces and $T : X \to Y$. Then T is said to be **linear** if for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

1. $T(x_1 + x_2) = T(x_1) + T(x_2)$,
2. $T(\lambda x_1) = \lambda T(x_1)$.

We define $L(X; Y) := \{T : X \to Y : \ T \text{ is linear}\}$.

Exercise C.1.0.7. Let X, Y be vector spaces and $T : X \to Y$. Then T is linear iff for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

$$T(x_1 + \lambda x_2) = T(x_1) + \lambda T(x_2)$$

Proof. Clear. (add details) $\square$

Definition C.1.0.8. define addition/scalar multiplication of linear maps

Exercise C.1.0.9. Let X, Y be vector spaces. Then $L(X;Y)$ is a $\mathbb{K}$-vector space.

Proof. Clear $\square$

Definition C.1.0.10. Let X be a vector space over $\mathbb{K}$ and $T : X \to \mathbb{K}$. Then T is said to be a **linear functional on** X if T is linear. We define the **dual space of** X, denoted X^*, by $X^* := \{T : X \to \mathbb{K} : T \text{ is linear}\}$.

Exercise C.1.0.11. Let X be a vector space. Then X^* is a vector space.

Proof. Clear. $\square$

C.2 Bases

Definition C.2.0.1. Let X be a vector space and $(e_\alpha)_{\alpha \in A} \subset X$. Then $(e_\alpha)_{\alpha \in A}$ is said to be

- **linearly independent** if for each $(\alpha_j)_{j=1}^n \subset A$, $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $\sum_{j=1}^n \lambda_j e_{\alpha_j} = 0$ implies that for each $j \in [n]$, $\lambda_j = 0$.
- a **Hamel basis for** X if $(e_\alpha)_{\alpha \in A}$ is linearly independent and $\text{span}(e_\alpha)_{\alpha \in A} = X$.

Exercise C.2.0.2. every vector space has a Hamel basis

Proof. $\square$

Exercise C.2.0.3.

Exercise C.2.0.4. Let X be a $\mathbb{K}$-vector space and $x \in X$. Then $x = 0$ iff for each $\phi \in X^*$, $\phi(x) = 0$.

Proof.

- $(\implies)$:
  Suppose that $x = 0$. Linearity implies that for each $\phi \in X^*$ $\phi(x) = 0$.
- $(\impliedby)$:
  Conversely, suppose that $x \neq 0$. Define $\epsilon_x : \text{span}(x) \to \mathbb{K}$ by $\epsilon_x(\lambda x) := \lambda$. Let $u, v \in \text{span}(x)$. Then there exists $\lambda_u, \lambda_v \in \mathbb{K}$ such that $u = \lambda_u x$ and $v = \lambda_v x$. Suppose that $u = v$. Then

$$\begin{aligned}(\lambda_u - \lambda_v)x &= \lambda_u x - \lambda_v x \\ &= u - v \\ &= 0\end{aligned}$$

  Since $x \neq 0$, we have that $\lambda_u - \lambda_v = 0$ and therefore $\lambda_u = \lambda_v$. Hence

$$\begin{aligned}\lambda_u &= \epsilon_x(u) \\ &= \epsilon_x(v) \\ &= \lambda_v.\end{aligned}$$

  Thus ϵ_x is well defined.

$\square$

C.3 Multilinear Maps

Definition C.3.0.1. Let $X_1, \cdots, X_n, Y$ be vector spaces and $T : \prod_{j=1}^{n} X_j \to \mathbb{K}$. Then T is said to be **multilinear** if for each $j_0 \in [n]$ and $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$, $T(x_1, \ldots, x_{j_0-1}, \cdot, x_{j_0+1})$ is linear.

$$L^n(X_1, \ldots, X_n; Y) = \left\{T : \prod_{j=1}^{n} X_j \to Y : T \text{ is multilinear}\right\}$$

If $X_1 = \cdots = X_n = X$, we write $L^n(X; Y)$ in place of $L^n(X, \ldots, X; Y)$.

Definition C.3.0.2. define addition and scalar mult of multilinear maps

Exercise C.3.0.3. Let $X_1, \cdots, X_n, Y$ be vector spaces. Then $L^n(X_1, \ldots, X_n; Y)$ is a $\mathbb{K}$-vector space.

Proof. content... $\square$

Exercise C.3.0.4. Let $X_1, \cdots, X_n, Y, Z$ be $\mathbb{K}$-vector spaces, $\alpha \in L^n(X_1, \ldots, X_n; Y)$ and $\phi \in L^1(Y; Z)$. Then $\phi \circ \alpha \in L^n(X_1, \ldots, X_n; Z)$.

Proof. Let $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$. Define $f : X_{j_0} \to Y$ by

$$f(a) := \alpha(x_1, \ldots, x_{j_0-1}, a, x_{j_0+1}, \ldots, x_n)$$

Since $\alpha \in L^n(X_1, \ldots, X_n; Y)$, f is linear. Since ϕ is linear, and $\phi \circ f$ is linear. Since $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$ are arbitrary, we have that $\phi \circ \alpha \in L^n(X_1, \ldots, X_n; Y)$. $\square$

C.4 Tensor Products

Definition C.4.0.1. Let X, Y and T be vector spaces over $\mathbb{K}$ and $\alpha \in L^2(X, Y; T)$. Then (T, α) is said to be a **tensor product of** X **and** Y if for each vector space Z and $\beta \in L^2(X, Y; Z)$, there exists a unique $\phi \in L^1(T; Z)$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & T \arrow[d, dashed, "\phi"] \\ & Z \end{tikzcd}$$

Exercise C.4.0.2. Let X, Y, S, T be vector spaces, $\alpha \in L^2(X, Y; S)$ and $\beta \in L^2(X, Y; T)$. Suppose that (S, α) and (T, β) are tensor products of X and Y. Then S and T are isomorphic.

