

# INTRODUCTION TO MEASURE AND INTEGRATION

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## PREFACE

## NOTES

- Replace the notation " $\operatorname{Im} f$ " with  $h$  where  $f = g + ih$  so that  $\operatorname{Im} f$  can refer to **image of  $f$** .

## 1. THE DARBOUX INTEGRAL

## 1.1. Definition and Properties.

**Definition 1.1.1.** Let  $a, b \in \mathbb{R}$ . Suppose that  $a < b$ . Define

$$B([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} : f \text{ is bounded}\}$$

**Definition 1.1.2.** Let  $a, b \in \mathbb{R}$ . Suppose that  $a < b$ . Let  $x_0, \dots, x_n \in [a, b]$ . Suppose that  $a = x_0 < x_1 < \dots < x_n = b$ . Put  $\mathcal{P} = \{x_0, \dots, x_n\}$ . Then  $\mathcal{P}$  is said to be a **partition** of  $[a, b]$ .

**Definition 1.1.3.** Let  $f \in B([a, b])$  and  $\mathcal{P} = \{x_0, \dots, x_n\}$  a partition of  $[a, b]$ . Suppose that  $f$  is bounded. For  $i = 1, \dots, n$ , put

$$M_i^f = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

and

$$m_i^f = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

We define the **upper Darboux sum** of  $f$  with respect to  $\mathcal{P}$ , denoted  $U_{\mathcal{P}}f$ , to be

$$U_{\mathcal{P}}f = \sum_{i=1}^n M_i^f (x_i - x_{i-1})$$

and we define the **lower Darboux sum** of  $f$  with respect to  $\mathcal{P}$ , denoted  $L_{\mathcal{P}}f$ , to be

$$L_{\mathcal{P}}f = \sum_{i=1}^n m_i^f (x_i - x_{i-1})$$

**Exercise 1.1.4.** Let  $f \in B([a, b])$  and  $\mathcal{P}$  a partition of  $[a, b]$ . Then

$$\left[ \inf_{x \in [a, b]} f(x) \right] (b - a) \leq L_{\mathcal{P}}f \leq U_{\mathcal{P}}f \leq \left[ \sup_{x \in [a, b]} f(x) \right] (b - a)$$

*Proof.* Clear. □

**Exercise 1.1.5.** Let  $f \in B([a, b])$  and  $\mathcal{P}, \mathcal{P}'$  partitions of  $[a, b]$ . If  $\mathcal{P} \subset \mathcal{P}'$ , then

- (1)  $U_{\mathcal{P}'}f \leq U_{\mathcal{P}}f$
- (2)  $L_{\mathcal{P}}f \leq L_{\mathcal{P}'}f$

*Proof.*

- (1) Assume that  $\mathcal{P} = \{x_0, \dots, x_n\}$  and  $\mathcal{P}' = \mathcal{P} \cup \{x'\}$ . Then there exists  $j \in \{1, \dots, n\}$  such that  $x_{j-1} < x' < x_j$ . Define

$$M'_1 = \sup_{x \in [x_{j-1}, x']} f(x), \quad M'_2 = \sup_{x \in [x', x_j]} f(x)$$

Since  $[x_{j-1}, x'], [x', x_j] \subset [x_{j-1}, x_j]$ , we have that  $M'_1, M'_2 \leq M_j^f$ . Then

$$\begin{aligned} U_{\mathcal{P}'}f &= \sum_{i=1}^{j-1} M_i^f (x_i - x_{i-1}) + M'_1 (x' - x_{j-1}) + M'_2 (x_j - x') + \sum_{i=j+1}^n M_i^f (x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n M_i^f (x_i - x_{i-1}) \\ &= U_{\mathcal{P}}f \end{aligned}$$

By induction, this is true for general partitions  $P \subset \mathcal{P}'$ .

(2) Similar to (1). □

**Exercise 1.1.6.** Let  $f, g \in B([a, b])$  and  $\mathcal{P} = \{x_0, \dots, x_n\}$  a partition of  $[a, b]$ . Then

- (1)  $U_{\mathcal{P}}(f + g) \leq U_{\mathcal{P}}f + U_{\mathcal{P}}g$
- (2)  $L_{\mathcal{P}}(f + g) \geq L_{\mathcal{P}}f + L_{\mathcal{P}}g$

*Proof.*

(1) For each  $i \in \{1, \dots, n\}$ ,  $M_i^{f+g} \leq M_i^f + M_i^g$ . So

$$\begin{aligned} U_{\mathcal{P}}(f + g) &= \sum_{i=1}^n M_i^{f+g}(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n (M_i^f + M_i^g)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n M_i^f(x_i - x_{i-1}) + \sum_{i=1}^n M_i^g(x_i - x_{i-1}) \\ &= U_{\mathcal{P}}f + U_{\mathcal{P}}g \end{aligned}$$

(2) Similar to (1). □

**Exercise 1.1.7.** Let  $f \in B([a, b])$  and  $\mathcal{P} = \{x_0, \dots, x_n\}$  a partition of  $[a, b]$ . Then

- (1)  $U_{\mathcal{P}}(-f) = -L_{\mathcal{P}}f$
- (2)  $L_{\mathcal{P}}(-f) = -U_{\mathcal{P}}f$

*Proof.*

(1) Since for  $i \in \{1, \dots, n\}$ ,  $M_i^{-f} = -m_i^f$  we see that

$$\begin{aligned} U_{\mathcal{P}}(-f) &= \sum_{i=1}^n M_i^{-f}(x_i - x_{i-1}) \\ &= - \sum_{i=1}^n m_i^f(x_i - x_{i-1}) \\ &= -L_{\mathcal{P}}f \end{aligned}$$

(2) Similar to (1). □

**Exercise 1.1.8.** Let  $f \in B([a, b])$ ,  $c > 0$  and  $\mathcal{P} = \{x_0, \dots, x_n\}$  a partition of  $[a, b]$ . Then

- (1)  $U_{\mathcal{P}}(cf) = cU_{\mathcal{P}}f$
- (2)  $L_{\mathcal{P}}(cf) = cL_{\mathcal{P}}f$

*Proof.*

(1) Since for  $i \in \{1, \dots, n\}$ ,  $M_i^{cf} = cM_i^f$ , we see that

$$\begin{aligned} U_{\mathcal{P}}(cf) &= \sum_{i=1}^n M_i^{cf}(x_i - x_{i-1}) \\ &= c \sum_{i=1}^n M_i^f(x_i - x_{i-1}) \\ &= cU_{\mathcal{P}}f \end{aligned}$$

(2) Similar to (1)

□

**Definition 1.1.9.** Let  $f \in B([a, b])$ . We define the **upper Darboux integral** of  $f$ , denoted  $Uf$ , to be

$$Uf = \inf\{U_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\}$$

and we define the **lower Darboux integral** of  $f$ , denoted  $Lf$ , to be

$$Lf = \sup\{L_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\}$$

**Exercise 1.1.10.** Let  $f \in B([a, b])$ . Then

$$\left[ \inf_{x \in [a, b]} f(x) \right] (b - a) \leq Lf \leq Uf \leq \left[ \sup_{x \in [a, b]} f(x) \right] (b - a)$$

*Proof.* Clearly

$$\left[ \inf_{x \in [a, b]} f(x) \right] (b - a) \leq Lf \quad \text{and} \quad Uf \leq \left[ \sup_{x \in [a, b]} f(x) \right] (b - a)$$

Let  $\epsilon > 0$ . Then there exist partitions  $\mathcal{P}_1$  and  $\mathcal{P}_2$  of  $[a, b]$  such that  $U_{\mathcal{P}_1}f < Uf + \epsilon/2$  and  $L_{\mathcal{P}_2}f > Lf - \epsilon/2$ . Define  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ . Then

$$\begin{aligned} Uf &\geq U_{\mathcal{P}_1}f - \epsilon/2 \\ &> U_{\mathcal{P}}f - \epsilon/2 \\ &\geq L_{\mathcal{P}}f - \epsilon/2 \\ &\geq L_{\mathcal{P}_2}f - \epsilon/2 \\ &> Lf - \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we have that  $Uf \geq Lf$ .

□

**Exercise 1.1.11.** Let  $f, g \in B([a, b])$ . Then

- (1)  $U(f + g) \leq Uf + Ug$
- (2)  $L(f + g) \geq Lf + Lg$

*Proof.*

- (1) Let  $\epsilon > 0$ . Then there exists a partitions  $\mathcal{P}_1$  of  $[a, b]$  such that  $U_{\mathcal{P}_1}f < Uf + \epsilon/2$  and  $U_{\mathcal{P}_2}g < Ug + \epsilon/2$ . Define  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ . Then

$$\begin{aligned} U_{\mathcal{P}}(f + g) &\leq U_{\mathcal{P}}f + U_{\mathcal{P}}g \\ &\leq U_{\mathcal{P}_1}f + U_{\mathcal{P}_2}g \\ &< Uf + \epsilon/2 + Ug + \epsilon/2 \\ &= Uf + Ug + \epsilon \end{aligned}$$

- Since  $\epsilon > 0$  is arbitrary,  $U_{\mathcal{P}}(f + g) \leq Uf + Ug$ .  
 (2) Similar to (1).

□

**Exercise 1.1.12.** Let  $f \in B([a, b])$ . Then

- (1)  $U(-f) = -Lf$   
 (2)  $L(-f) = -Uf$

*Proof.*

- (1) Using a previous exercise, we have that

$$\begin{aligned} U(-f) &= \inf\{U_{\mathcal{P}}(-f) : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= \inf\{-L_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= -\sup\{L_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= -Lf \end{aligned}$$

- (2) Similar to (1)

□

**Exercise 1.1.13.** Let  $f \in B([a, b])$  and  $c \geq 0$ . Then

- (1)  $U(cf) = cUf$   
 (2)  $L(cf) = cLf$

*Proof.*

- (1) Using a previous exercise, we have that

$$\begin{aligned} U(cf) &= \inf\{U_{\mathcal{P}}(cf) : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= \inf\{cU_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= c \inf\{U_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= cUf \end{aligned}$$

- (2) Similar to (1)

□

**Definition 1.1.14.** Let  $f \in B([a, b])$ . Then  $f$  is said to be **Darboux integrable** if  $Uf = Lf$ . If  $f$  is Darboux integrable, we define the **Darboux integral** of  $f$ , denoted by

$$\int f \text{ or } \int f(x)dx$$

to be

$$\int f = Uf = Lf$$

The set of bounded, Darboux integrable functions is denoted by  $D([a, b])$ .

**Exercise 1.1.15.** Let  $f \in B([a, b])$ . Then  $f \in D([a, b])$  iff for each  $\epsilon > 0$ , there exists a partition  $\mathcal{P}$  of  $[a, b]$  such that  $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$ .

*Proof.* Suppose that  $f \in D([a, b])$ . Let  $\epsilon > 0$ . Then there exist partitions  $\mathcal{P}_1, \mathcal{P}_2$  of  $[a, b]$  such that  $U_{\mathcal{P}_1}f < Uf + \epsilon/2$  and  $L_{\mathcal{P}_2}f > Lf - \epsilon/2$ . Define  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ . Then  $U_{\mathcal{P}}f \leq U_{\mathcal{P}_1}f$  and  $L_{\mathcal{P}}f \geq L_{\mathcal{P}_2}f$ . So

$$\begin{aligned} U_{\mathcal{P}}f - L_{\mathcal{P}}f &< Uf - Lf + \epsilon \\ &= \epsilon \end{aligned}$$

Conversely, suppose that for each  $\epsilon > 0$ , there exists a partition  $\mathcal{P}$  of  $[a, b]$  such that  $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$ . For the sake of contradiction, suppose that  $Uf - Lf > 0$ . Choose  $\epsilon = Uf - Lf$ . Then there exists a partition  $\mathcal{P}$  of  $[a, b]$  such that  $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$ . Since  $Uf \leq U_{\mathcal{P}}f$  and  $Lf \geq L_{\mathcal{P}}f$ , we have that

$$\begin{aligned} \epsilon &> U_{\mathcal{P}}f - L_{\mathcal{P}}f \\ &\geq Uf - Lf \\ &= \epsilon \end{aligned}$$

which is a contradiction. Hence  $Uf = Lf$  and  $f \in D([a, b])$ . □

**Exercise 1.1.16.** Let  $f, g \in D([a, b])$ . Then  $f + g \in D([a, b])$  and

$$\int (f + g) = \int f + \int g$$

*Proof.* Clearly  $f + g \in B([a, b])$ . Using some previous results, we have that

$$\begin{aligned} \int f + \int g &= Lf + Lg \\ &\leq L(f + g) \\ &\leq U(f + g) \\ &\leq Uf + Ug \\ &= \int f + \int g \end{aligned}$$

So  $U(f + g) = L(f + g) = \int f + \int g$ . Therefore  $f + g \in D([a, b])$  and

$$\int (f + g) = \int f + \int g$$

.

□

**Exercise 1.1.17.** Let  $f \in D([a, b])$  and  $c \in \mathbb{R}$ . Then  $cf \in D([a, b])$  and

$$\int (cf) = c \int f$$

*Proof.* Clearly  $cf \in B([a, b])$ . If  $c \geq 0$ , then

$$\begin{aligned} L(cf) &= cLf \\ &= c \int f \\ &= cUf \\ &= U(cf) \end{aligned}$$



So

$$L(cf) = U(cf) = c \int f$$

If  $c < 0$ , then

$$\begin{aligned} L(cf) &= L(-|c|f) \\ &= -U(|c|f) \\ &= -|c|Uf \\ &= c \int f \\ &= -|c|Lf \\ &= -L(|c|f) \\ &= U(-|c|f) \\ &= U(cf) \end{aligned}$$

So

$$L(cf) = U(cf) = c \int f$$

Therefore  $cf \in D([a, b])$  and

$$\int (cf) = c \int f$$

□

**Corollary 1.1.18.** We have that  $D([a, b])$  is a vector space and the map  $I : D([a, b]) \rightarrow \mathbb{R}$  given by  $If = \int f$  is linear.

*Proof.* Clear. □

**Exercise 1.1.19.** Let  $f : [a, b] \rightarrow \mathbb{R}$ . If  $f$  is continuous, then  $f \in D([a, b])$ .

*Proof.* Suppose that  $f$  is continuous. Then  $f$  is uniformly continuous. Let  $\epsilon > 0$ . Uniform continuity implies that there exists  $\delta > 0$  such that for each  $x, y \in [a, b]$ ,  $|x - y| < \delta$  implies that  $|f(x) - f(y)| < \epsilon/(b - a)$ . Choose  $n \in \mathbb{N}$  such that  $(b - a)/n < \delta$ . For  $i \in \{0, \dots, n\}$ , define  $x_i = a + i(b - a)/n$ . Put  $\mathcal{P} = \{x_0, \dots, x_n\}$ . Continuity implies that for each  $i \in \{1, \dots, n\}$ , there exists  $x_i^M, x_i^m \in [x_{i-1}, x_i]$  such that  $f(x_i^M) = M_i^f$  and  $f(x_i^m) = m_i^f$ .

Then

$$\begin{aligned}
U_{\mathcal{P}}f - L_{\mathcal{P}}f &= \sum_{i=1}^n M_i^f(x_i - x_{i-1}) - \sum_{i=1}^n m_i^f(x_i - x_{i-1}) \\
&= \sum_{i=1}^n (M_i^f - m_i^f)(x_i - x_{i-1}) \\
&= \sum_{i=1}^n [f(x_i^M) - f(x_i^m)](x_i - x_{i-1}) \\
&< \sum_{i=1}^n \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\
&= \epsilon
\end{aligned}$$

So for each  $\epsilon > 0$ , there exists a partition  $\mathcal{P}$  of  $[a, b]$  such that  $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$ . Hence  $f \in D([a, b])$ .  $\square$

**Exercise 1.1.20.** Let  $f : [a, b] \rightarrow \mathbb{R}$ . If  $f$  is monotonic, then  $f \in D([a, b])$ .

*Proof.* Suppose that  $f$  is increasing. Let  $\epsilon > 0$ . Choose  $n \in \mathbb{N}$  such that  $(b-a)[f(b) - f(a)]/n < \epsilon$ . For  $i \in \{0, \dots, n\}$ , define  $x_i = a + i(b-a)/n$ . Put  $\mathcal{P} = \{x_0, \dots, x_n\}$ . Then

$$\begin{aligned}
U_{\mathcal{P}}f - L_{\mathcal{P}}f &= \sum_{i=1}^n M_i^f(x_i - x_{i-1}) - \sum_{i=1}^n m_i^f(x_i - x_{i-1}) \\
&= \sum_{i=1}^n (M_i^f - m_i^f)(x_i - x_{i-1}) \\
&= \frac{b-a}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \\
&= \frac{b-a}{n} [f(b) - f(a)] \\
&< \epsilon
\end{aligned}$$

So for each  $\epsilon > 0$ , there exists a partition  $\mathcal{P}$  of  $[a, b]$  such that  $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$ . Hence  $f \in D([a, b])$ . The case is similar if  $f$  is decreasing.  $\square$

**Exercise 1.1.21.** Define  $\chi_{\mathbb{Q}} : [0, 1] \rightarrow \mathbb{R}$  by

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Then  $\chi_{\mathbb{Q}} \notin D([a, b])$ .

*Proof.* Let  $\mathcal{P} = \{x_0, \dots, x_n\}$  be a partition of  $[0, 1]$ . Then for each  $i \in \{1, \dots, n\}$ ,  $M_i^{\chi_{\mathbb{Q}}} = 1$  and  $m_i^{\chi_{\mathbb{Q}}} = 0$ . So  $U_{\mathcal{P}}\chi_{\mathbb{Q}} = 1$  and  $L_{\mathcal{P}}\chi_{\mathbb{Q}} = 0$ . Since  $\mathcal{P}$  is arbitrary, we have that  $U\chi_{\mathbb{Q}} = 1$  and  $L\chi_{\mathbb{Q}} = 0$ .  $\square$

## 2. MEASURE SPACES

## 2.1. Elementary Families and Algebras.

**Definition 2.1.1.** Let  $X$  be a set and  $\mathcal{E} \subset \mathcal{P}(X)$ . Then  $X$  is said to be an **elementary family on  $X$**  if

- (1)  $\emptyset \in \mathcal{E}$
- (2) for each  $A, B \in \mathcal{E}$ ,  $A \cap B \in \mathcal{E}$
- (3) for each  $A \in \mathcal{E}$ , there exist  $(A_j)_{j=1}^n \subset \mathcal{E}$  such that  $(A_j)_{j=1}^n$  is disjoint and  $A^c = \bigcup_{j=1}^n A_j$

**Exercise 2.1.2.** Define

$$\mathcal{E} = \{(a, b] : a, b \in \overline{\mathbb{R}}\}$$

where we take  $(a, \infty] = (a, \infty)$ . Then  $\mathcal{E}$  is an elementary family on  $\mathbb{R}$

*Proof.*

- (1)  $\emptyset = (0, 0] \in \mathcal{E}$
- (2) Let  $a_1, a_2, b_1, b_2 \in \overline{\mathbb{R}}$ . Then

$$(a_1, b_1] \cap (a_2, b_2] = \begin{cases} \emptyset & b_1 \leq a_2 \\ (a_2, b_1] & b_1 > a_2 \end{cases}$$

So  $(a_1, b_1] \cap (a_2, b_2] \in \mathcal{E}$ .

- (3) Let  $a, b \in \mathbb{R}$ . Suppose that  $a < b$ . Then  $(a, b]^c = (-\infty, a] \cup (b, \infty) \in \mathcal{E}$ .

□

**Definition 2.1.3.** Let  $X$  be a set and  $\mathcal{A}_0 \subset \mathcal{P}(X)$ . Then  $\mathcal{A}_0$  is said to be an **algebra on  $X$**  if

- (1)  $\mathcal{A}_0 \neq \emptyset$
- (2) for each  $A \in \mathcal{A}_0$ ,  $A^c \in \mathcal{A}_0$
- (3) for each  $A, B \in \mathcal{A}_0$ ,  $A \cup B \in \mathcal{A}_0$

**Exercise 2.1.4.** Let  $X$  be a set and  $\mathcal{E}$  an elementary family on  $X$ . Define

$$\mathcal{A}_0^\mathcal{E} = \left\{ \bigcup_{j=1}^n A_j : (A_j)_{j=1}^n \text{ is disjoint and } (A_j)_{j=1}^n \subset \mathcal{E} \right\}$$

Then  $\mathcal{A}_0^\mathcal{E}$  is an algebra on  $X$ .

*Proof.*

- (1) By definition,  $\emptyset \in \mathcal{E} \subset \mathcal{A}_0^\mathcal{E}$ . So  $\mathcal{A}_0^\mathcal{E} \neq \emptyset$ .
- (2) Let  $A \in \mathcal{A}_0^\mathcal{E}$ , there exists  $(A_j)_{j=1}^n \subset \mathcal{E}$  such that  $(A_j)_{j=1}^n$  is disjoint and  $A = \bigcup_{j=1}^n A_j$ . By definition of  $\mathcal{E}$ , for each  $j \in \{1, \dots, n\}$ , there exist  $(B_{j,k})_{k=1}^{n_j} \subset \mathcal{E}$  such that  $(B_{j,k})_{k=1}^{n_j}$

is disjoint and  $A_j^c = \bigcup_{k=1}^{n_j} B_{j,k}$ . Then

$$\begin{aligned} A^c &= \bigcap_{j=1}^n A_j^c \\ &= \bigcap_{j=1}^n \left( \bigcup_{k=1}^{n_j} B_{j,k} \right) \\ &= \bigcup \end{aligned}$$

(3) Let  $A, B \in \mathcal{A}_0^\mathcal{E}$ . Then there exist  $(A_j)_{j=1}^n, (B_j)_{j=1}^m \subset \mathcal{E}$  such that  $A = \bigcup_{j=1}^n A_j$  and

$B = \bigcup_{j=1}^m B_j$ . Then

$$A \cup B = \left( \bigcup_{j=1}^n A_j \right) \cup \left( \bigcup_{j=1}^m B_j \right)$$

**FINISH!!!**

□

## 2.2. Sigma Algebras.

**Definition 2.2.1.** Let  $X$  be a set and  $\mathcal{A} \subset \mathcal{P}(X)$ . Then  $\mathcal{A}$  is said to be a  $\sigma$ -algebra on  $X$  if

- (1)  $\mathcal{A} \neq \emptyset$
- (2) for each  $A \in \mathcal{A}$ ,  $A^c \in \mathcal{A}$
- (3) for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ ,  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

**Exercise 2.2.2.** Let  $X$  be a set and  $\mathcal{A}$  a  $\sigma$ -algebra on  $X$ . Then

- (1)  $X, \emptyset \in \mathcal{A}$
- (2) for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ ,  $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{A}$
- (3) For each  $A, B \in \mathcal{A}$ ,  $A \setminus B \in \mathcal{A}$

*Proof.*

- (1) Since  $\mathcal{A} \neq \emptyset$ , there exists  $A \in \mathcal{A}$ . Then  $A^c \in \mathcal{A}$ . Hence  $X = A \cup A^c \in \mathcal{A}$  and  $\emptyset = X^c \in \mathcal{A}$ .
- (2) Let  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Then  $(A_n^c)_{n \in \mathbb{N}} \subset \mathcal{A}$ . So  $\bigcup_{n \in \mathbb{N}} A_n^c \in \mathcal{A}$ . Therefore

$$\bigcap_{n \in \mathbb{N}} A_n = \left( \bigcup_{n \in \mathbb{N}} A_n^c \right)^c \in \mathcal{A}$$

- (3) Let  $A, B \in \mathcal{A}$ . Then  $A \setminus B = A \cap B^c \in \mathcal{A}$ .

□

**Exercise 2.2.3.** Let  $X$  be a set and  $(\mathcal{A}_i)_{i \in I}$  a collection of  $\sigma$ -algebras (resp. algebra) on  $X$ . Then  $\bigcap_{i \in I} \mathcal{A}_i$  is a  $\sigma$ -algebra (resp. algebra) on  $X$ .

*Proof.*

- (1) For each  $i \in I$ ,  $X \in \mathcal{A}_i$ . Thus  $X \in \bigcap_{i \in I} \mathcal{A}_i$  and  $\bigcap_{i \in I} \mathcal{A}_i \neq \emptyset$ .
- (2) Let  $A \in \bigcap_{i \in I} \mathcal{A}_i$ . Then for each  $i \in I$ ,  $A \in \mathcal{A}_i$ . Hence for each  $i \in I$ ,  $A^c \in \mathcal{A}_i$ . Thus  $A^c \in \bigcap_{i \in I} \mathcal{A}_i$ .
- (3) Let  $(A_n)_{n \in \mathbb{N}} \subset \bigcap_{i \in I} \mathcal{A}_i$ . Then for each  $i \in I$ ,  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}_i$ . Thus for each  $i \in I$ ,  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_i$ . So  $\bigcup_{n \in \mathbb{N}} A_n \in \bigcap_{i \in I} \mathcal{A}_i$ .

□

**Definition 2.2.4.** Let  $X$  be a set and  $\mathcal{C} \subset \mathcal{P}(X)$ . Put

$$\mathcal{S} = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ and } \mathcal{C} \subset \mathcal{A}\}$$

We define the  $\sigma$ -algebra generated by  $\mathcal{C}$  on  $X$ ,  $\sigma(\mathcal{C})$ , by

$$\sigma(\mathcal{C}) = \bigcap_{\mathcal{A} \in \mathcal{S}} \mathcal{A}$$

**Note 2.2.5.** Let  $X$  be a set,  $\mathcal{C} \subset \mathcal{P}(X)$  and  $\mathcal{A}$  a  $\sigma$ -alg on  $X$ . By definition, if  $\mathcal{C} \subset \mathcal{A}$ , then  $\sigma(\mathcal{C}) \subset \mathcal{A}$ .

**Note 2.2.6.** Let  $X$  be a set,  $\mathcal{T}$  an ordered set and  $(\mathcal{A}_t)_{t \in \mathcal{T}}$  a collection of  $\sigma$ -algebras on  $X$ . Suppose that for each  $s, t \in \mathcal{T}$ , if  $s \leq t$ , then  $\mathcal{A}_s \subset \mathcal{A}_t$ . If there exists  $t \in \mathcal{T}$  such that  $\mathcal{A}_t = \bigcup_{t \in \mathcal{T}} \mathcal{A}_t$ , then  $\bigcup_{t \in \mathcal{T}} \mathcal{A}_t$  is a  $\sigma$ -algebra on  $X$ . So if  $\mathcal{T}$  is finite or if  $(\mathcal{A}_t)_{t \in \mathcal{T}}$  terminates, the union is  $\sigma$ -algebra.

**Definition 2.2.7.** Let  $(X, \mathcal{T})$  be a topological space. We define the **Borel  $\sigma$ -algebra** on  $X$ ,  $\mathcal{B}(X)$ , to be

$$\mathcal{B}(X) = \sigma(\mathcal{T})$$

Let  $E \subset X$ . Then  $E$  is said to be **Borel** if  $E \in \mathcal{B}(X)$ .

**Exercise 2.2.8.** The Borel  $\sigma$ -algebra on  $\mathbb{R}$  with the standard topology is given by

$$\mathcal{B}(\mathbb{R}) = \begin{cases} \sigma(\{(a, b) : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{[a, b) : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{(a, b] : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{[a, b] : a, b \in \mathbb{R} \text{ and } a < b\}) \end{cases}$$

*Proof.* Define

$$(1) \mathcal{C}_{lo} = \{(a, b) : a, b \in \mathbb{R} \text{ and } a < b\}$$

$$(2) \mathcal{C}_c = \{[a, b) : a, b \in \mathbb{R} \text{ and } a < b\}$$

$$(3) \mathcal{C}_{ro} = \{(a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$$

$$(4) \mathcal{C}_o = \{[a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$$

Recall that for each open set  $A \subset \mathbb{R}$ , there exist  $(a_i)_{i \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}} \subset \mathbb{R}$  such that for each  $i \in \mathbb{N}$ ,  $a_i < b_i$ , for each  $i, j \in \mathbb{N}$ , if  $i \neq j$ , then  $(a_i, b_i) \cap (a_j, b_j) = \emptyset$  and  $A = \bigcup_{i \in \mathbb{N}} (a_i, b_i)$ . This implies that  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o)$ .

Now, let  $a, b \in \mathbb{R}$ . Suppose that  $a < b$ . Then

$$(1) [a, b] = \bigcap_{n \in \mathbb{N}} (a - \frac{1}{n}, b], \text{ so } \sigma(\mathcal{C}_c) \subset \sigma(\mathcal{C}_{lo})$$

$$(2) [a, b) = \bigcup_{n \in \mathbb{N}} [a, b - \frac{1}{n}], \text{ so } \sigma(\mathcal{C}_{ro}) \subset \sigma(\mathcal{C}_c)$$

$$(3) (a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b), \text{ so } \sigma(\mathcal{C}_o) \subset \sigma(\mathcal{C}_{ro})$$

$$(4) (a, b] = \bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n}), \text{ so } \sigma(\mathcal{C}_{lo}) \subset \sigma(\mathcal{C}_o)$$

Hence  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o) = \sigma(\mathcal{C}_{ro}) = \sigma(\mathcal{C}_c) = \sigma(\mathcal{C}_{lo}) = \sigma(\mathcal{C}_o)$ . □

**Exercise 2.2.9.** Let  $(X, \mathcal{T})$  be a topological space and  $\mathcal{E} \subset \mathcal{T}$  a basis for  $\mathcal{T}$ . If  $\mathcal{E}$  is countable, then  $\mathcal{B}(X) = \sigma(\mathcal{E})$ .

*Proof.* Since  $\mathcal{E} \subset \mathcal{T}$ ,

$$\begin{aligned} \sigma(\mathcal{E}) &\subset \sigma(\mathcal{T}) \\ &= \mathcal{B}(X) \end{aligned}$$

Let  $U \in \mathcal{T}$ . Since  $\mathcal{E}$  is a countable basis, there exists  $\mathcal{C}_U \subset \mathcal{E}$  such that  $\mathcal{C}_U$  is countable and  $U = \bigcup_{C \in \mathcal{C}_U} C$ . Hence  $U \in \sigma(\mathcal{E})$ . Since  $U \in \mathcal{T}$  is arbitrary,  $\mathcal{T} \subset \sigma(\mathcal{E})$ . Thus

$$\begin{aligned}\mathcal{B}(X) &= \sigma(\mathcal{T}) \\ &\subset \sigma(\mathcal{E})\end{aligned}$$

Therefore  $\mathcal{B}(X) = \sigma(\mathcal{E})$ . □

**Exercise 2.2.10.** Let  $X$  be a set. Define  $\mathcal{A} = \{A \in \mathcal{A} : A \text{ is countable or } A^c \text{ is countable}\}$ . Then  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ .

*Proof.*

- (1) Since  $X^c = \emptyset$  is countable,  $X \in \mathcal{A}$ .
- (2) Let  $A \in \mathcal{A}$ . Suppose that  $A^c$  is uncountable. Then by assumption,  $A = (A^c)^c$  is countable. Hence  $A^c \in \mathcal{A}$ .
- (3) Let  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Then for each  $n \in \mathbb{N}$ ,  $A_n$  is countable or  $A_n^c$  is countable. Suppose that  $\bigcup_{n \in \mathbb{N}} A_n$  is uncountable. Then there exists  $N \in \mathbb{N}$  such that  $A_N$  is uncountable. Hence  $A_N^c$  is countable. Thus

$$\begin{aligned}\left(\bigcup_{n \in \mathbb{N}} A_n\right)^c &= \bigcap_{n \in \mathbb{N}} A_n^c \\ &\subset A_N^c\end{aligned}$$

So  $\left(\bigcup_{n \in \mathbb{N}} A_n\right)^c$  is countable and  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ . □

**Definition 2.2.11.** Let  $X$  be a set and  $\mathcal{A}$  be a  $\sigma$ -algebra on  $X$ . Then  $(X, \mathcal{A})$  is called a **measurable space**.

### 2.3. Measurable Functions.

**Definition 2.3.1.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . Then  $f$  is said to be  $(\mathcal{A}, \mathcal{B})$ -**measurable** if for each  $B \in \mathcal{B}$ ,  $f^{-1}(B) \in \mathcal{A}$ . When  $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  we say that  $f$  is  $\mathcal{A}$ -**measurable**. If  $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and  $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  or  $(\mathbb{R}, \mathcal{L})$ , then we say that  $f$  is **Borel measurable** or **Lebsegue measurable** respectively.

**Definition 2.3.2.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. Define

- $L^+(X, \mathcal{A}) = \{f : X \rightarrow [0, \infty] : f \text{ is measurable}\}$
- $L^0(X, \mathcal{A}) = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable}\}$

**Definition 2.3.3.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and  $\phi : X \rightarrow Y$ . Then  $\phi$  is said to be a **isomorphism** if

- (1)  $\phi$  is a bijection
- (2)  $\phi$  is  $(\mathcal{A}, \mathcal{B})$ -measurable and  $\phi^{-1}$  is  $(\mathcal{B}, \mathcal{A})$ -measurable

**Definition 2.3.4.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. Then  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  are said to be **isomorphic** if there exists  $\phi : X \rightarrow Y$  such that  $\phi$  is an isomorphism.

**Definition 2.3.5.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . We define the

- (1) **pushforward of  $\mathcal{A}$** , denoted  $f_*\mathcal{A}$ , by

$$f_*\mathcal{A} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$$

- (2) **pullback of  $\mathcal{B}$** , denoted  $f^*\mathcal{B}$ , by

$$f^*\mathcal{B} = \{f^{-1}(B) : B \in \mathcal{B}\}$$

**Note 2.3.6.** It is also common to write  $\sigma(f)$  or  $f^{-1}(\mathcal{B})$  in place of  $f^*\mathcal{B}$ .

**Exercise 2.3.7.** Let  $(X, \mathcal{A}), (Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . Then

- (1)  $f_*\mathcal{A}$  is a  $\sigma$ -algebra on  $Y$
- (2)  $f^*\mathcal{B}$  is a  $\sigma$ -algebra on  $X$

*Proof.*

- (1)
  - Since  $f^{-1}(Y) = X \in \mathcal{A}$ ,  $Y \in f_*\mathcal{A}$  and  $f_*\mathcal{A} \neq \emptyset$ .
  - Let  $B \in f_*\mathcal{A}$ . Then  $f^{-1}(B) \in \mathcal{A}$ . Hence

$$f^{-1}(B^c) = (f^{-1}(B))^c \in \mathcal{A}$$

Thus  $B^c \in f_*\mathcal{A}$ .

- Now, let  $(B_n)_{n \in \mathbb{N}} \subset f_*\mathcal{A}$ . Then for each  $n \in \mathbb{N}$ ,  $f^{-1}(B_n) \in \mathcal{A}$ . Thus

$$f^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \bigcup_{n \in \mathbb{N}} f^{-1}(B_n) \in \mathcal{A}$$

Hence  $\bigcup_{n \in \mathbb{N}} B_n \in f_*\mathcal{A}$ .

- (2) Similar to (1).

□

**Exercise 2.3.8.** Let  $(X, \mathcal{A}), (Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . If  $f$  is an isomorphism, then



- (1)  $f^*(\mathcal{B}) = \mathcal{A}$
- (2)  $f_*(\mathcal{A}) = \mathcal{B}$

*Proof.* Suppose that  $f$  is an isomorphism.

- (1) Since  $f$  is  $(\mathcal{A}, \mathcal{B})$ -measurable,  $f^*(\mathcal{B}) \subset \mathcal{A}$ . Let  $A \in \mathcal{A}$ . Set  $B = f(A)$ . Since  $f^{-1}$  is  $(\mathcal{B}, \mathcal{A})$ -measurable,  $B \in \mathcal{B}$ . By definition,

$$\begin{aligned} A &= f^{-1}(B) \\ &\in f^*(\mathcal{B}) \end{aligned}$$

Since  $A \in \mathcal{A}$  is arbitrary,  $\mathcal{A} \subset f^*(\mathcal{B})$ . Hence  $f^*(\mathcal{B}) = \mathcal{A}$ .

- (2) Since  $f$  is  $(\mathcal{A}, \mathcal{B})$ -measurable,  $\mathcal{B} \subset f_*(\mathcal{A})$ . Let  $B \in f_*(\mathcal{A})$ . By definition,  $f^{-1}(B) \in \mathcal{A}$ . Set  $A = f^{-1}(B)$ . Since  $f^{-1}$  is  $(\mathcal{B}, \mathcal{A})$ -measurable,

$$\begin{aligned} B &= f(A) \\ &\in \mathcal{B} \end{aligned}$$

Since  $B \in f_*(\mathcal{A})$  is arbitrary,  $f_*(\mathcal{A}) \subset \mathcal{B}$ . Hence  $f_*(\mathcal{A}) = \mathcal{B}$ .

□

**Exercise 2.3.9.** Let  $(X, \mathcal{A}), (Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . If  $f$  is constant, then

- (1)  $f^*(\mathcal{B}) = \{\emptyset, X\}$
- (2)  $f_*(\mathcal{A}) = \mathcal{P}(Y)$

*Proof.* Suppose that  $f$  is constant. Then there exists  $y \in Y$  such that for each  $x \in X$ ,  $f(x) = y$ . Then for each  $B \subset Y$ ,

$$f^{-1}(B) = \begin{cases} X, & y \in B \\ \emptyset, & y \notin B \end{cases}$$

- (1) Clearly  $\{\emptyset, X\} \subset f^*(\mathcal{B})$ . Let  $A \in f^*(\mathcal{B})$ . Then there exists  $B \in \mathcal{B}$  such that  $A = f^{-1}(B)$ . Then

$$\begin{aligned} A &= f^{-1}(B) \\ &\in \{\emptyset, X\} \end{aligned}$$

Since  $A \in f^*(\mathcal{B})$  is arbitrary,  $f^*(\mathcal{B}) \subset \{\emptyset, X\}$ . Hence  $f^*(\mathcal{B}) = \{\emptyset, X\}$ .

- (2) Clearly  $f_*(\mathcal{A}) \subset \mathcal{P}(Y)$ . Let  $B \in \mathcal{P}(Y)$ . Since  $\{\emptyset, X\} \subset \mathcal{A}$ , we have that

$$\begin{aligned} f^{-1}(B) &= X \\ &\in \{\emptyset, X\} \\ &\subset \mathcal{A} \end{aligned}$$

Hence  $B \in f_*(\mathcal{A})$ . Since  $B \in \mathcal{P}(Y)$  is arbitrary,  $\mathcal{P}(Y) \subset f_*(\mathcal{A})$ . Hence  $f_*(\mathcal{A}) = \mathcal{P}(Y)$ .

□

**Exercise 2.3.10.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. Suppose that there exists  $\mathcal{E} \subset Y$  such that  $\sigma(\mathcal{E}) = \mathcal{B}$ . Let  $f : X \rightarrow Y$ . Then  $f$  is  $(\mathcal{A}, \mathcal{B})$  measurable iff for each  $B \in \mathcal{E}$ ,  $f^{-1}(B) \in \mathcal{A}$ .

*Proof.* By definition, if  $f$  is  $\mathcal{A}$ - $\mathcal{B}$  measurable, then for each  $B \in \mathcal{E}$ ,  $f^{-1}(B) \in \mathcal{A}$ . Conversely, suppose that for each  $B \in \mathcal{E}$ ,  $f^{-1}(B) \in \mathcal{A}$ . The previous exercise tells us that  $f_*\mathcal{A}$  is a  $\sigma$ -algebra on  $Y$ . Since  $\mathcal{E} \subset f_*\mathcal{A}$ , we have that  $\mathcal{B} = \sigma(\mathcal{E}) \subset f_*\mathcal{A}$ . So  $f$  is  $(\mathcal{A}, \mathcal{B})$  measurable.  $\square$

**Exercise 2.3.11.** Let  $X, Y$  be sets,  $f : X \rightarrow Y$  and  $\mathcal{E} \subset \mathcal{P}(Y)$ . Then  $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$ .

*Proof.* Clearly  $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$ . Since  $f^{-1}(\sigma(\mathcal{E}))$  is a  $\sigma$ -algebra, we have that  $\sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$ . Since  $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$ , the previous exercise tells us that  $f$  is  $f^{-1}(\sigma(\mathcal{E}))- \sigma(\mathcal{E})$  measurable. Then  $f^{-1}(\sigma(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$ . So  $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$ .  $\square$

**FINISH!!!**

**Definition 2.3.12.** Let  $X$  be a set,  $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$  a collection of measurable spaces and  $\mathcal{F} \in \prod_{\alpha \in A} Y_\alpha^X$  (i.e.  $\mathcal{F} = (f_\alpha)_{\alpha \in A}$  where for each  $\alpha \in A$ ,  $f_\alpha : X \rightarrow Y_\alpha$ ). We define the **initial  $\sigma$ -algebra generated by  $\mathcal{F}$  on  $X$** , denoted  $\sigma_X(\mathcal{F})$ , by

$$\sigma_X(\mathcal{F}) = \sigma(\{f_\alpha^{-1}(B) : B \in \mathcal{B}_\alpha \text{ and } \alpha \in A\})$$

**Note 2.3.13.** If  $\mathcal{F} = \{f\}$ , then  $\sigma_X(\mathcal{F}) = f^*\mathcal{B}$ .

**Note 2.3.14.** Essentially,  $\sigma_X(\mathcal{F})$  is the smallest  $\sigma$ -algebra on  $X$  such that for each  $\alpha \in A$ ,  $f_\alpha : X \rightarrow Y_\alpha$  is measurable.

**Exercise 2.3.15.** Let  $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$  be a collection of measurable spaces,  $X$  a set,  $(Z, \mathcal{C})$  a measurable space,  $\mathcal{F} = (f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} Y_\alpha^X$  and  $g : Z \rightarrow X$ . Then  $g$  is  $\mathcal{C}-\tau_X(\mathcal{F})$  measurable iff for each  $\alpha \in A$ ,  $f_\alpha \circ g$  is  $\mathcal{C}-\mathcal{B}_\alpha$  measurable:

$$\begin{array}{ccc} Y_\alpha & \xleftarrow{f_\alpha} & X \\ & \nwarrow g \circ f_\alpha & \uparrow g \\ & & Z \end{array}$$

*Proof.* If  $g$  is  $\mathcal{C}-\tau_X(\mathcal{F})$  measurable, then clearly for each  $\alpha \in A$ ,  $f_\alpha \circ g$  is  $\mathcal{C}-\mathcal{B}_\alpha$  measurable. Conversely, suppose that for each  $\alpha \in A$ ,  $f_\alpha \circ g$  is  $\mathcal{C}-\mathcal{B}_\alpha$  measurable. Let  $\alpha \in A$  and  $V \in \mathcal{B}_\alpha$ . Measurability implies that,

$$\begin{aligned} g^{-1}(f_\alpha^{-1}(V)) &= (f_\alpha \circ g)^{-1}(V) \\ &\in \mathcal{C} \end{aligned}$$

Since  $\alpha \in A$  and  $V \in \mathcal{B}_\alpha$  are arbitrary, we have that for each  $\alpha \in A$  and  $V \in \mathcal{B}_\alpha$ ,  $g^{-1}(f_\alpha^{-1}(V)) \in \mathcal{C}$ . Since  $\tau_X(\mathcal{F}) = \tau(\{f_\alpha^{-1}(V) : \alpha \in A \text{ and } V \in \mathcal{B}_\alpha\})$ , a previous exercise implies that  $g$  is  $\mathcal{C}-\tau_X(\mathcal{F})$  measurable.  $\square$

**Definition 2.3.16.** Let  $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  be a collection of measurable spaces,  $Y$  a set and  $\mathcal{F} \in \prod_{\alpha \in A} Y^{X_\alpha}$  (i.e.  $\mathcal{F} = (f_\alpha)_{\alpha \in A}$  where for each  $\alpha \in A$ ,  $f_\alpha : X_\alpha \rightarrow Y$ ). We define the **final  $\sigma$ -algebra generated by  $\mathcal{F}$  on  $X$** , denoted  $\sigma_Y(\mathcal{F})$ , by

$$\sigma_Y(\mathcal{F}) = \sigma(\{V \subset Y : \text{for each } \alpha \in A, f_\alpha^{-1}(V) \in \mathcal{A}_\alpha\})$$

**Note 2.3.17.** If  $\mathcal{F} = \{f\}$ , then  $\sigma_Y(\mathcal{F}) = f_*\mathcal{A}$ .

**Note 2.3.18.** Essentially,  $\sigma_Y(\mathcal{F})$  is the largest  $\sigma$ -algebra on  $Y$  such that for each  $\alpha \in A$ ,  $f_\alpha : X_\alpha \rightarrow Y$  is measurable.

**Exercise 2.3.19.** Let  $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  be a collection of measurable spaces,  $Y$  a set,  $(Z, \mathcal{C})$  a measurable space,  $\mathcal{F} = (f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} Y^{X_\alpha}$  and  $g : Y \rightarrow Z$ . Then  $g$  is  $\tau_Y(\mathcal{F})$ - $\mathcal{C}$  measurable iff for each  $\alpha \in A$ ,  $g \circ f_\alpha$  is  $X_\alpha$ - $\mathcal{C}$  measurable, i.e. for each  $\alpha \in A$ , the following diagram commutes in the category of measurable spaces:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_\alpha} & Y \\ & \searrow g \circ f_\alpha & \downarrow g \\ & & Z \end{array}$$

*Proof.* If  $g$  is  $\tau_Y(\mathcal{F})$ - $\mathcal{C}$  measurable, then clearly for each  $\alpha \in A$ ,  $g \circ f_\alpha$  is  $X_\alpha$ - $\mathcal{C}$  measurable. Conversely, suppose that for each  $\alpha \in A$ ,  $g \circ f_\alpha$  is  $X_\alpha$ - $\mathcal{C}$  measurable. Let  $V \in \mathcal{C}$ . Measurability implies that for each  $\alpha \in A$ ,  $f_\alpha^{-1}(g^{-1}(V)) \in \mathcal{A}_\alpha$ . By definition,  $g^{-1}(V) \in \tau_Y(\mathcal{F})$ . So  $g$  is  $\tau_Y(\mathcal{F})$ - $\mathcal{C}$  measurable.  $\square$

**Exercise 2.3.20.** Let  $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$  be topological spaces and  $f : X \rightarrow Y$ . If  $f$  is continuous, then  $f$  is  $\mathcal{B}(X)$ - $\mathcal{B}(Y)$  measurable.

*Proof.* Recall that  $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$  and continuity tells us that for each  $U \in \mathcal{T}_2$ ,  $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$ .  $\square$

**Definition 2.3.21.** Let  $X$  be a set and  $f : X \rightarrow \mathbb{C}$ . Then  $f$  is said to be **simple** if  $f(X)$  is finite.

**Definition 2.3.22.** Let  $(X, \mathcal{A})$  be a measurable space. We define  $S^+(X, \mathcal{A}) = \{f : X \rightarrow [0, \infty) : f \text{ is simple, measurable}\}$  and  $S(X, \mathcal{A}) = \{f : X \rightarrow \mathbb{C} : f \text{ is simple, measurable}\}$

**Theorem 2.3.23.** Let  $(X, \mathcal{A})$  be a measurable space. Then

- (1) If  $f : X \rightarrow [0, \infty]$  is measurable, then there exists a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \rightarrow f$  pointwise and  $\phi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.
- (2) If  $f : X \rightarrow \mathbb{C}$  is measurable, then there exists a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset S$  such that for each  $n \in \mathbb{N}$ ,  $|\phi_n| \leq |\phi_{n+1}| \leq |f|$  and  $\phi_n \rightarrow f$  pointwise and  $\phi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.

**Exercise 2.3.24.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and  $f : X \rightarrow Y$ . If  $f$  is  $\mathcal{A}$ - $\mathcal{B}$  measurable iff  $f$  is  $\mathcal{A}$ - $\mathcal{B} \cap f(X)$  measurable.

*Proof.* Suppose that  $f$  is  $\mathcal{A}$ - $\mathcal{B}$  measurable. Let  $E \in \mathcal{B} \cap f(X)$ . Then there exists  $B \in \mathcal{B}$  such that  $E = B \cap f(X)$ . Then

$$\begin{aligned} f^{-1}(E) &= f^{-1}(B \cap f(X)) \\ &= f^{-1}(B) \cap f^{-1}(f(X)) \\ &= f^{-1}(B) \cap X \\ &= f^{-1}(B) \\ &\in \mathcal{A} \end{aligned}$$

Conversely, suppose that  $f$  is  $\mathcal{A}$ - $\mathcal{B} \cap f(X)$  measurable. Let  $B \in \mathcal{B}$ . Then as before,

$$\begin{aligned} f^{-1}(B) &= f^{-1}(B \cap f(X)) \\ &\in \mathcal{A} \end{aligned}$$

□

**Exercise 2.3.25. Doob-Dynkin Lemma:**

Let  $(X_1, \mathcal{A}_1)$ ,  $(X_2, \mathcal{A}_2)$  and  $(X_3, \mathcal{A}_3)$  be measurable spaces and  $f : X_1 \rightarrow X_2$  and  $g : X_1 \rightarrow X_3$ . Suppose that  $f$  is surjective and  $\mathcal{A}_1$ - $\mathcal{A}_2$  measurable and  $g$  is  $\mathcal{A}_1$ - $\mathcal{A}_3$  measurable and for each  $t \in X_3$ ,  $\{t\} \in \mathcal{A}_3$ . Then  $g$  is  $f^*\mathcal{A}_2$ - $\mathcal{A}_3$  measurable iff there exists a unique  $\phi : X_2 \rightarrow X_3$  such that  $\phi$  is  $\mathcal{A}_2$ - $\mathcal{A}_3$  measurable and  $g = \phi \circ f$ .

**Hint:** For each  $t \in X_3$ , set  $A_t = g^{-1}(\{t\}) \in f^*\mathcal{A}_2$  and choose  $B_t \in \mathcal{A}_2$  such that  $A_t = f^{-1}(B_t)$ . Set  $\phi(y) = t$  for  $y \in B_t \cap f(X_1)$  and  $t \in g(X_1)$ .

*Proof.* Suppose that there exists a unique  $\phi : X_2 \rightarrow X_3$  such that  $\phi$  is  $\mathcal{A}_2$  -  $\mathcal{A}_3$  measurable and  $g = \phi \circ f$ . Since  $f$  is  $f^*\mathcal{A}_2$  -  $\mathcal{A}_2$  measurable, we have that  $g = \phi \circ f$  is  $f^*\mathcal{A}_2$ - $\mathcal{A}_3$  measurable.

Conversely, suppose that  $g$  is  $f^*\mathcal{A}_2$ - $\mathcal{A}_3$  measurable.

- **(Existence)**

For each  $t \in X_3$ , set  $A_t = g^{-1}(\{t\}) \in f^*\mathcal{A}_2$  and choose  $B_t \in \mathcal{A}_2$  such that  $A_t = f^{-1}(B_t)$ .

Note that

- for each  $t \in g(X_1)$ , there exists  $x \in A_t$  such that  $g(x) = t$ . Hence  $f(x) \in B_t$ .
- for  $t_1, t_2 \in g(X_1)$ ,  $t_1 \neq t_2$  implies that

$$\begin{aligned} f^{-1}(B_{t_1} \cap B_{t_2}) &= A_{t_1} \cap A_{t_2} \\ &= g^{-1}(\{t_1\} \cap \{t_2\}) \\ &= \emptyset \end{aligned}$$

and since  $f$  is surjective,

$$\begin{aligned} B_{t_1} \cap B_{t_2} &= f(f^{-1}(B_{t_1} \cap B_{t_2})) \\ &= f(\emptyset) \\ &= \emptyset \end{aligned}$$

- we have that

$$\begin{aligned} f^{-1}\left(\bigcup_{t \in g(X_1)} B_t\right) &= \bigcup_{t \in g(X_1)} A_t \\ &= \bigcup_{t \in g(X_1)} g^{-1}(\{t\}) \\ &= g^{-1}(g(X_1)) \\ &= X_1 \end{aligned}$$

Since  $f$  is surjective, we have that

$$\begin{aligned} X_2 &= f(X_1) \\ &= f\left(f^{-1}\left(\bigcup_{t \in g(X_1)} B_t\right)\right) \\ &= \bigcup_{t \in g(X_1)} B_t \end{aligned}$$

Therefore,

- for each  $t \in g(X_1)$ ,  $B_t \neq \emptyset$
- $(A_t)_{t \in g(X_1)}$  is a partition of  $X_1$
- $(B_t)_{t \in g(X_1)}$  is a partition of  $X_2$

Define  $\phi : X_2 \rightarrow X_3$  by  $\phi(y) = t$  for  $t \in g(X_1)$  and  $y \in B_t$ . Then the previous observations imply that  $\phi$  is well defined and  $\phi(X_2) = g(X_1)$ . Since for each  $t \in g(X_1)$  and  $x \in A_t$ ,  $f(x) \in B_t$  and  $g(x) = t$ , we have that  $\phi \circ f(x) = t = g(x)$ . So  $\phi \circ f = g$ .

To show that  $\phi$  is measurable, let  $C \in \mathcal{A}_3$ . Choose  $B \in \mathcal{A}_2$  such that  $g^{-1}(C) = f^{-1}(B)$ . Let  $y \in \phi^{-1}(C) \subset X_2$ . Set  $t = \phi(y) \in C$  and choose  $x \in X_1$  such that  $y = f(x)$ . Since

$$\begin{aligned} g(x) &= \phi \circ f(x) \\ &= \phi(y) \\ &= t \\ &\in C \end{aligned}$$

$x \in g^{-1}(C) = f^{-1}(B)$ . Therefore,  $y = f(x) \in B$ . So  $\phi^{-1}(C) \subset B$ .

Let  $y \in B$ . Choose  $x \in X_1$  such that  $f(x) = y$ . Then  $x \in f^{-1}(B) = g^{-1}(C)$ . So

$$\begin{aligned} \phi(y) &= \phi \circ f(x) \\ &= g(x) \\ &\in C \end{aligned}$$

and  $y \in \phi^{-1}(C)$ . So  $B \subset \phi^{-1}(C)$ . Hence  $\phi^{-1}(C) = B \in \mathcal{A}_2$  and  $\phi$  is  $\mathcal{A}_2$  -  $\mathcal{A}_3$  measurable.

• **(Uniqueness)**

Let  $\psi : X_2 \rightarrow X_3$ . Suppose that  $\psi$  is  $\mathcal{A}_2$ - $\mathcal{A}_3$  measurable and  $g = \psi \circ f$ . Let  $y \in X_2$ . Then there exists  $x \in X_1$  such that  $y = f(x)$ . Then

$$\begin{aligned} \psi(y) &= \psi \circ f(x) \\ &= g(x) \\ &= \phi \circ f(x) \\ &= \phi(y) \end{aligned}$$

So  $\psi = \phi$ .

□

**Exercise 2.3.26.** Let  $(X_1, \mathcal{A}_1)$ ,  $(X_2, \mathcal{A}_2)$  and  $(X_3, \mathcal{A}_3)$  be measurable spaces and  $f : X_1 \rightarrow X_2$  and  $g : X_1 \rightarrow X_3$ . Suppose that  $f$  is  $\mathcal{A}_1$ - $\mathcal{A}_2$  measurable and  $g$  is  $\mathcal{A}_1$ - $\mathcal{A}_3$  measurable and for each  $t \in X_3$ ,  $\{t\} \in \mathcal{A}_3$ . Then  $g$  is  $f^*\mathcal{A}_2$ - $\mathcal{A}_3$  measurable iff there exists a unique  $\phi : f(X_1) \rightarrow X_3$  such that  $\phi$  is  $\mathcal{A}_2 \cap f(X_1)$  -  $\mathcal{A}_3$  measurable and  $g = \phi \circ f$ .

*Proof.* A previous exercise implies that  $f : X_1 \rightarrow f(X_1)$  is  $\mathcal{A}_1$  -  $\mathcal{A}_2 \cap f(X_1)$  measurable. Now apply the previous exercise.  $\square$

## 2.4. Subspace Sigma Algebras.

**Definition 2.4.1.** Let  $X$  be a set,  $\mathcal{C} \subset \mathcal{P}(X)$  and  $E \subset X$ . We define  $\mathcal{C} \cap E \subset \mathcal{P}(X)$  by

$$\mathcal{C} \cap E = \{S \cap E : S \in \mathcal{C}\}$$

**Exercise 2.4.2.** Let  $X$  be a set,  $\mathcal{A}$  a  $\sigma$ -algebra on  $X$  and  $E \subset X$ . Then  $\mathcal{A} \cap E$  is a  $\sigma$ -algebra on  $E$ .

*Proof.*

- (1) Clearly  $\emptyset, E \in \mathcal{A} \cap E$ .
- (2) Let  $B \in \mathcal{A} \cap E$ . Then there exists  $A \in \mathcal{A}$  such that  $B = A \cap E$ . Since  $A^c \in \mathcal{A}$ , we have that

$$\begin{aligned} E \setminus B &= E \cap (A \cap E)^c \\ &= E \cap (A^c \cup E^c) \\ &= (E \cap A^c) \cup (E \cap E^c) \\ &= A^c \cap E \\ &\in \mathcal{A} \cap E \end{aligned}$$

- (3) Let  $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A} \cap E$ . Then for each  $n \in \mathbb{N}$ , there exists  $A_n \in \mathcal{A}$  such that  $B_n = A_n \cap E$ . So  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ . Hence

$$\begin{aligned} \bigcup_{n \in \mathbb{N}} (B_n) &= \bigcup_{n \in \mathbb{N}} (A_n \cap E) \\ &= \left( \bigcup_{n \in \mathbb{N}} A_n \right) \cap E \\ &\in \mathcal{A} \cap E \end{aligned}$$

□

**Exercise 2.4.3.** Let  $X$  be a set,  $\mathcal{C} \subset \mathcal{P}(X)$  and  $A \subset X$ . Let  $\sigma_A(\mathcal{C} \cap A)$  be the  $\sigma$ -algebra on  $A$  generated by  $\mathcal{C} \cap A$ . Define

$$\mathcal{G} = \{S \subset X : S \cap A \in \sigma_A(\mathcal{C} \cap A)\}$$

Then  $\mathcal{G}$  is a  $\sigma$ -algebra on  $X$ .

*Proof.*

- (1) Clearly  $\emptyset, X \in \mathcal{G}$ .
- (2) Let  $S \in \mathcal{G}$ . Then  $S \cap A \in \sigma_A(\mathcal{C} \cap A)$ . Hence

$$\begin{aligned} S^c \cap A &= A \setminus (S \cap A) \\ &\in \sigma_A(\mathcal{C} \cap A) \end{aligned}$$

So  $S^c \in \mathcal{G}$ .

- (3) Let  $(S_n)_{n \in \mathbb{N}} \subset \mathcal{G}$ . Then for each  $n \in \mathbb{N}$ ,  $S_n \cap A \in \sigma_A(\mathcal{C} \cap A)$ . Thus

$$\left( \bigcup_{n \in \mathbb{N}} S_n \right) \cap A = \bigcup_{n \in \mathbb{N}} (S_n \cap A) \in \sigma_A(\mathcal{C} \cap A)$$

Thus  $\bigcup_{n \in \mathbb{N}} S_n \in \mathcal{G}$ .

□

**Exercise 2.4.4.** Let  $X$  be a set,  $\mathcal{C} \subset \mathcal{P}(X)$  and  $A \subset X$ . Then

$$\sigma(\mathcal{C}) \cap A = \sigma_A(\mathcal{C} \cap A)$$

*Proof.* Clearly  $\mathcal{C} \cap A \subset \sigma(\mathcal{C}) \cap A$ . A previous exercise tells us that  $\sigma(\mathcal{C}) \cap A$  is a  $\sigma$ -algebra on  $A$ . Thus  $\sigma_A(\mathcal{C} \cap A) \subset \sigma(\mathcal{C}) \cap A$ .

