INTRODUCTION TO ANALYSIS

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Contents

Preface

1. Real and Complex Numbers

Note 1.0.1. As a starting point, we will take as fact the existence of the **natural numbers**

$$\mathbb{N} = \{1, 2, \cdots\}$$

the **integers**

$$\mathbb{Z} = \{\cdots, -2, -2, 0, 1, 2, \cdots\}$$

and the **rational numbers**

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \right\}$$

1.1. Real Numbers.

Definition 1.1.1. Let X be a set and $\leq$ a relation on X. Then $\leq$ is said to be a **total order** if for each $a, b, c \in X$,

1. $a \leq a$
2. $a \leq b$ and $b \leq c$ implies that $a \leq c$
3. $a \leq b$ and $b \leq a$ implies that $a = b$
4. $a \leq b$ or $b \leq a$

Exercise 1.1.2. We define the relation $\leq$ on $\mathbb{Q}$ defined by

$$\frac{a}{b} \leq \frac{c}{d} \text{ iff } ad \leq bc$$

Then $\leq$ is a total order of $\mathbb{Q}$.

Proof. Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$. Then

1. $\frac{a}{b} \leq \frac{a}{b}$ since $ab \leq ab$.
2. if $\frac{a}{b} \leq \frac{c}{d}$ and $\frac{c}{d} \leq \frac{e}{f}$, then $ad \leq bc$ and $cf \leq de$. Multiplying the first inequality by f and the second inequality by b, we obtain $adf \leq bcf \leq bde$. Dividing both sides by d yields $af \leq be$. Hence $\frac{a}{b} \leq \frac{e}{f}$.
3. if $\frac{a}{b} \leq \frac{c}{d}$ and $\frac{c}{d} \leq \frac{a}{b}$, then $ad \leq bc$ and $bc \leq ab$. This implies that $ad = bc$. Hence $\frac{a}{b} = \frac{c}{d}$.
4.

$\square$

2. Metric Spaces

2.1. Introduction.

3. Topology

Definition 3.0.1. Let X be a topological space and $S, N \subset X$. Then N is said to be a **neighborhood** of S if there exists $U \subset X$ such that U is open and $S \subset U \subset N$. For $S \in X$, we denote the set of neighborhoods of S by $\mathcal{N}_S$

Exercise 3.0.2. Let X be a topological space and $A \subset X$. Then A is open iff for each $a \in A$, there exists $U_a \in \mathcal{N}_a$ such that U_a is an open of a and $U_a \subset A$.

Proof. Suppose that A is open. Let $a \in A$. Then $A \in \mathcal{N}_a$, A is an open and $A \subset A$. Conversely, suppose that or each $a \in A$, there exists $U_a \in \mathcal{N}_a$ such that U_a is open and $U_a \subset A$. Then $A = \bigcup_{a \in A} U_a$ is open. □

Definition 3.0.3. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then

1. f is said to be **continuous** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$.
2. f is said to be open if for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$.
3. f is said to be **closed** if for each $A \subset X$, if $A^c \in \mathcal{A}$, then $f(A)^c \in \mathcal{B}$.

Exercise 3.0.4. Let X, Y be topological spaces and $\phi : X \to Y$ a homeomorphism. Then for each $A \subset X$,

1. $\overline{\phi(A)} = \phi(\overline{A})$
2. $\phi(A)^\circ = \phi(A^\circ)$

Proof.

1. Let $A \subset X$. Since $A \subset \overline{A}$, we have that $\phi(A) \subset \phi(\overline{A})$. Since $\overline{A}$ is closed, $\phi(\overline{A})$ is closed and thus $\overline{\phi(A)} \subset \phi(\overline{A})$. Conversely, let $x \in \phi(\overline{A})$. Then $\phi^{-1}(x) \in \overline{A}$. Then there exists a net $\langle y_\alpha \rangle \subset A$ such that $y_\alpha \to \phi^{-1}(x)$. Then $\langle \phi(y_\alpha) \rangle \subset \phi(A)$ and $\phi(y_\alpha) \to x$. Thus $x \in \overline{\phi(A)}$ and $\phi(\overline{A}) \subset \overline{\phi(A)}$.
2. Similar

□

3.1. **Semi-continuity.**

Definition 3.1.1. Let X be a topological space, $f : X \to (\infty, \infty]$ and $x_0 \in X$. Then f is said to be **lower semicontinuous (l.s.c.) at** x_0 if

$$\liminf_{x \to x_0} f(x) \geq f(x_0)$$

and f is said to be **lower semicontinuous (l.s.c.)** if for each $x_0 \in X$, f is lower semicontinuous at x_0.

Exercise 3.1.2. Let X be a topological space and $f : X \to (\infty, \infty]$. Then f is l.s.c. iff for each $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty])$ is open.

Proof. Suppose that f is l.s.c. Let $\alpha \in \mathbb{R}$ and $x_0 \in f^{-1}(\alpha, \infty]$. Put $\epsilon = f(x_0) - \alpha$. By definition,

$$\sup_{V \in N_{x_0}} \inf_{x \in V \setminus \{x_0\}} f(x) \geq f(x_0)$$

Choose $V_\epsilon \in N_{x_0}$ such that

$$\begin{aligned} \inf_{x \in V_\epsilon} f(x) &> f(x_0) - \epsilon \\ &= \alpha \end{aligned}$$

Then $V_\epsilon^o \in \mathcal{N}_{x_0}$ is open and

$$\begin{aligned} V_\epsilon^o &\subset V_\epsilon \\ &\subset f^{-1}((\alpha, \infty]) \end{aligned}$$

So $f^{-1}((\alpha, \infty])$ is open.
Conversely, suppose that for each $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty])$ is open. Let $x_0 \in X$. Put $\alpha = f(x_0)$. For $n \in \mathbb{N}$, define $V_n = f^{-1}((f(x_0) - 1/n, \infty])$. Then for each $n \in \mathbb{N}$, $V_n \in \mathcal{N}_{x_0}$ and

$$\begin{aligned} \liminf_{x \to x_0} f(x) &= \sup_{V \in \mathcal{N}_{x_0}} \inf_{x \in V \setminus \{x_0\}} f(x) \\ &\geq \sup_{n \in \mathbb{N}} \inf_{x \in V_n \setminus \{x_0\}} f(x) \\ &\geq \sup_{n \in \mathbb{N}} f(x_0) - 1/n \\ &= f(x_0) \end{aligned}$$

So f is l.s.c. $\square$

4. Banach Spaces

4.1. Introduction.

Note 4.1.1. In the following, we will consider vector spaces over $\mathbb{C}$. There are analogous results for real vector spaces as well, just replace every $\mathbb{C}$ with $\mathbb{R}$.

Definition 4.1.1. Let X be a normed vector space. Then X is said to be a **Banach space** if X is complete.

Definition 4.1.2. Let X be a normed vector space and $(x_i)_{i=1}^n \subset X$. The series $\sum_{i=1}^\infty x_i$ is said to **converge** if the sequence $s_n := \sum_{i=1}^n x_i$ converges. The series $\sum_{i=1}^\infty x_i$ is said to **converge absolutely** if $\sum_{i\in\mathbb{N}} \|x_i\| < \infty$.

Theorem 4.1.1. *Let X be a normed vector space. Then X is complete iff for each $(x_i)_{i\in\mathbb{N}} \subset X$, $\sum_{i=1}^\infty x_i$ converges absolutely implies that $\sum_{i=1}^\infty x_i$ converges.*

Proof. Suppose that X is complete. Let $(x_i)_{i\in\mathbb{N}} \subset X$. Suppose that $\sum_{i=1}^\infty x_i$ converges absolutely. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq N$ and $m < n$, then $\sum_{m+1}^n \|x_i\| < \epsilon$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then

$$\begin{aligned}
\|s_n - s_m\| &= \left\| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right\| \\
&= \left\| \sum_{i=m+1}^n x_i \right\| \\
&\leq \sum_{i=m+1}^n \|x_i\| \\
&< \epsilon
\end{aligned}$$

Thus $(s_n)_{n\in N}$ is cauchy. Since X is complete, $\sum_{i=1}^\infty x_i$ converges.
Conversely, Suppose that for each $(x_i)_{i\in\mathbb{N}} \subset X$, $\sum_{i=1}^\infty x_i$ converges absolutely implies that $\sum_{i=1}^\infty x_i$ converges. Let $(x_i)_{i\in\mathbb{N}} \subset X$ be cauchy. Proceed inductively to create a strictly increasing sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq n_i$, then $\|x_m - x_n\| < 2^{-i}$. Define $(y_i)_{i\in\mathbb{N}} \subset X$ by

$$y_i = \begin{cases} x_{n_1} & i = 1 \\ x_{n_i} - x_{n_{i-1}} & i \geq 2 \end{cases}$$

Then $\sum_{i=1}^k y_i = x_{n_k}$ and

$$\begin{aligned}
\sum_{i\in\mathbb{N}} \|y_i\| &= \|x_{n_1}\| + \sum_{i\in\mathbb{N}} \|x_{n_i} - x_{n_{i-1}}\| \\
&\leq \|x_{n_1}\| + \sum_{i\in\mathbb{N}} 2^{-i} \\
&= \|x_{n_1}\| + 1
\end{aligned}$$

Hence $(x_{n_k})_{k\in\mathbb{N}} = (\sum_{i=1}^k y_i)_{i\in\mathbb{N}}$ converges. Since $(x_i)_{i\in\mathbb{N}}$ is cauchy and has a convergent subsequence, it converges. So X is complete. □

Definition 4.1.3. Let X, Y be a normed vector spaces. A linear map $T : X \to Y$ is said to be **bounded** if there exists $C \geq 0$ such that for each $x \in X$,

$$\|Tx\| \leq C\|x\|$$

We define

$$L(X, Y) = \{T : X \to Y : T \text{ is bounded}\}$$

Exercise 4.1.4. Let X, Y be a normed vector spaces and $T : X \to Y$ a linear map. Then T is bounded iff there exists $r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$

Proof. Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Thus $T(B(0, 1)) \subset B(0, C + 1)$. Conversely. Suppose that there exists $r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$. Define $C = \frac{2s}{r}$. Let $x \in X$. Put $\alpha = \frac{r}{2\|x\|}$ Then $\alpha x \in B(0, r)$. So $T(\alpha x) = \alpha T(x) \in B(0, s)$. Hence

$$\begin{aligned}
\|T(\alpha x)\| &= \|\alpha T(x)\| \\
&= |\alpha|\|T(x)\| \\
&= \frac{r}{2\|x\|}\|T(x)\| \\
&< s.
\end{aligned}$$

Thus

$$\|Tx\| < \frac{2s}{r}\|x\| = C\|x\|$$

So T is bounded. □

Theorem 4.1.2. *Let X, Y be normed vector spaces and $T : X \to Y$ a linear map. Then the following are equivalent:*

(1) T is continuous
(2) T is continuous at $x = 0$
(3) T is bounded

Proof.

- (1) $\implies$ (2):
  Trivial
- (2) $\implies$ (3):
  Suppose that T is continuous at $x = 0$. Then there exists $\delta > 0$ such that for each $x \in X$, if $\|x\| < \delta$, then $\|Tx\| < 1$. Choose $C = \frac{2}{\delta}$. If $x = 0$, then $\|Tx\| \leq C\|x\|$. Suppose that $\|x\| \neq 0$. Define $y = \frac{\delta}{2\|x\|}x$. Then $\|y\| < \delta$. So

  $$\begin{aligned}
  1 &> \|Ty\| \\
  &= \frac{\delta}{2\|x\|}\|Tx\|
  \end{aligned}$$

  Thus

  $$\begin{aligned}
  \|Tx\| &< \frac{2}{\delta}\|x\| \\
  &= C\|x\|
  \end{aligned}$$

  Hence T is bounded.

- (3) $\implies$ (1)
  Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{C+1}$. Let $x, y \in X$ Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned} \|Tx - Ty\| &= \|T(x-y)\| \\ &\leq C\|x-y\| \\ &< (C+1)\delta \\ &= \epsilon \end{aligned}$$

  So T is continuous.

$\square$

Definition 4.1.5. Let X, Y be normed vector spaces. Define $\|\cdot\| : L(X,Y) \to [0, \infty)$ by

$$\|T\| = \inf\{C \geq 0 : \text{for each } x \in X,\ \|Tx\| \leq C\|x\|\}$$

We call $\|\cdot\|$ the **operator norm on** $L(X,Y)$

Exercise 4.1.6. Let X, Y be normed vector spaces. If $X \neq \{0\}$, then the operater norm on $L(X,Y)$ is given by:

(1) $\|T\| = \sup\limits_{\|x\|=1} \|Tx\|$
(2) $\|T\| = \sup\limits_{x \neq 0} \|x\|^{-1}\|Tx\|$
(3) $\|T\| = \inf\{C \geq 0 : \text{for each } x \in X,\ \|Tx\| \leq C\|x\|\}$

Proof. Since $X \neq \{0\}$, the supremums in (1) and (2) are well defined. Let $T \in L(X,Y)$. By linearity of T, the sets over which the supremums are taken in (1) and (2) are the same. So (1) and (2) are equal.

Now, put $M = \sup\limits_{\|x\|=1} \|Tx\|$, $m = \inf\{C \geq 0 : \text{ for each } x \in X,\ \|Tx\| \leq C\|x\|\}$ and let $x \in X$. If $\|x\| = 0$, then $\|Tx\| \leq M\|x\|$. Suppose that $\|x\| \neq 0$. Then

$$\begin{aligned} \|Tx\| &= \bigg(\Big\|T(x/\|x\|)\Big\|\bigg)\|x\| \\ &\leq M||x|| \end{aligned}$$

Hence $M \in \{C \geq 0 : \text{ for each } x \in X,\ \|Tx\| \leq C\|x\|\}$. Therefore $m \leq M$

Let $C \in \{C \geq 0 : \text{ for each } x \in X,\ \|Tx\| \leq C\|x\|\}$. Suppose that $\|x\| = 1$. Then $\|Tx\| \leq C\|x\| = C$. So $M \leq C$. Therefore $M \leq m$. So $M = m$ and the supremum in (1) is the same as the infimum in (3). $\square$

Note 4.1.2. From here on, unless stated otherwise, we assume $X \neq 0$.

Exercise 4.1.7. Let X, Y be normed vector spaces and $T \in L(X,Y)$. Then for each $x \in X$, $\|Tx\| \leq \|T\|\|x\|$

Proof. This is just part of the previous exercise. Let $x \in X$. If $x = 0$, then $\|Tx\| \leq \|T\|\|x\|$. Suppose that $x \neq 0$. Then $\|Tx\| = T(x/\|x\|)\|x\| \leq \|T\|\|x\|$ $\square$

Exercise 4.1.8. Let X, Y be normed vector spaces. Then the operator norm is a norm on $L(X, Y)$.

Proof. Let $S, T \in L(X, Y)$ and $\alpha \in \mathbb{C}$. For each $x \in X$, we have that

$$\begin{aligned} \|(S+T)x\| &= \|Sx + Tx\| \\ &\leq \|Sx\| + \|Tx\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= \big(\|S\| + \|T\|\big)\|x\| \end{aligned}$$

So $\|S+T\| \leq \|S\| + \|T\|$.

Using the definition of $\|T\|$, we see that

$$\begin{aligned} \|\alpha T\| &= \sup_{\|x\|=1} \|(\alpha T)x\| \\ &= \sup_{\|x\|=1} |\alpha|\|Tx\| \\ &= |\alpha| \sup_{\|x\|=1} \|Tx\| \\ &= |\alpha|\|T\| \end{aligned}$$

So $\|\alpha S\| = |\alpha|\|S\|$.

Suppose that $\|T\| = 0$. Let $x \in X$. Then $\|Tx\| \leq \|T\|\|x\| = 0$. So $Tx = 0$. Since $x \in X$ is arbitrary, we have that $T = 0$. $\square$

Exercise 4.1.9. Let X be a normed vector space. Then addition and scalar multiplication are continuous on $X \times X$ and $\|\cdot\| : X \to [0, \infty)$ is continuous.

Proof. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Let $(x_1, y_1), (x_2, y_2) \in X \times X$. Suppose that $\|(x_1, y_1) - (x_2, y_2)\| = \max\{\|x_1 - x_2\|, \|y_1 - y_2\|\} < \delta$. Then

$$\begin{aligned} \|(x_1 + y_1) - (x_2 + y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\ &\leq \|x_1 - x_2\| + \|y_1 - y_2\| \\ &< 2\delta \\ &= \epsilon \end{aligned}$$

Hence addition is uniformly continuous.

Let $(\lambda_1, x_1) \in \mathbb{C} \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(|\lambda_1| + \|x_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(\lambda_2, x_2) \in \mathbb{C} \times X$.

Suppose that $\|(\lambda_1, x_1) - (\lambda_2, x_2)\| = \max\{|\lambda_1 - \lambda_2|, \|x_1 - x_2\|\} < \delta$. Then

$$\begin{aligned}
\|\lambda_1 x_1 - \lambda_2 x_2\| &= \|\lambda_1 x_1 - \lambda_1 x_2 + \lambda_1 x_2 - \lambda_2 x_2\| \\
&= \|\lambda_1(x_1 - x_2) + (\lambda_1 - \lambda_2)x_2\| \\
&\leq |\lambda_1|\|x_1 - x_2\| + |\lambda_1 - \lambda_2|\|x_2\| \\
&\leq |\lambda_1|\|x_1 - x_2\| + |\lambda_1 - \lambda_2|(\|x_1 - x_2\| + \|x_1\|) \\
&< |\lambda_1|\delta + \delta(\delta + \|x_1\|) \\
&= (|\lambda_1| + \|x_1\|)\delta + \delta^2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Since $(\lambda_1, x_1) \in \mathbb{C} \times X$ is arbitrary, scalar multiplication is continuous.

Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x, y \in X$. Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned}
\big|\|x\| - \|y\|\big| &\leq \|x - y\| \\
&< \delta \\
&= \epsilon
\end{aligned}$$

So $\|\cdot\| : X \to [0, \infty)$ is uniformly continuous. □

Exercise 4.1.10. Let X, Y be normed vector spaces. If Y is complete, then so is $L(X, Y)$.

Proof. Suppose that Y is complete. Let $(T_n)_{n\in\mathbb{N}} \subset L(X, Y)$. Suppose that $(T_n)_{n\in\mathbb{N}}$ is Cauchy. Since for each $m, n \in \mathbb{N}$, $\big|\|T_m\| - \|T_n\|\big| \leq \|T_m - T_n\|$, we have that $(\|T_n\|)_{n\in\mathbb{N}} \subset [0, \infty)$ is Cauchy. Hence $\lim\limits_{n\to\infty} \|T_n\|$ exists.

Let $x \in X$ and $m, n \in \mathbb{N}$. Then

$$\begin{aligned}
\|T_m x - T_n x\| &= \|(T_m - T_n)x\| \\
&\leq \|T_m - T_n\|\|x\|
\end{aligned}$$

So $(T_n x)_{n\in\mathbb{N}} \subset Y$ is Cauchy and hence converges. Define $T : X \to Y$ by $Tx = \lim\limits_{n\to\infty} T_n x$.

Since addition and scalar multiplication are continuous, T is linear. Let $x \in X$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in N$, if $n \geq N$, then $\|Tx - T_n x\| < \epsilon$. Then for each $n \in \mathbb{N}$, if $n \geq N$ we have that

$$\begin{aligned}
\|Tx\| &\leq \|Tx - T_n x\| + \|T_n x\| \\
&< \epsilon + \|T_n x\| \\
&\leq \epsilon + \|T_n\|\|x\|
\end{aligned}$$

Thus $\|Tx\| \leq \epsilon + (\lim\limits_{n\to\infty} \|T_n\|)\|x\|$. Since $\epsilon > 0$ is arbitrary, $\|Tx\| \leq (\lim\limits_{n\to\infty} \|T_n\|)\|x\|$. Thus $T \in L(X, Y)$ and $\|T\| \leq \lim\limits_{n\to\infty} \|T_n\|$.

Note that since addition, scalar multiplication and $\|\cdot\|$ are continuous, we have that for each $n \in \mathbb{N}$ and $x \in X$, $\|(T_n - T_m)x\|$ converges to $\|(T_n - T)x\|$ because

$$\begin{aligned}\lim_{m\to\infty} \|(T_n - T_m)x\| &= \lim_{m\to\infty} \|T_n x - T_m x\| \\ &= \|T_n x - \lim_{m\to\infty} T_m x\| \\ &= \|T_n x - Tx\| \\ &= \|(T_n - T)x\|\end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ if $n, m \geq N$, then $\|T_n - T_m\| < \epsilon$. Then for each $n \in \mathbb{N}$ if $n \geq N$, then for each $x \in X$,

$$\|(T_n - T_m)x\| \leq \|(T_n - T_m)\|\|x\| < \epsilon\|x\|$$

Combining this with the previous fact, we see that for each $n \in N$, if $n \geq N$, then for each $x \in X$,

$$\|(T_n - T)x\| \leq \epsilon\|x\|$$

In particular, for each $n \in \mathbb{N}$, if $n \geq N$, then

$$\|T_n - T\| = \sup_{\|x\|=1} \|(T_n - T)x\| \leq \epsilon$$

This implies that T_n converges to T in $L(X, Y)$. Since

$$\big|\|T_n\| - \|T\|\big| \leq \|T_n - T\|$$

it is clear that $\lim\limits_{n\to\infty} \|T_n\| = \|T\|$ □

Definition 4.1.11. Let X be a normed vector space and $M \subset X$ a closed subspace. Define $\|\cdot\| : X/M \to [0, \infty)$ by

$$\|x + M\| := \inf_{y\in M} \|x + y\|$$

We call $\|\cdot\|$ the **subspace norm on** X/M

Exercise 4.1.12. Let X be a normed vector space and $M \subsetneq X$ a proper, closed subspace of M. Then

1. The previously defined subspace norm on X/M is well defined and is a norm.
2. For each $\epsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.
3. The projection map $\pi : X \to X/M$ defined by $\pi(x) = x + M$ is continuous and $\|\pi\| = 1$.
4. If X is complete, then X/M is complete.

Proof. (1) Let $x, y \in X$ and $\alpha \in \mathbb{C}$. Suppose that $x + M = y + M$. Then there exists $m \in M$ such that $x = y + m$. Since M is a subspace, the map $T : M \to M$ given by $Tx = x + m$ is a bijection. So

$$\inf_{z\in M} \|y + m + z\| = \inf_{z\in M} \|y + z\|$$

which implies that

$$\begin{aligned}\|x+M\| &= \inf_{z\in M}\|x+z\| \\ &= \inf_{z\in M}\|y+m+z\| \\ &= \inf_{z\in M}\|y+z\| \\ &= \|y+M\|\end{aligned}$$

So $\|\cdot\| : X/M \to [0,\infty)$ is well defined.

We observe that for each $z, w \in M$,

$$\|x+y+z\| \le \|x+w\| + \|y+w+z\|$$

Taking infimums over M with respect to z in this inequality implies that for each $w \in M$,

$$\begin{aligned}\inf_{z\in M}\|x+y+z\| &\le \inf_{z\in M}\Big(\|x+w\| + \|y+w+z\|\Big) \\ &= \|x+w\| + \inf_{z\in M}\|y+w+z\|\end{aligned}$$

Again we use the fact that for each $w \in M$,

$$\inf_{z\in M}\|y+w+z\| = \inf_{z\in M}\|y+z\|$$

This implies that for each $w \in M$,

$$\inf_{z\in M}\|x+y+z\| \le \|x+w\| + \inf_{z\in M}\|y+z\|$$

Therefore, taking infimums over M with respect to w in this inequality yields

$$\begin{aligned}\|x+y+M\| &= \inf_{z\in M}\|x+y+z\| \\ &\le \inf_{w\in M}\Big(\|x+w\| + \inf_{z\in M}\|y+z\|\Big) \\ &= \inf_{w\in M}\|x+w\| + \inf_{z\in M}\|y+z\| \\ &= \|x+M\| + \|y+M\|\end{aligned}$$

If $\alpha = 0$, then $\alpha x = 0$. Choosing $z = 0 \in M$ gives $\|\alpha x + M\| = 0 = |\alpha|\|x+M\|$. Suppose that $\alpha \neq 0$. Then the map $T : M \to M$ given by $Tx = \alpha^{-1}x$ is a bijection and thus $\inf_{z\in M}\|x+\alpha^{-1}z\| = \inf_{z\in M}\|x+z\|$. Hence we have that

$$\begin{aligned}\|\alpha x+M\| &= \inf_{z\in M}\|\alpha x+z\| \\ &= \inf_{z\in M}|\alpha|\|x+\alpha^{-1}z\| \\ &= |\alpha|\inf_{z\in M}\|x+\alpha^{-1}z\| \\ &= |\alpha|\inf_{z\in M}\|x+z\| \\ &= |\alpha|\|x+M\|\end{aligned}$$

Suppose that $\|x\| = 0$. Choose a sequence $(z_n)_{n\in N} \subset M$ such that

$$\begin{aligned}\lim_{n\to\infty} \|x - z_n\| &= \inf_{z\in M} \|x + z\| \\ &= 0\end{aligned}$$

Then $\lim_{n\to\infty} z_n = x$. Since M is closed, $x \in M$. Hence $x + M = 0 + M$.

(2) Since M is a proper subspace, there exists $v \in X$ such that $v \notin M$. Then $\|v+M\| \neq 0$. Let $\epsilon > 0$. Then $(1-\epsilon)^{-1}\|v + M\| > \|v + M\|$. So there exists $z \in M$ such that

$$0 < \|v + M\| \leq \|v + z\| < (1-\epsilon)^{-1}\|v + M\|$$

Choose $x = \|v + z\|^{-1}(v + z)$. Then $\|x\| = 1$ and

$$\begin{aligned}\|x + M\| &= \|v + z\|^{-1}\|v + z + M\| \\ &= \|v + z\|^{-1}\|v + M\| \\ &> 1 - \epsilon\end{aligned}$$

(3) Let $x \in X$. Taking $z = 0$, we we see that $\|\pi(x)\| = \|x + M\| \leq \|x + z\| = \|x\|$. So π is bounded and in particular,

$$\sup_{\|x\|=1} \|\pi(x)\| \leq 1$$

From (2) we see that

$$\sup_{\|x\|=1} \|\pi(x)\| \geq 1$$

Hence $\|\pi\| = 1$.

(4) Suppose that X is complete. Let $(x_i + M)_{i\in\mathbb{N}} \subset X/M$. Suppose that $\sum_{i\in\mathbb{N}} \|x_i + M\| < \infty$. Let $\epsilon > 0$. Then for each $i \in \mathbb{N}$, there exists $z_i \in M$ such that $\|x_i + z_i\| < \|x_i + M\| + \epsilon 2^{-i}$. Define the sequence $(a_i)_{i\in\mathbb{N}} \subset X$ by $a_i = x_i + z_i$. Then we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|a_i\| &= \sum_{i\in N} \|x_i + z_i\| \\ &\leq \sum_{i\in N} \left(\|x_i + M\| + \epsilon 2^{-i}\right) \\ &= \sum_{i\in\mathbb{N}} \|x_i + M\| + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\sum_{i\in\mathbb{N}} \|a_i\| \leq \sum_{i\in\mathbb{N}} \|x_i + M\| < \infty$$

Since X is complete, $\sum\limits_{i=1}^{\infty} a_i$ converges in X. Define $(s_n)_{n\in\mathbb{N}} \subset X$ and $s \in X$ by $s_n = \sum\limits_{i=1}^{n} a_i$ and $s = \sum\limits_{i=1}^{\infty} a_i$. Since $\lim\limits_{n\to\infty} s_n = s$, and $\pi : X \to X/M$ is continuous, it follows that $\lim\limits_{n\to\infty} \pi(s_n) = \pi(s)$. Since

$$\begin{aligned}\pi(s_n) &= \sum_{i=1}^{n} a_i + M \\ &= \sum_{i=1}^{n} x_i + M\end{aligned}$$

We have that $\sum\limits_{i=1}^{\infty} x_i + M$ converges which implies that X/M is complete.

□

Exercise 4.1.13. Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then

(1) $\ker T$ is closed
(2) there exists a unique map $S : X/\ker T \to T(X)$ such that $T = S \circ \pi$. Furthermore S is a bounded linear bijection and $\|S\| = \|T\|$.

Proof. (1) Since T is continuous and $\ker T = T^{-1}(\{0\})$, we have that $\ker T$ is closed.

(2) Suppose that there exists $S_1, S_2 \in L(X/\ker T, T(X))$ such that $T = S_1 \circ \pi$ and $T = S_2 \circ \pi$. Let $x \in X$. Then

$$S_1(x + \ker T) = S_1(\pi(x)) = T(x) = S_2(\pi(x)) = S_2(x + \ker T)$$

So $S_1 = S_2$. Therefore such a map is unique.
Define $S : X/\ker T \to T(X)$ by $S(x + \ker T) = T(x)$. Then S is clearly a linear bijection that satisfies $T = S \circ \pi$. Let $x \in X$ and $z \in \ker T$. Then

$$\begin{aligned}\|S(x + \ker T)\| &= \|T(x)\| \\ &= \|T(x + z)\| \\ &\leq \|T\|\|x + z\|\end{aligned}$$

Thus

$$\|S(x + \ker T)\| \leq \|T\| \inf_{z\in\ker T} \|x + z\| = \|T\|\|x + \ker T\|$$

So S is bounded and $\|S\| \leq \|T\|$. This implies that

$$\|T\| = \|S \circ \pi\| \leq \|S\|\|\pi\| = \|S\|$$

Thus $\|S\| = \|T\|$.

□

Exercise 4.1.14. Let X, Y be normed vector spaces. Define $\phi : L(X, Y) \times X \to Y$ by $\phi(T, x) = Tx$. Then ϕ is continuous.

Proof. Let $(T_1, x_1) \in L(X, Y) \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|x_1\|+\|T_1\|+1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(t_2, x_2) \in L(X, Y) \times X$. Suppose that

$$\|(T_1, x_1) - (T_2, x_2)\| = \max\{\|T_1 - T_2\|, \|x_1 - x_2\|\} < \delta$$

Then

$$\begin{aligned}\|\phi(T_1, x_1) - \phi(T_2 - x_2)\| &= \|T_1 x_- T_2 x_2\| \\ &= \|T_1 x_1 - T_2 x_1 + T_2 x_1 - T_2 x_2\| \\ &\leq \|(T_1 - T_2)x_1\| + \|T_2(x_1 - x_2)\| \\ &\leq \|T_1 - T_2\|\|x_1\| + \|T_2\|\|x_1 - x_2\| \\ &\leq \|T_1 - T_2\|\|x_1\| + \big(\|T_1 - T_2\| + \|T_1\|\big)\|x_1 - x_2\| \\ &< \delta\|x_1\| + (\delta + \|T_1\|)\delta \\ &= \delta(\|T_1\| + \|x_1\|) + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

So ϕ is continuous. □

Exercise 4.1.15. Let X be a normed vector space and $M \subset X$ a subspace. Then $\overline{M}$ is a subspace.

