INTRODUCTION TO FOURIER ANALYSIS

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Contents

1. Fourier Analysis on $\mathbb{R}$

1.1. Schwartz Space.

Definition 1.1.1. Let $f \in C^\infty(\mathbb{R}, \mathbb{C})$, and $\alpha, N \in \mathbb{N}_0$. We define $\|\cdot\|_{\alpha,N} : C^\infty(\mathbb{R}, \mathbb{C}) \to [0, \infty]$ by

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |\partial^\alpha f(x)|\bigg]$$

We define **Schwartz space** on $\mathbb{R}$, denoted $\mathcal{S}$, by

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}, \mathbb{C}) : \text{ for each } \alpha, N \in \mathbb{N}_0,\ \|f\|_{\alpha,N} < \infty\}$$

Exercise 1.1.2. We have that $\mathcal{S}$ is a vector space and for each $\alpha, N \in \mathbb{N}_0$, $\|\cdot\|_{\alpha,N} : \mathcal{S} \to [0, \infty)$ is a seminorm on $\mathcal{S}$.

Proof. Let $f, g \in \mathcal{S}$ and $\lambda \in \mathbb{C}$.

(1)

$$\begin{aligned}
\|\lambda f\|_{\alpha,N} &= \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |\partial^\alpha [\lambda f](x)|\bigg] \\
&= \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |\lambda \partial^\alpha f(x)|\bigg] \\
&= \sup_{x \in \mathbb{R}} \bigg[|\lambda|(1 + |x|)^N |\partial^\alpha f(x)|\bigg] \\
&= |\lambda| \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |\partial^\alpha f(x)|\bigg] \\
&= |\lambda| \|f\|_{\alpha,N}
\end{aligned}$$

Thus $\lambda f \in \mathcal{S}$ and $\|\lambda f\|_{\alpha,N} = |\lambda| \|f\|_{\alpha,N}$.

(2)

$$\begin{aligned}
\|f + g\|_{\alpha,N} &= \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |\partial^\alpha [f + g](x)|\bigg] \\
&= \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |[\partial^\alpha f + \partial g](x)|\bigg] \\
&\leq \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |\partial^\alpha f(x)| + (1 + |x|)^N |\partial g(x)|\bigg] \\
&\leq \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |\partial^\alpha f(x)|\bigg] + \sup_{x \in \mathbb{R}} \bigg[(1 + |x|)^N |\partial g(x)|\bigg] \\
&= \|f\|_{\alpha,N} + \|g\|_{\alpha,N}
\end{aligned}$$

Hence $f + g \in \mathcal{S}$ and $\|f + g\|_{\alpha,N} \leq \|f\|_{\alpha,N} + \|g\|_{\alpha,N}$.

So $\mathcal{S}$ is a vector space and $\|\cdot\|_{\alpha,N}$ is a seminorm on $\mathcal{S}$. $\square$

Exercise 1.1.3. We have that $\mathcal{S}$ is a algebra under pointwise multiplication and for each $\alpha, N \in \mathbb{N}_0$,

$$\|fg\|_{\alpha,N} \leq \sum_{\beta=0}^{\alpha} \|f\|_{\beta,N} \|g\|_{\alpha-\beta,0}$$

Hint: $\partial^\alpha[fg] = \sum\limits_{\beta=0}^{\alpha} (\partial^\beta f)(\partial^{\alpha-\beta} g)$

Proof. Let $f, g \in \mathcal{S}$ and $\alpha, N \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|fg\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N |\partial^\alpha[fg](x)|\bigg] \\
&= \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N \bigg|\sum_{\beta=0}^{\alpha} \partial^\beta f(x) \partial^{\alpha-\beta} g(x)\bigg|\bigg] \\
&\leq \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N \bigg(\sum_{\beta=0}^{\alpha} |\partial^\beta f(x)||\partial^{\alpha-\beta} g(x)|\bigg)\bigg] \\
&= \sup_{x\in\mathbb{R}} \bigg[\sum_{\beta=0}^{\alpha} (1+|x|)^N |\partial^\beta f(x)||\partial^{\alpha-\beta} g(x)|\bigg] \\
&\leq \sum_{\beta=0}^{\alpha} \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N |\partial^\beta f(x)||\partial^{\alpha-\beta} g(x)|\bigg] \\
&\leq \sum_{\beta=0}^{\alpha} \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N |\partial^\beta f(x)|\bigg] \sup_{x\in\mathbb{R}} \bigg[|\partial^{\alpha-\beta} g(x)|\bigg] \\
&= \sum_{\beta=0}^{\alpha} \|f\|_{\beta,N} \|g\|_{\alpha-\beta,0} \\
&< \infty
\end{aligned}$$

So $fg \in \mathcal{S}$. □

Definition 1.1.4. Set $\mathcal{P} = (\|\cdot\|_{\alpha,N})_{\alpha,N\in\mathbb{N}_0}$. Then $\mathcal{P}$ is a countable family of seminorms on $\mathcal{S}$. We equip $\mathcal{S}$ with the topology $\mathcal{T}$ induced by the family of projections

$$\pi_{\|\cdot\|_{\alpha,N}} : \mathcal{S} \to \mathcal{S}/\ker \|\cdot\|_{\alpha,N}$$

i.e. $\mathcal{T} = \tau_{\mathcal{S}}((\pi_{\|\cdot\|_{\alpha,N}})_{\alpha,N\in\mathbb{N}_0})$.
Explicitly, for a net $(f_\alpha)_{\alpha\in A} \subset \mathcal{S}$ and $f \in \mathcal{S}$, $f_\alpha \to f$ iff for each $\alpha, N \in \mathbb{N}_0$, $\|f_\alpha - f\|_{\alpha,N} \to 0$. Hence $(\mathcal{S}, \mathcal{T})$ is a locally convex space. Since $\mathcal{P}$ is countable, we may write $\mathcal{P} = (p_j)_{j\in\mathbb{N}}$ and thus $(\mathcal{S}, \mathcal{T})$ is metrizable with metric

$$d_{\mathcal{S}}(f, g) = \sum_{j\in\mathbb{N}} 2^{-j} \frac{p_j(f-g)}{1+p_j(f-g)}$$

Exercise 1.1.5. Let $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0$. Then $\partial^\alpha f \in \mathcal{S}$ and for each $\beta, N \in \mathbb{N}_0$,

$$\|\partial^\alpha f\|_{\beta,N} \leq \|f\|_{\alpha+\beta,N}$$

Proof. Let $f \in \mathcal{S}$, and β, $N \in \mathbb{N}_0$. By definition,

$$\begin{aligned}\|\partial^\alpha f\|_{\beta,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\beta[\partial^\alpha f](x)|\right] \\ &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^{\alpha+\beta} f(x)|\right] \\ &= \|f\|_{\alpha+\beta,N} \\ &< \infty\end{aligned}$$

So $\partial^\alpha f \in \mathcal{S}$. $\square$

Exercise 1.1.6. Let $f \in \mathcal{S}$. Then for each α, $N \in \mathbb{N}_0$,

$$\|f\|_{\alpha,N} = \|\partial^\alpha f\|_{0,N}$$

Proof. Clear by preceding exercise. $\square$

Exercise 1.1.7. Let $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}$. Define $g : \mathbb{R} \to \mathbb{C}$ by $g(x) = xf(x)$. Then for each $x \in \mathbb{R}$, $\partial^\alpha g(x) = x\partial^\alpha f(x) + \alpha\partial^{\alpha-1} f(x)$.

Proof. The claim is clear if $\alpha = 1$. Suppose that $\alpha > 1$ and that the claim is true for $\alpha - 1$ so that for each $x \in \mathbb{R}$, $\partial^{\alpha-1} g(x) = x\partial^{\alpha-1} f(x) + (\alpha - 1)\partial^{\alpha-2} f(x)$. Then

$$\begin{aligned}\partial^\alpha g(x) &= \partial[\partial^{\alpha-1} g(x)] \\ &= \partial[x\partial^{\alpha-1} f(x) + (\alpha-1)\partial^{\alpha-2} f(x)] \\ &= \partial[x\partial^{\alpha-1} f(x)] + \partial[(\alpha-1)\partial^{\alpha-2} f(x)] \\ &= [x\partial^\alpha f(x) + \partial^{\alpha-1} f(x)] + [(\alpha-1)\partial^{\alpha-1} f(x)] \\ &= x\partial^\alpha f(x) + \alpha\partial^{\alpha-1} f(x)\end{aligned}$$

So the claim is true for α. $\square$

Exercise 1.1.8. Let $f \in \mathcal{S}$ and $N \in \mathbb{N}_0$. Define $g : \mathbb{R} \to \mathbb{C}$ by $g(x) = xf(x)$. Then $g \in \mathcal{S}$ and for each α, $N \in \mathbb{N}_0$,

$$\|g\|_{\alpha,N} \le \|f\|_{\alpha,N+1} + \alpha\|f\|_{\alpha-1,N}$$

Proof. Let $\alpha, N \in \mathbb{N}_0$. The previous exercise implies that

$$\begin{aligned}\|g\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha xf(x)|\right] \\ &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|x\partial^\alpha f(x) + \alpha\partial^{\alpha-1} f(x)|\right] \\ &\le \sup_{x\in\mathbb{R}}\left[(1+|x|)^{N+1}|\partial^\alpha f(x)|\right] + \alpha\sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^{\alpha-1} f(x)|\right] \\ &= \|f\|_{\alpha,N+1} + \alpha\|f\|_{\alpha-1,N}\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $g \in \mathcal{S}$. $\square$

Definition 1.1.9. We define the

- **position operator**, denoted $X : \mathcal{S} \to \mathcal{S}$, by

$$Xf(x) = xf(x)$$

- **momentum operator**, denoted $D : \mathcal{S} \to \mathcal{S}$, by
$$Df(x) = -i\partial f(x)$$

Exercise 1.1.10. We have that
(1) $X : \mathcal{S} \to \mathcal{S}$ and $D : \mathcal{S} \to \mathcal{S}$ are linear
(2) $X : \mathcal{S} \to \mathcal{S}$ and $D : \mathcal{S} \to \mathcal{S}$ are continuous.

