# INTRODUCTION TO FOURIER ANALYSIS

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## Contents

# 1. Fourier Analysis on $\mathbb{R}$

## 1.1. Schwartz Space and Bump Functions.

**Definition 1.1.1.** Let $f \in C^\infty(\mathbb{R})$, and $\alpha,\ N \in \mathbb{N}_0$. We define $\|\cdot\|_{\alpha,N} : C^\infty(\mathbb{R},\mathbb{C}) \to [0,\infty]$ by

$$\|f\|_{\alpha,N} = \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N |\partial^\alpha f(x)|\bigg]$$

We define **Schwartz space** on $\mathbb{R}$, denoted $\mathcal{S}$, by

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}) : \text{ for each } \alpha, N \in \mathbb{N}_0,\ \|f\|_{\alpha,N} < \infty\}$$

**Exercise 1.1.2.** Let $f \in \mathcal{S}$. Then

(1) $f$ is Lipschitz
(2) $f \in L^1(m)$

*Proof.*

(1) There exists $M > 0$ such that for each $x \in \mathbb{R}$,

$$\begin{aligned}|\partial f(x)| &\leq M(1+|x|)^{-1} \\ &\leq M\end{aligned}$$

Thus $\partial f$ is bounded which implies that $\partial^\alpha f$ is Lipschitz.

(2) There exists $C \geq 0$ such that for each $x \in \mathbb{R}$,

$$\begin{aligned}|f(x)| &\leq C(1+|x|)^{-2} \\ &\leq C(1+|x|^2)^{-1}\end{aligned}$$

Define $g : \mathbb{R} \to [0,\infty)$ defined by $g(x) = (1+|x|^2)^{-1}$. Then $g \in L^1(m)$ which implies that $f \in L^1(m)$.

□

**Exercise 1.1.3.** We have that $\mathcal{S}$ is a vector space and for each $\alpha, N \in \mathbb{N}_0$, $\|\cdot\|_{\alpha,N} : \mathcal{S} \to [0,\infty)$ is a seminorm on $\mathcal{S}$.

*Proof.* Let $f, g \in \mathcal{S}$ and $\lambda \in \mathbb{C}$.

(1)

$$\begin{aligned}\|\lambda f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N |\partial^\alpha[\lambda f](x)|\bigg] \\ &= \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N |\lambda \partial^\alpha f(x)|\bigg] \\ &= \sup_{x\in\mathbb{R}} \bigg[|\lambda|(1+|x|)^N |\partial^\alpha f(x)|\bigg] \\ &= |\lambda| \sup_{x\in\mathbb{R}} \bigg[(1+|x|)^N |\partial^\alpha f(x)|\bigg] \\ &= |\lambda| \|f\|_{\alpha,N}\end{aligned}$$

Thus $\lambda f \in \mathcal{S}$ and $\|\lambda f\|_{\alpha,N} = |\lambda| \|f\|_{\alpha,N}$.

(2)

$$\begin{aligned}
\|f+g\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha[f+g](x)|\right] \\
&= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|[\partial^\alpha f+\partial g](x)|\right] \\
&\le \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha f(x)|+(1+|x|)^N|\partial g(x)|\right] \\
&\le \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha f(x)|\right]+\sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial g(x)|\right] \\
&= \|f\|_{\alpha,N}+\|g\|_{\alpha,N}
\end{aligned}$$

Hence $f+g\in\mathcal{S}$ and $\|f+g\|_{\alpha,N}\le\|f\|_{\alpha,N}+\|g\|_{\alpha,N}$.

So $\mathcal{S}$ is a vector space and $\|\cdot\|_{\alpha,N}$ is a seminorm on $\mathcal{S}$. $\square$

**Exercise 1.1.4.** We have that $\mathcal{S}$ is a algebra under pointwise multiplication and for each $\alpha, N\in\mathbb{N}_0$,

$$\|fg\|_{\alpha,N}\le\sum_{\beta=0}^{\alpha}\|f\|_{\beta,N}\|g\|_{\alpha-\beta,0}$$

**Hint:** $\partial^\alpha[fg]=\sum\limits_{\beta=0}^{\alpha}(\partial^\beta f)(\partial^{\alpha-\beta}g)$

*Proof.* Let $f,g\in\mathcal{S}$ and $\alpha,N\in\mathbb{N}_0$. Then

$$\begin{aligned}
\|fg\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha[fg](x)|\right] \\
&= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N\left|\sum_{\beta=0}^{\alpha}\partial^\beta f(x)\partial^{\alpha-\beta}g(x)\right|\right] \\
&\le \sup_{x\in\mathbb{R}}\left[(1+|x|)^N\left(\sum_{\beta=0}^{\alpha}|\partial^\beta f(x)||\partial^{\alpha-\beta}g(x)|\right)\right] \\
&= \sup_{x\in\mathbb{R}}\left[\sum_{\beta=0}^{\alpha}(1+|x|)^N|\partial^\beta f(x)||\partial^{\alpha-\beta}g(x)|\right] \\
&\le \sum_{\beta=0}^{\alpha}\sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\beta f(x)||\partial^{\alpha-\beta}g(x)|\right] \\
&\le \sum_{\beta=0}^{\alpha}\sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\beta f(x)|\right]\sup_{x\in\mathbb{R}}\left[|\partial^{\alpha-\beta}g(x)|\right] \\
&= \sum_{\beta=0}^{\alpha}\|f\|_{\beta,N}\|g\|_{\alpha-\beta,0} \\
&< \infty
\end{aligned}$$

So $fg\in\mathcal{S}$. $\square$

**Definition 1.1.5.** Set $\mathcal{P} = (\|\cdot\|_{\alpha,N})_{\alpha,N\in\mathbb{N}_0}$. Then $\mathcal{P}$ is a countable family of seminorms on $\mathcal{S}$. We equip $\mathcal{S}$ with the topology $\mathcal{T}$ induced by the family of projections

$$\pi_{\|\cdot\|_{\alpha,N}} : \mathcal{S} \to \mathcal{S}/\ker\|\cdot\|_{\alpha,N}$$

i.e. $\mathcal{T} = \tau_{\mathcal{S}}((\pi_{\|\cdot\|_{\alpha,N}})_{\alpha,N\in\mathbb{N}_0})$.
Explicitly, for a net $(f_\alpha)_{\alpha\in A} \subset \mathcal{S}$ and $f \in \mathcal{S}$, $f_\alpha \to f$ iff for each $\alpha, N \in \mathbb{N}_0$, $\|f_\alpha - f\|_{\alpha,N} \to 0$. Hence $(\mathcal{S},\mathcal{T})$ is a locally convex space. Since $\mathcal{P}$ is countable, we may write $\mathcal{P} = (p_j)_{j\in\mathbb{N}}$ and thus $(\mathcal{S},\mathcal{T})$ is metrizable with metric

$$d_{\mathcal{S}}(f,g) = \sum_{j\in\mathbb{N}} 2^{-j}\frac{p_j(f-g)}{1+p_j(f-g)}$$

**Exercise 1.1.6.** Let $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0$. Then $\partial^\alpha f \in \mathcal{S}$ and for each $\beta, N \in \mathbb{N}_0$,

$$\|\partial^\alpha f\|_{\beta,N} \leq \|f\|_{\alpha+\beta,N}$$

*Proof.* Let $f \in \mathcal{S}$, and $\beta$, $N \in \mathbb{N}_0$. By definition,

$$\begin{aligned}
\|\partial^\alpha f\|_{\beta,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\beta[\partial^\alpha f](x)|\right] \\
&= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^{\alpha+\beta} f(x)|\right] \\
&= \|f\|_{\alpha+\beta,N} \\
&< \infty
\end{aligned}$$

So $\partial^\alpha f \in \mathcal{S}$. □

**Exercise 1.1.7.** Let $f \in \mathcal{S}$. Then for each $\alpha$, $N \in \mathbb{N}_0$,

$$\|f\|_{\alpha,N} = \|\partial^\alpha f\|_{0,N}$$

*Proof.* Clear by preceding exercise. □

**Exercise 1.1.8.** Let $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}$. Define $g : \mathbb{R} \to \mathbb{C}$ by $g(x) = xf(x)$. Then for each $x \in \mathbb{R}$, $\partial^\alpha g(x) = x\partial^\alpha f(x) + \alpha\partial^{\alpha-1}f(x)$.

*Proof.* The claim is clear if $\alpha = 1$. Suppose that $\alpha > 1$ and that the claim is true for $\alpha - 1$ so that for each $x \in \mathbb{R}$, $\partial^{\alpha-1}g(x) = x\partial^{\alpha-1}f(x) + (\alpha-1)\partial^{\alpha-2}f(x)$. Then

$$\begin{aligned}
\partial^\alpha g(x) &= \partial[\partial^{\alpha-1}g(x)] \\
&= \partial[x\partial^{\alpha-1}f(x) + (\alpha-1)\partial^{\alpha-2}f(x)] \\
&= \partial[x\partial^{\alpha-1}f(x)] + \partial[(\alpha-1)\partial^{\alpha-2}f(x)] \\
&= [x\partial^\alpha f(x) + \partial^{\alpha-1}f(x)] + [(\alpha-1)\partial^{\alpha-1}f(x)] \\
&= x\partial^\alpha f(x) + \alpha\partial^{\alpha-1}f(x)
\end{aligned}$$

So the claim is true for $\alpha$. □

**Exercise 1.1.9.** Let $f \in \mathcal{S}$ and $N \in \mathbb{N}_0$. Define $g : \mathbb{R} \to \mathbb{C}$ by $g(x) = xf(x)$. Then $g \in \mathcal{S}$ and for each $\alpha$, $N \in \mathbb{N}_0$,

$$\|g\|_{\alpha,N} \leq \|f\|_{\alpha,N+1} + \alpha\|f\|_{\alpha-1,N}$$

*Proof.* Let $\alpha, N \in \mathbb{N}_0$. The previous exercise implies that

$$\begin{aligned}
\|g\|_{\alpha,N} &= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha x f(x)|\right] \\
&= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |x\partial^\alpha f(x) + \alpha \partial^{\alpha-1} f(x)|\right] \\
&\leq \sup_{x\in\mathbb{R}} \left[(1+|x|)^{N+1} |\partial^\alpha f(x)|\right] + \alpha \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^{\alpha-1} f(x)|\right] \\
&= \|f\|_{\alpha,N+1} + \alpha \|f\|_{\alpha-1,N}
\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $g \in \mathcal{S}$. □

**Definition 1.1.10.** We define the

- **position operator**, denoted $X : \mathcal{S} \to \mathcal{S}$, by
$$Xf(x) = xf(x)$$
- **momentum operator**, denoted $D : \mathcal{S} \to \mathcal{S}$, by
$$Df(x) = -i\partial f(x)$$

**Exercise 1.1.11.** We have that

(1) (a) $X : \mathcal{S} \to \mathcal{S}$ is linear
 (b) $D : \mathcal{S} \to \mathcal{S}$ is linear
(2) (a) $X : \mathcal{S} \to \mathcal{S}$ is continuous
 (b) $D : \mathcal{S} \to \mathcal{S}$ is continuous

*Proof.*

(1) Clear.
(2) Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$. Suppose that $f_n \to 0$. Then for each $\alpha, N \in \mathbb{N}_0$, $\|f_n\|_{\alpha,N} \to 0$.
 (a) A previous exercise implies that
$$\begin{aligned}
\|Xf_n\|_{\alpha,N} &\leq \|f_n\|_{\alpha,N+1} + \alpha\|f_n\|_{\alpha-1,N} \\
&\to 0
\end{aligned}$$
 So $Xf_n \to 0$ and $X$ is continuous at 0. Since $X$ is linear, $X$ is continuous.
 (b) A previous exercise implies that
$$\begin{aligned}
\|Df_n\|_{\alpha,N} &= \|\partial f_n\|_{\alpha,N} \\
&\leq \|f_n\|_{\alpha+1,N} \\
&\to 0
\end{aligned}$$
 So $Df_n \to 0$ and $D$ is continuous at 0. Since $D$ is linear, $D$ is continuous.

□

**Definition 1.1.12.** Let $f \in \mathcal{S}$ and $y \in \mathbb{R}$. Then

- for each e$y \in \mathbb{R}$ we define the **translation of** $f$ **by** $y$, denoted $\tau_y f : \mathbb{R} \to \mathbb{C}$, by $\tau_y f(x) = f(x-y)$
- for each $\xi \in \mathbb{R}$, we define the **rotation of** $f$ by $\xi$, denoted $\rho_\xi f : \mathbb{R} \to \mathbb{C}$ by $\rho_\xi f(x) = e^{-i\xi x} f(x)$
- for each $t \neq 0$, we define the **dilation of** $f$ by $t$, denoted $\delta_t f : \mathbb{R} \to \mathbb{C}$ by $\delta_t f(x) = f(tx)$

**Exercise 1.1.13.** Let $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0$. Then

(1) for each $y \in \mathbb{R}$, $\partial^\alpha \tau_y f = \tau_y \partial^\alpha f$
(2) for each $\xi \in \mathbb{R}$ ,

$$\begin{aligned}\partial^\alpha \rho_\xi f &= \rho_\xi[(-i\xi + \partial)^\alpha f] \\ &= \sum_{k=0}^{\alpha} \binom{\alpha}{k} (-i\xi)^{\alpha-k} \rho_\xi \partial^k f\end{aligned}$$

(3) for each $t \neq 0$, $\partial^\alpha \delta_t f = t^\alpha \delta_t \partial^\alpha f$

*Proof.*

(1) Clear by chain rule.
(2) Let $\xi \in \mathbb{R}$. The claim is clear for $\alpha = 0$ and $\alpha = 1$. Suppose that $\alpha > 1$ and the claim is true for $\alpha - 1$ so that $\partial^{\alpha-1}\rho_\xi f = \rho_\xi[(-i\xi + \partial)^{\alpha-1} f]$. Set $g = (-i\xi + \partial)^{\alpha-1} f$. Then

$$\begin{aligned}\partial^\alpha \rho_\xi f &= \partial[\partial^{\alpha-1}\rho_\xi f] \\ &= \partial \rho_\xi[(-i\xi + \partial)^{\alpha-1} f] \\ &= \partial \rho_\xi g \\ &= \rho_\xi[(-i\xi + \partial) g] \\ &= \rho_\xi[(-i\xi + \partial)^\alpha f]\end{aligned}$$

Since $-i\xi\, \mathrm{id}_{\mathcal{S}}$ and $\partial$ commute, the binomial theorem implies that

$$\begin{aligned}\rho_\xi[(-i\xi + \partial)^\alpha f] &= \rho_\xi[\sum_{k=0}^{\alpha} \binom{\alpha}{k}](-i\xi)^{\alpha-k} \partial^k f \\ &= \sum_{k=0}^{\alpha} \binom{\alpha}{k}](-i\xi)^{\alpha-k} \rho_\xi \partial^k f\end{aligned}$$

(3) Clear by chain rule

□

**Exercise 1.1.14.** Let $y \in \mathbb{R}$ and $t \neq 0$. Then

(1) for each $x \in \mathbb{R}$, $(1 + |x|) \leq (1 + |y|)(1 + |x - y|)$
(2) there exists $C > 0$ such that for each $x \in \mathbb{R}$, $1 + |x| \leq C(1 + |tx|)^2$

*Proof.*

(1) Let $x \in \mathbb{R}$. Then

$$\begin{aligned}(1 + |y|)(1 + |x - y|) &= 1 + |x - y| + |y| + |y||x - y| \\ &\geq 1 + |x| + |y||x - y| \\ &\geq 1 + |x|\end{aligned}$$

(2) Choose $C = \max(1/(2|t|), 1)$. Let $x \in \mathbb{R}$. Then

$$\begin{aligned} C(1+|tx|)^2 - (1+|x|) &= C + 2C|tx| + C(tx)^2 - 1 - |x| \\ &= C + (2C|t| - 1)|x| + C(tx)^2 - 1 \\ &= (C-1) + (2C|t|-1)|x| + C(tx)^2 \\ &\geq 0 \end{aligned}$$

So $1 + |x| \leq C(1+|tx|)^2$.

□

**Exercise 1.1.15.** Let $f \in \mathcal{S}$. Then

(1) for each $y \in \mathbb{R}$, $\tau_y f \in \mathcal{S}$ and for each $\alpha, N \in \mathbb{N}_0$, $\|\tau_y f\|_{\alpha,N} \leq (1+|y|)^N \|f\|_{\alpha,N}$
(2) for each $\xi \in \mathbb{R}$, $\rho_\xi f \in \mathcal{S}$ and for each $\alpha, N \in \mathbb{N}_0$,

$$\|\rho_\xi f\|_{\alpha,N} \leq \sum_{k=0}^{\alpha} \binom{\alpha}{k} |\xi|^{\alpha-k} \|f\|_{k,N}$$

(3) for each $t \neq 0$, $\delta_t f \in \mathcal{S}$ and $\kappa_t f \in \mathcal{S}$ and there exists $C_t > 0$ such that for each $\alpha, N \in \mathbb{N}_0$, $\|\delta_t f\|_{\alpha,N} \leq |t|^\alpha C_t^N \|f\|_{\alpha,2N}$

*Proof.*

(1) Let $y \in \mathbb{R}$ and $\alpha, N \in \mathbb{N}_0$. Then

$$\begin{aligned} \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha \tau_y f(x)|\right] &= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\tau_y \partial^\alpha f(x)|\right] \\ &= \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha f(x-y)|\right] \\ &\leq \sup_{x\in\mathbb{R}} \left[(1+|y|)^N (1+|x-y|)^N |\partial^\alpha f(x-y)|\right] \\ &= (1+|y|)^N \sup_{x\in\mathbb{R}} \left[(1+|x-y|)^N |\partial^\alpha f(x-y)|\right] \\ &= (1+|y|)^N \sup_{x\in\mathbb{R}} \left[(1+|x|)^N |\partial^\alpha f(x)|\right] \\ &= (1+|y|)^N \|f\|_{\alpha,N} \end{aligned}$$

(2) Let $\xi \in \mathbb{R}$ and $\alpha, N \in \mathbb{N}_0$. Then for each $x \in \mathbb{R}$, we have that

$$\begin{aligned}(1+|x|)^N|\partial^\alpha \rho_\xi f(x)| &= (1+|x|)^N\left|\sum_{k=0}^{\alpha}\binom{\alpha}{k}(-i\xi)^{\alpha-k}\rho_\xi \partial^k f(x)\right| \\ &= (1+|x|)^N\left|\sum_{k=0}^{\alpha}\binom{\alpha}{k}(-i\xi)^{\alpha-k}e^{-i\xi x}\partial^k f(x)\right| \\ &\leq (1+|x|)^N\sum_{k=0}^{\alpha}\binom{\alpha}{k}|\xi|^{\alpha-k}|\partial^k f(x)| \\ &= \sum_{k=0}^{\alpha}\binom{\alpha}{k}|\xi|^{\alpha-k}(1+|x|)^N|\partial^k f(x)| \\ &\leq \sum_{k=0}^{\alpha}\binom{\alpha}{k}|\xi|^{\alpha-k}\|f\|_{k,N}\end{aligned}$$

Therefore,

$$\begin{aligned}\|\rho_\xi f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha \rho_\xi f(x)|\right] \\ &\leq \sum_{k=0}^{\alpha}\binom{\alpha}{k}|\xi|^{\alpha-k}\|f\|_{k,N}\end{aligned}$$

(3) Let $t \neq 0$ and $\alpha, N \in \mathbb{N}_0$. The previous exercise implies that there exists $C_t > 0$ such that for each $x \in \mathbb{R}$, $1+|x| \leq C_t(1+|tx|)^2$. Then for each $x \in \mathbb{R}$, we have that

$$\begin{aligned}(1+|x|)^N|\partial^\alpha \delta_t f(x)| &= (1+|x|)^N|t|^\alpha|\delta_t\partial^\alpha f(x)| \\ &= |t|^\alpha(1+|x|)^N|\partial^\alpha f(tx)| \\ &\leq |t|^\alpha C_t^N(1+|tx|)^{2N}|\partial^\alpha f(tx)| \\ &\leq |t|^\alpha C_t^N\|f\|_{\alpha,2N}\end{aligned}$$

Therefore

$$\begin{aligned}\|\delta_t f\|_{\alpha,N} &= \sup_{x\in\mathbb{R}}\left[(1+|x|)^N|\partial^\alpha \delta_t f(x)|\right] \\ &\leq |t|^\alpha C_t^N\|f\|_{\alpha,2N}\end{aligned}$$

□

**Exercise 1.1.16.** For each $y, \xi \in \mathbb{R}$, $t \neq 0$, we have that $\tau_y : \mathcal{S} \to \mathcal{S}$, $\rho_\xi : \mathcal{S} \to \mathcal{S}$ and $\delta_t : \mathcal{S} \to \mathcal{S}$ are

(1) linear
(2) continuous

*Proof.* Let $y, \xi \in \mathbb{R}$ and $t \neq 0$.

(1) Clear.
(2) Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$. Suppose that $f_n \to 0$. Then for each $\alpha, N \in \mathcal{N}_0$, $\|f_n\|_{\alpha,N} \to 0$.

- Let $\alpha, N \in \mathcal{N}_0$. Then
$$\begin{aligned}\|\tau_y f_n\|_{\alpha,N} &\leq (1+|y|)^N \|f_n\|_{\alpha,N} \\ &\to 0\end{aligned}$$
Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\tau_y f_n \to 0$. So $\tau_y$ is continuous at 0. Since $\tau_y$ is linear, $\tau_y$ is continuous.
- Let $\alpha, N \in \mathcal{N}_0$. Then
$$\begin{aligned}\|\rho_\xi f_n\|_{\alpha,N} &\leq \sum_{k=0}^{\alpha} \binom{\alpha}{k} |\xi|^{\alpha-k} \|f_n\|_{k,N} \\ &\to 0\end{aligned}$$
Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\rho_\xi f_n \to 0$. So $\rho_\xi$ is continuous at 0. Since $\rho_\xi$ is linear, $\rho_\xi$ is continuous.
- Let $\alpha, N \in \mathcal{N}_0$. Define $C_t$ as in the previous exercise. Then
$$\begin{aligned}\|\delta_t f_n\|_{\alpha,N} &\leq |t|^\alpha C_t^N \|f_n\|_{\alpha,2N} \\ &\to 0\end{aligned}$$
Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\delta_t f_n \to 0$. So $\delta_t$ is continuous at 0. Since $\delta_t$ is linear, $\delta_t$ is continuous.

□

**Definition 1.1.17.** Define $\tau : \mathbb{R} \to \mathcal{S}$ by $\tau f(y) = \tau_y f$

**Exercise 1.1.18.** Let $f \in \mathcal{S}$. Then $\tau f : \mathbb{R} \to \mathcal{S}$ is a homomorphism

**Note 1.1.19.** Let $f, g \in \mathcal{S}$ and $x \in \mathbb{R}$, Define $h : \mathbb{R} \to \mathbb{R}$ defined by $h_x(y) = f(x-y)g(y)$. A previous exercise implies that $h_x = (\delta_{-1}\tau_x f)g \in \mathcal{S} \subset L^1(m)$ and for each $\alpha, N \in \mathbb{N}_0$, $\|h_x\|_{\alpha,N} \leq \sum_{\beta=0}^{\alpha}(1+|x|)^N \|f\|_{\beta,N}\|g\|_{\alpha-\beta,0}$
**FINISH FIX THIS!!!**

**Definition 1.1.20.** Let $f, g \in \mathcal{S}$. We define the **convolution of $f$ and $g$**, denoted $f * g : \mathbb{R} \to \mathbb{R}$ by
$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dm(y)$$

**Exercise 1.1.21.** Let $f, g \in \mathcal{S}$. Then for each $\alpha \in \mathbb{N}_0$, $\partial^\alpha (f * g) = (\partial^\alpha f) * g$.

*Proof.* The claim is clear if $\alpha = 0$. Suppose that $\alpha > 0$ and that the claim is true for $\alpha - 1$ so that $\partial^{\alpha-1}(f * g) = (\partial^{\alpha-1} f) * g$. Define $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(x,y) = \partial_x^{\alpha-1} f(x-y)g(y)$. Then for each $x, y \in \mathbb{R}$,
$$\begin{aligned}|h(x,y)| &= |\partial_x^{\alpha-1} f(x-y)g(y)| \\ &\leq \|\tau_y f\|_{\alpha-1,0}|g(y)| \\ &\leq \|f\|_{\alpha-1,0}|g(y)|\end{aligned}$$

Since $g \in L^1(m)$, we may differentiate under the integral to obtain that

$$\begin{aligned}
[\partial_x^\alpha (f * g)](x) &= \partial_x [\partial_x^{\alpha-1}(f * g)](x) \\
&= \partial_x [(\partial_x^{\alpha-1} f) * g](x) \\
&= \partial_x \int_{\mathbb{R}} \partial_x^{\alpha-1} f(x-y) g(y) \, dm(y) \\
&= \int_{\mathbb{R}} \partial_x [\partial_x^{\alpha-1} f(x-y) g(y)] \, dm(y) \\
&= \int_{\mathbb{R}} \partial_x^{\alpha} f(x-y) g(y) \, dm(y) \\
&= [(\partial_x^\alpha f) * g](x)
\end{aligned}$$

So the claim is true for $\alpha$. □

**Exercise 1.1.22.** Let $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$ and there exists $C > 0$ such that for each $\alpha, N \in \mathbb{N}_0$, $\|f * g\|_{\alpha,N} \leq C \|f\|_{\alpha,N} \|g\|_{0,N+2}$.

*Proof.* Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|y|)^2} \, dm(y)$$

Let $\alpha, N \in \mathbb{N}_0$ and $x \in \mathbb{R}$. Then

$$\begin{aligned}
(1+|x|)^N |\partial^\alpha (f * g)(x)| &= (1+|x|)^N |(\partial^\alpha f) * g(x)| \\
&= (1+|x|)^N \left| \int_{\mathbb{R}} \partial^\alpha f(x-y) g(y) \, dm(y) \right| \\
&\leq \int_{\mathbb{R}} (1+|x|)^N |\partial^\alpha f(x-y) g(y)| \, dm(y) \\
&\leq \int_{\mathbb{R}} (1+|y|)^N (1+|x-y|)^N |\partial^\alpha f(x-y)| |g(y)| \, dm(y) \\
&\leq \|f\|_{\alpha,N} \int_{\mathbb{R}} (1+|y|)^N |g(y)| \, dm(y) \\
&= \|f\|_{\alpha,N} \int_{\mathbb{R}} (1+|y|)^{N+2} \frac{|g(y)|}{(1+|y|)^2} \, dm(y) \\
&\leq \|f\|_{\alpha,N} \|g\|_{0,N+2} \int_{\mathbb{R}} \frac{1}{(1+|y|)^2} \, dm(y) \\
&= C \|f\|_{\alpha,N} \|g\|_{0,N+2}
\end{aligned}$$

Since $x \in \mathbb{R}$ is arbitrary, we have that

$$\begin{aligned}
\|f * g\|_{\alpha,N} &= \sup_{x \in \mathbb{R}} \left[ (1+|x|)^N |\partial^\alpha (f * g)(x)| \right] \\
&\leq C \|f\|_{\alpha,N} \|g\|_{0,N+2}
\end{aligned}$$

□

**Exercise 1.1.23.** The convolution $* : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$

(1) is bilinear
(2) is continuous

*Proof.*

(1) Clear.

(2) Let $(f_n, g_n)_{n\in\mathbb{N}} \subset \mathcal{S} \times \mathcal{S}$ and $(f, g) \in \mathcal{S} \times \mathcal{S}$. Suppose that $(f_n, g_n) \to (f, g)$. Then $f_n \to f$ and $g_n \to g$. Hence for each $\alpha, N \in \mathbb{N}_0$, $\|f_n - f\|_{\alpha,N} \to 0$ and $\|g_n - g\|_{\alpha,N} \to 0$. In particular

$$\begin{aligned}\Big| \|g_n\|_{0,N+2} - \|g\|_{0,N+2} \Big| &\leq \|g_n - g\|_{0,N+2} \\ &\to 0\end{aligned}$$

So that $(\|g_n\|_{0,N+2})_{n\in\mathbb{N}}$ is bounded. Let $\alpha, N \in \mathbb{N}_0$. Define $C > 0$ as in the previous exercise. Then

$$\begin{aligned}\|f_n * g_n - f * g\|_{\alpha,N} &= \|f_n * g_n - f * g_n + f * g_n - f * g\|_{\alpha,N} \\ &\leq \|(f_n - f) * g_n\|_{\alpha,N} + \|f_*(g_n - g)\|_{\alpha,N} \\ &\leq C\|f_n - f\|_{\alpha,N}\|g_n\|_{0,N+2} + C\|f\|_{\alpha,N}\|g_n - g\|_{0,N+2} \\ &\to 0\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $f_n * g_n \to f * g$. Thus $* : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is continuous.

□

**Exercise 1.1.24.** Let $f, g \in \mathcal{S}$. Then $\|f * g\|_1 \leq \|f\|_1\|g\|_1$.

*Proof.* Tonelli's theorem implies that

$$\begin{aligned}\|f * g\|_1 &= \int_{\mathbb{R}} |f * g(x)|\, dm(x) \\ &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)g(y)\, dm(y) \right| dm(x) \\ &\leq \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |f(x-y)g(y)|\, dm(y) \right] dm(x) \\ &= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |f(x-y)g(y)|\, dm(x) \right] dm(y) \\ &= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |f(x-y)|\, dm(x) \right] |g(y)|\, dm(y) \\ &= \|f\|_1 \int_{\mathbb{R}} |g(y)|\, dm(y) \\ &= \|f\|_1\|g\|_1\end{aligned}$$

□

**Exercise 1.1.25.** Let $f, g \in \mathcal{S}$, then $f * g = g * f$.

*Proof.* Let $x \in R$. Define $a, b : \mathbb{R} \to \mathbb{R}$ by $a(z) = f(z)g(x - z)$ and $b(y) = x - y$. Then for each $A \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}b_*m(A) &= m(b^{-1}(A)) \\ &= m(x - A) \\ &= m(A)\end{aligned}$$

So $b_*m = m$ and

$$\begin{aligned} f * g(x) &= \int_{\mathbb{R}} f(x-y)g(y)\, dm(y) \\ &= \int_{b^{-1}(\mathbb{R})} a \circ b \, dm \\ &= \int_{\mathbb{R}} a \, db_* m \\ &= \int_{\mathbb{R}} a \, dm \\ &= \int_{\mathbb{R}} g(x-z) f(z)\, dm(z) \\ &= g * f(x) \end{aligned}$$

Since $x \in \mathbb{R}$ is arbitrary, $f * g = g * f$. □

**Definition 1.1.26.** We define the **bump functions** on $\mathbb{R}$, denoted $C_c^\infty(\mathbb{R})$, by

$$C_c^\infty(\mathbb{R}) = C_c(\mathbb{R}) \cap C^\infty(\mathbb{R})$$

**Exercise 1.1.27.** Let $f \in C_c^\infty(\mathbb{R})$. Then $f \in \mathcal{S}$.

*Proof.* Let $\alpha, N \in \mathbb{N}^0$. Define $g : \mathbb{R} \to \mathbb{C}$ by

$$g(x) = (1 + |x|)^N |\partial^\alpha f(x)|$$

Then $g$ is continuous. Since $\mathrm{supp}(\partial^\alpha f) \subset \mathrm{supp}(f)$, we have that $g \in C_c(\mathbb{R})$ and

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left[ (1 + |x|)^N |\partial^\alpha f| \right] &= \sup_{x \in \mathbb{R}} g(x) \\ &= \|g\| \\ &< \infty \end{aligned}$$

□

**Exercise 1.1.28.** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-x^2}$. Then $f \in \mathcal{S}$.

*Proof.* meh... □

**Exercise 1.1.29.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & x \in (-1, 1) \\ 0 & x \notin (-1, 1) \end{cases}$$

Then $f \in \mathcal{S}$.

*Proof.* meh... □

**Exercise 1.1.30.** Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Then for each $\epsilon > 0$, there exists $f \in \mathcal{S}$ such that $\chi_{[a,b]} \leq f \leq \chi_{[a-\epsilon, b+\epsilon]}$.

*Proof.* Set $f(x) =$ □

**Exercise 1.1.31.** Let $f \in \mathcal{S}$. Define

## 1.2. The Fourier Transform on $\mathcal{S}$.

**Exercise 1.2.1.** Let $\phi : \mathbb{R} \to S^1$ be a measurable homomorphism.

(1) Then $\phi \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $a > 0$ such that

$$\int_{(0,a]} \phi dm \neq 0$$

(2) Define

$$c = \left[\int_{(0,a]} \phi dm\right]^{-1}$$

Then For each $x \in \mathbb{R}$,

$$\phi(x) = c\int_{(x,x+a]} \phi dm$$

(3) $\phi \in C^\infty(\mathbb{R})$ and $\phi' = c(\phi(a) - 1)\phi$

(4) Define $b = c(\phi(a) - 1)$ and $g \in C^\infty(\mathbb{R})$ by $g(x) = e^{-bx}\phi(x)$. Then $g$ is constant and there exists $\xi \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

*Proof.*

(1) Let $K \subset \mathbb{R}$ be compact. Then

$$\int_K |\phi| dm = m(K) < \infty$$

So $\phi \in L^1_{\text{loc}}(\mathbb{R})$. For the sake of contradiction, suppose that for each $a > 0$,

$$\int_{(0,a]} \phi dm = 0$$

Then the FTC implies that $\phi = 0$ a.e. on $[0, \infty)$, which is a contradiction. So there exists $a > 0$ such that

$$\int_{(0,a]} \phi dm \neq 0$$

(2) For $x \in \mathbb{R}$,

$$\begin{aligned}\phi(x) &= c\int_{(0,a]} \phi(x)\phi(t)dm(t)\\ &= c\int_{(0,a]} \phi(x+t)dm(t)\\ &= c\int_{(x,x+a]} \phi dm\end{aligned}$$

(3) Part (2) and the FTC imply that $\phi$ is continuous. Let $d \in \mathbb{R}$. Define $f_d \in C((d, \infty))$ by

$$f_d(x) = \int_{(d,x]} \phi dm$$

Since $\phi$ is continuous, the FTC implies that $f_d$ is differentiable and for each $x > d$ $f_d'(x) = \phi(x)$. Part (2) implies that for each $x > d$,

$$\begin{aligned}\phi(x) &= c\int_{(x,x+a]} \phi dm \\ &= c(f_d(x+a) - f_d(x))\end{aligned}$$

So for each $x > d$, $\phi$ is differentiable at $x$ and

$$\begin{aligned}\phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x)\end{aligned}$$

Since $d \in \mathbb{R}$ is arbitrary, $\phi$ is differentiable and $\phi' = c(\phi(a) - 1)\phi$. This implies that $\phi \in C^\infty(\mathbb{R})$.

(4) Let $x \in \mathbb{R}$. Then

$$\begin{aligned}g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0\end{aligned}$$

So $g' = 0$ and $g$ is constant. Hence there exists $k \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = ke^{bx}$. Since $\phi(0) = 1$, $k = 1$. Since $|\phi| = 1$, there exists $\xi \in \mathbb{R}$ such that $b = 2\pi i\xi$.

□

**Note 1.2.2.** To summarize, for each measurable homomorphism $\phi : \mathbb{R} \to S^1$, there exists $\xi \in \mathbb{R}$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

**Definition 1.2.3.** Let $f \in \mathcal{S}$. We define the **Fourier transform of** $f$, denoted $\hat{f} : \mathbb{R} \to \mathbb{C}$, by

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x)$$

**Exercise 1.2.4.** Let $f \in \mathcal{S}$. Then $\hat{f} \in C_b(\mathbb{R})$.

*Proof.* Since $f \in \mathcal{S}$, $f \in L^1(m)$. Then for each $\xi \in \mathbb{R}$,

$$\begin{aligned}|\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x)\right| \\ &\leq \int_{\mathbb{R}} |e^{-i\xi x} f(x)|\, dm(x) \\ &= \int_{\mathbb{R}} |f(x)|\, dm(x) \\ &= \|f\|_1\end{aligned}$$

So $f$ is bounded. Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. Suppose that $\xi_n \to \xi$. Define $(\phi_n)_{n\in\mathbb{N}} \subset L^1(m)$ and $\phi \in L^1(m)$ by $\phi_n(x) = e^{-i\xi_n x} f(x)$ and $\phi(x) = e^{-i\xi x} f(x)$. Then $\phi_n \xrightarrow{\text{p.w.}} \phi$ and for each $n \in \mathbb{N}$,

$$\begin{aligned}|\phi_n| &= |f| \\ &\in L^1(m)\end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}
\hat{f}(\xi_n) &= \int_{\mathbb{R}} e^{-i\xi_n x} f(x)\, dm(x) \\
&= \int_{\mathbb{R}} \phi_n\, dm \\
&\to \int_{\mathbb{R}} \phi\, dm \\
&= \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x) \\
&= \hat{f}(\xi)
\end{aligned}$$

So $\hat{f}$ is continuous. Hence $\hat{f} \in C_b(\mathbb{R})$. $\square$

**Definition 1.2.5.** We define the **Fourier transform on $\mathcal{S}$**, denoted $\mathcal{F} : \mathcal{S} \to C_b(\mathbb{R})$, by

$$\mathcal{F}(f) = \hat{f}$$

**Exercise 1.2.6.** We have that $\mathcal{F} : \mathcal{S} \to C_b(\mathbb{R})$ is linear.

*Proof.* Let $f, g \in \mathcal{S}$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}
\mathcal{F}(f + \lambda g) &= \int_{\mathbb{R}} e^{-i\xi x}[f(x) + \lambda g(x)]\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i\xi x} f(x) + \lambda e^{-i\xi x} g(x)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x) + \lambda \int_{\mathbb{R}} e^{-i\xi x} g(x)\, dm(x) \\
&= \mathcal{F}(f) + \lambda \mathcal{F}(g)
\end{aligned}$$

$\square$

**Exercise 1.2.7.** Let $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}^0$. Then

(1) $\mathcal{F}(X^\alpha f) = (-1)^\alpha D^\alpha \mathcal{F}(f)$
(2) $\mathcal{F}(D^\alpha f) = X^\alpha \mathcal{F}(f)$

*Proof.*

(1) The claim is clear for $\alpha = 0$. Suppose that $\alpha > 0$ and that the claim is true for $\alpha - 1$ so that $\mathcal{F}(X^{\alpha-1} f) = (-1)^{\alpha-1} D^{\alpha-1} \mathcal{F}(f)$. Define $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\phi(\xi, x) = e^{-i\xi x} x^{\alpha-1} f(x)$. Then for each $\xi, x \in \mathbb{R}$,

$$\begin{aligned}
|\partial_\xi \phi(\xi, x)| &= |-ixe^{-i\xi x} x^{\alpha-1} f(x)| \\
&= |x^\alpha f(x)| \\
&= |(X^\alpha f)(x)|
\end{aligned}$$

Since $X^\alpha f \in \mathcal{S} \subset L^1$, we may switch the order of differentiation and integration to obtain

$$\begin{aligned}\mathcal{F}(X^\alpha f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} x^\alpha f(x) dm(x) \\ &= \int_{\mathbb{R}} i\partial_\xi \left[ e^{-i\xi x} x^{\alpha-1} f(x) \right] dm(x) \\ &= i\partial_\xi \left[ \int_{\mathbb{R}} e^{-i\xi x} x^{\alpha-1} f(x) dm(x) \right] \\ &= i\partial_\xi \mathcal{F}(X^{\alpha-1} f)(\xi) \\ &= -D\mathcal{F}(X^{\alpha-1} f)(\xi) \\ &= (-1)^\alpha D^\alpha \mathcal{F}(f)(\xi)\end{aligned}$$

So the claim is true for $\alpha$.

(2) The claim is clear for $\alpha = 0$. Suppose that $\alpha > 0$ and that the claim is true for $\alpha - 1$ so that $\mathcal{F}(D^{\alpha-1} f) = X^{\alpha-1}\mathcal{F}(f)$. Then integration by parts yields

$$\begin{aligned}\mathcal{F}(D^\alpha f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} [-i\partial_x D^{\alpha-1} f(x)]\, dm(x) \\ &= -\int_{\mathbb{R}} -i\xi e^{-i\xi x} [-iD^{\alpha-1} f(x)]\, dm(x) \\ &= \xi \int_{\mathbb{R}} e^{-i\xi x} D^{\alpha-1} f(x)\, dm(x) \\ &= X\mathcal{F}(D^{\alpha-1} f)(\xi) \\ &= X^\alpha \mathcal{F}(f)(\xi)\end{aligned}$$

So the claim is true for $\alpha$.

□

**Exercise 1.2.8.** Let $P()$

*Proof.* content... □

**Exercise 1.2.9.** There exists $C > 0$ such that for each $f \in \mathcal{S}$, $\|\hat{f}\|_{0,0} \le C\|f\|_{0,2}$.
**Hint:** Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

*Proof.* Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

Let $f \in \mathcal{S}$. Let $\xi \in \mathbb{R}$. Then

$$\begin{aligned} |\hat{f}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dm(x) \right| \\ &\leq \int_{\mathbb{R}} |f(x)|\, dm(x) \\ &= \int_{\mathbb{R}} \frac{(1+|x|)^2 |f(x)|}{(1+|x|)^2}\, dm(x) \\ &\leq \|f\|_{0,2} \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x) \\ &= C\|f\|_{0,2} \end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\|\hat{f}\|_{0,0} \leq \|f\|_{0,2}$. □

**Exercise 1.2.10.** Let $a, b \in \mathbb{R}$ and $N \in \mathbb{N}_0$. Then $(a+b)^N \leq 2^{N-1}(a^N + b^N)$.
**Hint:** Jensen's inequality

*Proof.* Jensen's inequality implies that

$$\begin{aligned} 2^{-N}(a+b)^N &= \left(\frac{a}{2} + \frac{b}{2}\right)^N \\ &\leq \left(\frac{a^N}{2} + \frac{b^N}{2}\right) \\ &= 2^{-1}(a^N + b^N) \end{aligned}$$

So $(a+b)^N \leq 2^{N-1}(a^N + b^N)$. □

**Exercise 1.2.11.** We have that $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$ and $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is continuous.

*Proof.* Let $f \in \mathcal{S}$ and $\alpha, N \in \mathbb{N}_0$. Then the previous exercise implies that for each $\xi \in \mathbb{R}$,

$$\begin{aligned} \xi^N \partial_\xi^\alpha \mathcal{F}(f)(\xi) &= (-i)^\alpha X^N D^\alpha \mathcal{F}(f)(\xi) \\ &= i^\alpha X^N \mathcal{F}(X^\alpha f)(\xi) \\ &= i^\alpha \mathcal{F}(D^N X^\alpha f)(\xi) \end{aligned}$$

Set

$$C = \int_{\mathbb{R}} \frac{1}{(1+|x|)^2}\, dm(x)$$

as in the previous exercise. Since $\mathcal{F}(X^\alpha f)$, $\mathcal{F}(D^N X^\alpha f) \in C_b(\mathbb{R})$, we have that

$$\begin{aligned}
\|\mathcal{F}(f)\|_{\alpha,N} &= \sup_{\xi\in\mathbb{R}} \left[(1+|\xi|)^N |\partial_\xi^\alpha \mathcal{F}(f)(\xi)|\right] \\
&\leq \sup_{\xi\in\mathbb{R}} \left[2^{N-1}(1+|\xi|^N)|\partial_\xi^\alpha \mathcal{F}(f)(\xi)|\right] \\
&= \sup_{\xi\in\mathbb{R}} \left[|2^{N-1}\partial_\xi^\alpha \mathcal{F}(f)(\xi)| + |2^{N-1}\xi^N \partial_\xi^\alpha \mathcal{F}(f)(\xi)|\right] \\
&= \sup_{\xi\in\mathbb{R}} \left[|\mathcal{F}(2^{N-1}X^\alpha f)(\xi)| + |\mathcal{F}(2^{N-1}D^N X^\alpha f)(\xi)|\right] \\
&\leq \|\mathcal{F}(2^{N-1}X^\alpha f)\|_{0,0} + \|\mathcal{F}(2^{N-1}D^N X^\alpha f)\|_{0,0} \\
&\leq C2^{N-1}\|X^\alpha f\|_{0,2} + C2^{N-1}\|D^N X^\alpha f\|_{0,2} \qquad < \infty
\end{aligned}$$

Since $\alpha, N \in \mathbb{N}_0$ are arbitrary, $\mathcal{F}(f) \in \mathcal{S}$ and since $f \in \mathcal{S}$ is arbitrary, $\mathcal{F}(\mathcal{S}) \subset \mathcal{S}$. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}$. Suppose that $f_n \to 0$. Since $X, D : \mathcal{S} \to \mathcal{S}$ are continuous, $X^\alpha f_n \to 0$ and $D^N X^\alpha f_n \to 0$. Therefore, $\|X^\alpha f_n\|_{0,2} \to 0$ and $\|D^N X^\alpha f_n\|_{0,2} \to 0$. From above, we see that

$$\begin{aligned}
\|\mathcal{F}(f_n)\|_{\alpha,N} &\leq C2^{N-1}\|X^\alpha f_n\|_{0,2} + C2^{N-1}\|D^N X^\alpha f_n\|_{0,2} \\
&\to 0
\end{aligned}$$

Hence $\mathcal{F}(f_n) \to 0$ and $\mathcal{F}$ is continuous. $\square$

**Exercise 1.2.12.** Let $f \in \mathcal{S}$. Then

1. for each $y \in \mathbb{R}$, $\mathcal{F}(\tau_y f) = \rho_y \mathcal{F}(f)$
2. for each $\eta \in \mathbb{R}$, $\mathcal{F}(\rho_\eta f) = \tau_{-\eta}\mathcal{F}(f)$
3. $\mathcal{F}(\delta_t f) = t^{-1}\delta_{t^{-1}}\mathcal{F}(f)$

*Proof.*

1. Let $y, \xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(\tau_y f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} f(x-y)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i\xi(z+y)} f(z)\, dm(z) \\
&= e^{-i\xi y}\int_{\mathbb{R}} e^{-i\xi z} f(z)\, dm(z) \\
&= e^{-i\xi y}\mathcal{F}(f)(\xi) \\
&= \rho_y \mathcal{F}(f)(\xi)
\end{aligned}$$

2. Let $\eta, \xi \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}(\rho_\eta f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} e^{-i\eta x} f(x)\, dm(x) \\
&= \int_{\mathbb{R}} e^{-i(\xi+\eta)x} f(x)\, dm(x) \\
&= \mathcal{F}(f)(\xi+\eta) \\
&= \tau_{-\eta}\mathcal{F}(f)(\xi)
\end{aligned}$$

(3) Let $\xi \in \mathbb{R}$. Then

$$\begin{aligned}\mathcal{F}(\delta_t f)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x} f(tx)\, dm(x) \\ &= \int_{\mathbb{R}} e^{-i\xi t^{-1} z} f(z) t^{-1}\, dm(z) \\ &= t^{-1}\mathcal{F}(f)(t^{-1}\xi) \\ &= t^{-1}\delta_{t^{-1}}\mathcal{F}(f)(\xi)\end{aligned}$$

□

**Exercise 1.2.13.** Let $f, g \in \mathcal{S}$. Then $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.

*Proof.* Let $\xi \in \mathbb{R}$. Tonelli's theorem implies that

$$\begin{aligned}\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |e^{-i\xi x} f(x-y) g(y)|\, dm(y)\right] dm(x) &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)g(y)|\, dm(y)\right] dm(x) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)g(y)|\, dm(x)\right] dm(y) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)|\, dm(x)\right] |g(y)|\, dm(y) \\ &= \|f\|_1 \int_{\mathbb{R}} |g(y)|\, dm(y) \\ &= \|f\|_1 \|g\|_1\end{aligned}$$

So we may apply Fubini's theorem and change the order of integration to obtain that

$$\begin{aligned}\mathcal{F}(f * g)(\xi) &= \int_{\mathbb{R}} e^{-i\xi x}(f * g)(x)\, dm(x) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y) g(y)\, dm(y)\right] dm(x) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y) g(y)\, dm(x)\right] dm(y) \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} e^{-i\xi x} f(x-y)\, dm(x)\right] g(y)\, dm(y) \\ &= \int_{\mathbb{R}} [\mathcal{F}(\tau_y f)(\xi)] g(y)\, dm(y) \\ &= \int_{\mathbb{R}} [e^{-i\xi y}\mathcal{F}(f)(\xi)] g(y)\, dm(y) \\ &= \mathcal{F}(f)(\xi) \int_{\mathbb{R}} e^{-i\xi y} g(y)\, dm(y) \\ &= \mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$ □

**Exercise 1.2.14.** Let $f, g \in \mathcal{S}$. Then

$$\int_{\mathbb{R}} \hat{f} g\, dm = \int_{\mathbb{R}} f \hat{g}\, dm$$

*Proof.* Tonelli's theorem implies that

$$\begin{aligned}\int_{\mathbb{R}}\left[\int_{\mathbb{R}}|e^{-i\xi x}f(x)g(\xi)|\,dm(x)\right]dm(\xi) &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}|f(x)|\,dm(x)\right]|g(\xi)|\,dm(\xi)\\ &= \|f\|_1\int_{\mathbb{R}}|g(\xi)|\,dm(\xi)\\ &= \|f\|_1\|g\|_1\end{aligned}$$

So we may apply Fubini's theorem and switch the order of integration to obtain that

$$\begin{aligned}\int_{\mathbb{R}}\hat{f}g\,dm &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}e^{-i\xi x}f(x)\,dm(x)\right]g(\xi)\,dm(\xi)\\ &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}e^{-i\xi x}f(x)g(\xi)\,dm(x)\right]dm(\xi)\\ &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}e^{-i\xi x}f(x)g(\xi)\,dm(\xi)\right]dm(x)\\ &= \int_{\mathbb{R}}f(x)\left[\int_{\mathbb{R}}e^{-i\xi x}g(\xi)\,dm(\xi)\right]dm(x)\\ &= \int_{\mathbb{R}}f(x)\hat{g}(x)\,dm(x)\\ &= \int_{\mathbb{R}}f\hat{g}\,dm\end{aligned}$$

□

**Exercise 1.2.15.** Define $f \in \mathcal{S}$ by $f(x) = e^{-x^2/2}$. Then $\mathcal{F}(f) = \sqrt{2\pi}f$.

*Proof.* Note that for each $\xi \in \mathbb{R}$,

$$\begin{aligned}\mathcal{F}(Df)(\xi) &= \int_{\mathbb{R}}e^{-i\xi x}ixe^{-x^2/2}\,dm(x)\\ &= -\int_{\mathbb{R}}\partial_\xi\left[e^{-i\xi x}e^{-x^2/2}\right]dm(x)\\ &= -\partial_\xi\mathcal{F}(f)(\xi)\end{aligned}$$

A previous exercise implies that $\mathcal{F}(Df) = X\mathcal{F}(f)$. So for each $\xi \in \mathbb{R}$, $\partial_\xi\hat{f}(\xi) = -\xi\hat{f}(\xi)$. Define $g \in \mathbb{C}^\infty(\mathbb{R})$ by $g(\xi) = e^{\xi^2/2}$. Then

$$\begin{aligned}\partial_\xi(\hat{f}g) &= (\partial_\xi\hat{f})g + \hat{f}(\partial_\xi g)\\ &= 0\end{aligned}$$

So there exists $C \in \mathbb{R}$ such that $\hat{f}g = C$. Hence for each $\xi \in \mathbb{R}$,

$$\begin{aligned}\hat{f}(\xi) &= Ce^{-\xi^2/2}\\ &= Cf(\xi)\end{aligned}$$

Therefore,

$$\begin{aligned} C &= Cf(0) \\ &= \hat{f}(0) \\ &= \int_{\mathbb{R}} e^{-x^2/2}\, dm(x) \\ &= \sqrt{2\pi} \end{aligned}$$

So $\hat{f} = \sqrt{2\pi} f$. □

**Exercise 1.2.16.** Let $f \in \mathcal{S}$. Define $g : \mathbb{R} \to L^1$ by $g(x) = \tau_x f$. Then $g$ is continuous.
**Hint:** approximate by functions in $C_c(\mathbb{R})$.

*Proof.* Suppose that $f \in C_c(\mathbb{R})$. Then □

**Definition 1.2.17.** Let $f \in \mathcal{S}$ and $t \neq 0$. We define $f_t \in \mathcal{S}$ by $f_t = t^{-1}\delta_{t^{-1}} f$.

**Exercise 1.2.18.** Let $\phi \in \mathcal{S}$ and $t \neq 0$. Then

$$\int_{\mathbb{R}} \phi_t \, dm = \int_{\mathbb{R}} \phi \, dm$$

*Proof.* We have that

$$\begin{aligned} \int_{\mathbb{R}} \phi_t \, dm &= \int_{\mathbb{R}} t^{-1}\phi(t^{-1}x)\, dm(x) \\ &= \int_{\mathbb{R}} \phi(z)\, dm(z) \\ &= \int_{\mathbb{R}} \phi \, dm \end{aligned}$$

□

**Exercise 1.2.19.** Let $\phi \in \mathcal{S}$. Set

$$\alpha = \int_{\mathbb{R}} \phi \, dm$$

Then for each $f \in \mathcal{S}$, $f * \phi_{1/n} \xrightarrow{L^1} \alpha f$.
**Hint:** for each $t \neq 0$ and $x \in \mathbb{R}$,

$$f * \phi_t(x) - \alpha f(x) = \int_{\mathbb{R}} [\tau_{tz} f(x) - f(x)]\phi(z)\, dm(z)$$

*Proof.* Let $t \neq 0$ and $x \in \mathbb{R}$. The previous exercise implies that

$$\begin{aligned} f * \phi_t(x) - \alpha f(x) &= \int_{\mathbb{R}} f(x-y)\phi_t(y)\,dm(y) - \int_{\mathbb{R}} \phi(y)\,dm(y) f(x) \\ &= \int_{\mathbb{R}} f(x-y)\phi_t(y)\,dm(y) - \int_{\mathbb{R}} \phi_t(y)\,dm(y) f(x) \\ &= \int_{\mathbb{R}} f(x-y)\phi_t(y) - f(x)\phi_t(y)\,dm(y) \\ &= \int_{\mathbb{R}} [f(x-y) - f(x)]\phi_t(y)\,dm(y) \\ &= \int_{\mathbb{R}} [f(x-y) - f(x)]t^{-1}\phi(t^{-1}y)\,dm(y) \\ &= \int_{\mathbb{R}} [f(x-tz) - f(x)]\phi(z)\,dm(z) \\ &= \int_{\mathbb{R}} [\tau_{tz} f(x) - f(x)]\phi(z)\,dm(z) \end{aligned}$$

Tonelli's theorem implies that

$$\begin{aligned} \|f * \phi_t - \alpha f\|_1 &= \int_{\mathbb{R}} |f * \phi_t(x) - \alpha f(x)|\,dm(x) \\ &\leq \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |\tau_{tz} f(x) - f(x)||\phi(z)|\,dm(z) \right] dm(x) \\ &= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |\tau_{tz} f(x) - f(x)||\phi(z)|\,dm(x) \right] dm(z) \\ &= \int_{\mathbb{R}} \|\tau_{tz} f - f\|_1 |\phi(z)|\,dm(z) \end{aligned}$$

For $n \in \mathbb{N}$, define $g_n \in \mathcal{S}$ by $g_n(z) = \|\tau_{n^{-1}z} f(x) - f(x)\|_1 \phi(z)$. Then $g_n \xrightarrow{\text{p.w.}} 0$ and

$$\begin{aligned} |g_n| &\leq 2\|f\|_1 |\phi| \\ &\in L^1(m) \end{aligned}$$

The dominated convergence theorem implies that

$\square$

**Definition 1.2.20.** content...

## 1.3. The Fourier Transform on $\mathcal{M}(\mathbb{R})$.

**Note 1.3.1.** Recall that

$$\mathcal{M}(\mathbb{R}) = \{\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{C} : \mu \text{ is a complex measure}\}$$

**Definition 1.3.2.** Let $\mu \in \mathcal{M}(\mathbb{R})$. We define the **Fourier transform of** $\mu$, denoted $\hat{\mu} : \mathbb{R} \to \mathbb{C}$, by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x)$$

**Exercise 1.3.3.** Let $\mu \in \mathcal{M}(\mathbb{R})$. Then Then $\hat{\mu} : \mathbb{R} \to \mathbb{C}$ is bounded.

*Proof.* Let $\xi \in \mathbb{R}$.

$$\begin{aligned}
|\hat{\mu}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) \right| \\
&\leq \int_{\mathbb{R}} |e^{-i\xi x}|\, d|\mu|(x) \\
&= |\mu|(\mathbb{R})
\end{aligned}$$

So $\hat{\mu}$ is bounded. □

**Exercise 1.3.4.** Let $\mu \in \mathcal{M}(\mathbb{R})$. Then $\hat{\mu} \in C_b(\mathbb{R})$.

*Proof.* Let $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\xi \in \mathbb{R}$. Define $(f_n)_{n\in\mathbb{N}} \subset L^1(\mu)$ and $f \in L^1(\mu)$ by $f_n(x) = e^{-i\xi_n x}$ and $f(x) = e^{-i\xi x}$. Suppose that $\xi_n \to \xi$. Then $f_n \xrightarrow{\text{p.w.}} f$ and for each $n \in N$ and $x \in \mathbb{R}$,

$$\begin{aligned}
|f_n(x)| &= |e^{-i\xi_n x}| \\
&= 1 \\
&\in L^1(|\mu|)
\end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}
|\hat{\mu}(\xi_n) - \hat{\mu}(\xi)| &= \left| \int_{\mathbb{R}} e^{-i\xi_n x}\, d\mu(x) - \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) \right| \\
&= \left| \int_{\mathbb{R}} e^{-i\xi_n x} - e^{-i\xi x}\, d\mu(x) \right| \\
&\leq \int_{\mathbb{R}} |e^{-i\xi_n x} - e^{-i\xi x}|\, d|\mu|(x) \\
&\to 0
\end{aligned}$$

So $\hat{\mu} : \mathbb{R} \to \mathbb{C}$ is continuous. Hence $\hat{\mu} \in C_b(\mathbb{R})$. □

**Definition 1.3.5.** Let $X$ be a real normed vector space. We define $\mathcal{F} : \mathcal{M}(\mathbb{R}) \to C_b(\mathbb{R})$ by

$$\mathcal{F}(\mu) = \hat{\mu}$$

**Exercise 1.3.6.** Let $X$ be a real normed vector space. Then $\mathcal{F} : \mathcal{M}(\mathbb{R}) \to C_b(\mathbb{R})$ is linear.

*Proof.* Let $\mu, \nu \in \mathcal{M}(\mathbb{R})$ and $\xi \in \mathbb{R}$. Then

$$\begin{aligned}\mathcal{F}[\mu+\nu](\xi) &= \int_{\mathbb{R}} e^{-i\xi x}\, d[\mu+\nu](x) \\ &= \int_{\mathbb{R}} e^{-i\xi x}\, d\mu(x) + \int_{\mathbb{R}} e^{-i\xi x}\, d\nu(x) \\ &= \mathcal{F}[\mu](\xi) + \mathcal{F}[\nu](\xi)\end{aligned}$$

Since $\xi \in \mathbb{R}$ is arbitrary, $\mathcal{F}(\mu+\nu) = \mathcal{F}(\mu) + \mathcal{F}(\nu)$ and $\mathcal{F}$ is linear. $\square$

**Exercise 1.3.7.** Let $X$ be a real normed vector space. If $X$ is separable, then $\mathcal{F}$ is injective.

*Proof.* Suppose that $X$ is separable. Let $\mu \in \mathcal{M}(X)$. Suppose that $\mu \in \ker \mathcal{F}$. Then $\hat{\mu} = 0$ and for each $\phi \in X^*$,

$$\begin{aligned}0 &= \hat{\mu}(\phi) \\ &= \int_X e^{-i\phi(x)}\, d\mu(x) \\ &= \int_{\mathbb{R}} e^{-ix}\, d[\phi_*\mu](x)\end{aligned}$$

$\square$

**Exercise 1.3.8.** Let $X$ be a real normed vector space. Then $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$.

*Proof.* For $\mu \in \mathcal{M}(X)$ and $\phi \in X^*$, we have that

$$\begin{aligned}|\mathcal{F}[\mu](\phi)| &= \left| \int_X e^{-i\phi(x)}\, d\mu(x) \right| \\ &\leq \int_X |e^{-i\phi(x)}|\, d|\mu|(x) \\ &= |\mu|(X) \\ &= \|\mu\|\end{aligned}$$

Hence

$$\begin{aligned}\|\mathcal{F}(\mu)\| &= \sup_{\phi \in X^*} |\mathcal{F}[\mu](\phi)| \\ &\leq \|\mu\|\end{aligned}$$

which implies that $\mathcal{F} \in L(\mathcal{M}(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$. $\square$

## 2. Fourier Analysis on $\mathbb{R}^n$

### 2.1. **Schwartz Space.**

**Definition 2.1.1.** Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

(1) $\langle x, y\rangle = \sum_j x_j y_j$
(2) $|x| = \langle x, x\rangle^{1/2}$
(3) $|\alpha| = \alpha_1 + \cdots + \alpha_n$
(4) $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
(5) $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

**Definition 2.1.2.** Let $f \in C^\infty(\mathbb{R}^n)$,$\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define

$$\|f\|_{\alpha,N} = \sup_{x\in\mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

We define Schwartz space, denoted $\mathcal{S}$, by

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n,\ N \in \mathbb{N}_0,\ \|f\|_{\alpha,N} < \infty\}$$

**Exercise 2.1.3.** For each $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$.

*Proof.* Let $f \in \mathcal{S}$, $\alpha \in \mathbb{N}_0^n$. Then there exists $C \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^{-1}$$

Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = (1 + |x|^2)^{-1}$. Then $g \in L^1(\mathbb{R}^n)$ which implies that $\partial^\alpha f \in L^1(\mathbb{R}^n)$. □

**Definition 2.1.4.**

## 2.2. **The Convolution.**

**Definition 2.2.1.** Let $f, g \in L^0(\mathbb{R}^n)$. If for a.e. $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) < \infty$$

we define the **convolution of $f$ with $g$**, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$, by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

**Exercise 2.2.2.** Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

*Proof.* Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = f(x-y)g(y)$. Tonelli's theorem implies that,

$$\begin{aligned}
\int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 &= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}^n} |g(y)| \left[ \int_{\mathbb{R}^n} |f(x-y)|dm(y) \right] dm(x) \\
&= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dm(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

Then $h \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Fubini's theorem implies that $f * g \in L^1(\mathbb{R}^n)$. Clearly

$$\begin{aligned}
\|f * g\|_1 &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |h| dm^2 \\
&\leq \|f\|_1 \|g\|_1
\end{aligned}$$

$\square$

**Exercise 2.2.3.** Let $f, g, h \in L^1(\mathbb{R}^n)$. Then $(f * g) * h = f * (g * h)$.
**Hint:** use the substitution $z \mapsto z - y$

*Proof.* Let $x \in \mathbb{R}^n$. Then using the substitution $z \mapsto z - y$ and Fubini's theorem, we obtain

$$\begin{aligned}
(f * g) * h(x) &= \int f * g(x-y)h(y)dm(y) \\
&= \int \left[ \int f(x-y-z)g(z)dm(z) \right] h(y)dm(y) \\
&= \int \left[ \int f(x-z)g(z-y)dm(z) \right] h(y)dm(y) \\
&= \int \left[ \int f(x-z)g(z-y)h(y)dm(z) \right] dm(y) \\
&= \int \left[ \int f(x-z)g(z-y)h(y)dm(y) \right] dm(z) \\
&= \int f(x-z) \left[ \int g(z-y)h(y)dm(y) \right] dm(z) \\
&= \int f(x-z)g * h(z)dm(z) \\
&= f * (g * h)(z)
\end{aligned}$$

So $(f * g) * h = f * (g * h)$. □

**Exercise 2.2.4.** Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g = g * f$.

*Proof.* Let $x \in \mathbb{R}^n$. Using the transformation $y \mapsto x - y$, we obtain that

$$\begin{aligned}
f * g(x) &= \int f(x-y)g(y)dm(y) \\
&= \int f(y)g(x-y)dm(y) \\
&= \int g(x-y)f(y)dm(y) \\
&= g * f(x)
\end{aligned}$$

So $f * g = g * f$. □

**Note 2.2.5.** To summarize, $(L^1(\mathbb{R}^n), *)$ is a commutative Banach algebra.

**Exercise 2.2.6. Young's Inequality:**
Let $p \in [1, \infty]$, $f \in L^1$ and $g \in L^p$. Then $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

*Proof.* Define $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $K(x, y) = f(x - y)$. Since for each $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
\int |K(x,y)|dm(x) &= \int |K(x,y)|dm(y) \\
&= \|f\|_p
\end{aligned}$$

an exercise in section 5.1 of [4] implies that $f * g \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. □

**Exercise 2.2.7.** Let $p, q \in [1, \infty]$ be conjugate, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then

(1) for each $x \in \mathbb{R}^n$, $f * g(x)$ exists.
(2) $\|f * g\|_u \leq \|f\|_p \|g\|_q$

(3)

*Proof.* (1) Let $x \in \mathbb{R}^n$. Holder's inequality implies that

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \leq \|f\|_p \|g\|_q$$

Then $f * g(x)$ exists.

(2) Let $x \in \mathbb{R}^n$. Then in part (1) we showed that

$$\begin{aligned} |f * g(x)| &= \left| \int_{\mathbb{R}^n} f(x-y)g(y)dm(y) \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \\ &\leq \|f\|_p \|g\|_q \end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, $\|f * g\|_u \leq \|f\|_p \|g\|_q$.

(3)

$\square$

**Exercise 2.2.8.** Let $f \in L^1(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $g \in C^k(\mathbb{R}^n)$. Suppose that for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $\partial^\alpha g \in L^\infty$. Then for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq k$ implies that $f * g \in C^k$ and

$$\partial^\alpha (f * g) = f * \partial^\alpha g$$

*Proof.* Let $\alpha \in \mathbb{N}_0^n$. Suppose that $|\alpha| = 1$. Define $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$ by $h(x, y) = g(x-y)f(y)$. Young's inequality imples that for a.e. $x \in \mathbb{R}^n$, $h(x, \cdot) \in L^1(m)$. For each $y \in \mathbb{R}^n$, $\partial^\alpha h(\cdot, y) = \partial^\alpha g(\cdot - y)f(y)$ and for each $x, y \in \mathbb{R}^n$, $|\partial^\alpha h(x, y)| \leq \|\partial^\alpha g\|_\infty |f(y)| \in L^1(\mathbb{R}^n)$. An exercise in section 3.3 of [4] implies that for a.e. $x \in \mathbb{R}^n$, $\partial^\alpha (g * f)(x)$ exists and

$$\begin{aligned} \partial^\alpha (f * g)(x) &= \partial^\alpha (g * f)(x) \\ &= \partial^\alpha \int_{\mathbb{R}^n} h(x, y)dm(y) \\ &= \int_{\mathbb{R}^n} \partial^\alpha g(x-y)f(y)dm(y) \\ &= (\partial^\alpha g) * f(x) \\ &= f * (\partial^\alpha g)(x) \end{aligned}$$

Now proceed by induction on $|\alpha|$. $\square$

## 2.3. **The Fourier Transform.**

**Definition 2.3.1.**

**Exercise 2.3.2.** Let $\phi : \mathbb{R} \to S^1$ be a measurable homomorphism.

(1) Then $\phi \in L^1_{\text{loc}}(\mathbb{R})$ and there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$

(2) Define
$$c = \left[ \int_{(0,a]} \phi dm \right]^{-1}$$
Then For each $x \in \mathbb{R}$,
$$\phi(x) = c \int_{(x,x+a]} \phi dm$$

(3) $\phi \in C^\infty(\mathbb{R})$ and $\phi' = c(\phi(a) - 1)\phi$

(4) Define $b = c(\phi(a) - 1)$ and $g \in C^\infty(\mathbb{R})$ by $g(x) = e^{-bx}\phi(x)$. Then $g$ is constant and there exists $\xi \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i \xi x}$.

*Proof.*

(1) Let $K \subset \mathbb{R}$ be compact. Then
$$\int_K |\phi| dm = m(K) < \infty$$
So $\phi \in L^1_{\text{loc}}(\mathbb{R})$. For the sake of contradiction, suppose that for each $a > 0$,
$$\int_{(0,a]} \phi dm = 0$$
Then the FTC implies that $\phi = 0$ a.e. on $[0, \infty)$, which is a contradiction. So there exists $a > 0$ such that
$$\int_{(0,a]} \phi dm \neq 0$$

(2) For $x \in \mathbb{R}$,
$$\begin{aligned}
\phi(x) &= c \int_{(0,a]} \phi(x)\phi(t) dm(t) \\
&= c \int_{(0,a]} \phi(x + t) dm(t) \\
&= c \int_{(x,x+a]} \phi dm
\end{aligned}$$

(3) Part (2) and the FTC imply that $\phi$ is continuous. Let $d \in \mathbb{R}$. Define $f_d \in C((d, \infty))$ by
$$f_d(x) = \int_{(d,x]} \phi dm$$

Since $\phi$ is continuous, the FTC implies that $f_d$ is differentiable and for each $x > d$ $f_d'(x) = \phi(x)$. Part (2) implies that for each $x > d$,

$$\begin{aligned}\phi(x) &= c\int_{(x,x+a]} \phi dm \\ &= c(f_d(x+a) - f_d(x))\end{aligned}$$

So for each $x > d$, $\phi$ is differentiable at $x$ and

$$\begin{aligned}\phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x)\end{aligned}$$

Since $d \in \mathbb{R}$ is arbitrary, $\phi$ is differentiable and $\phi' = c(\phi(a) - 1)\phi$. This implies that $\phi \in C^\infty(\mathbb{R})$.

(4) Let $x \in \mathbb{R}$. Then

$$\begin{aligned}g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0\end{aligned}$$

So $g' = 0$ and $g$ is constant. Hence there exists $k \in \mathbb{R}$ such that for each $x \in \mathbb{R}$, $\phi(x) = ke^{bx}$. Since $\phi(0) = 1$, $k = 1$. Since $|\phi| = 1$, there exists $\xi \in \mathbb{R}$ such that $b = 2\pi i\xi$.

□

**Note 2.3.3.** To summarize, for each measurable homomorphism $\phi : \mathbb{R} \to S^1$, there exists $\xi \in \mathbb{R}$ such such that for each $x \in \mathbb{R}$, $\phi(x) = e^{2\pi i\xi x}$.

**Exercise 2.3.4.** Let $\phi : \mathbb{R}^n \to S^1$ be a measurable homomorphism. Then there exists $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi(x) = e^{2\pi i\langle \xi, x\rangle}$.

*Proof.* When done in the category of measurable groups, an exercise in the section on direct products of groups of [**?**] implies that there exist measurable homomorphism $(\phi_j)_{j=1}^n \subset (S^1)^{\mathbb{R}}$ such that $\phi = \bigotimes_{j=1}^n \phi_j$. The previous exercise imples that there exist $\xi \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $\phi_j(x_j) = e^{2\pi i\xi_j x_j}$. Then for each $x \in \mathbb{R}^n$,

$$\begin{aligned}\phi(x) &= \prod_{j=1}^n \phi_j(x_j) \\ &= \prod e^{2\pi i\xi_j x_j} \\ &= e^{2\pi i\sum\limits_{j=1}^n \xi_j x_j} \\ &= e^{2\pi i\langle \xi, x\rangle}\end{aligned}$$

□

**Definition 2.3.5.** Let $f \in L^1(\mathbb{R}^n)$. We define the **Fourier transform of** $f$, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle \xi, x\rangle} dm(x)$$

# 3. Fourier Analysis on LCA Groups

## 3.1. **The Convolution.**

**Note 3.1.1.** For the remainder of the section, we fix a locally compact abelian group $G$ and a Haar measure $\mu$ on $G$.

**Definition 3.1.2.** Let $f, g \in L^1(\mu)$. We define the **convolution of** $f$ **with** $g$, denoted $f * g : G \to \mathbb{C}$, by

$$f * g(x) = \int_X f(x-y)g(y)d\mu(y)$$

**Exercise 3.1.3.** Let $f, g \in L^1(\mu)$. Then $f * g \in L^1(\mu)$.

*Proof.* By Tonelli's theorem,

$$\begin{aligned}
\int_X |f * g| d\mu &\leq \int_X \left[ \int_X |f(x-y)g(y)| d\mu(y) \right] d\mu(x) \\
&= \int_X |g(y)| \left[ \int_X |f(x-y)| d\mu(y) \right] d\mu(x) \\
&= \|f\|_1 \int_X |g(y)| d\mu(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

$\square$

## 4. Fourier Analysis on Banach Spaces

## References

[1] Introduction to Algebra
[2] Introduction to Analysis
[3] Introduction to Fourier Analysis
[4] Introduction to Measure and Integration