Proof. Let $x, y \in \overline{M}$ and $\alpha \in \mathbb{C}$. Then there exist sequences $(x_n)_{n\in\mathbb{N}} \subset M$ and $(y_n)_{n\in\mathbb{N}} \subset M$ such that $x_n \to x$ and $y_n \to y$. Since M is a subspace, $(x_n + y_n)_{n\in\mathbb{N}} \subset M$ and $(\alpha x_n)_{n\in\mathbb{N}} \subset M$. Since addition and scalar multiplication are continuous, we have that $x_n + y_n \to x + y$ and $\alpha x_n \to \alpha x$. Thus $x + y \in \overline{M}$ and $\alpha x \in \overline{M}$ and hence $\overline{M}$ is a subspace. □

Exercise 4.1.16. Let X, Y, Z be normed vector spaces, $T \in L(X, Y)$ and $S \in L(Y, Z)$. Define $ST : X \to Z$ by $STx = S(Tx)$. Then $ST \in L(X, Z)$ and $\|ST\| \leq \|S\|\|T\|$.

Proof. Clearly ST is linear. Let $x \in X$. Then

$$\begin{aligned}\|STx\| &= \|S(Tx)\| \\ &\leq \|S\|\|Tx\| \\ &\leq \|S\|\|T\|\|x\|\end{aligned}$$

So $\|ST\| \leq \|S\|\|T\|$. □

Definition 4.1.17. Let X, Y be a normed vector spaces and $T \in L(X, Y)$. Then T is said to be **invertible** or an **isomorphism** if T is a bijection and $T^{-1} \in L(Y, X)$.

Definition 4.1.18. Let X be a Banach space. Define $GL(X) := \{T \in L(X, X) : T \text{ is invertible}\}$.

Exercise 4.1.19. Let X be a Banach space. Then

(1) For each $T \in L(X, X)$, if $\|I - T\| < 1$, then T is invertible and

$$T^{-1} = \sum_{n=0}^{\infty} (I - T)^n$$

(2) For each $S, T \in L(X, X)$, if S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$, then T is invertible.

(3) $GL(X)$ is open.

Proof.

(1) Let $T \in L(X,X)$. Suppose that $\|I-T\| < 1$. Then
$$\sum_{n=0}^{\infty} \|(I-T)^n\| \leq \sum_{n=0}^{\infty} \|I-T\|^n < \infty$$
Since X is a complete, so is $L(X,X)$ and thus $\sum\limits_{n=0}^{\infty}(I-T)^n$ converges in $L(X,X)$.
Define $(S_k)_{k=0}^{\infty} \subset L(X,X)$ and $S \in L(X,X)$ by $S_k = \sum\limits_{n=0}^{k}(I-T)^n$ and
$S = \sum\limits_{n=0}^{\infty}(I-T)^n$. Then for each $k \in \mathbb{N}$,
$$\begin{aligned} S_k T &= S_k - S_k(I-T) \\ &= (I-T)^0 - (I-T)^{k+1} \\ &= I - (I-T)^{k+1} \end{aligned}$$
and $\|S_k T - I\| \leq \|I-T\|^{k+1}$. Since multiplication on Banach algebras is continuous, we have that
$$ST = (\lim_{k\to\infty} S_k)T = \lim_{k\to\infty} S_k T = I$$
Similarly $TS = I$. Thus T is invertible and $T^{-1} = S \in L(X,X)$.

(2) Let $S, T \in L(X,X)$. Suppose that S is invertible and $\|S-T\| < \|S^{-1}\|^{-1}$. Then
$$\begin{aligned} \|I - S^{-1}T\| &= \|S^{-1}(S-T)\| \\ &\leq \|S^{-1}\|\|S-T\| \\ &< 1 \end{aligned}$$
So $S^{-1}T$ is invertible. Thus $T = S(S^{-1}T)$ is invertible.

(3) Let $T \in GL(X)$. Choose $\delta = \|T^{-1}\|^{-1}$. By (2), $B(T,\delta) \subset GL(X)$.

$\square$

Definition 4.1.20. Let $(X_n)_{n\in\mathbb{N}}$ be a collection of normed vector spaces. Put $X = \bigoplus\limits_{n\in\mathbb{N}} X_n$.
Let $p \in [1,\infty]$ and define $\|\cdot\|_p : X \to [0,\infty)$ by
$$\|(x_n)_{n\in\mathbb{N}}\|_p = \begin{cases} \left(\sum\limits_{n\in\mathbb{N}} \|x_n\|^p\right)^{1/p} & p < \infty \\ \sup\limits_{n\in\mathbb{N}} \|x_n\| & p = \infty \end{cases}$$
We define
$$\bigoplus_{n\in\mathbb{N}}^{p} X_n = \{x \in X : \|x\|_p < \infty\}$$
and
$$\bigoplus_{n\in\mathbb{N}}^{0} X_n = \left\{x \in \bigoplus_{n\in\mathbb{N}}^{\infty} X_n : \lim_{n\to\infty} \|x_n\| = 0\right\}$$

Exercise 4.1.21. Let $(X_n)_{n\in\mathbb{N}}$ be a collection of Banach spaces. Then for each $p \in [1,\infty]\cup\{0\}$, $\bigoplus_{n\in\mathbb{N}}^p X_n$ is a Banach space.

Definition 4.1.22. Let $X_1,\cdots,X_n,Y$ be vector spaces and $T : \bigoplus_{i=1}^n X_i \to Y$. Then T is said to be **multilinear** if for each $x_1 \in X_1,\cdots,x_n \in X_n$, and $i \in \{1,\cdots,n\}$ the maps $T_i : X_i \to Y$ defined by

$$T_i(x) = T(x_1,\cdots,x_{i-1},x,x_{i+1},\cdots,x_n)$$

are linear.

Definition 4.1.23. Let $X_1,\cdots,X_n,Y$ be normed vector spaces and $T : \bigoplus_{i=1}^n X_i \to Y$ multilinear. Then T is said to be **bounded** if there exists $C \geq 0$ such that for each $x_1,\cdots,x_n \in X$,

$$\|T(x_1,\cdots,x_n)\| \leq C\|x_1\|\cdots\|x_n\|$$

Exercise 4.1.24. Let $X_1,\cdots,X_n,Y$ be normed vector spaces and $T : \bigoplus_{i=1}^n X_i \to Y$ multilinear. Then the following are equivalent:

(1)
(2)
(3)

4.2. Linear and Sublinear Functionals.

Definition 4.2.1.

(1) Let X be a $\mathbb{C}$-vector space and $T : X \to \mathbb{C}$. Then T is said to be a **linear functional on** X if T is linear. We define the **dual space of** X, denoted X^*, by $X^* = \{T : X \to \mathbb{C} : T \text{ is linear}\}$

(2) If X is a normed $\mathbb{C}$-vector space, then T is said to be a **bounded linear functional on** X if $T \in L(X, \mathbb{C})$. We define the **dual space of** X, denoted X^*, by $X^* = L(X, \mathbb{C})$.

Note 4.2.1. We define X^* similarly when X is an $\mathbb{R}$-vector space or normed $\mathbb{R}$-vector space.

Definition 4.2.2. Let X be a normed vector space and $p : X \to \mathbb{R}$. Then p is said to be a **sublinear functional** if for each $x, y \in X$, $\lambda \geq 0$,

(1) $p(x + y) \leq p(x) + p(y)$
(2) $p(\lambda x) = \lambda p(x)$

Exercise 4.2.3. Let X be a vector space and $\|\cdot\| : X \to [0, \infty)$ be a seminorm, then $\|\cdot\|$ is a sublinear functional.

Proof. Clear $\square$

Exercise 4.2.4. Let X be a vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then for each $x, y \in X$

(1) $-p(-x) \leq p(x)$
(2) $-p(y - x) \leq p(x) - p(y) \leq p(x - y)$

Proof. Let $x, y \in X$.

(1) We have
$$\begin{aligned} 0 &= p(0) \\ &= p(x - x) \\ &\leq p(x) + p(-x) \end{aligned}$$
So $-p(-x) \leq p(x)$.

(2) We have
$$\begin{aligned} p(x) &= p(x - y + y) \\ &\leq p(x - y) + p(y) \end{aligned}$$
So $p(x) - p(y) \leq p(x - y)$. Switching x and y gives us $p(y) - p(x) \leq p(y - x)$ and multiplying both sides by -1 yields $-p(y - x) \leq p(x) - p(y)$
Putting these two together, we see that
$$-p(y - x) \leq p(x) - p(y) \leq p(x - y)$$

$\square$

Definition 4.2.5. Let X be a normed vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then p is said to be **bounded** if there exists $M > 0$ such that for each $x \in X$, $p(x) \leq M\|x\|$.

Exercise 4.2.6. Let X be a normed vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then p is bounded iff p is Lipschitz.

Proof. Suppose that p is bounded. Then there exists $M > 0$ such that for each $x \in X$, $p(x) \leq M\|x\|$. Let $x, y \in X$. Then the previous exercise implies that

$$\begin{aligned} -M\|x-y\| &= -M\|y-x\| \\ &\leq -p(y-x) \\ &\leq p(x)-p(y) \\ &\leq p(x-y) \\ &\leq M\|x-y\| \end{aligned}$$

So that

$$|p(x)-p(y)| \leq M\|x-y\|$$

and p is Lipschitz. Conversely, suppose that p is Lipschitz. Then there exists $M > 0$ such that for each $x, y \in X$, $|p(x) - p(y)| \leq M\|x - y\|$. Let $x \in X$. Then

$$\begin{aligned} p(x) &\leq |p(x)| \\ &= |p(x)-p(0)| \\ &\leq M\|x-0\| \\ &\leq M\|x\| \end{aligned}$$

So p is bounded. □

Theorem 4.2.1. ***Hahn-Banach Theorem:*** *Let X be a vector space, $p : X \to \mathbb{R}$ a sublinear functional, $M \subset X$ a subspace and $f : M \to \mathbb{R}$ a linear functional. If for each $x \in M$, $f(x) \leq p(x)$, then there exists a linear functional $F : X \to \mathbb{R}$ such that for each $x \in X$, $F(x) \leq p(x)$ and $F|_M = f$.*

Exercise 4.2.7. Let X be a vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then there exists $F : X \to \mathbb{R}$ such that for each $x \in X$, $F(x) \leq p(x)$.

Proof. Take $M = \{0\}$ and $f \equiv 0$ and apply the Hahn-Banach theorem. □

Exercise 4.2.8. Equivalency of linearity (General Case) Let X be a vector space and $p : X \to \mathbb{R}$ a sublinear functional. Then the following are equivalent:

(1) there exists a unique $F \in X^*$ such that $F \leq p$
(2) for each $x \in X$, $-p(-x) = p(x)$
(3) p is linear

Hint: If there exists $x \in X$ such that $-p(-x) \neq p(x)$, define $f_1, f_2 : \mathrm{span}(x) \to \mathbb{R}$ by $f_1(tx) = tp(x)$ and $f_2(tx) = -tp(-x)$

Proof.

- (1) $\Rightarrow$ (2):
  Suppose that there exists a unique $F \in X^*$ such that $F \leq p$. For the sake of contradiction, suppose that there exists $x \in X$ such that $-p(-x) \neq p(x)$. Define $f_1, f_2 : \mathrm{span}(x) \to \mathbb{R}$ by
  $$f_1(tx) = tp(x)$$
  and
  $$f_2(tx) = -tp(-x)$$

Let $y \in \text{span}(x)$. Then there exists $t \in \mathbb{R}$ such that $y = tx$. Then for each $k \in \mathbb{R}$,

$$\begin{aligned} f_1(ky) &= f_1(ktx) \\ &= ktp(x) \\ &= kf_1(tx) \\ &= kf_1(y) \end{aligned}$$

Similarly, $f_2(ky) = kf_2(y)$ and so $f_1, f_2 \in \text{span}(x)^*$. If $t \geq 0$, then

$$\begin{aligned} f_1(y) &= f_1(tx) \\ &= tp(x) \\ &= p(tx) \\ &= p(y) \end{aligned}$$

If $t < 0$, then

$$\begin{aligned} f_1(y) &= f_1(tx) \\ &= tp(x) \\ &= -|t|p(x) \\ &= -p(|t|x) \\ &= -p(-tx) \\ &\leq p(tx) \\ &= p(y) \end{aligned}$$

So $f_1 \leq p$ on $\text{span}(x)$. Similarly, $f_2 \leq p$ on $\text{span}(x)$. The Hahn-Banach theorem implies that there exist $F_1, F_2 \in X^*$ such that $F_1, F_2 \leq p$ and $F_1 = f_1, F_2 = f_2$ on $\text{span}(x)$. By the assumption of uniqueness, $F_1 = F_2$. This is a contradiction since

$$\begin{aligned} F_1(x) &= p(x) \\ &\neq -p(-x) \\ &= F_2(x) \end{aligned}$$

So for each $x \in X$, $-p(-x) = p(x)$.

- (2) $\Rightarrow$ (3):
Suppose that for each $x \in X$, $-p(-x) = p(x)$. The previous exercise implies that there exists $F \in X^*$ such that $F \leq p$. Let $x \in X$. Then

$$\begin{aligned} -F(x) &= F(-x) \\ &\leq p(-x) \\ &= -p(x) \end{aligned}$$

So $p(x) \leq F(x)$ and $p \leq F$. Therefore $p = F$ and p is linear.
- (3) $\Rightarrow$ (1):
Suppose that p is linear. Let $F \in X^*$. Suppose that $F \leq p$. Let $x \in X$. Then as in

the case for (2) $\Rightarrow$ (3), we have that

$$\begin{aligned} -F(x) &= F(-x) \\ &\leq p(-x) \\ &= -p(x) \end{aligned}$$

which implies that $p = F$. So p is the unique linear function $F \in X^*$ such that $F \leq p$. $\square$

Exercise 4.2.9. Let X be a normed vector space, $p : X \to \mathbb{R}$ a bounded sublinear functional and $\phi : X \to \mathbb{R}$ a linear functional. If $\phi \leq p$, then $\phi \in X^*$.

Proof. Since p is Lipschitz, there exists $M > 0$ such that for each $x \in X$, $|p(x)| \leq M\|x\|$. Let $x \in X$. Then

$$\begin{aligned} \phi(x) &\leq p(x) \\ &\leq |p(x)| \\ &\leq M\|x\| \end{aligned}$$

and therefore

$$\begin{aligned} -M\|x\| &= -M\|-x\| \\ &\leq -p(-x) \\ &\leq -\phi(-x) \\ &= \phi(x) \end{aligned}$$

So that $|\phi(x)| \leq M\|x\|$ and $\phi \in X^*$. $\square$

Exercise 4.2.10. Let X be a normed vector space and $p : X \to \mathbb{R}$ a bounded sublinear functional. Then there exists $\phi \in X^*$ such that for each $x \in X$, $\phi(x) \leq p(x)$.

Proof. A previous exercise implies there exists $\phi : X \to \mathbb{R}$ such that ϕ is linear and $\phi \leq p$. The previous exercise implies that $\phi \in X^*$. $\square$

Exercise 4.2.11. Equivalency of linearity (Bounded Case) Let X be a normed vector space and $p : X \to \mathbb{R}$ a bounded sublinear functional. Then the following are equivalent:

(1) there exists a unique $\phi \in X^*$ such that $\phi \leq p$
(2) for each $x \in X$, $-p(-x) = p(x)$
(3) p is linear

Proof. Basically the same as last time. $\square$

Theorem 4.2.2. *Complex Hahn-Banach Theorem:* *Let X be a vector space, $p : X \to \mathbb{R}$ a seminorm, $M \subset X$ a subspace and $f : M \to \mathbb{C}$ a linear functional. If for each $x \in M$, $|f(x)| \leq p(x)$, then there exists a linear functional $F : X \to \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x)$ and $F|_M = f$.*

Exercise 4.2.12. Let X be a normed vector space, $M \subset X$ a subspace and $f \in M^*$. Then there exists $F \in X^*$ such that $\|F\| = \|f\|$ and $F|_M = f$.

Proof. If $f = 0$, Choose $F = 0$. Suppose $f \neq 0$. Then $\|f\| \neq 0$ and there exists $x_0 \in M$ such that $x_0 \neq 0$. Thus $\|f\| = \sup\{|f(x)| : x \in M \text{ and } \|x\| = 1\}$. Define $p : X \to [0, \infty)$ by $p(x) = \|f\|\|x\|$. Then p is a sublinear functional on X and for each $x \in M$, $|f(x)| \leq p(x)$. So there exists a linear functional $F : X \to \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x) = \|f\|\|x\|$ and $F|_M = f$. Thus $F \in X^*$ with $\|F\| \leq \|f\|$. Also

$$\|F\| = \sup_{\substack{x\in X\\ \|x\|=1}} |F(x)| \geq \sup_{\substack{x\in M\\ \|x\|=1}} |F(x)| = \sup_{\substack{x\in M\\ \|x\|=1}} |f(x)| = \|f\|$$

So $\|F\| = \|f\|$. ◻

Exercise 4.2.13. Let X be a normed vector space, $M \subsetneq X$ a proper closed subspace and $x \in X \setminus M$. Then there exists $F \in X^*$ such that $F|_M = 0$, $\|F\| = 1$ and $F(x) = \|x+M\| \neq 0$. (**Hint:** Consider $f : M + \mathbb{C}x \to \mathbb{C}$ defined by $f(m + \lambda x) = \lambda\|x + M\|$.)

Proof. Define $f : M + \mathbb{C}x \to \mathbb{C}$ as above. Clearly f is linear and $f|M = 0$. Let $m \in M$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $|f(m + \lambda x)| = 0 \leq \|m + \lambda x\|$. Suppose that $\lambda \neq 0$. Then

$$\begin{aligned}|f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \inf_{z\in M} \|z + \lambda x\| \\ &\leq \|m + \lambda x\|\end{aligned}$$

So $f \in (M + \mathbb{C}x)^*$ and $\|f\| \leq 1$. Let $\epsilon > 0$. A previous exercise tells us that there exist $m \in M, \lambda \in \mathbb{C}$ such that $\|m + \lambda x\| = 1$ and $\|m + \lambda x + M\| > 1 - \epsilon$. Then

$$\begin{aligned}|f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \|m + \lambda x + M\| \\ &> 1 - \epsilon\end{aligned}$$

So

$$\|f\| = \sup_{\substack{z\in M+\mathbb{C}x\\ \|z\|=1}} |f(z)| \geq 1$$

Hence $\|f\| = 1$. The same exercise also tells us that $f(x) = \|x+M\| \neq 0$. Using the previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{M+\mathbb{C}x} = f$. ◻

Exercise 4.2.14. Let X be a normed vector space and $x \in X$. If $x \neq 0$, then there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$.

Proof. Define $f : \mathbb{C}x \to \mathbb{C}$ by $f(\lambda x) = \lambda\|x\|$. Then f is linear and $f(x) = \|x\|$. Clearly

$$\sup_{\substack{z\in\mathbb{C}x\\ \|z\|=1}} |f(z)| = 1$$

So $f \in (\mathbb{C}x)^*$ and $\|f\| = 1$. By a previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{\mathbb{C}x} = f$. ◻

Exercise 4.2.15. Let X be a normed vector space. Then X^* separates the points of X.

Proof. Let $x, y \in X$. Suppose that $x \neq y$. Then $x - y \neq 0$. The previous exercies implies that there exists $F \in X^*$ such that $\|F\| = 1$ and

$$F(x) - F(y) = F(x - y) = \|x - y\| \neq 0$$

Thus $F(x) \neq F(y)$ and X^* separates the points of X. $\square$

Definition 4.2.16. Let X, Y be metric spaces and $T : X \to Y$. Then T is said to be an **isometry** if for each $x_1, x_2 \in X$, $d(Tx_1, Tx_2) = d(x_1, x_2)$.

Exercise 4.2.17. Let X, Y be metric spaces and $T : X \to Y$ and isometry. Then T is injective.

Proof. Let $x_1, x_2 \in X$. Suppose that $Tx_1 = Tx_2$. Then $0 = d(Tx_1, Tx_2) = d(x_1, x_2)$. So $x_1 = x_2$. Hence T is injective. $\square$

Note 4.2.2. Let X, Y be metric spaces and $T : X \to Y$ an isometry. Then T is clearly continuous. If T is surjective, then T^{-1} is an isometry and therefore continuous. Hence T is a homeomorphism.

Exercise 4.2.18. Let X be a normed vector space and $x \in X$. Define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| = \|x\|$.

Proof. Let $f, g \in X^*$ and $\lambda \in \mathbb{C}$. Then

$$\hat{x}(f + \lambda g) = (f + \lambda g)(x) = f(x) + \lambda g(x) = \hat{x}(f) + \lambda \hat{x}(g)$$

So $\hat{x}$ is linear. For each $f \in X^*$,

$$|\hat{x}(f)| = |f(x)| \leq \|x\|\|f\|$$

Hence $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leq \|x\|$. If $x = 0$, then $\hat{x} = 0$ and $\|\hat{x}\| = \|x\|$. Suppose that $x \neq 0$. Then a previous exercise implies that there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$. Then we have that

$$\sup_{\substack{f \in X^* \\ \|f\|=1}} |\hat{x}(f)| = \sup_{\substack{f \in X^* \\ \|f\|=1}} |f(x)| \geq |F(x)| = \|x\|$$

Hence $\|\hat{x}\| = \|x\|$. $\square$

Exercise 4.2.19. Let X be a normed vector space. Define $\phi : X \to X^{**}$ by $\phi(x) = \hat{x}$. Then ϕ is a linear isometry.

Proof. Let $x, y \in X$ and $\lambda \in \mathbb{C}$. Then for each $f \in X^*$, we have that

$$\begin{aligned}
\phi(x + \lambda y)(f) &= \widehat{x + \lambda y}(f) \\
&= f(x + \lambda y) \\
&= f(x) + \lambda f(y) \\
&= \hat{x}(f) + \lambda \hat{y}(f) \\
&= \phi(x)(f) + \lambda \phi(y)(f)
\end{aligned}$$

So $\phi(x + \lambda y) = \phi(x) + \lambda \phi(y)$ and ϕ is linear. The previous exercise tells us that

$$\begin{aligned}
\|\phi(x) - \phi(y)\| &= \|\phi(x - y)\| \\
&= \|\widehat{x - y}\| = \|x - y\|
\end{aligned}$$

So ϕ is an isometry. $\square$

Definition 4.2.20. Let X be a normed vector space and define $\phi : X \to X^{**}$ as above. We define $\widehat{X} = \phi(X) \subset X^{**}$. Since $\widehat{X}$ and X are isomorphic, we may identify X as a subset of X^{**}.

Definition 4.2.21. Let X be a normed vector space and define $\phi : X \to X^{**}$ as above. Then X is said to be reflexive if ϕ is surjective. In this case ϕ is then an isomorphism

Exercise 4.2.22. Let X be a normed vector space and $f : X \to \mathbb{C}$ a linear functional on X. Then f is bounded iff $\ker f$ is closed.

Proof. Suppose that f is continuous. Since $\{0\}$ is closed, we have that $\ker f = f^{-1}(\{0\})$ is closed. Conversely, suppose that $\ker f$ is closed. If $\ker f = X$, then $f = 0$ and f is continuous. Suppose that $\ker f \neq X$. Then $\ker f$ is a proper, closed subspace of X. A previous exercise tells us that there exists $x \in X$ such that $\|x\| = 1$ and $\|x + \ker f\| > \frac{1}{2}$. Let $y \in X$. Suppose that $\|y\| < \frac{1}{2}$. Then for each $z \in \ker f$,

$$\begin{aligned}\|z - (x + y)\| &= \|(z - x) - y\| \\ &\geq \|z - x\| - \|y\| \\ &> \frac{1}{2} - \frac{1}{2} \\ &= 0\end{aligned}$$

So $x + y \notin \ker f$. Therefore $f(B(x, \frac{1}{2})) \cap \{0\} = \varnothing$. If $f(B(x, \frac{1}{2}))$ is unbounded, then $f(B(x, \frac{1}{2})) = \mathbb{C}$ by linearity. This is a contradiction since $0 \notin f(B(x, \frac{1}{2}))$. So There exists $s > 0$ such that $f(B(x, \frac{1}{2})) \subset B(0, s)$ and thus f is bounded. □

Exercise 4.2.23. Let X be a normed vector space.

(1) Let $M \subsetneq X$ be a proper closed subspace of X and $x \in X \setminus M$. Then $M + \mathbb{C}x$ is closed.
(2) Let $M \subset X$ be a finite dimensional subspace of X. Then M is closed.

Proof. (1) Let $y \in X$ and $(y_n)_{n\in\mathbb{N}} \subset M + \mathbb{C}x$. Suppose that $y_n \to y$. If $y \in M$, then $y \in M + \mathbb{C}x$. Suppose that $y \notin M$. For each $n \in \mathbb{N}$, there exists $m_n \in M$ and $\lambda_n \in \mathbb{C}$ such that $y_n = m_n + \lambda_n x$. A previous exercise tells us that there exists $F \in X^*$ such that $\|F\| = 1$, $F|_M = 0$ and $F(x) = \|x + M\| \neq 0$. Since F is continuous, $F(y_n) \to F(y)$. Since for each $n \in \mathbb{N}$,

$$F(y_n) = F(m_n + \lambda_n x) = F(m_n) + \lambda_n(F_x) = \lambda_n F(x)$$

we have that $\lambda_n F(x) \to F(y)$. Since $F(x) \neq 0$, this implies that $\lambda_n \to F(x)^{-1}F(y)$. It follows that $\lambda_n x \to F(x)^{-1}F(y)x$. Since for each $n \in \mathbb{N}$, $m_n = y_n - \lambda_n x$, we know that $m_n \to y - F(x)^{-1}F(y)x$. Since $(m_n)_{n\in\mathbb{N}} \subset M$ and M is closed, we have that $y - F(x)^{-1}F(y)x \in M$ and therefore $y \in M + \mathbb{C}x$. Hence $M + \mathbb{C}x$ is closed.

(2) If $M = X$, then M is closed. Suppose that $M \neq X$. Let $(x_i)_{i=1}^n$ be a basis for M. Define $N_0 = \{0\}$ and for each $i = 1, 2, \cdots, n$, define $N_i = N_{i-1} + \mathbb{C}x_i$. Since N_0 is a proper closed subpace of X and $x_1 \in X \setminus N_0$, (1) implies that N_1 is closed. Proceed inductively to obtain that $M = N_n$ is closed.

□

Exercise 4.2.24. Let X be an infinite-dimensional normed vector space.

(1) There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that for each $m, n \in \mathbb{N}$, $\|x_n\| = 1$ and if $m \neq n$, then $\|x_m - x_n\| > \frac{1}{2}$.
(2) X is not locally compact.

Proof. (1) Define $N_0 = \{0\}$. Then N_0 is a closed proper subspace of X. Choose $x_1 \in X$ such that $\|x_1\| = 1$. Using the results of previous exercises, we proceed inductively. For each $n \geq 2$ we define $N_{n-1} = \text{span}(x_1, x_2, \cdots, x_{n-1})$. Then N_{n-1} is a closed proper subspace of X. Thus we may choose $x_n \in X$ such that $\|x_n\| = 1$ and $\|x_n + N_{n-1}\| > \frac{1}{2}$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then $x_m \in N_{n-1}$. Thus $\|x_n - x_m\| \geq \|x_n + N_{n-1}\| > \frac{1}{2}$

(2) Suppose that X is locally compact. Then $\overline{B(0,1)}$ is compact and therefore sequentially compact. Using $(x_n)_{n\in\mathbb{N}} \subset \overline{B(0,1)}$ defined in (1), we see that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, $x \in \overline{B(0,1)}$ such that $x_{n_k} \to x$. Then $(x_{n_k})_{k\in\mathbb{N}}$ is Cauchy. So there exists $N \in N$ such that for each $j, k \in \mathbb{N}$, if $j, k \geq N$, then $\|x_{n_j} - x_{n_k}\| < \frac{1}{2}$. Then $\|x_{n_N} - x_{n_{N+1}}\| < \frac{1}{2}$. This is a contradiction since by construction, $\|x_{n_N} - x_{n_{N+1}}\| > \frac{1}{2}$. Thus X is not locally compact.

□

Exercise 4.2.25. Let X, Y be normed vector spaces and $T \in L(X, Y)$.

(1) Define the **adjoint of** T, denoted $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$. Then $T^* \in L(Y^*, X^*)$.
(2) Applying the result from (1) twice, we have that $T^{**} \in L(X^{**}, Y^{**})$. We have that for each $x \in X$, $T^{**}(\hat{x}) = \widehat{T(x)}$.
(3) T^* is injective iff $T(X)$ is dense in Y.
(4) If $T^*(Y^*)$ is dense in X^*, then T is injective. The converse is true if X is reflexive.

Proof. (1) Let $f \in Y^*$. Then $\|T^*(f)\| = \|f \circ T\| \leq \|T\|\|f\|$. So $T^* \in L(Y^*, X^*)$ with $\|T^*\| \leq \|T\|$.

(2) Let $x \in X$. Let $f \in Y^*$. Then

$$\begin{aligned} T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= \widehat{T(x)}(f) \end{aligned}$$

Hence $T^{**}(\hat{x}) = \widehat{T(x)}$.

(3) Suppose that $T(X)$ is not dense in Y. Then $\overline{T(X)} \neq Y$. So $T(X)$ is a proper closed subspace of Y and there exists $y \in Y$ such that $y \notin \overline{T(X)}$. By a previous exercise, there exists $f \in Y^*$ such that $f(y) = \|y + \overline{T(X)}\| \neq 0$, $\|f\| = 1$ and $f|_{\overline{T(X)}} = 0$. Let $x \in X$. Then $T^*(f)(x) = f \circ T(x) = 0$. Hence $T^*(f) = 0 = T^*(0)$. Since $f \neq 0$, T^* is not injective.
Now suppose that $T(X)$ is dense in Y. Let $f, g \in Y^*$. Define $h \in Y^*$ by $h = f - g$ Suppose that $T * (f) = T^*(g)$ Then $T^*(h) = 0$. So for each $x \in X$, $h(T(x)) = 0$. Let $y \in Y$ and $\epsilon > 0$. By continuity, there exists $\delta > 0$ such that for each $y' \in Y$, if $\|y - y'\| < \delta$, then $\|h(y) - h(y')\| < \epsilon$. Since $T(X)$ is dense in Y, there exists $x \in X$ such that $\|y - T(x)\| < \delta$. Thus

$$\begin{aligned}\|h(y)\| &\leq \|h(y) - h(T(x))\| + \|h(T(x))\| \\ &= \|h(y) - h(T(x))\| \\ &< \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|h(y)\| = 0$. This implies that $h(y) = 0$ and therefore $f(y) = g(y)$. Since $y \in Y$ is arbitrary, $f = g$ and T^* is injective.

(4) For the sake of contradiction, suppose that $T^*(Y^*)$ is dense in X^* and T is not injective. Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $T(x_1) = T(x_2)$. Define $x = x_1 - x_2$. Then $x \neq 0$ and $T(x) = 0$. A previous exercise implies that there exists $F \in X^*$ such that $F(x) = \|x\| \neq 0$ and $\|F\| = 1$. Let $\epsilon > 0$. Choose $g \in Y^*$ such that $\|F - T^*(g)\| < \epsilon$. Then

$$\begin{aligned}\|x\| &= |F(x)| \\ &\leq |F(x) - T^*(g)(x)| + |T^*(g)(x)| \\ &< \epsilon\|x\| + |g(T(x))| \\ &= \epsilon\|x\|\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\|x\| = 0$ which is a contradiction. Hence if $T^*(Y^*)$ is dense in X^*, then T is injective.

Now, suppose that X is reflexive and T is injective. Let $\phi_1, \phi_2 \in X^{**}$. Suppose that $T^{**}(\phi_1) = T^{**}(\phi_2)$. Then $T^{**}(\phi_1 - \phi_2) = 0$. Since X is reflexive, there exist $x_1, x_2 \in X$ such that $\phi_1 = \hat{x_1}$ and $\phi_2 = \hat{x_2}$. Define $x = x_1 - x_2$. Then $T^{**}(\hat{x}) = 0$. So for each $f \in Y^*$,

$$\begin{aligned}T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= 0\end{aligned}$$

Suppose that $T(x) \neq 0$. Then a previous exercise implies that there exists $g \in Y^*$ such that $g(T(x)) = \|T(x)\| \neq 0$ and $\|g\| = 1$. This is a contradiction since $g(T(x)) = 0$.

So $T(x) = 0$. Since T is injective, this implies that $x = 0$. Hence $\hat{x} = 0$ and thus $\phi_1 = \phi_2$. Thus T^{**} is injective. By (3), we have that $T^*(Y^*)$ is dense in X^*.

$\square$

Exercise 4.2.26. Let X be a normed vector space. Then X is reflexive iff X^* is reflexive.

Proof. Suppose that X is reflexive. Let $\alpha \in X^{***}$. Define $f : X \to \mathbb{C}$ by $f(x) = \alpha(\hat{x})$. Clearly f is linear and a previous exercise tells us that for each $x \in X$,

$$\begin{aligned} |f(x)| &\leq \|\alpha\|\|\hat{x}\| \\ &= \|\alpha\|\|x\| \end{aligned}$$

So $f \in X^*$. Let $\phi \in X^{**}$. Since X is reflexive, there exists $x \in X$ such that $\phi = \hat{x}$. Then

$$\begin{aligned} \alpha(\phi) &= \alpha(\hat{x}) \\ &= f(x) \\ &= \hat{x}(f) \\ &= \hat{f}(\hat{x}) \\ &= \hat{f}(\phi) \end{aligned}$$

Hence $\alpha = \hat{f}$. Thus the map $X^* \to X^{***}$ given by $f \mapsto \hat{f}$ is surjective and so X^* is reflexive.

Conversely, suppose that X^* is reflexive. Since $\phi : X \to X^{**}$ given by $\phi(x) = \hat{x}$ is an isometry, $\widehat{X} \subset X^{**}$ is closed. For the sake of contradiction, suppose that $\widehat{X} \neq X^{**}$. Then there exists $\alpha \in X^{**}$ such that $\alpha \notin \widehat{X}$. Thus there exists $F \in X^{***}$ such that $\|F\| = 1$, $F(\alpha) = \|\alpha + \widehat{X}\| \neq 0$ and $F|_{\widehat{X}} = 0$. Since X^* is reflexive, there exists $f \in X^*$ such that $F = \hat{f}$. A previous exercise tells us that $\|f\| = \|\hat{f}\| = \|F\| = 1$. Since for each $x \in X$, $f(x) = \hat{x}(f) = \hat{f}(\hat{x}) = F(\hat{x}) = 0$, we have that $f = 0$. Thus $\|f\| = 0$, a contradiction. So $\widehat{X} = X^{**}$ and X is reflexive.

$\square$

4.3. **The Baire Category and Closed Graph Theorems.**

Theorem 4.3.1. *Let X, Y be Banach spaces and $T \in L(X,Y)$. If T is surjective, then T is open.*

Corollary 4.3.2. *Let X, Y be Banach spaces and $T \in L(X,Y)$. If T is a bijection, then $T^{-1} \in L(X,Y)$.*

Definition 4.3.1. Let X, Y be sets and $f : X \to Y$. We define the **graph of f**, $\Gamma(f)$, by $\Gamma(f) = \{(x,y) \in X \times Y : f(x) = y\}$.

Theorem 4.3.3. *Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If $\Gamma(T)$ is closed, then $T \in L(X,Y)$.*

Note 4.3.1. We recall that $\Gamma(T)$ is closed iff for each $(x_n)_{n\in\mathbb{N}} \subset X$, $x \in X$ and $y \in Y$, $x_n \to x$ and $T(x_n) \to y$ implies that $T(x) = y$.

Theorem 4.3.4. *Let X, Y be Banach spaces and $S \subset L(X,Y)$. If for each $x \in X$,*

$$\sup_{T\in S} \|Tx\| < \infty$$

then

$$\sup_{T\in S} \|T\| < \infty$$

Exercise 4.3.2. Let μ be counting measure on $(N, \mathcal{P}(\mathbb{N}))$. Define $h : \mathbb{N} \to \mathbb{N}$ and ν on $(N, \mathcal{P}(\mathbb{N}))$ by $h(n) = n$ and $d\nu = hd\mu$. Define $X = L^1(\nu)$ and $Y = L^1(\mu)$. Equip both X and Y with the L^1 norm with respect to μ.

(1) We have that X is a proper subspace of Y and therefore X is not complete.
(2) Define $T : X \to Y$ by $Tf(n) = nf(n)$. Then T is linear, $\Gamma(T)$ is closed, and T is unbounded.
(3) Define $S : Y \to X$ by $Sg(n) = \frac{1}{n}g(n)$. Then $S \in L(Y,X)$, S is surjective and S is not open.

Proof.

(1) Note that for each $f : \mathbb{N} \to \mathbb{C}$,

$$\begin{aligned} \|f\|_{\mu,1} &= \sum_{n=1}^{\infty} |f(n)| \\ &\leq \sum_{n=1}^{\infty} n|f(n)| \\ &= \|f\|_{\nu,1} \end{aligned}$$

Hence X is a subspace of Y. Define $f : \mathbb{N} \to \mathbb{C}$ by $f(n) = \frac{1}{n^2}$. Then

$$\|f\|_{\mu,1} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

So $f \in Y$. However

$$\|f\|_{\nu,1} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

So $f \notin X$. Thus X is a proper subspace of Y. Let $g \in Y$ and $\epsilon > 0$. Since the simple functions are dense in $L^1(\mu)$, there exists $\phi \in L^1(\mu)$ such that ϕ is simple and $\|g - \phi\|_{\mu,1} < \epsilon$. Then there exist $(c_i)_{i=1}^k \subset \mathbb{C}$ and $(E_i)_{i=1}^k \subset \mathcal{P}(\mathbb{N})$ such that for each $i, j \in \{1, 2, \cdots, k\}$, E_i is finite, $i \neq j$ implies that $E_i \cap E_j = \varnothing$ and

$$\phi = \sum_{i=1}^k c_i \chi_{E_i}$$

Define $c = \max\{|c_i| : i = 1, 2, \cdots k\}$ and $m = \max \left[\bigcup_{i=1}^k E_i\right]$. Then

$$\begin{aligned}
\|\phi\|_{\nu,1} &= \sum_{n=1}^m n|\phi(n)| \\
&\leq \sum_{n=1}^m mc \\
&= cm^2 \\
&< \infty
\end{aligned}$$

Hence $\phi \in X$ and X is dense in Y. Since X is a dense, proper subspace, it is not closed. Since Y is complete and $X \subset Y$ is not closed, we have that X is not complete.

(2) Clearly T is linear. Let $(f_j)_{j\in\mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_j \xrightarrow{L^1(\mu)} f$ and $Tf_j \xrightarrow{L^1(\mu)} g$.

Note that for each $j \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$|f_j(n) - f(n)| \leq \sum_{n=1}^\infty |f_j(n) - f(n)| = \|f_j - f\|_{\mu,1}$$

and

$$|nf_j(n) - g(n)| \leq \sum_{n=1}^\infty |nf_j(n) - g(n)| = \|Tf_j - g\|_{\mu,1}$$

Thus for each $n \in \mathbb{N}$, $f_j(n) \xrightarrow{j} f(n)$ and $nf_j(n) \xrightarrow{j} g(n)$. This implies that for each $n \in \mathbb{N}$, $nf(n) = g(n)$. Thus $Tf = g$ which implies that $\Gamma(T)$ is closed. Suppose, for the sake of contradiction, that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\|_{\mu,1} \leq C\|f\|_{\mu,1}$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f : \mathbb{N} \to \mathbb{C}$ by $f = \chi_{\{n\}}$. As established above, $S^+ \subset L^1(\mu)$. Then $\|f\|_{\mu,1} = 1$ and

$$\begin{aligned}
\|Tf\|_{\mu,1} &= n \\
&> C \\
&= C\|f\|_{\mu,1}
\end{aligned}$$

which is a contradiction. So T is unbounded.

(3) Clearly S is linear. Let $g \in Y$. Then

$$\begin{aligned}\|Sg\|_{\mu,1} &= \sum_{n=1}^{\infty} \frac{1}{n}|g(n)| \\ &\leq \sum_{n=1}^{\infty} |g(n)| \\ &= \|g\|_{\mu,1}\end{aligned}$$

So S is bounded and $\|S\| \leq 1$. Thus $S \in L(Y, X)$. Let $f \in X$. Define $g : \mathbb{N} \to \mathbb{C}$ by $g(n) = nf(n)$. By defnition, $g \in Y$ and we have that

$$\begin{aligned}Sg(n) &= \frac{1}{n}g(n) \\ &= f(n)\end{aligned}$$

Hence $Sg = f$ and thus S is surjective. Let $g \in Y$. Suppsose that $Sg = 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n}|g(n)| = \|Sg\| = 0$$

Thus for each $n \in \mathbb{N}$, $g(n) = 0$. Hence $\ker S = \{0\}$ and S is injective. Note that for each $A \subset Y$, $S(A) = T^{-1}(A)$. If S is open, then T is continuous which as shown above is a contradiction. So g is not open.

□

Exercise 4.3.3. Let $X = C^1([0,1])$ and $Y = C([0,1])$. Equip both X and Y with the uniform norm.

(1) Then X is not complete
(2) Define $T : X \to Y$ by $Tf = f'$. Then $\Gamma(T)$ is closed and T is not bounded.

Proof. (1) Recall that for each $a, b \geq 0$ and $p \in \mathbb{N}$,

$$(a^{\frac{1}{p}} + b^{\frac{1}{p}})^p = \sum_{n=0}^{p} \binom{p}{n} a^{\frac{n}{p}} b^{\frac{p-n}{p}} \geq a + b$$

Thus $(a+b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$.

For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{C}$ by $f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n^2}}$. Then $(f_n)_{n\in\mathbb{N}} \subset X$. Define $f : [0,1] \to \mathbb{C}$ by $f(x) = |x - \frac{1}{2}|$. Then $f \in Y \cap X^c$. Note that for each $n \in \mathbb{N}$, $f \leq f_n$. Our observation above implies that for each $x \in X$,

$$\begin{aligned}f_n(x) &= \left[(x - \frac{1}{2})^2 + \frac{1}{n^2}\right]^{\frac{1}{2}} \\ &\leq |x - \frac{1}{2}| + \frac{1}{n}\end{aligned}$$

Thus $0 \leq f_n - f \leq \frac{1}{n}$. This implies that $f_n \xrightarrow{\text{u}} f$. Since $f \notin X$, X is not complete.

(2) Let $(f_n)_{n\in\mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_n \xrightarrow{\text{u}} f$ and $Tf_n \xrightarrow{\text{u}} g$. Let $x \in [0,1]$. Then $f_n(x) \to f(x)$ and $f_n(0) \to f(0)$ and $f_n' \xrightarrow{u} g$. Applying the DCT to this sequence of integrable functions that converges uniformly to an integrable function on a finite measure space (a previous exercise) we have that

$$\begin{aligned} f_n(x) - f_n(0) &= \int_{[0,x]} f_n' dm \\ &\to \int_{[0,x]} g dm \end{aligned}$$

Since $f_n(x) - f_n(0) \to f(x) - f(0)$, we know that

$$f(x) - f(0) = \int_{[0,x]} g dm$$

. Thus $Tf = g$ and $\Gamma(T)$ is closed.
Suppose for the sake of contradiction that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\| \leq C\|f\|$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f \in X$ by $f(x) = x^n$. Then $\|f\| = 1$ and

$$\begin{aligned} \|Tf\| &= \|f'\| \\ &= n \\ &> C \\ &= C\|f\| \end{aligned}$$

which is a contradiction. So T is not bounded.

□

Exercise 4.3.4. Let X, Y be Banach spaces and $T \in L(X,Y)$. Then $X/\ker T \cong T(X)$ iff $T(X)$ is closed.

Proof. Since X is a banach space and T is continuous, we have that $\ker T$ is closed and $X/\ker T$ is a Banach space. Suppose that $X/\ker T \cong T(X)$. Then $T(X)$ is complete. Since Y is complete, this implies that $T(X)$ is closed.
Conversely Suppose that $T(X)$ is closed. Then $T(X)$ is complete. Define $S : X/\ker T \to T(X)$ by $S(x+\ker T) = T(x)$. A previous exercise tells us that the map $S : X/\ker T \to T(X)$ defined by $S(x + \ker T) = T(x)$ is a bounded linear bijection. Since $T(X)$ is complete and S is surjective, S^{-1} is bounded and thus S is an isomorphism. □

Exercise 4.3.5. Let X be a separable Banach space. Define $B_X = \{x \in X : \|x\| < 1\}$. Let $(x_n)_{n\in\mathbb{N}} \subset B_X$ a dense subset of the unit ball and μ the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Define $T : L^1(\mu) \to X$ by

$$Tf = \sum_{n=1}^{\infty} f(n)x_n$$

Then

(1) T is well defined and $T \in L(L^1(\mu), X)$
(2) T is surjective
(3) There exists a closed subspace $K \subset L^1(\mu)$ such that $L^1(\mu)/K \cong X$

Proof. (1) Let $f \in L^1(\mu)$. Since X is complete and

$$\begin{aligned}\sum_{n=1}^{\infty} \|f(n)x_n\| &= \sum_{n=1}^{\infty} |f(n)|\|x_n\| \\ &\leq \sum_{n=1}^{\infty} |f(n)| \\ &< \infty\end{aligned}$$

we have that $\sum_{n=1}^{\infty} f(n)x_n$ converges and thus $Tf \in X$. Hence T is well defined.

Clearly T is linear. Let $f \in L^1(\mu)$. Then

$$\begin{aligned}\|Tf\| &= \|\sum_{n=1}^{\infty} f(n)x_n\| \\ &\leq \sum_{n=1}^{\infty} \|f(n)x_n\| \\ &\leq \sum_{n=1}^{\infty} |f(n)| \\ &= \|f\|_1\end{aligned}$$

So T is bounded with $\|T\| \leq 1$.

(2) Let $x \in X$. Suppose that $\|x\| < 1$. Then $x \in B_X$. So there exists $n_1 \in \mathbb{N}$ such that $\|x - x_{n_1}\| < \frac{1}{2}$. Then $2(x - x_{n_1}) \in B_X$. Since for each $j \in \mathbb{N}$, $B_X \setminus (x_n)_{n=1}^{j}$ is dense in B_X, there exists $n_2 \in \mathbb{N}$ such that $x_{n_2} \notin (x_n)_{n=1}^{n_1}$ and $\|2(x - x_{n_1}) - x_{n_2}\| < \frac{1}{2}$ which implies that $\|x - (x_{n_1} - \frac{1}{2}x_{n_2})\| < \frac{1}{4}$.

Proceed inductively to obtain a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that for each $k \geq 2$, $x_{n_k} \notin (x_n)_{n=1}^{n_{k-1}}$ and $\|x - \sum_{j=1}^{k} 2^{1-j}x_{n_j}\| < \frac{1}{2^k}$. Then $x = \sum_{k=1}^{\infty} 2^{1-k}x_{n_k}$.

Define $f : \mathbb{N} \to \mathbb{C}$ by $f = \sum_{k=1}^{\infty} 2^{1-k}\chi_{\{n_k\}}$. Then $\|f\|_1 = \sum_{k=1}^{\infty} 2^{1-k} < \infty$, so $f \in L^1(\mu)$ and $Tf = \sum_{k=1}^{\infty} 2^{1-k}x_{n_k} = x$. Now, suppose that $\|x\| \geq 1$, then $\frac{1}{2\|x\|}x \in B_X$. The above argument shows that there exists $f \in L^1(\mu)$ such that $Tf = \frac{1}{2\|x\|}x$. Then $2\|x\|f \in L^1(\mu)$ and $T(2\|x\|f) = 2\|x\|Tf = x$.
So for each $x \in X$, there exists $f \in L^1(\mu)$ such that $Tf = x$ and thus T is surjective.

(3) Since X is a Banach space and T is surjective, the previous exercise implies that $L^1(\mu)/\ker T \cong X$.

□

Exercise 4.3.6. Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If for each $f \in Y^*$, $f \circ T \in X^*$, then $T \in L(X, Y)$.

Proof. Suppose that for each $f \in Y^*$, $f \circ T \in X^*$. Let $x \in X$, □

4.4. **Banach Algebras.**

Definition 4.4.1. Let X be a Banach space and an associative algebra. Then X is said to be a **Banach algebra** if for each $S, T \in X$, $\|ST\| \leq \|S\|\|T\|$. If there exists $I \in X$ such that $I \neq 0$ and for each $T \in X$, $IT = TI = T$, then X is said to be **unital** with identity I. An element $T \in X$ is said to be **invertible** if there exists $S \in X$ such that $TS = ST = I$.

Exercise 4.4.2. Let X be a unital Banach algebra. Then $\|I\| \leq 1$.

Proof. Since $I \neq 0$, $\|I\| \neq 0$. By definition,

$$\|I\| = \|II\| \leq \|I\|\|I\|$$

Hence $1 \leq \|I\|$. □

Note 4.4.1. If X is a Banach space, then a previous exercise implies that $L(X, X)$ equipped with composition is a unital Banach algebra where I is the identity operator. It is easy to see that $\|I\| = 1$.

Note 4.4.2. Let X be a Banach algebra. Then the set of invertible elements in X is a group.

Exercise 4.4.3. Let X be a Banach algebra. Then mulitplication is continuous.

Proof. Let $(S_1, T_1) \in X \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|S_1\|+\|T_1\|+1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(S_2, T_2) \in X \times X$. Suppose that

$$\|(S_1, T_1) = (S_2, T_2)\| = \max\{\|S_2 - S_2\|, \|T_1 - T_2\|\} < \delta$$

Then

$$\begin{aligned}
\|S_1T_1 - S_2T_2\| &= \|S_1T_1 - S_2T_1 + S_2T_1 - S_2T_2\| \\
&\leq \|S_1 - S_2\|\|T_1\| + \|S_2\|\|T_1 - T_2\| \\
&\leq \|S_1 - S_2\|\|T_1\| + \big(\|S_1 - S_2\| + \|S_1\|\big)\|T_1 - T_2\| \\
&\leq \delta\|T_1\| + (\delta + \|S_1\|)\delta \\
&= \delta(\|S_1\| + \|T_1\|) + \delta^2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

4.5. **Differentiability.**

Note 4.5.1. In this section, we assume all Banach spaces to be over $\mathbb{R}$.

Definition 4.5.1. Let X, Y be a Banach spaces, $A \subset X$ open, $f : A \to Y$, $x_0 \in A$ and $x \in X$. Then f is said to be

(1) **right-hand-differentiable at x_0 in the direction** x if the limit
$$\lim_{t \to 0^+} \frac{f(x_0 + tx) - f(x_0)}{t}$$
exists. If f is right-hand-differentiable at x_0 in the direction x, we define the **right-hand derivative** of f at x_0 in the direction x, denoted by $d^+f(x_0; x)$, to be the above limit.

(2) **left-hand-differentiable at x_0 in the direction** x if the limit
$$\lim_{t \to 0^-} \frac{f(x_0 + tx) - f(x_0)}{t}$$
exists. If f is right-hand-differentiable at x_0 in the direction x, we define the **left-hand derivative** of f at x_0 in the direction x, denoted by $d^-f(x_0; x)$, to be the above limit.

(3) **differentiable at x_0 in the direction** x if the limit
$$\lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t}$$
exists. If f is differentiable at x_0 in the direction x, we define the **derivative** of f at x_0 in the direction x, denoted by $df(x_0; x)$, to be the above limit.

Exercise 4.5.2. Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to \mathbb{R}$ and $x_0 \in A$. Then $df(x_0; 0) = 0$.

Proof. Clear. □

Definition 4.5.3. Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Then f is said to be

(1) **right-hand Gateaux differentiable** at x_0 if for each $x \in X$, $d^+f(x_0; x)$ exits. We define the **right-hand Gateaux derivative** of f at x_0, denoted $d^+f(x_0) : X \to \mathbb{R}$, to be
$$d^+f(x_0)(x) = d^+f(x_0; x)$$

(2) **left-hand Gateaux differentiable** at x_0 if for each $x \in X$, $d^-f(x_0; x)$ exits. We define the **left-hand Gateaux derivative** of f at x_0, denoted $d^-f(x_0) : X \to \mathbb{R}$, to be
$$d^-f(x_0)(x) = d^-f(x_0; x)$$

(3) **Gateaux differentiable** at x_0 if for each $x \in X$, $df(x_0; x)$ exits. We define the **Gateaux derivative** of f at x_0, denoted $df(x_0) : X \to \mathbb{R}$, to be
$$df(x_0)(x) = df(x_0; x)$$

Exercise 4.5.4. Let X, Y be Banach spaces, $A \subset X$ open, $f, g : A \to Y$, $\lambda \in \mathbb{R}$ and $x_0 \in A$. If f, g are Gateaux differentiable at x_0, then $f + \lambda g$ Gateaux differentiable at x_0 and $d[f + \lambda g](x_0) = df(x_0) + \lambda dg(x_0)$.

Proof. Similar to the case of the derivative from Calc I. □

Exercise 4.5.5. Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Suppose that f is Gateaux differentiable at x_0. Then for each $\lambda \in \mathbb{R}$ and $x \in X$,

$$df(x_0)(\lambda x) = \lambda df(x_0)(x) \in X^*$$

Proof. Let $\lambda \in \mathbb{R}$ and $x \in X$. Then

$$\begin{aligned} df(x_0)(\lambda x) &= \lim_{t \to 0} \frac{f(x_0 + t\lambda x) - f(x_0)}{t} \\ &= \lim_{t \to 0} \lambda \frac{f(x_0 + t\lambda x) - f(x_0)}{\lambda t} \\ &= \lambda \lim_{t \to 0} \frac{f(x_0 + t\lambda x) - f(x_0)}{\lambda t} \\ &= \lambda \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= \lambda df(x_0)(x) \end{aligned}$$

□

Exercise 4.5.6. Let X be a Banach space, $A \subset X$ open, $f : A \to \mathbb{R}$ and $x_0 \in A$. If f is Gateaux differentiable at x_0 and f has a local extremum at x_0, then $df(x_0) = 0$.

Proof. Suppose that f is Gateaux differentiable at x_0 and f has a local minimum at x_0. Then there exists $\delta > 0$ such that $B(x_0, \delta) \subset A$ and for each $y \in B(x_0, \delta)$, $f(x_0) \leq f(y)$.
For the sake of contradiction, suppose that $df(x_0) \neq 0$. Then there exists $x \in X$ such that $x \neq 0$ and $df(x_0)(x) \neq 0$.
First, suppose that $df(x_0)(x) < 0$. Choose $\epsilon = -df(x_0)(x) > 0$. Then there exists $t_0 > 0$ such that for each $t \in B^*(0, t_0)$, $x_0 + tx \in B(x_0, \delta)$ and

$$\left| \frac{f(x_0 + tx) - f(x_0)}{t} - df(x_0)(x) \right| < \epsilon$$

This implies that for each $t \in B^*(0, t_0)$,

$$\begin{aligned} \frac{f(x_0 + tx) - f(x_0)}{t} &< \epsilon + df(x_0)(x) \\ &= 0 \end{aligned}$$

and hence $f(x_0 + tx) < f(x_0)$, which is a contradiction.
Now, suppose that $df(x_0)(x) > 0$. Then

$$\begin{aligned} df(x_0)(-x) &= -df(x_0)(x) \\ &< 0 \end{aligned}$$

Similarly to above, this implies that there exists $t_0 > 0$ such that for each $t \in B^*(0, t_0)$, $x_0 - tx \in B(x_0, \delta)$ and $f(x_0 - tx) < f(x_0)$ which is a contradiction. So $df(x_0)(x) = 0$ and $df(x_0) = 0$.
If f has a local maximum at x_0, then $-f$ has a local minimum at x_0. Then

$$\begin{aligned} df(x_0) &= -d[-f](x_0) \\ &= -0 \\ &= 0 \end{aligned}$$

□

Exercise 4.5.7. Let X, Y be a normed vector spaces and $\phi : X \to Y$ linear. If $\phi(h) = o(\|h\|)$ as $h \to 0$, then $\phi = 0$.

Proof. Let $h_0 \in X$. If $h_0 = 0$, then $\phi(h_0) = 0$. Suppose that $h_0 \neq 0$. Define $(h_n)_{n\in\mathbb{N}} \subset X$ by

$$h_n = \frac{h_0}{n}$$

Then $h_n \to 0$. By continuity of ϕ and our initial assumption we have that

$$\begin{aligned}\|h_0\|^{-1}\phi(h_0) &= \phi\left(\frac{h_0}{\|h_0\|}\right) \\ &= \phi\left(\frac{h_n}{\|h_n\|}\right) \\ &= \frac{\phi(h_n)}{\|h_n\|} \\ &\to 0\end{aligned}$$

which implies that $\|h_0\|^{-1}\phi(h_0) = 0$. So $\phi(h_0) = 0$ and hence $\phi = 0$. □

Exercise 4.5.8. Let X, Y be a normed vector spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Suppose that there exists $\phi : X \to Y$ such that ϕ is linear and

$$f(x_0 + h) = f(x_0) + \phi(h) + o(\|h\|) \quad \text{as } h \to 0$$

then ϕ is unique.

Proof. Suppose that there exists $\psi : X \to Y$ such that ψ is linear and such that

$$f(x_0 + h) = f(x_0) + \psi(h) + o(\|h\|) \quad \text{as } h \to 0$$

Then $\phi(h) - \psi(h) = o(h)$. Since $\phi - \psi$ is linear, the previous exercise implies that $\phi = \psi$. □

Definition 4.5.9. Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Then f is said to be **Frechet differentiable at** x_0 if there exists $Df(x_0) \in L(X, Y)$ such that,

$$f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|) \quad \text{as } h \to 0$$

If f is Frechet differentiable at x_0, we define the **Frechet derivative of** f **at** x_0 to be $Df(x_0)$.

Exercise 4.5.10. Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. If f is Frechet differentiable at x_0, then f is Gateaux differentiable at x_0 and $df(x_0) = Df(x_0)$.

Proof. Suppose that f is Frechet differentiable at x_0. Then $f(x_0+h) = f(x_0)+Df(x_0)(h)+o(\|h\|)$ as $h \to 0$. Let $x \in X$. Then $f(x_0 + tx) - f(x_0) = tDf(x_0)(x) + o(t)$ as $t \to 0$. This implies that f is differentiable at x_0 in the direction x and

$$\begin{aligned}df(x_0)(x) &= \lim_{t\to 0} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= Df(x_0)(x)\end{aligned}$$

Since $x \in X$ is arbitrary, f is Gateaux differentiable at x_0 and $df(x_0) = Df(x_0)$. □

Exercise 4.5.11. Let X be a Banach space, $A \subset X$ open, $f : A \to \mathbb{R}$ and $x_0 \in A$. If f is Frechet differentiable at x_0 and f has a local extremum at x_0, then $df(x_0) = 0$.

Proof. Suppose that f is Frechet differentiable at x_0 and f has a local extremum at x_0, then $df(x_0) = 0$. Two previous exercises imply that f is Gateaux differentiable at x_0 and

$$\begin{aligned} Df(x_0) &= df(x_0) \\ &= 0 \end{aligned}$$

□

Note 4.5.2. Recall that for Banach spaces X and Y, there isomorphic isometry $L(X, L(X, \cdots, L(X, Y)) \cdots)$ $L^n(X, Y)$ given by $\phi \mapsto \psi_\phi$ where

$$\psi_\phi(x_1, x_2, \cdots, x_n) = \phi(x_1)(x_2), \cdots, (x_n)$$

Definition 4.5.12. Let X, Y be a Banach spaces, $A \subset X$ open and $f : A \to Y$. Then f is said to be **Frechet differentiable** (or 1**-st order Frechet differentiable**) if for each $x \in A$, f is Frechet differentiable at x.
If f is Frechet differentiable, we define the **(first order) Frechet derivative of** f, denoted $D^{(1)}f : A \to L(X, Y)$, by $x \mapsto D^{(1)}f(x)$. We define higher order Frechet derivatives inductively:
Let $x_0 \in A$ and $n \geq 2$. Then f is said to be n**-th order Frechet differentiable at** x_0 if f is $(n-1)$-th order Frechet differentiable and $D^{n-1}f$ is Frechet differentiable at x_0. If f is n-th order Frechet differentiable at x_0, we define $D^n f(x_0) \in L^n(\bigoplus_{i=1}^n X, Y)$ by $D^n f(x_0) = D[D^{n-1}f](x_0)$. Finally, f is said to be n**-th order Frechet differentiable** if f is $(n-1)$-th order Frechet differentiable and for each $x \in A$, $D^{n-1}f$ is Frechet differentiable at x. If f is n-th order Frechet differentiable, we define the n**-th order Frechet derivative of** f, denoted $D^n f : A \to L^n(\bigoplus_{i=1}^n X, Y)$ by $D^n f = D[D^{n-1}f]$.

Exercise 4.5.13. Let X, Y be a Banach spaces, $A \subset X$ open, $f : A \to Y$, $x_0 \in A$ and $n \in \mathbb{N}$. Then f is n-th order Frechet differentiable at x_0 iff for each $i \in \{1, \cdots, n\}$, there exists $\phi_i \in L^i(\bigoplus_{j=1}^i X, Y)$ such that

$$f(x+h) = \sum_{i=1}^n \phi_i(h, \cdots, h) + o(\|h\|^n)$$

Proof. □

4.6. l^p Spaces.

Definition 4.6.1. Let $p \in [1, \infty] \cup \{0\}$. We define

$$l^p(\mathbb{N}) = \begin{cases} \mathbb{C}^{\mathbb{N}} & p = 0 \\ \left\{ f \in l^0(\mathbb{N}) : \sum_{n \in \mathbb{N}} |f(n)|^p < \infty \right\} & p \in [1, \infty) \\ \left\{ f \in l^0(\mathbb{N}) : \sup_{n \in \mathbb{N}} |f(n)| < \infty \right\} & p = \infty \end{cases}$$

So $l^0(\mathbb{N})$ consists of the sequences in $\mathbb{C}$ and $l^\infty(\mathbb{N})$ consists of the bounded sequences in $\mathbb{C}$.

For $p \in [1, \infty]$, we define $\| \cdot \|_p : l^p(\mathbb{N}) \to [0, \infty)$ by

$$\|f\|_p = \begin{cases} \left(\sum_{n \in \mathbb{N}} |f(n)|^p \right)^{1/p} & p \in [1, \infty) \\ \sup_{n \in \mathbb{N}} |f(n)| & p = \infty \end{cases}$$

5. Hilbert Spaces

Definition 5.0.1. Let H be a vector space and $\langle \cdot, \cdot \rangle : H \to \mathbb{C}$. Then $\langle \cdot, \cdot \rangle$ is said to be an **inner product** on H if for each $x, y, z \in H$and $c \in \mathbb{C}$

(1) $\langle x, y + cz \rangle = \langle x, y \rangle + c\langle x, z \rangle$
(2) $\langle x, y \rangle = \langle y, x \rangle^*$
(3) $\langle x, x \rangle \geq 0$
(4) if $\langle x, x \rangle = 0$, then $x = 0$.

Exercise 5.0.2. Let H be an inner product space, $(x_j)_{j=1}^n$, $(y_j)_{j=1}^n \subset H$ and $(\alpha_j)_{j=1}^n$, $(\beta_j)_{j=1}^n \subset \mathbb{C}$. Then

$$\left\langle \sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^n \beta_j y_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^n \alpha_i^* \beta_j \langle x_i, y_j \rangle$$

Proof. Clear. □

Definition 5.0.3. Let H be an inner product space. Define the **induced norm**, denoted $\| \cdot \| : H \to \mathbb{C}$, by

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

Exercise 5.0.4. Let H be an inner product space. Then the induced norm, $\| \cdot \| : H \to \mathbb{C}$, is a norm.

Proof. Let $x, y \in H$ and $c \in \mathbb{C}$. Then

(1) $\|x + y\|$
(2) Note that

$$\begin{aligned} \|cx\|^2 &= \langle cx, cx \rangle \\ &= c * c\langle x, x \rangle \\ &= |c|^2 \|x\|^2 \end{aligned}$$

So $\|cx\| = |c|\|x\|$

□

Definition 5.0.5. Let $x_1, x_2 \in H$ and $S \subset H$. Then x_1 and x_2 are said to be **orthogonal** if $\langle x_1, x_2 \rangle = 0$ and S is said to be **orthogonal** if for each $x_1, x_2 \in S$, x_1, x_2 are orthogonal.

6. Convexity

6.1. Introduction.

Note 6.1.1. In this section, we assume all vector spaces are real.

Definition 6.1.1. Let X be a vector space and $A \subset X$. Then A is said to be **convex** if for each $x, y \in A$, and $t \in [0,1]$, $tx + (1-t)y \in A$.

Definition 6.1.2. Let X be a vector space and $f : A \to R$. Then f is said to be **convex** if for each $x, y \in A$, $t \in [0,1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Exercise 6.1.3. Let X be a vector space, $f \in X^*$ and $g : X \to \mathbb{R}$ constant. Then f and g are convex.

Proof. Let $x, y \in X$ and $t \in [0,1]$. Put $c = g(0)$. Then

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

and

$$\begin{aligned} g(tx + (1-t)y) &= c \\ &= tc + (1-t)c \\ &= tg(x) + (1-t)g(y) \end{aligned}$$

So f and g are convex. $\square$

Exercise 6.1.4. Let X be a vector space, $A \subset X$ convex, $f, g : A \to \mathbb{R}$ and $\lambda \geq 0$. If f, g are convex, then

(1) $f + g$ is convex
(2) λf is convex

Proof. Suppose that f and g are convex. Let $x, y \in A$ and $t \in [0,1]$. Then

$$\begin{aligned} (f + \lambda g)(tx + (1-t)y) &= f(tx + (1-t)y) + \lambda g(tx + (1-t)y) \\ &\leq tf(x) + (1-t)f(y) + t\lambda g(x) + (1-t)\lambda g(y) \\ &= t(f(x) + \lambda g(x)) + (1-t)(f(y) + \lambda g(y)) \\ &= t(f + \lambda g)(x) + (1-t)(f + \lambda g)(y) \end{aligned}$$

$\square$

Definition 6.1.5. Let X be a vector space and $f : X \to \mathbb{R}$. Then f is said to be **affine** if there exists $\phi \in X^*$, $a \in R$ constant such that $f = \phi + a$.

Exercise 6.1.6. Let X be a vector space and $f : X \to \mathbb{R}$. If f is affine, then f is convex.

Proof. Suppose that f is affine. Then there exists $\phi \in X^*$, $a \in R$ constant such that $f = \phi + a$. Then ϕ is convex and $g : X \to \mathbb{R}$ defined by $g(x) = a$ is convex. So $f = \phi + g$ is convex. $\square$

Exercise 6.1.7. Let X be a vector space, $A \subset X$ convex, $f : \mathbb{R} \to \mathbb{R}$ and $g : A \to \mathbb{R}$. If f is convex and increasing and g is convex, then $f \circ g$ is convex.

Proof. Let $t \in [0,1]$ and $x, y \in A$. Then convexity of g implies that

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$$

and we have

$$\begin{aligned} f \circ g(tx + (1-t)y) &= f(g(tx + (1-t)y)) \\ &\leq f(tg(x) + (1-t)g(y)) && (f \text{ increasing}) \\ &\leq tf(g(x)) + (1-t)f(g(y)) && (f \text{ convex}) \\ &= tf \circ g(x) + (1-t)f \circ g(y) \end{aligned}$$

So $f \circ g$ is convex. $\square$

Exercise 6.1.8. Let X be a vector space, $A \subset X$ convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Then f has a local minimum at x_0 iff f has a global minimum at x_0.

Proof. If f has a global minimum at x_0, then f has a local minimum at x_0. Conversely, suppose that f has a local minimum at x_0. Then there exists $\delta > 0$ such that for each $x \in B(x_0, \delta)$, $f(x_0) \leq f(x)$. For the sake of contradiction, suppose that f does not have a global minimum at x_0. Then there exits $x' \in A$ such that $f(x') < f(x_0)$. Put $t_0 = \min(\frac{\delta}{\|x'-x_0\|+1}, 1) > 0$. Let $t \in (0, t_0)$, then

$$\begin{aligned} \|(tx' + (1-t)x_0) - x_0\| &= t\|x' - x_0\| \\ &< \frac{\|x' - x_0\|\delta}{\|x' - x_0\| + 1} \\ &< \delta \end{aligned}$$

so that $tx' + (1-t)x_0 \in B(x_0, \delta)$ and hence $f(x_0) \leq f(tx' + (1-t)x_0)$. Therefore

$$\begin{aligned} f(x_0) &\leq f(tx' + (1-t)x_0) \\ &\leq tf(x') + (1-t)f(x_0) \quad (\text{convexity of } f) \\ &< tf(x_0) + (1-t)f(x_0) \\ &= f(x_0) \end{aligned}$$

which is a contradiction. Hence f has a global minimum at x_0. $\square$

Definition 6.1.9. Let X, Y be vector spaces, $A \subset X \oplus Y$. For $y \in Y$, define

$$A^y = \{x \in X : (x, y) \in A\}$$

and $f^y : A^y \to \mathbb{R}$ by

$$f^y(x) = f(x, y)$$

Exercise 6.1.10. Let X, Y be vector spaces, $A \subset X \oplus Y$ convex and $f : A \to \mathbb{R}$ convex. Then for each $y \in \pi_2(A)$,

(1) A^y is convex
(2) f^y is convex

where $\pi_2 : X \times Y \to Y$, the canonical projection of $X \times Y$ onto Y given by $\pi_2(x, y) = y$.

Proof. Let $y \in \pi_2(A)$, $x_1, x_2 \in A^y$ and $t \in [0,1]$. Then by definition, $(x_1, y), (x_2, y) \in A$.

(1) Convexity of A implies that $(tx_1 + (1-t)x_2, y) \in A$. Hence $tx_1 + (1-t)x_2 \in A^y$ and A^y is convex.

(2) Convexity of f implies that

$$\begin{aligned} f^y(tx_1+(1-t)x_2) &= f(tx_1+(1-t)x_2,y) \\ &= f(t(x_1,y)+(1-t)(x_2,y)) \\ &\leq tf(x_1,y)+(1-t)f(x_2,y) \\ &= tf^y(x_1)+(t-t)f^y(x_2) \end{aligned}$$

and so f^y is convex.

□

Exercise 6.1.11. Let X, Y be vector spaces and $A \subset X, B \subset Y$. If A and B are convex, then $A \times B \subset X \oplus Y$ is convex.

Proof. Suppose that A and B are convex. Let $(x_1,y_1),(x_2,y_2) \in A \times B$ and $t \in [0,1]$. Convexity of A and B implies that $tx_1+(1-t)x_2 \in A$ and $ty_1+(1-t)y_2 \in B$. Therefore

$$\begin{aligned} t(x_1,y_1)+(1-t)(x_2,y_2) &= (tx_1+(1-t)x_2, ty_1+(1-t)y_2) \\ &\in A \times B \end{aligned}$$

□

Exercise 6.1.12. Let X,Y be vector spaces and $A \subset X$, $B \subset Y$ convex (implying that $A \times B$ is convex) and $f : A \times B \to \mathbb{R}$ convex. Suppose that for each $y \in B$, $\{f(x,y) : x \in A\}$ is bounded below. Then $\inf\limits_{y \in B} f^y$ is convex

Proof. Put $g = \inf\limits_{y \in B} f^y$. Let $x_1,x_2 \in A$, $y_1,y_2 \in B$ and $t \in [0,1]$. Put $y' = ty_1+(1-t)y_2$. Then convexity of f implies that

$$\begin{aligned} g(tx_1+(1-t)x_2) &\leq f^{y'}(tx_1+(1-t)x_2) \\ &= f(tx_1+(1-t)x_2, ty_1+(1-t)y_2) \\ &= f(t(x_1,y_1)+(1-t)(x_2,y_2)) \\ &\leq tf(x_1,y_1)+(1-t)f(x_2,y_2) \\ &= tf^{y_1}(x_1)+(1-t)f^{y_2}(x_2) \end{aligned}$$

Since $y_1 \in B$ is arbitrary, we have that

$$g(tx_1+(1-t)x_2) \leq tg(x_1)+(1-t)f^{y_2}(x_2)$$

Similarly, since $y_2 \in B$ is arbitrary, we have that

$$g(tx_1+(1-t)x_2) \leq tg(x_1)+(1-t)g(x_2)$$

and f is convex. □

Exercise 6.1.13. Let X be a vector space, $A \subset X$ convex and $(f_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}^A$. Suppose that for each $\lambda \in \Lambda$, f_λ is convex. Then $\sup\limits_{\lambda \in \Lambda} f_\lambda$ is convex.

Proof. Define $f = \sup\limits_{\lambda \in \Lambda} f_\lambda$. Let $x,y \in A$, $t \in [0,1]$ and $\lambda \in \Lambda$. Then

$$\begin{aligned} f_\lambda(tx+(1-t)y) &\leq tf_\lambda(x)+(1-t)f_\lambda(y) \\ &\leq tf(x)+(1-t)f(y) \end{aligned}$$

Since $\lambda \in \Lambda$ is arbitrary, $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$. $\square$

Exercise 6.1.14. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then f is locally Lipschitz at x_0. (**Hint:** Given x_1, x_2 near x_0 Choose a z near x_0 s.t. x_1 is a convex combination of x_2 and z. Then repeat but with x_2 as a convex combination of x_1 and z)

Proof. By continuity, f is locally bounded at x_0. So there exist $M, \delta > 0$ such that $B(x_0, \delta) \subset A$ and for each $x \in B(x_0, \delta)$, $|f(x)| \leq M$. Put $\delta' = \frac{\delta}{2}$ and choose $U = B(x_0, \delta')$. Then $U \subset A$, U is open and $U \in N_{x_0}$.
Let $x_1, x_2 \in U$. Suppose that $x_1 \neq x_2$. Define $\alpha = \|x_1 - x_2\| > 0$, $p = \frac{\alpha}{\alpha+\delta'}$, $q = 1-p$ and $z = p^{-1}(x_1 - qx_2)$. Then $x_1 = pz + qx_2$ and

$$\begin{aligned}\|z - x_1\| &= \|(p^{-1}-1)x_1 - p^{-1}qx_2\| \\ &= \frac{1-p}{p}\alpha \\ &= \frac{\delta'}{\alpha}\alpha \\ &= \delta'\end{aligned}$$

Therefore

$$\begin{aligned}\|z - x_0\| &\leq \|z - x_1\| + \|x_1 - x_0\| \\ &< \delta' + \delta' \\ &= \delta\end{aligned}$$

So $z \in B(x_0, \delta)$, which implies that

$$\begin{aligned}f(z) - f(x_2) &\leq |f(z) - f(x_2)| \\ &\leq |f(z)| + |f(x_2)| \\ &\leq 2M\end{aligned}$$

Since $x_1 = pz + qx_2$, convexity of f implies that $f(x_1) \leq pf(z) + qf(x_2)$. Hence

$$\begin{aligned}f(x_1) - f(x_2) &\leq pf(z) - pf(x_2) \\ &= p(f(z) - f(x_2)) \\ &\leq p2M \\ &= \frac{\alpha}{\alpha+\delta'}2M \\ &\leq \alpha 2M \\ &= 2M\|x_1 - x_2\|\end{aligned}$$

Similarly, choosing $z = p^{-1}(x_2 - qx_1)$, yields $f(x_2) - f(x_1) \leq 2M\|x_1 - x_2\|$ which implies that

$$|f(x_1) - f(x_2)| \leq 2M\|x_1 - x_2\|$$

and f is Lipschitz on U. $\square$

6.2. **Differentiability.**

Exercise 6.2.1. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $x_0 \in A$ and $x \in X$. Define $T = \{t \in \mathbb{R} : x_0 + tx \in A\}$. Then there exist $a, b \in (0, \infty]$ such that $T = (-a, b)$.

Proof. Continuity of scalar multiplication and addition implies that T is an open neighborhood of 0. Let $t > 0$ and $s \in [0, t]$. Then $\frac{s}{t} \in [0, 1]$ and by convexity of A, $x_0 + tx \in A$ implies that

$$\begin{aligned} x_0 + sx &= \frac{s}{t}(x_0 + tx) + \left(1 - \frac{s}{t}\right)x_0 \\ &\in A \end{aligned}$$

Thus $[0, t] \subset T$. Similarly, $x_0 - tx \in A$ implies that $[-t, 0] \subset T$.
Define $a, b \in (0, \infty]$ by $a = \sup\{t > 0 : x_0 - tx \in A\}$ and $b = \sup\{t > 0 : x_0 + tx \in A\}$. Then $(-a, b) = T$. □

Definition 6.2.2. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $x_0 \in A$ and $x \in X$. Define T as in the previous exercise and choose $t_0 > 0$ such that $(-t_0, t_0) \subset T$. For $t \in (0, t_0)$, define the difference quotient $q : (-t_0, t_0) \setminus \{0\} \to \mathbb{R}$ by

$$q(t) = \frac{f(x_0 + tx) - f(x_0)}{t}$$

Exercise 6.2.3. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $x_0 \in A$ and $x \in X$. Define t_0 as above. Then

(1) $q(t)$ is increasing on $(0, t_0)$
(2) $q(-t)$ decreasing on $(0, t_0)$

(**Hint:** As an example, look at the graph of $f(x) = x^2$. For the algebra, start at the desired end inequality and work backwards)

Proof. Let $s, t \in (0, t_0)$ and suppose that $s \leq t$. Then $x_0 + sx$, $x_0 + tx \in A$. Note that since $0 < s \leq t$, $\frac{s}{t} \in (0, 1]$ and $1 - \frac{s}{t} = \frac{t-s}{t} \in (0, 1]$. Also, since A is convex, we have that

$$\left(\frac{t-s}{t}\right)x_0 + \left(\frac{s}{t}\right)(x_0 + tx) \in A$$

Convexity of f implies that

$$\begin{aligned} f(x_0 + sx) &= f\left(\left(\frac{t-s}{t}\right)x_0 + \left(\frac{s}{t}\right)(x_0 + tx)\right) \\ &\leq \left(\frac{t-s}{t}\right)f(x_0) + \left(\frac{s}{t}\right)f(x_0 + tx) \end{aligned}$$

This implies that

$$tf(x_0 + sx) \leq (t - s)f(x_0) + sf(x_0 + tx)$$

and after rearranging, we get

$$tf(x_0 + sx) - tf(x_0) \leq sf(x_0 + tx) - sf(x_0)$$

and so finally, dividing both sides by st, we obtain

$$\begin{aligned} q(s) &= \frac{f(x_0+sx)-f(x_0)}{s} \\ &\leq \frac{f(x_0+tx)-f(x_0)}{t} \\ &= q(t) \end{aligned}$$

as desired.