Proof. Since (T, β) is a tensor product of X and Y, $\beta \in L^2(X, Y; T)$ there exists a unique $f \in L^1(T; T)$ such that $f\ circ\beta = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dd, dashed, "f"] \\ X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T \end{tikzcd}$$

Since $\mathrm{id}_T \in L^1(T; T)$ and $\mathrm{id}_T \circ \beta = \beta$, we have that $f = \mathrm{id}_T$. Since (S, α) is a tensor product of X and Y, there exists a unique $\phi : S \to T$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] \\ & T \end{tikzcd}$$

Similarly, since (T, β) is a tensor product of X and Y, there exists a unique $\psi : T \to S$ such that $\psi \circ \beta = \alpha$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & T \arrow[d, dashed, "\psi"] \\ & S \end{tikzcd}$$

Therefore

$$\begin{aligned} (\phi \circ \psi) \circ \beta &= \phi \circ (\psi \circ \beta) \\ &= \phi \circ \alpha \\ &= \beta, \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, dashed, "\psi"] & & & T \arrow[dd, dashed, "\phi\circ\psi"] \\ X \times Y \arrow[ur, "\beta"] \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] & \Longrightarrow & X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T & & & T \end{tikzcd}$$

By uniqueness of $f \in L^1(T; T)$, we have that

$$\begin{aligned} \mathrm{id}_T &= f \\ &= \phi \circ \psi \end{aligned}$$

A similar argument implies that $\psi \circ \phi = \mathrm{id}_S$. Hence ϕ and ψ are isomorphisms with $\phi^{-1} = \psi$. Hence S and T are isomorphic. $\square$

Definition C.4.0.3. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. We define $x \otimes y : X^* \times Y^* \to \mathbb{K}$ by $x \otimes y(\phi, \psi) := \phi(x)\psi(y)$.

Exercise C.4.0.4. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. Then $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$.

Proof. Let $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} x \otimes y(\phi_1 + \lambda\phi_2, \psi) &= [\phi_1 + \lambda\phi_2(x)]\psi(y) \\ &= \phi_1(x)\psi(y) + \lambda\phi_2(x)\psi(y) \\ &= x \otimes y(\phi_1, \psi) + \lambda x \otimes y(\phi_2, \psi) \end{aligned}$$

Since $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$ are arbitrary, we have that for each $\psi \in Y^*$, $x \otimes y(\cdot, \psi)$ is linear. Similarly for each $\phi \in X^*$, $x \otimes y(\phi, \cdot)$ is linear. Hence $x \otimes y$ is bilinear and $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$. $\square$

Definition C.4.0.5. Let X, Y be vector spaces. We define

- the **tensor product of** X **and** Y, denoted $X \otimes Y \subset L^2(X^*, Y^*; \mathbb{K})$, by

$$X \otimes Y := \mathrm{span}(x \otimes y : x \in X \text{ and } y \in Y),$$

- the **tensor map**, denoted $\otimes : X \times Y \to X \otimes Y$, by $\otimes(x, y) := x \otimes y$.

Exercise C.4.0.6. Let X, Y be vector spaces, $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$. The following are equivalent:

1. $\sum_{j=1}^n x_j \otimes y_j = 0$
2. for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum_{j=1}^n \phi(x_j)\psi(y_j) = 0$
3. for each $\phi \in X^*$, $\sum_{j=1}^n \phi(x_j) y_j = 0$
4. for each $\psi \in Y^*$, $\sum_{j=1}^n \psi(y_j) x_j = 0$

Proof.

1. (1) $\implies$ (2) :
   Suppose that $\sum_{j=1}^n x_j \otimes y_j = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Then

$$\begin{aligned} \sum_{j=1}^n \phi(x_j)\psi(y_j) &= \phi\bigg(\sum_{j=1}^n \psi(y_j)x_j\bigg) \\ &= \end{aligned}$$

2.
3.

$\square$

Exercise C.4.0.7. Let X, Y be vector spaces. Then $(X \otimes Y, \otimes)$ is a tensor product of X and Y.

Proof. Let Z be a vector space and $\alpha \in L^2(X, Y; Z)$. Define $\phi : X \otimes Y \to Z$ by $\phi\Big(\sum_{j=1}^{n} \lambda_j x_j \otimes y_j) := \sum_{j=1}^{n} \lambda_j \alpha(x_j, y_j)$.

- **(well defined):**
  Let $u \in X \otimes Y$. Then there exist $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $(x_j)_{j=1}^n \subset X$, $(y_j)_{j=1}^n \subset Y$ such that $u = \sum_{j=1}^{n} \lambda_j x_j \otimes y_j$. Suppose that $u = 0$. Let $\phi \in Z^*$. Then $\phi \circ \alpha \in L^2(X, Y; Z)$.

$\square$

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