INTRODUCTION TO FOURIER ANALYSIS

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Contents

1. The Fourier Transform on $\mathbb{R}^n$

1.1. Schwartz Space.

Definition 1.1.1. Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

(1) $\langle x, y \rangle = \sum_j x_j y_j$
(2) $|x| = \langle x, x \rangle^{1/2}$
(3) $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
(4) $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

Definition 1.1.2. Let $f \in C^\infty(\mathbb{R}^n)$,$\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}^n} (1 + |x|^N)|\partial^\alpha f(x)|$$

We define Schwartz space, denoted $\mathcal{S}$, by

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n, \, N \in \mathbb{N}_0, \, \|f\|_{\alpha,N} < \infty\}$$

Exercise 1.1.3. For each $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}$, $\alpha \in \mathbb{N}_0^n$. Then there exists $C \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^{-1}$$

Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = (1 + |x|^2)^{-1}$. Then $g \in L^1(\mathbb{R}^n)$ which implies that $\partial^\alpha f \in L^1(\mathbb{R}^n)$. □

Definition 1.1.4.

1.2. **The Convolution.**

Exercise 1.2.1. Let $f, g \in L^1(\mathbb{R}^n)$. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = f(x-y)g(y)$. Then $h \in L^1(m^2)$ and the function

$$x \mapsto \int f(x-y)g(y)dm(y)$$

is well defined in $L^1(\mathbb{R}^n)$.

Proof. By Tonelli's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n \times \mathbb{R}^n} |h|dm^2 &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}^n} |g(y)| \left[\int_{\mathbb{R}^n} |f(x-y)|dm(y) \right] dm(x) \\
&= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dm(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

By Fubini's theorem, the map

$$x \mapsto \int f(x-y)g(y)dm(y)$$

is defined a.e. □

Definition 1.2.2. Let $f, g \in L^1(\mathbb{R}^n)$. We define the **convolution of** f **with** g, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$, by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

Exercise 1.2.3. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.

Proof. By Tonelli's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f * g|dm &\le \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}^n} |g(y)| \left[\int_{\mathbb{R}^n} |f(x-y)|dm(y) \right] dm(x) \\
&= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dm(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

□

Exercise 1.2.4. Let $f, g, h \in L^1(\mathbb{R}^n)$. Then $(f * g) * h = f * (g * h)$.

Proof. Let $x \in \mathbb{R}^n$. Then using the substitution $z \mapsto z - y$ and Fubini's theorem, we obtain

$$\begin{aligned}
(f * g) * h(x) &= \int f * g(x-y)h(y)dm(y) \\
&= \int \bigg[\int f(x-y-z)g(z)dm(z)\bigg]h(y)dm(y) \\
&= \int \bigg[\int f(x-z)g(z-y)dm(z)\bigg]h(y)dm(y) \\
&= \int \bigg[\int f(x-z)g(z-y)h(y)dm(z)\bigg]dm(y) \\
&= \int \bigg[\int f(x-z)g(z-y)h(y)dm(y)\bigg]dm(z) \\
&= \int f(x-z)\bigg[\int g(z-y)h(y)dm(y)\bigg]dm(z) \\
&= \int f(x-z)g * h(z)dm(z) \\
&= f * (g * h)(z)
\end{aligned}$$

So $(f * g) * h = f * (g * h)$. $\square$

Exercise 1.2.5. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g = g * f$.

Proof. Let $x \in \mathbb{R}^n$. Using the transformation $y \mapsto x - y$, we obtain that

$$\begin{aligned}
f * g(x) &= \int f(x-y)g(y)dm(y) \\
&= \int f(y)g(x-y)dm(y) \\
&= \int g(x-y)f(y)dm(y) \\
&= g * f(x)
\end{aligned}$$

So $f * g = g * f$. $\square$

Note 1.2.6. To summarize, $(L^1(\mathbb{R}^n), *)$ is a commutative Banach algebra.

1.3. **The Fourier Transform on $L^1(\mathbb{R}^n)$.**

Definition 1.3.1. Let $f \in L^1(\mathbb{R}^n)$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-i}$$