Proof.
(1) Clear.
(2) Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$. Suppose that $f_n \to 0$. Then for each $\alpha, N \in \mathbb{N}_0$, $\|f_n\|_{\alpha,N} \to 0$.
  - A previous exercise implies that
$$\begin{aligned}\|Xf_n\|_{\alpha,N} &\leq \|f_n\|_{\alpha,N+1} + \alpha\|f_n\|_{\alpha-1,N} \\ &\to 0\end{aligned}$$
So $Xf_n \to 0$ and X is continuous at 0. Since X is linear, X is continuous.
  - A previous exercise implies that
$$\begin{aligned}\|Df_n\|_{\alpha,N} &= \|\partial f_n\|_{\alpha,N} & \leq \|f_n\|_{\alpha+1,N} \\ &\to 0\end{aligned}$$
So $Df_n \to 0$ and D is continuous at 0. Since D is linear, D is continuous.

□

Exercise 1.1.11. We have that $\mathcal{S} \subset L^1(m)$.

Proof. Let $f \in \mathcal{S}$. Then there exists $C \geq 0$ such that for each $x \in \mathbb{R}$,
$$|f(x)| \leq C(1 + |x|^2)^{-1}$$
Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = (1 + |x|^2)^{-1}$. Then $g \in L^1(\mathbb{R})$ which implies that $f \in L^1(\mathbb{R})$. □

Definition 1.1.12. Let $f \in \mathcal{S}$ and $y \in \mathbb{R}$. Define
- $L_y f : \mathbb{R} \to \mathbb{C}$ by $L_y f(x) = f(x - y)$
- $If : \mathbb{R} \to \mathbb{C}$ by $If(x) = f(-x)$.

Exercise 1.1.13. Let $x, t \in \mathbb{R}$. Then $(1 + |x|) \leq (1 + |x - t|)(1 + |t|)$.

Proof. We have that
$$\begin{aligned}(1 + |x - t|)(1 + |t|) &= 1 + |x - t| + |t| + |x - t||t| \\ &\geq 1 + |x| + |x - t||t| \\ &\geq 1 + |x|\end{aligned}$$

□

Exercise 1.1.14. Let $f \in \mathcal{S}$, then for each $y \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0$,
- $\partial^\alpha L_y f = L_y \partial^\alpha f$
- $\partial^\alpha If = (-1)^\alpha I\partial^\alpha f$

Proof. Clear by chain rule. □

Exercise 1.1.15. Let $f \in \mathcal{S}$. Then

(1) for each $y \in \mathbb{R}$, $L_y f \in \mathcal{S}$
(2) $If \in \mathcal{S}$

Proof.

(1)
(2)

□

Note 1.1.16. Let $f, g \in \mathcal{S}$ and $x \in \mathbb{R}$, Define $h : \mathbb{R} \to \mathbb{R}$ defined by $h_x(y) = f(x - y)g(y)$. A previous exercise implies that $h_x \in \mathcal{S}$ and for each $\alpha, N \in \mathbb{N}_0$, $\|h_x\|_{\alpha,N} \leq \sum_{\beta=0}^{\alpha} \|f\|_{\beta,N} \|g\|_{\alpha-\beta,0}$

Definition 1.1.17. Let $f, g \in \mathcal{S}$. We define the **convolution of** f **and** g, denoted $f * g$ by

$$f * g(x) = \int_{\mathbb{R}} f(x - y)g(y)dm(y)$$

Exercise 1.1.18. Let $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$.

Proof. □

Exercise 1.1.19. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-x^2}$. Then $f \in \mathcal{S}$.

Proof. meh... □

Exercise 1.1.20. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & x \in (-1, 1) \\ 0 & x \notin (-1, 1) \end{cases}$$

Then $f \in \mathcal{S}$.

Proof. meh... □

Exercise 1.1.21. Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Then for each $\epsilon > 0$, there exists $f \in \mathcal{S}$ such that $\chi_{[a,b]} \leq f \leq \chi_{[a-\epsilon,b+\epsilon]}$.

Proof. Set $f(x) =$ □

Exercise 1.1.22. Let $f \in \mathcal{S}$. Define

$$Then$$

1.2. **The Fourier Transform on $\mathcal{S}$.**

Definition 1.2.1. Let $f \in \mathcal{S}$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R} \to \mathbb{C}$, by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x)$$

Exercise 1.2.2. Let $f \in \mathcal{S}$. Then $\hat{f} \in C_b(\mathbb{R})$.

Proof. Since $f \in \mathcal{S}$, $f \in L^1(m)$. Then for each $\xi \in \mathbb{R}$,

$$\begin{aligned} |\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x) \right| \\ &\leq \int_{\mathbb{R}} |e^{-i\xi x} f(x)|\, dm(x) \\ &= \int_{\mathbb{R}} |f(x)|\, dm(x) \\ &= \|f\|_1 \end{aligned}$$

So f is bounded. Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. Suppose that $\xi_n \to \xi$. Define $(\phi_n)_{n\in\mathbb{N}} \subset L^1(m)$ and $\phi \in L^1(m)$ by $\phi_n(x) = e^{-i\xi_n x} f(x)$ and $\phi(x) = e^{-i\xi x} f(x)$. Then $\phi_n \xrightarrow{\text{p.w.}} \phi$ and for each $n \in \mathbb{N}$,

$$\begin{aligned} |\phi_n| &= |f| \\ &\in L^1(m) \end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned} \hat{f}(\xi_n) &= \int_{\mathbb{R}} e^{-i\xi_n x} f(x)\, dm(x) \\ &= \int_{\mathbb{R}} \phi_n\, dm \\ &\to \int_{\mathbb{R}} \phi\, dm \\ &= \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x) \\ &= \hat{f}(\xi) \end{aligned}$$

So $\hat{f}$ is continuous. Hence $\hat{f} \in C_b(\mathbb{R})$. □

Definition 1.2.3. We define the **Fourier transform on** $\mathcal{S}$, denoted $\mathcal{F} : \mathcal{S} \to C_b(\mathbb{R})$, by

$$\mathcal{F}(f) = \hat{f}$$

Exercise 1.2.4. We have that $\mathcal{F} : \mathcal{S} \to C_b(\mathbb{R})$ is linear.

Proof. Let $f, g \in \mathcal{S}$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\mathcal{F}(f+\lambda g) &= \int_{\mathbb{R}} e^{-i\xi x}[f(x)+\lambda g(x)]\,dm(x)\\ &= \int_{\mathbb{R}} e^{-i\xi x}f(x)+\lambda e^{-i\xi x}g(x)\,dm(x)\\ &= \int_{\mathbb{R}} e^{-i\xi x}f(x)\,dm(x)+\lambda\int_{\mathbb{R}} e^{-i\xi x}g(x)\,dm(x)\\ &= \mathcal{F}(f)+\lambda\mathcal{F}(g)\end{aligned}$$

□

Exercise 1.2.5. Let $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}^0$. Then

(1) $\mathcal{F}(X^\alpha f) = (-1)^\alpha D^\alpha \mathcal{F}(f)$
(2) $\mathcal{F}(D^\alpha f) = X^\alpha \mathcal{F}(f)$

Proof.

(1) The claim is clear for $\alpha = 0$. Suppose that $\alpha > 0$ and that the claim is true for $\alpha - 1$ so that $\mathcal{F}(X^{\alpha-1}f) = (-1)^{\alpha-1}D^{\alpha-1}\mathcal{F}(f)$. Define $\phi : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $\phi(\xi, x) = e^{-i\xi x}x^{\alpha-1}f(x)$. Then for each $\xi, x \in \mathbb{R}$,

$$\begin{aligned}|\partial_\xi\phi(\xi,x)| &= |-ixe^{-i\xi x}x^{\alpha-1}f(x)|\\ &= |x^\alpha f(x)|\\ &= |(X^\alpha f)(x)|\end{aligned}$$

Since $X^\alpha f \in \mathcal{S} \subset L^1$, we may switch the order of differentiation and integration to obtain

$$\begin{aligned}\mathcal{F}(X^\alpha f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x}x^\alpha f(x)dm(x)\\ &= \int_{\mathbb{R}} i\partial_\xi\left[e^{-i\xi x}x^{\alpha-1}f(x)\right]dm(x)\\ &= i\partial_\xi\left[\int_{\mathbb{R}} e^{-i\xi x}x^{\alpha-1}f(x)dm(x)\right]\\ &= i\partial_\xi\mathcal{F}(X^{\alpha-1}f)(\xi)\\ &= -D\mathcal{F}(X^{\alpha-1}f)(\xi)\\ &= (-1)^\alpha D^\alpha\mathcal{F}(f)(\xi)\end{aligned}$$

So the claim is true for α.

(2) The claim is clear for $\alpha = 0$. Suppose that $\alpha > 0$ and that the claim is true for $\alpha - 1$ so that $\mathcal{F}(D^{\alpha-1}f) = X^{\alpha-1}\mathcal{F}(f)$. Then integration by parts yields

$$\begin{aligned}\mathcal{F}(D^\alpha f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x}[-i\partial_x D^{\alpha-1}f(x)]\, dm(x) \\ &= -\int_{\mathbb{R}} -i\xi e^{-i\xi x}[-iD^{\alpha-1}f(x)]\, dm(x) \\ &= \xi \int_{\mathbb{R}} e^{-i\xi x} D^{\alpha-1}f(x)\, dm(x) \\ &= X\mathcal{F}(D^{\alpha-1}f)(\xi) \\ &= X^\alpha \mathcal{F}(f)(\xi)\end{aligned}$$

So the claim is true for α.

□

Exercise 1.2.6. There exists $C > 0$ such that for each $f \in \mathcal{S}$, $\|\hat{f}\|_{0,0} \leq C\|f\|_{0,2}$.
Hint: Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

Proof. Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

Let $f \in \mathcal{S}$. Let $\xi \in \mathbb{R}$. Then

$$\begin{aligned}|\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x) \right| \\ &\leq \int_{\mathbb{R}} |f(x)|\, dm(x) \\ &= \int_{\mathbb{R}} \frac{(1+|x|)^2|f(x)|}{(1+|x|)^2}\, dm(x) \\ &\leq \|f\|_{0,2} \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x) \\ &= C\|f\|_{0,2}\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\|\hat{f}\|_{0,0} \leq \|f\|_{0,2}$. □

Exercise 1.2.7. Let $a, b \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Then $(a+b)^N \leq 2^{N-1}(a^N + b^N)$.
Hint: Jensen's inequality

Proof. Jensen's inequality implies that

$$\begin{aligned}2^{-N}(a+b)^N &= \left(\frac{a}{2} + \frac{b}{2}\right)^N \\ &\leq \left(\frac{a^N}{2} + \frac{b^N}{2}\right) \\ &= 2^{-1}(a^N + b^N)\end{aligned}$$

So $(a+b)^N \leq 2^{N-1}(a^N + b^N)$. $\square$

Exercise 1.2.8. We have that $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$ and $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous.

Proof. Let $f \in \mathcal{S}$ and $\alpha, N \in \mathbb{N}_0$. Then the previous exercise implies that for each $\xi \in \mathbb{R}$,

$$\begin{aligned}
\xi^N \partial_\xi^\alpha \mathcal{F}(f)(\xi) &= (-i)^\alpha X^N D^\alpha \mathcal{F}(f)(\xi) \\
&= i^\alpha X^N \mathcal{F}(X^\alpha f)(\xi) \\
&= i^\alpha \mathcal{F}(D^N X^\alpha f)(\xi)
\end{aligned}$$

Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

as in the previous exercise. Since $\mathcal{F}(X^\alpha f)$, $\mathcal{F}(D^N X^\alpha f) \in C_b(\mathbb{R})$, we have that

$$\begin{aligned}
\|\mathcal{F}(f)\|_{\alpha,N} &= \sup_{\xi \in \mathbb{R}} \left[(1+|\xi|)^N |\partial_\xi^\alpha \mathcal{F}(f)(\xi)|\right] \\
&\leq \sup_{\xi \in \mathbb{R}} \left[2^{N-1}(1+|\xi|^N)|\partial_\xi^\alpha \mathcal{F}(f)(\xi)|\right] \\
&= \sup_{\xi \in \mathbb{R}} \left[|2^{N-1}\partial_\xi^\alpha \mathcal{F}(f)(\xi)| + |2^{N-1}\xi^N \partial_\xi^\alpha \mathcal{F}(f)(\xi)|\right] \\
&= \sup_{\xi \in \mathbb{R}} \left[|\mathcal{F}(2^{N-1}X^\alpha f)(\xi)| + |\mathcal{F}(2^{N-1}D^N X^\alpha f)(\xi)|\right] \\
&\leq \|\mathcal{F}(2^{N-1}X^\alpha f)\|_{0,0} + \|\mathcal{F}(2^{N-1}D^N X^\alpha f)\|_{0,0} \\
&\leq C2^{N-1}\|X^\alpha f\|_{0,2} + C2^{N-1}\|D^N X^\alpha f\|_{0,2} \qquad < \infty
\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\mathcal{F}(f) \in \mathcal{S}$ and since $f \in \mathcal{S}$ is arbitrary, $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$. Suppose that $f_n \to 0$. Since $X, D : \mathcal{S} \to \mathcal{S}$ are continuous, $X^\alpha f_n \to 0$ and $D^N X^\alpha f_n \to 0$. Therefore, $\|X^\alpha f_n\|_{0,2} \to 0$ and $\|D^N X^\alpha f_n\|_{0,2} \to 0$. From above, we see that

$$\begin{aligned}
\|\mathcal{F}(f_n)\|_{\alpha,N} &\leq C2^{N-1}\|X^\alpha f_n\|_{0,2} + C2^{N-1}\|D^N X^\alpha f_n\|_{0,2} \\
&\to 0
\end{aligned}$$

Hence $\mathcal{F}(f_n) \to 0$ and $\mathcal{F}$ is continuous. $\square$

Exercise 1.2.9. Define $f \in \mathcal{S}$ by $f(x) = e^{-x^2/2}$. Then $\mathcal{F}(f) = \sqrt{2\pi} f$.

Proof. Note that for each $\xi \in \mathbb{R}$,

$$\begin{aligned}
\mathcal{F}(Df)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} i x e^{-x^2/2}\, dm(x) \\
&= -\int_{\mathbb{R}} \partial_\xi \left[e^{-i\xi x} e^{-x^2/2}\right] dm(x) \\
&= -\partial_\xi \mathcal{F}(f)(\xi)
\end{aligned}$$

A previous exercise implies that $\mathcal{F}(Df) = X\mathcal{F}(f)$. So for each $\xi \in \mathbb{R}$, $\partial_\xi \hat{f}(\xi) = -\xi \hat{f}(\xi)$. Define $g \in \mathbb{C}^\infty(\mathbb{R})$ by $g(\xi) = e^{\xi^2/2}$. Then

$$\begin{aligned}\partial_\xi(\hat{f}g) &= (\partial_\xi \hat{f})g + \hat{f}(\partial_\xi g)\\ &= 0\end{aligned}$$

So there exists $C \in \mathbb{R}$ such that $\hat{f}g = C$. Hence for each $\xi \in \mathbb{R}$,

$$\begin{aligned}\hat{f}(\xi) &= Ce^{-\xi^2/2}\\ &= Cf(\xi)\end{aligned}$$

Therefore,

$$\begin{aligned}C &= Cf(0)\\ &= \hat{f}(0)\\ &= \int_{\mathbb{R}} e^{-x^2/2}\, dm(x)\\ &= \sqrt{2\pi}\end{aligned}$$

So $\hat{f} = \sqrt{2\pi} f$. □

Definition 1.2.10. content...

1.3. **The Fourier Transform on $\mathcal{M}(\mathbb{R})$.**

Note 1.3.1. Recall that

$$\mathcal{M}(\mathbb{R}) = \{\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{C} : \mu \text{ is a complex measure}\}$$

Definition 1.3.2. Let $\mu \in \mathcal{M}(\mathbb{R})$. We define the **Fourier transform of** μ, denoted $\hat{\mu} : \mathbb{R} \to \mathbb{C}$, by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x)$$

Exercise 1.3.3. Let $\mu \in \mathcal{M}(\mathbb{R})$. Then Then $\hat{\mu} : \mathbb{R} \to \mathbb{C}$ is bounded.

Proof. Let $\xi \in \mathbb{R}$.

$$\begin{aligned} |\hat{\mu}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) \right| \\ &\leq \int_{\mathbb{R}} |e^{-i\xi x}|\, d|\mu|(x) \\ &= |\mu|(\mathbb{R}) \end{aligned}$$

So $\hat{\mu}$ is bounded. □

Exercise 1.3.4. Let $\mu \in \mathcal{M}(\mathbb{R})$. Then $\hat{\mu} \in C_b(\mathbb{R})$.

Proof. Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. Define $(f_n)_{n\in\mathbb{N}} \subset L^1(\mu)$ and $f \in L^1(\mu)$ by $f_n(x) = e^{-i\xi_n x}$ and $f(x) = e^{-i\xi x}$. Suppose that $\xi_n \to \xi$. Then $f_n \xrightarrow{\text{p.w.}} f$ and for each $n \in N$ and $x \in \mathbb{R}$,

$$\begin{aligned} |f_n(x)| &= |e^{-i\xi_n x}| && = 1 \\ &\in L^1(|\mu|) \end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned} |\hat{\mu}(\xi_n) - \hat{\mu}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi_n x}\, d\mu(x) - \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) \right| \\ &= \left| \int_{\mathbb{R}} e^{-i\xi_n x} - e^{-i\xi x}\, d\mu(x) \right| \\ &\leq \int_{\mathbb{R}} |e^{-i\xi_n x} - e^{-i\xi x}|\, d|\mu|(x) \\ &\to 0 \end{aligned}$$

So $\hat{\mu} : \mathbb{R} \to \mathbb{C}$ is continuous. Hence $\hat{\mu} \in C_b(\mathbb{R})$. □

Definition 1.3.5. Let X be a real normed vector space. We define $\mathcal{F} : \mathcal{M}(\mathbb{R}) \to C_b(\mathbb{R})$ by

$$\mathcal{F}(\mu) = \hat{\mu}$$

Exercise 1.3.6. Let X be a real normed vector space. Then $\mathcal{F} : \mathcal{M}(\mathbb{R}) \to C_b(\mathbb{R})$ is linear.

Proof. Let $\mu, \nu \in \mathcal{M}(\mathbb{R})$ and $\xi \in \mathbb{R}$. Then

$$\begin{aligned}\mathcal{F}[\mu+\nu](\xi) &= \int_{\mathbb{R}} e^{-i\xi x}\, d[\mu+\nu](x) \\ &= \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) + \int_{\mathbb{R}} e^{-i\xi x}\, d\nu(x) \\ &= \mathcal{F}[\mu](\xi) + \mathcal{F}[\nu](\xi)\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\mathcal{F}(\mu+\nu) = \mathcal{F}(\mu) + \mathcal{F}(\nu)$ and $\mathcal{F}$ is linear. $\square$

Exercise 1.3.7. Let X be a real normed vector space. If X is separable, then $\mathcal{F}$ is injective.

Proof. Suppose that X is separable. Let $\mu \in \mathcal{M}(X)$. Suppose that $\mu \in \ker \mathcal{F}$. Then $\hat{\mu} = 0$ and for each $\phi \in X^*$,

$$\begin{aligned}0 &= \hat{\mu}(\phi) \\ &= \int_X e^{-i\phi(x)}\, d\mu(x) \\ &= \int_{\mathbb{R}} e^{-ix}\, d[\phi_*\mu](x)\end{aligned}$$

$\square$

Exercise 1.3.8. Let X be a real normed vector space. Then $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$.

Proof. For $\mu \in \mathcal{M}(X)$ and $\phi \in X^*$, we have that

$$\begin{aligned}|\mathcal{F}[\mu](\phi)| &= \left| \int_X e^{-i\phi(x)}\, d\mu(x) \right| \\ &\leq \int_X |e^{-i\phi(x)}|\, d|\mu|(x) \\ &= |\mu|(X) \\ &= \|\mu\|\end{aligned}$$

Hence

$$\begin{aligned}\|\mathcal{F}(\mu)\| &= \sup_{\phi \in X^*} |\mathcal{F}[\mu](\phi)| \\ &\leq \|\mu\|\end{aligned}$$

which implies that $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$. $\square$

2. Fourier Analysis on $\mathbb{R}^n$

2.1. Schwartz Space.

Definition 2.1.1. Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

(1) $\langle x, y \rangle = \sum_j x_j y_j$
(2) $|x| = \langle x, x \rangle^{1/2}$
(3) $|\alpha| = \alpha_1 + \cdots + \alpha_n$
(4) $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
(5) $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

Definition 2.1.2. Let $f \in C^\infty(\mathbb{R}^n)$,$\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

We define Schwartz space, denoted $\mathcal{S}$, by

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n,\, N \in \mathbb{N}_0,\, \|f\|_{\alpha,N} < \infty\}$$

Exercise 2.1.3. For each $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}$, $\alpha \in \mathbb{N}_0^n$. Then there exists $C \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^{-1}$$

Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = (1 + |x|^2)^{-1}$. Then $g \in L^1(\mathbb{R}^n)$ which implies that $\partial^\alpha f \in L^1(\mathbb{R}^n)$. $\square$

Definition 2.1.4.

2.2. **The Convolution.**

Definition 2.2.1. Let $f, g \in L^0(\mathbb{R}^n)$. If for a.e. $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) < \infty$$

we define the **convolution of f with g**, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$, by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

Exercise 2.2.2. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = f(x-y)g(y)$. Tonelli's theorem implies that,

$$\begin{aligned}
\int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}^n} |g(y)| \left[\int_{\mathbb{R}^n} |f(x-y)|dm(y) \right] dm(x) \\
&= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dm(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

Then $h \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Fubini's theorem implies that $f * g \in L^1(\mathbb{R}^n)$. Clearly

$$\begin{aligned}
\|f * g\|_1 &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 \\
&\leq \|f\|_1 \|g\|_1
\end{aligned}$$

□

Exercise 2.2.3. Let $f, g, h \in L^1(\mathbb{R}^n)$. Then $(f * g) * h = f * (g * h)$.
Hint: use the substitution $z \mapsto z - y$

Proof. Let $x \in \mathbb{R}^n$. Then using the substitution $z \mapsto z - y$ and Fubini's theorem, we obtain

$$\begin{aligned}
(f * g) * h(x) &= \int f * g(x-y)h(y)dm(y) \\
&= \int \left[\int f(x-y-z)g(z)dm(z) \right] h(y)dm(y) \\
&= \int \left[\int f(x-z)g(z-y)dm(z) \right] h(y)dm(y) \\
&= \int \left[\int f(x-z)g(z-y)h(y)dm(z) \right] dm(y) \\
&= \int \left[\int f(x-z)g(z-y)h(y)dm(y) \right] dm(z) \\
&= \int f(x-z) \left[\int g(z-y)h(y)dm(y) \right] dm(z) \\
&= \int f(x-z)g * h(z)dm(z) \\
&= f * (g * h)(z)
\end{aligned}$$

So $(f * g) * h = f * (g * h)$. □

Exercise 2.2.4. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g = g * f$.

Proof. Let $x \in \mathbb{R}^n$. Using the transformation $y \mapsto x - y$, we obtain that

$$\begin{aligned}
f * g(x) &= \int f(x-y)g(y)dm(y) \\
&= \int f(y)g(x-y)dm(y) \\
&= \int g(x-y)f(y)dm(y) \\
&= g * f(x)
\end{aligned}$$

So $f * g = g * f$. □

Note 2.2.5. To summarize, $(L^1(\mathbb{R}^n), *)$ is a commutative Banach algebra.

Exercise 2.2.6. Young's Inequality:
Let $p \in [1, \infty]$, $f \in L^1$ and $g \in L^p$. Then $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Proof. Define $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $K(x, y) = f(x - y)$. Since for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
\int |K(x,y)|dm(x) &= \int |K(x,y)|dm(y) \\
&= \|f\|_p
\end{aligned}$$

an exercise in section 5.1 of [4] implies that $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. □

Exercise 2.2.7. Let $p, q \in [1, \infty]$ be conjugate, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then

(1) for each $x \in \mathbb{R}^n$, $f * g(x)$ exists.
(2) $\|f * g\|_u \leq \|f\|_p \|g\|_q$

(3)

Proof. (1) Let $x \in \mathbb{R}^n$. Holder's inequality implies that

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \leq \|f\|_p \|g\|_q$$

Then $f * g(x)$ exists.

(2) Let $x \in \mathbb{R}^n$. Then in part (1) we showed that

$$\begin{aligned}|f * g(x)| &= \left| \int_{\mathbb{R}^n} f(x-y)g(y)dm(y) \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \\ &\leq \|f\|_p \|g\|_q\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, $\|f * g\|_u \leq \|f\|_p \|g\|_q$.

(3)

□

Exercise 2.2.8. Let $f \in L^1(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $g \in C^k(\mathbb{R}^n)$. Suppose that for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $\partial^\alpha g \in L^\infty$. Then for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $f * g \in C^k$ and

$$\partial^\alpha (f * g) = f * \partial^\alpha g$$

Proof. Let $\alpha \in \mathbb{N}_0^n$. Suppose that $|\alpha| = 1$. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x,y) = g(x-y)f(y)$. Young's inequality imples that for a.e. $x \in \mathbb{R}^n$, $h(x, \cdot) \in L^1(m)$. For each $y \in \mathbb{R}^n$, $\partial^\alpha h(\cdot, y) = \partial^\alpha g(\cdot - y)f(y)$ and for each $x, y \in \mathbb{R}^n$, $|\partial^\alpha h(x,y)| \leq \|\partial^\alpha g\|_\infty |f(y)| \in L^1(\mathbb{R}^n)$. An exercise in section 3.3 of [4] implies that for a.e. $x \in \mathbb{R}^n$, $\partial^\alpha (g * f)(x)$ exists and

$$\begin{aligned}\partial^\alpha (f * g)(x) &= \partial^\alpha (g * f)(x) \\ &= \partial^\alpha \int_{\mathbb{R}^n} h(x,y)dm(y) \\ &= \int_{\mathbb{R}^n} \partial^\alpha g(x-y)f(y)dm(y) \\ &= (\partial^\alpha g) * f(x) \\ &= f * (\partial^\alpha g)(x)\end{aligned}$$

Now proceed by induction on $|\alpha|$. □

2.3. **The Fourier Transform.**

Definition 2.3.1.

Exercise 2.3.2. Let $\phi : \mathbb{R} \to S^1$ be a measurable homomorphism.

(1) Then $\phi \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$

(2) Define
$$c = \left[\int_{(0,a]} \phi dm \right]^{-1}$$
Then For each $x \in \mathbb{R}$,
$$\phi(x) = c \int_{(x,x+a]} \phi dm$$

(3) $\phi \in C^\infty(\mathbb{R})$ and $\phi' = c(\phi(a) - 1)\phi$

(4) Define $b = c(\phi(a) - 1)$ and $g \in C^\infty(\mathbb{R})$ by $g(x) = e^{-bx}\phi(x)$. Then g is constant and there exists $\xi \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i \xi x}$.

Proof.

(1) Let $K \subset \mathbb{R}$ be compact. Then
$$\int_K |\phi| dm = m(K) < \infty$$
So $\phi \in L^1_{\text{loc}}(\mathbb{R})$. For the sake of contradiction, suppose that for each $a > 0$,
$$\int_{(0,a]} \phi dm = 0$$
Then the FTC implies that $\phi = 0$ a.e. on $[0, \infty)$, which is a contradiction. So there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$

(2) For $x \in \mathbb{R}$,
$$\begin{aligned}
\phi(x) &= c \int_{(0,a]} \phi(x)\phi(t) dm(t) \\
&= c \int_{(0,a]} \phi(x + t) dm(t) \\
&= c \int_{(x,x+a]} \phi dm
\end{aligned}$$

(3) Part (2) and the FTC imply that ϕ is continuous. Let $d \in \mathbb{R}$. Define $f_d \in C((d, \infty))$ by
$$f_d(x) = \int_{(d,x]} \phi dm$$

Since ϕ is continuous, the FTC implies that f_d is differentiable and for each $x > d$ $f_d'(x) = \phi(x)$. Part (2) implies that for each $x > d$,

$$\begin{aligned}\phi(x) &= c\int_{(x,x+a]} \phi dm \\ &= c(f_d(x+a) - f_d(x))\end{aligned}$$

So for each $x > d$, ϕ is differentiable at x and

$$\begin{aligned}\phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x)\end{aligned}$$

Since $d \in \mathbb{R}$ is arbitrary, ϕ is differentiable and $\phi' = c(\phi(a) - 1)\phi$. This implies that $\phi \in C^\infty(\mathbb{R})$.

(4) Let $x \in \mathbb{R}$. Then

$$\begin{aligned}g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0\end{aligned}$$

So $g' = 0$ and g is constant. Hence there exists $k \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = ke^{bx}$. Since $\phi(0) = 1$, $k = 1$. Since $|\phi| = 1$, there exists $\xi \in \mathbb{R}$ such that $b = 2\pi i\xi$.

$\square$

Note 2.3.3. To summarize, for each measurable homomorphism $\phi : \mathbb{R} \to S^1$, there exists $\xi \in \mathbb{R}$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

Exercise 2.3.4. Let $\phi : \mathbb{R}^n \to S^1$ be a measurable homomorphism. Then there exists $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi(x) = e^{2\pi i\langle \xi, x\rangle}$.

Proof. When done in the category of measurable groups, an exercise in the section on direct products of groups of [**?**] implies that there exist measurable homomorphism $(\phi_j)_{j=1}^n \subset (S^1)^{\mathbb{R}}$ such that $\phi = \bigotimes_{j=1}^n \phi_j$. The previous exercise imples that there exist $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi_j(x_j) = e^{2\pi i\xi_j x_j}$. Then for each $x \in \mathbb{R}^n$,

$$\begin{aligned}\phi(x) &= \prod_{j=1}^n \phi_j(x_j) \\ &= \prod e^{2\pi i\xi_j x_j} \\ &= e^{2\pi i \sum\limits_{j=1}^n \xi_j x_j} \\ &= e^{2\pi i\langle \xi, x\rangle}\end{aligned}$$

$\square$

Definition 2.3.5. Let $f \in L^1(\mathbb{R}^n)$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle \xi, x\rangle}dm(x)$$

3. Fourier Analysis on LCA Groups

3.1. The Convolution.

Note 3.1.1. For the remainder of the section, we fix a locally compact abelian group G and a Haar measure μ on G.

Definition 3.1.2. Let $f, g \in L^1(\mu)$. We define the **convolution of** f **with** g, denoted $f * g : G \to \mathbb{C}$, by

$$f * g(x) = \int_X f(x-y)g(y)d\mu(y)$$

Exercise 3.1.3. Let $f, g \in L^1(\mu)$. Then $f * g \in L^1(\mu)$.

Proof. By Tonelli's theorem,

$$\begin{aligned}
\int_X |f * g| d\mu &\leq \int_X \left[\int_X |f(x-y)g(y)| d\mu(y) \right] d\mu(x) \\
&= \int_X |g(y)| \left[\int_X |f(x-y)| d\mu(y) \right] d\mu(x) \\
&= \|f\|_1 \int_X |g(y)| d\mu(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

□

4. Fourier Analysis on Banach Spaces

References

[1] Introduction to Algebra
[2] Introduction to Analysis
[3] Introduction to Fourier Analysis
[4] Introduction to Measure and Integration