Similar to (1).

□

Exercise 6.2.4. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $x_0 \in A$ and $x \in X$. Define t_0 as before. Then for each $t \in (0, t_0)$,

$$q(-t) \leq q(t)$$

(**Hint:** for sufficiently small t, convexity of f implies that $f(x_0) \leq \frac{1}{2}f(x_0-2tx)+\frac{1}{2}f(x_0+2tx)$)

(1) *Proof.* Choose t_0 as in the previous exercise. Since convexity of f implies that for each $t \in (0, t_0/2)$,

$$f(x_0) \leq \frac{1}{2}f(x_0-2tx)+\frac{1}{2}f(x_0+2tx)$$

we have that for each $t \in (0, t_0/2)$,

$$\begin{aligned} q(-2t) &= \frac{f(x_0-2tx)-f(x_0)}{-2t} \\ &\leq \frac{f(x_0+2tx)-f(x_0)}{2t} \\ &= q(2t) \end{aligned}$$

So for each $t \in (0, t_0)$, $q(-t) \leq q(t)$. □

Exercise 6.2.5. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Then

(1) f is left-hand and right-hand Gateaux differentiable at x_0 with $d^- f(x_0) \leq d^+ f(x_0)$
(2) for each $x \in X$, $d^- f(x_0)(x) = -d^+ f(x_0)(-x)$

Proof.

(1) Let $x \in X$. Choose $t_0 > 0$ as in the previous two exercises. Let $t, u \in (0, t_0)$. Choose $s \in (0, \min(u, t))$. The previous two exercises imply that

$$\begin{aligned} q(-u) &\leq q(-s) \\ &\leq q(s) \\ &\leq q(t) \end{aligned}$$

and therefore $q(t)$ is an upper bound for $\{q(-u) : u \in (0, t_0)\}$ and $d^- f(x_0)(x) = \sup\limits_{u\in(0,t_0)} q(-u)$ exists with $d^- f(x_0)(x) \leq q(t)$.
Since $t \in (0, t_0)$ is arbitrary, $d^- f(x_0)(x)$ is a lower bound for $\{q(t) : t \in (0, t_0)\}$. Therefore

$$d^+ f(x_0)(x) = \inf_{t\in(0,t_0)} q(t)$$

exists with $d^+f(x_0)(x) \geq d^-f(x_0)(x)$.

(2) By definition, we have

$$\begin{aligned} d^-f(x_0)(x) &= \lim_{t \to 0^+} \frac{f(x_0 + -tx) - f(x_0)}{-t} \\ &= -\lim_{t \to 0^+} \frac{f(x_0 + -tx) - f(x_0)}{t} \\ &= -d^+f(x_0)(-x) \end{aligned}$$

□

Exercise 6.2.6. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Then $d^+f(x_0) : X \to \mathbb{R}$ is a sublinear functional.

Proof. Let $x, y \in X$ and $k \geq 0$. If $k = 0$, then clearly

$$d^+f(x_0)(kx) = kd^+(x_0)(x)$$

If $k > 0$. Then

$$\begin{aligned} d^+f(x_0)(kx) &= \lim_{t \to 0^+} \frac{f(x_0 + tkx) - f(x_0)}{t} \\ &= k \lim_{t \to 0^+} \frac{f(x_0 + tkx) - f(x_0)}{tk} \\ &= kd^+f(x_0)(x) \end{aligned}$$

Define $t_0 > 0$ as before and let $t \in (0, \frac{t_0}{2})$. Note that

$$x_0 + tx + ty = \frac{1}{2}(x_0 + 2tx) + \frac{1}{2}(x_0 + 2ty)$$

Convexity of f implies that

$$f(x_0 + tx + ty) \leq \frac{1}{2}f(x_0 + 2tx) + \frac{1}{2}f(x_0 + 2ty)$$

which implies that

$$\frac{f(x_0 + tx + ty) - f(x_0)}{t} \leq \frac{f(x_0 + 2tx) - f(x_0)}{2t} + \frac{f(x_0 + 2ty) - f(x_0)}{2t}$$

Therefore

$$\begin{aligned} d^+f(x_0)(x + y) &= \lim_{t \to 0^+} \frac{f(x_0 + t(x + y)) - f(x_0)}{t} \\ &= \lim_{t \to 0^+} \frac{f(x_0 + tx + ty) - f(x_0)}{t} \\ &\leq \lim_{t \to 0^+} \left[\frac{f(x_0 + 2tx) - f(x_0)}{2t} + \frac{f(x_0 + 2ty) - f(x_0)}{2t} \right] \\ &= \lim_{t \to 0^+} \frac{f(x_0 + 2tx) - f(x_0)}{2t} + \lim_{t \to 0^+} \frac{f(x_0 + 2ty) - f(x_0)}{2t} \\ &= d^+f(x_0)(x) + d^+f(x_0)(y) \end{aligned}$$

□

Exercise 6.2.7. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. Then for each $x \in A$,

$$d^+ f(x_0)(x - x_0) \leq f(x) - f(x_0)$$

Proof. Let $x \in A$. Define $T = \{t \in \mathbb{R} : x_0 + t(x - x_0) \in A\}$ similarly to earlier. Clearly $1 \in T$ and

$$\begin{aligned} d^+ f(x_0)(x - x_0) &= \inf_{t \in (0,1]} \frac{f(x_0 + t(x - x_0)) - f(x_0)}{t} \\ &\leq f(x) - f(x_0) \end{aligned}$$

□

Exercise 6.2.8. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then $d^+ f(x_0)$ is Lipschitz (equivalently bounded).

Proof. Suppose that f is continuous at x_0. A previous exercise about convex functions tells us that f is locally Lipschitz at x_0, so there exists $\delta, M > 0$ such that for each $x_1, x_2 \in B(x_0, \delta)$, $|f(x_1) - f(x_2)| \leq M\|x_1 - x_2\|$. Let $x \in X$ and define $t_0 = \frac{\delta}{\|x\|+1}$ so that for each $t \in (0, t_0)$,

$$\begin{aligned} \|(x_0 + tx) - x_0\| &= t\|x\| \\ &\leq t_0\|x\| \\ &= \frac{\delta\|x\|}{\|x\| + 1} \\ &< \delta \end{aligned}$$

and $x_0 + tx \in B(x_0, \delta)$. Then for each $t \in (0, t_0)$,

$$\begin{aligned} d^+ f(x_0)(x) &\leq \frac{f(x_0 + tx) - f(x_0)}{t} \\ &\leq \frac{|f(x_0 + tx) - f(x_0)|}{t} \\ &\leq t^{-1} M\|(x_0 + tx) - x_0\| \\ &= M\|x\| \end{aligned}$$

Thus $d^+ f(x_0)$ is a bounded sublinear functional and a previous exercise in the section on sublinear functionals implies this is eqivalent to $d^+ f(x_0)$ being Lipschitz. □

Exercise 6.2.9. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then there exists $\phi \in X^*$ such that $\phi \leq d^+ f(x_0)$.

Proof. Suppose that f is continuous at x_0. The previous exercise implies that $d^+ f(x_0)$ is Lipschitz (equivalently bounded). A previous exercise in the section discussing sublinear functionals tells us that boundedness of $d^+ f(x_0)$ implies that there exists $\phi \in X^*$ such that $\phi \leq d^+ f(x_0)$. □

Definition 6.2.10. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. We define the **subdifferential of f at x_0**, denoted $\partial f(x_0)$, to be

$$\partial f(x_0) = \{\phi \in X^* : \text{for each } x \in A, f(x_0) + \phi(x - x_0) \leq f(x)\}$$

Exercise 6.2.11. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then $\partial f(x_0) \neq \varnothing$.

Proof. Suppose that f is continuous at x_0. The previous exercise tells us that there exists $\phi \in X^*$ such that $\phi \leq d^+f(x_0)$. Let $x \in A$. A previous exercise implies that

$$\begin{aligned}\phi(x - x_0) &\leq d^+f(x_0)(x - x_0)\\ &\leq f(x) - f(x_0)\end{aligned}$$

Then $f(x_0) + \phi(x - x_0) \leq f(x)$. ∎

Exercise 6.2.12. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex, $\phi \in X^*$ and $x_0 \in A$. Then

(1) for each $x \in A$,
$$\phi(x - x_0) \leq f(x) - f(x_0)$$
iff
$$\phi \leq d^+f(x_0)$$
(2) $\partial f(x_0) = \{\phi \in X^* : \phi \leq d^+f(x_0)\}$

Proof.

(1) Suppose that for each $x \in A$, $\phi(x - x_0) \leq f(x) - f(x_0)$. Let $x \in X$. Define t_0 as before. Then for each $t \in (0, t_0)$,
$$\begin{aligned}t\phi(x) &= \phi((x_0 + tx) - x_0)\\ &\leq f(x_0 + tx) - f(x_0)\end{aligned}$$
This implies that $\phi(x) \leq d^+f(x_0)(x)$.
Conversely, suppose that $\phi \leq d^+f(x_0)$. Let $x \in A$. A previous exercise implies that,
$$\begin{aligned}\phi(x - x_0) &\leq d^+f(x_0)(x - x_0)\\ &\leq f(x) - f(x_0)\end{aligned}$$
(2) Clear.

∎

Exercise 6.2.13. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then the following are equivalent:

(1) f is Gateaux differentiable at x_0
(2) $d^+f(x_0)$ is linear
(3) $\#\partial f(x_0) = 1$

Proof. Suppose that f is continuous at x_0. Then $d^+f(x_0)$ is Lipschitz and bounded.

- (1) $\Rightarrow$ (2):
Suppose that f is Gateaux differentiable at x_0. Let $x \in X$. Then a previous exercise implies that
$$\begin{aligned}-df^+(x_0)(-x) &= df^-f(x_0)(x)\\ &= df^+f(x_0)(x)\end{aligned}$$
An exercise in the section on sublinear functionals implies that $df^+f(x_0)$ is linear.
- (2) $\Rightarrow$ (3):
Suppose that $df^+f(x_0)$ is linear. Let $\phi \in \partial f(x_0)$. The previous exercise implies that $\phi \leq df^+f(x_0)$. Equivalence of linearity in the section on sublinear functionals implies that $d^+f(x_0) = \phi$.

- (3) $\Rightarrow$ (1):
  Suppose that $\#\partial f(x_0) = 1$. Since $\partial f(x_0) = \{\phi \in X^* : \phi \leq d^+ f(x_0)\}$, equivalence of linearity in the section on sublinear functionals implies that $d^+ f(x_0)$ is linear. This implies that $d^+ f(x_0) = d^- f(x_0)$ and which implies that f is Gateaux differentiable at x_0.

$\square$

Exercise 6.2.14. Let X be a Banach space, $A \subset X$ open and convex, $f : A \to \mathbb{R}$ convex and $x_0 \in A$. If f is continuous at x_0, then f has a global minimum at x_0 iff $0 \in \partial f(x_0)$.

Proof. Suppose that f has a global minimum at x_0 iff $0 \in \partial f(x_0)$ Let $x \in X$. Then

$$\begin{aligned} d^+ f(x_0)(x) &= \lim_{t \to 0^+} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &\geq 0 \end{aligned}$$

So $0 \leq df^+(x_0)$ and $0 \in \partial f(x_0)$.
Conversely, suppose that $0 \in \partial f(x_0)$. Let $x \in A$. Then

$$\begin{aligned} 0 &= 0(x - x_0) \\ &\leq f(x) - f(x_0) \end{aligned}$$

So that $f(x_0) \leq f(x)$ which implies that f has a global minimum at x_0. $\square$

6.3. **Conjugacy.**

Definition 6.3.1. Let X be a Banach space, $A \subset X$ and $f : A \to \mathbb{R}$. Define $A^* \subset X^*$ and $f^* : A^* \to \mathbb{R}$ by

$$A^* = \left\{\phi \in X^* : \sup_{x \in A} \left[\phi(x) - f(x)\right] < \infty\right\}$$

and

$$f^*(\phi) = \sup_{x \in A} \left[\phi(x) - f(x)\right]$$

If X is a Hilbert space, we may define $A^* \subset X$ and $f^* : A^* \to \mathbb{R}$ via the Riesz representation theorem by

$$A^* = \left\{y \in X : \sup_{x \in A} \left[\langle y, x\rangle - f(x)\right] < \infty\right\}$$

and $f^* : A^* \to \mathbb{R}$ and

$$f^*(y) = \sup_{x \in A} \left[\langle y, x\rangle - f(x)\right]$$

Exercise 6.3.2. Let X be a Banach space, $A \subset X$ and $f : A \to \mathbb{R}$. Then f^* is convex.

Proof. For $x \in A$, define $g_x : X^* \to [\infty, \infty)$ by $g_x(\phi) = \phi(x) - f(x)$. Then for each $x \in A$, g_x is convex since it is affine. Thus $f^* = \sup\limits_{x \in A} g_x$ is convex. □

Exercise 6.3.3. Let X be a Banach space, $A \subset X$ and $f : A \to \mathbb{R}$. Then for each $x \in X$ and $\phi \in X^*$, $f(x) \geq \phi(x) - f^*(\phi)$.

Proof. Clear □

Exercise 6.3.4.

Definition 6.3.5. Let

Definition 6.3.6. ∂f

Exercise 6.3.7.

6.4. **Functional Optimization.**

Exercise 6.4.1. Let X be a Banach space, $(S, \mathcal{S}, \mu)$ a measure space, $A \subset X$, $K \in L^0(A, \mathbb{R})$ and $\Lambda \subset L^0(S, A) \cap \{f : S \to A : K \circ f \in L^1(\mu)\}$. Suppose that A and Λ are convex. Define $\phi : \Lambda \to \mathbb{R}$ by

$$\phi f = \int K \circ f d\mu$$

Then K is convex implies that ϕ is convex.

Proof. Suppose that K is convex. Let $t \in [0, 1]$ and $f, g \in \Lambda$. Convexity of K implies that for each $s \in S$,

$$K[tf(s) + (1-t)g(s)] \leq tK[f(s)] + (1-t)K[g(s)]$$

So

$$K \circ [tf + (1-t)g] \leq tK \circ f + (1-t)K \circ g$$

Therefore

$$\begin{aligned}
\phi[tf + (1-t)g] &= \int K \circ [tf + (1-t)g] d\mu \\
&\leq \int tK \circ f + (1-t)K \circ g d\mu \\
&= t \int K \circ f d\mu + (1-t) \int K \circ g d\mu \\
&= t\phi f + (1-t)\phi g
\end{aligned}$$

and ϕ is convex. □

7. Appendix

7.1. **Asymptotic Notation.**

Definition 7.1.1. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = o(g) \quad \text{as } x \to x_0$$

if for each $\epsilon > 0$, there exists $U \in \mathcal{N}_{x_0}$ such that U is open and for each $x \in U$,

$$\|f(x)\| \leq \epsilon \|g(x)\|$$

Exercise 7.1.2. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. If there exists $U \in \mathcal{N}_{x_0}$ such that U is open and for each $x \in U \setminus \{x_0\}$, $g(x) > 0$, then

$$f = o(g) \text{ as } x \to x_0 \quad \text{iff} \quad \lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = 0$$