





# Introduction to Differential Geometry

Carson James



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# Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
$v$	velocity





# Preface

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# Chapter 1

## Review of Fundamentals

### 1.1 Set Theory

**Definition 1.1.0.1.** Let  $\{A_i\}_{i \in I}$  be a collection of sets. The **disjoint union of**  $\{A_i\}_{i \in I}$ , denoted  $\coprod_{i \in I} A_i$ , is defined by

$$\coprod_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$$

We define the **natural projection map**, denoted  $\pi : \coprod_{i \in I} A_i \rightarrow I$ , by  $\pi(i, a) = i$ .

**Definition 1.1.0.2.** Let  $E$  and  $M$  be sets,  $\pi : E \rightarrow B$  a surjection and  $\sigma : B \rightarrow E$ . Then  $\sigma$  is said to be a section of  $(E, M, \pi)$  if  $\pi \circ \sigma = \text{id}_M$ .

**Note 1.1.0.3.** Let  $\{A_i\}_{i \in I}$  be a collection of sets and  $\sigma : I \rightarrow \coprod_{i \in I} A_i$ . We will typically be interested in sections  $\sigma$  of  $\left( \coprod_{i \in I} A_i, I, \pi \right)$ .

**Exercise 1.1.0.4.** Let  $\{A_i\}_{i \in I}$  be a collection of sets and  $\sigma : I \rightarrow \coprod_{i \in I} A_i$ . Then  $\sigma$  is a section of  $\coprod_{i \in I} A_i$  iff for each  $i \in I$ ,  $\sigma(i) \in A_i$

*Proof.* Clear. □

## 1.2 Linear Algebra

**Note 1.2.0.1.** We denote the standard basis on  $\mathbb{R}^n$  by  $(e_1, \dots, e_n)$ .

**Definition 1.2.0.2.** Let  $A \in \mathbb{R}^{n \times n}$ . Then  $A$  is said to be **invertible** if  $\det(A) \neq 0$ . We denote the set of  $n \times n$  invertible matrices by  $GL(n, \mathbb{R})$ .

$$O(n)$$

**Exercise 1.2.0.3.** Let  $A, B \in \mathbb{R}^{n \times n}$ . Then  $AB = I$  iff  $BA = I$ .

*Proof.*

- ( $\implies$ ):  
Suppose that  $AB = I$ . Then

$$\begin{aligned} \ker B &\subset \ker AB \\ &= \ker I \\ &= \{0\} \end{aligned}$$

so that  $\ker B = \{0\}$ . Hence  $\text{Im } B = \mathbb{R}^n$  and  $B$  is surjective. Then

$$\begin{aligned} IB &= BI \\ &= B(AB) \\ &= (BA)B \end{aligned}$$

Since  $B$  is surjective,  $I = BA$ .

- ( $\impliedby$ ):  
Immediate by the previous part.

□

**Definition 1.2.0.4.** Let  $A \in \mathbb{R}^{n \times p}$ . Then  $A$  is said to be an **orthogonal matrix** if  $A^*A = I$ . We denote the set of  $n \times p$  orthogonal matrices by  $O(n, p)$ . We write  $O(n)$  in place of  $O(n, n)$ .

$$O(n)$$

**Exercise 1.2.0.5.** Define  $\phi : S_n \rightarrow GL(n, \mathbb{R})$  by

$$\phi(\sigma) = \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix}$$

Then

1. for each  $A \in \mathbb{R}^{n \times p}$ ,

$$(\phi(\sigma)A)_{i,j} = A_{\sigma(i),j}$$

i.e. left multiplying  $A$  by  $\phi(\sigma)$  the the same as permuting the rows of  $A$  by  $\sigma$

2.  $\phi$  is a group homomorphism

*Proof.* 1. Let  $A \in \mathbb{R}^{n \times p}$ . Then

$$\begin{aligned} (\phi(\sigma)A)_{i,j} &= \langle e_{\sigma(i)}^*, Ae_j \rangle \\ &= A_{\sigma(i),j} \end{aligned}$$

2. Let  $\sigma, \tau \in S_n$ . Part (1) implies that

$$\begin{aligned}\phi(\sigma\tau) &= \begin{pmatrix} e_{\sigma\tau(1)}^* \\ \vdots \\ e_{\sigma\tau(n)}^* \end{pmatrix} \\ &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} \begin{pmatrix} e_{\tau(1)}^* \\ \vdots \\ e_{\tau(n)}^* \end{pmatrix} \\ &= \phi(\sigma)\phi(\tau)\end{aligned}$$

Since  $\sigma, \tau \in S_n$  are arbitrary,  $\phi$  is a group homomorphism. □

**Definition 1.2.0.6.** Define  $\phi : S_n \rightarrow GL(n, \mathbb{R})$  as in the previous exercise. Let  $P \in GL(n, \mathbb{R})$ . Then  $P$  is said to be a **permutation matrix** if there exists  $\sigma \in S_n$  such that  $P = \phi(\sigma)$ . We denote the set of  $n \times n$  permutation matrices by  $\text{Perm}(n)$ .

**Exercise 1.2.0.7.** We have that

1.  $\text{Perm}(n)$  is a subgroup of  $GL(n, \mathbb{R})$
2.  $\text{Perm}(n)$  is a subgroup of  $O(n)$

*Proof.*

1. By definition,  $\text{Perm}(n) = \text{Im } \phi$ . Since  $\phi : S_n \rightarrow GL(n, \mathbb{R})$  is a group homomorphism,  $\text{Im } \phi$  is a subgroup of  $GL(n, \mathbb{R})$ . Hence  $\text{Perm}(n)$  is a subgroup of  $GL(n, \mathbb{R})$ .
2. Let  $P \in \text{Perm}(n)$ . Then there exists  $\sigma \in S_n$  such that  $P = \phi(\sigma)$ . Then

$$\begin{aligned}PP^* &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix}^* \\ &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} (e_{\sigma(1)} \quad \cdots \quad e_{\sigma(n)}) \\ &= (\langle e_{\sigma(i)}, e_{\sigma(j)} \rangle)_{i,j} \\ &= I\end{aligned}$$

A previous exercise implies that  $P^*P = I$ . Hence  $P \in O(n)$ . Since  $P \in \text{Perm}(n)$  is arbitrary,  $\text{Perm}(n) \subset O(n)$ . Part (1) implies that  $\text{Perm}(n)$  is a group. Hence  $\text{Perm}(n)$  is a subgroup of  $O(n)$  □

**Note 1.2.0.8.** We will write  $P_\sigma$  in place of  $\phi(\sigma)$ .

**Exercise 1.2.0.9.** Let  $Z \in \mathbb{R}^{p \times n}$ . If  $\text{rank } Z = k$ , then there exist  $\sigma \in S_n$ ,  $\tau \in S_p$  and  $A \in GL(k, \mathbb{R})$ , such that for each  $i, j \in \{1, \dots, k\}$ ,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

*Proof.* Suppose that  $\text{rank } Z = k$ . Then there exist  $i_1, \dots, i_k \in \{1, \dots, p\}$  such that  $i_1 < \dots < i_k$  and  $\{e_{i_1}^* Z, \dots, e_{i_k}^* Z\}$  is linearly independent. Set

$$Z' = \begin{pmatrix} e_{i_1}^* Z \\ \vdots \\ e_{i_k}^* Z \end{pmatrix}$$

Then  $\text{rank } Z' = k$ . Hence there exist  $j_1, \dots, j_k \in \{1, \dots, n\}$  such that  $j_1 < \dots < j_k$ , and  $\{Z'e_{i_1}, \dots, Z'e_{i_k}\}$  is linearly independent. Set

$$A = (Z'e_{i_1} \quad \dots \quad Z'e_{i_k})$$

Then  $A \in \mathbb{R}^{k \times k}$  and  $\text{rank } A = k$ . Thus  $A \in GL(k, \mathbb{R})$ . Choose  $\sigma \in S_n$  and  $\tau \in S_p$  such that  $\sigma(1) = j_1, \dots, \sigma(k) = j_k$  and  $\tau(1) = i_1, \dots, \tau(k) = i_k$ . Let  $a, b \in \{1, \dots, k\}$ . By construction,

$$\begin{aligned} (P_\tau Z P_\sigma^*)_{a,b} &= Z_{\tau(a), \sigma(b)} \\ &= Z_{i_a, j_b} \\ &= A_{a,b} \end{aligned}$$

□

**Definition 1.2.0.10.** Let  $A \in \mathbb{R}^{n \times p}$ . Then  $A$  is said to be a **diagonal matrix** if for each  $i \in [n]$  and  $j \in [p]$ ,  $i \neq j$  implies that  $A_{i,j} = 0$ . We denote the set of  $n \times p$  diagonal matrices by  $D(n, p, \mathbb{R})$ . We write  $D(n, \mathbb{R})$  in place of  $D(n, n, \mathbb{R})$ .

**Definition 1.2.0.11.** For  $(n, k), (m, l)$   $\text{diag}_{p, (n \times p)} : \mathbb{R}^p \rightarrow \mathbb{R}^{n \times p}$  and  $\text{diag}_{n, (n \times p)} : \mathbb{R}^p \rightarrow \mathbb{R}^{n \times p}$  by  $\text{diag}(v)$   
**FINISH!!!**

**Definition 1.2.0.12.** Let  $A \in \mathbb{R}^{n \times n}$  and  $\lambda \in \sigma(A)$ . Suppose that  $A$  is symmetric. We define the **geometric multiplicity** of  $\lambda$ , denoted  $\mu(\lambda)$ , by

$$\mu(\lambda) = \dim \ker([\phi_\alpha] - \lambda I)$$

**Definition 1.2.0.13.** Let  $V$  be an  $n$ -dimensional vector space,  $U \subset V$  a  $k$ -dimensional subspace and  $(e_j)_{j=1}^n \subset V$  a basis. Then  $(e_j)_{j=1}^n$  is said to be **adapted to**  $U$  if  $(e_j)_{j=1}^k$  is a basis for  $U$ .

## 1.3 Calculus

### 1.3.1 Differentiation

**Definition 1.3.1.1.** Let  $n \geq 1$ . For  $i = 1, \dots, n$ , define  $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $x^i(a^1, \dots, a^n) = a^i$ . The functions  $(x^i)_{i=1}^n$  are called the **standard coordinate functions on  $\mathbb{R}^n$** .

**Definition 1.3.1.2.** Let  $U \subset \mathbb{R}^n$  be open,  $f : U \rightarrow \mathbb{R}$  and  $a \in U$ . Then  $f$  is said to be **differentiable with respect to  $x^i$  at  $a$**  if

$$\lim_{h \rightarrow 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If  $f$  is differentiable with respect to  $x^i$  at  $a$ , we define the **partial derivative of  $f$  with respect to  $x^i$  at  $a$** , denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \frac{\partial}{\partial x^i} f$$

to be the limit above.

**Definition 1.3.1.3.** Let  $U \subset \mathbb{R}^n$  be open and  $f : U \rightarrow \mathbb{R}$ . Then  $f$  is said to be **differentiable with respect to  $x^i$**  if for each  $a \in U$ ,  $f$  is differentiable with respect to  $x^i$  at  $a$ .

**Exercise 1.3.1.4.** Let  $U \subset \mathbb{R}^n$  be open,  $f : U \rightarrow \mathbb{R}$  and  $a \in U$ . Suppose that  $\frac{\partial^2 f}{\partial x^i \partial x^j}$  and  $\frac{\partial^2 f}{\partial x^j \partial x^i}$  exist and are continuous at  $a$ . Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(a) = \frac{\partial^2 f}{\partial x^j \partial x^i}(a)$$

*Proof.* □

**Definition 1.3.1.5.** Let  $U \subset \mathbb{R}^n$  be open and  $f : U \rightarrow \mathbb{R}$ . Then  $f$  is said to be **smooth** if for each  $i_1, \dots, i_k \in \{1, \dots, n\}$ ,  $\frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}$  exists and is continuous on  $U$ .

**Definition 1.3.1.6.** Let  $U \subset \mathbb{R}^n$ ,  $f : U \rightarrow \mathbb{R}$ . Then  $f$  is said to be **smooth** if there exists  $U' \subset \mathbb{R}^n$  and  $f' : U' \rightarrow \mathbb{R}$  such that  $U \subset U'$ ,  $U'$  is open,  $f'|_U = f$  and  $f'$  is smooth. The set of smooth functions on  $U$  is denoted  $C^\infty(U)$ .

**Theorem 1.3.1.7. Taylor's Theorem:**

Let  $U \subset \mathbb{R}^n$  be open and convex,  $p \in U$ ,  $f \in C^\infty(U)$  and  $T \in \mathbb{N}$ . Then there exist  $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$  such that for each  $x \in U$ ,

$$f(x) = \sum_{k=0}^T \left[ \sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each  $|\alpha| = T+1$ ,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

*Proof.* See analysis notes □

**Definition 1.3.1.8.** Let  $U \subset \mathbb{R}^n$  and  $F : U \rightarrow \mathbb{R}^m$ . Let  $x^1, \dots, x^n$  be the standard coordinate functions on  $\mathbb{R}^n$  and  $y_1, \dots, y_m$  be the standard coordinate functions on  $\mathbb{R}^m$ . For  $i \in \{1, \dots, m\}$ , we define the  **$i$ th component of  $F$** , denoted  $F^i : U \rightarrow \mathbb{R}$ , by

$$F^i = y^i \circ F$$

Thus  $F = (F_1, \dots, F_m)$

**Definition 1.3.1.9.** Let  $U \subset \mathbb{R}^n$  be open and  $F : U \rightarrow \mathbb{R}^m$ . Then  $F$  is said to be **smooth** if for each  $i \in \{1, \dots, m\}$ , the  $i$ th component of  $F$ ,  $F^i : U \rightarrow \mathbb{R}$ , is smooth.

**Definition 1.3.1.10.** Let  $U \subset \mathbb{R}^n$  and  $F : U \rightarrow \mathbb{R}^m$ . Then  $F$  is said to be **smooth** if for each  $x \in U$ , there exists  $U_x \in \mathcal{N}_x$  and  $\tilde{F} : U_x \rightarrow \mathbb{R}^m$  such that  $U_x$  is open,  $\tilde{F}$  is smooth and  $\tilde{F}|_{U \cap U_x} = F|_{U \cap U_x}$ .

**Definition 1.3.1.11.** Let  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  and  $F : U \rightarrow V$ . Then  $F$  is said to be a **diffeomorphism** if  $F$  is a bijection and  $F, F^{-1}$  are smooth.

**Exercise 1.3.1.12.** Let  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  and  $F : U \rightarrow V$ . If  $F$  is a diffeomorphism, then  $F$  is a homeomorphism.

*Proof.* Suppose that  $F$  is a diffeomorphism. By definition,  $F$  is a bijection and  $F$  and  $F^{-1}$  are smooth. Thus,  $F$  and  $F^{-1}$  are continuous and  $F$  is a homeomorphism.  $\square$

**Definition 1.3.1.13.** Let  $U \subset \mathbb{R}^n$  be open,  $p \in U$  and  $F : U \rightarrow \mathbb{R}^m$ . We define the **Jacobian of  $F$  at  $p$** , denoted  $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$ , by

$$\left( \frac{\partial F}{\partial x}(p) \right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

**Exercise 1.3.1.14. Inverse Function Theorem:**

Let  $U, V \subset \mathbb{R}^n$  be open and  $F : U \rightarrow V$ .

**Exercise 1.3.1.15.** Let  $U, V \subset \mathbb{R}^n$  and  $F : U \rightarrow V$ . Then  $F$  is a diffeomorphism iff for each  $p \in U$ , there exists a relatively open neighborhood  $N \subset U$  of  $p$  such that  $F|_N : N \rightarrow F(N)$  is a diffeomorphism

*Proof.* content...  $\square$

**Exercise 1.3.1.16.** Let  $\sigma \in S_n$ . Define  $\phi_\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $\phi(x^1, \dots, x^n) = \phi(x^{\sigma(1)}, \dots, x^{\sigma(n)})$ . Then  $D\phi = P_\sigma$

**Definition 1.3.1.17.** Let  $\sigma \in S_n$  and  $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ . We define  $\sigma x \in \mathbb{R}^n$  by

$$\sigma x = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of  $S_n$  on  $\mathbb{R}^n$  to be the map  $S_n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $(\sigma, x) \mapsto \sigma x$

**Definition 1.3.1.18.** Let  $\sigma \in S_n$ ,  $U$  a set,  $V \subset \mathbb{R}^n$  and  $\phi : U \rightarrow \mathbb{R}^n$  with  $\phi = (x^1, \dots, x^n)$ . We define  $\sigma\phi : U \rightarrow \mathbb{R}^n$  by

$$\sigma\phi = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of  $S_n$  on  $(\mathbb{R}^n)^U$  to be the map  $S_n \times (\mathbb{R}^n)^U \rightarrow (\mathbb{R}^n)^U$  given by  $(\sigma, \phi) \mapsto \sigma\phi$ .

**Exercise 1.3.1.19.** Let  $\sigma \in S_m$ . Then for each  $p \in \mathbb{R}^n$ ,  $D(\sigma \text{id}_{\mathbb{R}^n})(p) = P_\sigma$ .

*Proof.* Note that since  $\text{id}_{\mathbb{R}^n} = (\pi_1, \dots, \pi_n)$ , we have that  $\sigma \text{id}_{\mathbb{R}^n} = (\pi_{\sigma(1)}, \dots, \pi_{\sigma(n)})$ . Let  $p \in \mathbb{R}^n$ . Then

$$\begin{aligned} D(\sigma \text{id}_{\mathbb{R}^n})(p) &= \left( \frac{\partial \pi_i \circ \sigma \text{id}_{\mathbb{R}^n}}{\partial x^j}(p) \right)_{i,j} \\ &= \left( \frac{\partial \pi_{\sigma(i)}}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma \left( \frac{\partial \pi_i}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma \left( \frac{\partial \pi_i \circ \text{id}_{\mathbb{R}^n}}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma D \text{id}_{\mathbb{R}^n}(p) \\ &= P_\sigma I \\ &= P_\sigma \end{aligned}$$

$\square$

## 1.3.2 Integration



## 1.4 Topology

**Definition 1.4.0.1.** Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is said to be **continuous** if for each  $U \in \mathcal{T}$ ,  $f^{-1}(U) \in \mathcal{T}_X$ .

**Definition 1.4.0.2.** Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is said to be a **homeomorphism** if  $f$  is a bijection and  $f, f^{-1}$  are continuous.

**Definition 1.4.0.3.** Let  $X, Y$  be topological spaces. Then  $X$  and  $Y$  are said to be **homeomorphic** if there exists  $f : X \rightarrow Y$  such that  $f$  is a homeomorphism. If  $X$  and  $Y$  are homeomorphic, we write  $X \cong Y$ .

**Theorem 1.4.0.4.** Let  $m, n \in \mathbb{N}$ . If  $m \neq n$ , then  $\mathbb{R}^m \not\cong \mathbb{R}^n$



## Chapter 2

# Multilinear Algebra

### 2.1 $(r, s)$ -Tensors

**Definition 2.1.0.1.** Let  $V_1, \dots, V_k, W$  be vector spaces and  $\alpha : \prod_{i=1}^n V_i \rightarrow W$ . Then  $\alpha$  is said to be **multilinear** if for each  $i \in \{1, \dots, k\}$ ,  $v \in V$ ,  $c \in \mathbb{R}$  and  $v_1, \dots, v_k \in V$ ,

$$\alpha(v_1, \dots, v_i + cv, \dots, v_k) = \alpha(v_1, \dots, v_i, \dots, v_k) + c\alpha(v_1, \dots, v, \dots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{ \alpha : \prod_{i=1}^n V_i \rightarrow W : \alpha \text{ is multilinear} \right\}$$

**Note 2.1.0.2.** For the remainder of this section we let  $V$  denote an  $n$ -dimensional vector space with basis  $\{e^1, \dots, e^n\}$  with dual space  $V^*$  and dual basis  $\{\epsilon_1, \dots, \epsilon_n\}$  defined by  $\epsilon^i(e^j) = \delta_{i,j}$ . We identify  $V$  with  $V^{**}$  by the isomorphism  $V \rightarrow V^{**}$  defined by  $v \mapsto \hat{v}$  where  $\hat{v}(\alpha) = \alpha(v)$  for each  $\alpha \in V^*$ .

**Definition 2.1.0.3.** Let  $\alpha : (V^*)^r \times V^s \rightarrow \mathbb{R}$ . Then  $\alpha$  is said to be an  $(r, s)$ -tensor on  $V$  if  $\alpha \in L(\underbrace{V^*, \dots, V^*}_r, \underbrace{V, \dots, V}_s; \mathbb{R})$ . The set of all  $(r, s)$ -tensors on  $V$  is denoted  $T_s^r(V)$ .

When  $r = s = 0$ , we set  $T_s^r = \mathbb{R}$ .

**Exercise 2.1.0.4.** We have that  $T_s^r(V)$  is a vector space.

*Proof.* Clear. □

**Exercise 2.1.0.5.** Under the identification of  $V$  with  $V^{**}$  as noted above, we have that  $V = T_0^1(V)$ .

*Proof.* By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T_0^1(V) \end{aligned}$$

□

**Definition 2.1.0.6.** Let  $\alpha \in T_{s_1}^{r_1}(V)$  and  $\beta \in T_{s_2}^{r_2}(V)$ . We define the **tensor product of  $\alpha$  with  $\beta$** , denoted  $\alpha \otimes \beta \in T_{s_1+s_2}^{r_1+r_2}(V)$ , by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each  $v^* \in (V^*)^{r_1}$ ,  $w^* \in (V^*)^{r_2}$ ,  $v \in V^{s_1}$  and  $w \in V^{s_2}$ .

When  $r_1 = s_1 = r_2 = s_2 = 0$  (so that  $\alpha, \beta \in \mathbb{R}$ ), we set  $\alpha \otimes \beta = \alpha\beta$ .

**Definition 2.1.0.7.** We define the **tensor product**, denoted  $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$  by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

**Exercise 2.1.0.8.** The tensor product  $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$  is well defined.

*Proof.* Tedious but straightforward. □

**Exercise 2.1.0.9.** The tensor product  $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$  is associative.

*Proof.* Let  $\alpha \in T_{s_1}^{r_1}(V)$ ,  $\beta \in T_{s_2}^{r_2}(V)$  and  $\gamma \in T_{s_3}^{r_3}(V)$ . Then for each  $u^* \in (V^*)^{r_1}$ ,  $v^* \in (V^*)^{r_2}$ ,  $w^* \in (V^*)^{r_3}$ ,  $u \in V^{s_1}$ ,  $v \in V^{s_2}$ ,  $w \in V^{s_3}$ ,

$$\begin{aligned} (\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v) \gamma(w^*, w) \\ &= [\alpha(u^*, u) \beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(u^*, u) [\beta(v^*, v) \gamma(w^*, w)] \\ &= \alpha(u^*, u) (\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w) \end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

**Exercise 2.1.0.10.** The tensor product  $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$  is bilinear.

*Proof.*

1. Linearity in the first argument:

Let  $\alpha, \beta \in T_{s_1}^{r_1}(V)$ ,  $\gamma \in T_{s_2}^{r_2}(V)$ ,  $\lambda \in \mathbb{R}$ ,  $v^* \in (V^*)^{r_1}$ ,  $w^* \in (V^*)^{r_2}$ ,  $v \in V^{s_1}$  and  $w \in V^{s_2}$ . To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned} [(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v) \gamma(w^*, w) \\ &= [\alpha(v^*, v) + \lambda\beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(v^*, v) \gamma(w^*, w) + \lambda\beta(v^*, v) \gamma(w^*, w) \\ &= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w) \end{aligned}$$

So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

2. Linearity in the second argument:

Similar to (1).

□

**Definition 2.1.0.11.**

1. Define  $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1, \dots, i_k \leq n\}$ . Each element  $I \in \mathcal{I}_k$  is called an **unordered multi-index of length  $k$** . Recall that  $\#\mathcal{I}_{\otimes k} = n^k$ .
2. Define  $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$ . Each element  $I \in \mathcal{I}_k$  is called an **ordered multi-index of length  $k$** . Recall that  $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$ .

**Note 2.1.0.12.** For the remainder of this section we will write  $\mathcal{I}_k$  in place of  $\mathcal{I}_{\otimes k}$ .

**Definition 2.1.0.13.** Let  $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k$ .

1. Define  $\epsilon^I \in (V^*)^k$  and  $e_I \in V^k$  by

$$\epsilon^I = (\epsilon^{i_1}, \dots, \epsilon^{i_k})$$

and

$$e^I = (e^{i_1}, \dots, e^{i_k})$$

2. Define  $e^{\otimes I} \in T_0^k(V)$  and  $\epsilon^{\otimes I} \in T_k^0(V)$  by

$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$

and

$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

**Exercise 2.1.0.14.** Let  $\alpha, \beta \in T_s^r(V)$ . If for each  $I \in \mathcal{I}_r, J \in \mathcal{I}_s$ ,  $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$ , then  $\alpha = \beta$ .

*Proof.* Suppose that for each  $I \in \mathcal{I}_r, J \in \mathcal{I}_s$ ,  $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$ . Let  $v_1^*, \dots, v_r^* \in V^*$  and  $v_1, \dots, v_s \in V$ . For each  $i \in \{1, \dots, r\}$  and  $j \in \{1, \dots, s\}$ , write

$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$

and

$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$

Then

$$\begin{aligned} \alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s) \end{aligned}$$

So that  $\alpha = \beta$ . □

**Exercise 2.1.0.15.** Let  $I, K \in \mathcal{I}_r$  and  $J, L \in \mathcal{I}_s$ . Then  $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K} \delta_{J,L}$ .

*Proof.* Write  $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$  and  $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$ . Then

$$\begin{aligned} e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K) \epsilon^{\otimes J}(e^L) \\ &= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r}) \epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\ &= \left[ \prod_{m=1}^r e^{i_m}(\epsilon^{k_m}) \right] \left[ \prod_{n=1}^s \epsilon^{j_n}(e^{l_n}) \right] \\ &= \left[ \prod_{m=1}^r \delta_{i_m, k_m} \right] \left[ \prod_{n=1}^s \delta_{j_n, l_n} \right] \\ &= \delta_{I,K} \delta_{J,L} \end{aligned}$$

□

**Exercise 2.1.0.16.** The set  $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$  is a basis for  $T_s^r(V)$  and  $\dim T_s^r(V) = n^{r+s}$ .

*Proof.* Let  $(a_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ . Let  $\alpha = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a_J^I e^{\otimes I} \otimes \epsilon^{\otimes J}$ . Suppose that  $\alpha = 0$ . Then for each  $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$ ,  $\alpha(\epsilon^I, e^J) = a_J^I = 0$ . Thus  $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$  is linearly independent. Let  $\beta \in T_s^r(V)$ . For  $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$ , put  $b_J^I = \beta(\epsilon^J, e^I)$ . Define  $\mu = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b_J^I e^{\otimes I} \otimes \epsilon^{\otimes J} \in T_s^r(V)$ . Then for each  $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$ ,  $\mu(\epsilon^I, e^J) = b_J^I = \beta(\epsilon^I, e^J)$ . Hence  $\mu = \beta$  and therefore  $\beta \in \text{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$ . □

## 2.2 Covariant $k$ -Tensors

### 2.2.1 Symmetric and Alternating Covariant $k$ -Tensors

**Definition 2.2.1.1.** Let  $\alpha : V^k \rightarrow \mathbb{R}$ . Then  $\alpha$  is said to be a **covariant  $k$ -tensor on  $V$**  if  $\alpha \in T_k^0(V)$ . We denote the set of covariant  $k$ -tensors by  $T_k(V)$ .

**Definition 2.2.1.2.** For  $\sigma \in S_k$  and  $\alpha \in T_k(V)$ , define the  $\sigma\alpha : V^k \rightarrow \mathbb{R}$  by

$$\sigma\alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

We define the **permutation action** of  $S_k$  on  $T_k(V)$  to be the map  $S_k \times T_k(V) \rightarrow T_k(V)$  given by  $(\sigma, \alpha) \mapsto \sigma\alpha$

**Exercise 2.2.1.3.** The permutation action of  $S_k$  on  $T_k(V)$  is a group action.

*Proof.*

1. Clearly for each  $\sigma \in S_k$  and  $\alpha \in T_k(V)$ ,  $\sigma\alpha \in T_k(V)$ .
2. Clearly for each  $\alpha \in T_k(V)$ ,  $e\alpha = \alpha$ .
3. Let  $\tau, \sigma \in S_k$  and  $\alpha \in T_k(V)$ . Then for each  $v_1, \dots, v_k \in V$ ,

$$\begin{aligned} (\tau\sigma)\alpha(v_1, \dots, v_k) &= \alpha(v_{\tau\sigma(1)}, \dots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \dots, v_k) \end{aligned}$$

□

**Exercise 2.2.1.4.** Let  $\sigma \in S_k$ . Then  $L_\sigma : T_k(V) \rightarrow T_k(V)$  given by  $L_\sigma(\alpha) = \sigma\alpha$  is a linear transformation.

*Proof.* Let  $\alpha, \beta \in T_k(V)$ ,  $c \in \mathbb{R}$  and  $v_1, \dots, v_k \in V$ . Then

$$\begin{aligned} \sigma(c\alpha + \beta)(v_1, \dots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \dots, v_k) + \sigma\beta(v_1, \dots, v_k) \end{aligned}$$

So  $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$ .

□

**Definition 2.2.1.5.** Let  $\alpha \in T_k(V)$ . Then  $\alpha$  is said to be

- **symmetric** if for each  $\sigma \in S_k$ ,  $\sigma\alpha = \alpha$
- **antisymmetric** if for each  $\sigma \in S_k$ ,  $\sigma\alpha = \text{sgn}(\sigma)\alpha$
- **alternating** if for each  $v_1, \dots, v_k \in V$ , if there exists  $i, j \in \{1, \dots, k\}$  such that  $v_i = v_j$ , then  $\alpha(v_1, \dots, v_k) = 0$ .

We denote the set of symmetric  $k$ -tensors on  $V$  by  $\Sigma^k(V)$ . We denote the set of alternating  $k$ -tensors on  $V$  by  $\Lambda^k(V)$ .

**Exercise 2.2.1.6.** Let  $\alpha \in T_k(V)$ . Then  $\alpha$  is antisymmetric iff  $\alpha$  is alternating.

*Proof.* Suppose that  $\alpha$  is antisymmetric. Let  $v_1, \dots, v_k \in V$ . Suppose that there exists  $i, j \in \{1, \dots, k\}$  such that  $v_i = v_j$ . Define  $\sigma \in S_k$  by  $\sigma = (i, j)$ . Then

$$\begin{aligned} \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) &= \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ &= \sigma(\alpha)(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\ &= \text{sgn}(\sigma)\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\ &= -\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \end{aligned}$$

Therefore  $2\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = 0$  which implies that  $\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = 0$ . Hence  $\alpha$  is alternating.

