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Introduction to Harmonic Analysis

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Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Fourier Analysis on $\mathcal{S}(\mathbb{R}^n)$

1.1 Schwartz Space

Definition 1.1.0.1. Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

1. $\langle x, y\rangle = \sum_j x_j y_j$
2. $|x| = \langle x, x\rangle^{1/2}$
3. $|\alpha| = \alpha_1 + \cdots + \alpha_n$
4. $\alpha! = \prod_{j=1}^{n} \alpha_j!$
5. $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
6. $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$
7. $\Omega_\alpha = \{(\beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n : \beta + \gamma = \alpha\}$

Exercise 1.1.0.2. Let $\alpha \in \mathbb{N}_0^n$ and $j \in \{1, \dots, n\}$. Suppose that $\alpha_j > 0$. Set $\eta = \alpha - e_j$. Then

1. $\Omega_\eta = \{(\beta - e_j, \gamma) : (\beta, \gamma) \in \Omega_\alpha \text{ and } \beta_j > 0\}$
2. $\Omega_\eta = \{(\beta, \gamma - e_j) : (\beta, \gamma) \in \Omega_\alpha \text{ and } \gamma_j > 0\}$

Proof.

1. Set $A = \{(\beta - e_j, \gamma) : (\beta, \gamma) \in \Omega_\alpha \text{ and } \beta_j > 0\}$. Let $(\mu, \nu) \in \Omega_\eta$. Set $\beta = \mu + e_j$ and $\gamma = \nu$. Then $\beta_j > 0$ and
$$\begin{aligned}\beta + \gamma &= \mu + e_j + \nu \\ &= \eta + e_j \\ &= \alpha\end{aligned}$$
So $(\beta, \gamma) \in \Omega_\alpha$. Hence
$$\begin{aligned}(\mu, \nu) &= (\beta - e_j, \gamma) \\ &\in A\end{aligned}$$
and $\Omega_\eta \subset A$.
Conversely, let $(\mu, \nu) \in A$. Then there exists $(\beta, \gamma) \in \Omega_\alpha$ such that $\beta_j > 0$ and $(\mu, \nu) = (\beta - e_j, \gamma)$. Then
$$\begin{aligned}\mu + \nu &= \beta - e_j + \gamma \\ &= \alpha - e_j \\ &= \eta\end{aligned}$$
So that $(\mu, \nu) \in \Omega_\eta$ and $A \subset \Omega_\eta$. Thus $\Omega_\eta = A$.

2. Similar to (1).

□

Exercise 1.1.0.3. Let $f, g \in C^\infty(\mathbb{R}^n)$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha (fg) = \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$$

Proof. Let $\alpha \in \mathbb{N}_0^n$. The claim is true if $|\alpha| = 0$. Let $k > 0$. Suppose that $|\alpha| > 0$ and that the claim is true for $|\alpha| = k-1$ so that for each $\eta \in \mathbb{N}_0^n$, $|\eta| = k-1$ implies that

$$\partial^\eta (fg) = \sum_{(\beta,\gamma)\in\Omega_\eta} \frac{\eta!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$$

Since $|\alpha| > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Define $\eta = \alpha - e_j$. Then the previous exercise implies that

$$\begin{aligned}
\partial^\alpha (fg) &= \partial_j[\partial^\eta (fg)] \\
&= \partial_j \left[\sum_{(\beta,\gamma)\in\Omega_\eta} \frac{\eta!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) \right] \\
&= \sum_{(\beta,\gamma)\in\Omega_\eta} \frac{\eta!}{\beta!\gamma!}(\partial^{\beta+e_j} f)(\partial^\gamma g) + \sum_{(\beta,\gamma)\in\Omega_\eta} \frac{\eta!}{\beta!\gamma!}(\partial^\beta f)(\partial^{\gamma+e_j} g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0}} \frac{(\alpha-e_j)!}{(\beta-e_j)!\gamma!}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \gamma_j>0}} \frac{(\alpha-e_j)!}{\beta!(\gamma-e_j)!}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\gamma_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0,\gamma_j=0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) \\
&\quad + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\gamma_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j=0,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0,\gamma_j=0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j+\gamma_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) \\
&\quad + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j=0,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0,\gamma_j=0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j=0,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)
\end{aligned}$$

So the claim is true for $|\alpha| = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$. □

Exercise 1.1.0.4. Let $\xi \in \mathbb{R}^n$. Define $f \in \mathbb{C}^\infty(\mathbb{R}^n)$ by $f(x) = e^{-i\langle \xi, x\rangle}$. Then for each $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f = (-i\xi)^\alpha f$

Proof. Let $\alpha \in \mathbb{N}_0^n$. The claim is true for $|\alpha| = 0$. Let $k > 0$. Suppose that the claim is true for $|\alpha| \leq k-1$ so that for each $\beta \in \mathbb{N}_0$, $|\beta| \leq k-1$ implies that $\partial^\beta f = (-i\xi)^\beta f$. Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Then

$$\begin{aligned}
\partial^\alpha f &= \partial_j(\partial^{\alpha - e_j} f) \\
&= \partial_j((-i\xi)^{\alpha - e_j} f) \\
&= (-i\xi)^{\alpha - e_j} \partial_j f \\
&= (-i\xi)^{\alpha - e_j} i\xi_j \\
&= (-i\xi)^\alpha f
\end{aligned}$$

So the claim is true for $|\alpha| = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$.

□

Definition 1.1.0.5. Let $f \in C^\infty(\mathbb{R})$, $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define $\|\cdot\|_{\alpha,N} : C^\infty(\mathbb{R}^n, \mathbb{C}) \to [0, \infty]$ by

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha f(x)| \right]$$

We define **Schwartz space** on $\mathbb{R}^n$, denoted $\mathcal{S}(\mathbb{R}^n)$, by

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n \text{ and } N \in \mathbb{N}_0, \|f\|_{\alpha,N} < \infty\}$$

Exercise 1.1.0.6. For each $p \in [1, \infty)$ and $x \in \mathbb{R}^n$,

$$(1 + |x|)^p \geq (1/2)(1 + |x|^p)$$

Proof. Let $p \in [1, \infty)$ and $x \in \mathbb{R}^n$. Suppose that $p \in \mathbb{Q}$. Then there exist $m, n \in \mathbb{N}$ such that $m \geq n$ and $p = m/n$. The binomial theorem implies that

$$\begin{aligned}
(1 + |x|)^m &= \sum_{j=0}^m \binom{m}{j} |x|^{m-j} \\
&\geq 1 + |x|^m
\end{aligned}$$

Jensen's inequality implies that

$$\begin{aligned}
(1 + |x|)^p &= [(1 + |x|)^m]^{1/n} \\
&\geq (1 + |x|^m)^{1/n} \\
&\geq (1/2)^{\frac{n-1}{n}} (1 + |x|^p) \\
&\geq (1/2)(1 + |x|^p)
\end{aligned}$$

Suppose that $p \notin \mathbb{Q}$. Choose a sequence $(p_j)_{j \in \mathbb{N}} \subset [1, \infty) \cap \mathbb{Q}$ such that $p_j \to p$. By continuity,

$$\begin{aligned}
(1 + |x|)^p &= \lim_{j \to \infty} (1 + |x|)^{p_j} \\
&\geq \lim_{j \to \infty} (1/2)(1 + |x|^{p_j}) \\
&= (1/2)(1 + |x|^p)
\end{aligned}$$

□

Exercise 1.1.0.7. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then f is Lipschitz.

Proof.

1. Set $M = \max\{\|f\|_{e_j,0} : j \in \{1, \dots, n\}\}$. By definition, for each $j \in \{1, \cdots, n\}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}|\partial_j f(x)| &\leq \|f\|_{e_j,0} \\ &\leq M\end{aligned}$$

   Let $x, h \in \mathbb{R}^n$. Jensen's inequality implies that

$$\begin{aligned}|Df(x)(h)| &= \left|\sum_{j=1}^n \partial_j f(x) h_j\right| \\ &\leq \sum_{j=1}^n |\partial_j f(x)||h_j| \\ &\leq M \sum_{j=1}^n |h_j| \\ &\leq \sqrt{n} M |h|\end{aligned}$$

   Since $h \in \mathbb{R}^n$ is arbitrary, $\|Df(x)\| \leq \sqrt{n} M$. Since $x \in \mathbb{R}^n$ is arbitrary, Df is bounded. Hence f is Lipschitz.

□

Exercise 1.1.0.8. We have that $\mathcal{S}(\mathbb{R}^n)$ is a vector space and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, $\|\cdot\|_{\alpha,N} : \mathcal{S}(\mathbb{R}^n) \to [0, \infty)$ is a seminorm on $\mathcal{S}(\mathbb{R}^n)$.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$.

1.

$$\begin{aligned}\|\lambda f\|_{\alpha,N} &= \sup_{x \in \mathbb{R}} \left[(1+|x|)^N |\partial^\alpha [\lambda f](x)|\right] \\ &= \sup_{x \in \mathbb{R}} \left[(1+|x|)^N |\lambda \partial^\alpha f(x)|\right] \\ &= \sup_{x \in \mathbb{R}} \left[|\lambda|(1+|x|)^N |\partial^\alpha f(x)|\right] \\ &= |\lambda| \sup_{x \in \mathbb{R}} \left[(1+|x|)^N |\partial^\alpha f(x)|\right] \\ &= |\lambda| \|f\|_{\alpha,N}\end{aligned}$$

   Thus $\lambda f \in \mathcal{S}(\mathbb{R}^n)$ and $\|\lambda f\|_{\alpha,N} = |\lambda| \|f\|_{\alpha,N}$.

2.

$$\begin{aligned}\|f+g\|_{\alpha,N} &= \sup_{x \in \mathbb{R}} \left[(1+|x|)^N |\partial^\alpha [f+g](x)|\right] \\ &= \sup_{x \in \mathbb{R}} \left[(1+|x|)^N |[\partial^\alpha f + \partial^\alpha g](x)|\right] \\ &\leq \sup_{x \in \mathbb{R}} \left[(1+|x|)^N |\partial^\alpha f(x)| + (1+|x|)^N |\partial^\alpha g(x)|\right] \\ &\leq \sup_{x \in \mathbb{R}} \left[(1+|x|)^N |\partial^\alpha f(x)|\right] + \sup_{x \in \mathbb{R}} \left[(1+|x|)^N |\partial^\alpha g(x)|\right] \\ &= \|f\|_{\alpha,N} + \|g\|_{\alpha,N}\end{aligned}$$

   Hence $f + g \in \mathcal{S}(\mathbb{R}^n)$ and $\|f+g\|_{\alpha,N} \leq \|f\|_{\alpha,N} + \|g\|_{\alpha,N}$.

So $\mathcal{S}(\mathbb{R}^n)$ is a vector space and $\|\cdot\|_{\alpha,N}$ is a seminorm on $\mathcal{S}(\mathbb{R}^n)$. $\square$

Exercise 1.1.0.9. We have that $\mathcal{S}(\mathbb{R}^n)$ is a algebra under pointwise multiplication and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|fg\|_{\alpha,N} \leq \sum_{\beta=0}^{\alpha} \|f\|_{\beta,N} \|g\|_{\alpha-\beta,0}$$

Hint: $\partial^\alpha(fg) = \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|fg\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha(fg)(x)|\right] \\
&= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N \left| \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!} \partial^\beta f(x) \partial^\gamma g(x) \right|\right] \\
&\leq \sup_{x\in\mathbb{R}} \left[(1+|x|)^N \left(\sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!} |\partial^\beta f(x)||\partial^\gamma g(x)| \right)\right] \\
&= \sup_{x\in\mathbb{R}} \left[\sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!} (1+|x|)^N |\partial^\beta f(x)||\partial^\gamma g(x)| \right] \\
&\leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!} \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\beta f(x)||\partial^\gamma g(x)|\right] \\
&\leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!} \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\beta f(x)|\right] \sup_{x\in\mathbb{R}} \left[|\partial^\gamma g(x)|\right] \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!} \|f\|_{\beta,N} \|g\|_{\gamma,0} \\
&< \infty
\end{aligned}$$

So $fg \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Definition 1.1.0.10. Set $\mathcal{P} = \{\|\cdot\|_{\alpha,N} : \alpha \in \mathbb{N}_0^n, N \in \mathbb{N}_0\}$. Then $\mathcal{P}$ is a countable family of seminorms on $\mathcal{S}(\mathbb{R}^n)$. We equip $\mathcal{S}(\mathbb{R}^n)$ with the topology $\mathcal{T}$ induced by the family of projections

$$\pi_{\|\cdot\|_{\alpha,N}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)/\ker\|\cdot\|_{\alpha,N}$$

i.e. $\mathcal{T} = \tau_{\mathcal{S}(\mathbb{R}^n)}((\pi_p)_{p\in\mathcal{P}})$.
Explicitly, for a net $(f_\gamma)_{\gamma\in\Gamma} \subset \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, $f_\gamma \to f$ iff for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, $\|f_\gamma - f\|_{\alpha,N} \to 0$.
Hence $(\mathcal{S}(\mathbb{R}^n), \mathcal{T})$ is a locally convex space. Since $\mathcal{P}$ is countable, we may write $\mathcal{P} = (p_j)_{j\in\mathbb{N}}$ and thus $(\mathcal{S}(\mathbb{R}^n), \mathcal{T})$ is metrizable with metric

$$d_{\mathcal{S}(\mathbb{R}^n)}(f,g) = \sum_{j\in\mathbb{N}} 2^{-j} \frac{p_j(f-g)}{1+p_j(f-g)}$$

Exercise 1.1.0.11. Let $f \in \mathcal{S}(\mathbb{R}^n)$. For each $p \in [1,\infty]$, $f \in L^p(\mathbb{R}^n)$

Proof. Let $p \in [1,\infty]$. Suppose that $p < \infty$. The previous exercise implies that for each $x \in \mathbb{R}$,

$$(1+|x|)^{2p} \geq (1/2)(1+|x|^{2p})$$

By definition, there exists $C \geq 0$ such that for each $x \in \mathbb{R}$,

$$|f(x)| \leq C(1+|x|)^{-2}$$

Then for each $x \in \mathbb{R}$,

$$
\begin{aligned}
|f(x)|^p &\leq C^p(1+|x|)^{-2p} \\
&\leq 2C^p(1+|x|^{2p})^{-1}
\end{aligned}
$$

Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = 2C^p(1+|x|^{2p})^{-1}$. Since $g \in L^1(m)$ and $|f|^p \leq g$, we have that $f \in L^p(\mathbb{R}^n)$. If $p = \infty$, then by definition,

$$
\begin{aligned}
\|f\|_\infty &= \|f\|_{0,0} \\
&< \infty
\end{aligned}
$$

So $f \in L^p(\mathbb{R}^n)$. □

Exercise 1.1.0.12. For each $p \in [1, \infty)$, the inclusion $\iota : \mathcal{S}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_j)_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_j \to f$. Then for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, $\|f_j - f\|_{\alpha,N} \to 0$. By definition, for each $x \in \mathbb{R}$,

$$
|f_j(x) - f(x)| \leq \|f_j - f\|_{0,2}(1+|x|)^{-2}
$$

Therefore, for each $x \in \mathbb{R}$,

$$
\begin{aligned}
\|f_j - f\|_p^p &= \int_{\mathbb{R}^n} |f_j - f|^p \, dm \\
&\leq \int_{\mathbb{R}^n} \|f_j - f\|_{0,2}^p (1+|x|)^{-2p} \, dm(x) \\
&\leq \|f_j - f\|_{0,2}^p \int_{\mathbb{R}^n} 2(1+|x|^{2p})^{-1} \, dm(x) \\
&= \|f_j - f\|_{0,2}^p \int_{\mathbb{R}^n} 2(1+|x|^{-2p})^{-1} \, dm(x) \\
&\to 0
\end{aligned}
$$

Hence $f_j \xrightarrow{L^p} f$ and $\iota : \mathcal{S}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is continuous. □

Exercise 1.1.0.13. For each $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $\alpha \in \mathbb{N}_0^n$. The claim is true for $|\alpha| = 0$ and $|\alpha| = 1$. Let $k > 1$. Suppose that the claim is true for $|\alpha| = k-1$ so that for each $\beta \in \mathbb{N}_0^n$, $|\beta| = k-1$ implies that $\partial^\beta : \mathcal{S}(\mathbb{R}^n) \to C^\infty$ is linear. Suppose that $|\alpha| = k$. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Then

$$
\begin{aligned}
\partial^\alpha(f + \lambda g) &= \partial_j(\partial^{\alpha - e_j}[f + \lambda g]) \\
&= \partial_j(\partial^{\alpha - e_j} f + \lambda \partial^{\alpha - e_j} g) \\
&= \partial_j(\partial^{\alpha - e_j} f) + \lambda \partial_j(\partial^{\alpha - e_j} g) \\
&= \partial^\alpha f + \lambda \partial^\alpha g
\end{aligned}
$$

Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$ are arbitrary, we have that ∂^α is linear. So the claim is true for $|\alpha| = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$. □

Exercise 1.1.0.14. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Then $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$
\|\partial^\alpha f\|_{\beta,N} \leq \|f\|_{\alpha+\beta,N}
$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. By definition,

$$\begin{aligned}\|\partial^\alpha f\|_{\beta,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\beta(\partial^\alpha f)(x)|\right]\\ &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^{\alpha+\beta} f(x)|\right]\\ &= \|f\|_{\alpha+\beta,N}\\ &< \infty\end{aligned}$$

So $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 1.1.0.15. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|f\|_{\alpha,N} = \|\partial^\alpha f\|_{0,N}$$

Proof. Clear by preceding exercise. □

Exercise 1.1.0.16. Let $\alpha \in \mathbb{N}_0^n$. Then $\partial^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathbb{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. Let $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}$. Then

$$\begin{aligned}\|\partial^\alpha f_k\|_{\beta,N} &\leq \|f_k\|_{\alpha+\beta,N}\\ &\to 0\end{aligned}$$

Since $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\partial^\alpha f_k \to 0$. Thus ∂^α is continuous at 0. Since ∂^α is linear, ∂^α is continuous. □

1.2 Position and Momentum Operators

Definition 1.2.0.1. Let $j \in \{1, \dots, n\}$. We define the **j-th position operator**, denoted $X_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$X_j f(x) = x_j f(x)$$

Exercise 1.2.0.2. Let $j \in \{1, \dots, n\}$. Then $X_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Then for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned} X_j(f + \lambda g)(x) &= x_j(f(x) + \lambda g(x)) \\ &= x_j f(x) + \lambda x_j g(x) \\ &= (X_j f + \lambda X_j g)(x) \end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that $X_j(f + \lambda g) = X_j f + \lambda X_j g$. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$ are arbitrary, we have that X_j is linear. $\square$

Exercise 1.2.0.3. For each $j \in \{1, \dots, n\}$ and $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha X_j = \begin{cases} X_j \partial^\alpha & \alpha_j = 0 \\ X_j \partial^\alpha + \alpha_j \partial^{\alpha - e_j} & \alpha_j > 0 \end{cases}$$

Proof. Let $j \in \{1, \dots, n\}$, $\alpha \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. The claim is true if $\alpha_j = 0$ or $\alpha_j = 1$. Let $k > 1$. Suppose that the claim is true for $\alpha_j = k - 1$ so that $\partial_j^{k-1}(X_j f) = X_j(\partial_j^{k-1} f) + (k-1)\partial_j^{k-2} f$. Suppose that $\alpha_j = k$. Then

$$\begin{aligned} (\partial_j^k X_j) f &= \partial_j^k (X_j f) \\ &= \partial_j(\partial_j^{k-1}[X_j f]) \\ &= \partial_j(X_j[\partial_j^{k-1} f] + (k-1)\partial_j^{k-2}) \\ &= \partial_j(X_j[\partial_j^{k-1} f]) + (k-1)\partial_j(\partial_j^{k-2} f) \\ &= (X_j[\partial_j^k f] + \partial_j^{k-1} f) + (k-1)\partial_j^{k-1} f \\ &= X_j(\partial_j^k f) + k\partial_j^{k-1} f \\ &= (X_j \partial_j^k + k\partial_j^{k-1}) f \end{aligned}$$

which implies that

$$\begin{aligned} (\partial^\alpha X_j) f &= \partial^\alpha (X_j f) \\ &= \partial^{\alpha - k e_j}(\partial_j^k [X_j f]) \\ &= \partial^{\alpha - k e_j}(X_j[\partial_j^k f] + k\partial_j^{k-1} f) \\ &= X_j(\partial^{\alpha - k e_j}[\partial_j^k f]) + k\partial^{\alpha - k e_j}(\partial_j^{k-1} f) \\ &= X_j(\partial^\alpha f) + \alpha_j \partial^{\alpha - e_j} f \\ &= (X_j \partial^\alpha + \alpha_j \partial^{\alpha - e_j}) f \end{aligned}$$

So the claim is true for $\alpha_j = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$. $\square$

Exercise 1.2.0.4. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \{1, \dots, n\}$. Then $X_j f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|X_j f\|_{\alpha, N} \leq \begin{cases} \|f\|_{\alpha, N+1} & \alpha_j = 0 \\ \|f\|_{\alpha, N+1} + \alpha_j \|f\|_{\alpha - e_j, N} & \alpha_j > 0 \end{cases}$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. If $\alpha_j = 0$, then the previous exercise implies that

$$\begin{aligned}
\|X_j f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha (X_j f)(x)|\right] \\
&= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |x_j \partial^\alpha f(x)|\right] \\
&\leq \sup_{x\in\mathbb{R}} \left[(1+|x|)^{N+1} |\partial^\alpha f(x)|\right] \\
&= \|f\|_{\alpha,N+1} \\
&< \infty
\end{aligned}$$

If $\alpha_j > 0$, then the previous exercise implies that

$$\begin{aligned}
\|X_j f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha (X_j f)(x)|\right] \\
&= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |x_j \partial^\alpha f(x) + \alpha_j \partial^{\alpha-e_j} f(x)|\right] \\
&\leq \sup_{x\in\mathbb{R}} \left[(1+|x|)^{N+1} |\partial^\alpha f(x)|\right] + \alpha_j \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^{\alpha-e_j} f(x)|\right] \\
&= \|f\|_{\alpha,N+1} + \alpha_j \|f\|_{\alpha-e_j,N} \\
&< \infty
\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $X_j f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 1.2.0.5. Let $j \in \{1,\dots,n\}$. Then $X_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathbb{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}$. If $\alpha_j = 0$, then

$$\begin{aligned}
\|X_j f_k\|_{\alpha,N} &\leq \|f_k\|_{\alpha,N+1} \\
&\to 0
\end{aligned}$$

If $\alpha_j > 0$, then

$$\begin{aligned}
\|X_j f_k\|_{\alpha,N} &\leq \|f_k\|_{\alpha,N+1} + \alpha_j \|f_k\|_{\alpha-e_j,N} \\
&\to 0
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $X_j f_k \to 0$. Thus X_j is continuous at 0. Since X_j is linear, X_j is continuous. □

Exercise 1.2.0.6. Let $j,k \in \{1,\dots,n\}$. Then $X_j X_k = X_k X_j$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}
([X_j X_k]f)(x) &= (X_j[X_k f])(x) \\
&= x_j (X_k f)(x) \\
&= x_j x_k f(x) \\
&= x_k x_j f(x) \\
&= x_k (X_j f)(x) \\
&= (X_k[X_j f])(x) \\
&= ([X_k X_j]f)(x)
\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ are arbitrary, $X_j X_k = X_k X_j$. □

Definition 1.2.0.7. Let $\alpha \in \mathbb{N}_0^n$. We define $X^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$

Definition 1.2.0.8. Let $j \in \{1, \ldots, n\}$. We define the **j-th momentum operator**, denoted $P_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$P_j = -i\partial_j$$

Exercise 1.2.0.9. Let $j \in \{1, \ldots, n\}$. Then $P_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Clear since $\partial_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear. □

Exercise 1.2.0.10. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \{1, \ldots, n\}$. Then $P_j f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|P_j f\|_{\alpha,N} \leq \|f\|_{\alpha+e_j,N}$$

Proof. Clear since $\partial_j f \in \mathcal{S}(\mathbb{R}^n)$ and $\|\partial_j f\|_{\alpha,N} \leq \|f\|_{\alpha+e_j,N}$. □

Exercise 1.2.0.11. Let $j \in \{1, \ldots, n\}$. Then $P_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Clear cince $\partial_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous. □

Exercise 1.2.0.12. Let $j, k \in \{1, \ldots, n\}$. Then $P_j P_k = P_k P_j$.

Proof. Clear since $\partial_j \partial_k = \partial_k \partial_j$. □

Definition 1.2.0.13. Let $\alpha \in \mathbb{N}_0^n$. We define $P^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by $P^\alpha = P_1^{\alpha_1} \cdots P_n^{\alpha_n}$

Exercise 1.2.0.14. Let $j, k \in \{1, \ldots, n\}$. Then $[X_j, P_k] = i\delta_{j,k}$.

Proof. A previous exercise implies that $\partial_k X_j = X_j \partial_k + \delta_{j,k} I$. Therefore

$$\begin{aligned}
[X_j, P_k] &= X_j P_k - P_k X_j \\
&= -i(X_j \partial_k - \partial_k X_j) \\
&= -i(X_j \partial_k - [X_j \partial_k + \delta_{j,k} I]) \\
&= -i\delta_{j,k} I
\end{aligned}$$

□

1.3 Translation and Rotation Operators

Definition 1.3.0.1. Let $y \in \mathbb{R}^n$. We define the **translation by** y **operator**, denoted $\tau_y : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}^\infty(\mathbb{R}^n)$, by $\tau_y f(x) = f(x-y)$.

Exercise 1.3.0.2. Let $y \in \mathbb{R}^n$. Then $\tau_y : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Then for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned}\tau_y(f+\lambda g)(x) &= (f+\lambda g)(x-y)\\ &= f(x-y) + \lambda g(x-y)\\ &= \tau_y f(x) + \lambda \tau_y g(x)\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that $\tau_y(f+\lambda g) = \tau_y f + \lambda \tau_y g$. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ are arbitary, τ_y is linear. □

Exercise 1.3.0.3. Let $\alpha \in \mathbb{N}_0$. Then for each $y \in \mathbb{R}^n$,

$$\partial^\alpha \tau_y = \tau_y \partial^\alpha$$

Proof. Let $y \in \mathbb{R}^n$. The claim is true if $|\alpha| = 0$. Let $k \geq 1$. Suppose that the claim is true for $|\alpha| \leq k-1$ so that for each $\beta \in \mathbb{N}_0^n$, $|\beta| \leq k-1$ implies that

$$\partial^\beta \tau_y = \tau_y \partial^\beta$$

Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define $g : \mathbb{R}^n \to \mathbb{R}^n$ and $g_k : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = x - y$ and $g_k = \pi_k \circ g$. Then the chain rule implies that

$$\begin{aligned}(\partial^\alpha \tau_y) f &= \partial^\alpha(\tau_y f)\\ &= \partial_j(\partial^{\alpha - e_j}[\tau_y f])\\ &= \partial_j(\tau_y[\partial^{\alpha-e_j} f])\\ &= \partial_j([\partial^{\alpha-e_j} f] \circ g)\\ &= \sum_{k=1}^n [\partial_k(\partial^{\alpha-e_j} f) \circ g]\partial_j g_k\\ &= \partial_j(\partial^{\alpha-e_j} f) \circ g\\ &= (\partial^\alpha f) \circ g\\ &= \tau_y(\partial^\alpha f)\\ &= (\tau_y \partial^\alpha) f\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ is arbitrary, $\partial^\alpha \tau_y = \tau_y \partial^\alpha$. Hence the claim is true for $|\alpha| = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$. □

Exercise 1.3.0.4. Let $y \in \mathbb{R}$. Then for each $x \in \mathbb{R}^n$, $(1+|x|) \leq (1+|y|)(1+|x-y|)$.

Proof. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}(1+|y|)(1+|x-y|) &= 1 + (|x-y| + |y|) + |y||x-y|\\ &\geq 1 + |x| + |y||x-y|\\ &\geq 1 + |x|\end{aligned}$$

□

Exercise 1.3.0.5. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$. Then $\tau_y f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\tau_y f\|_{\alpha,N} \leq (1+|y|)^N \|f\|_{\alpha,N}$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|\tau_y f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N|\partial^\alpha \tau_y f(x)|\bigg] \\
&= \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N|\tau_y \partial^\alpha f(x)|\bigg] \\
&= \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N|\partial^\alpha f(x-y)|\bigg] \\
&\leq \sup_{x\in\mathbb{R}} \bigg[(1+|y|)^N(1+|x-y|)^N|\partial^\alpha f(x-y)|\bigg] \\
&= (1+|y|)^N \sup_{x\in\mathbb{R}} \bigg[(1+|x-y|)^N|\partial^\alpha f(x-y)|\bigg] \\
&= (1+|y|)^N \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N|\partial^\alpha f(x)|\bigg] \\
&= (1+|y|)^N \|f\|_{\alpha,N} \\
&< \infty
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\tau_y f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 1.3.0.6. Let $y \in \mathbb{R}^n$. Then $\tau_y : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathcal{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. Let $\alpha, N \in \mathcal{N}_0$. Then

$$\begin{aligned}
\|\tau_y f_k\|_{\alpha,N} &\leq (1+|y|)^N \|f_k\|_{\alpha,N} \\
&\to 0
\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\tau_y f_k \to 0$. So τ_y is continuous at 0. Since τ_y is linear, τ_y is continuous. □

Definition 1.3.0.7. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define $\tau f : \mathbb{R}^n \to \mathcal{S}(\mathbb{R}^n)$ by $\tau f(y) = \tau_y f$.

Exercise 1.3.0.8. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\tau f : \mathbb{R}^n \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. content... □

Definition 1.3.0.9. Let $\xi \in \mathbb{R}^n$. We define the **rotation by ξ operator**, denoted $\rho_\xi : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, by $\rho_\xi f(x) = e^{-i\langle \xi, x\rangle} f(x)$.

Exercise 1.3.0.10. Let $\xi \in \mathbb{R}^n$. Then $\rho_\xi : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Then for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned}
\rho_\xi(f+\lambda g)(x) &= e^{-i\langle \xi, x\rangle}(f+\lambda g)(x) \\
&= e^{-i\langle \xi, x\rangle} f(x) + \lambda e^{-i\langle \xi, x\rangle} g(x) \\
&= \rho_\xi f(x) + \lambda \rho_\xi g(x)
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that $\rho_\xi(f+\lambda g) = \rho_\xi f + \lambda \rho_\xi g$. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ are arbitary, ρ_ξ is linear. □

Exercise 1.3.0.11. Let $\xi \in \mathbb{R}^n$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha \rho_\xi = \rho_\xi \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Define $g \in C^\infty(\mathbb{R}^n)$ by $g(x) = e^{-i\langle \xi, x\rangle}$. A previous exercise implies that

$$\begin{aligned}
(\partial^\alpha \rho_\xi)f &= \partial^\alpha(\rho_\xi f) \\
&= \partial^\alpha(gf) \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta g)(\partial^\gamma f) \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}((-i\xi)^\beta g)(\partial^\gamma f) \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \rho_\xi(\partial^\gamma f) \\
&= \rho_\xi\bigg(\sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma f\bigg) \\
&= \bigg(\rho_\xi \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma\bigg) f
\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ is arbitrary,

$$\partial^\alpha \rho_\xi = \rho_\xi \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma$$

□

Exercise 1.3.0.12. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Then $\rho_\xi f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\rho_\xi f\|_{\alpha,N} \leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta| \|f\|_{\gamma,N}$$

Proof. Let $\alpha \in \mathbb{N}_0^n$, $N \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
(1+|x|)^N|\partial^\alpha(\rho_\xi f)(x)| &= (1+|x|)^N\bigg|\rho_\xi\bigg(\sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma f\bigg)(x)\bigg| \\
&= (1+|x|)^N\bigg|e^{-i\langle\xi,x\rangle} \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma f(x)\bigg| \\
&\leq (1+|x|)^N \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta||\partial^\gamma f(x)| \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta||(1+|x|)^N \partial^\gamma f(x)| \\
&\leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta| \|f\|_{\gamma,N}
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that

$$\begin{aligned}
\|\rho_\xi f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\Big[(1+|x|)^N|\partial^\alpha(\rho_\xi f)(x)|\Big] \\
&\leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta| \|f\|_{\gamma,N} \\
&< \infty
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\rho_\xi f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 1.3.0.13. Let $\xi \in \mathbb{R}^n$. Then $\rho_\xi : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathcal{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. Let $\alpha, N \in \mathcal{N}_0$. Then

$$\begin{aligned}\|\rho_\xi f_k\|_{\alpha,N} &\leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta|\|f_k\|_{\gamma,N} \\ &\to 0\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\rho_\xi f_k \to 0$. So ρ_ξ is continuous at 0. Since ρ_ξ is linear, ρ_ξ is continuous. $\square$

1.4 Dilation and Concentration Operators

Definition 1.4.0.1. Let $\xi \in \mathbb{R}^n$. We define the **dilation by t operator**, denoted $\gamma_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, by $\gamma_t f(x) = f(tx)$.

Exercise 1.4.0.2. Let $t \neq 0$. Then $\gamma_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Then for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned}
\gamma_t(f + \lambda g)(x) &= (f + \lambda g)(tx) \\
&= f(tx) + \lambda g(tx) \\
&= \gamma_t f(x) + \lambda \gamma_t g(x)
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that $\gamma_t(f + \lambda g) = \gamma_t f + \lambda \gamma_t g$. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ are arbitary, γ_t is linear. $\square$

Exercise 1.4.0.3. Let $t \neq 0$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha \gamma_t = t^{|\alpha|} \gamma_t \partial^\alpha$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. The chain rule implies that the claim is true if $|\alpha| = 0$ or $|\alpha| = 1$. Let $k > 1$. Suppose the claim is true for $|\alpha| = k - 1$ so that for each $\beta \in \mathbb{N}_0$, $|\beta| = k - 1$ implies that $\partial^\beta(\gamma_t f) = t^{|\beta|}\gamma_t(\partial^\beta f)$. Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \ldots, n\}$ such that $\alpha_j > 0$. The chain rule implies that

$$\begin{aligned}
(\partial^\alpha \gamma_t) f &= \partial^\alpha(\gamma_t f) \\
&= \partial_j(\partial^{\alpha - e_j}[\gamma_t f]) \\
&= \partial_j(t^{|\alpha - e_j|}\gamma_t[\partial^{\alpha - e_j} f]) \\
&= t^{|\alpha - e_j|}\partial_j(\gamma_t[\partial^{\alpha - e_j} f]) \\
&= t^{|\alpha - e_j|} t \gamma_t(\partial_j[\partial^{\alpha - e_j} f]) \\
&= t^{|\alpha|}\gamma_t(\partial^\alpha f) \\
&= (t^{|\alpha|}\gamma_t \partial^\alpha) f
\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ is arbitrary, $\partial^\alpha \gamma_t = t^{|\alpha|}\gamma_t \partial^\alpha$. So the claim is true for $|\alpha| = k$. By induction the claim is true for each $\alpha \in \mathbb{N}_0^n$. $\square$

Exercise 1.4.0.4. Let $y \in \mathbb{R}$ and $t \neq 0$. Then there exists $C > 0$ such that for each $x \in \mathbb{R}^n$, $1 + |x| \leq C(1 + |tx|)^2$.

Proof. Choose $C = \max(1/(2|t|), 1)$. Let $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
C(1 + |tx|)^2 - (1 + |x|) &= C + 2C|tx| + C|tx|^2 - 1 - |x| \\
&= C + (2C|t| - 1)|x| + C|tx|^2 - 1 \\
&= (C - 1) + (2C|t| - 1)|x| + C|tx|^2 \\
&\geq 0
\end{aligned}$$

So $1 + |x| \leq C(1 + |tx|)^2$. $\square$

Exercise 1.4.0.5. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$. Then $\gamma_t f \in \mathcal{S}(\mathbb{R}^n)$ and there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\gamma_t f\|_{\alpha,N} \leq |t|^{|\alpha|} C^N \|f\|_{\alpha,2N}$$

Proof. The previous exercise implies that there exists $C > 0$ such that for each $x \in \mathbb{R}^n$, $1+|x| \leq C(1+|tx|)^2$. Let $\alpha \in \mathbb{N}_0^n$, $N \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$. Then

$$\begin{aligned}(1+|x|)^N|\partial^\alpha(\gamma_t f)(x)| &= (1+|x|)^N|t^{|\alpha|}(\gamma_t\partial^\alpha f)(x)| \\ &\leq C(1+|tx|)^{2N}|t|^{|\alpha|}|(\gamma_t\partial^\alpha f)(x)| \\ &= C(1+|tx|)^{2N}|t|^{|\alpha|}|\partial^\alpha f(tx)| \\ &\leq C|t|^{|\alpha|}\|f\|_{\alpha,2N}\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that

$$\begin{aligned}\|\gamma_t f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\bigg[(1+|x|)^N|\partial^\alpha(\gamma_t f)(x)|\bigg] \\ &\leq Ct^{|\alpha|}\|f\|_{\alpha,2N} \\ &< \infty\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\gamma_t f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 1.4.0.6. Let $t \neq 0$. Then $\gamma_t : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathcal{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. The previous exercise implies that there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\gamma_t f\|_{\alpha,N} \leq |t|^{|\alpha|}C^N\|f\|_{\alpha,2N}$$

Let $\alpha, N \in \mathcal{N}_0$. Then

$$\begin{aligned}\|\gamma_t f_k\|_{\alpha,N} &\leq C|t|^{|\alpha|}\|f\|_{\alpha,2N} \\ &\to 0\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\gamma_t f_k \to 0$. So γ_t is continuous at 0. Since γ_t is linear, ρ_ξ is continuous. □

Definition 1.4.0.7. Let $\xi \in \mathbb{R}^n$. We define the **concentration by** t **operator**, denoted $\kappa_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, by $\kappa_t f(x) = t^{-1}\gamma_{t^{-1}}f$.

Exercise 1.4.0.8. Let $t \neq 0$. Then $\kappa_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Clear since $\gamma_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear. □

Exercise 1.4.0.9. Let $t \neq 0$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha\kappa_t = t^{-|\alpha|}\kappa_t\partial^\alpha$$

Proof. Let $\alpha \in \mathbb{N}_0^n$. Then

$$\begin{aligned}\partial^\alpha\kappa_t &= \partial^\alpha t^{-1}\gamma_{t^{-1}} \\ &= t^{-1}\partial^\alpha\gamma_{t^{-1}} \\ &= t^{-1}(t^{-1})^{|\alpha|}\gamma_{t^{-1}}\partial^\alpha \\ &= t^{-|\alpha|}\kappa_t\partial^\alpha\end{aligned}$$

□

Exercise 1.4.0.10. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$. Then $\kappa_t f \in \mathcal{S}(\mathbb{R}^n)$ and there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\kappa_t f\|_{\alpha,N} \leq |t|^{-(|\alpha|+1)}C^N\|f\|_{\alpha,2N}$$

Proof. A previous exercise implies that there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\gamma_t f\|_{\alpha,N} \leq |t|^{|\alpha|} C^N \|f\|_{\alpha,2N}$$

Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|\kappa_t f\|_{\alpha,N} &= \|t^{-1}\gamma_{t^{-1}} f\|_{\alpha,N} \\
&= |t^{-1}| \|\gamma_{t^{-1}} f\|_{\alpha,N} \\
&\leq |t^{-1}| |t^{-1}|^{|\alpha|} C^N \|f\|_{\alpha,2N} \\
&= |t|^{-(|\alpha|+1)} C^N \|f\|_{\alpha,2N} \\
&< \infty
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\kappa_t f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 1.4.0.11. Let $t \neq 0$. Then $\kappa_t : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Since $\gamma_{t^{-1}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, $\kappa_t = t^{-1}\gamma_{t^{-1}}$ is continuous. □

Exercise 1.4.0.12. Let $t \neq 0$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}} \kappa_t f \, dm = \int_{\mathbb{R}} f \, dm$$

Proof. We have that

$$\begin{aligned}
\int_{\mathbb{R}} \kappa_t f \, dm &= \int_{\mathbb{R}} t^{-1}\gamma_{t^{-1}} f \, dm \\
&= \int_{\mathbb{R}} t^{-1} f(t^{-1}y) \, dm(y) \\
&= \int_{\mathbb{R}} f(z) \, dm(z)
\end{aligned}$$

□

1.5 The Convolution on $\mathcal{S}(\mathbb{R}^n)$

Definition 1.5.0.1. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. We define the **convolution of f and g**, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$ by

$$f * g(x) = \int_{\mathbb{R}^n} \tau_y f(x) g(y) \, dm(y)$$

Exercise 1.5.0.2. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $f * g \in C^\infty(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g$$

Hint: exchange integration and differentiation

Proof. Let $\alpha \in \mathbb{N}_0^n$. We proceed by induction on $|\alpha|$.

- Suppose that $|\alpha| = 0$. Then $\alpha = 0$. Define $h_0 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = \tau_y f(x) g(y)$. We observe that for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned} |h(x,y)| &= |\tau_y f(x)||g(y)| \\ &\leq \|\tau_y f\|_{0,0} |g(y)| \\ &\leq \|f\|_{0,0} |g(y)| \end{aligned}$$

  Since $\|f\|_{0,0}|g| \in L^1(\mathbb{R}^n)$ and for each $y \in \mathbb{R}^n$, $h(x, y) \to h(x_0, y)$ as $x \to x_0$, we have that

$$\begin{aligned} f * g &= \int_{\mathbb{R}^n} \tau_y f(\cdot) g(y) \, dm(y) \\ &= \int_{\mathbb{R}^n} h(\cdot, y) \, dm(y) \end{aligned}$$

  is continuous. Therefore, $f * g \in C(\mathbb{R}^n)$ and $\partial^\alpha (f * g) = (\partial^\alpha f) * g$.

- Let $k > 0$. Suppose that for each $\beta \in \mathbb{N}_0^n$, $|\beta| = k - 1$ implies that $f * g \in C^{|\beta|}(\mathbb{R}^n)$ and

$$\partial^\beta (f * g) = (\partial^\beta f) * g$$

  Suppose that $|\alpha| = k$. Then there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Define $h \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = \tau_y [\partial_x^{\alpha - e_j} f](x) g(y)$. By hypothesis,

$$\begin{aligned} [\partial^{\alpha - e_j} (f * g)](x) &= [(\partial^{\alpha - e_j} f) * g](x) \\ &= \int_{\mathbb{R}^n} \tau_y [\partial_x^{\alpha - e_j} f](x) g(y) \, dm(y) \\ &= \int_{\mathbb{R}^n} h(x, y) \, dm(y) \end{aligned}$$

  We observe that for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \partial_x^{e_j} h(x,y) &= \partial_x^{e_j} [\tau_y (\partial_x^{\alpha - e_j} f)](x) g(y) \\ &= \partial_x^\alpha [\tau_y f](x) g(y) \end{aligned}$$

  which implies that

$$\begin{aligned} |\partial_x^{e_j} h(x,y)| &= |\partial_x^\alpha [\tau_y f](x) g(y)| \\ &\leq \|\tau_y f\|_{\alpha,0} |g(y)| \\ &\leq \|f\|_{\alpha,0} |g(y)| \end{aligned}$$

Since $g \in L^1(\mathbb{R}^n)$, $\partial^{e_j}[\partial^{\alpha - e_j}(f * g)]$ exists and we may exchange the order of integration and differentiation to obtain that

$$\begin{aligned}
[\partial_x^\alpha (f * g)](x) &= \partial_x^{e_j}[\partial_x^{\alpha - e_j}(f * g)](x) \\
&= \partial_x^{e_j} \int_{\mathbb{R}^n} h(x, y)\, dm(y) \\
&= \int_{\mathbb{R}^n} \partial_x^{e_j} h(x, y)\, dm(y) \\
&= \int_{\mathbb{R}^n} \partial_x^{e_j}[\tau_y(\partial_x^{\alpha - e_j} f)](x) g(y)\, dm(y) \\
&= \int_{\mathbb{R}^n} \tau_y[\partial_x^\alpha f](x) g(y)\, dm(y) \\
&= [(\partial_x^\alpha f) * g](x)
\end{aligned}$$

So $f * g \in C^{|\alpha|}(\mathbb{R}^n)$ and $\partial^\alpha (f * g) = (\partial^\alpha f) * g$.

- By induction, for each $\alpha \in \mathbb{N}_0$, $f * g \in C^{|\alpha|}(\mathbb{R}^n)$ and $\partial^\alpha (f * g) = (\partial^\alpha f) * g$.

Since for each $\alpha \in \mathbb{N}_0^n$, $f * g \in C^{|\alpha|}(\mathbb{R}^n)$, we have that $f * g \in C^\infty(\mathbb{R}^n)$. □

Exercise 1.5.0.3. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $f * g \in \mathcal{S}(\mathbb{R}^n)$ and there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|f * g\|_{\alpha, N} \leq C \|f\|_{\alpha, N} \|g\|_{0, N+2}$$

Proof. Set

$$C = \int_{\mathbb{R}} \frac{1}{(1 + |y|)^2}\, dm(y)$$

Let $\alpha \in \mathbb{N}_0^n$, $N \in \mathbb{N}_0$ and $x \in \mathbb{R}$. Then

$$\begin{aligned}
(1 + |x|)^N |\partial^\alpha (f * g)(x)| &= (1 + |x|)^N |(\partial^\alpha f) * g(x)| \\
&= (1 + |x|)^N \left| \int_{\mathbb{R}} \tau_y[\partial_x^\alpha f](x) g(y)\, dm(y) \right| \\
&= \left| \int_{\mathbb{R}} (1 + |x|)^N \partial_x^\alpha [\tau_y f](x) g(y)\, dm(y) \right| \\
&\leq \int_{\mathbb{R}} (1 + |x|)^N |\partial_x^\alpha [\tau_y f](x)| |g(y)|\, dm(y) \\
&\leq \int_{\mathbb{R}} \|\tau_y f\|_{\alpha, N} |g(y)|\, dm(y) \\
&\leq \int_{\mathbb{R}} (1 + |y|)^N \|f\|_{\alpha, N} |g(y)|\, dm(y) \\
&= \|f\|_{\alpha, N} \int_{\mathbb{R}} (1 + |y|)^{N+2} |g(y)| (1 + |y|)^{-2}\, dm(y) \\
&\leq \|f\|_{\alpha, N} \int_{\mathbb{R}} \|g\|_{0, N+2} (1 + |y|)^{-2}\, dm(y) \\
&= \|f\|_{\alpha, N} \|g\|_{0, N+2} \int_{\mathbb{R}} (1 + |y|)^{-2}\, dm(y) \\
&= C \|f\|_{\alpha, N} \|g\|_{0, N+2}
\end{aligned}$$

Since $x \in \mathbb{R}$ is arbitrary, we have that

$$\begin{aligned}
\|f * g\|_{\alpha, N} &= \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha (f * g)(x)| \right] \\
&\leq C \|f\|_{\alpha, N} \|g\|_{0, N+2} \\
&< \infty
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, we have that $f * g \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Exercise 1.5.0.4. The convolution $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is bilinear.

Proof. Let $f, g, h \in \mathcal{S}(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}^n$. Since $\tau_y : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear, we have that

$$\begin{aligned}
[(f + \lambda g) * h](x) &= \int_{\mathbb{R}^n} \tau_y[f + \lambda g](x) h(y)\, dm(y) \\
&= \int_{\mathbb{R}^n} \Big(\tau_y[f](x) + \lambda \tau_y[g](x)\Big) h(y)\, dm(y) \\
&= \int_{\mathbb{R}^n} \tau_y[f](x) h(y)\, dm(y) + \lambda \int_{\mathbb{R}^n} \tau_y[g](x) h(y)\, dm(y) \\
&= [f * h](x) + [\lambda g * h](x)
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, $(f + \lambda g) * h = f * h + \lambda g * h$. Similarly, $f * (g + \lambda h) = f * g + \lambda f * h$. $\square$

Exercise 1.5.0.5. The convolution $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is commutative.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
f * g(x) &= \int_{\mathbb{R}} f(x - y) g(y)\, dm(y) \\
&= \int_{\mathbb{R}} f(z) g(x - z)\, dm(z) \\
&= \int_{\mathbb{R}} g(x - z) f(z)\, dm(z) \\
&= g * f(x)
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, $f * g = g * f$. $\square$

Exercise 1.5.0.6. The convolution $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_n, g_n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ and $(f, g) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$. Suppose that $(f_n, g_n) \to (f, g)$. Then $f_n \to f$ and $g_n \to g$. Hence for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, $\|f_n - f\|_{\alpha,N} \to 0$ and $\|g_n - g\|_{\alpha,N} \to 0$. In particular

$$\begin{aligned}
\Big| \|g_n\|_{0,N+2} - \|g\|_{0,N+2} \Big| &\leq \|g_n - g\|_{0,N+2} \\
&\to 0
\end{aligned}$$

So that $(\|g_n\|_{0,N+2})_{n \in \mathbb{N}}$ is bounded. Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Define $C > 0$ as in the previous exercise. Then

$$\begin{aligned}
\|f_n * g_n - f * g\|_{\alpha,N} &= \|f_n * g_n - f * g_n + f * g_n - f * g\|_{\alpha,N} \\
&\leq \|(f_n - f) * g_n\|_{\alpha,N} + \|f_*(g_n - g)\|_{\alpha,N} \\
&\leq C\|f_n - f\|_{\alpha,N} \|g_n\|_{0,N+2} + C\|f\|_{\alpha,N} \|g_n - g\|_{0,N+2} \\
&\to 0
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $f_n * g_n \to f * g$. Thus $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous. $\square$

Exercise 1.5.0.7. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. Tonelli's theorem implies that

$$\begin{aligned}
\|f * g\|_1 &= \int_{\mathbb{R}} |f * g(x)| \, dm(x) \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)g(y) \, dm(y) \right| dm(x) \\
&\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)g(y)| \, dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)g(y)| \, dm(x) \right] dm(y) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)| \, dm(x) \right] |g(y)| \, dm(y) \\
&= \|f\|_1 \int_{\mathbb{R}} |g(y)| \, dm(y) \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

□

Definition 1.5.0.8. We define the **bump functions** on $\mathbb{R}$, denoted $C_c^\infty(\mathbb{R})$, by

$$C_c^\infty(\mathbb{R}) = C_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$$

Exercise 1.5.0.9. Let $f \in C_c^\infty(\mathbb{R})$. Then $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $\alpha, N \in \mathbb{N}^0$. Define $g : \mathbb{R}^n \to \mathbb{C}$ by

$$g(x) = (1 + |x|)^N |\partial^\alpha f(x)|$$

Then g is continuous. Since $\operatorname{supp}(\partial^\alpha f) \subset \operatorname{supp}(f)$, we have that $g \in C_c(\mathbb{R})$ and

$$\begin{aligned}
\sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha f| \right] &= \sup_{x \in \mathbb{R}} g(x) \\
&= \|g\| \\
&< \infty
\end{aligned}$$

□

Exercise 1.5.0.10. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = e^{-x^2}$. Then $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. meh... □

Exercise 1.5.0.11. Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & x \in (-1, 1) \\ 0 & x \notin (-1, 1) \end{cases}$$

Then $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. meh... □

Exercise 1.5.0.12. Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Then for each $\epsilon > 0$, there exists $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\chi_{[a,b]} \leq f \leq \chi_{[a-\epsilon, b+\epsilon]}$.

Proof. Set $f(x) =$ □

Exercise 1.5.0.13. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define

1.6 The Fourier Transform on $\mathcal{S}(\mathbb{R}^n)$

Exercise 1.6.0.1. Let $\phi : \mathbb{R} \to S^1$ be a measurable homomorphism.

1. Then $\phi \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
2. Define
$$c = \left[\int_{(0,a]} \phi dm\right]^{-1}$$
Then For each $x \in \mathbb{R}$,
$$\phi(x) = c\int_{(x,x+a]} \phi dm$$
3. $\phi \in C^\infty(\mathbb{R})$ and $\phi' = c(\phi(a) - 1)\phi$
4. Define $b = c(\phi(a) - 1)$ and $g \in C^\infty(\mathbb{R})$ by $g(x) = e^{-bx}\phi(x)$. Then g is constant and there exists $\xi \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

Proof.

1. Let $K \subset \mathbb{R}$ be compact. Then
$$\int_K |\phi| dm = m(K) < \infty$$
So $\phi \in L^1_{\text{loc}}(\mathbb{R})$. For the sake of contradiction, suppose that for each $a > 0$,
$$\int_{(0,a]} \phi dm = 0$$
Then the FTC implies that $\phi = 0$ a.e. on $[0, \infty)$, which is a contradiction. So there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
2. For $x \in \mathbb{R}$,
$$\begin{aligned}\phi(x) &= c\int_{(0,a]} \phi(x)\phi(t)dm(t)\\ &= c\int_{(0,a]} \phi(x + t)dm(t)\\ &= c\int_{(x,x+a]} \phi dm\end{aligned}$$
3. Part (2) and the FTC imply that ϕ is continuous. Let $d \in \mathbb{R}$. Define $f_d \in C((d, \infty))$ by
$$f_d(x) = \int_{(d,x]} \phi dm$$
Since ϕ is continuous, the FTC implies that f_d is differentiable and for each $x > d$ $f'_d(x) = \phi(x)$. Part (2) implies that for each $x > d$,
$$\begin{aligned}\phi(x) &= c\int_{(x,x+a]} \phi dm\\ &= c(f_d(x + a) - f_d(x))\end{aligned}$$

So for each $x > d$, ϕ is differentiable at x and

$$\begin{aligned}\phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x)\end{aligned}$$

Since $d \in \mathbb{R}$ is arbitrary, ϕ is differentiable and $\phi' = c(\phi(a) - 1)\phi$. This implies that $\phi \in C^\infty(\mathbb{R})$.

4. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0\end{aligned}$$

So $g' = 0$ and g is constant. Hence there exists $k \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = ke^{bx}$. Since $\phi(0) = 1$, $k = 1$. Since $|\phi| = 1$, there exists $\xi \in \mathbb{R}$ such that $b = 2\pi i \xi$.

□

Note 1.6.0.2. To summarize, for each measurable homomorphism $\phi : \mathbb{R} \to S^1$, there exists $\xi \in \mathbb{R}$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i \xi x}$.

Exercise 1.6.0.3. Let $\phi : \mathbb{R}^n \to S^1$ be a measurable homomorphism. Then there exists $\xi \in \mathbb{R}^n$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i \langle \xi, x\rangle}$.

Definition 1.6.0.4. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$, by

$$\hat{f}(\xi) = \int_{\mathbb{R}} \rho_\xi f \, dm$$

Exercise 1.6.0.5. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\hat{f} \in C_b(\mathbb{R}^n)$.

Proof. Since $f \in \mathcal{S}(\mathbb{R}^n)$, $f \in L^1(\mathbb{R}^n)$. Then for each $\xi \in \mathbb{R}$,

$$\begin{aligned}|\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} \rho_\xi f \, dm \right| \\ &\leq \int_{\mathbb{R}} |\rho_\xi f| \, dm \\ &= \int_{\mathbb{R}} |e^{-i\langle \xi, x\rangle} f(x)| \, dm(x) \\ &= \int_{\mathbb{R}} |f(x)| \, dm(x) \\ &= \|f\|_1\end{aligned}$$

So f is bounded. Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. Suppose that $\xi_n \to \xi$. Define $(\phi_n)_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^n)$ and $\phi \in L^1(\mathbb{R}^n)$ by $\phi_n(x) = \rho_{\xi_n} f(x)$ and $\phi(x) = \rho_\xi f(x)$. Then $\phi_n \xrightarrow{\text{p.w.}} \phi$ and for each $n \in \mathbb{N}$,

$$\begin{aligned}|\phi_n| &= |f| \\ &\in L^1(\mathbb{R}^n)\end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}\hat{f}(\xi_n) &= \int_{\mathbb{R}} \phi_n \, dm \\ &\to \int_{\mathbb{R}} \phi \, dm \\ &= \hat{f}(\xi)\end{aligned}$$

So $\hat{f}$ is continuous. Hence $\hat{f} \in C_b(\mathbb{R})$. □

Definition 1.6.0.6. We define the **Fourier transform on** $\mathcal{S}(\mathbb{R}^n)$, denoted $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$, by

$$\mathcal{F}(f) = \hat{f}$$

Exercise 1.6.0.7. We have that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{R}^n$. Since $\rho_\xi : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear, we have that

$$\begin{aligned}
\mathcal{F}(f + \lambda g)(\xi) &= \int_{\mathbb{R}} \rho_\xi (f + \lambda g)\, dm \\
&= \int_{\mathbb{R}} \rho_\xi f + \lambda \rho_\xi g\, dm \\
&= \int_{\mathbb{R}} \rho_\xi f\, dm + \lambda \int_{\mathbb{R}} \rho_\xi g\, dm \\
&= \mathcal{F}(f)(\xi) + \lambda \mathcal{F}(g)(\xi)
\end{aligned}$$

$\square$

Exercise 1.6.0.8. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Then

1. $\mathcal{F}(X^\alpha f) = (-1)^{|\alpha|} P^\alpha \mathcal{F}(f)$
2. $\mathcal{F}(P^\alpha f) = X^\alpha \mathcal{F}(f)$

Proof.

1. Let $\alpha \in \mathbb{N}_0^n$. The claim is true if $|\alpha| = 0$. Let $k > 0$. Suppose that the claim is true for $|\alpha| = k - 1$ so that for each $\beta \in \mathbb{N}_0^n$, $|\beta| = k - 1$ implies that $\mathcal{F}(X^\beta f) = (-1)^{|\beta|} P^\beta \mathcal{F}(f)$. Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \ldots, n\}$ such that $\alpha_j > 0$. Define $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $\phi(\xi, x) = \rho_\xi X^{\alpha - e_j} f(x)$. Then for each $\xi, x \in \mathbb{R}$,

$$\begin{aligned}
\partial_\xi^{e_j} \phi(\xi, x) &= -i x^{e_j} \phi(x) \\
&= -i \rho_\xi X^\alpha f(x)
\end{aligned}$$

Hence for each $x, \xi \in \mathbb{R}^n$,

$$\begin{aligned}
|\partial_\xi^{e_j} \phi(\xi, x)| &= |-i \rho_\xi X^\alpha f(x)| \\
&= |X^\alpha f(x)|
\end{aligned}$$

Since $X^\alpha f \in \mathcal{S}(\mathbb{R}^n) \subset L^1$, we may exchange the order of integration and differentiation to obtain that

$$\begin{aligned}
\mathcal{F}(X^\alpha f)(\xi) &= \int_{\mathbb{R}} \rho_\xi X^\alpha f(x) dm(x) \\
&= \int_{\mathbb{R}^n} i \partial_\xi^{e_j} \phi(\xi, x)\, dm(x) \\
&= i \partial^{e_j} \int_{\mathbb{R}} e^{-i\xi x} x^{\alpha - e_j} f(x) dm(x) \\
&= -P^{e_j} \mathcal{F}(X^{\alpha - e_j} f)(\xi) \\
&= -P^{e_j} \left[(-1)^{|\alpha| - 1} P^{\alpha - e_j} \mathcal{F}(f) \right](\xi) \\
&= (-1)^{|\alpha|} P^\alpha \mathcal{F}(f)(\xi)
\end{aligned}$$

So the claim is true for α. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$.

2. Let $\alpha \in \mathbb{N}_0^n$. The claim is true if $|\alpha| = 0$. Let $k > 0$. Suppose that the claim is true for $|\alpha| = k - 1$ so that for each $\beta \in \mathbb{N}_0^n$, $|\beta| = k - 1$ implies that $\mathcal{F}(P^\beta f) = X^\beta \mathcal{F}(f)$. Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Then integration by parts yields

$$\begin{aligned}
\mathcal{F}(P^\alpha f)(\xi) &= \int_{\mathbb{R}} e^{-i\langle \xi, x\rangle} [-i\partial_x P^{\alpha - e_j} f(x)]\, dm(x) \\
&= -\int_{\mathbb{R}} -i\xi^{e_j} e^{-i\langle \xi, x\rangle} [-iP^{\alpha - e_j} f(x)]\, dm(x) \\
&= \xi^{e_j} \int_{\mathbb{R}} e^{-i\langle \xi, x\rangle} P^{\alpha - e_j} f(x)\, dm(x) \\
&= X^{e_j} \mathcal{F}(P^{\alpha - e_j} f)(\xi) \\
&= X^{e_j} \Big[X^{\alpha - e_j} \mathcal{F}(f) \Big](\xi) \\
&= X^\alpha \mathcal{F}(f)(\xi)
\end{aligned}$$

So the claim is true for α. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$.

□

Exercise 1.6.0.9. There exists $C > 0$ such that for each $f \in \mathcal{S}(\mathbb{R}^n)$, $\|\hat{f}\|_{0,0} \le C\|f\|_{0,2}$.
Hint: Set

$$C = \int_{\mathbb{R}} \frac{1}{(1 + |x|)^2}\, dm(x)$$

Proof. Set

$$C = \int_{\mathbb{R}} \frac{1}{(1 + |x|)^2}\, dm(x)$$

Let $f \in \mathcal{S}(\mathbb{R}^n)$. Let $\xi \in \mathbb{R}$. Then

$$\begin{aligned}
|\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} \rho_\xi f(x)\, dm(x) \right| \\
&\le \int_{\mathbb{R}} |f(x)|\, dm(x) \\
&= \int_{\mathbb{R}} \frac{(1 + |x|)^2 |f(x)|}{(1 + |x|)^2}\, dm(x) \\
&\le \|f\|_{0,2} \int_{\mathbb{R}} \frac{1}{(1 + |x|)^2}\, dm(x) \\
&= C\|f\|_{0,2}
\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\|\hat{f}\|_{0,0} \le C\|f\|_{0,2}$. □

Exercise 1.6.0.10. Let $a, b \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Then $(a + b)^N \le 2^{N-1}(a^N + b^N)$.
Hint: Jensen's inequality

Proof. Jensen's inequality implies that

$$\begin{aligned}
2^{-N}(a + b)^N &= \left(\frac{a}{2} + \frac{b}{2} \right)^N \\
&\le \left(\frac{a^N}{2} + \frac{b^N}{2} \right) \\
&= 2^{-1}(a^N + b^N)
\end{aligned}$$

So $(a + b)^N \le 2^{N-1}(a^N + b^N)$. □

Exercise 1.6.0.11. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^n)$ and there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\mathcal{F}(f)\|_{\alpha,N} \leq C2^{N-1}\|X^\alpha f\|_{0,2} + C2^{N-1}\|P^N X^\alpha f\|_{0,2}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Then the previous exercise implies that for each $\xi \in \mathbb{R}$,

$$\begin{aligned}
\xi^N \partial^\alpha \mathcal{F}(f)(\xi) &= (-i)^N X^N P^\alpha \mathcal{F}(f)(\xi) \\
&= i^N X^N \mathcal{F}(X^\alpha f)(\xi) \\
&= i^N \mathcal{F}(P^N X^\alpha f)(\xi)
\end{aligned}$$

Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

as in the previous exercise. Since $\mathcal{F}(X^\alpha f)$, $\mathcal{F}(P^N X^\alpha f) \in C_b(\mathbb{R})$, we have that

$$\begin{aligned}
\|\mathcal{F}(f)\|_{\alpha,N} &= \sup_{\xi\in\mathbb{R}} \bigg[(1+|\xi|)^N |\partial^\alpha \mathcal{F}(f)(\xi)|\bigg] \\
&\leq \sup_{\xi\in\mathbb{R}} \bigg[2^{N-1}(1+|\xi|^N)|\partial^\alpha \mathcal{F}(f)(\xi)|\bigg] \\
&= \sup_{\xi\in\mathbb{R}} \bigg[|2^{N-1}\partial^\alpha \mathcal{F}(f)(\xi)| + |2^{N-1}\xi^N \partial^\alpha \mathcal{F}(f)(\xi)|\bigg] \\
&= \sup_{\xi\in\mathbb{R}} \bigg[|\mathcal{F}(2^{N-1}X^\alpha f)(\xi)| + |\mathcal{F}(2^{N-1}P^N X^\alpha f)(\xi)|\bigg] \\
&\leq \|\mathcal{F}(2^{N-1}X^\alpha f)\|_{0,0} + \|\mathcal{F}(2^{N-1}P^N X^\alpha f)\|_{0,0} \\
&\leq C2^{N-1}\|X^\alpha f\|_{0,2} + C2^{N-1}\|P^N X^\alpha f\|_{0,2} \\
&< \infty
\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 1.6.0.12. We have that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_n \to 0$. Since $X, P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ are continuous, $X^\alpha f_n \to 0$ and $P^N X^\alpha f_n \to 0$. Therefore, $\|X^\alpha f_n\|_{0,2} \to 0$ and $\|P^N X^\alpha f_n\|_{0,2} \to 0$. The previous exercise implies there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\begin{aligned}
\|\mathcal{F}(f_n)\|_{\alpha,N} &\leq C2^{N-1}\|X^\alpha f_n\|_{0,2} + C2^{N-1}\|P^N X^\alpha f_n\|_{0,2} \\
&\to 0
\end{aligned}$$

Hence $\mathcal{F}(f_n) \to 0$ and $\mathcal{F}$ is continuous at 0. Since $\mathcal{F}$ is linear, $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous. □

Exercise 1.6.0.13. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

1. for each $y \in \mathbb{R}$, $\mathcal{F}(\tau_y f) = \rho_y \mathcal{F}(f)$
2. for each $\eta \in \mathbb{R}$, $\mathcal{F}(\rho_\eta f) = \tau_{-\eta}\mathcal{F}(f)$
3. $\mathcal{F}(\gamma_t f) = \kappa_t \mathcal{F}(f)$

Proof.

1. Let $y, \xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(\tau_y f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} f(x-y)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i\xi(z+y)} f(z)\, dm(z) \\
&= e^{-i\xi y} \int_{\mathbb{R}} e^{-i\xi z} f(z)\, dm(z) \\
&= e^{-i\xi y} \mathcal{F}(f)(\xi) \\
&= \rho_y \mathcal{F}(f)(\xi)
\end{aligned}$$

2. Let $\eta, \xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(\rho_\eta f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} e^{-i\eta x} f(x)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i(\xi+\eta)x} f(x)\, dm(x) \\
&= \mathcal{F}(f)(\xi+\eta) \\
&= \tau_{-\eta} \mathcal{F}(f)(\xi)
\end{aligned}$$

3. Let $\xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(\gamma_t f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} f(tx)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i\xi t^{-1} z} f(z) t^{-1}\, dm(z) \\
&= t^{-1} \mathcal{F}(f)(t^{-1}\xi) \\
&= t^{-1} \gamma_{t^{-1}} \mathcal{F}(f)(\xi)
\end{aligned}$$

$\square$

Exercise 1.6.0.14. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.

Proof. Let $\xi \in \mathbb{R}$. Tonelli's theorem implies that

$$\begin{aligned}
\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |e^{-i\xi x} f(x-y) g(y)|\, dm(y) \right] dm(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y) g(y)|\, dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y) g(y)|\, dm(x) \right] dm(y) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)|\, dm(x) \right] |g(y)|\, dm(y) \\
&= \|f\|_1 \int_{\mathbb{R}} |g(y)|\, dm(y) \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

So we may apply Fubini's theorem and change the order of integration to obtain that

$$\begin{aligned}
\mathcal{F}(f*g)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x}(f*g)(x)\,dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y)g(y)\,dm(y)\right] dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y)g(y)\,dm(x)\right] dm(y) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y)\,dm(x)\right] g(y)\,dm(y) \\
&= \int_{\mathbb{R}} [\mathcal{F}(\tau_y f)(\xi)] g(y)\,dm(y) \\
&= \int_{\mathbb{R}} [e^{-i\xi y}\mathcal{F}(f)(\xi)] g(y)\,dm(y) \\
&= \mathcal{F}(f)(\xi)\int_{\mathbb{R}} e^{-i\xi y} g(y)\,dm(y) \\
&= \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)
\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$ □

Exercise 1.6.0.15. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}} \hat{f} g\,dm = \int_{\mathbb{R}} f\hat{g}\,dm$$

Proof. Tonelli's theorem implies that

$$\begin{aligned}
\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |e^{-i\xi x} f(x)g(\xi)|\,dm(x)\right] dm(\xi) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x)|\,dm(x)\right] |g(\xi)|\,dm(\xi) \\
&= \|f\|_1 \int_{\mathbb{R}} |g(\xi)|\,dm(\xi) \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

So we may apply Fubini's theorem and switch the order of integration to obtain that

$$\begin{aligned}
\int_{\mathbb{R}} \hat{f} g\,dm &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x)\,dm(x)\right] g(\xi)\,dm(\xi) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x)g(\xi)\,dm(x)\right] dm(\xi) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x)g(\xi)\,dm(\xi)\right] dm(x) \\
&= \int_{\mathbb{R}} f(x)\left[\int_{\mathbb{R}} e^{-i\xi x} g(\xi)\,dm(\xi)\right] dm(x) \\
&= \int_{\mathbb{R}} f(x)\hat{g}(x)\,dm(x) \\
&= \int_{\mathbb{R}} f\hat{g}\,dm
\end{aligned}$$

□

Exercise 1.6.0.16. Define $f \in \mathcal{S}(\mathbb{R}^n)$ by $f(x) = e^{-x^2/2}$. Then $\mathcal{F}(f) = \sqrt{2\pi} f$.

Proof. Note that for each $\xi \in \mathbb{R}$,

$$\begin{aligned}\mathcal{F}(Df)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} i x e^{-x^2/2}\, dm(x) \\ &= -\int_{\mathbb{R}} \partial_\xi \left[e^{-i\xi x} e^{-x^2/2}\right] dm(x) \\ &= -\partial_\xi \mathcal{F}(f)(\xi)\end{aligned}$$

A previous exercise implies that $\mathcal{F}(Df) = X\mathcal{F}(f)$. So for each $\xi \in \mathbb{R}$, $\partial_\xi \hat{f}(\xi) = -\xi \hat{f}(\xi)$. Define $g \in \mathbb{C}^\infty(\mathbb{R})$ by $g(\xi) = e^{\xi^2/2}$. Then

$$\begin{aligned}\partial_\xi(\hat{f}g) &= (\partial_\xi \hat{f})g + \hat{f}(\partial_\xi g) \\ &= 0\end{aligned}$$

So there exists $C \in \mathbb{R}$ such that $\hat{f}g = C$. Hence for each $\xi \in \mathbb{R}$,

$$\begin{aligned}\hat{f}(\xi) &= Ce^{-\xi^2/2} \\ &= Cf(\xi)\end{aligned}$$

Therefore,

$$\begin{aligned}C &= Cf(0) \\ &= \hat{f}(0) \\ &= \int_{\mathbb{R}} e^{-x^2/2}\, dm(x) \\ &= \sqrt{2\pi}\end{aligned}$$

So $\hat{f} = \sqrt{2\pi} f$. □

Exercise 1.6.0.17. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define $g : \mathbb{R}^n \to L^1$ by $g(x) = \tau_x f$. Then g is continuous.
Hint: approximate by functions in $C_c(\mathbb{R})$.

Proof. Suppose that $f \in C_c(\mathbb{R})$. Then □

Definition 1.6.0.18. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$. We define $f_t \in \mathcal{S}(\mathbb{R}^n)$ by $f_t = t^{-1}\gamma_{t^{-1}} f$.

Exercise 1.6.0.19. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$. Then

$$\int_{\mathbb{R}} \phi_t\, dm = \int_{\mathbb{R}} \phi\, dm$$

Proof. We have that

$$\begin{aligned}\int_{\mathbb{R}} \phi_t\, dm &= \int_{\mathbb{R}} t^{-1}\phi(t^{-1}x)\, dm(x) \\ &= \int_{\mathbb{R}} \phi(z)\, dm(z) \\ &= \int_{\mathbb{R}} \phi\, dm\end{aligned}$$

□

Exercise 1.6.0.20. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Set

$$\alpha = \int_{\mathbb{R}} \phi\, dm$$

Then for each $f \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_{1/n} \xrightarrow{L^1} \alpha f$.
Hint: for each $t \neq 0$ and $x \in \mathbb{R}$,

$$f * \phi_t(x) - \alpha f(x) = \int_{\mathbb{R}} [\tau_{tz} f(x) - f(x)] \phi(z) \, dm(z)$$

Proof. Let $t \neq 0$ and $x \in \mathbb{R}$. The previous exercise implies that

$$\begin{aligned}
f * \phi_t(x) - \alpha f(x) &= \int_{\mathbb{R}} f(x-y)\phi_t(y) \, dm(y) - \int_{\mathbb{R}} \phi(y) \, dm(y) f(x) \\
&= \int_{\mathbb{R}} f(x-y)\phi_t(y) \, dm(y) - \int_{\mathbb{R}} \phi_t(y) \, dm(y) f(x) \\
&= \int_{\mathbb{R}} f(x-y)\phi_t(y) - f(x)\phi_t(y) \, dm(y) \\
&= \int_{\mathbb{R}} [f(x-y) - f(x)]\phi_t(y) \, dm(y) \\
&= \int_{\mathbb{R}} [f(x-y) - f(x)] t^{-1} \phi(t^{-1} y) \, dm(y) \\
&= \int_{\mathbb{R}} [f(x-tz) - f(x)] \phi(z) \, dm(z) \\
&= \int_{\mathbb{R}} [\tau_{tz} f(x) - f(x)] \phi(z) \, dm(z)
\end{aligned}$$

Tonelli's theorem implies that

$$\begin{aligned}
\|f * \phi_t - \alpha f\|_1 &= \int_{\mathbb{R}} |f * \phi_t(x) - \alpha f(x)| \, dm(x) \\
&\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\tau_{tz} f(x) - f(x)| |\phi(z)| \, dm(z) \right] dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\tau_{tz} f(x) - f(x)| |\phi(z)| \, dm(x) \right] dm(z) \\
&= \int_{\mathbb{R}} \|\tau_{tz} f - f\|_1 |\phi(z)| \, dm(z)
\end{aligned}$$

For $n \in \mathbb{N}$, define $g_n \in \mathcal{S}(\mathbb{R}^n)$ by $g_n(z) = \|\tau_{n^{-1}z} f(x) - f(x)\|_1 \phi(z)$. Then $g_n \xrightarrow{\text{p.w.}} 0$ and

$$\begin{aligned}
|g_n| &\leq 2\|f\|_1 |\phi| \\
&\in L^1(\mathbb{R}^n)
\end{aligned}$$

The dominated convergence theorem implies that

□

Definition 1.6.0.21. content...

1.7 Tempered Distributions

1.8 The Fourier Transform on $\mathcal{M}(\mathbb{R})$

Note 1.8.0.1. Recall that

$$\mathcal{M}(\mathbb{R}) = \{\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{C} : \mu \text{ is a complex measure}\}$$

Definition 1.8.0.2. Let $\mu \in \mathcal{M}(\mathbb{R})$. We define the **Fourier transform of** μ, denoted $\hat{\mu} : \mathbb{R} \to \mathbb{C}$, by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x)$$

Exercise 1.8.0.3. Let $\mu \in \mathcal{M}(\mathbb{R})$. Then Then $\hat{\mu} : \mathbb{R} \to \mathbb{C}$ is bounded.

Proof. Let $\xi \in \mathbb{R}$.

$$\begin{aligned}
|\hat{\mu}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) \right| \\
&\leq \int_{\mathbb{R}} |e^{-i\xi x}|\, d|\mu|(x) \\
&= |\mu|(\mathbb{R})
\end{aligned}$$

So $\hat{\mu}$ is bounded. □

Exercise 1.8.0.4. Let $\mu \in \mathcal{M}(\mathbb{R})$. Then $\hat{\mu} \in C_b(\mathbb{R})$.

Proof. Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. Define $(f_n)_{n\in\mathbb{N}} \subset L^1(\mu)$ and $f \in L^1(\mu)$ by $f_n(x) = e^{-i\xi_n x}$ and $f(x) = e^{-i\xi x}$. Suppose that $\xi_n \to \xi$. Then $f_n \xrightarrow{\text{p.w.}} f$ and for each $n \in N$ and $x \in \mathbb{R}$,

$$\begin{aligned}
|f_n(x)| &= |e^{-i\xi_n x}| \\
&= 1 \\
&\in L^1(|\mu|)
\end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}
|\hat{\mu}(\xi_n) - \hat{\mu}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi_n x}\, d\mu(x) - \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) \right| \\
&= \left| \int_{\mathbb{R}} e^{-i\xi_n x} - e^{-i\xi x}\, d\mu(x) \right| \\
&\leq \int_{\mathbb{R}} |e^{-i\xi_n x} - e^{-i\xi x}|\, d|\mu|(x) \\
&\to 0
\end{aligned}$$

So $\hat{\mu} : \mathbb{R} \to \mathbb{C}$ is continuous. Hence $\hat{\mu} \in C_b(\mathbb{R})$. □

Definition 1.8.0.5. Let X be a real normed vector space. We define $\mathcal{F} : \mathcal{M}(\mathbb{R}) \to C_b(\mathbb{R})$ by

$$\mathcal{F}(\mu) = \hat{\mu}$$

Exercise 1.8.0.6. Let X be a real normed vector space. Then $\mathcal{F} : \mathcal{M}(\mathbb{R}) \to C_b(\mathbb{R})$ is linear.

Proof. Let $\mu, \nu \in \mathcal{M}(\mathbb{R})$ and $\xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}[\mu + \nu](\xi) &= \int_{\mathbb{R}} e^{-i\xi x}\, d[\mu + \nu](x) \\
&= \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) + \int_{\mathbb{R}} e^{-i\xi x}\, d\nu(x) \\
&= \mathcal{F}[\mu](\xi) + \mathcal{F}[\nu](\xi)
\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\mathcal{F}(\mu + \nu) = \mathcal{F}(\mu) + \mathcal{F}(\nu)$ and $\mathcal{F}$ is linear. □

Exercise 1.8.0.7. Let X be a real normed vector space. If X is separable, then $\mathcal{F}$ is injective.

Proof. Suppose that X is separable. Let $\mu \in \mathcal{M}(X)$. Suppose that $\mu \in \ker \mathcal{F}$. Then $\hat{\mu} = 0$ and for each $\phi \in X^*$,

$$\begin{aligned} 0 &= \hat{\mu}(\phi) \\ &= \int_X e^{-i\phi(x)}\, d\mu(x) \\ &= \int_{\mathbb{R}} e^{-ix}\, d[\phi_* \mu](x) \end{aligned}$$

□

Exercise 1.8.0.8. Let X be a real normed vector space. Then $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$.

Proof. For $\mu \in \mathcal{M}(X)$ and $\phi \in X^*$, we have that

$$\begin{aligned} |\mathcal{F}[\mu](\phi)| &= \left| \int_X e^{-i\phi(x)}\, d\mu(x) \right| \\ &\leq \int_X |e^{-i\phi(x)}|\, d|\mu|(x) \\ &= |\mu|(X) \\ &= \|\mu\| \end{aligned}$$

Hence

$$\begin{aligned} \|\mathcal{F}(\mu)\| &= \sup_{\phi \in X^*} |\mathcal{F}[\mu](\phi)| \\ &\leq \|\mu\| \end{aligned}$$

which implies that $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$. □

Chapter 2

Fourier Analysis on $\mathbb{R}^n$

2.1 Schwartz Space

Definition 2.1.0.1. Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

1. $\langle x, y \rangle = \sum_j x_j y_j$
2. $|x| = \langle x, x \rangle^{1/2}$
3. $|\alpha| = \alpha_1 + \cdots + \alpha_n$
4. $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
5. $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

Definition 2.1.0.2. Let $f \in C^\infty(\mathbb{R}^n)$,$\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

We define Schwartz space, denoted $\mathcal{S}(\mathbb{R}^n)$, by

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n,\, N \in \mathbb{N}_0,\, \|f\|_{\alpha,N} < \infty\}$$

Exercise 2.1.0.3. For each $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$. Then there exists $C \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^{-1}$$

Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = (1 + |x|^2)^{-1}$. Then $g \in L^1(\mathbb{R}^n)$ which implies that $\partial^\alpha f \in L^1(\mathbb{R}^n)$. □

Definition 2.1.0.4.

2.2 The Convolution

Definition 2.2.0.1. Let $f, g \in L^0(\mathbb{R}^n)$. If for a.e. $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) < \infty$$

we define the **convolution of** f **with** g, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$, by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

Exercise 2.2.0.2. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x,y) = f(x-y)g(y)$. Tonelli's theorem implies that,

$$\begin{aligned}
\int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}^n} |g(y)| \left[\int_{\mathbb{R}^n} |f(x-y)|dm(y) \right] dm(x) \\
&= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dm(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

Then $h \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Fubini's theorem implies that $f * g \in L^1(\mathbb{R}^n)$. Clearly

$$\begin{aligned}
\|f * g\|_1 &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 \\
&\leq \|f\|_1 \|g\|_1
\end{aligned}$$

□

Exercise 2.2.0.3. Let $f, g, h \in L^1(\mathbb{R}^n)$. Then $(f * g) * h = f * (g * h)$.
Hint: use the substitution $z \mapsto z - y$

Proof. Let $x \in \mathbb{R}^n$. Then using the substitution $z \mapsto z - y$ and Fubini's theorem, we obtain

$$\begin{aligned}
(f * g) * h(x) &= \int f * g(x-y)h(y)dm(y) \\
&= \int \left[\int f(x-y-z)g(z)dm(z) \right] h(y)dm(y) \\
&= \int \left[\int f(x-z)g(z-y)dm(z) \right] h(y)dm(y) \\
&= \int \left[\int f(x-z)g(z-y)h(y)dm(z) \right] dm(y) \\
&= \int \left[\int f(x-z)g(z-y)h(y)dm(y) \right] dm(z) \\
&= \int f(x-z) \left[\int g(z-y)h(y)dm(y) \right] dm(z) \\
&= \int f(x-z) g * h(z)dm(z) \\
&= f * (g * h)(z)
\end{aligned}$$

So $(f * g) * h = f * (g * h)$. □

Exercise 2.2.0.4. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g = g * f$.

Proof. Let $x \in \mathbb{R}^n$. Using the transformation $y \mapsto x - y$, we obtain that

$$\begin{aligned} f * g(x) &= \int f(x-y)g(y)dm(y) \\ &= \int f(y)g(x-y)dm(y) \\ &= \int g(x-y)f(y)dm(y) \\ &= g * f(x) \end{aligned}$$

So $f * g = g * f$. □

Note 2.2.0.5. To summarize, $(L^1(\mathbb{R}^n), *)$ is a commutative Banach algebra.

Exercise 2.2.0.6. Young's Inequality:
Let $p \in [1, \infty]$, $f \in L^1$ and $g \in L^p$. Then $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof. Define $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $K(x, y) = f(x - y)$. Since for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \int |K(x,y)|dm(x) &= \int |K(x,y)|dm(y) \\ &= \|f\|_p \end{aligned}$$

an exercise in section 5.1 of [4] implies that $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. □

Exercise 2.2.0.7. Let $p, q \in [1, \infty]$ be conjugate, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then

1. for each $x \in \mathbb{R}^n$, $f * g(x)$ exists.
2. $\|f * g\|_u \leq \|f\|_p \|g\|_q$
3.

Proof.

1. Let $x \in \mathbb{R}^n$. Holder's inequality implies that

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \leq \|f\|_p \|g\|_q$$

Then $f * g(x)$ exists.

2. Let $x \in \mathbb{R}^n$. Then in part (1) we showed that

$$\begin{aligned} |f * g(x)| &= \left| \int_{\mathbb{R}^n} f(x-y)g(y)dm(y) \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \\ &\leq \|f\|_p \|g\|_q \end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, $\|f * g\|_u \leq \|f\|_p \|g\|_q$.

3.

□

Exercise 2.2.0.8. Let $f \in L^1(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $g \in C^k(\mathbb{R}^n)$. Suppose that for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $\partial^\alpha g \in L^\infty$. Then for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $f * g \in C^k$ and

$$\partial^\alpha (f * g) = f * \partial^\alpha g$$

Proof. Let $\alpha \in \mathbb{N}_0^n$. Suppose that $|\alpha| = 1$. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x,y) = g(x-y)f(y)$. Young's inequality imples that for a.e. $x \in \mathbb{R}^n$, $h(x,\cdot) \in L^1(\mathbb{R}^n)$. For each $y \in \mathbb{R}^n$, $\partial^\alpha h(\cdot, y) = \partial^\alpha g(\cdot - y)f(y)$ and for each $x, y \in \mathbb{R}^n$, $|\partial^\alpha h(x,y)| \leq \|\partial^\alpha g\|_\infty |f(y)| \in L^1(\mathbb{R}^n)$. An exercise in section 3.3 of [4] implies that for a.e. $x \in \mathbb{R}^n$, $\partial^\alpha (g * f)(x)$ exists and

$$\begin{aligned}
\partial^\alpha (f * g)(x) &= \partial^\alpha (g * f)(x) \\
&= \partial^\alpha \int_{\mathbb{R}^n} h(x,y)dm(y) \\
&= \int_{\mathbb{R}^n} \partial^\alpha g(x-y)f(y)dm(y) \\
&= (\partial^\alpha g) * f(x) \\
&= f * (\partial^\alpha g)(x)
\end{aligned}$$

Now proceed by induction on $|\alpha|$. $\square$

2.3 The Fourier Transform

Definition 2.3.0.1.

Exercise 2.3.0.2. Let $\phi : \mathbb{R} \to S^1$ be a measurable homomorphism.

1. Then $\phi \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
2. Define
$$c = \left[\int_{(0,a]} \phi dm\right]^{-1}$$
Then For each $x \in \mathbb{R}$,
$$\phi(x) = c\int_{(x,x+a]} \phi dm$$
3. $\phi \in C^\infty(\mathbb{R})$ and $\phi' = c(\phi(a) - 1)\phi$
4. Define $b = c(\phi(a) - 1)$ and $g \in C^\infty(\mathbb{R})$ by $g(x) = e^{-bx}\phi(x)$. Then g is constant and there exists $\xi \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

Proof.

1. Let $K \subset \mathbb{R}$ be compact. Then
$$\int_K |\phi| dm = m(K) < \infty$$
So $\phi \in L^1_{\text{loc}}(\mathbb{R})$. For the sake of contradiction, suppose that for each $a > 0$,
$$\int_{(0,a]} \phi dm = 0$$
Then the FTC implies that $\phi = 0$ a.e. on $[0, \infty)$, which is a contradiction. So there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
2. For $x \in \mathbb{R}$,
$$\begin{aligned}\phi(x) &= c\int_{(0,a]} \phi(x)\phi(t)dm(t) \\ &= c\int_{(0,a]} \phi(x+t)dm(t) \\ &= c\int_{(x,x+a]} \phi dm\end{aligned}$$
3. Part (2) and the FTC imply that ϕ is continuous. Let $d \in \mathbb{R}$. Define $f_d \in C((d, \infty))$ by
$$f_d(x) = \int_{(d,x]} \phi dm$$
Since ϕ is continuous, the FTC implies that f_d is differentiable and for each $x > d$ $f_d'(x) = \phi(x)$. Part (2) implies that for each $x > d$,
$$\begin{aligned}\phi(x) &= c\int_{(x,x+a]} \phi dm \\ &= c(f_d(x+a) - f_d(x))\end{aligned}$$

So for each $x > d$, ϕ is differentiable at x and

$$\begin{aligned} \phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x) \end{aligned}$$

Since $d \in \mathbb{R}$ is arbitrary, ϕ is differentiable and $\phi' = c(\phi(a) - 1)\phi$. This implies that $\phi \in C^\infty(\mathbb{R})$.

4. Let $x \in \mathbb{R}$. Then

$$\begin{aligned} g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0 \end{aligned}$$

So $g' = 0$ and g is constant. Hence there exists $k \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = ke^{bx}$. Since $\phi(0) = 1$, $k = 1$. Since $|\phi| = 1$, there exists $\xi \in \mathbb{R}$ such that $b = 2\pi i \xi$.

□

Note 2.3.0.3. To summarize, for each measurable homomorphism $\phi : \mathbb{R} \to S^1$, there exists $\xi \in \mathbb{R}$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i \xi x}$.

Exercise 2.3.0.4. Let $\phi : \mathbb{R}^n \to S^1$ be a measurable homomorphism. Then there exists $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi(x) = e^{2\pi i \langle \xi, x \rangle}$.

Proof. When done in the category of measurable groups, an exercise in the section on direct products of groups of [**?**] implies that there exist measurable homomorphism $(\phi_j)_{j=1}^n \subset (S^1)^{\mathbb{R}}$ such that $\phi = \bigotimes_{j=1}^n \phi_j$. The previous exercise imples that there exist $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi_j(x_j) = e^{2\pi i \xi_j x_j}$. Then for each $x \in \mathbb{R}^n$,

$$\begin{aligned} \phi(x) &= \prod_{j=1}^n \phi_j(x_j) \\ &= \prod e^{2\pi i \xi_j x_j} \\ &= e^{2\pi i \sum\limits_{j=1}^n \xi_j x_j} \\ &= e^{2\pi i \langle \xi, x \rangle} \end{aligned}$$

□

Definition 2.3.0.5. Let $f \in L^1(\mathbb{R}^n)$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle \xi, x \rangle} dm(x)$$

Chapter 3

Fourier Analysis on LCA Groups

3.1 The Convolution

Note 3.1.0.1. For the remainder of the section, we fix a locally compact abelian group G and a Haar measure μ on G.

Definition 3.1.0.2. Let $f, g \in L^1(\mu)$. We define the **convolution of** f **with** g, denoted $f * g : G \to \mathbb{C}$, by

$$f * g(x) = \int_X f(x-y)g(y)d\mu(y)$$

Exercise 3.1.0.3. Let $f, g \in L^1(\mu)$. Then $f * g \in L^1(\mu)$.

Proof. By Tonelli's theorem,

$$\begin{aligned}
\int_X |f * g| d\mu &\leq \int_X \left[\int_X |f(x-y)g(y)| d\mu(y) \right] d\mu(x) \\
&= \int_X |g(y)| \left[\int_X |f(x-y)| d\mu(y) \right] d\mu(x) \\
&= \|f\|_1 \int_X |g(y)| d\mu(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

$\square$

Chapter 4

Fourier Analysis on Banach Spaces

Appendix A

Summation

Appendix B

Asymptotic Notation

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration