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Introduction to Harmonic Analysis

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Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Prelimiaries

1.1 Category Theory

- **Hilb**:
  - $\mathrm{Obj}(\mathbf{Hilb}) := \{H : H \text{ is a Hilbert space}\}$
  - $\mathrm{Hom}_{\mathbf{Hilb}}(H_1, H_2) := \mathrm{Hom}_{\mathbf{Ban}}(H_1, H_2)$
- **Mon**

1.1.1 The Unitary Group

Definition 1.1.1.1. Let $H_1, H_2 \in \mathrm{Obj}(\mathbf{Hilb})$. We define the **unitary group from** H_1 **to** H_2, denoted $U(H_1, H_2)$, by

$$U(H_1, H_2) = \{T \in \mathrm{Iso}_{\mathbf{Hilb}}(H_1, H_2) : T^* = T^{-1}\}$$

We write $U(H)$ in place of $U(H, H)$. We equip $U(H_1, H_2)$ with the strong operator topology.

Exercise 1.1.1.2. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$. Then $\mathcal{T}^s_{U(H)} = \mathcal{T}^w_{U(H)}$. strong weak operator topologies coincide

Exercise 1.1.1.3. Let $H \in \mathrm{Obj}(\mathbf{Hilb})$. Then $U(H)$ is a topological group.

Proof. content... $\square$

Chapter 2

Representation Theory

2.1 Group Representations

2.1.1 Unitary representations

Definition 2.1.1.1. Let $G \in \text{Obj}(\mathbf{TopGrp})$, $H \in \text{Obj}(\mathbf{Hilb})$ and $\pi \in \text{Hom}_{\mathbf{TopGrp}}(G, U(H))$. Then (H, π) is said to be a **unitary representation of** G. We define the **dimension of** (H, π), denoted $\dim(H, \pi)$, by $\dim(H, \pi) := \dim V$.

Definition 2.1.1.2. Let $G \in \text{Obj}(\mathbf{TopGrp})$, (H_π, π), (H_ρ, ρ) unitary representations of G and $T \in \text{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho)$. Then T is said to be (π, ρ)**-equivariant** if for each $g \in G$, $T \circ \pi(g) = \rho(g) \circ T$, i.e. the following diagram commutes:

$$\begin{tikzcd} H_\pi \arrow[r, "T"] \arrow[d, "\pi(g)"'] & H_\rho \arrow[d, "\rho(g)"] \\ H_\pi \arrow[r, "T"'] & H_\rho \end{tikzcd}$$

Definition 2.1.1.3. Let $G \in \text{Obj}(\mathbf{TopGrp})$. We define $\mathbf{URep}(G)$ by

- $\text{Obj}(\mathbf{URep}(G)) = \{(H, \pi) : (H, \pi) \text{ is a unitary representation of } G\}$.
- for $(H_\pi, \pi), (H_\rho, \rho) \in \text{Obj}(\mathbf{URep}(G))$,

$$\text{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) = \{T \in \text{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho) : T \text{ is } (\pi, \rho)\text{-equivariant}\}$$

- for $(H_\pi, \pi), (H_\rho, \rho), (H_\mu, \mu) \in \text{Obj}(\mathbf{URep}(G))$, $T \in \text{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$ and $S \in \text{Hom}_{\mathbf{URep}(G)}((H_\rho, \rho), (H_\mu, \mu))$,

$$S \circ_{\mathbf{URep}(G)} T = S \circ T$$

Exercise 2.1.1.4. Let $G \in \text{Obj}(\mathbf{TopGrp})$. Then $\mathbf{URep}(G)$ is a category.

Proof. FINISH!!! □

Exercise 2.1.1.5. Let $G \in \text{Obj}(\mathbf{TopGrp})$ and $(H_\pi, \pi), (H_\rho, \rho) \in \text{Obj}(\mathbf{URep}(G))$. Then $\text{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) \in \text{Obj}(\mathbf{Vect}_{\mathbb{C}})$.

Proof. Let $S, T \in \text{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$ and $\lambda \in \mathbb{C}$. Then for each $g \in G$,

$$\begin{aligned} (S + \lambda T) \circ \pi(g) &= S \circ \pi(g) + \lambda T \circ \pi(g) \\ &= \rho(g) \circ S + \rho(g) \circ (\lambda T) \\ &= \rho(g) \circ (S + \lambda T). \end{aligned}$$

Hence $S + \lambda T \in \text{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$. Since $S, T \in \text{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$ and $\lambda \in \mathbb{C}$ is arbitrary, we have that $\text{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) \in \text{Obj}(\mathbf{Vect}_{\mathbb{C}})$. □

Definition 2.1.1.6. Let $G \in \text{Obj}(\mathbf{TopGrp})$ and $(H_\pi, \pi), (H_\rho, \rho) \in \mathbf{URep}(G)$. Then (H_π, π) is said to be **unitarily equivalent** to (H_ρ, ρ), denoted $(H_\pi, \pi) \equiv (H_\rho, \rho)$, if $\text{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) \cap U(H_\pi, H_\rho) \neq \varnothing$.

Note 2.1.1.7. Let $\pi \in \text{Hom}_{\mathbf{TopGrp}}(G, U(H))$. Since $U(H)$ is equipped with the strong operator topology, we have that for each $u \in H$, the map $g \mapsto \pi(g)u$ is continuous.

Definition 2.1.1.8. Let $G \in \text{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \text{Obj}(\mathbf{URep}(G))$. We define the **induced group action of G on H**, denoted $\phi_{(H,\pi)} : G \times H \to H$, by

$$\phi_{(H,\pi)}(g, v) = \pi(g)v$$

Note 2.1.1.9. When the context is clear, we write $g \cdot v$ in place of $\phi_{(H,\pi)}(g, v)$.

Exercise 2.1.1.10. Let $G \in \text{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \text{Obj}(\mathbf{URep}(G))$. Then

1. $\phi_{(H,\pi)}$ is a linear group action.
2. G is locally compact implies that $\phi_{(H,\pi)}$ is continuous

Proof.

1. - Let $g, h \in G$ and $v \in H$.
     (a) Since $\pi \in \text{Hom}_{\mathbf{TopGrp}}(G, U(H))$,

     $$\begin{aligned} e \cdot v &= \pi(e)v \\ &= \text{id}_H v \\ &= v \end{aligned}$$

     (b) Since $\pi \in \text{Hom}_{\mathbf{TopGrp}}(G, U(H))$,

     $$\begin{aligned} g \cdot (h \cdot v) &= \pi(g)[\pi(h)v] \\ &= [\pi(g)\pi(h)]v \\ &= \pi(gh)v \\ &= (gh) \cdot v \end{aligned}$$

     Since $g, h \in G$ and $v \in H$ are arbitrary, $\phi_{(H,\pi)}$ is a group action of G on H.
   - Let $g \in G$, $\lambda \in \mathbb{C}$ and $v, w \in H$. Then

     $$\begin{aligned} g \cdot (\lambda v + w) & \\ &= \pi(g)(\lambda v + w) \\ &= \lambda\pi(g)v + \pi(g)w \\ &= \lambda g \cdot v + g \cdot w \end{aligned}$$

     Since $g \in G$, $\lambda \in \mathbb{C}$ and $v, w \in H$ are arbitrary, $\phi_{(H,\pi)}$ is a linear action.
2. Suppose that G is locally compact. Let $(g_0, v_0) \in G \times H$ and $\epsilon > 0$. Since G is locally compact, there exists $K \subset G$ such that $g_0 \in \text{Int}\, K$ and K is compact. Let $v \in H$. Define $f_v : G \to H$ by $f_v(g) = g \cdot v$. Since $\pi : G \to U(H)$ is continuous, f_v is continuous. Thus $\|f_v\|$ is continuous. Since K is compact, $\|f_v\|(K)$ is compact. Thus

   $$\begin{aligned} \sup_{g \in K} \|g \cdot v\| &= \sup_{g \in K} \|f_v(g)\| \\ &< \infty \end{aligned}$$

   Since $v \in H$ is arbitrary, we have that for each $v \in H$, $\sup_{g \in K} \|g \cdot v\| < \infty$. The uniform boundedness principle implies that there exists $M > 0$ such that $\sup_{g \in K} \|\pi(g)\| \leq M$. Since f_{v_0} is continuous, there exists $U \subset K$ such that U is open, $g_0 \in U$, and $f_{v_0}(U) \subset B(f_{v_0}(g_0), \epsilon/2)$. Let $(g_1, v_1) \in U \times B(v_0, (2M)^{-1}\epsilon)$. Then

$$\begin{aligned}\|\phi_{(H,\pi)}(g_0, v_0) - \phi_{(H,\pi)}(g_1, v_1)\| &= \|g_0 \cdot v_0 - g_1 \cdot v_1\| \\ &\leq \|g_0 \cdot v_0 - g_1 \cdot v_0\| + \|g_1 \cdot v_0 - g_1 \cdot v_1\| \\ &= \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + \|\pi(g_1)(v_0 - v_1)\| \\ &\leq \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + \|\pi(g_1)\|\|v_0 - v_1\| \\ &\leq \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + M\|v_0 - v_1\| \\ &\leq \frac{\epsilon}{2} + M\frac{\epsilon}{2M} \\ &= \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\phi_{(H,\pi)}$ is continuous at (g_0, v_0). Since $(g_0, v_0) \in G \times H$ is arbitrary, we have that $\phi_{(H,\pi)} : G \times H \to H$ is continuous.

□

2.1.2 Subrepresentations

Definition 2.1.2.1. Let $G \in \text{Obj}(\textbf{TopGrp})$, $(H, \pi) \in \text{Obj}(\textbf{URep}(G))$ and $E \subset H$ a closed subspace. Then E is said to be

- **nontrivial** if $E \neq H, \varnothing$
- (H, π)**-invariant** if for each $g \in G$, $\pi(g)(E) \subset E$

Exercise 2.1.2.2. Let $G \in \text{Obj}(\textbf{TopGrp})$, $(H, \pi) \in \text{Obj}(\textbf{URep}(G))$ and $E \subset H$ a closed subspace. Suppose that E is (H, π)-invariant. Then for each $g, h \in G$,

1. $\pi(g)|_E \in \text{Aut}_{\textbf{Hilb}}(E)$, $\pi(g)|_E^{-1} = \pi(g^{-1})|_E$ and $\pi(g)(E) = E$,
2. $\pi(g)|_E \in U(E)$ and $\pi(g)|_E^* = \pi(g^{-1})|_E$,
3. $\pi(gh)|_E = \pi(g)|_E \circ \pi(h)|_E$.

Proof. Let $g, h \in G$.

1. Let $x \in E$. Since E is (H, π)-invariant, we have that $\pi(g)(x) \in E$. Therefore

$$\begin{aligned}[\pi(g^{-1})|_E \circ \pi(g)|_E](x) &= \pi(g^{-1})|_E[\pi(g)|_E(x)] \\ &= \pi(g^{-1})|_E[\pi(g)(x)] \\ &= \pi(g^{-1})[\pi(g)(x)] \\ &= [\pi(g^{-1}) \circ \pi(g)](x) \\ &= \pi(g^{-1}g)(x) \\ &= \pi(e)(x) \\ &= I(x) \\ &= I_E(x).\end{aligned}$$

Similarly, $\pi(g^{-1})(x) \in E$ and $[\pi(g)|_E \circ \pi(g^{-1})|_E](x) = I|_E(x)$. Since $x \in E$ is arbitrary, we have that $\pi(g)|_E \in \text{Aut}_{\textbf{Hilb}}(E)$ and $\pi(g^{-1})|_E = \pi(g)|_E^{-1}$. Since $\pi(g)|_E \in \text{Aut}_{\textbf{Hilb}}(E)$, we have that

$$\begin{aligned}\pi(g)(E) &= \pi(g)|_E(E) \\ &= E.\end{aligned}$$

2. Let $x, y \in E$. Then

$$\begin{aligned}\langle \pi(g)|_E x, y\rangle &= \langle \pi(g)x, y\rangle \\ &= \langle x, \pi(g)^* y\rangle \\ &= \langle x, \pi(g)^*|_E y\rangle\end{aligned}$$

Since $x, y \in E$ are arbitrary, we have that $\pi(g)|_E^* = \pi(g)^*|_E$. The previous part then implies that

$$\begin{aligned}\pi(g)|_E^* &= \pi(g)^*|_E \\ &= \pi(g)^{-1}|_E \\ &= \pi(g^{-1})|_E \\ &= \pi(g)|_E^{-1}.\end{aligned}$$

Since $\pi(g)|_E^* = \pi(g)|_E^{-1}$, we have that $\pi(g)|_E \in U(E)$.

3. Let $x \in E$. Since E is (H, π)-invariant, we have that $\pi(h)(x) \in E$ and therefore

$$\begin{aligned}\pi(gh)|_E(x) &= \pi(gh)(x) \\ &= [\pi(g) \circ \pi(g)](x) \\ &= \pi(g)[\pi(h)(x)] \\ &= \pi(g)|_E[\pi(h)(x)] \\ &= \pi(g)|_E[\pi(h)|_E(x)] \\ &= [\pi(g)|_E \circ \pi(g)|_E](x).\end{aligned}$$

Since $x \in E$ is arbitrary, we have that $\pi(gh)|_E = \pi(g)|_E \circ \pi(g)|_E$.

$\square$

Definition 2.1.2.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $\mathbb{K} \in \mathrm{Obj}(\mathbf{Field})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then

- (H, π) is said to be **reducible** if there exists a closed subspace $E \subset H$ such that E is not trivial and E is (H, π)-invariant
- (H, π) is said to be **irreducible** if (H, π) is not reducible.

Definition 2.1.2.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. Suppose that E is (H, π)-invariant.

- We define $\pi^E \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(E))$ by $\pi^E(g) := \pi(g)|_E$
- We define the **restriction** (H, π) **to** E, denoted $(H, \pi)|_E$, by $(H, \pi)|_E := (E, \pi^E)$

Exercise 2.1.2.5. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace.

1. If E is nontrivial, then $E^\perp$ is nontrivial.
2. If E is (H, π)-invariant, then $E^\perp$ is (H, π)-invariant.

Proof.

1. Suppose that E is nontrivial. Then $E \neq \{0\}, H$. Then $E^\perp \neq \{0\}, H$. Thus $E^\perp$ is nontrivial.
2. Suppose that E is (H, π)-invariant. Let $g \in G$. Since $\pi(g) \in U(H)$ and $\pi(g)(E) = E$, An exercise in the analysis notes section on Hilbert spaces implies that $\pi(g)(E^\perp) = E^\perp$. Since $g \in G$ is arbitrary, $E^\perp$ is (H, π)-invariant.

$\square$

Definition 2.1.2.6. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $u \in H$. We define the **cyclic subspace of H generated by u under** (H, π), denoted $\mathrm{cyc}_{(H,\pi)}(u)$, by

$$\mathrm{cyc}_{(H,\pi)}(u) := \mathrm{cl}\,\mathrm{span}(\phi_{(H,\pi)}(G, u))$$

replace $\phi(G, u)$ with $G \triangleright u$

Note 2.1.2.7. When the context is clear, we write $\mathrm{cyc}(u)$ in place of $\mathrm{cyc}_{(H,\pi)}(u)$.

Exercise 2.1.2.8. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $u \in H$. Then $\mathrm{cyc}(u)$ is (H, π)-invariant. this should largely be a result about linear group actions.