Conversely, suppose that  $\alpha$  is alternating. Let  $i, j \in \{1, \dots, k\}$  and  $v_1, \dots, v_k \in V$ . Then

$$\begin{aligned} 0 &= \alpha(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) \\ &= \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \end{aligned}$$

Since  $i, j \in \{1, \dots, k\}$  and  $v_1, \dots, v_k \in V$  are arbitrary, we have that for each  $\tau \in S_k$ ,  $\tau$  is a transposition implies that

$$\begin{aligned} \tau\alpha &= -\alpha \\ &= \text{sgn}(\tau)\alpha \end{aligned}$$

Let  $n \in \mathbb{N}$ . Suppose that for each  $\tau_1, \dots, \tau_{n-1} \in S_k$  if for each  $j \in \{1, \dots, n-1\}$ ,  $\tau_j$  is a transposition, then  $(\tau_1 \cdots \tau_{n-1})\alpha = \text{sgn}(\tau_1 \cdots \tau_{n-1})\alpha$ . Let  $\tau_1, \dots, \tau_n \in S_k$ . Suppose that for each  $j \in \{1, \dots, n\}$ ,  $\tau_j$  is a transposition. Then

$$\begin{aligned} (\tau_1 \cdots \tau_n)\alpha &= (\tau_1 \cdots \tau_{n-1})(\tau_n\alpha) \\ &= (\tau_1 \cdots \tau_{n-1})(\text{sgn}(\tau_n)\alpha) \\ &= (\text{sgn}(\tau_n)(\tau_1 \cdots \tau_{n-1})\alpha) \\ &= (\text{sgn}(\tau_n) \text{sgn}(\tau_1 \cdots \tau_{n-1})\alpha) \\ &= \text{sgn}(\tau_1 \cdots \tau_n)\alpha \end{aligned}$$

By induction, for each  $n \in \mathbb{N}$  and  $\tau_1, \dots, \tau_n \in S_k$ , if for each  $j \in \{1, \dots, n\}$ ,  $\tau_j$  is a transposition, then  $(\tau_1 \cdots \tau_n)\alpha = \text{sgn}(\tau_1 \cdots \tau_n)\alpha$ . Now let  $\sigma \in S_k$ . Then there exist  $n \in \mathbb{N}$  and  $\tau_1, \dots, \tau_n \in S_k$  such that  $\sigma = \tau_1 \cdots \tau_n$  and for each  $j \in \{1, \dots, n\}$ ,  $\tau_j$  is a transposition. Hence

$$\begin{aligned} \sigma\alpha &= (\tau_1 \cdots \tau_n)\alpha \\ &= \text{sgn}(\tau_1 \cdots \tau_n)\alpha \\ &= \text{sgn}(\sigma)\alpha \end{aligned}$$

Therefore  $\alpha$  is antisymmetric. □

**Definition 2.2.1.7.** Define the **symmetric operator**  $S : T_k(V) \rightarrow \Sigma^k(V)$  by

$$\text{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator**  $A : T_k(V) \rightarrow \Lambda^k(V)$  by

$$\text{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma\alpha$$

**Exercise 2.2.1.8.**

1. For  $\alpha \in T_k(V)$ ,  $\text{Sym}(\alpha)$  is symmetric.
2. For  $\alpha \in T_k(V)$ ,  $\text{Alt}(\alpha)$  is alternating.

*Proof.*

1. Let  $\alpha \in T_k(V)$  and  $\sigma \in S_k$ . Then

$$\begin{aligned} \sigma \text{Sym}(\alpha) &= \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \\ &= \text{Sym}(\alpha) \end{aligned}$$

2. Let  $\alpha \in T_k(V)$  and  $\sigma \in S_k$ . Then

$$\begin{aligned}
 \sigma \operatorname{Alt}(\alpha) &= \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \right] \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \sigma \tau \alpha \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \operatorname{Alt}(\alpha)
 \end{aligned}$$

□

**Exercise 2.2.1.9.**

1. For  $\alpha \in \Sigma^k(V)$ ,  $\operatorname{Sym}(\alpha) = \alpha$ .
2. For  $\alpha \in \Lambda^k(V)$ ,  $\operatorname{Alt}(\alpha) = \alpha$ .

*Proof.*

1. Let  $\alpha \in \Sigma^k(V)$ . Then

$$\begin{aligned}
 \operatorname{Sym}(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} \alpha \\
 &= \alpha
 \end{aligned}$$

2. Let  $\alpha \in \Lambda^k(V)$ . Then

$$\begin{aligned}
 \operatorname{Alt}(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma \alpha \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2 \alpha \\
 &= \alpha
 \end{aligned}$$

□

**Exercise 2.2.1.10.** The symmetric operator  $S : T_k(V) \rightarrow \Sigma^k(V)$  and the alternating operator  $A : T_k(V) \rightarrow \Lambda^k(V)$  are linear.

*Proof.* Clear.

□

**Exercise 2.2.1.11.** Let  $\alpha \in T_k(V)$  and  $\beta \in T_l(V)$ . Then

1.  $\operatorname{Alt}(\operatorname{Alt}(\alpha) \otimes \beta) = \operatorname{Alt}(\alpha \otimes \beta)$
2.  $\operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta)) = \operatorname{Alt}(\alpha \otimes \beta)$



*Proof.* First note that if we fix  $\mu \in S_{k+1}$ , then for each  $\tau \in S_k$ , choosing  $\sigma = \mu\tau^{-1}$  yields  $\sigma\tau = \mu$ . For each  $\mu \in S_{k+l}$ , the map  $\phi_\mu : S_k \rightarrow S_{k+l}$  given by  $\phi_\mu(\tau) = \mu\tau^{-1}$  is injective. Thus for each  $\mu \in S_{k+l}$ , we have that  $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

1. Then

$$\begin{aligned}
 \text{Alt}(\text{Alt}(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[ \text{Alt}(\alpha) \otimes \beta \right] \\
 &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[ \left( \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \right) \otimes \beta \right] \\
 &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) (\tau \alpha) \otimes \beta \right] \\
 &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau (\alpha \otimes \beta) \right] \\
 &= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \text{sgn}(\sigma\tau) \sigma\tau (\alpha \otimes \beta) \\
 &= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
 &= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
 &= \text{Alt}(\alpha \otimes \beta)
 \end{aligned}$$

2. Similar to (1).

□

### 2.2.2 Exterior Product

**Definition 2.2.2.1.** Let  $\alpha \in \Lambda^k(V)$  and  $\beta \in \Lambda^l(V)$ . The **exterior product** of  $\alpha$  and  $\beta$  is defined to be the map  $\alpha \wedge \beta \in \Lambda^{k+l}(V)$  given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta)$$

Thus  $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$ .

**Exercise 2.2.2.2.** The exterior product  $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$  is bilinear.

*Proof.* Clear.

□

**Exercise 2.2.2.3.** The exterior product  $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$  is associative.

*Proof.* Let  $\alpha \in \Lambda^k(V)$ ,  $\beta \in \Lambda^l(V)$  and  $\gamma \in \Lambda^m(V)$ . Then

$$\begin{aligned}
 (\alpha \wedge \beta) \wedge \gamma &= \left[ \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \right] \wedge \gamma \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \text{Alt} \left( \left[ \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \right] \otimes \gamma \right) \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} \text{Alt}(\text{Alt}(\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} \text{Alt}((\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \text{Alt}(\alpha \otimes (\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \text{Alt}(\alpha \otimes \text{Alt}(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \text{Alt}(\alpha \otimes \frac{(l+m)!}{l!m!} \text{Alt}(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \text{Alt}(\alpha \otimes (\beta \wedge \gamma)) \\
 &= \alpha \wedge (\beta \wedge \gamma)
 \end{aligned}$$

□

**Exercise 2.2.2.4.** Let  $\alpha_i \in \Lambda^{k_i}(V)$  for  $i = 1, \dots, m$ . Then

$$\bigwedge_{i=1}^m \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} \text{Alt} \left( \bigotimes_{i=1}^m \alpha_i \right)$$

*Proof.* To see that the statement is true in the case  $m = 3$ , the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} \text{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each  $3 \leq m \leq m_0$ . Then the proof of the previous exercise tells us the

$$\begin{aligned}
 \bigwedge_{i=1}^{m_0+1} \alpha_i &= \left( \bigwedge_{i=1}^{m_0-1} \alpha_i \right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
 &= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \text{Alt} \left( \left[ \bigwedge_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \text{Alt} \left( \left[ \frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} \text{Alt} \left( \bigotimes_{i=1}^{m_0-1} \alpha_i \right) \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left( \text{Alt} \left[ \bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left( \left[ \bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left( \bigotimes_{i=1}^{m_0+1} \alpha_i \right)
 \end{aligned}$$

□

**Exercise 2.2.2.5.** Define  $\tau \in S_{k+l}$  by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of  $\tau$  is  $kl$ . (Hint: inversion number)

*Proof.*

$$\begin{aligned} N(\tau) &= \sum_{i=1}^l k \\ &= kl \end{aligned}$$

Since  $\text{sgn}(\tau) = (-1)^{N(\tau)}$  we know that  $\text{sgn}(\tau) = (-1)^{kl}$ . □

**Exercise 2.2.2.6.** Let  $\alpha \in \Lambda^k(V)$ ,  $\beta \in \Lambda^l(V)$ . Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

*Proof.* Define  $\tau \in S_{k+l}$  as in the previous exercise. Note that For  $\sigma \in S_{k+l}$  and  $v_1, \dots, v_{k+l} \in V$ , we have that

$$\begin{aligned} \sigma\tau(\beta \otimes \alpha)(v_1, \dots, v_l, v_{l+1}, \dots, v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)})\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\ &= \alpha \otimes \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}, v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\ &= \sigma(\alpha \otimes \beta)(v_1, \dots, v_k, v_{1+k}, \dots, v_{l+k}) \end{aligned}$$

Thus  $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$ . Then

$$\begin{aligned} \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} \text{Alt}(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau) \sigma\tau(\beta \otimes \alpha) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha \otimes \beta) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \\ &= \text{sgn}(\tau) \alpha \wedge \beta \\ &= (-1)^{kl} \alpha \wedge \beta \end{aligned}$$

□

**Exercise 2.2.2.7.** Let  $\alpha \in \Lambda^k(V)$ . If  $k$  is odd, then  $\alpha \wedge \alpha = 0$ .

*Proof.* Suppose that  $k$  is odd. The previous exercise tells us that

$$\begin{aligned} \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\ &= -\alpha \wedge \alpha \end{aligned}$$

Thus  $\alpha \wedge \alpha = 0$ . □

**Exercise 2.2.2.8. Fundamental Example:**

Let  $\alpha_1, \dots, \alpha_m \in \Lambda^1(V)$  and  $v_1, \dots, v_m \in V$ . Then

$$\left( \bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) = \det(\alpha_i(v_j))$$

*Proof.* The previous exercises tell us that

$$\begin{aligned} \left( \bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) &= m! \operatorname{Alt} \left( \bigotimes_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) \\ &= m! \left[ \frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \sigma \left( \bigotimes_{i=1}^m \alpha_i \right) \right] (v_1, \dots, v_m) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \left( \bigotimes_{i=1}^m \alpha_i \right) (v_{\sigma(1)}, \dots, v_{\sigma(m)}) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\ &= \det(\alpha_i(v_j)) \end{aligned}$$

□

**Note 2.2.2.9.** Recall that  $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$  and that  $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$ . For the remainder of this section, we will write  $\mathcal{I}_k$  in place of  $\mathcal{I}_{\wedge k}$ .

**Definition 2.2.2.10.** Let  $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k\}$ . Define  $\epsilon^{\wedge I} \in \Lambda^k(V)$  by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \dots \wedge \epsilon^{i_k}$$

**Exercise 2.2.2.11.** Let  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_k) \in \mathcal{I}_k$ . Then  $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$ .

*Proof.* Put  $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \dots & \epsilon^{i_1}(e^{j_k}) \\ \vdots & & \vdots \\ \epsilon^{i_k}(e^{j_1}) & \dots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$ . A previous exercise tells us that  $\epsilon^{\wedge I}(e^J) = \det A$ . If  $I = J$ , then

$A = I_{k \times k}$  and therefore  $\epsilon^{\wedge I}(e^J) = 1$ . Suppose that  $I \neq J$ . Put  $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$ . If  $i_{l_0} < j_{l_0}$ , then all entries on the  $l_0$ -th row of  $A$  are 0. If  $i_{l_0} > j_{l_0}$ , then all entries on the  $l_0$ -th column of  $A$  are 0. □

**Exercise 2.2.2.12.** Let  $\alpha, \beta \in \Lambda^k(V)$ . If for each  $I \in \mathcal{I}_k$ ,  $\alpha(e^I) = \beta(e^I)$ , then  $\alpha = \beta$ .

*Proof.* Suppose that for each  $I \in \mathcal{I}_k$ ,  $\alpha(e^I) = \beta(e^I)$ . Let  $v_1, \dots, v_k \in V$ . For  $i = 1, \dots, k$ , write  $v_i =$

$\sum_{j_i=1}^n a_{i,j_i} e^{j_i}$ . Then

$$\begin{aligned}
 \alpha(v_1, \dots, v_k) &= \sum_{j_1, \dots, j_k=1}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\
 &= \sum_{j_1 \neq \dots \neq j_k}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\
 &= \sum_{J \in \mathcal{I}_k} \left[ \sum_{\sigma \in S_J} \text{sgn}(\sigma) \left( \prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \alpha(e^J) \\
 &= \sum_{J \in \mathcal{I}_k} \left[ \sum_{\sigma \in S_J} \text{sgn}(\sigma) \left( \prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \beta(e^J) \\
 &= \sum_{j_1, \dots, j_k=1}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \beta(e^{j_1}, \dots, e^{j_k}) \\
 &= \beta(v_1, \dots, v_k)
 \end{aligned}$$

□

**Exercise 2.2.2.13.** The set  $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$  is a basis for  $\Lambda^k(V)$  and  $\dim \Lambda^k(V) = \binom{n}{k}$ .

*Proof.* Let  $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$ . Let  $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$ . Suppose that  $\alpha = 0$ . Then for each  $J \in \mathcal{I}_k$ ,  $\alpha(e^J) = a_J = 0$ .

Thus  $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$  is linearly independent. Let  $\beta \in \Lambda^k(V)$ . For  $I \in \mathcal{I}_k$ , put  $b_I = \beta(e^I)$ . Define  $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda^k(V)$ . Then for each  $J \in \mathcal{I}_k$ ,  $\mu(e^J) = b_J = \beta(e^J)$ . Hence  $\mu = \beta$  and therefore  $\beta \in \text{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ .

□

### 2.2.3 Interior Product

**Definition 2.2.3.1.** Let  $V$  be a finite dimensional vector space and  $v \in V$ . We define **interior multiplication by  $v$** , denoted  $\iota_v : T_k \rightarrow T_{k-1}$ , by

$$\iota_v \alpha(w_1, \dots, w_{k-1}) = \alpha(v, w_1, \dots, w_{k-1})$$

**Exercise 2.2.3.2.** Let  $V$  be a finite dimensional vector space and  $v \in V$ . Then  $\iota_v|_{\Lambda^k(V)} : \Lambda^k(V) \rightarrow \Lambda^{k-1}(V)$ .

*Proof.* Let  $\alpha \in \Lambda^k(V)$ . Define  $\beta \in \Lambda^k(V)$  by  $\beta(w_1, \dots, w_k) = \alpha(w_k, w_1, \dots, w_{k-1})$ . Let  $\sigma \in S_{k-1}$ . Define  $\tau \in S_k$  by  $\tau(j) = \begin{cases} 1 & j = k \\ \sigma(j) & j \neq k \end{cases}$ . Let  $w_1, \dots, w_{k-1} \in V$ . Set  $w_k = v$ . Then

$$\begin{aligned}
 \sigma(\iota_v \alpha)(w_1, \dots, w_{k-1}) &= \iota_v \alpha(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}) \\
 &= \alpha(v, w_{\sigma(1)}, \dots, w_{\sigma(k-1)}) \\
 &= \beta(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}, v) \\
 &= \beta(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}, w_k) \\
 &= \beta(w_{\tau(1)}, \dots, w_{\tau(k-1)}, w_{\tau(k)}) \\
 &= \text{sgn}(\tau) \beta(w_1, \dots, w_{k-1}, w_k) \\
 &= \text{sgn}(\sigma) \beta(w_1, \dots, w_{k-1}, v) \\
 &= \text{sgn}(\sigma) \alpha(v, w_1, \dots, w_{k-1}) \\
 &= \text{sgn}(\sigma) (\iota_v \alpha)(w_1, \dots, w_{k-1})
 \end{aligned}$$

Since  $w_1, \dots, w_{k-1} \in V$  are arbitrary,  $\sigma(\iota_v \alpha) = \text{sgn}(\sigma) \iota_v \alpha$ . Hence  $\iota_v \alpha \in \Lambda^{k-1}(V)$ .

□

## 2.3 $(0, 2)$ -Tensors

**Definition 2.3.0.1.** Let  $V$  be a finite dimensional vector space,  $v \in V$  and  $\alpha \in T_2^0(V)$ . Then  $\alpha$  is said to be **degenerate** if there exists  $v \in V$  such that for each  $w \in V$ ,  $\alpha(v, w) = 0$  and  $v \neq 0$ .

**Definition 2.3.0.2.** Let  $V$  be a finite dimensional vector space,  $\alpha \in T_2^0(V)$ . We define  $\phi_\alpha : V \rightarrow V^*$  by

$$\phi_\alpha(v) = \iota_v \alpha$$

**Exercise 2.3.0.3.** Let  $V$  be a finite dimensional vector space,  $\alpha \in T_2^0(V)$ . Then  $\phi_\alpha \in L(V; V^*)$ .

*Proof.* Let  $v_1, v_2 \in V$  and  $\lambda \in \mathbb{R}$ . Then for each  $w \in V$ ,

$$\begin{aligned} \phi_\alpha(v_1 + \lambda v_2)(w) &= (\iota_{v_1 + \lambda v_2} \alpha)(w) \\ &= \alpha(v_1 + \lambda v_2, w) \\ &= \alpha(v_1, w) + \lambda \alpha(v_2, w) \\ &= (\iota_{v_1} \alpha)(w) + \lambda (\iota_{v_2} \alpha)(w) \\ &= \phi_\alpha(v_1)(w) + \lambda \phi_\alpha(v_2)(w) \\ &= [\phi_\alpha(v_1) + \lambda \phi_\alpha(v_2)](w) \end{aligned}$$

Therefore,  $\phi_\alpha(v_1 + \lambda v_2) = \phi_\alpha(v_1) + \lambda \phi_\alpha(v_2)$ . Thus  $\phi_\alpha \in L(V; V^*)$ . □

**Exercise 2.3.0.4.** Let  $V$  be a finite dimensional vector space and  $\alpha \in T_2^0(V)$ . Then  $\alpha$  is nondegenerate iff  $\phi_\alpha$  is an isomorphism.

*Proof.*

- ( $\implies$  :)

Suppose that  $\alpha$  is nondegenerate. Let  $v \in \ker \phi_\alpha$ . Then for each  $w \in V$ ,

$$\begin{aligned} \alpha(v, w) &= (\iota_v \alpha)(w) \\ &= \phi_\alpha(v)(w) \\ &= 0 \end{aligned}$$

Since  $\alpha$  is nondegenerate,  $v = 0$ . Since  $v \in \ker \phi_\alpha$  is arbitrary,  $\ker \phi_\alpha = \{0\}$ . Hence  $\phi_\alpha$  is injective. Since  $\dim V = \dim V^*$ ,  $\phi_\alpha$  is surjective. Hence  $\phi_\alpha$  is an isomorphism.

- ( $\impliedby$  :)

Suppose that  $\phi_\alpha$  is an isomorphism. Let  $v \in V$ . Suppose that for each  $w \in V$ ,  $\alpha(v, w) = 0$ . Then for each  $w \in V$ ,

$$\begin{aligned} \phi_\alpha(v)(w) &= (\iota_v \alpha)(w) \\ &= \alpha(v, w) \\ &= 0 \end{aligned}$$

Thus  $\phi_\alpha(v) = 0$  which implies that  $v \in \ker \phi_\alpha$ . Since  $\phi_\alpha$  is an isomorphism,  $v = 0$ . Hence  $\alpha$  is nondegenerate. □

**Exercise 2.3.0.5.** Let  $V$  be a finite dimensional vector space and  $\alpha \in T_2^0(V)$ . Then

1.  $[\phi_\alpha]_{i,j} = \alpha(e_j, e_i)$

2. for each  $v, w \in V$ ,

$$\alpha(v, w) = [w]^* [\phi_\alpha] [v]$$

*Proof.* 1. Set  $A = [\phi_\alpha]$ . Let  $i, j \in \{1, \dots, n\}$ . By definition,

$$\phi_\alpha(e_j) = \sum_{k=1}^n A_{k,j} \epsilon^k$$

Then

$$\begin{aligned} \phi_\alpha(e_j)(e_i) &= \sum_{k=1}^n A_{k,j} \epsilon^k(e_i) \\ &= \sum_{k=1}^n A_{k,j} \delta_{k,i} \\ &= A_{i,j} \end{aligned}$$

2. Let  $v, w \in V$ . Then there exist  $(v^i)_{i=1}^n, (w^j)_{j=1}^n \subset \mathbb{R}$  such that  $v = \sum_{i=1}^n v^i e_i$  and  $w = \sum_{j=1}^n w^j e_j$ . Part (1) implies that

$$\begin{aligned} \alpha(v, w) &= \sum_{i=1}^n \sum_{j=1}^n v^i w^j \alpha(e_i, e_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n v^i w^j [\phi_\alpha]_{j,i} \\ &= \sum_{i=1}^n \sum_{j=1}^n [v]_i [w]_j [\phi_\alpha]_{j,i} \\ &= [w]^* [\phi_\alpha] [v] \end{aligned}$$

□

### 2.3.1 Scalar Product Spaces

**Definition 2.3.1.1.** Let  $V$  be a finite dimensional vector space and  $\alpha \in \Sigma^2(V)$ . Then  $\alpha$  is said to be

- **positive semidefinite** if for each  $v \in V$ ,  $\alpha(v, v) \geq 0$
- **positive definite** if for each  $v \in V$ ,  $v \neq 0$  implies that  $\alpha(v, v) > 0$
- **negative semidefinite** if  $-\alpha$  is positive semidefinite
- **negative definite** if  $-\alpha$  is positive definite

**Exercise 2.3.1.2.** Let  $V$  be a finite dimensional vector space and  $\alpha \in \Sigma^2(V)$ . Then

1.  $\alpha$  is positive definite iff for each  $\lambda \in \sigma([\phi_\alpha])$ ,  $\lambda > 0$
2.  $\alpha$  is positive definite iff for each  $\lambda \in \sigma([\phi_\alpha])$ ,  $\lambda \geq 0$

*Proof.*

1. Suppose that  $\alpha$  is positive definite. Write  $\sigma(\phi_\alpha) = \{\lambda_1, \dots, \lambda_n\}$ . Define  $\Lambda \in \mathbb{R}^{n \times n}$  by  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Since  $\alpha$  is symmetric,  $[\phi_\alpha]$  is symmetric. There exists  $U \in O(n)$  such that  $[\phi_\alpha] = U \Lambda U^*$ . **FINISH!!!**

□

**Definition 2.3.1.3.** Let  $V$  be a finite dimensional vector space and  $\alpha \in \Sigma^2(V)$ . Then  $\alpha$  is said to be a **scalar product** if  $\alpha$  is nondegenerate. In this case,  $(V, \alpha)$  is said to be a **scalar product space**.

**Definition 2.3.1.4.** Let  $V$  be a finite dimensional vector space and  $\alpha \in \Sigma^2(V)$  a scalar product on  $V$ . We define the **index** of  $\alpha$ , denoted  $\text{ind } \alpha$  by

$$\text{ind } \alpha = \max\{\dim W : W \text{ is a subspace of } V \text{ and } \alpha|_{W \times W} \text{ is negative definite}\}$$

**Definition 2.3.1.5.** Let  $(V, \alpha)$  be a scalar product space.

- Let  $v_1, v_2 \in V$ . Then  $v_1$  and  $v_2$  are said to be **orthogonal** if  $\alpha(v_1, v_2) = 0$ .
- Let  $U \subset V$  be a subspace. We define the **orthogonal subspace of  $U$** , denoted by  $U^\perp$ , by

$$U^\perp = \{v \in V : \text{for each } u \in U, \alpha(u, v) = 0\}$$

**Exercise 2.3.1.6.** Let  $(V, \alpha)$  be a scalar product space and  $U \subset V$  a subspace. Then  $U^\perp$  is a subspace of  $V$ .

*Proof.* We note that since  $U^\perp = \bigcap_{u \in U} \ker \phi_\alpha(u)$ ,  $U^\perp$  is a subspace of  $V$ . □

**Exercise 2.3.1.7.** Let  $(V, \alpha)$  be an  $n$ -dimensional scalar product space,  $U \subset V$  a  $k$ -dimensional subspace and  $(e_j)_{j=1}^n \subset V$  a basis for  $V$ . Suppose that  $(e_j)_{j=1}^k$  is a basis for  $U$ . Then for each  $v \in V$ ,  $v \in U^\perp$  iff for each  $j \in [k]$ ,  $\alpha(v, e_j) = 0$ .

*Proof.* Let  $v \in V$ .

- ( $\implies$ ): Suppose that  $v \in U^\perp$ . Since  $(e_j)_{j=1}^k \subset U$ , we have that for each  $j \in [k]$ ,  $\alpha(v, e_j) = 0$ .
- ( $\impliedby$ ): Suppose that for each  $j \in [k]$ ,  $\alpha(v, e_j) = 0$ . Let  $u \in U$ . Then there exist  $(a^j)_{j=1}^k \subset \mathbb{R}$  such that  $u = \sum_{j=1}^k a^j u_j$ . This implies that

$$\begin{aligned} \alpha(v, u) &= \sum_{j=1}^k a^j \alpha(v, u_j) \\ &= 0 \end{aligned}$$

Since  $u \in U$  is arbitrary, we have that  $v \in U^\perp$ . □

**Exercise 2.3.1.8.** Let  $(V, \alpha)$  be a scalar product space and  $U \subset V$  a subspace. Then

1.  $\dim V = \dim U + \dim U^\perp$
2.  $(U^\perp)^\perp = U$

*Proof.* 1. Set  $n = \dim V$  and  $k = \dim U$ . Choose a basis  $(e_j)_{j=1}^n$  such that  $(e_j)_{j=1}^k$  is a basis for  $U$ .

2. □

**Exercise 2.3.1.9.** Let  $V$  be a finite dimensional vector space and  $\alpha \in \Sigma^2(V)$ . Set  $\sigma([\phi_\alpha])^- = \{\lambda \in \sigma([\phi_\alpha]) : \lambda < 0\}$ . Then

$$\text{ind } \alpha = \sum_{\lambda \in \sigma([\phi_\alpha])^-} \mu(\lambda)$$



*Proof.* Since  $\alpha$  is symmetric, there exist  $U \in O(n)$  and  $\Lambda \in D(n, \mathbb{R})$  such that  $[\phi_\alpha] = U\Lambda U^*$ . Define  $(u_j)_{j=1}^n \subset V$  by  $u_j = \sum_{i=1}^n U_{i,j} e_i$ . Define  $J^- = \{j \in [n] : \Lambda_{j,j} < 0\}$ ,  $n^- = \#J^-$  and  $V^- = \text{span}\{u_j : j \in J^-\}$ . Let  $v \in V^-$ . Then there exist  $(a^j)_{j \in J^-}$  such that  $v = \sum_{j \in J^-} a^j u_j$ . We note that

$$\begin{aligned} U^*[\phi_\alpha]U &= U^*(U\Lambda U^*)U \\ &= (U^*U)\Lambda(U^*U) \\ &= I\Lambda I \\ &= \Lambda \end{aligned}$$

A previous exercise implies that

$$\begin{aligned} \alpha(v, v) &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k \alpha(u_j, u_k) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k [u_j]^* [\phi_\alpha] [u_k] \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k ([e_j]^* U^*) [\phi_\alpha] (U[e_k]) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (U^*[\phi_\alpha]U)_{j,k} \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (\Lambda)_{j,k} \\ &= \sum_{j \in J^-} |a^j|^2 \Lambda_{j,j} \\ &< 0 \end{aligned}$$

Since  $v \in V^-$  is arbitrary,  $\alpha|_{V^- \times V^-}$  is negative definite. Thus

$$\begin{aligned} \text{ind } \alpha &\geq \dim V^- \\ &= n^- \end{aligned}$$

Set  $J^+ = (J^-)^c$ . Let  $W \subset V$  be a subspace. Suppose that  $\alpha|_{W \times W}$  is negative definite. For the sake of contradiction, suppose that there exists  $j_0 \in J^+$  such that  $u_{j_0} \in W$ . Then

$$\begin{aligned} \alpha(u_{j_0}, u_{j_0}) &= [u_{j_0}]^* [\phi_\alpha] [u_{j_0}] \\ &= [u_{j_0}]^* U\Lambda U^* [u_{j_0}] \\ &= \Lambda_{j_0, j_0} \\ &\geq 0 \end{aligned}$$

which is a contradiction since  $\alpha|_{W \times W}$  is negative definite. Thus for each  $j \in J^+$ ,  $u_j \notin W$ . □

### 2.3.2 Symplectic Vector Spaces

**Definition 2.3.2.1.** Let  $V$  be a finite dimensional vector space and  $\omega \in \Lambda^2(V)$ . Then  $\omega$  is said to be a **symplectic form** if  $\omega$  is nondegenerate. In this case  $(V, \omega)$  is said to be a **symplectic space**.

**Exercise 2.3.2.2.** Let  $V$  be a  $2n$ -dimensional vector space with basis  $(a_j, b_j)_{j=1}^n$  and corresponding dual basis  $(\alpha^j, \beta^j)_{j=1}^n$ . Define  $\omega \in \Lambda^2(V)$  by

$$\omega = \sum_{j=1}^n \alpha^j \wedge \beta^j$$

Then

1. for each  $j, k \in \{1, \dots, n\}$ ,

(a)  $\omega(a_j, a_k) = 0$

(b)  $\omega(b_j, b_k) = 0$

(c)  $\omega(a_j, b_k) = \delta_{j,k}$

2.  $(V, \omega)$  is a symplectic space

*Proof.*

1. Let  $j, k \in \{1, \dots, n\}$ .

(a)

$$\begin{aligned} \omega(a_j, a_k) &= \sum_{l=1}^n \alpha^l \wedge \beta^l(a_j, a_k) \\ &= \sum_{l=1}^n [\alpha^l(a_j)\beta^l(a_k) - \alpha^l(a_k)\beta^l(a_j)] \\ &= 0 \end{aligned}$$

(b) Similar to (a)

(c)

$$\begin{aligned} \omega(a_j, b_k) &= \sum_{l=1}^n \alpha^l \wedge \beta^l(a_j, b_k) \\ &= \sum_{l=1}^n [\alpha^l(a_j)\beta^l(b_k) - \alpha^l(b_k)\beta^l(a_j)] \\ &= \sum_{l=1}^n \alpha^l(a_j)\beta^l(b_k) \\ &= \sum_{l=1}^n \delta_{j,l}\delta_{l,k} \\ &= \delta_{j,k} \end{aligned}$$

2. Let  $v \in V$ . Then there exist  $(q^j, p^j)_{j=1}^n \subset \mathbb{R}$  such that  $v = \sum_{j=1}^n q^j a_j + p^j b_j$ . Suppose that for each  $w \in V$ ,  $\omega(v, w) = 0$ . Let  $k \in \{1, \dots, n\}$ . Then

$$\begin{aligned} 0 &= \omega(v, a_k) \\ &= \sum_{j=1}^n q^j \omega(a_j, a_k) + p^j \omega(b_j, a_k) \\ &= \sum_{j=1}^n p^j \delta_{j,k} \\ &= p^k \end{aligned}$$

Similarly,

$$\begin{aligned}
 0 &= \omega(v, b_k) \\
 &= \sum_{j=1}^n q^j \omega(a_j, b_k) + p^j \omega(b_j, b_k) \\
 &= \sum_{j=1}^n q^j \delta_{j,k} \\
 &= q^k
 \end{aligned}$$

Since  $k \in \{1, \dots, n\}$  is arbitrary,  $v = 0$ . Hence  $\omega$  is nondegenerate. Therefore  $(V, \omega)$  is symplectic.  $\square$

**Exercise 2.3.2.3.** Let  $(V, \omega)$  be a symplectic space. Then  $\dim V$  is even.

*Proof.* Set  $n = \dim V$ . Let  $(e_j)_{j=1}^n$  be a basis for  $V$ . Define  $[\omega] \in \mathbb{R}^{n \times n}$  by  $[\omega]_{i,j} = \omega(e_i, e_j)$ . Since  $\omega \in \Lambda^2(V)$ ,  $[\omega]^* = -[\omega]$ . Therefore

$$\begin{aligned}
 \det[\omega] &= \det[\omega]^* \\
 &= \det(-[\omega]) \\
 &= (-1)^n \det[\omega]
 \end{aligned}$$

For the sake of contradiction, suppose that  $n$  is odd. Then  $\det[\omega] = -\det[\omega]$  which implies that  $\det[\omega] = 0$ . Since  $\omega$  is nondegenerate,  $[\omega] \in GL(n, \mathbb{R})$ . This is a contradiction. Hence  $n$  is even.  $\square$

**Definition 2.3.2.4.** Let  $(V, \omega)$  be a symplectic space and  $S \subset V$  a subspace. We define the **symplectic complement of  $V$** , denoted  $S^\perp$ , by

$$S^\perp = \{v \in V : \text{for each } w \in S, \omega(v, w) = 0\}$$

**Exercise 2.3.2.5.** Let  $(V, \omega)$  be a symplectic space and  $S \subset V$  a subspace. Then  $S^\perp$  is a subspace.

*Proof.* We note that

$$S^\perp = \bigcap_{v \in S} \ker \iota_v \omega$$

Hence  $S^\perp$  is a subspace.  $\square$

**Exercise 2.3.2.6.** Let  $(V, \omega)$  be a symplectic space and  $S \subset V$  a subspace. Then

$$\dim V = \dim S + \dim S^\perp$$

*Proof.*  $\square$

**Exercise 2.3.2.7.** Let  $(V, \omega)$  be a symplectic space and  $S \subset V$  a subspace. Then  $(S^\perp)^\perp = S$ .

*Proof.* Let  $v \in (S^\perp)^\perp$ . Then for each  $w \in S^\perp$ ,  $\omega(v, w) = 0$ .  $\square$



# Chapter 3

## Smooth Manifolds

### 3.1 Topological Manifolds

**Exercise 3.1.0.1.** We have that  $\mathbb{R}$  is homeomorphic to  $(0, \infty)$

*Proof.* Define  $f : \mathbb{R} \rightarrow (0, \infty)$  by  $f(x) = e^x$ . Then  $f$  is a homeomorphism. □

**Definition 3.1.0.2.** Let  $n \in \mathbb{N}$ . We define the **upper half space** of  $\mathbb{R}^n$ , denoted  $\mathbb{H}^n$ , by

$$\mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and we define

$$\partial\mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$$

$$\text{Int } \mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

We endow  $\mathbb{H}^n$ ,  $\partial\mathbb{H}^n$  and  $\text{Int } \mathbb{H}^n$  with the subspace topology inherited from  $\mathbb{R}^n$ .

We define the projection map  $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$  by

$$\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$$

**Definition 3.1.0.3.** We define  $\mathbb{R}^0 = \{0\}$  and  $\mathbb{H}^0 = \emptyset$  endowed with the discrete topology.

**Exercise 3.1.0.4.** Let  $n \in \mathbb{N}$ .

1.  $\partial\mathbb{H}^n$  is homeomorphic to  $\mathbb{R}^{n-1}$
2.  $\text{Int } \mathbb{H}^n$  is homeomorphic to  $\mathbb{R}^n$

*Proof.*

1. Let  $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$  be the projection map given by

$$\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$$

Then  $\pi$  is a homeomorphism.

2. Define  $f : \mathbb{R}^n \rightarrow \text{Int } \mathbb{H}^n$  by  $f(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, e^{x_n})$ . Then  $f$  is a homeomorphism. □

**Definition 3.1.0.5.** Let  $(M, \mathcal{T})$  be a topological space and  $n \in \mathbb{N}_0$ . Let  $U \subset M$  and  $V \subset \mathbb{H}^n$  and  $\phi : U \rightarrow V$ . Then  $(U, \phi)$  is said to be a  **$n$ -coordinate chart on  $(M, \mathcal{T})$**  if

- $U \in \mathcal{T}$
- $V \in \mathcal{T}_{\mathbb{H}^n}$

- $\phi$  is a  $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n} \cap V)$ -homeomorphism

We denote the set of all  $n$ -coordinate charts on  $M$  by  $X^n(M, \mathcal{T})$ .

**Note 3.1.0.6.** We will write  $X^n(M)$  in place of  $X^n(M, \mathcal{T})$  when the topology is not ambiguous.

**Definition 3.1.0.7.** Let  $M$  be a topological space and  $n \in \mathbb{N}$ . Then  $M$  is said to be **locally Euclidean of dimension  $n$**  if for each  $p \in M$ , there exists  $(U, \phi) \in X^n(M)$  such that  $p \in U$ .

**Definition 3.1.0.8.** Let  $M$  be a topological space and  $n \in \mathbb{N}$ . Then  $M$  is said to be an  **$n$ -dimensional topological manifold** if

1.  $M$  is Hausdorff
2.  $M$  is second-countable
3.  $M$  is locally Euclidean of dimension  $n$

**Theorem 3.1.0.9. Topological Invariance of Dimension:**

Let  $M$  be an  $n$ -dimensional topological manifold and  $N$  a  $p$ -dimensional topological manifold. If  $M$  and  $N$  are homeomorphic, then  $n = p$ .

**Note 3.1.0.10.** In light of the previous theorem, we write  $X(M)$  in place of  $X^n(M)$  and refer to  $n$ -coordinate charts as coordinate charts when the context is clear.

**Definition 3.1.0.11.** Let  $M$  be an  $n$ -dimensional topological manifold and  $(U, \phi) \in X(M)$ . Then  $(U, \phi)$  is said to be an

- **interior chart** if  $\phi(U)$  is open in  $\mathbb{R}^n$
- **boundary chart** if  $\phi(U)$  is open in  $\mathbb{H}^n$  and  $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$

We denote the set of all interior charts on  $M$  and the set of all boundary charts on  $M$  by  $X_{\text{Int}}(M)$  and  $X_{\partial}(M)$  respectively.

**Exercise 3.1.0.12.** Let  $M$  be an  $n$ -dimensional topological manifold. Then

1.  $X(M) = X_{\text{Int}}(M) \cup X_{\partial}(M)$
2.  $X_{\text{Int}}(M) \cap X_{\partial}(M) = \emptyset$

*Proof.*

1. By definition,  $X_{\text{Int}}(M) \cup X_{\partial}(M) \subset X(M)$ . Let  $(U, \phi) \in X(M)$ . Since  $(U, \phi)$  is a coordinate chart on  $M$ ,  $\phi(U)$  is open in  $\mathbb{R}^n$  or  $\phi(U)$  is open in  $\mathbb{H}^n$ . If  $\phi(U)$  is open in  $\mathbb{R}^n$ , then

$$\begin{aligned} (U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

Suppose that  $\phi(U)$  is open in  $\mathbb{H}^n$ . If  $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$ , then  $\phi(U)$  is open in  $\mathbb{R}^n$  and

$$\begin{aligned} (U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

Suppose that  $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$ . Then

$$\begin{aligned} (U, \phi) &\in X_{\partial}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

Since  $(U, \phi) \in X(M)$  is arbitrary,  $X(M) \subset X_{\text{Int}}(M) \cup X_{\partial}(M)$ . Therefore  $X(M) = X_{\text{Int}}(M) \cup X_{\partial}(M)$ .

2. For the sake of contradiction, suppose that  $X_{\text{Int}}(M) \cup X_{\partial}(M) \neq \emptyset$ . Then there exists  $(U, \phi) \in X(M)$  such that  $(U, \phi) \in X_{\text{Int}}(M)$  and  $(U, \phi) \in X_{\partial}(M)$ . Therefore  $\phi(U)$  is open in  $\mathbb{R}^n$ ,  $\phi(U)$  is open in  $\mathbb{H}^n$  and  $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$ . Since  $\phi(U)$  is open in  $\mathbb{R}^n$  and  $\phi(U) \subset \mathbb{H}^n$ ,  $\phi(U) \subset \text{Int } \mathbb{H}^n$  and therefore  $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$  which is a contradiction.

□

**Definition 3.1.0.13.** Let  $M$  be an  $n$ -dimensional topological manifold. We define the

- **interior** of  $M$ , denoted  $\text{Int } M$ , by

$$\text{Int } M = \{p \in M : \text{there exists } (U, \phi) \in X_{\text{Int}}(M) \text{ such that } p \in U\}$$

- **boundary** of  $M$ , denoted  $\partial M$ , by

$$\partial M = \{p \in M : \text{there exists } (V, \psi) \in X_{\partial}(M) \text{ such that } p \in V \text{ and } \psi(p) \in \partial\mathbb{H}^n\}$$

**Exercise 3.1.0.14.** Let  $M$  be an  $n$ -dimensional topological manifold. Let  $(U, \phi) \in X_{\text{Int}}(M)$ . Then  $U \subset \text{Int } M$ .

*Proof.* Let  $p \in U$ . Since  $(U, \phi) \in X_{\text{Int}}(M)$  and  $p \in U$ , by definition,  $p \in \text{Int } M$ . Since  $p \in U$  is arbitrary,  $U \subset \text{Int } M$ . □

**Exercise 3.1.0.15.** Let  $M$  be an  $n$ -dimensional topological manifold,  $(U, \phi) \in X_{\partial}(M)$  and  $p \in U$ . If  $\phi(p) \notin \partial\mathbb{H}^n$ , then  $p \in \text{Int } M$ .

*Proof.* Suppose that  $\phi(p) \notin \partial\mathbb{H}^n$ . Then  $\phi(p) \in \text{Int } \mathbb{H}^n$ . Hence there exists  $B' \subset \phi(U)$  such that  $B'$  is open in  $\mathbb{R}^n$  and  $\phi(p) \in B'$ . Set  $U' = \phi^{-1}(B')$  and  $\phi' = \phi|_{U'}$ . Then  $U'$  is open in  $M$  and  $\phi' : U' \rightarrow B'$  is a homeomorphism. Hence  $(U', \phi') \in X_{\text{Int}}(M)$ . Since  $\phi(p) \in B'$ , we have that  $p \in U'$ . By definition,  $p \in \text{Int } M$ . □

**Exercise 3.1.0.16.** Let  $M$  be an  $n$ -dimensional topological manifold. Then

$$1. M = \text{Int } M \cup \partial M$$

$$2. \text{Int } M \cap \partial M = \emptyset$$

**Hint:** simply connected

*Proof.*

1. By definition,  $\text{Int } M \cup \partial M \subset M$ . Let  $p \in M$ . Since  $M$  is a manifold, there exists  $(U, \phi) \in X(M)$  such that  $p \in U$ . A previous exercise implies that  $(U, \phi) \in X_{\text{Int}}(M) \cup X_{\partial}(M)$ . If  $(U, \phi) \in X_{\text{Int}}(M)$ , then by definition,

$$\begin{aligned} p &\in \text{Int } M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Suppose that  $(U, \phi) \in X_{\partial}(M)$ . If  $\phi(p) \in \partial\mathbb{H}^n$ , then by definition,

$$\begin{aligned} p &\in \partial M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Suppose that  $\phi(p) \notin \partial\mathbb{H}^n$ . The previous exercise implies that  $p \in \text{Int } M$ . Therefore,

$$\begin{aligned} p &\in \text{Int } M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Since  $p \in M$  is arbitrary,  $M \subset \text{Int } M \cup \partial M$ . Therefore  $M = \text{Int } M \cup \partial M$ .

2. For the sake of contradiction, suppose that  $\text{Int } M \cap \partial M \neq \emptyset$ . Then there exists  $p \in M$  such that  $p \in \text{Int } M \cap \partial M$ . By definition, there exists  $(U, \phi) \in X_{\text{Int}}(M)$ ,  $(V, \psi) \in X_{\partial}(M)$  such that  $p \in U \cap V$  and  $\psi(p) \in \partial \mathbb{H}^n$ . Note that  $\psi(U \cap V)$  is open in  $\mathbb{H}^n$ ,  $\phi(U \cap V)$  is open in  $\mathbb{R}^n$  and  $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$  is a homeomorphism.

Since  $\psi(U \cap V)$  is open in  $\mathbb{H}^n$ , there exists an  $B_\psi \subset \psi(U \cap V)$  such that  $B_\psi$  is open in  $\mathbb{H}^n$ ,  $B_\psi$  is simply connected and  $\psi(p) \in B_\psi$ . Set  $B_\phi = \phi \circ \psi^{-1}(B_\psi)$ . Since  $\phi(U \cap V)$  is open in  $\mathbb{R}^n$ ,  $B_\phi$  is open in  $\mathbb{R}^n$ . Since  $B_\psi$  is simply connected and  $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$  is a homeomorphism,  $B_\phi$  is simply connected.

Set  $B'_\phi = B_\phi \setminus \{\phi(p)\}$  and  $B'_\psi = B_\psi \setminus \{\psi(p)\}$ . Then  $\phi \circ \psi^{-1} : B'_\psi \rightarrow B'_\phi$  is a homeomorphism. Since  $\psi(p) \in \partial \mathbb{H}^n$ ,  $B'_\psi$  is simply connected. Since  $B_\phi$  is open in  $\mathbb{R}^n$ ,  $B'_\phi$  is not simply connected. This is a contradiction since  $B'_\phi$  is homeomorphic to  $B'_\psi$ . So  $\partial M \cap \text{Int } M = \emptyset$ .

□

**Exercise 3.1.0.17.** Let  $M$  be an  $n$ -dimensional topological manifold. Then

1.  $\text{Int } M$  is open
2.  $\partial M$  is closed

*Proof.*

1. Let  $p \in \text{Int } M$ . Then there exists  $(U, \phi) \in X_{\text{Int}}(M)$  such that  $p \in U$ . By definition,  $U$  is open and a previous exercise implies that  $U \subset \text{Int } M$ . Since  $p \in \text{Int } M$  is arbitrary, we have that for each  $p \in \text{Int } M$ , there exists  $U \subset \text{Int } M$  such that  $U$  is open. Hence  $\text{Int } M$  is open.
2. Since  $\partial M = (\text{Int } M)^c$ , and  $\text{Int } M$  is open, we have that  $\partial M$  is closed.

□

**Exercise 3.1.0.18.** Let  $M$  be an  $n$ -dimensional topological manifold,  $(U, \phi) \in X(M)$  and  $p \in U$ . If  $p \in \partial M$ , then  $(U, \phi) \in X_{\partial}(M)$ .

**Hint:** simply connected

*Proof.* Suppose that  $p \in \partial M$ . Then there exists a  $(V, \psi) \in X_{\partial}(M)$  such that  $p \in V$  and  $\psi(p) \in \partial \mathbb{H}^n$ . Note that  $\psi(U \cap V)$  is open in  $\mathbb{H}^n$ ,  $\phi(U \cap V)$  is open in  $\mathbb{R}^n$  and  $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$  is a homeomorphism.

Since  $\psi(U \cap V)$  is open in  $\mathbb{H}^n$ , there exists  $B_\psi \subset \psi(U \cap V)$  such  $B_\psi$  is open in  $\mathbb{H}^n$ ,  $B_\psi$  is simply connected and  $\psi(p) \in B_\psi$ . Set  $B_\phi = \phi \circ \psi^{-1}(B_\psi)$ .

For the sake of contradiction, suppose that  $(U, \phi) \in X_{\text{Int}}(M)$ . Then  $\phi(U)$  is open in  $\mathbb{R}^n$ . Hence  $\phi(U \cap V)$  is open in  $\mathbb{R}^n$  and  $B_\phi$  is open in  $\mathbb{R}^n$ . Since  $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$  is a homeomorphism,  $B_\phi$  is simply connected. Set  $B'_\phi = B_\phi \setminus \{\phi(p)\}$  and  $B'_\psi = B_\psi \setminus \{\psi(p)\}$ . Since  $\psi(p) \in \partial \mathbb{H}^n$ ,  $B'_\psi$  is simply connected. Since  $B_\phi$  is open in  $\mathbb{R}^n$ ,  $B'_\phi$  is not simply connected. This is a contradiction since  $B'_\phi$  is homeomorphic to  $B'_\psi$ . So  $(U, \phi) \notin X_{\text{Int}}(M)$ . Since  $(X_{\text{Int}}(M))^c = X_{\partial}(M)$ , we have that  $(U, \phi) \in X_{\partial}(M)$ .

□

**Exercise 3.1.0.19.** Let  $M$  be an  $n$ -dimensional topological manifold,  $(U, \phi) \in X_{\partial}(M)$  and  $p \in U$ . Then

1.  $p \in \partial M$  iff  $\phi(p) \in \partial \mathbb{H}^n$
2.  $p \in \text{Int } M$  iff  $\phi(p) \in \text{Int } \mathbb{H}^n$

*Proof.*

1. Suppose that  $p \in \partial M$ . For the sake of contradiction, suppose that  $\phi(p) \notin \partial \mathbb{H}^n$ . Then  $\phi(p) \in \text{Int } \mathbb{H}^n$ . Hence there exists  $B' \subset \phi(U)$  such that  $B'$  is open in  $\mathbb{R}^n$  and  $\phi(p) \in B'$ . Set  $U' = \phi^{-1}(B')$  and  $\phi' = \phi|_{U'}$ . Then  $p \in U'$  and  $(U', \phi') \in X_{\text{Int}}(M)$ . Since  $p \in U'$ , the previous exercise implies that  $(U', \phi') \in X_{\partial}(M)$ . This is a contradiction since  $X_{\text{Int}}(M) \cap X_{\partial}(M) = \emptyset$ . So  $\phi(p) \in \partial \mathbb{H}^n$ . Conversely, suppose that  $\phi(p) \in \partial \mathbb{H}^n$ . By definition,  $p \in \partial M$ .



2. A previous exercise implies that  $\text{Int } M = (\partial M)^c$ . Part (1) implies that

$$\begin{aligned} p &\in (\partial M)^c \\ &= \text{Int } M \end{aligned}$$

if and only if

$$\begin{aligned} \phi(p) &\in (\partial \mathbb{H}^n)^c \\ &= \text{Int } \mathbb{H}^n \end{aligned}$$

□

**Exercise 3.1.0.20.** Let  $M$  be an  $n$ -dimensional topological manifold and  $p \in M$ . Then  $p \in \partial M$  iff for each  $(U, \phi) \in X(M)$ ,  $p \in U$  implies that  $(U, \phi) \in X_{\partial}(M)$  and  $\phi(p) \in \partial \mathbb{H}^n$ .

*Proof.* Suppose that  $p \in \partial M$ . Let  $(U, \phi) \in X(M)$ . Suppose that  $p \in U$ . The previous two exercises imply that  $(U, \phi) \in X_{\partial}(M)$  and  $\phi(p) \in \partial \mathbb{H}^n$ .

Conversely, suppose that for each  $(U, \phi) \in X(M)$ ,  $p \in U$  implies that  $(U, \phi) \in X_{\partial}(M)$  and  $\phi(p) \in \partial \mathbb{H}^n$ . Since  $M$  is a manifold, there exists  $(U, \phi) \in X(M)$  such that  $p \in U$ . By assumption,  $(U, \phi) \in X_{\partial}(M)$  and  $\phi(p) \in \partial \mathbb{H}^n$ . By definition,  $p \in \partial M$ . □

**Exercise 3.1.0.21.** Let  $M$  be an  $n$ -dimensional topological manifold. Let  $(U, \phi) \in X_{\partial}(M)$ . Then

1.  $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$
2.  $\phi(U \cap \text{Int } M) = \phi(U) \cap \text{Int } \mathbb{H}^n$

*Proof.*

1. Since  $(U, \phi) \in X_{\partial}(M)$ , a previous exercise implies that for each  $p \in U$ ,  $p \in \partial M$  iff  $\phi(p) \in \partial \mathbb{H}^n$ . Let  $q \in \phi(U \cap \partial M)$ . Then there exists  $p \in U \cap \partial M$  such that  $\phi(p) = q$ . Since  $p \in \partial M$ ,  $\phi(p) \in \partial \mathbb{H}^n$ . Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \partial \mathbb{H}^n \end{aligned}$$

Since  $q \in \phi(U \cap \partial M)$  is arbitrary,  $\phi(U \cap \partial M) \subset \phi(U) \cap \partial \mathbb{H}^n$ .

Let  $q \in \phi(U) \cap \partial \mathbb{H}^n$ . Then there exists  $p \in U$  such that  $q = \phi(p)$ . Since  $\phi(p) \in \partial \mathbb{H}^n$ , we have that  $p \in \partial M$ . Hence  $p \in U \cap \partial M$  and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since  $q \in \phi(U) \cap \partial \mathbb{H}^n$  is arbitrary,  $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \partial M)$ . Thus  $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$ .

2. Since  $(U, \phi) \in X_{\partial}(M)$ , a previous exercise implies that for each  $p \in U$ ,  $p \in \text{Int } M$  iff  $\phi(p) \in \text{Int } \mathbb{H}^n$ . Let  $q \in \phi(U \cap \text{Int } M)$ . Then there exists  $p \in U \cap \text{Int } M$  such that  $\phi(p) = q$ . Since  $p \in \text{Int } M$ ,  $\phi(p) \in \text{Int } \mathbb{H}^n$ . Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \text{Int } \mathbb{H}^n \end{aligned}$$

Since  $q \in \phi(U \cap \text{Int } M)$  is arbitrary,  $\phi(U \cap \text{Int } M) \subset \phi(U) \cap \text{Int } \mathbb{H}^n$ .

Let  $q \in \phi(U) \cap \text{Int } \mathbb{H}^n$ . Then there exists  $p \in U$  such that  $q = \phi(p)$ . Since  $\phi(p) \in \text{Int } \mathbb{H}^n$ , we have that  $p \in \text{Int } M$ . Hence  $p \in U \cap \text{Int } M$  and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \text{Int } M) \end{aligned}$$

Since  $q \in \phi(U) \cap \text{Int } \mathbb{H}^n$  is arbitrary,  $\phi(U) \cap \text{Int } \mathbb{H}^n \subset \phi(U \cap \text{Int } M)$ . Thus  $\phi(U \cap \text{Int } M) = \phi(U) \cap \text{Int } \mathbb{H}^n$ .

□

**Definition 3.1.0.22.** Let  $M$  be an  $n$ -dimensional topological manifold and  $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$  the projection map. For  $(U, \phi) \in X_\partial(M)$ , we define  $\bar{U} \subset \partial M$  and  $\bar{\phi} : \bar{U} \rightarrow \pi(\phi(\bar{U}))$  by  $\bar{U} = U \cap \partial M$  and  $\bar{\phi} = \pi \circ \phi|_{\bar{U}}$  respectively.

**Exercise 3.1.0.23.** Let  $M$  be an  $n$ -dimensional topological manifold, and  $\lambda : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$  a homeomorphism. Then  $\{(\bar{U}, \bar{\phi}) : (U, \phi) \in X_\partial(M)\} \subset X_{\text{Int}}^{n-1}(\partial M)$ .

*Proof.* Let  $(U, \phi) \in X_\partial(M)$ .

1. Since  $U$  is open in  $M$ ,  $\bar{U} = U \cap \partial M$  is open in  $\partial M$ .
2. Since  $(U, \phi) \in X_\partial(M)$ ,  $\phi(U)$  is open in  $\mathbb{H}^n$ . A previous exercise implies that  $\phi(\bar{U}) = \phi(U) \cap \partial\mathbb{H}^n$  which is open in  $\partial\mathbb{H}^n$ . Since  $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$  is a homeomorphism, we have that  $\pi(\phi(\bar{U}))$  is open in  $\mathbb{R}^{n-1}$ .
3. Since  $\phi|_{\bar{U}} : \bar{U} \rightarrow \phi(U) \cap \partial\mathbb{H}^n$  and  $\pi|_{\phi(\bar{U})} : \phi(\bar{U}) \rightarrow \pi(\phi(\bar{U}))$  are homeomorphisms, we have that  $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$  is a homeomorphism.

Hence  $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$ . □

**Exercise 3.1.0.24.** Let  $M$  be an  $n$ -dimensional topological manifold. Then

1.  $\partial M$  is an  $(n-1)$ -dimensional topological manifold
2.  $\partial(\partial M) = \emptyset$

*Proof.*

1. (a) Since  $M$  is Hausdorff,  $\partial M$  is Hausdorff.
- (b) Since  $M$  is second-countable,  $\partial M$  is second countable.
- (c) Let  $p \in \partial M$ . Then there exists  $(U, \phi) \in X_\partial(M)$  such that  $\phi(p) \in \partial\mathbb{H}^n$ . Then  $p \in \bar{U}$  and the previous exercise implies that  $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$ . Thus  $\partial M$  is locally Euclidean of dimension  $n-1$ .

Hence  $\partial M$  is an  $(n-1)$ -dimensional topological manifold.

2. Let  $p \in \partial M$ . Part (1) implies that there exists  $(U, \phi) \in X_{\text{Int}}^{n-1}(\partial M)$  such that  $p \in U$ . Thus  $p \in \text{Int } \partial M$ . Since  $p \in \partial M$  is arbitrary,  $\text{Int } \partial M = \partial M$ . Hence

$$\begin{aligned} \partial(\partial M) &= (\text{Int}(\partial M))^c \\ &= (\partial M)^c \\ &= \emptyset \end{aligned}$$

□

**Exercise 3.1.0.25.** Let  $M$  be an  $n$ -dimensional topological manifold,  $(U, \phi) \in X(M)$  and  $U' \subset U$ . If  $U'$  is open in  $M$ , then  $(U', \phi|_{U'}) \in X^n(M)$ .

*Proof.* Suppose that  $U'$  is open in  $M$ . Set  $\phi' = \phi|_{U'}$ .

- By assumption  $U'$  is open in  $M$ .
- Since  $U'$  is open in  $M$ , we have that  $U' = U' \cap U$  is open in  $U$ . Since  $\phi$  is a homeomorphism and  $U'$  is open in  $U$ , we have that  $\phi(U')$  is open in  $\phi(U)$ . By assumption  $\phi(U)$  is open in  $\mathbb{R}^n$  or  $\phi(U)$  is open in  $\mathbb{H}^n$ . Therefore  $\phi'(U')$  is open in  $\mathbb{R}^n$  or  $\phi'(U')$  is open in  $\mathbb{H}^n$ .
- Since  $\phi : U \rightarrow V$  is a homeomorphism,  $\phi' : U' \rightarrow \phi'(U')$  is a homeomorphism.

So  $(U', \phi') \in X^n(M)$ . □

**Note 3.1.0.26.** Since  $U$  is open in  $M$ ,  $U'$  being open in  $U$  is equivalent to  $U'$  being open in  $M$ , so we could have also assumed that  $U'$  is open in  $U$ .

**Exercise 3.1.0.27.** Let  $M$  be an  $n$ -dimensional topological manifold and  $U \subset M$ . If  $U$  is open, then

$$X^n(U) = \{(V, \psi) \in X^n(M) : V \subset U\}$$

*Proof.* Suppose that  $U$  is open and set  $A = \{(V, \psi) \in X^n(M) : V \subset U\}$ . Let  $(V, \psi) \in X^n(U)$ . By definition of  $X^n(U)$ ,  $V$  is open in  $U$ . Thus, there exists  $W \subset M$  such that  $W$  is open in  $M$  and  $V = U \cap W$ . Since  $U$  is open in  $M$ , we have that  $V = U \cap W$  is open in  $M$ . Hence  $(V, \psi) \in X^n(M)$  which implies that  $(V, \psi) \in A$ . Since  $(V, \psi) \in X^n(U)$  is arbitrary,  $X^n(U) \subset A$ .

Conversely, suppose that  $(V, \psi) \in A$ . Then  $(V, \psi) \in X^n(M)$  and  $V \subset U$ . By definition of  $X^n(M)$ ,  $V$  is open in  $M$ . Since  $V \subset U$ , we have that  $V = V \cap U$  is open in  $U$ . Hence  $(V, \psi) \in X^n(U)$ . Since  $(V, \psi) \in A$  is arbitrary,  $A \subset X^n(U)$ . Hence  $X^n(U) = A$ .  $\square$

**Exercise 3.1.0.28.** Let  $M$  be an  $n$ -dimensional topological manifold,  $(U, \phi) \in X(M)$  and  $U' \subset U$ . If  $U'$  is open in  $M$ , then  $(U', \phi|_{U'}) \in X^n(U)$ .

*Proof.* Suppose that  $U'$  is open in  $M$ . A previous exercise implies that  $(U', \phi') \in X^n(M)$ . The previous exercise implies that  $(U', \phi') \in X^n(U)$ .  $\square$

**Exercise 3.1.0.29. Topological Open Submanifolds:**

Let  $M$  be an  $n$ -dimensional topological manifold and  $U \subset M$  open. Then  $U$  is an  $n$ -dimensional topological manifold.

*Proof.*

1. Since  $M$  is Hausdorff,  $U$  is Hausdorff.
2.  $M$  is second-countable,  $U$  is second countable.
3. Let  $p \in U$ . Since then there exists  $(V, \psi) \in X^n(M)$  such that  $p \in V$ . Set  $V' = U \cap V$  and  $\psi' = \psi|_{U \cap V}$ . The previous exercise implies that  $(V', \psi') \in X^n(U)$ . Therefore  $U$  is locally Euclidean of dimension  $n$ .

Hence  $U$  is an  $n$ -dimensional topological manifold.  $\square$

**Exercise 3.1.0.30.** Let  $M$  be an  $n$ -dimensional topological manifold and  $U \subset M$ . If  $U$  is open, then

1.  $X_{\text{Int}}(U) = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$
2.  $X_{\partial}(U) = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$

*Proof.* Suppose that  $U$  is open in  $M$ .

1. Set  $A = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$ . Let  $(V, \psi) \in X_{\text{Int}}(U)$ . By definition of  $X_{\text{Int}}(U)$ ,  $V$  is open in  $U$  and  $\phi(V)$  is open in  $\mathbb{R}^n$ . Since  $U$  is open in  $M$ ,  $V$  is open in  $M$ . Hence  $(V, \psi) \in X_{\text{Int}}(M)$ . Since  $U$  is open in  $M$ ,  $V$  is open in  $M$ . Hence  $(V, \psi) \in X_{\text{Int}}(M)$  which implies that  $(V, \psi) \in A$ . Since  $(V, \psi) \in X_{\text{Int}}(U)$  is arbitrary,  $X_{\text{Int}}(U) \subset A$ .  
Conversely, let  $(V, \psi) \in A$ . Then  $(V, \psi) \in X_{\text{Int}}(M)$  and  $V \subset U$ . By definition of  $X_{\text{Int}}(M)$ ,  $V$  is open in  $M$  and  $\phi(V)$  is open in  $\mathbb{R}^n$ . Thus  $V = V \cap U$  is open in  $U$ . So  $(V, \psi) \in X_{\text{Int}}(U)$ . Since  $(V, \psi) \in A$  is arbitrary,  $A \subset X_{\text{Int}}(U)$ . Thus  $X_{\text{Int}}(U) = A$ .
2. Set  $B = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$ . Let  $(V, \psi) \in X_{\partial}(U)$ . By definition of  $X_{\partial}(U)$ ,  $V$  is open in  $U$ ,  $\phi(V)$  is open in  $\mathbb{H}^n$  and  $\partial\mathbb{H}^n \cap \phi(V) \neq \emptyset$ . Since  $U$  is open in  $M$ ,  $V$  is open in  $M$ . Hence  $(V, \psi) \in X_{\partial}(M)$ , which implies that  $(V, \psi) \in B$ . Since  $(V, \psi) \in X_{\partial}(U)$  is arbitrary,  $X_{\partial}(U) \subset B$ .  
Conversely, let  $(V, \psi) \in B$ . Then  $(V, \psi) \in X_{\partial}(M)$  and  $V \subset U$ . By definition of  $X_{\partial}(M)$ ,  $V$  is open in  $M$ ,  $\phi(V)$  is open in  $\mathbb{H}^n$  and  $\partial\mathbb{H}^n \cap \phi(V) \neq \emptyset$ . Thus  $V = V \cap U$  is open in  $U$ . So  $(V, \psi) \in X_{\partial}(U)$ . Since  $(V, \psi) \in B$  is arbitrary,  $B \subset X_{\partial}(U)$ . Thus  $X_{\partial}(U) = B$ .

$\square$

**Exercise 3.1.0.31.** Let  $M$  be an  $n$ -dimensional topological manifold and  $U \subset M$ . If  $U$  is open, then  $\partial U = \partial M \cap U$ .

*Proof.* Suppose that  $U$  is open. Let  $p \in \partial U$ . Then there exists  $(V, \psi) \in X_{\partial}(U)$  such that  $p \in V$  and  $\psi(p) \in \partial \mathbb{H}^n$ . Since  $U$  is open, the previous exercise implies that  $(V, \psi) \in X_{\partial}(M)$ . Thus  $p \in \partial M$ . Since  $p \in \partial U$  is arbitrary,  $\partial U \subset \partial M$ . Since  $\partial U \subset U$ , we have that  $\partial U \subset \partial M \cap U$ .

Conversely, let  $p \in \partial M \cap U$ . Since  $p \in \partial M$ , there exists  $(V, \psi) \in X_{\partial}(M)$  such that  $p \in V$  and  $\psi(p) \in \partial \mathbb{H}^n$ . Set  $V' = V \cap U$  and  $\psi' = \psi|_{V'}$ . Then  $p \in V'$  since  $V$  and  $U$  are open in  $M$ ,  $V'$  is open in  $M$ . A previous exercise implies that  $(V', \psi') \in X(M)$ . Since  $p \in \partial M$ , a previous exercise implies that  $(V', \psi') \in X_{\partial}(M)$ . The previous exercise implies that  $(V', \psi') \in X_{\partial}(U)$ . Since  $\psi'(p) \in \partial \mathbb{H}^n$ ,  $p \in \partial U$ . Since  $p \in \partial M \cap U$  is arbitrary,  $\partial M \cap U \subset \partial U$ . Hence  $\partial U = \partial M \cap U$ .

label exercises and reference them!!! □

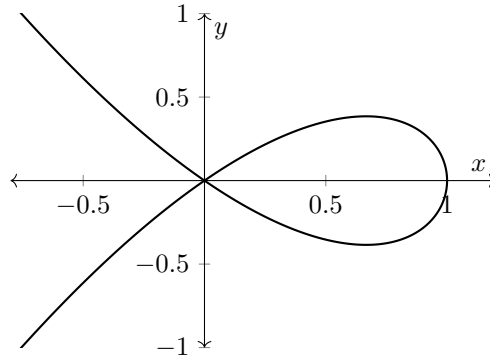
**Exercise 3.1.0.32. Graph of Continuous Function:**

Let  $f \in C(\mathbb{R})$ . Set  $M = \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$  (i.e. the graph of  $f$ ). Then  $M$  is a 1-dimensional manifold.

*Proof.* Set  $U = \mathbb{R}$  and define  $\phi : U \rightarrow M$  by  $\phi(x) = (x, f(x))$ . Then  $\phi^{-1} = \pi_1$ . Since  $f$  is continuous,  $\phi$  is continuous. Since  $\pi_1$  is continuous,  $\phi$  is a homeomorphism. □

**Exercise 3.1.0.33. Nodal Cubic:**

Let  $M = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2 - x^3\}$ . We equip  $M$  with the subspace topology.



Then  $M$  is not a 1-dimensional topological manifold.

**Hint:** connected components

*Proof.* Suppose that  $M$  is a 1-dimensional manifold. Set  $p = (0, 0)$ . Then there exists  $(U, \phi) \in X(M)$  such that  $p \in U$ . Since  $\phi(U)$  is open (in  $\mathbb{R}$  or  $\mathbb{H}$ ), there exists a  $B \subset \phi(U)$  such that  $B$  is open (in  $\mathbb{R}$  or  $\mathbb{H}$ ),  $B$  is connected and  $\phi(p) \in B$ . Set  $V = \phi^{-1}(B)$ ,  $V' = V \setminus \{p\}$  and  $B' = B \setminus \{\phi(p)\}$ . Then  $\phi : V \rightarrow B$  and  $\phi' : V' \rightarrow B'$  are homeomorphisms. Since  $B$  is open (in  $\mathbb{R}$  or  $\mathbb{H}$ ) and connected,  $B'$  has at most two connected components. Then  $V'$  This is a contradiction since  $V'$  has four connected components and  $B'$  and  $V'$  are homeomorphic. □

**Exercise 3.1.0.34. Topological Manifold Chart Lemma:**

Let  $M$  be a set,  $\Gamma$  an index set and for each  $\alpha \in \Gamma$ ,  $U_{\alpha} \subset M$  and  $\phi_{\alpha} : U_{\alpha} \rightarrow \mathbb{H}^n$ . Suppose that

- for each  $\alpha \in \Gamma$ ,  $\phi_{\alpha}(U_{\alpha}) \in \mathcal{T}_{\mathbb{H}^n}$
- for each  $\alpha, \beta \in \Gamma$ ,  $\phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \in \mathcal{T}_{\mathbb{H}^n}$
- for each  $\alpha \in \Gamma$ ,  $\phi_{\alpha} : U_{\alpha} \rightarrow \phi_{\alpha}(U_{\alpha})$  is a bijection
- for each  $\alpha, \beta \in \Gamma$ ,  $\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}} \circ (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}})^{-1} : \phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow \phi_{\beta}(U_{\alpha} \cap U_{\beta})$  is continuous
- there exists  $\Gamma' \subset \Gamma$  such that  $\Gamma'$  is countable and  $M \subset \bigcup_{\alpha \in \Gamma'} U_{\alpha}$

- for each  $p, q \in M$ , there exists  $\alpha \in \Gamma$  such that  $p, q \in U_\alpha$  or there exist  $\alpha, \beta \in \Gamma$  such that  $p \in U_\alpha$ ,  $q \in U_\beta$  and  $U_\alpha \cap U_\beta = \emptyset$

Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : V \in \mathcal{T}_{\mathbb{H}^n} \text{ and } \alpha \in \Gamma\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$

Then

1.  $\mathcal{B}$  is a basis for  $\mathcal{T}_M$   
**Hint:** For  $B_1, B_2 \subset \mathbb{H}^n$ ,  $\phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) = \phi_{\alpha_1}^{-1}(B_1 \cap [\phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}} \circ (\phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}})^{-1}(B_2)])$
2.  $(M, \mathcal{T}_M)$  is an  $n$ -dimensional topological manifold
3.  $\mathcal{T}_M$  is the unique topology  $\mathcal{T}$  on  $M$  such that  $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T})$

*Proof.*

1.
  - By assumption,  $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
  - Let  $A_1, A_2 \in \mathcal{B}$  and  $p \in A_1 \cap A_2$ . By definition, there exist  $\alpha_1, \alpha_2 \in \Gamma$  and  $B_1, B_2 \subset \mathbb{H}^n$  such that  $B_1, B_2$  are open in  $\mathbb{H}^n$  and

$$\begin{aligned} A_1 &= \phi_{\alpha_1}^{-1}(B_1) & A_2 &= \phi_{\alpha_2}^{-1}(B_2) \\ &\subset U_{\alpha_1} & &\subset U_{\alpha_2} \end{aligned}$$

Set  $\psi_1 = \phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}}$  and  $\psi_2 = \phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}}$ . We note that

$$\begin{aligned} \psi_1^{-1}(B_1) &= U_{\alpha_2} \cap \phi_{\alpha_1}^{-1}(B_1) & \psi_2^{-1}(B_2) &= U_{\alpha_1} \cap \phi_{\alpha_2}^{-1}(B_2) \\ &= U_{\alpha_2} \cap A_1 & &= U_{\alpha_1} \cap A_2 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} & &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned}$$

Let  $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$ . Then  $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$ . Hence  $\phi_{\alpha_1}(q) \in B_1$  and  $\phi_{\alpha_1}(q) \in \psi_1 \circ \psi_2^{-1}(B_2)$ . This implies that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(B_1) \\ &= A_1 \end{aligned}$$

and since  $\psi_2^{-1}(B_2) \subset U_{\alpha_1} \cap U_{\alpha_2}$  and  $\phi_{\alpha_1} : U_{\alpha_1} \rightarrow \phi_{\alpha_1}(U_{\alpha_1})$  is a bijection, we have that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(\psi_1 \circ \psi_2^{-1}(B_2)) \\ &= \psi_2^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \end{aligned}$$

Thus

$$\begin{aligned} q &\in A_1 \cap (U_{\alpha_1} \cap A_2) \\ &= A_1 \cap A_2 \end{aligned}$$

Since  $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$  is arbitrary, we have that  $\phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \subset A_1 \cap A_2$ . Conversely, let

$$\begin{aligned} q &\in A_1 \cap A_2 \\ &= \phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) \end{aligned}$$

Then  $\phi_{\alpha_1}(q) \in B_1$  and  $\phi_{\alpha_2}(q) \in B_2$ . Since  $A_1 \cap A_2 \subset U_{\alpha_1} \cap U_{\alpha_2}$ , we have that

$$\begin{aligned}\psi_2(q) &= \phi_{\alpha_2}(q) \\ &\in B_2\end{aligned}$$

which implies that  $q \in \psi_2^{-1}(B_2)$ . Therefore

$$\begin{aligned}\phi_{\alpha_1}(q) &= \psi_1(q) \\ &\in \psi_1(\psi_2^{-1}(B_2)) \\ &= \psi_1 \circ \psi_2^{-1}(B_2)\end{aligned}$$

Hence  $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$ . This implies that  $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$ . Since  $q \in A_1 \cap A_2$  is arbitrary, we have that  $A_1 \cap A_2 \subset \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$ . Thus

$$\begin{aligned}A_1 \cap A_2 &= \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \\ &\in \mathcal{B}\end{aligned}$$

Thus  $\mathcal{B}$  is a basis for  $\mathcal{T}_M$ .

2. (a) **(locally Euclidean of dimension  $n$ ):**

Let  $\alpha \in \Gamma$ . By definition, for each  $B \subset \mathcal{T}_{\mathbb{H}^n}$ ,

$$\begin{aligned}\phi_{\alpha}^{-1}(B) &\in \mathcal{B} \\ &\subset \mathcal{T}_M\end{aligned}$$

Hence  $\phi_{\alpha}$  is continuous.

Let  $A \in \mathcal{T}_{U_{\alpha}}$ . Then there exists  $U \subset \mathcal{T}_M$  such that  $A = U \cap U_{\alpha}$ . Since  $\mathcal{B}$  is a basis for  $\mathcal{T}_M$ , there exists  $\Gamma' \subset \Gamma$ ,  $(V_{\beta})_{\beta \in \Gamma'} \subset \mathcal{T}_{\mathbb{H}^n}$  such that  $U = \bigcup_{\beta \in \Gamma'} \phi_{\beta}^{-1}(V_{\beta})$ . Thus

$$\begin{aligned}A &= U \cap U_{\alpha} \\ &= \left[ \bigcup_{\beta \in \Gamma'} \phi_{\beta}^{-1}(V_{\beta}) \right] \cap U_{\alpha} \\ &= \bigcup_{\beta \in \Gamma'} [\phi_{\beta}^{-1}(V_{\beta}) \cap U_{\alpha}]\end{aligned}$$

Let  $\beta \in \Gamma'$ . Since  $\phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \subset \phi_{\alpha}(U_{\alpha})$  and  $\phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \in \mathcal{T}_{\mathbb{H}^n}$ , we have that

$$\begin{aligned}\phi_{\alpha}(U_{\alpha} \cap U_{\beta}) &= \phi_{\alpha}(U_{\alpha}) \cap \phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \\ &\in \mathcal{T}_{\phi_{\alpha}(U_{\alpha})}\end{aligned}$$

Therefore  $\mathcal{T}_{\phi_{\alpha}(U_{\alpha} \cap U_{\beta})} \subset \mathcal{T}_{\phi_{\alpha}(U_{\alpha})}$ . Since  $(\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}}) \circ (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}})^{-1} : \phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow \phi_{\beta}(U_{\alpha} \cap U_{\beta})$  is continuous, we have that  $(\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}}) \circ (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}})^{-1} : \phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow \mathbb{H}^n$  is continuous and therefore

$$\begin{aligned}[(\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}}) \circ (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}})^{-1}]^{-1}(V_{\beta}) &\in \mathcal{T}_{\phi_{\alpha}(U_{\alpha} \cap U_{\beta})} \\ &\subset \mathcal{T}_{\phi_{\alpha}(U_{\alpha})}\end{aligned}$$

Since  $\beta \in \Gamma'$  is arbitrary, we have that

$$\begin{aligned}
 \phi_\alpha(A) &= \phi_\alpha\left(\bigcup_{\beta \in \Gamma'} [\phi_\beta^{-1}(V_\beta) \cap U_\alpha]\right) \\
 &= \bigcup_{\beta \in \Gamma'} \phi_\alpha(\phi_\beta^{-1}(V_\beta) \cap U_\alpha) \\
 &= \bigcup_{\beta \in \Gamma'} (\phi_\alpha|_{U_\alpha \cap U_\beta}) \circ (\phi_\beta|_{U_\alpha \cap U_\beta})^{-1}(V_\beta) \\
 &= \bigcup_{\beta \in \Gamma'} [(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1}]^{-1}(V_\beta) \\
 &\in \mathcal{T}_{\phi_\alpha(U_\alpha)}
 \end{aligned}$$

Since  $A \in \mathcal{T}_{U_\alpha}$  is arbitrary,  $\phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \rightarrow U_\alpha$  is continuous. Hence  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a homeomorphism and  $(U_\alpha, \phi_\alpha) \in X^n(M)$ . Since  $M = \bigcup_{\alpha \in \Gamma} U_\alpha$ , we have that  $M$  is locally Euclidean of dimension  $n$ .

(b) **(Hausdorff):**

Let  $p, q \in M$ . Suppose that  $p \neq q$ . Then there exists  $\alpha \in \Gamma$  such that  $p, q \in U_\alpha$  or there exist  $\alpha, \beta \in \Gamma$  such that  $p \in U_\alpha$ ,  $q \in U_\beta$  and  $U_\alpha \cap U_\beta = \emptyset$ .

- Suppose that there exists  $\alpha \in \Gamma$  such that  $p, q \in U_\alpha$ . Since  $p \neq q$ ,  $\phi_\alpha(p) \neq \phi_\alpha(q)$ . Since  $\mathbb{H}^n$  is Hausdorff, there exist  $V_p, V_q \subset \phi(U_\alpha)$  such that  $V_p$  and  $V_q$  are open in  $\mathbb{H}^n$ ,  $p \in V_p$ ,  $q \in V_q$  and  $V_p \cap V_q = \emptyset$ . Set  $U_p = \phi_\alpha^{-1}(V_p)$  and  $U_q = \phi_\alpha^{-1}(V_q)$ . Then  $U_p, U_q$  are open,  $p \in U_p$ ,  $q \in U_q$  and  $U_p \cap U_q = \emptyset$ .
- Suppose that there exist  $\alpha, \beta \in \Gamma$  such that  $p \in U_\alpha$ ,  $q \in U_\beta$  and  $U_\alpha \cap U_\beta = \emptyset$ . Set  $U_p = U_\alpha$  and  $U_q = U_\beta$ . Then  $U_p, U_q$  are open,  $p \in U_p$ ,  $q \in U_q$  and  $U_p \cap U_q = \emptyset$ .

Thus for each  $p, q \in M$  there exist  $U_p, U_q \subset M$  such that  $U_p, U_q$  are open,  $p \in U_p$ ,  $q \in U_q$  and  $U_p \cap U_q = \emptyset$ . Hence

(c) **(second-countable):**

By assumption, there exists  $\Gamma' \subset \Gamma$  such that  $\Gamma'$  is countable and  $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$ . Let  $\alpha \in \Gamma'$ .

Since  $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$  and  $\mathbb{H}^n$  is second-countable, we have that  $\phi_\alpha(U_\alpha)$  is second-countable. Since  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a homeomorphism, we have that  $U_\alpha$  is second-countable. Since  $M = \bigcup_{\alpha \in \Gamma'} U_\alpha$ ,

an exercise in topology [cite](#) implies that  $M$  is second-countable.

3. Let  $\mathcal{T}$  be a topology on  $M$ . Suppose that  $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T})$ . Then for each  $\alpha \in \Gamma$ ,  $U_\alpha \in \mathcal{T}$  and  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a  $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$ -homeomorphism.

Let  $U \in \mathcal{B}$ . By definition, there exists  $\alpha \in \Gamma$  and  $V \in \mathcal{T}_{\mathbb{H}^n}$  such that  $U = \phi_\alpha^{-1}(V)$ . Since  $U_\alpha \in \mathcal{T}$ , we have that  $\mathcal{T} \cap U_\alpha \subset \mathcal{T}$ . Since  $V \cap \phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$ , and  $\phi_\alpha$  is a  $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$ -homeomorphism, we have that

$$\begin{aligned}
 U &= \phi_\alpha^{-1}(V) \\
 &= \phi_\alpha^{-1}(V \cap \phi_\alpha(U_\alpha)) \\
 &\in \mathcal{T} \cap U_\alpha \\
 &\subset \mathcal{T}
 \end{aligned}$$

Since  $U \in \mathcal{B}$  is arbitrary,  $\mathcal{B} \subset \mathcal{T}$ . Therefore

$$\begin{aligned}
 \mathcal{T}_M &= \tau(\mathcal{B}) \\
 &\subset \tau(\mathcal{T}) \\
 &= \mathcal{T}
 \end{aligned}$$

Conversely, Let  $U \in \mathcal{T}$  and  $\alpha \in \Gamma$ . Then  $U \cap U_\alpha \in \mathcal{T} \cap U_\alpha$ . Since  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a  $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$ -homeomorphism, we have that  $\phi_\alpha(U \cap U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$ . Since  $U_\alpha \in \mathcal{T}_M$ ,  $\mathcal{T}_M \cap U_\alpha \subset \mathcal{T}_M$ . Since  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a  $(\mathcal{T}_M \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$ -homeomorphism, we have that

$$\begin{aligned} U \cap U_\alpha &= \phi_\alpha^{-1}(\phi_\alpha(U \cap U_\alpha)) \\ &\in \mathcal{T}_M \cap U_\alpha \\ &\subset \mathcal{T}_M \end{aligned}$$

Then

$$\begin{aligned} U &= U \cap M \\ &= U \cap \left( \bigcup_{\alpha \in \Gamma} U_\alpha \right) \\ &= \bigcup_{\alpha \in \Gamma} (U \cap U_\alpha) \\ &\in \mathcal{T}_M \end{aligned}$$

Since  $U \in \mathcal{T}$  is arbitrary,  $\mathcal{T} \subset \mathcal{T}_M$ . Thus  $\mathcal{T} = \mathcal{T}_M$ . □

**Exercise 3.1.0.35.** Let  $M$  be a set,  $\Gamma$  an index set and for each  $\alpha \in \Gamma$ ,  $U_\alpha \subset M$  and  $\phi_\alpha : U_\alpha \rightarrow \mathbb{H}^n$ . Suppose that

- for each  $\alpha \in \Gamma$ ,  $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$
- for each  $\alpha, \beta \in \Gamma$ ,  $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$
- for each  $\alpha \in \Gamma$ ,  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a bijection
- for each  $\alpha, \beta \in \Gamma$ ,  $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$  is continuous
- there exists  $\Gamma' \subset \Gamma$  such that  $\Gamma'$  is countable and  $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$
- for each  $p, q \in M$ , there exists  $\alpha \in \Gamma$  such that  $p, q \in U_\alpha$  or there exist  $\alpha, \beta \in \Gamma$  such that  $p \in U_\alpha$ ,  $q \in U_\beta$  and  $U_\alpha \cap U_\beta = \emptyset$

Then there exists a unique topology  $\mathcal{T}_M$  on  $M$  such that  $(M, \mathcal{T}_M)$  is an  $n$ -dimensional topological manifold and  $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T}_M)$ .

*Proof.* Immediate by previous exercise. □



## 3.2 Smooth Manifolds

**Definition 3.2.0.1.** Let  $M$  be an  $n$ -dimensional topological manifold and  $(U, \phi), (V, \psi) \in X(M)$ . Then  $(U, \phi)$  and  $(V, \psi)$  are said to be **smoothly compatible** if

$$\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V) \text{ is a diffeomorphism}$$

**Definition 3.2.0.2.** Let  $M$  be an  $n$ -dimensional topological manifold.

- Let  $\mathcal{A} \subset X(M)$ . Then  $\mathcal{A}$  is said to be an **atlas on  $M$**  if  $M \subset \bigcup_{(U, \phi) \in \mathcal{A}} U$ .
- Let  $\mathcal{A}$  be an atlas on  $M$ . Then  $\mathcal{A}$  is said to be **smooth** if for each  $(U, \phi), (V, \psi) \in \mathcal{A}$ ,  $(U, \phi)$  and  $(V, \psi)$  are smoothly compatible.
- Let  $\mathcal{A}$  be a smooth atlas on  $M$ . Then  $\mathcal{A}$  is said to be **maximal** if for each smooth atlas  $\mathcal{B}$  on  $M$ ,  $\mathcal{A} \subset \mathcal{B}$  implies that  $\mathcal{A} = \mathcal{B}$ . A maximal smooth atlas on  $M$  is called a **smooth structure on  $M$** .
- Let  $\mathcal{A}$  be an atlas on  $M$ . Then  $(M, \mathcal{A})$  is said to be an  **$n$ -dimensional smooth manifold** if  $\mathcal{A}$  is a smooth structure on  $M$ .

**Exercise 3.2.0.3.** Let  $M$  be an  $n$ -dimensional topological manifold and  $\mathcal{B}$  a smooth atlas on  $M$ . Then there exists a unique smooth structure  $\mathcal{A}$  on  $M$  such that  $\mathcal{B} \subset \mathcal{A}$ .

*Proof.* Define

$$\mathcal{A} = \{(U, \phi) \in X(M) : \text{for each } (V, \psi) \in \mathcal{B}, (U, \phi) \text{ and } (V, \psi) \text{ are smoothly compatible}\}$$

Clearly  $\mathcal{B} \subset \mathcal{A}$ . Let  $(U, \phi)$  and  $(V, \psi) \in \mathcal{A}$ . Define  $F : \phi(U \cap V) \rightarrow \psi(U \cap V)$  by

$$F = \psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$$

Let  $q \in \phi(U \cap V)$ . Set  $p = \phi^{-1}(q)$ . Since  $p \in U \cap V \subset M$ , there exists  $(W, \chi) \in \mathcal{B}$  such that  $p \in W$ . By definition of  $\mathcal{A}$ ,  $\psi|_{W \cap V} \circ (\chi|_{W \cap V})^{-1} : \chi(W \cap V) \rightarrow \psi(W \cap V)$  and  $\chi|_{U \cap W} \circ (\phi|_{U \cap W})^{-1} : \phi(U \cap W) \rightarrow \chi(U \cap W)$  are diffeomorphisms. Set  $N = U \cap W \cap V$ . Then  $q \in \phi(N) \subset \phi(U \cap V)$  and

$$\begin{aligned} F|_{\phi(N)} &= \psi|_N \circ (\phi|_N)^{-1} \\ &= [\psi|_N \circ (\chi|_N)^{-1}] \circ [\chi|_N \circ (\phi|_N)^{-1}] \end{aligned}$$

is a diffeomorphism. Thus, for each  $q \in \phi(U \cap V)$ , there exists  $N' \subset \phi(U \cap V)$  such that  $F|_{N'}$  is a diffeomorphism. Hence  $F$  is a diffeomorphism and  $(U, \phi), (V, \psi)$  are smoothly compatible. Therefore  $\mathcal{A}$  is a smooth atlas.

To see that  $\mathcal{A}$  is maximal, let  $\mathcal{B}'$  be a smooth atlas on  $M$ . Suppose that  $\mathcal{A} \subset \mathcal{B}'$  and let  $(U, \phi) \in \mathcal{B}'$ . By definition, for each chart  $(V, \psi) \in \mathcal{B}'$ ,  $(U, \phi)$  and  $(V, \psi)$  are smoothly compatible. Since  $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$ , we have that  $(U, \phi) \in \mathcal{A}$ . So  $\mathcal{A} = \mathcal{B}'$  and  $\mathcal{A}$  is a maximal smooth atlas on  $M$ .  $\square$

**Exercise 3.2.0.4.** Let  $(M, \mathcal{A})$  be an  $n$ -dimensional smooth manifold,  $(U, \phi) \in \mathcal{A}$  and  $U' \subset U$ . If  $U'$  is open, then  $(U', \phi|_{U'}) \in \mathcal{A}$ .

*Proof.* Set  $\phi' = \phi|_{U'}$ . A previous exercise implies that  $(U', \phi') \in X(U)$ . Define  $\mathcal{B} = \mathcal{A} \cup \{(U', \phi')\}$ . Let  $(V, \psi) \in \mathcal{B}$ . If  $(V, \psi) = (U', \phi')$ , then

$$\phi' \circ \psi^{-1} = \text{id}_{U'}$$

which is a diffeomorphism. Thus  $(U', \phi'), (V, \psi)$  are smoothly compatible. Suppose that  $(V, \psi) \in \mathcal{A}$ . Since  $\mathcal{A}$  is smooth,  $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$  is a diffeomorphism. Therefore  $\psi|_{U' \cap V} \circ (\phi'|_{U' \cap V})^{-1} : \phi'(U' \cap V) \rightarrow \psi(U' \cap V)$  is a diffeomorphism and  $(U', \phi'), (V, \psi)$  are smoothly compatible. Since  $(V, \psi) \in \mathcal{A}$  is arbitrary,  $\mathcal{B}$  is smooth. Since  $\mathcal{A}$  is maximal and  $\mathcal{A} \subset \mathcal{B}$ , we have that  $\mathcal{A} = \mathcal{B}$  and  $(U', \phi') \in \mathcal{A}$ .  $\square$

**Exercise 3.2.0.5.** Let  $(M, \mathcal{A})$  be a  $n$ -dimensional smooth manifold and  $U \subset M$  open. Set  $\mathcal{B} = \{(V, \psi) \in \mathcal{A} : V \subset U\}$ . Then  $\mathcal{B}$  is a smooth atlas on  $U$ .

*Proof.*

- Some previous exercises imply that  $U$  is an  $n$ -dimensional topological manifold and  $X(U) = \{(V, \psi) \in X(M) : V \subset U\}$ . Since

$$\begin{aligned} \mathcal{B} &\subset \mathcal{A} \\ &\subset X(M) \end{aligned}$$

we have that  $\mathcal{B} \subset X(U)$ . Let  $p \in U$ . Then there exists  $(V, \psi) \in \mathcal{A}$  such that  $p \in V$ . Set  $V' = U \cap V$  and  $\psi' = \psi|_{V'}$ . The previous exercise implies that  $(V', \psi') \in \mathcal{A}$ . By definition,  $(V', \psi') \in \mathcal{B}$ . Since  $p \in U$  is arbitrary, we have that for each  $p \in U$ , there exists  $(V', \psi') \in \mathcal{B}$  such that  $p \in V'$ . Hence  $\mathcal{B}$  is an atlas on  $U$ .

- Let  $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ . Then  $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{A}$ . Since  $\mathcal{A}$  is smooth,  $(V_1, \psi_1)$  and  $(V_2, \psi_2)$  are smoothly compatible. Since  $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$  are arbitrary,  $\mathcal{B}$  is smooth. □

**Definition 3.2.0.6. Smooth Open Submanifold:**

Let  $(M, \mathcal{A})$  be an  $n$ -dimensional smooth manifold and  $U \subset M$  open. A previous exercise implies that  $U$  is an  $n$ -dimensional topological manifold. We define  $\mathcal{A}|_U \subset X(U)$  to be the unique smooth structure on  $U$  such that  $\{(V, \psi) \in \mathcal{A} : V \subset U\} \subset \mathcal{A}|_U$ . Then  $(U, \mathcal{A}|_U)$  is said to be a **smooth open submanifold of  $(M, \mathcal{A})$** .

**Exercise 3.2.0.7.** Let  $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$  be the projection map given by  $\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$ . Then  $\pi$  is a diffeomorphism.

*Proof.* Define projection map  $\pi' : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  by  $\pi'(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1})$ . Then  $\mathbb{R}^n$  is an open neighborhood of  $\partial\mathbb{H}^n$ ,  $\pi'|_{\partial\mathbb{H}^n} = \pi$  and  $\pi'$  is smooth. Then by definition,  $\pi$  is smooth. Clearly,  $\pi^{-1}$  is smooth. So  $\pi$  is a diffeomorphism. □

**Definition 3.2.0.8.** Let  $(M, \mathcal{A})$  be a  $n$ -dimensional smooth manifold and  $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$  the projection map. Recall that for  $(U, \phi) \in X_{\partial}^n(M)$ , the  $(n-1)$ -coordinate chart  $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$  is defined by  $\bar{U} = U \cap \partial M$  and  $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$ .

We define

$$\bar{\mathcal{A}} = \{(\bar{U}, \bar{\phi}) : (U, \phi) \in \mathcal{A} \cap X_{\partial}^n(M)\}$$

**Exercise 3.2.0.9.** Let  $(M, \mathcal{A})$  be a  $n$ -dimensional smooth manifold. Then  $\bar{\mathcal{A}}$  is a smooth atlas on  $\partial M$ .

*Proof.*

- A previous exercise implies that  $\partial M$  is an  $(n-1)$ -dimensional topological manifold. Let  $p \in \partial M$ . Then there exists  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . Since  $\mathcal{A} \subset X^n(M)$  and  $p \in \partial M$ , we have that  $p \in \bar{U}$  and a previous exercise implies that  $(U, \phi) \in X_{\partial}^n(M)$ . By definition of  $\bar{\mathcal{A}}$ ,  $(\bar{U}, \bar{\phi}) \in \bar{\mathcal{A}}$ . Since  $p \in \partial M$  is arbitrary,  $\bar{\mathcal{A}}$  is an atlas on  $\partial M$ .
- Let  $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \bar{\mathcal{A}}$ . Since  $(U, \phi)$  and  $(V, \psi)$  are smoothly compatible,  $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$  is a diffeomorphism. Thus  $\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}$  is a diffeomorphism. Since  $\pi|_{\phi(U \cap V)}$  and  $\pi|_{\psi(U \cap V)}$  are diffeomorphisms,  $\pi|_{\phi(\bar{U} \cap \bar{V})}$  and  $\pi|_{\psi(\bar{U} \cap \bar{V})}$  are diffeomorphisms. Then

$$\begin{aligned} \bar{\psi}|_{\bar{U} \cap \bar{V}} \circ (\bar{\phi}|_{\bar{U} \cap \bar{V}})^{-1} &= \left[ \pi|_{\psi(\bar{U} \cap \bar{V})} \circ \psi|_{\bar{U} \cap \bar{V}} \right] \circ \left[ (\phi|_{\bar{U} \cap \bar{V}})^{-1} \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \right] \\ &= \pi|_{\psi(\bar{U} \cap \bar{V})} \circ \left[ \psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1} \right] \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \end{aligned}$$

is a diffeomorphism. Therefore  $(\bar{U}, \bar{\phi})$  and  $(\bar{V}, \bar{\psi})$  are smoothly compatible. Since  $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \bar{\mathcal{A}}$  are arbitrary,  $\bar{\mathcal{A}}$  is smooth. □

**Definition 3.2.0.10.** Let  $(M, \mathcal{A})$  be a  $n$ -dimensional smooth manifold. We define  $\mathcal{A}|_{\partial M}$  to be the unique smooth structure on  $\partial M$  such that  $\overline{\mathcal{A}} \subset \mathcal{A}|_{\partial M}$ . We define the **smooth boundary submanifold of  $M$**  to be  $(\partial M, \mathcal{A}|_{\partial M})$ .

**Exercise 3.2.0.11. Smooth Manifold Chart Lemma:**

Let  $M$  be a set,  $\Gamma$  an index set and for each  $\alpha \in \Gamma$ ,  $U_\alpha \subset M$  and  $\phi_\alpha : U_\alpha \rightarrow \mathbb{H}^n$ . Suppose that

- (a) for each  $\alpha \in \Gamma$ ,  $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$
- (b) for each  $\alpha, \beta \in \Gamma$ ,  $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$
- (c) for each  $\alpha \in \Gamma$ ,  $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$  is a bijection
- (d) for each  $\alpha, \beta \in \Gamma$ ,  $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$  is smooth
- (e) there exists  $\Gamma' \subset \Gamma$  such that  $\Gamma'$  is countable and  $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$
- (f) for each  $p, q \in M$ , there exists  $\alpha \in \Gamma$  such that  $p, q \in U_\alpha$  or there exist  $\alpha, \beta \in \Gamma$  such that  $p \in U_\alpha$ ,  $q \in U_\beta$  and  $U_\alpha \cap U_\beta \neq \emptyset$

Then there exists a unique smooth structure  $\mathcal{A}$  on  $M$  such that  $(M, \mathcal{A})$  is an  $n$ -dimensional smooth manifold and  $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset \mathcal{A}$ .

*Proof.* Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : \alpha \in \Gamma \text{ and } V \in \mathcal{T}_{\mathbb{H}^n}\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$
- $\mathcal{A}' = \{(U_\alpha, \phi_\alpha) : \alpha \in \Gamma\}$ .

The topological manifold chart lemma implies that  $(M, \mathcal{T}_M)$  is an  $n$ -dimensional topological manifold and  $\mathcal{A}' \subset X^n(M, \mathcal{T}_M)$ . Since  $M = \bigcup_{\alpha \in \Gamma} U_\alpha$ ,  $\mathcal{A}'$  is an atlas on  $M$ . Since for each  $\alpha, \beta \in \Gamma$ ,  $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$  is smooth, we have that  $\mathcal{A}'$  is smooth. A previous exercise implies that there exists a unique smooth structure  $\mathcal{A}$  on  $M$  such that  $\mathcal{A}' \subset \mathcal{A}$ .  $\square$

### 3.3 Smooth Maps

**Definition 3.3.0.1.** Let  $(M, \mathcal{A})$  be a smooth manifold and  $f : M \rightarrow \mathbb{R}$ . Then  $f$  is said to be smooth if for each coordinate chart  $(U, \phi) \in \mathcal{A}$ ,  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$  is smooth. The set of all smooth functions on  $M$  is denoted  $C^\infty(M)$ .

**Exercise 3.3.0.2.** Let  $(M, \mathcal{A})$  be a smooth manifold. Then  $C^\infty(M)$  is a vector space.

*Proof.* Let  $f, g \in C^\infty(M)$ ,  $\lambda \in \mathbb{R}$  and  $(U, \phi) \in \mathcal{A}$ . By assumption,  $f \circ \phi^{-1}$  and  $g \circ \phi^{-1}$  are smooth. Hence

$$(f + \lambda g) \circ \phi^{-1} = f \circ \phi^{-1} + \lambda g \circ \phi^{-1}$$

is smooth. Since  $(U, \phi) \in \mathcal{A}$  is arbitrary,  $f + \lambda g \in C^\infty(M)$ . Since  $f, g \in C^\infty(M)$  and  $\lambda \in \mathbb{R}$  are arbitrary,  $C^\infty(M)$  is a vector space.  $\square$

**Exercise 3.3.0.3.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $\mathcal{B}$  an atlas on  $M$  and  $f : M \rightarrow \mathbb{R}$ . Suppose that  $\mathcal{B} \subset \mathcal{A}$ . Then  $f$  is smooth iff for each  $(U, \phi) \in \mathcal{B}$ ,  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$  is smooth.

*Proof.*

- $(\implies)$ :  
Suppose that  $f$  is smooth. Let  $(U, \phi) \in \mathcal{B}$ . Since  $\mathcal{B} \subset \mathcal{A}$ ,  $(U, \phi) \in \mathcal{A}$ . Since  $f$  is smooth,  $f \circ \phi^{-1}$  is smooth. Since  $(U, \phi) \in \mathcal{B}$  is arbitrary, we have that for each  $(U, \phi) \in \mathcal{B}$ ,  $f \circ \phi^{-1}$  is smooth.
- $(\impliedby)$ :  
Suppose that for each  $(V, \psi) \in \mathcal{B}$ ,  $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$  is smooth. Let  $(U, \phi) \in \mathcal{A}$  and  $q \in \phi(U)$ . Set  $p = \phi^{-1}(q)$ . Since  $\mathcal{B}$  is an atlas, there exists  $(V, \psi) \in \mathcal{B}$  such that  $p \in V$ . Since  $\mathcal{B} \subset \mathcal{A}$ ,  $(V, \psi) \in \mathcal{A}$ . Set  $W = U \cap V$  and  $\tilde{\phi} = \phi|_W$  and  $\tilde{\psi} = \psi|_W$ . We note that  $\phi(W) \in \mathcal{N}_q$  and  $\phi(W)$  is open. An exercise in the section on smooth manifolds implies that  $(W, \tilde{\phi}), (W, \tilde{\psi}) \in \mathcal{A}$ . Therefore  $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(W) \rightarrow \psi(W)$  is smooth. By assumption,  $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$  is smooth. This implies that  $(f \circ \psi^{-1})|_{\psi(W)} : \psi(W) \rightarrow \mathbb{R}$  is smooth. Hence

$$\begin{aligned} (f \circ \phi^{-1})|_{\phi(W)} &= f \circ \tilde{\phi}^{-1} \\ &= f \circ (\tilde{\psi}^{-1} \circ \tilde{\psi}) \circ \tilde{\phi}^{-1} \\ &= (f \circ \tilde{\psi}^{-1}) \circ (\tilde{\psi} \circ \tilde{\phi}^{-1}) \end{aligned}$$

is smooth. Since  $q \in \phi(U)$  is arbitrary, for each  $q \in \phi(U)$ , there exists  $A \in \mathcal{N}_q$  such that  $A$  is open and  $(f \circ \phi^{-1})|_A : A \rightarrow \mathbb{R}$  is smooth. This implies that  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$  is smooth. Since  $(U, \phi) \in \mathcal{A}$  is arbitrary,  $f$  is smooth.  $\square$

**Exercise 3.3.0.4.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ ,  $p \in U$  and  $f \in C^\infty(M)$ . Then  $f|_U \in C^\infty(U)$ .

*Proof.* Let  $\square$

**Definition 3.3.0.5.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ ,  $f \in C^\infty(U)$  and  $i \in \{1, \dots, n\}$ . We define the **partial derivative of  $f$  with respect to  $x^i$** , denoted

$$\partial f / \partial x^i : U \rightarrow \mathbb{R} \quad \text{or} \quad \partial_i f : U \rightarrow \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left( \frac{\partial}{\partial u^i}[f \circ \phi^{-1}] \right) \circ \phi$$

**Exercise 3.3.0.6.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ ,  $f \in C^\infty(U)$  and  $i \in \{1, \dots, n\}$ . Then  $\partial/\partial x^i : C^\infty(U) \rightarrow C^\infty(U)$  is linear.

*Proof.* **FINISH!!!** □

**Exercise 3.3.0.7.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ ,  $f \in C^\infty(U)$  and  $i, j \in \{1, \dots, n\}$ . Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \left( \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi$$

*Proof.*

$$\begin{aligned} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial x^j} f \right) \\ &= \frac{\partial}{\partial x^i} \left( \left[ \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \circ \phi \right) \\ &= \left( \frac{\partial}{\partial u^i} \left[ \left( \left[ \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \circ \phi \right) \circ \phi^{-1} \right] \right) \circ \phi \\ &= \left( \frac{\partial}{\partial u^i} \left[ \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \right) \circ \phi \\ &= \left( \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi \end{aligned}$$

□

**Exercise 3.3.0.8.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $i, j \in \{1, \dots, n\}$ . Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i}$$

*Proof.* Let  $f \in C^\infty(U)$ . Since  $f \circ \phi^{-1}$  is smooth,

$$\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] = \frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i} [f \circ \phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \left( \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi \\ &= \left( \frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i} [f \circ \phi^{-1}] \right) \circ \phi \\ &= \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f \end{aligned}$$

□

**Exercise 3.3.0.9.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $f \in C^\infty(U)$ . Then for each  $\alpha \in \mathbb{N}_0^n$ ,

$$\partial^\alpha f = (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi$$

*Proof.* The claim is clearly true when  $|\alpha| = 0$  or by definition if  $|\alpha| = 1$ . Let  $n \in \mathbb{N}$  and suppose the claim is true for each  $|\alpha| \in \{1, \dots, n-1\}$ . Then there exists  $i \in \{1, \dots, n\}$  such that  $\alpha_i \geq 1$ . Hence

$$\begin{aligned} \partial^\alpha f &= \partial^{e^i} (\partial^{\alpha-e^i} f) \\ &= \partial^{e^i} (\partial^{\alpha-e^i} [f \circ \phi^{-1}] \circ \phi) \\ &= (\partial^{e^i} [(\partial^{\alpha-e^i} [f \circ \phi^{-1}] \circ \phi) \circ \phi^{-1}]) \circ \phi \\ &= (\partial^{e^i} [\partial^{\alpha-e^i} [f \circ \phi^{-1}]]) \circ \phi \\ &= (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi \end{aligned}$$

□

**Exercise 3.3.0.10. Taylor's Theorem:**

Let  $(M, \mathcal{A})$  be a smooth manifold,  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $\phi(U)$  convex,  $p \in U$ ,  $f \in C^\infty(U)$  and  $T \in \mathbb{N}$ . Then there exist  $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$  such that

$$f = \sum_{k=0}^T \left[ \sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(x_0) \right] + \sum_{|\alpha|=T+1} (x^i - x^i(p))^\alpha g_\alpha$$

and for each  $|\alpha| = T+1$ ,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

*Proof.* Since  $\phi(U)$  is open and convex and  $f \circ \phi^{-1} \in C^\infty(\phi(U))$ , Taylors thorem in section 2.1 implies that there exist  $(\tilde{g}_\alpha)_{|\alpha|=T+1} \subset C^\infty(\phi(U))$  such that for each  $q \in U$ ,

$$f \circ \phi^{-1}(\phi(q)) = \sum_{k=0}^T \left[ \sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q))$$

and for each  $|\alpha| = T+1$ ,

$$\begin{aligned} \tilde{g}_\alpha(\phi(p)) &= \frac{1}{(T+1)!} \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \\ &= \frac{1}{(T+1)!} \partial^\alpha f(p) \end{aligned}$$

For  $|\alpha| = T+1$ , set  $g_\alpha = \tilde{g}_\alpha \circ \phi$ . Then

$$\begin{aligned} f(q) &= f \circ \phi^{-1}(\phi(q)) \\ &= \sum_{k=0}^T \left[ \sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\ &= \sum_{k=0}^T \left[ \sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q) \end{aligned}$$

□

**Definition 3.3.0.11.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F : M \rightarrow N$ . Then  $F$  is said to be

- **smooth** if for each  $(U, \phi) \in \mathcal{A}$  and  $(V, \psi) \in \mathcal{B}$ ,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \psi(F(U) \cap V)$$

is smooth

- a **diffeomorphism** if  $F$  is a bijection and  $F, F^{-1}$  are smooth.

**Exercise 3.3.0.12.** Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifold and  $F : M \rightarrow N$ . If  $F$  is smooth, then  $F$  is continuous.

*Proof.* Suppose that  $F$  is smooth. Let  $p \in M$ . Choose  $(U, \phi) \in \mathcal{A}$  and  $(V, \psi) \in \mathcal{B}$  such that  $p \in U$  and  $F(p) \in V$ . Put  $\tilde{U} = U \cap F^{-1}(V)$  and  $\tilde{V} = F(U) \cap V$ . Define  $\tilde{\phi} : \tilde{U} \rightarrow \phi(\tilde{U})$  and  $\tilde{\psi} : \tilde{V} \rightarrow \psi(\tilde{V})$  by

$$\tilde{\phi} = \phi|_{\tilde{U}}, \quad \tilde{\psi} = \psi|_{\tilde{V}}$$

Then  $\tilde{\phi}$  and  $\tilde{\psi}$  are homeomorphisms,  $p \in \tilde{U}$  and  $F(\tilde{U}) \subset \tilde{V}$ . Define  $\tilde{F} : \phi(\tilde{U}) \rightarrow \psi(\tilde{V})$  by

$$\tilde{F} = \tilde{\psi} \circ F \circ \tilde{\phi}^{-1}$$

By definition,  $\tilde{F}$  is smooth and therefore continuous. Since  $\phi$  and  $\psi$  are homeomorphisms and  $F|_{\tilde{U}} = \tilde{\psi}^{-1} \circ \tilde{F} \circ \tilde{\phi}$ , we have that  $F|_{\tilde{U}}$  is continuous. In particular,  $F$  is continuous at  $p$  and since  $p \in M$  is arbitrary,  $F$  is continuous.  $\square$

**Exercise 3.3.0.13.** Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifold and  $F : M \rightarrow N$ . If  $F$  is a diffeomorphism, then  $F$  is a homeomorphism.

*Proof.* Suppose that  $F$  is a diffeomorphism. By definition,  $F$  and  $F^{-1}$  are smooth. The previous exercise implies that  $F$  and  $F^{-1}$  are continuous. Hence  $F$  is a homeomorphism.  $\square$

**Exercise 3.3.0.14.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F : M \rightarrow N$  a diffeomorphism. Then for each  $(U, \phi) \in \mathcal{A}$ ,  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ .

*Proof.* Let  $(V, \psi) \in \mathcal{B}$ .

1. Since  $\phi$  and  $F^{-1}$  are homeomorphisms,  $\phi \circ F^{-1} : F(U) \cap V \rightarrow \phi(U \cap F^{-1}(V))$  is a homeomorphism
2. Since  $F$  is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \rightarrow \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \psi(V \cap F(U))$$

are smooth.

Therefore  $(F(U), \phi \circ F^{-1})$  and  $(V, \psi)$  are smoothly compatible. Since  $\mathcal{B}$  is maximal,  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ .  $\square$

**Definition 3.3.0.15.** Let  $(N, \mathcal{B})$  be a smooth  $n$ -dimensional manifold,  $F : M \rightarrow N$  smooth and  $(V, \psi) \in \mathcal{B}$  with  $\psi = (y^1, \dots, y^n)$ . For  $i \in \{1, \dots, n\}$ , We define the  **$i$ -th component of  $F$  with respect to  $(V, \psi)$** , denoted  $F^i : V \rightarrow \mathbb{R}$ , by

$$F^i = y^i \circ F$$

### 3.4 Partitions of Unity

**Definition 3.4.0.1.** Let  $p \in M$ ,  $U \in \mathcal{N}_a$  open and  $\rho \in C_c^\infty(M)$ . Then  $\rho$  is said to be a **bump function at  $p$  supported in  $U$**  if

1.  $\rho \geq 0$
2. there exists  $V \in \mathcal{N}_p$  such that  $V$  is open and  $\rho|_V = 1$
3.  $\text{supp } \rho \subset U$

**Exercise 3.4.0.2.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then  $f \in C_c^\infty(\mathbb{R})$ .

*Proof.*

□



### 3.5 The Tangent Space

**Definition 3.5.0.1.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $p \in U$ . For  $i \in \{1, \dots, n\}$ , define the partial derivative with respect to  $x^i$  at  $p$ , denoted

$$\left. \frac{\partial}{\partial x^i} \right|_p : C^\infty(M) \rightarrow \mathbb{R}, \text{ or } \partial_i|_p : C^\infty(M) \rightarrow \mathbb{R}$$

by

$$\left. \frac{\partial}{\partial x^i} \right|_p f = \frac{\partial f}{\partial x^i}(p)$$

**Exercise 3.5.0.2.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $p \in U$ . Then for each  $i, j \in \{1, \dots, n\}$ , we have that

$$\left. \frac{\partial}{\partial x^i} \right|_p x^j(p) = \delta_{i,j}$$

*Proof.* Let  $i, j \in \{1, \dots, n\}$ . Then

$$\begin{aligned} \left. \frac{\partial}{\partial x^i} \right|_p x^j &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} x^j \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} u^j \circ \phi \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} u^j \\ &= \delta_{i,j} \end{aligned}$$

□

**Exercise 3.5.0.3. Change of Coordinates:**

Let  $(U, \phi), (V, \psi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $\psi = (y^1, \dots, y^n)$ ,  $p \in U \cap V$  and  $f \in C^\infty(M)$ . Then for each  $i \in \{1, \dots, n\}$ ,

$$\left. \frac{\partial}{\partial y^i} \right|_p = \sum_{j=1}^n \left. \frac{\partial}{\partial x^j} \right|_p y^j(p) \left. \frac{\partial}{\partial x^i} \right|_p$$

*Proof.* Put  $h = \phi \circ \psi^{-1}$  and write  $h = (h_1, \dots, h_n)$ . Then  $\phi = h \circ \psi$  and  $\psi^{-1} = \phi^{-1} \circ h$ . By definition and the chain rule, we have that

$$\begin{aligned} \left. \frac{\partial}{\partial y^i} \right|_p f &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \psi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \phi^{-1} \circ h \\ &= \sum_{j=1}^n \left( \left. \frac{\partial}{\partial u^j} \right|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left( \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} h_j \right) \\ &= \sum_{j=1}^n \left( \left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} f \circ \phi^{-1} \right) \left( \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} x^j \circ \psi^{-1} \right) \\ &= \sum_{j=1}^n \left( \left. \frac{\partial}{\partial x^j} \right|_p f \right) \left( \left. \frac{\partial}{\partial y^i} \right|_p x^j \right) \end{aligned}$$

□

**Definition 3.5.0.4.** Let  $p \in M$  and  $v : C^\infty(M) \rightarrow \mathbb{R}$ . Then  $v$  is said to be **Leibnizian** if for each  $f, g \in C^\infty(M)$ ,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and  $v$  is said to be a **derivation at  $p$**  if for each  $f, g \in C^\infty(M)$  and  $a \in \mathbb{R}$ ,

1.  $v$  is linear
2.  $v$  is Leibnizian

We define the **tangent space of  $M$  at  $p$** , denoted  $T_pM$ , by

$$T_pM = \{v : C^\infty(M) \rightarrow \mathbb{R} : v \text{ is a derivation at } p\}$$

**Exercise 3.5.0.5.** Let  $f \in C^\infty(M)$  and  $v \in T_pM$ . If  $f$  is constant, then  $vf = 0$ .

*Proof.* Suppose that  $f = 1$ . Then  $f^2 = f$  and  $v(f^2) = 2v(f)$ . So  $v(f) = 2v(f)$  which implies that  $v(f) = 0$ . If  $f \neq 1$ , then there exists  $c \in \mathbb{R}$  such that  $f = c$ . Since  $v$  is linear,  $v(f) = cv(1) = 0$ .  $\square$

**Exercise 3.5.0.6.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $p \in U$ . Then

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

is a basis for  $T_pM$  and  $\dim T_pM = n$ .

*Proof.* Clearly  $\frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \in T_pM$ . Let  $a_1, \dots, a_n \in \mathbb{R}$ . Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p x^j \\ &= a_j \end{aligned}$$

Hence  $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$  is independent.

Now, let  $v \in T_pM$  and  $f \in C^\infty(M)$ . By Taylor's theorem, there exist  $g_1, \dots, g_n \in C_p^\infty(M)$  such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \frac{\partial}{\partial x^i} \Big|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p f \\ &= \left[ \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p \right] f \end{aligned}$$

So

$$v = \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p$$

and

$$v \in \text{span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

□

**Definition 3.5.0.7.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F : M \rightarrow N$  smooth and  $p \in M$ . We define the **differential of  $F$  at  $p$** , denoted  $DF_p : T_p M \rightarrow T_{F(p)} N$ , by

$$\left[ DF_p(v) \right] (f) = v(f \circ F)$$

for  $v \in T_p M$  and  $f \in C^\infty(N)$ .

**Exercise 3.5.0.8.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F : M \rightarrow N$  smooth and  $p \in M$ . Then for each  $v \in T_p M$ ,  $DF_p(v)$  is a derivation.

*Proof.* Let  $v \in T_p M$ ,  $f, g \in C_{F(p)}^\infty(N)$  and  $c \in \mathbb{R}$ . Then

1.

$$\begin{aligned} DF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= DF_p(v)(f) + cDF_p(v)(g) \end{aligned}$$

So  $DF_p(v)$  is linear.

2.

$$\begin{aligned} DF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= DF_p(v)(f) * g(F(p)) + f(F(p)) * DF_p(v)(g) \end{aligned}$$

So  $DF_p(v)$  is Leibnizian and hence  $DF_p(v) \in T_{F(p)} N$

□

**Exercise 3.5.0.9.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F : M \rightarrow N$  smooth and  $p \in M$ . If  $F$  is a diffeomorphism, then  $DF_p$  is an isomorphism.

*Proof.* Suppose that  $F$  is a diffeomorphism. Since  $F$  is a homeomorphism,  $\dim N = n$ . Choose  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . A previous exercise tells us that  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ . Write  $\phi = (x^1, \dots, x^n)$  and  $\phi \circ F^{-1} = (y^1, \dots, y^n)$ . Let  $f \in C^\infty(N)$ . Then

$$\begin{aligned} \frac{\partial}{\partial y^i} \Big|_{F(p)} f &= \frac{\partial}{\partial u^i} \Big|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \frac{\partial}{\partial x^i} \Big|_p f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[ DF_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) \right] (f) &= \frac{\partial}{\partial x^i} \Big|_p f \circ F \\ &= \frac{\partial}{\partial y^i} \Big|_{F(p)} f \end{aligned}$$

Hence

$$DF_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) = \frac{\partial}{\partial y^i} \Big|_{F(p)}$$

Since  $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$  is a basis for  $T_p M$  and  $\left\{ \frac{\partial}{\partial y^1} \Big|_{F(p)}, \dots, \frac{\partial}{\partial y^n} \Big|_{F(p)} \right\}$  is a basis for  $T_{F(p)} N$ ,  $DF_p$  is an isomorphism.  $\square$

**Exercise 3.5.0.10.** Let  $(M, \mathcal{A})$  be a smooth  $m$ -dimensional manifold,  $(N, \mathcal{B})$  a  $n$ -dimensional smooth manifold,  $F : M \rightarrow N$  smooth,  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^m)$  and  $(V, \psi) \in \mathcal{B}$  with  $\psi = (y^1, \dots, y^n)$ . Suppose that  $p \in U$  and  $F(p) \in V$ . Define the ordered bases  $B_\phi = \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^m} \Big|_p \right\}$  and  $B_\psi = \left\{ \frac{\partial}{\partial y^1} \Big|_{F(p)}, \dots, \frac{\partial}{\partial y^n} \Big|_{F(p)} \right\}$ . Then the matrix representation of  $DF_p$  with respect to the bases  $B_\phi$  and  $B_\psi$  is

$$DF_p^{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

*Proof.* Let  $(DF_p)_{B_\phi, B_\psi} = (a_{i,j})_{i,j} \in \mathbb{R}^{n \times m}$ . Then for each  $j \in \{1, \dots, m\}$ ,

$$DF_p \left( \frac{\partial}{\partial x^j} \Big|_p \right) = \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i} \Big|_{F(p)}$$

This implies that

$$\begin{aligned} DF_p \left( \frac{\partial}{\partial x^j} \Big|_p \right) (y^k) &= \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i} \Big|_{F(p)} (y^k) \\ &= \sum_{i=1}^n a_{i,j} \delta_{i,k} \\ &= a_{k,j} \end{aligned}$$

By definition,

$$\begin{aligned} DF_p \left( \frac{\partial}{\partial x^j} \Big|_p \right) (y^k) &= \frac{\partial}{\partial x^j} \Big|_p y^k \circ F \\ &= \frac{\partial}{\partial x^j} \Big|_p F^k \\ &= \frac{\partial F^k}{\partial x^j}(p) \end{aligned}$$

$\square$

**Note 3.5.0.11.** Since  $\text{rank } DF_p$  is independent of basis, it is independent of coordinate charts  $(U, \phi) \in \mathcal{A}$  and  $(V, \psi) \in \mathcal{B}$ .

### 3.6 The Cotangent Space

**Definition 3.6.0.1.** Let  $p \in M$ . We define the **cotangent space of  $M$  at  $p$** , denoted  $T_p^*M$ , by

$$T_p^*M = (T_pM)^*$$

**Definition 3.6.0.2.** Let  $f \in C^\infty(M)$ . We define the **differential of  $f$  at  $p$** , denoted  $df_p : T_pM \rightarrow \mathbb{R}$ , by

$$df_p(v) = vf$$

**Exercise 3.6.0.3.** Let  $f \in C^\infty(M)$  and  $p \in M$ . Then  $df_p \in T_p^*M$ .

*Proof.* Let  $v_1, v_2 \in T_pM$  and  $\lambda \in \mathbb{R}$ . Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1f + \lambda v_2f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that  $df_p$  is linear and hence  $df_p \in T_p^*M$ . □

**Exercise 3.6.0.4.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $p \in U$ . Then for each  $i, j \in \{1, \dots, n\}$ ,

$$dx_p^i \left( \frac{\partial}{\partial x^j} \Big|_p \right) = \delta_{i,j}$$

In particular,  $\{dx_p^1, \dots, dx_p^n\}$  is the dual basis to  $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$  and  $T_p^*M = \text{span}\{dx_p^1, \dots, dx_p^n\}$ .

*Proof.* Let  $i, j \in \{1, \dots, n\}$ . Then by definition,

$$\begin{aligned} \left[ dx_p^i \left( \frac{\partial}{\partial x^i} \Big|_p \right) \right]_p &= \frac{\partial}{\partial x^i} \Big|_p x^i \\ &= \delta_{i,i} \end{aligned}$$

□

**Exercise 3.6.0.5.** Let  $f \in C^\infty(M)$ ,  $(U, \phi)$  a chart on  $M$  with  $\phi = (x^1, \dots, x^n)$  and  $p \in U$ . Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

*Proof.* Since  $\{dx_p^1, \dots, dx_p^n\}$  is a basis for  $T_p^*M$ , for each there exist  $a_1(p), \dots, a_n(p) \in \mathbb{R}$  such that  $df_p = \sum_{i=1}^n a_i(p) dx_p^i$ . Therefore, we have that

$$\begin{aligned} df_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left( \frac{\partial}{\partial x^i} \Big|_p \right) \\ &= a_i(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) &= \frac{\partial f}{\partial x^i} \Big|_p \\ &= \frac{\partial f}{\partial x^i}(p) \end{aligned}$$

So  $a_j(p) = \frac{\partial f}{\partial x^j}(p)$  and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

□

## Chapter 4

# Submersions and Immersions

### 4.1 Maps of Constant Rank

**Definition 4.1.0.1.** Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifolds,  $F : M \rightarrow N$  a smooth map. We define the **rank map of  $F$** , denoted  $\text{rank } F : M \rightarrow \mathbb{N}_0$  by

$$\text{rank}_p F = \dim \text{Im } DF(p)$$

and  $F$  is said to have **constant rank** if for each  $p, q \in M$ ,  $\text{rank}_p F = \text{rank}_q F$ . If  $F$  has constant rank, we define the **rank of  $F$** , denoted  $\text{rank } F$ , by  $\text{rank } F = \text{rank}_p F$  for  $p \in M$ .

**Exercise 4.1.0.2.** Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifolds of dimensions  $m$  and  $n$  respectively,  $F \in C^\infty(M, N)$  and  $p \in M$ . Suppose that  $\text{rank}_p F = k$ . Then there exist  $(U, \phi) \in \mathcal{A}_M$ ,  $(V, \psi) \in \mathcal{A}_N$  and  $A \in GL(k, \mathbb{R})$  such that for each  $i, j \in \{1, \dots, k\}$ ,

$$([DF(p)]_{\phi, \psi})_{i,j} = A_{i,j}$$

*Proof.* Define  $q \in V$  by  $q = F(p)$ . Choose  $(U', \phi') \in \mathcal{A}$  and  $(V', \psi') \in \mathcal{B}$  such that  $p \in U'$  and  $q \in V'$ . Set  $Z = [DF(p)]_{\phi', \psi'}$ . By assumption,  $\text{rank } Z = k$ . An exercise in the subsection on linear algebra implies that there exist  $\sigma \in S_m$ ,  $\tau \in S_n$  and  $A \in GL(k, \mathbb{R})$  such that for each  $i, j \in \{1, \dots, k\}$ ,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Define  $\phi : U \rightarrow \sigma\phi(U)$  and  $\psi : V \rightarrow \tau\psi(V)$  by

$$\phi = \sigma\phi', \quad \psi = \tau\psi'$$

A previous exercise implies that

$$[DF(p)]_{\phi, \psi} = P_\tau Z P_\sigma^*$$

□

**Exercise 4.1.0.3. Constant Rank Theorem:**

Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifolds of dimensions  $m$  and  $n$  respectively,  $F \in C^\infty(M, N)$ . Suppose that  $F$  has constant rank and  $\text{rank } F = k$ . Then for each  $p \in M$ , there exist  $(U, \phi) \in \mathcal{A}$  and  $(V, \psi) \in \mathcal{B}$  such that  $p \in U$ ,  $F(p) \in V$  and

$$\psi \circ F \circ \phi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

*Proof.* Let  $p \in M$ . The previous exercise implies that there exist  $(U_0, \phi_0) \in \mathcal{A}$ ,  $(V_0, \psi_0) \in \mathcal{B}$  and  $L \in GL(k, \mathbb{R})$  such that  $p \in U$ ,  $F(p) \in V_0$  and for each  $i, j \in \{1, \dots, k\}$ ,

$$([DF(p)]_{\phi_0, \psi_0})_{i,j} = L_{i,j}$$

Define  $\hat{M} \subset \mathbb{R}^m$ ,  $\hat{N} \subset \mathbb{R}^n$  and  $\hat{F} : \hat{M} \rightarrow \hat{N}$  by  $\hat{M} = \phi_0(U_0)$ ,  $\hat{N} = \psi_0(V_0)$  and  $\hat{F} = \psi_0 \circ F \circ \phi_0^{-1}$ . Set  $\hat{p} = \phi_0(p)$ . Let  $(x, y)$  be the standard coordinates on  $\mathbb{R}^m$ , with  $\pi_x : \mathbb{R}^m \rightarrow \mathbb{R}^k$  and  $\pi_y : \mathbb{R}^m \rightarrow \mathbb{R}^{m-k}$  the standard projection maps. Write  $\hat{p} = (x_0, y_0)$ . There exist  $Q : \hat{M} \rightarrow \mathbb{R}^k$  and  $R : \hat{M} \rightarrow \mathbb{R}^{n-k}$  such that  $\hat{F} = (Q, R)$ . By construction,  $[D_x Q(x_0, y_0)] = A$ . Define  $G : \hat{M} \rightarrow \mathbb{R}^m$  by  $G(x, y) = (Q(x, y), y)$ . Then

$$\begin{aligned} [DG(x_0, y_0)] &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_x Q(x_0, y_0)] \\ [D_x \pi_y(x_0, y_0)] & [D_y \pi_y(x_0, y_0)] \end{pmatrix} \\ &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} L & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \det([DG(x_0, y_0)]) &= \det(L) \det(I) \\ &= \det(L) \\ &\neq 0 \end{aligned}$$

The inverse function theorem implies that there exist  $\hat{U} \subset \hat{M}$  such that  $\hat{U}$  is open,  $\hat{p} \in \hat{U}$  and  $G|_{\hat{U}} : \hat{U} \rightarrow G(\hat{U})$  is a diffeomorphism. Since

$$\{U_1 \times U_2 : U_1 \subset \mathbb{R}^k, U_2 \subset \mathbb{R}^{m-k} \text{ and } U_1, U_2 \text{ are open}\}$$

is a basis for the topology on  $\mathbb{R}^m$ , there exist  $\hat{U}_1 \subset \mathbb{R}^k$  and  $\hat{U}_2 \subset \mathbb{R}^{m-k}$  such that  $\hat{U}_1, \hat{U}_2$  are open,  $\hat{p} \in \hat{U}_1 \times \hat{U}_2$  and  $\hat{U}_1 \times \hat{U}_2 \subset \hat{U}$ . Set  $\hat{U}_{12} = \hat{U}_1 \times \hat{U}_2$ . Since  $G(\hat{U}_1 \times \hat{U}_2) = Q(\hat{U}_{12}) \times \hat{U}_2$ , we have that  $G|_{\hat{U}_{12}} : \hat{U}_{12} \rightarrow Q(\hat{U}_{12}) \times \hat{U}_2$  is a diffeomorphism. Since  $\pi_x$  is open,  $Q(\hat{U}_{12})$  is open. There exist  $A : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_1$  and  $B : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_2$  such that  $G^{-1} = (A, B)$ . Define  $\tilde{R} : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \mathbb{R}^{n-k}$  by  $\tilde{R}(x, y) = R(A(x, y), y)$ . Let  $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$ . Then

$$\begin{aligned} (x, y) &= G \circ G^{-1}(x, y) \\ &= G(A(x, y), B(x, y)) \\ &= (Q(A(x, y), B(x, y)), B(x, y)) \end{aligned}$$

This implies that  $B(x, y) = y$ ,

$$\begin{aligned} x &= Q(A(x, y), B(x, y)) \\ &= Q(A(x, y), y) \end{aligned}$$

Hand

$$\begin{aligned} G^{-1}(x, y) &= (A(x, y), B(x, y)) \\ &= (A(x, y), y) \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{F} \circ G^{-1}(x, y) &= \hat{F}(A(x, y), y) \\ &= (Q(A(x, y), y), R(A(x, y), y)) \\ &= (x, R(A(x, y), y)) \\ &= (x, \tilde{R}(x, y)) \end{aligned}$$

We note that

$$\begin{aligned} [D(\hat{F} \circ G^{-1})(x, y)] &= \begin{pmatrix} [D_x \pi_x(x, y)] & [D_y \pi_x(x, y)] \\ [D_x \tilde{R}(x, y)] & [D_y \tilde{R}(x, y)] \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ [D_x \tilde{R}(x, y)] & [D_y \tilde{R}(x, y)] \end{pmatrix} \end{aligned}$$



Since  $G^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_{12}$  is a diffeomorphism, we have that  $[DG^{-1}(x, y)] \in GL(m, \mathbb{R})$ . Since  $\hat{F}$  has constant rank and  $\text{rank } \hat{F} = k$ , we have that

$$\begin{aligned} \text{rank}[D(\hat{F} \circ G^{-1})(x, y)] &= \text{rank}([D\hat{F}(G^{-1}(x, y))][DG^{-1}(x, y)]) \\ &= \text{rank}[D\hat{F}(G^{-1}(x, y))] \\ &= k \end{aligned}$$

Since  $\text{rank} \begin{pmatrix} I \\ [D_x \tilde{R}(x, y)] \end{pmatrix} = k$ , we have that  $\text{rank} \begin{pmatrix} 0 \\ [D_y \tilde{R}(x, y)] \end{pmatrix} = 0$ . Thus  $[D_y \tilde{R}(x, y)] = 0$ . Since  $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$  is arbitrary, for each  $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$ ,

$$\tilde{R}(x, y) = \tilde{R}(x, y_0)$$

Define  $\tilde{S} : Q(\hat{U}_{12}) \rightarrow \mathbb{R}^{n-k}$  by  $\tilde{S}(x) = \tilde{R}(x, y_0)$ . Then for each  $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$ ,

$$\hat{F} \circ G^{-1}(x, y) = (x, \tilde{S}(x))$$

Let  $(a, b)$  be the standard coordinates on  $\mathbb{R}^n$ , with  $\pi_a : \mathbb{R}^n \rightarrow \mathbb{R}^k$  and  $\pi_b : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$  the standard projection maps. Write  $\hat{F}(\hat{p}) = (a_0, b_0)$ . Set

$$\begin{aligned} \hat{V} &= [(\pi_a)|_{\hat{N}}]^{-1}(Q(\hat{U}_{12})) \\ &= \pi_a^{-1}(Q(\hat{U}_{12})) \cap \hat{N} \end{aligned}$$

Since  $Q(\hat{U}_{12})$  is open,  $\hat{N}$  is open and  $\pi_a$  is continuous, we have that  $\hat{V}$  is open. Since

$$\begin{aligned} Q(\hat{U}_{12}) &= (\pi_a)|_{\hat{N}} \circ \hat{F} \circ G^{-1}(Q(\hat{U}_{12}) \times \hat{U}_2) \\ &= (\pi_a)|_{\hat{N}} \circ \hat{F}(\hat{U}_{12}) \end{aligned}$$

we have that  $\hat{F}(\hat{U}_{12}) \subset \hat{V}$ . In particular,  $\hat{F}(\hat{p}) \in \hat{V}$ . Define  $H : \hat{V} \rightarrow \mathbb{R}^n$  by  $H(a, b) = (a, b - \tilde{S}(a))$ . Then  $H \circ \hat{F} \circ G^{-1}(x, y) = (x, 0)$ . Define  $(U, \phi) \in \mathcal{A}$  and  $(V, \psi) \in \mathcal{N}$  by  $U = \phi_0^{-1}(\hat{U}_{12})$ ,  $V = \psi_0^{-1}(\hat{V})$ ,  $\phi = G \circ \phi_0$  and  $\psi = H \circ \psi_0$ . Then for each  $(x, y) \in \phi(U)$ ,

$$\begin{aligned} \psi \circ F \circ \phi^{-1}(x, y) &= H \circ \psi_0 \circ F \circ \phi_0^{-1} \circ G^{-1}(x, y) \\ &= H \circ \hat{F} \circ G^{-1}(x, y) \\ &= (x, 0) \end{aligned}$$

□

**Definition 4.1.0.4.** Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifolds,  $F : M \rightarrow N$  a smooth map. Then  $F$  is said to be

- an **immersion** if for each  $p \in M$ ,  $DF(p) : T_p M \rightarrow T_{F(p)} N$  is injective
- a **submersion** if for each  $p \in M$ ,  $DF(p) : T_p M \rightarrow T_{F(p)} N$  is surjective

**Exercise 4.1.0.5.** Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifolds,  $F : M \rightarrow N$  a smooth map.

**Definition 4.1.0.6.** Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifolds and  $F : M \rightarrow N$  smooth. Then  $F$  is said to be an **embedding** if

1.  $F$  is an immersion
2.  $F : M \rightarrow F(M)$ .

**Note 4.1.0.7.** Here the topology on  $F(M)$  is the subspace topology.

## 4.2 Submanifolds

**Exercise 4.2.0.1.** Let  $(M, \mathcal{A})$  be a smooth manifold and  $S \subset M$  open. For  $(U, \phi) \in \mathcal{A}$ , define  $\tilde{U} \subset S$  and  $\tilde{\phi} : \tilde{U} \rightarrow \phi(\tilde{U})$  by  $\tilde{U} = U \cap S$  and  $\tilde{\phi} = \phi|_{U \cap S}$ . Set  $\mathcal{B} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \in \mathcal{A}\}$ . Then  $\mathcal{B}$  is a smooth structure on  $S$ .

*Proof.*

□

**Definition 4.2.0.2.** Let  $(M, \mathcal{A})$  and  $(N, \mathcal{B})$  be smooth manifolds. Suppose that  $M \subset N$ . Then  $(M, \mathcal{A})$  is said to be

1. an **immersed submanifold** of  $(N, \mathcal{B})$  if  $\text{id} : M \rightarrow N$  is a smooth immersion
2. an **embedded submanifold** of  $(N, \mathcal{B})$  if  $\text{id} : M \rightarrow N$  is a smooth embedding

**Note 4.2.0.3.** Essentially, embedded submanifolds are immersed submanifolds with the subspace topology.

**Note 4.2.0.4.** For the remainder of this section, we assume that  $k \leq n$ .

**Definition 4.2.0.5.** Let  $U \subset \mathbb{R}^n$  and  $S \subset U$ . Then  $S$  is said to be a  **$k$ -slice** of  $U$  if  $S = \{u \in U : u^{k+1}, \dots, u^n = 0\}$ .

**Exercise 4.2.0.6.** Let  $U \subset \mathbb{R}^n$  and  $S \subset U$ . Suppose that  $S$  is a  $k$ -slice of  $U$ . Define  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$  by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then  $\pi|_S \rightarrow \pi(S)$  is a diffeomorphism.

*Proof.* Clear. □

**Definition 4.2.0.7.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $(U, \phi) \in \mathcal{A}$  and  $S \subset U$ . Then  $S$  is said to be a  **$k$ -slice** of  $U$  if  $\phi(S)$  is a  $k$ -slice of  $\phi(U)$ .

**Definition 4.2.0.8.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $S \subset M$  and  $(U, \phi) \in \mathcal{A}$ . Then  $(U, \phi)$  is said to be a  **$k$ -slice chart for  $S$**  if  $U \cap S$  is a  $k$ -slice of  $U$ .

**Exercise 4.2.0.9.** Let  $(M, \mathcal{A})$  be a smooth manifold,  $S \subset M$  and  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ . If  $(U, \phi)$  is a  $k$ -slice chart for  $S$ , then  $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$ .

*Proof.* Clear. □

**Definition 4.2.0.10.** Let  $(M, \mathcal{A})$  be a smooth manifold and  $S \subset M$ . Then  $S$  is said to satisfy the **local  $k$ -slice condition** if for each  $p \in S$ , there exists  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$  and  $(U, \phi)$  is a  $k$ -slice chart of  $S$ .

**Exercise 4.2.0.11.** Let  $(M, \mathcal{A})$  be a  $n$ -dimensional smooth manifold and  $S \subset M$  a subspace. If  $S$  satisfies the local  $k$ -slice condition, then there exists a smooth structure  $\tilde{\mathcal{A}}$  on  $S$  such that  $(S, \tilde{\mathcal{A}})$  is an embedded submanifold of  $M$ .

*Proof.* Suppose that  $S$  satisfies the local  $k$ -slice condition. Define  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$  as above. Let  $(U, \phi) \in \mathcal{A}$ . Suppose that  $(U, \phi)$  is a  $k$ -slice chart for  $S$ . Define  $\tilde{U} = U \cap S$  and  $\tilde{\phi} : \tilde{U} \rightarrow \pi \circ \phi(\tilde{U})$  by

$$\tilde{\phi} = \pi \circ \phi|_{\tilde{U}}$$

By definition,  $\phi(\tilde{U})$  is a  $k$ -slice of  $\phi(U)$ . A previous exercise implies that  $\pi|_{\phi(\tilde{U})} \rightarrow \pi \circ \phi(\tilde{U})$  is a diffeomorphism and hence a homeomorphism. Thus  $\tilde{\phi}$  is a homeomorphism.

Define

$$\tilde{\mathcal{B}} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \text{ is a } k\text{-slice for } S\}$$

Let  $p \in S$ . By assumption, there exists  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$  and  $(U, \phi)$  is a  $k$ -slice chart of  $S$ . Then  $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{B}}$  and  $\tilde{\mathcal{A}}$  is an atlas on  $S$ . By construction of  $\tilde{\mathcal{B}}$ ,  $S$  is locally half Euclidean of dimension  $k$ . Since  $M$  is second countable Hausdorff, so is  $S$  in the subspace topology. Thus  $(S, \tilde{\mathcal{B}})$  is a  $k$ -dimensional manifold. Let  $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$ . Then

$$\tilde{\phi} \circ \tilde{\psi}^{-1}|_{\tilde{U} \cap \tilde{V}} = \pi|_{\phi(\tilde{U} \cap \tilde{V})} \circ \phi|_{\tilde{U} \cap \tilde{V}} \circ \psi|_{\tilde{U} \cap \tilde{V}}^{-1} \circ \pi|_{\psi(\tilde{U} \cap \tilde{V})}^{-1}$$

which is a diffeomorphism. So  $(\tilde{U}, \tilde{\phi})$  and  $(\tilde{V}, \tilde{\psi})$  smoothly compatible. Hence  $\tilde{\mathcal{B}}$  is smooth. An exercise in section 4.1 implies that there exists a unique smooth structure  $\tilde{\mathcal{A}}$  on  $S$  such that  $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$ . So  $(S, \tilde{\mathcal{A}})$  is a smooth  $k$ -dimensional manifold.

Clearly  $\text{id} : S \rightarrow S$  is a homeomorphism. Let  $(V, \psi) \in \mathcal{A}$  and  $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$ .

**Finish!!** □

**Definition 4.2.0.12.**

**Exercise 4.2.0.13.**



# Chapter 5

## Vector Fields

### 5.1 The Tangent Bundle

**Definition 5.1.0.1.** Let  $(M, \mathcal{A}_M)$  be an  $n$ -dimensional smooth manifold. We define the **tangent bundle** of  $M$ , denoted  $TM$ , by

$$TM = \coprod_{p \in M} T_p M$$

and we define the **tangent bundle projection**, denoted  $\pi : TM \rightarrow M$ , by

$$\pi(p, v) = p$$

Let  $(U, \phi) \in \mathcal{A}_M$  with  $\phi = (x^1, \dots, x^n)$ . We define  $\Phi_\phi : \pi^{-1}(U) \rightarrow \mathbb{R}^{2n}$  by

$$\Phi_\phi \left( p, \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j} \Big|_p \right) = (\phi(p), \xi^1, \dots, \xi^n)$$

We define  $\mathcal{T}_{TM} = \tau_{TM}(\iota_p : p \in M)$ .

**Exercise 5.1.0.2.**  $\psi : \bigcup_{p \in U} T_p M \rightarrow \mathbb{R}^n$  is given by

$$\psi \left( \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j} \Big|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i} \Big|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i} \Big|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned}
\left. \frac{\partial}{\partial \tilde{y}^i} \right|_{(p, \xi)} [x^k \circ \pi] &= \left. \frac{\partial}{\partial v^i} \right|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \left. \frac{\partial}{\partial v^i} \right|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \left. \frac{\partial}{\partial v^i} \right|_{\phi(p)} [x^k \circ \phi^{-1}] \\
&= 0
\end{aligned}$$

This implies that for each  $i \in \{1, \dots, n\}$ , we have that

$$\begin{aligned}
D\pi(p, \xi) \left( \left. \frac{\partial}{\partial \tilde{x}^i} \right|_{(p, \xi)} \right) (f) &= \left. \frac{\partial}{\partial \tilde{x}^i} \right|_{(p, \xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i} (p, \xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) \delta_{i,k} \\
&= \frac{\partial f}{\partial x^i} (p)
\end{aligned}$$

and

$$\begin{aligned}
D\pi(p, \xi) \left( \left. \frac{\partial}{\partial \tilde{y}^i} \right|_{(p, \xi)} \right) (f) &= \left. \frac{\partial}{\partial \tilde{y}^i} \right|_{(p, \xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i} (p, \xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) 0 \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p, \xi) \in \pi^{-1}(U)} \ker D\pi(p, \xi) \\
&= \coprod_{(p, \xi) \in \pi^{-1}(U)} \text{span} \left\{ \left. \frac{\partial}{\partial \tilde{y}^j} \right|_{(p, \xi)} : j \in \{1, \dots, n\} \right\}
\end{aligned}$$

# Chapter 6

## Lie Theory

### 6.1 Lie Groups

**Definition 6.1.0.1.** Let  $G$  be a smooth manifold and group. Then  $G$  is said to be a **Lie group** if

- multiplication  $G \times G \rightarrow G$  given by  $(g, h) \mapsto gh$  is smooth
- inversion  $G \rightarrow G$  given by  $g \mapsto g^{-1}$  is smooth

**Definition 6.1.0.2.** Let  $\mathfrak{g}$  be a vector space and  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ . Then  $[\cdot, \cdot]$  is said to be a **Lie bracket** on  $\mathfrak{g}$  if

1.  $[\cdot, \cdot]$  is bilinear
2.  $[\cdot, \cdot]$  is antisymmetric
3.  $[\cdot, \cdot]$  satisfies the Jacobi identity:  
for each  $x, w, y \in \mathcal{F}g$ ,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

In this case,  $(\mathfrak{g}, [\cdot, \cdot])$  is said to be a **Lie algebra**.

**Definition 6.1.0.3.** Let  $X \in$





# Chapter 7

## Bundles and Sections

### 7.1 Fiber Bundles

**Note 7.1.0.1.** Let  $U, F$  be sets, we write  $\text{proj}_1 : U \times F \rightarrow U$  to denote the projection onto  $U$ .

**Definition 7.1.0.2.** Let  $E, M, F \in \text{Obj}(\mathbf{Set})$ ,  $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$  a surjection,  $U \subset M$  and  $\Phi : \pi^{-1}(U) \rightarrow U \times F$ . Then  $(U, \Phi)$  is said to be a **local trivialization of  $E$  over  $U$  with fiber  $F$**  if

1.  $\Phi$  is a bijection
2.  $\text{proj}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ & \searrow \pi & \downarrow \text{proj}_1 \\ & & U \end{array}$$

**Exercise 7.1.0.3.** Let  $E, M, F \in \text{Obj}(\mathbf{Set})$  and  $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$  a surjection,  $U \subset M$  and  $\Phi : \pi^{-1}(U) \rightarrow U \times F$  a local trivialization of  $E$  over  $U$  with fiber  $F$ . Then for each  $A \subset U$ ,

$$\Phi(\pi^{-1}(A)) = A \times F$$

**Hint:** consider  $\Phi^{-1}(A \times F)$

*Proof.* Let  $A \subset U$ . Since  $\text{proj}_1^{-1}(A) = A \times F$ , we have that

$$\begin{aligned} \Phi^{-1}(A \times F) &= \Phi^{-1}(\text{proj}_1^{-1}(A)) \\ &= (\text{proj}_1 \circ \Phi)^{-1}(A) \\ &= (\pi|_{\pi^{-1}(U)})^{-1}(A) \\ &= \pi^{-1}(A) \cap \pi^{-1}(U) \\ &= \pi^{-1}(A \cap U) \\ &= \pi^{-1}(A) \end{aligned}$$

Since  $\Phi$  is a bijection, we have that

$$\begin{aligned} \Phi(\pi^{-1}(A)) &= \Phi \circ \Phi^{-1}(A \times F) \\ &= A \times F \end{aligned}$$

□

### 7.1.1 Topological Fiber Bundles

**Definition 7.1.1.1.** Let  $E, M, F \in \text{Obj}(\mathbf{Man}^0)$  and  $\pi \in \text{Hom}_{\mathbf{Man}^0}(E, M)$  a surjection,  $U \subset M$  and  $\Phi : \pi^{-1}(U) \rightarrow U \times F$ . Then  $(U, \Phi)$  is said to be a **continuous local trivialization of  $E$  over  $U$  with fiber  $F$**  if

1.  $U$  is open
2.  $(U, \Phi)$  is a local trivialization of  $E$  over  $U$  with fiber  $F$
3.  $\Phi$  is a homeomorphism

**Exercise 7.1.1.2. Topological Fiber Bundle Chart Lemma:**

Let  $E \in \text{Obj}(\mathbf{Set})$ ,  $M, F \in \text{Obj}(\mathbf{Man}^0)$ ,  $\pi : E \rightarrow M$  a surjection,  $\Gamma$  an index set and for each  $\alpha \in \Gamma$ ,  $U_\alpha \subset M$  and  $\Phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ . Set  $n = \dim M$  and  $k = \dim F$ . Suppose that

- for each  $\alpha \in \Gamma$ ,  $U_\alpha \in \mathcal{T}_M$
- $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
- for each  $\alpha \in \Gamma$ ,  $(U_\alpha, \Phi_\alpha)$  is a local trivialization of  $E$  over  $U_\alpha$
- for each  $\alpha, \beta \in \Gamma$ ,  $\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times F \rightarrow (U_\alpha \cap U_\beta) \times F$  is continuous.

Then there exist a unique topology,  $\mathcal{T}_E$ , on  $E$  such that

1.  $(E, \mathcal{T}_E)$  is a  $n + k$ -dimensional topological manifold
2.  $(E, \mathcal{T}_E)$

$(U_\alpha, \Phi_\alpha)$  a local trivialization of  $E$  over  $U$ . Suppose that

*Proof.* For  $\alpha \in \Gamma$ , we define  $X_\alpha^n(M, \mathcal{T}_M) \subset X^n(M, \mathcal{T}_M)$  by

$$X_\alpha^n(M, \mathcal{T}_M) = \{(V^M, \psi^M) \in X^n(M, \mathcal{T}_M) : V^M \subset U_\alpha\}$$

Choose index sets  $(\Pi_\alpha^M)_{\alpha \in \Gamma}$  and  $\Pi^F$  such that for each  $\alpha \in \Gamma$ ,  $X_\alpha^n(M, \mathcal{T}_M) = (V_\mu^M, \psi_\mu^M)_{\mu \in \Pi_\alpha^M}$  and  $X^k(F, \mathcal{T}_F) = (V_\nu^F, \psi_\nu^F)_{\nu \in \Pi^F}$ . Set  $\Pi^M = \coprod_{\alpha \in \Gamma} \Pi_\alpha^M$ . For  $(\alpha, \mu) \in \Pi^M$ ,  $\nu \in \Pi^F$ , we define  $V_{\mu, \nu}^E \subset E$  and  $\psi_{\mu, \nu}^E : V_{\mu, \nu}^E \rightarrow \psi_\mu^M(V_\mu^M) \times \psi_\nu^F(V_\nu^F)$  by

- $V_{\mu, \nu}^E = \Phi_\alpha^{-1}(V_\mu^M \times V_\nu^F)$
- $\psi_{\mu, \nu}^E = (\psi_\mu^M \times \psi_\nu^F) \circ \Phi_\alpha|_{V_{\mu, \nu}^E}$

We have the following:

- $(\alpha, \mu) \in \Pi^M$  and  $\nu \in \Pi^F$ . Then  $\psi_{\mu, \nu}^E(V_{\mu, \nu}^E) = \psi_\mu^M(V_\mu^M) \times \psi_\nu^F(V_\nu^F)$  and thus  $\psi_{\mu, \nu}^E(V_{\mu, \nu}^E) \in \mathcal{T}_{\mathbb{H}^{n+k}}$
- for each  $(\alpha_1, \mu_1), (\alpha_2, \mu_2) \in \Pi^M$  and  $\nu_1, \nu_2 \in \Pi^F$ ,

$$\begin{aligned} \psi_{\mu_1, \nu_1}^E(V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E) &= (\psi_{\mu_1}^M \times \psi_{\nu_1}^F) \circ \Phi_{\alpha_1}|_{V_{\mu_1, \nu_1}^E} (\Phi_{\alpha_1}^{-1}([V_{\mu_1}^M \times V_{\nu_1}^F]) \cap [V_{\mu_2}^M \times V_{\nu_2}^F]) \\ &= (\psi_{\mu_1}^M \times \psi_{\nu_1}^F)([V_{\mu_1}^M \times V_{\nu_1}^F] \cap [V_{\mu_2}^M \times V_{\nu_2}^F]) \\ &= (\psi_{\mu_1}^M \times \psi_{\nu_1}^F)([V_{\mu_1}^M \cap V_{\mu_2}^M] \times [V_{\nu_1}^F \cap V_{\nu_2}^F]) \\ &= \psi_{\mu_1}^M(V_{\mu_1}^M \cap V_{\mu_2}^M) \times \psi_{\nu_1}^F(V_{\nu_1}^F \cap V_{\nu_2}^F) \\ &\in \mathcal{T}_{\mathbb{H}^{n+k}} \end{aligned}$$

- Let  $(\alpha, \mu) \in \Pi^M$  and  $\nu \in \Pi^F$ . Then  $\psi_{\mu, \nu}^E : V_{\mu, \nu}^E \rightarrow \psi_\mu^M(V_\mu^M) \times \psi_\nu^F(V_\nu^F)$  is a bijection

- Let  $(\alpha_1, \mu_1), (\alpha_2, \mu_2) \in \Pi^M$  and  $\nu_1, \nu_2 \in \Pi^F$ . For notational convenience, set  $\psi_1^E = \psi_{\mu_1, \nu_1}^E$ ,  $\psi_2^E = \psi_{\mu_2, \nu_2}^E$ ,  $V^E = V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E$ ,  $V^M = V_{\mu_1}^M \cap V_{\mu_2}^M$  and  $V^F = V_{\nu_1}^F \cap V_{\nu_2}^F$ . Then  $\psi_2|_{V^E} \circ (\psi_1|_{V^E})^{-1} : \psi_1(V^E) \rightarrow \psi_2(V^E)$  is given by

$$\begin{aligned} \psi_2^E|_{V^E} \circ (\psi_1^E|_{V^E})^{-1} &= [(\psi_{\mu_2}^M|_{V^M} \times \psi_{\nu_2}^F|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\psi_{\mu_1}^M|_{V^M} \times \psi_{\nu_1}^F|_{V^F}) \circ \Phi_{\alpha_1}|_{V^E}]^{-1} \\ &= [(\psi_{\mu_2}^M|_{V^M} \times \psi_{\nu_2}^F|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\Phi_{\alpha_1}|_{V^E})^{-1} \circ (\psi_{\mu_1}^M|_{V^M} \times \psi_{\nu_1}^F|_{V^F})^{-1}] \\ &= (\psi_{\mu_2}^M|_{V^M} \times \psi_{\nu_2}^F|_{V^F}) \circ [\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}] \circ (\psi_{\mu_1}^M|_{V^M} \times \psi_{\nu_1}^F|_{V^F})^{-1} \end{aligned}$$

Since  $\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}$  is continuous, we have that

$$\psi_{\mu_2, \nu_2}^E|_{V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E} \circ (\psi_{\mu_1, \nu_1}^E|_{V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E})^{-1} : \psi_{\mu_1, \nu_1}^E(V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E) \rightarrow \psi_{\mu_2, \nu_2}^E(V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E)$$

is continuous.

- A previous exercise in the section on topological manifolds implies that  $(V_\mu^M)_{(\alpha, \mu) \in \Pi^M}$  is an open cover of  $M$  and  $(V_\nu^F)_{\nu \in \Pi^F}$  is an open cover of  $F$ . Since  $M, F$  are second-countable  $M, F$  are Lindelöf and there exists  $S^M \subset \Pi^M$ ,  $S^F \subset \Pi^F$  such that  $S^M, S^F$  are countable and  $(V_\mu^M)_{(\alpha, \mu) \in S^M}$  is an open cover of  $M$  and  $(V_\nu^F)_{\nu \in S^F}$  is an open cover of  $F$ . Then  $S^M \times S^F$  is countable and  $(V_\mu^M \times V_\nu^F)_{(\alpha, \mu, \nu) \in S^M \times S^F}$  is an open cover of  $M \times F$ .

Let  $a \in E$ . Set  $p = \pi(a)$ . Choose  $(\alpha, \mu) \in S^M$  such that  $p \in V_\mu^M$ . Since  $V_\mu^M \subset U_\alpha$ ,  $a \in \pi^{-1}(U_\alpha)$  which implies that

$$\begin{aligned} p &= \pi(a) \\ &= \text{proj}_1 \circ \Phi_\alpha(a) \end{aligned}$$

Set  $q = \text{proj}_2 \circ \Phi_\alpha(a)$ . Choose  $\nu \in S^F$  such that  $q \in V_\nu^F$ . Then

$$\begin{aligned} \Phi_{\alpha_p}(a) &= (\text{proj}_1 \circ \Phi_{\alpha_p}(a), \text{proj}_2 \circ \Phi_{\alpha_p}(a)) \\ &= (p, q) \\ &\in V_\mu^M \times V_\nu^F \end{aligned}$$

Thus

$$\begin{aligned} a &\in \Phi_{\alpha_p}^{-1}(V_p^M \times V_q^F) \\ &= V_{\mu, \nu}^E \end{aligned}$$

Since  $a \in E$  is arbitrary, we have that for each  $a \in E$ , there exists  $(\alpha, \mu, \nu) \in S^M \times S^F$  such that  $a \in V_{\mu, \nu}^E$ . Thus

$$E \subset \bigcup_{(\alpha, \mu, \nu) \in S^M \times S^F} V_{\mu, \nu}^E$$

- Let  $a_1, a_2 \in E$ . Set  $p_1 = \pi(a_1)$  and  $p_2 = \pi(a_2)$ . Suppose that  $p_1 \neq p_2$ . Since  $M$  is Hausdorff, there exist  $(\alpha_1, \mu_1), (\alpha_2, \mu_2) \in \Pi^M$  such that  $p_1 \in V_{\mu_1}^M$ ,  $p_2 \in V_{\mu_2}^M$  and  $V_{\mu_1}^M \cap V_{\mu_2}^M = \emptyset$ . Set  $q_1 = \text{proj}_2 \circ \Phi_{\alpha_1}(a_1)$  and  $q_2 = \text{proj}_2 \circ \Phi_{\alpha_2}(a_2)$ . Choose  $\nu_1, \nu_2 \in \Pi^F$  such that  $q_1 \in V_{\nu_1}^F$  and  $q_2 \in V_{\nu_2}^F$ . Then similarly to the previous part,  $a_1 \in V_{\mu_1, \nu_1}^E$  and  $a_2 \in V_{\mu_2, \nu_2}^E$ . Then

$$\begin{aligned} V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E &= \Phi_{\alpha_1}^{-1}(V_{\mu_1}^M \times V_{\nu_1}^F) \cap \Phi_{\alpha_2}^{-1}(V_{\mu_2}^M \times V_{\nu_2}^F) \\ &\subset \pi^{-1}(V_{\mu_1}^M) \cap \pi^{-1}(V_{\mu_2}^M) \\ &= \pi^{-1}(V_{\mu_1}^M \cap V_{\mu_2}^M) \\ &= \pi^{-1}(\emptyset) \\ &= \emptyset \end{aligned}$$

Suppose that  $p_1 = p_2$ .

$$\begin{aligned}\Phi_{\alpha_1}(V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E) &= \Phi_{\alpha_1} \left[ \Phi_{\alpha_1}^{-1}(V_{\mu_1}^M \times V_{\nu_2}^F) \cap \Phi_{\alpha_2}^{-1}(V_{\mu_2}^M \times V_{\nu_2}^F) \right] \\ &= (V_{\mu_1}^M \times V_{\nu_2}^F) \cap \Phi_{\alpha_1}(\Phi_{\alpha_2}^{-1}(V_{\mu_2}^M \times V_{\nu_2}^F))\end{aligned}$$

Using the hint on the [Topological Manifold Chart Lemma](#), we have that

$$\begin{aligned}V_{\mu_1, \nu_1}^E \cap V_{\mu_2, \nu_2}^E &= \Phi_{\alpha_1} \left[ (V_{\mu_1}^M \times V_{\nu_2}^F) \cap \Phi_{\alpha_1}^{-1} \Phi_{\alpha_2}^{-1}(V_{\mu_2}^M \times V_{\nu_2}^F) \right] \\ &= \Phi_{\alpha_1}^{-1}(V_{\mu_1}^M \times V_{\nu_2}^F) \cap \Phi_{\alpha_2}^{-1}(V_{\mu_2}^M \times V_{\nu_2}^F)\end{aligned}$$

FINISH!!! □

### 7.1.2 Smooth Fiber Bundles

**Definition 7.1.2.1.** Let  $E, M, F \in \text{Obj}(\mathbf{Man}^\infty)$  and  $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$  a surjection,  $U \subset M$  and  $\Phi : \pi^{-1}(U) \rightarrow U \times F$ . Then  $(U, \Phi)$  is said to be a **smooth local trivialization of  $E$  over  $U$  with fiber  $F$**  if

1.  $U$  is open
2.  $(U, \Phi)$  is a local trivialization of  $E$  over  $U$  with fiber  $F$
3.  $\Phi$  is a diffeomorphism

**Definition 7.1.2.2.** Let  $E, M, F \in \text{Obj}(\mathbf{Man}^\infty)$  and  $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$  a surjection. Then  $(E, M, \pi, F)$  is said to be a **smooth fiber bundle with total space  $E$ , base space  $M$ , fiber  $F$  and projection  $\pi$**  if for each  $p \in M$ , there exist  $U \in \mathcal{N}_p$  and  $\Phi : \pi^{-1}(U) \rightarrow U \times F$  such that  $U$  is open and  $(U, \Phi)$  is a smooth local trivialization of  $E$  over  $U$  with fiber  $F$ . For  $p \in M$ , we define the **fiber over  $p$** , denoted  $E_p$ , by  $E_p = \pi^{-1}(\{p\})$ .

**Note 7.1.2.3.** When the context is clear, we will suppress the fiber manifold  $F$ .

**Definition 7.1.2.4.** Let  $(E_1, M_1, \pi_1, F_1)$  and  $(E_2, M_2, \pi_2, F_2)$  be smooth fiber bundles,  $\Phi \in \text{Hom}_{\mathbf{Man}^\infty}(E_1, E_2)$  and  $\phi \in \text{Hom}_{\mathbf{Man}^\infty}(M_1, M_2)$ . Then  $(\Phi, \phi)$  is said to be a **smooth bundle morphism** from  $(E_1, M_1, \pi_1, F_1)$  to  $(E_2, M_2, \pi_2, F_2)$  if  $\pi_2 \circ \Phi = \phi \circ \pi_1$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{\phi} & M_2 \end{array}$$

**Definition 7.1.2.5.** We define the category of smooth fiber bundles, denoted  $\mathbf{Bun}^\infty$ , by

- $\text{Obj}(\mathbf{Bun}^\infty) = \{(E, M, \pi, F) : (E, M, \pi, F) \text{ is a smooth fiber bundle}\}$
- For  $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \text{Obj}(\mathbf{Bun}^\infty)$ ,

$$\begin{aligned}\text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) &= \\ \{(\Phi, \phi) : (\Phi, \phi) \text{ is a smooth bundle morphism from } (E_1, M_1, \pi_1, F_1) \text{ to } (E_2, M_2, \pi_2, F_2)\} &\end{aligned}$$

- For
  - $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3) \in \text{Obj}(\mathbf{Bun}^\infty)$
  - $(\Phi_{12}, \phi_{12}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$
  - $(\Phi_{23}, \phi_{23}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3))$

we define  $(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_3, M_3, \pi_3))$  by

$$(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) = (\Phi_{23} \circ \Phi_{12}, \phi_{23} \circ \phi_{12})$$

**Exercise 7.1.2.6.** We have that  $\mathbf{Bun}^\infty$  is a full subcategory of  $(\text{id}_{\mathbf{Man}^\infty} \downarrow \text{id}_{\mathbf{Man}^\infty})$ .

*Proof.* Set  $\mathcal{C} = (\text{id}_{\mathbf{Man}^\infty} \downarrow \text{id}_{\mathbf{Man}^\infty})$ . We note that

- $\text{Obj}(\mathbf{Bun}^\infty) \subset \text{Obj}(\mathcal{C})$
- for each  $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \text{Obj}(\mathbf{Bun}^\infty)$ ,

$$\text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) = \text{Hom}_{\mathcal{C}}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$$

So  $\mathbf{Bun}^\infty$  is a full subcategory of  $\mathcal{C}$ . □

**Exercise 7.1.2.7.** Let  $(E, M, \pi, F) \in \mathbf{Bun}^\infty$  and  $(U, \Phi)$  a local trivialization of  $E$  over  $U$  and  $(V, \Psi)$  a local trivialization of  $E$  over  $V$ . Then

1.  $\text{proj}_{U \cap V} \circ \Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1} = \text{proj}_1$
2. there exists  $\sigma \in \text{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$  such that for each  $p \in U \cap V$ ,  $\sigma(p, \cdot) : F \rightarrow F$  is a diffeomorphism.

*Proof.*

1. By definition, the following diagram commutes:

$$\begin{array}{ccc} (U \cap V) \times F & \xleftarrow{\Phi} \pi^{-1}(U \cap V) & \xrightarrow{\Psi} (U \cap V) \times F \\ & \searrow \text{proj}_1 \quad \downarrow \pi \quad \swarrow \text{proj}_1 & \\ & N & \end{array}$$

$$\text{proj}_1 \circ \Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1} = \text{proj}_1$$

2. there exists  $\sigma \in \text{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$  such that for each  $p \in U \cap V$  and  $x \in F$ ,

$$\Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1}(p, x) = (p, \sigma(p, x))$$

and  $\sigma(p, \cdot) : F \rightarrow F$  is a diffeomorphism. □

**Definition 7.1.2.8.** Let  $(E, M, \pi, F) \in \mathbf{Bun}^\infty$  and  $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$  a collection of smooth local trivializations of  $E$ . Then  $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$  is said to be a **fiber bundle atlas** if for each  $p \in M$ , there exists  $\alpha \in A$  such that  $p \in U_\alpha$ . For  $\alpha, \beta \in A$ , we define  $\phi$

## 7.2 $G$ -Bundles

**Definition 7.2.0.1.** Let  $G$  be a Lie group and  $(E, M, \pi, F) \in \text{Obj}(\mathbf{Bun}^\infty)$ . Then

## 7.3 Vector Bundles

**Note 7.3.0.1.** Let  $M$  be a set and  $p \in M$ . We endow  $\{p\} \times \mathbb{R}^n$  with the natural vector space structure such that  $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$ .

**Definition 7.3.0.2.** Let  $E, M \in \text{Obj}(\mathbf{Man}^\infty)$  and  $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$  a surjection. Then  $(E, M, \pi)$  is said to be a **rank  $k$  smooth vector bundle** if

1.  $(E, M, \pi, \mathbb{R}^k) \in \text{Obj}(\mathbf{Bun}^\infty)$
2. for each  $p \in M$ ,  $E_p$  is a  $k$ -dimensional real vector space
3. for each smooth local trivialization  $(U, \Phi)$  of  $E$  over  $U$  with fiber  $\mathbb{R}^k$  and  $p \in U$ ,

$$\Phi|_{E_p} : E_p \rightarrow \{p\} \times \mathbb{R}^k$$

is a vector space isomorphism

In this case we define the **rank of**  $(E, M, \pi)$ , denoted  $\text{rank}(E, M, \pi)$ , by  $\text{rank}(E, M, \pi) = k$ .

**Definition 7.3.0.3.** We define the category of smooth vector bundles, denoted  $\mathbf{VecBun}^\infty$ , by

- $\text{Obj}(\mathbf{VecBun}^\infty) = \{(E, M, \pi) : (E, M, \pi) \text{ is a smooth vector bundle}\}$
- For  $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{VecBun}^\infty)$  with  $\text{rank}(E_1, M_1, \pi_1) = k_1$  and  $\text{rank}(E_2, M_2, \pi_2) = k_2$ ,

$$\text{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

**Exercise 7.3.0.4.** We have that  $\mathbf{VecBun}^\infty$  is a full subcategory of  $\mathbf{Bun}^\infty$ .

*Proof.* We note that

- $\text{Obj}(\mathbf{VecBun}^\infty) \subset \text{Obj}(\mathbf{Bun}^\infty)$
- for each  $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{Bun}^\infty)$  with  $\text{rank}(E_1, M_1, \pi_1) = k_1$  and  $\text{rank}(E_2, M_2, \pi_2) = k_2$ ,

$$\text{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

So  $\mathbf{Bun}^\infty$  is a full subcategory of  $\mathcal{C}$ . □

**Exercise 7.3.0.5.** Let  $M \in \text{Obj}(\mathbf{Man}^\infty)$ . Set  $n = \dim M$ ,  $E = M \times \mathbb{R}^k$  and define  $\pi : E \rightarrow M$  by  $\pi(p, x) = p$ . Then  $(E, M, \pi)$  is a rank  $k$  smooth vector bundle.

*Proof.*

1. For each  $p \in M$ ,  $\pi_1^{-1}(\{p\}) = \{p\} \times \mathbb{R}^k$  is an  $n$ -dimensional real vector space.
2. Let  $p \in M$ . Set  $U = M$ . Then  $\pi^{-1}(U) = E$ . Define  $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$  by  $\Phi = \text{id}_E$ . Then  $(U, \Phi)$  is a smooth local trivialization of  $E$  over  $U$ .
3. Let  $p \in M$ . Then  $\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \rightarrow \{p\} \times \mathbb{R}^k$  is clearly an isomorphism.

□

**Exercise 7.3.0.6. Smooth Vector Bundle Chart Lemma:**

Let  $M \in \text{Obj}(\mathbf{Man}^\infty)$ . Denote the topology on  $M$  by  $\mathcal{T}_M$ . Suppose that for each  $p \in M$ , there exists  $E_p \in \text{Obj}(\mathbf{Vect}_{\mathbb{R}})$  such that  $\dim E_p = k$ . We define  $E \in \text{Obj}(\mathbf{Set})$  and  $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$  by

$$E = \coprod_{p \in M} E_p$$

and  $\pi(p, v) = p$ . Let  $\Gamma$  be an index set and  $(U_\alpha)_{\alpha \in \Gamma} \subset \mathcal{T}_M$ . Suppose that

1.  $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
2. for each  $\alpha \in \Gamma$ , there exists  $\Phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$  such that
  - $\Phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$  is a bijection
  - $\Phi_\alpha|_{E_p} : E_p \rightarrow \{p\} \times \mathbb{R}^k$  is a vector space isomorphism
3. for each  $\alpha, \beta \in \Gamma$ , there exists  $\tau_{\alpha, \beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R})$  such that
  - $\tau_{\alpha, \beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{R})$  is smooth
  - $\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k$  is given by



## 7.4 Bundle Morphisms

**Definition 7.4.0.1.** Let  $(E, M, \pi_E)$  and  $(F, N, \pi_F)$  be smooth fiber bundles and  $\Phi : E \rightarrow F$  and  $\phi : M \rightarrow N$ . Then  $(\Phi, \phi)$  is said to be a **smooth fiber bundle morphism** from  $(E, M, \pi_E)$  to  $(F, N, \pi_F)$  if  $\Phi$  is smooth,  $\phi$  is smooth and  $\pi_F \circ \Phi = \phi \circ \pi_E$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ \pi_E \downarrow & & \downarrow \pi_F \\ M & \xrightarrow{\phi} & N \end{array}$$

and we write  $(\Phi, \phi) : (E, M, \pi_E) \rightarrow (F, N, \pi_F)$ .

**Exercise 7.4.0.2.** Let  $(E, M, \pi_E)$  and  $(F, N, \pi_F)$  be smooth fiber bundles and  $(\Phi, \phi) : (E, M, \pi_E) \rightarrow (F, N, \pi_F)$ . Suppose that  $(\Phi, \phi)$  is smooth. Then for each  $p \in M$ ,

$$\Phi^{-1}(F_{\phi(p)}) = E_p$$

*Proof.* Let  $p \in M$ . Set  $q = \phi(p)$ . Then

$$\begin{aligned} \Phi^{-1}(F_q) &= \Phi^{-1}(\pi_F^{-1}(\{q\})) \\ &= (\pi_F \circ \Phi)^{-1}(\{q\}) \\ &= (\phi \circ \pi_E)^{-1}(\{q\}) \\ &= \pi_E^{-1}(\phi^{-1}(\{\phi(p)\})) \end{aligned}$$

FINISH!!!, multiple fibers get mapped to same fiber

□

## 7.5 Subbundles

## 7.6 Vertical and Horizontal Subbundles

**Definition 7.6.0.1.** Let  $(E, M, \pi_M) \in \text{Obj}(\mathbf{Bun}^\infty)$ . We define the **vertical bundle associated to**  $(E, M, \pi_M)$ , denoted  $(VE, M, \pi_V) \in \mathbf{Bun}^\infty$ , by

$$VE = \coprod_{q \in E} \ker D\pi(q)$$

relocate this to after tangent bundle is introduced

**Exercise 7.6.0.2.** Let  $(M, \mathcal{A})$  be an  $n$ -dimensional smooth manifold and  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ ,  $(\pi^{-1}(U), \Phi_\phi) \in \mathcal{A}_{TM}$  the induced chart on  $TM$  with  $\Phi_\phi = (\tilde{x}^1, \dots, \tilde{x}^n, \tilde{y}^1, \dots, \tilde{y}^n)$ . Then

$$V(TM)|_{\pi^{-1}(U)} = \coprod_{(p, \xi) \in \pi^{-1}(U)} \text{span} \left\{ \frac{\partial}{\partial \tilde{y}^j} \Big|_{(p, \xi)} : j \in \{1, \dots, n\} \right\}$$

Split into smaller exercises

*Proof.* Let  $f \in C^\infty(M)$  and  $(u^1, \dots, u^n, v^1, \dots, v^n)$  the standard coordinates on  $\mathbb{R}^n \times \mathbb{R}^n$ . We note that by definition,  $\Phi_\phi(p, \xi) = (\phi(p), \psi(\xi))$  where  $\psi : \bigcup_{p \in U} T_p M \rightarrow \mathbb{R}^n$  is given by

$$\psi \left( \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j} \Big|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i} \Big|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i} \Big|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} [x^k \circ \pi] &= \frac{\partial}{\partial v^i} \Big|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i} \Big|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i} \Big|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= 0 \end{aligned}$$

This implies that for each  $i \in \{1, \dots, n\}$ , we have that

$$\begin{aligned}
 D\pi(p, \xi) \left( \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} \right) (f) &= \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} f \circ \pi \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i} (p, \xi) \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) \delta_{i,k} \\
 &= \frac{\partial f}{\partial x^i} (p)
 \end{aligned}$$

and

$$\begin{aligned}
 D\pi(p, \xi) \left( \frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} \right) (f) &= \frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} f \circ \pi \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i} (p, \xi) \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) 0 \\
 &= 0
 \end{aligned}$$

Hence

$$\begin{aligned}
 V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p, \xi) \in \pi^{-1}(U)} \ker D\pi(p, \xi) \\
 &= \coprod_{(p, \xi) \in \pi^{-1}(U)} \text{span} \left\{ \frac{\partial}{\partial \tilde{y}^j} \Big|_{(p, \xi)} : j \in \{1, \dots, n\} \right\}
 \end{aligned}$$

□

## 7.7 The Tangent Bundle

**Definition 7.7.0.1.** We define the **tangent bundle of  $M$** , denoted  $TM$ , by

$$TM = \coprod_{p \in M} T_p M$$

We denote the natural projection map by  $\pi : TM \rightarrow M$ .

**Definition 7.7.0.2.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ . Define  $\tilde{U} \subset TM$  and  $\tilde{\phi} : \tilde{U} \rightarrow \phi(U) \times \mathbb{R}^n$  by

- $\tilde{U} = \pi^{-1}(U)$
- 

$$\begin{aligned} \tilde{\phi} \left( \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p \right) &= (\phi(p), v) \\ &= (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \end{aligned}$$

**Exercise 7.7.0.3.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ . Then  $\tilde{\phi} : \tilde{U} \rightarrow \phi(U) \times \mathbb{R}^n$  is a bijection.

## 7.8 The cotangent Bundle

**Definition 7.8.0.1.** We define the **cotangent bundle of  $M$** , denoted  $T^*M$ , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

## 7.9 The $(r, s)$ -Tensor Bundle

**Definition 7.9.0.1.** 1. the **cotangent bundle of  $M$** , denoted  $T^*M$ , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

2. the  **$(r, s)$ -tensor bundle of  $M$** , denoted  $T_s^r M$ , by

$$T_s^r M = \coprod_{p \in M} T_s^r(T_p M)$$

3. the  **$k$ -alternating tensor bundle of  $M$** , denoted  $\Lambda^k(M)$ , by

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k(T_p M)$$

## 7.10 Vector Fields

**Definition 7.10.0.1.** Let  $X : M \rightarrow TM$ . Then  $X$  is said to be a **vector field on  $M$**  if for each  $p \in M$ ,  $X_p \in T_p M$ .

For  $f \in C^\infty(M)$ , we define  $Xf : M \rightarrow \mathbb{R}$  by

$$(Xf)_p = X_p(f)$$

and  $X$  is said to be **smooth** if for each  $f \in C^\infty(M)$ ,  $Xf$  is smooth.

We denote the set of smooth vector fields on  $M$  by  $\Gamma^1(M)$ .

**Definition 7.10.0.2.** Let  $f \in C^\infty(M)$  and  $X, Y \in \Gamma^1(M)$ . We define

- $fX \in \Gamma^1(M)$  by

$$(fX)_p = f(p)X_p$$

- $X + Y \in \Gamma^1(M)$  by

$$(X + Y)_p = X_p + Y_p$$

**Exercise 7.10.0.3.** The set  $\Gamma^1(M)$  is a  $C^\infty(M)$ -module.

*Proof.* Clear. □

**Exercise 7.10.0.4.** Let  $X \in \Gamma^1(M)$  and  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ . Then

$$X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$$

*Proof.* Let  $p \in M$ . Then  $X_p \in T_p M$  and  $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$  is a basis of  $T_p M$ . So there exist  $f_1(p), \dots, f_n(p) \in \mathbb{R}$  such that  $X_p = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i} \Big|_p$ . Let  $j \in \{1, \dots, n\}$ . Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i} x^j(p) \\ &= f_j(p) \end{aligned}$$

Hence  $Xx^j = f_j$  and  $X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$ . □

**Exercise 7.10.0.5.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ . Then for each  $i \in \{1, \dots, n\}$ ,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

*Proof.* Let  $i \in \{1, \dots, n\}$  and  $f \in C^\infty(M)$ . Define  $g : M \rightarrow \mathbb{R}$  by  $g = \frac{\partial}{\partial x^i} f$ . Let  $(V, \psi) \in \mathcal{A}$ . Then for each  $x \in \psi(U \cap V)$ ,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i} \Big|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i} [f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since  $f \circ \phi^{-1}$  and  $\phi \circ \psi^{-1}$  are smooth,  $g \circ \psi^{-1}$  is smooth and hence  $g$  is smooth. Since  $f \in C^\infty(M)$  was arbitrary, by definition,  $\frac{\partial}{\partial x^i}$  is smooth. □

## 7.11 1-Forms

**Definition 7.11.0.1.** Let  $\omega : M \rightarrow T^*M$ . Then  $\omega$  is said to be a **1-form on  $M$**  if for each  $p \in M$ ,  $\omega_p \in T_p^*M$ .

For each  $X \in \Gamma^1(M)$ , we define  $\omega(X) : M \rightarrow \mathbb{R}$  by

$$\omega(X)_p = \omega_p(X_p)$$

and  $\omega$  is said to be **smooth** if for each  $X \in \Gamma^1(M)$ ,  $\omega(X)$  is smooth. The set of smooth 1-forms on  $M$  is denoted  $\Gamma_1(M)$ .

**Definition 7.11.0.2.** Let  $f \in C^\infty(M)$  and  $\alpha, \beta \in \Gamma_1(M)$ . We define

- $f\alpha \in \Gamma_1(M)$  by

$$(f\omega)_p = f(p)\omega_p$$

- $\alpha + \beta \in \Gamma_1(M)$  by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

**Exercise 7.11.0.3.** The set  $\Gamma_1(M)$  is a  $C^\infty(M)$ -module.

*Proof.* Clear. □

**Exercise 7.11.0.4.**



## 7.12 $(r, s)$ -Tensor Fields

**Definition 7.12.0.1.** Let  $\alpha : M \rightarrow T_s^r M$ . Then  $\alpha$  is said to be an  $(r, s)$ -**tensor field on**  $M$  if for each  $p \in M$ ,  $\alpha_p \in T_p^r(T_p M)$ .

For each  $\omega \in \Gamma_1(M)^r$  and  $X \in \Gamma^1(M)^s$ , we define  $\alpha(\omega, X) : M \rightarrow \mathbb{R}$  by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and  $\alpha$  is said to be **smooth** if for each  $\omega \in \Gamma_1(M)^r$  and  $X \in \Gamma^1(M)^s$ ,  $\alpha(\omega, X)$  is smooth. The set of smooth  $(r, s)$ -tensor fields on  $M$  is denoted  $T_s^r(M)$ .

**Definition 7.12.0.2.** Let  $f \in C^\infty(M)$  and  $\alpha, \beta \in T_s^r(M)$ . We define

- $f\alpha : M \rightarrow T_s^r M$  by

$$(f\alpha)_p = f(p)\alpha_p$$

- $\alpha + \beta : M \rightarrow T_s^r M$  by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

**Exercise 7.12.0.3.** Let  $f \in C^\infty(M)$  and  $\alpha, \beta \in T_s^r(M)$ . Then

1.  $f\alpha \in T_s^r(M)$  by

$$(f\alpha)_p = f(p)\alpha_p$$

2.  $\alpha + \beta \in T_s^r(M)$  by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

*Proof.* Clear. □

**Exercise 7.12.0.4.** The set  $T_s^r(M)$  is a  $C^\infty(M)$ -module.

*Proof.* Clear. □

**Definition 7.12.0.5.** Let  $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$  and  $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$ . We define the **tensor product of**  $\alpha$  **with**  $\beta$ , denoted  $\alpha \otimes \beta : M \rightarrow T_{s_1+s_2}^{r_1+r_2} M$ , by

$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

**Exercise 7.12.0.6.** Let  $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$  and  $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$ . Then  $\alpha_1 \otimes \alpha_2 \in \Gamma_{s_1+s_2}^{r_1+r_2}(M)$

*Proof.* Let  $\omega_1 \in \Gamma_1(M)^{r_1}$ ,  $\omega_2 \in \Gamma_1(M)^{r_2}$ ,  $X_1 \in \Gamma^1(M)^{s_1}$  and  $X_2 \in \Gamma^1(M)^{s_2}$ . By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that  $\alpha_1 \otimes \alpha_2$  is smooth since  $\alpha_1$  and  $\alpha_2$  are smooth by assumption. □

**Definition 7.12.0.7.** We define the **tensor product**, denoted  $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$  by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

**Exercise 7.12.0.8.** The tensor product  $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$  is associative.

*Proof.* Clear. □

**Exercise 7.12.0.9.** The tensor product  $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$  is  $C^\infty(M)$ -bilinear.

*Proof.* Clear. □

**Definition 7.12.0.10.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F : M \rightarrow N$  a smooth map and  $\alpha \in \Gamma_k^0(N)$ . We define the **pullback of  $\alpha$  by  $F$** , denoted  $F^*\alpha \in \Gamma_k^0(M)$ , by

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k))$$

for  $p \in M$  and  $v_1, \dots, v_k \in T_p M$

**Exercise 7.12.0.11.** Let  $(M, \mathcal{A})$ ,  $(N, \mathcal{B})$  and  $(L, \mathcal{C})$  be smooth manifolds,  $F : M \rightarrow N$  and  $G : N \rightarrow L$  smooth maps,  $\alpha \in \Gamma_k^0(N)$ ,  $\beta \in \Gamma_l^0(N)$ ,  $\gamma \in \Gamma_k^0(L)$  and  $f \in C^\infty(N)$ . Then

1.  $F^*(f\alpha) = (f \circ F)F^*\alpha$
2.  $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
3.  $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
4.  $(G \circ F)^*\gamma = F^*(G^*\gamma)$
5.  $id_N^*\alpha = \alpha$

*Proof.*

1.

$$\begin{aligned} [F^*(f\alpha)]_p(v_1, \dots, v_k) &= (f\alpha)_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\ &= f(F(p))\alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\ &= (f \circ F)(p)(F^*\alpha)_p(v_1, \dots, v_k) \end{aligned}$$

So that  $F^*(f\alpha) = (f \circ F)F^*\alpha$

2.

$$F^*$$

□

**Definition 7.12.0.12.**

**Exercise 7.12.0.13.**

*Proof.*

□

**Exercise 7.12.0.14.** Let  $\alpha \in T_s^r(M)$  and  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$ . Then there exist  $(f_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$  such that

$$\alpha|_U = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

*Proof.* Let  $p \in M$ . Then  $\omega_p \in T_s^r(T_p M)$  and  $\left\{ \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J} \right\}$  is a basis of  $T_s^r(T_p M)$ . So there exist  $(f_J^I(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$  such that

$$\omega_p = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let  $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$ . Then

$$\begin{aligned} \alpha_p(dx_p^K, \partial_{x^L}|_p) &= \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}(dx_p^K, \partial_{x^L}|_p) \\ &= \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p(dx_p^K) dx_p^{\otimes J}(\partial_{x^L}|_p) \\ &= f_L^K(p) \end{aligned}$$

By assumption, the map  $p \mapsto \alpha(dx_p^K, \partial_{x^L}|_p)$  is smooth, so that  $f_L^K \in C^\infty(U)$ .

□

**Definition 7.12.0.15.**

## 7.13 Differential Forms

**Definition 7.13.0.1.** We define

$$\Lambda^k(TM) = \coprod_{p \in M} \Lambda^k(T_p M)$$

**Definition 7.13.0.2.** Let  $\omega : M \rightarrow \Lambda^k(TM)$ . Then  $\omega$  is said to be a  **$k$ -form on  $M$**  if for each  $p \in M$ ,  $\omega_p \in \Lambda^k(T_p M)$ .

For each  $X \in \Gamma^1(M)^k$ , we define  $\omega(X) : M \rightarrow \mathbb{R}$  by

$$\omega(X)_p = \omega_p(X_p)$$

and  $\omega$  is said to be **smooth** if for each  $X \in \Gamma^1(M)^k$ ,  $\omega(X)$  is smooth. The set of smooth  $k$ -forms on  $M$  is denoted  $\Omega^k(M)$ .

**Note 7.13.0.3.** Observe that

1.  $\Omega^k(M) \subset \Gamma_k^0(M)$
2.  $\Omega^0(M) = C^\infty(M)$

**Exercise 7.13.0.4.** The set  $\Omega^k(M)$  is a  $C^\infty(M)$ -submodule of  $\Gamma_k^0(M)$ .

*Proof.* Clear. □

**Definition 7.13.0.5.** Define the **exterior product**

$$\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

**Note 7.13.0.6.** For  $f \in \Omega^0(M)$  and  $\alpha \in \Omega^k(M)$ , we have that  $f \wedge \alpha = f\alpha$ .

**Exercise 7.13.0.7.** The exterior product  $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$  is well defined.

*Proof.* Let  $\alpha \in \Omega^k(M)$ ,  $\beta \in \Omega^l(M)$ ,  $(x^i)_{i=1}^k \subset \Gamma^1(M)$ ,  $(y^j)_{j=1}^l \subset \Gamma^1(M)$  and  $p \in M$ . Then

$$\begin{aligned} \alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{(k+l)!}{k!l!} \text{Alt}(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) (\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \end{aligned}$$

□

**Exercise 7.13.0.8.** The exterior product  $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$  is  $C^\infty(M)$ -bilinear.

*Proof.*

1.  $C^\infty(M)$ -linearity in the first argument:

Let  $\alpha \in \Omega^k(M)$ ,  $\beta, \gamma \in \Omega^l(M)$ ,  $f \in C^\infty(M)$  and  $p \in M$ . Bilinearity of  $\wedge : \Lambda^k(T_p M) \times \Lambda^l(T_p M) \rightarrow \Lambda^{k+l}(T_p M)$  implies that

$$\begin{aligned} [(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p \end{aligned}$$

So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

and  $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$  is  $C^\infty(M)$ -linear in the first argument.

2.  $C^\infty(M)$ -linearity in the second argument:

Similar to (1).

□

**Note 7.13.0.9.** All of the results from multilinear algebra apply here.

**Definition 7.13.0.10.** We define the **exterior derivative**  $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  inductively by

1.  $d(d\alpha) = 0$  for  $\alpha \in \Omega^p(M)$
2.  $df(X) = Xf$  for  $f \in \Omega^0(M)$
3.  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$  for  $\alpha \in \Omega^p(M)$  and  $\beta \in \Omega^q(M)$
4. extending linearly

**Exercise 7.13.0.11.** Let  $(U, \phi)$  be a chart on  $M$  with  $\phi = (x^1, \dots, x^n)$ . Then on  $U$ , for each  $i, j \in \{1, \dots, n\}$ ,

$$dx^i \left( \frac{\partial}{\partial x^j} \right) = \delta_{i,j}$$

In particular, for each  $p \in U$ ,  $\{dx_p^1, \dots, dx_p^n\}$  is the dual basis to  $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$  and  $T_p^* M = \text{span}\{dx_p^1, \dots, dx_p^n\}$ .

*Proof.* Let  $p \in U$  and  $i, j \in \{1, \dots, n\}$ . Then by definition,

$$\begin{aligned} \left[ dx^i \left( \frac{\partial}{\partial x^j} \right) \right]_p &= \left( \frac{\partial}{\partial x^j} x^i \right)_p \\ &= \frac{\partial}{\partial x^j} x^i \Big|_p \\ &= \delta_{i,j} \end{aligned}$$

□

**Exercise 7.13.0.12.** Let  $f \in C^\infty(M)$  and  $(U, \phi)$  be a chart on  $M$  with  $\phi = (x^1, \dots, x^n)$ . Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

*Proof.* Let  $p \in U$ . Since  $\{dx^1, \dots, dx^n\}$  is a basis for  $\Lambda(T_p M)$ , for each there exist  $a_1(p), \dots, a_n(p) \in \mathbb{R}$  such that  $df_p = \sum_{i=1}^n a^i(p) dx_p^i$ . Therefore, we have that

$$\begin{aligned} df_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) &= \sum_{i=1}^n a^i(p) dx_p^i \left( \frac{\partial}{\partial x^i} \Big|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) &= \frac{\partial}{\partial x^i} \Big|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So  $a_j(p) = \frac{\partial f}{\partial x^j}(p)$  and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx_p^i$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

□

**Exercise 7.13.0.13.** Let  $f \in \Omega^0(M)$ . If  $f$  is constant, then  $df = 0$ .

*Proof.* Suppose that  $f$  is constant. Let  $p \in M$ . Choose  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . Write  $\phi = (x_1, \dots, x_n)$ . Then for each  $i \in \{1, \dots, n\}$ ,

$$\frac{\partial}{\partial x^i} \Big|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx_p^i \\ &= 0 \end{aligned}$$

□

**Exercise 7.13.0.14.**

**Definition 7.13.0.15.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x^1, \dots, x^n)$  and  $I = (i_1, \dots, i_k) \in \mathcal{I}_k$ . We define

$$dx^i = dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Omega^k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right)$$

**Note 7.13.0.16.** We have that

1.

$$dx^i \left( \frac{\partial}{\partial x^j} \right) = \delta_{I,J}$$

2. Since  $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$ , by definition, for each  $\omega \in \Omega^k(U)$ ,

$$\omega\left(\frac{\partial}{\partial x^i}\right) \in C^\infty(U)$$

**Exercise 7.13.0.17.** Let  $\omega \in \Omega^k(M)$  and  $(U, \phi)$  be a chart on  $M$  with  $\phi = (x^1, \dots, x^n)$ . Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega\left(\frac{\partial}{\partial x^i}\right) dx^i$$

*Proof.* Let  $p \in U$ . Since  $\{dx_p^i : I \in \mathcal{I}_k\}$  is a basis for  $\Lambda^k(T_p M)$ , there exists  $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$  such that  $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p) dx_p^i$ . So for each  $J \in \mathcal{I}_k$ ,

$$\begin{aligned} \omega\left(\frac{\partial}{\partial x^j}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j}\right) \\ &= f_J \end{aligned}$$

□

**Exercise 7.13.0.18.** Let  $\omega \in \Omega^k(M)$  and  $(U, \phi)$  be a chart on  $M$  with  $\phi = (x^1, \dots, x^n)$ . If  $\omega = \sum_{I \in \mathcal{I}_k} f_I dx^i$ , then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

*Proof.* First we note that

$$\begin{aligned} d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f_I d(dx^i) \\ &= df_I \wedge dx^i \\ &= \left( \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \right) \wedge dx^i \\ &= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i \end{aligned}$$

Then we extend linearly. □

**Definition 7.13.0.19.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F : M \rightarrow N$  be a diffeomorphism. Define the **pullback of  $F$** , denoted  $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$  by

$$(F^* \omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(DF_p(v_1), \dots, DF_p(v_k))$$

for  $\omega \in \Omega^k(N)$ ,  $p \in M$  and  $v_1, \dots, v_k \in T_p M$

# Chapter 8

## Connections

### 8.1 Koszul Connections

**Definition 8.1.0.1.** Let  $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$  and  $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ . Then  $\nabla$  is said to be a **Koszul connection on  $E$  in the first representation** if

1. for each  $\sigma \in \Gamma(E)$ ,  $\nabla(\cdot, \sigma)$  is  $C^\infty(M)$ -linear
2. for each  $X \in \mathfrak{X}(M)$ ,  $\nabla(X, \cdot)$  is  $\mathbb{R}$ -linear
3. for each  $X \in \mathfrak{X}(M)$ ,  $\sigma \in \Gamma(E)$  and  $f \in C^\infty(M)$ ,

$$\nabla(X, f\sigma) = f \nabla(X, \sigma) + X(f)\sigma$$

**Definition 8.1.0.2.** Let  $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$  be a smooth vector bundle and  $\nabla : \Gamma(E) \rightarrow T^*M \otimes \Gamma(E)$ . Then  $\nabla$  is said to be a **Koszul connection on  $E$  in the second representation** if

1.  $\nabla$  is  $\mathbb{R}$ -linear
2. for each  $\sigma \in \Gamma(E)$  and  $f \in C^\infty(M)$ ,

$$\nabla(f\sigma) = f \nabla \sigma + df \otimes \sigma$$

**Note 8.1.0.3.** When the context is clear, we will write  $\nabla_X Y$  in place of  $\nabla(X, Y)$  and we will refer to  $\nabla$  as a connection.

**Exercise 8.1.0.4.** Define  $\phi : \Gamma(E)^{\mathfrak{X}(M) \times \Gamma(E)} \rightarrow [T^*M \otimes \Gamma(E)]^{\Gamma(E)}$  by

$$\phi(\nabla)(X) = \nabla_X \sigma$$

Then  $\nabla$  is a Koszul connection on  $E$  in the first representation iff  $\phi(\nabla)$  Koszul connection on  $E$  in the second representation.

*Proof.* □

**Exercise 8.1.0.5.** Let  $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$ ,  $\nabla$  a connection on  $E$ ,  $X \in \mathfrak{X}(M)$  and  $Y \in \Gamma(E)$ . If  $X = 0$  or  $Y = 0$ , then  $\nabla_X Y = 0$ .

*Proof.*

- If  $X = 0$ , then

$$\begin{aligned} \nabla_X Y &= \nabla_{0X} Y \\ &= 0 \nabla_X Y \\ &= 0 \end{aligned}$$

- Similarly, if  $Y = 0$ , then  $\nabla_X Y = 0$ .

□

**Exercise 8.1.0.6.** Let  $(E, M, \pi)$  be a smooth vector bundle,  $\nabla$  a connection on  $E$ ,  $X \in \mathfrak{X}(M)$ ,  $Y \in \Gamma(E)$  and  $p \in M$ . If  $X \sim_p 0$  or  $Y \sim_p 0$ , then  $[\nabla_X Y]_p = 0$ .

*Proof.*

- Suppose that  $X \sim_p 0$ . Then there exists  $U \subset M$  such that  $U$  is open and  $X|_U = 0$ . Choose  $\phi \in C^\infty(M)$  such that  $\text{supp } \phi \subset U$  and  $\phi \sim_p 1$ . Then  $\phi X = 0$ . The previous exercise implies that  $\nabla_{\phi X} Y = 0$ . Therefore

$$\begin{aligned} \nabla_X Y &= \nabla_{\phi X + (1-\phi)X} Y \\ &= \nabla_{\phi X} Y + \nabla_{(1-\phi)X} Y \\ &= 0 + (1-\phi) \nabla_X Y \\ &= (1-\phi) \nabla_X Y \end{aligned}$$

Hence

$$\begin{aligned} [\nabla_X Y]_p &= [(1-\phi) \nabla_X Y]_p \\ &= (1-\phi(p))[\nabla_X Y]_p \\ &= 0 \end{aligned}$$

- Suppose that  $Y \sim_p 0$ . Then there exists  $U \subset M$  such that  $U$  is open and  $Y|_U = 0$ . Choose  $\phi \in C^\infty(M)$  such that  $\text{supp } \phi \subset U$  and  $\phi \sim_p 1$ . Then  $\phi Y = 0$ . The previous exercise implies that  $\nabla_X \phi Y = 0$ . Since  $\phi \sim_p 1$ , we have that  $1-\phi \sim_p 0$ . Thus  $X(1-\phi) \sim_p 0$  and

$$\begin{aligned} \nabla_X Y &= \nabla_X [\phi Y + (1-\phi)Y] \\ &= \nabla_X [\phi Y] + \nabla_X [(1-\phi)Y] \\ &= \nabla_X [(1-\phi)Y] \\ &= (1-\phi) \nabla_X Y + [X(1-\phi)] \nabla_X Y \end{aligned}$$

Hence

$$\begin{aligned} [\nabla_X Y]_p &= (1-\phi(p))[\nabla_X Y]_p + [X(1-\phi)](p)[\nabla_X Y]_p \\ &= 0 \end{aligned}$$

□

**Exercise 8.1.0.7.** Let  $(E, M, \pi)$  be a smooth vector bundle and  $\nabla$  a connection on  $E$ . Then for each  $X_1, X_2 \in \mathfrak{X}(M)$  and  $Y_1, Y_2 \in \Gamma(E)$ ,  $X_1 \sim_p X_2$  and  $Y_1 \sim_p Y_2$  implies that  $[\nabla_{X_1} Y_1]_p = [\nabla_{X_2} Y_2]_p$ .

*Proof.* Let  $X_1, X_2 \in \mathfrak{X}(M)$  and  $Y_1, Y_2 \in \Gamma(E)$ . Suppose that  $X_1 \sim_p X_2$  and  $Y_1 \sim_p Y_2$ . Define  $X \in \mathfrak{X}(M)$  and  $Y \in \Gamma(E)$  by  $X = X_2 - X_1$  and  $Y = Y_2 - Y_1$ . Then  $X \sim_p 0$  and  $Y \sim_p 0$ . The previous exercise implies



that  $[\nabla_X Y_1]_p = 0$  and  $[\nabla_{X_2} Y]_p = 0$ . Therefore

$$\begin{aligned}
 [\nabla_{X_1} Y_1]_p &= [\nabla_{X_1} Y_1]_p + [\nabla_X Y_1]_p \\
 &= [\nabla_{X_1} Y_1 + \nabla_X Y_1]_p \\
 &= [\nabla_{X_1+X} Y_1]_p \\
 &= [\nabla_{X_2} Y_1]_p \\
 &= [\nabla_{X_2} Y_1]_p + [\nabla_{X_2} Y]_p \\
 &= [\nabla_{X_2} Y_1 + \nabla_{X_2} Y]_p \\
 &= [\nabla_{X_2} (Y_1 + Y)]_p \\
 &= [\nabla_{X_2} Y_2]_p
 \end{aligned}$$

□

**Exercise 8.1.0.8.** Let  $(E, M, \pi)$  be a smooth vector bundle,  $\nabla$  a connection on  $E$  and  $U \subset M$ . If  $U$  is open, then there exists a unique connection  $\nabla^U : \mathfrak{X}(U) \times \Gamma(E|_U) \rightarrow \Gamma(E|_U)$  such that for each  $X \in \mathfrak{X}(M)$  and  $Y \in \Gamma(E)$ ,

$$\nabla^U_{X|_U} Y|_U = (\nabla_X Y)|_U$$



## Chapter 9

# Semi-Riemannian Geometry

**Definition 9.0.0.1.** Let  $M$  be a manifold and  $g \in \Gamma(\Sigma^2 M)$ . Then  $g$  is said to be nondegenerate if for each  $p \in M$ ,  $g_p$  is nondegenerate.

**Definition 9.0.0.2.** Let  $M$  be a manifold and  $g \in \Gamma(\Sigma^2 M)$ . Then  $g$  is said to be a **metric tensor field** on  $M$  if

1.  $g$  is nondegenerate
2.  $g$  has constant index

In this case  $(M, g)$  is said to be a **semi-Riemannian manifold**

**Definition 9.0.0.3.** [Define Interval](#)  
[FINISH!!!](#)

**Definition 9.0.0.4.** Let  $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$ ,  $I \subset \mathbb{R}$  an interval,  $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$  and  $\gamma \in \text{Hom}_{\mathbf{Man}^\infty}(I, E)$ . Then  $\gamma$  is said to be a **section of  $E$  over  $\alpha$**  if  $\pi \circ \gamma = \alpha$ . We denote the set of sections of  $E$  over  $\alpha$  by  $\Gamma(E, \alpha)$ .

**Definition 9.0.0.5.** Let  $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$ ,  $I \subset \mathbb{R}$  an interval,  $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$  and  $\gamma \in \Gamma(E, \alpha)$ . Then  $\gamma$  is said to be **extendible** if there exists  $U \in \mathcal{N}_{\alpha(I)}$  and  $\tilde{\gamma} \in \Gamma(E|_U)$  such that  $U$  is open and  $\tilde{\gamma} \circ \alpha = \gamma$ .

**Exercise 9.0.0.6.** figure 8 not extendible [FINISH!!!](#)

**Exercise 9.0.0.7.** Let  $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$ ,  $\nabla$  a connection on  $E$ ,  $I \subset \mathbb{R}$  an interval and  $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ . There exists a unique  $D_\alpha : \Gamma(E, \alpha) \rightarrow \Gamma(E, \alpha)$  such that

1. for each  $\lambda \in \mathbb{R}$  and  $\gamma, \sigma \in \Gamma(E, \alpha)$ ,

$$D_\alpha(\gamma + \lambda\sigma) = D_\alpha\gamma + \lambda D_\alpha\sigma$$

2. for each  $f \in C^\infty(I)$  and  $\gamma \in \Gamma(E, \alpha)$ ,

$$D_\alpha(f\gamma) = f'\gamma + fD_\alpha\gamma$$

3. for each  $\gamma \in \Gamma(E)$ , if  $\tilde{\gamma}$  extends  $\gamma$ , then

$$D_\alpha\gamma = \nabla_{\alpha'}\gamma$$

*Proof.*

□



# Chapter 10

## Riemannian Geometry

**Definition 10.0.0.1.** Let  $M$  be a smooth manifold and  $g \in T_2^0(M)$  a metric tensor on  $M$ . We define  $\hat{g} \in T_0^2(M)$  by  $\hat{g}(\omega, \eta) = g(\phi_g^{-1}(\omega), \phi_g^{-1}(\eta))$ .

**Exercise 10.0.0.2.** content...

**Exercise 10.0.0.3.** Let  $(M, g)$  be a semi-Riemannian manifold and  $(U, \phi) \in \mathcal{A}$ . Then the induced metric  $\langle \rangle_{T^*M \otimes TM}$  on  $T^*M \otimes TM$  is given by

$$\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} = g^{i,j} g_{kl}$$

*Proof.* We have that

$$\begin{aligned} \left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} &= \langle dx^i, dx^j \rangle_{T^*M} \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle_{TM} \\ &= g^{i,j} g_{kl} \end{aligned}$$

□

**Exercise 10.0.0.4.** Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold.

1. There exists  $\lambda \in \Omega^n(M)$  such that for each orthonormal frame  $e_1, \dots, e_n$ ,

$$\lambda(e_1, \dots, e_n) = 1$$

**Hint:** Choose a frame  $z_1, \dots, z_n$  on  $M$  with corresponding dual frame  $\zeta^1, \dots, \zeta^n$ . Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n$$

2. Let  $N \in \mathfrak{X}(M)$  be the outward pointing normal to  $\partial M$  and  $X \in \mathfrak{X}(M)$ . Then

$$\int_M \operatorname{div} X \lambda = \int_{\partial M} g(X, N) \tilde{\lambda}$$

3. For each  $u \in C^\infty(M)$  and  $X \in \mathfrak{X}(M)$ , we have that

$$\operatorname{div}(uX) = u \operatorname{div}(X) + du(X)$$

and therefore

$$\int_M du(X) \lambda = \int_{\partial M} u g(X, N) \tilde{\lambda} - \int_M u \operatorname{div}(X) \lambda$$

*Proof.*

1. Let  $z_1, \dots, z_n$  be a frame on  $M$  and  $\zeta^1, \dots, \zeta^n$  with corresponding dual frame  $\zeta^1, \dots, \zeta^n$ . Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n$$

Let  $e_1, \dots, e_n$ , be an orthonormal frame on  $M$  with corresponding dual coframe  $\epsilon^1, \dots, \epsilon^n$ . Let  $i, j \in \{1, \dots, n\}$ . Then there exist  $(a_{k,i}) \subset \mathbb{R}$  such that  $\zeta^i = \sum_{k=1}^n a_{k,i} \epsilon^k$ . Then

$$\begin{aligned} \hat{g}(\epsilon^j, \zeta^i) &= \sum_{k=1}^n a_{k,i} \hat{g}(\epsilon^j, \epsilon^k) \\ &= \sum_{k=1}^n a_{k,i} g(\phi_g^{-1}(\epsilon^j), \phi_g^{-1}(\epsilon^k)) \\ &= \sum_{k=1}^n a_{k,i} g(e_j, e_k) \\ &= \sum_{k=1}^n a_{k,i} \delta_{j,k} \\ &= a_{j,i} \end{aligned}$$

which implies that

$$\begin{aligned} \delta_{i,j} &= \zeta^i(z_j) \\ &= \sum_{k=1}^n a_{k,i} \epsilon^k(z_j) \\ &= \sum_{k=1}^n a_{k,i} g(e_k, z_j) \\ &= \sum_{k=1}^n \hat{g}(\epsilon^k, \zeta^i) g(e_k, z_j) \end{aligned}$$

Define  $U, V \in \mathbb{R}^{n \times n}$  by  $U_{i,k} = \hat{g}(\zeta^i, \epsilon^k)$  and  $V_{k,j} = g(e_k, z_j)$ . Then from above, we have that  $UV = I$ . Since  $U, V \in \mathbb{R}^{n \times n}$ ,  $VU = I$ . Hence  $U = V^{-1}$ . Since

$$\begin{aligned} \zeta^i(e_j) &= \sum_{k=1}^n a_{k,i} \epsilon^k(e_j) \\ &= \sum_{k=1}^n a_{k,i} \delta_{k,j} \\ &= a_{j,i} \\ &= \hat{g}(\epsilon^j, \zeta^i) \\ &= U_{i,j} \end{aligned}$$

and

$$\begin{aligned}
g(z_i, z_j) &= \left( \sum_{k=1}^n g(e_k, z_i) e_k, \sum_{l=1}^n g(e_l, z_j) e_l \right) \\
&= \sum_{k=1}^n \sum_{l=1}^n g(e_k, z_i) g(e_l, z_j) g(e_k, e_l) \\
&= \sum_{k=1}^n \sum_{l=1}^n g(e_k, z_i) g(e_l, z_j) \delta_{k,l} \\
&= \sum_{k=1}^n g(e_k, z_i) g(e_k, z_j) \\
&= (V^* V)_{i,j}
\end{aligned}$$

we have that

$$\begin{aligned}
\lambda(e_1, \dots, e_n) &= \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n(e_1, \dots, e_n) \\
&= \det[g(z_i, z_j)]^{1/2} \det[\zeta^i(e_j)] \\
&= \det(V^* V)^{1/2} \det U \\
&= \det V (\det V)^{-1} \\
&= 1
\end{aligned}$$

2. Choose an orthonormal frame  $e_1, \dots, e_{n-1} \in \mathfrak{X}(\partial M)$  with dual coframe  $\epsilon^1, \dots, \epsilon^{n-1}$ . Define  $\nu \in \Omega^1(M)$  to be the dual covector to  $N$ . We note that  $N, e_1, \dots, e_{n-1}$  is an orthonormal frame on  $\mathfrak{X}(M)$ . Let  $X_1, \dots, X_{n-1} \in \mathfrak{X}(\partial M)$ . Since for each  $j \in \{1, \dots, n-1\}$ ,  $X_j \in \mathfrak{X}(\partial M)$  and for each  $p \in \partial M$ ,  $N_p \in (T_p \partial M)^\perp$ , we have that for each  $j \in \{1, \dots, n-1\}$ ,  $g(X_j, N) = 0$ . This implies that

$$\begin{aligned}
\iota^* \iota_X \lambda(X_1, \dots, X_{n-1}) &= \lambda(X, X_1, \dots, X_{n-1}) \\
&= \nu \wedge \epsilon^1 \wedge \dots \wedge \epsilon^{n-1}(X, X_1, \dots, X_{n-1}) \\
&= \det \begin{pmatrix} \nu(X) & \nu(X_1) & \dots & \nu(X_{n-1}) \\ \epsilon^1(X) & \epsilon^1(X_1) & \dots & \epsilon^1(X_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \dots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= \det \begin{pmatrix} g(X, N) & g(X_1, N) & \dots & g(X_{n-1}, N) \\ \epsilon^1(X) & \epsilon^1(X_1) & \dots & \epsilon^1(X_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \dots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= g(X, N) \det(\epsilon^i(X_j)) \\
&= g(X, N) \epsilon^1 \wedge \dots \wedge \epsilon^{n-1}(X_1, \dots, X_{n-1}) \\
&= g(X, N) \tilde{\lambda}(X_1, \dots, X_{n-1})
\end{aligned}$$

Therefore  $\iota^* \iota_X \lambda = g(X, N) \tilde{\lambda}$  and

$$\begin{aligned}
\int_M \operatorname{div} X \lambda &= \int_M d(\iota_X \lambda) \\
&= \int_{\partial M} \iota^* (\iota_X \lambda) \\
&= \int_{\partial M} g(X, N) \tilde{\lambda}
\end{aligned}$$

3. We note that

$$\begin{aligned}
 0 &= \iota_X(du \wedge \lambda) \\
 &= \iota_X(du) \wedge \lambda - du \wedge (\iota_X \lambda) \\
 &= du(X)\lambda - du \wedge (\iota_X \lambda)
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \operatorname{div}(uX)\lambda &= d(\iota_{uX}\lambda) \\
 &= d(u\iota_X\lambda) \\
 &= du \wedge (\iota_X\lambda) + u d(\iota_X\lambda) \\
 &= du(X)\lambda + u \operatorname{div}(X)\lambda \\
 &= [du(X) + u \operatorname{div}(X)]\lambda
 \end{aligned}$$

This implies that  $\operatorname{div}(uX) = du(X) + u \operatorname{div}(X)$ . From before, we have that

$$\begin{aligned}
 \int_M du(X)\lambda &= \int_M \operatorname{div}(uX)\lambda - \int_M u \operatorname{div}(X)\lambda \\
 &= \int_{\partial M} g(uX, N)\tilde{\lambda} - \int_M u \operatorname{div}(X)\lambda \\
 &= \int_{\partial M} u g(X, N)\tilde{\lambda} - \int_M u \operatorname{div}(X)\lambda
 \end{aligned}$$

□

**Exercise 10.0.0.5.**

$$\operatorname{div}(X) = \sum_{j=1}^n (\nabla_{\partial_j} X)^j$$

*Proof.* We have that

$$\begin{aligned}
 \nabla_{\partial_i}(X) &= \sum_{j=1}^n \nabla_{\partial_i}(X^j \partial_j) \\
 &= \sum_{j=1}^n \left[ X^j \nabla_{\partial_i} \partial_j + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n \left[ X^j \left( \sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n \left[ X^j \left( \sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n X^j \left( \sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \sum_{j=1}^n \partial_i(X^j) \partial_j \\
 &= \sum_{k=1}^n \left( \sum_{j=1}^n X^j \Gamma_{i,j}^k \right) \partial_k + \sum_{k=1}^n \partial_i(X^k) \partial_k \\
 &= \sum_{k=1}^n \left[ \left( \sum_{j=1}^n X^j \Gamma_{i,j}^k \right) + \partial_i(X^k) \right] \partial_k
 \end{aligned}$$



so that  $(\nabla_{\partial_i}(X))^i = \left( \sum_{j=1}^n X^j \Gamma_{i,j}^i \right) + \partial_i(X^i)$ . We note that

$$\begin{aligned} \operatorname{div}(X) &= \sum_{i=1}^n \operatorname{div}(X^i \partial_i) \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + dx^i(\partial_i)] \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + 1] \end{aligned}$$

Since  $\lambda = [\det g(\partial_i, \partial_j)]^{1/2} dx^1 \wedge \cdots \wedge dx^n = (\det g)^{1/2} dx$ , we have that

$$\begin{aligned} d(\iota_{\partial_i} \lambda) &= d((\det g)^{1/2} \iota_{\partial_i} dx) \\ &= d[(\det g)^{1/2}] \iota_{\partial_i} dx + (\det g)^{1/2} d(\iota_{\partial_i} dx) \\ &= d[(\det g)^{1/2}] \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n + (\det g)^{1/2} \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n \end{aligned}$$

FINISH!!!

□

**Exercise 10.0.0.6.** Let  $(M, g)$  be a Riemannian manifold.

1. For each  $u, v \in C^\infty(M)$ . Then

(a)

$$\int_M u \Delta v \lambda + \int_M g(\nabla u, \nabla v) \lambda = \int_{\partial M} u N(v) \tilde{\lambda}$$

(b)

$$\int_M [u \Delta v - v \Delta u] \lambda = \int_{\partial M} [u N(v) - v N(u)] \tilde{\lambda}$$

2. (a) If  $\partial M \neq \emptyset$ , then for each  $u, v \in C^\infty(M)$ ,  $u$  and  $v$  are harmonic and  $u|_{\partial M} = v|_{\partial M}$  implies that  $u = v$ .

(b) If  $\partial M = \emptyset$ , then for each  $u \in C^\infty(M)$ ,  $u$  is harmonic implies that  $u$  is constant.

*Proof.*

1. Let  $u, v \in C^\infty(M)$ . Then

(a)

$$\begin{aligned} \int_M u \Delta v \lambda &= \int_M u \operatorname{div}(\nabla v) \lambda \\ &= \int_{\partial M} u g(\nabla v, N) \tilde{\lambda} - \int_M du(\nabla v) \lambda \\ &= \int_{\partial M} u dv(N) \tilde{\lambda} - \int_M g(\nabla u, \nabla v) \lambda \\ &= \int_{\partial M} u N(v) \tilde{\lambda} - \int_M g(\nabla u, \nabla v) \lambda \end{aligned}$$

(b) From above, we have that

$$\begin{aligned}
 \int_M [u\Delta v - v\Delta u]\lambda &= \int_M u\Delta v\lambda - \int_M v\Delta u\lambda \\
 &= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda - \left( \int_{\partial M} vN(u)\tilde{\lambda} - \int_M g(\nabla v, \nabla u)\lambda \right) \\
 &= \int_{\partial M} uN(v)\tilde{\lambda} - \int_{\partial M} vN(u)\tilde{\lambda} \\
 &= \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}
 \end{aligned}$$

2. (a) Suppose that  $\partial M \neq \emptyset$ . Let  $u, v \in C^\infty(M)$ . Suppose that  $u$  and  $v$  are harmonic and  $u|_{\partial M} = v|_{\partial M}$ . Then  $u - v$  is harmonic and

$$\begin{aligned}
 \int_M \|\nabla(u - v)\|_g^2 \lambda &= \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
 &= 0 + \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
 &= \int_M (u - v)\Delta(u - v)\lambda + \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
 &= \int_{\partial M} (u - v)N(u - v)\tilde{\lambda} \\
 &= 0
 \end{aligned}$$

Thus  $\nabla(u - v) = 0$  and  $u - v$  is constant. Since  $u|_{\partial M} = v|_{\partial M}$ , we have that  $u - v = 0$  and thus  $u = v$ .

- (b) Suppose that  $\partial M = \emptyset$ . Let  $u \in C^\infty(M)$ . Suppose that  $u$  is harmonic. Then

$$\begin{aligned}
 \int_M \|\nabla u\|_g^2 \lambda &= \int_M g(\nabla u, \nabla u)\lambda \\
 &= 0 + \int_M g(\nabla u, \nabla u)\lambda \\
 &= \int_M u\Delta u\lambda + \int_M g(\nabla u, \nabla u)\lambda \\
 &= \int_{\partial M} (u - v)g(\nabla(u - v), N)\tilde{\lambda} \\
 &= 0
 \end{aligned}$$

Therefore  $\nabla u = 0$  and  $u$  is constant.

□

## Chapter 11

# Symplectic Geometry

## 11.1 Symplectic Manifolds

**Definition 11.1.0.1.** Let  $M \in \text{Obj}(\mathbf{Man}^\infty)$  and  $\omega \in \Omega^2(M)$ . Then  $\omega$  is said to be **symplectic** if

1.  $\omega$  is nondegenerate
2.  $\omega$  is closed

# Chapter 12

## Extra

**Definition 12.0.0.1.** When working in  $\mathbb{R}^n$ , we introduce the formal objects  $dx^1, dx_2, \dots, dx^n$ . Let  $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_{k,n}$  and  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ . Write  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ . We formally define  $dx^i = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$  and  $\phi_I = (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$ .

**Definition 12.0.0.2.** Let  $k \in \{0, 1, \dots, n\}$ . We define a  $C^\infty(\mathbb{R}^n)$ -module of dimension  $\binom{n}{k}$ , denoted  $\Gamma^k(\mathbb{R}^n)$  to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx^i : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each  $\omega \in \Phi_k(\mathbb{R}^n)$  and  $\chi \in \Gamma^l(\mathbb{R}^n)$ , we may form their **exterior product**, denoted by  $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$ . Thus the exterior product is a map  $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \rightarrow \Gamma^{k+l}(\mathbb{R}^n)$ . The exterior product is characterized by the following properties:

1. the exterior product is bilinear
2. for each  $\omega \in \Phi_k(\mathbb{R}^n)$  and  $\chi \in \Gamma^l(\mathbb{R}^n)$ ,  $\omega \wedge \chi = -\chi \wedge \omega$
3. for each  $\omega \in \Phi_k(\mathbb{R}^n)$ ,  $\omega \wedge \omega = 0$
4. for each  $f \in C^\infty(\mathbb{R}^n)$  and  $\omega \in \Phi_k(\mathbb{R}^n)$ ,  $f \wedge \omega = f\omega$

We call  $\Phi_k(\mathbb{R}^n)$  the differential  $k$ -forms on  $\mathbb{R}^n$ . Let  $\omega$  be a  $k$ -form on  $\mathbb{R}^n$ . If  $k \geq 1$ , then for each  $I \in \mathcal{I}_{k,n}$ , there exists  $f_I \in C^\infty(\mathbb{R}^n)$  such that  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$

**Note 12.0.0.3.** The terms  $dx^1, dx_2, \dots, dx^n$  are a sort of place holder for the coordinates of a point  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ . When we work with functions  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ , we will have different coordinates and to avoid confusion, we will write  $\{du^1, du_2, \dots, du_k\}$  when referencing the coordinates on  $\mathbb{R}^k$  and  $\{dx^1, dx_2, \dots, dx^n\}$  when referencing the coordinates on  $\mathbb{R}^n$ .

**Exercise 12.0.0.4.** Let  $B_{n \times n} = (b_{i,j}) \in [C^\infty(M)]^{n \times n}$  be an  $n \times n$  matrix. Then

$$\bigwedge_{i=1}^n \left( \sum_{j=1}^n b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \dots \wedge dx^n$$

*Proof.* Bilinearity of the exterior product implies that

$$\begin{aligned}
 \bigwedge_{i=1}^n \left( \sum_{j=1}^n b_{i,j} dx^j \right) &= \left( \sum_{j=1}^n b_{1,j} dx^j \right) \wedge \left( \sum_{j=1}^n b_{2,j} dx^j \right) \wedge \cdots \wedge \left( \sum_{j=1}^n b_{n,j} dx^j \right) \\
 &= \sum_{j_1, \dots, j_n=1}^n \left( \prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\
 &= \sum_{j_1 \neq \dots \neq j_n} \left( \prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\
 &= \left[ \sum_{\sigma \in S_n} \text{sgn}(\sigma) \left( \prod_{i=1}^n b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\
 &= (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n
 \end{aligned}$$

□

**Definition 12.0.0.5.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a 0-form on  $\mathbb{R}^n$ . We define a 1-form, denoted  $df$ , on  $\mathbb{R}^n$  by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Let  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$  be a  $k$ -form on  $\mathbb{R}^n$ . We can define a differential  $k+1$ -form, denoted  $d\omega$ , on  $\mathbb{R}^n$  by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx^I$$

**Exercise 12.0.0.6.** On  $\mathbb{R}^3$ , put

1.  $\omega_0 = f_0$ ,
2.  $\omega_1 = f_1 dx^1 + f_2 dx_2 + f_3 dx_3$ ,
3.  $\omega_2 = f_1 dx_2 \wedge dx_3 - f_2 dx^1 \wedge dx_3 + f_3 dx^1 \wedge dx_2$

Show that

1.  $d\omega_0 = \frac{\partial f_0}{\partial x^1} dx^1 + \frac{\partial f_0}{\partial x^2} dx_2 + \frac{\partial f_0}{\partial x^3} dx_3$
2.  $d\omega_1 = \left( \frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) dx_2 \wedge dx_3 + \left( \frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3} \right) dx^1 \wedge dx_3 + \left( \frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) dx^1 \wedge dx_2$
3.  $d\omega_2 = \left( \frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3} \right) dx^1 \wedge dx_2 \wedge dx_3$

*Proof.* Straightforward. □

**Exercise 12.0.0.7.** Let  $I \in \mathcal{I}_{k,n}$ . Then there is a unique  $I_* \in \mathcal{I}_{n-k,n}$  such that  $dx^I \wedge dx_{I_*} = dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$ .

**Definition 12.0.0.8.** We define a linear map  $*$  :  $\Phi_k(\mathbb{R}^n) \rightarrow \Gamma^{n-k}(\mathbb{R}^n)$  called the **Hodge \*-operator** by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

**Definition 12.0.0.9.** Let  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$  be smooth. Write  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ . We define  $\phi^* : \Phi_k(\mathbb{R}^n) \rightarrow \Phi_k(\mathbb{R}^k)$  via the following properties:

1. for each 0-form  $f$  on  $\mathbb{R}^n$ ,  $\phi^* f = f \circ \phi$
2. for  $i = 1, \dots, n$ ,  $\phi^* dx^i = d\phi_i$
3. for an  $s$ -form  $\omega$ , and a  $t$ -form  $\chi$  on  $\mathbb{R}^n$ ,  $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
4. for  $l$ -forms  $\omega, \chi$  on  $\mathbb{R}^n$ ,  $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

**Exercise 12.0.0.10.** Let  $M \subset \mathbb{R}^n$  be a  $k$ -dimensional smooth submanifold of  $\mathbb{R}^n$ ,  $\phi : U \rightarrow V$  a smooth parametrization of  $M$ ,  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$  an  $k$ -form on  $\mathbb{R}^n$ . Then

$$\phi^*\omega = \left( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \dots \wedge du_k$$

*Proof.* By definition,

$$\begin{aligned} \phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx^i \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \end{aligned}$$

A previous exercise tells us that for each  $I \in \mathcal{I}_{k,n}$ ,

$$\begin{aligned} d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \dots \wedge d\phi_{i_n} \\ &= \left( \sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u^j} du^j \right) \wedge \left( \sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u^j} du^j \right) \wedge \dots \wedge \left( \sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u^j} du^j \right) \\ &= (\det v\phi_I) du^1 \wedge du_2 \wedge \dots \wedge du_k \end{aligned}$$

Therefore

$$\begin{aligned} \phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) du^1 \wedge du_2 \wedge \dots \wedge du_k \\ &= \left( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \dots \wedge du_k \end{aligned}$$

□

## 12.1 Integration of Differential Forms

**Definition 12.1.0.1.** Let  $U \subset \mathbb{R}^k$  be open and  $\omega = f dx^1 \wedge dx_2 \wedge \dots \wedge dx_k$  a  $k$ -form on  $\mathbb{R}^k$ . Define

$$\int_U \omega = \int_U f dx$$

**Definition 12.1.0.2.** Let  $M \subset \mathbb{R}^n$  be a  $k$ -dimensional oriented smooth submanifold of  $\mathbb{R}^n$ ,  $\omega$  a  $k$ -form on  $\mathbb{R}^n$  and  $\phi : U \rightarrow V$  a local smooth, orientation-preserving parametrization of  $M$ . Define

$$\int_V \omega = \int_U \phi^*\omega$$

**Exercise 12.1.0.3.****Theorem 12.1.0.4. Stokes Theorem:**

Let  $M \subset \mathbb{R}^n$  be a  $k$ -dimensional oriented smooth submanifold of  $\mathbb{R}^n$  and  $\omega$  a  $k-1$ -form on  $\mathbb{R}^n$ . Then

$$\int_{\partial M} \omega = \int_M d\omega$$



# Appendix A

## Summation



## Appendix B

# Asymptotic Notation



# Bibliography

- [1] [Introduction to Algebra](#)
- [2] [Introduction to Analysis](#)
- [3] [Introduction to Fourier Analysis](#)
- [4] [Introduction to Measure and Integration](#)