Conversely, from the previous exercise, we have that  $\mathcal{G} = \{S \subset X : S \cap A \in \sigma_A(\mathcal{C} \cap A)\}$  is a  $\sigma$ -algebra on  $X$ . Clearly  $\mathcal{C} \subset \mathcal{G}$ . Then  $\sigma(\mathcal{C}) \subset \mathcal{G}$ . The definition of  $\mathcal{G}$  implies that  $\sigma(\mathcal{C}) \cap A \subset \sigma_A(\mathcal{C} \cap A)$ . Hence  $\sigma(\mathcal{C}) \cap A = \sigma_A(\mathcal{C} \cap A)$ . □



## 2.5. Product Sigma Algebras.

**Definition 2.5.1.** Let  $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  be a collection of measurable spaces. We define the **product  $\sigma$ -algebra** on  $\prod_{\alpha \in A} X_\alpha$ , denoted by  $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha$ , by

$$\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\pi_\alpha : \alpha \in A)$$

**Exercise 2.5.2.** Let  $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  be a collection of measurable spaces and for each  $\alpha \in A$ ,  $\mathcal{E}_\alpha \subset \mathcal{A}_\alpha$ . Suppose that for each  $\alpha \in A$ ,  $\mathcal{A}_\alpha = \sigma(\mathcal{E}_\alpha)$ . Then

$$\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{E}_\alpha)$$

**Hint:** set  $\mathcal{G} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{E}_\alpha\}$  and for  $\alpha \in A$ , consider the pushforward  $\sigma$ -algebra on  $X_\alpha$ ,  $\pi_{\alpha*}\sigma(\mathcal{G})$

*Proof.* Set

- $\mathcal{F} = \{\pi_\alpha^{-1}(V_\alpha) : \alpha \in A \text{ and } V_\alpha \in \mathcal{A}_\alpha\}$
- $\mathcal{G} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{E}_\alpha\}$

Clearly,  $\mathcal{G} \subset \mathcal{F}$ . By definition,  $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{F})$ . Therefore,

$$\begin{aligned} \sigma(\mathcal{G}) &\subset \sigma(\mathcal{F}) \\ &= \bigotimes_{\alpha \in A} \mathcal{A}_\alpha \end{aligned}$$

Let  $\alpha \in A$ . By definition, for each  $V \subset X_\alpha$ ,  $V \in \pi_{\alpha*}\sigma(\mathcal{G})$  iff  $\pi_\alpha^{-1}(V) \in \sigma(\mathcal{G})$ . Thus  $\mathcal{E}_\alpha \subset \pi_\alpha^*\sigma(\mathcal{G})$  which implies that

$$\begin{aligned} \mathcal{A}_\alpha &= \sigma(\mathcal{E}_\alpha) \\ &\subset \pi_\alpha^*\sigma(\mathcal{G}) \end{aligned}$$

Since  $\alpha \in A$  is arbitrary,  $\mathcal{F} \subset \sigma(\mathcal{G})$ . Hence

$$\begin{aligned} \bigotimes_{\alpha \in A} \mathcal{A}_\alpha &= \sigma(\mathcal{F}) \\ &\subset \sigma(\mathcal{G}) \end{aligned}$$

Thus  $\sigma(\mathcal{G}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$ . □

**Exercise 2.5.3.** Let  $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  be a collection of measurable spaces. Define

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} B_\alpha : \text{for each } \alpha \in A, B_\alpha \in \mathcal{A}_\alpha \right\}$$

If  $A$  is countable, then  $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{B})$ .

*Proof.* Suppose that  $A$  is countable. Set  $\mathcal{C} = \{\pi_\alpha^{-1}(B_\alpha) : \alpha \in A, B_\alpha \in \mathcal{A}_\alpha\}$ . By definition,  $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{C})$ . Let  $\alpha \in A$  and  $B_\alpha \in \mathcal{A}_\alpha$ . For  $\beta \in A$ , set

$$C_\beta = \begin{cases} B_\beta & \beta = \alpha \\ X_\beta & \beta \neq \alpha \end{cases}$$

Then

$$\begin{aligned}\pi_\alpha^{-1}(B_\alpha) &= \prod_{\beta \in A} C_\beta \\ &\in \mathcal{B}\end{aligned}$$

So  $\mathcal{C} \subset \mathcal{B}$  and

$$\begin{aligned}\bigotimes_{\alpha \in A} \mathcal{A}_\alpha &= \sigma(\mathcal{C}) \\ &\subset \sigma(\mathcal{B})\end{aligned}$$

For each  $\alpha \in A$ , let  $B_\alpha \in \mathcal{A}_\alpha$ . Since  $A$  is countable, we have that

$$\begin{aligned}\prod_{\alpha \in A} B_\alpha &= \bigcap_{\alpha \in A} \pi_\alpha^{-1}(B_\alpha) \\ &\in \sigma(\mathcal{C})\end{aligned}$$

Thus  $\mathcal{B} \subset \sigma(\mathcal{C})$  and

$$\begin{aligned}\sigma(\mathcal{B}) &\subset \sigma(\mathcal{C}) \\ &= \bigotimes_{\alpha \in A} \mathcal{A}_\alpha\end{aligned}$$

Hence  $\sigma(\mathcal{B}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$ . □

**Exercise 2.5.4.** Let  $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  be a collection of measurable spaces and for each  $\alpha \in A$ ,  $\mathcal{E}_\alpha \subset \mathcal{A}_\alpha$ . Suppose that for each  $\alpha \in A$ ,  $X_\alpha \in \mathcal{E}_\alpha$  and  $\mathcal{A}_\alpha = \sigma(\mathcal{E}_\alpha)$ . Set

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} E_\alpha : \text{for each } \alpha \in A, E_\alpha \in \mathcal{E}_\alpha \right\}$$

If  $A$  is countable, then  $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{B})$ .

*Proof.* Suppose that  $A$  is countable. Set  $\mathcal{C} = \left\{ (\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{E}_\alpha) \right\}$ . A previous exercise implies that  $\sigma(\mathcal{C}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$ . Let  $\alpha \in A$  and  $E_\alpha \in \mathcal{E}_\alpha$ . For  $\beta \in A$ , set

$$C_\beta = \begin{cases} E_\beta & \beta = \alpha \\ X_\beta & \beta \neq \alpha \end{cases}$$

Then for each  $\beta \in A$ ,  $C_\beta \in \mathcal{E}_\beta$  and

$$\begin{aligned}\pi_\alpha^{-1}(E_\alpha) &= \prod_{\beta \in A} C_\beta \\ &\in \mathcal{B}\end{aligned}$$

So  $\mathcal{C} \subset \mathcal{B}$  and

$$\begin{aligned}\bigotimes_{\alpha \in A} \mathcal{A}_\alpha &= \sigma(\mathcal{C}) \\ &\subset \sigma(\mathcal{B})\end{aligned}$$

For each  $\alpha \in A$ , let  $E_\alpha \in \mathcal{E}_\alpha$ . Since  $A$  is countable, we have that

$$\begin{aligned} \prod_{\alpha \in A} E_\alpha &= \bigcap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha) \\ &\in \sigma(\mathcal{C}) \end{aligned}$$

Thus  $\mathcal{B} \subset \sigma(\mathcal{C})$  and

$$\begin{aligned} \sigma(\mathcal{B}) &\subset \sigma(\mathcal{C}) \\ &\subset \bigotimes_{\alpha \in A} \mathcal{A}_\alpha \end{aligned}$$

Hence  $\sigma(\mathcal{B}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$ . □

**Exercise 2.5.5.** Let  $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$  be a collection of topological spaces. Then

(1)

$$\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) \subset \mathcal{B}\left(\prod_{\alpha \in A} X_\alpha\right)$$

(2) if  $A$  is countable and for each  $\alpha \in A$ ,  $X_\alpha$  is second-countable, then

$$\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) = \mathcal{B}\left(\prod_{\alpha \in A} X_\alpha\right)$$

*Proof.* Set  $X = \prod_{j=1}^n X_j$  and denote the product topology on  $X$  by  $\mathcal{T}_X$ .

(1) By definition,  $\mathcal{B}(X) = \sigma(\mathcal{T}_X)$  and for each  $\alpha \in A$ ,  $X_\alpha \in \mathcal{T}_\alpha$  and  $\mathcal{B}(X_\alpha) = \sigma(\mathcal{T}_\alpha)$ . Set

$$\mathcal{E} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{T}_\alpha\}$$

A previous exercise implies that  $\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) = \sigma(\mathcal{E})$ . Since  $\mathcal{E} \subset \mathcal{T}_X$ , we have that

$$\begin{aligned} \bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) &= \sigma(\mathcal{E}) \\ &\subset \sigma(\mathcal{T}_X) \\ &= \mathcal{B}(X) \end{aligned}$$

(2) Suppose that  $A$  is countable and for each  $\alpha \in A$ ,  $X_\alpha$  is second-countable. Then for each  $\alpha \in A$ , there exists  $\mathcal{B}_\alpha \subset \mathcal{T}_\alpha$  such that  $\mathcal{B}_\alpha$  is a countable basis for  $\mathcal{T}_\alpha$ . Set

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} U_\alpha : \text{there exists } J \subset A \text{ such that } \#J < \infty, \right. \\ \left. \text{for each } \alpha \in J, U_\alpha \in \mathcal{B}_\alpha \text{ and for each } \alpha \in J^c, U_\alpha = X_\alpha \right\}$$

Since  $A$  is countable,  $\mathcal{B}$  is a countable basis for  $\mathcal{T}_X$ . An exercise in the section on  $\sigma$ -algebras implies that  $\mathcal{B}(X) = \sigma(\mathcal{B})$ . The previous exercise implies that  $\mathcal{B} \subset \bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha)$ . Hence

$$\begin{aligned} \mathcal{B}(X) &= \sigma(\mathcal{B}) \\ &\subset \bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) \end{aligned}$$

□

**Exercise 2.5.6.** Let  $(X, \mathcal{A})$  be a measurable space,  $(Y_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  a collection of measurable spaces and  $f : X \rightarrow \prod_{\alpha \in A} Y_\alpha$ . Then  $f$  is  $(\mathcal{A}, \bigotimes_{\alpha \in A} \mathcal{A}_\alpha)$ -measurable iff for each  $\alpha \in A$ ,  $\pi_\alpha \circ f$  is  $(\mathcal{A}, \mathcal{A}_\alpha)$ -measurable.

*Proof.* Immediate by a previous exercise about the initial  $\sigma$ -algebra. □

**Definition 2.5.7.** Let  $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  and  $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$  be collections of measurable spaces and  $(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} Y_\alpha^{X_\alpha}$ , i.e. for each  $\alpha \in A$ ,  $f_\alpha : X_\alpha \rightarrow Y_\alpha$ . Set  $X = \prod_{\alpha \in A} X_\alpha$  and  $Y = \prod_{\alpha \in A} Y_\alpha$ . We define  $\prod_{\alpha \in A} f_\alpha : X \rightarrow Y$  by  $\prod_{\alpha \in A} f_\alpha((x_\alpha)_{\alpha \in A}) = (f_\alpha(x_\alpha))_{\alpha \in A}$ .

**Exercise 2.5.8.** Let  $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$  and  $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$  be collections of measurable spaces and  $(f_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} Y_\alpha^{X_\alpha}$ , i.e. for each  $\alpha \in A$ ,  $f_\alpha : X_\alpha \rightarrow Y_\alpha$ . Set  $X = \prod_{\alpha \in A} X_\alpha$  and  $Y = \prod_{\alpha \in A} Y_\alpha$ . If for each  $\alpha \in A$ ,  $f_\alpha$  is  $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ -measurable, then  $\prod_{\alpha \in A} f_\alpha$  is  $(\bigotimes_{\alpha \in A} \mathcal{A}_\alpha, \bigotimes_{\alpha \in A} \mathcal{B}_\alpha)$ -measurable.

*Proof.* Suppose that for each  $\alpha \in A$ ,  $f_\alpha$  is  $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ -measurable. Set  $f = \prod_{\alpha \in A} f_\alpha$ . Denote the  $\alpha$ -th projection maps on  $X$  and  $Y$  by  $\pi_\alpha^X$  and  $\pi_\alpha^Y$  respectively. Let  $\alpha \in A$  and  $x \in X$ . Then

$$\begin{aligned} \pi_\alpha^Y \circ f(x) &= (f(x))_\alpha \\ &= f_\alpha(x_\alpha) \\ &= f_\alpha \circ \pi_\alpha^X(x) \end{aligned}$$

Since  $x \in X$  are arbitrary,  $\pi_\alpha^Y \circ f = f_\alpha \circ \pi_\alpha^X$ . Since  $f_\alpha \circ \pi_\alpha^X$  is  $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ -measurable, we have that  $\pi_\alpha^Y \circ f$  is  $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ -measurable. Since  $\alpha \in A$  is arbitrary, a previous exercise implies that  $f$  is  $(\bigotimes_{\alpha \in A} \mathcal{A}_\alpha, \bigotimes_{\alpha \in A} \mathcal{B}_\alpha)$ -measurable. □

**Definition 2.5.9.** Let  $X, Y$  be sets,  $x \in X$  and  $y \in Y$ . We define the **slice maps at  $x$  and  $y$** , denoted  $\iota_X^y : X \rightarrow X \times Y$  and  $\iota_Y^x : Y \rightarrow X \times Y$  respectively, by  $\iota_X^y(\cdot) = (\cdot, y)$  and  $\iota_Y^x(\cdot) = (x, \cdot)$  respectively.

**Exercise 2.5.10.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces,  $x \in X$  and  $y \in Y$ . Then  $\iota_X^y$  is  $(\mathcal{A}, \mathcal{A} \otimes \mathcal{B})$ -measurable and  $\iota_Y^x$  is  $(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$ -measurable.

*Proof.* Since  $\pi_1 \circ \iota_X^y = \text{id}_X$  and  $\pi_2 \circ \iota_X^y$  is constant, we have that  $\pi_1 \circ \iota_X^y = \text{id}_X$  is  $(\mathcal{A}, \mathcal{A})$ -measurable and  $\pi_2 \circ \iota_X^y$  is  $(\mathcal{B}, \mathcal{B})$ -measurable. Since  $\mathcal{A} \otimes \mathcal{B} = \sigma_{X \times Y}(\pi_1, \pi_2)$ , an exercise in the section on measurable functions implies that  $\iota_X^y$  is  $(\mathcal{A}, \mathcal{A} \otimes \mathcal{B})$ -measurable. Similarly,  $\iota_Y^x$  is  $(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$ -measurable. □

**Definition 2.5.11.** Let  $X, Y$ , and  $Z$  be sets,  $E \subset X \times Y$ ,  $f : X \times Y \rightarrow Z$ ,  $x \in X$  and  $y \in Y$ . Then

- we define the **sections of  $E$  at  $x$  and  $y$** , denoted  $E_x$  and  $E^y$  respectively, by  $E_x = \{y \in Y : (x, y) \in E\}$  and  $E^y = \{x \in X : (x, y) \in E\}$  respectively
- we define the **sections of  $f$  at  $x$  and  $y$** , denoted  $f_x : Y \rightarrow Z$  and  $f^y : X \rightarrow Z$  respectively, by  $f_x(\cdot) = f(x, \cdot)$  and  $f^y(\cdot) = f(\cdot, y)$  respectively

**Exercise 2.5.12.** Let  $(X, \mathcal{A}), (Y, \mathcal{B})$  be measurable spaces,  $E \in \mathcal{A} \otimes \mathcal{B}$ ,  $x \in X$  and  $y \in Y$ . Then  $E_x \in \mathcal{B}$  and  $E^y \in \mathcal{A}$ .

*Proof.* Since  $\iota_Y^x$  is  $(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$ -measurable, we have that

$$\begin{aligned} E_x &= (\iota_Y^x)^{-1}(E) \\ &\in \mathcal{B} \end{aligned}$$

Similarly,  $E^y \in \mathcal{A}$ . □

**Exercise 2.5.13.** Let  $(X, \mathcal{A})$ ,  $(Y, \mathcal{B})$ ,  $(Z, \mathcal{C})$  be measurable spaces,  $f : X \times Y \rightarrow Z$ ,  $x \in X$  and  $y \in Y$ . Suppose that  $f$  is  $(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$ -measurable. Then  $f_x$  is  $(\mathcal{B}, \mathcal{C})$ -measurable and  $f^y$  is  $(\mathcal{A}, \mathcal{C})$ -measurable.

*Proof.* Since  $\iota_Y^x$  is  $(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$ -measurable,  $f$  is  $(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$ -measurable and  $f_x = f \circ \iota_Y^x$ , we have that  $f_x$  is  $(\mathcal{B}, \mathcal{C})$ -measurable. Similarly,  $f^y$  is  $(\mathcal{A}, \mathcal{C})$ -measurable. □

**Exercise 2.5.14.** Let  $X_1, X_2, Y_1, Y_2$  be topological spaces and  $f_1 : X_1 \rightarrow Y_1$ ,  $f_2 : X_2 \rightarrow Y_2$ . If  $f_1$  and  $f_2$  are open, then  $f_1 \times f_2$  is open.

*Proof.* Let  $A_1 \subset X_1, A_2 \subset X_2$  be open. Then  $f_1 \times f_2(A_1 \times A_2) = f_1(A_1) \times f_2(A_2)$  which is open in  $Y_1 \times Y_2$ . Since  $\mathcal{B} = \{A_1 \times A_2 : A_1 \subset X_1 \text{ and } A_2 \subset X_2 \text{ are open}\}$  is a basis for the product topology on  $X_1 \times X_2$ , an exercise in the section on continuous maps implies that  $f_1 \times f_2$  is open. □

**Exercise 2.5.15.** Let  $X$  and  $Y$  be topological spaces and  $U \subset X \times Y$  open. Then for each  $(x_0, y_0) \in U$ ,  $U^{x_0}$  and  $U^{y_0}$  are open.

*Proof.* Let  $(x_0, y_0) \in U$ . Define  $\phi : X \rightarrow X \times Y$  by  $\phi(x) = (x, y_0)$ . Since  $\pi_X \circ \phi = \text{id}_X$  and  $\pi_Y \circ \phi$  is constant,  $\pi_X \circ \phi$  and  $\pi_Y \circ \phi$  are continuous. Therefore,  $\phi$  is continuous. Then  $U^{y_0}$  is open since  $U$  is open and  $\phi^{-1}(U) = U^{y_0}$ . Similarly,  $U_{x_0}$  is open. □

**Exercise 2.5.16.** Let  $X, Y$  and  $Z$  be topological spaces,  $U \subset X \times Y$  open and  $f : U \rightarrow Z$ . Equip  $U$  with the subspace topology. Suppose that  $f$  is continuous. Let  $(x_0, y_0) \in U$ . Equip  $U_{x_0}$  and  $U^{y_0}$  with the subspace topology. Then  $f_{x_0} : U_{x_0} \rightarrow Z$  and  $f^{y_0} : U^{y_0} \rightarrow Z$  are continuous.

*Proof.* Let  $(x_0, y_0) \in U$ . Let  $V \subset Z$ . Suppose that  $V$  is open. Continuity of  $f$  implies that  $f^{-1}(V)$  is open in  $U$ . Since  $U$  is open in  $X \times Y$ ,  $f^{-1}(V)$  is open in  $X \times Y$ . A previous exercise in the section on product sets implies that  $(f^{y_0})^{-1}(V) = (f^{-1}(V))^{y_0}$ . The previous exercise implies that  $(f^{-1}(V))^{y_0}$  is open in  $X$ . So  $(f^{y_0})^{-1}(V)$  is open in  $X$ . Since  $(f^{y_0})^{-1}(V) \subset U^{y_0}$ ,  $(f^{y_0})^{-1}(V)$  is open in  $U^{y_0}$ . Thus  $f^{y_0} : U^{y_0} \rightarrow Z$  is continuous. Similarly,  $f_{x_0} : U_{x_0} \rightarrow Z$  is continuous. □

## 2.6. Quotient Sigma Algebras.

**Definition 2.6.1.** Let  $X, Y$  be sets,  $\sim$  an equivalence relation on  $X$  and  $f : X \rightarrow Y$ . Then  $f$  is said to be **invariant under**  $\sim$  if for each  $a, b \in X$ ,  $\bar{a} = \bar{b}$  implies that  $f(a) = f(b)$ .

**Exercise 2.6.2.** Let  $(X, \mathcal{A})$ ,  $(Y, \mathcal{B})$  be topological spaces,  $\sim$  an equivalence relation on  $X$ ,  $\pi : X \rightarrow X/\sim$  the projection map and  $f : X \rightarrow Y$  measurable. If  $f$  is invariant under  $\sim$ , then there exists a unique  $\bar{f} : X/\sim \rightarrow Y$  such that

- (1)  $\bar{f} \circ \pi = f$
- (2)  $\bar{f}$  is  $\mathcal{A}/\pi_*\mathcal{A}$  measurable

*Proof.* Suppose that  $f$  is invariant under  $\sim$ . Define  $\bar{f} : X/\sim \rightarrow Y$  by  $\bar{f}(\bar{x}) = f(x)$ . By assumption, for each  $a, b \in X$ ,  $\bar{a} = \bar{b}$  implies that  $f(a) = f(b)$ . Thus  $\bar{f}$  is well defined. By construction,  $f = \bar{f} \circ \pi$ . Let  $V \in \mathcal{B}$ . Measurability of  $f$  implies that  $f^{-1}(V) \in \mathcal{A}$ . Since

$$\begin{aligned} f^{-1}(V) &= \pi^{-1}(\bar{f}^{-1}(V)) \\ &\in \mathcal{A} \end{aligned}$$

by definition of  $\pi_*\mathcal{A}$ ,  $\bar{f}^{-1}(V) \in \pi_*\mathcal{A}$ . So  $\bar{f}$  is  $\mathcal{A}/\pi_*\mathcal{A}$  measurable. □

## 2.7. Dynkin's Lemma.

**Definition 2.7.1.** Let  $X$  be a set and  $\mathcal{P} \subset \mathcal{P}(X)$ . Then  $\mathcal{P}$  is said to be a  $\pi$ -system on  $X$  if for each  $A, B \in \mathcal{P}$ ,  $A \cap B \in \mathcal{P}$ .

**Definition 2.7.2.** Let  $X$  be a set and  $\mathcal{L} \subset \mathcal{P}(X)$ . Then  $\mathcal{L}$  is said to be a  $\lambda$ -system on  $X$  if

- (1)  $\mathcal{L} \neq \emptyset$
- (2) for each  $A \in \mathcal{L}$ ,  $A^c \in \mathcal{L}$
- (3) for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}$ , if  $(A_n)_{n \in \mathbb{N}}$  is disjoint, then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}$

**Exercise 2.7.3.** Let  $X$  be a set and  $\mathcal{L}$  a  $\lambda$ -system on  $X$ . Then

- (1)  $X, \emptyset \in \mathcal{L}$

*Proof.* Straightforward. □

**Definition 2.7.4.** Let  $X$  be a set and  $\mathcal{C} \subset \mathcal{P}(X)$ . Put

$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(X) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the  $\lambda$ -system on  $X$  generated by  $\mathcal{C}$ ,  $\lambda(\mathcal{C})$ , to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L} \in \mathcal{S}} \mathcal{L}$$

**Exercise 2.7.5.** Let  $X$  be a set and  $\mathcal{A} \subset \mathcal{P}(X)$ . If  $\mathcal{A}$  is a  $\lambda$ -system and  $\mathcal{A}$  is a  $\pi$ -system, then  $\mathcal{A}$  is a  $\sigma$ -algebra.

*Proof.* Suppose that  $\mathcal{A}$  is a  $\lambda$ -system and  $\mathcal{A}$  is a  $\pi$ -system. Then we need only verify the third axiom in the definition of a  $\sigma$ -algebra. Let  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Define  $B_1 = A_1$  and for  $n \geq 2$ , define  $B_n = A_n \cap \left( \bigcup_{k=1}^{n-1} A_k \right)^c = A_n \cap \left( \bigcap_{k=1}^{n-1} A_k^c \right) \in \mathcal{A}$ . Then  $(B_n)_{n \in \mathbb{N}}$  is disjoint and therefore  $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{A}$ . □

### Theorem 2.7.6. Dynkin's Lemma:

Let  $X$  be a set,  $\mathcal{P}$  be a  $\pi$ -system on  $X$  and  $\mathcal{L}$  a  $\lambda$ -system on  $X$ . Then

- (1)  $\mathcal{P} \subset \mathcal{L}$  implies that  $\sigma(\mathcal{P}) \subset \mathcal{L}$
- (2)  $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

**Exercise 2.7.7.** Define  $\mathcal{P} \subset \mathcal{B}(\mathbb{R})$  by

$$\mathcal{P} = \{(a, b] : a, b \in \mathbb{R}\} \cup \{\emptyset, X\}$$

Then  $\mathcal{P}$  is a  $\pi$ -system on  $X$ .

*Proof.* Let  $a_1, a_2, b_1, b_2 \in \mathbb{R}$ . Then

$$\begin{aligned} (a_1, b_1] \cap (a_2, b_2] &= (a_2, b_1] \\ &\in \mathcal{P} \end{aligned}$$

□

## 2.8. Borel Spaces.

**Exercise 2.8.1.** For each  $x \in [0, 1]$ , there exists  $(x_n)_{n \in \mathbb{N}} \subset \{0, 1\}^{\mathbb{N}}$  such that  $x = \sum_{j \in \mathbb{N}} x_j 2^{-j}$ .

**Hint:** Set  $x_1 = \begin{cases} 0, & x < 1/2 \\ 1, & x \geq 1/2 \end{cases}$  and proceed inductively.

*Proof.* Let  $x \in [0, 1]$ . Set

$$x_1 = \begin{cases} 0, & x < 1/2 \\ 1, & x \geq 1/2 \end{cases}$$

and for  $j \geq 2$ , set

$$x_j = \begin{cases} 0, & x - \sum_{k=1}^{j-1} x_k 2^{-k} < 2^{-j} \\ 1, & x - \sum_{k=1}^{j-1} x_k 2^{-k} \geq 2^{-j} \end{cases}$$

Note that for each  $j \in \mathbb{N}$ ,  $x - \sum_{k=1}^j x_k 2^{-k} \in [0, 2^{-j}]$ . Hence  $x = \sum_{j \in \mathbb{N}} x_j 2^{-j}$  □

**Exercise 2.8.2.** Let  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subset \{0, 1\}^{\mathbb{N}}$ . Suppose that  $(x_n)_{n \in \mathbb{N}} \neq (y_n)_{n \in \mathbb{N}}$ . Set  $N = \min\{j \in \mathbb{N} : x_j \neq y_j\}$ . Suppose that  $x_N = 0$  and  $y_N = 1$ . Then

$$\sum_{j \in \mathbb{N}} x_j 2^{-j} = \sum_{j \in \mathbb{N}} y_j 2^{-j}$$

iff for each  $j \in \mathbb{N}$ ,  $j > N$  implies that  $x_j = 1$  and  $y_j = 0$ .

*Proof.* Suppose that

$$\sum_{j \in \mathbb{N}} x_j 2^{-j} = \sum_{j \in \mathbb{N}} y_j 2^{-j}$$

By definition of  $N$ , for each  $j \in \mathbb{N}$ ,  $j < N$  implies that  $x_j = y_j$ . Hence

$$\sum_{j=N} x_j 2^{-j} = \sum_{j=N} y_j 2^{-j}$$

Since  $x_N = 0$  and  $y_N = 1$ , we have that

$$\sum_{j=N+1} x_j 2^{-j} = 2^{-N} + \sum_{j=N+1} y_j 2^{-j}$$

Thus  $2^{-N} = \sum_{j=N+1} (x_j - y_j) 2^{-j}$ . For the sake of contradiction, suppose that there exists  $m > N$  and  $x_m \neq 1$  or  $y_m \neq 0$ . Then

$$\begin{aligned} 2^{-N} &= \sum_{j=N+1} (x_j - y_j) 2^{-j} \\ &< \sum_{j=N+1} (1 - 0) 2^{-j} \\ &= 2^{-N} \end{aligned}$$

which is a contradiction. Hence for each  $m \in \mathbb{N}$ ,  $m > N$  implies that  $x_m = 1$  and  $y_m = 0$ .



Conversely, suppose that for each  $j \in \mathbb{N}$ ,  $j > N$  implies that  $x_j = 1$  and  $y_j = 0$ . Then

$$\begin{aligned}
 \sum_{j \in \mathbb{N}} x_j 2^{-j} &= \sum_{j=1}^{N-1} x_j + \sum_{j \geq N+1} 2^{-j} \\
 &= \sum_{j=1}^{N-1} x_j + 2^{-N} \sum_{j \in \mathbb{N}} 2^{-j} \\
 &= \sum_{j=1}^{N-1} y_j + 2^{-N} \\
 &= \sum_{j=1}^N y_j 2^{-j} \\
 &= \sum_{j \in \mathbb{N}} y_j 2^{-j}
 \end{aligned}$$

□

**Definition 2.8.3.** We define  $Z \subset \{0, 1\}^{\mathbb{N}}$  by

$$Z = \left\{ (x_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}} : \#\{n \in \mathbb{N} : x_n = 0\} = \infty \right\} \cup \{(1, 1, 1, \dots)\}$$

**Exercise 2.8.4.** Consider  $\{0, 1\}^{\mathbb{N}}$  equipped with the product topology. Then  $Z \in \mathcal{B}(\{0, 1\}^{\mathbb{N}})$ .

**Hint:** Note that  $Z^c$  is countable.

*Proof.* Since the product of  $T_1$  spaces is  $T_1$ ,  $\{0, 1\}^{\mathbb{N}}$  is  $T_1$ . Since  $\{0, 1\}^{\mathbb{N}}$  is  $T_1$ , for each  $x \in Z^c$ ,  $\{x\}$  is closed. Since  $Z^c$  is countable, we have that

$$\begin{aligned}
 Z^c &= \bigcup_{x \in Z^c} \{x\} \\
 &\in \mathcal{B}(\{0, 1\}^{\mathbb{N}})
 \end{aligned}$$

Therefore  $Z \in \mathcal{B}(\{0, 1\}^{\mathbb{N}})$ . □

**Exercise 2.8.5.** Consider  $\{0, 1\}^{\mathbb{N}}$  equipped with the product topology. Define  $\phi : Z \rightarrow [0, 1]$  by

$$\phi((x_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} x_n 2^{-n}$$

Then

- (1)  $\phi$  is a bijection
- (2)  $\phi$  is  $(\mathcal{B}(Z), \mathcal{B}([0, 1]))$ -measurable
- (3)  $\phi^{-1}$  is  $(\mathcal{B}([0, 1]), \mathcal{B}(Z))$ -measurable

Then  $(Z, \mathcal{B}(Z))$  is isomorphic to  $([0, 1], \mathcal{B}([0, 1]))$ .

**Hint:**

- (1) straightforward
- (2) Weierstrass M-test.
- (3) Show  $\phi(\pi_j^{-1}(\{1\}) \cap Z) \in \mathcal{B}([0, 1])$  by showing it is a countable

*Proof.*

- (1) Let  $x \in [0, 1]$ . Then Exercise 2.8.1 implies that there exists  $(x_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$  such that  $x = \sum_{n \in \mathbb{N}} x_n 2^{-n}$ . If for each  $n \in \mathbb{N}$ ,  $x_n = 1$ , then  $(x_n)_{n \in \mathbb{N}} \in Z$ . Suppose that there exists  $n \in \mathbb{N}$  such that  $x_n = 0$ . If  $\#\{n \in \mathbb{N} : x_n = 0\} = \infty$ , then  $(x_n)_{n \in \mathbb{N}} \in Z$ . Suppose that  $\#\{n \in \mathbb{N} : x_n = 0\} < \infty$ . Set  $N = \max\{n \in \mathbb{N} : x_n = 0\}$ . Define  $(y_n)_{n \in \mathbb{N}} \in Z$  by

$$y_n = \begin{cases} x_n, & n \in \{1, \dots, N-1\} \\ 1, & n = N \\ 0, & n > N \end{cases}$$

Then Exercise 2.8.2 implies that  $\phi((y_n)_{n \in \mathbb{N}}) = x$ . Since  $x \in [0, 1]$  is arbitrary,  $\phi$  is surjective.

Let  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in Z$ . Suppose that  $(x_n)_{n \in \mathbb{N}} \neq (y_n)_{n \in \mathbb{N}}$ . If  $\phi((x_n)_{n \in \mathbb{N}}) = \phi((y_n)_{n \in \mathbb{N}})$ , then Exercise 2.8.2 implies that  $(x_n)_{n \in \mathbb{N}} \notin Z$  or  $(y_n)_{n \in \mathbb{N}} \notin Z$ , which is a contradiction. Hence  $\phi((x_n)_{n \in \mathbb{N}}) \neq \phi((y_n)_{n \in \mathbb{N}})$ . Since  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in Z$  are arbitrary,  $\phi$  is injective. So  $\phi$  is a bijection.

- (2) For  $n \in \mathbb{N}$ , define  $f_n : Z \rightarrow [0, 1]$  by  $f_n = 2^{-n} \pi_n$ . Then  $\phi = \sum_{n \in \mathbb{N}} \phi_n$  and  $\|f_n\|_{\infty} = 2^{-n}$ .

The Weierstrass M-test implies that  $\phi$  is continuous. Thus  $\phi$  is  $(\mathcal{B}(Z), \mathcal{B}([0, 1]))$ -measurable.

(3)

□

**Definition 2.8.6.** Let  $(X, \mathcal{A})$  be a measurable space. Then  $(X, \mathcal{A})$  is said to be a **Borel space** if there exists  $S \in \mathcal{B}(\mathbb{R})$  such that  $(X, \mathcal{A})$  is isomorphic to  $(S, \mathcal{B}(\mathbb{R}) \cap S)$ .

## 2.9. Limits of Sets.

**Definition 2.9.1.** Let  $X$  be a set and  $\mathcal{A} \subset \mathcal{P}(X)$ . We define

$$\inf \mathcal{A} = \bigcap_{A \in \mathcal{A}} A, \quad \sup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

**Definition 2.9.2.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  a sequence of subsets. We define

$$\liminf_{n \rightarrow \infty} A_n = \sup_{n \in \mathbb{N}} \left( \inf_{k \geq n} A_k \right), \quad \limsup_{n \rightarrow \infty} A_n = \inf_{n \in \mathbb{N}} \left( \sup_{k \geq n} A_k \right)$$

**Note 2.9.3.**

- (1)  $\liminf_{n \rightarrow \infty} A_n$  is the set of elements that are in all  $A_n$  except for finitely many.
- (2)  $\limsup_{n \rightarrow \infty} A_n$  is the set of elements that are in infinitely many  $A_n$ .

**Exercise 2.9.4.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  a sequence of subsets. Then

- (1)  $\liminf_{n \rightarrow \infty} A_n = \left\{ x \in X : \liminf_{n \rightarrow \infty} \chi_{A_n}(x) = 1 \right\}$
- (2)  $\limsup_{n \rightarrow \infty} A_n = \left\{ x \in X : \limsup_{n \rightarrow \infty} \chi_{A_n}(x) = 1 \right\}$

*Proof.*

- (1) Let  $x \in \liminf_{n \rightarrow \infty} A_n$ . Then there exists  $n^* \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ ,  $k \geq n^*$  implies that  $x \in A_k$ . So for each  $k \in \mathbb{N}$ ,  $k \geq n^*$  implies that  $\chi_{A_k}(x) = 1$ . Then  $\inf_{k \geq n^*} \chi_{A_k}(x) = 1$  and thus

$$1 = \sup_{n \in \mathbb{N}} \left( \inf_{k \geq n} \chi_{A_k}(x) \right) = \liminf_{n \rightarrow \infty} \chi_{A_n}(x)$$

Conversely, if  $1 = \liminf_{n \rightarrow \infty} \chi_{A_n}(x)$ , then choosing  $\epsilon = \frac{1}{2}$ , there exists  $n \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ ,  $k \geq n$  implies that  $\chi_{A_k}(x) > 1 - \epsilon$ . Hence for each  $k \in \mathbb{N}$ ,  $k \geq n$  implies that  $\chi_{A_k}(x) = 1$ . So for each  $k \in \mathbb{N}$ ,  $k \geq n$  implies that  $x \in A_k$ . So  $x \in \liminf_{n \rightarrow \infty} A_n$ .

- (2) Similar to (1).

□

**Exercise 2.9.5.** Let  $A_k = [0, \frac{k}{k+1})$ . Then

- (1)  $\inf_{k \geq n} A_k = [0, \frac{n}{n+1})$
- (2)  $\sup_{k \geq n} A_k = [0, 1)$
- (3)  $\liminf_{n \rightarrow \infty} A_n = [0, 1)$
- (4)  $\limsup_{n \rightarrow \infty} A_n = [0, 1)$

*Proof.* Straightforward.

□

**Exercise 2.9.6.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  a sequence of subsets. Then

$$\liminf_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} A_n$$

*Proof.* Let  $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$ . Then there exists  $n^* \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq n^*$ , then  $x \in A_k$ . Let  $n \in \mathbb{N}$ . Choose  $k = \max\{n^*, n\} \geq n^*$ . Then  $x \in A_k$ . Hence for each  $n \in \mathbb{N}$ , there exists  $k \in \mathbb{N}$  such that  $k \geq n$  and  $x \in A_k$ . So  $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ . Thus  $\liminf_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} A_n$ .  $\square$

**Definition 2.9.7.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  a sequence of subsets. If

$$\liminf_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} A_n$$

then we define

$$\lim_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} A_n$$

**Exercise 2.9.8.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  sequences of subsets. Suppose that for each  $n \in \mathbb{N}$ ,  $A_n \subset A_{n+1}$  and  $B_{n+1} \subset B_n$ . Then

$$(1) \lim_{n \rightarrow \infty} A_n = \sup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$$

$$(2) \lim_{n \rightarrow \infty} B_n = \inf_{n \in \mathbb{N}} B_n = \bigcap_{n=1}^{\infty} B_n$$

*Proof.*

(1) Let  $n \in \mathbb{N}$ . Then

$$\begin{aligned} \inf_{k \geq n} A_k &= \bigcap_{k=n}^{\infty} A_k \\ &= A_n \end{aligned}$$

Thus

$$\begin{aligned} \liminf_{n \rightarrow \infty} A_n &= \bigcup_{n=1}^{\infty} \inf_{k \geq n} A_k \\ &= \bigcup_{n=1}^{\infty} A_n \end{aligned}$$

In addition,

$$\begin{aligned} \sup_{n \geq k} A_k &= \bigcup_{k=n}^{\infty} A_k \\ &= \bigcup_{k=1}^{\infty} A_k \end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{n \rightarrow \infty} A_n &= \bigcap_{n=1}^{\infty} \inf_{k \geq n} A_k \\ &= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_k \\ &= \bigcup_{n=1}^{\infty} A_n\end{aligned}$$

So

$$\lim_{n \rightarrow \infty} A_n = \sup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$$

(2) Similar

□

**Exercise 2.9.9.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  a sequence of subsets and  $(A_{n_k})_{k \in \mathbb{N}}$  a subsequence of  $(A_n)_{n \in \mathbb{N}}$ . Then

- (1)  $\limsup_{k \rightarrow \infty} A_{n_k} \subset \limsup_{n \rightarrow \infty} (A_n)$
- (2)  $\liminf_{n \rightarrow \infty} A_n \subset \liminf_{k \rightarrow \infty} (A_{n_k})$

*Proof.*

- (1) The elements that are in  $A_{n_k}$  for infinitely many  $k$  are in  $A_n$  for infinitely many  $n$ .
- (2) Similar.

□

**Exercise 2.9.10.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  a sequence of subsets,  $(A_{n_k})_{k \in \mathbb{N}}$  a subsequence of  $(A_n)_{n \in \mathbb{N}}$  and  $A \subset X$ . If  $A_{n_k} \rightarrow A$ , then

$$\liminf_{n \rightarrow \infty} A_n \subset A \subset \limsup_{n \rightarrow \infty} A_n$$

*Proof.* The previous exercises tells us that

$$\begin{aligned}\liminf_{n \rightarrow \infty} A_n &\subset \liminf_{k \rightarrow \infty} A_{n_k} \\ &= A \\ &= \limsup_{k \rightarrow \infty} A_{n_k} \\ &\subset \limsup_{n \rightarrow \infty} A_n\end{aligned}$$

□

**Exercise 2.9.11.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  sequences of subsets. Suppose that for each  $n \in \mathbb{N}$ ,  $A_n \subset B_n$ . Then

- (1)  $\limsup_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} B_n$
- (2)  $\liminf_{n \rightarrow \infty} A_n \subset \liminf_{n \rightarrow \infty} B_n$

*Proof.*

- (1) Let  $x \in \limsup A_n$ . Then for infinitely many  $n \in \mathbb{N}$ ,  $x \in A_n \subset B_n$ . So for infinitely many  $n \in \mathbb{N}$ ,  $x \in B_n$ . Hence  $x \in \limsup_{n \rightarrow \infty} B_n$ . Therefore  $\limsup_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} B_n$ .
- (2) Similar.

□

**Exercise 2.9.12.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  a sequence of subsets. Then

- (1)  $\limsup_{n \rightarrow \infty} A_n = \left( \liminf_{n \rightarrow \infty} A_n^c \right)^c$
- (2)  $\liminf_{n \rightarrow \infty} A_n = \left( \limsup_{n \rightarrow \infty} A_n^c \right)^c$

*Proof.*

(1)

$$\begin{aligned} \left( \liminf_{n \rightarrow \infty} A_n^c \right)^c &= \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c \right)^c \\ &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \\ &= \limsup_{n \rightarrow \infty} A_n \end{aligned}$$

(2) Similar.

□

**Exercise 2.9.13.** For  $n \in \mathbb{N}$ , define

$$A_n = \left\{ \frac{m}{n} : m \in \mathbb{N} \right\}$$

Then

- (1)  $\liminf_{n \rightarrow \infty} A_n = \mathbb{N}$
- (2)  $\limsup_{n \rightarrow \infty} A_n = \mathbb{Q} \cap (0, \infty)$

*Proof.*

- (1) For each  $x \in \mathbb{N}$  and  $n \in \mathbb{N}$ ,  $x = \frac{nx}{n} \in A_n$ . Hence  $\mathbb{N} \subset \liminf_{n \rightarrow \infty} A_n$ . Conversely, let  $x \in \liminf_{n \rightarrow \infty} A_n$ . Then there exists  $n \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq n$ , then  $x \in A_k$ . In particular,  $x \in A_n$ . Hence there exists  $m_n \in \mathbb{N}$  such that  $x = \frac{m_n}{n}$ . Choose  $s, t \in \mathbb{N}$  such that  $x = \frac{s}{t}$  and  $\gcd(s, t) = 1$ . Choose a prime  $p > n$ . By assumption,  $x \in A_p$ . Then there exist  $m_p \in \mathbb{N}$  such that  $x = \frac{m_p}{p}$ . Hence  $\frac{s}{t} = \frac{m_p}{p}$  and  $tm_p = sp$ . Since  $t \mid sp$  and  $\gcd(s, t) = 1$ , we see that  $t \mid p$ . If  $t > 1$ , then  $p$  is not prime, which is a contradiction. So  $t = 1$ . Hence  $x \in \mathbb{N}$ . Thus  $\liminf_{n \rightarrow \infty} A_n \subset \mathbb{N}$ .

- (2) Let  $x \in \mathbb{Q} \cap (0, \infty)$ . Then there exist  $s, t \in \mathbb{N}$  such that  $x = \frac{s}{t}$ . Define the subsequence  $(A_{n_k})_{k \in \mathbb{N}}$  by  $A_{n_k} = A_{tk}$ . Then for each  $k \in \mathbb{N}$ ,  $x = \frac{sk}{tk} \in A_{tk} = A_{n_k}$ . Thus

$$\begin{aligned} x &\in \inf_{k \in \mathbb{N}} A_{n_k} \\ &\subset \liminf_{n \rightarrow \infty} A_{n_k} \\ &\subset \limsup_{n \rightarrow \infty} A_{n_k} \\ &\subset \limsup_{n \rightarrow \infty} A_n \end{aligned}$$

Conversely, clearly  $\limsup_{n \rightarrow \infty} A_n \subset \mathbb{Q} \cap (0, \infty)$

□

**Exercise 2.9.14.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  sequences of subsets. Then

$$\limsup_{n \rightarrow \infty} A_n \cup B_n = \limsup_{n \rightarrow \infty} A_n \cup \limsup_{n \rightarrow \infty} B_n$$

*Proof.* Let  $x \in \limsup_{n \rightarrow \infty} A_n \cup B_n$ . Suppose that  $x \notin \limsup_{n \rightarrow \infty} A_n$ . Then there exists  $n^* \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$  if  $k \geq n^*$ , then  $x \notin A_k$ . Let  $n \in \mathbb{N}$ . Then there exists  $k$  such that  $k \geq \max\{n, n^*\}$  and  $x \in A_k \cup B_k$ . Since  $k \geq n^*$ ,  $x \notin A_k$ . Thus  $x \in B_k$ . So for each  $n \in \mathbb{N}$ , there exists  $k \in \mathbb{N}$  such that  $k \geq n$  and  $x \in B_k$ . Therefore  $x \in \limsup_{n \rightarrow \infty} B_n$  and

$$\limsup_{n \rightarrow \infty} A_n \cup B_n \subset \limsup_{n \rightarrow \infty} A_n \cup \limsup_{n \rightarrow \infty} B_n$$

Conversely, a previous exercise tells us that  $\limsup_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} A_n \cup B_n$  and  $\limsup_{n \rightarrow \infty} B_n \subset \limsup_{n \rightarrow \infty} A_n \cup B_n$ . Thus

$$\limsup_{n \rightarrow \infty} A_n \cup \limsup_{n \rightarrow \infty} B_n \subset \limsup_{n \rightarrow \infty} A_n \cup B_n$$

□

**Exercise 2.9.15.** Let  $X$  be a set and  $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$  sequences of subsets. Then

$$\liminf_{n \rightarrow \infty} A_n \cap B_n = \liminf_{n \rightarrow \infty} A_n \cap \liminf_{n \rightarrow \infty} B_n$$

*Proof.* A previous exercise tells us that

$$\begin{aligned} \liminf_{n \rightarrow \infty} A_n \cap B_n &= \left( \limsup_{n \rightarrow \infty} A_n^c \cup B_n^c \right)^c \\ &= \left( \limsup_{n \rightarrow \infty} A_n^c \cup \limsup_{n \rightarrow \infty} B_n^c \right)^c \\ &= \left( \limsup_{n \rightarrow \infty} A_n^c \right)^c \cap \left( \limsup_{n \rightarrow \infty} B_n^c \right)^c \\ &= \liminf_{n \rightarrow \infty} A_n \cap \liminf_{n \rightarrow \infty} B_n \end{aligned}$$

□

## 2.10. Measures.

**Definition 2.10.1.** Let  $(X, \mathcal{A})$  be a measurable space and  $\mu : \mathcal{A} \rightarrow [0, \infty]$ . Then  $\mu$  is said to be a **measure** on  $(X, \mathcal{A})$  if

- (1) there exists  $A \in \mathcal{A}$  such that  $\mu(A) < \infty$
- (2) for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . If  $(A_n)_{n \in \mathbb{N}}$  is disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

**Definition 2.10.2.** Let  $(X, \mathcal{A})$  be a measurable space and  $\mu$  a measure on  $(A, \mathcal{A})$ . Then  $(A, \mathcal{A}, \mu)$  is called a **measure space**.

**Exercise 2.10.3.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $A$  and index set and  $(E_\alpha)_{\alpha \in A} \subset \mathcal{A}$ . Suppose that  $\mu(X) < \infty$  and  $(E_\alpha)_{\alpha \in A}$  is disjoint. Then  $\{\alpha \in A : \mu(E_\alpha) > 0\}$  is countable.

**Hint:** set  $A_n = \{\alpha \in A : \mu(E_\alpha) \geq 1/n\}$

*Proof.* For  $n \in \mathbb{N}$ , set  $A_n = \{\alpha \in A : \mu(E_\alpha) \geq 1/n\}$  and define  $A_{>} = \{\alpha \in A : \mu(E_\alpha) > 0\}$ . Then

$$A_{>} = \bigcup_{n \in \mathbb{N}} A_n$$

For the sake of contradiction, suppose that  $A_{>}$  is uncountable. Then there exists  $N \in \mathbb{N}$  such that  $A_N$  is uncountable. So there exists a sequence  $(\alpha_j)_{j \in \mathbb{N}} \subset A_N$ . Then

$$\begin{aligned} \infty &> \mu(X) \\ &\geq \mu\left(\bigcup_{j \in \mathbb{N}} E_{\alpha_j}\right) \\ &= \sum_{j \in \mathbb{N}} \mu(E_{\alpha_j}) \\ &\geq \sum_{j \in \mathbb{N}} \frac{1}{N} \\ &= \infty \end{aligned}$$

which is a contradiction. So  $A_{>}$  is countable. □

**Exercise 2.10.4.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Then

- (1) (monotonicity): for each  $A, B \in \mathcal{A}$ , if  $A \subset B$ , then  $\mu(A) \leq \mu(B)$ .
- (2) (subadditivity): for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ ,

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$$

- (3) (continuity from below): for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ , if for each  $n \in \mathbb{N}$ ,  $A_n \subset A_{n+1}$ , then

$$\mu\left(\sup_{n \in \mathbb{N}} A_n\right) = \sup_{n \in \mathbb{N}} \mu(A_n)$$

- (4) (continuity from above): for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ , if for each  $n \in \mathbb{N}$ ,  $A_{n+1} \subset A_n$  and  $\mu(A_1) < \infty$ , then

$$\mu\left(\inf_{n \in \mathbb{N}} A_n\right) = \inf_{n \in \mathbb{N}} \mu(A_n)$$



*Proof.*

(1) Let  $A, B \in \mathcal{A}$ . Suppose that  $A \subset B$ . Then

$$\begin{aligned}\mu(B) &= \mu\left((B \cap A) \cup (B \cap A^c)\right) \\ &= \mu(B \cap A) + \mu(B \cap A^c) \\ &= \mu(A) + \mu(B \cap A^c) \\ &\geq \mu(A)\end{aligned}$$

(2) Let  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Define  $B_1 = A_1$  and for  $n \geq 2$ ,  $B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right)$ . Then  $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ ,  $\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n$ ,  $(B_n)_{n \in \mathbb{N}}$  disjoint and for each  $n \in \mathbb{N}$ ,  $B_n \subset A_n$ . Thus

$$\begin{aligned}\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) \\ &= \sum_{n \in \mathbb{N}} \mu(B_n) \\ &\leq \sum_{n \in \mathbb{N}} \mu(A_n)\end{aligned}$$

(3) Let  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Suppose that for each  $n \in \mathbb{N}$ ,  $A_n \subset A_{n+1}$ . Then for each  $n \in \mathbb{N}$ ,  $\mu(A_n) \leq \mu(A_{n+1})$  and  $\lim_{n \rightarrow \infty} \mu(A_n) = \sup_{n \in \mathbb{N}} \mu(A_n)$ . Recall that  $\sup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A_n$ . Define  $B_1 = A_1$  and for  $n \geq 2$ ,  $B_n = A_n \setminus A_{n-1}$ . Then  $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ ,  $(B_n)_{n \in \mathbb{N}}$  is disjoint,  $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$  and for each  $n \in \mathbb{N}$ ,  $\bigcup_{k=1}^n B_k = A_n$ . Then

$$\begin{aligned}\mu\left(\sup_{n \in \mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \\ &= \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) \\ &= \sum_{n \in \mathbb{N}} \mu(B_n) \\ &= \lim_{k \rightarrow \infty} \sum_{n=1}^k \mu(B_n) \\ &= \lim_{k \rightarrow \infty} \mu\left(\bigcup_{n=1}^k B_n\right) \\ &= \lim_{k \rightarrow \infty} \mu(A_k) \\ &= \sup_{n \in \mathbb{N}} \mu(A_n)\end{aligned}$$

(4) Let  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Suppose that for each  $n \in \mathbb{N}$ ,  $A_{n+1} \subset A_n$  and  $\mu(A_1) < \infty$ . Then for each  $n \in \mathbb{N}$   $\mu(A_{n+1}) \leq \mu(A_n) \leq \mu(A_1) < \infty$  and the arithmetic that follows is

well defined. Recall that  $\inf_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} A_n$ . For each  $n \in \mathbb{N}$ , define  $B_n = A_1 \cap A_n$ . Then for each  $n \in \mathbb{N}$ ,  $B_n \subset B_{n+1}$  and

$$\begin{aligned} \sup_{n \in \mathbb{N}} B_n &= \bigcup_{n \in \mathbb{N}} B_n \\ &= A_1 \setminus \bigcap_{n \in \mathbb{N}} A_n \\ &= A_1 \setminus \inf_{n \in \mathbb{N}} A_n \end{aligned}$$

So (3) implies that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mu(B_n) &= \mu\left(\sup_{n \in \mathbb{N}} B_n\right) \\ &= \mu\left(A_1 \setminus \inf_{n \in \mathbb{N}} A_n\right) \\ &= \mu(A_1) - \mu\left(\inf_{n \in \mathbb{N}} A_n\right) \end{aligned}$$

On the other hand,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mu(B_n) &= \sup_{n \in \mathbb{N}} \mu(A_1 \setminus A_n) \\ &= \sup_{n \in \mathbb{N}} \left[ \mu(A_1) - \mu(A_n) \right] \\ &= \mu(A_1) - \inf_{n \in \mathbb{N}} \mu(A_n) \end{aligned}$$

Therefore

$$\mu\left(\inf_{n \in \mathbb{N}} A_n\right) = \inf_{n \in \mathbb{N}} \mu(A_n)$$

□

**Exercise 2.10.5.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  and  $A \in \mathcal{A}$ . Then

- (1)  $\mu\left(\liminf_{n \rightarrow \infty} A_n\right) \leq \liminf_{n \rightarrow \infty} \mu(A_n)$
- (2) If  $\mu\left(\sup_{n \in \mathbb{N}} A_n\right) < \infty$ , then  $\limsup_{n \rightarrow \infty} \mu(A_n) \leq \mu\left(\limsup_{n \rightarrow \infty} A_n\right)$

*Proof.*

- (1) Since  $\left(\inf_{k \geq n} A_k\right)_{n \in \mathbb{N}}$  is an increasing sequence and for each  $n \in \mathbb{N}$   $\inf_{k \geq n} A_k \subset A_n$ , we have that

$$\begin{aligned} \mu\left(\liminf_{n \rightarrow \infty} A_n\right) &= \mu\left[\sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} A_k\right)\right] \\ &= \sup_{n \in \mathbb{N}} \mu\left(\inf_{k \geq n} A_k\right) \\ &= \liminf_{n \rightarrow \infty} \mu\left(\inf_{k \geq n} A_k\right) \\ &\leq \liminf_{n \rightarrow \infty} \mu(A_n) \end{aligned}$$

- (2) Since  $\mu\left(\sup_{k \geq 1} A_k\right) < \infty$ ,  $\left(\sup_{k \geq n} A_k\right)_{n \in \mathbb{N}}$  is a decreasing and for each  $n \in \mathbb{N}$ ,  $A_n \subset \sup_{k \geq n} A_k$ , we have that

$$\begin{aligned} \mu\left(\limsup_{n \rightarrow \infty} A_n\right) &= \mu\left[\inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} A_k\right)\right] \\ &= \inf_{n \in \mathbb{N}} \mu\left(\sup_{k \geq n} A_k\right) \\ &= \limsup_{n \rightarrow \infty} \mu\left(\sup_{k \geq n} A_k\right) \\ &\geq \limsup_{n \rightarrow \infty} \mu(A_n) \end{aligned}$$

□

**Exercise 2.10.6.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  and  $A \in \mathcal{A}$ . Suppose that  $\mu\left(\sup_{n \in \mathbb{N}} A_n\right) < \infty$ . Then  $A_n \rightarrow A$  implies that  $\mu(A_n) \rightarrow \mu(A)$ .

*Proof.* Suppose that  $A_n \rightarrow A$ . Then the previous exercise tells us that

$$\begin{aligned} \mu(A) &= \mu\left(\liminf_{n \rightarrow \infty} A_n\right) \\ &\leq \liminf_{n \rightarrow \infty} \mu(A_n) \\ &\leq \limsup_{n \rightarrow \infty} \mu(A_n) \\ &\leq \mu(\limsup_{n \rightarrow \infty} A_n) \\ &= \mu(A) \end{aligned}$$

Thus  $\mu(A) = \limsup_{n \rightarrow \infty} \mu(A_n) = \liminf_{n \rightarrow \infty} \mu(A_n)$  and  $\mu(A_n) \rightarrow \mu(A)$

□

**Definition 2.10.7.** Let  $(X, \mathcal{A})$  be a measurable space and  $\mu : \mathcal{A} \rightarrow [0, \infty]$  a measure. Then  $\mu$  is said to be

- **finite** if  $\mu(X) < \infty$

- **$\sigma$ -finite** if there exists  $(E_j)_{j \in \mathbb{N}} \subset \mathcal{A}$  such that
  - (1)  $X = \bigcup_{j \in \mathbb{N}} E_j$
  - (2) for each  $j \in \mathbb{N}$ ,  $\mu(E_j) < \infty$
- **semifinite** if for each  $F \in \mathcal{A}$ ,  $\mu(F) = \infty$  implies that there exists  $E \in \mathcal{A}$  such that  $E \subset F$  and  $\mu(E) < \infty$ .

**Exercise 2.10.8.** Let  $(X, \mathcal{A})$  be a measurable space and  $\mu : \mathcal{A} \rightarrow [0, \infty]$  a measure.

- (1) If  $\mu$  is finite, then  $\mu$  is  $\sigma$ -finite.
- (2) If  $\mu$  is  $\sigma$ -finite, then  $\mu$  is semifinite.

*Proof.*

- Suppose that  $\mu$  is finite. Define  $(E_j)_{j \in \mathbb{N}} \subset \mathcal{A}$  by

$$E_j = \begin{cases} X & j = 1 \\ \emptyset & j > 1 \end{cases}$$

Then  $X = \bigcup_{j \in \mathbb{N}} E_j$  and for each  $j \in \mathbb{N}$ ,  $0 < \mu(E_j) < \infty$ .

- Suppose that  $\mu$  is  $\sigma$ -finite. Then there exists  $(E_j)_{j \in \mathbb{N}} \subset \mathcal{A}$  such that  $X = \bigcup_{j \in \mathbb{N}} E_j$  and for each  $j \in \mathbb{N}$ ,  $\mu(E_j) < \infty$ . Let  $F \in \mathcal{A}$ . Suppose that  $\mu(F) = \infty$ . Define  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  by

$$A_n = \bigcup_{j=1}^n E_j$$

Note that  $X = \bigcup_{n \in \mathbb{N}} A_n$  and for each  $n \in \mathbb{N}$ ,  $F \cap A_n \subset F \cap A_{n+1}$  and

$$\begin{aligned} \mu(F \cap A_n) &= \mu\left(F \cap \left[\bigcup_{j=1}^n E_j\right]\right) \\ &\leq \mu\left(\bigcup_{j=1}^n E_j\right) \\ &\leq \sum_{j=1}^n \mu(E_j) \\ &< \infty \end{aligned}$$

For the sake of contradiction, suppose that for each  $n \in \mathbb{N}$ ,  $\mu(F \cap A_n) = 0$ . Then

$$\begin{aligned} \infty &= \mu(F) \\ &= \mu(F \cap X) \\ &= \mu\left(F \cap \left[\bigcup_{n \in \mathbb{N}} A_n\right]\right) \\ &= \mu\left(\bigcup_{n \in \mathbb{N}} [F \cap A_n]\right) \\ &= \sup_{n \in \mathbb{N}} \mu(F \cap A_n) \\ &= 0 \end{aligned}$$

which is a contradiction. So there exists  $N \in \mathbb{N}$  such that  $\mu(F \cap A_N) > 0$ . Set  $E = F \cap A_N$ . Then  $E \subset F$  and  $0 < \mu(E) < \infty$ . Hence  $\mu$  is semifinite.  $\square$

**Definition 2.10.9.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $(f_\alpha)_{\alpha \in A} \subset L^0(X, \mathcal{A})$  a net. Suppose that for each  $\alpha \in A$ ,  $f_\alpha : X \rightarrow \mathbb{R}$ . For each  $\alpha, \beta \in A$ , define  $M_{\alpha, \beta}, N_{\alpha, \beta} \in \mathcal{A}$  by

$$M_{\alpha, \beta} = \{x \in X : f_\alpha(x) \leq f_\beta(x)\}$$

and

$$N_{\alpha, \beta} = \{x \in X : f_\alpha(x) \geq f_\beta(x)\}$$

respectively. Define  $M, N \subset X$  by  $M = \bigcap_{\substack{(\alpha, \beta) \in A^2 \\ \alpha \leq \beta}} M_{\alpha, \beta}$  and  $N = \bigcap_{\substack{(\alpha, \beta) \in A^2 \\ \alpha \leq \beta}} N_{\alpha, \beta}$  respectively.

Then  $(f_\alpha)_{\alpha \in A}$  is said to be

- **increasing  $\mu$ -a.e.** if  $M^c$  is a  $\mu$ -null set
- **decreasing  $\mu$ -a.e.** if  $N^c$  is a  $\mu$ -null set
- **monotonic  $\mu$ -a.e.** if  $(f_n)_{n \in \mathbb{N}}$  is increasing  $\mu$ -a.e. or  $(f_n)_{n \in \mathbb{N}}$  is decreasing  $\mu$ -a.e.

**Exercise 2.10.10.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $(f_\alpha)_{\alpha \in A} \subset L^0(X, \mathcal{A})$  a net. Suppose that for each  $\alpha \in A$ ,  $f_\alpha : X \rightarrow \mathbb{R}$ . If  $A$  is countable, then

- (1)  $(f_\alpha)_{\alpha \in A}$  is increasing  $\mu$ -a.e. iff for each  $\alpha, \beta \in A$ ,  $\alpha \leq \beta$  implies that  $f_\alpha \leq f_\beta$   $\mu$ -a.e.
- (2)  $(f_\alpha)_{\alpha \in A}$  is decreasing  $\mu$ -a.e. iff for each  $\alpha, \beta \in A$ ,  $\alpha \leq \beta$  implies that  $f_\alpha \geq f_\beta$   $\mu$ -a.e.

*Proof.* Suppose that  $A$  is countable. For each  $\alpha, \beta \in A$ , define  $M_{\alpha, \beta}, N_{\alpha, \beta}, M, N \in \mathcal{A}$  as in the previous definition. Since  $A$  is countable,  $M, N \in \mathcal{A}$ .

- (1) Suppose that  $(f_\alpha)_{\alpha \in \mathbb{N}}$  is increasing  $\mu$ -a.e. By definition,  $M^c$  is a  $\mu$ -null set. Since  $M^c \in \mathcal{A}$ ,  $\mu(M^c) = 0$ . Let  $\alpha, \beta \in A$ . Suppose that  $\alpha \leq \beta$ . Since  $M \subset M_{\alpha, \beta}$ ,  $M_{\alpha, \beta}^c \subset M^c$ . Hence  $\mu(M_{\alpha, \beta}^c) = 0$ . By definition,  $f_\alpha \leq f_\beta$   $\mu$ -a.e. Conversely, suppose that for each  $\alpha, \beta \in A$ ,  $\alpha \leq \beta$  implies that  $f_\alpha \leq f_\beta$   $\mu$ -a.e. Then

for each  $\alpha, \beta \in A$ ,  $\mu(M_{\alpha,\beta}^c) = 0$ . Since  $A$  is countable, we have that

$$\begin{aligned}\mu(M^c) &= \mu\left(\bigcup_{\substack{(\alpha,\beta) \in A^2 \\ \alpha \leq \beta}} M_{\alpha,\beta}^c\right) \\ &\leq \sum_{\substack{(\alpha,\beta) \in A^2 \\ \alpha \leq \beta}} \mu(M_{\alpha,\beta}^c) \\ &= 0\end{aligned}$$

Thus  $(f_n)_{n \in \mathbb{N}}$  is increasing  $\mu$ -a.e.

(2) Similar to (1).

□

### 2.11. Outer Measures.

**Definition 2.11.1.** Let  $X$  be a set and  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ . Then  $\mu^*$  is said to be an **outer measure on  $X$**  if

- (1)  $\mu^*(\emptyset) = 0$
- (2) for each  $A, B \subset X$ , if  $A \subset B$ , then  $\mu^*(A) \leq \mu^*(B)$ .
- (3) for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ ,

$$\mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$$

**Definition 2.11.2.** Let  $X$  be a set,  $\mu^*$  an outer measure on  $X$  and  $A \subset X$ . Then  $A$  is said to be  **$\mu^*$ -outer measurable** if for each  $E \subset X$ ,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

**Exercise 2.11.3.** Let  $X$  be a set,  $\mu^*$  an outer measure on  $X$  and  $A \subset X$ . Then  $A$  is  $\mu^*$ -outer measurable iff for each  $E \subset X$ ,  $\mu^*(E) < \infty$  implies that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

*Proof.* Suppose that  $A$  is  $\mu^*$ -outer measurable. Let  $E \subset X$ , Suppose that  $\mu^*(E) < \infty$ . By definition  $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ .

Conversely, suppose that for each  $E \subset X$ ,  $\mu^*(E) < \infty$  implies that  $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ . Let  $E \subset X$ .

- If  $\mu^*(E) < \infty$ , then by assumption,

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

If  $\mu^*(E) = \infty$ , then trivially,

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

So  $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$

- Since  $E = (E \cap A) \cup (E \cap A^c)$ , by definition,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

So  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$  and  $A$  is  $\mu^*$ -outer measurable. □

**Theorem 2.11.4. Construction of Outer Measures:**

Let  $X$  be a set and  $\mathcal{E} \subset \mathcal{P}(X)$  and  $\rho : \mathcal{E} \rightarrow [0, \infty]$ . Suppose that  $\emptyset, X \in \mathcal{E}$  and  $\rho(\emptyset) = 0$ . Define  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  by

$$\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \rho(E_n) : (E_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } A \subset \bigcup_{n \in \mathbb{N}} E_n \right\}$$

Then  $\mu^*$  is an outer measure on  $X$ .

**Note 2.11.5.** In particular, for each  $A \in \mathcal{E}$ ,  $\mu^*(A) = \rho(A)$ .

**Definition 2.11.6.** Let  $X$  be a set and  $\mathcal{E} \subset \mathcal{P}(X)$  and  $\rho : \mathcal{E} \rightarrow [0, \infty]$ . Suppose that  $\emptyset, X \in \mathcal{E}$  and  $\rho(\emptyset) = 0$ . Let  $\mu^*$  be the outer measure on  $X$  defined as in the last theorem. Then  $\mu^*$  is called the **outer measure on  $X$  induced by  $\rho$** .

**Theorem 2.11.7.** Let  $X$  be a set and  $\mu^*$  an outer measure on  $X$ . Define  $\mathcal{A} = \{A \subset X : A \text{ is } \mu^*\text{-measurable}\}$ . Then  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$  and  $\mu^*|_{\mathcal{A}}$  is a complete measure on  $(X, \mathcal{A})$ .

**Definition 2.11.8.** Let  $X$  be a set,  $\mathcal{A}_0$  be an algebra on  $X$  and  $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty]$ . Then  $\mu_0$  is said to be a **premeasure on**  $(X, \mathcal{A}_0)$  if

- (1) there exists  $A \in \mathcal{A}_0$  such that  $\mu_0(A) < \infty$
- (2) for each  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}_0$ , if  $(A_n)_{n \in \mathbb{N}}$  is disjoint and  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_0$ , then

$$\mu_0\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu_0(A_n)$$

**Note 2.11.9.** The same reasoning applied to measures shows that  $\mu_0(\emptyset) = 0$ .

**Theorem 2.11.10.** Let  $X$  be a set,  $\mathcal{A}_0$  an algebra on  $X$ ,  $\mu_0$  a premeasure on  $(X, \mathcal{A}_0)$  and  $\mu^*$  the outer measure on  $X$  induced by  $\mu_0$ . Put  $\mathcal{A} = \sigma(\mathcal{A}_0)$ . If  $\mu_0$  is  $\sigma$ -finite, then there exists a unique measure  $\mu$  on  $(X, \mathcal{A})$  such that  $\mu|_{\mathcal{A}_0} = \mu^*|_{\mathcal{A}_0} = \mu_0$ .



### 2.12. Product Measures.

**Definition 2.12.1.** Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measurable spaces. Put  $\mathcal{E} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$ . Then  $\mathcal{E}$  is an elementary family and thus  $\mathcal{M}_0 = \{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n \subset \mathcal{E} \text{ are disjoint}\}$  is an algebra on  $X \times Y$ . We define  $\pi_0 : \mathcal{M}_0 \rightarrow [0, \infty]$  by

$$\pi_0\left(\bigcup_{i=1}^n A_i \times B_i\right) = \sum_{i=1}^n \mu(A_i)\nu(B_i)$$

Then  $\pi_0$  is a premeasure on  $(X \times Y, \mathcal{M}_0)$ . Since  $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{M}_0)$ , we define the **product measure**,  $\mu \otimes \nu$  on  $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ , to be the unique extension of  $\pi_0$  to  $\mathcal{A} \otimes \mathcal{B}$ . The existence of which is guaranteed by a theorem in the previous section. In particular,

$$\begin{aligned} \mu \otimes \nu(E) &= \inf \left\{ \sum_{n \in \mathbb{N}} \pi_0(E_i) : (E_i)_{i \in \mathbb{N}} \subset \mathcal{M}_0 \text{ and } E \subset \bigcup_{i \in \mathbb{N}} E_i \right\} \\ &= \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_i)\nu(B_i) : (A_i \times B_i)_{i \in \mathbb{N}} \subset \mathcal{E} \text{ and } E \subset \bigcup_{i \in \mathbb{N}} A_i \times B_i \right\} \end{aligned}$$

### 2.13. Pushforward Measures.

**Definition 2.13.1.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(Y, \mathcal{B})$  a measurable space and  $f : X \rightarrow Y$ . Suppose that  $f$  is  $(\mathcal{A}, \mathcal{B})$  measurable. We define the **pushforward of  $\mu$  by  $f$  on  $(Y, \mathcal{B})$** , denoted  $f_*\mu : \mathcal{B} \rightarrow [0, \infty]$ , by

$$f_*\mu(B) = \mu(f^{-1}(B))$$

**Exercise 2.13.2.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(Y, \mathcal{B})$  a measurable space and  $f : X \rightarrow Y$ . Suppose that  $f$  is  $(\mathcal{A}, \mathcal{B})$  measurable. Then  $f_*\mu : \mathcal{B} \rightarrow [0, \infty]$  is a measure.

*Proof.*

(1) Since  $f^{-1}(\emptyset) = \emptyset$ ,

$$\begin{aligned} f_*\mu(\emptyset) &= \mu(f^{-1}(\emptyset)) \\ &= \mu(\emptyset) \\ &= 0 \end{aligned}$$

(2) Let  $(B_j)_{j \in \mathbb{N}} \subset \mathcal{B}$ . Suppose that  $(B_j)_{j \in \mathbb{N}}$  is disjoint. Then  $(f^{-1}(B_j))_{j \in \mathbb{N}}$  is disjoint. Hence

$$\begin{aligned} f_*\mu\left(\bigcup_{j \in \mathbb{N}} B_j\right) &= \mu\left(\bigcup_{j \in \mathbb{N}} f^{-1}(B_j)\right) \\ &= \sum_{j \in \mathbb{N}} \mu(f^{-1}(B_j)) \\ &= \sum_{j \in \mathbb{N}} f_*\mu(B_j) \end{aligned}$$

Hence  $f_*\mu$  is a measure. □

## 3. THE LEBESGUE INTEGRAL

## 3.1. Integration of Nonnegative Functions.

**Theorem 3.1.1. Monotone Convergence Theorem:**

Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$ . Then

$$\sup_{n \in \mathbb{N}} \int f_n = \int \sup_{n \in \mathbb{N}} f_n$$

.

**Exercise 3.1.2.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$ ,  $\lambda \geq 0$  and  $f \in L^+$ . Then

$$\int f d(\mu_1 + \lambda \mu_2) = \int f d\mu_1 + \lambda \int f d\mu_2$$

.

*Proof.* Suppose that  $f$  is simple. Then there exist  $(a_n)_{i=1}^n \subset [0, \infty)$  and  $(E_i)_{i=1}^n \subset \mathcal{A}$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d(\mu_1 + \lambda \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \lambda \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \lambda \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + \lambda \sum_{i=1}^n a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \lambda \int f d\mu_2 \end{aligned}$$

Now for a general  $f$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that  $\phi_n \rightarrow f$  pointwise and for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1} \leq f$ . Then monotone convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \lambda \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \lambda \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \lambda \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \lambda \int f d\mu_2 \end{aligned}$$

□

**Exercise 3.1.3.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$ . Suppose that  $\mu_1 \leq \mu_2$ . Then for each  $f \in L^+$ ,

$$\int f d\mu_1 \leq \int f d\mu_2$$

*Proof.* First suppose that  $f$  is simple. Then there exist  $(a_n)_{n=1}^\infty \subset [0, \infty)$  and  $(E_i)_{i=1}^\infty \subset \mathcal{A}$  such that  $f = \sum_{i=1}^\infty a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d\mu_1 &= \sum_{i=1}^\infty a_i \mu_1(E_i) \\ &\leq \sum_{i=1}^\infty a_i \mu_2(E_i) \\ &= \int f d\mu_2 \end{aligned}$$

for general  $f$ ,

$$\begin{aligned} \int f d\mu_1 &= \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_1 \\ &\leq \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_2 \\ &= \int f d\mu_2 \end{aligned}$$

□

**Theorem 3.1.4. Fatou's Lemma:**

Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

**Theorem 3.1.5.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$ . Then

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n.$$

**Exercise 3.1.6.** Let  $f \in L^+$  and suppose that  $\int f < \infty$ . Put  $N = \{x \in X : f(x) = \infty\}$  and  $S = \{x \in X : f(x) > 0\}$ . Then  $\mu(N) = 0$  and  $S$  is  $\sigma$ -finite.

*Proof.* Suppose that  $\mu(N) > 0$ . Define  $f_n = n \chi_N \in L^+$ . Then for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1} \leq f$  on  $N$ . So

$$\begin{aligned} \int f &\geq \int_N f \\ &= \lim_{n \rightarrow \infty} \int_N f_n \\ &= \lim_{n \rightarrow \infty} n \mu(N) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

Hence  $N$  is a null set. Now, put  $S_n = \{x \in X : f(x) > 1/n\}$ . Then  $S = \bigcup_{n \in \mathbb{N}} S_n$ . Suppose that there exists some  $n \in \mathbb{N}$  such that  $\mu(S_n) = \infty$ . Then

$$\begin{aligned} \int f &\geq \int_{S_n} f \\ &\geq \frac{1}{n} \mu(S_n) \\ &= \infty, \text{ a contradiction.} \end{aligned}$$

So for each  $n \in \mathbb{N}$ ,  $\mu(S_n) < \infty$  and  $S$  is  $\sigma$ -finite. □

**Exercise 3.1.7.** Let  $f \in L^+$ . Then  $f = 0$  a.e. iff for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$ .

*Proof.*  $f = 0$  a.e. implies that for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$  is clear. Conversely, suppose that for each  $E \in \mathcal{A}$ ,  $\int_E f = 0$ . For  $n \in \mathbb{N}$  put  $N_n = \{x \in X : f(x) > 1/n\}$  and define  $N = \{x \in X : f(x) > 0\}$ . So  $N = \bigcup_{n \in \mathbb{N}} N_n$ . Let  $n \in \mathbb{N}$ . Then our assumption tells us that

$$\begin{aligned} 0 &= \int_{N_n} f \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

Hence for each  $n \in \mathbb{N}$ ,  $\mu(N_n) = 0$ . Thus  $\mu(N) = 0$  and  $f = 0$  a.e. as required. □

**Exercise 3.1.8.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$  and  $f \in L^+$ . Suppose that  $f_n \xrightarrow{\text{p.w.}} f$ ,  $\lim_{n \rightarrow \infty} \int f_n = \int f$  and  $\int f < \infty$ . Then for each  $E \in \mathcal{A}$ ,  $\lim_{n \rightarrow \infty} \int_E f_n = \int_E f$ . This result may fail to be true if  $\int f = \infty$

*Proof.* Let  $E \in \mathcal{A}$ . By Fatou's lemma,  $\int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$ . Note that since  $\int f < \infty$ , we have that  $\int_{E^c} f \leq \int f < \infty$ . Thus we may write

$$\begin{aligned} \int_E f &= \int f - \int_{E^c} f \\ &\geq \int f - \liminf_{n \rightarrow \infty} \int_{E^c} f_n \\ &= \int f - \liminf_{n \rightarrow \infty} \left( \int f_n - \int_E f_n \right) \\ &= \int f - \int f + \limsup_{n \rightarrow \infty} \int_E f_n \\ &= \limsup_{n \rightarrow \infty} \int_E f_n. \end{aligned}$$

Hence

$$\limsup_{n \rightarrow \infty} \int_E f_n \leq \int_E f \leq \liminf_{n \rightarrow \infty} \int_E f_n$$

and therefore

$$\lim_{n \rightarrow \infty} \int_E f_n = \int_E f.$$

If we drop the assumption that  $\int f < \infty$ , then the result would fail to be true for the functions  $f = \infty \chi_{(0,1)}$  and  $f_n = \infty \chi_{(0,1)} + n \chi_{(1,1+1/n)}$ . Here  $f_n \xrightarrow{\text{p.w.}} f$ ,  $\lim_{n \rightarrow \infty} \int f_n = \int f = \infty$  and  $\lim_{n \rightarrow \infty} \int_{(1,\infty)} f_n = 1$  while  $\int_{(1,\infty)} f = 0$ .

□

**Exercise 3.1.9.** Let  $f \in L^+$ . Define  $\lambda : \mathcal{A} \rightarrow [0, \infty]$  by  $\lambda(E) = \int_E f d\mu$  for  $E \in \mathcal{A}$ . Then  $\lambda$  is a measure on  $(X, \mathcal{A})$  and for each  $g \in L^+$ ,  $\int g d\lambda = \int g f d\mu$ .

*Proof.* Clearly  $\lambda(\emptyset) = 0$ . Let  $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$  and suppose that for each  $i, j \in \mathbb{N}$ , if  $i \neq j$ , then  $A_i \cap A_j = \emptyset$ . For now, suppose that  $f$  is simple. Then there exist  $E_1, E_2, \dots, E_n \in \mathcal{A}$  and  $a_1, a_2, \dots, a_n \in [0, \infty)$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Then

$$\begin{aligned} \lambda\left(\bigcup_{j \in \mathbb{N}} A_j\right) &= \int_{\bigcup_{j \in \mathbb{N}} A_j} f \\ &= \sum_{i=1}^n a_i \mu\left(E_i \cap \left(\bigcup_{j \in \mathbb{N}} A_j\right)\right) \\ &= \sum_{i=1}^n a_i \mu\left(\bigcup_{j \in \mathbb{N}} E_i \cap A_j\right) \\ &= \sum_{i=1}^n a_i \sum_{j \in \mathbb{N}} \mu(E_i \cap A_j) \\ &= \sum_{j \in \mathbb{N}} \sum_{i=1}^n a_i \mu(E_i \cap A_j) \\ &= \sum_{j \in \mathbb{N}} \int_{A_j} f \\ &= \sum_{j \in \mathbb{N}} \lambda(A_j) \end{aligned}$$

Hence  $\lambda$  is a measure on  $(X, \mathcal{A})$ . Now, for a general  $f$ , there exist  $(\phi_n)_{n \in \mathbb{N}} \subset L^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n$  is simple,  $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \xrightarrow{\text{p.w.}} f$ . Put  $A = \bigcup_{j \in \mathbb{N}} A_j$  and define the measures  $\lambda_n$  by  $\lambda_n(E) = \int_E \phi_n$ . Note that we may define a monotonically increasing sequence of functions  $g_n : \mathbb{N} \rightarrow [0, \infty]$  by  $g_n(j) = \int_{A_j} \phi_n$ . Using monotone convergence three times and a nice application of the counting measure on  $\mathbb{N}$ , we may write

$$\begin{aligned}
\lambda(A) &= \int_A f \\
&= \lim_{n \rightarrow \infty} \int_A \phi_n \\
&= \lim_{n \rightarrow \infty} \sum_{j \in \mathbb{N}} \int_{A_j} \phi_n \\
&= \sum_{j \in \mathbb{N}} \lim_{n \rightarrow \infty} \int_{A_j} \phi_n \quad (\text{by the above}) \\
&= \sum_{j \in \mathbb{N}} \int_{A_j} f \\
&= \sum_{j \in \mathbb{N}} \lambda(A_j).
\end{aligned}$$

Hence  $\lambda$  is a measure on  $(X, \mathcal{A})$ . Let  $g \in L^+$ . First assume that  $g$  is simple. Then there exist  $E_1, E_2, \dots, E_n \in \mathcal{A}$  and  $a_1, a_2, \dots, a_n \in [0, \infty)$  such that  $g = \sum_{i=1}^n a_i \chi_{E_i}$ . In this case, we have that

$$\begin{aligned}
\int g d\lambda &= \sum_{i=1}^n a_i \lambda(E_i) \\
&= \sum_{i=1}^n a_i \int_{E_i} f d\mu \\
&= \int \left( \sum_{i=1}^n a_i \chi_{E_i} \right) f d\mu \\
&= \int g f d\mu.
\end{aligned}$$

Now for a general  $g \in L^+$ , there exist  $(\psi_n)_{n \in \mathbb{N}} \subset L^+$  such that for each  $n \in \mathbb{N}$ ,  $\psi_n$  is simple,  $\psi_n \leq \psi_{n+1} \leq f$  and  $\psi_n \xrightarrow{\text{p.w.}} g$ . Monotone convergence then gives us

$$\begin{aligned}
\int g d\lambda &= \lim_{n \rightarrow \infty} \int \psi_n d\lambda \\
&= \lim_{n \rightarrow \infty} \int \psi_n f d\mu \\
&= \int g f d\mu \text{ as required.}
\end{aligned}$$

□

**Exercise 3.1.10.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^+$  and  $f \in L^+$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \geq f_{n+1}$ ,  $f_n \xrightarrow{\text{p.w.}} f$  and  $\int f_1 < \infty$ . Then  $\lim_{n \rightarrow \infty} \int f_n = \int f$ .

*Proof.* First we note that since  $\int f_1 < \infty$ ,  $f_1 < \infty$  a.e., for each  $n \in \mathbb{N}$ ,  $f_1 - f_n$  and  $\int f_1 - \int f_n$  are well defined and  $\int f_n \leq \int f_1 < \infty$ . Also, for  $n \in \mathbb{N}$ ,  $f_1 - f_n \in L^+$ . So we may write

$$\begin{aligned} \int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n \\ &= \int [(f_1 - f_n) + f_n] - \int f_n \\ &= \int f_1 - \int f_n \end{aligned}$$

Put  $g_n = f + (f_1 - f_n)$ . Then  $g_n \in L^+$ , for each  $n \in \mathbb{N}$ ,  $g_n \leq g_{n+1}$  and  $g_n \xrightarrow{\text{p.w.}} f_1$ . Monotone convergence tells us that

$$\begin{aligned} \int f_1 &= \lim_{n \rightarrow \infty} \int g_n \\ &= \lim_{n \rightarrow \infty} \left[ \int f + (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \int f + \int (f_1 - f_n) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \int f + \int f_1 - \int f_n \right] \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \int f$  and  $\lim_{n \rightarrow \infty} \int f_1$  exist,  $\lim_{n \rightarrow \infty} \int f_n = \int f$  as required. □

**Exercise 3.1.11.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(Y, \mathcal{B})$  a measurable space and  $f : X \rightarrow Y$ . Suppose that  $f$  is  $(\mathcal{A}, \mathcal{B})$ -measurable. Then for each  $g \in L^+(Y, \mathcal{B})$  and  $B \in \mathcal{B}$ ,

$$\int_{f^{-1}(B)} g \circ f \, d\mu = \int_B g \, df_* \mu$$

*Proof.* Let  $g \in L^+(X, \mathcal{A})$  and  $B \in \mathcal{B}$ . Suppose that there exists  $E \in \mathcal{B}$  such that  $g = \chi_E$ . Then  $g \circ f = \chi_{f^{-1}(E)}$  and

$$\begin{aligned} \int_{f^{-1}(B)} g \circ f \, d\mu &= \int_{f^{-1}(B)} \chi_{f^{-1}(E)} \, d\mu \\ &= \mu(f^{-1}(E) \cap f^{-1}(B)) \\ &= \mu(f^{-1}(E \cap B)) \\ &= f_* \mu(E \cap B) \\ &= \int_B \chi_E \, df_* \mu \\ &= \int_B g \, df_* \mu \end{aligned}$$



Suppose that  $g$  is simple. Then there exist  $(a_j)_{j=1}^n \subset [0, \infty)$  and  $(E_j)_{j=1}^n \subset \mathcal{B}$  such that  $g = \sum_{j=1}^n a_j \chi_{E_j}$ . Then

$$\begin{aligned} g \circ f &= \left( \sum_{j=1}^n a_j \chi_{E_j} \right) \circ f \\ &= \sum_{j=1}^n a_j \chi_{E_j} \circ f \end{aligned}$$

and

$$\begin{aligned} \int_{f^{-1}(B)} g \circ f \, d\mu &= \int_{f^{-1}(B)} \sum_{j=1}^n a_j \chi_{E_j} \circ f \, d\mu \\ &= \sum_{j=1}^n a_j \int_{f^{-1}(B)} \chi_{E_j} \circ f \, d\mu \\ &= \sum_{j=1}^n a_j \int_B \chi_{E_j} \, df_* \mu \\ &= \int_B \sum_{j=1}^n a_j \chi_{E_j} \, df_* \mu \\ &= \int_B g \, df_* \mu \end{aligned}$$

Suppose that  $g \in L^+(Y, \mathcal{B})$ . Then there exists  $(\phi_n)_{n \in \mathbb{N}} \subset S^+(Y, \mathcal{B})$  such that  $\phi_n \xrightarrow{\text{p.w.}} g$  and for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1}$ . Then  $\phi_n \circ f \xrightarrow{\text{p.w.}} g \circ f$  and for each  $n \in \mathbb{N}$ ,  $\phi_n \circ f \leq \phi_{n+1} \circ f$ . The monotone convergence theorem implies that

$$\begin{aligned} \int_{f^{-1}(B)} g \circ f \, d\mu &= \lim_{n \rightarrow \infty} \int_{f^{-1}(B)} \phi_n \circ f \, d\mu \\ &= \lim_{n \rightarrow \infty} \int_B \phi_n \, df_* \mu \\ &= \int_B g \, df_* \mu \end{aligned}$$

□

**Exercise 3.1.12.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(Y, \mathcal{B})$  a measurable space and  $f : X \rightarrow Y$ . Suppose that  $f$  is  $(\mathcal{A}, \mathcal{B})$ -measurable. Let  $g, h \in L^0(Y, \mathcal{B})$ . If  $g \circ f = h \circ f$   $\mu$ -a.e., then  $g = h$   $f_*\mu$ -a.e.

*Proof.* Suppose that  $g \circ f = h \circ f$   $\mu$ -a.e. Then  $|(g - h) \circ f| = 0$   $\mu$ -a.e. The previous exercise implies that

$$\begin{aligned}\int_Y |g - h| df_*\mu &= \int_X |g - h| \circ f d\mu \\ &= \int_X |(g - h) \circ f| d\mu \\ &= 0\end{aligned}$$

Hence  $|g - h| = 0$   $f_*\mu$ -a.e. and  $g = h$   $f_*\mu$ -a.e. □

### 3.2. Integration of Complex Valued Functions.

**Definition 3.2.1.** Let  $f : X \rightarrow \mathbb{C}$  be measurable. Then  $f$  is said to be **integrable** if

$$\int |f| d\mu < \infty$$

**Definition 3.2.2.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Define

$$L^1(X, \mathcal{A}, \mu) = \left\{ f : X \rightarrow \mathbb{C} : f \text{ is measurable and } \int |f| < \infty \right\}$$

**Lemma 3.2.3.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is integrable iff  $f^+$  and  $f^-$  are integrable.

*Proof.*  $f^+, f^- \leq |f| = f^+ + f^-$  □

**Definition 3.2.4.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is said to be **extended integrable** if

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

**Lemma 3.2.5.** Let  $f : X \rightarrow \mathbb{R}$  be measurable. Then  $f$  is integrable iff  $Re(f)$  and  $Im(f)$  are integrable.

*Proof.*  $|Re(f)|, |Im(f)| \leq |f| \leq |Re(f)| + |Im(f)|$  □

**Exercise 3.2.6. Dominated Convergence Theorem:**

Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ ,  $f \in L^0$  and  $g \in L^1$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$  and there exists  $g \in L^1$  such that for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then  $f \in L^1$  and

$$\int_X |f_n - f| d\mu \rightarrow 0$$

**Hint:** Fatou's lemma

*Proof.* Continuity implies that  $|f| \leq g$  a.e. Since

$$\begin{aligned} |f_n - f| &\leq |f_n| + |f| \\ &\leq 2g \end{aligned}$$

Fatou's lemma implies that

$$\begin{aligned} \int 2g d\mu &= \int \liminf_{n \rightarrow \infty} (2g - |f_n - f|) d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int 2g - |f_n - f| d\mu \\ &= \int 2g d\mu - \limsup_{n \rightarrow \infty} \int |f_n - f| d\mu \end{aligned}$$

Hence

$$\limsup_{n \rightarrow \infty} \int |f_n - f| d\mu \leq 0$$

and thus

$$\int |f_n - f| d\mu \rightarrow 0$$

□

**Exercise 3.2.7.** Let  $\mu_1, \mu_2$  be measures on  $(X, \mathcal{A})$ . Then

- (1)  $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$
- (2) for each  $f \in L^1(\mu_1 + \mu_2)$ , we have that

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

*Proof.* (1) The first part is clear since similar exercise from the section on nonnegative functions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f| d\mu_1 + \int |f| d\mu_2$$

- (2) Suppose that  $f$  is simple. Then there exist  $(a_n)_{n=1}^n \subset \mathbb{C}$  and  $(E_i)_{i=1}^n \subset \mathcal{A}$  such that  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Then

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

Now for general  $f$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S$  such that  $\phi_n \rightarrow f$  pointwise and for each  $n \in \mathbb{N}$ ,  $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ . Then dominated convergence tells us that

$$\begin{aligned} \int f d(\mu_1 + \mu_2) &= \lim_{n \rightarrow \infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu_1 + \lim_{n \rightarrow \infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2 \end{aligned}$$

□

**Theorem 3.2.8.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$ . Suppose that

$$\sum_{n \in \mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero,  $\sum_{n \in \mathbb{N}} f_n \in L^1$  and

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

**Theorem 3.2.9.** Let  $f \in L^1$ . Then for each  $\epsilon > 0$ , there exists  $\phi \in L^1$  such that  $\phi$  is simple and  $\int |f - \phi| < \epsilon$ .

**Exercise 3.2.10. Generalized Fatou's Lemma:** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $(f_n)_{n \in \mathbb{N}} \subset L^0(X, \mathcal{A})$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f : X \rightarrow \mathbb{R}$ , there exists  $g \in L^1$  such that  $g \geq 0$  and for each  $n \in \mathbb{N}$ ,  $f_n \geq -g$ . Then

$$\int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu$$

What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?

*Proof.* First note that for each  $n \in \mathbb{N}$ ,  $\int f_n$  is well defined since  $f_n^- \leq g \in L^1$ . Since  $g + f_n \geq 0$ , we may use Fatou's lemma to write

$$\begin{aligned} \int g d\mu + \int \liminf_{n \rightarrow \infty} f_n d\mu &= \int \liminf_{n \rightarrow \infty} (g + f_n) d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int (g + f_n) d\mu \\ &= \int g d\mu + \liminf_{n \rightarrow \infty} \int f_n d\mu \end{aligned}$$

Since  $\int g < \infty$ ,  $\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n$  as required. The analogue is as follows: Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of measurable real valued functions. Suppose that there exists  $g \in L^1$  such that  $g \geq 0$  and for each  $n \in \mathbb{N}$ ,  $f_n \leq g$ . Then  $\limsup_{n \rightarrow \infty} \int f_n \leq \int \limsup_{n \rightarrow \infty} f_n$ . To show this, just use the result from above with the sequence  $(g_n)_{n \in \mathbb{N}}$  given by  $g_n = -f_n$ . □

**Exercise 3.2.11.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$  and  $f : X \rightarrow \mathbb{C}$ . Suppose that  $f_n \xrightarrow{u} f$ . Then

- (1) if  $\mu(X) < \infty$ , then  $f \in L^1(X, \mathcal{A}, \mu)$  and  $\lim_{n \rightarrow \infty} \int f_n = \int f$
- (2) if  $\mu(X) = \infty$ , then the conclusion of (1) may fail (find an example on  $\mathbb{R}$  with Lebesgue measure).

*Proof.* Choose  $N \in \mathbb{N}$  such that for  $n \geq N$  and  $x \in X$ ,  $|f(x) - f_n(x)| < 1$ . Then  $||f| - |f_N|| < 1$  and so  $|f| < |f_N| + 1$ . Thus  $\int |f| \leq \int |f_N| + \mu(X) < \infty$  and  $f \in L^1$ . Similarly for  $n \geq N$ ,  $|f_n| < |f| + 1$ . Dominated convergence then gives us that  $\lim_{n \rightarrow \infty} \int f_n = \int f$  as required. To see the necessity that  $\mu(X) < \infty$ , consider  $f \equiv 0$  and  $f_n = (1/n)\chi_{(0,n)}$ . Then  $f_n \xrightarrow{u} f$ , but  $1 = \lim_{n \rightarrow \infty} \int f_n \neq \int f = 0$ . □

**Exercise 3.2.12.** Generalized Dominated Convergence Let  $f_n, g_n, f, g \in L^1$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$ ,  $g_n \xrightarrow{\text{a.e.}} g$  and for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g_n$ . If

$$\int g_n d\mu \rightarrow \int g d\mu$$

then

$$\int f_n d\mu \rightarrow \int f d\mu$$

*Proof.* We simply use Fatou's lemma. Put  $h_n = (g + g_n) - |f_n - f|$ . Since for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g_n$ , we know that  $|f| \leq g$ . So  $h_n \geq 0$  and  $h_n \xrightarrow{\text{p.w.}} 2g$ . Thus

$$\begin{aligned} 2 \int g &= \int \liminf_{n \rightarrow \infty} h_n \\ &\leq \liminf_{n \rightarrow \infty} \left[ \left( \int g + \int g_n \right) - \int |f_n - f| \right] \\ &= 2 \int g + \liminf_{n \rightarrow \infty} \left( - \int |f_n - f| \right) \\ &= 2 \int g - \limsup_{n \rightarrow \infty} \int |f_n - f| \end{aligned}$$

Hence  $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$  which implies that  $\int |f_n - f| \rightarrow 0$  and  $\int f_n \rightarrow \int f$  as required.  $\square$

**Exercise 3.2.13.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$  and  $f \in L^1$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$ . Then  $\int |f_n - f| \rightarrow 0$  iff  $\int |f_n| \rightarrow \int |f|$ .

*Proof.* Suppose that  $\int |f_n - f| \rightarrow 0$ . Since

$$\begin{aligned} \left| \int |f_n| - \int |f| \right| &= \left| \int (|f_n| - |f|) \right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|, \end{aligned}$$

we see that  $\int |f_n| \rightarrow \int |f|$ . Conversely, suppose that  $\int |f_n| \rightarrow \int |f|$ . Put  $h_n = |f_n - f|$ ,  $g_n = |f_n| + |f|$ ,  $h \equiv 0$  and  $g = 2f$ . Then  $h_n \xrightarrow{\text{a.e.}} h$ ,  $g_n \xrightarrow{\text{a.e.}} g$  and for each  $n \in \mathbb{N}$ ,  $h_n \leq g_n$ . Our assumption implies that  $\int g_n \rightarrow \int g$ . Thus the last exercise tells us that  $\int h_n \rightarrow \int h$  as required.  $\square$

**Exercise 3.2.14.** Let  $(r_n)_{n \in \mathbb{N}}$  be an enumeration of the rationals. Define  $f : \mathbb{R} \rightarrow [0, \infty)$  by

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}$$

and define  $g : X \rightarrow [0, \infty]$  by

$$g(x) = \sum_{n \in \mathbb{N}} 2^{-n} f(x - r_n).$$

Then

- (1)  $g \in L^1$  (perhaps after redefinition on a null set) and particularly  $g < \infty$  a.e.
- (2)  $g^2 < \infty$  a.e., but  $g^2$  is not integrable on any subinterval of  $\mathbb{R}$
- (3) Taking  $g \in L^1$ ,  $g$  is unbounded on each subinterval of  $\mathbb{R}$  and discontinuous everywhere and remains so after redefinition on a null set

*Proof.* For convenience, define  $f_n : \mathbb{R} \rightarrow [0, \infty)$  by  $f_n(x) = f(x - r_n)$  for  $x \in \mathbb{R}$ . To show (1) we note that for each  $n \in \mathbb{N}$ ,  $f_n \in L^1$  and

$$\begin{aligned} \int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1} \end{aligned}$$

Hence

$$\sum_{n \in \mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set,  $g \in L^1$ . In particular  $\int |g| < \infty$  and so  $|g|$  (and hence  $g$ ) are finite almost everywhere. For (2), since  $g < \infty$  a.e., so too is  $g^2$ . Let  $a, b \in \mathbb{R}$  and suppose that  $a < b$ . Choose  $N \in \mathbb{N}$  such that  $r_N \in (a, b)$ . Since all the terms in the sum are nonnegative,  $g^2 \geq \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2$  and so

$$\begin{aligned} \int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2 \\ &= \sum_{n \in \mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\ &\geq 2^{-2N} \int_{(a,b)} f_N^2 \\ &\geq 2^{-2N} \int_{r_N}^{b \wedge (r_N+1)} \frac{1}{x - r_N} dx \\ &= \infty \end{aligned}$$

So  $g^2$  is not integrable on any subinterval of  $\mathbb{R}$ . For (3), note that redefining  $g$  on a null set does not change the result of (2). Suppose that there is a finite subinterval  $I \subset \mathbb{R}$  such that  $g$  is bounded on  $I$ . Hence there exists  $M > 0$  such that for each  $x \in I$ ,  $g(x)^2 \leq M$ . Then

$$\begin{aligned} \int_I g^2 &\leq M^2 m(I) \\ &< \infty \end{aligned}$$

which is a contradiction. So  $g$  is not bounded on any subinterval of  $\mathbb{R}$ . Now, suppose that there exists  $x_0 \in \mathbb{R}$  such that  $g$  is continuous at  $x_0$ . Choose  $\delta > 0$  such that for each  $x \in \mathbb{R}$ , if  $|x - x_0| < \delta$ , then  $|g(x) - g(x_0)| < 1$ . The reverse triangle inequality tells us that for each  $x \in (x_0 - \delta, x_0 + \delta)$ ,  $|g(x)| < 1 + |g(x_0)|$ . Hence  $g$  is bounded on  $(x_0 - \delta, x_0 + \delta)$  which is a contradiction. So  $g$  is discontinuous everywhere.  $\square$

**Exercise 3.2.15.** Let  $f \in L^1$ .

- (1) If  $f$  is bounded, then for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ , if  $\mu(E) < \delta$ , then  $\int_E |f| < \epsilon$ .
- (2) The same conclusion holds for general  $f \in L^1$ .

*Proof.* (1) Since  $f$  is bounded, there exists  $M > 0$  such that  $|f| \leq M$ . Let  $\epsilon > 0$ . Choose  $\delta = \epsilon/2M$ . Let  $E \in \mathcal{A}$ . Suppose that  $\mu(A) < \delta$ . Then

$$\begin{aligned} \int_E |f| &\leq M\mu(E) \\ &= M \frac{\epsilon}{2M} \\ &= \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

(2) Suppose that  $f$  is unbounded. Let  $\epsilon > 0$ . Then there exists  $\phi \in L^1$  such that  $\phi$  is simple and  $\int |f - \phi| < \epsilon/2$ . Since  $\phi$  is bounded, there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ , if  $\mu(E) < \delta$ , then  $\int_E |\phi| < \epsilon/2$ . Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \delta$ . Then

$$\begin{aligned} \int_E |f| &\leq \int_E |f - \phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

□

**Exercise 3.2.16.** Let  $f \in L^1(\mathbb{R}, \mathcal{L}, m)$ . Define  $F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x) = \int_{(-\infty, x]} f \, dm$$

Then  $F$  is continuous.

*Proof.* Let  $x_0 \in \mathbb{R}$  and  $\epsilon > 0$ . Since  $f \in L^1$ , there exists  $\delta > 0$  such that for  $x \in \mathbb{R}$ , if  $|x - x_0| < \delta$ , then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| \, dm < \epsilon.$$

Let  $x \in \mathbb{R}$ . Suppose that  $|x - x_0| < \delta$ . Then

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f \, dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f| \, dm \\ &< \epsilon \end{aligned}$$

So  $F$  is continuous.

□

**Exercise 3.2.17.** Let  $x \in X$  and denote by  $\delta_x$  the point mass measure at  $x \in X$  on measurable space  $(X, \mathcal{P}(X))$ . Let  $f : X \rightarrow \mathbb{C}$ . Then

$$\int f d\delta_x = f(x)$$



*Proof.* First assume that  $f$  is simple. Then there exist  $(a_j)_{j=1}^n \subset \mathbb{C}$  and  $(E_j)_{j=1}^n \subset \mathcal{P}(X)$  such that  $(E_j)_{j=1}^n$  is disjoint and  $f = \sum_{i=1}^n a_i \chi_{E_i}$ . Choose  $j^* \in \{1, \dots, n\}$  such that  $x \in E_{j^*}$ . Thus

$$\begin{aligned} \int f d\delta_x &= \int \sum_{j=1}^n c_j \chi_{E_j} d\delta_x \\ &= \sum_{j=1}^n c_j \delta_x(E_j) \\ &= c_{j^*} \delta_x(E_{j^*}) \\ &= c_{j^*} \\ &= f(x) \end{aligned}$$

Now for  $f \in L^+$ , choose a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1}$  and  $\phi_n \xrightarrow{\text{p.w.}} f$ . Then monotone convergence implies that

$$\begin{aligned} \int f d\delta_x &= \int \lim_{n \rightarrow \infty} \phi_n d\delta_x \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\delta_x \\ &= \lim_{n \rightarrow \infty} \phi_n(x) \\ &= f(x) \end{aligned}$$

Now just extend to complex valued functions. □

**Exercise 3.2.18.** Denote by  $\#$  the counting measure on the measurable space  $(X, \mathcal{P}(X))$ . Let  $f : X \rightarrow \mathbb{C}$  and suppose that  $f \in L^1$ . Then

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if  $f$  is integrable, then  $\{x \in X : f(x) \neq 0\}$  is countable.

*Proof.* Please refer to the definition of the sum in the appendix. First suppose that  $f(X) \subset [0, \infty)$ . For  $n \in \mathbb{N}$ , put  $X_n = \{x \in X : f(x) > 1/n\}$  and define  $X_+ = \{x \in X : f(x) > 0\}$ ,  $X_0 = \{x \in X : f(x) = 0\}$ . Then  $X_+ = \bigcup_{n \in \mathbb{N}} X_n$ . Since  $f \in L^1$ , we have that for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} \infty &> \int f d\# \\ &\geq \int_{X_n} f d\# \\ &\geq \frac{1}{n} \#(X_n). \end{aligned}$$

Thus for each  $n \in \mathbb{N}$ ,  $X_n$  is finite and  $X_+$  is countable. Thus there exists  $\{x_n\}_{n \in \mathbb{N}} \subset X$  such that  $X_+ = \{x_n\}_{n \in \mathbb{N}}$ . For  $n \in \mathbb{N}$ , define  $E_n = \{x_1, x_2, \dots, x_n\}$  and

$$\begin{aligned} f_n &= f \chi_{E_n} \\ &= \sum_{i=1}^n f(x_i) \chi_{\{x_i\}} \end{aligned}$$

Then  $f_n \xrightarrow{\text{p.w.}} f \chi_{X_+} = f$  and for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$ . So

$$\begin{aligned} \int f &= \sup_{n \in \mathbb{N}} \int f_n \\ &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n f(x_i) \\ &= \sum_{x \in X_+} f(x) \\ &= \sum_{x \in X} f(x). \end{aligned}$$

For  $f : X \rightarrow \mathbb{C}$ , our  $L^1$  assumption and the result above tell us that

$$\sum_{x \in X} |f(x)| < \infty.$$

Thus writing  $f = g + ih$ , we see that the same is true for  $f^+, f^-, g^+, g^-$ . Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f d\# = \sum_{x \in X} f(x).$$

□

**Exercise 3.2.19.** Let  $f, g : X \rightarrow \mathbb{R}$ . Suppose that  $f, g \in L^1$ . Then  $f \leq g$  a.e. iff for each  $E \in \mathcal{A}$ ,

$$\int_E f \leq \int_E g$$

*Proof.* Suppose  $f \leq g$  a.e. Put  $N = \{x \in X : f(x) > g(x)\} \subset N$ . Then  $\mu(N) = 0$  and  $g - f \geq 0$  on  $N^c$ . So for each  $E \in \mathcal{A}$ ,

$$\begin{aligned} \int_E g d\mu - \int_E f d\mu &= \int_E (g - f) d\mu \\ &= \int_{E \cap N^c} (g - f) d\mu \\ &\geq 0 \end{aligned}$$

Conversely, suppose that for each  $E \in \mathcal{A}$ ,

$$\int_E f d\mu \leq \int_E g d\mu$$

Put  $N_n = \{x \in X : f(x) - g(x) > 1/n\}$  and  $N = \{x \in X : f(x) > g(x)\}$ . Then  $N = \bigcup_{n \in \mathbb{N}} N_n$ . Let  $n \in \mathbb{N}$ . Then our assumption tells us that

$$\begin{aligned} 0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n} \mu(N_n) \\ &\geq 0. \end{aligned}$$

So that  $\mu(N_n) = 0$ . Thus for each  $n \in \mathbb{N}$ ,  $\mu(N_n) = 0$  which implies  $\mu(N) = 0$ . Therefore  $f \leq g$  a.e. as required.  $\square$

**Exercise 3.2.20.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $f : X \times \mathbb{R} \rightarrow \mathbb{C}$ . Suppose that for each  $t \in \mathbb{R}$ ,  $f(\cdot, t) \in L^1(\mu)$ . Define  $F : \mathbb{R} \rightarrow \mathbb{C}$  by

$$F(t) = \int_X f(x, t) d\mu(x)$$

- (1) Suppose that there exists  $g \in L^1(\mu)$  such that for each  $(x, t) \in X \times \mathbb{R}$ ,  $|f(x, t)| \leq g(x)$ . Let  $t_0 \in \mathbb{R}$ . If for each  $x \in X$ ,  $f(x, \cdot)$  is continuous at  $t_0$ , then  $F$  is continuous at  $t_0$ .
- (2) Suppose that  $\partial f / \partial t$  exists and there exists  $g \in L^1(\mu)$  such that for each  $(x, t) \in X \times \mathbb{R}$ ,  $|\partial f / \partial t(x, t)| \leq g(x)$ . Then  $F$  is differentiable and for each  $t \in \mathbb{R}$ ,

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t) d\mu(x)$$

*Proof.*

- (1) Suppose that for each  $x \in X$ ,  $f(x, \cdot)$  is continuous at  $t_0$ . Let  $(t_n) \subset \mathbb{R}$ . Suppose that  $t_n \rightarrow t_0$ . Then  $f(\cdot, t_n) \xrightarrow{\text{p.w.}} f(\cdot, t_0)$ . Since for each  $n \in \mathbb{N}$ ,  $|f(x, t_n)| \leq g(x)$ , the dominated convergence theorem implies that  $F(t_n) \rightarrow F(t_0)$ .
- (2) Let  $t_0 \in \mathbb{R}$ . Choose  $(t_n)_{n \in \mathbb{N}}$  such that  $t_n \rightarrow t_0$  and for each  $n \in \mathbb{N}$ ,  $t_n < t_0$ . For  $n \in \mathbb{N}$ , define  $q_n : X \rightarrow \mathbb{R}$  by

$$q_n(x) = \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}$$

So  $q_n(\cdot) \xrightarrow{\text{p.w.}} \partial f / \partial t(\cdot, t_0)$ . The mean value theorem implies that for each  $x \in X$  and  $n \in \mathbb{N}$ , there exists  $c_{n,x} \in (t_n, t_0)$  such that  $q_n(x) = \partial f / \partial t(x, c_{n,x})$ . Then for each  $n \in \mathbb{N}$ ,  $|q_n| \leq g$ . The dominated convergence theorem then implies that  $\partial f / \partial t(\cdot, t_0) \in L^1(\mu)$  and

$$\begin{aligned} \int \frac{\partial f}{\partial t}(x, t_0) d\mu(x) &= \lim_{n \rightarrow \infty} \int_X q_n d\mu \\ &= \lim_{n \rightarrow \infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} \\ &= F'(t_0^-) \end{aligned}$$

So that  $F$  is differentiable at  $t_0$  from the left. Similarly,  $F$  is differentiable at  $t_0$  from the right.  $\square$

**Exercise 3.2.21.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(Y, \mathcal{B})$  a measurable space and  $f : X \rightarrow Y$ . Suppose that  $f$  is  $(\mathcal{A}, \mathcal{B})$ -measurable. Then for each  $g \in L^0(Y, \mathcal{B})$  and  $B \in \mathcal{B}$ ,

- (1)  $g \circ f \in L^1(X, \mathcal{A})$  iff  $g \in L^1(Y, \mathcal{B}, f_*\mu)$
- (2) if  $g \circ f \in L^1(X, \mathcal{A}, \mu)$ ,

$$\int_{f^{-1}(B)} g \circ f \, d\mu = \int_B g \, df_*\mu$$

*Proof.* Let  $g \in L^0(Y, \mathcal{B})$  and  $B \in \mathcal{B}$ .

- (1) Suppose that  $g \circ f \in L^1(X, \mathcal{A}, \mu)$ . Since  $|g| \in L^+(X, \mathcal{A})$  and  $|g \circ f| = |g| \circ f$ , an exercise in the previous section implies that

$$\begin{aligned} \int_B |g| \, df_*\mu &= \int_{f^{-1}(B)} |g| \circ f \, d\mu \\ &= \int_{f^{-1}(B)} |g \circ f| \, d\mu \\ &< \infty \end{aligned}$$

Hence  $g \in L^1(Y, \mathcal{B}, f_*\mu)$ .

Conversely, suppose that  $g \in L^1(Y, \mathcal{B}, f_*\mu)$ . Since  $|g \circ f| \in L^+(X, \mathcal{B})$ , we have that

$$\begin{aligned} \int_{f^{-1}(B)} |g \circ f| \, d\mu &= \int_{f^{-1}(B)} |g| \circ f \, d\mu \\ &= \int_B |g| \, df_*\mu \\ &< \infty \end{aligned}$$

Hence  $g \circ f \in L^1(X, \mathcal{A}, \mu)$ .

- (2) Suppose that  $g \circ f \in L^1(X, \mathcal{A}, \mu)$ . Write  $g = h_1^+ - h_1^- + i(h_2^+ - h_2^-)$ . Since  $h_1^+, h_1^-, h_2^+, h_2^- \in L^+(Y, \mathcal{B})$ , an exercise in the previous section implies that

$$\begin{aligned} \int_{f^{-1}(B)} g \circ f \, d\mu &= \int_{f^{-1}(B)} \left[ h_1^+ - h_1^- + i(h_2^+ - h_2^-) \right] \circ f \, d\mu \\ &= \int_{f^{-1}(B)} h_1^+ \circ f \, d\mu - \int_{f^{-1}(B)} h_1^- \circ f \, d\mu \\ &\quad + i \int_{f^{-1}(B)} h_2^+ \circ f \, d\mu - i \int_{f^{-1}(B)} h_2^- \circ f \, d\mu \\ &= \int_B h_1^+ \, df_*\mu - \int_B h_1^- \, df_*\mu + i \int_B h_2^+ \, df_*\mu - i \int_B h_2^- \, df_*\mu \\ &= \int_B h_1^+ - h_1^- + i(h_2^+ - h_2^-) \, df_*\mu \\ &= \int_B g \, df_*\mu \end{aligned}$$

□

**Definition 3.2.22.** Let  $\mathcal{F} \subset L^1$ . Then  $\mathcal{F}$  is said to be **uniformly integrable** if for each  $\epsilon > 0$ , there exists  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then  $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$ . (i.e.  $\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$ ).

**Exercise 3.2.23.** Suppose that  $\mu$  is finite. Let  $\mathcal{F} \subset L^1$ . Then  $\mathcal{F}$  is uniformly integrable iff

- (1) there exists  $M > 0$  such that  $\sup_{f \in \mathcal{F}} \int |f| \leq M$
- (2) for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $E \in \mathcal{A}$ , if  $\mu(E) < \delta$ , then  $\sup_{f \in \mathcal{F}} \int_E |f| < \epsilon$ .

*Proof.* ( $\implies$ ): (1) Suppose that  $\mathcal{F}$  is uniformly integrable. Then there exists  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then  $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < 1$ . Choose  $M = \mu(X)K + 1$ . Then for each  $f \in \mathcal{F}$ ,

$$\begin{aligned} \int |f| &= \int_{\{|f| > K\}} |f| + \int_{\{|f| \leq K\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M \end{aligned}$$

- (2) Let  $\epsilon > 0$ . Then choose  $K \in \mathbb{N}$  such that  $\sup_{f \in \mathcal{F}} \int_{\{|f| > K\}} |f| < \epsilon/2$  and choose  $\delta = \epsilon/2K$ .

Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \delta$ . Then for  $f \in \mathcal{F}$ ,

$$\begin{aligned} \int_E |f| &= \int_{E \cap \{|f| > K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon \end{aligned}$$

( $\impliedby$ ): Choose  $M > 0$  as in (1). Suppose that there exists  $\epsilon > 0$  such that for each  $K \in \mathbb{N}$ , there exists  $f \in \mathcal{F}$  such that  $\mu(\{|f| > K\}) \geq \epsilon$ . Choose  $K \in \mathbb{N}$  such that  $K > M/\epsilon$ . Then choose  $f_K \in \mathcal{F}$  such that  $\mu(\{|f_K| > K\}) \geq \epsilon$ . Then

$$\begin{aligned} \int |f_K| &\geq \int_{\{|f_K| > K\}} |f| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M, \end{aligned}$$

which is a contradiction. Hence for each  $\epsilon > 0$ , there exists  $K \in \mathbb{N}$  such that for each  $f \in \mathcal{F}$ ,  $\mu(\{|f| > K\}) < \epsilon$ . Since  $\mu(\{|f| > k\})$  is a decreasing sequence in  $k$ , we have that  $\lim_{k \rightarrow \infty} \mu(\{|f| > k\}) = 0$ . Now, let  $\epsilon > 0$ . Choose  $\delta > 0$  as in (2). Choose  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then for each  $f \in \mathcal{F}$ ,  $\mu(\{|f| > k\}) < \delta$ . Then for each  $k \in \mathbb{N}$ , if  $k \geq K$ , then for each  $f \in \mathcal{F}$ ,

$$\int_{\{|f| > k\}} |f| < \epsilon.$$

Thus

$$\lim_{k \rightarrow \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$$

as required. □

**Definition 3.2.24.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Define  $\|\cdot\|_* : L^1(\mu) \rightarrow [0, \infty)$  by

$$\|f\|_* = \sup_{A \in \mathcal{A}} \left| \int_A f \, d\mu \right|$$

**Exercise 3.2.25.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Then  $\|\cdot\|_*$  is a norm on  $L^1(\mu)$  and there exists  $C > 0$  such that  $C\|\cdot\|_1 \leq \|\cdot\|_* \leq \|\cdot\|_1$ .

### 3.3. Integration on Product Spaces.

**Note 3.3.1.** Recall the definition of the sections of  $E$  and  $f$  from the section on product  $\sigma$ -algebras. It is often helpful to observe that  $(\chi_E)_x = \chi_{E_x}$  and  $(\chi_E)^y = \chi_{E^y}$ .

**Theorem 3.3.2.** Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces. Then for each  $E \in \mathcal{A} \otimes \mathcal{B}$ , the maps  $\phi : X \rightarrow [0, \infty]$  and  $\psi : Y \rightarrow [0, \infty]$  defined by  $\phi(x) = \nu(E_x)$  and  $\psi(y) = \mu(E^y)$  are  $\mathcal{A}$ -measurable and  $\mathcal{B}$ -measurable, respectively and

$$\mu \otimes \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

**Theorem 3.3.3. Fubini, Tonelli:** Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces.

- (1) (Tonelli) For each  $f \in L^+(X \times Y)$ , the functions  $g : X \rightarrow [0, \infty]$ ,  $h : Y \rightarrow [0, \infty]$  defined by  $g(x) = \int_Y f_x(y) d\nu(y)$  and  $h(y) = \int_X f^y(x) d\mu(x)$  are  $\mathcal{A}$ -measurable and  $\mathcal{B}$ -measurable respectively and

$$\int_{X \times Y} f d\mu \otimes \nu = \int_X g d\mu = \int_Y h d\nu$$

- (2) (Fubini) For each  $f \in L^1(X \times Y)$ ,  $f_x \in L^1(\nu)$  for  $\mu$ -a.e.  $x \in X$  and  $f^y \in L^1(\mu)$  for  $\nu$ -a.e.  $y \in Y$ , respectively and (after redefinition of  $f$  on a null set) the functions  $g : X \rightarrow \mathbb{C}$ ,  $h : Y \rightarrow \mathbb{C}$  defined by  $g(x) = \int_Y f_x(y) d\nu(y)$  and  $h(y) = \int_X f^y(x) d\mu(x)$  are in  $L^1(\mu)$  and  $L^1(\nu)$  respectively. Furthermore

$$\int_{X \times Y} f d\mu \otimes \nu = \int_X g d\mu = \int_Y h d\nu$$

**Note 3.3.4.** We usually just write

$$\int \int f d\mu d\nu \text{ and } \int \int f d\nu d\mu$$

instead of

$$\int h d\nu$$

and

$$\int g d\mu$$

respectively. We have a similar result for complete product measure spaces. See

**Exercise 3.3.5.** Take  $X = Y = [0, 1]$ ,  $\mathcal{A} = \mathcal{B}([0, 1])$ ,  $\mathcal{B} = \mathcal{P}([0, 1])$  and  $\mu, \nu$  to be Lebesgue measure and counting measure respectively. Define  $D = \{(x, y) \in [0, 1]^2 : x = y\}$  Show that

$$\int \chi_D d\mu \otimes \nu, \int \int \chi_D d\mu d\nu \text{ and } \int \int \chi_D d\nu d\mu$$

are all different. (Hint: for the first integral, use the definition of  $\mu \otimes \nu$ )

*Proof.* Let  $x, y \in [0, 1]$ . Then  $(\chi_D)_x = \chi_{D_x} = \chi_x$  and  $(\chi_D)^y = \chi_{D^y} = \chi_y$ . Thus

$$\begin{aligned}
\int \int \chi_D d\mu d\nu &= \int \mu(\{y\}) d\nu \\
&= \int 0 d\nu \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\int \int \chi_D d\mu d\nu &= \int \nu(\{x\}) d\mu \\
&= \int 1 d\mu \\
&= 1
\end{aligned}$$

Now, Observe that  $\int \chi_D d\mu \otimes \nu = \mu \otimes \nu(D)$ . Recall from the section on product measures that  $\mu \otimes \nu(D) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) : (A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n\}$ . Let  $(A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ . Suppose that  $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n$ . Then for each  $x \in [0, 1]$ ,  $(x, x) \in \bigcup_{n \in \mathbb{N}} A_n \times B_n$ . So for each  $x \in [0, 1]$ , there exists  $n \in \mathbb{N}$ , such that  $x \in A_n \cap B_n$ . Thus  $[0, 1] \subset \bigcup_{n \in \mathbb{N}} A_n \cap B_n$ . Since  $1 = \mu([0, 1]) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap B_n)$ , we know that there exists  $n \in \mathbb{N}$  such that  $0 < \mu(A_n \cap B_n)$ . Thus  $\mu(A_n) > 0$  and  $\mu(B_n) > 0$ . Since  $\mu(B_n) > 0$ ,  $B_n$  must be infinite and therefore  $\nu(B_n) = \infty$ . So  $\sum_{n \in \mathbb{N}} \mu(A_n) \nu(B_n) = \infty$ .  $\square$

**Exercise 3.3.6.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $f : X \rightarrow [0, \infty) \in L^+$ . Show that  $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$  and  $\mu \times m(G) = \int_X f d\mu$ . The same is true if we replace " $\geq$ " with " $>$ ". (Hint: to show that  $G$  is measurable, split up  $(x, y) \mapsto f(x) - y$  into the composition of measurable functions.

*Proof.* Define  $\phi : X \times [0, \infty) \rightarrow [0, \infty)^2$  and  $\psi : [0, \infty)^2 \rightarrow [0, \infty)$  by  $\phi(x, y) = (f(x), y)$  and  $\psi(z, y) = z - y$ . Then  $G = \{(x, y) \in X \times [0, \infty) : \psi \circ \phi(x, y) \geq 0\}$ . Let  $A, B \in \mathcal{B}([0, \infty))$ . Then  $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0, \infty))$ . Since  $\mathcal{B}([0, \infty)^2) = \mathcal{B}([0, \infty)) \otimes \mathcal{B}([0, \infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0, \infty))\})$ , we have that  $\phi$  is  $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty)^2)$  measurable. Since  $\psi$  is continuous, we have that  $\psi$  is  $\mathcal{B}([0, \infty)^2)$ - $\mathcal{B}([0, \infty))$  measurable. This implies that  $\psi \circ \phi$  is  $\mathcal{A} \otimes \mathcal{B}([0, \infty))$ - $\mathcal{B}([0, \infty))$  measurable. Thus  $G = \psi \circ \phi^{-1}([0, \infty)) \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ . Now for  $x \in X$ ,  $G_x = \{y \in [0, \infty) : f(x) \geq y\} = [0, f(x)]$ . Thus

$$\begin{aligned}
\mu \times m(G) &= \int \chi_G d\mu \times m \\
&= \int_X \int_{[0, \infty)} \chi_{G_x} dm d\mu(x) \\
&= \int_X f(x) d\mu(x)
\end{aligned}$$

The same reasoning holds if we replace " $\geq$ " with " $>$ ".  $\square$

**Exercise 3.3.7.** Let  $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces and  $f : X \rightarrow \mathbb{C}, g : Y \rightarrow \mathbb{C}$ . Define  $h : X \times Y \rightarrow \mathbb{C}$  by  $h(x, y) = f(x)g(y)$ .



- (1) If  $f$  is  $\mathcal{A}$ -measurable and  $g$  is  $\mathcal{B}$ -measurable, then  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable.  
 (2) If  $f \in L^1(\mu)$  and  $g \in L^1(\nu)$ , then  $h \in L^1(\mu \otimes \nu)$  and

$$\int_{X \times Y} h \, d\mu \otimes \nu = \int_X f \, d\mu \int_Y g \, d\nu$$

*Proof.*

- (1) First suppose that  $f, g$  are simple. Then there exist  $(A_i)_{i=1}^n \subset \mathcal{A}$ ,  $(B_j)_{j=1}^m \subset \mathcal{B}$  and  $(a_i)_{i=1}^n, (b_j)_{j=1}^m \subset \mathbb{C}$  such that  $f = \sum_{i=1}^n a_i \chi_{A_i}$  and  $g = \sum_{j=1}^m b_j \chi_{B_j}$ . Then  $h = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{A_i \times B_j}$ . So  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable. For general  $f, g$ , there exist  $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$  and  $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$  such that  $f_n \rightarrow f$  pointwise,  $g_n \rightarrow g$  pointwise and for each  $n \in \mathbb{N}$ ,  $|f_n| \leq |f_{n+1}| \leq |f|$  and  $|g_n| \leq |g_{n+1}| \leq |g|$ . For  $n \in \mathbb{N}$ , define  $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$  by  $h_n = f_n g_n$ . Then  $h_n \rightarrow h$  pointwise and for each  $n \in \mathbb{N}$ ,  $|h_n| \leq |h_{n+1}| \leq |h|$ . Thus  $h$  is  $\mathcal{A} \otimes \mathcal{B}$ -measurable.  
 (2) First suppose  $f$  and  $g$  are simple as before. Then

$$\begin{aligned} \int_{X \times Y} |h| \, d\mu \otimes \nu &\leq \sum_{i=1}^n \sum_{j=1}^m |a_i b_j| \mu(A_i) \nu(B_j) \\ &= \left( \sum_{i=1}^n |a_i| \mu(A_i) \right) \left( \sum_{j=1}^m |b_j| \nu(B_j) \right) \\ &= \int_X |f| \, d\mu \int_Y |g| \, d\nu \\ &< \infty \end{aligned}$$

So  $h \in L^1(\mu \otimes \nu)$ . Furthermore,

$$\begin{aligned} \int_{X \times Y} h \, d\mu \otimes \nu &= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mu(A_i) \nu(B_j) \\ &= \left( \sum_{i=1}^n a_i \mu(A_i) \right) \left( \sum_{j=1}^m b_j \nu(B_j) \right) \\ &= \int_X f \, d\mu \int_Y g \, d\nu \end{aligned}$$

For general  $f \in L^1(\mu), g \in L^1(\nu)$ , take  $(h_n)_{n \in \mathbb{N}}$  as before. Monotone convergence and the result above say that

$$\begin{aligned} \int_{X \times Y} |h| \, d\mu \otimes \nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} |h_n| \, d\mu \otimes \nu \\ &= \lim_{n \rightarrow \infty} \left( \int_X |f_n| \, d\mu \int_Y |g_n| \, d\nu \right) \\ &= \int_X |f| \, d\mu \int_Y |g| \, d\nu \\ &< \infty \end{aligned}$$

So  $h \in L^1(\mu \otimes \nu)$ . Dominated convergence and the result above then tell us that

$$\begin{aligned}
\int_{X \times Y} h \, d\mu \times d\nu &= \lim_{n \rightarrow \infty} \int_{X \times Y} h_n \, d\mu \times d\nu \\
&= \lim_{n \rightarrow \infty} \left( \int_X f_n \, d\mu \int_Y g_n \, d\nu \right) \\
&= \int_X f \, d\mu \int_Y g \, d\nu
\end{aligned}$$

□

**Note 3.3.8.** In the above exercise part (2), we can replace  $L^1$  with  $L^+$  and get the same result by the same method.

**Exercise 3.3.9.** Let  $f : \mathbb{R} \rightarrow [0, \infty) \in L^+$ . Show that

$$\int_{\mathbb{R}} f \, dm = \int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) \, dm(t)$$

*Proof.* Note that

$$\int_{[0, \infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) \, dm(t) = \int_{[0, \infty)} \left[ \int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}} \, dm \right] dm(t)$$

Comparing this with Tonelli's theorem, we can put  $\chi_{\{x \in \mathbb{R} : f(x) \geq t\}} = (\chi_E)^t = \chi_{E^t}$ . Then  $E = \{(x, t) \in \mathbb{R} \times [0, \infty) : f(x) \geq t\}$  and  $E_x = \{t \in [0, \infty) : f(x) \geq t\} = [0, f(x)]$ . Tonelli's theorem tells us that

$$\begin{aligned}
\int_{[0, \infty)} \left[ \int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}}(x) \, dm(x) \right] dm(t) &= \int_{\mathbb{R}} \left[ \int_{[0, \infty)} \chi_{[0, f(x)]}(t) \, dm(t) \right] dm(x) \\
&= \int_{\mathbb{R}} f(x) \, dm(x)
\end{aligned}$$

□

### 3.4. Modes of Convergence.

**Definition 3.4.1.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(Y, d)$  a metric space,  $(f_n)_{n \in \mathbb{N}} \subset L_Y^0(X, \mathcal{A}, \mu)$  and  $f \in L_Y^0(X, \mathcal{A}, \mu)$ . Then  $(f_n)_{n \in \mathbb{N}}$  is said to **converge to  $f$  in measure**, denoted  $f_n \xrightarrow{\mu} f$ , if for each  $\epsilon > 0$ ,

$$\mu(\{x \in X : d(f_n(x), f(x)) \geq \epsilon\}) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

**Definition 3.4.2.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ . Then  $(f_n)_{n \in \mathbb{N}}$  is said to be **Cauchy in measure** if for each  $\epsilon > 0$ ,

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

i.e. for each  $\epsilon, \delta > 0$ , there exists  $N \in \mathbb{N}$  such that for each  $n, m \in \mathbb{N}$ ,  $n, m \geq N$  implies that  $\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) < \delta$ .

**Note 3.4.3.** It is useful to observe that

$$\bigcup_{\epsilon > 0} \limsup_{n \rightarrow \infty} \{x \in X : |f_n(x) - f(x)| \geq \epsilon\} = \{x \in X : f_n(x) \not\rightarrow f(x)\}$$

and

$$\bigcap_{\epsilon > 0} \liminf_{n \rightarrow \infty} \{x \in X : |f_n(x) - f(x)| < \epsilon\} = \{x \in X : f_n(x) \rightarrow f(x)\}$$

**Exercise 3.4.4.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . If  $f_n \xrightarrow{\mu} f$ , then  $(f_n)_{n \in \mathbb{N}}$  is Cauchy in measure.

*Proof.* Suppose that  $f_n \xrightarrow{\mu} f$ . For  $\epsilon > 0$  and  $n, m \in \mathbb{N}$ , set

$$A_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$$

and

$$B_{n,m,\epsilon} = \{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}$$

Let  $\epsilon > 0$ ,  $n, m \in \mathbb{N}$  and  $x \in A_{n,\frac{\epsilon}{2}}^c \cap A_{m,\frac{\epsilon}{2}}^c$ . Then

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

and  $x \in B_{n,m,\epsilon}^c$ . Therefore  $A_{n,\frac{\epsilon}{2}}^c \cap A_{m,\frac{\epsilon}{2}}^c \subset B_{n,m,\epsilon}^c$ . This implies that  $B_{n,m,\epsilon} \subset A_{n,\frac{\epsilon}{2}} \cup A_{m,\frac{\epsilon}{2}}$ . Let  $\delta > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ ,  $n \geq N$  implies that  $\mu(A_{n,\frac{\epsilon}{2}}) < \delta/2$ . Then for each  $n, m \in \mathbb{N}$ ,  $n, m \geq N$  implies that

$$\begin{aligned} \mu(B_{n,m,\epsilon}) &\leq \mu(A_{n,\frac{\epsilon}{2}}) + \mu(A_{m,\frac{\epsilon}{2}}) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta \end{aligned}$$

So for each  $\epsilon > 0$ ,

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

and  $(f_n)_{n \in \mathbb{N}}$  is Cauchy in measure. □

**Exercise 3.4.5.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f, g \in L^0$ . Suppose that  $f_n \xrightarrow{\mu} f$  and  $f_n \xrightarrow{\mu} g$ . Then  $f = g$  a.e.

*Proof.* Set  $B = \{x \in X : |f(x) - g(x)| \geq 0\}$  and for  $n, k \in \mathbb{N}$ , set

- $B_k = \{x \in X : |f(x) - g(x)| \geq \frac{1}{k}\}$
- $A_{f,n,k} = \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{k}\}$
- $A_{g,n,k} = \{x \in X : |f_n(x) - g(x)| \geq \frac{1}{k}\}$

As in the proof of Exercise 3.4.4, for each  $n, k \in \mathbb{N}$

$$\mu(B_k) \leq \mu(A_{f,n,2k}) + \mu(A_{g,n,2k})$$

Let  $\epsilon > 0$ . Convergence in measure implies that for each  $k \in \mathbb{N}$ , there exists  $N_k \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ ,  $n \geq N$  implies that  $\mu(A_{f,n,2k}), \mu(A_{g,n,2k}) < \epsilon 2^{-(1+k)}$ . Then

$$\begin{aligned} \mu(B) &= \mu\left(\bigcup_{k \in \mathbb{N}} B_k\right) \\ &\leq \sum_{k \in \mathbb{N}} \mu(B_k) \\ &\leq \sum_{k \in \mathbb{N}} \mu(A_{f,N_k,2k}) + \sum_{k \in \mathbb{N}} \mu(A_{g,N_k,2k}) \\ &\leq \sum_{k \in \mathbb{N}} \epsilon 2^{-(1+k)} + \sum_{k \in \mathbb{N}} \epsilon 2^{-(1+k)} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary,  $\mu(B) = 0$  and  $f = g$  a.e. □

**Exercise 3.4.6.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ . Suppose that  $(f_n)_{n \in \mathbb{N}}$  is Cauchy in measure.

- (1) There exists a subsequence  $(f_{n_j})_{j \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$  such that for each  $j \in \mathbb{N}$ ,

$$\mu(\{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}) < 2^{-j}$$

- (2) For  $j, k \in \mathbb{N}$  set

$$E_j = \{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}$$

and

$$F_k = \bigcup_{j \geq k} E_j$$

Then  $(F_k)_{k \in \mathbb{N}}$  is decreasing and for each  $k \in \mathbb{N}$ ,  $\mu(F_k) \leq 2^{1-k}$  and for each  $i, j, k \in \mathbb{N}$ ,  $i \geq j \geq k$  implies that for each  $x \in F_k^c$ ,

$$|f_{n_i}(x) - f_{n_j}(x)| \leq 2^{1-k}$$

So for each  $k \in \mathbb{N}$ ,  $(f_{n_j})_{j \in \mathbb{N}}$  is uniformly Cauchy on  $F_k^c$  and therefore  $(f_{n_j})_{j \in \mathbb{N}}$  is pointwise Cauchy on  $F_k^c$ .

**Hint:** get a telescoping sum via the triangle inequality

(3) Set

$$F = \bigcap_{k \in \mathbb{N}} F_k$$

Then  $\mu(F) = 0$  and there exists  $f \in L^0$  such that  $f_{n_j} \xrightarrow{\text{a.e.}} f$ .

(4) Finally,  $f_{n_j} \xrightarrow{\mu} f$ ,  $f_n \xrightarrow{\mu} f$

**Hint:** consider showing  $\{x \in X : |f_{n_k}(x) - f(x)| \geq \epsilon\} \subset F_k$  and use something similar to the proof of Exercise 3.4.4

*Proof.*

(1) By definition, for each  $j \in \mathbb{N}$ , there exists  $N_j \in \mathbb{N}$  such that for each  $n, m \in \mathbb{N}$ ,  $n, m \geq N_j$  implies that

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq 2^{-j}\}) < 2^{-j}$$

Setting  $n_1 = N_1$  and for  $j \geq 2$ , setting  $n_j = \max(n_{j-1} + 1, N_j)$ , we may obtain a subsequence  $(f_{n_j})$  such that for each  $j \in \mathbb{N}$ ,

$$\mu(\{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}) < 2^{-j}$$

(2) Clearly  $(F_k)_{k \in \mathbb{N}}$  is decreasing. Let  $k \in \mathbb{N}$ . Part (1) implies that

$$\begin{aligned} \mu(F_k) &\leq \sum_{j \geq k} 2^{-j} \\ &= 2^{1-k} \sum_{j \geq 1} 2^{-j} \\ &= 2^{1-k} \end{aligned}$$

Let  $i, j \in \mathbb{N}$ . Suppose that  $i \geq j \geq k$ . Let  $x \in F_k^c$ . Then

$$\begin{aligned} |f_{n_i}(x) - f_{n_j}(x)| &\leq \sum_{l=j}^{i-1} |f_{n_{l+1}}(x) - f_{n_l}(x)| \\ &< \sum_{l=j}^{i-1} 2^{-l} \\ &< \sum_{l \geq j} 2^{-l} \\ &= 2^{1-j} \\ &\leq 2^{1-k} \end{aligned}$$

Let  $\epsilon > 0$ . Choose  $k' \in \mathbb{N}$  such that  $k' \geq k$  and  $2^{1-k'} < \epsilon$ . Let  $i, j \in \mathbb{N}$ . Suppose that  $i, j \geq k'$ . Let  $x \in F_{k'}^c \subset F_k^c$ . Then

$$\begin{aligned} |f_{n_i}(x) - f_{n_j}(x)| &< 2^{1-k'} \\ &< \epsilon \end{aligned}$$

So  $(f_{n_j})_{j \in \mathbb{N}}$  is uniformly Cauchy on  $F_k^c$

(3) Since  $\mu(F_1) < \infty$ ,  $(F_k)_{k \in \mathbb{N}}$  is decreasing and  $F = \inf_{k \in \mathbb{N}} F_k$ , we have that

$$\begin{aligned}\mu(F) &= \inf_{k \in \mathbb{N}} \mu(F_k) \\ &\leq \inf_{k \in \mathbb{N}} 2^{1-k} \\ &= 0\end{aligned}$$

Since for each  $k \in \mathbb{N}$ ,  $(f_{n_j})_{j \in \mathbb{N}}$  is pointwise Cauchy on  $F_k^c$ ,  $(f_{n_j})_{j \in \mathbb{N}}$  is pointwise Cauchy on  $F^c$ . Then  $(f_{n_j} \chi_{F^c})_{j \in \mathbb{N}}$  is pointwise Cauchy.

Define  $f : X \rightarrow \mathbb{C}$  pointwise by

$$f = \lim_{j \rightarrow \infty} f_{n_j} \chi_{F^c}$$

Then  $f \in L^0$  since  $(f_{n_j} \chi_{F^c})_{j \in \mathbb{N}} \subset L^0$  and  $f_{n_j} \chi_{F^c} \xrightarrow{\text{p.w.}} f$ . Since  $\mu(F) = 0$  and  $\{x \in X : f_{n_j}(x) \not\rightarrow f(x)\} \subset F$ , we have that  $f_{n_j} \xrightarrow{\text{a.e.}} f$ .

(4) For  $n, m \in \mathbb{N}$  and  $\epsilon > 0$ , set

$$A_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$$

and

$$B_{m,n,\epsilon} = \{x \in X : |f_m(x) - f_n(x)| \geq \epsilon\}$$

Let  $\epsilon, \delta > 0$ . Choose  $k \in \mathbb{N}$  such that  $2^{2-k} < \epsilon$  and  $\mu(F_k) < \delta$ . Let  $x \in F_k^c$ . Since  $f_{n_j}(x) \rightarrow f(x)$ , there exists  $J \in \mathbb{N}$  such that  $J \geq k$  and for each  $j \in \mathbb{N}$ ,  $j \geq J$  implies that  $|f_{n_j}(x) - f(x)| < 2^{1-k}$ . Let  $l \in \mathbb{N}$ . Suppose that  $l \geq k$ . Then part (2) implies that

$$\begin{aligned}|f_{n_l}(x) - f(x)| &\leq |f_{n_l}(x) - f_{n_J}(x)| + |f_{n_J}(x) - f(x)| \\ &\leq 2^{1-k} + 2^{1-k} \\ &\leq 2^{2-k} \\ &< \epsilon\end{aligned}$$

So  $x \in A_{n_l,\epsilon}^c$ . Hence  $A_{n_l,\epsilon} \subset F_k$  and  $\mu(A_{n_l,\epsilon}) < \delta$ . So  $f_{n_j} \xrightarrow{\mu} f$ .

Let  $\epsilon > 0$ ,  $\delta > 0$ . Since  $(f_n)_{n \in \mathbb{N}}$  is Cauchy in measure, there exists  $J_1 \in \mathbb{N}$  such that for each  $m, n \in \mathbb{N}$ ,  $m, n \geq J_1$  implies that  $\mu(B_{m,n,\frac{\epsilon}{2}}) < \frac{\delta}{2}$ . Since  $f_{n_j} \xrightarrow{\mu} f$ , there exists  $J_2$  such that for each  $j \in \mathbb{N}$ ,  $j \geq J_2$  implies that  $\mu(A_{n_j,\frac{\epsilon}{2}}) < \frac{\delta}{2}$ . Set  $J = \max(J_1, J_2)$ . Let  $j \in \mathbb{N}$ . Suppose that  $j \geq J$ . Since  $n_j \geq j$ , the proof of Exercise 3.4.4 implies that,

$$\begin{aligned}\mu(A_{j,\epsilon}) &\leq \mu(B_{j,n_j,\frac{\epsilon}{2}}) + \mu(A_{n_j,\frac{\epsilon}{2}}) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta\end{aligned}$$

So that  $f_n \xrightarrow{\mu} f$ .

□

**Exercise 3.4.7.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ .

- (1) If  $(f_n)_{n \in \mathbb{N}}$  is Cauchy in measure, then there exists a  $f_0 \in L^0$  and a subsequence  $(f_{n_j})_{j \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$  such that  $f_n \xrightarrow{\mu} f_0$  and  $f_{n_j} \xrightarrow{\text{a.e.}} f_0$ .
- (2) If  $f_n \xrightarrow{\mu} f$ , then there exists a subsequence  $(f_{n_j})_{j \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$  such that  $f_n \xrightarrow{\text{a.e.}} f$ .

*Proof.*

- (1) Previous exercise.
- (2) Suppose that  $f_n \xrightarrow{\mu} f$ . Then  $(f_n)_{n \in \mathbb{N}}$  is Cauchy in measure. Part (1) implies that there exists a  $f_0 \in L^0$  and a subsequence  $(f_{n_j})_{j \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$  such that  $f_n \xrightarrow{\mu} f_0$  and  $f_{n_j} \xrightarrow{\text{a.e.}} f_0$ . Since  $f_n \xrightarrow{\mu} f$  and  $f_n \xrightarrow{\mu} f_0$ ,  $f = f_0$  a.e. Hence  $f_{n_j} \xrightarrow{\text{a.e.}} f$ .

□

**Exercise 3.4.8.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(f_n)_{n \in \mathbb{N}} \subset L^0(X, \mathcal{A})$  and  $f \in L^0(X, \mathcal{A})$ . Suppose that  $f_n \xrightarrow{\mu} f$ .

- (1) If for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$  a.e., then  $f_n \xrightarrow{\text{a.e.}} f$ .
- (2) If for each  $n \in \mathbb{N}$ ,  $f_n \geq f_{n+1}$  a.e., then  $f_n \xrightarrow{\text{a.e.}} f$ .

*Proof.*

- (1) Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$  a.e. Define  $N_1 \in \mathcal{A}$  by

$$N_1 = \bigcap_{n \in \mathbb{N}} \{x \in X : f_n(x) \leq f_{n+1}(x)\}$$

By assumption,  $\mu(N_1^c) = 0$ . Since  $f_n \xrightarrow{\mu} f$ , there exists a subsequence  $(f_{n_k})_{k \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$  such that  $f_{n_k} \xrightarrow{\text{a.e.}} f$ . Hence there exists  $N_2 \in \mathcal{A}$  such that  $\mu(N_2^c) = 0$  and  $f_{n_k} \chi_{N_2} \xrightarrow{\text{p.w.}} f \chi_{N_2}$ . Set  $N = N_1 \cap N_2$ . Then

$$\begin{aligned} \mu(N^c) &= \mu(N_1^c \cup N_2^c) \\ &\leq \mu(N_1^c) + \mu(N_2^c) \\ &= 0 \end{aligned}$$

By construction,  $f \chi_N = \sup_{k \in \mathbb{N}} f_{n_k} \chi_N$  which implies that for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} f_n \chi_N &\leq f_{n_n} \chi_N \\ &\leq f \chi_N \end{aligned}$$

Let  $x \in N$  and  $\epsilon > 0$ . Choose  $K \in \mathbb{N}$  such that for each  $k \in \mathbb{N}$ ,  $k \geq K$  implies that  $|f_{n_k}(x) - f(x)| < \epsilon$ . Let  $n \in \mathbb{N}$ . Suppose that  $n \geq n_K$ . Then

$$\begin{aligned} |f_n(x) - f(x)| &= f(x) - f_n(x) \\ &\leq f(x) - f_{n_K}(x) \\ &= |f_{n_K}(x) - f(x)| \\ &< \epsilon \end{aligned}$$

Hence  $f_n(x) \rightarrow f(x)$ . Since  $x \in N$  is arbitrary,  $f_n \chi_N \xrightarrow{\text{p.w.}} f \chi_N$ . Since  $\mu(N^c) = 0$ ,  $f_n \xrightarrow{\text{a.e.}} f$ .

- (2) Similar to (1).

□

**Definition 3.4.9.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Then  $(f_n)_{n \in \mathbb{N}}$  is said to **converge to  $f$  almost uniformly**, denoted  $f_n \xrightarrow{\text{a.u.}} f$ , if for each  $\epsilon > 0$ , there exists  $N \in \mathcal{A}$  such that  $\mu(N) < \epsilon$  and  $f_n \xrightarrow{u} f$  on  $N^c$ .

**Exercise 3.4.10. Egoroff's Theorem:** Suppose that  $\mu(X) < \infty$ . Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$ . Then  $f_n \xrightarrow{\text{a.u.}} f$ .

*Proof.* For each  $n, k \in \mathbb{N}$ , define  $E_{n,k} = \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{k}\}$  and  $F_{n,k} = \bigcup_{m \geq n} E_{m,k}$ .

Then  $F_{n,k}$  is decreasing in  $n$  and

$$\bigcap_{n \in \mathbb{N}} F_{n,k} \subset \{x : f_n(x) \not\xrightarrow{u} f(x)\}$$

Thus  $\mu(\bigcap_{n \in \mathbb{N}} F_{n,k}) = 0$ . Since  $\mu(X) < \infty$ ,  $\inf_{n \in \mathbb{N}} \mu(F_{n,k}) = 0$ . Let  $\epsilon > 0$ . We may choose a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}} \subset \mathbb{N}$  such that  $\mu(F_{n_k,k}) \leq \frac{\epsilon}{2^k}$ . Put  $N = \bigcup_{k \in \mathbb{N}} F_{n_k,k}$ . Then

$$\begin{aligned} \mu(N) &\leq \sum_{k \in \mathbb{N}} \mu(F_{n_k,k}) \\ &\leq \sum_{k \in \mathbb{N}} \frac{\epsilon}{2^k} \\ &= \epsilon \end{aligned}$$

Let  $\delta > 0$ . Choose  $K \in \mathbb{N}$  such that  $\frac{1}{K} < \delta$ . Then for each  $m \geq n_K$  and  $x \in N^c = \bigcap_{k \in \mathbb{N}} \bigcap_{m \geq n_k} E_{m,k}^c$ ,  $|f_m(x) - f(x)| < \frac{1}{K} < \delta$ . So  $f_n \xrightarrow{u} f$  on  $N^c$ .  $\square$

**Exercise 3.4.11.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^1$  and  $f \in L^1$ . If  $f_n \xrightarrow{L^1} f$ , then  $f_n \xrightarrow{\mu} f$ .

*Proof.* Let  $\epsilon > 0$ . for  $n \in \mathbb{N}$ , define  $E_{\epsilon,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$ . Then for  $n \in \mathbb{N}$ ,

$$\begin{aligned} \int |f - f_n| &\geq \int_{E_{\epsilon,n}} |f - f_n| \\ &\geq \epsilon \mu(E_{\epsilon,n}). \end{aligned}$$

So for each  $n \in \mathbb{N}$ ,  $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$ . Since  $\int |f - f_n| \rightarrow 0$ , we have that  $\mu(E_{\epsilon,n}) \rightarrow 0$ . Since  $\epsilon > 0$  is arbitrary,  $f_n \xrightarrow{\mu} f$  as required.  $\square$

**Exercise 3.4.12.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Suppose  $\mu(X) < \infty$ . Define  $d : L^0 \times L^0 \rightarrow [0, \infty)$  by

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu$$

Then  $d$  is a metric on  $L^0$  if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each  $f, g \in L^0$ ,  $d(f, g) \leq \mu(X)$ .

*Proof.* Let  $f, g \in L^0$ . Clearly  $d(f, g) = d(g, f)$ . If  $f = g$  a.e. then clearly  $d(f, g) = 0$ . Conversely, if  $d(f, g) = 0$ , then  $\frac{|f - g|}{1 + |f - g|} = 0$  a.e and so  $|f - g| = 0$  a.e. which implies  $f = g$  a.e. It is not hard to show that  $\phi : [0, \infty) \rightarrow [0, \infty)$  given by  $\phi(x) = \frac{x}{1+x}$  satisfies  $\phi(x + y) \leq \phi(x) + \phi(y)$ . Thus satisfies the triangle inequality. Now, let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ . Suppose that  $f_n \not\xrightarrow{\mu} f$ . Then there exists  $\epsilon > 0, \delta > 0$  and a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  such that



for each  $k \in \mathbb{N}$ ,  $\mu(E_{\epsilon, n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$ . It is not hard to show that  $\phi$  from earlier is increasing. Thus for each  $k \in \mathbb{N}$ ,

$$\begin{aligned} d(f_{n_k}, f) &= \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon, n_k}} \frac{\epsilon}{1 + \epsilon} \\ &\geq \frac{\epsilon \delta}{1 + \epsilon} \end{aligned}$$

So  $f_{n_k} \not\stackrel{d}{\rightarrow} f$ . Hence  $f_{n_k} \stackrel{d}{\rightarrow} f$  implies that  $f_{n_k} \stackrel{\mu}{\rightarrow} f$ . Conversely, suppose that  $f_{n_k} \stackrel{\mu}{\rightarrow} f$ . Let  $\epsilon > 0$ . Then  $\delta = \frac{\epsilon}{1 + \mu(X)} > 0$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then  $\mu(E_{\delta, n}) < \frac{\delta}{1 + \delta}$ . Let  $n \in \mathbb{N}$ . Suppose that  $n \geq N$ . Since  $\phi$  is increasing and  $\phi \leq 1$ , we have that

$$\begin{aligned} d(f_n, f) &= \int \frac{|f_n - f|}{1 + |f_n - f|} \\ &= \int_{E_{\delta, n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E_{\delta, n}^c} \frac{|f_n - f|}{1 + |f_n - f|} \\ &\leq \mu(E_{\delta, n}) + \mu(X) \frac{\delta}{1 + \delta} \\ &< \frac{\delta}{1 + \delta} (1 + \mu(X)) \\ &\leq \delta (1 + \mu(X)) \\ &= \epsilon \end{aligned}$$

□

**Exercise 3.4.13.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that for each  $n \in \mathbb{N}$ ,  $f_n \geq 0$  and  $f_n \stackrel{\mu}{\rightarrow} f$ . Then  $f \geq 0$  a.e. and

$$\int f \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu$$

*Proof.* Since  $f_n \stackrel{\mu}{\rightarrow} f$ , there is a subsequence converging to  $f$  a.e. So clearly  $f \geq 0$  a.e. Now, choose a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  such that  $\int f_{n_k} \rightarrow \liminf_{n \rightarrow \infty} \int f_n$ . Since  $f_n \stackrel{\mu}{\rightarrow} f$  so does  $(f_{n_k})_{k \in \mathbb{N}}$ . Therefore there exists a subsequence  $(f_{n_{k_j}})_{j \in \mathbb{N}}$  of  $(f_{n_k})_{k \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ . Thus  $f \geq 0$  a.e. and Fatou's lemma tells us that

$$\begin{aligned} \int f &\leq \liminf_{j \in \mathbb{N}} \int f_{n_{k_j}} \\ &= \liminf_{n \rightarrow \infty} \int f_n. \end{aligned}$$

□

**Exercise 3.4.14.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that there exists  $g \in L^1$  such that for each  $n \in \mathbb{N}$ ,  $|f_n| \leq g$ . Then  $f_n \xrightarrow{\mu} f$  implies that  $f \in L^1$  and  $f_n \xrightarrow{L^1} f$ .

*Proof.* Clearly  $(f_n)_{n \in \mathbb{N}} \subset L^1$ . Since  $f_n \xrightarrow{\mu} f$ , there exists a subsequence  $(f_{n_k})_{k \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$  such that  $f_{n_k} \xrightarrow{\text{a.e.}} f$ . This implies that  $|f| \leq g$  a.e. and so  $f \in L^1$ . For  $n \in \mathbb{N}$ , put  $h_n = 2g - |f_n - f|$ . Then for each  $n \in \mathbb{N}$ ,  $h_n \geq 0$  and  $h_n \xrightarrow{\mu} 2g$ . By the previous exercise

$$\begin{aligned} \int 2g &\leq \liminf_{n \rightarrow \infty} \int (2g - |f_n - f|) \\ &= \int 2g - \limsup_{n \rightarrow \infty} \int |f_n - f|. \end{aligned}$$

So  $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$  which implies that  $\int |f_n - f| \rightarrow 0$  and  $f_n \xrightarrow{L^1} f$  as required.  $\square$

**Exercise 3.4.15.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ ,  $f \in L^0$  and  $\phi : \mathbb{C} \rightarrow \mathbb{C}$ .

- (1) If  $\phi$  is continuous, and  $f_n \xrightarrow{\text{a.e.}} f$  then  $\phi \circ f_n \xrightarrow{\text{a.e.}} \phi \circ f$ .
- (2) If  $\phi$  is uniformly continuous and  $f_n \rightarrow f$  uniformly, almost uniformly or in measure, then  $\phi \circ f_n \rightarrow \phi \circ f$  uniformly, almost uniformly or in measure, respectively.
- (3) Find a counter example to (2) if we drop the word "uniform".

*Proof.*

(1) Clear

(2) Suppose that  $\phi$  is uniformly continuous.

- uniformly:

Suppose that  $f_n \xrightarrow{u} f$ . Let  $\epsilon > 0$ . Choose  $\delta > 0$  such that for each  $z, w \in \mathbb{C}$ , if  $|z - w| < \delta$ , then  $|\phi(z) - \phi(w)| < \epsilon$ . Now choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$  if  $n \geq N$  then for each  $x \in X$ ,  $|f_n(x) - f(x)| < \delta$ . Let  $n \in \mathbb{N}$ , suppose  $n \geq N$ . Let  $x \in X$ . Then  $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$ . Thus  $\phi \circ f_n \xrightarrow{u} \phi \circ f$ .

- almost uniformly:

Suppose that  $f_n \xrightarrow{\text{a.u.}} f$ . Let  $\epsilon > 0$ . Choose  $N \in \mathcal{A}$  such  $\mu(N) < \epsilon$  and  $f_n \xrightarrow{u} f$  on  $N^c$ . Then from above, we know that  $\phi \circ f_n \xrightarrow{u} \phi \circ f$  on  $N^c$ . Thus  $\phi \circ f_n \xrightarrow{\text{a.u.}} \phi \circ f$ .

- in measure:

Suppose that  $f_n \xrightarrow{\mu} f$ . Let  $\epsilon > 0$ . Choose  $\delta > 0$  such that for each  $z, w \in \mathbb{C}$ , if  $|z - w| < \delta$ , then  $|\phi(z) - \phi(w)| < \epsilon$ . Observe that for  $x \in X$ , if  $|f_n(x) - f(x)| < \delta$ , then  $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$ . Hence  $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$ . By definition of convergence in measure,  $\mu(F_{n,\delta}) \rightarrow 0$ . Thus  $\mu(E_{n,\epsilon}) \rightarrow 0$ . Hence  $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$ .

(3)

$\square$

**Exercise 3.4.16.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that  $f_n \xrightarrow{\text{a.u.}} f$ . Then  $f_n \xrightarrow{\mu} f$  and  $f_n \xrightarrow{\text{a.e.}} f$ .

*Proof.* (measure) Let  $\epsilon > 0$ ,  $\delta > 0$ . Choose  $M \in \mathcal{A}$  such that  $\mu(M) < \delta$  and  $f_n \xrightarrow{u} f$  on  $M^c$ . Choose  $N \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N$ , then for each  $x \in M^c$ ,  $|f_n(x) - f(x)| < \epsilon$ . Let  $n \in \mathbb{N}$ . Suppose  $n \geq N$ . Then  $E_{\epsilon,n} \subset M$  and  $\mu(E_{\epsilon,n}) < \delta$ . Thus  $\mu(E_{\epsilon,n}) \rightarrow 0$  and  $f_n \xrightarrow{\mu} f$ .

(a.e.) For each  $n \in \mathbb{N}$ , Choose  $N_n \in \mathcal{A}$  such that  $\mu(N_n) < 1/n$  and  $f_n \xrightarrow{u} f$  on  $N_n^c$ . Observe that for  $x \in X$ , if  $x \in \bigcup_{n \in \mathbb{N}} N_n^c$ , then  $f_n(x) \rightarrow f(x)$ . Thus  $N = \{x \in X : f_n(x) \not\rightarrow f(x)\} \subset \bigcap_{n \in \mathbb{N}} N_n$ . Therefore  $\mu(N) = 0$  and  $f_n \xrightarrow{\text{a.e.}} f$ .  $\square$

**Exercise 3.4.17.** Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset L^0$  and  $f, g \in L^0$ . Suppose that  $f_n \xrightarrow{\mu} f$  and  $g_n \xrightarrow{\mu} g$ . Then

- (1)  $f_n + g_n \xrightarrow{\mu} f + g$
- (2) if  $\mu(X) < \infty$ , then  $f_n g_n \xrightarrow{\mu} fg$

*Proof.* (1) Let  $\epsilon > 0$ . For convenience, put  $F_{n, \epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$ ,  $G_{n, \epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$ , and  $(F + G)_{n, \epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g(x))| \geq \epsilon\}$ . Observe that for  $x \in X$ ,  $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$ . Thus  $(F + G)_{n, \epsilon} \subset F_{n, \epsilon/2} \cup G_{n, \epsilon/2}$ . Since  $\mu(F_{n, \epsilon/2} \cup G_{n, \epsilon/2}) \leq \mu(F_{n, \epsilon/2}) + \mu(G_{n, \epsilon/2}) \rightarrow 0$ , we have that  $\mu((F + G)_{n, \epsilon}) \rightarrow 0$ . Hence  $f_n + g_n \xrightarrow{\mu} f + g$ .

- (2) Suppose that  $\mu(X) < \infty$ . Let  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  be a subsequence of  $(f_n g_n)_{n \in \mathbb{N}}$ . Choose a subsequence  $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$  and  $g_{n_{k_j}} \xrightarrow{\text{a.e.}} g$ . Then  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.e.}} fg$ . Egoroff's theorem tells us that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.u.}} fg$ , which implies that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$ . Thus for each subsequence  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  of  $(f_n g_n)_{n \in \mathbb{N}}$ , there exists a subsequence  $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$  of  $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$  such that  $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$ . Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that  $f_n g_n \xrightarrow{\mu} fg$ .  $\square$

**Exercise 3.4.18.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that  $\mu(X) < \infty$ . Then  $f_n \xrightarrow{\mu} f$  iff for each subsequence  $(f_{n_k})_{k \in \mathbb{N}}$ , there exists a subsequence  $(f_{n_{k_j}})_{j \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ .

*Proof.* Suppose that  $f_n \xrightarrow{\mu} f$ . Let  $(f_{n_k})_{k \in \mathbb{N}}$  be a subsequence. Then  $f_{n_k} \xrightarrow{\mu} f$ . By a previous theorem, there exists a subsequence  $(f_{n_{k_j}})_{j \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ . Conversely, suppose that for each subsequence  $(f_{n_k})_{k \in \mathbb{N}}$ , there exists a subsequence  $(f_{n_{k_j}})_{j \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ . Let  $\epsilon > 0$ . For  $n \in \mathbb{N}$ , define  $E_n = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$  and define  $E = \{x \in X : f_n(x) \not\rightarrow f(x)\}$ . Let  $(f_{n_k})_{k \in \mathbb{N}}$  be a subsequence. Choose a subsequence  $(f_{n_{k_j}})_{j \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ . Since  $\left\{x \in X : \limsup_{j \rightarrow \infty} \chi_{E_{n_{k_j}}}(x) = 1\right\} = \limsup_{j \rightarrow \infty} E_{n_{k_j}} \subset E$  and  $\mu(E) = 0$ , we have that  $\limsup_{j \rightarrow \infty} \chi_{E_{n_{k_j}}} = 0$  a.e. and  $\chi_{E_{n_{k_j}}} \xrightarrow{\text{a.e.}} 0$ . Since  $\mu(X) < \infty$ , the dominated convergence theorem implies that

$$\mu(E_{n_{k_j}}) = \int \chi_{E_{n_{k_j}}} d\mu \rightarrow 0$$

So for each subsequence  $(\mu(E_{n_k}))_{k \in \mathbb{N}}$ , there exists a subsequence  $(\mu(E_{n_{k_j}}))_{j \in \mathbb{N}}$  such that  $\mu(E_{n_{k_j}}) \rightarrow 0$ . Thus  $\mu(E_n) \rightarrow 0$  and  $f_n \xrightarrow{\mu} f$ .  $\square$

**Exercise 3.4.19.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$ ,  $f \in L^0$  and  $\phi : \mathbb{C} \rightarrow \mathbb{C}$ . Suppose that  $\mu(X) < \infty$ . If  $\phi$  is continuous and  $f_n \xrightarrow{\mu} f$ , then  $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$ .

*Proof.* Suppose that  $\phi$  is continuous and  $f_n \xrightarrow{\mu} f$ . Let  $(\phi \circ f_{n_k})_{k \in \mathbb{N}}$  be a subsequence of  $(\phi \circ f_n)_{n \in \mathbb{N}}$ . Then  $(f_{n_k})_{k \in \mathbb{N}}$  is a subsequence of  $(f_n)_{n \in \mathbb{N}}$ . Since  $f_n \xrightarrow{\mu} f$ , the previous exercise tells us that there exists a subsequence  $(f_{n_{k_j}})_{j \in \mathbb{N}}$  such that  $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$ . A previous exercise implies that  $\phi \circ f_{n_{k_j}} \xrightarrow{\text{a.e.}} \phi \circ f$ . The previous exercise implies that  $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$ .  $\square$

**Exercise 3.4.20.** Let  $(f_n)_{n \in \mathbb{N}} \subset L^0$  and  $f \in L^0$ . Suppose that for each  $\epsilon > 0$ ,

$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then  $f_n \xrightarrow{\text{a.e.}} f$ .

*Proof.* Let  $\epsilon > 0$ . By assumption we know that

$$\begin{aligned} \int \left[ \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} \right] d\mu &= \sum_{n \in \mathbb{N}} \int \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} d\mu \\ &= \sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \\ &< \infty \end{aligned}$$

Thus we also know that  $\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} < \infty$  a.e. Equivalently, we could say that for a.e.  $x \in X$ ,  $|\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}| < \infty$ . For  $k \in \mathbb{N}$ , define  $N_k = \{x \in X : \sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > 1/k\}} = \infty\}$ . Then for each  $k \in \mathbb{N}$ ,  $\mu(N_k) = 0$ . Define  $N = \bigcup_{k \in \mathbb{N}} N_k$ . Then  $\mu(N) = 0$ . Let  $x \in N^c$  and  $\epsilon > 0$ . Choose  $k \in \mathbb{N}$  such that  $1/k < \epsilon$ . Then  $\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\} \subset \{n \in \mathbb{N} : |f_n(x) - f(x)| > 1/k\}$  which is finite because  $x \in N_k^c$ . Put  $M = \max\{n \in \mathbb{N} : |f_n(x) - f(x)| > \epsilon\}$ . Then for  $m \geq M$ ,  $|f_m(x) - f(x)| \leq \epsilon$ . Thus  $f_n(x) \rightarrow f(x)$ . Hence  $f_n \xrightarrow{\text{a.e.}} f$ .  $\square$

## 4. THE RADON-NIKODYM DERIVATIVE

## 4.1. Signed Measures.

**Definition 4.1.1.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$ . Then  $\nu$  is said to be a **signed measure** if

- (1) for each  $E \in \mathcal{A}$ ,  $\nu(E) < \infty$  or for each  $E \in \mathcal{A}$ ,  $\nu(E) > -\infty$ .
- (2)  $\nu(\emptyset) = 0$
- (3) for each  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  if  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is disjoint, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$  and if  $|\sum_{n \in \mathbb{N}} \nu(E_n)| < \infty$ , then  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely.

**Exercise 4.1.2.** Let  $\nu : \mathcal{A} \rightarrow [0, \infty]$  be a signed measure and  $(E_n)_{n \in \mathbb{N}}, (F_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . If  $(E_n)_{n \in \mathbb{N}}$  is increasing, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \rightarrow \infty} \nu(E_n)$ . If  $(F_n)_{n \in \mathbb{N}}$  is decreasing and  $|\nu(E_1)| < \infty$ , then  $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$ .

*Proof.* Put  $E'_1 = E_1$ ,  $F'_1 = F_1$  and for  $n \in \mathbb{N}$ ,  $n \geq 2$ , put  $E'_n = E_n \setminus E_{n-1}$  and  $F'_n = F_1 \setminus F_n$ . Then  $(E'_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is disjoint. Thus

$$\begin{aligned} \nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) &= \nu\left(\bigcup_{n \in \mathbb{N}} E'_n\right) \\ &= \sum_{n \in \mathbb{N}} \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \sum_{n=1}^n \nu(E'_n) \\ &= \lim_{n \rightarrow \infty} \nu(E_n) \end{aligned}$$

Since  $(F'_n)_{n \in \mathbb{N}}$  is increasing, we now know that

$$\begin{aligned} \nu(F_1) - \nu\left(\bigcap_{n \in \mathbb{N}} F_n\right) &= \nu(F_1 \setminus \bigcap_{n \in \mathbb{N}} F_n) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} F'_n\right) \\ &= \lim_{n \rightarrow \infty} \nu(F'_n) \\ &= \lim_{n \rightarrow \infty} \nu(F_1 \setminus F_n) \\ &= \nu(F_1) - \lim_{n \rightarrow \infty} \nu(F_n) \end{aligned}$$

Since  $|\nu(F_1)| < \infty$ , we see that  $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \rightarrow \infty} \nu(F_n)$ . □

**Definition 4.1.3.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$  a signed measure and  $E \in \mathcal{A}$ . Then  $E$  is said to be  $\nu$ -**positive**,  $\nu$ -**negative** and  $\nu$ -**null** if for each  $F \in \mathcal{A}$ ,  $F \subset E$  implies that  $\nu(F) \geq 0$ ,  $\nu(F) \leq 0$ ,  $\nu(F) = 0$  respectively.

**Exercise 4.1.4.** Let  $E \subset \mathcal{A}$ . If  $E$  is positive, negative or null, then for each  $F \in \mathcal{A}$ , if  $F \subset E$ , then  $F$  is positive, negative or null respectively.

*Proof.* Clear □

**Exercise 4.1.5.** Let  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  be positive, negative or null. Then  $\bigcup_{n \in \mathbb{N}} E_n$  is positive, negative or null respectively.

*Proof.* Suppose that  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$  is positive. Let  $F \in \mathcal{A}$ . Suppose that  $F \subset \bigcup_{n \in \mathbb{N}} E_n$ . Put

$P_1 = E_1$  and for  $n \in \mathbb{N}$ ,  $n \geq 2$ , put  $P_n = E_n \setminus (\bigcup_{j=1}^{n-1} E_j)$ . So  $\bigcup_{n \in \mathbb{N}} P_n = \bigcup_{n \in \mathbb{N}} E_n$  and  $(P_n)_{n \in \mathbb{N}}$  is disjoint. Thus

$$\begin{aligned} \nu(F) &= \nu(F \cap \bigcup_{n \in \mathbb{N}} P_n) \\ &= \nu(\bigcup_{n \in \mathbb{N}} (F \cap P_n)) \\ &= \sum_{n \in \mathbb{N}} \nu(F \cap P_n) \\ &\geq 0 \end{aligned}$$

The process is the same if  $(E_n)_{n \in \mathbb{N}}$  is negative and null.  $\square$

**Theorem 4.1.6.** Hahn Decomposition: Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then there exist  $P, N \in \mathcal{A}$  such that  $P$  is positive,  $N$  is negative,  $X = N \cup P$  and  $N \cap P = \emptyset$ . Furthermore, these two sets are unique in the following sense: For any  $P', N' \in \mathcal{A}$ , if  $N, P$  satisfy the properties above,  $P' \Delta P = N' \Delta N$  is  $\nu$ -null.

**Definition 4.1.7.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$  and  $P, N \in \mathcal{A}$ . Then  $P$  and  $N$  are said to form a **Hahn decomposition** of  $X$  with respect to  $\nu$  if  $P, N$  satisfy the results in the above theorem.

**Definition 4.1.8.** Let  $\mu, \nu$  be signed measures on  $(X, \mathcal{A})$ . Then  $\mu$  and  $\nu$  are said to be **mutually singular** if there exist  $E, F \in \mathcal{A}$  such that  $X = E \cup F$ ,  $E \cap F = \emptyset$  and  $E$  is  $\mu$ -null and  $F$  is  $\nu$ -null. We will denote this by  $\mu \perp \nu$ .

**Theorem 4.1.9.** Jordan Decomposition: Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then there exist unique positive measures  $\nu^+$  and  $\nu^-$  on  $(X, \mathcal{A})$  such that  $\nu = \nu^+ - \nu^-$  and  $\nu^+ \perp \nu^-$ .

*Proof.* Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Define  $\nu^+, \nu^-$  by  $\nu^+(E) = \nu(E \cap P)$  and  $\nu^-(E) = \nu(E \cap N)$ .  $\square$

**Definition 4.1.10.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then  $\nu^+$  and  $\nu^-$  from the last theorem are called the **positive** and **negative variations** of  $\nu$  respectively. We define the **total variation of  $\nu$** , denoted  $|\nu| : \mathcal{A} \rightarrow [0, \infty]$  by

$$|\nu| = \nu^+ + \nu^-$$

**Definition 4.1.11.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then  $\nu$  is said to be  $\sigma$ -finite if  $|\nu|$  is  $\sigma$ -finite.

**Exercise 4.1.12.** Let  $\nu$  be a signed measure and  $\lambda, \mu$  positive measures on  $(X, \mathcal{A})$ . Suppose that  $\nu = \lambda - \mu$ . Then  $\lambda \geq \nu^+$  and  $\mu \geq \nu^-$ .

*Proof.* Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Let  $E \in \mathcal{A}$ . Then

$$\begin{aligned} \lambda(E \cap P) - \mu(E \cap P) &= \nu(E \cap P) \\ &= \nu^+(E \cap P) \end{aligned}$$

So  $\lambda(E \cap P) \geq \nu^+(E \cap P)$  and therefore

$$\begin{aligned}\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) \\ &= \nu^+(E)\end{aligned}$$

Similarly  $\mu(E \cap N) \geq \nu^-(E \cap N)$  and  $\mu(E) \geq \nu^-(E)$ .  $\square$

**Exercise 4.1.13.** Let  $\nu_1, \nu_2$  be signed measures on  $(X, \mathcal{A})$ . Suppose that  $\nu_1 + \nu_2$  is a signed measure. Then  $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ . (Hint: use the last exercise)

*Proof.* Since

$$\begin{aligned}\nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\ &= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)\end{aligned}$$

the previous exercise tells us that  $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$  and  $\mu = \nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$ . Therefore

$$\begin{aligned}|\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ &\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\ &= |\nu_1| + |\nu_2|\end{aligned}$$

$\square$

**Note 4.1.14.** Recall that a previous exercise from the section on complex valued functions tells us that  $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$ .

**Definition 4.1.15.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then we define  $L^1(\nu) = L^1(|\nu|)$ . For  $f \in L^1(\nu)$ , we define

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

**Exercise 4.1.16.** Let  $\nu_1, \nu_2$  be signed measures on  $(X, \mathcal{A})$ . Suppose that  $\nu_1 + \nu_2$  is a signed measure. Then  $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$

*Proof.* The previous exercise tells us that  $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$ . Two previous exercises from the section on nonnegative functions tells us that

$$\begin{aligned}\int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\ &= \int |f| d|\nu_1| + \int |f| d|\nu_2|\end{aligned}$$

$\square$

**Exercise 4.1.17.** Let  $\nu, \mu$  be signed measures on  $(X, \mathcal{A})$  and  $E \in \mathcal{A}$ . Then

- (1)  $E$  is  $\nu$ -null iff  $|\nu|(E) = 0$
- (2)  $\nu \perp \mu$  iff  $|\nu| \perp \mu$  iff  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ .

- Proof.* (1) Suppose that  $E$  is  $\nu$ -null. Choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Then  $\nu^+(E) = \nu(E \cap P) = 0$  and  $\nu^-(E) = \nu(E \cap N) = 0$ . Therefore  $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$ . Conversely, suppose that  $|\nu|(E) = 0$ . Then  $\nu^+(E) = \nu^-(E) = 0$ . Let  $F \in \mathcal{A}$ . Suppose that  $F \subset E$ . Then  $\nu^+(F) = 0$  and  $\nu^-(F) = 0$ . Therefore  $\nu(F) = \nu^+(F) - \nu^-(F) = 0$ . So  $E$  is  $\nu$ -null.
- (2) Suppose that  $\nu \perp \mu$ . Then there exist  $E, F \in \mathcal{A}$  such that  $E \cup F = X$ ,  $E \cap F = \emptyset$ ,  $E$  is  $\mu$ -null and  $F$  is  $\nu$ -null. By (1),  $F$  is  $|\nu|$ -null and thus  $|\nu| \perp \mu$ . If  $|\nu| \perp \mu$ , choose  $E, F \in \mathcal{A}$  as before. Since  $F$  is  $|\nu|$ -null, we know that  $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$ . This implies that  $F$  is  $\nu^+$ -null and  $F$  is  $\nu^-$ -null. So  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ . Finally assume that  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$ . **FINISH!!!!**

□

**Exercise 4.1.18.** Let  $\nu$  be a signed measure on  $(X, \mathcal{A})$ . Then

- (1) for  $f \in L^1(\nu)$ ,  $|\int f d\nu| \leq \int |f| d|\nu|$   
(2) if  $\nu$  is finite, then for each  $E \in \mathcal{A}$ ,

$$|\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| : f \text{ is measurable and } |f| \leq 1 \right\}$$

*Proof.* (1) Let  $f \in L^1(\nu)$ . Then

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f d\nu^+ - \int f d\nu^- \right| \\ &\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \\ &= \int |f| d(\nu^+ + \nu^-) \\ &= \int |f| d|\nu| \end{aligned}$$

- (2) Let  $E \in \mathcal{A}$ . Let  $f : X \rightarrow \mathbb{R}$  be measurable and suppose that  $|f| \leq 1$ . Since  $\nu$  is finite, so is  $|\nu|$  and thus  $f \in L^1(\nu)$ . Then (1) tells us that

$$\begin{aligned} \left| \int_E f d\nu \right| &\leq \int_E |f| d|\nu| \\ &\leq |\nu|(E) \end{aligned}$$

Now, choose a Hahn decomposition  $P, N$  of  $X$  with respect to  $\nu$ . Define  $f = \chi_P - \chi_N$ . Then  $|f| \leq 1$ ,  $f$  is measurable and

$$\begin{aligned} \left| \int_E f d\nu \right| &= \left| \int_E f d\nu^+ - \int_E f d\nu^- \right| \\ &= |\nu^+(E \cap P) + \nu^-(E \cap N)| \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E). \end{aligned}$$

□



**Exercise 4.1.19.** Let  $\mu$  be a positive measure on  $(X, \mathcal{A})$  and  $f \in L^0(X, \mathcal{A})$  extended  $\mu$ -integrable. Define  $\nu$  on  $(X, \mathcal{A})$  by

$$\nu(E) = \int_E f \, d\mu$$

Then

- (1)  $\nu$  is a signed measure
- (2) for each  $E \in \mathcal{A}$ ,

$$|\nu|(E) = \int_E |f| \, d\mu$$

*Proof.* (1) Clearly  $\nu(\emptyset) = 0$  and  $\nu$  is finite by assumption. Let  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Suppose that  $(E_n)_{n \in \mathbb{N}}$  is disjoint. Then

$$\begin{aligned} \nu\left(\bigcup_{n \in \mathbb{N}} E_n\right) &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f \, d\mu \\ &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ \, d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- \, d\mu \\ &= \sum_{n \in \mathbb{N}} \int_{E_n} f^+ \, d\mu - \sum_{n \in \mathbb{N}} \int_{E_n} f^- \, d\mu \\ &= \sum_{n \in \mathbb{N}} \left[ \int_{E_n} f^+ \, d\mu - \int_{E_n} f^- \, d\mu \right] \\ &= \sum_{n \in \mathbb{N}} \int_{E_n} f \, d\mu \\ &= \sum_{n \in \mathbb{N}} \nu(E_n) \end{aligned}$$

If  $|\nu(\bigcup_{n \in \mathbb{N}} E_n)| < \infty$ , then  $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ \, d\mu < \infty$  and  $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^- \, d\mu < \infty$  because

$$\begin{aligned} |\nu(\bigcup_{n \in \mathbb{N}} E_n)| &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f \, d\mu \right| \\ &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ \, d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- \, d\mu \right| \end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\sum_{n \in \mathbb{N}} |\nu(E_n)| &= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f d\mu \right| \\
&= \sum_{n \in \mathbb{N}} \left| \int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right| \\
&\leq \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu + \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
&= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu + \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
&< \infty
\end{aligned}$$

So the sum  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely and  $\nu$  is a signed measure.

- (2) Put  $P = \{x \in X : f(x) \geq 0\}$  and  $N = \{x \in X : f(x) < 0\}$ . Then  $P, N$  form a Hahn decomposition of  $X$  with respect to  $\nu$ . Thus for  $E \in \mathcal{A}$ ,

$$\nu^+(E) = \int_{E \cap P} f d\mu = \int_E f^+ d\mu$$

and

$$\nu^-(E) = \int_{E \cap N} f d\mu = \int_E f^- d\mu$$

So for  $E \in \mathcal{A}$ ,

$$|\nu|(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$$

□

**Definition 4.1.20.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  be a signed measure on  $(X, \mathcal{A})$  and  $\mu$  a measure on  $(X, \mathcal{A})$ . Then  $\nu$  is said to be **absolutely continuous** with respect to  $\mu$ , denoted  $\nu \ll \mu$ , if for each  $E \in \mathcal{A}$ ,  $\mu(E) = 0$  implies that  $\nu(E) = 0$ .

**Note 4.1.21.** If there exists an extended  $\mu$ -integrable  $f \in L^0(X, \mathcal{A})$  such that for each  $E \in \mathcal{A}$ ,  $\nu(E) = \int_E f d\mu$ , then we write  $d\nu = f d\mu$ .

**Exercise 4.1.22.** Let  $(X, \mathcal{A})$ ,  $(Y, \mathcal{B})$  be measurable spaces,  $f : X \rightarrow Y$   $\mathcal{A}$ - $\mathcal{B}$  measurable,  $\nu$  be a signed measure on  $(X, \mathcal{A})$  and  $\mu$  a measure on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \mu$ . Then  $f_*\nu \ll f_*\mu$ .

*Proof.* Let  $E \in \mathcal{B}$ . Suppose that  $f_*\mu(E) = 0$ . By definition,  $\mu(f^{-1}(E)) = 0$ . Since  $\nu \ll \mu$ ,  $\nu(f^{-1}(E)) = 0$ . Hence  $f_*\nu(E) = 0$  and  $f_*\nu \ll f_*\mu$ . □

**Theorem 4.1.23.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu$  be a  $\sigma$ -finite signed measure on  $(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measure on  $(X, \mathcal{A})$ . Then there exist unique  $\sigma$ -finite signed measures  $\lambda, \rho$  on  $(X, \mathcal{A})$  such that  $\lambda \perp \mu$ ,  $\rho \ll \mu$  and  $\nu = \lambda + \rho$ , and there exists an extended  $\mu$ -integrable  $f \in L^0(X, \mathcal{A})$  such that  $d\rho = f d\mu$  and  $f$  is unique  $\mu$ -a.e.

**Definition 4.1.24.** The decomposition  $\nu = \lambda + \rho$  is referred to as the **Lebesgue decomposition of  $\nu$  with respect to  $\mu$** . In the case  $\nu \ll \mu$ , we have  $\lambda = 0$  and  $\rho = \nu$  and we define the **Radon-Nikodym derivative of  $\nu$  with respect to  $\mu$** , denoted by  $d\nu/d\mu$ , to be  $d\nu/d\mu = f$  where  $d\nu = f d\mu$ .

**Theorem 4.1.25.** Let  $\nu$  be a  $\sigma$ -finite signed measure on  $(X, \mathcal{A})$  and  $\mu, \lambda$   $\sigma$ -finite measures on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \mu$  and  $\mu \ll \lambda$ . Then

- (1) for each  $g \in L^1(\nu)$ ,  $g(d\nu/d\mu) \in L^1(\mu)$  and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

- (2)  $\nu \ll \lambda$  and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

**Exercise 4.1.26.** Let  $(\nu_n)_{n \in \mathbb{N}}$  be a sequence of measures and  $\mu$  a measure.

- (1) If for each  $n \in \mathbb{N}$ ,  $\nu_n \ll \mu$ , then  $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$ .  
(2) If for each  $n \in \mathbb{N}$ ,  $\nu_n \perp \mu$ , then  $\sum_{n \in \mathbb{N}} \nu_n \perp \mu$ .

*Proof.* (1) Let  $E \in \mathcal{A}$ . Suppose that  $\mu(E) = 0$ . Then for each  $n \in \mathbb{N}$ ,  $\nu_i(E) = 0$  and thus  $\sum_{n \in \mathbb{N}} \nu_n(E) = 0$ . Hence  $\sum_{n \in \mathbb{N}} \nu_n \ll \mu$ .  
(2) For each  $n \in \mathbb{N}$ , there exist  $N_i, M_i \in \mathcal{A}$  such that  $N_i \cap M_i = \emptyset$ ,  $N_i \cup M_i = X$  and  $\nu_i(M_i) = \mu(N_i) = 0$ . Put  $N = \bigcup_{n \in \mathbb{N}} N_i$  and  $M = N^c$ . Note that for each  $n \in \mathbb{N}$ ,  $M \subset N_i^c = M_i$ . So  $\mu(N) \leq \sum_{n \in \mathbb{N}} \mu(N_i) = 0$  and  $(\sum_{n \in \mathbb{N}} \nu_i)(M) \leq \sum_{n \in \mathbb{N}} \nu_i(M_i) = 0$ . Thus  $\sum_{n \in \mathbb{N}} \nu_i \perp \mu$ . □

**Exercise 4.1.27.** Choose  $X = [0, 1]$ ,  $\mathcal{A} = \mathcal{B}_{[0,1]}$ . Let  $m$  be Lebesgue measure and  $\mu$  the counting measure.

Then

- (1)  $m \ll \mu$  but for each  $f \in L^+$ ,  $dm \neq f d\mu$   
(2) There is no Lebesgue decomposition of  $\mu$  with respect to  $m$ .

*Proof.* (1) Let  $E \in \mathcal{A}$ . If  $\mu(E) = 0$ , then  $E = \emptyset$  and  $m(E) = 0$ . So  $m \ll \mu$ . Suppose for the sake of contradiction that there exists  $f \in L^+$  such that  $dm = f d\mu$ . Then

$$\begin{aligned} 1 &= m(X) \\ &= \sum_{x \in X} f(x) \end{aligned}$$

Put  $Z = \{x \in X : f(x) \neq 0\}$ . Then  $Z$  is countable. So

$$\begin{aligned} 1 &= m(X \setminus Z) \\ &= \sum_{x \in X \setminus Z} f(x) \\ &= 0 \end{aligned}$$

This is a contradiction, so no such  $f$  exists.

- (2) Suppose for the sake of contradiction that there is a Lebesgue decomposition for  $\mu$  with respect to  $m$  given by  $\mu = \lambda + \rho$  where  $\lambda \perp m$  and  $\rho \ll m$ . We may assume  $\lambda$  and  $\rho$  are positive. Then for each  $x \in X$ ,  $m(\{x\}) = 0$  which implies that  $\rho(\{x\}) = 0$ . Let  $E \subset X$ , if  $E$  is countable, then  $\lambda(E) = \mu(E)$ . If  $E$  is uncountable, choose  $F \subset E$

such that  $F$  is countable. Then

$$\begin{aligned}\lambda(E) &\geq \lambda(F) \\ &= \mu(F) \\ &= \infty\end{aligned}$$

So  $\lambda = \mu$ . This is a contradiction since  $\mu \not\ll m$ .

□

**Exercise 4.1.28.** Let  $(X, \mathcal{A})$  be a measurable space and  $\mu, \nu$   $\sigma$ -finite measures on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \mu$ . Then  $d\nu/d\mu > 0$   $\mu$ -a.e. iff for each  $E \in \mathcal{A}$ ,  $\mu(E) \neq 0$  implies that  $\nu(E) > 0$ .

*Proof.* Since  $\nu$  is a measure, there exists  $f \in L^+(X, \mathcal{A})$  such that  $f = d\nu/d\mu$   $\mu$ -a.e. Suppose that there exists  $E \in \mathcal{A}$  such that  $\mu(E) > 0$  and  $\nu(E) = 0$ . Then

$$\begin{aligned}\int_E f \, d\mu &= \nu(E) \\ &= 0\end{aligned}$$

Hence

$$\begin{aligned}\frac{d\nu}{d\mu} \chi_E &= f \chi_E \\ &= 0 \text{ } \mu\text{-a.e.}\end{aligned}$$

Therefore  $d\nu/d\mu \not> 0$   $\mu$ -a.e.

Conversely, suppose that  $d\nu/d\mu \not> 0$   $\mu$ -a.e. Then there exists  $E \in \mathcal{A}$  such that  $\mu(E) > 0$  and  $(d\nu/d\mu)\chi_E = 0$   $\mu$ -a.e. Therefore

$$\begin{aligned}\nu(E) &= \int_E \frac{d\nu}{d\mu} \chi_E \, d\mu \\ &= 0\end{aligned}$$

□

## 4.2. Complex Measures.

**Definition 4.2.1.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu : \mathcal{A} \rightarrow \mathbb{C}$ . Then  $\nu$  is said to be a **complex measure** if

- (1)  $\nu(\emptyset) = 0$
- (2) for each sequence  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ , if  $(E_n)_{n \in \mathbb{N}}$  is disjoint, then  $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$  and  $\sum_{n \in \mathbb{N}} \nu(E_n)$  converges absolutely.

**Definition 4.2.2.** Let  $(X, \mathcal{A})$  be a measurable space. We define

$$\mathcal{M}(X, \mathcal{A}) = \{\mu : \mathcal{A} \rightarrow \mathbb{C} : \mu \text{ is a complex measure}\}$$

When  $X$  is a topological space, we write  $\mathcal{M}(X)$  in place of  $\mathcal{M}(X, \mathcal{B}(X))$ .

**Exercise 4.2.3.** Let  $(X, \mathcal{A})$  be a measurable space and  $\mu, \nu \in \mathcal{M}(X, \mathcal{A})$ . Set  $\mathcal{L}_{\mu, \nu} = \{A \in \mathcal{A} : \mu(A) = \nu(A)\}$ . If  $X \in \mathcal{L}_{\mu, \nu}$ , then  $\mathcal{L}_{\mu, \nu}$  is a  $\lambda$ -system on  $X$ .

*Proof.* Suppose that  $X \in \mathcal{L}_{\mu, \nu}$ .

- (1) Since  $X \in \mathcal{L}_{\mu, \nu}$ ,  $\mathcal{L}_{\mu, \nu} \neq \emptyset$ .
- (2) Let  $A \in \mathcal{L}_{\mu, \nu}$ . Then  $\mu(A) = \nu(A)$ . Thus

$$\begin{aligned} \mu(A^c) &= \mu(X) - \mu(A) \\ &= \nu(X) - \nu(A) \\ &= \nu(A^c) \end{aligned}$$

So  $A^c \in \mathcal{L}_{\mu, \nu}$ .

- (3) Let  $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}_{\mu, \nu}$ . So for each  $n \in \mathbb{N}$ ,  $\mu(A_n) = \nu(A_n)$ . Suppose that  $(A_n)_{n \in \mathbb{N}}$  is disjoint. Then

$$\begin{aligned} \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \sum_{n \in \mathbb{N}} \mu(A_n) \\ &= \sum_{n \in \mathbb{N}} \nu(A_n) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \end{aligned}$$

Hence  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}_{\mu, \nu}$ .

□

**Exercise 4.2.4.** Let  $(X, \mathcal{A})$  be a measurable space,  $\mu, \nu \in \mathcal{M}(X, \mathcal{A})$  and  $\mathcal{P} \subset \mathcal{A}$  a  $\pi$ -system on  $X$ . Suppose that  $X \in \mathcal{P}$  and that for each  $A \in \mathcal{P}$ ,  $\mu(A) = \nu(A)$ . Then for each  $A \in \sigma(\mathcal{P})$ ,  $\mu(A) = \nu(A)$ .

*Proof.* The previous exercise implies that  $\mathcal{L}_{\mu, \nu}$  is a  $\lambda$ -system on  $X$ . By assumption,  $\mathcal{P} \subset \mathcal{L}_{\mu, \nu}$ . Dynkin's theorem implies that  $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu, \nu}$ . So for each  $A \in \sigma(\mathcal{P})$ ,  $\mu(A) = \nu(A)$ . □

**Exercise 4.2.5.** Let  $(X, \mathcal{T})$  be a topological space and  $\mu, \nu \in \mathcal{M}(X)$ . If for each  $A \in \mathcal{T}$ ,  $\mu(A) = \nu(A)$ , then  $\mu = \nu$ .

*Proof.* Since  $\mathcal{T} \subset \mathcal{B}(X)$  is a  $\pi$ -system on  $X$  and  $X \in \mathcal{T}$ , the previous exercise implies that for each  $A \in \sigma(\mathcal{T})$ ,  $\mu(A) = \nu(A)$ . Since  $\sigma(\mathcal{T}) = \mathcal{B}(X)$ ,  $\mu = \nu$ . □

**Note 4.2.6.** We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.

**Definition 4.2.7.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu \in \mathcal{M}(X, \mathcal{A})$  with  $\nu = \nu_1 + i\nu_2$ . We define  $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$ . For  $f \in L^1(\nu)$ , we define

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

**Theorem 4.2.8. Lebesgue-Radon-Nikodym Theorem:**

Let  $(X, \mathcal{A})$  be a measurable space,  $\nu \in \mathcal{M}(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measure on  $(X, \mathcal{A})$ . Then there exists unique  $\lambda, \rho \in \mathcal{M}(X, \mathcal{A})$  such that  $\lambda \perp \mu$ ,  $\rho \ll \mu$  and  $\nu = \lambda + \rho$ , and there exists  $f \in L^1(\mu)$  such that  $d\rho = f d\mu$  and  $f$  is unique  $\mu$ -a.e.

**Exercise 4.2.9.** Let  $\nu \in \mathcal{M}(X, \mathcal{A})$  and  $\mu, \lambda$   $\sigma$ -finite measures on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \mu$  and  $\mu \ll \lambda$ . Then

(1) for each  $g \in L^1(\nu)$ ,  $g(d\nu/d\mu) \in L^1(\mu)$  and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2)  $\nu \ll \lambda$  and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

**Definition 4.2.10.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu \in \mathcal{M}(X, \mathcal{A})$  with  $\nu = \nu_1 + i\nu_2$ . Define  $\mu = |\nu_1| + |\nu_2|$ . Then  $\nu \ll \mu$  and thus there exists  $f \in L^1(\mu)$  such that  $d\nu = f d\mu$ . We define the **total variation of  $\nu$** , denoted  $|\nu| : \mathcal{A} \rightarrow [0, \infty)$ , by

$$|\nu|(E) = \int_E |f| d\mu$$

**Exercise 4.2.11.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu \in \mathcal{M}(X, \mathcal{A})$  and  $\lambda$  a  $\sigma$ -finite measure on  $(X, \mathcal{A})$ . Suppose that  $\nu \ll \lambda$ . Set  $g = d\nu/d\lambda$ . Then for each  $E \in \mathcal{A}$ ,

$$|\nu|(E) = \int_E |g| d\lambda$$

*Proof.* Write  $\nu = \nu_1 + i\nu_2$ . Then  $\nu_1, \nu_2 \ll \lambda$ . Set  $f_1 = d\nu_1/d\lambda$  and  $f_2 = d\nu_2/d\lambda$ . Then Exercise 4.1.19 implies that  $d|\nu_1| = |f_1| d\lambda$  and  $d|\nu_2| = |f_2| d\lambda$ . Set  $\mu = |\nu_1| + |\nu_2|$  and  $f = d\nu/d\mu$  as in Definition 4.2.10. Then by construction,

$$\begin{aligned} d\mu &= d|\nu_1| + d|\nu_2| \\ &= |f_1| d\lambda + |f_2| d\lambda \\ &= (|f_1| + |f_2|) d\lambda \end{aligned}$$

So that  $\mu \ll \lambda$  with  $d\mu/d\lambda = |f_1| + |f_2|$ . Then Exercise 4.2.9 implies that  $\nu \ll \lambda$  with

$$\begin{aligned} \frac{d\nu}{d\lambda} &= \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \\ &= f(|f_1| + |f_2|) \\ &= g \end{aligned}$$

and for each  $E \in \mathcal{A}$ ,

$$\begin{aligned} |\nu|(E) &= \int_E |f| d\mu \\ &= \int_E |f|(|f_1| + |f_2|) d\lambda \\ &= \int_E |g| d\lambda \end{aligned}$$

□

**Exercise 4.2.12.** Let  $\nu \in \mathcal{M}(X, \mathcal{A})$  and  $\mu$  a  $\sigma$ -finite measures on  $(X, \mathcal{A})$ . If  $\nu \ll \mu$ , then  $\{x \in X : d\nu/d\mu(x) = 0\}$  is  $\nu$ -null.

*Proof.* Define  $f = d\nu/d\mu$  and  $E = \{x : f(x) = 0\}$ . Let  $A \in \mathcal{A}$  and suppose that  $A \subset E$ . Then

$$\begin{aligned} \nu(A) &= \int_A f d\mu \\ &= 0 \end{aligned}$$

□

**Exercise 4.2.13.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu \in \mathcal{M}(X, \mathcal{A})$  with  $\nu = \nu_1 + i\nu_2$ . Then  $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu_2|$ .

*Proof.* Let  $\mu$  and  $f = f_1 + if_2$  be as in the definition of  $|\nu|$ . Since for each  $E \in \mathcal{A}$ , we have

$$\begin{aligned} \nu(E) &= \int_E f d\mu \\ &= \int_E f_1 d\mu + i \int_E f_2 d\mu \end{aligned}$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E)$$

we know that  $\nu_1 = f_1 d\mu$  and  $\nu_2 = f_2 d\mu$ .

A previous exercise tells us that  $d|\nu_1| = |f_1| d\mu$  and  $d|\nu_2| = |f_2| d\mu$ . Since  $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$ , we have that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \end{aligned}$$

□

**Exercise 4.2.14.** Let  $(X, \mathcal{A})$  be a measurable space,  $\nu \in \mathcal{M}(X, \mathcal{A})$  and  $c \in \mathbb{C}$ . Then  $|c\nu| = |c||\nu|$ .

*Proof.* Define  $\mu$  and  $f$  as before so that  $d\nu = f d\mu$ . Then  $d(c\nu) = cf d\mu$ . Hence

$$\begin{aligned} d|c\nu| &= |cf| d\mu \\ &= |c||f| d\mu \\ &= |c|d|\nu| \end{aligned}$$

So  $|c\nu| = |c||\nu|$ .

□

**Exercise 4.2.15.** Define  $\|\cdot\| : \mathcal{M}(X, \mathcal{A}) \rightarrow [0, \infty)$  by

$$\|\mu\| = |\mu|(X)$$

Then  $\|\cdot\|$  is a norm on  $\mathcal{M}(X, \mathcal{A})$ .

*Proof.* Let  $\mu, \nu \in \mathcal{M}(X, \mathcal{A})$  and  $\alpha \in \mathbb{C}$ . The previous exercises tell us that  $|\mu + \nu| \leq |\mu| + |\nu|$  and  $|\alpha\mu| = |\alpha||\mu|$ . So clearly  $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$  and  $\|c\mu\| = |c|\|\mu\|$ . If  $\|\mu\| = 0$ , then  $X$  is  $\mu$ -null and  $\mu$  is the zero measure.  $\square$

**Exercise 4.2.16.** Let  $(X, \mathcal{A})$  be a measurable space and  $\nu \in \mathcal{M}(X, \mathcal{A})$ . Then

- (1) for each  $E \in \mathcal{A}$ ,  $|\nu(E)| \leq |\nu|(E)$ .
- (2)  $\nu \ll |\nu|$  and  $|d\nu/d|\nu|| = 1$   $|\nu|$ -a.e.
- (3)  $L^1(\nu) = L^1(|\nu|)$  and for each  $g \in L^1(\nu)$ ,

$$\left| \int g d\nu \right| \leq \int |g| d|\nu|$$

*Proof.* Let  $\mu, f \in L^1(\mu)$  be as in the definition of  $|\nu|$ .

- (1) Let  $E \in \mathcal{A}$ . Then

$$\begin{aligned} |\nu(E)| &= \left| \int_E f d\mu \right| \\ &\leq \int_E |f| d\mu \\ &= |\nu|(E) \end{aligned}$$

- (2) Let  $E \in \mathcal{A}$  and suppose that  $|\nu|(E) = 0$ . The previous part implies  $|\nu(E)| = 0$  and  $\nu \ll |\nu|$ . Put  $g = d\nu/d|\nu|$ . Then

$$\begin{aligned} f &= \frac{d\nu}{d\mu} \\ &= g|f| \quad \mu\text{-a.e.} \end{aligned}$$

Hence  $|f| = |g||f|$   $\mu$ -a.e. Since  $|\nu| \ll \mu$ ,  $|f| = |g||f|$   $|\nu|$ -a.e.

A previous exercise tells us that  $|f| \neq 0$   $|\nu|$ -a.e. Thus  $|g| = 1$   $|\nu|$ -a.e.

- (3) Write  $\nu = \nu_1 + i\nu_2$  and  $f = f_1 + if_2$ . First we observe that

$$\begin{aligned} L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\ &= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\ &= L^1(|\nu_1| + |\nu_2|) \\ &= L^1(\mu) \end{aligned}$$

The previous exercise tells us that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \\ &= \mu \end{aligned}$$



Let  $g \in L^1(\mu)$ . Then

$$\begin{aligned} \int |g|d|\nu| &\leq \int |g|d\mu \\ &< \infty \end{aligned}$$

So  $g \in L^1(|\nu|)$ . Conversely, let  $g \in L^1(|\nu|)$ . Then

$$\begin{aligned} \int |g|d|\nu_1|, \int |g|d|\nu_2| &\leq \int |g|d|\nu| \\ &< \infty \end{aligned}$$

So

$$\begin{aligned} \int |g|d\mu &= \int |g|d|\nu_1| + \int |g|d|\nu_2| \\ &< \infty \end{aligned}$$

and  $g \in L^1(\mu)$ . Hence  $L^1(\nu) = L^1(|\nu|)$ . Now, let  $g \in L^1(\nu) = L^1(|\nu|)$ , then

$$\begin{aligned} \left| \int g d\nu \right| &= \left| \int g f d\mu \right| \\ &\leq \int |g||f| d\mu \\ &= \int |g|d|\nu| \end{aligned}$$

□

**Exercise 4.2.17.** Let  $(X, \mathcal{A})$  be a measurable space,  $\mu_1, \mu_2 \in \mathcal{M}(X, \mathcal{A})$  and  $\lambda \in \mathbb{C}$ . Then for each  $f \in L^1(\mu_1 + \lambda\mu_2)$ ,

$$\int f d(\mu_1 + \lambda\mu_2) = \int f d\mu_1 + \lambda \int f d\mu_2$$

*Proof.* Clear by an exercise in section 3.2.

□

### 4.3. Differentiation on $\mathbb{R}^n$ .

**Definition 4.3.1.** Let  $B \subset \mathbb{R}^n$ . Then  $B$  is said to be a **ball** if there exists  $x \in \mathbb{R}^n$  and  $r > 0$  such that  $B = B(x, r)$ .

**Definition 4.3.2.** Let  $f \in L^0(\mathbb{R}^n)$ . Then  $f$  is said to be **locally integrable** (with respect to Lebesgue measure) if  $f$  is measurable and for each  $K \subset \mathbb{R}^n$ ,  $K$  compact implies  $\int_K |f| dm < \infty$ . We define  $L^1_{\text{loc}}(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{C} : f \text{ is locally integrable}\}$

**Definition 4.3.3.** For  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ ,  $r > 0$ ,  $x \in \mathbb{R}^n$ , we define the **average of  $f$  over  $B(x, r)$** , denoted by  $Af(x, r)$ , to be

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f dm$$

**Exercise 4.3.4.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ . Define

$$H^*f(x) = \sup \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\} \quad (x \in \mathbb{R}^n)$$

Then  $Hf \leq H^*f \leq 2^n Hf$ .

*Proof.* Let  $x \in \mathbb{R}^n$ . Then

$$\left\{ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f| dm : r > 0 \right\} \subset \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\}$$

So  $Hf(x) \leq H^*f(x)$ . Let  $B$  be a ball. Then there exists  $y \in \mathbb{R}^n$ ,  $R > 0$  such that  $B = B(y, R)$ . Suppose that  $x \in B$ . Then  $B \subset B(x, 2R)$ . Since  $m(B(x, 2R)) = 2^n m(B(y, R))$ , we have that

$$\begin{aligned} \frac{1}{m(B)} \int_B |f| dm &\leq \frac{1}{m(B)} \int_{m(B(x, 2R))} |f| dm \\ &= \frac{2^n}{m(B(x, 2R))} \int_{m(B(x, 2R))} |f| dm \end{aligned}$$

Thus  $H^*f(x) \leq 2^n Hf(x)$ . □

**Lemma 4.3.5.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ , then  $Af : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  is continuous.

**Definition 4.3.6.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ . We define its **Hardy Littlewood maximal function**, denoted by  $Hf$  to be

$$Hf(x) = \sup_{r>0} Af(x, r) \quad x \in \mathbb{R}^n$$

**Theorem 4.3.7.** There exists  $C > 0$  such that for each  $f \in L^1(m)$  and  $\alpha > 0$ ,

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f| dm$$

**Exercise 4.3.8.** Let  $f \in L^1(\mathbb{R}^n)$ . Suppose that  $\|f\|_1 > 0$ . Then there exist  $C, R > 0$  such that for each  $x \in \mathbb{R}^n$ , if  $|x| > R$ , then  $Hf(x) \geq C|x|^{-n}$ . Hence there exists  $C' > 0$  such that for each  $\alpha > 0$ ,  $m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) > C'/\alpha$  when  $\alpha$  is small.

*Proof.* Since  $\|f\|_1 > 0$ , there exists  $R > 0$  such that  $\int_{B(0,R)} |f| dm > 0$ . Recall that there exists  $K > 0$  such that for each  $x \in \mathbb{R}^n$  and  $r > 0$ ,  $m(B(x,r)) = Kr^n$ . Choose

$$C = \frac{1}{K2^n} \int_{B(0,R)} |f| dm$$

. Let  $x \in \mathbb{R}^n$ . Suppose that  $|x| > R$ . Then  $B(0,R) \subset B(x, 2|x|)$ . Thus

$$\begin{aligned} Hf(x) &\geq \frac{1}{m(B(x, 2|x|))} \int_{B(x, 2|x|)} |f| dm \\ &= \frac{1}{K2^n|x|^n} \int_{B(x, 2|x|)} |f| dm \\ &\geq \frac{1}{K2^n|x|^n} \int_{B(0,R)} |f| dm \\ &= \frac{C}{|x|^n} \end{aligned}$$

Let  $a < \frac{C}{2R^n}$ . Then  $R^n < \frac{C}{2a}$ . Choose  $C' = \frac{KC}{2}$ . Let  $A = \{x \in \mathbb{R}^n : R < |x| < (\frac{C}{a})^{\frac{1}{n}}\}$ . For  $x \in A$ ,

$$\begin{aligned} Hf(x) &\geq \frac{C}{|x|^n} \\ &> a \end{aligned}$$

Thus  $A \subset m(\{x \in \mathbb{R}^n : Hf(x) > a\})$  and therefore

$$\begin{aligned} m(\{x \in \mathbb{R}^n : Hf(x) > a\}) &\geq m(A) \\ &= m(B(0, (C/a)^{1/n})) - m(B(0, R)) \\ &= K \left[ \frac{C}{a} - R^n \right] \\ &> K \left[ \frac{C}{a} - \frac{C}{2a} \right] \\ &= \frac{KC}{2a} \\ &= \frac{C'}{a} \end{aligned}$$

□

**Theorem 4.3.9.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ , then for a.e.  $x \in \mathbb{R}^n$ ,

$$\lim_{r \rightarrow 0} Af(x, r) = f(x)$$

Equivalently, for a.e.  $x \in \mathbb{R}^n$ ,

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} [f(y) - f(x)] dm(y) \right] = 0$$

**Note 4.3.10.** We can a stronger result of the same flavor.

**Definition 4.3.11.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ . We define the **Lebesgue set of  $f$** , denoted by  $L_f$ , to be

$$\begin{aligned} L_f &= \{x \in \mathbb{R}^n : \lim_{r \rightarrow 0} A|f - f(x)|(x, r) = 0\} \\ &= \left\{x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0 \right\} \end{aligned}$$

**Exercise 4.3.12.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ . If  $f$  is continuous at  $x$ , then  $x \in L_f$ .

*Proof.* Suppose that  $f$  is continuous at  $x$ . Let  $\epsilon > 0$ . By assumption, there exists  $\delta > 0$  such that for each  $y \in \mathbb{R}^n$ , if  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ . Let  $r > 0$ . Suppose that  $r < \delta$ . Then for each  $y \in \mathbb{R}^n$ ,  $y \in B(x, r)$  implies that  $|f(x) - f(y)| < \epsilon$  and thus

$$\begin{aligned} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(B(x, r))} \epsilon m(B(x, r)) \\ &= \epsilon \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| dm(y) \right] = 0$$

and  $x \in L_f$ . □

**Theorem 4.3.13.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ . Then  $m((L_f)^c) = 0$

**Definition 4.3.14.** Let  $x \in \mathbb{R}^n$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ . Then  $(E_r)_{r>0}$  is said to **shrink nicely to  $x$**  if

- (1) for each  $r > 0$ ,  $E_r \subset B(x, r)$
- (2) there exists  $\alpha > 0$  such that for each  $r > 0$ ,  $m(E_r) > \alpha m(B(x, r))$

**Theorem 4.3.15.** Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ . Then for each  $x \in L_f$ ,

$$\lim_{r \rightarrow 0} \left[ \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \right] = 0$$

and

$$\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

**Definition 4.3.16.** Let  $\mu : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, \infty]$  be a Borel measure. Then  $\mu$  is said to be **regular** if

- (1) for each  $K \subset \mathbb{R}^n$ , if  $K$  is compact, then  $\mu(K) < \infty$
- (2) for each  $E \in \mathcal{B}(\mathbb{R}^n)$ ,  $\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subset U\}$

Let  $\nu$  be a signed or complex Borel measure on  $\mathbb{R}^n$ . Then  $\nu$  is said to be regular if  $|\nu|$  is regular.

**Theorem 4.3.17.** Let  $\nu$  be a regular signed or complex measure on  $\mathbb{R}^n$ . Let  $d\nu = d\lambda + f dm$  be the Lebesgue decomposition of  $\nu$  with respect to  $m$ . Then for  $m$ -a.e.  $x \in \mathbb{R}^n$  and  $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$ , if  $(E_r)_{r>0}$  shrinks nicely to  $x$ , then

$$\lim_{r \rightarrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

#### 4.4. Functions of Bounded Variation.

**Definition 4.4.1.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Define  $F_+ : \mathbb{R} \rightarrow \mathbb{R}$  and  $F_- : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F_+(x) = \lim_{t \rightarrow x^+} F(t) = \inf\{F(t) : t > x\}$$

and

$$F_-(x) = \lim_{t \rightarrow x^-} F(t) = \sup\{F(t) : t < x\}$$

respectively.

**Exercise 4.4.2.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then

- (1) (a)  $F \leq F_+$
- (b)  $F_+$  is increasing
- (2) (a)  $F_- \leq F$
- (b)  $F_-$  is increasing

*Proof.*

- (1) (a) Let  $x \in \mathbb{R}$ . Since  $F$  is increasing, for each  $t > x$ ,  $F(x) \leq F(t)$ . Hence

$$\begin{aligned} F(x) &\leq \inf\{F(t) : t > x\} \\ &= F_+(x) \end{aligned}$$

Since  $x \in \mathbb{R}$  is arbitrary,  $F \leq F_+$ .

- (b) Let  $x, y \in \mathbb{R}$ . Suppose that  $x \leq y$ . Then  $\{F(t) : t > y\} \subset \{F(t) : t > x\}$ . Thus

$$\begin{aligned} F_+(x) &= \inf\{F(t) : t > x\} \\ &\leq \inf\{F(t) : t > y\} \\ &= F_+(y) \end{aligned}$$

- (2) Similar to (1).

□

**Exercise 4.4.3.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing and  $x \in \mathbb{R}$ . Then  $F$  is discontinuous at  $x$  iff  $F_-(x) < F_+(x)$ .

*Proof.* Since  $F$  is continuous at  $x$  iff  $\lim_{t \rightarrow x^+} F(t) = F(x)$  and  $\lim_{t \rightarrow x^-} F(t) = F(x)$ , by definition,  $F$  is continuous at  $x$  iff  $F_+(x) = F(x)$  and  $F_-(x) = F(x)$ . Then the previous exercise implies that  $F$  is discontinuous at  $x$  iff  $F_+(x) > F(x)$  or  $F_-(x) < F(x)$ . Since  $F_+(x) > F(x)$  implies that  $F_-(x) < F_+(x)$  and  $F_-(x) < F(x)$  implies that  $F_-(x) < F_+(x)$ , we have that  $F$  is discontinuous at  $x$  iff  $F_-(x) < F_+(x)$ . □

**Exercise 4.4.4.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then for each  $x \in \mathbb{R}$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each  $y \in (x, x + \delta)$ ,  $0 \leq F_+(y) - F(y) \leq \epsilon$ .

*Proof.* For the sake of contradiction, suppose not. Then there exists  $x \in \mathbb{R}$  and  $\epsilon > 0$  such that for each  $\delta > 0$ , there exist  $y \in (x, x + \delta)$  such that  $F_+(y) - F(y) > \epsilon$ . Then there exists a sequence  $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  such that for each  $n \in \mathbb{N}$ ,  $y_n \in (x, x + \frac{1}{n})$ ,  $y_n > y_{n+1}$  and  $F_+(y_n) - F(y_n) > \epsilon$ . Choose  $N \in \mathbb{N}$  such that  $(N - 1)\epsilon > F(y_1) - F(x)$ . Note that for each  $n \in \mathbb{N}$ ,  $(y_n + y_{n+1})/2 < y_n$  which implies that

$$\begin{aligned} F_+(y_{n+1}) &\leq F((y_n + y_{n+1})/2) \\ &\leq F(y_n) \end{aligned}$$

Therefore

$$\begin{aligned}
 F(y_1) - F(x) &= \sum_{j=1}^{N-1} \left[ F(y_j) - F_+(y_{j+1}) + F_+(y_{j+1}) - F(y_{j+1}) \right] + F(y_N) - F(x) \\
 &= \sum_{j=1}^{N-1} \left[ F(y_j) - F_+(y_{j+1}) \right] + \sum_{j=1}^{N-1} \left[ F_+(y_{j+1}) - F(y_{j+1}) \right] + F(y_N) - F(x) \\
 &\geq \sum_{j=1}^{N-1} \left[ F_+(y_{j+1}) - F(y_{j+1}) \right] \\
 &\geq (N-1)\epsilon \\
 &> F(y_1) - F(x)
 \end{aligned}$$

This is a contradiction, so the claim holds.  $\square$

**Exercise 4.4.5.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then  $F_+$  is right continuous.

*Proof.* Let  $x \in \mathbb{R}$ . Let  $\epsilon > 0$ . By definition, there exists  $\delta_1 > 0$  such that for each  $y \in (x, x + \delta_1)$   $0 \leq F(y) - F_+(x) < \epsilon/2$ . The previous exercise implies that there exists  $\delta_2 > 0$  such that for each  $y \in (x, x + \delta_2)$ ,  $0 \leq F_+(y) - F(y) < \epsilon/2$ . Choose  $\delta = \min\{\delta_1, \delta_2\}$ . Let  $y \in (x, x + \delta)$ .

$$\begin{aligned}
 |F_+(x) - F_+(y)| &\leq |F_+(x) - F(y)| + |F(y) - F_+(y)| \\
 &= (F(y) - F_+(x)) + (F_+(y) - F(y)) \\
 &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon
 \end{aligned}$$

So  $\lim_{t \rightarrow x^+} F_+(t) = F_+(x)$  and  $F_+$  is right continuous.  $\square$

**Exercise 4.4.6.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing. Then

- (1)  $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$  is countable
- (2)  $F$  and  $F_+$  are differentiable a.e. and  $F' = F'_+$  a.e.

*Proof.*

- (1)
- (2)

$\square$

**Definition 4.4.7.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Define  $T_F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$T_F(x) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \quad (x \in \mathbb{R})$$

$T_F$  is called the **total variation function of  $F$** .

**Exercise 4.4.8.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $T_F$  is increasing.

*Proof.* Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ .

Define  $A_x = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$  and  $A_y = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = y \right\}$ . Let  $z \in A_x$ . Then

there exists  $(x_i)_{i=0}^n \subset \mathbb{R}$  such that  $(x_i)_{i=0}^n$  is increasing,  $x_n = x$  and  $z = \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$ . Then

$$\begin{aligned} z &\leq z + |F(y) - F(x)| \\ &= \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \\ &\in A_y \end{aligned}$$

So  $z \leq \sup A_y = T_F(y)$  and thus  $F_T(x) = \sup A_x \leq T_F(y)$   $\square$

**Lemma 4.4.9.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $T_F + F$  and  $T_F - F$  are increasing.

**Exercise 4.4.10.** For each  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $T_{|F|} \leq T_F$ .

*Proof.* Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then by the reverse triangle inequality,

$$\sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

Thus

$$\begin{aligned} T_{|F|}(x) &= \sup \left\{ \sum_{i=1}^n ||F(x_i)| - |F(x_{i-1})|| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\ &\leq \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\ &= T_F(x) \end{aligned}$$

Hence  $T_{|F|} \leq T_F$   $\square$

**Definition 4.4.11.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is said to have **bounded variation** if  $\lim_{x \rightarrow \infty} T_F(x) < \infty$ . The **total variation of**  $F$ , denoted by  $\text{TV}(F)$ , is defined to be  $\text{TV}(F) = \lim_{x \rightarrow \infty} T_F(x)$ . We define  $\text{BV} = \{F : \mathbb{R} \rightarrow \mathbb{C} : \text{TV}(F) < \infty\}$ .

**Definition 4.4.12.** Let  $F : [a, b] \rightarrow \mathbb{C}$ . Define  $G_F : \mathbb{R} \rightarrow \mathbb{C}$  by  $G_F = F(a)\chi_{(-\infty, a)} + F\chi_{[a, b]} + F(b)\chi_{(b, \infty)}$ . Then  $F$  is said to have **bounded variation on**  $[a, b]$  if  $G_F \in \text{BV}$ . The **total variation of**  $F$ , denoted  $\text{TV}(F)$ , is defined to be  $\text{TV}(F) = \text{TV}(G_F)$ . We define  $\text{BV}(a, b) = \{F : [a, b] \rightarrow \mathbb{C} : \text{TV}(F) < \infty\}$ .

**Note 4.4.13.** Equivalently,  $\text{TV}(F) = \sup \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset [a, b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$  and  $F \in \text{BV}(a, b)$  iff  $\text{TV}(F) < \infty$ . In general,

**Exercise 4.4.14.** Let  $F \in \text{BV}$ . Then  $F$  is bounded.

*Proof.* If  $F$  is unbounded, then the supremum in the previous definition is clearly infinite.  $\square$

**Exercise 4.4.15.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$ . If  $F$  is bounded and increasing, then  $F \in \text{BV}$ .

*Proof.* Suppose that  $F$  is bounded and increasing. Then  $-\infty < \inf_{x \in \mathbb{R}} F(x) \leq \sup_{x \in \mathbb{R}} F(x) < \infty$ . Let  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n F(x_i) - F(x_{i-1}) \\ &= F(x) - F(x_0) \end{aligned}$$

Thus

$$T_F(x) = F(x) - \inf_{x \in \mathbb{R}} F(x)$$

This implies that

$$\begin{aligned} \text{TV}(F) &= \sup_{x \in \mathbb{R}} F(x) - \inf_{x \in \mathbb{R}} F(x) \\ &< \infty \end{aligned}$$

Hence  $F \in \text{BV}$ . □

**Exercise 4.4.16.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . If  $F$  is differentiable and  $F'$  is bounded on  $[a, b]$ , then,  $F \in \text{BV}(a, b)$ .

*Proof.* Suppose that  $F$  is differentiable and  $F'$  is bounded on  $[a, b]$ . Then there exists  $M > 0$  such that for each  $x \in [a, b]$ ,  $|F'(x)| \leq M$ . Let  $(x_i)_{i=1}^n \subset [a, b]$ . Suppose that  $(x_i)_{i=1}^n$  is strictly increasing,  $x_0 = a$  and  $x_n = b$ . By the mean value theorem, for each  $i = 1, 2, \dots, n$ , there exists  $c_i \in (x_{i-1}, x_i)$  such that  $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n |F'(c_i)(x_i - x_{i-1})| \\ &\leq \sum_{i=1}^n M(x_i - x_{i-1}) \\ &= M(b - a) \end{aligned}$$

Hence  $\text{TV}(F) \leq M(b - a)$ . □

**Exercise 4.4.17.** Define  $F, G : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x) = \begin{cases} x^2 \sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 \sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then  $F$  and  $G$  are differentiable,  $F \in \text{BV}(-1, 1)$  and  $G \notin \text{BV}(-1, 1)$ .

*Proof.* On  $\mathbb{R} \setminus \{0\}$ ,

$$\begin{aligned} F'(x) &= 2x \sin(x^{-1}) - \sin(x^{-1}) \\ &= \sin(x^{-1})(2x - 1) \end{aligned}$$



We see that  $F$  is also differentiable at  $x = 0$  since

$$\begin{aligned} F'(0) &= \lim_{x \rightarrow 0} \frac{F(x) - F(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-1})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-1}) \\ &= 0 \end{aligned}$$

Therefore for each  $x \in [-1, 1]$ ,  $|F'(x)| \leq 3$ . Which by a previous exercise implies that  $F \in \text{BV}(-1, 1)$ .

On  $\mathbb{R} \setminus \{0\}$ ,

$$\begin{aligned} G'(x) &= 2x \sin(x^{-2}) - \frac{2 \sin(x^{-2})}{x} \\ &= \sin(x^{-2}) \left( 2x - \frac{2}{x} \right) \end{aligned}$$

We see that  $G$  is also differentiable at  $x = 0$  since

$$\begin{aligned} G'(0) &= \lim_{x \rightarrow 0} \frac{G(x) - G(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin(x^{-2})}{x} \\ &= \lim_{x \rightarrow 0} x \sin(x^{-2}) \\ &= 0 \end{aligned}$$

For  $n \in \mathbb{N}$ , define  $(x_i)_{i=0}^n \subset [-1, 1]$  by

$$x_i = \frac{-1}{\sqrt{\frac{\pi}{2} + i\pi}}$$

Then for each  $n \in \mathbb{N}$ ,  $(x_i)_{i=1}^n$  is strictly increasing and for each  $i = 1, 2, \dots, n$  we have that

$$\begin{aligned} |G(x_i) - G(x_{i-1})| &= \frac{1}{\frac{\pi}{2} + i\pi} + \frac{1}{\frac{\pi}{2} + (i-1)\pi} \\ &= \frac{2}{\pi} \left[ \frac{(2i-1) + (2i+1)}{(2i+1)(2i-1)} \right] \\ &= \frac{2}{\pi} \left[ \frac{4i}{4i^2 - 1} \right] \\ &> \frac{2}{i\pi} \end{aligned}$$

Hence for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} \text{TV}(G, [-1, 1]) &\geq \sum_{i=1}^n |G(x_i) - G(x_{i-1})| \\ &> \frac{2}{\pi} \sum_{i=1}^n \frac{1}{i} \end{aligned}$$

Therefore  $G \notin \text{BV}([-1, 1])$ . □

**Exercise 4.4.18.** The following is stated for BV, but is also true for  $\text{BV}(a, b)$ .

- (1) For each  $F, G \in \text{BV}$ ,  $T_{F+G} \leq T_F + T_G$  and therefore BV is a vector space.
- (2) For each  $F : \mathbb{R} \rightarrow \mathbb{C}$ ,  $F \in \text{BV}$  iff  $\text{Re}(f) \in \text{BV}$  and  $\text{Im}(F) \in \text{BV}$ .
- (3) For each  $F : \mathbb{R} \rightarrow \mathbb{R}$ ,  $F \in \text{BV}$  iff there exist functions  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $F_1, F_2$  are bounded, increasing and  $F = F_1 - F_2$ .
- (4) For each  $F \in \text{BV}$  and  $x \in \mathbb{R}$ ,  $\lim_{t \rightarrow x^+} F(t)$  and  $\lim_{t \rightarrow x^-} F(t)$  exist.
- (5) For each  $F \in \text{BV}$ ,  $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$  is countable.
- (6) For each  $F \in \text{BV}$ ,  $F$  and  $F_+$  are differentiable a.e. and  $F' = (F_+)'$  a.e.
- (7) For each  $F \in \text{BV}$ ,  $c \in \mathbb{R}$ ,  $F - c \in \text{BV}$ .

*Proof.* (1) Let  $F, G \in \text{BV}$ ,  $x \in \mathbb{R}$  and  $\epsilon > 0$ . Since  $T_{F+G}(x) < \infty$ ,  $T_{F+G}(x) - \epsilon < T_{F+G}(x)$ . Thus there exists  $(x_i)_{i=0}^n \subset \mathbb{R}$  such that  $(x_i)_{i=0}^n$  is increasing,  $x_n = x$  and  $T_{F+G}(x) < \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1}))| + \epsilon$ . Therefore

$$\begin{aligned} T_{F+G}(x) &< \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1}))| + \epsilon \\ &\leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})| + \sum_{i=1}^n |G(x_i) - G(x_{i-1})| + \epsilon \\ &\leq T_F(x) + T_G(x) + \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary,  $T_{F+G}(x) \leq T_F(x) + T_G(x)$ . Therefore  $\text{TV}(F+G) \leq \text{TV}(F) + \text{TV}(G) < \infty$ . Thus  $F+G \in \text{BV}$ . It is straight forward to verify the other requirements needed to show that BV is a vector space.

- (2) Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Write  $F = F_1 + iF_2$  with  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$ . Suppose that  $F \in \text{BV}$ . Note that for each  $x_1, x_2 \in \mathbb{R}$  and  $j = 1, 2$ ,  $|F_j(x_1) - F_j(x_2)| \leq |F(x_1) - F(x_2)|$ . Let  $x \in \mathbb{R}$  and  $(x_i)_{i=0}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=0}^n$  is increasing and  $x_n = x$ . Then for  $j = 1, 2$

$$\sum_{i=1}^n |F_j(x_i) - F_j(x_{i-1})| \leq \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$$

Thus for  $j = 1, 2$  we have that  $T_{F_j}(x) \leq T_F(x)$  which implies that  $\text{Re}(f), \text{Im}(F) \in \text{BV}$ . Conversely, Suppose that  $\text{Re}(f), \text{Im}(F) \in \text{BV}$ . Then  $F = \text{Re}(f) + i\text{Im}(f) \in \text{BV}$  by (1).

- (3) Suppose that  $F \in \text{BV}$ . Choose  $F_1 = \frac{1}{2}(T_F - F)$  and  $F_2 = \frac{1}{2}(T_F + F)$ . Then  $F_1, F_2$  are bounded, increasing and  $F = F_1 - F_2$ . Conversely, if there exist  $F_1, F_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $F_1, F_2$  are bounded, increasing and  $F = F_1 - F_2$ , then  $F_1, F_2 \in \text{BV}$ . By (1)  $F \in \text{BV}$ .
- (4) This is clear by previous results and (3)

- (5) This is clear by previous results and (3)
- (6) This is clear by previous results and (3)
- (7) Clearly constant functions have zero total variation. The rest is implied by (1).  $\square$

**Lemma 4.4.19.** Let  $F \in BV$ . Then  $\lim_{x \rightarrow -\infty} T_F(x) = 0$  and if  $F$  is right continuous, then  $T_F$  is right continuous.

**Definition 4.4.20.** Define  $NBV = \{F \in BV : F \text{ is right continuous and } \lim_{x \rightarrow -\infty} F(x) = 0\}$ .

**Theorem 4.4.21.** Let  $M(\mathbb{R})$  be the set of complex Borel measures on  $\mathbb{R}$ . For  $F \in NBV$ , define  $\mu_F \in M(\mathbb{R})$  by  $\mu_F((-\infty, x]) = F(x)$ . Then  $F \mapsto \mu_F$  defines a bijection  $NBV \rightarrow M(\mathbb{R})$ . In addition,  $|\mu_F| = \mu_{T_F}$ .

**Theorem 4.4.22.** Let  $F \in NBV$ . Then  $F' \in L^1(m)$ ,  $\mu_F \perp m$  iff  $F' = 0$  a.e. and  $\mu_F \ll m$  iff for each  $x \in \mathbb{R}$ ,

$$\int_{(-\infty, x]} F' dm = F(x)$$

**Definition 4.4.23.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is said to be **absolutely continuous** if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each disjoint  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$ ,  $\sum_{i=1}^n b_i - a_i < \delta$  implies that  $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$ .

**Definition 4.4.24.** Let  $F : [a, b] \rightarrow \mathbb{C}$ . Then  $F$  is said to be **absolutely continuous** if for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that for each disjoint  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$ ,  $\sum_{i=1}^n b_i - a_i < \delta$  implies that  $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$ .

**Exercise 4.4.25.** Let  $F : [a, b] \rightarrow \mathbb{C}$ . If  $F$  is absolutely continuous, then  $F \in BV$ .

*Proof.* Suppose that  $F$  is absolutely continuous. Then for each  $j \in \mathbb{N}$ , there exists  $\delta > 0$  such that for each disjoint  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$ ,  $\sum_{i=1}^n b_i - a_i < \delta$  implies that  $\sum_{i=1}^n |F(b_i) - F(a_i)| < 1$ . Define Choose  $n^* \in \mathbb{N}$  such that  $(b-a)/n < \delta$  and define  $(x_j^*)_{j=0}^{n^*} \subset [a, b]$  by

$$x_j^* = a + \frac{b-a}{n} j$$

Let  $(x_j)_{j=1}^n \subset [a, b]$  be increasing. Consider the refinement

$$(x'_j)_{j=0}^{n'} = (x_j)_{j=0}^n \cup (x_j^*)_{j=0}^{n^*}$$

For  $j \in \{1, \dots, n\}$ , set  $k_0 = 0$  and  $k_j = \max\{k : x'_k \in [x_{j-1}^*, x_j^*]\}$ . Then for each  $k \in \{k_{j-1} + 1, \dots, k_j\}$ ,  $x'_k - x'_{k-1} < \delta$ . Then

$$\begin{aligned} \sum_{j=1}^{n'} |F(x'_j) - F(x'_{j-1})| &= \sum_{j=1}^n \sum_{k=k_{j-1}+1}^{k_j} |F(x'_k) - F(x'_{k-1})| \\ &< \sum_{j=1}^n 1 \\ &= n \end{aligned}$$

So  $TV(F) \leq n < \infty$  and  $F \in BV$ .  $\square$

**Exercise 4.4.26.** There exists  $F : \mathbb{R} \rightarrow \mathbb{C}$  such that  $F$  is absolutely continuous and  $F \notin \text{BV}$ .

*Proof.* Define  $F : \mathbb{R} \rightarrow \mathbb{C}$  by  $F(x) = x$ . □

**Exercise 4.4.27.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Suppose that there exists  $f \in L^1(m)$  such that for each  $x \in \mathbb{R}$ ,

$$F(x) = \int_{(-\infty, x]} f \, dm$$

Then  $F \in \text{NBV}$ .

*Proof.* Let  $x \in \mathbb{R}$  and  $(x_i)_{i=1}^n \subset \mathbb{R}$ . Suppose that  $(x_i)_{i=1}^n$  is increasing and  $x_n = x$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \left| \int_{(x_{i-1}, x_i]} f \, dm \right| \\ &\leq \sum_{i=1}^n \int_{(x_{i-1}, x_i]} |f| \, dm \\ &= \int_{(x_0, x]} |f| \, dm \\ &< \int |f| \, dm \end{aligned}$$

Hence  $T_F(x) \leq \int |f| \, dm$ . Since  $x \in \mathbb{R}$  is arbitrary,  $\text{TV}(F) \leq \int |f| \, dm$ . Therefore  $F \in \text{BV}$ . By the continuity from above and below for measures and the fact that  $m(x) = 0$  for each  $x \in \mathbb{R}$ ,  $F$  is continuous. By continuity from above for measures,  $\lim_{x \rightarrow -\infty} F(x) = 0$ . So  $F \in \text{NBV}$ . □

**Lemma 4.4.28.** Let  $F \in \text{NBV}$ . Then  $F$  is absolutely continuous iff  $\mu_F \ll m$ .

**Exercise 4.4.29. The Fundamental Theorem of Calculus:**

Let  $F : [a, b] \rightarrow \mathbb{C}$ . The following are equivalent:

- (1)  $F$  is absolutely continuous on  $[a, b]$ .
- (2) there exists  $f \in L^1([a, b], m)$  such that for each  $x \in [a, b]$ ,

$$F(x) - F(a) = \int_{(a, x]} f \, dm$$

- (3)  $F$  is differentiable a.e. on  $[a, b]$ ,  $F' \in L^1([a, b], m)$  and for each  $x \in [a, b]$ ,

$$F(x) - F(a) = \int_{(a, x]} F' \, dm$$

*Proof.* (1)  $\implies$  (3)

Suppose that  $F$  is absolutely continuous on  $[a, b]$ . Then  $F \in \text{BV}[a, b]$ . Extend  $F$  to  $\mathbb{R}$  by setting  $F(x) = F(a)$  for  $x < a$  and  $F(x) = F(b)$  for  $x > b$ . Then  $G = F - F(a) \in \text{NBV}$  and is absolutely continuous. The previous lemma implies that there exists  $f \in L^1(m)$  such that  $d\mu_G = f \, dm$ . A previous theorem implies that for a.e.  $x \in [a, b]$

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow x} \frac{\mu_G((x, x+r])}{m((x, x+r])} \\ &= f(x) \end{aligned}$$

So  $F$  is differentiable a.e. on  $[a, b]$ ,  $F' \in L^1([a, b], m)$  and by construction, for each  $x \in [a, b]$ , we have that

$$\begin{aligned} F(x) - F(a) &= \mu_G((a, x]) \\ &= \int_{(a, x]} f \, dm \\ &= \int_{(a, x]} F' \, dm \end{aligned}$$

(3)  $\implies$  (2)

Trivial.

(2)  $\implies$  (1)

Suppose that there exists  $f \in L^1([a, b], m)$  such that for each  $x \in [a, b]$ ,  $F(x) - F(a) = \int_{(a, x]} f \, dm$ . Extend  $F$  as before and obtain  $G$  as before. Note that a previous exercise implies that  $G \in \text{NBV}$ . Since  $\mu_G \ll m$ , the previous lemma implies that  $G$  is absolutely continuous.  $\square$

**Exercise 4.4.30.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . If  $F$  is absolutely continuous. Then  $F$  is differentiable a.e.

*Proof.* Let  $n \in \mathbb{N}$ . Since  $F$  is absolutely continuous on  $\mathbb{R}$ ,  $F$  is absolutely continuous on  $[-n, n]$ . The FTC implies that  $F$  is differentiable a.e. on  $[-n, n]$ . Since  $n \in \mathbb{N}$  is arbitrary,  $F$  is differentiable a.e. on  $\mathbb{R}$ .  $\square$

**Exercise 4.4.31.** Let  $F : \mathbb{R} \rightarrow \mathbb{C}$ . Then  $F$  is Lipschitz continuous iff  $F$  is absolutely continuous and  $F'$  is bounded a.e.

*Proof.* Suppose that  $F$  is Lipschitz continuous. Then there exists  $M > 0$  such that for each  $x, y \in \mathbb{R}$ ,  $|F(x) - F(y)| \leq M|x - y|$ . Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{M}$ . Let  $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$ , Suppose that  $\sum_{i=1}^n b_i - a_i < \delta$ . Then

$$\begin{aligned} \sum_{i=1}^n |F(b_i) - F(a_i)| &\leq \sum_{i=1}^n M(b_i - a_i) \\ &< M\delta \\ &= \epsilon \end{aligned}$$

Hence  $F$  is absolutely continuous. For each  $x, y \in \mathbb{R}$ , if  $x \neq y$ , then  $\left| \frac{F(x) - F(y)}{x - y} \right| \leq M$ . Hence for a.e.  $x \in \mathbb{R}$ ,  $|F'(x)| \leq M$ . Conversely, suppose that  $F$  is absolutely continuous and  $F'$  is bounded a.e. Then there exists  $M > 0$  such that for a.e.  $x \in \mathbb{R}$ ,  $|F'(x)| \leq M$ . Let  $x, y \in \mathbb{R}$ . Suppose  $x < y$ . Then the FTC implies that

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_{(x, y]} F' \, dm \right| \\ &\leq \int_{(x, y]} |F'| \, dm \\ &= M|y - x| \end{aligned}$$

and  $F$  is Lipschitz continuous.  $\square$

**Exercise 4.4.32.** Construct an increasing function  $F : \mathbb{R} \rightarrow \mathbb{R}$  whose discontinuities is  $\mathbb{Q}$ .

*Proof.* Let  $(q_n)_{n \in \mathbb{N}}$  be an enumeration of  $\mathbb{Q}$ . Define  $F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F = \sum_{n \in \mathbb{N}} 2^{-n} \chi_{[q_n, \infty)}$$

Equivalently, if we define  $S_x = \{n \in \mathbb{N} : q_n \leq x\}$ , then we may write

$$F(x) = \sum_{n \in S_x} 2^{-n}$$

Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ . Then  $S_x \subsetneq S_y$ . So  $F(x) < F(y)$  and therefore  $F$  is strictly increasing.

For each  $x, y \in \mathbb{R}$  with  $x < y$ , define  $S_{x,y} = \{n \in \mathbb{N} : x < q_n \leq y\}$ . Note that  $\lim_{y \rightarrow x^+} \min(S_{x,y}) = \infty$  and if  $y \in \mathbb{R} \setminus \mathbb{Q}$ , then  $\lim_{x \rightarrow y^-} \min(S_{x,y}) = \infty$ .

Now, let  $x \in \mathbb{R}$  and  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that  $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$ . Choose  $\delta > 0$  such that  $\min(S_{x, x+\delta}) \geq N$ . Let  $y \in [x, \infty)$ . Suppose that  $|x - y| < \delta$ . Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_y} 2^{-n} - \sum_{n \in S_x} 2^{-n} \\ &= \sum_{n \in S_{x,y}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence  $F$  is right continuous. Now let  $x \in \mathbb{R} \setminus \mathbb{Q}$  and  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  as before and  $\delta > 0$  such that  $\min(S_{x-\delta, x}) \geq N$ . Let  $y \in (-\infty, x]$ . Suppose that  $|x - y| < \delta$ . Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_x} 2^{-n} - \sum_{n \in S_y} 2^{-n} \\ &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence  $F$  is left continuous on  $\mathbb{R} \setminus \mathbb{Q}$ .

Now, let  $x \in \mathbb{Q}$ . Then there exists  $j \in \mathbb{N}$  such that  $q_j = x$ . Choose  $\epsilon = 2^{-j}$ . Let  $\delta > 0$ . Choose  $y = x - \frac{\delta}{2}$ . Then  $|x - y| < \delta$  and

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\geq 2^{-j} \\ &= \epsilon \end{aligned}$$

Hence  $F$  is discontinuous from the left at  $x$ . Since  $x \in \mathbb{Q}$  is arbitrary,  $F$  is discontinuous from the left on  $\mathbb{Q}$ .  $\square$

**Exercise 4.4.33.** Let  $(F_n)_{n \in \mathbb{N}} \in \text{NBV}$  be a sequence of nonnegative, increasing functions. If for each  $x \in \mathbb{R}$ ,  $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$ , then for a.e.  $x \in \mathbb{R}$ ,  $F$  is differentiable at  $x$  and  $F'(x) = \sum_{n \in \mathbb{N}} F'_n(x)$ .

*Proof.* Define  $\mu = \sum_{n \in \mathbb{N}} \mu_{F_n}$ . Note that

$$\begin{aligned} \mu((-\infty, x]) &= \sum_{n \in \mathbb{N}} \mu_{F_n}((-\infty, x]) \\ &= \sum_{n \in \mathbb{N}} F_n(x) \\ &= F(x) \end{aligned}$$

Hence  $F \in \text{NBV}$  and  $\mu = \mu_F$ . For each  $n \in \mathbb{N}$ , there exist  $\lambda_n \in M(\mathbb{R})$  and  $f_n \in L^1(\mathbb{R})$  such that  $d\mu_{F_n} = d\lambda_n + f_n dm$  and  $\lambda_n \perp m$ . Since for each  $n \in \mathbb{N}$ ,  $\lambda_n, f_n$  are nonnegative, we have that  $d\mu_F = \sum_{n \in \mathbb{N}} d\lambda_n + (\sum_{n \in \mathbb{N}} f_n) dm$ . By a previous theorem, for a.e.  $x \in \mathbb{R}$ ,

$$\begin{aligned} F'(x) &= \lim_{r \rightarrow 0} \frac{\mu_F((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} f_n(x) \\ &= \sum_{n \in \mathbb{N}} \lim_{r \rightarrow 0} \frac{\mu_{F_n}((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} F'_n(x) \end{aligned}$$

$\square$

**Exercise 4.4.34.** Let  $F : [0, 1] \rightarrow [0, 1]$  be the Cantor function. Extend  $F$  to  $\mathbb{R}$  by setting  $F(x) = 0$  for  $x < 0$  and  $F(x) = 1$  for  $x > 1$ . Let  $([a_n, b_n])_{n \in \mathbb{N}}$  be an enumeration of the closed subintervals of  $[0, 1]$  with rational endpoints. For  $n \in \mathbb{N}$ , define  $F_n : \mathbb{R} \rightarrow [0, 1]$  by  $F_n(x) = F(\frac{x-a_n}{b_n-a_n})$ . Define  $G : \mathbb{R} \rightarrow \mathbb{R}$  by  $G = \sum_{n \in \mathbb{N}} 2^{-n} F_n$ . Then  $G$  is continuous, strictly increasing on  $[0, 1]$  and  $G' = 0$  a.e.

*Proof.* Since  $F$  is continuous on  $\mathbb{R}$ , we have that for each  $n \in \mathbb{N}$ ,  $F_n$  is continuous on  $\mathbb{R}$ . We observe that for each  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ ,  $|2^{-n} F_n(x)| \leq 2^{-n}$ . Thus the Weierstrass M-test implies that  $G$  converges uniformly on  $\mathbb{R}$  and is therefore continuous. Since  $F$  is increasing, for each  $n \in \mathbb{N}$ ,  $F_n$  is increasing. Let  $x, y \in \mathbb{R}$ . Suppose that  $x < y$ . Choose  $j \in \mathbb{N}$  such that

$x < a_j < y < b_j$ . Then

$$\begin{aligned}
 G(x) &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(x) \\
 &= \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(x) + 0 \\
 &< \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(y) + 2^{-j} F_n(y) \\
 &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(y) \\
 &= G(y)
 \end{aligned}$$

So  $G$  is strictly increasing.

Now we observe that for each  $n \in \mathbb{N}$ ,  $F_n \in \text{NBV}$ . The previous exercise implies that

$$G' = \sum 2^{-n} F'_n = 0 \text{ a.e.}$$

□



#### 4.5. Disintegration of Measures.

**Exercise 4.5.1.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $\mathcal{B} \subset \mathcal{A}$  a sub  $\sigma$ -algebra. Define  $\mu_{\mathcal{B}} = \mu|_{\mathcal{B}}$ . Let  $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$ . Then  $L^1(X, \mathcal{B}, \mu_{\mathcal{B}}) \subset L^1(X, \mathcal{A}, \mu)$  and for each  $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  and  $B \in \mathcal{B}$ ,

$$\int_B f d\mu_{\mathcal{B}} = \int_B f d\mu$$

*Proof.* Let  $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  and  $B \in \mathcal{B}$ . Clearly  $f$  is  $\mathcal{A}$ -measurable. If  $f$  is simple, then there exist  $(b_i)_{i=1}^n \subset [0, \infty)$  and  $(B_i)_{i=1}^n \subset \mathcal{B}$  such that

$$f = \sum_{i=1}^n b_i \chi_{B_i}$$

such that for each  $i \in \{1, \dots, n\}$ ,

$$\begin{aligned} \infty &> \mu_{\mathcal{B}}(B_i) \\ &= \mu(B_i) \end{aligned}$$

So  $f \in L^1(X, \mathcal{A}, \mu)$  and

$$\begin{aligned} \int_B f d\mu_{\mathcal{B}} &= \int_B \sum_{i=1}^n b_i \chi_{B_i} d\mu_{\mathcal{B}} \\ &= \sum_{i=1}^n b_i \mu_{\mathcal{B}}(B_i \cap B) \\ &= \sum_{i=1}^n b_i \mu(B_i \cap B) \\ &= \int_B \sum_{i=1}^n b_i \chi_{B_i} d\mu \\ &= \int_B f d\mu \end{aligned}$$

If  $f \geq 0$ , then there exist  $(\phi_n)_{n \in \mathbb{N}} \subset S^+(X, \mathcal{B})$  such that for each  $n \in \mathbb{N}$ ,  $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \xrightarrow{\text{p.w.}} f$ . The monotone convergence theorem implies that for each  $B \in \mathcal{B}$ ,

$$\begin{aligned} \int_B f d\mu &= \lim_{n \rightarrow \infty} \int_B \phi_n d\mu \\ &= \lim_{n \rightarrow \infty} \int_B \phi_n d\mu_{\mathcal{B}} \\ &= \int_B f d\mu_{\mathcal{B}} \\ &< \infty \end{aligned}$$

So  $f \in L^1(X, \mathcal{A}, \mu)$ . Similarly, the statement also holds for general  $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  by writing  $f = g + ih$  and applying the above to  $g^+$ ,  $g^-$ ,  $h^+$  and  $h^-$ .  $\square$

**Note 4.5.2.** Denote the  $L^1$  norms on  $L^1(X, \mathcal{A}, \mu)$  and  $L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  by  $N$  and  $N_{\mathcal{B}}$  respectively. The previous exercise implies that  $L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  is a subspace of  $L^1(X, \mathcal{A}, \mu)$  and  $N|_{L^1(X, \mathcal{B}, \mu_{\mathcal{B}})} = N_{\mathcal{B}}$ .

**Exercise 4.5.3.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{A}$  and  $f \in L^1(X, \mathcal{A}, \mu)$ . Define  $\mu_{\mathcal{B}} : \mathcal{B} \rightarrow [0, \infty]$  and  $\nu_f : \mathcal{B} \rightarrow [0, \infty)$  by  $\mu_{\mathcal{B}} = \mu|_{\mathcal{B}}$  and

$$\nu_f(B) = \int_B f d\mu$$

Then  $\nu_f \ll \mu_{\mathcal{B}}$ .

*Proof.* Let  $B \in \mathcal{B}$ . Suppose that  $\mu_{\mathcal{B}}(B) = 0$ . By definition,  $\mu(B) = 0$ . So  $\nu(B) = 0$  and  $\nu \ll \mu_{\mathcal{B}}$ .  $\square$

**Note 4.5.4.** Since  $\nu_f \ll \mu_{\mathcal{B}}$  and  $\nu_f(X) < \infty$ , if  $\mu$  is  $\sigma$ -finite, then  $d\nu_f/d\mu_{\mathcal{B}}$  exists and

$$\begin{aligned} d\nu_f/d\mu_{\mathcal{B}} &\in L^1(X, \mathcal{B}, \mu_{\mathcal{B}}) \\ &\subset L^1(X, \mathcal{A}, \mu) \end{aligned}$$

**Definition 4.5.5.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{A}$ . We define the **projection from  $L^1(X, \mathcal{A}, \mu)$  to  $L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$** , denoted  $P_{\mathcal{B}} : L^1(X, \mathcal{A}, \mu) \rightarrow L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  by

$$P_{\mathcal{B}}f = \frac{d\nu_f}{d\mu_{\mathcal{B}}}$$

**Exercise 4.5.6.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space and  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{A}$ . Then

- (1)  $P_{\mathcal{B}} \in L(L^1(X, \mathcal{A}, \mu))$  and  $\|P_{\mathcal{B}}\| = 1$
- (2)  $P_{\mathcal{B}}|_{L^1(X, \mathcal{B}, \mu_{\mathcal{B}})} = \text{id}_{L^1(X, \mathcal{B}, \mu_{\mathcal{B}})}$
- (3)  $P_{\mathcal{B}}$  is idempotent

*Proof.*

- (1) Let  $f, g \in L^1(X, \mathcal{A}, \mu)$  and  $\lambda \in \mathbb{C}$ . For each  $B \in \mathcal{B}$ , we have that

$$\begin{aligned} \nu_{f+\lambda g}(B) &= \int_B f + \lambda g d\mu \\ &= \int_B f d\mu + \lambda \int_B g d\mu \\ &= \nu_f(B) + \lambda \nu_g(B) \\ &= (\nu_f + \lambda \nu_g)(B) \end{aligned}$$

Hence  $\nu_{f+\lambda g} = \nu_f + \lambda \nu_g$ . Thus

$$\begin{aligned} P_{\mathcal{B}}(f + \lambda g) &= \frac{d\nu_{f+\lambda g}}{d\mu_{\mathcal{B}}} \\ &= \frac{d\nu_f}{d\mu_{\mathcal{B}}} + \lambda \frac{d\nu_g}{d\mu_{\mathcal{B}}} \\ &= P_{\mathcal{B}}f + \lambda P_{\mathcal{B}}g \end{aligned}$$

So  $P_{\mathcal{B}}$  is linear. Since  $|P_{\mathcal{B}}f| \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$ , a previous exercise implies that

$$\begin{aligned}\|P_{\mathcal{B}}f\|_1 &= \int |P_{\mathcal{B}}f| d\mu \\ &= \int |P_{\mathcal{B}}f| d\mu_{\mathcal{B}} \\ &= |\nu_f|(X) \\ &= \int |f| d\mu \\ &= \|f\|_1\end{aligned}$$

Hence  $\|P_{\mathcal{B}}f\|_1 = \|f\|_1$  and  $P_{\mathcal{B}} \in L(L^1(X, \mathcal{A}, \mu))$ .

(2) Let  $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$ . Then for each  $B \in \mathcal{B}$ ,

$$\begin{aligned}\nu_f(B) &= \int_B f d\mu \\ &= \int_B f d\mu_{\mathcal{B}}\end{aligned}$$

Uniqueness of the Radon-Nikodym derivative implies that  $P_{\mathcal{B}}f = f$ . Since  $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  is arbitrary,  $P_{\mathcal{B}}|_{L^1(X, \mathcal{B}, \mu_{\mathcal{B}})} = \text{id}_{L^1(X, \mathcal{B}, \mu_{\mathcal{B}})}$ .

(3) Let  $f \in L^1(X, \mathcal{A}, \mu)$ . Since  $P_{\mathcal{B}}f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  and  $P_{\mathcal{B}}|_{L^1(X, \mathcal{B}, \mu_{\mathcal{B}})} = \text{id}_{L^1(X, \mathcal{B}, \mu_{\mathcal{B}})}$ , we have that

$$\begin{aligned}P_{\mathcal{B}}^2f &= P_{\mathcal{B}}(P_{\mathcal{B}}f) \\ &= \text{id}_{L^1(X, \mathcal{B}, \mu_{\mathcal{B}})}(P_{\mathcal{B}}f) \\ &= P_{\mathcal{B}}f\end{aligned}$$

Since  $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$  is arbitrary,  $P_{\mathcal{B}}^2 = P_{\mathcal{B}}$  and  $P_{\mathcal{B}}$  is idempotent. □

**Exercise 4.5.7.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space,  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{A}$ ,  $f \in L^1(X, \mathcal{A}, \mu)$  and  $g \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$ . Then  $g = P_{\mathcal{B}}f$  iff for each  $B \in \mathcal{B}$ ,

$$\int_B g d\mu = \int_B f d\mu$$

*Proof.* Suppose that  $g = P_{\mathcal{B}}f$ . Let  $B \in \mathcal{B}$ . Then

$$\begin{aligned}\int_B g d\mu &= \int_B g d\mu_{\mathcal{B}} \\ &= \nu_f(B) \\ &= \int_B f d\mu\end{aligned}$$

Since  $B \in \mathcal{B}$  is arbitrary, for each  $B \in \mathcal{B}$ ,

$$\int_B g d\mu = \int_B f d\mu$$

Conversely, suppose that for each  $B \in \mathcal{B}$ ,

$$\int_B g \, d\mu = \int_B f \, d\mu$$

Then for each  $B \in \mathcal{B}$ ,

$$\begin{aligned} \int_B g \, d\mu_{\mathcal{B}} &= \int_B g \, d\mu \\ &= \int_B f \, d\mu \\ &= \nu_f(B) \end{aligned}$$

By definition,

$$\begin{aligned} P_{\mathcal{B}}f &= \frac{d\nu_f}{d\mu_{\mathcal{B}}} \\ &= g \end{aligned}$$

□

**Exercise 4.5.8.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space,  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{A}$  and  $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ . Suppose that  $(A_j)_{j \in \mathbb{N}}$  is disjoint and  $\mu\left(\bigcup_{j \in \mathbb{N}} A_j\right) < \infty$ . Then

- (1)  $\chi_{\bigcup_{j \in \mathbb{N}} A_j} \in L^1(X, \mathcal{A}, \mu)$
- (2)  $P_{\mathcal{B}}\chi_{\bigcup_{j \in \mathbb{N}} A_j} = \sum_{j \in \mathbb{N}} P_{\mathcal{B}}\chi_{A_j}$

*Proof.*

- (1) Since  $(A_j)_{j \in \mathbb{N}}$  is disjoint, we have that

$$\begin{aligned} \|\chi_{\bigcup_{j \in \mathbb{N}} A_j}\|_1 &= \int \chi_{\bigcup_{j \in \mathbb{N}} A_j} \, d\mu \\ &= \mu\left(\bigcup_{j \in \mathbb{N}} A_j\right) \\ &< \infty \end{aligned}$$

So  $\chi_{\bigcup_{j \in \mathbb{N}} A_j} \in L^1(X, \mathcal{A}, \mu)$ .

- (2) Since  $(A_j)_{j \in \mathbb{N}}$  is disjoint, we have that

$$\chi_{\bigcup_{j \in \mathbb{N}} A_j} = \sum_{j \in \mathbb{N}} \chi_{A_j}$$

For each  $n \in \mathbb{N}$ , define  $f_n = \sum_{j=1}^n \chi_{A_j}$ . Set  $f = \chi_{\bigcup_{j \in \mathbb{N}} A_j}$ . Then for each  $n \in \mathbb{N}$ ,  $f_n \leq f$  and  $f_n \xrightarrow{\text{p.w.}} f$ . Since  $f \in L^1(X, \mathcal{A}, \mu)$ , the dominated convergence theorem implies

that  $f_n \xrightarrow{L^1(\mu)} f$ . Since  $P_{\mathcal{B}} \in L(L^1(X, \mathcal{A}, \mu))$ ,

$$\begin{aligned} \sum_{j=1}^n P_{\mathcal{B}} \chi_{A_j} &= P_{\mathcal{B}} \sum_{j=1}^n \chi_{A_j} \\ &= P_{\mathcal{B}} f_n \\ &\xrightarrow{L^1(\mu)} P_{\mathcal{B}} f \\ &= P_{\mathcal{B}} \chi_{\bigcup_{j \in \mathbb{N}} A_j} \end{aligned}$$

Hence  $P_{\mathcal{B}} \chi_{\bigcup_{j \in \mathbb{N}} A_j} = \sum_{j \in \mathbb{N}} P_{\mathcal{B}} \chi_{A_j}$ .

□

**Exercise 4.5.9.** Let  $(X, \mathcal{A}, \mu)$  be a  $\sigma$ -finite measure space,  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{A}$  and  $f \in L^1(X, \mathcal{A}, \mu)$ . If  $f \geq 0$ , then  $P_{\mathcal{B}} f \geq 0$   $\mu_{\mathcal{B}}$ -a.e.

*Proof.* Suppose that  $f \geq 0$ . Then  $\nu_f : \mathcal{B} \rightarrow [0, \infty)$  is a finite measure. Hence

$$\begin{aligned} P_{\mathcal{B}} f &= \frac{d\nu_f}{d\mu_{\mathcal{B}}} \\ &\geq 0 \text{ } \mu_{\mathcal{B}}\text{-a.e.} \end{aligned}$$

□

**Exercise 4.5.10.** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a finite measure space and  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{B}(\mathbb{R})$ . For each  $z \in \mathbb{R}$ , define  $h_z \in L^1(X, \mathcal{A}, \mu)$  by  $h_z = \chi_{(-\infty, z]}$  and choose  $f_z \in L_0(X, \mathcal{A})$  such that  $f_z = P_{\mathcal{B}} h_z$   $\mu$ -a.e. Then there exists  $M \in \mathcal{B}$  such that  $\mu_{\mathcal{B}}(M^c) = 0$  and for each  $x \in M$ ,  $(f_q(x))_{q \in \mathbb{Q}}$  is increasing.

*Proof.* Let  $q, r \in \mathbb{Q}$ . Suppose that  $q < r$ . Then  $\chi_{(-\infty, r]} - \chi_{(-\infty, q]} \geq 0$  and

$$\begin{aligned} f_r - f_q &= P_{\mathcal{B}} \chi_{(-\infty, r]} - P_{\mathcal{B}} \chi_{(-\infty, q]} \\ &= P_{\mathcal{B}} \left[ \chi_{(-\infty, r]} - \chi_{(-\infty, q]} \right] \\ &\geq 0 \text{ } \mu_{\mathcal{B}}\text{-a.e.} \end{aligned}$$

Hence  $f_q \leq f_r$   $\mu_{\mathcal{B}}$ -a.e. An exercise in the section on measures implies that  $(f_q)_{q \in \mathbb{Q}}$  is increasing  $\mu_{\mathcal{B}}$ -a.e. and thus there exists  $M \in \mathcal{B}$  such that  $\mu_{\mathcal{B}}(M^c) = 0$  and for each  $x \in M$ ,  $(f_q(x))_{q \in \mathbb{Q}}$  is increasing. □

**Exercise 4.5.11.** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a finite measure space and  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{B}(\mathbb{R})$ . Define  $(h_z)_{z \in \mathbb{R}} \subset L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ ,  $(f_z)_{z \in \mathbb{R}} \subset L^0(\mathbb{R}, \mathcal{B})$  and  $M \in \mathcal{B}$  as in the previous exercise. Choose  $g \in \text{NBV}(\mathbb{R})$  such that  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g$  is increasing and  $\sup_{z \in \mathbb{R}} g(z) = 1$ . Define

$G : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$G(z, x) = \begin{cases} \inf_{\substack{q \in \mathbb{Q} \\ q > z}} f_q(x) & x \in M \\ g(z) & x \in M^c \end{cases}$$

Then for each  $x \in \mathbb{R}$ ,  $G(\cdot, x)$  is increasing and right continuous.

*Proof.* Let  $x \in \mathbb{R}$ . If  $x \in M^c$ , by definition,  $G(\cdot, x)$  is increasing and right continuous. Suppose that  $x \in M$ . Since  $(f_q(x))_{q \in \mathbb{Q}}$  is increasing, slightly modifying the statement and proof of an exercise in the section on functions of bounded variation implies that  $G(\cdot, x)$  is increasing and right continuous.  $\square$

**Exercise 4.5.12.** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a finite measure space and  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{B}(\mathbb{R})$ . Define  $(h_z)_{z \in \mathbb{R}} \subset L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ ,  $(f_z)_{z \in \mathbb{R}} \subset L^0(\mathbb{R}, \mathcal{B})$ ,  $M \in \mathcal{B}$  and  $G : \mathbb{R}^2 \rightarrow \mathbb{R}$  as in the previous exercise.

- (1) for each  $z \in \mathbb{R}$ ,  $G(z, \cdot) \in L^0(X, \mathcal{B})$  and  $G(z, \cdot) = f_z$   $\mu_B$ -a.e.
- (2)  $\sup_{z \in \mathbb{R}} G(z, \cdot) = 1$   $\mu_B$ -a.e.
- (3)  $\inf_{z \in \mathbb{R}} G(z, \cdot) = 0$   $\mu_B$ -a.e.

*Proof.*

- (1) Let  $z \in \mathbb{R}$ . By definition,

$$G(z, \cdot) = \inf_{\substack{q \in \mathbb{Q} \\ q > z}} [f_q \chi_M](\cdot) + g(z) \chi_{M^c}(\cdot)$$

Since  $(f_q \chi_M)_{q \in \mathbb{Q} \cap (z, \infty)} \subset L^0(X, \mathcal{B})$  and is point-wise bounded below,  $\inf_{\substack{q \in \mathbb{Q} \\ q > z}} f_q \chi_M \in L^0(X, \mathcal{B})$ . Hence  $G(z, \cdot) \in L^0(X, \mathcal{B})$ . Choose  $(q_n)_{n \in \mathbb{N}} \subset \mathbb{Q}$  such that for each  $n \in \mathbb{N}$ ,  $q_n \geq q_{n+1} > z$  and  $q_n \rightarrow z$ . Since for each  $n \in \mathbb{N}$ ,  $h_{q_n} - h_z = \chi_{(z, q_n]}$ ,  $(z, q_{n+1}] \subset (z, q_n]$  and  $\mu$  is finite, we have that

$$\begin{aligned} \|h_{q_n} - h_z\|_1 &= \|\chi_{(z, q_n]}\|_1 \\ &= \mu((z, q_n]) \\ &\rightarrow \mu(\emptyset) \\ &= 0 \end{aligned}$$

So that  $h_{q_n} \xrightarrow{L^1(\mu)} h_z$ . Therefore

$$\begin{aligned} f_{q_n} &= P_{\mathcal{B}} h_{q_n} \\ &\xrightarrow{L^1(\mu_{\mathcal{B}})} P_{\mathcal{B}} h_z \\ &= f_z \end{aligned}$$

This implies that  $f_{q_n} \xrightarrow{\mu_{\mathcal{B}}} f_z$ . Since  $(f_{q_n})_{n \in \mathbb{N}}$  is decreasing  $\mu_B$ -a.e., an exercise in the section on modes of convergence implies that  $f_{q_n} \xrightarrow{\mu_B\text{-a.e.}} f_z$ . So there exists  $N_1 \in \mathcal{B}$  such that  $\mu_B(N_1^c) = 0$  and  $f_{q_n} \chi_{f_i} \xrightarrow{\text{p.w.}} f_z \chi_{N_1}$ . Set  $E = M \cap N_1$ . Then

$$\begin{aligned} \mu_B(E^c) &= \mu_B(M^c \cup N_1^c) \\ &\leq \mu_B(M^c) + \mu_B(N_1^c) \\ &= 0 \end{aligned}$$

and for each  $x \in E$ ,  $f_{q_n}(x) \rightarrow f_z(x)$  and  $f_{q_n}(x) \rightarrow G(z, x)$ . Hence  $G(z, \cdot) \chi_E(\cdot) = f_z \chi_E(\cdot)$  which implies that  $G(z, \cdot) = f_z$   $\mu_B$ -a.e.

- (2) Part (1) implies that for each  $n \in \mathbb{N}$ , there exists  $E_n \in \mathcal{B}$  such that  $E_n \subset M$ ,  $\mu(E_n^c) = 0$  and  $G(n, \cdot)\chi_{E_n}(\cdot) = f_n(\cdot)\chi_{E_n}(\cdot)$ . Set  $E = \bigcap_{n \in \mathbb{N}} E_n$ . Since for each  $n \in \mathbb{N}$ ,  $\chi_{\mathbb{R}} - h_n = \chi_{(n, \infty)}$ ,  $(n+1, \infty) \subset (n, \infty)$  and  $\mu$  is finite, we have that

$$\begin{aligned} \|h_n - \chi_{\mathbb{R}}\|_1 &= \mu((n, \infty)) \\ &\rightarrow \mu(\emptyset) \\ &= 0 \end{aligned}$$

So that  $h_n \xrightarrow{L^1(\mu)} \chi_{\mathbb{R}}$ . Therefore

$$\begin{aligned} f_n &= P_{\mathcal{B}} h_n \\ &\xrightarrow{L^1(\mu_{\mathcal{B}})} P_{\mathcal{B}} \chi_{\mathbb{R}} \\ &= \chi_{\mathbb{R}} \end{aligned}$$

This implies that  $f_n \xrightarrow{\mu_{\mathcal{B}}} \chi_{\mathbb{R}}$ . Since  $(f_n)_{n \in \mathbb{N}}$  is increasing  $\mu_{\mathcal{B}}$ -a.e., an exercise in the section on modes of convergence implies that  $f_n \xrightarrow{\mu_{\mathcal{B}}\text{-a.e.}} \chi_{\mathbb{R}}$ . So there exists  $N_2 \in \mathcal{B}$  such that  $\mu_{\mathcal{B}}(N_2^c) = 0$  and  $f_n \chi_{N_2} \xrightarrow{\text{p.w.}} \chi_{N_2}$ . Set  $M^+ = E \cap N_2$ . Then  $M^+ \subset E \subset M$  and

$$\begin{aligned} \mu_{\mathcal{B}}((M^+)^c) &= \mu_{\mathcal{B}}(E^c \cup N_2^c) \\ &\leq \mu_{\mathcal{B}}(E^c) + \mu_{\mathcal{B}}(N_2^c) \\ &= \mu_{\mathcal{B}}\left(\bigcup_{n \in \mathbb{N}} E_n^c\right) + \mu_{\mathcal{B}}(N_2^c) \\ &\leq \left[\sum_{n \in \mathbb{N}} \mu_{\mathcal{B}}(E_n^c)\right] + \mu_{\mathcal{B}}(N_2^c) \\ &= 0 \end{aligned}$$

Since  $M^+ \subset M$ , for each  $x \in M^+$ ,  $(f_n(x))_{n \in \mathbb{N}}$  is increasing. Hence for each  $x \in M^+$ ,

$$\begin{aligned} \sup_{z \in \mathbb{R}} G(z, x) &= \sup_{n \in \mathbb{N}} G(n, x) \\ &= \sup_{n \in \mathbb{N}} f_n(x) \\ &= 1 \end{aligned}$$

Thus  $\sup_{z \in \mathbb{R}} G(z, \cdot) = 1$   $\mu_{\mathcal{B}}$ -a.e.

- (3) Part (2) implies that for each  $n \in \mathbb{N}$ , there exists  $E_n \in \mathcal{B}$  such that  $E_n \subset M$ ,  $\mu(E_n^c) = 0$  and  $G(n, \cdot)\chi_{E_n}(\cdot) = f_n(\cdot)\chi_{E_n}(\cdot)$ . Set  $E = \bigcap_{n \in \mathbb{N}} E_n$ . Since for each  $n \in \mathbb{N}$ ,  $h_{-n} = \chi_{(-\infty, -n)}$ ,  $(-\infty, -(n+1)) \subset (-\infty, -n)$  and  $\mu$  is finite, we have that

$$\begin{aligned} \|h_{-n}\|_1 &= \mu((-\infty, -n)) \\ &\rightarrow \mu(\emptyset) \\ &= 0 \end{aligned}$$

So that  $h_{-n} \xrightarrow{L^1(\mu)} 0$ . Therefore

$$\begin{aligned} f_{-n} &= P_{\mathcal{B}} h_{-n} \\ &\xrightarrow{L^1(\mu_{\mathcal{B}})} P_{\mathcal{B}} 0 \\ &= 0 \end{aligned}$$

This implies that  $f_n \xrightarrow{\mu_{\mathcal{B}}} 0$ . Since  $(f_{-n})_{n \in \mathbb{N}}$  is decreasing  $\mu_{\mathcal{B}}$ -a.e., an exercise in the section on modes of convergence implies that  $f_{-n} \xrightarrow{\mu_{\mathcal{B}}\text{-a.e.}} 0$ . So there exists  $N_3 \in \mathcal{B}$  such that  $\mu_{\mathcal{B}}(N_3^c) = 0$  and  $f_{-n} \chi_{N_3} \xrightarrow{\text{p.w.}} 0$ . Set  $M^- = E \cap N_3$ . Then  $M^- \subset E \subset M$  and

$$\begin{aligned} \mu_{\mathcal{B}}((M^-)^c) &= \mu_{\mathcal{B}}(E^c \cup N_3^c) \\ &\leq \mu_{\mathcal{B}}(E^c) + \mu_{\mathcal{B}}(N_3^c) \\ &= \mu_{\mathcal{B}}\left(\bigcup_{n \in \mathbb{N}} E_n^c\right) + \mu_{\mathcal{B}}(N_3^c) \\ &\leq \left[\sum_{n \in \mathbb{N}} \mu_{\mathcal{B}}(E_n^c)\right] + \mu_{\mathcal{B}}(N_3^c) \\ &= 0 \end{aligned}$$

Since  $M^- \subset M$ , for each  $x \in M^-$ ,  $(f_{-n}(x))_{n \in \mathbb{N}}$  is decreasing. Hence for each  $x \in M^-$ ,

$$\begin{aligned} \inf_{z \in \mathbb{R}} G(z, x) &= \inf_{n \in \mathbb{N}} G(-n, x) \\ &= \inf_{n \in \mathbb{N}} f_{-n}(x) \\ &= 0 \end{aligned}$$

Thus  $\inf_{z \in \mathbb{R}} G(z, \cdot) = 0$   $\mu_{\mathcal{B}}$ -a.e. □

**Exercise 4.5.13.** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a finite measure space and  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{B}(\mathbb{R})$ . Then there exists  $F : \mathbb{R}^2 \rightarrow [0, 1]$  such that

- (1) for each  $z \in \mathbb{R}$ ,  $F(z, \cdot) \in L^0(X, \mathcal{B})$  and  $F(z, \cdot) = P_{\mathcal{B}} \chi_{(-\infty, z]} \mu_{\mathcal{B}}$ -a.e.
- (2) for each  $x \in \mathbb{R}$ ,  $F(\cdot, x) \in \text{NBV}(\mathbb{R})$ ,  $F(\cdot, x)$  increasing and  $\sup_{z \in \mathbb{R}} F(z, \cdot) = 1$ .

*Proof.* Define  $(h_z)_{z \in \mathbb{R}} \subset L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ ,  $(f_z)_{z \in \mathbb{R}} \subset L^0(\mathbb{R}, \mathcal{B})$  as in the previous exercises. Choose  $g \in \text{NBV}(\mathbb{R})$  such that  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g$  is increasing and  $\sup_{z \in \mathbb{R}} g(z) = 1$ . Define  $M, M^+, M^- \in \mathcal{B}$  and  $G : \mathbb{R}^2 \rightarrow \mathbb{R}$  as in the previous exercises. Set  $E = M \cap M^+ \cap M^-$ . Define  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$F(z, x) = G(z, x) \chi_E(x) + g(z) \chi_{E^c}(x)$$

- (1) Let  $z \in \mathbb{R}$ . Then  $F(z, \cdot) = G(z, \cdot) \chi_E(\cdot) + g(z) \chi_{E^c}(\cdot)$ . Since  $G(z, \cdot) \in L^0(X, \mathcal{B})$ ,  $F(z, \cdot) \in L^0(X, \mathcal{B})$ . Note that

$$\begin{aligned} \mu(E^c) &= \mu(M^c \cup (M^+)^c \cup (M^-)^c) \\ &\leq \mu(M^c) + \mu((M^+)^c) + \mu((M^-)^c) \\ &= 0 \end{aligned}$$



Since  $E \subset M$ , by definition of  $G$  and  $F$ , we have that for each  $x \in E$ ,  $F(z, x) = G(z, x)$ . Hence  $\{x \in \mathbb{R} : F(z, x) \neq G(z, x)\} \subset E^c$ . Thus

$$\begin{aligned} F(z, \cdot) &= G(z, \cdot) \\ &= f_z \mu_{\mathcal{B}\text{-a.e.}} \end{aligned}$$

- (2) Let  $x \in \mathbb{R}$ . Suppose that  $x \in E$ . The previous exercise implies that  $G(\cdot, x) \in \text{NBV}(\mathbb{R})$ ,  $G(\cdot, x)$  is increasing and  $\sup_{z \in \mathbb{R}} G(z, x) = 1$ . Since  $F(\cdot, x) = G(\cdot, x)$ , we have that  $F(\cdot, x) \in \text{NBV}(\mathbb{R})$ ,  $F(\cdot, x)$  is increasing and  $\sup_{z \in \mathbb{R}} F(z, x) = 1$ .

If  $x \in E^c$ , then  $F(\cdot, x) = g$ . By definition of  $g$ ,  $F(\cdot, x) \in \text{NBV}(\mathbb{R})$ ,  $F(\cdot, x)$  is increasing and  $\sup_{z \in \mathbb{R}} F(z, \cdot) = 1$ .

□

**Definition 4.5.14.** Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and  $\kappa : X \times \mathcal{B} \rightarrow [0, 1]$ . Then  $\kappa$  is said to be a **Markov kernel from  $(X, \mathcal{A})$  to  $(Y, \mathcal{B})$**  if

- (1) for each  $x \in X$ ,  $\kappa(x, \cdot)$  is a probability measure on  $(Y, \mathcal{B})$
- (2) for each  $B \in \mathcal{B}$ ,  $\kappa(\cdot, B)$  is  $\mathcal{A}$ -measurable

**Exercise 4.5.15.** Let  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$  be a finite measure space and  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{B}(\mathbb{R})$ . Then there exists  $\kappa : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$  such that

- (1)  $\kappa$  is a Markov kernel from  $(\mathbb{R}, \mathcal{B})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .
- (2) For each  $A \in \mathcal{B}(\mathbb{R})$ ,  $\kappa(\cdot, A) = P_{\mathcal{B}} \chi_A \mu_{\mathcal{B}\text{-a.e.}}$

**Hint:**

- (1) Consider  $F : \mathbb{R}^2 \rightarrow [0, 1]$  defined in the previous exercise and  $\mu_x((a, b]) = F(b, x) - F(a, x)$ .
- (2) Consider Dynkin's lemma with

$$\nu_B(A) = \int_B \kappa(x, A) d\mu_{\mathcal{B}}(x) \quad \text{and} \quad \lambda_B(A) = \mu(A \cap B)$$

*Proof.* Define  $F : \mathbb{R}^2 \rightarrow [0, 1]$  as in the previous exercise. For each  $x \in \mathbb{R}$ , define  $\mu_x : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$  to be the unique measure such that for each  $a, b \in \mathbb{R}$ ,  $a \leq b$  implies that  $\mu_x((a, b]) = F(b, x) - F(a, x)$ . Define  $\kappa : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$  by  $\kappa(A, x) = \mu_x(A)$ .

- (1)
  - (a) Let  $x \in \mathbb{R}$ . By definition,  $\kappa(x, \cdot) = \mu_x$  is a measure and

$$\begin{aligned} \kappa(x, \mathbb{R}) &= \sup_{n \in \mathbb{N}} \mu_x((-\infty, n]) \\ &= \sup_{n \in \mathbb{N}} F(n, x) \\ &= 1 \end{aligned}$$

- (b) Let  $A \in \mathcal{B}(\mathbb{R})$ . Recall that for each  $x \in \mathbb{R}$ ,

$$\mu_x(A) = \inf \left\{ \sum_{j \in \mathbb{N}} F(b_j, x) - F(a_j, x) : \text{for each } j \in \mathbb{N}, a_j, b_j \in \mathbb{R} \text{ and } A \subset \bigcup_{j \in \mathbb{N}} (a_j, b_j] \right\}$$

Therefore, for each  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ , there exist  $(a_{n,j}^x)_{j \in \mathbb{N}}, (b_{n,j}^x)_{j \in \mathbb{N}} \subset \mathbb{R}$  such that  $A \subset \bigcup_{j \in \mathbb{N}} (a_{n,j}^x, b_{n,j}^x]$  and

$$\mu_x(A) \leq \sum_{j \in \mathbb{N}} F(b_{n,j}^x, x) - F(a_{n,j}^x, x) < \mu_x(A) + \frac{1}{n}$$

Define  $(f_n)_{n \in \mathbb{N}} \subset L^0(X, \mathcal{B})$  by

$$f_n(x) = \sum_{j \in \mathbb{N}} F(b_{n,j}^x, x) - F(a_{n,j}^x, x)$$

Then  $f_n \xrightarrow{\text{p.w.}} \kappa(\cdot, A)$  which implies that  $\kappa(\cdot, A) \in L^0(X, \mathcal{B})$ . Hence  $\kappa$  is a markov kernel from  $(\mathbb{R}, \mathcal{B})$  to  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

(2) Let  $B \in \mathcal{B}$ . Define  $\nu_B, \lambda_B : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty)$  by

$$\nu_B(A) = \int_B \kappa(x, A) d\mu_{\mathcal{B}}(x)$$

and

$$\lambda_B(A) = \mu(A \cap B)$$

Let  $a, b \in \mathbb{R}$ . Then

$$\begin{aligned} \nu_B((a, b]) &= \int_B \kappa(x, (a, b]) d\mu_{\mathcal{B}}(x) \\ &= \int_B F(b, x) - F(a, x) d\mu_{\mathcal{B}}(x) \\ &= \int_B P_{\mathcal{B}}\chi_{(-\infty, b]} - P_{\mathcal{B}}\chi_{(-\infty, a]} d\mu_{\mathcal{B}} \\ &= \int_B P_{\mathcal{B}}\chi_{(a, b]} d\mu_{\mathcal{B}} \\ &= \int_B \chi_{(a, b]} d\mu \\ &= \mu((a, b] \cap B) \\ &= \lambda_B((a, b]) \end{aligned}$$

Define  $\mathcal{P} \subset \mathcal{B}(\mathbb{R})$  by  $\mathcal{P} = \{(a, b] : a, b \in \mathbb{R}\} \cup \{\emptyset, X\}$ . A previous exercise in the sections on Dynkin's lemma implies that  $\mathcal{P}$  is a  $\pi$ -system. Since  $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$ , an exercise in the section on complex measures implise that  $\nu_B = \lambda_B$ . Let  $A \in \mathcal{B}(\mathbb{R})$ .

Then

$$\begin{aligned}
 \int_B \kappa(x, A) d\mu_{\mathcal{B}}(x) &= \nu_B(A) \\
 &= \lambda_B(A) \\
 &= \mu(A \cap B) \\
 &= \int_B \chi_A d\mu \\
 &= \int_B P_{\mathcal{B}} \chi_A d\mu \\
 &= \int_B P_{\mathcal{B}} \chi_A d\mu_{\mathcal{B}}
 \end{aligned}$$

Since  $B \in \mathcal{B}$  is arbitrary,  $\kappa(\cdot, A) = P_{\mathcal{B}} \chi_A$   $\mu_{\mathcal{B}}$ -a.e. Since  $A \in \mathcal{B}(\mathbb{R})$  is arbitrary, we have that for each  $A \in \mathcal{B}(\mathbb{R})$ ,  $\kappa(\cdot, A) = P_{\mathcal{B}} \chi_A$   $\mu_{\mathcal{B}}$ -a.e. □

**Exercise 4.5.16.** Let  $(X, \mathcal{A}, \mu)$  be a finite measure space and  $\mathcal{B}$  a sub  $\sigma$ -algebra of  $\mathcal{A}$ . If  $(X, \mathcal{A})$  is a Borel space, then there exists  $\kappa : X \times \mathcal{A} \rightarrow [0, 1]$  such that

- (1)  $\kappa$  is a markov kernel from  $(X, \mathcal{B})$  to  $(X, \mathcal{A})$ .
- (2) For each  $A \in \mathcal{A}$ ,  $\kappa(\cdot, A) = P_{\mathcal{B}} \chi_A$   $\mu_{\mathcal{B}}$ -a.e.

*Proof.* Suppose that  $(X, \mathcal{A})$  is a Borel space. Then there exists  $S \in \mathcal{B}(\mathbb{R})$  and  $\phi : X \rightarrow S$  such that  $\phi$  is an isomorphism. □

**Exercise 4.5.17.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $(Y, \mathcal{B})$  a measurable space,  $f \in L^1(X, \mathcal{A}, \mu)$  and  $g : X \rightarrow Y$ . Suppose that for each  $y \in Y$ ,  $\{y\} \in \mathcal{B}$ ,  $g$  is surjective and  $g$  is  $(\mathcal{A}, \mathcal{B})$ -measurable. Then there exists a  $\phi \in L^0(Y, \mathcal{B})$  such that  $\phi \circ g = \mathcal{P}_{g^* \mathcal{B}} f$   $\mu$ -a.e. and  $\phi$  is unique  $g_* \mu$ -a.e.

**Hint:** Doob-Dynkin lemma

*Proof.*

• **Existence:**

Since  $P_{g^* \mathcal{B}} f \in L^1(X, g^* \mathcal{B}, \mu_{g^* \mathcal{B}})$  and  $\mathcal{B}$ , the Doob-Dynkin lemma implies that there exists a  $\phi \in L^0(Y, \mathcal{B})$  such that  $\phi \circ g = P_{g^* \mathcal{B}} f$ .

• **Uniqueness:**

Suppose that there exists  $\psi \in L^0(Y, \mathcal{B})$  such that  $\psi \circ g = \mathcal{P}_{g^* \mathcal{B}} f$   $\mu$ -a.e. Then  $\phi \circ g = \psi \circ g$   $\mu$ -a.e. An exercise in the section on integration of nonnegative functions implies that  $\phi = \psi$   $g_* \mu$ -a.e. □

**Exercise 4.5.18.** Let  $(X, \mathcal{A}, \mu)$  be a finite measure space,  $(Y, \mathcal{B})$  a measurable space and  $g : X \rightarrow Y$ . Suppose that for each  $y \in Y$ ,  $\{y\} \in \mathcal{B}$ ,  $g$  is surjective and  $g$  is  $(\mathcal{A}, \mathcal{B})$ -measurable. Then there exists  $\kappa : Y \times \mathcal{A} \rightarrow [0, \infty)$  such that  $\kappa$  is a transition kernel from  $(Y, \mathcal{B})$  to  $(X, \mathcal{A})$ .

**Hint:** For  $A \in \mathcal{A}$ , define  $\phi_A \in L^0(Y, \mathcal{B})$  to be the  $g_* \mu$ -a.e. unique  $\phi \in L^0(Y, \mathcal{B})$  such that  $\phi \circ g = P_{g^* \mathcal{B}} \chi_A$ . Define  $\kappa' : Y \times \mathcal{A} \rightarrow [0, \infty)$  by  $\kappa'(y, A) = \phi_A(y)$ . For each  $A \in \mathcal{A}$ , define  $\kappa(\cdot, A)$  by redefining  $\kappa'(\cdot, A)$  on a  $g_* \mu$ -null set.

*Proof.*

- Since  $\chi_\emptyset = 0$ ,  $P_{g^*\mathcal{B}}\chi_\emptyset = 0$   $\mu$ -a.e. Therefore

$$\begin{aligned} 0 \circ g &= 0 \\ &= P_{g^*\mathcal{B}}\chi_\emptyset \text{ } \mu\text{-a.e.} \end{aligned}$$

Uniqueness of  $\phi_\emptyset$  implies that  $\phi_\emptyset = 0$   $g_*\mu$ -a.e. Thus there exists  $N_1 \in \mathcal{B}$  such that  $g_*\mu(N_1) = 0$  and for each  $y \in N_1^c$ ,

$$\begin{aligned} \kappa'(y, \emptyset) &= \phi_\emptyset(y) \\ &= 0 \end{aligned}$$

- Let  $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ . Suppose that  $(A_j)_{j \in \mathbb{N}}$  is disjoint. Since  $\mu$  is finite,  $\mu\left(\bigcup_{j \in \mathbb{N}} A_j\right) < \infty$ .

A previous exercise implies that

- (1)  $\chi_{\bigcup_{j \in \mathbb{N}} A_j} \in L^1(X, \mathcal{A}, \mu)$
- (2)  $P_{\mathcal{B}}\chi_{\bigcup_{j \in \mathbb{N}} A_j} = \sum_{j \in \mathbb{N}} P_{\mathcal{B}}\chi_{A_j}$

Therefore

$$\begin{aligned} \phi_{\bigcup_{j \in \mathbb{N}} A_j} \circ g &= P_{\mathcal{B}}\chi_{\bigcup_{j \in \mathbb{N}} A_j} \\ &= \sum_{j \in \mathbb{N}} P_{\mathcal{B}}\chi_{A_j} \\ &= \sum_{j \in \mathbb{N}} \phi_{A_j} \circ g \text{ } \mu\text{-a.e.} \end{aligned}$$

Uniqueness of  $\phi_{\bigcup_{j \in \mathbb{N}} A_j}$  implies that  $\phi_{\bigcup_{j \in \mathbb{N}} A_j} = \sum_{j \in \mathbb{N}} \phi_{A_j}$   $g_*\mu$ -a.e.  $\phi_\emptyset$  implies that  $\phi_\emptyset = 0$   $g_*\mu$ -a.e. Thus there exists  $N_2 \in \mathcal{B}$  such that  $g_*\mu(N_2) = 0$  and for each  $y \in N_2^c$ ,

$$\begin{aligned} \kappa'\left(y, \bigcup_{j \in \mathbb{N}} A_j\right) &= \phi_{\bigcup_{j \in \mathbb{N}} A_j}(y) \\ &= \sum_{j \in \mathbb{N}} \phi_{A_j}(y) \\ &= \sum_{j \in \mathbb{N}} \mu_y(A_j) \\ &= \sum_{j \in \mathbb{N}} \kappa'(y, A_j) \end{aligned}$$

Set  $N = N_1 \cup N_2$ . Then  $g_*\mu(N) = 0$  and for each  $y \in N^c$ ,  $\kappa'(y, \cdot) : \mathcal{A} \rightarrow [0, \infty)$  is a measure on  $(X, \mathcal{A})$ . Choose  $x \in X$ . Define  $\kappa : Y \times \mathcal{A} \rightarrow [0, \infty)$  by  $\kappa(y, A) = \chi_N(y)\delta_x(A) + \chi_{N^c}(y)\kappa'(y, A)$ .

- (1) Let  $A \in \mathcal{A}$ . Then

$$\begin{aligned} \kappa(\cdot, A) &= \chi_N(\cdot)\delta_x(A) + \chi_{N^c}(\cdot)\kappa'(\cdot, A) \\ &= \chi_N(\cdot)\delta_x(A) + \chi_{N^c}(\cdot)\phi_A(\cdot) \end{aligned}$$

Hence for each  $A \in \mathcal{A}$ ,  $\kappa(\cdot, A)$  is  $\mathcal{B}$ -measurable.

(2) Let  $y \in Y$ . Then

$$\kappa(y, \cdot) = \begin{cases} \delta_x(\cdot) & y \in N \\ \kappa'(y, \cdot) & y \in N^c \end{cases}$$

Hence for each  $y \in Y$ ,  $\kappa(y, \cdot)$  is a measure on  $(X, \mathcal{A})$ .

Thus  $\kappa$  is a transition kernel from  $(Y, \mathcal{B}, g_*\mu)$  to  $(X, \mathcal{A})$ . □

**Definition 4.5.19.** Let  $(X, \mathcal{A}, \mu)$  be a finite measure space,  $(Y, \mathcal{B})$  a measurable space and  $g : X \rightarrow Y$ . Suppose that for each  $y \in Y$ ,  $\{y\} \in \mathcal{B}$ ,  $g$  is surjective and  $g$  is  $(\mathcal{A}, \mathcal{B})$ -measurable. For  $A \in \mathcal{A}$ , define  $\phi_A \in L^0(Y, \mathcal{B})$  to be the  $g_*\mu$ -a.e. unique  $\phi \in L^0(Y, \mathcal{B})$  such that  $\phi \circ g = P_{g^*\mathcal{B}}\chi_A$ . For  $y \in Y$ , we define the **conditional of  $\mu$  on  $y$** , denoted  $\mu_y : \mathcal{A} \rightarrow [0, \infty)$ , by  $\mu_y(A) = \phi_A(y)$ .

**Exercise 4.5.20. Disintegration of Measure:**

Let  $(X, \mathcal{A}, \mu)$  be a finite measure space,  $(Y, \mathcal{B})$  a measurable space and  $g : X \rightarrow Y$ . Suppose that for each  $y \in Y$ ,  $\{y\} \in \mathcal{B}$ ,  $g$  is surjective and  $g$  is  $(\mathcal{A}, \mathcal{B})$ -measurable. Then there exists a collection of measures  $(\mu_y)_{y \in Y}$  such that

(1) for each  $A \in \mathcal{A}$ ,

$$\mu(A) = \int \mu_y(A) dg_*\mu(y)$$

(2) for each  $f \in L^1(X, \mathcal{A}, \mu)$ ,

$$\int f d\mu = \int \left[ \int f d\mu_y(x) \right] dg_*\mu(y)$$

5.  $L^p$  SPACES

## 5.1. Introduction.

**Definition 5.1.1.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $p \in (0, \infty]$ . Define  $\|\cdot\|_p : L^0(X, \mathcal{A}, \mu) \rightarrow [0, \infty]$  by

$$\|f\|_p = \left( \int |f|^p d\mu \right)^{\frac{1}{p}} \quad (p < \infty)$$

and

$$\|f\|_\infty = \inf \left\{ \lambda > 0 : \mu(\{x \in X : \lambda < |f(x)|\}) = 0 \right\}$$

We define

$$L^p(X, \mathcal{A}, \mu) = \{f \in L^0(X, \mathcal{A}, \mu) : \|f\|_p < \infty\}$$

**Exercise 5.1.2.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $p \in (0, \infty]$  and  $f, g \in L^p(X, \mathcal{A}, \mu)$ . If  $|f| \leq |g|$   $\mu$ -a.e., then  $\|f\|_p \leq \|g\|_p$ .

*Proof.* Suppose that  $|f| \leq |g|$   $\mu$ -a.e. Then  $|f|^p \leq |g|^p$   $\mu$ -a.e. This implies that

$$\int |f|^p d\mu \leq \int |g|^p d\mu$$

Hence  $\|f\|_p \leq \|g\|_p$ . □

**Theorem 5.1.3. Hölder's Inequality:** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $p, q \in [1, \infty)$  and  $f, g \in L^0$ . Suppose that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

**Exercise 5.1.4. Minkowski Inequality:** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $p \in [1, \infty)$  and  $f, g \in L^p$ . Then  $f + g \in L^p$  and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

*Proof.* Define  $\phi : \mathbb{R} \rightarrow [0, \infty)$  by  $\phi(x) = |x|^p$ . Then  $\phi$  is convex because it is the composition of an increasing convex function with a convex function. By Jensen's inequality, we have that

$$\phi\left(\frac{1}{2}[f + g]\right) \leq \frac{1}{2}[\phi(f) + \phi(g)]$$

This implies that

$$\frac{1}{2^p}|f + g|^p \leq \frac{1}{2}(|f|^p + |g|^p)$$

Hence

$$\begin{aligned} \int |f + g|^p d\mu &\leq 2^{p-1} \int |f|^p + |g|^p d\mu \\ &= 2^{p-1} \left( \int |f|^p d\mu + \int |g|^p d\mu \right) \\ &= 2^{p-1} \left( \|f\|_p^p + \|g\|_p^p \right) \\ &< \infty \end{aligned}$$

So  $f + g \in L^p$ . Now, it is not hard to see that  $|f + g|^p \leq (|f| + |g|)|f + g|^{p-1}$ . Let  $q$  be the conjugate of  $p$ , so that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then  $q(p-1) = p$ . We use Hölder's inequality to show that

$$\begin{aligned}
 \|f + g\|_p^p &= \int |f + g|^p d\mu \\
 &\leq \int |f||f + g|^{p-1} d\mu + \int |g||f + g|^{p-1} d\mu \\
 &\leq \|f\|_p \left( \int |f + g|^{(p-1)q} d\mu \right)^{\frac{1}{q}} + \|g\|_p \left( \int |f + g|^{(p-1)q} d\mu \right)^{\frac{1}{q}} \\
 &= \|f\|_p \left( \int |f + g|^p d\mu \right)^{\frac{1}{q}} + \|g\|_p \left( \int |f + g|^p d\mu \right)^{\frac{1}{q}} \\
 &= (\|f\|_p + \|g\|_p) \left( \int |f + g|^p d\mu \right)^{\frac{1}{q}} \\
 &= (\|f\|_p + \|g\|_p) \|f + g\|_p^{p/q}
 \end{aligned}$$

Since  $\|f + g\|_p < \infty$ , we see that

$$\begin{aligned}
 \|f\|_p + \|g\|_p &\geq \|f + g\|_p^{p-p/q} \\
 &= \|f + g\|_p^{p(1-1/q)} \\
 &= \|f + g\|_p^{p/p} \\
 &= \|f + g\|_p
 \end{aligned}$$

□

**Exercise 5.1.5.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $p, q \in (0, \infty]$ . Suppose that  $\mu(X) < \infty$  and  $p < q$ . Then  $L^q \subset L^p$ . In particular, if  $\mu(X) = 1$ , then for each  $f \in L^q$ ,  $\|f\|_p \leq \|f\|_q$ .

*Proof.* Suppose that  $q = \infty$ . Let  $f \in L^q$ . Then

$$\begin{aligned}
 \|f\|_p &= \left( \int |f|^p d\mu \right)^{\frac{1}{p}} \\
 &\leq \left( \int \|f\|_\infty^p d\mu \right)^{\frac{1}{p}} \\
 &= \|f\|_\infty \mu(X)^{\frac{1}{p}}
 \end{aligned}$$

If  $q < \infty$ , then  $\frac{q}{p} > 1$  and the conjugate of  $\frac{q}{p}$  is  $\frac{1}{1-p/q}$ . By Hölder's inequality, we have that

$$\begin{aligned} \|f\|_p^p &= \|f^p\|_1 \\ &\leq \|f^p\|_{\frac{q}{p}} \|1\|_{\frac{1}{1-p/q}} \\ &= \left( \int |f|^{\frac{pq}{p}} d\mu \right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}} \\ &= \left( \int |f|^q d\mu \right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}} \\ &= \|f\|_q^p \mu(X)^{1-\frac{p}{q}} \end{aligned}$$

Hence

$$\begin{aligned} \|f\|_p &\leq \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}} \\ &< \infty \end{aligned}$$

□

**Exercise 5.1.6.** Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  be  $\sigma$ -finite measure spaces and  $K \in L^0(X \times Y)$ . Suppose that there exists  $C > 0$  such that for  $\mu$ -a.e  $x \in X$ ,

$$\int_Y |K(x, y)| d\nu(y) < C$$

and for  $\nu$ -a.e  $y \in Y$ ,

$$\int_X |K(x, y)| d\mu(x) < C$$

Let  $f \in L^p(\nu)$ .

(1) Then for  $\mu$ -a.e.  $x \in X$ ,

$$\int_Y K(x, y) f(y) d\nu(y)$$

exists.

**Hint:** Note that  $|K(x, y) f(y)| = (|K(x, y)|^{1/q}) (|K(x, y)|^{1/p} |f(y)|)$

(2) Define  $Tf \in L^0(X)$  by

$$Tf(x) = \int_Y K(x, y) f(y) d\nu(y)$$

Then  $Tf \in L^p(\mu)$  and  $\|Tf\|_p \leq C\|f\|_p$ .

*Proof.* Let  $p, q \in (0, \infty)$  be conjugate.

(1) Define  $h \in L^0(X \times Y)$  by  $h(x, y) = K(x, y) f(y)$ . By assumption, there exists  $N \in \mathcal{A}$  such that  $\mu(N) = 0$  and

$$\left\{ x \in X : \int_Y |K(x, y)| d\nu(y) < C \right\} \subset N^c$$



Let  $x \in N^c$ . Then Holder's inequality implies that

$$\begin{aligned} \int_Y |h(x, y)| d\nu(y) &= \int_Y (|K(x, y)|^{1/q}) (|K(x, y)|^{1/p} |f(y)|) d\nu(y) \\ &\leq \left( \int_Y |K(x, y)| d\nu(y) \right)^{1/q} \left( \int_Y |K(x, y)| |f(y)|^p d\nu(y) \right)^{1/p} \\ &\leq C^{1/q} \left( \int_Y |K(x, y)| |f(y)|^p d\nu(y) \right)^{1/p} \end{aligned}$$

Tonelli's theorem implies that the map

$$x \mapsto \int_Y |h(x, y)| d\nu(y)$$

is measurable and that

$$\begin{aligned} \int_X \left[ \int_Y |h(x, y)| d\nu(y) \right]^p d\mu(x) &\leq C^{p/q} \int_X \left[ \int_Y |K(x, y)| |f(y)|^p d\nu(y) \right] d\mu(x) \\ &= C^{p/q} \int_Y \left[ \int_X |K(x, y)| |f(y)|^p d\mu(x) \right] d\nu(y) \\ &= C^{p/q} \int_Y \left[ \int_X |K(x, y)| d\mu(x) \right] |f(y)|^p d\nu(y) \\ &\leq C^{1+p/q} \int_Y |f(y)|^p d\nu(y) \\ &= C^{1+p/q} \|f\|_p^p \end{aligned}$$

So for  $\mu$ -a.e.  $x \in X$ ,

$$\int_Y |h(x, y)| d\nu(y) < \infty$$

which implies that for  $\mu$ -a.e.  $x \in X$ ,  $h(x, \cdot) \in L^1(\nu)$ . Therefore, for  $\mu$ -a.e.  $x \in X$ ,

$$\int_Y h(x, y) d\nu(y)$$

exists. The case is similar when  $p \in \{1, \infty\}$ .

(2) Let  $x \in X$ . Then

$$|Tf(x)| \leq \int_Y |K(x, y)f(y)| d\nu(y)$$

which implies that

$$|Tf(x)|^p \leq \left( \int_Y |K(x, y)f(y)| d\nu(y) \right)^p$$

By part (1),

$$\int_X |Tf|^p d\mu \leq C^{1+p/q} \|f\|_p^p$$

So  $Tf \in L^p(\mu)$  and  $\|Tf\|_p \leq C\|f\|_p$ . The case is similar when  $p \in \{1, \infty\}$ .

□

## 6. BOREL MEASURES

## 6.1. Radon Measures.

**Definition 6.1.1.** Let  $X$  be a topological space,  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a measure and  $E \in \mathcal{B}(X)$ . Then  $\mu$  is said to be

- (1) **inner regular on  $E$**  if

$$\mu(E) = \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}$$

- (2) **outer regular on  $E$**  if

$$\mu(E) = \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$$

- (3) **regular on  $E$**  if  $\mu$  is inner regular on  $E$  and  $\mu$  is outer regular on  $E$

**Definition 6.1.2.** Let  $X$  be a topological space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a measure. Then  $\mu$  is said to be

- (1) **inner regular** if for each  $E \in \mathcal{A}$ ,  $\mu$  is inner regular on  $E$
- (2) **outer regular** if for each  $E \in \mathcal{A}$ ,  $\mu$  is outer regular on  $E$
- (3) **regular** if  $\mu$  is inner regular and  $\mu$  is outer regular

**Definition 6.1.3.** Let  $X$  be a topological space,  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a measure and  $E \in \mathcal{B}(X)$ . Then  $\mu$  is said to be a **Radon measure** if for each  $E \in \mathcal{B}(X)$ ,

- (1)  $E$  is compact implies that  $\mu(E) < \infty$
- (2)  $\mu$  is outer regular on  $E$
- (3)  $E$  is open implies that  $\mu$  is inner regular on  $E$

**Definition 6.1.4.** Let  $X$  be a topological space,  $\mu \in \mathcal{M}(X)$ . Then  $\mu$  is said to be **Radon** if  $\|\mu\|$  is Radon.

**Exercise 6.1.5.** Let  $X$  be a topological space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure. Set

$$\mathcal{N}_\mu = \{U \subset X : U \text{ is open and } \mu(U) = 0\}$$

and

$$N_\mu = \bigcup_{U \in \mathcal{N}_\mu} U$$

Then  $N_\mu$  is open,  $N_\mu^c$  is closed and  $\mu(N_\mu) = 0$ .

**Hint:** use inner regularity and compactness

*Proof.* Since  $N_\mu$  is the union of open sets, it is open and  $N_\mu^c$  is closed since  $N_\mu$  is open. Let  $K \subset N_\mu$ . Suppose that  $K$  is compact. Since  $\mathcal{N}_\mu$  is an open cover for  $K$ , there exist  $U_1, \dots, U_n \in \mathcal{N}_\mu$  such that

$$K \subset \bigcup_{j=1}^n U_j$$

This implies that

$$\begin{aligned}\mu(K) &\leq \mu\left(\bigcup_{j=1}^n U_j\right) \\ &\leq \sum_{j=1}^n \mu(U_j) \\ &= 0\end{aligned}$$

Inner regularity implies that

$$\begin{aligned}\mu(N_\mu) &= \sup\{\mu(K) : K \subset N_\mu \text{ and } K \text{ is compact}\} \\ &= 0\end{aligned}$$

□

**Definition 6.1.6.** Let  $X$  be a topological space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure. Define  $\mathcal{N}_\mu$  and  $N_\mu$  as in the previous exercise. We define the **support of  $\mu$** , denoted  $\text{supp}(\mu)$ , by

$$\text{supp}(\mu) = N_\mu^c$$

**Exercise 6.1.7.** Let  $X$  be a topological space,  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure and  $E \in \mathcal{B}(X)$ . If  $\mu(E) < \infty$ , then for each  $\epsilon > 0$ ,

- (1) there exists  $U \in \mathcal{B}(X)$  such that  $U$  is open,  $E \subset U$  and  $\mu(U \setminus E) < \epsilon$
- (2) there exists  $C \in \mathcal{B}(X)$  such that  $C$  is compact,  $C \subset U$  and  $\mu(U) - \epsilon < \mu(C)$
- (3) there exists  $V \in \mathcal{B}(X)$  such that  $V$  is open,  $U \setminus E \subset V$  and  $\mu(V) < \epsilon$

*Proof.* Suppose that  $\mu(E) < \infty$ . Let  $\epsilon > 0$ .

- (1) Outer regularity on  $E$  implies that there exists  $U \in \mathcal{B}(X)$  such that  $U$  is open,  $E \subset U$  and  $\mu(U \setminus E) < \epsilon$ .
- (2) Inner regularity on  $U$  implies that there exists  $C \in \mathcal{B}(X)$  such that  $C$  is compact,  $C \subset U$  and  $\mu(U) - \epsilon < \mu(C)$ .
- (3) Outer regularity on  $U \setminus E$  implies that there exists  $V \in \mathcal{B}(X)$  such that  $U$  and  $V$  are open,  $U \setminus E \subset V$  and  $\mu(V) < \epsilon$ .

□

**Exercise 6.1.8.** Let  $X$  be a topological space,  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure and  $E \in \mathcal{A}$ . Suppose that  $\mu(E) < \infty$ . Let  $\epsilon > 0$ . Define  $U$ ,  $C$  and  $V$  as in the previous exercise. Set  $K = C \setminus V$ . Then  $K$  is compact,  $K \subset E$  and  $\mu(K) > \mu(E) - 2\epsilon$

*Proof.* Since  $C$  is closed and  $V$  is open,  $C \setminus V = C \cap V^c$  is closed. Since  $C$  is compact and  $C \setminus V \subset C$ , we have that  $K = C \setminus V$  is compact. Set algebra implies that

$$\begin{aligned}K &= C \cap V^c \\ &\subset U \cap V^c \\ &\subset U \cap (U^c \cup E) \\ &= (U \cap U^c) \cup (U \cap E) \\ &= U \cap E \\ &\subset E\end{aligned}$$

The previous exercise implies that

$$\begin{aligned}
 \mu(K) &= \mu(C \cap V^c) \\
 &= \mu(C) - \mu(C \cap V) \\
 &> \mu(U) - \epsilon - \mu(V) \\
 &> \mu(E) - 2\epsilon
 \end{aligned}$$

□

**Exercise 6.1.9.** Let  $X$  be a topological space,  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure and  $E \in \mathcal{B}(X)$ . If  $E$  is  $\sigma$ -finite, then  $\mu$  is inner regular on  $E$ .

**Hint:** use the previous exercise

*Proof.* Suppose that  $E$  is  $\sigma$ -finite.

If  $\mu(E) < \infty$ , the previous exercise implies that for each  $\epsilon > 0$ , there exists  $K \in \mathcal{B}(X)$  such that  $K$  is compact,  $K \subset E$  and  $\mu(K) > \mu(E) - \epsilon$ . Hence  $\mu$  is inner regular on  $E$ .

If  $\mu(E) = \infty$ , then  $\sigma$ -finiteness implies that there exists  $(E_j)_{j \in \mathbb{N}} \subset \mathcal{B}(X)$  such that  $E = \bigcup_{j \in \mathbb{N}} E_j$ , for each  $j \in \mathbb{N}$ ,  $\mu(E_j) < \infty$  and  $\mu(E_j) \rightarrow \infty$ . Let  $N \in \mathbb{N}$ . Choose  $J \in \mathbb{N}$  such that  $\mu(E_J) > N$ . The above argument implies that there exists  $K \in \mathcal{B}(X)$  such that  $K$  is compact,  $K \subset E_J \subset E$  and  $\mu(K) > N$ . So

$$\begin{aligned}
 \mu(E) &= \infty \\
 &= \sup_{\substack{K \subset E \\ K \text{ is compact}}} \mu(K)
 \end{aligned}$$

and  $\mu$  is inner regular on  $E$ . □

**Exercise 6.1.10.** Let  $X$  be a topological space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure. If  $\mu$  is  $\sigma$ -finite, then  $\mu$  is regular.

*Proof.* Clear by previous exercise. □

**Exercise 6.1.11.** Let  $X$  be a topological space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure. If  $X$  is  $\sigma$ -compact, then  $\mu$  is  $\sigma$ -finite. The previous exercise implies that  $\mu$  is regular.

*Proof.* If  $X$  is  $\sigma$ -compact, then  $\mu$  is  $\sigma$ -finite. Hence  $\mu$  is regular. □

**Exercise 6.1.12.** Let  $X$  be a topological space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure. Then for each  $p \in [1, \infty]$ ,  $C_c(X) \subset L^p(\mu)$ .

*Proof.* Let  $p \in [1, \infty]$  and  $f \in C_c(X)$ . Then  $|f|^p \in C_c(X)$  and

$$\begin{aligned}
 \|f\|_p &= \int |f|^p d\mu \\
 &\leq \|f\|_\infty^p \mu(\text{supp}(f)) \\
 &< \infty
 \end{aligned}$$

□

## 6.2. Radon Measures on LCH Spaces.

**Definition 6.2.1.** Let  $X$  be a topological space and  $I : C_c(X) \rightarrow \mathbb{C}$  a linear functional. Then  $I$  is said to be **positive** if for each  $f \in C_c(X, \mathbb{R})$ ,  $f \geq 0$  implies that  $I(f) \geq 0$ .

**Exercise 6.2.2.** Let  $X$  be a topological space,  $I : C_c(X) \rightarrow \mathbb{C}$  a positive linear functional and  $f, g \in C_c(X, \mathbb{R})$ . If  $f \geq g$ , then  $I(f) \geq I(g)$ .

*Proof.* Suppose that  $f \geq g$ . Then  $f - g \geq 0$ . So

$$\begin{aligned} I(f) - I(g) &= I(f - g) \\ &\geq 0 \end{aligned}$$

□

**Exercise 6.2.3.** Let  $X$  be a LCH space,  $I : C_c(X) \rightarrow \mathbb{C}$  a positive linear functional. Then for each  $K \subset X$ ,  $K$  is compact implies that there exists  $C_K \geq 0$  such that for each  $f \in C_c(X)$ , if  $\text{supp}(f) \subset K$ , then  $I(f) \leq C_K \|f\|_\infty$ .

**Hint:** Urysohn's lemma

*Proof.* Let  $K \subset X$ . Suppose that  $K$  is compact. Then Urysohn's lemma implies that there exists  $\phi \in C_c(X)$  such that  $0 \leq \phi \leq 1$  and  $\phi|_K = 1$ . Then  $I(\phi) \geq 0$ . Choose  $C_K = I(\phi)$ . Let  $f \in C_c(X)$ . Suppose that  $\text{supp}(f) \subset K$ . Then

$$\begin{aligned} f, -f &\leq |f| \\ &\leq \|f\|_\infty \phi \end{aligned}$$

The previous exercise implies that  $I(f), -I(f) \leq \|f\|_\infty I(\phi)$ . So

$$\begin{aligned} |I(f)| &\leq \|f\|_\infty I(\phi) \\ &\leq C_K \|f\|_\infty \end{aligned}$$

□

**Note 6.2.4.** Let  $X$  be a LCH space,  $U \subset X$  open and  $f \in C_c(X)$ . We write  $f \prec U$  to mean  $0 \leq f \leq 1$  and  $\text{supp}(f) \subset U$ .

**Exercise 6.2.5.** Let  $X$  be a LCH space,  $I : C_c(X) \rightarrow \mathbb{C}$  a positive linear functional and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  a Radon measure. Suppose that for each  $f \in C_c(X)$ ,

$$I(f) = \int f d\mu$$

Then

(1) for each  $U \subset X$ ,  $U$  is open implies that

$$\mu(U) = \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\}$$

(2)  $\mu$  is the unique Radon measure such that for each  $f \in C_c(X)$ ,

$$I(f) = \int f d\mu$$

*Proof.*

(1) Let  $U \subset X$ . Suppose that  $U$  is open. For  $f \in C_c(X)$ , if  $f \prec U$ , then

$$\begin{aligned} I(f) &= \int f \, d\mu \\ &\leq \mu(U) \end{aligned}$$

Let  $K \subset U$ . Suppose that  $K$  is compact. Then Urysohn's lemma implies that there exists  $f \in C_c(X)$  such that  $f \prec U$  and  $f|_K = 1$ . Then

$$\begin{aligned} \mu(K) &\leq \int f \, d\mu \\ &= I(f) \end{aligned}$$

Inner regularity implies that

$$\begin{aligned} \mu(U) &= \sup\{\mu(K) : K \subset U \text{ and } K \text{ is compact}\} \\ &\leq \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\} \\ &\leq \mu(U) \end{aligned}$$

(2) Let  $\nu : \mathcal{B}(X) \rightarrow [0, \infty]$  be a Radon measure. Suppose that for each  $f \in C_c(X)$ ,

$$I(f) = \int f \, d\nu$$

Part (1) implies that for each  $U \subset X$ , if  $U$  is open, then

$$\begin{aligned} \nu(U) &= \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\} \\ &= \mu(U) \end{aligned}$$

Outer regularity implies that for each  $E \in \mathcal{B}(X)$ ,

$$\begin{aligned} \nu(E) &= \inf\{\nu(U) : E \subset U \text{ and } U \text{ is open}\} \\ &= \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\} \\ &= \mu(E) \end{aligned}$$

So  $\nu = \mu$  and  $\mu$  is unique.

□

**Theorem 6.2.6. Representation Theorem 1:**

Let  $X$  be a LCH space and  $I : C_c(X) \rightarrow \mathbb{C}$  a positive linear functional. Then there exists a unique Radon measure  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  such that for each  $f \in C_c(X)$ ,

$$I(f) = \int f \, d\mu$$

In addition,

(1) for each  $U \subset X$ ,  $U$  is open implies that

$$\mu(U) = \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\}$$

(2) for each  $K \subset X$ ,  $K$  is compact implies that

$$\mu(K) = \inf\{I(f) : f \in C_c(X) \text{ and } f \geq \chi_K\}$$

**Note 6.2.7.** Let  $X$  be a topological space. Recall from section (4.3) that we define

$$\mathcal{M}(X) = \{\mu : \mathcal{B}(X) \rightarrow \mathbb{C} : \mu \text{ is a complex measure}\}$$

and that  $\mu \mapsto |\mu|(X)$  is a norm on  $\mathcal{M}(X)$ .

**Definition 6.2.8.** Let  $X$  be a topological space. For  $\mu \in \mathcal{M}(X)$ , define  $I_\mu : C_0(X) \rightarrow \mathbb{C}$  by

$$I_\mu(f) = \int f d\mu$$

**Exercise 6.2.9.** Let  $X$  be a topological space. For each  $\mu \in \mathcal{M}(X)$ ,  $I_\mu \in C_0(X)^*$ .

*Proof.* Let  $\mu \in \mathcal{M}(X)$  and  $f \in C_0(X)$ . An exercise in section (4.3) implies that

$$\begin{aligned} |I_\mu(f)| &= \left| \int f d\mu \right| \\ &\leq \int |f| d|\mu| \\ &\leq \|\mu\| \|f\|_\infty \end{aligned}$$

So  $I_\mu$  is bounded and  $I_\mu \in C_0(X)^*$ . □

**Theorem 6.2.10.** Let  $I \in C_0(X, \mathbb{R})^*$ , then there exist positive linear functionals  $I^+, I^- \in C_0(X, \mathbb{R})^*$  such that  $I = I^+ - I^-$

**Exercise 6.2.11.** Let  $X$  be a LCH space. Then the map  $\phi : \mathcal{M}(X) \rightarrow C_0(X)^*$  given by  $\phi(\mu) = I_\mu$  is a linear surjection.

*Proof.* An exercise in section (4.3) implies that  $\phi$  is linear. Let  $I \in C_0(X)^*$ . Then there exists positive linear functionals  $I^\pm, J^\pm \in C_0(X)^*$  such that  $I = I^+ - I^- + i(J^+ - J^-)$ . The first representation theorem implies that there exist Radon measures  $\mu^\pm, \nu^\pm$  such that  $I^\pm = I_{\mu^\pm}$  and  $J^\pm = I_{\nu^\pm}$ . Set  $\mu = \mu^+ - \mu^- + i(\nu^+ - \nu^-)$ . Then  $I = \phi(\mu)$  □

**Theorem 6.2.12. Representation Theorem 2:**

Let  $X$  be a LCH space. Then the map  $\phi : \mathcal{M}(X) \rightarrow C_0(X)^*$  given by  $\phi(\mu) = I_\mu$  is an isometric linear isomorphism.

**Definition 6.2.13.** Let  $X$  be a LCH space,  $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(X)$  and  $\mu \in \mathcal{M}(X)$ . Then  $\mu_n$  is said to **converge to  $\mu$  in weak-\***, denoted  $\mu_n \xrightarrow{w^*} \mu$ , if  $I_{\mu_n} \xrightarrow{w^*} I_\mu$ , i.e. for each  $f \in C_0(X)$ ,

$$\int f d\mu_n \rightarrow \int f d\mu$$

**Exercise 6.2.14.**

### 6.3. Borel Measures on Metric Spaces.

**Note 6.3.1.** Let  $X$  be a metric space and  $A \subset X$ . For  $\epsilon > 0$ , we write  $A_\epsilon = \{x \in X : d(x, A) < \epsilon\}$  and recall that  $A_\epsilon$  is open.

**Exercise 6.3.2.** Let  $X$  be a metric space,  $\mu : \mathcal{B}(X) \rightarrow [0, \infty)$  be a finite measure and  $E \in \mathcal{B}(X)$ . Then  $\mu(E) = \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$  iff  $\mu(E^c) = \sup\{\mu(C) : C \subset E^c \text{ and } C \text{ is closed}\}$

*Proof.* Suppose that  $\mu(E) = \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$ . Let  $\epsilon > 0$ . Then there exists  $U \subset X$  such that  $E \subset U$ ,  $U$  is open and  $\mu(U) < \mu(E) + \epsilon$ . Choose  $C = U^c$ . Then  $C \subset E^c$ ,  $E^c$  is closed and

$$\begin{aligned} \mu(E^c) - \epsilon &= \mu(E^c \cap C) + \mu(E^c \cap C^c) - \epsilon \\ &= \mu(C) + \mu(E^c \cap U) - \epsilon \\ &= \mu(C) + [\mu(U) - \mu(E)] - \epsilon \\ &< \mu(C) + \epsilon - \epsilon \\ &= \mu(C) \end{aligned}$$

So for each  $\epsilon > 0$ , there exists  $C \subset E^c$  such that  $C$  is closed and  $\mu(C) < \mu(E^c) - \epsilon$ . is arbitrary,  $\mu(E^c) = \sup\{\mu(C) : C \subset E^c \text{ and } C \text{ is closed}\}$ .

The converse is similar. □

**Exercise 6.3.3.** Let  $X$  be a metric space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty)$  be a finite measure. Then for each  $C \subset X$ , if  $C$  is closed, then  $\mu$  is outer regular on  $C$ .

**Hint:** For  $\epsilon > 0$ , consider  $C_\epsilon = \{x \in X : d(x, C) < \epsilon\}$ .

*Proof.* Let  $n \in \mathbb{N}$ . Set  $V_n = C_{1/n}$ . Then  $V_n$  is open and  $C \subset V_n$ . Since  $C$  is closed,  $C = \bigcap_{n \in \mathbb{N}} V_n$ . Since for each  $n \in \mathbb{N}$ ,  $V_{n+1} \subset V_n$  and  $\mu$  is finite, we have that  $\mu(C) = \inf_{n \in \mathbb{N}} \mu(V_n)$ . So for each  $\epsilon > 0$ , there exists  $n \in \mathbb{N}$  such that  $\mu(V_n) < \mu(C) + \epsilon$ . Hence  $\mu(C) = \inf\{\mu(U) : C \subset U \text{ and } U \text{ is open}\}$  and  $\mu$  is outer regular on  $C$ . □

**Exercise 6.3.4.** Let  $X$  be a metric space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty)$  be a finite measure. Set

$$\mathcal{A} = \left\{ E \in \mathcal{B}(X) : \mu \text{ is outer regular on } E \text{ and } E^c \right\}$$

Then  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ .

*Proof.*

- (1) Clearly,  $\emptyset \in \mathcal{A}$ .
- (2) Let  $E \in \mathcal{A}$ . Since  $(E^c)^c = E$ , by definition,  $E^c \in \mathcal{A}$ .
- (3) Let  $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ . Set  $E = \bigcup_{n \in \mathbb{N}} E_n$ . Let  $\epsilon > 0$ .



- For each  $n \in \mathbb{N}$ , there exists  $U_n \subset X$  such that  $U_n$  is open,  $E_n \subset U_n$  and  $\mu(U_n) < \mu(E_n) + \epsilon 2^{-n-1}$ . Set  $U = \bigcup_{n \in \mathbb{N}} U_n$ . Then  $U$  is open,  $E \subset U$  and

$$\begin{aligned}
 U \setminus E &= \left( \bigcup_{n \in \mathbb{N}} U_n \right) \cap E^c \\
 &= \left( \bigcup_{n \in \mathbb{N}} U_n \cap E^c \right) \\
 &= \left( \bigcup_{n \in \mathbb{N}} U_n \cap \left[ \bigcap_{j \in \mathbb{N}} E_j^c \right] \right) \\
 &= \left( \bigcup_{n \in \mathbb{N}} \left[ \bigcap_{j \in \mathbb{N}} (U_n \cap E_j^c) \right] \right) \\
 &\subset \bigcup_{n \in \mathbb{N}} (U_n \cap E_n^c) \\
 &= \bigcup_{n \in \mathbb{N}} (U_n \setminus E_n)
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \mu(U) - \mu(E) &= \mu(U \setminus E) \\
 &\leq \mu\left(\bigcup_{n \in \mathbb{N}} [U_n \setminus E_n]\right) \\
 &\leq \sum_{n \in \mathbb{N}} \mu(U_n \setminus E_n) \\
 &= \sum_{n \in \mathbb{N}} [\mu(U_n) - \mu(E_n)] \\
 &\leq \sum_{n \in \mathbb{N}} \epsilon 2^{-n-1} \\
 &= \frac{\epsilon}{2} \\
 &< \epsilon
 \end{aligned}$$

So for each  $\epsilon > 0$ , there exists  $U \subset X$  such that  $U$  is open,  $\bigcup_{n \in \mathbb{N}} E_n \subset U$  and  $\mu(U) < \mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) + \epsilon$ . Therefore

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \inf \left\{ \mu(U) : \bigcup_{n \in \mathbb{N}} E_n \subset U \text{ and } U \text{ is open} \right\}$$

and  $\mu$  is outer regular on  $\bigcup_{n \in \mathbb{N}} E_n$ .

- A previous exercise implies that for each  $n \in \mathbb{N}$ , there exists  $C_n \subset E_n$  such that  $C_n$  is closed and  $\mu(C_n) > \mu(E_n) - 2^{-n-1}\epsilon$ . Since

$$\mu\left(\bigcup_{n \in \mathbb{N}} C_n\right) = \sup_{K \in \mathbb{N}} \mu\left(\bigcup_{n=1}^K C_n\right)$$

there exists  $K \in \mathbb{N}$  such that  $\mu\left(\bigcup_{n=1}^K C_n\right) > \mu\left(\bigcup_{n \in \mathbb{N}} C_n\right) - \epsilon/2$ . Set  $C = \bigcup_{n=1}^K C_n$ . Then  $C$  is closed,  $C \subset E$  and similar to the previous part, we have that

$$\begin{aligned} \mu(E) - \mu(C) &< \mu(E) - \mu\left(\bigcup_{n \in \mathbb{N}} C_n\right) + \frac{\epsilon}{2} \\ &= \mu\left(E \setminus \bigcup_{n \in \mathbb{N}} C_n\right) + \frac{\epsilon}{2} \\ &= \mu\left(\bigcup_{n \in \mathbb{N}} \left[\bigcap_{j \in \mathbb{N}} (E_n \cap C_j^c)\right]\right) + \frac{\epsilon}{2} \\ &\leq \mu\left(\bigcup_{n \in \mathbb{N}} (E_n \cap C_n^c)\right) + \frac{\epsilon}{2} \\ &\leq \left[\sum_{n \in \mathbb{N}} \mu(E_n \cap C_n^c)\right] + \frac{\epsilon}{2} \\ &= \left[\sum_{n \in \mathbb{N}} \mu(E_n) - \mu(C_n)\right] + \frac{\epsilon}{2} \\ &\leq \left[\sum_{n \in \mathbb{N}} 2^{-n-1}\epsilon\right] + \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So for each  $\epsilon > 0$ , there exists  $C \subset X$  such that  $C$  is closed,  $C \subset \bigcup_{n \in \mathbb{N}} E_n$  and  $\mu(C) > \mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) - \epsilon$ . Therefore

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sup \left\{ \mu(C) : C \subset \bigcup_{n \in \mathbb{N}} E_n \text{ and } C \text{ is closed} \right\}$$

which implies that

$$\mu\left(\left[\bigcup_{n \in \mathbb{N}} E_n\right]^c\right) = \inf \left\{ \mu(U) : \left[\bigcup_{n \in \mathbb{N}} E_n\right]^c \subset U \text{ and } U \text{ is open} \right\}$$

and  $\mu$  is outer regular on  $\left(\bigcup_{n \in \mathbb{N}} E_n\right)^c$ .

Hence  $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{A}$ .

Therefore  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ . □

**Exercise 6.3.5.** Let  $X$  be a metric space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty)$  be a finite measure. Then  $\mu$  is outer regular.

*Proof.* Set  $\mathcal{T} = \{U \subset X : U \text{ is open}\}$  and define  $\mathcal{A}$  as in the previous exercise. The previous exercises imply that  $\mathcal{T} \subset \mathcal{A}$ . Since  $\mathcal{B}(X) = \sigma(\mathcal{T})$ , we have that  $\mathcal{B}(X) \subset \mathcal{A}$ . Therefore  $\mathcal{B}(X) = \mathcal{A}$  and  $\mu$  is outer regular.  $\square$

**Exercise 6.3.6.** Let  $X$  be a metric space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty)$  a finite measure. If  $\mu$  is inner regular on  $X$ , then  $\mu$  is inner regular.

*Proof.* Suppose that  $\mu$  is inner regular on  $X$ . Let  $E \in \mathcal{B}(X)$  and  $\epsilon > 0$ . Then there exists  $K_0 \subset X$  such that  $K_0$  is compact and  $\mu(K_0) > \mu(X) - \epsilon/2$ . The previous exercise implies that there exists  $C \subset E$  such that  $C$  is closed and  $\mu(C) > \mu(E) - \epsilon/2$ . Set  $K = K_0 \cap C$ . Then  $K \subset E$ ,  $K$  is compact and

$$\begin{aligned} \mu(E) &< \mu(C) + \frac{\epsilon}{2} \\ &= [\mu(C \cap K_0) + \mu(C \cap K_0^c)] + \frac{\epsilon}{2} \\ &\leq \mu(C \cap K_0) + \mu(X \cap K_0^c) + \frac{\epsilon}{2} \\ &= \mu(K) + [\mu(X) - \mu(K_0)] + \frac{\epsilon}{2} \\ &< \mu(K) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \mu(K) + \epsilon \end{aligned}$$

So for each  $\epsilon > 0$ , there exists  $K \subset E$  such that  $K$  is compact and  $\mu(K) > \mu(E) - \epsilon$ . Hence  $\mu(E) = \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}$  and  $\mu$  is inner regular on  $E$ . Since  $E \in \mathcal{B}(X)$  is arbitrary,  $\mu$  is inner regular.  $\square$

**Exercise 6.3.7.** Let  $X$  be a Polish space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty)$  a finite measure. Then  $\mu$  is inner regular.

**Hint:** If  $(x_n)_{n \in \mathbb{N}}$  is a countable dense of  $X$ , consider  $K \subset X$  of the form

$$K = \bigcap_{m \in \mathbb{N}} \bigcup_{n=1}^{n_m} \text{cl } B(x_n, 1/m)$$

*Proof.* Let  $\epsilon > 0$ . Since  $X$  is separable, there exists a countable dense subset  $(x_n)_{n \in \mathbb{N}}$  of  $X$ . Let  $m \in \mathbb{N}$ . Then  $X = \bigcup_{n \in \mathbb{N}} \text{cl } B(x_n, 1/m)$ . This implies that there exists  $n_m \in \mathbb{N}$  such that

$$\mu\left(\bigcup_{n=1}^{n_m} \text{cl } B(x_n, 1/m)\right) > \mu(X) - 2^{-m-1}\epsilon$$

Set

$$K = \bigcap_{m \in \mathbb{N}} \bigcup_{n=1}^{n_m} \text{cl } B(x_n, 1/m)$$

Then  $K$  is closed. Let  $\delta > 0$ . Choose  $m_\delta \in \mathbb{N}$  such that  $1/m_\delta < \delta$ . Then

$$\begin{aligned} K &= \bigcap_{m \in \mathbb{N}} \bigcup_{n=1}^{n_m} \text{cl } B(x_n, 1/m) \\ &\subset \bigcup_{n=1}^{n_{m_\delta}} \text{cl } B(x_n, 1/m_\delta) \\ &\subset \bigcup_{n=1}^{n_{m_\delta}} B(x_n, \delta) \end{aligned}$$

Hence  $K$  is totally bounded. Since  $X$  is complete,  $K$  is compact. Finally, we have that

$$\begin{aligned} \mu(X) - \mu(K) &= \mu(K^c) \\ &= \mu\left(\bigcup_{m \in \mathbb{N}} \left[\bigcup_{n=1}^{n_m} \text{cl } B(x_n, 1/m)\right]^c\right) \\ &\leq \sum_{m \in \mathbb{N}} \mu\left(\left[\bigcup_{n=1}^{n_m} \text{cl } B(x_n, 1/m)\right]^c\right) \\ &= \sum_{m \in \mathbb{N}} \left[\mu(X) - \mu\left(\bigcup_{n=1}^{n_m} \text{cl } B(x_n, 1/m)\right)\right] \\ &\leq \sum_{m \in \mathbb{N}} 2^{-m-1} \epsilon \\ &= \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

So for each  $\epsilon > 0$ , there exists  $K \subset X$  such that  $K$  is compact and  $\mu(K) > \mu(X) - \epsilon$ . Thus

$$\mu(X) = \sup\{\mu(K) : K \subset X \text{ and } K \text{ is compact}\}$$

and  $\mu$  is inner regular on  $X$ . The previous exercise implies that  $\mu$  is inner regular.  $\square$

**Exercise 6.3.8. Ulam's Theorem:**

Let  $X$  be a Polish space and  $\mu : \mathcal{B}(X) \rightarrow [0, \infty)$  a finite measure. Then  $\mu$  is regular and Radon.

*Proof.* Clear by preceding exercises.  $\square$

**Definition 6.3.9.** Let  $X$  be a topological space. For  $f \in C_b(X)$ , define  $\lambda_f : \mathcal{M}(X) \rightarrow \mathbb{C}$  by

$$\lambda_f(\mu) = \int f d\mu$$

.

**Exercise 6.3.10.** Let  $X$  be a topological space. For each  $f \in C_b(X)$ ,  $\lambda_f \in \mathcal{M}(X)^*$ .

*Proof.* Let  $f \in C_b(X)$  and  $\mu \in \mathcal{M}(X)$ . Then

$$\begin{aligned} |\lambda_f(\mu)| &= \left| \int f d\mu \right| \\ &\leq \int |f| d|\mu| \\ &\leq \|f\|_u \|\mu\| \end{aligned}$$

Exercise 4.2.17 implies that  $\lambda_f$  is linear. So  $\lambda_f \in \mathcal{M}(X)^*$ .  $\square$

**Definition 6.3.11.** Let  $X$  be a topological space. We define the **weak topology on  $\mathcal{M}(X)$**  to be the weak topology generated by  $\{\lambda_f \in \mathcal{M}(X)^* : f \in C_b(X)\}$ .

**Definition 6.3.12.** Let  $X$  be a topological space and  $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(X)$  and  $\mu \in \mathcal{M}(X)$ . Then  $(\mu_n)_{n \in \mathbb{N}}$  is said to **converge weakly** to  $\mu$ , denoted  $\mu_n \xrightarrow{w} \mu$ , if  $(\mu_n)_{n \in \mathbb{N}}$  converges to  $\mu$  in the weak topology, i.e. for each  $f \in C_b(X)$ ,

$$\int f d\mu_n \rightarrow \int f d\mu$$

**Exercise 6.3.13. Portmanteau Theorem:** Let  $X$  be a topological space and  $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(X)$  and  $\mu \in \mathcal{M}(X)$ . Suppose that for each  $n \in \mathbb{N}$ ,  $\mu_n(X) = \mu(X)$ . Then the following are equivalent:

- (1)  $\mu_n \xrightarrow{w} \mu$
- (2) for each  $A \in \mathcal{B}(X)$ ,  $A$  is open implies that  $\mu(A) \leq \liminf_{n \rightarrow \infty} \mu_n(A)$
- (3) for each  $A \in \mathcal{B}(X)$ ,  $A$  is closed implies that  $\mu(A) \geq \limsup_{n \rightarrow \infty} \mu_n(A)$
- (4) for each  $A \in \mathcal{B}(X)$ ,  $\mu(\partial A) = 0$  implies that  $\mu_n(A) \rightarrow \mu(A)$

*Proof.*

- (2)  $\iff$  (3):

Suppose (2). Let  $A \in \mathcal{B}(X)$ . Suppose that  $A$  is closed. Then  $A^c$  is open. By assumption,  $\mu(A^c) \leq \liminf_{n \rightarrow \infty} \mu_n(A^c)$ . Hence

$$\begin{aligned} \mu(A) &= \mu(X) - \mu(A^c) \\ &\geq \mu(X) - \liminf_{n \rightarrow \infty} \mu_n(A^c) \\ &= \mu(X) + \limsup_{n \rightarrow \infty} [-\mu_n(A^c)] \\ &= \limsup_{n \rightarrow \infty} [\mu(X) - \mu_n(A^c)] \\ &= \limsup_{n \rightarrow \infty} [\mu_n(X) - \mu_n(A^c)] \\ &= \limsup_{n \rightarrow \infty} \mu_n(A) \end{aligned}$$

So (3) holds. Similarly, (3) implies (2).

- (2)  $\iff$  (4):

Suppose (2). From above, (3) holds. Let  $A \in \mathcal{B}(X)$ . Then

$$\begin{aligned}
 \mu(A^\circ) &\leq \liminf_{n \rightarrow \infty} \mu_n(A^\circ) \\
 &\leq \liminf_{n \rightarrow \infty} \mu_n(A) \\
 &\leq \limsup_{n \rightarrow \infty} \mu_n(A) \\
 &\leq \limsup_{n \rightarrow \infty} \mu_n(\overline{A}) \\
 &\leq \mu(\overline{A})
 \end{aligned}$$

Suppose that  $\mu(\partial A) = 0$ . Then

$$\begin{aligned}
 \mu(A^\circ) &\leq \mu(A) \\
 &\leq \mu(\overline{A}) \\
 &= \mu(A^\circ) + \mu(\partial A) \\
 &= \mu(A^\circ)
 \end{aligned}$$

which implies that  $\mu_n(A) \rightarrow \mu(A)$ . Conversely, suppose (4).

•

□

## 7. HAAR MEASURE

## 7.1. Introduction.

**Note 7.1.1.** This section assumes familiarity with topological groups. See section 8.1 of [2] for details.

**Definition 7.1.2.** Let  $G$  be a group and  $g \in G$ . Define  $l_g : G \rightarrow G$  and  $r_g : G \rightarrow G$  by  $l_g(x) = gx$  and  $r_g(x) = xg^{-1}$ .

**Definition 7.1.3.** Let  $G$  be a topological group,  $y \in G$  and  $f \in L^0$ . Define  $L_y, R_y : L^0(G) \rightarrow L^0(G)$  by  $L_y f = f \circ l_y^{-1}$  and  $R_y f = f \circ r_y^{-1}$ , that is,  $L_y f(x) = f(y^{-1}x)$  and  $R_y f(x) = f(xy)$ .

**Definition 7.1.4.** Let  $G$  be a topological group and  $\mu$  a Radon measure on  $G$ . Then  $\mu$  is said to be a **left Haar measure on  $G$**  if

- (1)  $\mu$  is nonzero
- (2) for each  $U \in \mathcal{B}(G)$  and  $g \in G$ ,  $\mu(gU) = \mu(U)$ .

Similarly,  $\mu$  is said to be a **right Haar measure on  $G$**  if

- (1)  $\mu$  is nonzero
- (2) for each  $U \in \mathcal{B}(G)$  and  $g \in G$ ,  $\mu(Ug) = \mu(U)$ .

**Exercise 7.1.5.** Let  $G$  be a topological group,  $\mu$  a Radon measure on  $G$ . Then  $\mu$  is a left Haar measure on  $G$  iff  $\iota_*\mu$  is a right Haar measure on  $G$ .

*Proof.* Suppose that  $\mu$  is a left Haar measure on  $G$ . Let  $U \in \mathcal{B}(G)$  and  $g \in G$ . Then

$$\begin{aligned} \iota_*\mu(Ug) &= \mu(\iota^{-1}(Ug)) \\ &= \mu(g^{-1}U^{-1}) \\ &= \mu(U^{-1}) \\ &= \mu(\iota^{-1}(U)) \\ &= \iota_*\mu(U) \end{aligned}$$

So  $\iota_*\mu$  is a right Haar measure on  $G$ . The converse is similar. □

**Exercise 7.1.6.** Let  $G$  be a topological group, and  $\mu$  a left Haar measure on  $G$ . Then for each  $g \in G$ ,  $r_{g*}\mu$  is a left Haar measure on  $G$ .

*Proof.* Let  $g \in G$  and  $U \in \mathcal{B}(G)$ . Observe that  $r_{g*}\mu(U) = \mu(Ug)$ . So for each  $h \in G$ ,

$$\begin{aligned} r_{g*}\mu(hU) &= \mu(hUg) \\ &= \mu(Ug) \\ &= r_{g*}\mu(U) \end{aligned}$$

□

**Exercise 7.1.7.** Let  $G$  be a topological group,  $\mu$  a left Haar measure on  $G$  and  $\nu$  a right Haar measure on  $G$ . Then for each  $f \in L^1 \cup L^+$  and  $y \in G$ ,

$$\begin{aligned} (1) \quad & \int L_y f \, d\mu = \int f \, d\mu \\ (2) \quad & \int R_y f \, d\nu = \int f \, d\nu \end{aligned}$$

*Proof.*

(1) Let  $y \in G$  and  $E \in \mathcal{B}(G)$ . Put  $f = \chi_E$ . Then

$$\begin{aligned} \int L_y f \, d\mu &= \int L_y \chi_E \, d\mu \\ &= \int \chi_{yE} \, d\mu \\ &= \mu(yE) \\ &= \mu(E) \\ &= \int \chi_E \, d\mu \\ &= \int f \, d\mu \end{aligned}$$

By linearity of  $L_y$ , for  $f \in S^+$  we have that,

$$\int L_y f \, d\mu = \int f \, d\mu$$

For  $f \in L^+$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that for each  $n \in \mathbb{N}$   $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \rightarrow f$ . Then for each  $n \in \mathbb{N}$   $L_y \phi_n \leq L_y \phi_{n+1} \leq L_y f$  and  $L_y \phi_n \rightarrow L_y f$ . So MCT implies that

$$\begin{aligned} \int L_y f \, d\mu &= \lim_{n \rightarrow \infty} \int L_y \phi_n \, d\mu \\ &= \lim_{n \rightarrow \infty} \int \phi_n \, d\mu \\ &= \int f \, d\mu \end{aligned}$$

Let  $f \in L^1$ . If  $f$  is real valued, write  $f = f^+ - f^-$ . Then  $L_y f = L_y f^+ - L_y f^-$  and

$$\begin{aligned} \int L_y f \, d\mu &= \int L_y f^+ \, d\mu - \int L_y f^- \, d\mu \\ &= \int f^+ \, d\mu - \int f^- \, d\mu \\ &= \int f \, d\mu \end{aligned}$$

If  $f$  is complex valued, write  $f = g + ih$  with  $g, h \in L^1$  real valued. Then

$$\begin{aligned} \int L_y f \, d\mu &= \int L_y g \, d\mu + i \int L_y h \, d\mu \\ &= \int g \, d\mu + i \int h \, d\mu \\ &= \int f \, d\mu \end{aligned}$$

(2) Similar



□

**Exercise 7.1.8.** Let  $G$  be a topological group and  $\mu$  a left Haar measure on  $G$ . Then for each  $U \subset G$ , if  $U$  is open and  $U \neq \emptyset$ , then  $\mu(U) > 0$

*Proof.* Let  $U \subset G$ . Suppose that  $U$  is open and  $U \neq \emptyset$ . Suppose that  $\mu(U) = 0$ . Since  $\mu$  is nonzero, inner regularity implies that there exists  $K \subset G$  such that  $K$  is compact and  $\mu(K) > 0$ . Then  $\{xU : x \in K\}$  is an open cover of  $K$ . Then there exist  $x_1, \dots, x_n \in K$  such that  $K \subset \bigcap_{k=1}^n x_k U$ . Then

$$(3) \quad \mu(K) \leq \sum_{k=1}^n \mu(x_k U)$$

$$(4) \quad = \sum_{k=1}^n \mu(U)$$

$$(5) \quad = 0$$

This is a contradiction. So  $\mu(U) > 0$ . □

**Exercise 7.1.9.** Let  $G$  be a locally compact group and  $\mu$  a left Haar measure on  $G$ . Then there exists  $S \in \mathcal{B}(G)$  such that  $S$  is symmetric,  $e \in S$  and  $\mu(E) > 0$

*Proof.* Since  $G$  is locally compact, there exists a compact neighborhood  $K$  of  $e$ . Then  $\mu(K) > 0$ . Put  $S = KK^{-1} \in \mathcal{B}(G)$ . Then  $S$  is symmetric. Since  $e \in K$ ,  $K \subset S$  and  $0 < \mu(K) \leq \mu(S)$ . □

**Exercise 7.1.10.** Let  $G$  be a locally compact group and  $\mu$  a left Haar measure on  $G$ . Then

- (1)  $\mu(\{e\}) > 0$  iff there exists  $\lambda > 0$  such that  $\mu = \lambda\#$ .
- (2)  $\mu$  is finite iff  $G$  is compact

*Proof.*

- (1) If there exists  $\lambda > 0$  such that  $\mu = \lambda\#$ , then  $\mu(\{e\}) > 0$ . Conversely, suppose that  $\mu(\{e\}) > 0$ . Define  $\lambda = \mu(\{e\}) > 0$ . Let  $B \in \mathcal{B}(G)$ . If  $B$  is finite, then

$$\begin{aligned} \mu(B) &= \sum_{x \in B} \mu(\{x\}) \\ &= \sum_{x \in B} \mu(x\{e\}) \\ &= \sum_{x \in B} \mu(\{e\}) \\ &= \sum_{x \in B} \lambda \\ &= \lambda\#(\{e\}) \end{aligned}$$

If  $B$  is infinite, then we may choose a countable subset and the same reasoning as above tells us that

$$\mu(B) = \infty = \lambda\#(B)$$

- (2) If  $G$  is compact, then  $\mu$  is finite since  $\mu$  is Radon. Conversely, suppose that  $\mu$  is finite. Then **FINISH**

□

**Theorem 7.1.11.** Let  $G$  be a locally compact group. Then there exists a left Haar measure on  $G$ .

**Theorem 7.1.12.** Let  $G$  be a locally compact group and  $\mu_1, \mu_2$  left Haar measures on  $G$ . Then there exists  $\lambda > 0$  such that  $\mu_1 = \lambda\mu_2$ .

**Definition 7.1.13.** Let  $G$  be a locally compact group and  $\mu$  a left Haar measure on  $G$ . A previous exercise tells us that for each  $g \in G$ ,  $r_{g*}\mu$  is a left Haar measure on  $G$ . The previous result tells us that for each  $g \in G$  there exists  $\lambda_g > 0$  such that  $r_{g*}\mu = \lambda_g\mu$ . Define  $\Delta : G \rightarrow (0, \infty)$  by  $\Delta(g) = \lambda_g$ . We call  $\Delta$  the **modular function of  $G$** .

**Exercise 7.1.14.** Let  $G$  be a locally compact group and  $\mu$  a left Haar measure on  $G$ . Then

- (1)  $\Delta$  is a homomorphism
- (2) for each  $f \in L^1 \cup L^+$ ,

$$\int R_{y^{-1}}f \, d\mu = \Delta(y) \int f \, d\mu$$

*Proof.*

- (1) Recall that for each  $g \in G$ ,  $\Delta(g)\mu(U) = r_{g*}\mu(U) = \mu(Ug)$ . Let  $g, h \in G$  and  $U \in \mathcal{B}(G)$ . Then  $\Delta(gh)\mu(U) = \mu(Ugh) = \Delta(h)\mu(Ug) = \Delta(g)\Delta(h)\mu(U)$ . So  $\Delta(gh) = \Delta(g)\Delta(h)$ .
- (2) Let  $y \in G$  and  $U \in \mathcal{B}(G)$ . Put  $f = \chi_U$ . Then

$$\begin{aligned} \int R_{y^{-1}}f \, d\mu &= \int R_{y^{-1}}\chi_U \, d\mu \\ &= \int \chi_{Uy} \, d\mu \\ &= \mu(Uy) \\ &= \mu(r_y^{-1}(U)) \\ &= r_{y*}\mu(U) \\ &= \Delta(y)\mu(U) \\ &= \Delta(y) \int \chi_U \, d\mu \\ &= \Delta(y) \int f \, d\mu \end{aligned}$$

By linearity of  $R_{y^{-1}}$ , for  $f \in S^+$ ,

$$\int R_{y^{-1}}f \, d\mu = \Delta(y) \int f \, d\mu$$

For  $f \in L^+$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S^+$  such that for each  $n \in \mathbb{N}$   $\phi_n \leq \phi_{n+1} \leq f$  and  $\phi_n \rightarrow f$ . Then for each  $n \in \mathbb{N}$   $R_{y^{-1}}\phi_n \leq R_{y^{-1}}\phi_{n+1} \leq R_{y^{-1}}f$  and  $R_{y^{-1}}\phi \rightarrow R_{y^{-1}}f$ . So

the monotone convergence theorem implies that

$$\begin{aligned} \int R_{y^{-1}} f \, d\mu &= \lim_{n \rightarrow \infty} \int R_{y^{-1}} \phi_n \, d\mu \\ &= \lim_{n \rightarrow \infty} \Delta(y) \int \phi_n \, d\mu \\ &= \Delta(y) \int f \, d\mu \end{aligned}$$

Let  $f \in L^1$ . If  $f$  is real valued, write  $f = f^+ - f^-$ . Then  $R_{y^{-1}} f = R_{y^{-1}} f^+ - R_{y^{-1}} f^-$  and

$$\begin{aligned} \int R_{y^{-1}} f \, d\mu &= \int R_{y^{-1}} f^+ \, d\mu - \int R_{y^{-1}} f^- \, d\mu \\ &= \Delta(y) \int f^+ \, d\mu - \Delta(y) \int f^- \, d\mu \\ &= \Delta(y) \int f \, d\mu \end{aligned}$$

If  $f$  is complex valued, write  $f = g + ih$  with  $g, h \in L^1$  real valued. Then

$$\begin{aligned} \int R_{y^{-1}} f \, d\mu &= \int R_{y^{-1}} g \, d\mu + i \int R_{y^{-1}} h \, d\mu \\ &= \Delta(y) \int g \, d\mu + i \Delta(y) \int h \, d\mu \\ &= \Delta(y) \int f \, d\mu \end{aligned}$$

□

**Definition 7.1.15.** Let  $G$  be a locally compact group. Then  $G$  is said to be **unimodular** if  $\ker \Delta = G$ .

**Exercise 7.1.16.** Let  $G$  be a locally compact group. Then the following are equivalent:

- (1)  $G$  is unimodular
- (2) there exists a left Haar measure  $\mu$  on  $G$  such that  $\mu$  is a right Haar measure on  $G$ .
- (3) for each nonzero Radon measure  $\mu$  on  $G$ ,  $\mu$  is a left Haar measure on  $G$  iff  $\mu$  is a right Haar measure on  $G$ .

*Proof.*

- (1)  $\implies$  (2):

Since  $G$  is a locally compact group, there exists a left Haar measure  $\mu$  on  $G$ . Let  $g \in G$  and  $U \in \mathcal{B}(G)$ . Then

$$\mu(Ug) = \Delta(g)\mu(U) = \mu(U)$$

Since  $G$  is unimodular,  $\Delta(g) = 1$ . Then  $\mu$  is a right Haar measure on  $G$ .

- (2)  $\implies$  (3):

By assumption, there exists a left Haar measure  $\mu'$  on  $G$  such that  $\mu'$  is a right Haar measure on  $G$ . Let  $\mu$  be a nonzero Radon measure on  $G$ . If  $\mu$  is a left Haar measure on  $G$ , then there exists  $\lambda > 0$  such that  $\mu = \lambda\mu'$  and therefore  $\mu$  is a right Haar

measure. The same reasoning implies that if  $\mu$  is a right Haar measure on  $G$ , then  $\mu$  is a left Haar measure on  $G$ .

• (3)  $\implies$  (1):

Since  $G$  is locally compact, there exists a left *Haar* measure  $\mu$  on  $G$ . By assumption,  $\mu$  is a right Haar measure on  $G$ . By inner regularity there exists  $K \in \mathcal{B}(G)$  such that  $\mu(K) > 0$ . Let  $g \in G$ . Then

$$\Delta(g)\mu(K) = \mu(Kg) = \mu(K)$$

So  $\Delta(g) = 1$ .

□

**Note 7.1.17.** If  $G$  is a locally compact abelian group, then  $G$  is unimodular.

**Exercise 7.1.18.** Let  $G$  be a locally compact group and  $\mu$  a left Haar measure on  $G$ . If  $G$  is unimodular then  $\iota_*\mu = \mu$ .

*Proof.* Suppose that  $G$  is unimodular. A previous exercise tells us that  $\iota_*\mu$  is a right Haar measure on  $G$ . The unimodularity of  $G$  implies that  $\iota_*\mu$  is a left Haar measure on  $G$ . Then there exists  $\lambda > 0$  such that  $\iota_*\mu = \lambda\mu$ . Since  $G$  is locally compact, there exists  $S \in \mathcal{B}(G)$  such that  $S$  is symmetric and  $\mu(S) > 0$ . Then

$$\begin{aligned} \mu(S) &= \mu(S^{-1}) \\ &= \iota_*\mu(S) \\ &= \lambda\mu(S) \end{aligned}$$

So  $\lambda = 1$  and  $\iota_*\mu = \mu$ .

it is also (Since  $G$  is locally compact, there exists  $S \in \mathcal{B}(G)$  such that  $S$  is symmetric and  $\mu(S) > 0$ . Then

$$\mu(S) = \mu(S^{-1}) = \iota_*\mu(S)$$

Since  $\iota_*\mu$  is a right Haar measure on  $G$  and  $G$  is unimodular,  $\iota_*\mu(S)$  is also a left Haar measure on  $G$ . Then there exists  $\lambda > 0$  such that  $\mu(S) = \lambda\iota_*\mu(S)$ . □

## 7.2. Fundamental Examples.

**Note 7.2.1.** The Haar measure on  $(\mathbb{R}^n, +)$  is  $m$ .

**Exercise 7.2.2.** The Haar measure on  $(\mathbb{R}^\times, \cdot)$  is

$$d\mu(x) = \frac{1}{|x|} dm(x)$$

*Proof.* Let  $0 < a < b$  and  $c > 0$ . Then

$$\begin{aligned} \mu(c(a, b)) &= \mu((ca, cb)) \\ &= \int_{(ca, cb)} \frac{1}{|x|} dm(x) \\ &= \int_{(ca, cb)} \frac{1}{x} dm(x) \\ &= \left[ \log |x| \right]_{ca}^{cb} \\ &= \log(cb) - \log(ca) \\ &= \log b - \log a \\ &= \left[ \log |x| \right]_a^b \\ &= \int_{(a, b)} \frac{1}{x} dm(x) \\ &= \mu((a, b)) \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mu(-c(a, b)) &= \mu((-cb, -ca)) \\ &= \int_{(-cb, -ca)} \frac{1}{|x|} dm(x) \\ &= - \int_{(-cb, -ca)} \frac{1}{x} dm(x) \\ &= - \left[ \log |x| \right]_{-cb}^{-ca} \\ &= \log(cb) - \log(ca) \\ &= \log b - \log a \\ &= \left[ \log |x| \right]_a^b \\ &= \int_{(a, b)} \frac{1}{x} dm(x) \\ &= \mu((a, b)) \end{aligned}$$

□

**Exercise 7.2.3.** Define  $f : [0, 1) \rightarrow \mathbb{T}$  by  $f(x) = e^{i2\pi x}$ . Let  $m$  be Lebesgue measure on  $[0, 1)$ , then the Haar measure on  $\mathbb{T}$  is  $f_*m$ .

*Proof.* Note that  $f$  is a bijection and the topology on  $\mathbb{T}$  is generated by sets of the form  $f((a, b))$  where  $a, b \in [0, 1)$  and  $a < b$ . Let  $a, b \in [0, 1)$  and suppose that  $a < b$ . Put  $A = f((a, b))$ . Let  $z \in \mathbb{T}$ . Then there exists  $\theta \in [0, 1)$  such that  $z = f(\theta)$ . If  $1 \notin zA$ , then  $f^{-1}(zA) = (\theta + a, \theta + b)$ . If  $1 \in zA$ , then  $f^{-1}(zA) = (\theta + a, 1) \cup [0, \theta + b - 1)$ . Suppose that  $1 \notin zA$ . Then

$$\begin{aligned} &= f_*m(zA) &&= m(f^{-1}(zA)) \\ &= m((\theta + a, \theta + b)) \\ &= b - a \\ &= m((a, b)) \\ &= m(f^{-1}(A)) \\ &= f_*m(A) \end{aligned}$$

Similarly if  $1 \in zA$ ,  $f_*m(zA) = f_*m(A)$ . □

**Exercise 7.2.4.** Let  $p$  be a prime. Define  $|\cdot|_p : \mathbb{Q} \rightarrow [0, \infty]$  by

$$\begin{cases} |\frac{a}{b}p^n|_p = p^{-n}, & \text{if } \gcd(a, p) = \gcd(b, p) = 1 \\ |0|_p = 0 \end{cases}$$

Then  $|\cdot|_p$  is an absolute value on  $\mathbb{Q}$ . Define  $\mathbb{Q}_p$  to be the completion of  $\mathbb{Q}$  with respect to the metric induced by  $|\cdot|_p$ . Define  $\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1\}$ . It is well known that  $\mathbb{Q}_p$  is a locally compact field and  $\mathbb{Z}_p$  is compact. Define  $P = \{0, 1, \dots, p-1\}$ . It is known that the topology is generated by

$$\{x + p^n\mathbb{Z}_p : \text{for } n \in \mathbb{Z}, x \in \mathbb{Q}_p\}$$

Another useful fact is that

$$\mathbb{Q}_p = \left\{ \sum_{j=-n}^{\infty} a_j p^j : a_j \in P, n \in \mathbb{N}_0 \right\}$$

and

$$\mathbb{Z}_p = \left\{ \sum_{j=0}^{\infty} a_j p^j : a_j \in P \right\}$$

Let  $\mu$  be the Haar measure on  $\mathbb{Q}_p$ . Then  $\mu$  is completely determined by the value  $\mu(\mathbb{Z}_p)$

*Proof.* We observe that for  $n \in \mathbb{Z}$ , we may write  $p^n\mathbb{Z}_p$  as the following disjoint union:

$$p^n\mathbb{Z}_p = \bigcup_{j \in P} jp^n + p^{n+1}\mathbb{Z}_p$$

Thus  $\mu(p^n\mathbb{Z}_p) = p\mu(p^{n+1}\mathbb{Z}_p)$ . If we set  $\mu(\mathbb{Z}_p) = 1$ , we obtain that  $\mu(\mathbb{Z}_p) = p^n\mu(p^n\mathbb{Z}_p)$ , which implies that

$$\mu(p^n\mathbb{Z}_p) = \frac{1}{p^n}\mu(\mathbb{Z}_p)$$

□

**Exercise 7.2.5.** Let  $\nu$  be the Haar measure on  $\mathbb{Q}_p$ . Then the Haar measure on  $\mathbb{Q}_p^\times$  is  $d\mu = \frac{1}{|x|_p} d\nu$ .

*Proof.* Let  $x, y \in P^\times$  and  $\alpha = xp^{n-1} + p^n\mathbb{Z}_p$ . Then

$$\alpha(y p^{k-1} + p^k\mathbb{Z}_p) = p^{(n-1)+(k-1)}(xy + p^{n+k}\mathbb{Z}_p)$$

□

### 7.3. Action on Measures.

**Exercise 7.3.1.** Let  $G$  be a locally compact group,  $\mu$  a left Haar measure on  $G$  and  $\nu \in \mathcal{M}(G)$ . If  $\nu \ll \mu$ , then  $l_{g*}\nu \ll \mu$ .

*Proof.* Suppose that  $\nu \ll \mu$ . Let  $A \in \mathcal{B}(G)$ . Then

$$\begin{aligned} \mu(A) = 0 &\implies \mu(g^{-1}A) = 0 \\ &\implies \nu(g^{-1}A) = 0 \\ &\implies \nu(l_{g^{-1}}(A)) = 0 \\ &\implies \nu(l_g^{-1}(A)) = 0 \\ &\implies l_{g*}\nu(A) = 0 \end{aligned}$$

So  $l_{g*}\nu \ll \mu$ .

□

**Definition 7.3.2.** Let  $G$  be a locally compact group and  $\mu$  a left Haar measure on  $G$ . Define  $\mathcal{M}_\mu \subset \mathcal{M}(G)$  by

$$\mathcal{M}_\mu = \{\nu \in \mathcal{M}(G) : \nu \ll \mu\}$$

We define an action  $\phi : G \times \mathcal{M}_\mu \rightarrow \mathcal{M}_\mu$  by

$$g \cdot \nu = l_{g*}\nu$$

**Exercise 7.3.3.** Let  $G$  be a locally compact group,  $\mu$  a  $\sigma$ -finite left Haar measure on  $G$ ,  $\nu \in \mathcal{M}_\mu$  and  $g \in G$ . Then

$$\frac{d(g \cdot \nu)}{d\mu} = L_g \frac{d\nu}{d\mu}$$

*Proof.* Set  $f = d\nu/d\mu$ . Let  $A \in \mathcal{B}(X)$ . Then

$$\begin{aligned}
 \int_A L_g f \, d\mu &= \int_A f \circ l_g^{-1} \, d\mu \\
 &= \int_A f \circ l_g^{-1} \, d\mu \\
 &= \int_{l_g^{-1}(A)} f \, d(l_g^{-1} \, {}_* \mu) \\
 &= \int_{l_g^{-1}(A)} f \, d(l_{g^{-1}} \, {}_* \mu) \\
 &= \int_{l_g^{-1}(A)} f \, d\mu \\
 &= \nu(l_g^{-1}(A)) \\
 &= l_{g*} \nu(A) \\
 &= g \cdot \nu(A)
 \end{aligned}$$

Since  $A$  is arbitrary, uniqueness implies that

$$\frac{d(g \cdot \nu)}{d\mu} = L_g \frac{d\nu}{d\mu}$$

□

**Exercise 7.3.4.** Let  $G$  be a locally compact group,  $\mu$  a  $\sigma$ -finite left Haar measure on  $G$ ,  $\nu \in \mathcal{M}_\mu$  and  $g \in G$ . Then  $\|g \cdot \nu\| = \|\nu\|$ .

*Proof.* Exercise 4.2.11 implies that

$$\begin{aligned}
 \|g \cdot \nu\| &= \int \left| \frac{d(g \cdot \nu)}{d\mu} \right| d\mu \\
 &= \int \left| L_g \frac{d\nu}{d\mu} \right| d\mu \\
 &= \int L_g \left| \frac{d\nu}{d\mu} \right| d\mu \\
 &= \int \left| \frac{d\nu}{d\mu} \right| d\mu \\
 &= \|\nu\|
 \end{aligned}$$

□



#### 7.4. Measures Invariant under Group Actions.

**Definition 7.4.1.** Let  $G$  be a group,  $X$  a set,  $\phi : G \times X \rightarrow X$  a group action and  $g \in G$ . Define  $l_g : X \rightarrow G$  by  $l_g(x) = g \cdot x$ .

**Definition 7.4.2.** Let  $G$  be a topological group,  $X$  a set,  $\phi : G \times X \rightarrow X$  a group action and  $g \in G$ . Define  $L_g : L^0(G) \rightarrow L^0(G)$  by

$$L_g f = f \circ l_g^{-1}$$

i.e.  $L_g f(x) = f(g^{-1} \cdot x)$

**Definition 7.4.3.** Let  $G$  be a group,  $(X, \mathcal{A}, \mu)$  a measure space,  $\phi : G \times X \rightarrow X$  a group action and  $\zeta : G \rightarrow (0, \infty)$ . Then  $\mu$  is said to be **relatively  $\phi$ -invariant with multiplier  $\zeta$**  if for each  $g \in G$  and  $U \in \mathcal{A}$   $\mu(g^{-1} \cdot U) = \zeta(g)\mu(U)$ . If for each  $g \in G$ ,  $\zeta(g) = e$ , then  $\mu$  is said to be  **$\phi$ -invariant**.

**Exercise 7.4.4.** Let  $G$  be a locally compact group and  $\mu : \mathcal{B}(G) \rightarrow [0, \infty]$  a left Haar measure. Define the actions  $\phi, \psi : G \times G \rightarrow G$  by  $\phi(g, x) = gx$  and  $\psi(g, x) = xg^{-1}$ . Then  $\mu$  is  $\phi$ -invariant and  $\mu$  is relatively  $\psi$ -invariant with multiplier  $\Delta$ .

*Proof.* Clear. □

**Exercise 7.4.5.** Let  $G$  be a group,  $(X, \mathcal{A}, \mu)$  a semifinite measure space,  $\phi : G \times X \rightarrow X$  a group action and  $\zeta : G \rightarrow (0, \infty)$ . Suppose that  $\mu \neq 0$ . If  $\mu$  is relatively  $\phi$ -invariant with multiplier  $\zeta$ , then

- (1)  $\zeta$  is a homomorphism
- (2) for each  $g \in G$ ,  $f \in L^1(\mu) \cup L^+$ ,

$$\int L_g f d\mu = \zeta(g) \int f d\mu$$

*Proof.*

- (1) Let  $g, h \in G$ . Choose  $U \in \mathcal{A}$  such that  $\mu(U) \in (0, \infty)$ . Then

$$\begin{aligned} \zeta(gh)\mu(U) &= \mu(gh \cdot U) \\ &= \mu(g \cdot (h \cdot U)) \\ &= \zeta(g)\mu(h \cdot U) \\ &= \zeta(g)\zeta(h)\mu(U) \end{aligned}$$

Then  $\zeta(gh) = \zeta(g)\zeta(h)$ . Since  $g, h \in G$  are arbitrary,  $\zeta$  is a homomorphism.

- (2) Let  $g \in G$  and  $U \in \mathcal{A}$ . Set  $f = \chi_U$ . Then

$$\begin{aligned} \int L_g f d\mu &= \int \chi_{gU} d\mu \\ &= \mu(gU) \\ &= \zeta(g)\mu(U) \\ &= \zeta(g) \int f d\mu \end{aligned}$$

Linearity of  $L_g$  implies that for each  $f \in S^+$ ,

$$\int L_g f \, d\mu = \zeta(g) \int f \, d\mu$$

Let  $f \in L^+$ . Then there exists a sequence  $(f_n)_{n \in \mathbb{N}} \subset S^+$  such that  $f_n \xrightarrow{\text{p.w.}} f$  and for each  $N \in \mathbb{N}$ ,  $f_n \leq f_{n+1}$ . Hence  $L_g f_n \xrightarrow{\text{p.w.}} L_g f$  and for each  $N \in \mathbb{N}$ ,  $L_g f_n \leq L_g f_{n+1}$ . The monotone convergence theorem then implies that

$$\begin{aligned} \int L_g f \, d\mu &= \lim_{n \rightarrow \infty} \int L_g f_n \, d\mu \\ &= \lim_{n \rightarrow \infty} \zeta(g) \int f_n \, d\mu \\ &= \zeta(g) \lim_{n \rightarrow \infty} \int f_n \, d\mu \\ &= \zeta(g) \int f \, d\mu \end{aligned}$$

Let  $f \in L^1(\mu)$ . If  $f : X \rightarrow \mathbb{R}$ , then  $f = f^+ - f^-$  and

$$\begin{aligned} \int L_g f \, d\mu &= \int L_g (f^+ - f^-) \, d\mu \\ &= \int L_g f^+ \, d\mu - \int L_g f^- \, d\mu \\ &= \zeta(g) \int f^+ \, d\mu - \zeta(g) \int f^- \, d\mu \\ &= \zeta(g) \int f^+ - f^- \, d\mu \\ &= \zeta(g) \int f \, d\mu \end{aligned}$$

If  $f : X \rightarrow \mathbb{C}$ , then there exist  $a, b : X \rightarrow \mathbb{R}$  such that  $f = a + ib$ . Then

$$\begin{aligned} \int L_g f \, d\mu &= \int L_g (a + ib) \, d\mu \\ &= \int L_g a \, d\mu + i \int L_g b \, d\mu \\ &= \zeta(g) \int a \, d\mu + i \zeta(g) \int b \, d\mu \\ &= \zeta(g) \int a + ib \, d\mu \\ &= \zeta(g) \int f \, d\mu \end{aligned}$$

□

**Definition 7.4.6.** Let  $X$  be a set,  $G$  a group,  $\phi : G \times X \rightarrow X$  a group action,  $f : X \rightarrow \mathbb{C}$  and  $x \in X$ . We define  $f^x : G \rightarrow \mathbb{C}$  by

$$f^x(g) = f(g^{-1} \cdot x)$$

**Exercise 7.4.7.** Let  $X$  be a LCH space,  $G$  a locally compact group  $\phi : G \times X \rightarrow X$  a proper group action and  $f \in C_c(X)$ . Then for each  $x \in X$ ,  $f^x \in C_c(G)$ .

*Proof.*

□

**Exercise 7.4.8.** Let  $X$  be a LCH space,  $G$  a locally compact group with left Haar measure  $\mu$ ,  $\phi : G \times X \rightarrow X$  a group action and  $f \in C_c(X)$ . Define  $f^* : X \rightarrow \mathbb{C}$  by

$$f^*(x) = \int f(g^{-1} \cdot x) d\mu(g)$$

## 8. HAUSDORFF MEASURE

## 8.1. Introduction.

**Definition 8.1.1.** Let  $X$  be a metric space and  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  an outer measure on  $X$ . Then  $\mu^*$  is said to be a **metric outer measure on  $X$**  if for each  $A, B \subset X$ ,  $d(A, B) > 0$  implies that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

**Exercise 8.1.2.** Let  $X$  be a metric space and  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  a metric outer measure on  $X$ . Then for each  $A \in \mathcal{B}(X)$ ,  $A$  is  $\mu^*$ -outer measurable.

*Proof.* □

**Definition 8.1.3.** Let  $X$  be a metric space,  $E \subset X$  and  $\delta > 0$ . Define  $\mathcal{A}_{E,\delta} \subset \mathcal{P}(X)^\mathbb{N}$  by

$$\mathcal{A}_{E,\delta} = \inf \left\{ (A_j)_{j \in \mathbb{N}} \subset \mathcal{P}(X) : E \subset \bigcup_{j \in \mathbb{N}} A_j \text{ and for each } j \in \mathbb{N}, \text{diam}(A_j) < \delta \right\}$$

**Exercise 8.1.4.** Let  $X$  be a metric space,  $E \subset X$  and  $\delta_1, \delta_2 > 0$ . If  $\delta_1 \leq \delta_2$ , then  $\mathcal{A}_{E,\delta_1} \subset \mathcal{A}_{E,\delta_2}$ .

*Proof.* Clear. □

**Definition 8.1.5.** Let  $X$  be a metric space,  $d \geq 0$  and  $\delta > 0$ . Define  $H_{d,\delta} : \mathcal{P}(X) \rightarrow [0, \infty]$  by

$$H_{d,\delta}(E) = \inf \left\{ \sum_{j \in \mathbb{N}} \text{diam}(A_j)^d : (A_j)_{j \in \mathbb{N}} \in \mathcal{A}_{E,\delta} \right\}$$

**Exercise 8.1.6.** Let  $X$  be a metric space,  $d \geq 0$  and  $\delta_1, \delta_2 > 0$ . If  $\delta_1 \leq \delta_2$ , then  $H_{d,\delta_2} \leq H_{d,\delta_1}$ .

*Proof.* Clear. □

**Definition 8.1.7.** Let  $X$  be a metric space and  $d \geq 0$ . We define the  **$d$ -dimensional Hausdorff outer measure**, denoted  $H_d : \mathcal{P}(X) \rightarrow [0, \infty]$ , by

$$\begin{aligned} H_d(E) &= \sup_{\delta > 0} H_{d,\delta}(E) \\ &= \lim_{\delta \rightarrow 0^+} H_{d,\delta}(E) \end{aligned}$$

**Exercise 8.1.8.** Let  $X$  be a metric space and  $d \geq 0$ . Then  $H_d : \mathcal{P}(X) \rightarrow [0, \infty]$  is an outer measure on  $X$ .

*Proof.* □

**Exercise 8.1.9.** Let  $X$  be a metric space and  $d \geq 0$ . Then  $H_d : \mathcal{P}(X) \rightarrow [0, \infty]$  is a metric outer measure on  $X$ .

*Proof.* □

**8.2. Hausdorff Measure on Smooth Manifolds.**

### 8.3. Induced Measures on Isometric Orbit Spaces.

**Note 8.3.1.** This section assumes familiarity with induced metrics on orbit spaces of metric spaces under isometric group actions. See section 9.1 of [2] for details.

**Note 8.3.2.**

**Definition 8.3.3.** Let  $(X, d)$  be a metric space,  $G$  a group, and  $\phi : G \times X \rightarrow X$  an isometric group action. Suppose that  $(X/G, \bar{d})$  is a metric space. Let  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  be a measure on  $X$ . We define  $\bar{\mu} : \mathcal{B}(X/G) \rightarrow [0, \infty]$  by  $\bar{\mu} = \pi_*\mu$ .

**Note 8.3.4.** If  $\mu \ll H_p^X$ , where  $X$  has Hausdorff dimension  $p$ , I want to be able to define  $\bar{\mu}$  in terms of  $H_q^{X/G}$  where  $X/G$  has Hausdorff dimension  $q$ . I was unable to do this. It might be possible with some manifold theory, for instance  $O(2)$  acting on  $\mathbb{R}^2$ .

**Definition 8.3.5.** Let  $(X, d)$  be a metric space,  $G$  a group, and  $\phi : G \times X \rightarrow X$  an isometric group action. Suppose that  $(X/G, \bar{d})$  is a metric space. Let  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  be a measure on  $X$ . Then  $\mu$  is said to be  $G$ -invariant if for each  $g \in G$ ,  $U \in \mathcal{B}(X)$ ,

$$\mu(g \cdot U) = \mu(U)$$

**Exercise 8.3.6.** Let  $X$  be a metric space,  $G$  a group, and  $\phi : G \times X \rightarrow X$  an isometric group action. Then for each  $p \geq 0$ ,  $H_p$  is  $G$ -invariant.

*Proof.* Clear. □

**Exercise 8.3.7.** Let  $X$  be a metric space,  $G$  a group, and  $\phi : G \times X \rightarrow X$  an isometric group action. Let  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  be a measure on  $X$ . Suppose that  $\mu \ll H_p$ . Then  $\mu$  is  $G$ -invariant iff  $d\mu/dH_p$  is  $G$ -invariant.

*Proof.* Suppose that  $\mu$  is  $G$ -invariant. Let  $g \in G$  and  $U \in \mathcal{B}(X)$ . Then

$$\begin{aligned} \int_U L_g \frac{d\mu}{dH_p}(x) dH_p(x) &= \int_U \frac{d\mu}{dH_p} \circ l_g^{-1}(x) dH_p(x) \\ &= \int_{l_g^{-1}(U)} \frac{d\mu}{dH_p}(x) d(l_g^{-1})_* H_p(x) \\ &= \int_{g^{-1} \cdot U} \frac{d\mu}{dH_p}(x) dH_p(x) \\ &= \mu(g^{-1} \cdot U) \\ &= \mu(U) \end{aligned}$$

So that

$$L_g \frac{d\mu}{dH_p} = \frac{d\mu}{dH_p}$$

The Converse is similar. □

**Exercise 8.3.8.** Let  $(X, d)$  be a metric space,  $G$  a group, and  $\phi : G \times X \rightarrow X$  an isometric group action. Suppose that  $\bar{d}$  is a metric. Let  $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$  be a measure on  $X$ . Suppose that  $\mu$  is  $G$ -invariant,  $\mu \ll H_p^X$  and  $d\mu/dH_p^X$  is continuous. Then  $\bar{\mu} \ll \bar{H}_p^X$ ,  $d\bar{\mu}/d\bar{H}_p^X$  is  $G$ -invariant,  $d\bar{\mu}/d\bar{H}_p^X$  is continuous and

$$\frac{d\bar{\mu}}{d\bar{H}_p^X} = \overline{\frac{d\mu}{dH_p^X}}$$

*Proof.* A previous exercise implies that  $\bar{\mu} \ll \bar{H}_p^X$ . Set  $f = d\mu/dH_p^X$ . Since  $\mu$  is  $G$ -invariant,  $f$  is  $G$ -invariant. Since  $f$  is continuous, an exercise in section 9.2 of [2] implies that  $\bar{f}$  is continuous and  $f = \bar{f} \circ \pi$ . Let  $E \in \mathcal{B}(X/G)$ . Then

$$\begin{aligned} \int_E \bar{f} d\bar{H}_p^X &= \int_{\pi^{-1}(E)} \bar{f} \circ \pi dH_p^X \\ &= \int_{\pi^{-1}(E)} f dH_p^X \\ &= \mu(\pi^{-1}(E)) \\ &= \bar{\mu}(E) \end{aligned}$$

Therefore, by definition, we have that

$$\frac{d\bar{\mu}}{d\bar{H}_p^X} = \bar{f} = \overline{\frac{d\mu}{dH_p^X}}$$

□

## 9. MEASURE AND INTEGRATION ON BANACH SPACES

## 9.1. Borel Measures on Banach Spaces.

**Definition 9.1.1.** Let  $X$  be a normed vector space. We define the **cylindrical  $\sigma$ -algebra** on  $X$ , denoted  $\mathcal{E}(X)$ , by

$$\mathcal{E}(X) = \sigma_X(X^*)$$

**Exercise 9.1.2.** Let  $(X, \mathcal{A})$  be a measurable space,  $Y$  a normed vector space and  $f : X \rightarrow Y$ . Then  $f$  is  $(\mathcal{A} - \mathcal{E}(Y))$  measurable iff for each  $\phi \in X^*$ ,  $\phi \circ f$  is  $(\mathcal{A} - \mathcal{B}(\mathbb{C}))$  measurable.

*Proof.* Immediate by exercise about initial  $\sigma$ -algebra.  $\square$

**Exercise 9.1.3.** Let  $X$  be a normed vector space. Then  $\mathcal{E}(X) \subset \mathcal{B}(X)$ .

*Proof.* Let  $\phi \in X^*$ . Since  $\phi$  is continuous,  $\phi$  is  $\mathcal{B}(X)$ -measurable. Hence for each  $E \in \mathcal{B}_{\mathbb{C}}$ ,  $\phi^{-1}(E) \in \mathcal{B}(X)$ . Thus  $\{\phi^{-1}(E) : E \in \mathcal{B}(\mathbb{C}) \text{ and } \phi \in X^*\} \subset \mathcal{B}(X)$ . This implies that

$$\begin{aligned} \mathcal{E}(X) &= \sigma_X(X^*) \\ &= \sigma(\{\phi^{-1}(E) : E \in \mathcal{B}(\mathbb{C}) \text{ and } \phi \in X^*\}) \\ &\subset \mathcal{B}(X) \end{aligned}$$

$\square$

**Exercise 9.1.4. Mourier's Theorem:**

Let  $X$  be a normed vector space. If  $X$  is separable, then  $\mathcal{E}(X) = \mathcal{B}(X)$ .

**Hint:** Let  $(x_n)_{n \in \mathbb{N}} \subset X$  be a dense subset. An exercise in the section on duality implies that there exist  $(\phi_n)_{n \in \mathbb{N}} \subset X^*$  such that for each  $n \in \mathbb{N}$ ,  $\|\phi_n\| = 1$  and  $\phi_n(x_n) = \|x_n\|$  and for each  $x \in X$ ,  $\|x\| = \sup_{n \in \mathbb{N}} |\phi_n(x)|$ . Then  $\text{cl } B(0, 1) \in \mathcal{E}(X)$ .

*Proof.* Suppose that  $X$  is separable. Then there exists  $(x_n)_{n \in \mathbb{N}} \subset X$  such that  $(x_n)_{n \in \mathbb{N}}$  is dense in  $X$ . An exercise from the section on duality in [2] implies that there exists  $(\phi_n)_{n \in \mathbb{N}} \subset X^*$  such that for each  $n \in \mathbb{N}$ ,  $\|\phi_n\| = 1$  and  $\phi_n(x_n) = \|x_n\|$ . A previous exercise implies that for each  $x \in X$ ,  $\|x\| = \sup_{n \in \mathbb{N}} |\phi_n(x)|$ . Let  $x \in X$  and  $r > 0$ . Then

$$r^{-1}\|x - y\| = \sup_{n \in \mathbb{N}} |r^{-1}\phi_n(x - y)| \text{ and}$$

$$\begin{aligned} \text{cl } B(x, r) &= \{y \in X : \|x - y\| \leq r\} \\ &= \{y \in X : r^{-1}\|x - y\| \leq 1\} \\ &= \bigcap_{n \in \mathbb{N}} \{y \in X : |r^{-1}\phi_n(x - y)| \leq 1\} \\ &= \bigcap_{n \in \mathbb{N}} \{y \in X : |\phi_n(x - y)| \leq r\} \\ &= \bigcap_{n \in \mathbb{N}} \{y \in X : |\phi_n(x) - \phi_n(y)| \leq r\} \\ &= \bigcap_{n \in \mathbb{N}} \phi_n^{-1}(\text{cl } B_{\mathbb{C}}(\phi_n(x), r)) \\ &\in \mathcal{E}(X) \end{aligned}$$



Let  $A \subset X$ . Suppose that  $A$  is open. Since  $X$  is separable, there exist  $(a_n)_{n \in \mathbb{N}} \subset A$  and  $(r_n)_{n \in \mathbb{N}} \subset (0, \infty)$  such that

$$A = \bigcup_{n \in \mathbb{N}} \text{cl } B(a_n, r_n) \\ \in \mathcal{E}(X)$$

Therefore,  $\mathcal{B}(X) \subset \mathcal{E}(X)$ .

The previous exercise implies that  $\mathcal{E}(X) \subset \mathcal{B}(X)$ . So  $\mathcal{E}(X) = \mathcal{B}(X)$ .  $\square$

**Exercise 9.1.5.** Let  $X$  be a separable normed vector space and  $\mu, \nu \in \mathcal{M}(X)$ . Then  $\mu = \nu$  iff for each  $\phi \in X^*$ ,  $\phi_*\mu = \phi_*\nu$ .

*Proof.* If  $\mu = \nu$ , then clearly for each  $\phi \in X^*$ ,  $\phi_*\mu = \phi_*\nu$ .

Conversely, suppose that for each  $\phi \in X^*$ ,  $\phi_*\mu = \phi_*\nu$ . Let  $E \in \mathcal{B}(\mathbb{C})$  and  $\phi \in X^*$ . Then

$$\begin{aligned} \mu(\phi^{-1}(E)) &= \phi_*\mu(E) \\ &= \phi_*\nu(E) \\ &= \nu(\phi^{-1}(E)) \end{aligned}$$

Set  $\mathcal{P} = \{\phi^{-1}(E) : \phi \in X^* \text{ and } E \in \mathcal{B}(\mathbb{C})\}$ . Then  $\mathcal{P}$  is a  $\pi$ -system. Since

$$\begin{aligned} \sigma(\mathcal{P}) &= \mathcal{E}(X) \\ &= \mathcal{B}(X) \end{aligned}$$

An exercise from the section on complex measures that uses Dynkin's lemma implies that  $\mu = \nu$ .  $\square$

**Definition 9.1.6.** Let  $X$  be a real normed vector space and  $\mu \in \mathcal{M}(X)$ . We define the **Fourier transform of  $\mu$** , denoted  $\hat{\mu} : X^* \rightarrow \mathbb{C}$ , by

$$\hat{\mu}(\phi) = \int_X e^{-i\phi(x)} d\mu(x)$$

**Exercise 9.1.7.** Let  $X$  be a real normed vector space and  $\mu \in \mathcal{M}(X)$ . Then  $\hat{\mu} : X^* \rightarrow \mathbb{C}$  is bounded.

*Proof.* Let  $\phi \in X^*$ .

$$\begin{aligned} |\hat{\mu}(\phi)| &= \left| \int_X e^{-i\phi(x)} d\mu(x) \right| \\ &\leq \int_X |e^{-i\phi(x)}| d|\mu|(x) \\ &= |\mu|(X) \end{aligned}$$

So  $\hat{\mu}$  is bounded.  $\square$

**Exercise 9.1.8.** Let  $X$  be a real normed vector space and  $\mu \in \mathcal{M}(X)$ . Then  $\hat{\mu} \in C_b(X^*)$ .

*Proof.* Let  $(\phi_n)_{n \in \mathbb{N}} \subset X^*$  and  $\phi \in X^*$ . Suppose that  $\phi_n \rightarrow \phi$ . Then  $e^{-i\phi_n} \xrightarrow{\text{p.w.}} e^{-i\phi}$  and for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} |e^{-i\phi_n}| &= 1 \\ &\in L^1(|\mu|) \end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}
 |\hat{\mu}(\phi_n) - \hat{\mu}(\phi)| &= \left| \int_X e^{-i\phi_n(x)} d\mu(x) - \int_X e^{-i\phi(x)} d\mu(x) \right| \\
 &= \left| \int_X e^{-i\phi_n(x)} - e^{-i\phi(x)} d\mu(x) \right| \\
 &\leq \int_X |e^{-i\phi_n(x)} - e^{-i\phi(x)}| d|\mu|(x) \\
 &\rightarrow 0
 \end{aligned}$$

So  $\hat{\mu} : X^* \rightarrow \mathbb{C}$  is continuous (in the norm topology). Hence  $\hat{\mu} \in C_b(X^*)$ .  $\square$

**Definition 9.1.9.** Let  $X$  be a real normed vector space. We define  $\mathcal{F} : \mathcal{M}(X) \rightarrow C_b(X^*)$  by

$$\mathcal{F}(\mu) = \hat{\mu}$$

**Exercise 9.1.10.** Let  $X$  be a real normed vector space. Then  $\mathcal{F} : \mathcal{M}(X) \rightarrow C_b(X^*)$  is linear.

*Proof.* Let  $\mu, \nu \in \mathcal{M}(X)$  and  $\phi \in X^*$ . Then

$$\begin{aligned}
 \mathcal{F}[\mu + \nu](\phi) &= \int_X e^{-i\phi(x)} d[\mu + \nu](x) \\
 &= \int_X e^{-i\phi(x)} d\mu(x) + \int_X e^{-i\phi(x)} d\nu(x) \\
 &= \mathcal{F}[\mu](\phi) + \mathcal{F}[\nu](\phi)
 \end{aligned}$$

Since  $\phi \in X^*$  is arbitrary,  $\mathcal{F}(\mu + \nu) = \mathcal{F}(\mu) + \mathcal{F}(\nu)$  and  $\mathcal{F}$  is linear.  $\square$

**Exercise 9.1.11.** Let  $X$  be a real normed vector space. If  $X$  is separable, then  $\mathcal{F}$  is injective.

*Proof.* Suppose that  $X$  is separable. Let  $\mu \in \mathcal{M}(X)$ . Suppose that  $\mu \in \ker \mathcal{F}$ . Then  $\hat{\mu} = 0$  and for each  $\phi \in X^*$ ,

$$\begin{aligned}
 0 &= \hat{\mu}(\phi) \\
 &= \int_X e^{-i\phi(x)} d\mu(x) \\
 &= \int_{\mathbb{R}} e^{-ix} d[\phi_*\mu](x)
 \end{aligned}$$

$\square$

**Exercise 9.1.12.** Let  $X$  be a real normed vector space. Then  $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$  and  $\|\mathcal{F}\| \leq 1$ .

*Proof.* For  $\mu \in \mathcal{M}(X)$  and  $\phi \in X^*$ , we have that

$$\begin{aligned}
 |\mathcal{F}[\mu](\phi)| &= \left| \int_X e^{-i\phi(x)} d\mu(x) \right| \\
 &\leq \int_X |e^{-i\phi(x)}| d|\mu|(x) \\
 &= |\mu|(X) \\
 &= \|\mu\|
 \end{aligned}$$

Hence

$$\begin{aligned}\|\mathcal{F}(\mu)\| &= \sup_{\phi \in X^*} |\mathcal{F}[\mu](\phi)| \\ &\leq \|\mu\|\end{aligned}$$

which implies that  $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$  and  $\|\mathcal{F}\| \leq 1$ . □

## 9.2. The Bochner Integral.

**Definition 9.2.1.** Let  $(X, \mathcal{A})$  be a measurable space,  $Y$  a Banach space and  $f : X \rightarrow Y$ . Then  $f$  is said to be **strongly measurable** if

- (1)  $f$  is  $(\mathcal{A}, \mathcal{B}(Y))$  measurable
- (2)  $f(X)$  is separable

We define  $L_Y^0(X, \mathcal{A}) = \{f : X \rightarrow Y : f \text{ is strongly measurable}\}$

**Exercise 9.2.2.** Let  $(X, \mathcal{A})$  be a measurable space,  $Y$  a Banach space and  $f : X \rightarrow Y$ . Then  $f$  is strongly measurable iff

- (1)  $f$  is  $(\mathcal{A}, \mathcal{E}(Y))$  measurable
- (2)  $f(X)$  is separable

*Proof.* □

**Exercise 9.2.3.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $Y$  a Banach space. Then  $L_Y^0(X, \mathcal{A})$  is a vector space.

*Proof.* Let  $f, g \in L_Y^0(X, \mathcal{A})$  and  $\lambda \in \mathbb{C}$ . By definition,  $f$  and  $g$  are measurable. Since  $f + \lambda g$  is a composition of measurable maps,  $f + \lambda g$  is measurable. Therefore  $f + \lambda g \in L_Y^0(X, \mathcal{A})$ . Clearly constant maps are measurable and hence  $0 \in L_Y^0(X, \mathcal{A})$ . So  $L_Y^0(X, \mathcal{A})$  is a vector space. □

**Definition 9.2.4.** Let  $(X, \mathcal{A})$  be a measurable space,  $Y$  a Banach space and  $\phi : X \rightarrow Y$ . Then  $\phi$  is said to be **simple** if

- (1)  $\phi$  is  $(\mathcal{A}, \mathcal{B}(X))$ -measurable
- (2)  $\phi(X)$  is finite

If  $\phi$  is simple then the **standard representation of  $\phi$**  is defined to be the sum

$$\phi = \sum_{j=1}^n \chi_{E_j} y_j$$

where  $(y_j)_{j=1}^n = \phi(X)$  and for each  $j \in \{1, \dots, n\}$ ,  $E_j = \phi^{-1}(y_j)$ . We define

$$S_Y(X, \mathcal{A}) = \{f \in L_Y^0(X, \mathcal{A}) : f \text{ is simple}\}$$

**Note 9.2.5.** If  $\phi = \sum_{j=1}^n \chi_{E_j} y_j$  is in the standard representation, then  $(E_j)_{j=1}^n$  are disjoint and

$$\bigcup_{j=1}^n E_j = X.$$

**Exercise 9.2.6.** Let  $(X, \mathcal{A})$  be a measurable space,  $Y$  a Banach space. Then

- (1)  $S_Y$  is a subspace of  $L_Y^0(X, \mathcal{A})$
- (2) Let  $\phi, \psi \in S_Y$ . Suppose that the standard representation of  $\phi$  is

$$\phi = \sum_{j=1}^n \chi_{A_j} a_j$$

and the standard representation of  $\psi$  is

$$\psi = \sum_{k=1}^m \chi_{B_k} b_k$$

Set

$$L = \{(j, k) \in \mathbb{N}^2 : j \leq n, k \leq m, \text{ and } A_j \cap B_k \neq \emptyset\}$$

Then the standard representation of  $\phi + \psi$  is

$$\phi + \psi = \sum_{(j,k) \in L} \chi_{A_j \cap B_k}(a_j + b_k)$$

*Proof.* Let  $\phi, \psi \in S_Y$  and  $\lambda \in \mathbb{C}$ . Then write  $\phi = \sum_{j=1}^n \chi_{A_j} a_j$  and  $\psi = \sum_{k=1}^m \chi_{B_k} b_k$  in the standard representation. Put

$$L = \{(j, k) \in \mathbb{N}^2 : j \leq n, k \leq m, \text{ and } A_j \cap B_k \neq \emptyset\}$$

Then the standard representation of  $\phi + \lambda\psi$  is given by  $\phi + \lambda\psi = \sum_{(j,k) \in L} \chi_{A_j \cap B_k}(a_j + \lambda b_k)$ . □

**Definition 9.2.7.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space and  $p \in [1, \infty]$ . Define  $\|\cdot\|_p : L_Y^0(X, \mathcal{A}, \mu) \rightarrow [0, \infty]$  by

$$\|f\|_p = \left( \int \|f\|^p d\mu \right)^{\frac{1}{p}} \quad (p < \infty)$$

and

$$\|f\|_\infty = \inf \left\{ \lambda > 0 : \mu(\{x \in X : \lambda < \|f(x)\|\}) = 0 \right\}$$

We define

$$L_Y^p(X, \mathcal{A}, \mu) = \{f \in L_Y^0(X, \mathcal{A}, \mu) : \|f\|_p < \infty\}$$

**Exercise 9.2.8.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space and  $p \in [1, \infty]$ . Then  $L_Y^p(X, \mathcal{A}, \mu)$  is a subspace of  $L_Y^0(X, \mathcal{A}, \mu)$ .

*Proof.* Let  $f, g \in L_Y^p(X, \mathcal{A}, \mu)$  and  $\lambda \in \mathbb{C}$ . Then  $\|f\|_p, \|g\|_p < \infty$ .

- (1) Clearly  $\|\lambda f\|_p = |\lambda| \|f\|_p < \infty$ . So  $\lambda f \in L_Y^p$ .
- (2) Let  $\|\cdot\|'_p : L^0(X, \mathcal{A}, \mu) \rightarrow [0, \infty]$  denote the usual  $L^p$  norm. Since  $\|f+g\| \leq \|f\| + \|g\|$ , we have that

$$\begin{aligned} \|f+g\|_p &= \| \|f+g\|'_p \| \\ &\leq \| \|f\|'_p + \|g\|'_p \| \\ &\leq \| \|f\|'_p \| + \| \|g\|'_p \| \\ &= \|f\|_p + \|g\|_p \\ &< \infty \end{aligned}$$

So  $f+g \in L_Y^p$ .

Hence  $L_Y^p$  is a subspace. □

**Exercise 9.2.9.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space and  $p \in [1, \infty]$ . Then

- (1)  $\|\cdot\|_p$  is a seminorm on  $L_Y^p(X, \mathcal{A}, \mu)$
- (2) if we identify functions that are equal  $\mu$ -a.e., then  $\|\cdot\|_p$  is a norm on  $L_Y^p(X, \mathcal{A}, \mu)$

*Proof.* Let  $f, g \in L_Y^p(X, \mathcal{A}, \mu)$  and  $\lambda \in \mathbb{C}$ .

- (1) The previous exercise implies that,  $\|\lambda f\|_p = |\lambda|\|f\|_p$  and  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ . So  $\|\cdot\|_p$  is a seminorm on  $L_Y^p$ .
- (2) If  $f = 0$   $\mu$ -a.e., then  $\|f\| = 0$   $\mu$ -a.e. Hence

$$\begin{aligned}\|f\|_p &= \|\|f\|\|_p' \\ &= 0\end{aligned}$$

So if we identify functions that are equal  $\mu$ -a.e.,  $\|\cdot\|_p$  becomes a norm on  $L_Y^p$ . □

**Note 9.2.10.** So for  $(f_n)_{n \in \mathbb{N}} \subset L_Y^p$  and  $f \in L_Y^p$ ,

$$f_n \xrightarrow{L_Y^p} f \text{ iff } \int \|f_n - f\|^p \rightarrow 0$$

**Definition 9.2.11.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space and  $\phi : X \rightarrow Y$ . Then  $\phi$  is said to be **simple** if  $\phi$  is measurable,  $\phi(X)$  is finite and for each  $y \in \phi(X) \setminus \{0\}$ ,  $\mu(\phi^{-1}(y)) < \infty$ . If  $\phi$  is simple then the **standard representation of  $\phi$**  is defined to be the sum

$$\phi = \sum_{j=1}^n \chi_{E_j} y_j$$

where  $(y_j)_{j=1}^n = \phi(X)$  and for each  $j \in \{1, \dots, n\}$ ,  $E_j = \phi^{-1}(y_j)$ . We define

$$S_Y(X, \mathcal{A}, \mu) = \{f \in L_Y^0(X, \mathcal{A}) : f \text{ is simple}\}$$

**Note 9.2.12.** If  $\phi = \sum_{j=1}^n \chi_{E_j} y_j$  is in the standard representation, then  $(E_j)_{j=1}^n$  are disjoint and  $\bigcup_{j=1}^n E_j = X$ .

**Exercise 9.2.13.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $Y$  a Banach space. Then  $S_Y \subset L_Y^1$ .

*Proof.* Let  $\phi \in S_Y$ . Write  $\phi = \sum_{j=1}^n \chi_{E_j} y_j$  in the standard representation. Then  $\|\phi\| = \sum_{j=1}^n \|y_j\| \chi_{E_j}$ . By definition, for each  $j \in \{1, \dots, n\}$ ,  $y_j \neq 0$  implies that  $\mu(E_j) < \infty$ . Then

$$\begin{aligned}\int \|\phi\| d\mu &= \sum_{j=1}^n \|y_j\| \mu(E_j) \\ &< \infty\end{aligned}$$

So  $\phi \in L_Y^1$ . □

**Exercise 9.2.14.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $Y$  a Banach space. Then  $S_Y(X, \mathcal{A}, \mu)$  is a subspace of  $L_Y^0(X, \mathcal{A})$

*Proof.* Clear. □

**Note 9.2.15.** For the remainder of this section, we will use the shorthand notation  $L_Y^0, L_Y^p$  and  $S_Y$  unless the context underlying measure space  $(X, \mathcal{A}, \mu)$  is unclear.

**Definition 9.2.16.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $Y$  a Banach space. Let  $\phi \in S_Y$ . Write  $\phi = \sum_{j=1}^n \chi_{E_j} y_j$  in the standard representation. With the convention that  $\infty \cdot 0_Y = 0_Y$ , we define

$$\int \phi d\mu = \sum_{j=1}^n \mu(E_j) y_j$$

For  $A \in \mathcal{A}$ , define

$$\int_A \phi d\mu = \int \chi_A \phi d\mu$$

**Exercise 9.2.17.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space,  $\phi \in S_Y$  and  $A \in \mathcal{A}$ . Write  $\phi = \sum_{j=1}^n \chi_{E_j} y_j$  in the standard representation. Then

$$\int_A \phi d\mu = \sum_{j=1}^n \mu(A \cap E_j) y_j$$

*Proof.* Note that  $\chi_A \phi = \sum_{j=1}^n \chi_{A \cap E_j} y_j$ . □

**Exercise 9.2.18.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space,  $\phi, \psi \in S_Y$  and  $\lambda \in \mathbb{C}$ . Then

$$\int \phi + \lambda \psi d\mu = \int \phi d\mu + \lambda \int \psi d\mu$$

*Proof.* If  $\lambda = 0$ , then the result clearly holds. Suppose that  $\lambda \neq 0$ . Write  $\phi = \sum_{j=1}^n \chi_{A_j} a_j$  and

$\psi = \sum_{k=1}^m \chi_{B_k} b_k$  in the standard representation. Put

$$L = \{(j, k) \in \mathbb{N}^2 : j \leq n, k \leq m, \text{ and } A_j \cap B_k \neq \emptyset\}$$

Then the standard representation of  $\phi + \lambda\psi$  is given by  $\phi + \lambda\psi = \sum_{(j,k) \in L} \chi_{A_j \cap B_k}(a_j + \lambda b_k)$ . So

$$\begin{aligned}
 \int \phi + \lambda\psi d\mu &= \int \sum_{(j,k) \in L} \chi_{A_j \cap B_k}(a_j + \lambda b_k) d\mu \\
 &= \sum_{(j,k) \in L} \mu(A_j \cap B_k)(a_j + \lambda b_k) \\
 &= \sum_{j=1}^n \sum_{k=1}^m \mu(A_j \cap B_k)(a_j + \lambda b_k) \\
 &= \sum_{j=1}^n \sum_{k=1}^m \mu(A_j \cap B_k)a_j + \lambda \sum_{j=1}^n \sum_{k=1}^m \mu(A_j \cap B_k)b_k \\
 &= \sum_{j=1}^n \mu(A_j)a_j + \lambda \sum_{k=1}^m \mu(B_k)b_k \\
 &= \int \phi d\mu + \lambda \int \psi d\mu
 \end{aligned}$$

□

**Exercise 9.2.19.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space,  $\phi \in S_Y$ . Then

$$\left\| \int \phi d\mu \right\| \leq \int \|\phi\| d\mu$$

*Proof.* Write  $\phi = \sum_{j=1}^n \chi_{E_j} y_j$  in the standard representation. Note that  $\|\phi\| = \sum_{j=1}^n \chi_{E_j} \|y_j\|$ .

Then

$$\begin{aligned}
 \left\| \int \phi d\mu \right\| &= \left\| \int \sum_{j=1}^n \chi_{E_j} y_j d\mu \right\| \\
 &= \left\| \sum_{j=1}^n \mu(E_j) y_j \right\| \\
 &\leq \sum_{j=1}^n \mu(E_j) \|y_j\| \\
 &= \int \sum_{j=1}^n \|y_j\| \chi_{E_j} d\mu \\
 &= \int \|\phi\| d\mu
 \end{aligned}$$

□

**Exercise 9.2.20.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space,  $f \in L_Y^1$  and  $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$ . If  $\phi_n \xrightarrow{L_Y^1} f$ , then

$$\lim_{n \rightarrow \infty} \int \phi_n d\mu = \int f d\mu$$



exists.

*Proof.* Suppose that  $\phi \xrightarrow{L_Y^1} f$ . Then by definition,

$$\int \|\phi_n - f\| d\mu \rightarrow 0$$

Let  $m, n \in \mathbb{N}$ . Then

$$\begin{aligned} \left\| \int \phi_m d\mu - \int \phi_n d\mu \right\| &= \left\| \int \phi_m - \phi_n d\mu \right\| \\ &\leq \int \|\phi_m - \phi_n\| d\mu \\ &\leq \int \|\phi_m - f\| d\mu + \int \|f - \phi_n\| d\mu \end{aligned}$$

Hence  $(\int \phi_n d\mu)_{n \in \mathbb{N}} \subset Y$  is Cauchy and  $\lim_{n \rightarrow \infty} \int \phi_n d\mu$  exists.  $\square$

**Exercise 9.2.21.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space,  $f \in L_Y^1$  and  $(\phi_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}} \subset S_Y$ . If  $\phi_n \xrightarrow{L_Y^1} f$  and  $\psi_n \xrightarrow{L_Y^1} f$ , then

$$\lim_{n \rightarrow \infty} \int \phi_n d\mu = \lim_{n \rightarrow \infty} \int \psi_n d\mu$$

*Proof.* Suppose that  $\phi_n \xrightarrow{L_Y^1} f$  and  $\psi_n \xrightarrow{L_Y^1} f$ . Let  $\epsilon > 0$ . By definition, there exist  $N_1 \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ ,  $n \geq N_1$  implies that  $\int \|\phi_n - f\| d\mu < \frac{\epsilon}{6}$  and  $\int \|\psi_n - f\| d\mu < \frac{\epsilon}{6}$ . Similarly to the previous exercise we have that for each  $n \in \mathbb{N}$ ,  $n \geq N_1$  implies that

$$\begin{aligned} \left\| \int \phi_n d\mu - \int \psi_n d\mu \right\| &= \left\| \int \phi_n - \psi_n d\mu \right\| \\ &\leq \int \|\phi_n - \psi_n\| d\mu \\ &\leq \int \|\phi_n - f\| d\mu + \int \|f - \psi_n\| d\mu \\ &< \frac{\epsilon}{6} + \frac{\epsilon}{6} \\ &= \frac{\epsilon}{3} \end{aligned}$$

Put  $I_\phi = \lim_{n \rightarrow \infty} \int \phi_n d\mu$  and  $I_\psi = \lim_{n \rightarrow \infty} \int \psi_n d\mu$ . Then there exists  $N_2 \in \mathbb{N}$  such that for each  $n \in \mathbb{N}$ , if  $n \geq N_2$ , then

$$\left\| \int \phi_n d\mu - I_\phi \right\| < \frac{\epsilon}{3}$$

and

$$\left\| \int \psi_n d\mu - I_\psi \right\| < \frac{\epsilon}{3}$$

Choose  $N = \max(N_1, N_2)$ . Then for each  $n \in \mathbb{N}$ ,  $n \geq N$  implies that

$$\begin{aligned} \|I_\phi - I_\psi\| &\leq \left\| I_\phi - \int \phi_n d\mu \right\| + \left\| \int \phi_n d\mu - \int \psi_n d\mu \right\| + \left\| \int \psi_n d\mu - I_\psi \right\| \\ &= < \frac{\epsilon}{3} + < \frac{\epsilon}{3} + < \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary,  $I_\phi = I_\psi$ . □

**Exercise 9.2.22.** Let  $Y$  be a Banach space and  $(y_n)_{n \in \mathbb{N}} \subset Y$  a countable dense subset. For  $\epsilon > 0$  and  $n \in \mathbb{N}$ , define  $B_n^\epsilon \in \mathcal{B}(Y)$  by

$$B_n^\epsilon = \{y \in Y : \|y - y_n\| < \epsilon\|y_n\|\}$$

Then for each  $\epsilon \geq 0$ ,

(1)

$$Y \setminus \{0\} \subset \bigcup_{n \in \mathbb{N}} B_n^\epsilon$$

(2) if  $\epsilon \leq 1$ ,

$$Y \setminus \{0\} = \bigcup_{n \in \mathbb{N}} B_n^\epsilon$$

*Proof.* Let  $\epsilon \geq 0$ .

(1) For the sake of contradiction, suppose that  $Y \setminus \{0\} \not\subset \bigcup_{n \in \mathbb{N}} B_n^\epsilon$ . Then there exists  $y \in Y$  such that  $y \neq 0$  and for each  $n \in \mathbb{N}$ ,  $\|y - y_n\| \geq \epsilon\|y_n\|$ . Since  $(y_n)_{n \in \mathbb{N}}$  is dense in  $Y$ , there exists a subsequence  $(y_{n_j})_{j \in \mathbb{N}} \subset (y_n)_{n \in \mathbb{N}}$  such that for each  $j \in \mathbb{N}$ ,  $\|y_{n_j} - y\| < 1/j$ . Then for each  $j \in \mathbb{N}$ ,

$$\begin{aligned} \|y_{n_j}\| &\leq \epsilon^{-1}\|y - y_{n_j}\| \\ &< \epsilon^{-1}1/j \end{aligned}$$

So that  $y_{n_j} \rightarrow y$  and  $y_{n_j} \rightarrow 0$ . Since  $y \neq 0$ , this is a contradiction and thus

$$Y \setminus \{0\} \subset \bigcup_{n \in \mathbb{N}} B_n^\epsilon$$

(2) Suppose that  $\epsilon \leq 1$ . For the sake of contradiction, suppose that  $0 \notin \bigcup_{n \in \mathbb{N}} B_n^\epsilon$ . Then there exists  $n \in \mathbb{N}$  such that  $0 \in B_n^\epsilon$ . By definition,

$$\begin{aligned} \|y_n\| &= \|0 - y_n\| \\ &< \epsilon\|y_n\| \\ &\leq \|y_n\| \end{aligned}$$

Which is a contradiction. So  $0 \notin \bigcup_{n \in \mathbb{N}} B_n^\epsilon$ . Hence  $\{0\} \subset \left(\bigcup_{n \in \mathbb{N}} B_n^\epsilon\right)^c$  and  $\bigcup_{n \in \mathbb{N}} B_n^\epsilon \subset \{0\}^c$ . Hence  $\bigcup_{n \in \mathbb{N}} B_n^\epsilon \subset Y \setminus \{0\}$  and  $Y \setminus \{0\} = \bigcup_{n \in \mathbb{N}} B_n^\epsilon$ . □

**Exercise 9.2.23.** Let  $(X, \mathcal{A})$  be a measurable space,  $Y$  a separable Banach space and  $f \in L_Y^0(X, \mathcal{A})$ . Let  $(y_n)_{n \in \mathbb{N}} \subset Y$  be a countable dense subset. For  $j \in \mathbb{N}$ , define  $(A_n^j)_{n \in \mathbb{N}} \subset \mathcal{B}(Y)$  and  $(E_n^j)_{n \in \mathbb{N}} \subset \mathcal{A}$  by

- $A_1^j = B_1^{1/j}$
- $A_n^j = B_n^{1/j} \setminus \left( \bigcup_{k=1}^{n-1} B_k^{1/j} \right)$
- $E_n^j = f^{-1}(A_n^j)$

Let  $j \in \mathbb{N}$ . Then

- (1)  $(A_n^j)_{n \in \mathbb{N}}$  is disjoint and

$$\bigcup_{n \in \mathbb{N}} A_n^j = Y \setminus \{0\}$$

- (2)  $(E_n^j)_{n \in \mathbb{N}}$  is disjoint and

$$\bigcup_{n \in \mathbb{N}} E_n^j = X \setminus f^{-1}(\{0\})$$

- (3) if  $j \geq 2$ , then for each  $n \in \mathbb{N}$  and  $x \in E_n^j$ ,

$$\|y_n\| < \frac{j}{j-1} \|f(x)\|$$

**Hint:** reverse triangle inequality

*Proof.*

- (1) Clear by previous exercise  
 (2) Clear  
 (3) Suppose that  $j \geq 2$ . Let  $n \in \mathbb{N}$  and  $x \in E_n^j$ . Then  $f(x) \in A_n^j \subset B_n^{1/j}$ . Hence

$$\begin{aligned} \|y_n\| - \|f(x)\| &\leq \left| \|y_n\| - \|f(x)\| \right| \\ &\leq \|y_n - f(x)\| \\ &< \frac{1}{j} \|y_n\| \end{aligned}$$

Thus  $(1 - 1/j)\|y_n\| < \|f(x)\|$ . Since  $j - 1 > 0$ , we have that

$$\|y_n\| < \frac{j}{j-1} \|f(x)\|$$

□

**Exercise 9.2.24.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a separable Banach space and  $f \in L_Y^1(X, \mathcal{A}, \mu)$ . Let  $(y_n)_{n \in \mathbb{N}} \subset Y$  be a countable dense subset. For  $j \in \mathbb{N}$ , define  $(E_n^j)_{n \in \mathbb{N}} \subset \mathcal{A}$  as in the previous exercise and  $(\psi_j)_{j \in \mathbb{N}} \subset L_Y^0(X, \mathcal{A})$  by

$$\psi_j = \sum_{n \in \mathbb{N}} \chi_{E_n^j} y_n$$

Then for each  $j \in \mathbb{N}$ ,  $j \geq 2$  implies that

- (1)  $\psi_j \in L^1(X, \mathcal{A}, \mu)$   
 (2)  $\|\psi_j - f\| < \frac{1}{j-1} \|f\|_1$

*Proof.* Let  $j \in \mathbb{N}$ . Suppose that  $j \geq 2$ . Then

(1)

$$\begin{aligned}
 \|\psi_j\|_1 &= \int \|\psi_j\| d\mu \\
 &= \int \sum_{n \in \mathbb{N}} \|y_n\| \chi_{E_n^j} d\mu \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n^j} \|y_n\| d\mu \\
 &\leq \frac{j}{j-1} \sum_{n \in \mathbb{N}} \int_{E_n^j} \|f\| d\mu \\
 &= \frac{j}{j-1} \int_{\bigcup_{n \in \mathbb{N}} E_n^j} \|f\| d\mu \\
 &= \frac{j}{j-1} \int \|f\| d\mu \\
 &= \frac{j}{j-1} \|f\|_1
 \end{aligned}$$

So  $\psi_j \in L_Y^1(X, \mathcal{A}, \mu)$ .

(2) Similarly, we have that

$$\begin{aligned}
 \|\psi_j - f\|_1 &= \int \|\psi_j - f\| d\mu \\
 &= \int_{f^{-1}(\{0\})} \|\psi_j - f\| d\mu + \sum_{n \in \mathbb{N}} \int_{E_n^j} \|\psi_j - f\| d\mu \\
 &= \sum_{n \in \mathbb{N}} \int_{E_n^j} \|y_n - f\| d\mu \\
 &\leq \sum_{n \in \mathbb{N}} \int_{E_n^j} \frac{1}{j-1} \|y_n\| d\mu \\
 &\leq \sum_{n \in \mathbb{N}} \int_{E_n^j} \frac{1}{j-1} \|f\| d\mu \\
 &= \frac{1}{j-1} \int \|f\| d\mu \\
 &= \frac{1}{j-1} \|f\|_1
 \end{aligned}$$

So  $\|\psi_j - f\| < \frac{1}{j-1} \|f\|_1$ .

□

**Exercise 9.2.25.** such that  $\phi_n \xrightarrow{\text{a.e.}} f$  and  $\phi_n \xrightarrow{L_Y^1} f$ .

**Hint:** Choose a countable dense subset  $(y_n)_{n \in \mathbb{N}} \subset f(X)$  and define

**Definition 9.2.26. Bochner Integral:**

Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a separable Banach space and  $f : X \rightarrow Y$ . Then  $f$  is said to be **Bochner** integrable if  $f \in L_Y^1$ . If  $f$  is Bochner integrable, then there exists  $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$  such that  $\phi_n \xrightarrow{\text{a.e.}} f$  and  $\phi_n \xrightarrow{L_Y^1} f$  and the **Bochner integral of  $f$**  with respect to  $\mu$ , denoted

$$\int f d\mu$$

is defined to be

$$\int f d\mu = \lim_{n \rightarrow \infty} \int \phi_n d\mu$$

**Exercise 9.2.27.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a separable Banach space,  $f, g \in L_Y^1$  and  $\lambda \in \mathbb{C}$ . Then

$$\int f + \lambda g d\mu = \int f d\mu + \lambda \int g d\mu$$

*Proof.* Choose  $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$  such that  $\phi_n \xrightarrow{L_Y^1} f$  and  $(\psi_n)_{n \in \mathbb{N}} \subset S_Y$  such that  $\psi_n \xrightarrow{L_Y^1} g$ . Since addition and scalar multiplication are continuous,  $\phi_n + \lambda \psi_n \xrightarrow{L_Y^1} f + \lambda g$ . By definition, we have that

$$\int \phi_n + \lambda \psi_n d\mu \rightarrow \int f + \lambda g d\mu$$

$$\int \phi_n d\mu \rightarrow \int f d\mu$$

and

$$\int \psi_n d\mu \rightarrow \int g d\mu$$

Hence

$$\begin{aligned} \int f + \lambda g d\mu &= \lim_{n \rightarrow \infty} \int \phi_n + \lambda \psi_n d\mu \\ &= \lim_{n \rightarrow \infty} \int \phi_n d\mu + \lambda \lim_{n \rightarrow \infty} \int \psi_n d\mu \\ &= \int f d\mu + \lambda \int g d\mu \end{aligned}$$

□

**Exercise 9.2.28.** Let  $(X, \mathcal{A}, \mu)$  be a measure space and  $Y$  a separable Banach space. Define  $I : L_Y^1 \rightarrow Y$  by

$$If = \int f d\mu$$

Then  $I \in L(L_Y^1, Y)$  and  $\|I\| \leq 1$ .

*Proof.* Let  $f \in L_Y^1$ . Choose  $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$  such that  $\phi_n \xrightarrow{L_Y^1} f$ . Then

$$\begin{aligned} \left| \int \|\phi_n\| d\mu - \int \|f\| d\mu \right| &= \left| \int \|\phi_n\| - \|f\| d\mu \right| \\ &\leq \int \|\phi_n - f\| d\mu \\ &\leq \int \|\phi_n - f\| d\mu \\ &\rightarrow 0 \end{aligned}$$

So

$$\int \|\phi_n\| d\mu \rightarrow \int \|f\| d\mu$$

By continuity of  $\|\cdot\| : Y \rightarrow [0, \infty)$ ,

$$\begin{aligned} \|If\| &= \left\| \int f d\mu \right\| \\ &= \left\| \lim_{n \rightarrow \infty} \int \phi_n d\mu \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \int \phi_n d\mu \right\| \\ &\leq \lim_{n \rightarrow \infty} \int \|\phi_n\| d\mu \\ &= \int \|f\| d\mu \\ &= \|f\|_1 \end{aligned}$$

□

**Exercise 9.2.29.** Let  $Y$  be a separable Banach space and  $f : [a, b] \rightarrow Y$  continuous. Then  $f$  is Banach-integrable.

*Proof.* Continuity implies that  $f \in L_Y^\infty$  and

$$\begin{aligned} \int \|f\| dm &\leq \|f\|_\infty (b - a) \\ &< \infty \end{aligned}$$

so that  $f \in L_Y^1$  and  $f$  is Bochner integrable. □

**Exercise 9.2.30. Dominated Convergence Theorem:**

Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a separable Banach space,  $(f_n)_{n \in \mathbb{N}} \subset L_Y^1$  and  $f \in L_Y^0$ . Suppose that  $f_n \xrightarrow{\text{a.e.}} f$  and there exists  $g \in L^1$  such that for each  $n \in \mathbb{N}$ ,  $\|f_n\| \leq g$ . Then  $f \in L_Y^1$  and  $f_n \xrightarrow{L^1} f$ .

*Proof.* Since  $f_n \xrightarrow{\text{a.e.}} f$ ,  $\|f\| \leq g$  a.e. and  $f \in L_Y^1$ . Also,

$$\begin{aligned} \|f_n - f\| &\leq \|f_n\| + \|f\| \\ &\leq 2g \text{ a.e.} \end{aligned}$$

Hence  $2g - \|f_n - f\| \geq 0$  a.e. Fatou's lemma implies that

$$\begin{aligned} \int 2g \, d\mu &= \int \liminf_{n \rightarrow \infty} (2g - \|f_n - f\|) \, d\mu \\ &\leq \liminf_{n \rightarrow \infty} \left[ \int 2g - \|f_n - f\| \, d\mu \right] \\ &= \int 2g \, d\mu - \limsup_{n \rightarrow \infty} \int \|f_n - f\| \, d\mu \end{aligned}$$

Hence

$$0 \leq \limsup_{n \rightarrow \infty} \int \|f_n - f\| \, d\mu \leq 0$$

and  $f_n \xrightarrow{L_Y^1} f$ . □

**Exercise 9.2.31.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y, Z$  separable Banach spaces and  $f \in L_Y^1$  and  $T \in L(Y, Z)$ . Then  $T \circ f \in L_Z^1$  and

$$\int T \circ f \, d\mu = T \left( \int f \, d\mu \right)$$

**Note 9.2.32.** The statement remains true if  $T$  is continuous and conjugate-linear.

*Proof.* Suppose that  $f \in S_Y$ . Write  $f = \sum_{j=1}^n \chi_{E_j} y_j$  in the standard representation. Then

$$T \circ f = \sum_{j=1}^n \chi_{E_j} T(y_j) \text{ and}$$

$$\begin{aligned} \int T \circ f \, d\mu &= \sum_{j=1}^n \mu(E_j) T(y_j) \\ &= T \left( \sum_{j=1}^n \mu(E_j) y_j \right) \\ &= T \left( \int f \, d\mu \right) \end{aligned}$$

For  $f \in L_Y^1$ , choose  $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$  such that  $\phi_n \xrightarrow{\text{a.e.}} f$  and  $\phi_n \xrightarrow{L_Y^1} f$ . Then

$$\begin{aligned} \|T \circ \phi_n - T \circ f\| &= \|T \circ (\phi_n - f)\| \\ &\leq \|T\| \|\phi_n - f\| \end{aligned}$$

So  $T \circ \phi_n \xrightarrow{\text{a.e.}} T \circ f$  and  $T \circ \phi_n \xrightarrow{L_Z^1} T \circ f$ . Thus

$$\begin{aligned} \int T \circ f d\mu &= \lim_{n \rightarrow \infty} \int T \circ \phi_n d\mu \\ &= \lim_{n \rightarrow \infty} T \left( \int \phi_n d\mu \right) \\ &= T \left( \lim_{n \rightarrow \infty} \int \phi_n d\mu \right) \\ &= T \left( \int f d\mu \right) \end{aligned}$$

□

**Note 9.2.33.** Recall that for a function  $f : X \times Y \rightarrow Z$ ,  $x \in X$  and  $y \in Y$ , the functions  $f_x : Y \rightarrow Z$  and  $f^y : X \rightarrow Z$  are defined by  $f_x(y) = f(x, y)$  and  $f^y(x) = f(x, y)$ .

**Exercise 9.2.34.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $Y$  a Banach space,  $A \subset Y$  open and  $f : X \times A \rightarrow Z$ . Suppose that for each  $y \in A$ ,  $f^y \in L^1(\mu)$ . Define  $F : Y \rightarrow \mathbb{C}$  by

$$F(y) = \int_X f^y d\mu$$

- (1) Suppose that there exists  $g \in L^1(\mu)$  such that for each  $(x, y) \in X \times A$ ,  $\|f(x, y)\| \leq g(x)$ . Let  $y_0 \in A$ . If for each  $x \in X$ ,  $f_x$  is continuous at  $y_0$ , then  $F$  is continuous at  $y_0$ .
- (2) Suppose that for each  $x \in X$ ,  $f_x : A \rightarrow Z$  is Gateaux differentiable and there exists  $g \in L^1(\mu)$  such that for each  $(x, y) \in X \times A$ ,  $h \in Y$ ,  $|df_x(y)(h)| \leq g(x)$ . Then  $F$  is Gateaux differentiable and for each  $y \in A$ ,  $h \in Y$ ,

$$dF(y)(h) = \int_X df_x(y)(h) d\mu(x)$$

*Proof.*

- (1) Suppose that for each  $x \in X$ ,  $f_x$  is continuous at  $y_0$ . Let  $(y_n) \subset A$ . Suppose that  $y_n \rightarrow y_0$ . Continuity implies that  $f^{y_n} \xrightarrow{\text{p.w.}} f^{y_0}$ . Since for each  $n \in \mathbb{N}$ ,  $|f^{y_n}| \leq g$ , the dominated convergence theorem implies that  $F(y_n) \rightarrow F(y_0)$ .
- (2) Let  $y_0 \in \mathbb{R}$ . Choose  $(y_n)_{n \in \mathbb{N}}$  such that  $y_n \rightarrow y_0$  and for each  $n \in \mathbb{N}$ ,  $y_n \neq y_0$ . For  $n \in \mathbb{N}$ , define  $q_n : X \rightarrow \mathbb{R}$  by

$$q_n(x) = \frac{f(x, y_n) - f(x, y_0)}{y_n - y_0}$$

So  $h_n(\cdot) \xrightarrow{\text{p.w.}} \partial f / \partial t(\cdot, y_0)$ . The mean value theorem implies that for each  $x \in X$  and  $n \in \mathbb{N}$ , there exists  $c_{n,x} \in (y_n, y_0)$  such that  $h_n(x) = \partial f / \partial t(x, c_{n,x})$ . Then for each  $n \in \mathbb{N}$ ,  $|h_n| \leq g$ . The dominated convergence theorem then implies that



$\partial f/\partial t(\cdot, t_0) \in L^1(\mu)$  and

$$\begin{aligned}\int \frac{\partial f}{\partial t}(x, t_0) d\mu(x) &= \lim_{n \rightarrow \infty} \int_X h_n d\mu \\ &= \lim_{n \rightarrow \infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} \\ &= F'(t_0^-)\end{aligned}$$

So that  $F$  is differentiable at  $t_0$  from the left. Similarly,  $F$  is differentiable at  $t_0$  from the right.

**FINISH!!!**

□

## 10. BANACH SPACE VALUED MEASURES

## 11. TODO

- Add background for banach space valued measures like riesz representation theorem and radon-nikodym derivatives to be able to talk about condition expectation of banach space valued random variables
- Discuss disintegration of measures independently of probability by discussing the projection of  $L^1(X, \mathcal{A})$  onto  $L^1(X, \mathcal{B})$  for  $\mathcal{B} \subset \mathcal{A}$  and the Doob-Dynkin Lemma. Use this to define the disintegration measure. Also do this for disintegration of vector measures.
- Talk about homology when conditioning measures on a value in relation to the entropy of that distribution (maybe make a new set of notes about entropy and put it there)
- Consider the category  $\mathcal{C}$  of measurable spaces with measurable singletons. Fix an object  $(X, \mathcal{A}) \in \mathcal{C}$ . Consider the coslice category of  $\mathcal{C}$  under  $(X, \mathcal{A})$ . Introduce an equivalence relation on objects in the coslice category by  $f : X \rightarrow (Y, \mathcal{F}) \sim g : X \rightarrow (Z, \mathcal{G})$  iff  $f^*\mathcal{F} = g^*\mathcal{G}$ . Introduce a partial order on the quotient by  $f : X \rightarrow (Y, \mathcal{F}) \leq g : X \rightarrow (Z, \mathcal{G})$  iff  $f^*\mathcal{F} \subset g^*\mathcal{G}$ . Describe the Doob-Dynkin Lemma in this context, i.e. that  $f \leq g$  implies that there is exactly one morphism from  $g$  to  $f$  in the coslice category.

**11.1. Applications to Hilbert Spaces.**

**Exercise 11.1.1.** Let  $(X, \mathcal{A}, \mu)$  be a measure space,  $H$  a separable Hilbert space,  $f \in L_H^1$  and  $a \in H$ . Then

$$\int \langle f(x), a \rangle d\mu(x) = \left\langle \int f(x) d\mu(x), a \right\rangle$$

*Proof.* Define  $T \in L^*(H, \mathbb{C})$  by  $T(x) = \langle x, a \rangle$  and apply a previous exercise. □

## 12. APPENDIX

## 12.1. Summation.

**Definition 12.1.1.** Let  $f : X \rightarrow [0, \infty)$ , Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when  $X$  is countable. For  $f : X \rightarrow \mathbb{C}$ , we can write  $f = g + ih$  where  $g, h : X \rightarrow \mathbb{R}$ . If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for  $g^+, g^-, h^+, h^-$ . In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation  $\sum_{x \in X} f(x)$  where  $f : X \rightarrow \mathbb{C}$ .

**Note 12.1.2.** Let  $f : X \rightarrow \mathbb{C}$  and  $\alpha : X \rightarrow X$  a bijection. If  $\sum_{x \in X} |f(x)| < \infty$ , then  $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$ .

## REFERENCES

- [1] [Introduction to Algebra](#)
- [2] [Introduction to Analysis](#)
- [3] [Introduction to Fourier Analysis](#)
- [4] [Introduction to Measure and Integration](#)