Proof. Let $g \in G$. Since G acts linearly and homeomorphically on H,

$$\begin{aligned}
g \cdot \mathrm{cyc}(u) &= g \cdot \mathrm{cl}\,\mathrm{span}(G \cdot u) \\
&= \mathrm{cl}\, g \cdot \mathrm{span}(G \cdot u) \\
&= \mathrm{cl}\,\mathrm{span}[g \cdot (G \cdot u)] \\
&= \mathrm{cl}\,\mathrm{span}(G \cdot u) \\
&= \mathrm{cyc}(u)
\end{aligned}$$

Since $g \in G$ is arbitrary, $\mathrm{cyc}(u)$ is G-invariant. $\square$

Definition 2.1.2.9. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$.

- Let $u \in H$. Then u is said to be (H, π)**-cyclic** if $\mathrm{cyc}(u) = H$.
- Then (H, π) is said to be **cyclic** if there exists $u \in H$ such that u is (H, π)-cyclic.

2.1.3 Direct Sum of Representations

Definition 2.1.3.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H_\alpha, \pi_\alpha)_{\alpha \in A} \subset \mathrm{Obj}(\mathbf{URep}(G))$.

- We define $\bigoplus\limits_{\alpha \in A} \pi_\alpha \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(\bigoplus\limits_{\alpha \in A} H_\alpha))$ by

$$\left[\bigoplus_{\alpha \in A} \pi_\alpha\right](g) = \bigoplus_{\alpha \in A} \pi_\alpha(g)$$

- We define the **direct sum** of $(H_\alpha, \pi_\alpha)_{\alpha \in A}$, denoted $\bigoplus\limits_{\alpha \in A} (H_\alpha, \pi_\alpha)$, by

$$\bigoplus_{\alpha \in A} (H_\alpha, \pi_\alpha) = \left(\bigoplus_{\alpha \in A} H_\alpha, \bigoplus_{\alpha \in A} \pi_\alpha\right)$$

Note 2.1.3.2. FINISH!!! the last definition works for internal or external direct sum, just need to define inner or external sum of H_α and π_α in either case. work out problems of unitary operators in analysis notes, eg, direct product of unitary ops is unitary, internal direct products, etc

Exercise 2.1.3.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. If E is (H, π)-invariant, then $(H, \pi) = (E \oplus E^\perp, \pi^E \oplus \pi^{E^\perp})$.

Proof. Suppose that E is (H, π)-invariant. A previous exercise implies that $E^\perp$ is (H, π)-invariant. Since $H = E \oplus E^\perp$. Let $g \in G$ and $u \in H$. Since $H = E \oplus E^\perp$, there exists $v \in E$ and $w \in E^\perp$ such that $u = v + w$.

Then

$$\begin{aligned}\pi(g)(u) &= \pi(g)(v+w)\\ &= \pi(g)(v)+\pi(g)(w)\\ &= \pi(g)|_E(v)+\pi(g)|_{E^\perp}(w)\\ &= \pi^E(g)(v)+\pi^{E^\perp}(g)(w)\\ &= [\pi^E(g)\oplus\pi^{E^\perp}(g)](v+w)\\ &= [\pi^E\oplus\pi^{E^\perp}](g)(v+w)\\ &= [\pi^E\oplus\pi^{E^\perp}](g)(u)\end{aligned}$$

Since $u \in H$ is arbitrary, $\pi(g) = [\pi^E \oplus \pi^{E^\perp}](g)$. Since $g \in G$ is arbitrary, $\pi = \pi^E \oplus \pi^{E^\perp}$. □

Definition 2.1.3.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H,\pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $\mathcal{E} \subset \mathcal{P}(H)$. Then $\mathcal{E}$ is said to be an **(H,π)-orthocyclic system** if for each $E, F \in \mathcal{E}$,

1. E is a closed subspace of H
2. $(H,\pi)|_E$ is cyclic
3. if $E \neq F$, then $E \perp F$

Exercise 2.1.3.5. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H,\pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then there exists $\mathcal{E} \subset \mathcal{P}(H)$ such that $\mathcal{E}$ is an (H,π)-orthocyclic system and $(H,\pi) = \bigoplus_{E\in\mathcal{E}} (H,\pi)|_E$.

Hint: Zorn's lemma

Proof. Define $\mathcal{P} = \{\mathcal{E} : \mathcal{E}$ is an (H,π)-orthocyclic system$\}$. We partially order $\mathcal{P}$ by inclusion. Let $\mathcal{C} \subset \mathcal{P}$ be a chain. Set $\mathcal{E}_0 = \bigcup_{\mathcal{E}\in\mathcal{C}} \mathcal{E}$. Let $E_1, E_2 \in \mathcal{E}_0$. Then there exist $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}$ such that $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$. Since $\mathcal{C}$ is a chain, $\mathcal{E}_1 \subset \mathcal{E}_2$ or $\mathcal{E}_2 \subset \mathcal{E}_1$. Suppose that $\mathcal{E}_1 \subset \mathcal{E}_2$. Then $E_1 \in \mathcal{E}_2$. Since $\mathcal{E}_2$ is an (H,π)-orthocyclic system, we have that E_1 is a closed subspaces of H, $(H,\pi)|_{E_1}$ is cyclic and if $E_1 \neq E_2$, then $E_1 \perp E_2$. Similarly, $\mathcal{E}_2 \subset \mathcal{E}_1$ implies the same conclusion. Since $E_1, E_2 \in \mathcal{E}_0$ are arbitrary, we have that for each $E_1, E_2 \in \mathcal{E}_0$

1. E_1 is a closed subspaces of H and E_1 is (H,π)-invariant
2. $(H,\pi)|_{E_1}$ is cyclic
3. if $E_1 \neq E_2$, then $E_1 \perp E_2$

Thus $\mathcal{E}_0$ is an (H,π)-orthocyclic system. Hence $\mathcal{E}_0 \in \mathcal{P}$. By construction, for each $\mathcal{E} \in \mathcal{C}$, $\mathcal{E} \subset \mathcal{E}_0$. So $\mathcal{E}_0$ is an upper bound of $\mathcal{C}$. Since $\mathcal{C} \subset \mathcal{P}$ such that $\mathcal{C}$ is a chain is arbitrary, we have that for each $\mathcal{C} \subset \mathcal{P}$, if $\mathcal{C}$ is a chain, then there exists $\mathcal{E}_0 \in \mathcal{P}$ such that $\mathcal{E}_0$ is an upper bound of $\mathcal{C}$. Zorn's lemma implies that there exists $\mathcal{E} \in \mathcal{P}$ such that $\mathcal{E}$ is maximal. Set $E = \bigoplus_{E_0\in\mathcal{E}} E_0$. For the sake of contradiction, suppose that $H \neq E$. Then $E^\perp \neq \{0\}$. Thus there exists $u \in E^\perp$ such that $u \neq 0$. Therefore $\mathrm{cyc}(u) \neq 0$ and $\mathrm{cyc}(u) \subset E^\perp$. Let $E_0 \in \mathcal{E}$. By construction, $E_0 \subset E$. Thus

$$\begin{aligned}\mathrm{cyc}(u) &\subset E^\perp\\ &\subset E_0^\perp\end{aligned}$$

Since $E_0 \in \mathcal{E}$ is arbitrary, we have that for each $E_0 \in \mathcal{E}$, $\mathrm{cyc}(u) \subset E_0^\perp$. Set $\mathcal{E}' = \mathcal{E} \cup \{\mathrm{cyc}(u)\}$. Then for each $E, F \in \mathcal{E}'$,

1. E is a closed subspaces of H and E is (H,π)-invariant
2. $(H,\pi)|_E$ is cyclic

3. if $E \neq F$, then $E \perp F$

Hence $\mathcal{E}' \in \mathcal{P}$. Since $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{E}$ □

Note 2.1.3.6. Let H be a Hilbert space and $E \subset H$ a closed subspace. We denote the orthogonal projection onto E by P_E.

Exercise 2.1.3.7. Let $G \in \text{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \text{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. Then E is (H, π)-invariant iff $P_E \in \text{End}_{\mathbf{URep}(G)}((H, \pi))$.

Proof.

- $(\implies)$:
  Suppose that E is (H, π)-invariant. Let $g \in G$ and $z \in H$. Then there exists $x \in E$ and $y \in E^\perp$ such that $z = x + y$. Since E is (H, π) invariant, $\pi(g)(x) \in E$. Thus

$$\begin{aligned} \pi(g) \circ P_E(x) &= \pi(g)(x) \\ &= P_E \circ \pi(g)(x). \end{aligned}$$

  Since E is (H, π)-invariant, ref previous ex here implies that $E^\perp$ is (H, π)-invariant. Therefore $\pi(g)(y) \in E^\perp$ and

$$\begin{aligned} \pi(g) \circ P_E(x) &= \pi(g)(0) \\ &= 0 \\ &= P_E \circ \pi(g)(y). \end{aligned}$$

  Hence

$$\begin{aligned} \pi(g) \circ P_E(z) &= \pi(g) \circ P_E(x + y) \\ &= \pi(g) \circ P_E(x) + \pi(g) \circ P_E(y) \\ &= P_E \circ \pi(g)(x) + P_E \circ \pi(g)(y) \\ &= P_E \circ \pi(g)(x + y) \\ &= P_E \circ \pi(g)(z). \end{aligned}$$

  Since $z \in H$ is arbitrary, we have that $\pi(g) \circ P_E = P_E \circ \pi(g)$. Since $g \in G$ is arbitrary, $P_E \in \text{End}_{\mathbf{URep}(G)}(H, \pi)$.

- $(\impliedby)$:
  Conversely, suppose that $P_E \in \text{End}_{\mathbf{URep}(G)}((H, \pi))$. Let $g \in G$ and $x \in E$. Then

$$\begin{aligned} &\pi(g)(x) \\ &\quad = \pi(g) \circ P_E(x) \\ &\quad = P_E \circ \pi(g)(x) \\ &\quad \in E. \end{aligned}$$

  Since $x \in E$ is arbitrary, $\pi(g)(E) \subset E$. Since $g \in G$ is arbitrary, E is (H, π)-invariant.

□

2.2 Tannaka Duality

Definition 2.2.0.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. We define the **forgetful functor from $\mathbf{URep}(G)$ to Hilb**, denoted $U : \mathbf{URep}(G) \to \mathbf{Hilb}$, by

- $U(H, \pi) = H, \quad (H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$
- $U(T) = T, \quad T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$.

Need to find out if quotienting by equivalence of isomorphism makes $\mathbf{URep}(G)$ a small category so that we can talk about the functor category $\mathbf{Hilb}^{\mathbf{URep}(G)}$ containing the forgetful functor as an object.

Definition 2.2.0.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $g \in G$. We define $\hat{g} : U \Rightarrow U$ by

$$\hat{g}_{(H,\pi)} = \pi(g)$$

Exercise 2.2.0.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $g \in G$. Then

1. $\hat{g} : U \Rightarrow U$ is a natural transformation.
2. $\hat{g} \in \mathrm{Aut}_{\mathbf{Hilb}^{\mathbf{URep}(G)}}(U)$

Proof.

1. (a) Let $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. By definition,

$$\begin{aligned} \hat{g}_{(H,\pi)} &= \pi(g) \\ &\in U(H) \\ &\subset \mathrm{Aut}_{\mathbf{Hilb}}(U(H, \pi)) \end{aligned}$$

   (b) Let $(H_\pi, \pi), (H_\rho, \rho) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$. By definition, $T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho)$ and T is (π, ρ)-equivariant. Therefore

$$\begin{aligned} U(T) \circ \hat{g}_{(H_\pi,\pi)} &= T \circ \pi(g) \\ &= \rho(g) \circ T \\ &= \hat{g}_{(H_\rho,\rho)} \circ U(T) \end{aligned}$$

   i.e. the following diagram commutes:

$$\begin{array}{ccc} U(H_\pi, \pi) & \xrightarrow{\hat{g}_{(H_\pi,\pi)}} & U(H_\pi, \pi) \\ {\scriptstyle U(T)}\downarrow & & \downarrow{\scriptstyle U(T)} \\ U(H_\rho, \rho) & \xrightarrow{\hat{g}_{(H_\rho,\rho)}} & U(H_\rho, \rho) \end{array} = \begin{array}{ccc} H_\pi & \xrightarrow{\pi(g)} & H_\pi \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ H_\rho & \xrightarrow{\rho(g)} & H_\rho \end{array}$$

   Thus $\hat{g} : U \Rightarrow U$ is a natural transformation.

2. Set $h = g^{-1}$. Part (1) implies that $\hat{g}, \hat{h} \in \mathrm{End}_{\mathbf{Hilb}^{\mathbf{URep}(G)}}(U)$. Let $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then

$$(\hat{g} \circ \hat{h})_{(H,\pi)} = \hat{g}_{(H,\pi)}$$

   The previous part implies that

$$\begin{aligned} \hat{g} &\in \mathrm{Hom}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U, U) \\ &= \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U) \end{aligned}$$

$\square$

Definition 2.2.0.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H,\pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. We define the (H,π)**-projection**, denoted $\pi_{(H,\pi)} : \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U) \to \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}}(V)$, by $\pi_{(H,\pi)}(\alpha) = \alpha_{(H,\pi)}$. We define the **topology of endomorphisms of** U, denoted $\mathcal{T}_{\mathcal{E}(U)}$, by

$$\mathcal{T}_{\mathcal{E}(U)} = \tau(\pi_{(H,\pi)} : (H,\pi) \in \mathbf{URep}(G))$$

Definition 2.2.0.5. define addition of endomorphisms of U pointwise

Exercise 2.2.0.6. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. Then $(\mathrm{Aut}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U), \mathcal{T}_{\mathcal{E}(U)})$ is a topological unital algebra.

Proof. □

Chapter 3

Fourier Analysis on $\mathcal{S}(\mathbb{R}^n)$

3.1 Schwartz Space

Definition 3.1.0.1. Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

1. $\langle x, y \rangle = \sum_j x_j y_j$
2. $|x| = \langle x, x \rangle^{1/2}$
3. $|\alpha| = \alpha_1 + \cdots + \alpha_n$
4. $\alpha! = \prod\limits_{j=1}^{n} \alpha_j!$
5. $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
6. $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$
7. $\Omega_\alpha = \{(\beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n : \beta + \gamma = \alpha\}$

Exercise 3.1.0.2. Let $\alpha \in \mathbb{N}_0^n$ and $j \in \{1, \ldots, n\}$. Suppose that $\alpha_j > 0$. Set $\eta = \alpha - e_j$. Then

1. $\Omega_\eta = \{(\beta - e_j, \gamma) : (\beta, \gamma) \in \Omega_\alpha \text{ and } \beta_j > 0\}$
2. $\Omega_\eta = \{(\beta, \gamma - e_j) : (\beta, \gamma) \in \Omega_\alpha \text{ and } \gamma_j > 0\}$

Proof.

