CALCULUS ON MANIFOLDS NOTES

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1. Review of Multivariable Calculus and Linear Algebra

1.1. **The derivative.**

2. Multilinear Algebra

Note 2.0.1. *For the remainder of this section we let V denote an n-dimensional vector space with basis $\{e_1, \cdots, e_n\}$ with dual space V^* and dual basis $\{\epsilon_1, \cdots, \epsilon_n\}$ defined by $\epsilon_i(e_j) = \delta_{i,j}$.*

2.1. k-Forms.

Definition 2.1.1. *Let $\alpha : V^k \to \mathbb{R}$. Then α is said to be **multilinear** or a **k-form on V** if for $i \in \{1, \cdots, k\}$, $w \in V$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$,*

$$\alpha(v_1, \cdots, v_i + cw, \cdots, v_k) = \alpha(v_1, \cdots, v_i, \cdots, v_k) + c\alpha(v_1, \cdots, w, \cdots, v_k)$$

The set of all k-forms on V is denoted by $T_k(V)$. Define $L_0(V) = \mathbb{R}$.

Exercise 2.1.2. *We have that $T_k(V)$ is a vector space.*

Proof. Clear. □

Definition 2.1.3. *For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \to \mathbb{R}$ by*

$$\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

*The map $\alpha \mapsto \sigma\alpha$ is called the **permutation action** of S_k on $T_k(V)$*

Exercise 2.1.4. *The permutation action of S_k on $T_k(V)$ is a group action.*

Proof.

(1) Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V), \sigma\alpha \in T_k(V)$.
(2) Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
(3) Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \cdots, v_k \in V$,

$$\begin{aligned}(\tau\sigma)\alpha(v_1, \cdots, v_k) &= \alpha(v_{\tau\sigma(1)}, \cdots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \cdots, v_k)\end{aligned}$$

□

Exercise 2.1.5. *Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \to T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.*

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$. Then

$$\begin{aligned}\sigma(c\alpha + \beta)(v_1, \cdots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \cdots, v_k) + \sigma\beta(v_1, \cdots, v_k)\end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

Definition 2.1.6. *Let α be a k-form on V. Then α is said to be **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$. and α is said to be **alternating** if for each $\sigma \in S_k$, $\sigma\alpha = \operatorname{sgn}(\sigma)\alpha$. The set of symmetric k-forms on V is denoted $\Gamma_k(V)$ and the set of alternating k-forms on V is denoted $\Lambda_k(V)$.*

Definition 2.1.7. *Define the **symmetric operator** $S : T_k(V) \to \Gamma_k(V)$ by*

$$S(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the ***alternating operator*** $A : T_k(V) \to \Lambda_k(V)$ *by*

$$A(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha$$

Exercise 2.1.8.

(1) *For* $\alpha \in T_k(V)$, $S(\alpha)$ *is symmetric.*
(2) *For* $\alpha \in T_k(V)$, $A(\alpha)$ *is alternating.*

Proof.

(1) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
\sigma S(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \tau\alpha\right] \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \sigma\tau\alpha \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \tau\alpha \\
&= S(\alpha)
\end{aligned}$$

(2) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
\sigma A(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right] \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\sigma\tau\alpha \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \\
&= \operatorname{sgn}(\sigma)A(\alpha)
\end{aligned}$$

□

Exercise 2.1.9.

(1) *For* $\alpha \in \Gamma_k(V)$, $S(\alpha) = \alpha$.
(2) *For* $\alpha \in \Lambda_k(V)$, $A(\alpha) = \alpha$.

Proof.

(1) Let $\alpha \in \Gamma_k(V)$. Then

$$\begin{aligned} S(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \alpha \\ &= \alpha \end{aligned}$$

(2) Let $\alpha \in \Lambda_k(V)$. Then

$$\begin{aligned} A(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2\alpha \\ &= \alpha \end{aligned}$$

□

Exercise 2.1.10. *The symmetric operator $S : T_k(V) \to \Gamma_k(V)$ and the alternating operator $A : T_k(V) \to \Lambda_k(V)$ are linear.*

Proof. Clear. □

Definition 2.1.11. *Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. The* ***tensor product*** *of α and β is defined to be the map $\alpha \otimes \beta \in T_{k+l}(V)$ given by*

$$\alpha \otimes \beta(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = \alpha(v_1, \cdots, v_k)\beta(v_{k+1}, \cdots, v_{k+l})$$

Thus $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$.

Exercise 2.1.12. *The tensor product $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$ is associative.*

Proof. Clear. □

Exercise 2.1.13. *The tensor product $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$ is bilinear.*

Proof. Clear. □

Definition 2.1.14. *Let $\alpha \in \Lambda_k(V)$ and $\beta \in \Lambda_l(V)$. The* ***exterior product*** *of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda_{k+l}(V)$ given by*

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to \Lambda_{k+l}(V)$.

Exercise 2.1.15. *Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then*

(1) $A(A(\alpha) \otimes \beta) = A(\alpha \otimes \beta)$
(2) $A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \to S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

(1) Then

$$\begin{aligned}
A(A(\alpha)\otimes\beta) &= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\sigma\left[A(\alpha)\otimes\beta\right]\\
&= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\sigma\left[\left(\frac{1}{k!}\sum_{\tau\in S_k}\operatorname{sgn}(\tau)\tau\alpha\right)\otimes\beta\right]\\
&= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\sigma\left[\frac{1}{k!}\sum_{\tau\in S_k}\operatorname{sgn}(\tau)(\tau\alpha)\otimes\beta\right]\\
&= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\sigma\left[\frac{1}{k!}\sum_{\tau\in S_k}\operatorname{sgn}(\tau)\tau(\alpha\otimes\beta)\right]\\
&= \frac{1}{k!(k+l)!}\sum_{\sigma\in S_{k+l}}\sum_{\tau\in S_k}\operatorname{sgn}(\sigma\tau)\sigma\tau(\alpha\otimes\beta)\\
&= \frac{k!}{k!(k+l)!}\sum_{\mu\in S_{k+l}}\operatorname{sgn}(\mu)\mu(\alpha\otimes\beta)\\
&= \frac{1}{(k+l)!}\sum_{\mu\in S_{k+l}}\operatorname{sgn}(\mu)\mu(\alpha\otimes\beta)\\
&= A(\alpha\otimes\beta)
\end{aligned}$$

(2) Similar to (1).

$\square$

Exercise 2.1.16. *The exterior product* $\wedge : \Lambda_k(V)\times\Lambda_l(V)\to\Lambda_{k+l}(V)$ *is associative.*

Proof. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$ and $\gamma \in \Lambda_m(V)$. Then

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)\right] \wedge \gamma \\
&= \frac{(k+l+m)!}{(k+l)!m!} A\left(\left[\frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)\right] \otimes \gamma\right) \\
&= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} A(A(\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} A((\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes (\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes A(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes \frac{(l+m)!}{l!m!} A(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes (\beta \wedge \gamma)) \\
&= \alpha \wedge (\beta \wedge \gamma))
\end{aligned}$$

□

Exercise 2.1.17. *Let $\alpha_i \in \Lambda_{k_i}(V)$ for $i = 1, \cdots, m$. Then*

$$\bigwedge_{i=1}^{m} \alpha_i = \frac{(\sum_{i=1}^{m} k_i)!}{\prod_{i=1}^{m} k_i!} A\left(\bigotimes_{i=1}^{m} \alpha_i\right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} A(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i\right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A\left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A\left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} A\left(\bigotimes_{i=1}^{m_0-1} \alpha_i\right)\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(A\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(\bigotimes_{i=1}^{m_0+1} \alpha_i\right)
\end{aligned}$$

□

Exercise 2.1.18. *Define $\tau \in S_{k+l}$ by*

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl. (Hint: inversion number)

Proof.

$$\begin{aligned}
N(\tau) &= \sum_{i=1}^{l} k \\
&= kl
\end{aligned}$$

Since $\operatorname{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\operatorname{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.1.19. *Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$. Then*

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Note that For $\sigma \in S_{k+l}$ and $v_1, \cdots, v_{k+l} \in V$, we have that

$$\begin{aligned}
\sigma\tau(\beta \otimes \alpha)(v_1, \cdots, v_l, v_{l+1}, \cdots v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)})\alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \\
&= \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)})\beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\
&= \alpha \otimes \beta(v_{\sigma(1)}, \cdots v_{\sigma(k)}, v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\
&= \sigma(\alpha \otimes \beta)(v_1, \cdots, v_k, v_{1+k}, \cdots v_{l+k})
\end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Define τ as in the previous exercise. Then

$$\begin{aligned}
\beta \wedge \alpha &= \frac{(k+l)!}{k!l!} A(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma\tau)\sigma\tau(\beta \otimes \alpha) \\
&= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha \otimes \beta) \\
&= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \\
&= \operatorname{sgn}(\tau) \alpha \wedge \beta \\
&= (-1)^{kl} \alpha \wedge \beta
\end{aligned}$$

□

Exercise 2.1.20. *Let $\alpha_{\in}\Lambda_k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.*

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned}
\alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\
&= -\alpha \wedge \alpha
\end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.1.21. ***(Fundamental Example)*** *Let $\alpha_1, \cdots, \alpha_m \in \Lambda_1(V)$ and $v_1, \cdots, v_m \in V$. Then*

$$\left(\bigwedge_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}
\left(\bigwedge_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) &= m! A\left(\bigotimes_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) \\
&= m! \left[\frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\sigma\left(\bigotimes_{i=1}^m \alpha_i \right) \right](v_1, \cdots, v_m) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \left(\bigotimes_{i=1}^m \alpha_i \right)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
&= \det(\alpha_i(v_j))
\end{aligned}$$

□

Definition 2.1.22. *Define $\mathcal{I}_k = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called a **multi-index**. Recall that $\#\mathcal{I}_k = \binom{n}{k}$.*

Definition 2.1.23. *Let $I = \{(i_1, i_2, \cdots, i_k) \in I_k$.*
Define $e_I \in V^k$ by

$$e_I = (e_{i_1}, \cdots, e_{i_k})$$

Define $\epsilon_I \in \Lambda_k(V)$ by

$$\epsilon_I = \epsilon_{i_1} \wedge \cdots, \wedge \epsilon_{i_k}$$

Definition 2.1.24. *Define the **permutation action** of S_k on $\mathcal{I}_k$ by $\sigma I = (i_{\sigma(1)}, \cdots, i_{\sigma(k)})$.*

Exercise 2.1.25. *The permutation action of S_k on $\mathcal{I}_k$ is a group action.*

Proof. Clear. □

Exercise 2.1.26. *Let $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k) \in \mathcal{I}_k$. Then $\epsilon_I(e_J) = \delta_{I,J}$.*

Proof. Put $A = \begin{pmatrix} \epsilon_{i_1}(e_{j_1}) & \cdots & \epsilon_{i_1}(e_{j_k}) \\ & \vdots & \\ \epsilon_{i_k}(e_{j_1}) & \cdots & \epsilon_{i_k}(e_{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon_I(e_J) = \det A$. If $I = J$, then $A = I_{k\times k}$ and therefore $\epsilon_I(e_J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the $l_0 th$ row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the $l_0 th$ column of A are 0. □

Exercise 2.1.27. *Let $\alpha, \beta \in \Lambda_k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e_I) = \beta(e_I)$, then $\alpha = \beta$.*

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e_I) = \beta(e_I)$. Let $v_1, \cdots, v_k \in V$. For $i = 1, \cdots, k$, write $v_i = \sum_{j_i=1}^n a_{i,j_i} e_{j_i}$. Then

$$\begin{aligned}
\alpha(v_1, \cdots, v_k) &= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^k a_{i,j_i}\right) \alpha(e_{j_1}, \cdots, e_{j_k}) \\
&= \sum_{j_1\neq\cdots\neq j_k}^{n} \left(\prod_{i=1}^k a_{i,j_i}\right) \alpha(e_{j_1}, \cdots, e_{j_k}) \\
&= \sum_{J\in\mathcal{I}_k} \left[\sum_{\sigma\in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^k a_{i,\sigma(j_i)}\right)\right] \alpha(e_J) \\
&= \sum_{J\in\mathcal{I}_k} \left[\sum_{\sigma\in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^k a_{i,\sigma(j_i)}\right)\right] \beta(e_J) \\
&= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^k a_{i,j_i}\right) \beta(e_{j_1}, \cdots, e_{j_k}) \\
&= \beta(v_1, \cdots, v_k)
\end{aligned}$$

□

Exercise 2.1.28. *The set $\{\epsilon_I : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(V)$ and $\dim \Lambda_k(V) = \binom{n}{k}$.*

Proof. Let $(a_I)_{I\in\mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I\in\mathcal{I}_k} a_I\epsilon_I$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e_J) = a_J = 0$. Thus $\{e_I : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda_k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e_I)$. define $\mu = \sum_{I\in\mathcal{I}_k} b_I\epsilon_I$. Then for each $J \in \mathcal{I}_k$, $\mu(e_J) = b_J = \beta(e_J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{\epsilon_I : I \in \mathcal{I}_k\}$. $\square$

2.2. (r, s)**-Tensors.**

3. Calculus on Manifolds

3.1. Submanifolds of $\mathbb{R}^n$.

Definition 3.1.1. *Let $\Omega \subset \mathbb{R}^n$ and $U \subset M$. Then U is said to be **open** if there exists $U' \subset \mathbb{R}^n$ such that U' is open in $\mathbb{R}^n$ and $U = M \cap U'$.*

Definition 3.1.2. *Define the **upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}_n$, by*

$$\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\partial\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$$

Definition 3.1.3. *Let $\Omega \subset \mathbb{R}^n, U \subset \Omega$, $V \subset \mathbb{H}_k$ and $\phi : U \to V$. Then ϕ is said to be a **coordinate chart** from Ω to $\mathbb{H}_k$ if U is open in Ω, V is open in $\mathbb{H}_k$ and ϕ is a homeomorphism (that is, ϕ is a bijection, continuous and ϕ^{-1} is continuous). We will typically denote a chart from Ω to $\mathbb{H}_k$ by the pair (ϕ, U). Let $\mathcal{A} = \{(\phi_\alpha, U_\alpha) : \alpha \in A\}$ be a set of coordinate charts from Ω to $\mathbb{H}_k$ indexed by A. Then $\mathcal{A}$ is said to be a **smooth** k**-atlas** on Ω if*

(1) $\Omega = \bigcup\limits_{\alpha \in A} U_\alpha$

(2) *for each $\alpha, \beta \in A$, $U_\alpha \cap U_\beta \neq \varnothing$ implies that*

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_\alpha \cap U_\beta) \to \phi_2(U_\alpha \cap U_\beta)$$

is smooth

*If $\mathcal{A}$ is a smooth k-atlas on Ω, and $(\phi, U) \in \mathcal{A}$, then ϕ is said to be a **smooth coordinate chart** from Ω to $\mathbb{H}_k$.*

Definition 3.1.4. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A}$ a smooth k-atlas on Ω. Then $(\Omega, \mathcal{A})$ is said to be a k-dimensional smooth submanifold of $\mathbb{R}^n$. Define the **boundary** of Ω, denoted $\partial\Omega$, by*

$$\partial\Omega = \bigcup_{\substack{\phi \in \mathcal{A} \\ \phi : U \to V}} \phi^{-1}(V \cap \partial\mathbb{H}_k)$$

Exercise 3.1.5. *Let Ω be a k-dimensional smooth submanifold of $\mathbb{R}^n$. Then $\partial\Omega$ is a $k-1$-dimensional smooth manifold of $\mathbb{R}^n$.*

Proof. Straightforward. $\square$

Definition 3.1.6. *Let Ω be a smooth k-dimensional smooth submanifold of $\mathbb{R}^n$, $U \subset \Omega$ open in Ω, $V \subset \mathbb{H}_k$ open in $\mathbb{H}_k$ and $\phi : U \to V$ a smooth coordinate chart on Ω. Then $\phi^{-1} : V \to U$ is called a **local smooth parametrization** of Ω.*

3.2. Differential Forms.

Note 3.2.1. *The definitions in this section will introduce a very slick book-keeping device for doing calculus on manifolds. Since we are not developing the theory from the ground up, it may feel abstract. Hopefully the many exercises facilitate becoming accostumed to this book-keeping tool.*

Definition 3.2.2. *When working in $\mathbb{R}^n$, we introduce the formal objects $dx_1, dx_2, \cdots, dx_n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.*

Definition 3.2.3. *Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be*

$$\Gamma^k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \operatorname{span}\{dx_I : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Gamma^k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their ***exterior product****, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Gamma^k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:*

1. *the exterior product is bilinear*
2. *for each $\omega \in \Gamma^k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$*
3. *for each $\omega \in \Gamma^k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$*
4. *for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Gamma^k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$*

We call $\Gamma^k(\mathbb{R}^n)$ the differential k-forms on $\mathbb{R}^n$. Let ω be a k-form on $\mathbb{R}^n$. If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx_I$

Note 3.2.4. *The terms $dx_1, dx_2, \cdots, dx_n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du_1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx_1, dx_2, \cdots, dx_n\}$ when referencing the coordinates on $\mathbb{R}^n$.*

Exercise 3.2.5. *Let $B = (b_{i,j})$ be an $n \times n$ matrix. Then*

$$\bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx_j \right) = (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

Proof. First we have

$$(*) \qquad \bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx_j \right) = \left(\sum_{j=1}^n b_{1,j} dx_j \right) \wedge \left(\sum_{j=1}^n b_{2,j} dx_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n b_{n,j} dx_j \right)$$

The expression on the right side of $(*)$ is just the sum all terms of the form

$$b_{1,j_1} b_{2,j_2} \cdots b_{n,j_n} dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n}$$

where $j_k \in \{1, 2, \cdots, n\}$. The terms in which for some j, dx_j appears more than once in the exterior product are zero. Thus the expression on the right of $(*)$ is just the sum of all terms of the form

$$b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{n,\sigma(n)} dx_{\sigma(1)} \wedge dx_{\sigma(2)} \wedge \cdots \wedge dx_{\sigma(n)} = \operatorname{sgn}(\sigma) b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{n,\sigma(n)} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where $\sigma \in S_n$. Explicitely writing this out, we see that

$$\begin{aligned}\bigwedge_{i=1}^{n}\left(\sum_{j=1}^{n} b_{i,j}dx_j\right) &= \left(\sum_{j=1}^{n} b_{1,j}dx_j\right)\wedge\left(\sum_{j=1}^{n} b_{2,j}dx_j\right)\wedge\cdots\wedge\left(\sum_{j=1}^{n} b_{n,j}dx_j\right)\\ &= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)b_{1,\sigma(1)}b_{2,\sigma(2)}\cdots b_{n,\sigma(n)}dx_1\wedge dx_2\wedge\cdots\wedge dx_n\\ &= \left(\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)b_{1,\sigma(1)}b_{2,\sigma(2)}\cdots b_{n,\sigma(n)}\right)dx_1\wedge dx_2\wedge\cdots\wedge dx_n\\ &= (\det B)dx_1\wedge dx_2\wedge\cdots\wedge dx_n\end{aligned}$$

□

Definition 3.2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted df, on $\mathbb{R}^n$ by*

$$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i$$

Let $\omega = \sum_{I\in\mathcal{I}_{k,n}} f_I dx_I$ be a k-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega = \sum_{I\in\mathcal{I}_{k,n}} df_I\wedge dx_I$$

Exercise 3.2.7. *On $\mathbb{R}^3$, put*

(1) $\omega_0 = f_0$,
(2) $\omega_1 = f_1dx_1 + f2dx_2 + f_2dx_3$,
(3) $\omega_2 = f_1dx_2\wedge dx_3 - f_2dx_1\wedge dx_3 + f_3dx_1\wedge dx_2$

Show that

(1) $d\omega_0 = \frac{\partial f_0}{\partial x_1}dx_1 + \frac{\partial f_0}{\partial x_2}dx_2 + \frac{\partial f_0}{\partial x_3}dx_3$
(2) $d\omega_1 = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)dx_2\wedge dx_3 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right)dx_1\wedge dx_3 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)dx_1\wedge dx_2$
(3) $d\omega_2 = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right)dx_1\wedge dx_2\wedge dx_3$

Proof. Straightforward. □

Exercise 3.2.8. *Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx_I\wedge dx_{I_*} = dx_1\wedge dx_2\wedge\cdots\wedge dx_n$.*

Definition 3.2.9. *We define a linear map $* : \Gamma^k(\mathbb{R}^n) \to \Gamma^{n-k}(\mathbb{R}^n)$ called the* ***Hodge $*$-operator*** *by*

$$*\sum_{I\in\mathcal{I}_{k,n}} f_I dx_I = \sum_{I\in\mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 3.2.10. *Let $\phi : \mathbb{R}^k \to \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We define $\phi^* : \Gamma^k(\mathbb{R}^n) \to \Gamma^k(\mathbb{R}^k)$ via the following properties:*

(1) *for each 0-form f on $\mathbb{R}^n$, $\phi^* f = f\circ\phi$*
(2) *for $i = 1, \cdots, n$, $\phi^* dx_i = d\phi_i$*

(3) *for an s-form* ω*, and a t-form* χ *on* $\mathbb{R}^n$*,* $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
(4) *for l-forms* ω, χ *on* $\mathbb{R}^n$*,* $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 3.2.11. *Let* $\Omega \subset \mathbb{R}^n$ *be a k-dimensional smooth submanifold of* $\mathbb{R}^n$*,* $\phi : U \to V$ *a smooth parametrization of* Ω*,* $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$ *an k-form on* $\mathbb{R}^n$*. Then*

$$\phi^*\omega = \bigg(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det D\phi_I)\bigg) du_1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof. Using the definitions, we see that

$$\begin{aligned} \phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I)\phi^* dx_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned} d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\ &= (\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u_j} du_j) \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u_j} du_j) \wedge \cdots \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u_j} du_j) \\ &= (\det D\phi_I) du_1 \wedge du_2 \wedge \cdots \wedge du_k \end{aligned}$$

Therefore

$$\begin{aligned} \phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det D\phi_I) du_1 \wedge du_2 \wedge \cdots \wedge du_k \\ &= \bigg(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det D\phi_I)\bigg) du_1 \wedge du_2 \wedge \cdots \wedge du_k \end{aligned}$$

□

3.3. **Integration of Differential Forms.**

Definition 3.3.1. *Let* $U \subset \mathbb{R}^k$ *be open and* $\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$ *a k-form on* $\mathbb{R}^k$*. Define*

$$\int_U \omega = \int_U f dx$$

Definition 3.3.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a k-dimensional oriented smooth submanifold of* $\mathbb{R}^n$*,* ω *a k-form on* $\mathbb{R}^n$ *and* $\phi : U \to V$ *a local smooth, orientation-preserving parametrization of* Ω*. Define*

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 3.3.3.

Theorem 3.3.4. ***(Stokes)*** *Let $\Omega \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$ and ω a $k-1$-form on $\mathbb{R}^n$. Then*

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$