

INTRODUCTION TO DIFFERENTIAL GEOMETRY

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1. FUNDAMENTAL DEFINITIONS AND RESULTS

1.1. Set Theory.

Definition 1.1.1. Let $\{A_i\}_{i \in I}$ be a collection of sets. The **disjoint union** of $\{A_i\}_{i \in I}$, denoted $\coprod_{i \in I} A_i$, is defined by

$$\coprod_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$$

Note 1.1.2. In these notes, we will identify $\{i\} \times A_i$ and A_i .

Definition 1.1.3. Let $\{A_i\}_{i \in I}$ be a collection of sets and $\sigma : I \rightarrow \coprod_{i \in I} A_i$. Then σ is said to be a **section of $\coprod_{i \in I} A_i$** if for each $i \in I$, $\sigma(i) \in A_i$.

2. CALCULUS

2.1. Differentiation.

Definition 2.1.1. Let $n \geq 1$. For $i = 1, \dots, n$, define $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$ by $x^i(a_1, \dots, a_n) = a_i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on \mathbb{R}^n** .

Definition 2.1.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to x^i at a** if

$$\lim_{h \rightarrow 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If f is differentiable with respect to x^i at a , we define the **partial derivative of f with respect to x^i at a** , denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \left. \frac{\partial}{\partial x^i} \right|_a f$$

to be the limit above.

Definition 2.1.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$. Then f is said to be **differentiable with respect to x^i** if for each $a \in U$, f is differentiable with respect to x^i at a .

Exercise 2.1.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i \partial x^j}$ and $\frac{\partial^2 f}{\partial x^j \partial x^i}$ exist and are continuous at a . Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(a) = \frac{\partial^2 f}{\partial x^j \partial x^i}(a)$$

Proof. □

Definition 2.1.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \dots, i_k \in \{1, \dots, n\}$, $\frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_k}}$ exists and is continuous on U .

Definition 2.1.6. Let $U \subset \mathbb{R}^n$, $f : U \rightarrow \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \rightarrow \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Definition 2.1.7. Let $U \subset \mathbb{R}^n$ and $p \in U$. Then U is said to be **star-shaped** if for each $q \in U$, $\{p + t(q - p) : 0 \leq t \leq 1\} \subset U$.

Theorem 2.1.1. (Taylor's Theorem) Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $f \in C^\infty(U)$. Suppose that U is star-shaped with respect to p . Then there exist $g_1, \dots, g_n \in C^\infty(U)$ such that for each $x \in U$,

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g_i(x)$$

and for each $i \in \{1, \dots, n\}$,

$$g_i(p) = \frac{\partial f}{\partial x^i}(p)$$

Proof. Let $x \in U$. Since U is star-shaped with respect to p , $\{p + t(x - p) : 0 \leq t \leq 1\} \subset U$. By the chain rule,

$$\frac{d}{dt} \left[f(p + t(x - p)) \right] = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p + t(x - p))(x^i - p^i)$$

Integrating both sides with respect to t from 0 to 1, we obtain

$$f(x) - f(p) = \sum_{i=1}^n (x^i - p^i) \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p)) dt$$

For $i \in \{1, \dots, n\}$, define $g_i \in C^\infty(U)$ by

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p)) dt$$

Then for each $i \in \{1, \dots, n\}$,

$$g_i(p) = \frac{\partial f}{\partial x^i}(p)$$

□

2.2. Smooth Maps.

Definition 2.2.1. Let $U \subset \mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^m$. Let x^1, \dots, x^n be the standard coordinate functions on \mathbb{R}^n and y_1, \dots, y_m be the standard coordinate functions on \mathbb{R}^m . For $i \in \{1, \dots, m\}$, we define the **i th component of F** , denoted $F^i : U \rightarrow \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \dots, F_m)$

Definition 2.2.2. Let $U \subset \mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \dots, m\}$, the i th component of F , $F^i : U \rightarrow \mathbb{R}$, is smooth.

Definition 2.2.3. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \rightarrow V$. Then F is said to be a **diffeomorphism** if F is a homeomorphism and F, F^{-1} are smooth.

Definition 2.2.4. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \rightarrow \mathbb{R}^m$. We define the **Jacobian of F at p** , denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p) \right)_{i,j} = \frac{\partial F^i}{\partial x^j}$$

Exercise 2.2.5. Inverse Function Theorem:

Let $U, V \subset \mathbb{R}^n$ be open and $F : U \rightarrow V$.

Exercise 2.2.6. Let $U, V \subset \mathbb{R}^n$ and $F : U \rightarrow V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \rightarrow F(N)$ is a diffeomorphism

Proof. content...

□

2.3. Topology.

Definition 2.3.1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \rightarrow Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 2.3.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \rightarrow Y$. Then f is said to be a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Definition 2.3.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \rightarrow Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 2.3.1. *Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \not\cong \mathbb{R}^n$*

3. MULTILINEAR ALGEBRA

3.1. (r, s) -Tensors.

Definition 3.1.1. Let V_1, \dots, V_k, W be vector spaces and $\alpha : \prod_{i=1}^n V_i \rightarrow W$. Then α is said to be **multilinear** if for each $i \in \{1, \dots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \dots, v_k \in V$,

$$\alpha(v_1, \dots, v_i + cv, \dots, v_k) = \alpha(v_1, \dots, v_i, \dots, v_k) + c\alpha(v_1, \dots, v, \dots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{ \alpha : \prod_{i=1}^n V_i \rightarrow W : \alpha \text{ is multilinear} \right\}$$

Note 3.1.2. For the remainder of this section we let V denote an n -dimensional vector space with basis $\{e^1, \dots, e^n\}$ with dual space V^* and dual basis $\{\epsilon_1, \dots, \epsilon_n\}$ defined by $\epsilon^i(e^j) = \delta_{i,j}$. We identify V with V^{**} by the isomorphism $V \rightarrow V^{**}$ defined by $v \mapsto \hat{v}$ where $\hat{v}(\alpha) = \alpha(v)$ for each $\alpha \in V^*$.

Definition 3.1.3. Let $\alpha : (V^*)^r \times V^s \rightarrow \mathbb{R}$. Then α is said to be an (r, s) -tensor on V if $\alpha \in L(\underbrace{V^*, \dots, V^*}_r, \underbrace{V, \dots, V}_s; \mathbb{R})$. The set of all (r, s) -tensors on V is denoted $T_s^r(V)$.

When $r = s = 0$, we set $T_s^r = \mathbb{R}$.

Exercise 3.1.4. We have that $T_s^r(V)$ is a vector space.

Proof. Clear. □

Exercise 3.1.5. Under the identification of V with V^{**} as noted above, we have that $V = T_0^1(V)$.

Proof. By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T_0^1(V) \end{aligned}$$

□

Definition 3.1.6. Let $\alpha \in T_{s_1}^{r_1}(V)$ and $\beta \in T_{s_2}^{r_2}(V)$. We define the **tensor product of α with β** , denoted $\alpha \otimes \beta \in T_{s_1+s_2}^{r_1+r_2}(V)$, by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$.

When $r_1 = s_1 = r_2 = s_2 = 0$ (so that $\alpha, \beta \in \mathbb{R}$), we set $\alpha \otimes \beta = \alpha\beta$.

Definition 3.1.7. We define the **tensor product**, denoted $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

Exercise 3.1.8. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is well defined.

Proof. Tedious but straightforward. □

Exercise 3.1.9. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is associative.

Proof. Let $\alpha \in T_{s_1}^{r_1}(V)$, $\beta \in T_{s_2}^{r_2}(V)$ and $\gamma \in T_{s_3}^{r_3}(V)$. Then for each $u^* \in (V^*)^{r_1}$, $v^* \in (V^*)^{r_2}$, $w^* \in (V^*)^{r_3}$, $u \in V^{s_1}$, $v \in V^{s_2}$, $w \in V^{s_3}$,

$$\begin{aligned} (\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v) \gamma(w^*, w) \\ &= [\alpha(u^*, u) \beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(u^*, u) [\beta(v^*, v) \gamma(w^*, w)] \\ &= \alpha(u^*, u) (\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w) \end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

Exercise 3.1.10. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is bilinear.

Proof.

(1) Linearity in the first argument:

Let $\alpha, \beta \in T_{s_1}^{r_1}(V)$, $\gamma \in T_{s_2}^{r_2}(V)$, $\lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned} [(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v) \gamma(w^*, w) \\ &= [\alpha(v^*, v) + \lambda\beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(v^*, v) \gamma(w^*, w) + \lambda\beta(v^*, v) \gamma(w^*, w) \\ &= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w) \end{aligned}$$

So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

(2) Linearity in the second argument:

Similar to (1).

□

Definition 3.1.11.

- (1) Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1, \dots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length k** . Recall that $\#\mathcal{I}_{\otimes k} = n^k$.
- (2) Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length k** . Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

Note 3.1.12. For the remainder of this section we will write \mathcal{I}_k in place of $\mathcal{I}_{\otimes k}$.

Definition 3.1.13. Let $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k\}$.

- (1) Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by

$$\epsilon^I = (\epsilon^{i_1}, \dots, \epsilon^{i_k})$$

and

$$e^I = (e^{i_1}, \dots, e^{i_k})$$

(2) Define $e^{\otimes I} \in T_0^k(V)$ and $\epsilon^{\otimes I} \in T_k^0(V)$ by

$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$

and

$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

Exercise 3.1.14. Let $\alpha, \beta \in T_s^r(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v_1^*, \dots, v_r^* \in V^*$ and $v_1, \dots, v_s \in V$. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, write

$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$

and

$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$

Then

$$\begin{aligned} \alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s) \end{aligned}$$

So that $\alpha = \beta$. □

Exercise 3.1.15. Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K} \delta_{J,L}$.

Proof. Write $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$ and $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$. Then

$$\begin{aligned} e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K) \epsilon^{\otimes J}(e^L) \\ &= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r}) \epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\ &= \left[\prod_{m=1}^r e^{i_m}(\epsilon^{k_m}) \right] \left[\prod_{n=1}^s \epsilon^{j_n}(e^{l_n}) \right] \\ &= \left[\prod_{m=1}^r \delta_{i_m, k_m} \right] \left[\prod_{n=1}^s \delta_{j_n, l_n} \right] \\ &= \delta_{I,K} \delta_{J,L} \end{aligned}$$

□

Exercise 3.1.16. The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T_s^r(V)$ and $\dim T_s^r(V) = n^{r+s}$.

Proof. Let $(a_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a_J^I e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a_J^I = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T_s^r(V)$. For $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b_J^I = \beta(\epsilon^J, e^I)$. Define

$\mu = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b_J^I e^{\otimes I} \otimes \epsilon^{\otimes J} \in T_s^r(V)$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b_J^I = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. \square

3.2. k -Tensors.

Definition 3.2.1. Let $\alpha : V^k \rightarrow \mathbb{R}$. Then α is said to be a **k -tensor on V** if $\alpha \in T_k^0(V)$. We will write $T_k(V)$ in place of $T_k^0(V)$.

Definition 3.2.2. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \rightarrow \mathbb{R}$ by

$$\sigma\alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

The map $\alpha \mapsto \sigma\alpha$ is called the **permutation action** of S_k on $T_k(V)$

Exercise 3.2.3. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

- (1) Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V)$, $\sigma\alpha \in T_k(V)$.
- (2) Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
- (3) Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \dots, v_k \in V$,

$$\begin{aligned} (\tau\sigma)\alpha(v_1, \dots, v_k) &= \alpha(v_{\tau\sigma(1)}, \dots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \dots, v_k) \end{aligned}$$

□

Exercise 3.2.4. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \rightarrow T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \dots, v_k \in V$. Then

$$\begin{aligned} \sigma(c\alpha + \beta)(v_1, \dots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \dots, v_k) + \sigma\beta(v_1, \dots, v_k) \end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$.

□

Definition 3.2.5. Let $\alpha \in T_k(V)$. Then α is said to be **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$. and α is said to be **alternating** if for each $\sigma \in S_k$, $\sigma\alpha = \text{sgn}(\sigma)\alpha$. The set of symmetric k -tensors on V is denoted $\Xi_k(V)$ and the set of alternating k -tensors on V is denoted $\Lambda_k(V)$.

Definition 3.2.6. Define the **symmetric operator** $S : T_k(V) \rightarrow \Xi_k(V)$ by

$$S(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \rightarrow \Lambda_k(V)$ by

$$A(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma\alpha$$

Exercise 3.2.7.

- (1) For $\alpha \in T_k(V)$, $S(\alpha)$ is symmetric.
- (2) For $\alpha \in T_k(V)$, $A(\alpha)$ is alternating.

Proof.

(1) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
 \sigma S(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \tau \alpha \right] \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma \tau \alpha \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \tau \alpha \\
 &= S(\alpha)
 \end{aligned}$$

(2) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
 \sigma A(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \right] \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \sigma \tau \alpha \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\sigma) \text{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \text{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \text{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \\
 &= \text{sgn}(\sigma) A(\alpha)
 \end{aligned}$$

□

Exercise 3.2.8.

- (1) For $\alpha \in \Xi_k(V)$, $S(\alpha) = \alpha$.
- (2) For $\alpha \in \Lambda_k(V)$, $A(\alpha) = \alpha$.

Proof.

(1) Let $\alpha \in \Xi_k(V)$. Then

$$\begin{aligned}
 S(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} \alpha \\
 &= \alpha
 \end{aligned}$$

(2) Let $\alpha \in \Lambda_k(V)$. Then

$$\begin{aligned} A(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma \alpha \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)^2 \alpha \\ &= \alpha \end{aligned}$$

□

Exercise 3.2.9. The symmetric operator $S : T_k(V) \rightarrow \Xi_k(V)$ and the alternating operator $A : T_k(V) \rightarrow \Lambda_k(V)$ are linear.

Proof. Clear.

□

Definition 3.2.10. Let $\alpha \in \Lambda_k(V)$ and $\beta \in \Lambda_l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda_{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$.

Exercise 3.2.11. The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$ is bilinear.

Proof. Clear.

□

Exercise 3.2.12. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

- (1) $A(A(\alpha) \otimes \beta) = A(\alpha \otimes \beta)$
- (2) $A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+l}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \rightarrow S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

(1) Then

$$\begin{aligned}
A(A(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[A(\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) (\tau \alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau (\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \text{sgn}(\sigma\tau) \sigma\tau (\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= A(\alpha \otimes \beta)
\end{aligned}$$

(2) Similar to (1).

□

Exercise 3.2.13. The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$ and $\gamma \in \Lambda_m(V)$. Then

$$\begin{aligned}
 (\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \right] \wedge \gamma \\
 &= \frac{(k+l+m)!}{(k+l)!m!} A \left(\left[\frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \right] \otimes \gamma \right) \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} A(A(\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} A((\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes (\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes A(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes \frac{(l+m)!}{l!m!} A(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes (\beta \wedge \gamma)) \\
 &= \alpha \wedge (\beta \wedge \gamma)
 \end{aligned}$$

□

Exercise 3.2.14. Let $\alpha_i \in \Lambda_{k_i}(V)$ for $i = 1, \dots, m$. Then

$$\bigwedge_{i=1}^m \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} A \left(\bigotimes_{i=1}^m \alpha_i \right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} A(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i \right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A \left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A \left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} A \left(\bigotimes_{i=1}^{m_0-1} \alpha_i \right) \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left(A \left[\bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left(\bigotimes_{i=1}^{m_0+1} \alpha_i \right)
\end{aligned}$$

□

Exercise 3.2.15. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl . (Hint: inversion number)

Proof.

$$\begin{aligned}
N(\tau) &= \sum_{i=1}^l k \\
&= kl
\end{aligned}$$

Since $\text{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\text{sgn}(\tau) = (-1)^{kl}$.

□

Exercise 3.2.16. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \dots, v_{k+l} \in V$, we have that

$$\begin{aligned}
\sigma\tau(\beta \otimes \alpha)(v_1, \dots, v_l, v_{l+1}, \dots, v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}) \alpha(v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\
&= \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\
&= \alpha \otimes \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}, v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\
&= \sigma(\alpha \otimes \beta)(v_1, \dots, v_k, v_{1+k}, \dots, v_{l+k})
\end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned}
\beta \wedge \alpha &= \frac{(k+l)!}{k!l!} A(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau) \sigma\tau(\beta \otimes \alpha) \\
&= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha \otimes \beta) \\
&= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \\
&= \text{sgn}(\tau) \alpha \wedge \beta \\
&= (-1)^{kl} \alpha \wedge \beta
\end{aligned}$$

□

Exercise 3.2.17. Let $\alpha \in \Lambda_k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned}
\alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\
&= -\alpha \wedge \alpha
\end{aligned}$$

Thus $\alpha \wedge \alpha = 0$.

□

Exercise 3.2.18. (Fundamental Example) Let $\alpha_1, \dots, \alpha_m \in \Lambda_1(V)$ and $v_1, \dots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}
\left(\bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) &= m! A \left(\bigotimes_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) \\
&= m! \left[\frac{1}{m!} \sum_{\sigma \in S_m} \text{sgn}(\sigma) \sigma \left(\bigotimes_{i=1}^m \alpha_i \right) \right] (v_1, \dots, v_m) \\
&= \sum_{\sigma \in S_m} \text{sgn}(\sigma) \left(\bigotimes_{i=1}^m \alpha_i \right) (v_{\sigma(1)}, \dots, v_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
&= \det(\alpha_i(v_j))
\end{aligned}$$

□

Note 3.2.19. Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write \mathcal{I}_k in place of $\mathcal{I}_{\wedge k}$.

Definition 3.2.20. Let $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_k$.

Define $\epsilon^{\wedge I} \in \Lambda_k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \dots \wedge \epsilon^{i_k}$$

Exercise 3.2.21. Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \dots & \epsilon^{i_1}(e^{j_k}) \\ \vdots & & \vdots \\ \epsilon^{i_k}(e^{j_1}) & \dots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$.

If $I = J$, then $A = I_{k \times k}$ and therefore $\epsilon^{\wedge I}(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the l_0 th row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the l_0 th column of A are 0. \square

Exercise 3.2.22. Let $\alpha, \beta \in \Lambda_k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \dots, v_k \in V$. For $i = 1, \dots, k$, write $v_i = \sum_{j_i=1}^n a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned} \alpha(v_1, \dots, v_k) &= \sum_{j_1, \dots, j_k=1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\ &= \sum_{j_1 \neq \dots \neq j_k}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\ &= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \text{sgn}(\sigma) \left(\prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \alpha(e^J) \\ &= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \text{sgn}(\sigma) \left(\prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \beta(e^J) \\ &= \sum_{j_1, \dots, j_k=1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \beta(e^{j_1}, \dots, e^{j_k}) \\ &= \beta(v_1, \dots, v_k) \end{aligned}$$

\square

Exercise 3.2.23. The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(V)$ and $\dim \Lambda_k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$. Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda_k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda_k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$. \square

4. MANIFOLDS

4.1. Smooth Manifolds.

Definition 4.1.1. Define the **upper half space** of \mathbb{R}^n , denoted \mathbb{H}_n , by

$$\mathbb{H}_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\begin{aligned}\partial\mathbb{H}_n &= \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n = 0\} \\ (\mathbb{H}^n)^\circ &= \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}\end{aligned}$$

Definition 4.1.2. Let M be a topological space and $n \geq 1$.

- (1) Let $U \subset M$, $V \subset \mathbb{H}^n$ open and $\phi : U \rightarrow V$. Then (U, ϕ) is said to be a **coordinate chart** on M if ϕ is a homeomorphism.
- (2) Let $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$ be a collection of coordinate charts on M . Then \mathcal{A} is said to be an **atlas** on M if $\bigcup_{a \in A} U_a = M$.
- (3) The space M is said to be **locally half Euclidean of dimension n** if there exists an atlas $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$ on M such that for each $a \in A$, $\phi_a(U_a) \subset \mathbb{H}^n$.
- (4) The space M is said to be an **n -dimensional manifold** if M is Hausdorff, second countable and locally half Euclidean of dimension n .

Note 4.1.3. For the remainder of this section, we assume M is an n -dimensional manifold.

Definition 4.1.4.

- (1) Define the **boundary** of M , denoted ∂M , by
$$\partial M = \{p \in M : \text{there exists a chart } (U, \phi) \text{ on } M \text{ such that } p \in U \text{ and } \phi(p) \in \partial\mathbb{H}^n\}$$
- (2) Define the **interior** of M , denoted M° , by

$$M^\circ = M \setminus \partial M$$

Exercise 4.1.5. Let $p \in M$. Then $p \in \partial M$ iff for each chart (U, ϕ) on M , $p \in U$ implies that $\phi(p) \in \partial\mathbb{H}^n$. (Hint: simply connected)

Proof. Supposet that $p \in \partial M$. Then there exists a coordinate chart (V, ψ) on M such that $\psi(p) \in \partial\mathbb{H}^n$. Let (U, ϕ) be a coordinate chart on M . Suppose that $p \in U$. Note that $\phi \circ \psi^{-1} : \psi(V \cap U) \rightarrow \phi(V \cap U)$ is a homeomorphism. Choose open n -balls $B_\phi, B_\psi \subset \mathbb{H}^n$ such that $B_\phi \subset \phi(V \cap U)$, $B_\psi \subset \psi(V \cap U)$, $\phi(p) \in B_\phi$ and $\psi(p) \in B_\psi$. For the sake of contradiction, suppose that $\phi(p) \notin \partial\mathbb{H}^n$. Put $U' = B_\phi \setminus \{\phi(p)\}$ and $V' = B_\psi \setminus \{\psi(p)\}$. Define $\lambda : V' \rightarrow U'$ by $\lambda = \phi \circ \psi|_{V'}$. Then λ is a homeomorphism. Note that V' is simply connected and U' is not. This is a contradiction. \square

Exercise 4.1.6. If $\partial M \neq \emptyset$, then

- (1) ∂M is an $n - 1$ -dimensional manifold
- (2) $\partial(\partial M) = \emptyset$.

Proof. (1) Since subspaces of Hausdorff, second countable spaces are Hausdorff and second countable, we need only show that ∂M is locally half euclidean of dimension $n - 1$. Let $p \in \partial M$. Then there exists a coordinate chart (U, ϕ) on M such that $p \in U$ and $\phi(p) \in \partial\mathbb{H}^n$.

Put $U' = U \cap \partial M$. Note that U' is open in ∂M and $\phi(U) \cap \partial\mathbb{H}^n$ is open in $\partial\mathbb{H}^n$.

Define $\phi' : U' \rightarrow \phi(U) \cap \partial\mathbb{H}^n$ by $\phi' = \phi|_{U'}$. Then ϕ' is a homeomorphism.

Since $\partial\mathbb{H}^n$ is homeomorphic to \mathbb{R}^{n-1} which is homeomorphic to $(\mathbb{H}^{n-1})^\circ$ there exists $\psi : \partial\mathbb{H}^n \rightarrow (\mathbb{H}^{n-1})^\circ$ such that ψ is a homeomorphism.

Define $V' = \psi(\phi(U) \cap \partial\mathbb{H}^n)$ and $\psi' : \phi(U) \cap \partial\mathbb{H}^n \rightarrow V'$ by $\psi' = \psi|_{\phi(U) \cap \partial\mathbb{H}^n}$. Then V' is open in $(\mathbb{H}^{n-1})^\circ$ and ψ' is a homeomorphism.

Define $\lambda : U' \rightarrow V'$ by $\lambda = \psi' \circ \phi'$. Then λ is a homeomorphism and (U', λ) is a coordinate chart on ∂M . So ∂M is locally Euclidean of dimension $n - 1$.

- (2) Let $p \in \partial M$. Define $(U \cap \partial M, \lambda \circ \psi)$ as in (1). Since $\lambda \circ \psi(p) \in (\mathbb{H}^{n-1})^\circ$, we have that $p \in M^\circ$. Thus $\partial M = (\partial M)^\circ$ and $\partial(\partial M) = \emptyset$.

□

Definition 4.1.7.

- (1) Let $(U, \phi), (V, \psi)$ be coordinate charts on M . Then (U, ϕ) and (V, ψ) are said to be **smoothly compatible** if

$$\phi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \phi(U \cap V) \text{ is a diffeomorphism}$$

- (2) Let $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$ be an atlas on M . Then \mathcal{A} is said to be **smooth** if for each $a, b \in A$, (U_a, ϕ_a) and (U_b, ϕ_b) are smoothly compatible.
- (3) Let \mathcal{A} be a smooth atlas on M . Then \mathcal{A} is said to be **maximal** if for each smooth atlas \mathcal{B} on M , $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on M** .
- (4) Let \mathcal{A} be a smooth structure on M . Then (M, \mathcal{A}) is said to be a **smooth n -dimensional manifold**.

Exercise 4.1.8. Let \mathcal{B} be a smooth atlas on M . Then there exists a unique smooth structure \mathcal{A} on M such that $\mathcal{B} \subset \mathcal{A}$.

Proof. Define \mathcal{A} to be the set of all coordinate charts (U, ϕ) on M such that for each coordinate chart $(V, \psi) \in \mathcal{B}$, (U, ϕ) and (V, ψ) are smoothly compatible.

Clearly $\mathcal{B} \subset \mathcal{A}$.

Let $(U, \phi), (V, \psi) \in \mathcal{A}$ and $p \in U \cap V$. Then there exists $(W, \chi) \in \mathcal{B}$ such that $p \in W$. By assumption, $\phi \circ \chi^{-1} : \chi(U \cap W) \rightarrow \phi(U \cap W)$ and $\chi \circ \psi^{-1} : \psi(W \cap V) \rightarrow \chi(W \cap V)$ are diffeomorphisms. Then $(\phi \circ \chi^{-1}) \circ (\chi \circ \psi^{-1}) = \phi \circ \psi^{-1} : \psi(U \cap W \cap V) \rightarrow \phi(U \cap W \cap V)$ is a diffeomorphism. Since for each $q \in \psi(U \cap V)$, there exists an open neighborhood $N \subset \psi(U \cap V)$ of q on which $\phi \circ \psi^{-1}$ are diffeomorphic, we have that $\phi \circ \psi^{-1}$ is a diffeomorphism on $\psi(U \cap V)$ and therefore (U, ϕ) and (V, ψ) are smoothly compatible. Hence \mathcal{A} is a smooth atlas.

To see that \mathcal{A} is maximal, let \mathcal{B}' be a smooth atlas on M . Suppose that $\mathcal{A} \subset \mathcal{B}'$ and let $(U, \phi) \in \mathcal{B}'$. By definition, for each chart $(V, \psi) \in \mathcal{B}'$, (U, ϕ) and (V, ψ) are smoothly compatible. Since $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$, we have that $(U, \phi) \in \mathcal{A}$. So $\mathcal{A} = \mathcal{B}'$ and \mathcal{A} is a maximal smooth atlas on M . □

Exercise 4.1.9. Let \mathcal{A} be a smooth atlas on M . Define $\lambda : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ by $\lambda(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$. Put $\mathcal{A}|_{\partial M} = \{(U \cap \partial M, \lambda \circ \phi_{U \cap \partial M}) : (U, \phi) \in \mathcal{A}\}$. Then

- (1) $\mathcal{A}|_{\partial M}$ is a smooth atlas on ∂M .
- (2) if \mathcal{A} is maximal, then $\mathcal{A}|_{\partial M}$ is maximal.

Proof.

□

Note 4.1.10. For the rest of this section, we assume that (M, \mathcal{A}) is a smooth n -dimensional manifold and we denote the standard coordinate functions on \mathbb{R}^n by u^1, \dots, u^n . For a

coordinate chart $(U, \phi) \in \mathcal{A}$ and $i \in \{1, \dots, n\}$, we will typically denote the i th coordinate of ϕ by x^i , that is, $x^i = u^i(\phi)$.

4.2. Smooth Maps.

Definition 4.2.1. Let $f : M \rightarrow \mathbb{R}$. Then f is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M)$.

Exercise 4.2.2. We have that $C^\infty(M)$ is a vector space.

Proof. Clear. □

Definition 4.2.3. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$. Then F is said to be **smooth** if for each $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \psi(F(U) \cap V)$$

is smooth and F is said to be a **diffeomorphism** if F is a homeomorphism and F, F^{-1} are smooth.

Exercise 4.2.4. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$ a diffeomorphism. Then for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

Proof. Let $(V, \psi) \in \mathcal{B}$.

- (1) Since ϕ and F^{-1} are homeomorphisms, $\phi \circ F^{-1} : F(U) \cap V \rightarrow \phi(U \cap F^{-1}(V))$ is a homeomorphism
- (2) Since F is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \rightarrow \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \psi(V \cap F(U))$$

are smooth.

Therefore $(F(U), \phi \circ F^{-1})$ and (V, ψ) are smoothly compatible. Since \mathcal{B} is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. □

Exercise 4.2.5. Let (M, \mathcal{A}) be smooth m -dimensional manifold, (N, \mathcal{B}) a smooth n -dimensional manifold and $F : M \rightarrow N$. If F is a diffeomorphism, then $m = n$.

Proof. Suppose that F is a diffeomorphism. Let $(U, \phi) \in \mathcal{A}$. The previous exercise implies that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Then

$$\begin{aligned} \phi(U) &= \phi \circ F^{-1}(F(U)) \\ &= \subset \mathbb{H}^n \end{aligned}$$

By definition, $\phi(U) \subset \mathbb{H}^m$. So $m = n$. □

4.3. Partitions of Unity.

Definition 4.3.1. Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then ρ is said to be a **bump function at p supported in U** if

- (1) $\rho \geq 0$
- (2) there exists $V \in \mathcal{N}_p$ such that V is open and $\rho|_V = 1$
- (3) $\text{supp } \rho \subset U$

Exercise 4.3.2. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

Proof.

□

4.4. The Tangent Space.

Definition 4.4.1. Let $p \in M$. Define the relation \sim_p on $C^\infty(M)$ by $f \sim_p g$ iff there exists $U \in \mathcal{N}_p$ such that U is open and $f|_U = g|_U$. Clearly \sim_p is an equivalence relation on $C^\infty(M)$. We denote $C^\infty(M)/\sim_p$ by $C_p^\infty(M)$. For $f \in C^\infty(M)$, we define the **germ of f at p** to be the equivalence class of f under \sim_p .

Exercise 4.4.2. Let $p \in M$. We have that $C_p^\infty(M)$ is a vector space.

Proof. Clear. □

Definition 4.4.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $p \in U$ and $f \in C_p^\infty(M)$. For $i \in \{1, \dots, n\}$, define the partial derivative of f with respect to x^i at p , denoted

$$\frac{\partial f}{\partial x^i}(p), \quad \left. \frac{\partial}{\partial x^i} \right|_p f, \quad \partial_{x^i} f(p) \quad \text{or} \quad \left. \partial_{x^i} \right|_p f$$

by

$$\left. \frac{\partial}{\partial x^i} \right|_p f = \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} f \circ \phi^{-1}$$

Exercise 4.4.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \dots, n\}$, we have that

$$\frac{\partial x^i}{\partial x^j}(p) = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \dots, n\}$. Then

$$\begin{aligned} \left. \frac{\partial}{\partial x^j} \right|_p x^i &= \left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} x^i \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} u^i \circ \phi \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} u^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 4.4.5. (Change of Coordinates): Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$, $p \in U \cap V$ and $f \in C_p^\infty(M)$. Then for each $i \in \{1, \dots, n\}$, we have

$$\frac{\partial f}{\partial y^i}(p) = \sum_{j=1}^n \frac{\partial f}{\partial x^j}(p) \frac{\partial x^j}{\partial y^i}(p)$$

Proof. Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \dots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned} \left. \frac{\partial}{\partial y^i} \right|_p f &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \psi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \phi^{-1} \circ h \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial u^j} \right|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left(\left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} h_j \right) \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} f \circ \phi^{-1} \right) \left(\left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} x^j \circ \psi^{-1} \right) \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial x^j} \right|_p f \right) \left(\left. \frac{\partial}{\partial y^i} \right|_p x^j \right) \end{aligned}$$

□

Exercise 4.4.6. Taylor's Theorem:

Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $p \in U$ and $f \in C_p^\infty(M)$. Then there exist $g_1, \dots, g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p)) g_i$$

and for each $i \in \{1, \dots, n\}$,

$$g_i(p) = \left. \frac{\partial}{\partial x^i} \right|_p f$$

Proof. Since we are interested in the germ of f at p , we may assume that $\phi(U)$ is star-shaped with respect to $\phi(p)$. Let $q \in U$. From Taylor's theorem in section 1, we know that there exist $\tilde{g}_1, \dots, \tilde{g}_n \in C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = f \circ \phi^{-1}(\phi(p)) + \sum_{i=1}^n [u^i \circ \phi(q) - u^i \circ \phi(p)] \tilde{g}_i(\phi(q))$$

and for each $i \in \{1, \dots, n\}$,

$$\tilde{g}_i(\phi(p)) = \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} f \circ \phi^{-1}$$

For each $i \in \{1, \dots, n\}$, define $g_i = \tilde{g}_i \circ \phi$. Then for each $q \in U$,

$$f(q) = f(p) + \sum_{i=1}^n [x^i(q) - x^i(p)] g_i(q)$$

and for each $i \in \{1, \dots, n\}$,

$$g_i(p) = \left. \frac{\partial}{\partial x^i} \right|_p f$$

□

Definition 4.4.7. Let $p \in M$ and $v : C_p^\infty(M) \rightarrow \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C_p^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation at p** if for each $f, g \in C_p^\infty(M)$ and $a \in \mathbb{R}$,

- (1) v is linear
- (2) v is Leibnizian

We define the **tangent space of M at p** , denoted $T_p M$, by

$$T_p M = \{v : C_p^\infty(M) \rightarrow \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 4.4.8. Let $f \in C_p^\infty(M)$ and $v \in T_p M$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. \square

Exercise 4.4.9. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

is a basis for $T_p M$ and $\dim T_p M = n$.

Proof. Clearly $\frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \in T_p M$. Let $a_1, \dots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p x^j \\ &= a_j \end{aligned}$$

Hence $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is independent.

Now, let $v \in T_p M$ and $f \in C_p^\infty(M)$. By Taylor's theorem, there exist $g_1, \dots, g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \dots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x^i} \Big|_p f$$

Then

$$\begin{aligned}
v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\
&= \sum_{i=1}^n v(x^i)g_i(p) \\
&= \sum_{i=1}^n v(x^i) \left. \frac{\partial}{\partial x^i} \right|_p f \\
&= \left[\sum_{i=1}^n v(x^i) \left. \frac{\partial}{\partial x^i} \right|_p \right] f
\end{aligned}$$

So

$$v = \sum_{i=1}^n v(x^i) \left. \frac{\partial}{\partial x^i} \right|_p$$

and

$$v \in \text{span} \left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right\}$$

□

Definition 4.4.10. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. We define the **differential of F at p** , denoted $dF_p : T_p M \rightarrow T_{F(p)} N$, by

$$\left[dF_p(v) \right] (f) = v(f \circ F)$$

for $v \in T_p M$ and $f \in C_{F(p)}^\infty(N)$.

Exercise 4.4.11. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. Then dF_p is well defined.

Proof. Let $v \in T_p M$, $f, g \in C_{F(p)}^\infty(N)$ and $c \in \mathbb{R}$. Then

(1)

$$\begin{aligned}
dF_p(v)(f + cg) &= v((f + cg) \circ F) \\
&= v(f \circ F + cg \circ F) \\
&= v(f \circ F) + cv(g \circ F) \\
&= dF_p(v)(f) + cdF_p(v)(g)
\end{aligned}$$

So $dF_p(v)$ is linear.

(2)

$$\begin{aligned}
dF_p(v)(fg) &= v(fg \circ F) \\
&= v((f \circ F) * (g \circ F)) \\
&= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\
&= dF_p(v)(f) * g(F(p)) + f(F(p)) * dF_p(v)(g)
\end{aligned}$$

So $dF_p(v)$ is Leibnizian and hence $dF_p(v) \in T_{F(p)}N$ \square

Exercise 4.4.12. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. If F is a diffeomorphism, then dF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \dots, x^n)$ and $\phi \circ F^{-1} = (y^1, \dots, y^n)$. Let $f \in C_{F(p)}^\infty(N)$. Then

$$\begin{aligned} \left. \frac{\partial}{\partial y^i} \right|_{F(p)} f &= \left. \frac{\partial}{\partial u^i} \right|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial x^i} \right|_p f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) \right] (f) &= \left. \frac{\partial}{\partial x^i} \right|_p f \circ F \\ &= \left. \frac{\partial}{\partial y^i} \right|_{F(p)} f \end{aligned}$$

Hence

$$dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \left. \frac{\partial}{\partial y^i} \right|_{F(p)}$$

Since $\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right\}$ is a basis for $T_p M$ and $\left\{ \left. \frac{\partial}{\partial y^1} \right|_{F(p)}, \dots, \left. \frac{\partial}{\partial y^n} \right|_{F(p)} \right\}$ is a basis for $T_{F(p)}N$, dF_p is an isomorphism. \square

Definition 4.4.13. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ a diffeomorphism. Define the **push forward of F** , denoted

$$F_* : M \rightarrow \coprod_{p \in M} \text{Iso}(T_p M, T_{F(p)}N)$$

by

$$p \mapsto dF_p$$

4.5. The Cotangent Space.

Definition 4.5.1. Let $p \in M$. We define the **cotangent space of M at p** , denoted T_p^*M , by

$$T_p^*M = (T_pM)^*$$

Definition 4.5.2. Let $f \in C^\infty(M)$. We define the **differential of f at p** , denoted $df_p : T_pM \rightarrow \mathbb{R}$, by

$$df_p(v) = vf$$

Exercise 4.5.3. Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

Proof. Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1f + \lambda v_2f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that df_p is linear and hence $df_p \in T_p^*M$. □

Exercise 4.5.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \dots, n\}$,

$$dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \delta_{i,j}$$

In particular, $\{dx_p^1, \dots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ and $T_p^*M = \text{span}\{dx_p^1, \dots, dx_p^n\}$.

Proof. Let $i, j \in \{1, \dots, n\}$. Then by definition,

$$\begin{aligned} \left[dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) \right]_p &= \frac{\partial}{\partial x^j} \Big|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 4.5.5. Let $f \in C^\infty(M)$, (U, ϕ) a chart on M with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

Proof. Since $\{dx_p^1, \dots, dx_p^n\}$ is a basis for T_p^*M , for each there exist $a_1(p), \dots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) &= \frac{\partial}{\partial x^j}\Big|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

□

4.6. Submanifolds.

5. TENSOR FIELDS

5.1. Tensor Bundles.

Definition 5.1.1. We define

- (1) the **tangent bundle of M** , denoted TM , by

$$TM = \coprod_{p \in M} T_p M$$

- (2) the **cotangent bundle of M** , denoted T^*M , by

$$T^*M = \coprod_{p \in M} T_p^* M$$

- (3) the **(r, s) -tensor bundle of M** , denoted $T_s^r M$

$$T_s^r M = \coprod_{p \in M} T_s^r(T_p M)$$

5.2. Vector Fields.

Definition 5.2.1. Let $X : M \rightarrow TM$. Then X is said to be a **vector field on M** if for each $p \in M$, $X_p \in T_p M$.

For $f \in C^\infty(M)$, we define $Xf : M \rightarrow \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in C^\infty(M)$, Xf is smooth. We denote the set of smooth vector fields on M by $\Gamma^1(M)$.

Definition 5.2.2. Let $f \in C^\infty(M)$ and $X, Y \in \Gamma^1(M)$. We define

- $fX \in \Gamma^1(M)$ by

$$(fX)_p = f(p)X_p$$

- $X + Y \in \Gamma^1(M)$ by

$$(X + Y)_p = X_p + Y_p$$

Exercise 5.2.3. The set $\Gamma^1(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Exercise 5.2.4. Let $X \in \Gamma^1(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then

$$X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$$

Proof. Let $p \in M$. Then $X_p \in T_p M$ and $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is a basis of $T_p M$. So there exist $f_1(p), \dots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i} \Big|_p$. Let $j \in \{1, \dots, n\}$. Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^n f_i(p) \frac{\partial x^j}{\partial x^i}(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx^j = f_j$ and $X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$. □

Exercise 5.2.5. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then for each $i \in \{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

Proof. Let $i \in \{1, \dots, n\}$ and $f \in C^\infty(M)$. Define $g : M \rightarrow \mathbb{R}$ by $g = \frac{\partial}{\partial x^i} f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i} \Big|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i} [f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence g is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x^i}$ is smooth. \square

5.3. 1-Forms.

Definition 5.3.1. Let $\omega : M \rightarrow T^*M$. Then ω is said to be a **1-form on M** if for each $p \in M$, $\omega_p \in T_p^*M$.

For each $X \in \Gamma^1(M)$, we define $\omega(X) : M \rightarrow \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)$, $\omega(X)$ is smooth.

The set of smooth 1-forms on M is denoted $\Gamma_1(M)$.

Definition 5.3.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma_1(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by

$$(f\alpha)_p = f(p)\alpha_p$$

- $\alpha + \beta \in \Gamma_1(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 5.3.3. The set $\Gamma_1(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Exercise 5.3.4.

5.4. (r, s) -Tensor Fields.

Definition 5.4.1. Let $\alpha : M \rightarrow T_s^r M$. Then α is said to be a (r, s) -**tensor field on M** if for each $p \in M$, $\alpha_p \in T_p^r(T_p M)$.

For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \rightarrow \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and α is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth. The set of smooth (r, s) -tensor fields on M is denoted $\Gamma_s^r(M)$.

Definition 5.4.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma_s^r(M)$. We define

- $f\alpha : M \rightarrow T_s^r M$ by

$$(f\alpha)_p = f(p)\alpha_p$$

- $\alpha + \beta : M \rightarrow T_s^r M$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 5.4.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma_s^r(M)$. Then

- (1) $f\alpha \in \Gamma_s^r(M)$ by

$$(f\alpha)_p = f(p)\alpha_p$$

- (2) $\alpha + \beta \in \Gamma_s^r(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Proof. Clear. □

Exercise 5.4.4. The set $\Gamma_s^r(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Definition 5.4.5. Let $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$ and $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$. We define the **tensor product of α with β** , denoted $\alpha \otimes \beta : M \rightarrow T_{s_1+s_2}^{r_1+r_2} M$, by

$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

Exercise 5.4.6. Let $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$ and $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma_{s_1+s_2}^{r_1+r_2}(M)$

Proof. Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since α_1 and α_2 are smooth by assumption. □

Definition 5.4.7. We define the **tensor product**, denoted $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

Exercise 5.4.8. Let $\alpha \in \Gamma_s^r(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then there exist $(f_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

Proof. Let $p \in M$. Then $\omega_p \in T_s^r(T_p M)$ and $\left\{ \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J} \right\}$ is a basis of $T_s^r(T_p M)$. So there exist $(f_J^I(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned} \alpha_p(dx_p^K, \partial_{x^L}|_p) &= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J} (dx_p^K, \partial_{x^L}|_p) \\ &= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p (dx_p^K) dx_p^{\otimes J} (\partial_{x^L}|_p) \\ &= f_L^K(p) \end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx^K, \partial_{x^L})_p$ is smooth, so that $f_L^K \in C^\infty(U)$.

□

5.5. Differential Forms.

Definition 5.5.1. We define

$$\Lambda_k(TM) = \coprod_{p \in M} \Lambda_k(T_p M)$$

Definition 5.5.2. Let $\omega : M \rightarrow \Lambda_k(TM)$. Then ω is said to be a **k -form on M** if for each $p \in M$, $\omega_p \in \Lambda_k(T_p M)$.

For each $X_1, \dots, X_k \in \Gamma^1(M)$, we define $\omega(X_1, \dots, X_k) : M \rightarrow \mathbb{R}$ by

$$\omega(X_1, \dots, X_k)_p = \omega_p(X_1(p), \dots, X_k(p))$$

and ω is said to be **smooth** if for each $X_1, \dots, X_k \in \Gamma^1(M)$, $\omega(X_1, \dots, X_k)$ is smooth. The set of smooth k -forms on M is denoted $\Omega_k(M)$.

Note 5.5.3. Observe that $\Omega_0(M) = C^\infty(M)$.

Definition 5.5.4. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Omega_k(M)$. We define

- $f\alpha \in \Omega_k(M)$ by

$$(f\alpha)_p = f(p)\alpha_p$$

- $\alpha + \beta \in \Omega_k(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 5.5.5. The set $\Omega_k(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Definition 5.5.6. Define the **exterior product**

$$\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Note 5.5.7. For $f \in \Omega_0(M)$ and $\alpha \in \Omega_k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 5.5.8. The exterior product $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$ is well defined.

Proof. Let $\alpha \in \Omega_k(M)$, $\beta \in \Omega_l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{j=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then

$$\begin{aligned} \alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{(k+l)!}{k!l!} A(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) (\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta_p(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \end{aligned}$$

□

Exercise 5.5.9. The exterior product $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$ is $C^\infty(M)$ -bilinear.

Proof.

(1) $C^\infty(M)$ -linearity in the first argument:

Let $\alpha \in \Omega_k(M)$, $\beta, \gamma \in \Omega_l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda_k(T_p M) \times \Lambda_l(T_p M) \rightarrow \Lambda_{k+l}(T_p M)$ implies that

$$\begin{aligned} [(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p \end{aligned}$$

So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

and $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$ is $C^\infty(M)$ -linear in the first argument.

(2) $C^\infty(M)$ -linearity in the second argument:

Similar to (1).

□

Note 5.5.10. All of the results from multilinear algebra apply here.

Definition 5.5.11. We define the **exterior derivative** $d : \Omega_k(M) \rightarrow \Omega_{k+1}(M)$ inductively by

- (1) $d(d\alpha) = 0$ for $\alpha \in \Omega_p(M)$
- (2) $df(X) = Xf$ for $f \in \Omega_0(M)$
- (3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega_p(M)$ and $\beta \in \Omega_q(M)$
- (4) extending linearly

Exercise 5.5.12. Let (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then on U , for each $i, j \in \{1, \dots, n\}$,

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx_p^1, \dots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ and $T_p^* M = \text{span}\{dx_p^1, \dots, dx_p^n\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \dots, n\}$. Then by definition,

$$\begin{aligned} \left[dx^i \left(\frac{\partial}{\partial x^j} \right) \right]_p &= \left(\frac{\partial}{\partial x^j} x^i \right)_p \\ &= \frac{\partial}{\partial x^j} \Big|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 5.5.13. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Proof. Let $p \in U$. Since $\{dx^1, \dots, dx^n\}$ is a basis for $\Lambda(T_p M)$, for each there exist $a_1(p), \dots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) &= \left. \frac{\partial}{\partial x^j} \right|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

□

Exercise 5.5.14. Let $f \in \Omega_0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \dots, x_n)$. Then for each $i \in \{1, \dots, n\}$,

$$\left. \frac{\partial}{\partial x^i} \right|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i \\ &= 0 \end{aligned}$$

□

Exercise 5.5.15.

Definition 5.5.16. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $I = (i_1, \dots, i_k) \in \mathcal{I}_k$. We define

$$dx^I = dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Omega_k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}} \right)$$

Note 5.5.17. We have that

(1)

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{I,J}$$

(2) Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega_k(U)$,

$$\omega \left(\frac{\partial}{\partial x^i} \right) \in C^\infty(U)$$

Exercise 5.5.18. Let $\omega \in \Omega_k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega \left(\frac{\partial}{\partial x^i} \right) dx^i$$

Proof. Let $p \in U$. Since $\{dx_p^i : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(T_p M)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p) dx_p^i$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned} \omega \left(\frac{\partial}{\partial x^j} \right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j} \right) \\ &= f_J \end{aligned}$$

□

Exercise 5.5.19. Let $\omega \in \Omega_k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. If $\omega = \sum_{I \in \mathcal{I}_k} f_I dx^i$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

.

Proof. First we note that

$$\begin{aligned} d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f_I d(dx^i) \\ &= df_I \wedge dx^i \\ &= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \right) \wedge dx^i \\ &= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i \end{aligned}$$

Then we extend linearly.

□

Definition 5.5.20. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$ be a diffeomorphism. Define the **pullback of F** , denoted $F^* : \Omega_k(N) \rightarrow \Omega_k(M)$ by

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))$$

for $\omega \in \Omega_k(N)$, $p \in M$ and $v_1, \dots, v_k \in T_pM$

6. EXTRA

Definition 6.0.1. When working in \mathbb{R}^n , we introduce the formal objects dx^1, dx_2, \dots, dx^n . Let $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. We formally define $dx^I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$.

Definition 6.0.2. Let $k \in \{0, 1, \dots, n\}$. We define a $C^\infty(\mathbb{R}^n)$ -module of dimension $\binom{n}{k}$, denoted $\Phi_k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx^I : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \rightarrow \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

- (1) the exterior product is bilinear
- (2) for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$
- (3) for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$
- (4) for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k -forms on \mathbb{R}^n . Let ω be a k -form on \mathbb{R}^n . If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$

Note 6.0.3. The terms dx^1, dx_2, \dots, dx^n are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \dots, du_k\}$ when referencing the coordinates on \mathbb{R}^k and $\{dx^1, dx_2, \dots, dx^n\}$ when referencing the coordinates on \mathbb{R}^n .

Exercise 6.0.4. Let $B_{n \times n} = (b_{i,j}) \in [C^\infty(M)]^{n \times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \dots \wedge dx^n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned} \bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) &= \left(\sum_{j=1}^n b_{1,j} dx^j \right) \wedge \left(\sum_{j=1}^n b_{2,j} dx^j \right) \wedge \dots \wedge \left(\sum_{j=1}^n b_{n,j} dx^j \right) \\ &= \sum_{j_1, \dots, j_n=1}^n \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_n} \\ &= \sum_{j_1 \neq \dots \neq j_n} \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_n} \\ &= \left[\sum_{\sigma \in S_n} \text{sgn}(\sigma) \left(\prod_{i=1}^n b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \\ &= (\det B) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \end{aligned}$$

□

Definition 6.0.5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a 0-form on \mathbb{R}^n . We define a 1-form, denoted df , on \mathbb{R}^n by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Let $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$ be a k -form on \mathbb{R}^n . We can define a differential $k+1$ -form, denoted $d\omega$, on \mathbb{R}^n by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx^I$$

Exercise 6.0.6. On \mathbb{R}^3 , put

- (1) $\omega_0 = f_0$,
- (2) $\omega_1 = f_1 dx^1 + f_2 dx_2 + f_3 dx_3$,
- (3) $\omega_2 = f_1 dx_2 \wedge dx_3 - f_2 dx^1 \wedge dx_3 + f_3 dx^1 \wedge dx_2$

Show that

- (1) $d\omega_0 = \frac{\partial f_0}{\partial x_1} dx^1 + \frac{\partial f_0}{\partial x_2} dx_2 + \frac{\partial f_0}{\partial x_3} dx_3$
- (2) $d\omega_1 = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \right) dx^1 \wedge dx_3 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx^1 \wedge dx_2$
- (3) $d\omega_2 = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx^1 \wedge dx_2 \wedge dx_3$

Proof. Straightforward. □

Exercise 6.0.7. Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx^I \wedge dx_{I_*} = dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$.

Definition 6.0.8. We define a linear map $*$: $\Phi_k(\mathbb{R}^n) \rightarrow \Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge ***-operator by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 6.0.9. Let $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. We define $\phi^* : \Phi_k(\mathbb{R}^n) \rightarrow \Phi_k(\mathbb{R}^k)$ via the following properties:

- (1) for each 0-form f on \mathbb{R}^n , $\phi^* f = f \circ \phi$
- (2) for $i = 1, \dots, n$, $\phi^* dx^i = d\phi_i$
- (3) for an s -form ω , and a t -form χ on \mathbb{R}^n , $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
- (4) for l -forms ω, χ on \mathbb{R}^n , $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 6.0.10. Let $M \subset \mathbb{R}^n$ be a k -dimensional smooth submanifold of \mathbb{R}^n , $\phi : U \rightarrow V$ a smooth parametrization of M , $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$ an k -form on \mathbb{R}^n . Then

$$\phi^*\omega = \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof. Using the definitions, we see that

$$\begin{aligned}\phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx^i \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\ &= \left(\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u^j} du^j \right) \wedge \left(\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u^j} du^j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n \frac{\partial \phi_{i_n}}{\partial u^j} du^j \right) \\ &= (\det v\phi_I) du^1 \wedge du^2 \wedge \cdots \wedge du^n\end{aligned}$$

Therefore

$$\begin{aligned}\phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det v\phi_I) du^1 \wedge du^2 \wedge \cdots \wedge du^n \\ &= \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det v\phi_I) \right) du^1 \wedge du^2 \wedge \cdots \wedge du^n\end{aligned}$$

□

6.1. Integration of Differential Forms.

Definition 6.1.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \cdots \wedge dx_k$ a k -form on \mathbb{R}^k . Define

$$\int_U \omega = \int_U f dx$$

Definition 6.1.2. Let $M \subset \mathbb{R}^n$ be a k -dimensional oriented smooth submanifold of \mathbb{R}^n , ω a k -form on \mathbb{R}^n and $\phi : U \rightarrow V$ a local smooth, orientation-preserving parametrization of M . Define

$$\int_V \omega = \int_U \phi^* \omega$$

Exercise 6.1.3.

Theorem 6.1.1. (Stokes Theorem) Let $M \subset \mathbb{R}^n$ be a k -dimensional oriented smooth submanifold of \mathbb{R}^n and ω a $k-1$ -form on \mathbb{R}^n . Then

$$\int_{\partial M} \omega = \int_M d\omega$$