1. Set $A = \{(\beta - e_j, \gamma) : (\beta, \gamma) \in \Omega_\alpha \text{ and } \beta_j > 0\}$. Let $(\mu, \nu) \in \Omega_\eta$. Set $\beta = \mu + e_j$ and $\gamma = \nu$. Then $\beta_j > 0$ and

$$\begin{aligned} \beta + \gamma &= \mu + e_j + \nu \\ &= \eta + e_j \\ &= \alpha \end{aligned}$$

So $(\beta, \gamma) \in \Omega_\alpha$. Hence

$$\begin{aligned} (\mu, \nu) &= (\beta - e_j, \gamma) \\ &\in A \end{aligned}$$

and $\Omega_\eta \subset A$.
Conversely, let $(\mu, \nu) \in A$. Then there exists $(\beta, \gamma) \in \Omega_\alpha$ such that $\beta_j > 0$ and $(\mu, \nu) = (\beta - e_j, \gamma)$. Then

$$\begin{aligned} \mu + \nu &= \beta - e_j + \gamma \\ &= \alpha - e_j \\ &= \eta \end{aligned}$$

So that $(\mu, \nu) \in \Omega_\eta$ and $A \subset \Omega_\eta$. Thus $\Omega_\eta = A$.

2. Similar to (1).

$\square$

Exercise 3.1.0.3. Let $f, g \in C^\infty(\mathbb{R}^n)$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha(fg) = \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$$

Proof. Let $\alpha \in \mathbb{N}_0^n$. The claim is true if $|\alpha| = 0$. Let $k > 0$. Suppose that $|\alpha| > 0$ and that the claim is true for $|\alpha| = k - 1$ so that for each $\eta \in \mathbb{N}_0^n$, $|\eta| = k - 1$ implies that

$$\partial^\eta(fg) = \sum_{(\beta,\gamma)\in\Omega_\eta} \frac{\eta!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$$

Since $|\alpha| > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Define $\eta = \alpha - e_j$. Then the previous exercise implies that

$$\begin{aligned}
\partial^\alpha(fg) &= \partial_j[\partial^\eta(fg)] \\
&= \partial_j\Bigg[\sum_{(\beta,\gamma)\in\Omega_\eta} \frac{\eta!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)\Bigg] \\
&= \sum_{(\beta,\gamma)\in\Omega_\eta} \frac{\eta!}{\beta!\gamma!}(\partial^{\beta+e_j} f)(\partial^\gamma g) + \sum_{(\beta,\gamma)\in\Omega_\eta} \frac{\eta!}{\beta!\gamma!}(\partial^\beta f)(\partial^{\gamma+e_j} g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0}} \frac{(\alpha-e_j)!}{(\beta-e_j)!\gamma!}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \gamma_j>0}} \frac{(\alpha-e_j)!}{\beta!(\gamma-e_j)!}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\gamma_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0,\gamma_j=0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) \\
&\qquad + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\gamma_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j=0,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0,\gamma_j=0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}\frac{\beta_j+\gamma_j}{\alpha_j}(\partial^\beta f)(\partial^\gamma g) \\
&\qquad + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j=0,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j>0,\gamma_j=0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) + \sum_{\substack{(\beta,\gamma)\in\Omega_\alpha \\ \beta_j=0,\gamma_j>0}} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)
\end{aligned}$$

So the claim is true for $|\alpha| = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$. $\square$

Exercise 3.1.0.4. Let $\xi \in \mathbb{R}^n$. Define $f \in \mathbb{C}^\infty(\mathbb{R}^n)$ by $f(x) = e^{-i\langle \xi, x\rangle}$. Then for each $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f = (-i\xi)^\alpha f$

Proof. Let $\alpha \in \mathbb{N}_0^n$. The claim is true for $|\alpha| = 0$. Let $k > 0$. Suppose that the claim is true for $|\alpha| \leq k-1$ so that for each $\beta \in \mathbb{N}_0$, $|\beta| \leq k-1$ implies that $\partial^\beta f = (-i\xi)^\beta f$. Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Then

$$\begin{aligned}
\partial^\alpha f &= \partial_j(\partial^{\alpha - e_j} f) \\
&= \partial_j((-i\xi)^{\alpha - e_j} f) \\
&= (-i\xi)^{\alpha - e_j} \partial_j f \\
&= (-i\xi)^{\alpha - e_j} i\xi_j \\
&= (-i\xi)^\alpha f
\end{aligned}$$

So the claim is true for $|\alpha| = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$.

□

Definition 3.1.0.5. Let $f \in C^\infty(\mathbb{R})$, $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define $\|\cdot\|_{\alpha,N} : C^\infty(\mathbb{R}^n, \mathbb{C}) \to [0, \infty]$ by

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha f(x)| \right]$$

We define **Schwartz space** on $\mathbb{R}^n$, denoted $\mathcal{S}(\mathbb{R}^n)$, by

$$\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n \text{ and } N \in \mathbb{N}_0, \|f\|_{\alpha,N} < \infty \}$$

Exercise 3.1.0.6. For each $p \in [1, \infty)$ and $x \in \mathbb{R}^n$,

$$(1 + |x|)^p \geq (1/2)(1 + |x|^p)$$

Proof. Let $p \in [1, \infty)$ and $x \in \mathbb{R}^n$. Suppose that $p \in \mathbb{Q}$. Then there exist $m, n \in \mathbb{N}$ such that $m \geq n$ and $p = m/n$. The binomial theorem implies that

$$\begin{aligned}
(1 + |x|)^m &= \sum_{j=0}^m \binom{m}{j} |x|^{m-j} \\
&\geq 1 + |x|^m
\end{aligned}$$

Jensen's inequality implies that

$$\begin{aligned}
(1 + |x|)^p &= [(1 + |x|)^m]^{1/n} \\
&\geq (1 + |x|^m)^{1/n} \\
&\geq (1/2)^{\frac{n-1}{n}} (1 + |x|^p) \\
&\geq (1/2)(1 + |x|^p)
\end{aligned}$$

Suppose that $p \notin \mathbb{Q}$. Choose a sequence $(p_j)_{j \in \mathbb{N}} \subset [1, \infty) \cap \mathbb{Q}$ such that $p_j \to p$. By continuity,

$$\begin{aligned}
(1 + |x|)^p &= \lim_{j \to \infty} (1 + |x|)^{p_j} \\
&\geq \lim_{j \to \infty} (1/2)(1 + |x|^{p_j}) \\
&= (1/2)(1 + |x|^p)
\end{aligned}$$

□

Exercise 3.1.0.7. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then f is Lipschitz.

Proof.

1. Set $M = \max\{\|f\|_{e_j,0} : j \in \{1, \dots, n\}\}$. By definition, for each $j \in \{1, \cdots, n\}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}|\partial_j f(x)| &\leq \|f\|_{e_j,0} \\ &\leq M\end{aligned}$$

Let $x, h \in \mathbb{R}^n$. Jensen's inequality implies that

$$\begin{aligned}|Df(x)(h)| &= \left|\sum_{j=1}^n \partial_j f(x) h_j\right| \\ &\leq \sum_{j=1}^n |\partial_j f(x)||h_j| \\ &\leq M \sum_{j=1}^n |h_j| \\ &\leq \sqrt{n} M |h|\end{aligned}$$

Since $h \in \mathbb{R}^n$ is arbitrary, $\|Df(x)\| \leq \sqrt{n}M$. Since $x \in \mathbb{R}^n$ is arbitrary, Df is bounded. Hence f is Lipschitz.

$\square$

Exercise 3.1.0.8. We have that $\mathcal{S}(\mathbb{R}^n)$ is a vector space and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, $\|\cdot\|_{\alpha,N} : \mathcal{S}(\mathbb{R}^n) \to [0, \infty)$ is a seminorm on $\mathcal{S}(\mathbb{R}^n)$.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$.

1.

$$\begin{aligned}\|\lambda f\|_{\alpha,N} &= \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha [\lambda f](x)|\right] \\ &= \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\lambda \partial^\alpha f(x)|\right] \\ &= \sup_{x \in \mathbb{R}} \left[|\lambda|(1 + |x|)^N |\partial^\alpha f(x)|\right] \\ &= |\lambda| \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha f(x)|\right] \\ &= |\lambda| \|f\|_{\alpha,N}\end{aligned}$$

Thus $\lambda f \in \mathcal{S}(\mathbb{R}^n)$ and $\|\lambda f\|_{\alpha,N} = |\lambda| \|f\|_{\alpha,N}$.

2.

$$\begin{aligned}\|f + g\|_{\alpha,N} &= \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha [f + g](x)|\right] \\ &= \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |[\partial^\alpha f + \partial^\alpha g](x)|\right] \\ &\leq \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha f(x)| + (1 + |x|)^N |\partial^\alpha g(x)|\right] \\ &\leq \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha f(x)|\right] + \sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha g(x)|\right] \\ &= \|f\|_{\alpha,N} + \|g\|_{\alpha,N}\end{aligned}$$

Hence $f + g \in \mathcal{S}(\mathbb{R}^n)$ and $\|f + g\|_{\alpha,N} \leq \|f\|_{\alpha,N} + \|g\|_{\alpha,N}$.

So $\mathcal{S}(\mathbb{R}^n)$ is a vector space and $\|\cdot\|_{\alpha,N}$ is a seminorm on $\mathcal{S}(\mathbb{R}^n)$. $\square$

Exercise 3.1.0.9. We have that $\mathcal{S}(\mathbb{R}^n)$ is a algebra under pointwise multiplication and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|fg\|_{\alpha,N} \leq \sum_{\beta=0}^{\alpha} \|f\|_{\beta,N}\|g\|_{\alpha-\beta,0}$$

Hint: $\partial^\alpha(fg) = \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|fg\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\bigg[(1+|x|)^N|\partial^\alpha(fg)(x)|\bigg] \\
&= \sup_{x\in\mathbb{R}}\bigg[(1+|x|)^N\bigg|\sum_{(\beta,\gamma)\in\Omega_\alpha}\frac{\alpha!}{\beta!\gamma!}\partial^\beta f(x)\partial^\gamma g(x)\bigg|\bigg] \\
&\leq \sup_{x\in\mathbb{R}}\bigg[(1+|x|)^N\bigg(\sum_{(\beta,\gamma)\in\Omega_\alpha}\frac{\alpha!}{\beta!\gamma!}|\partial^\beta f(x)||\partial^\gamma g(x)|\bigg)\bigg] \\
&= \sup_{x\in\mathbb{R}}\bigg[\sum_{(\beta,\gamma)\in\Omega_\alpha}\frac{\alpha!}{\beta!\gamma!}(1+|x|)^N|\partial^\beta f(x)||\partial^\gamma g(x)|\bigg] \\
&\leq \sum_{(\beta,\gamma)\in\Omega_\alpha}\frac{\alpha!}{\beta!\gamma!}\sup_{x\in\mathbb{R}}\bigg[(1+|x|)^N|\partial^\beta f(x)||\partial^\gamma g(x)|\bigg] \\
&\leq \sum_{(\beta,\gamma)\in\Omega_\alpha}\frac{\alpha!}{\beta!\gamma!}\sup_{x\in\mathbb{R}}\bigg[(1+|x|)^N|\partial^\beta f(x)|\bigg]\sup_{x\in\mathbb{R}}\bigg[|\partial^\gamma g(x)|\bigg] \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha}\frac{\alpha!}{\beta!\gamma!}\|f\|_{\beta,N}\|g\|_{\gamma,0} \\
&< \infty
\end{aligned}$$

So $fg \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Definition 3.1.0.10. Set $\mathcal{P} = \{\|\cdot\|_{\alpha,N} : \alpha \in \mathbb{N}_0^n, N \in \mathbb{N}_0\}$. Then $\mathcal{P}$ is a countable family of seminorms on $\mathcal{S}(\mathbb{R}^n)$. We equip $\mathcal{S}(\mathbb{R}^n)$ with the topology $\mathcal{T}$ induced by the family of projections

$$\pi_{\|\cdot\|_{\alpha,N}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)/\ker\|\cdot\|_{\alpha,N}$$

i.e. $\mathcal{T} = \tau_{\mathcal{S}(\mathbb{R}^n)}((\pi_p)_{p\in\mathcal{P}})$.
Explicitly, for a net $(f_\gamma)_{\gamma\in\Gamma} \subset \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, $f_\gamma \to f$ iff for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, $\|f_\gamma - f\|_{\alpha,N} \to 0$.
Hence $(\mathcal{S}(\mathbb{R}^n), \mathcal{T})$ is a locally convex space. Since $\mathcal{P}$ is countable, we may write $\mathcal{P} = (p_j)_{j\in\mathbb{N}}$ and thus $(\mathcal{S}(\mathbb{R}^n), \mathcal{T})$ is metrizable with metric

$$d_{\mathcal{S}(\mathbb{R}^n)}(f,g) = \sum_{j\in\mathbb{N}} 2^{-j}\frac{p_j(f-g)}{1+p_j(f-g)}$$

Exercise 3.1.0.11. Let $f \in \mathcal{S}(\mathbb{R}^n)$. For each $p \in [1,\infty]$, $f \in L^p(\mathbb{R}^n)$

Proof. Let $p \in [1,\infty]$. Suppose that $p < \infty$. The previous exercise implies that for each $x \in \mathbb{R}$,

$$(1+|x|)^{2p} \geq (1/2)(1+|x|^{2p})$$

By definition, there exists $C \geq 0$ such that for each $x \in \mathbb{R}$,

$$|f(x)| \leq C(1+|x|)^{-2}$$

Then for each $x \in \mathbb{R}$,

$$\begin{aligned}|f(x)|^p &\leq C^p(1+|x|)^{-2p} \\ &\leq 2C^p(1+|x|^{2p})^{-1}\end{aligned}$$

Define $g : \mathbb{R}^n \to [0,\infty)$ defined by $g(x) = 2C^p(1+|x|^{2p})^{-1}$. Since $g \in L^1(m)$ and $|f|^p \leq g$, we have that $f \in L^p(\mathbb{R}^n)$. If $p = \infty$, then by definition,

$$\begin{aligned}\|f\|_\infty &= \|f\|_{0,0} \\ &< \infty\end{aligned}$$

So $f \in L^p(\mathbb{R}^n)$. □

Exercise 3.1.0.12. For each $p \in [1,\infty)$, the inclusion $\iota : \mathcal{S}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_j)_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_j \to f$. Then for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, $\|f_j - f\|_{\alpha,N} \to 0$. By definition, for each $x \in \mathbb{R}$,

$$|f_j(x) - f(x)| \leq \|f_j - f\|_{0,2}(1+|x|)^{-2}$$

Therefore, for each $x \in \mathbb{R}$,

$$\begin{aligned}\|f_j - f\|_p^p &= \int_{\mathbb{R}^n} |f_j - f|^p \, dm \\ &\leq \int_{\mathbb{R}^n} \|f_j - f\|_{0,2}^p (1+|x|)^{-2p} \, dm(x) \\ &\leq \|f_j - f\|_{0,2}^p \int_{\mathbb{R}^n} 2(1+|x|^{2p})^{-1} \, dm(x) \\ &= \|f_j - f\|_{0,2}^p \int_{\mathbb{R}^n} 2(1+|x|^{-2p})^{-1} \, dm(x) \\ &\to 0\end{aligned}$$

Hence $f_j \xrightarrow{L^p} f$ and $\iota : \mathcal{S}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is continuous. □

Exercise 3.1.0.13. For each $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $\alpha \in \mathbb{N}_0^n$. The claim is true for $|\alpha| = 0$ and $|\alpha| = 1$. Let $k > 1$. Suppose that the claim is true for $|\alpha| = k-1$ so that for each $\beta \in \mathbb{N}_0^n$, $|\beta| = k-1$ implies that $\partial^\beta : \mathcal{S}(\mathbb{R}^n) \to C^\infty$ is linear. Suppose that $|\alpha| = k$. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Since $k > 0$, there exists $j \in \{1,\dots,n\}$ such that $\alpha_j > 0$. Then

$$\begin{aligned}\partial^\alpha(f+\lambda g) &= \partial_j(\partial^{\alpha-e_j}[f+\lambda g]) \\ &= \partial_j(\partial^{\alpha-e_j} f + \lambda \partial^{\alpha-e_j} g) \\ &= \partial_j(\partial^{\alpha-e_j} f) + \lambda \partial_j(\partial^{\alpha-e_j} g) \\ &= \partial^\alpha f + \lambda \partial^\alpha g\end{aligned}$$

Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$ are arbitrary, we have that ∂^α is linear. So the claim is true for $|\alpha| = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$. □

Exercise 3.1.0.14. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Then $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\partial^\alpha f\|_{\beta,N} \leq \|f\|_{\alpha+\beta,N}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. By definition,

$$\begin{aligned}\|\partial^\alpha f\|_{\beta,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\beta(\partial^\alpha f)(x)|\right]\\ &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^{\alpha+\beta} f(x)|\right]\\ &= \|f\|_{\alpha+\beta,N}\\ &< \infty\end{aligned}$$

So $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 3.1.0.15. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|f\|_{\alpha,N} = \|\partial^\alpha f\|_{0,N}$$

Proof. Clear by preceding exercise. □

Exercise 3.1.0.16. Let $\alpha \in \mathbb{N}_0^n$. Then $\partial^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathbb{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. Let $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}$. Then

$$\begin{aligned}\|\partial^\alpha f_k\|_{\beta,N} &\leq \|f_k\|_{\alpha+\beta,N}\\ &\to 0\end{aligned}$$

Since $\beta \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\partial^\alpha f_k \to 0$. Thus ∂^α is continuous at 0. Since ∂^α is linear, ∂^α is continuous. □

3.2 Position and Momentum Operators

Definition 3.2.0.1. Let $j \in \{1, \dots, n\}$. We define the **j-th position operator**, denoted $X_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$X_j f(x) = x_j f(x)$$

Exercise 3.2.0.2. Let $j \in \{1, \dots, n\}$. Then $X_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Then for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned} X_j(f + \lambda g)(x) &= x_j(f(x) + \lambda g(x)) \\ &= x_j f(x) + \lambda x_j g(x) \\ &= (X_j f + \lambda X_j g)(x) \end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that $X_j(f + \lambda g) = X_j f + \lambda X_j g$. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$ are arbitrary, we have that X_j is linear. $\square$

Exercise 3.2.0.3. For each $j \in \{1, \dots, n\}$ and $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha X_j = \begin{cases} X_j \partial^\alpha & \alpha_j = 0 \\ X_j \partial^\alpha + \alpha_j \partial^{\alpha - e_j} & \alpha_j > 0 \end{cases}$$

Proof. Let $j \in \{1, \dots, n\}$, $\alpha \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. The claim is true if $\alpha_j = 0$ or $\alpha_j = 1$. Let $k > 1$. Suppose that the claim is true for $\alpha_j = k - 1$ so that $\partial_j^{k-1}(X_j f) = X_j(\partial_j^{k-1} f) + (k-1)\partial_j^{k-2} f$. Suppose that $\alpha_j = k$. Then

$$\begin{aligned} (\partial_j^k X_j) f &= \partial_j^k (X_j f) \\ &= \partial_j(\partial_j^{k-1}[X_j f]) \\ &= \partial_j(X_j[\partial_j^{k-1} f] + (k-1)\partial_j^{k-2}) \\ &= \partial_j(X_j[\partial_j^{k-1} f]) + (k-1)\partial_j(\partial_j^{k-2} f) \\ &= (X_j[\partial_j^k f] + \partial_j^{k-1} f) + (k-1)\partial_j^{k-1} f \\ &= X_j(\partial_j^k f) + k\partial_j^{k-1} f \\ &= (X_j \partial_j^k + k\partial_j^{k-1}) f \end{aligned}$$

which implies that

$$\begin{aligned} (\partial^\alpha X_j) f &= \partial^\alpha (X_j f) \\ &= \partial^{\alpha - k e_j}(\partial_j^k [X_j f]) \\ &= \partial^{\alpha - k e_j}(X_j[\partial_j^k f] + k\partial_j^{k-1} f) \\ &= X_j(\partial^{\alpha - k e_j}[\partial_j^k f]) + k\partial^{\alpha - k e_j}(\partial_j^{k-1} f) \\ &= X_j(\partial^\alpha f) + \alpha_j \partial^{\alpha - e_j} f \\ &= (X_j \partial^\alpha + \alpha_j \partial^{\alpha - e_j}) f \end{aligned}$$

So the claim is true for $\alpha_j = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$. $\square$

Exercise 3.2.0.4. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \{1, \dots, n\}$. Then $X_j f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|X_j f\|_{\alpha, N} \leq \begin{cases} \|f\|_{\alpha, N+1} & \alpha_j = 0 \\ \|f\|_{\alpha, N+1} + \alpha_j \|f\|_{\alpha - e_j, N} & \alpha_j > 0 \end{cases}$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. If $\alpha_j = 0$, then the previous exercise implies that

$$\begin{aligned}\|X_j f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha(X_j f)(x)|\right]\\ &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|x_j\partial^\alpha f(x)|\right]\\ &\leq \sup_{x\in\mathbb{R}}\left[(1+|x|)^{N+1}|\partial^\alpha f(x)|\right]\\ &= \|f\|_{\alpha,N+1}\\ &< \infty\end{aligned}$$

If $\alpha_j > 0$, then the previous exercise implies that

$$\begin{aligned}\|X_j f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha(X_j f)(x)|\right]\\ &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|x_j\partial^\alpha f(x) + \alpha_j\partial^{\alpha-e_j}f(x)|\right]\\ &\leq \sup_{x\in\mathbb{R}}\left[(1+|x|)^{N+1}|\partial^\alpha f(x)|\right] + \alpha_j\sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^{\alpha-e_j}f(x)|\right]\\ &= \|f\|_{\alpha,N+1} + \alpha_j\|f\|_{\alpha-e_j,N}\\ &< \infty\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $X_j f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 3.2.0.5. Let $j \in \{1,\dots,n\}$. Then $X_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathbb{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}$. If $\alpha_j = 0$, then

$$\begin{aligned}\|X_j f_k\|_{\alpha,N} &\leq \|f_k\|_{\alpha,N+1}\\ &\to 0\end{aligned}$$

If $\alpha_j > 0$, then

$$\begin{aligned}\|X_j f_k\|_{\alpha,N} &\leq \|f_k\|_{\alpha,N+1} + \alpha_j\|f_k\|_{\alpha-e_j,N}\\ &\to 0\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $X_j f_k \to 0$. Thus X_j is continuous at 0. Since X_j is linear, X_j is continuous. □

Exercise 3.2.0.6. Let $j, k \in \{1,\dots,n\}$. Then $X_j X_k = X_k X_j$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}([X_j X_k]f)(x) &= (X_j[X_k f])(x)\\ &= x_j(X_k f)(x)\\ &= x_j x_k f(x)\\ &= x_k x_j f(x)\\ &= x_k(X_j f)(x)\\ &= (X_k[X_j f])(x)\\ &= ([X_k X_j]f)(x)\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ are arbitrary, $X_j X_k = X_k X_j$. □

Definition 3.2.0.7. Let $\alpha \in \mathbb{N}_0^n$. We define $X^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$

Definition 3.2.0.8. Let $j \in \{1, \dots, n\}$. We define the **j-th momentum operator**, denoted $P_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$P_j = -i\partial_j$$

Exercise 3.2.0.9. Let $j \in \{1, \dots, n\}$. Then $P_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Clear since $\partial_j : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear. □

Exercise 3.2.0.10. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \{1, \dots, n\}$. Then $P_j f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|P_j f\|_{\alpha,N} \leq \|f\|_{\alpha+e_j,N}$$

Proof. Clear since $\partial_j f \in \mathcal{S}(\mathbb{R}^n)$ and $\|\partial_j f\|_{\alpha,N} \leq \|f\|_{\alpha+e_j,N}$. □

Exercise 3.2.0.11. Let $j \in \{1, \dots, n\}$. Then $P_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Clear cince $\partial_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous. □

Exercise 3.2.0.12. Let $j, k \in \{1, \dots, n\}$. Then $P_j P_k = P_k P_j$.

Proof. Clear since $\partial_j \partial_k = \partial_k \partial_j$. □

Definition 3.2.0.13. Let $\alpha \in \mathbb{N}_0^n$. We define $P^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ by $P^\alpha = P_1^{\alpha_1} \cdots P_n^{\alpha_n}$

Exercise 3.2.0.14. Let $j, k \in \{1, \dots, n\}$. Then $[X_j, P_k] = i\delta_{j,k}$.

Proof. A previous exercise implies that $\partial_k X_j = X_j \partial_k + \delta_{j,k} I$. Therefore

$$\begin{aligned}
[X_j, P_k] &= X_j P_k - P_k X_j \\
&= -i(X_j \partial_k - \partial_k X_j) \\
&= -i(X_j \partial_k - [X_j \partial_k + \delta_{j,k} I]) \\
&= -i\delta_{j,k} I
\end{aligned}$$

□

3.3 Translation and Rotation Operators

Definition 3.3.0.1. Let $y \in \mathbb{R}^n$. We define the **translation by y operator**, denoted $\tau_y : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}^\infty(\mathbb{R}^n)$, by $\tau_y f(x) = f(x-y)$.

Exercise 3.3.0.2. Let $y \in \mathbb{R}^n$. Then $\tau_y : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Then for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned}\tau_y(f+\lambda g)(x) &= (f+\lambda g)(x-y)\\ &= f(x-y)+\lambda g(x-y)\\ &= \tau_y f(x) + \lambda \tau_y g(x)\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that $\tau_y(f+\lambda g) = \tau_y f + \lambda \tau_y g$. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ are arbitary, τ_y is linear. □

Exercise 3.3.0.3. Let $\alpha \in \mathbb{N}_0$. Then for each $y \in \mathbb{R}^n$,

$$\partial^\alpha \tau_y = \tau_y \partial^\alpha$$

Proof. Let $y \in \mathbb{R}^n$. The claim is true if $|\alpha| = 0$. Let $k \geq 1$. Suppose that the claim is true for $|\alpha| \leq k-1$ so that for each $\beta \in \mathbb{N}_0^n$, $|\beta| \leq k-1$ implies that

$$\partial^\beta \tau_y = \tau_y \partial^\beta$$

Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define $g : \mathbb{R}^n \to \mathbb{R}^n$ and $g_k : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = x - y$ and $g_k = \pi_k \circ g$. Then the chain rule implies that

$$\begin{aligned}(\partial^\alpha \tau_y) f &= \partial^\alpha(\tau_y f)\\ &= \partial_j(\partial^{\alpha-e_j}[\tau_y f])\\ &= \partial_j(\tau_y[\partial^{\alpha-e_j} f])\\ &= \partial_j([\partial^{\alpha-e_j} f]\circ g)\\ &= \sum_{k=1}^n [\partial_k(\partial^{\alpha-e_j} f)\circ g]\partial_j g_k\\ &= \partial_j(\partial^{\alpha-e_j} f)\circ g\\ &= (\partial^\alpha f)\circ g\\ &= \tau_y(\partial^\alpha f)\\ &= (\tau_y \partial^\alpha) f\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ is arbitrary, $\partial^\alpha \tau_y = \tau_y \partial^\alpha$. Hence the claim is true for $|\alpha| = k$. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$. □

Exercise 3.3.0.4. Let $y \in \mathbb{R}$. Then for each $x \in \mathbb{R}^n$, $(1+|x|) \leq (1+|y|)(1+|x-y|)$.

Proof. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}(1+|y|)(1+|x-y|) &= 1 + (|x-y|+|y|) + |y||x-y|\\ &\geq 1+|x|+|y||x-y|\\ &\geq 1+|x|\end{aligned}$$

□

Exercise 3.3.0.5. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$. Then $\tau_y f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\tau_y f\|_{\alpha,N} \leq (1+|y|)^N \|f\|_{\alpha,N}$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|\tau_y f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha \tau_y f(x)|\right] \\
&= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\tau_y \partial^\alpha f(x)|\right] \\
&= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha f(x-y)|\right] \\
&\leq \sup_{x\in\mathbb{R}} \left[(1+|y|)^N (1+|x-y|)^N |\partial^\alpha f(x-y)|\right] \\
&= (1+|y|)^N \sup_{x\in\mathbb{R}} \left[(1+|x-y|)^N |\partial^\alpha f(x-y)|\right] \\
&= (1+|y|)^N \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha f(x)|\right] \\
&= (1+|y|)^N \|f\|_{\alpha,N} \\
&< \infty
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\tau_y f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 3.3.0.6. Let $y \in \mathbb{R}^n$. Then $\tau_y : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathcal{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. Let $\alpha, N \in \mathcal{N}_0$. Then

$$\begin{aligned}
\|\tau_y f_k\|_{\alpha,N} &\leq (1+|y|)^N \|f_k\|_{\alpha,N} \\
&\to 0
\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\tau_y f_k \to 0$. So τ_y is continuous at 0. Since τ_y is linear, τ_y is continuous. □

Definition 3.3.0.7. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define $\tau f : \mathbb{R}^n \to \mathcal{S}(\mathbb{R}^n)$ by $\tau f(y) = \tau_y f$.

Exercise 3.3.0.8. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\tau f : \mathbb{R}^n \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. content... □

Definition 3.3.0.9. Let $\xi \in \mathbb{R}^n$. We define the **rotation by ξ operator**, denoted $\rho_\xi : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, by $\rho_\xi f(x) = e^{-i\langle \xi, x\rangle} f(x)$.

Exercise 3.3.0.10. Let $\xi \in \mathbb{R}^n$. Then $\rho_\xi : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Then for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned}
\rho_\xi(f+\lambda g)(x) &= e^{-i\langle \xi, x\rangle}(f+\lambda g)(x) \\
&= e^{-i\langle \xi, x\rangle} f(x) + \lambda e^{-i\langle \xi, x\rangle} g(x) \\
&= \rho_\xi f(x) + \lambda \rho_\xi g(x)
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that $\rho_\xi(f+\lambda g) = \rho_\xi f + \lambda \rho_\xi g$. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ are arbitary, ρ_ξ is linear. □

Exercise 3.3.0.11. Let $\xi \in \mathbb{R}^n$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha \rho_\xi = \rho_\xi \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Define $g \in C^\infty(\mathbb{R}^n)$ by $g(x) = e^{-i\langle \xi, x\rangle}$. A previous exercise implies that

$$\begin{aligned}
(\partial^\alpha \rho_\xi) f &= \partial^\alpha(\rho_\xi f) \\
&= \partial^\alpha(gf) \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta g)(\partial^\gamma f) \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}((-i\xi)^\beta g)(\partial^\gamma f) \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \rho_\xi(\partial^\gamma f) \\
&= \rho_\xi\bigg(\sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma f\bigg) \\
&= \bigg(\rho_\xi \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma\bigg) f
\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ is arbitrary,

$$\partial^\alpha \rho_\xi = \rho_\xi \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma$$

□

Exercise 3.3.0.12. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Then $\rho_\xi f \in \mathcal{S}(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\rho_\xi f\|_{\alpha,N} \leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta| \|f\|_{\gamma,N}$$

Proof. Let $\alpha \in \mathbb{N}_0^n$, $N \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
(1+|x|)^N |\partial^\alpha(\rho_\xi f)(x)| &= (1+|x|)^N \bigg|\rho_\xi\bigg(\sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma f\bigg)(x)\bigg| \\
&= (1+|x|)^N \bigg| e^{-i\langle\xi,x\rangle} \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}(-i\xi)^\beta \partial^\gamma f(x)\bigg| \\
&\leq (1+|x|)^N \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta||\partial^\gamma f(x)| \\
&= \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta||(1+|x|)^N \partial^\gamma f(x)| \\
&\leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta| \|f\|_{\gamma,N}
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that

$$\begin{aligned}
\|\rho_\xi f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N |\partial^\alpha(\rho_\xi f)(x)|\bigg] \\
&\leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta| \|f\|_{\gamma,N} \\
&< \infty
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\rho_\xi f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 3.3.0.13. Let $\xi \in \mathbb{R}^n$. Then $\rho_\xi : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathcal{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. Let $\alpha, N \in \mathcal{N}_0$. Then

$$\begin{aligned}\|\rho_\xi f_k\|_{\alpha,N} &\leq \sum_{(\beta,\gamma)\in\Omega_\alpha} \frac{\alpha!}{\beta!\gamma!}|\xi^\beta|\|f_k\|_{\gamma,N} \\ &\to 0\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\rho_\xi f_k \to 0$. So ρ_ξ is continuous at 0. Since ρ_ξ is linear, ρ_ξ is continuous. $\square$

3.4 Dilation and Concentration Operators

Definition 3.4.0.1. Let $\xi \in \mathbb{R}^n$. We define the **dilation by t operator**, denoted $\gamma_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, by $\gamma_t f(x) = f(tx)$.

Exercise 3.4.0.2. Let $t \neq 0$. Then $\gamma_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. Then for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned}\gamma_t(f+\lambda g)(x) &= (f+\lambda g)(tx)\\ &= f(tx) + \lambda g(tx)\\ &= \gamma_t f(x) + \lambda \gamma_t g(x)\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that $\gamma_t(f+\lambda g) = \gamma_t f + \lambda \gamma_t g$. Since $f, g \in \mathcal{S}(\mathbb{R}^n)$ are arbitary, γ_t is linear. □

Exercise 3.4.0.3. Let $t \neq 0$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha \gamma_t = t^{|\alpha|}\gamma_t \partial^\alpha$$

Proof. Let $\alpha \in \mathbb{N}_0^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. The chain rule implies that the claim is true if $|\alpha| = 0$ or $|\alpha| = 1$. Let $k > 1$. Suppose the claim is true for $|\alpha| = k-1$ so that for each $\beta \in \mathbb{N}_0$, $|\beta| = k-1$ implies that $\partial^\beta(\gamma_t f) = t^{|\beta|}\gamma_t(\partial^\beta f)$. Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. The chain rule implies that

$$\begin{aligned}(\partial^\alpha \gamma_t) f &= \partial^\alpha(\gamma_t f)\\ &= \partial_j(\partial^{\alpha - e_j}[\gamma_t f])\\ &= \partial_j(t^{|\alpha-e_j|}\gamma_t[\partial^{\alpha-e_j} f])\\ &= t^{|\alpha-e_j|}\partial_j(\gamma_t[\partial^{\alpha-e_j} f])\\ &= t^{|\alpha-e_j|}t\gamma_t(\partial_j[\partial^{\alpha-e_j} f])\\ &= t^{|\alpha|}\gamma_t(\partial^\alpha f)\\ &= (t^{|\alpha|}\gamma_t\partial^\alpha) f\end{aligned}$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ is arbitrary, $\partial^\alpha\gamma_t = t^{|\alpha|}\gamma_t\partial^\alpha$. So the claim is true for $|\alpha| = k$. By induction the claim is true for each $\alpha \in \mathbb{N}_0^n$. □

Exercise 3.4.0.4. Let $y \in \mathbb{R}$ and $t \neq 0$. Then there exists $C > 0$ such that for each $x \in \mathbb{R}^n$, $1 + |x| \leq C(1+|tx|)^2$.

Proof. Choose $C = \max(1/(2|t|), 1)$. Let $x \in \mathbb{R}^n$. Then

$$\begin{aligned}C(1+|tx|)^2 - (1+|x|) &= C + 2C|tx| + C|tx|^2 - 1 - |x|\\ &= C + (2C|t| - 1)|x| + C|tx|^2 - 1\\ &= (C-1) + (2C|t|-1)|x| + C|tx|^2\\ &\geq 0\end{aligned}$$

So $1 + |x| \leq C(1+|tx|)^2$. □

Exercise 3.4.0.5. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$. Then $\gamma_t f \in \mathcal{S}(\mathbb{R}^n)$ and there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\gamma_t f\|_{\alpha,N} \leq |t|^{|\alpha|}C^N\|f\|_{\alpha,2N}$$

Proof. The previous exercise implies that there exists $C > 0$ such that for each $x \in \mathbb{R}^n$, $1+|x| \leq C(1+|tx|)^2$. Let $\alpha \in \mathbb{N}_0^n$, $N \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$. Then

$$\begin{aligned}(1+|x|)^N|\partial^\alpha(\gamma_t f)(x)| &= (1+|x|)^N|t^{|\alpha|}(\gamma_t \partial^\alpha f)(x)| \\ &\leq C(1+|tx|)^{2N}|t|^{|\alpha|}|(\gamma_t \partial^\alpha f)(x)| \\ &= C(1+|tx|)^{2N}|t|^{|\alpha|}|\partial^\alpha f(tx)| \\ &\leq C|t|^{|\alpha|}\|f\|_{\alpha,2N}\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we have that

$$\begin{aligned}\|\gamma_t f\|_{\alpha,N} &= \sup_{x \in \mathbb{R}}\bigg[(1+|x|)^N|\partial^\alpha(\gamma_t f)(x)|\bigg] \\ &\leq Ct^{|\alpha|}\|f\|_{\alpha,2N} \\ &< \infty\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\gamma_t f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 3.4.0.6. Let $t \neq 0$. Then $\gamma_t : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_k \to 0$. Then for each $\alpha, N \in \mathcal{N}_0$, $\|f_k\|_{\alpha,N} \to 0$. The previous exercise implies that there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\gamma_t f\|_{\alpha,N} \leq |t|^{|\alpha|}C^N\|f\|_{\alpha,2N}$$

Let $\alpha, N \in \mathcal{N}_0$. Then

$$\begin{aligned}\|\gamma_t f_k\|_{\alpha,N} &\leq C|t|^{|\alpha|}\|f\|_{\alpha,2N} \\ &\to 0\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\gamma_t f_k \to 0$. So γ_t is continuous at 0. Since γ_t is linear, ρ_ξ is continuous. □

Definition 3.4.0.7. Let $\xi \in \mathbb{R}^n$. We define the **concentration by** t **operator**, denoted $\kappa_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, by $\kappa_t f(x) = t^{-1}\gamma_{t^{-1}}f$.

Exercise 3.4.0.8. Let $t \neq 0$. Then $\kappa_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear.

Proof. Clear since $\gamma_t : \mathcal{S}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear. □

Exercise 3.4.0.9. Let $t \neq 0$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha \kappa_t = t^{-|\alpha|}\kappa_t\partial^\alpha$$

Proof. Let $\alpha \in \mathbb{N}_0^n$. Then

$$\begin{aligned}\partial^\alpha \kappa_t &= \partial^\alpha t^{-1}\gamma_{t^{-1}} \\ &= t^{-1}\partial^\alpha\gamma_{t^{-1}} \\ &= t^{-1}(t^{-1})^{|\alpha|}\gamma_{t^{-1}}\partial^\alpha \\ &= t^{-|\alpha|}\kappa_t\partial^\alpha\end{aligned}$$

□

Exercise 3.4.0.10. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$. Then $\kappa_t f \in \mathcal{S}(\mathbb{R}^n)$ and there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\kappa_t f\|_{\alpha,N} \leq |t|^{-(|\alpha|+1)}C^N\|f\|_{\alpha,2N}$$

Proof. A previous exercise implies that there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\gamma_t f\|_{\alpha,N} \leq |t|^{|\alpha|} C^N \|f\|_{\alpha,2N}$$

Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|\kappa_t f\|_{\alpha,N} &= \|t^{-1}\gamma_{t^{-1}} f\|_{\alpha,N} \\
&= |t^{-1}| \|\gamma_{t^{-1}} f\|_{\alpha,N} \\
&\leq |t^{-1}| |t^{-1}|^{|\alpha|} C^N \|f\|_{\alpha,2N} \\
&= |t|^{-(|\alpha|+1)} C^N \|f\|_{\alpha,2N} \\
&< \infty
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $\kappa_t f \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 3.4.0.11. Let $t \neq 0$. Then $\kappa_t : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Since $\gamma_{t^{-1}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, $\kappa_t = t^{-1}\gamma_{t^{-1}}$ is continuous. □

Exercise 3.4.0.12. Let $t \neq 0$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}} \kappa_t f \, dm = \int_{\mathbb{R}} f \, dm$$

Proof. We have that

$$\begin{aligned}
\int_{\mathbb{R}} \kappa_t f \, dm &= \int_{\mathbb{R}} t^{-1}\gamma_{t^{-1}} f \, dm \\
&= \int_{\mathbb{R}} t^{-1} f(t^{-1}y) \, dm(y) \\
&= \int_{\mathbb{R}} f(z) \, dm(z)
\end{aligned}$$

□

3.5 The Convolution on $\mathcal{S}(\mathbb{R}^n)$

Definition 3.5.0.1. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. We define the **convolution of f and g**, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$ by

$$f * g(x) = \int_{\mathbb{R}^n} \tau_y f(x) g(y)\, dm(y)$$

Exercise 3.5.0.2. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $f * g \in C^\infty(\mathbb{R}^n)$ and for each $\alpha \in \mathbb{N}_0^n$

$$\partial^\alpha (f * g) = (\partial^\alpha f) * g$$

Hint: exchange integration and differentiation

Proof. Let $\alpha \in \mathbb{N}_0^n$. We proceed by induction on $|\alpha|$.

- Suppose that $|\alpha| = 0$. Then $\alpha = 0$. Define $h_0 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = \tau_y f(x) g(y)$. We observe that for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned} |h(x,y)| &= |\tau_y f(x)||g(y)| \\ &\leq \|\tau_y f\|_{0,0} |g(y)| \\ &\leq \|f\|_{0,0} |g(y)| \end{aligned}$$

  Since $\|f\|_{0,0}|g| \in \mathrm{L}^1(\mathbb{R}^n)$ and for each $y \in \mathbb{R}^n$, $h(x, y) \to h(x_0, y)$ as $x \to x_0$, we have that

$$\begin{aligned} f * g &= \int_{\mathbb{R}^n} \tau_y f(\cdot) g(y)\, dm(y) \\ &= \int_{\mathbb{R}^n} h(\cdot, y)\, dm(y) \end{aligned}$$

  is continuous. Therefore, $f * g \in C(\mathbb{R}^n)$ and $\partial^\alpha (f * g) = (\partial^\alpha f) * g$.

- Let $k > 0$. Suppose that for each $\beta \in \mathbb{N}_0^n$, $|\beta| = k - 1$ implies that $f * g \in C^{|\beta|}(\mathbb{R}^n)$ and

$$\partial^\beta (f * g) = (\partial^\beta f) * g$$

  Suppose that $|\alpha| = k$. Then there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Define $h \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = \tau_y [\partial_x^{\alpha - e_j} f](x) g(y)$. By hypothesis,

$$\begin{aligned} [\partial^{\alpha - e_j}(f * g)](x) &= [(\partial^{\alpha - e_j} f) * g](x) \\ &= \int_{\mathbb{R}^n} \tau_y [\partial_x^{\alpha - e_j} f](x) g(y)\, dm(y) \\ &= \int_{\mathbb{R}^n} h(x, y)\, dm(y) \end{aligned}$$

  We observe that for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \partial_x^{e_j} h(x, y) &= \partial_x^{e_j} [\tau_y (\partial_x^{\alpha - e_j} f)](x) g(y) \\ &= \partial_x^\alpha [\tau_y f](x) g(y) \end{aligned}$$

  which implies that

$$\begin{aligned} |\partial_x^{e_j} h(x, y)| &= |\partial_x^\alpha [\tau_y f](x) g(y)| \\ &\leq \|\tau_y f\|_{\alpha, 0} |g(y)| \\ &\leq \|f\|_{\alpha, 0} |g(y)| \end{aligned}$$

Since $g \in L^1(\mathbb{R}^n)$, $\partial^{e_j}[\partial^{\alpha-e_j}(f*g)]$ exists and we may exchange the order of integration and differentiation to obtain that

$$\begin{aligned}
[\partial_x^\alpha(f*g)](x) &= \partial_x^{e_j}[\partial_x^{\alpha-e_j}(f*g)](x) \\
&= \partial_x^{e_j}\int_{\mathbb{R}^n} h(x,y)\,dm(y) \\
&= \int_{\mathbb{R}^n} \partial_x^{e_j} h(x,y)\,dm(y) \\
&= \int_{\mathbb{R}^n} \partial_x^{e_j}[\tau_y(\partial_x^{\alpha-e_j} f)](x)g(y)\,dm(y) \\
&= \int_{\mathbb{R}^n} \tau_y[\partial_x^\alpha f](x)g(y)\,dm(y) \\
&= [(\partial_x^\alpha f)*g](x)
\end{aligned}$$

So $f*g \in C^{|\alpha|}(\mathbb{R}^n)$ and $\partial^\alpha(f*g) = (\partial^\alpha f)*g$.

- By induction, for each $\alpha \in \mathbb{N}_0$, $f*g \in C^{|\alpha|}(\mathbb{R}^n)$ and $\partial^\alpha(f*g) = (\partial^\alpha f)*g$.

Since for each $\alpha \in \mathbb{N}_0^n$, $f*g \in C^{|\alpha|}(\mathbb{R}^n)$, we have that $f*g \in C^\infty(\mathbb{R}^n)$. □

Exercise 3.5.0.3. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $f*g \in \mathcal{S}(\mathbb{R}^n)$ and there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|f*g\|_{\alpha,N} \leq C\|f\|_{\alpha,N}\|g\|_{0,N+2}$$

Proof. Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|y|)^2}\,dm(y)$$

Let $\alpha \in \mathbb{N}_0^n$, $N \in \mathbb{N}_0$ and $x \in \mathbb{R}$. Then

$$\begin{aligned}
(1+|x|)^N|\partial^\alpha(f*g)(x)| &= (1+|x|)^N|(\partial^\alpha f)*g(x)| \\
&= (1+|x|)^N\left|\int_{\mathbb{R}} \tau_y[\partial_x^\alpha f](x)g(y)\,dm(y)\right| \\
&= \left|\int_{\mathbb{R}} (1+|x|)^N\partial_x^\alpha[\tau_y f](x)g(y)\,dm(y)\right| \\
&\leq \int_{\mathbb{R}} (1+|x|)^N|\partial_x^\alpha[\tau_y f](x)||g(y)|\,dm(y) \\
&\leq \int_{\mathbb{R}} \|\tau_y f\|_{\alpha,N}|g(y)|\,dm(y) \\
&\leq \int_{\mathbb{R}} (1+|y|)^N\|f\|_{\alpha,N}|g(y)|\,dm(y) \\
&= \|f\|_{\alpha,N}\int_{\mathbb{R}} (1+|y|)^{N+2}|g(y)|(1+|y|)^{-2}\,dm(y) \\
&\leq \|f\|_{\alpha,N}\int_{\mathbb{R}} \|g\|_{0,N+2}(1+|y|)^{-2}\,dm(y) \\
&= \|f\|_{\alpha,N}\|g\|_{0,N+2}\int_{\mathbb{R}} (1+|y|)^{-2}\,dm(y) \\
&= C\|f\|_{\alpha,N}\|g\|_{0,N+2}
\end{aligned}$$

Since $x \in \mathbb{R}$ is arbitrary, we have that

$$\begin{aligned}
\|f*g\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha(f*g)(x)|\right] \\
&\leq C\|f\|_{\alpha,N}\|g\|_{0,N+2} \\
&< \infty
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, we have that $f * g \in \mathcal{S}(\mathbb{R}^n)$. $\square$

Exercise 3.5.0.4. The convolution $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is bilinear.

Proof. Let $f, g, h \in \mathcal{S}(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}^n$. Since $\tau_y : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear, we have that

$$\begin{aligned}
[(f + \lambda g) * h](x) &= \int_{\mathbb{R}^n} \tau_y[f + \lambda g](x) h(y)\, dm(y) \\
&= \int_{\mathbb{R}^n} \bigg(\tau_y[f](x) + \lambda \tau_y[g](x)\bigg) h(y)\, dm(y) \\
&= \int_{\mathbb{R}^n} \tau_y[f](x) h(y)\, dm(y) + \lambda \int_{\mathbb{R}^n} \tau_y[g](x) h(y)\, dm(y) \\
&= [f * h](x) + [\lambda g * h](x)
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, $(f + \lambda g) * h = f * h + \lambda g * h$. Similarly, $f * (g + \lambda h) = f * g + \lambda f * h$. $\square$

Exercise 3.5.0.5. The convolution $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is commutative.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Then

$$\begin{aligned}
f * g(x) &= \int_{\mathbb{R}} f(x - y) g(y)\, dm(y) \\
&= \int_{\mathbb{R}} f(z) g(x - z)\, dm(z) \\
&= \int_{\mathbb{R}} g(x - z) f(z)\, dm(z) \\
&= g * f(x)
\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, $f * g = g * f$. $\square$

Exercise 3.5.0.6. The convolution $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_n, g_n)_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ and $(f, g) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$. Suppose that $(f_n, g_n) \to (f, g)$. Then $f_n \to f$ and $g_n \to g$. Hence for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$, $\|f_n - f\|_{\alpha,N} \to 0$ and $\|g_n - g\|_{\alpha,N} \to 0$. In particular

$$\begin{aligned}
\bigg| \|g_n\|_{0,N+2} - \|g\|_{0,N+2} \bigg| &\leq \|g_n - g\|_{0,N+2} \\
&\to 0
\end{aligned}$$

So that $(\|g_n\|_{0,N+2})_{n \in \mathbb{N}}$ is bounded. Let $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Define $C > 0$ as in the previous exercise. Then

$$\begin{aligned}
\|f_n * g_n - f * g\|_{\alpha,N} &= \|f_n * g_n - f * g_n + f * g_n - f * g\|_{\alpha,N} \\
&\leq \|(f_n - f) * g_n\|_{\alpha,N} + \|f_*(g_n - g)\|_{\alpha,N} \\
&\leq C\|f_n - f\|_{\alpha,N}\|g_n\|_{0,N+2} + C\|f\|_{\alpha,N}\|g_n - g\|_{0,N+2} \\
&\to 0
\end{aligned}$$

Since $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ are arbitrary, $f_n * g_n \to f * g$. Thus $* : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous. $\square$

Exercise 3.5.0.7. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. Tonelli's theorem implies that

$$\begin{aligned}
\|f * g\|_1 &= \int_{\mathbb{R}} |f * g(x)| \, dm(x) \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)g(y) \, dm(y) \right| dm(x) \\
&\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)g(y)| \, dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)g(y)| \, dm(x) \right] dm(y) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)| \, dm(x) \right] |g(y)| \, dm(y) \\
&= \|f\|_1 \int_{\mathbb{R}} |g(y)| \, dm(y) \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

□

Definition 3.5.0.8. We define the **bump functions** on $\mathbb{R}$, denoted $C_c^\infty(\mathbb{R})$, by

$$C_c^\infty(\mathbb{R}) = C_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$$

Exercise 3.5.0.9. Let $f \in C_c^\infty(\mathbb{R})$. Then $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $\alpha, N \in \mathbb{N}^0$. Define $g : \mathbb{R}^n \to \mathbb{C}$ by

$$g(x) = (1 + |x|)^N |\partial^\alpha f(x)|$$

Then g is continuous. Since $\operatorname{supp}(\partial^\alpha f) \subset \operatorname{supp}(f)$, we have that $g \in C_c(\mathbb{R})$ and

$$\begin{aligned}
\sup_{x \in \mathbb{R}} \left[(1 + |x|)^N |\partial^\alpha f| \right] &= \sup_{x \in \mathbb{R}} g(x) \\
&= \|g\| \\
&< \infty
\end{aligned}$$

□

Exercise 3.5.0.10. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = e^{-x^2}$. Then $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. meh... □

Exercise 3.5.0.11. Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & x \in (-1, 1) \\ 0 & x \notin (-1, 1) \end{cases}$$

Then $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. meh... □

Exercise 3.5.0.12. Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Then for each $\epsilon > 0$, there exists $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\chi_{[a,b]} \leq f \leq \chi_{[a-\epsilon, b+\epsilon]}$.

Proof. Set $f(x) =$ □

Exercise 3.5.0.13. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define

3.6 The Fourier Transform on $\mathcal{S}(\mathbb{R}^n)$

Exercise 3.6.0.1. Let $\phi : \mathbb{R} \to S^1$ be a measurable homomorphism.

1. Then $\phi \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
2. Define
$$c = \left[\int_{(0,a]} \phi dm\right]^{-1}$$
Then For each $x \in \mathbb{R}$,
$$\phi(x) = c\int_{(x,x+a]} \phi dm$$
3. $\phi \in C^\infty(\mathbb{R})$ and $\phi' = c(\phi(a) - 1)\phi$
4. Define $b = c(\phi(a) - 1)$ and $g \in C^\infty(\mathbb{R})$ by $g(x) = e^{-bx}\phi(x)$. Then g is constant and there exists $\xi \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

Proof.

1. Let $K \subset \mathbb{R}$ be compact. Then
$$\int_K |\phi| dm = m(K) < \infty$$
So $\phi \in L^1_{\text{loc}}(\mathbb{R})$. For the sake of contradiction, suppose that for each $a > 0$,
$$\int_{(0,a]} \phi dm = 0$$
Then the FTC implies that $\phi = 0$ a.e. on $[0, \infty)$, which is a contradiction. So there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
2. For $x \in \mathbb{R}$,
$$\begin{aligned}\phi(x) &= c\int_{(0,a]} \phi(x)\phi(t)dm(t)\\ &= c\int_{(0,a]} \phi(x+t)dm(t)\\ &= c\int_{(x,x+a]} \phi dm\end{aligned}$$
3. Part (2) and the FTC imply that ϕ is continuous. Let $d \in \mathbb{R}$. Define $f_d \in C((d, \infty))$ by
$$f_d(x) = \int_{(d,x]} \phi dm$$
Since ϕ is continuous, the FTC implies that f_d is differentiable and for each $x > d$ $f_d'(x) = \phi(x)$. Part (2) implies that for each $x > d$,
$$\begin{aligned}\phi(x) &= c\int_{(x,x+a]} \phi dm\\ &= c(f_d(x+a) - f_d(x))\end{aligned}$$

So for each $x > d$, ϕ is differentiable at x and

$$\begin{aligned}\phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x)\end{aligned}$$

Since $d \in \mathbb{R}$ is arbitrary, ϕ is differentiable and $\phi' = c(\phi(a) - 1)\phi$. This implies that $\phi \in C^\infty(\mathbb{R})$.

4. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0\end{aligned}$$

So $g' = 0$ and g is constant. Hence there exists $k \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = ke^{bx}$. Since $\phi(0) = 1$, $k = 1$. Since $|\phi| = 1$, there exists $\xi \in \mathbb{R}$ such that $b = 2\pi i\xi$.

□

Note 3.6.0.2. To summarize, for each measurable homomorphism $\phi : \mathbb{R} \to S^1$, there exists $\xi \in \mathbb{R}$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

Exercise 3.6.0.3. Let $\phi : \mathbb{R}^n \to S^1$ be a measurable homomorphism. Then there exists $\xi \in \mathbb{R}^n$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\langle \xi, x\rangle}$.

Definition 3.6.0.4. Let $f \in \mathcal{S}(\mathbb{R}^n)$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$, by

$$\hat{f}(\xi) = \int_{\mathbb{R}} \rho_\xi f \, dm$$

Exercise 3.6.0.5. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\hat{f} \in C_b(\mathbb{R}^n)$.

Proof. Since $f \in \mathcal{S}(\mathbb{R}^n)$, $f \in L^1(\mathbb{R}^n)$. Then for each $\xi \in \mathbb{R}$,

$$\begin{aligned}|\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} \rho_\xi f \, dm \right| \\ &\leq \int_{\mathbb{R}} |\rho_\xi f| \, dm \\ &= \int_{\mathbb{R}} |e^{-i\langle \xi, x\rangle} f(x)| \, dm(x) \\ &= \int_{\mathbb{R}} |f(x)| \, dm(x) \\ &= \|f\|_1\end{aligned}$$

So f is bounded. Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. Suppose that $\xi_n \to \xi$. Define $(\phi_n)_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^n)$ and $\phi \in L^1(\mathbb{R}^n)$ by $\phi_n(x) = \rho_{\xi_n} f(x)$ and $\phi(x) = \rho_\xi f(x)$. Then $\phi_n \xrightarrow{\text{p.w.}} \phi$ and for each $n \in \mathbb{N}$,

$$\begin{aligned}|\phi_n| &= |f| \\ &\in L^1(\mathbb{R}^n)\end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}\hat{f}(\xi_n) &= \int_{\mathbb{R}} \phi_n \, dm \\ &\to \int_{\mathbb{R}} \phi \, dm \\ &= \hat{f}(\xi)\end{aligned}$$

So $\hat{f}$ is continuous. Hence $\hat{f} \in C_b(\mathbb{R})$. □

Definition 3.6.0.6. We define the **Fourier transform on** $\mathcal{S}(\mathbb{R}^n)$, denoted $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$, by

$$\mathcal{F}(f) = \hat{f}$$

Exercise 3.6.0.7. We have that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to C_b(\mathbb{R}^n)$ is linear.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{R}^n$. Since $\rho_\xi : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear, we have that

$$\begin{aligned}
\mathcal{F}(f + \lambda g)(\xi) &= \int_{\mathbb{R}} \rho_\xi(f + \lambda g)\, dm \\
&= \int_{\mathbb{R}} \rho_\xi f + \lambda \rho_\xi g \, dm \\
&= \int_{\mathbb{R}} \rho_\xi f \, dm + \lambda \int_{\mathbb{R}} \rho_\xi g \, dm \\
&= \mathcal{F}(f)(\xi) + \lambda \mathcal{F}(g)(\xi)
\end{aligned}$$

$\square$

Exercise 3.6.0.8. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Then

1. $\mathcal{F}(X^\alpha f) = (-1)^{|\alpha|} P^\alpha \mathcal{F}(f)$
2. $\mathcal{F}(P^\alpha f) = X^\alpha \mathcal{F}(f)$

Proof.

1. Let $\alpha \in \mathbb{N}_0^n$. The claim is true if $|\alpha| = 0$. Let $k > 0$. Suppose that the claim is true for $|\alpha| = k - 1$ so that for each $\beta \in \mathbb{N}_0^n$, $|\beta| = k - 1$ implies that $\mathcal{F}(X^\beta f) = (-1)^{|\beta|} P^\beta \mathcal{F}(f)$. Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Define $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $\phi(\xi, x) = \rho_\xi X^{\alpha - e_j} f(x)$. Then for each $\xi, x \in \mathbb{R}$,

$$\begin{aligned}
\partial_\xi^{e_j} \phi(\xi, x) &= -i x^{e_j} \phi(x) \\
&= -i \rho_\xi X^\alpha f(x)
\end{aligned}$$

Hence for each $x, \xi \in \mathbb{R}^n$,

$$\begin{aligned}
|\partial_\xi^{e_j} \phi(\xi, x)| &= |-i \rho_\xi X^\alpha f(x)| \\
&= |X^\alpha f(x)|
\end{aligned}$$

Since $X^\alpha f \in \mathcal{S}(\mathbb{R}^n) \subset L^1$, we may exchange the order of integration and differentiation to obtain that

$$\begin{aligned}
\mathcal{F}(X^\alpha f)(\xi) &= \int_{\mathbb{R}} \rho_\xi X^\alpha f(x) dm(x) \\
&= \int_{\mathbb{R}^n} i \partial_\xi^{e_j} \phi(\xi, x)\, dm(x) \\
&= i \partial^{e_j} \int_{\mathbb{R}} e^{-i\xi x} x^{\alpha - e_j} f(x) dm(x) \\
&= -P^{e_j} \mathcal{F}(X^{\alpha - e_j} f)(\xi) \\
&= -P^{e_j} \left[(-1)^{|\alpha| - 1} P^{\alpha - e_j} \mathcal{F}(f) \right](\xi) \\
&= (-1)^{|\alpha|} P^\alpha \mathcal{F}(f)(\xi)
\end{aligned}$$

So the claim is true for α. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$.

2. Let $\alpha \in \mathbb{N}_0^n$. The claim is true if $|\alpha| = 0$. Let $k > 0$. Suppose that the claim is true for $|\alpha| = k-1$ so that for each $\beta \in \mathbb{N}_0^n$, $|\beta| = k-1$ implies that $\mathcal{F}(P^\beta f) = X^\beta \mathcal{F}(f)$. Suppose that $|\alpha| = k$. Since $k > 0$, there exists $j \in \{1, \dots, n\}$ such that $\alpha_j > 0$. Then integration by parts yields

$$\begin{aligned}
\mathcal{F}(P^\alpha f)(\xi) &= \int_{\mathbb{R}} e^{-i\langle \xi, x\rangle}[-i\partial_x P^{\alpha - e_j} f(x)]\, dm(x) \\
&= -\int_{\mathbb{R}} -i\xi^{e_j} e^{-i\langle \xi, x\rangle}[-iP^{\alpha - e_j} f(x)]\, dm(x) \\
&= \xi^{e_j} \int_{\mathbb{R}} e^{-i\langle \xi, x\rangle} P^{\alpha - e_j} f(x)\, dm(x) \\
&= X^{e_j} \mathcal{F}(P^{\alpha - e_j} f)(\xi) \\
&= X^{e_j} \Big[X^{\alpha - e_j} \mathcal{F}(f)\Big](\xi) \\
&= X^\alpha \mathcal{F}(f)(\xi)
\end{aligned}$$

So the claim is true for α. By induction, the claim is true for each $\alpha \in \mathbb{N}_0^n$.

□

Exercise 3.6.0.9. There exists $C > 0$ such that for each $f \in \mathcal{S}(\mathbb{R}^n)$, $\|\hat{f}\|_{0,0} \le C\|f\|_{0,2}$.
Hint: Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

Proof. Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

Let $f \in \mathcal{S}(\mathbb{R}^n)$. Let $\xi \in \mathbb{R}$. Then

$$\begin{aligned}
|\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} \rho_\xi f(x)\, dm(x) \right| \\
&\le \int_{\mathbb{R}} |f(x)|\, dm(x) \\
&= \int_{\mathbb{R}} \frac{(1+|x|)^2 |f(x)|}{(1+|x|)^2}\, dm(x) \\
&\le \|f\|_{0,2} \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x) \\
&= C\|f\|_{0,2}
\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\|\hat{f}\|_{0,0} \le C\|f\|_{0,2}$. □

Exercise 3.6.0.10. Let $a, b \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Then $(a+b)^N \le 2^{N-1}(a^N + b^N)$.
Hint: Jensen's inequality

Proof. Jensen's inequality implies that

$$\begin{aligned}
2^{-N}(a+b)^N &= \left(\frac{a}{2} + \frac{b}{2}\right)^N \\
&\le \left(\frac{a^N}{2} + \frac{b^N}{2}\right) \\
&= 2^{-1}(a^N + b^N)
\end{aligned}$$

So $(a+b)^N \le 2^{N-1}(a^N + b^N)$. □

Exercise 3.6.0.11. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^n)$ and there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\|\mathcal{F}(f)\|_{\alpha,N} \leq C2^{N-1}\|X^\alpha f\|_{0,2} + C2^{N-1}\|P^N X^\alpha f\|_{0,2}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. Then the previous exercise implies that for each $\xi \in \mathbb{R}$,

$$\begin{aligned}
\xi^N \partial^\alpha \mathcal{F}(f)(\xi) &= (-i)^N X^N P^\alpha \mathcal{F}(f)(\xi) \\
&= i^N X^N \mathcal{F}(X^\alpha f)(\xi) \\
&= i^N \mathcal{F}(P^N X^\alpha f)(\xi)
\end{aligned}$$

Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

as in the previous exercise. Since $\mathcal{F}(X^\alpha f)$, $\mathcal{F}(P^N X^\alpha f) \in C_b(\mathbb{R})$, we have that

$$\begin{aligned}
\|\mathcal{F}(f)\|_{\alpha,N} &= \sup_{\xi\in\mathbb{R}} \left[(1+|\xi|)^N |\partial^\alpha \mathcal{F}(f)(\xi)|\right] \\
&\leq \sup_{\xi\in\mathbb{R}} \left[2^{N-1}(1+|\xi|^N)|\partial^\alpha \mathcal{F}(f)(\xi)|\right] \\
&= \sup_{\xi\in\mathbb{R}} \left[|2^{N-1}\partial^\alpha \mathcal{F}(f)(\xi)| + |2^{N-1}\xi^N \partial^\alpha \mathcal{F}(f)(\xi)|\right] \\
&= \sup_{\xi\in\mathbb{R}} \left[|\mathcal{F}(2^{N-1}X^\alpha f)(\xi)| + |\mathcal{F}(2^{N-1}P^N X^\alpha f)(\xi)|\right] \\
&\leq \|\mathcal{F}(2^{N-1}X^\alpha f)\|_{0,0} + \|\mathcal{F}(2^{N-1}P^N X^\alpha f)\|_{0,0} \\
&\leq C2^{N-1}\|X^\alpha f\|_{0,2} + C2^{N-1}\|P^N X^\alpha f\|_{0,2} \\
&< \infty
\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^n)$. □

Exercise 3.6.0.12. We have that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$. Suppose that $f_n \to 0$. Since $X, P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ are continuous, $X^\alpha f_n \to 0$ and $P^N X^\alpha f_n \to 0$. Therefore, $\|X^\alpha f_n\|_{0,2} \to 0$ and $\|P^N X^\alpha f_n\|_{0,2} \to 0$. The previous exercise implies there exists $C > 0$ such that for each $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$,

$$\begin{aligned}
\|\mathcal{F}(f_n)\|_{\alpha,N} &\leq C2^{N-1}\|X^\alpha f_n\|_{0,2} + C2^{N-1}\|P^N X^\alpha f_n\|_{0,2} \\
&\to 0
\end{aligned}$$

Hence $\mathcal{F}(f_n) \to 0$ and $\mathcal{F}$ is continuous at 0. Since $\mathcal{F}$ is linear, $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous. □

Exercise 3.6.0.13. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

1. for each $y \in \mathbb{R}$, $\mathcal{F}(\tau_y f) = \rho_y \mathcal{F}(f)$
2. for each $\eta \in \mathbb{R}$, $\mathcal{F}(\rho_\eta f) = \tau_{-\eta}\mathcal{F}(f)$
3. $\mathcal{F}(\gamma_t f) = \kappa_t \mathcal{F}(f)$

Proof.

1. Let $y, \xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(\tau_y f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} f(x-y)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i\xi(z+y)} f(z)\, dm(z) \\
&= e^{-i\xi y} \int_{\mathbb{R}} e^{-i\xi z} f(z)\, dm(z) \\
&= e^{-i\xi y} \mathcal{F}(f)(\xi) \\
&= \rho_y \mathcal{F}(f)(\xi)
\end{aligned}$$

2. Let $\eta, \xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(\rho_\eta f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} e^{-i\eta x} f(x)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i(\xi+\eta)x} f(x)\, dm(x) \\
&= \mathcal{F}(f)(\xi+\eta) \\
&= \tau_{-\eta} \mathcal{F}(f)(\xi)
\end{aligned}$$

3. Let $\xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(\gamma_t f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} f(tx)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i\xi t^{-1} z} f(z) t^{-1}\, dm(z) \\
&= t^{-1} \mathcal{F}(f)(t^{-1}\xi) \\
&= t^{-1} \gamma_{t^{-1}} \mathcal{F}(f)(\xi)
\end{aligned}$$

□

Exercise 3.6.0.14. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.

Proof. Let $\xi \in \mathbb{R}$. Tonelli's theorem implies that

$$\begin{aligned}
\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |e^{-i\xi x} f(x-y) g(y)|\, dm(y) \right] dm(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y) g(y)|\, dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y) g(y)|\, dm(x) \right] dm(y) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)|\, dm(x) \right] |g(y)|\, dm(y) \\
&= \|f\|_1 \int_{\mathbb{R}} |g(y)|\, dm(y) \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

So we may apply Fubini's theorem and change the order of integration to obtain that

$$\begin{aligned}
\mathcal{F}(f*g)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x}(f*g)(x)\,dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y)g(y)\,dm(y)\right] dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y)g(y)\,dm(x)\right] dm(y) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y)\,dm(x)\right] g(y)\,dm(y) \\
&= \int_{\mathbb{R}} [\mathcal{F}(\tau_y f)(\xi)] g(y)\,dm(y) \\
&= \int_{\mathbb{R}} [e^{-i\xi y}\mathcal{F}(f)(\xi)] g(y)\,dm(y) \\
&= \mathcal{F}(f)(\xi) \int_{\mathbb{R}} e^{-i\xi y} g(y)\,dm(y) \\
&= \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)
\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\mathcal{F}(f*g) = \mathcal{F}(f)\mathcal{F}(g)$ □

Exercise 3.6.0.15. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}} \hat{f} g\,dm = \int_{\mathbb{R}} f \hat{g}\,dm$$

Proof. Tonelli's theorem implies that

$$\begin{aligned}
\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |e^{-i\xi x} f(x)g(\xi)|\,dm(x)\right] dm(\xi) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x)|\,dm(x)\right] |g(\xi)|\,dm(\xi) \\
&= \|f\|_1 \int_{\mathbb{R}} |g(\xi)|\,dm(\xi) \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

So we may apply Fubini's theorem and switch the order of integration to obtain that

$$\begin{aligned}
\int_{\mathbb{R}} \hat{f} g\,dm &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x)\,dm(x)\right] g(\xi)\,dm(\xi) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x)g(\xi)\,dm(x)\right] dm(\xi) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x)g(\xi)\,dm(\xi)\right] dm(x) \\
&= \int_{\mathbb{R}} f(x)\left[\int_{\mathbb{R}} e^{-i\xi x} g(\xi)\,dm(\xi)\right] dm(x) \\
&= \int_{\mathbb{R}} f(x)\hat{g}(x)\,dm(x) \\
&= \int_{\mathbb{R}} f\hat{g}\,dm
\end{aligned}$$

□

Exercise 3.6.0.16. Define $f \in \mathcal{S}(\mathbb{R}^n)$ by $f(x) = e^{-x^2/2}$. Then $\mathcal{F}(f) = \sqrt{2\pi} f$.

Proof. Note that for each $\xi \in \mathbb{R}$,

$$\begin{aligned}\mathcal{F}(Df)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} ixe^{-x^2/2}\, dm(x) \\ &= -\int_{\mathbb{R}} \partial_\xi \left[e^{-i\xi x}e^{-x^2/2}\right] dm(x) \\ &= -\partial_\xi \mathcal{F}(f)(\xi)\end{aligned}$$

A previous exercise implies that $\mathcal{F}(Df) = X\mathcal{F}(f)$. So for each $\xi \in \mathbb{R}$, $\partial_\xi \hat{f}(\xi) = -\xi \hat{f}(\xi)$. Define $g \in \mathbb{C}^\infty(\mathbb{R})$ by $g(\xi) = e^{\xi^2/2}$. Then

$$\begin{aligned}\partial_\xi(\hat{f}g) &= (\partial_\xi \hat{f})g + \hat{f}(\partial_\xi g) \\ &= 0\end{aligned}$$

So there exists $C \in \mathbb{R}$ such that $\hat{f}g = C$. Hence for each $\xi \in \mathbb{R}$,

$$\begin{aligned}\hat{f}(\xi) &= Ce^{-\xi^2/2} \\ &= Cf(\xi)\end{aligned}$$

Therefore,

$$\begin{aligned}C &= Cf(0) \\ &= \hat{f}(0) \\ &= \int_{\mathbb{R}} e^{-x^2/2}\, dm(x) \\ &= \sqrt{2\pi}\end{aligned}$$

So $\hat{f} = \sqrt{2\pi} f$. □

Exercise 3.6.0.17. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Define $g : \mathbb{R}^n \to L^1$ by $g(x) = \tau_x f$. Then g is continuous.
Hint: approximate by functions in $C_c(\mathbb{R})$.

Proof. Suppose that $f \in C_c(\mathbb{R})$. Then □

Definition 3.6.0.18. Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$. We define $f_t \in \mathcal{S}(\mathbb{R}^n)$ by $f_t = t^{-1}\gamma_{t^{-1}} f$.

Exercise 3.6.0.19. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $t \neq 0$. Then

$$\int_{\mathbb{R}} \phi_t\, dm = \int_{\mathbb{R}} \phi\, dm$$

Proof. We have that

$$\begin{aligned}\int_{\mathbb{R}} \phi_t\, dm &= \int_{\mathbb{R}} t^{-1}\phi(t^{-1}x)\, dm(x) \\ &= \int_{\mathbb{R}} \phi(z)\, dm(z) \\ &= \int_{\mathbb{R}} \phi\, dm\end{aligned}$$

□

Exercise 3.6.0.20. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Set

$$\alpha = \int_{\mathbb{R}} \phi\, dm$$

Then for each $f \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_{1/n} \xrightarrow{L^1} \alpha f$.
Hint: for each $t \neq 0$ and $x \in \mathbb{R}$,

$$f * \phi_t(x) - \alpha f(x) = \int_{\mathbb{R}} [\tau_{tz} f(x) - f(x)]\phi(z)\, dm(z)$$

Proof. Let $t \neq 0$ and $x \in \mathbb{R}$. The previous exercise implies that

$$\begin{aligned}
f * \phi_t(x) - \alpha f(x) &= \int_{\mathbb{R}} f(x-y)\phi_t(y)\, dm(y) - \int_{\mathbb{R}} \phi(y)\, dm(y) f(x) \\
&= \int_{\mathbb{R}} f(x-y)\phi_t(y)\, dm(y) - \int_{\mathbb{R}} \phi_t(y)\, dm(y) f(x) \\
&= \int_{\mathbb{R}} f(x-y)\phi_t(y) - f(x)\phi_t(y)\, dm(y) \\
&= \int_{\mathbb{R}} [f(x-y) - f(x)]\phi_t(y)\, dm(y) \\
&= \int_{\mathbb{R}} [f(x-y) - f(x)]t^{-1}\phi(t^{-1}y)\, dm(y) \\
&= \int_{\mathbb{R}} [f(x-tz) - f(x)]\phi(z)\, dm(z) \\
&= \int_{\mathbb{R}} [\tau_{tz} f(x) - f(x)]\phi(z)\, dm(z)
\end{aligned}$$

Tonelli's theorem implies that

$$\begin{aligned}
\|f * \phi_t - \alpha f\|_1 &= \int_{\mathbb{R}} |f * \phi_t(x) - \alpha f(x)|\, dm(x) \\
&\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\tau_{tz} f(x) - f(x)||\phi(z)|\, dm(z) \right] dm(x) \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\tau_{tz} f(x) - f(x)||\phi(z)|\, dm(x) \right] dm(z) \\
&= \int_{\mathbb{R}} \|\tau_{tz} f - f\|_1 |\phi(z)|\, dm(z)
\end{aligned}$$

For $n \in \mathbb{N}$, define $g_n \in \mathcal{S}(\mathbb{R}^n)$ by $g_n(z) = \|\tau_{n^{-1}z} f(x) - f(x)\|_1 \phi(z)$. Then $g_n \xrightarrow{\text{p.w.}} 0$ and

$$\begin{aligned}
|g_n| &\leq 2\|f\|_1|\phi| \\
&\in L^1(\mathbb{R}^n)
\end{aligned}$$

The dominated convergence theorem implies that

$\square$

Definition 3.6.0.21. content...

3.7 Tempered Distributions

3.8 The Fourier Transform on $\mathcal{M}(\mathbb{R})$

Note 3.8.0.1. Recall that

$$\mathcal{M}(\mathbb{R}) = \{\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{C} : \mu \text{ is a complex measure}\}$$

Definition 3.8.0.2. Let $\mu \in \mathcal{M}(\mathbb{R})$. We define the **Fourier transform of** μ, denoted $\hat{\mu} : \mathbb{R} \to \mathbb{C}$, by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} \, d\mu(x)$$

Exercise 3.8.0.3. Let $\mu \in \mathcal{M}(\mathbb{R})$. Then Then $\hat{\mu} : \mathbb{R} \to \mathbb{C}$ is bounded.

Proof. Let $\xi \in \mathbb{R}$.

$$\begin{aligned}
|\hat{\mu}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi x} \, d\mu(x) \right| \\
&\leq \int_{\mathbb{R}} |e^{-i\xi x}| \, d|\mu|(x) \\
&= |\mu|(\mathbb{R})
\end{aligned}$$

So $\hat{\mu}$ is bounded. □

Exercise 3.8.0.4. Let $\mu \in \mathcal{M}(\mathbb{R})$. Then $\hat{\mu} \in C_b(\mathbb{R})$.

Proof. Let $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. Define $(f_n)_{n \in \mathbb{N}} \subset L^1(\mu)$ and $f \in L^1(\mu)$ by $f_n(x) = e^{-i\xi_n x}$ and $f(x) = e^{-i\xi x}$. Suppose that $\xi_n \to \xi$. Then $f_n \xrightarrow{\text{p.w.}} f$ and for each $n \in N$ and $x \in \mathbb{R}$,

$$\begin{aligned}
|f_n(x)| &= |e^{-i\xi_n x}| \\
&= 1 \\
&\in L^1(|\mu|)
\end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}
|\hat{\mu}(\xi_n) - \hat{\mu}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi_n x} \, d\mu(x) - \int_{\mathbb{R}} e^{-i\xi x} \, d\mu(x) \right| \\
&= \left| \int_{\mathbb{R}} e^{-i\xi_n x} - e^{-i\xi x} \, d\mu(x) \right| \\
&\leq \int_{\mathbb{R}} |e^{-i\xi_n x} - e^{-i\xi x}| \, d|\mu|(x) \\
&\to 0
\end{aligned}$$

So $\hat{\mu} : \mathbb{R} \to \mathbb{C}$ is continuous. Hence $\hat{\mu} \in C_b(\mathbb{R})$. □

Definition 3.8.0.5. Let X be a real normed vector space. We define $\mathcal{F} : \mathcal{M}(\mathbb{R}) \to C_b(\mathbb{R})$ by

$$\mathcal{F}(\mu) = \hat{\mu}$$

Exercise 3.8.0.6. Let X be a real normed vector space. Then $\mathcal{F} : \mathcal{M}(\mathbb{R}) \to C_b(\mathbb{R})$ is linear.

Proof. Let $\mu, \nu \in \mathcal{M}(\mathbb{R})$ and $\xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}[\mu + \nu](\xi) &= \int_{\mathbb{R}} e^{-i\xi x} \, d[\mu + \nu](x) \\
&= \int_{\mathbb{R}} e^{-i\xi x} \, d\mu(x) + \int_{\mathbb{R}} e^{-i\xi x} \, d\nu(x) \\
&= \mathcal{F}[\mu](\xi) + \mathcal{F}[\nu](\xi)
\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\mathcal{F}(\mu + \nu) = \mathcal{F}(\mu) + \mathcal{F}(\nu)$ and $\mathcal{F}$ is linear. □

Exercise 3.8.0.7. Let X be a real normed vector space. If X is separable, then $\mathcal{F}$ is injective.

Proof. Suppose that X is separable. Let $\mu \in \mathcal{M}(X)$. Suppose that $\mu \in \ker \mathcal{F}$. Then $\hat{\mu} = 0$ and for each $\phi \in X^*$,

$$\begin{aligned} 0 &= \hat{\mu}(\phi) \\ &= \int_X e^{-i\phi(x)}\, d\mu(x) \\ &= \int_{\mathbb{R}} e^{-ix}\, d[\phi_* \mu](x) \end{aligned}$$

□

Exercise 3.8.0.8. Let X be a real normed vector space. Then $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$.

Proof. For $\mu \in \mathcal{M}(X)$ and $\phi \in X^*$, we have that

$$\begin{aligned} |\mathcal{F}[\mu](\phi)| &= \left| \int_X e^{-i\phi(x)}\, d\mu(x) \right| \\ &\leq \int_X |e^{-i\phi(x)}|\, d|\mu|(x) \\ &= |\mu|(X) \\ &= \|\mu\| \end{aligned}$$

Hence

$$\begin{aligned} \|\mathcal{F}(\mu)\| &= \sup_{\phi \in X^*} |\mathcal{F}[\mu](\phi)| \\ &\leq \|\mu\| \end{aligned}$$

which implies that $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$. □

Chapter 4

Fourier Analysis on $\mathbb{R}^n$

4.1 Schwartz Space

Definition 4.1.0.1. Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

1. $\langle x, y \rangle = \sum_j x_j y_j$
2. $|x| = \langle x, x \rangle^{1/2}$
3. $|\alpha| = \alpha_1 + \cdots + \alpha_n$
4. $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
5. $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

Definition 4.1.0.2. Let $f \in C^\infty(\mathbb{R}^n)$,$\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

We define Schwartz space, denoted $\mathcal{S}(\mathbb{R}^n)$, by

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n, N \in \mathbb{N}_0, \|f\|_{\alpha,N} < \infty\}$$

Exercise 4.1.0.3. For each $f \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$. Then there exists $C \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^{-1}$$

Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = (1 + |x|^2)^{-1}$. Then $g \in L^1(\mathbb{R}^n)$ which implies that $\partial^\alpha f \in L^1(\mathbb{R}^n)$. $\square$

Definition 4.1.0.4.

4.2 The Convolution

Definition 4.2.0.1. Let $f, g \in L^0(\mathbb{R}^n)$. If for a.e. $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) < \infty$$

we define the **convolution of** f **with** g, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$, by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

Exercise 4.2.0.2. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = f(x-y)g(y)$. Tonelli's theorem implies that,

$$\begin{aligned}
\int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}^n} |g(y)| \left[\int_{\mathbb{R}^n} |f(x-y)|dm(y) \right] dm(x) \\
&= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dm(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

Then $h \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Fubini's theorem implies that $f * g \in L^1(\mathbb{R}^n)$. Clearly

$$\begin{aligned}
\|f * g\|_1 &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 \\
&\leq \|f\|_1 \|g\|_1
\end{aligned}$$

□

Exercise 4.2.0.3. Let $f, g, h \in L^1(\mathbb{R}^n)$. Then $(f * g) * h = f * (g * h)$.
Hint: use the substitution $z \mapsto z - y$

Proof. Let $x \in \mathbb{R}^n$. Then using the substitution $z \mapsto z - y$ and Fubini's theorem, we obtain

$$\begin{aligned}
(f * g) * h(x) &= \int f * g(x-y)h(y)dm(y) \\
&= \int \left[\int f(x-y-z)g(z)dm(z) \right] h(y)dm(y) \\
&= \int \left[\int f(x-z)g(z-y)dm(z) \right] h(y)dm(y) \\
&= \int \left[\int f(x-z)g(z-y)h(y)dm(z) \right] dm(y) \\
&= \int \left[\int f(x-z)g(z-y)h(y)dm(y) \right] dm(z) \\
&= \int f(x-z) \left[\int g(z-y)h(y)dm(y) \right] dm(z) \\
&= \int f(x-z) g * h(z)dm(z) \\
&= f * (g * h)(z)
\end{aligned}$$

So $(f * g) * h = f * (g * h)$. □

Exercise 4.2.0.4. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g = g * f$.

Proof. Let $x \in \mathbb{R}^n$. Using the transformation $y \mapsto x - y$, we obtain that

$$\begin{aligned} f * g(x) &= \int f(x-y)g(y)dm(y) \\ &= \int f(y)g(x-y)dm(y) \\ &= \int g(x-y)f(y)dm(y) \\ &= g * f(x) \end{aligned}$$

So $f * g = g * f$. □

Note 4.2.0.5. To summarize, $(L^1(\mathbb{R}^n), *)$ is a commutative Banach algebra.

Exercise 4.2.0.6. Young's Inequality:
Let $p \in [1, \infty]$, $f \in L^1$ and $g \in L^p$. Then $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof. Define $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $K(x, y) = f(x - y)$. Since for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \int |K(x,y)|dm(x) &= \int |K(x,y)|dm(y) \\ &= \|f\|_p \end{aligned}$$

an exercise in section 5.1 of [4] implies that $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. □

Exercise 4.2.0.7. Let $p, q \in [1, \infty]$ be conjugate, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then

1. for each $x \in \mathbb{R}^n$, $f * g(x)$ exists.
2. $\|f * g\|_u \leq \|f\|_p \|g\|_q$
3.

Proof.

1. Let $x \in \mathbb{R}^n$. Holder's inequality implies that

   $$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \leq \|f\|_p \|g\|_q$$

   Then $f * g(x)$ exists.

2. Let $x \in \mathbb{R}^n$. Then in part (1) we showed that

   $$\begin{aligned} |f * g(x)| &= \left| \int_{\mathbb{R}^n} f(x-y)g(y)dm(y) \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \\ &\leq \|f\|_p \|g\|_q \end{aligned}$$

   Since $x \in \mathbb{R}^n$ is arbitrary, $\|f * g\|_u \leq \|f\|_p \|g\|_q$.

3.

□

Exercise 4.2.0.8. Let $f \in L^1(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $g \in C^k(\mathbb{R}^n)$. Suppose that for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $\partial^\alpha g \in L^\infty$. Then for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $f * g \in C^k$ and

$$\partial^\alpha (f * g) = f * \partial^\alpha g$$

Proof. Let $\alpha \in \mathbb{N}_0^n$. Suppose that $|\alpha| = 1$. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x,y) = g(x-y)f(y)$. Young's inequality imples that for a.e. $x \in \mathbb{R}^n$, $h(x,\cdot) \in L^1(\mathbb{R}^n)$. For each $y \in \mathbb{R}^n$, $\partial^\alpha h(\cdot,y) = \partial^\alpha g(\cdot - y)f(y)$ and for each $x,y \in \mathbb{R}^n$, $|\partial^\alpha h(x,y)| \leq \|\partial^\alpha g\|_\infty |f(y)| \in L^1(\mathbb{R}^n)$. An exercise in section 3.3 of [4] implies that for a.e. $x \in \mathbb{R}^n$, $\partial^\alpha(g*f)(x)$ exists and

$$\begin{aligned}
\partial^\alpha(f*g)(x) &= \partial^\alpha(g*f)(x) \\
&= \partial^\alpha \int_{\mathbb{R}^n} h(x,y)dm(y) \\
&= \int_{\mathbb{R}^n} \partial^\alpha g(x-y)f(y)dm(y) \\
&= (\partial^\alpha g) * f(x) \\
&= f * (\partial^\alpha g)(x)
\end{aligned}$$

Now proceed by induction on $|\alpha|$. $\square$

4.3 The Fourier Transform

Definition 4.3.0.1.

Exercise 4.3.0.2. Let $\phi : \mathbb{R} \to S^1$ be a measurable homomorphism.

1. Then $\phi \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
2. Define
$$c = \left[\int_{(0,a]} \phi dm\right]^{-1}$$
Then For each $x \in \mathbb{R}$,
$$\phi(x) = c\int_{(x,x+a]} \phi dm$$
3. $\phi \in C^\infty(\mathbb{R})$ and $\phi' = c(\phi(a) - 1)\phi$
4. Define $b = c(\phi(a) - 1)$ and $g \in C^\infty(\mathbb{R})$ by $g(x) = e^{-bx}\phi(x)$. Then g is constant and there exists $\xi \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i \xi x}$.

Proof.

1. Let $K \subset \mathbb{R}$ be compact. Then
$$\int_K |\phi| dm = m(K) < \infty$$
So $\phi \in L^1_{\text{loc}}(\mathbb{R})$. For the sake of contradiction, suppose that for each $a > 0$,
$$\int_{(0,a]} \phi dm = 0$$
Then the FTC implies that $\phi = 0$ a.e. on $[0, \infty)$, which is a contradiction. So there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$
2. For $x \in \mathbb{R}$,
$$\begin{aligned}\phi(x) &= c\int_{(0,a]} \phi(x)\phi(t) dm(t) \\ &= c\int_{(0,a]} \phi(x+t) dm(t) \\ &= c\int_{(x,x+a]} \phi dm\end{aligned}$$
3. Part (2) and the FTC imply that ϕ is continuous. Let $d \in \mathbb{R}$. Define $f_d \in C((d, \infty))$ by
$$f_d(x) = \int_{(d,x]} \phi dm$$
Since ϕ is continuous, the FTC implies that f_d is differentiable and for each $x > d$ $f_d'(x) = \phi(x)$. Part (2) implies that for each $x > d$,
$$\begin{aligned}\phi(x) &= c\int_{(x,x+a]} \phi dm \\ &= c(f_d(x+a) - f_d(x))\end{aligned}$$

So for each $x > d$, ϕ is differentiable at x and

$$\begin{aligned}\phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x)\end{aligned}$$

Since $d \in \mathbb{R}$ is arbitrary, ϕ is differentiable and $\phi' = c(\phi(a) - 1)\phi$. This implies that $\phi \in C^\infty(\mathbb{R})$.

4. Let $x \in \mathbb{R}$. Then

$$\begin{aligned}g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0\end{aligned}$$

So $g' = 0$ and g is constant. Hence there exists $k \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = ke^{bx}$. Since $\phi(0) = 1$, $k = 1$. Since $|\phi| = 1$, there exists $\xi \in \mathbb{R}$ such that $b = 2\pi i\xi$.

□

Note 4.3.0.3. To summarize, for each measurable homomorphism $\phi : \mathbb{R} \to S^1$, there exists $\xi \in \mathbb{R}$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

Exercise 4.3.0.4. Let $\phi : \mathbb{R}^n \to S^1$ be a measurable homomorphism. Then there exists $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi(x) = e^{2\pi i\langle \xi, x\rangle}$.

Proof. When done in the category of measurable groups, an exercise in the section on direct products of groups of [**?**] implies that there exist measurable homomorphism $(\phi_j)_{j=1}^n \subset (S^1)^{\mathbb{R}}$ such that $\phi = \bigotimes_{j=1}^n \phi_j$. The previous exercise imples that there exist $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi_j(x_j) = e^{2\pi i\xi_j x_j}$. Then for each $x \in \mathbb{R}^n$,

$$\begin{aligned}\phi(x) &= \prod_{j=1}^n \phi_j(x_j) \\ &= \prod e^{2\pi i\xi_j x_j} \\ &= e^{2\pi i \sum_{j=1}^n \xi_j x_j} \\ &= e^{2\pi i\langle \xi, x\rangle}\end{aligned}$$

□

Definition 4.3.0.5. Let $f \in L^1(\mathbb{R}^n)$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle \xi, x\rangle}dm(x)$$

Chapter 5

Fourier Analysis on LCA Groups

5.1 The Convolution

Note 5.1.0.1. For the remainder of the section, we fix a locally compact abelian group G and a Haar measure μ on G.

Definition 5.1.0.2. Let $f, g \in L^1(\mu)$. We define the **convolution of** f **with** g, denoted $f * g : G \to \mathbb{C}$, by

$$f * g(x) = \int_X f(x-y)g(y)d\mu(y)$$

Exercise 5.1.0.3. Let $f, g \in L^1(\mu)$. Then $f * g \in L^1(\mu)$.

Proof. By Tonelli's theorem,

$$\begin{aligned}
\int_X |f * g| d\mu &\leq \int_X \left[\int_X |f(x-y)g(y)| d\mu(y) \right] d\mu(x) \\
&= \int_X |g(y)| \left[\int_X |f(x-y)| d\mu(y) \right] d\mu(x) \\
&= \|f\|_1 \int_X |g(y)| d\mu(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

□

Chapter 6

Fourier Analysis on Banach Spaces

Chapter 7

Fourier Analysis on Banach Spaces

Appendix A

Summation

Appendix B

Asymptotic Notation

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration