Introduction to Probability

Carson James

1

Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Probability

1.1 Distributions

Definition 1.1.0.1. Let Ω be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then $\mathcal{P}$ is said to be a **π-system** on Ω if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Definition 1.1.0.2. Let Ω be a set and $\mathcal{L} \subset \mathcal{P}(\Omega)$. Then $\mathcal{L}$ is said to be a **λ-system** on Ω if

1. $\mathcal{L} \neq \varnothing$
2. for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$
3. for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}$, if $(A_n)_{n\in\mathbb{N}}$ is disjoint, then $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{L}$

Exercise 1.1.0.3. Let Ω be a set and $\mathcal{L}$ a λ-system on Ω. Then

1. $\Omega, \varnothing \in \mathcal{L}$

Proof. Straightforward. □

Definition 1.1.0.4. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. Put

$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(\Omega) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the **λ-system on Ω generated by** $\mathcal{C}$, $\lambda(\mathcal{C})$, to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L}\in\mathcal{S}} \mathcal{L}$$

Exercise 1.1.0.5. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. If $\mathcal{C}$ is a λ-system and $\mathcal{C}$ is a π-system, then $\mathcal{C}$ is a σ-algebra.

Proof. Suppose that $\mathcal{C}$ is a λ-system and $\mathcal{C}$ is a π-system. Then we need only verify the third axiom in the definition of a σ-algebra. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. Define $B_1 = A_1$ and for $n \geq 2$, define $B_n = A_n \cap \left(\bigcup\limits_{k=1}^{n-1} A_k\right)^c = A_n \cap \left(\bigcap\limits_{k=1}^{n-1} A_k^c\right) \in \mathcal{C}$. Then $(B_n)_{n\in\mathbb{N}}$ is disjoint and therefore $\bigcup\limits_{n\in\mathbb{N}} A_n = \bigcup\limits_{n\in\mathbb{N}} B_n \in \mathcal{C}$. □

Theorem 1.1.0.6. (Dynkin's Theorem)
Let Ω be a set.

1. Let $\mathcal{P}$ be a π-system on Ω and $\mathcal{L}$ a λ-system on Ω. If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
2. Let $\mathcal{P}$ be a π-system on Ω. Then $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

Exercise 1.1.0.7. Let $(\Omega, \mathcal{F})$ be a measurable space and μ, ν probability measures on $(\Omega, \mathcal{F})$. Put $\mathcal{L}_{\mu,\nu} = \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$. Then $\mathcal{L}_{\mu,\nu}$ is a λ-system on Ω.

Proof.

1. $\varnothing \in \mathcal{L}_{\mu,\nu}$.
2. Let $A \in \mathcal{L}_{\mu,\nu}$. Then $\mu(A) = \nu(A)$. Thus
$$\begin{aligned}\mu(A^c) &= 1 - \mu(A) \\ &= 1 - \nu(A) \\ &= \nu(A^c)\end{aligned}$$
So $A^c \in \mathcal{L}_{\mu,\nu}$.
3. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}_{\mu,\nu}$. So for each $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Then
$$\begin{aligned}\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) &= \sum_{n\in\mathbb{N}} \mu(A_n) \\ &= \sum_{n\in\mathbb{N}} \nu(A_n) \\ &= \nu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)\end{aligned}$$
Hence $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}_{\mu,\nu}$.

□

Exercise 1.1.0.8. Let $(\Omega, \mathcal{F})$ be a measurable space, μ, ν probability measures on $(\Omega, \mathcal{F})$ and $\mathcal{P} \subset \mathcal{F}$ a π-system on Ω. Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Proof. Using the previous exercise, we see that $\mathcal{P} \subset \mathcal{L}_{\mu,\nu}$. Dynkin's theorem implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu,\nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$. □

Definition 1.1.0.9. Let $F : \mathbb{R} \to \mathbb{R}$. Then F is said to be a **probability distribution function** if

1. F is right continuous
2. F is increasing
3. $F(-\infty) = 0$ and $F(\infty) = 1$

Definition 1.1.0.10. Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We define $F_P : \mathbb{R} \to \mathbb{R}$, by
$$F_P(x) = P((-\infty, x])$$
We call F_P the **probability distribution function of** P.

Exercise 1.1.0.11. Let $(\Omega, \mathcal{F}, P)$ be a probability measure. Then F_P is a probability distribution function.

Proof.

1. Let $x \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}} \subset [x, \infty)$. Suppose that $x_n \to x$. Then $(x, x_n] \to \varnothing$ because $\limsup_{n\to\infty}(x, x_n] = \varnothing$. Thus
$$F(x_n) - F(x) = P((x, x_n]) \to P(\varnothing) = 0$$
This implies that
$$F(x_n) \to F(x)$$
So F is right continuous.
2. Clearly F_P is increasing.

3. Continuity from below tells us that

$$F(-\infty) = \lim_{n\to-\infty} F(n) = \lim_{n\to-\infty} P((-\infty, n]) = 0$$

and continuity from above tell us that

$$F(\infty) = \lim_{n\to\infty} F(n) = \lim_{n\to\infty} P((-\infty, n]) = 1$$

□

Exercise 1.1.0.12. Let μ, ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $F_\mu = F_\nu$ iff $\mu = \nu$.

Proof. Clearly if $\mu = \nu$, then $F_\mu = F_\nu$. Conversely, suppose that $F_\mu = F_\nu$. Then for each $x \in \mathbb{R}$,

$$\begin{aligned}\mu((-\infty, x]) &= F_\mu(x)\\ &= F_\nu(x)\\ &= \nu((-\infty, x])\end{aligned}$$

Put $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then $\mathcal{C}$ is a π-system and for each $A \in \mathcal{C}$, $\mu(A) = \nu(A)$. Hence for each $A \in \sigma(C) = \mathcal{B}(\mathbb{R})$, $\mu(A) = \nu(A)$. So $\mu = \nu$. □

Definition 1.1.0.13. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}^n$. Then X is said to be a **random vector** on $(\Omega, \mathcal{F})$ if X is $\mathcal{F}$-$\mathcal{B}(\mathbb{R}^n)$ measurable. If $n = 1$, then X is said to be a **random variable**. We define

$$L^0_n(\Omega, \mathcal{F}, P) = \{X : \Omega \to \mathbb{R}^n : X \text{ is a random vector}\}$$

and

$$L^p_n(\Omega, \mathcal{F}, P) = \left\{X \in L^0_n : \int \|X\|^p dP < \infty\right\}$$

Definition 1.1.0.14. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable on $(\Omega, \mathcal{F})$. We define the **probability distribution** of X, $P_X : \mathcal{B}(R) \to [0, 1]$, to be the measure

$$P_X = X_* P$$

That is, for each $A \in \mathcal{B}(\mathbb{R})$,

$$P_X(A) = P(X^{-1}(F))$$

We define the **probability distribution function** of X, $F_X : \mathbb{R} \to [0, 1]$, to be

$$F_X = F_{P_X}$$

Definition 1.1.0.15. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable on $(\Omega, \mathcal{F})$. If $P_X \ll m$, we define the **probability density** of X, $f_X : \mathbb{R} \to \mathbb{R}$, by

$$f_X = \frac{dP_X}{dm}$$

Exercise 1.1.0.16. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{F})$. Then for each $x \in \mathbb{R}$,

$$\P\left(\liminf_{n\to\infty} X_n > x\right) \leq \liminf_{n\to\infty} P(X_n > x)$$

Proof. Let $\omega \in \left\{\liminf_{n\to\infty} X_n > x\right\}$. Then $x < \liminf_{n\to\infty} X_n(\omega) = \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} X_k(\omega)\right)$. So there exists $n^* \in \mathbb{N}$ such that $x < \inf_{k\geq n^*} X_k(\omega)$. Then for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x < X_k(\omega)$. So there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$,

$k \geq n^*$ implies that $\mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Hence $\inf_{k \geq n^*} \mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Thus $\liminf_{n\to\infty} \mathbf{1}_{\{X_k > x\}}(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k\geq n} \mathbf{1}_{\{X_k > x\}}(\omega) \right) = 1$. Therefore $\omega \in \liminf_{n\to\infty} \{X_k > x\}$ and we have shown that

$$\left\{ \liminf_{n\to\infty} X_n > x \right\} \subset \liminf_{n\to\infty} \{X_k > x\}$$

Then

$$\begin{aligned} P\left(\liminf_{n\to\infty} X_n > x \right) &\leq P\left(\liminf_{n\to\infty} \{X_k > x\} \right) \\ &\leq \liminf_{n\to\infty} P(\{X_k > x\}) \end{aligned}$$

□

Definition 1.1.0.17. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^+(\Omega) \cup L^1$. Define the **expectation of X**, $E(X)$, to be

$$E(X) = \int X \, dP$$

.

1.2 Independence

Definition 1.2.0.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C} \subset \mathcal{F}$. Then $\mathcal{C}$ is said to be **independent** if for each $(A_i)_{i=1}^n \subset \mathcal{C}$,

$$P\left(\bigcap_{k=1}^n A_k \right) = \prod_{k=1}^n P(A_k)$$

Definition 1.2.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C}_1, \cdots, \mathcal{C}_n \subset \mathcal{F}$. Then $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are said to be **independent** if for each $A_1 \in \mathcal{C}_1, \cdots, A_n \in \mathcal{C}_n$, $A_1, \cdots, A_n$ are independent.

Note 1.2.0.3. We will explicitely say that for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is independent when talking about the independence of the elements of $\mathcal{C}_i$ to avoid ambiguity.

Definition 1.2.0.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_2$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are said to be **independent** if for each $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R})$, $X_1^{-1}B_1, \cdots, X_n^{-1}B_n$ are independent.

Exercise 1.2.0.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are independent iff $\sigma(X_1), \cdots, \sigma(X_n)$ are independent.

Proof. Suppose that $X_1, \cdots, X_n$ are independent. Let $A_1, \in \sigma(X_1), \cdots, A_n \in \sigma(A_n)$. Then for each $i = 1, \cdots, n$, there exists $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X_i^{-1}(B_i)$. Then $A_1, \cdots, A_n$ are independent. Hence $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Conversely, suppose that $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Let $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R})$. Then for each $i = 1, \cdots, n$, $X_i^{-1}B_i \in \sigma(X_i)$. Then $X_1^{-1}B_1, \cdots, X_n^{-1}B_n$ are independent. Hence $X_1, \cdots, X_n$ are independent. □

Exercise 1.2.0.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$ and $\mathcal{F}_1, \cdots, \mathcal{F}_n \subset \mathcal{F}$ a collection of σ-algebras on Ω. Suppose that for each $i = 1, \cdots, n$, X_i is $\mathcal{F}_i$-measurable. If $\mathcal{F}_1, \cdots, \mathcal{F}_n$ are independent, then $X_1, \cdots, X_n$ are independent.

Proof. For each $i = 1, \cdots, n$, $\sigma(X_i) \subset \mathcal{F}_i$. So $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Hence $X_1, \cdots, X_n$ are independent. □

Exercise 1.2.0.7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C}_1, \cdots, \mathcal{C}_n \subset \mathcal{F}$. Suppose that for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is a π-system and $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are independent, then $\sigma(\mathcal{C}_1), \cdots, \sigma(\mathcal{C}_n)$ are independent.

Proof. Let $A_2 \in \mathcal{C}_2$. Define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap A_2) = P(A)P(A_2)\}$. Then

1. $\Omega \in \mathcal{L}$

2. If $A \in \mathcal{L}$, then
$$\begin{aligned} P(A^c \cap A_2) &= P(A_2) - P(A_2 \cap A) \\ &= P(A_2) - P(A_2)P(A) \\ &= (1 - P(A))P(A_2) \\ &= P(A^c)P(A_2) \end{aligned}$$
So $A^c \in \mathcal{L}$

3. If $(B_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ is disjoint, then
$$\begin{aligned} P\bigg(\bigg[\bigcup_{n\in\mathbb{N}} B_n\bigg] \cap A_2\bigg) &= P\bigg(\bigcup_{n\in\mathbb{N}} B_n \cap A_2\bigg) \\ &= \sum_{n\in\mathbb{N}} P(B_n \cap A_2) \\ &= \sum_{n\in\mathbb{N}} P(B_n)P(A_2) \\ &= \bigg[\sum_{n\in\mathbb{N}} P(B_n)\bigg]P(A_2) \\ &= P\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)P(A_2) \end{aligned}$$
So $\bigcup\limits_{n\in\mathbb{N}} B_n \in \mathcal{L}$.

Thus $\mathcal{L}$ is a λ-system. Since $\mathcal{C}_1 \subset \mathcal{L}$ is a π-system, Dynkin's theorem tells us that $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Since $A_2 \in \mathcal{C}_2$ is arbitrary $\sigma(\mathcal{C}_1)$ and $\mathcal{C}_2$ are independent. The same reasoning implies that $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent. Let $A_2 \in \mathcal{C}_1, \cdots, A_n \in \mathcal{C}_n$ We may do the same process with
$$\mathcal{L} = \bigg\{A \in \mathcal{F} : P\bigg(A \cap \bigg(\bigcap_{i=2}^{n} A_i\bigg)\bigg) = P(A)\prod_{i=2}^{n} P(A_i)\bigg\}$$
and conclude that $\sigma(\mathcal{C}_1), \mathcal{C}_2, \cdots, \mathcal{C}_n$ are independent. Which, using the same reasoning would imply that $\sigma(\mathcal{C}_1), \cdots, \sigma(\mathcal{C}_n)$ are independent. □

Exercise 1.2.0.8. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are independent iff for each $x_1, \cdots, x_n \in \mathbb{R}$,
$$P(X_1 \le x_1, \cdots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i)$$

Proof. Suppose that $X_1, \cdots, X_n$ are independent. Then $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Let $x_1, \cdots, x_n \in \mathbb{R}$. Then for each $i = 1, \cdots, n$, $\{X_i \le x_i\} \in \sigma(X_i)$. Hence
$P(X_1 \le x_1, \cdots, X_n \le x_n) = \prod\limits_{i=1}^{n} P(X_i \le x_i)$. Conversely, suppose that for each
$x_1, \cdots, x_n \in \mathbb{R}$, $P(X_1 \le x_1, \cdots, X_n \le x_n) = \prod\limits_{i=1}^{n} P(X_i \le x_i)$. Define $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$. For each $i = 1, \cdots, n$, define $\mathcal{C}_i = X_i^{-1}\mathcal{C}$. Then for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is a π-system and
$$\begin{aligned} \sigma(\mathcal{C}_i) &= \sigma(X^{-1}(\mathcal{C})) \\ &= X_i^{-1}(\sigma(\mathcal{C})) \\ &= X_i^{-1}(\mathcal{B}(\mathbb{R})) \\ &= \sigma(X_i) \end{aligned}$$
By assumption, $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are independent. The previous exercies tells us that $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Then $X_1, \cdots, X_n$ are independent. □

Exercise 1.2.0.9. Let Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Define $X = (X_1, \cdots, X_n)$. If $X_1, \cdots, X_n$ are independent, then

$$P_X = \prod_{i=1}^{n} P_{X_i}$$

.

Proof. Let $A_1, \cdots, A_n \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned} P_X(A_1 \times \cdots \times A_n) &= P(X \in A_1 \times \cdots \times \in A_n) \\ &= P(X_1 \in A_1, \cdots, X_n \in A_n) \\ &= P(X_1 \in A_1) \cdots P(X_n \in A_n) \\ &= P_{X_1}(A_1) \cdots P_{X_n}(A_n) \\ &= \prod_{i=1}^{n} P_{X_i}(A_1 \times \cdots \times A_n) \end{aligned}$$

Put

$$\mathcal{P} = \{A_1 \times \cdots \times A_n : A_1 \in \mathcal{B}(R), \cdots, A_n \in \mathcal{B}(R)\}$$

Then $\mathcal{P}$ is a π-system and

$$\sigma(\mathcal{P}) = \mathcal{B}(R) \otimes \cdots \otimes \mathcal{B}(R) = \mathcal{B}(\mathbb{R}^n)$$

A previous exercise then tells us that $P_X = \prod_{i=1}^{n} P_{X_i}$ $\square$

Exercise 1.2.0.10. Let Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$ and $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R} \in L^0$. Suppose that $f_1 \circ X_1, \cdots, f_n \circ X_n \in L^+(\Omega)$ or $f_1 \circ X_1, \cdots, f_n \circ X_n \in L^1(\Omega)$. If $X_1, \cdots, X_n$ are independent, then

$$E(f_1(X_1) \cdots f_n(X_n)) = \prod_{i=1}^{n} E(f_i(X_i))$$

Proof. Define the random vector $X : \Omega \to \mathbb{R}^n$ by $X = (X_1, \cdots, X_n)$ and $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x_1, \cdots, x_n) = f_1(x_1) \cdots f_n(x_n)$. Suppose that for each $i = 1, \cdots, n$, $f_i \in L^+(\mathbb{R})$. Then $g \in L^+(\mathbb{R}^n)$ and by change of variables,

$$\begin{aligned} E(f_1(X_1) \cdots f_n(X_n)) &= E(g(X)) \\ &= \int_{\Omega} g \circ X \, dP \\ &= \int_{\mathbb{R}^n} g(x) \, dP_X(x) \\ &= \int_{R^n} g(x) d \prod_{i=1}^{n} P_{X_i}(x) \\ &= \prod_{i=1}^{n} \int_{\mathbb{R}} f_i(x) dP_{X_i}(x) \\ &= \prod_{i=1}^{n} \int_{\Omega} f_i \circ X dP \\ &= \prod_{i=1}^{n} E(f_i(X_i)) \end{aligned}$$

If for each $i = 1, \cdots, n$, $f_i \in L^1(\mathbb{R}, P_{X_i})$, then following the above reasoning with $|g|$ tells us that $g \in L^1(\mathbb{R}^n, P_X)$ and we use change of variables and Fubini's theorem to get the same result. $\square$

1.3 L^p Spaces for Probability

Note 1.3.0.1. Recall that for a probability space $(\Omega, \mathcal{F}, P)$ and $1 \leq p \leq q \leq \infty$ we have $L^q \subset L^p$ and for each $X \in L^q$, $\|X\|_p \leq \|X\|_q$. Also recall that for $X, Y \in \mathrm{L}^2$, we have that $\|XY\|_1 \leq \|X\|_2 \|X\|_2$.

Definition 1.3.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Define the **variance of X**, $Var(X)$, to be

$$Var(X) = E(\big[\big(X - E(X)\big)^2\big)$$

.

Definition 1.3.0.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Define the

Definition 1.3.0.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Define the **covariance of** X **and** Y, $Cov(X, Y)$, to be

$$Cov(X, Y) = E([X - E(X)][Y - E(Y)])$$

Exercise 1.3.0.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Then the covariance is well defined and $Cov(X, Y)^2 \leq Var(X)Var(Y)$

Proof. By Holder's inequality,

$$\begin{aligned}
|Cov(X, Y)| &= \left| \int (X - E(X))(Y - E(Y))\, dP \right| \\
&\leq \int |(X - E(X))(Y - E(Y))|\, dP \\
&= \|(X - E(X))(Y - E(Y))\|_1 \\
&\leq \|X - E(X)\|_2 \|(Y - E(Y)\|_2 \\
&= \left(\int |X - E(X)|^2\, dP \right)^{\frac{1}{2}} \left(|Y - E(Y)|^2 \right)^{\frac{1}{2}} \\
&= Var(X)^{\frac{1}{2}} Var(Y)^{\frac{1}{2}}
\end{aligned}$$

So $Cov(X, Y)^2 \leq Var(X)Var(Y)$. □

Exercise 1.3.0.6. Let $(\Omega, \mathcal{F}, P)$ be a measure space and $X, Y \in L^2$. Then

1. $Cov(X, Y) = E(XY) - E(X)E(Y)$
2. If X, Y are independent, then $Cov(X, Y) = 0$
3. $Var(X) = E(X^2) - E(X)^2$
4. for each $a, b \in \mathbb{R}$, $Var(aX + b) = a^2 Var(X)$.
5. $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Proof.

1. We have that

$$\begin{aligned}
Cov(X, Y) &= E\Big[(X - E(X))(Y - E(Y))\Big] \\
&= E(XY - E(Y)X - E(X)Y + E(X)E(Y)) \\
&= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\
&= E(XY) - E(X)E(Y)
\end{aligned}$$

2. Suppose that X, Y are independent. Then $E(XY) = E(X)E(Y)$. Hence

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= E(X)E(Y) - E(X)E(Y) \\ &= 0 \end{aligned}$$

3. Part (1) implies that

$$\begin{aligned} Var(X) &= Cov(X,X) \\ &= E(X^2) - E(X)^2 \end{aligned}$$

4. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned} Var(aX+b) &= E[(aX+b)^2] - E(aX+b)^2 \\ &= E[a^2X^2 + 2abX + b^2] - (aE(X)+b)^2 \\ &= a^2E(X^2) + 2abE(X) + b^2 - (a^2E(X)^2 + 2abE(X) + b^2) \\ &= a^2(E(X^2) - E(X)^2) \\ &= a^2Var(X) \end{aligned}$$

5. We have that

$$\begin{aligned} Var(X+Y) &= E[(X+Y)^2] - E[X+Y]^2 \\ &= E[X^2 + 2XY + Y^2] - (E(X) + E[Y])^2 \\ &= E(X^2) + 2E[XY] + E[Y^2] - (E(X)^2 + 2E(X)E[Y] + E[Y]^2) \\ &= (E(X^2) - E(X)^2) + (E[Y^2] - E[Y]^2) + 2(E[XY] - E(X)E[Y]) \\ &= Var(X) + Var(Y) + 2Cov(X,Y) \end{aligned}$$

□

Definition 1.3.0.7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. The **correlation of X and Y**, $Cor(X,Y)$, is defined to be

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Exercise 1.3.0.8.

Exercise 1.3.0.9. Jensen's Inequality:
Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X \in L^1$ and $\phi : \mathbb{R} \to \mathbb{R}$. If ϕ is convex, then

$$\phi(E(X)) \leq E[\phi(X)]$$

Proof. Put $x_0 = E(X)$. Since ϕ is convex, there exist $a, b \in \mathbb{R}$ such that $\phi(x_0) = ax_0 + b$ and for each $x \in \mathbb{R}$, $\phi(x) \geq ax + b$. Then

$$\begin{aligned} E[\phi(X)] &= \int \phi(X)\, dP \\ &\geq \int [aX + b]\, dP \\ &= a \int X\, dP + b \\ &= aE(X) + b \\ &= ax_0 + b \\ &= \phi(x_0) \\ &= \phi(E(X)) \end{aligned}$$

□

Exercise 1.3.0.10. Markov's Inequality: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^+$. Then for each $a \in (0, \infty)$,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Proof. Let $a \in (0, \infty)$. Then $a\mathbf{1}_{\{X \geq a\}} \leq X\mathbf{1}_{\{X \geq a\}}$. Thus

$$\begin{aligned} aP(X \geq a) &= \int a\mathbf{1}_{\{X \geq a\}} \, dP \\ &= \int X\mathbf{1}_{\{X \geq a\}} \, dP \\ &\leq \int X \, dP \\ &= E(X) \end{aligned}$$

Therefore

$$P(X \geq a) \leq \frac{E(X)}{a}$$

. □

Exercise 1.3.0.11. Chebychev's Inequality: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Then for each $a \in (0, \infty)$,

$$P(|X - E(X)| \geq a) \leq \frac{Var(X)}{a^2}$$

Proof. Let $a \in (0, \infty)$. Then

$$\begin{aligned} P(|X - E(X)| \geq a) &= P((X - E(X))^2 \geq a^2) \\ &\leq \frac{E[(X - E(X))^2]}{a^2} \\ &= \frac{Var(X)}{a^2} \end{aligned}$$

□

Exercise 1.3.0.12. Chernoff's Bound: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Then for each $a, t \in (0, \infty)$,

$$P(X \geq a) \leq e^{-ta} E[e^{tX}]$$

Proof. Let $a, t \in (0, \infty)$. Then

$$\begin{aligned} P(X \geq a) &= P(tX \geq ta) \\ &= P(e^{tX} \geq e^{ta}) \\ &\leq e^{-ta} E[e^{tX}] \end{aligned}$$

□

Exercise 1.3.0.13. Weak Law of Large Numbers: Let $(\Omega, \mathcal{F}, P)$ be a probability space $(X_i)_{i \in \mathbb{N}} \subset L^2$. Suppose that $(X_i)_{i \in \mathbb{N}}$ are iid. Then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} E[X_1]$$

Proof. Put $\mu = E[X_1]$ and $\sigma^2 = Var(X_1)$. Then

$$\begin{aligned} E[\frac{1}{n} \sum_{i=1}^{n} X_i] &= \frac{1}{n} \sum_{i=1}^{n} E[X_i] \\ &= \frac{1}{n} \sum_{i=1}^{n} \mu \\ &= \mu \end{aligned}$$

and

$$
\begin{aligned}
Var(\frac{1}{n}\sum_{i=1}^{n} X_i) &= \frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}
$$

Let $\epsilon > 0$. Then

$$
\begin{aligned}
P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - E[X_1]\right| \geq \epsilon\right) &= P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) \\
&= P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - E(\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]\right| \geq \epsilon\right) \\
&\leq \frac{Var(\frac{1}{n}\sum_{i=1}^{n} X_i)}{\epsilon^2} \\
&= \frac{\sigma^2/n}{\epsilon^2} \\
&= \frac{\sigma^2}{n\epsilon^2} \to 0
\end{aligned}
$$

So

$$
\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} E[X_1]
$$

□

1.4 Borel Cantelli Lemma

Exercise 1.4.0.1. Borel Cantelli Lemma:
Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$.

1. If $\sum\limits_{n\in\mathbb{N}} P(A_n) < \infty$, then $P(\limsup\limits_{n\to\infty} A_n) = 0$.
2. If $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum\limits_{n\in\mathbb{N}} P(A_n) = \infty$, then $P(\limsup\limits_{n\to\infty} A_n) = 1$

Proof.

1. Suppose that $\sum\limits_{n\in\mathbb{N}} P(A_n) < \infty$. Recall that

$$\limsup_{n\to\infty} A_n = \left\{\omega \in \Omega : \sum_{n\in\mathbb{N}} 1_{A_n}(\omega) = \infty\right\}$$

Then

$$\begin{aligned}\infty &> \sum_{n\in\mathbb{N}} P(A_n)\\ &= \sum_{n\in\mathbb{N}} \int 1_{A_n}\, dP\\ &= \int \sum_{n\in\mathbb{N}} 1_{A_n}\, dP\end{aligned}$$

Thus $\sum\limits_{n\in\mathbb{N}} 1_{A_n} < \infty$ a.e. and $P(\limsup\limits_{n\to\infty} A_n) = 0$.

2. Suppose that $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum\limits_{n\in\mathbb{N}} P(A_n) = \infty$.

$\square$

Exercise 1.4.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n)_{n\in\mathbb{N}} \subset L^0$ and $X \in L^0$.

1. If for each $\epsilon > 0$, $\sum\limits_{n\in\mathbb{N}} P(|X_n - X| \geq \epsilon) < \infty$, then $X_n \to X$ a.e.
2. If $(X_n)_{n\in\mathbb{N}}$ are independent and there exists $\epsilon > 0$ such that $\sum\limits_{n\in\mathbb{N}} P(|X_n - X| \geq \epsilon) = \infty$, then $X_n \not\to X$ a.e.

Proof.

1. For $\epsilon > 0$ and $n \in \mathbb{N}$, set $A_n(\epsilon) = \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}$. Suppose that for each $\epsilon > 0$, $\sum\limits_{n\in\mathbb{N}} P(|X_n - X| \geq \epsilon) < \infty$.

The Borel-Cantelli lemma implies that for each $m \in \mathbb{N}$,

$$P(\limsup_{n\to\infty} A_n(1/m)) = 0$$

Let $\omega \in \Omega$. Then $X_n(\omega) \not\to X(\omega)$ iff

$$\omega \in \bigcup_{m\in\mathbb{N}} \limsup_{n\to\infty} A_n(1/m)$$

So

$$\begin{aligned}P(X_n \not\to X) &= P\bigg(\bigcup_{m\in\mathbb{N}} \limsup_{n\to\infty} A_n(1/m)\bigg)\\ &\leq \sum_{m\in\mathbb{N}} P(\limsup_{n\to\infty} A_n(1/m))\\ &= 0\end{aligned}$$

Hence $X_n \to X$ a.e.

2.

□

Chapter 2

Probability on locally compact Groups

Note 2.0.0.1. In this section, familiarity with Haar measure will be assumed. This section is intended as a continuation of section 7 of [4].

2.1 Action on Probability Measures

Note 2.1.0.1. We recall some notation from section 7.1 of [4].

- $l_g \in \text{Homeo}(G)$, $l_g(x) = gc$
- $L_g \in \text{Sym}(L_0(G))$, $L_g f = f \circ l_g^{-1}$ We continue from section 7

Note 2.1.0.2. The next exercise generalizes the notion of a scale-family.

Exercise 2.1.0.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, G a locally compact group, μ a left Haar measure on G, $X \in L^0_G$ and $g \in G$. If $P_X \ll \mu$, then $f_{gX} = L_g f_X$.

Proof. Suppose that $P_X \ll \mu$. Let $A \in \mathcal{B}(G)$. Then

$$\begin{aligned} P_{gX}(A) &= P(gX \in A) \\ &= P(X \in g^{-1}A) \\ &= P_X(g^{-1}A) \\ &= P_X(l_g^{-1}(A)) \\ &= l_{g*}P_X(A) \\ &= g \cdot P_X(A) \end{aligned}$$

The previous exercise tells us that $f_{gX} = L_g f_X$. □

Chapter 3

Gaussian Measures

3.1 Gaussian Measures on $\mathbb{R}$

Definition 3.1.0.1. Let $\mu \in \mathcal{M}_1(\mathbb{R})$. Then μ is said to be **Gaussian** if

3.2 Gaussian Measures on Topological Vector Spaces

Definition 3.2.0.1. Let W be a topological vector space and $\mu \in \mathcal{M}_1(W)$. Then μ is said to be **Gaussian** if for each $\phi \in W^*$, $\phi_*\mu$ is Gaussian.

need exercise in analysis notes showing $\mathbb{R}^{\mathbb{N}}$ is a topological vector space.

Definition 3.2.0.2. For $n \in \mathbb{N}$, define $\mu_n \in \mathcal{M}_1(\mathbb{R})$ by

$$\mu_n(A) := \int_A \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dm(x).$$

An exercise in measure theory notes section on projective limits of measure spaces implies that there exists a unique $\mu \in \mathcal{M}_1(\mathbb{R}^{\mathbb{N}})$ such that

- for each $n \in \mathbb{N}$, $\pi_n^*\mu = \mu_n$ and for each,
- for each $(A_n)_{n\in\mathbb{N}} \in \mathcal{B}(\mathbb{R})^{\mathbb{N}}$,

$$\mu\bigg(\prod_{n\in\mathbb{N}} A_n\bigg) = \prod_{n\in\mathbb{N}} \mu_n(A_n).$$

We refer to μ as the **standard Gaussian measure** on $\mathbb{R}^{\mathbb{N}}$.

Exercise 3.2.0.3. μ is Gaussian.

Proof. Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Define $\eta_\lambda : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $\eta_\lambda(x) := \lambda x$. Then

$$\begin{aligned}(\lambda\pi_n)_*\mu &= (\eta_\lambda \circ \pi_n)_*\mu \\ &= (\eta_\lambda)_*((\pi_n)_*\mu) \\ &= (\eta_\lambda)_*\mu_n\end{aligned}$$

Define $\nu \in \mathcal{M}_1(\mathbb{R})$ by $\nu := (\eta_\lambda)_*\mu_n$. If $\lambda = 0$, then $\nu = \delta_0$. Suppose that $\lambda \neq 0$. Since $\mu_n \ll m$, (check/give details) we have that $\nu \ll m$. The Radon-Nikodym-Lebesgue theorem implies that there exists $f \in L^1(\mathbb{R}, \mathcal{B}(R), m)$ such that $d\nu = f\, dm$. An exercise in analysis notes (need to rework in the newer general framerwork) implies that for m-a.e. $x \in \mathbb{R}$,

$$\widehat{\nu}(x) = \int_{\mathbb{R}} e^{itx}\, d\nu(t) \qquad\qquad =$$

$$\begin{aligned}f(x) &= \lim_{r\to 0} \frac{\nu((x-r, x+r))}{m((x-r,x+r))} \\ &= \lim_{r\to 0} \frac{\mu_n(\eta_\lambda^{-1}(x-r,x+r))}{m((x-r,x+r))} \\ &= \lim_{r\to 0} \frac{\mu_n(\lambda^{-1}(x-r,x+r))}{m((x-r,x+r))} \\ &= \lim_{r\to 0} \frac{\mu_n((\lambda^{-1}x-\lambda^{-1}r, \lambda^{-1}x+\lambda^{-1}r))}{m((x-r,x+r))} \\ &= \lim_{r\to 0} \frac{1}{2r} \int_{\lambda^{-1}x-\lambda^{-1}r}^{\lambda^{-1}x+\lambda^{-1}r} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dm(x) \\ &= \lim_{r\to 0} \frac{1}{2r} \int_u\end{aligned}$$

$f(x)$ Let $\phi \in (\mathbb{R}^{\mathbb{N}})^*$. An exercise in the analysis notes section on topological vector spaces subsection on products implies that $(\mathbb{R}^{\mathbb{N}})^* = \mathrm{span}\{\pi_j : j \in \mathbb{N}\}$.

Then show that the pushforward of a linear combos of projection maps is Gaussian $\square$

3.3 Gaussian Measures on Hilbert Spaces

Note 3.3.0.1. recall definitions of $(Y, (\tau_n)_{n\in\mathbb{N}})$ and $((Y_n)_{n\in\mathbb{N}}, (\tau_{n,k})_{(n\leq k)})$ from measure and integration notes section on projective systems of complex measure spaces exercise on existence and uniqueness of product measures. We need an exercise to show that $(Y, (\tau_n)_{n\in\mathbb{N}})$ is a **Top**-projective limit of $((Y_n)_{n\in\mathbb{N}}, (\tau_{n,k})_{(n\leq k)})$ in the analysis notes.

Exercise 3.3.0.2. Let $a \in \mathbb{C}^{\mathbb{N}}$. Then $l^{2,a} \in \mathcal{B}(\mathbb{C}^{\mathbb{N}})$.

Chapter 4

Weak Convergence of Measures

Chapter 5

Concentration Inequalities

5.1 Introduction

Exercise 5.1.0.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$. Then for each $s, t \in \mathbb{R}$,

$$P(X + Y \geq s + t) \leq P(X \geq s) + P(Y \geq t)$$

Proof. For $Z \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ and $t \in \mathbb{R}$, define $A^t_Z \in \mathcal{F}$ by

$$A^t_Z = \{\omega \in \Omega : Z(\omega) \geq t\}$$

Let $s, t \in \mathbb{R}$. Since $(A^s_X)^c \cap (A^t_Y)^c \subset (A^{s+t}_{X+Y})^c$, we have that $A^{s+t}_{X+Y} \subset A^s_X \cup A^t_Y$. Then

$$\begin{aligned} P(X + Y \geq s + t) &= P(A^{s+t}_{X+Y}) \\ &\leq P(A^s_X) + P(A^t_Y) \\ &= P(X \geq s) + P(Y \geq t) \end{aligned}$$

□

Exercise 5.1.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^+(\Omega, \mathcal{F}, P)$. Then for each $s, t \geq 0$,

$$P(XY \geq st) \leq P(X \geq s) + P(Y \geq t)$$

Proof. For $Z \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ and $t \in \mathbb{R}$, define define $A^t_Z \in \mathcal{F}$ by

$$A^t_Z = \{\omega \in \Omega : Z(\omega) \geq t\}$$

Let $s, t \in \mathbb{R}$. Since $(A^s_X)^c \cap (A^t_Y)^c \subset (A^{st}_{XY})^c$, we have that $A^{st}_{XY} \subset A^s_X \cup A^t_Y$. Then

$$\begin{aligned} P(XY \geq st) &= P(A^{st}_{XY}) \\ &\leq P(A^s_X) + P(A^t_Y) \\ &= P(X \geq s) + P(Y \geq t) \end{aligned}$$

□

5.2 Sub α-Exponential Random Variables

Definition 5.2.0.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then X is said to be **sub α-exponential** if there exist $M, K > 0$ such that for each $t \geq 0$,

$$P(|X| \geq t) \leq M e^{-Kt^\alpha}$$

Exercise 5.2.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then the following are equivalent:

1. X is sub α-exponential
2. there exists $K > 0$ such that for each $p \geq 1$, $\|X\|_p \leq K p^{1/\alpha}$
3.
4.

Proof.

- (1) $\implies$ (2):
  Choose $C_\alpha > 0$ such that for each $x \geq \alpha^{-1}$, $\Gamma(x) \leq C_\alpha x^x$. Since X is sub α-exponential, there exist $M, K_0 > 0$ such that for each $t \geq 0$, $P(|X| \geq t) \leq M e^{-Kt^\alpha}$. Set $K = \max(M\alpha^{-1}C_\alpha, 1)2K_0^{-1/\alpha}\alpha^{-1/\alpha}$. Let $p \geq 1$. Then $p\alpha^{-1} \geq \alpha^{-1}$

$$\begin{aligned}
\|X\|_p^p &= E(|X|^p) \\
&= \int_0^\infty P(|X|^p \geq t)dt \\
&= \int_0^\infty P(|X| \geq t^{1/p})dt \\
&\leq \int_0^\infty M e^{-K_0 t^{\alpha/p}} dt \\
&= M p \alpha^{-1} \int_0^\infty u^{p/\alpha - 1} e^{-K_0 u} du \\
&= M p \alpha^{-1} \Gamma(p/\alpha) K_0^{-p/\alpha} \\
&\leq M p \alpha^{-1} C_\alpha (p\alpha^{-1})^{p/\alpha} K_0^{-p/\alpha}
\end{aligned}$$

  Therefore

$$\begin{aligned}
\|X\|_p &\leq (M\alpha^{-1}C_\alpha)^{1/p} p^{1/p} K_0^{-1/\alpha} \alpha^{-1/\alpha} p^{1/\alpha} \\
&\leq \max(M\alpha^{-1}C_\alpha, 1) 2 K_0^{-1/\alpha} \alpha^{-1/\alpha} p^{1/\alpha} \\
&= K p^{1/\alpha}
\end{aligned}$$

- (2) $\implies$ (3):

□

Definition 5.2.0.3. Let $\psi : [0, \infty) \to [0, \infty)$. Then ψ is said to be an **Orlicz function** if

1. ψ is convex
2. ψ is increasing
3. $\psi(x) \to \infty$ as $x \to \infty$
4. $\psi(0) = 0$

Definition 5.2.0.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\psi : [0, \infty) \to [0, \infty)$ and Orlicz function. We define the **Orlicz ψ-norm**, denoted $\| \cdot \|_\psi : L^0(\Omega, \mathcal{F}, P) \to [0, \infty]$, by

$$\|X\|_\psi = \inf\{t > 0 : E[\psi(|X|/t)] \leq 1\}$$

We define the **Orlicz ψ-space**, denoted $L^\psi(\Omega, \mathcal{F}, P)$, by

$$L^\psi(\Omega, \mathcal{F}, P) = \{X \in L^1(\Omega, \mathcal{F}, P) : \|X\|_\psi < \infty\}$$

Exercise 5.2.0.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\psi : [0,\infty) \to [0,\infty)$ an Orlicz function. Then L^ψ is a vector space and $\|\cdot\|_\psi : L^\psi \to [0,\infty)$ is a norm.
Hint: note that

- for $s, t > 0$,
$$\frac{|X|}{s+t} + \frac{|Y|}{s+t} = \frac{s}{s+t}\frac{|X|}{s} + \frac{t}{s+t}\frac{|Y|}{t}$$
- ψ is star-shaped, i.e. for each $x, t \geq 0$ and $f(tx) \leq tf(x)$.

Proof. For $X \in L^0(\Omega, \mathcal{F}, P)$, define $A_X \in \mathcal{F}$ by

$$A_X = \{t > 0 : E[\psi(|X|/t)] \leq 1\}$$

Let $X, Y \in L^\psi(\Omega, \mathcal{F}, P)$ and $\lambda \in \mathbb{C}$. Since $\|X\|_\psi < \infty$ and $\|Y\|_\psi < \infty$, we have that $A_X \neq \varnothing$ and $A_Y \neq \varnothing$,

1. **subadditivity:** Let $\epsilon > 0$. Then there exists $s \in A_X$ and $t \in A_Y$ such that $s < \inf A_X + \epsilon/2$ and $t < \inf A_Y + \epsilon/2$. Since is convex and increasing, we have that
$$\begin{aligned}
\psi\bigg(\frac{|X+Y|}{s+t}\bigg) &\leq \psi\bigg(\frac{|X|}{s+t} + \frac{|Y|}{s+t}\bigg) \\
&= \psi\bigg(\frac{s}{s+t}\frac{|X|}{s} + \frac{t}{s+t}\frac{|Y|}{t}\bigg) \\
&\leq \frac{s}{s+t}\psi\bigg(\frac{|X|}{s}\bigg) + \frac{t}{s+t}\psi\bigg(\frac{|Y|}{t}\bigg)
\end{aligned}$$
Therefore,
$$\begin{aligned}
E\bigg[\psi\bigg(\frac{|X+Y|}{s+t}\bigg)\bigg] &\leq \frac{s}{s+t}E\bigg[\psi\bigg(\frac{|X|}{s}\bigg)\bigg] + \frac{t}{s+t}E\bigg[\psi\bigg(\frac{|Y|}{t}\bigg)\bigg] \\
&\leq \frac{s}{s+t} + \frac{t}{s+t} \\
&\leq 1
\end{aligned}$$
Hence $s + t \in A_{X+Y}$. Thus $A_{X+Y} \neq \varnothing$. Since $s < \inf A_X + \epsilon/2$ and $t < \inf A_Y + \epsilon/2$, we have that
$$\begin{aligned}
\|X+Y\|_\psi &= \inf A_{X+Y} \\
&\leq s + t \\
&< \inf A_X + \inf A_Y + \epsilon \\
&= \|X\|_\psi + \|Y\|_\psi + \epsilon
\end{aligned}$$
Since $\epsilon > 0$ is arbitrary, $\|X+Y\|_\psi \leq \|X\|_\psi + \|Y\|_\psi$. So $X + Y \in L^\psi(\Omega, \mathcal{F}, P)$

2. **absolute homogeneity:** Suppose that $\lambda = 0$. Then for each $t > 0$,
$$\begin{aligned}
E[\psi(|\lambda X|/t)] &= E[\psi(0)] \\
&= E[0] \\
&= 0 \\
&\leq 1
\end{aligned}$$
Thus
$$\begin{aligned}
\|\lambda X\|_\psi &= 0 \\
&= |\lambda|\|X\|_\psi
\end{aligned}$$

Suppose that $\lambda \neq 0$. Since for each $t > 0$,

$$\psi\left(\frac{|X|}{t}\right) = \psi\left(\frac{|\lambda X|}{|\lambda| t}\right)$$

we have that for each $t > 0$, $t \in A_X$ iff $|\lambda| t \in A_{\lambda X}$. Therefore $A_{\lambda X} = |\lambda| A_X$ and

$$\begin{aligned}\|\lambda X\|_\psi &= \inf A_{\lambda X} \\ &= |\lambda| \inf A_X \\ &= |\lambda| \|X\|_\psi\end{aligned}$$

So $|\lambda X\| \in L^\psi(\Omega, \mathcal{F}, P)$.

3. **positive definiteness:** Note that since ψ is increasing, for each $t_0 > 0$, $t_0 \in A_X$ implies that $[t_0, \infty) \subset A_X$. Suppose that $\|X\|_\psi = 0$. Then $(0, \infty) \subset A_X$. Jensen's inequality implies that for each $t > 0$,

$$\begin{aligned}\psi\left(\frac{E|X|}{t}\right) &\leq E\left[\psi\left(\frac{|X|}{t}\right)\right] \\ &\leq 1\end{aligned}$$

For the sake of contradiction, suppose that $E|X| > 0$. Since $\psi(x) \to \infty$ as $x \to \infty$, Choose $x > 0$ such that $\psi(x) > 1$. Set $t = E|X|/x$. Then $t > 0$ and

$$\begin{aligned}1 &< \psi(x) \\ &= \psi\left(\frac{E|X|}{t}\right) \\ &\leq 1\end{aligned}$$

which is a contradiction. Hence $E|X| = 0$. Thus $X = 0$ a.s.

□

Exercise 5.2.0.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\psi : [0, \infty) \to [0, \infty)$ an Orlicz function. Then $L^\psi(\Omega, \mathcal{F}, P)$ is a Banach space.

Proof. □

Chapter 6

Conditional Expectation and Probability

6.1 Conditional Expectation

Exercise 6.1.0.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub σ-algebra of $\mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F}, P)$. Define $P_{\mathcal{G}} = P|_{\mathcal{G}}$ and $Q : \mathcal{G} \to [0, \infty)$ by $Q(G) = \int_G X dP$. Then $Q \ll P_{\mathcal{G}}$.

Proof. Let $G \in \mathcal{G}$. Suppose that $P_{\mathcal{G}}(G) = 0$. By definition, $P(G) = 0$. So $Q(G) = 0$ and $Q \ll P_{\mathcal{G}}$. □

Definition 6.1.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub σ-algebra of $\mathcal{F}$ and $X, Y \in L^1(\Omega, \mathcal{F}, P)$. Then Y is said to be a **conditional expectation of X given $\mathcal{G}$** if

1. Y is $\mathcal{G}$-measurable
2. for each $G \in \mathcal{G}$,

$$\int_G Y\, dP = \int_G X\, dP$$

Since (2) implies that conditional expectations of X given $\mathcal{G}$ are equal $P_{\mathcal{G}}$-a.e., we write $Y = E(X|\mathcal{G})$.

Note 6.1.0.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(S, \mathcal{S})$ a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. We typically write $E(X|Y)$ instead of $E(X|Y^*\mathcal{S})$.

Exercise 6.1.0.4. Existence of Conditional Expectation:
Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub σ-algebra of $\mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F}, P)$. Define Q and $P_{\mathcal{G}}$ as in the previous exercise. Define $Y = dQ/dP_{\mathcal{G}}$. Then Y is a conditional expectation of X given $\mathcal{G}$.

Proof. The Radon-Nikodym theorem implies that Y is $\mathcal{G}$-measurable. Since Q is finite, so is $|Q|$. Since $d|Q| = |Y|\, dP_{\mathcal{G}}$, we have that $Y \in L^1(\Omega, \mathcal{G}, P_{\mathcal{G}})$. An exercise in section 3.3 of [4], implies that for each $G \in \mathcal{G}$

$$\begin{aligned}\int_G Y\, dP &= \int_G Y\, dP_{\mathcal{G}} \\ &= Q(G) \\ &= \int_G X\, dP\end{aligned}$$

□

Definition 6.1.0.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(S, \mathcal{S})$ a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. Let $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$. Then ϕ is said to be a **conditional expectation function of X given Y** if for each $B \in \mathcal{S} \cap Y(\Omega)$,

$$\int_{Y^{-1}(B)} X\, dP = \int_B \phi\, dP_Y$$

To denote this, we write $\phi(y) = E[X|Y = y]$.

Exercise 6.1.0.6. Existence of Conditional Expectation Function:
Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(S, \mathcal{S})$ a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. Suppose that for each $y \in S$, $\{y\} \in \mathcal{S}$. Then there exists $\phi \in L^0(Y(S), \mathcal{S} \cap Y(\Omega))$ such that ϕ is a conditional expectation function of X given Y.
Hint: Doob-Dynkin lemma

Proof. Since $E[X|Y] \in L^0(\Omega, Y^*\mathcal{S})$, the Doob-Dynkin lemma implies that there exists $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$ such that $\phi \circ Y = E(X|Y)$. Let $B \in \mathcal{S} \cap Y(\Omega)$. Then

$$\begin{aligned}
\int_B \phi \, dP_Y &= \int_{Y^{-1}(B)} \phi \circ Y \, dP \\
&= \int_{Y^{-1}(B)} E(X|Y) \, dP \\
&= \int_{Y^{-1}(B)} X \, dP
\end{aligned}$$

□

6.2 Conditional Probability

Definition 6.2.0.1. Let $(A, \mathcal{A})$ be a measurable space, $(B, \mathcal{B}, P_Y)$ a probability space and $Q : B \times \mathcal{A} \to [0, 1]$. Then Q is said to be a **stochastic transition kernel from** $(B, \mathcal{B}, P)$ **to** $(A, \mathcal{A})$ if

1. for each $E \in \mathcal{A}$, $Q(\cdot, E)$ is $\mathcal{B}$-measurable
2. for P-a.e. $b \in B$, $Q(b, \cdot)$ is a probability measure on $(A, \mathcal{A})$

Definition 6.2.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L^0_n(\Omega, \mathcal{F}, P)$ and $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$. Then Q is said to be a **conditional probability distribution of** X **given** Y if

1. Q is a stochastic transition kernel from $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_Y)$ to $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$
2. for each $A, B \in \mathcal{F}$,
$$\int_B Q(y, A) dP_Y(y) = P(X \in A, Y \in B)$$

Note 6.2.0.3. It is helpful to connect this notion of conditional probability with the elementary one by writing $Q(y, A) = P(X \in A|Y = y)$. If $P_Y \ll \mu$, then property (2) in the definition becomes

$$\begin{aligned}
P(X \in A, Y \in B) &= \int_B Q(y, A) dP_Y(y) \\
&= \int_B P(X \in A|Y = y) f_Y(y) d\mu(y)
\end{aligned}$$

as in a first course on probability.

Exercise 6.2.0.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L^0_n$ and $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$. Suppose that for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$-measurable, for P_Y-a.e. $y \in \mathbb{R}^n$, $P_{X|Y}(y, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$ and $Q(Y, A) = P(X \in A|Y)$ a.e. Then Q is a conditional probability of X given Y.

Proof. By assumption, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$-measurable and for P_Y-a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$. Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned}
\int_B Q(y, A) dP_Y(y) &= \int_{Y^{-1}(B)} Q(Y(\omega), A) dP(\omega) \\
&= \int_{Y^{-1}(B)} P(X \in A|Y) dP \\
&= \int_{Y^{-1}(B)} E[1_{X^{-1}(A)}|Y] dP \\
&= \int_{Y^{-1}(B)} 1_{X^{-1}(A)} dP \\
&= \int 1_{X^{-1}(A)} 1_{Y^{-1}(B)} dP \\
&= \int 1_{X^{-1}(A) \cap Y^{-1}(B)} dP \\
&= P(X \in A, Y \in B)
\end{aligned}$$

So Q is a conditional probability distribution of X given Y. $\square$

Definition 6.2.0.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L^0_n$ and μ a σ-finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Then $P_{X,Y} \ll \mu^2$. Let $f_X = dP_X/d\mu$, $f_Y = dP_Y/d\mu$ and $f_{X,Y} = dP_{X,Y}/d\mu^2$. Define $f_{X|Y} : \mathbb{R}^n \times \mathbb{R}^n$ by

$$f_{X|Y}(x, y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)}, & y \in \operatorname{supp} Y \\ 0, & y \notin \operatorname{supp} Y \end{cases}$$

Then $f_{X|Y}$ is called the **conditional probability density of X given Y**.

Exercise 6.2.0.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L^0_n$ and μ a σ-finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Define $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$ by

$$Q(y, A) = \int_A f_{X|Y}(x, y) d\mu(x)$$

Then Q is a conditional probability distribution of X given Y.

Proof. By the Fubini-Tonelli Theorem, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$-measurable and for P_Y-a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$. Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned}
\int_B Q(y, A) dP_Y(y) &= \int_B \left[\int_A f_{X|Y}(x, y) d\mu(x) \right] dP_Y(y) \\
&= \int_{B \cap \operatorname{supp} Y} \left[\int_{A \cap \operatorname{supp} Y} \frac{f_{X,Y}(x, y)}{f_Y(y)} d\mu(x) f_Y(y) \right] d\mu(y) \\
&= \int_{B \cap \operatorname{supp} Y} \left[\int_A f_{X,Y}(x, y) d\mu(x) \right] d\mu(y) \\
&= P(X \in A, Y \in B \cap \operatorname{supp} Y) \\
&= P(X \in A, Y \in B)
\end{aligned}$$

$\square$

Theorem 6.2.0.7. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L^1_n(\Omega, \mathcal{F}, P)$. Suppose that $\operatorname{Img} X \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a conditional probability distribution of Y given X.

Chapter 7

Markov Chains

Definition 7.0.0.1. Define Markov kernels, their composition and

Definition 7.0.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n)_{n\in\mathbb{N}_0} \in L_n^0$. Then $(X_n)_{n\in\mathbb{N}_0}$ is said to be a **homogeneous Markov chain** if for each $A \in \mathcal{F}$ and $n \in \mathbb{N}$, $P(X_n \in A|X_1, \cdots, X_{n-1}) = P(X_1 \in A|X_0)$ a.e.

Chapter 8

Probabilities Induced by Isometric Group Actions

8.1 Applications to Bayesian Statistics

Exercise 8.1.0.1. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \to \Theta$ an isometric group action. Suppose that $\bar{d}$ is a metric on Θ/G. Let

- H_p^Θ be the Hausdorff measure on Θ, $\mu_\mathcal{X}$ a measure on $\mathcal{X}$,
- p a denisty on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on $\mathcal{X}$.
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose that μ_Θ is G-invariant, p is G-invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G-invariant and continuous on Θ. For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \dots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \to [0, 1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)})\, dH_p^\Theta(\theta)$$

Then there exists a density $\bar{p}(\cdot|X^{(n)})$ on Θ/G such that

$$d\bar{P}_{\Theta|X^{(n)}}(\theta) = \bar{p}(\theta|X^{(n)})\, d\bar{H}^\Theta(\theta)$$

Proof. Clear from previous work. $\square$

Exercise 8.1.0.2. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \to \Theta$ an isometric group action. Suppose that $\bar{d}$ is a metric on Θ/G. Let

- H_p^Θ be the Hausdorff measure on Θ, $\mu_\mathcal{X}$ a measure on $\mathcal{X}$,
- p a denisty on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on $\mathcal{X}$.
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose p is G-invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G-invariant and continuous on Θ. For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \dots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \to [0, 1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)})\, dH_p^\Theta(\theta)$$

Suppose that $(P_{\Theta|X^{(n)}})_{n\in\mathbb{N}}$ concentrates on $\bar{\theta}_0 \subset \Theta$ a.s. or in probability. Then $(\bar{P}_{\Theta|X^{(n)}})_{n\in\mathbb{N}}$ concentrates a.s. or in probability on $\{\bar{\theta}_0\} \subset \Theta/G$ (i.e. is consistent a.s. or in probability)

Proof. Let $V \in \mathcal{N}_{\bar{\theta}_0}$. Then $\pi^{-1}(V) \in \mathcal{N}_{\bar{\theta}_0}$. By definition,

$$\begin{aligned}
\bar{P}_{\Theta|X^{(n)}}(V^c) &= P_{\Theta|X^{(n)}}(\pi^{-1}(V^c)) \\
&= P_{\Theta|X^{(n)}}(\pi^{-1}(V)^c) \\
&\xrightarrow{\text{a.s.}/P} 0
\end{aligned}$$

$\square$

Note 8.1.0.3. Some examples of G-invariant priors would be the uniform distribution, or $N_n(0, \sigma^2 I)$ on $\mathbb{R}^n$ when acted on by $O(n)$. An example of a G-invariant likelihood would be $f(A|Z) \sim \mathrm{Ber}(ZZ^T)$ as in a latent position random graph model where $Z \in \mathbb{R}^{n\times d}$ is the parameter is invariant under right multiplication by $U \in O_d$.

Chapter 9

Projective Systems

Chapter 10

Stochastic Integration

Exercise 10.0.0.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, X a set $\mathcal{A}_0$ an algebra, $\mu_0 : \mathcal{A}_0 \to \mathbb{C}$ and $B : \mathcal{A}_0 \to L^2(\Omega, \mathcal{F}, P)$. Suppose that

1. $B(\varnothing) = 0$
2. for each $E, F \in \mathcal{A}_0$, if $E \cap F = \varnothing$, then $B(E \cup F) = B(E) + B(F)$
3. $E(B(E)B(F)^*) = \mu_0(E \cap F)$

Then

1. for each $E \in \mathcal{A}_0$, $\mu_0(E) = E(|B(E)|^2)$.
2. for each $E \in \mathcal{A}_0$, $0 \leq \mu_0(E) < \infty$
3. for each $E, F \in \mathcal{A}_0$, if $E \cap F = \varnothing$, then $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$

Proof.

1. Clear
2. Clear
3. Let $E, F \in \mathcal{A}_0$. Suppose that $E \cap F = \varnothing$. Then

$$\begin{aligned} E(B(E)B(F)^*) &= \mu_0(E \cap F) \\ &= \mu_0(\varnothing) \\ &= E(|B(\varnothing)|^2) \\ &= E(0) \\ &= 0 \end{aligned}$$

This implies that

$$\begin{aligned} \mu_0(E \cup F) &= E(|B(E \cup F)|^2) \\ &= E(|B(E) + B(F)|^2) \\ &= E(|B(E)|^2) + E(|B(F)|^2) + 2\mathrm{Re}\, E(B(E)B(F)^*) \\ &= \mu_0(E) + \mu_0(F) + 0 \\ &= \mu_0(E) + \mu_0(F) \end{aligned}$$

$\square$

Definition 10.0.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, X a set $\mathcal{A}_0$ an algebra, $\mu_0 : \mathcal{A}_0 \to [0, \infty)$ a premeasure and $B : \mathcal{A}_0 \to L^2(\Omega, \mathcal{F}, P)$. Suppose that

1. $B(\varnothing) = 0$
2. for each $E, F \in \mathcal{A}_0$, if $E \cap F = \varnothing$, then $B(E \cup F) = B(E) + B(F)$
3. $E(B(E)B(F)^*) = \mu_0(E \cap F)$

Then B is said to be a **stochastic premeasure with sturcture** μ_0

Chapter 11

TODO

Incorporate vector measures to have conditional expectations of Banach space valued functions given a σ-algebra

Appendix A

Summation

Definition A.0.0.1. Let $f : X \to [0, \infty)$, Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note A.0.0.2. Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.

Appendix B

Asymptotic Notation

Definition B.0.0.1. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = o(g) \qquad \text{as } x \to x_0$$

if for each $\epsilon > 0$, there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U$,

$$\|f(x)\| \leq \epsilon \|g(x)\|$$

Exercise B.0.0.2. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. If there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U \setminus \{x_0\}$, $g(x) > 0$, then

$$f = o(g) \text{ as } x \to x_0 \quad \text{iff} \quad \lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = 0$$

Exercise B.0.0.3. Let X and Y a be normed vector spaces, $A \subset X$ open and $f : A \to Y$. Suppose that $0 \in A$. If $f(h) = o(\|h\|)$ as $h \to 0$, then for each $h \in X$, $f(th) = o(|t|)$ as $t \to 0$.

Proof. Suppose that $f(h) = o(\|h\|)$ as $h \to 0$. Let $h \in X$ and $\epsilon > 0$. Choose $\delta' > 0$ such that for each $h' \in B(0, \delta')$, $h' \in A$ and

$$\|f(h')\| \leq \frac{\epsilon}{\|h\| + 1} \|h'\|$$

Choose $\delta > 0$ such that for each $t \in B(0, \delta)$, $th \in B(0, \delta')$. Let $t \in B(0, \delta)$. Then

$$\begin{aligned} \|f(th)\| &\leq \frac{\epsilon}{\|h\| + 1} |t| \|h\| \\ &< \epsilon |t| \end{aligned}$$

So $f(th) = o(|t|)$ as $t \to 0$. $\square$

Definition B.0.0.4. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = O(g) \qquad \text{as } x \to x_0$$

if there exists $U \in \mathcal{N}(x_0)$ and $M \geq 0$ such that for each $x \in U$,

$$\|f(x)\| \leq M \|g(x)\|$$

Appendix C

Categories

move to notation?

Definition C.0.0.1. We define the category of topological measure spaces, denoted $\mathbf{TopMsr}_+$, by

- $\mathrm{Obj}(\mathbf{TopMsr}_+) := \{(X, \mu) : X \in \mathrm{Obj}(\mathbf{Top}) \text{ and } \mu \in M(X)\}$
- $\mathrm{Hom}_{\mathbf{TopMsr}_+}((X, \mu), (Y, \nu)) := \mathrm{Hom}_{\mathbf{Top}}(X, Y) \cap \mathrm{Hom}_{\mathbf{Msr}_+}((X, \mathcal{B}(X), \mu), (Y, \mathcal{B}(Y), \nu))$

Appendix D

Rings

Appendix E

Introduction

Definition E.0.0.1. Let R be a ring and $I, J \subset R$ ideals. We define

- $I + J \subset R$ by $I + J := \{i + j : i \in I \text{ and } j \in J\}$,
- $IJ \subset R$ by $IJ := \{i_1 j_1 + i_2 j_2 : i_1, i_2 \in I \text{ and } j_1, j_2 \in J\}$.

Exercise E.0.0.2. Let R be a ring and $I, J \subset R$ ideals. Then $I + J$ and IJ are ideals.

Proof. FINISH!!! $\square$

Appendix F

Quotient Rings

Exercise F.0.0.1. Let R be a commutative unital ring and $I \subset R$ and ideal. Then I is maximal iff R/I is a field.

Proof. A previous exercise implies that R/I is a commutative unital ring.

- $(\implies)$:
  Suppose that I is maximal. Let $k + I \in R/I$. Suppose that $k + I \neq 0 + I$. Then $k \notin I$. Define $J \subset R$ by $J := (k) + I$. A previous exercise implies that J is an ideal. By construction $I \subset J$ and $k \in J$. Hence $I \neq J$. Since I is maximal, $J = R$. Since $1 \in J$, there exists $r \in R$ and $i \in I$ such that $1 = rk + i$. Hence

$$\begin{aligned} 1 + I &= rk + i + I \\ &= rk + I \\ &= (r + I)(k + I). \end{aligned}$$

  Thus $(k + I)$ is invertible. Since $k + I \in R/I$ such that $k + I \neq 0 + I$ is arbitrary, we have that for each $k + I \in R/I$, $k + I \neq 0 + I$ implies that $k + I$ is invertible. Thus R/I is a field.

- $(\impliedby)$:
  Suppose that R/I is a field. Let $J \subset R$. Suppose that J is an ideal, $I \subset J$ and $I \neq J$. Then there exists $j \in J$ such that $j \notin I$. Thus $j + I \neq 0 + I$. Since R/I is a field, $j + I$ is invertible. Hence there exists $k \in R$ such that $jk + I = 1 + I$. Hence

$$\begin{aligned} 1 - jk &\in I \\ &\subset J. \end{aligned}$$

  Since J is an ideal, $jk \in J$ and therefore

$$\begin{aligned} 1 &= (1 - jk) + jk \\ &\in J. \end{aligned}$$

  Since $1 \in J$, $J = R$. Since $J \subset R$ with J an ideal and $I \subset J$ is arbitrary, we have that for each $J \subset R$, J an ideal and $I \subset J$ implies that $J = I$ or $J = X$. Hence I is maximal.

$\square$

Appendix G

Vector Spaces

it might be better to cover some category theory and write everything in terms of $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}$ and $\mathrm{Obj}(\mathbf{Vect}_{\mathbb{K}})$

Appendix H

Introduction

Definition H.0.0.1. Let X be a set, $\mathbb{K}$ a field, $+ : X \times X \to X$ and $\cdot : \mathbb{K} \times X \to X$. Then $(X, +, \cdot)$ is said to be a **$\mathbb{K}$-vector space** if

1. $(X, +)$ is an abelian group
2.

Definition H.0.0.2. Let $(X, +, \cdot) \in \text{Obj}(\mathbf{Vect}_{\mathbb{C}})$. We define the **conjugate of** $(X, +, \cdot)$, denoted $(\bar{H}, \bar{+}, \bar{\cdot})$, by

- $\bar{X} := X$,
- $x \bar{+} y := x + y$,
- $\lambda \bar{\cdot} x := \lambda^* \cdot x$.

Exercise H.0.0.3. Let $(X, +, \cdot) \in \text{Obj}(\mathbf{Vect}_{\mathbb{C}})$. Then $(\bar{H}, \bar{+}, \bar{\cdot}) \in \text{Obj}(\mathbf{Vect}_{\mathbb{C}})$.

Proof. FINISH!!! □

Definition H.0.0.4. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$. Then $(E, +_E, \cdot_E)$ is said to be a subspace of X if

1. $+_E = +_X|_{E \times E}$
2. $\cdot_E = \cdot_X|_{\mathbb{K} \times E}$

Exercise H.0.0.5. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$.

Exercise H.0.0.6. Let $(X, +, \cdot)$ be a vector space and $E \subset X$. Then E is a subspace of X

Definition H.0.0.7. Let X be a vector space and $(E_j)_{j \in J}$ a collection of subspaces of X. Then $\bigcap_{j \in J} E_j$ is a subspace of X.

Proof. Set $E := \bigcap_{j \in J} E_j$. Let $x, y \in E$ and $\lambda \in \mathbb{K}$. Then for each $j \in J$, $x, y \in E_j$. Since for each $j \in J$, E_j is a subspace of X, we have that for each $j \in J$, $x + \lambda y \in E_j$. Thus $x + \lambda y \in E$. Since $x, y \in E$ and $\lambda \in \mathbb{K}$ are arbitrary, (cite exercise here) we have that E is a subspace of X. □

Definition H.0.0.8. Let X, Y be vector spaces and $T : X \to Y$. Then T is said to be **linear** if for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

1. $T(x_1 + x_2) = T(x_1) + T(x_2)$,
2. $T(\lambda x_1) = \lambda T(x_1)$.

We define $\mathcal{L}(X; Y) := \{T : X \to Y : \ T \text{ is linear}\}$.

Exercise H.0.0.9. Let X, Y be vector spaces and $T : X \to Y$. Then T is linear iff for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

$$T(x_1 + \lambda x_2) = T(x_1) + \lambda T(x_2)$$

Proof. Clear. (add details) □

Exercise H.0.0.10. $\mathcal{L}(X;Y)$ is a vector space FINISH!!!

Proof. content... □

Definition H.0.0.11. define addition/scalar multiplication of linear maps

Exercise H.0.0.12. Let X, Y be vector spaces over $\mathbb{K}$. Then $\mathcal{L}(X;Y)$ is a $\mathbb{K}$-vector space.

Proof. Clear □

Exercise H.0.0.13. Let X, Y, Z be vector spaces over $\mathbb{K}$, $T \in \mathcal{L}(X.Y)$ and $S \in \mathcal{L}(Y, Z)$. Then $S \circ T \in \mathcal{L}(X, Z)$.

Proof. FINISH!!! □

Appendix I

Bases

Definition I.0.0.1. Let X be a vector space and $(e_\alpha)_{\alpha \in A} \subset X$. Then $(e_\alpha)_{\alpha \in A}$ is said to be

- **linearly independent** if for each $(\alpha_j)_{j=1}^n \subset A$, $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $\sum\limits_{j=1}^n \lambda_j e_{\alpha_j} = 0$ implies that for each $j \in [n]$, $\lambda_j = 0$.
- a **Hamel basis for** X if $(e_\alpha)_{\alpha \in A}$ is linearly independent and $\operatorname{span}(e_\alpha)_{\alpha \in A} = X$.

Exercise I.0.0.2. every vector space has a Hamel basis

Proof. $\square$

Exercise I.0.0.3.

Exercise I.0.0.4. Let X be a $\mathbb{K}$-vector space and $x \in X$. Then $x = 0$ iff for each $\phi \in X^*$, $\phi(x) = 0$.

Proof.

- $(\implies)$:
  Suppose that $x = 0$. Linearity implies that for each $\phi \in X^*$ $\phi(x) = 0$.
- $(\impliedby)$:
  Conversely, suppose that $x \neq 0$. Define $\epsilon_x : \operatorname{span}(x) \to \mathbb{K}$ by $\epsilon_x(\lambda x) := \lambda$. Let $u, v \in \operatorname{span}(x)$. Then there exists $\lambda_u, \lambda_v \in \mathbb{K}$ such that $u = \lambda_u x$ and $v = \lambda_v x$. Suppose that $u = v$. Then

  $$\begin{aligned}(\lambda_u - \lambda_v)x &= \lambda_u x - \lambda_v x \\ &= u - v \\ &= 0\end{aligned}$$

  Since $x \neq 0$, we have that $\lambda_u - \lambda_v = 0$ and therefore $\lambda_u = \lambda_v$. Hence

  $$\begin{aligned}\lambda_u &= \epsilon_x(u) \\ &= \epsilon_x(v) \\ &= \lambda_v.\end{aligned}$$

  Thus ϵ_x is well defined.

$\square$

Appendix J

Duality

Definition J.0.0.1. Let X be a vector space over $\mathbb{K}$. We define the **algebraic dual space of** X, denoted X', by $X' := \mathcal{L}(X;\mathbb{K})$.

Exercise J.0.0.2. Let X be a vector space. Then X' is a vector space.

Proof. Clear. $\square$

Definition J.0.0.3. Let X, Y be vector spaces over $\mathbb{K}$. We define the algebraic adjoint of $T \in \mathcal{L}(X;Y)$, denoted $T' \in \mathcal{L}(Y', X')$, by $T'(\phi) := \phi \circ T$.

Appendix K

Multilinear Maps

Definition K.0.0.1. Let $X_1, \cdots, X_n, Y$ be vector spaces and $T : \prod_{j=1}^{n} X_j \to \mathbb{K}$. Then T is said to be **multilinear** if for each $j_0 \in [n]$ and $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$, $T(x_1, \ldots, x_{j_0-1}, \cdot, x_{j_0+1})$ is linear.

$$L^n(X_1, \ldots, X_n; Y) = \left\{ T : \prod_{j=1}^{n} X_j \to Y : T \text{ is multilinear} \right\}$$

If $X_1 = \cdots = X_n = X$, we write $L^n(X; Y)$ in place of $L^n(X, \ldots, X; Y)$.

Definition K.0.0.2. define addition and scalar mult of multilinear maps

Exercise K.0.0.3. Let $X_1, \cdots, X_n, Y$ be vector spaces. Then $L^n(X_1, \ldots, X_n; Y)$ is a $\mathbb{K}$-vector space.

Proof. content... □

Exercise K.0.0.4. Let $X_1, \cdots, X_n, Y, Z$ be $\mathbb{K}$-vector spaces, $\alpha \in L^n(X_1, \ldots, X_n; Y)$ and $\phi \in L^1(Y; Z)$. Then $\phi \circ \alpha \in L^n(X_1, \ldots, X_n; Z)$.

Proof. Let $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$. Define $f : X_{j_0} \to Y$ by

$$f(a) := \alpha(x_1, \ldots, x_{j_0-1}, a, x_{j_0+1}, \ldots, x_n)$$

Since $\alpha \in L^n(X_1, \ldots, X_n; Y)$, f is linear. Since ϕ is linear, and $\phi \circ f$ is linear. Since $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$ are arbitrary, we have that $\phi \circ \alpha \in L^n(X_1, \ldots, X_n; Y)$. □

Appendix L

Tensor Products

Definition L.0.0.1. Let X, Y and T be vector spaces over $\mathbb{K}$ and $\alpha \in L^2(X, Y; T)$. Then (T, α) is said to be a **tensor product of** X **and** Y if for each vector space Z and $\beta \in L^2(X, Y; Z)$, there exists a unique $\phi \in L^1(T; Z)$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & T \arrow[d, dashed, "\phi"] \\ & Z \end{tikzcd}$$

Exercise L.0.0.2. Let X, Y, S, T be vector spaces, $\alpha \in L^2(X, Y; S)$ and $\beta \in L^2(X, Y; T)$. Suppose that (S, α) and (T, β) are tensor products of X and Y. Then S and T are isomorphic.

Proof. Since (T, β) is a tensor product of X and Y, $\beta \in L^2(X, Y; T)$ there exists a unique $f \in L^1(T; T)$ such that $f \circ \beta = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dd, dashed, "f"] \\ X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T \end{tikzcd}$$

Since $\mathrm{id}_T \in L^1(T; T)$ and $\mathrm{id}_T \circ \beta = \beta$, we have that $f = \mathrm{id}_T$. Since (S, α) is a tensor product of X and Y, there exists a unique $\phi : S \to T$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] \\ & T \end{tikzcd}$$

Similarly, since (T, β) is a tensor product of X and Y, there exists a unique $\psi : T \to S$ such that $\psi \circ \beta = \alpha$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & T \arrow[d, dashed, "\psi"] \\ & S \end{tikzcd}$$

Therefore

$$\begin{aligned} (\phi \circ \psi) \circ \beta &= \phi \circ (\psi \circ \beta) \\ &= \phi \circ \alpha \\ &= \beta, \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, dashed, "\psi"] \\ X \times Y \arrow[ur, "\beta"] \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] \\ & T \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} & T \arrow[dd, dashed, "\phi\circ\psi"] \\ X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T \end{tikzcd}$$

By uniqueness of $f \in L^1(T;T)$, we have that

$$\begin{aligned} \mathrm{id}_T &= f \\ &= \phi \circ \psi \end{aligned}$$

A similar argument implies that $\psi \circ \phi = \mathrm{id}_S$. Hence ϕ and ψ are isomorphisms with $\phi^{-1} = \psi$. Hence S and T are isomorphic. □

Definition L.0.0.3. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. We define $x \otimes y : X^* \times Y^* \to \mathbb{K}$ by $x \otimes y(\phi, \psi) := \phi(x)\psi(y)$.

Exercise L.0.0.4. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. Then $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$.

Proof. Let $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} x \otimes y(\phi_1 + \lambda\phi_2, \psi) &= [\phi_1 + \lambda\phi_2(x)]\psi(y) \\ &= \phi_1(x)\psi(y) + \lambda\phi_2(x)\psi(y) \\ &= x \otimes y(\phi_1, \psi) + \lambda x \otimes y(\phi_2, \psi) \end{aligned}$$

Since $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$ are arbitrary, we have that for each $\psi \in Y^*$, $x \otimes y(\cdot, \psi)$ is linear. Similarly for each $\phi \in X^*$, $x \otimes y(\phi, \cdot)$ is linear. Hence $x \otimes y$ is bilinear and $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$. □

Definition L.0.0.5. Let X, Y be vector spaces. We define

- the **tensor product of X and Y**, denoted $X \otimes Y \subset L^2(X^*, Y^*; \mathbb{K})$, by

$$X \otimes Y := \mathrm{span}(x \otimes y : x \in X \text{ and } y \in Y),$$

- the **tensor map**, denoted $\otimes : X \times Y \to X \otimes Y$, by $\otimes(x, y) := x \otimes y$.

Exercise L.0.0.6. Let X, Y be vector spaces, $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$. The following are equivalent:

1. $\sum\limits_{j=1}^n x_j \otimes y_j = 0$
2. for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum\limits_{j=1}^n \phi(x_j)\psi(y_j) = 0$
3. for each $\phi \in X^*$, $\sum\limits_{j=1}^n \phi(x_j) y_j = 0$
4. for each $\psi \in Y^*$, $\sum\limits_{j=1}^n \psi(y_j) x_j = 0$

Proof.

1. (1) $\implies$ (2) :
   Suppose that $\sum\limits_{j=1}^n x_j \otimes y_j = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Then

$$\begin{aligned} \sum_{j=1}^n \phi(x_j)\psi(y_j) &= \phi\bigg(\sum_{j=1}^n \psi(y_j) x_j\bigg) \\ &= \end{aligned}$$

2.

3.

$\square$

Exercise L.0.0.7. Let X, Y be vector spaces. Then $(X \otimes Y, \otimes)$ is a tensor product of X and Y.

Proof. Let Z be a vector space and $\alpha \in L^2(X, Y; Z)$. Define $\phi : X \otimes Y \to Z$ by $\phi\Big(\sum_{j=1}^{n} \lambda_j x_j \otimes y_j) := \sum_{j=1}^{n} \lambda_j \alpha(x_j, y_j)$.

- **(well defined):**
  Let $u \in X \otimes Y$. Then there exist $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $(x_j)_{j=1}^n \subset X$, $(y_j)_{j=1}^n \subset Y$ such that $u = \sum_{j=1}^{n} \lambda_j x_j \otimes y_j$. Suppose that $u = 0$. Let $\phi \in Z^*$. Then $\phi \circ \alpha \in L^2(X, Y; Z)$.

$\square$

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration