1

Introduction to Group Theory

Carson James

1

Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

0.1 Category Theory

- **Hilb**:
  - $\mathrm{Obj}(\mathbf{Hilb}) = \{H : H \text{ is a Hilbert space}\}$
  - $\mathrm{Hom}_{\mathbf{Hilb}}(H_1, H_2) = \{T \in \mathbf{Vect}_{\mathbb{C}}(H_1, H_2) : T \text{ is continuous}\}$
- **Mon**

Chapter 1

Representation Theory

1.1 Representations of Groups

1.1.1 The Unitary Group

Definition 1.1.1.1. Let $H_1, H_2 \in \mathrm{Obj}(\mathbf{Hilb})$. We define the **unitary group from** H_1 **to** H_2, denoted $U(H_1, H_2)$, by

$$U(H_1, H_2) = \{T \in \mathrm{Iso}_{\mathbf{Hilb}}(H_1, H_2) : T^* = T^{-1}\}$$

We write $U(H)$ in place of $U(H, H)$. We equip $U(H_1, H_2)$ with the strong operator topology.

Exercise 1.1.1.2. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$. Then $\mathcal{T}^s_{U(H)} = \mathcal{T}^w_{U(H)}$. strong weak operator topologies coincide

Exercise 1.1.1.3. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$. Then $U(H)$ is a topological group.

Proof. content... □

1.1.2 Unitary representations

Definition 1.1.2.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $H \in \mathrm{Obj}(\mathbf{Hilb})$ and $\pi \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(H))$. Then (H, π) is said to be a **unitary representation of** G. We define the **dimension of** (H, π), denoted $\dim(H, \pi)$, by $\dim(H, \pi) := \dim V$.

Definition 1.1.2.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, (H_π, π), (H_ρ, ρ) unitary representations of G and $T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho)$. Then T is said to be (π, ρ)**-equivariant** if for each $g \in G$, $T \circ \pi(g) = \rho(g) \circ T$, i.e. the following diagram commutes:

$$\begin{tikzcd}
H_\pi \arrow[r, "T"] \arrow[d, "\pi(g)"'] & H_\rho \arrow[d, "\rho(g)"] \\
H_\pi \arrow[r, "T"'] & H_\rho
\end{tikzcd}$$

Definition 1.1.2.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. We define $\mathbf{URep}(G)$ by

- $\mathrm{Obj}(\mathbf{URep}(G)) = \{(H, \pi) : (H, \pi) \text{ is a unitary representation of } G\}$.
- for $(H_\pi, \pi), (H_\rho, \rho) \in \mathrm{Obj}(\mathbf{URep}(G))$,

$$\mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) = \{T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho) : T \text{ is } (\pi, \rho)\text{-equivariant}\}$$

- for $(H_\pi, \pi), (H_\rho, \rho), (H_\mu, \mu) \in \mathrm{Obj}(\mathbf{URep}(G))$, $T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$ and $S \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\rho, \rho), (H_\mu, \mu))$,

$$S \circ_{\mathbf{URep}(G)} T = S \circ T$$

Exercise 1.1.2.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. Then $\mathbf{URep}(G)$ is a category.

Proof. FINISH!!! $\square$

Definition 1.1.2.5. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a subspace. Then E is said to be

- **nontrivial** if $E \neq H, \varnothing$
- (H, π)**-invariant** if for each $g \in G$, $\pi(g)(E) \subset E$

Definition 1.1.2.6. Let $G \in \mathrm{Obj}(\mathbf{Grp})$ and $\mathbb{K} \in \mathrm{Obj}(\mathbf{Field})$ and $(V, \pi) \in \mathrm{Obj}(\mathbf{Rep}(G, \mathbb{K}))$. Then (V, π) is said to be **irriducible** if for each subspace $E \subset V$, E is trivial or E is not (V, π)-invariant.

Definition 1.1.2.7. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. Let $(H_\pi, \pi), (H_\rho, \rho) \in \mathbf{URep}(G)$. Then (H_π, π) and (H_ρ, ρ) are said to be **unitarily equivalent** if there exists $\mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) \cap U(H_{\pi, H_\rho}) \neq \varnothing$.

1.2 Tannaka Duality

Definition 1.2.0.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. We define the **forgetful functor from $\mathbf{URep}(G)$ to $\mathbf{Hilb}$**, denoted $U : \mathbf{Rep}(G) \to \mathbf{Hilb}$, by

- $U(H,\pi) = H, \quad (H,\pi) \in \mathrm{Obj}(\mathbf{URep}(G))$
- $U(T) = T, \quad T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi,\pi),(H_\rho,\rho))$.

Need to find out if quotienting by equivalence of isomorphism makes $\mathbf{URep}(G)$ a small category so that we can talk about the functor category $\mathbf{Hilb}^{\mathbf{URep}(G)}$ containing the forgetful functor as an object.

Definition 1.2.0.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $g \in G$. We define $\hat{g} : U \Rightarrow U$ by

$$\hat{g}_{(H,\pi)} = \pi(g)$$

Exercise 1.2.0.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $g \in G$. Then

1. $\hat{g} : U \Rightarrow U$ is a natural transformation.
2. $\hat{g} \in \mathrm{Aut}_{\mathbf{Hilb}^{\mathbf{URep}(G)}}(U)$

Proof.

1. (a) Let $(H,\pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. By definition,

$$\begin{aligned}\hat{g}_{(H,\pi)} &= \pi(g) \\ &\in U(H) \\ &\subset \mathrm{Aut}_{\mathbf{Hilb}}(U(H,\pi))\end{aligned}$$

   (b) Let $(H_\pi,\pi),(H_\rho,\rho) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi,\pi),(H_\rho,\rho))$. By definition, $T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi,H_\rho)$ and T is (π,ρ)-equivariant. Therefore

$$\begin{aligned}U(T) \circ \hat{g}_{(H_\pi,\pi)} &= T \circ \pi(g) \\ &= \rho(g) \circ T \\ &= \hat{g}_{(H_\rho,\rho)} \circ U(T)\end{aligned}$$

   i.e. the following diagram commutes:

$$\begin{array}{ccc} U(H_\pi,\pi) & \xrightarrow{\hat{g}_{(H_\pi,\pi)}} & U(H_\pi,\pi) \\ {\scriptstyle U(T)}\downarrow & & \downarrow{\scriptstyle U(T)} \\ U(H_\rho,\rho) & \xrightarrow{\hat{g}_{(H_\rho,\rho)}} & U(H_\rho,\rho) \end{array} = \begin{array}{ccc} H_\pi & \xrightarrow{\pi(g)} & H_\pi \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ H_\rho & \xrightarrow{\rho(g)} & H_\rho \end{array}$$

   Thus $\hat{g} : U \Rightarrow U$ is a natural transformation.

2. Set $h = g^{-1}$. Part (1) implies that $\hat{g},\hat{h} \in \mathrm{End}_{\mathbf{Hilb}^{\mathbf{URep}(G)}}(U)$. Let $(H,\pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then

$$(\hat{g} \circ \hat{h})_{(H,\pi)} = \hat{g}_{(H,\pi)}$$

   The previous part implies that

$$\begin{aligned}\hat{g} &\in \mathrm{Hom}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U,U) \\ &= \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U)\end{aligned}$$

$\square$

Definition 1.2.0.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(V, \pi) \in \mathrm{Obj}(\mathbf{Rep}(G, \mathbb{C}))$. We define the (V, π)**-projection**, denoted $\pi_{(V,\pi)} : \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U) \to \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}}(V)$, by $\pi_{(V,\pi)}(\alpha) = \alpha_{(V,\pi)}$. We define the **topology of endomorphisms of** U, denoted $\mathcal{T}_{\mathcal{E}(U)}$, by

$$\mathcal{T}_{\mathcal{E}(U)} = \tau(\pi_{(V,\pi)} : (V, \pi) \in \mathbf{Rep}(G, \mathbb{C}))$$

Definition 1.2.0.5. define addition of endomorphisms of U pointwise

Exercise 1.2.0.6. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. Then $(\mathrm{Aut}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{Rep}(G,\mathbb{C})}}(U), \mathcal{T}_{\mathcal{E}(U)})$ is a topological unital algebra.

Proof. □

Chapter 2

Groupoids

Definition 2.0.0.1.

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration