Gradient Descent in Hilbert Space

Carson James

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Outline

Banach Spaces

Definition

Let X be a normed vector space. Then X is said to be a **Banach space** if X is complete.

Definition

Let X, Y be a normed vector spaces and $T : X \to Y$ a linear map. Then T is said to be **bounded** if there exists $C \geq 0$ such that for each $x \in X$,

$$\|Tx\| \leq C\|x\|$$

We define

$$L(X, Y) = \{T : X \to Y : T \text{ is linear and bounded}\}$$

Definition

Let $X_1, \ldots, X_n$ and Y be a normed vector spaces and $T : \prod_{j=1}^{n} X_j \to Y$ a multilinear linear map. Then T is said to be **bounded** if there exists $C \geq 0$ such that for each $(x_j)_{j=1}^{n} \in \prod_{j=1}^{n} X_j$,

$$\|T(x_1, \ldots, x_n)\| \leq C\|x_1\| \ldots \|x_n\|$$

We define

$$L^n\left(\prod_{j=1}^{n} X_j, Y\right) = \{T : X \to Y : T \text{ is multilinear and bounded}\}$$

If $X_1, \ldots, X_n = X$, we write $L^n(X, Y)$ in place of $L^n(X^n, Y)$.

Remark

Let X and Y be normed vector spaces. We may identify $L(X, L(X, \ldots, L(X, Y)) \ldots)$ and $L^n(X, Y)$ via the isometric isomorphism given by $\phi \mapsto \psi_\phi$ where

$$\psi_\phi(x_1, x_2, \ldots, x_n) = \phi(x_1)(x_2) \ldots (x_n)$$

Definition

Let X be a normed vector space over $\mathbb{R}$. We define the **dual space of** X, denoted X^*, by $X^* = L(X, \mathbb{R})$. Let $T : X \to \mathbb{R}$. Then T is said to be a **bounded linear functional on** X if $T \in X^*$.

Definition

Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$ and $x_0 \in A$. Then f is said to be **(1-st order) Frechet differentiable at** x_0 if there exists $Df(x_0) \in L(X, Y)$ such that,

$$f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|) \qquad \text{as } h \to 0$$

If f is Frechet differentiable at x_0, we define the **Frechet derivative of** f **at** x_0 to be $Df(x_0)$. We say that f is **(1-st order) Frechet differentiable** if for each $x_0 \in A$, f is Frechet differentiable at x_0.

If f is Frechet differentiable, we define the **Frechet derivative** of f, denoted $Df : A \to L(X, Y)$, by

$$x \mapsto Df(x)$$

Continuing inductively,

Definition

Let X, Y be a banach spaces, $A \subset X$ open, $f : A \to Y$. We define n-**th** order Frechet differentiablility inductively.
If f is $n-1$-th order Frechet differentiable, f is said to be n-th order Frechet differentiable at x_0 if $D^{n-1}f$ is Frechet differentiable at x_0. We define $D^n f(x_0) = D(D^{n-1}f)(x_0)$.

Remark

Note that $D^n f(x_0) \in L^n(X, Y)$.

Calculus

Remark

The tools used to obtain the following results:

- Frechet Derivative
- Bochner Integral
- Hahn-Banach Theorem

Result

Let X, Y be Banach spaces and $f \in L(X, Y)$. Then f is Frechet differentiable and for each $x_0 \in X$, $Df(x_0) = f$.

Result

Let X, Y, Z be Banach spaces, $f : X \to Y$, $g : Y \to Z$ and $x_0 \in X$. If f is Frechet differentiable at x_0 and g is Frechet differentiable at $f(x_0)$, then $g \circ f$ is Frechet differentiable at x_0 and

$$D(g \circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$$

Result

Let X, Y be a Banach spaces, $A \subset X$ open and convex and $f : A \to Y$. If f is Frechet differentiable, then for each $x, y \in A$, there exists $t \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \|Df(tx + (1 - t)y)\|\|x - y\|$$

Result

Let X, Y be a Banach spaces, $A \subset X$ open and convex and $f : A \to Y$. Suppose that f is Frechet differentiable. If for each $x \in A$, $Df(x) = 0$, then f is constant.

Result

Let X, Y be a Banach spaces, $A \subset X$ open and convex and $f, g : A \to Y$. Suppose that f and g are Frechet differentiable. If $Df = Dg$, then there exists $c \in Y$ such that $f = g + c$.

Result

Let Y be a separable Banach space and $f \in C^1_Y(a,b)$. Then for each $x, x_0 \in (a,b)$, $x_0 < x$ implies that

1. *f' is Bochner integrable on $(x_0, x]$*
2. $$f(x) - f(x_0) = \int_{(x_0,x]} f' dm$$

Result

Let Y be a separable Banach space, $A \subset X$ open and convex, $f \in C^n_Y(A)$ and $x_0 \in A$. Then

$$f(x_0 + h) = \sum_{k=0}^{n} D^k f(x_0)(h, \dots, h) + o(\|h\|^n) \qquad \text{as } h \to 0$$

Hilbert Spaces

Definition

Let H be an inner product space. Then H is said to be a **Hilbert space** if H is complete with respect to the norm induced by the inner product.

Remark

We will be assuming the Hilbert space is real.

Definition

Let H be a Hilbert space. Define $\phi : H \to H^*$ by $x \mapsto x^*$ where

$$x^* y = \langle x, y \rangle$$

Result

Let H be a Hilbert space. Then $\phi : H \to H^*$ defined above is an isometric isomorphism.

Let H be a Hilbert space, $f : H \to \mathbb{R}$ and $x_0 \in H$. Suppose that f is Frechet differentiable at x_0 so that $Df(x_0) \in H^*$. We define the **gradient of f at x_0**, denoted $\nabla f(x_0) \in H$, by

$$\nabla f(x_0) = \phi^{-1} Df(x_0)$$

That is, $\nabla f(x_0)$ is the unique element of H such that for each $y \in H$,

$$\langle \nabla f(x_0), y \rangle = Df(x_0)(y)$$

Convex Analysis

Result