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Introduction to Differential Geometry

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Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

cc-by-nc-sa

Chapter 1

Review of Fundamentals

1.1 Set Theory

merge with set theory from analysis notes

Definition 1.1.0.1. Let $\{A_i\}_{i\in I}$ be a collection of sets. The **disjoint union of** $\{A_i\}_{i\in I}$, denoted $\coprod_{i\in I} A_i$, is defined by

$$\coprod_{i\in I} A_i = \bigcup_{i\in I} \{i\} \times A_i$$

We define the **natural projection map**, denoted $\pi : \coprod_{i\in I} A_i \to I$, by $\pi(i, a) = i$.

Definition 1.1.0.2. Let E and M be sets, $\pi : E \to M$ a surjection and $\sigma : M \to E$. Then σ is said to be a section of (E, M, π) if $\pi \circ \sigma = \mathrm{id}_M$.

Note 1.1.0.3. Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod_{i\in I} A_i$. We will typically be interested in sections σ of $\left(\coprod_{i\in I} A_i, I, \pi\right)$.

Exercise 1.1.0.4. Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod_{i\in I} A_i$. Then σ is a section of $\coprod_{i\in I} A_i$ iff for each $i \in I$, $\sigma(i) \in A_i$

Proof. Clear. $\square$

1.2 Linear Algebra

Note 1.2.0.1. We denote the standard basis on $\mathbb{R}^n$ by $(e_1, \ldots, e_n)$.

Definition 1.2.0.2. Let $A \in \mathbb{R}^{n\times n}$. Then A is said to be **invertible** if $\det(A) \neq 0$. We denote the set of $n \times n$ invertible matrices by $GL(n, \mathbb{R})$.

$$O(n)$$

Exercise 1.2.0.3. Let $A, B \in \mathbb{R}^{n\times n}$. Then $AB = I$ iff $BA = I$.

Proof.

- $(\implies)$:
  Suppose that $AB = I$. Then

$$\begin{aligned} \ker B &\subset \ker AB \\ &= \ker I \\ &= \{0\} \end{aligned}$$

  so that $\ker B = \{0\}$. Hence $\operatorname{Im} B = \mathbb{R}^n$ and B is surjective. Then

$$\begin{aligned} IB &= BI \\ &= B(AB) \\ &= (BA)B \end{aligned}$$

  Since B is surjective, $I = BA$.

- $(\impliedby)$:
  Immediate by the previous part.

$\square$

Definition 1.2.0.4. Let $A \in \mathbb{R}^{n\times p}$. Then A is said to be an **orthogonal matrix** if $A^*A = I$. We denote the set of $n \times p$ orthogonal matrices by $O(n, p)$. We write $O(n)$ in place of $O(n, n)$.

$$O(n)$$

Exercise 1.2.0.5. Define $\phi : S_n \to GL(n, \mathbb{R})$ by

$$\phi(\sigma) = \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix}$$

Then

1. for each $A \in \mathbb{R}^{n\times p}$,
$$(\phi(\sigma)A)_{i,j} = A_{\sigma(i),j}$$
   i.e. left multiplying A by $\phi(\sigma)$ the the same as permuting the rows of A by σ
2. ϕ is a group homomorphism

Proof. 1. Let $A \in \mathbb{R}^{n\times p}$. Then

$$\begin{aligned} (\phi(\sigma)A)_{i,j} &= \langle e^*_{\sigma(i)}, Ae_j\rangle \\ &= A_{\sigma(i),j} \end{aligned}$$

2. Let $\sigma, \tau \in S_n$. Part (1) implies that

$$\begin{aligned}\phi(\sigma\tau) &= \begin{pmatrix} e^*_{\sigma\tau(1)} \\ \vdots \\ e^*_{\sigma\tau(n)} \end{pmatrix} \\ &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e^*_{\tau(1)} \\ \vdots \\ e^*_{\tau(n)} \end{pmatrix} \\ &= \phi(\sigma)\phi(\tau)\end{aligned}$$

Since $\sigma, \tau \in S_n$ are arbitrary, ϕ is a group homomorphism.

□

Definition 1.2.0.6. Define $\phi : S_n \to GL(n, \mathbb{R})$ as in the previous exercise. Let $P \in GL(n, \mathbb{R})$. Then P is said to be a **permutation matrix** if there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. We denote the set of $n \times n$ permutation matrices by $\mathrm{Perm}(n)$.

Exercise 1.2.0.7. We have that

1. $\mathrm{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$
2. $\mathrm{Perm}(n)$ is a subgroup of $O(n)$

Proof.

1. By definition, $\mathrm{Perm}(n) = \mathrm{Im}\,\phi$. Since $\phi : S_n \to GL(n, \mathbb{R})$ is a group homomorphism, $\mathrm{Im}\,\phi$ is a subgroup of $GL(n, \mathbb{R})$. Hence $\mathrm{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$.
2. Let $P \in \mathrm{Perm}(n)$. Then there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. Then

$$\begin{aligned}PP^* &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix}^* \\ &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e_{\sigma(1)} & \cdots & e_{\sigma(n)} \end{pmatrix} \\ &= (\langle e_{\sigma(i)}, e_{\sigma(j)} \rangle)_{i,j} \\ &= I\end{aligned}$$

A previous exercise implies that $P^*P = I$. Hence $P \in O(n)$. Since $P \in \mathrm{Perm}(n)$ is arbitrary, $\mathrm{Perm}(n) \subset O(n)$. Part (1) implies that $\mathrm{Perm}(n)$ is a group. Hence $\mathrm{Perm}(n)$ is a subgroup of $O(n)$

□

Note 1.2.0.8. We will write P_σ in place of $\phi(\sigma)$.

Exercise 1.2.0.9. Let $Z \in \mathbb{R}^{p \times n}$. If $\mathrm{rank}\, Z = k$, then there exist $\sigma \in S_n$, $\tau \in S_p$ and $A \in GL(k, \mathbb{R})$, such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P^*_\sigma)_{i,j} = A_{i,j}$$

Proof. Suppose that $\mathrm{rank}\, Z - k$. Then there exist $i_1, \dots, i_k \in \{1, \dots, p\}$ such that $i_1 < \cdots < i_k$ and $\{e^*_{i_1} Z, \dots, e^*_{i_k} Z\}$ is linearly independent. Set

$$Z' = \begin{pmatrix} e^*_{i_1} Z \\ \vdots \\ e^*_{i_k} Z \end{pmatrix}$$

Then $\operatorname{rank} Z' = k$. Hence there exist $j_1, \dots, j_k \in \{1, \dots, n\}$ such that $j_1 < \cdots < j_k$, and $\{Z'e_{i_1}, \dots, Z'e_{i_k}\}$ is linearly independent. Set

$$A = \begin{pmatrix} Z'e_{i_1} & \cdots & Z'e_{i_k} \end{pmatrix}$$

Then $A \in \mathbb{R}^{k \times k}$ and $\operatorname{rank} A = k$. Thus $A \in GL(k, \mathbb{R})$. Choose $\sigma \in S_n$ and $\tau \in S_p$ such that $\sigma(1) = j_1, \dots, \sigma(k) = j_k$ and $\tau(1) = i_1, \dots, \tau(k) = i_k$. Let $a, b \in \{1, \dots, k\}$. By construction,

$$\begin{aligned}(P_\tau Z P_\sigma^*)_{a,b} &= Z_{\tau(a),\sigma(b)} \\ &= Z_{i_a, j_b} \\ &= A_{a,b}\end{aligned}$$

□

Definition 1.2.0.10. Let $A \in \mathbb{R}^{n \times p}$. Then A is said to be a **diagonal matrix** if for each $i \in [n]$ and $j \in [p]$, $i \neq j$ implies that $A_{i,j} = 0$. We denote the set of $n \times p$ diagonal matrices by $D(n, p, \mathbb{R})$. We write $D(n, \mathbb{R})$ in place of $D(n, n, \mathbb{R})$.

Definition 1.2.0.11. For (n, k), (m, l) $\operatorname{diag}_{p,(n \times p)} : \mathbb{R}^p \to \mathbb{R}^{n \times p}$ and $\operatorname{diag}_{n,(n \times p)} : \mathbb{R}^p \to \mathbb{R}^{n \times p}$ by $\operatorname{diag}(v)$ FINISH!!!

Definition 1.2.0.12. Let $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \sigma(A)$. Suppose that A is symmetric. We define the **geometric multiplicity** of λ, denoted $\mu(\lambda)$, by

$$\mu(\lambda) = \dim \ker([\phi_\alpha] - \lambda I)$$

Definition 1.2.0.13. Let V be an n-dimensional vector space, $U \subset V$ a k-dimensional subspace and $(e_j)_{j=1}^n \subset V$ a be a basis. Then $(e_j)_{j=1}^n$ is said to be **adapted to** U if $(e_j)_{j=1}^k$ is a basis for U.

1.3 Calculus

1.3.1 Differentiation

Definition 1.3.1.1. Let $n \geq 1$. For $i = 1, \cdots, n$, define $x^i : \mathbb{R}^n \to \mathbb{R}$ by $x^i(a^1, \cdots, a^n) = a^i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on** $\mathbb{R}^n$.

Definition 1.3.1.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to** x^i **at** a if

$$\lim_{h \to 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If f is differentiable with respect to x^i at a, we define the **partial derivative of** f **with respect to** x^i **at** a, denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \frac{\partial}{\partial x^i} f$$

to be the limit above.

Definition 1.3.1.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **differentiable with respect to** x^i if for each $a \in U$, f is differentiable with respect to x^i at a.

Exercise 1.3.1.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i x^j}$ and $\frac{\partial^2 f}{\partial x^j x^i}$ exist and are continuous at a. Then

$$\frac{\partial^2 f}{\partial x^i x^j}(a) = \frac{\partial^2 f}{\partial x^j x^i}(a)$$

Proof. □

Definition 1.3.1.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $\frac{\partial^k f}{\partial i_1 \cdots i_k}$ exists and is continuous on U.

Definition 1.3.1.6. Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \to \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Theorem 1.3.1.7. Taylor's Theorem:
Let $U \subset \mathbb{R}^n$ be open and convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that for each $x \in U$,

$$f(x) = \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each $|\alpha| = T + 1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. See analysis notes □

Definition 1.3.1.8. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Let $x^1, \cdots, x^n$ be the standard coordinate functions on $\mathbb{R}^n$ and $y_1, \cdots, y_m$ be the standard coordinate functions on $\mathbb{R}^m$. For $i \in \{1, \cdots, m\}$, we define the i**th component of** F, denoted $F^i : U \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \cdots, F_m)$

Definition 1.3.1.9. Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \cdots, m\}$, the ith component of F, $F^i : U \to \mathbb{R}$, is smooth.

Definition 1.3.1.10. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Then F is said to be **smooth** if for each $x \in U$, there exists $U_x \in \mathcal{N}_x$ and $\tilde{F} : U_x \to \mathbb{R}^m$ such that U_x is open, $\tilde{F}$ is smooth and $\tilde{F}|_{U \cap U_x} = F|_{U \cap U_x}$.

Definition 1.3.1.11. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. Then F is said to be a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 1.3.1.12. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F is a bijection and F and F^{-1} are smooth. Thus, F and F^{-1} are continuous and F is a homeomorphism. □

Definition 1.3.1.13. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \to \mathbb{R}^m$. We define the **Jacobian of F at p**, denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p)\right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Exercise 1.3.1.14. Inverse Function Theorem:
Let $U, V \subset \mathbb{R}^n$ be open and $F : U \to V$.

1.3.2 Differentiation on Subspaces

Definition 1.3.2.1. Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$. Then f is said to be **smooth** if for each $a \in A$, there exists $B \subset \mathbb{R}^m$ and $g : B \to \mathbb{R}^n$ such that $a \in B$, B is open in $\mathbb{R}^m$, g is smooth and $g|_{A\cap B} = f|_{A\cap B}$.

Exercise 1.3.2.2. Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$. If f is smooth, then f is continuous.

Proof. Suppose that f is smooth. Let $a \in A$. Since f is smooth, there exists $B \subset \mathbb{R}^m$ such that $a \in B$, B is open in $\mathbb{R}^m$, g is smooth and $g|_{A\cap B} = f|_{A\cap B}$. Since g is smooth, g is continuous. Let $V \subset \mathbb{R}^n$. Suppose that V is open in $\mathbb{R}^n$ and $f(a) \in V$. Since $f(a) = g(a)$ and g is continuous, there exists $U_g \subset B$ such that U_g is open in B, $a \in U_g$ and $g(U_g) \subset V$. Since B is open in $\mathbb{R}^m$ and U_g is open in B, we have that U_g is open in $\mathbb{R}^m$. Set $U_f = U_g \cap A$. Then $a \in U_f$, U_f is open in A and

$$\begin{aligned} f(U_f) &= f(U_g \cap A) \\ &= g(U_g \cap A) \\ &\subset g(U_g) \\ &\subset V \end{aligned}$$

Since $V \subset \mathbb{R}^n$ such that V is open in $\mathbb{R}^n$ and $f(a) \in V$ is arbitrary, we have that for each $V \subset \mathbb{R}^n$, if V is open in $\mathbb{R}^n$ and $f(a) \in V$, then there exists $U_f \subset A$ such that U_f is open in A, $a \in U_f$ and $f(U_f) \subset V$. Thus f is continuous at a. Since $a \in A$ is arbitrary, f is continuous. □

Exercise 1.3.2.3. Let $A \subset \mathbb{R}^m$, $B \subset A$ and $f : A \to \mathbb{R}^n$. If f is smooth, then $f|_B$ is smooth.

Proof. Suppose that f is smooth. Let $b \in B$. Since $B \subset A$, $b \in A$. Since $b \in A$ and f is smooth, there exists $U \subset \mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ such that $b \in U$, U is open in $\mathbb{R}^m$, F is smooth and $F|_{U\cap A} = f|_{U\cap A}$. Define $g : B \to \mathbb{R}^n$ by $g := f|_B$. Since $B \subset A$,

$$\begin{aligned} F|_{U\cap B} &= f|_{U\cap B} \\ &= g|_{U\cap B} \end{aligned}$$

Since $b \in B$ is arbitrary, we have that for each $b \in B$, there exists $U \subset \mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ such that $b \in U$, U is open in $\mathbb{R}^m$, F is smooth and $F|_{U\cap B} = g|_{U\cap B}$. Thus g is smooth. □

Exercise 1.3.2.4. Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$. Then f is smooth iff for each $a \in A$, there exists $U \subset A$ such that $a \in U$, U is open in A and $f|_U$ is smooth.

Proof.

- $(\implies)$:
  Suppose that f is smooth. Let $a \in A$. Set $U := A$. Then $a \in U$, U is open in A and $f|_U = f$ which is smooth.

- $(\impliedby)$:
  Suppose that for each $a \in A$, there exists $U \subset A$ such that $a \in U$ and $f|_U$ is smooth. Let $a \in A$. By assumption, there exists $U \subset A$ such that $a \in U$, U is open in A and $f|_U$ is smooth. Define $h : U \to \mathbb{R}^n$ by $h := f|_U$. Since $a \in U$ and h is smooth, there exists $U_0 \subset \mathbb{R}^m$ and $g_0 : U_0 \to \mathbb{R}^n$ such that $a \in U_0$, U_0 is open in $\mathbb{R}^m$ and $g_0|_{U\cap U_0} = h|_{U\cap U_0}$. Since U is open in A, there exists $\tilde{U} \subset \mathbb{R}^m$ such that $\tilde{U}$ is open in $\mathbb{R}^m$ and $U = \tilde{U} \cap A$. Define $B \subset \mathbb{R}^m$ and $g : B \to \mathbb{R}^n$ by $B := U_0 \cap \tilde{U}$ and $g = g_0|_B$. Then $a \in B$ and B is open in $\mathbb{R}^m$. The previous exercise implies that g is smooth. Furthermore,

$$\begin{aligned} g|_{B\cap A} &= g|_{U_0\cap\tilde{U}\cap A} \\ &= g|_{U_0\cap U} \\ &= h|_{U_0\cap U} \\ &= f|_{U_0\cap U} \\ &= f|_{U_0\cap\tilde{U}\cap A} \\ &= f|_{B\cap A} \end{aligned}$$

Since $a \in A$ is arbitrary, we have that for each $a \in A$, there exists $B \subset \mathbb{R}^m$ and $g : B \to \mathbb{R}^n$ such that $a \in B$, B is open in $\mathbb{R}^m$, g is smooth and $g|_{A\cap B} = f|_{A\cap B}$. Hence f is smooth.

□

Exercise 1.3.2.5. Let $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$, $f : A \to B$ and $g : B \to \mathbb{R}^p$. If f and g are smooth, then $g \circ f$ is smooth.

Proof. Suppose that f and g are smooth. Let $a \in A$. Set $b = f(a)$. Then $b \in B$. Since f is smooth, there exists $U \subset \mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ such that $a \in U$, U is open in $\mathbb{R}^m$, F is smooth and $F|_{U\cap A} = f|_{U\cap A}$. Since g is smooth, there exists $V \subset \mathbb{R}^n$ and $G : V \to \mathbb{R}^p$ such that $b \in V$, V is open in $\mathbb{R}^n$, G is smooth and $G|_{V\cap B} = g|_{V\cap B}$. We define $W \subset \mathbb{R}^m$ and $H : W \to \mathbb{R}^p$ by $W := U \cap F^{-1}(V)$ and $H := G \circ F|_W$.

- By construction, $a \in W$.
- Since F is smooth, F is continuous. Thus $F^{-1}(V)$ is open in $\mathbb{R}^m$ which implies that W is open in $\mathbb{R}^m$.
- Since F is smooth, an exercise in the section on differentiation implies that $F|_W$ is smooth. Since $F|_W$ and G are smooth, a previous exercise in the section on differentiation implies that H is smooth.
- Let $x \in W \cap A$. Since $W \cap A \subset A \cap U$, $f(x) = F(x)$. Since $f(x) \in B$ and $W \subset F^{-1}(V)$, we have that $F(x) \in V \cap B$. Thus

$$\begin{aligned} g \circ f(x) &= g(F(x)) \\ &= G(F(x)) \\ &= H(x) \end{aligned}$$

  Since $x \in W \cap A$ is arbitrary, we have that $H|_{W\cap A} = (g \circ f)|_{W\cap A}$.

Thus $g \circ f$ is smooth. □

1.3.3 Calculus and Permutations

Exercise 1.3.3.1. Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \to F(N)$ is a diffeomorphism

Proof. content... FIX or get rid □

Definition 1.3.3.2.

- Let $\sigma \in S_n$ and $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. We define $\sigma \cdot x \in \mathbb{R}^n$ by

$$\sigma \cdot x = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

- We define the **permutation action** of S_n on $\mathbb{R}^n$ to be the map $S_n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(\sigma, x) \mapsto \sigma \cdot x$.
- Let $\sigma \in S_n$. We define $\Phi_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by $\Phi_\sigma(x) := \sigma \cdot x$.

Exercise 1.3.3.3. Let $\sigma \in S_n$. Then

1. $D\Phi_\sigma = P_\sigma$.
2. $\Phi_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism,

Proof.

1.

$$\begin{aligned}
D(\Phi_\sigma)(p) &= \left(\frac{\partial \pi_i \circ \Phi_\sigma}{\partial x^j}(p)\right)_{i,j} \\
&= \left(\frac{\partial \pi_{\sigma(i)}}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma \left(\frac{\partial \pi_i}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma \left(\frac{\partial \pi_i \circ \mathrm{id}_{\mathbb{R}^n}}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma D\,\mathrm{id}_{\mathbb{R}^n}(p) \\
&= P_\sigma I \\
&= P_\sigma
\end{aligned}$$

2. Clear.

□

Definition 1.3.3.4.

- Let $\sigma \in S_n$, U a set, $V \subset \mathbb{R}^n$ and $\phi : U \to \mathbb{R}^n$ with $\phi = (x^1, \ldots, x^m)$. We define $\sigma \cdot \phi : U \to \mathbb{R}^n$ by

$$(\sigma \cdot \phi)(x) := \phi(\sigma \cdot x)$$

- We define the **permutation action** of S_n on $(\mathbb{R}^n)^U$ to be the map $S_n \times (\mathbb{R}^n)^U \to \mathbb{R}^n$ given by $(\sigma, \phi) \mapsto \sigma \cdot \phi$.

Exercise 1.3.3.5. Let $\sigma \in S_m$. Then for each $p \in \mathbb{R}^n$, $D(\sigma\,\mathrm{id}_{\mathbb{R}^n})(p) = P_\sigma$.

Proof. Note that since $\mathrm{id}_{\mathbb{R}^n} = (\pi_1, \ldots, \pi_n)$, we have that $\sigma\,\mathrm{id}_{\mathbb{R}^n} = (\pi_{\sigma(1)}, \ldots, \pi_{\sigma(n)})$. Let $p \in \mathbb{R}^n$. Then

□

1.3.4 Integration

1.4 Topology

Definition 1.4.0.1. Let $(X, \mathbb{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 1.4.0.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Definition 1.4.0.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \to Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 1.4.0.4. Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \ncong \mathbb{R}^n$

1.5 Group Actions

1.5.1 Subactions

Exercise 1.5.1.1. Let X be a set, G a group and $\triangleleft : G \times X \to X$ a group action. Then

1. for each $x \in X$, $\triangleright(\bar{x} \times G) = \bar{x}$,
2. for each $x \in X$, $\triangleright|_{\bar{x} \times G} : \bar{x} \times G \to \bar{x}$ is a group action.

Proof. content... □

Definition 1.5.1.2. Let X be a set, G a group and $\triangleleft : G \times X \to X$ a group action. For each $x \in X$, we define **action of G on $\bar{x}$ induced by** $\triangleleft$ $\triangleright_x : G \times \bar{x} \to \bar{x}$ by $g \triangleright_x := g \triangleright x$.

Exercise 1.5.1.3. Let X be a set, G a group and $\triangleleft : G \times X \to X$ a group action.
is free iff for each $x \in M$, $\triangleleft|_{P_x \times G}$ is free. given a left action $\triangleright : G \times X \to X$ and $x \in X$, such that $\triangleright(\times G) \subset Y$, show that $\triangleright(Y \times G) = Y$ and $\triangleright|_{Y \times G}$ is a group action and $\triangleright|_{Y \times G}$ is free iff

Proof. Suppose that $\triangleleft$ is free. Let $x \in M$, $p \in P_x$ and $g \in G$. Suppose that $p \triangleleft_x g = p$. Then $p \triangleleft g = p$. Thus $g = e$. Since $p \in P_x$ and $g \in G$ are arbitrary, $\triangleleft$ is free

Conversely, suppose that for each $x \in M$, $\triangleleft|_{P_x \times G}$ is free. Let $g \in G$ and $p \in P$. □

Chapter 2

Multilinear Algebra

2.1 Tensor Products

Let V and W be vector spaces.

2.2 (r, s)-Tensors

Definition 2.2.0.1. Let $V_1, \ldots, V_k, W$ be vector spaces and $\alpha : \prod_{i=1}^n V_i \to W$. Then α is said to be **multilinear** if for each $i \in \{1, \cdots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$,

$$\alpha(v_1, \cdots, v_i + cv, \cdots, v_k) = \alpha(v_1, \cdots, v_i, \cdots, v_k) + c\alpha(v_1, \cdots, v, \cdots, v_k)$$

We define

$$L(V_1, \ldots, V_k; W) = \left\{ \alpha : \prod_{i=1}^n V_i \to W : \alpha \text{ is multilinear} \right\}$$

Note 2.2.0.2. For the remainder of this section we let V denote an n-dimensional vector space with basis $\{e^1, \cdots, e^n\}$ with dual space V^* and dual basis $\{\epsilon_1, \cdots, \epsilon_n\}$ defined by $\epsilon^i(e^j) = \delta_{i,j}$. We identify V with V^{**} by the isomorphism $V \to V^{**}$ defined by $v \mapsto \hat{v}$ where $\hat{v}(\alpha) = \alpha(v)$ for each $\alpha \in V^*$.

Definition 2.2.0.3. Let $\alpha : (V^*)^r \times V^s \to \mathbb{R}$. Then α is said to be an (r, s)-tensor on V if $\alpha \in L(\underbrace{V^*, \ldots, V^*}_{r}, \underbrace{V, \ldots, V}_{s}; \mathbb{R})$. The set of all (r, s)-tensors on V is denoted $T^r_s(V)$.
When $r = s = 0$, we set $T^r_s = \mathbb{R}$.

Exercise 2.2.0.4. We have that $T^r_s(V)$ is a vector space.

Proof. Clear. $\square$

Exercise 2.2.0.5. Under the identification of V with V^{**} as noted above, we have that $V = T^1_0(V)$.

Proof. By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T^1_0(V) \end{aligned}$$

$\square$

Definition 2.2.0.6. Let $\alpha \in T^{r_1}_{s_1}(V)$ and $\beta \in T^{r_2}_{s_2}(V)$. We define the **tensor product of** α **with** β, denoted $\alpha \otimes \beta \in T^{r_1+r_2}_{s_1+s_2}(V)$, by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$.
When $r_1 = s_1 = r_2 = s_2 = 0$ (so that $\alpha, \beta \in \mathbb{R}$), we set $\alpha \otimes \beta = \alpha\beta$.

Definition 2.2.0.7. We define the **tensor product**, denoted $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

Exercise 2.2.0.8. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is well defined.

Proof. Tedious but straightforward. □

Exercise 2.2.0.9. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is associative.

Proof. Let $\alpha \in T^{r_1}_{s_1}(V)$, $\beta \in T^{r_2}_{s_2}(V)$ and $\gamma \in T^{r_3}_{s_3}(V)$. Then for each $u^* \in (V^*)^{r_1}, v^* \in (V^*)^{r_2}, w^* \in (V^*)^{r_3}, u \in V^{s_1}, v \in V^{s_2}, w \in V^{s_3}$,

$$\begin{aligned}
(\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v)\gamma(w^*, w) \\
&= [\alpha(u^*, u)\beta(v^*, v)]\gamma(w^*, w) \\
&= \alpha(u^*, u)[\beta(v^*, v)\gamma(w^*, w)] \\
&= \alpha(u^*, u)(\beta \otimes \gamma)(v^*, w^*, v, w) \\
&= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w)
\end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

Exercise 2.2.0.10. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is bilinear.

Proof.

1. Linearity in the first argument:
   Let $\alpha, \beta \in T^{r_1}_{s_1}(V)$, $\gamma \in T^{r_2}_{s_2}(V), \lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $vinV^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned}
[(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v)\gamma(w^*, w) \\
&= [\alpha(v^*, v) + \lambda\beta(v^*, v)]\gamma(w^*, w) \\
&= \alpha(v^*, v)\gamma(w^*, w) + \lambda\beta(v^*, v)\gamma(w^*, w) \\
&= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\
&= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w)
\end{aligned}$$

   So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

2. Linearity in the second argument:
   Similar to (1).

□

Definition 2.2.0.11.

1. Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1, \cdots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length** k. Recall that $\#\mathcal{I}_{\otimes k} = n^k$.

2. Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length** k. Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

Note 2.2.0.12. For the remainder of this section we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\otimes k}$.

Definition 2.2.0.13. Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.

1. Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by
$$\epsilon^I = (\epsilon^{i_1}, \cdots, \epsilon^{i_k})$$
and
$$e^I = (e^{i_1}, \cdots, e^{i_k})$$

2. Define $e^{\otimes I} \in T_0^k(V)$ and $\epsilon^{\otimes I} \in T_k^0(V)$ by
$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$
and
$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

Exercise 2.2.0.14. Let $\alpha, \beta \in T_s^r(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v_1^*, \ldots, v_r^* \in V^*$ and $v_1, \ldots, v_s \in V$. For each $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$, write
$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$
and
$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$
Then
$$\begin{aligned}
\alpha(v_1^*, \ldots, v_r^*, v_1, \ldots, v_s) &= \sum_{k_1,\ldots,k_r=1}^n \sum_{l_1,\ldots,l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \ldots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \sum_{k_1,\ldots,k_r=1}^n \sum_{l_1,\ldots,l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \ldots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \beta(v_1^*, \ldots, v_r^*, v_1, \ldots, v_s)
\end{aligned}$$
So that $\alpha = \beta$. □

Exercise 2.2.0.15. Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K}\delta_{J,L}$.

Proof. Write $I = (i_1, \ldots, i_r), K = (k_1, \ldots, k_r)$ and $J = (j_1, \ldots, j_s), L = (l_1, \ldots, l_s)$. Then
$$\begin{aligned}
e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K)\epsilon^{\otimes J}(e^L) \\
&= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \ldots, \epsilon^{k_r})\epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \ldots, e^{l_s}) \\
&= \left[\prod_{m=1}^r e^{i_m}(\epsilon^{k_m})\right]\left[\prod_{n=1}^s \epsilon^{j_n}(e^{l_n})\right] \\
&= \left[\prod_{m=1}^r \delta_{i_m,k_m}\right]\left[\prod_{n=1}^s \delta_{j_n,l_n}\right] \\
&= \delta_{I,K}\delta_{J,L}
\end{aligned}$$
□

Exercise 2.2.0.16. The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T_s^r(V)$ and $\dim T_s^r(V) = n^{r+s}$.

Proof. Let $(a_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum\limits_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a_J^I e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a_J^I = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T_s^r(V)$. For $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b_J^I = \beta(\epsilon^J, e^I)$. Define $\mu = \sum\limits_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b_J^I e^{\otimes I} \otimes \epsilon^{\otimes J} \in T_s^r(V)$. Then for each $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b_J^I = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. □

2.3 Covariant k-Tensors

2.3.1 Symmetric and Alternating Covariant k-Tensors

Definition 2.3.1.1. Let $\alpha : V^k \to \mathbb{R}$. Then α is said to be a **covariant k-tensor on V** if $\alpha \in T^0_k(V)$. We denote the set of covariant k-tensors by $T_k(V)$.

Definition 2.3.1.2. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \to \mathbb{R}$ by

$$\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

We define the **permutation action** of of S_k on $T_k(V)$ to be the map $S_k \times T_k(V) \to T_k(V)$ given by $(\sigma, \alpha) \mapsto \sigma\alpha$

Exercise 2.3.1.3. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

1. Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V), \sigma\alpha \in T_k(V)$.
2. Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
3. Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \cdots, v_k \in V$,

$$\begin{aligned}(\tau\sigma)\alpha(v_1, \cdots, v_k) &= \alpha(v_{\tau\sigma(1)}, \cdots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \cdots, v_k)\end{aligned}$$

□

Exercise 2.3.1.4. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \to T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$. Then

$$\begin{aligned}\sigma(c\alpha + \beta)(v_1, \cdots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \cdots, v_k) + \sigma\beta(v_1, \cdots, v_k)\end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

Definition 2.3.1.5. Let $\alpha \in T_k(V)$. Then α is said to be

- **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$
- **antisymmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \operatorname{sgn}(\sigma)\alpha$
- **alternating** if for each $v_1, \dots, v_k \in V$, if there exists $i, j \in \{1, \dots, k\}$ such that $v_i = v_j$, then $\alpha(v_1, \cdots, v_k) = 0$.

We denote the set of symmetric k-tensors on V by $\Sigma^k(V)$. We denote the set of alternating k-tensors on V by $\Lambda^k(V)$.update language here

Exercise 2.3.1.6. Let $\alpha \in T_k(V)$. Then α is antisymmetric iff α is alternating.

Proof. Suppose that α is antisymmetric. Let $v_1, \dots, v_k \in V$. Suppose that there exists $i, j \in \{1, \dots, k\}$ such that $v_i = v_j$. Define $\sigma \in S_k$ by $\sigma = (i, j)$. Then

$$\begin{aligned}\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) &= \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \\ &= \sigma(\alpha)(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &= \operatorname{sgn}(\sigma)\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &= -\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k)\end{aligned}$$

Therefore $2\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = 0$ which implies that $\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = 0$. Hence α is alternating.
Conversely, suppose that α is alternating. Let $i, j \in \{1, \dots, k\}$ and $v_1, \dots, v_k \in V$. Then

$$\begin{aligned} 0 &= \alpha(v_1, \cdots, v_i + v_j, \cdots, v_i + v_j, \cdots, v_k) \\ &= \alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) + \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \end{aligned}$$

Since $i, j \in \{1, \dots, k\}$ and $v_1, \dots, v_k \in V$ are arbitrary, we have that for each $\tau \in S_k$, τ is a transposition implies that

$$\begin{aligned} \tau\alpha &= -\alpha \\ &= \operatorname{sgn}(\tau)\alpha \end{aligned}$$

Let $n \in \mathbb{N}$. Suppose that for each $\tau_1, \dots, \tau_{n-1} \in S_k$ if for each $j \in \{1, \dots, n-1\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_{n-1})\alpha = \sigma(\tau_1 \cdots \tau_{n-1})\alpha$. Let $\tau_1, \dots, \tau_n \in S_k$. Suppose that for each $j \in \{1, \dots, n\}$, τ_j is a transposition. Then

$$\begin{aligned} (\tau_1 \cdots \tau_n)\alpha &= (\tau_1 \cdots \tau_{n-1})(\tau_n\alpha) \\ &= (\tau_1 \cdots \tau_{n-1})(\operatorname{sgn}(\tau_n)\alpha) \\ &= (\operatorname{sgn}(\tau_n)(\tau_1 \cdots \tau_{n-1})\alpha \\ &= (\operatorname{sgn}(\tau_n) \operatorname{sgn}((\tau_1 \cdots \tau_{n-1})\alpha \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha \end{aligned}$$

By induction, for each $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in S_k$, if for each $j \in \{1, \dots, n\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_n)\alpha = \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha$. Now let $\sigma \in S_k$. Then there exist $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in S_k$ such that $\sigma = \tau_1 \cdots \tau_n$ and for each $j \in \{1, \dots n\}$, τ_j is a transposition. Hence

$$\begin{aligned} \sigma\alpha &= (\tau_1 \cdots \tau_n)\alpha \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha \\ &= \operatorname{sgn}(\sigma)\alpha \end{aligned}$$

Therefore α is antisymmetric. □

Definition 2.3.1.7. Define the **symmetric operator** $S : T_k(V) \to \Sigma^k(V)$ by

$$\operatorname{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \to \Lambda^k(V)$ by

$$\operatorname{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha$$

Exercise 2.3.1.8.

1. For $\alpha \in T_k(V)$, $\operatorname{Sym}(\alpha)$ is symmetric.
2. For $\alpha \in T_k(V)$, $\operatorname{Alt}(\alpha)$ is alternating.

Proof.

1. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned} \sigma \operatorname{Sym}(\alpha) &= \sigma\left[\frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha\right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \\ &= \operatorname{Sym}(\alpha) \end{aligned}$$

2. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
\sigma \operatorname{Alt}(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right] \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\sigma\tau\alpha \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \\
&= \operatorname{sgn}(\sigma)\operatorname{Alt}(\alpha)
\end{aligned}$$

□

Exercise 2.3.1.9.

1. For $\alpha \in \Sigma^k(V)$, $\operatorname{Sym}(\alpha) = \alpha$.
2. For $\alpha \in \Lambda^k(V)$, $\operatorname{Alt}(\alpha) = \alpha$.

Proof.

1. Let $\alpha \in \Sigma^k(V)$. Then

$$\begin{aligned}
\operatorname{Sym}(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} \alpha \\
&= \alpha
\end{aligned}$$

2. Let $\alpha \in \Lambda^k(V)$. Then

$$\begin{aligned}
\operatorname{Alt}(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2\alpha \\
&= \alpha
\end{aligned}$$

□

Exercise 2.3.1.10. The symmetric operator $S : T_k(V) \to \Sigma^k(V)$ and the alternating operator $A : T_k(V) \to \Lambda^k(V)$ are linear.

Proof. Clear. □

Exercise 2.3.1.11. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

1. $\operatorname{Alt}(\operatorname{Alt}(\alpha) \otimes \beta) = \operatorname{Alt}(\alpha \otimes \beta)$
2. $\operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta)) = \operatorname{Alt}(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \to S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma,\tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

1. Then

$$\begin{aligned}
\mathrm{Alt}(\mathrm{Alt}(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\mathrm{Alt}(\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)\tau\alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)(\tau\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)\tau(\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \mathrm{sgn}(\sigma\tau)\sigma\tau(\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \mathrm{Alt}(\alpha \otimes \beta)
\end{aligned}$$

2. Similar to (1).

□

2.3.2 Exterior Product

Definition 2.3.2.1. Let $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$.

Exercise 2.3.2.2. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is bilinear.

Proof. Clear. □

Exercise 2.3.2.3. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$ and $\gamma \in \Lambda^m(V)$. Then

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)\right] \wedge \gamma \\
&= \frac{(k+l+m)!}{(k+l)!m!} \operatorname{Alt}\left(\left[\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)\right] \otimes \gamma\right) \\
&= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} \operatorname{Alt}(\operatorname{Alt}(\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} \operatorname{Alt}((\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \operatorname{Alt}(\alpha \otimes (\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \operatorname{Alt}(\alpha \otimes \frac{(l+m)!}{l!m!} \operatorname{Alt}(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \operatorname{Alt}(\alpha \otimes (\beta \wedge \gamma)) \\
&= \alpha \wedge (\beta \wedge \gamma))
\end{aligned}$$

□

Exercise 2.3.2.4. Let $\alpha_i \in \Lambda^{k_i}(V)$ for $i = 1, \cdots, m$. Then

$$\bigwedge_{i=1}^{m} \alpha_i = \frac{(\sum_{i=1}^{m} k_i)!}{\prod_{i=1}^{m} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m} \alpha_i\right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} \operatorname{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i\right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \operatorname{Alt}\left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \operatorname{Alt}\left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m_0-1} \alpha_i\right)\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\operatorname{Alt}\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m_0+1} \alpha_i\right)
\end{aligned}$$

□

Exercise 2.3.2.5. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl. (Hint: inversion number)

Proof.

$$\begin{aligned} N(\tau) &= \sum_{i=1}^{l} k \\ &= kl \end{aligned}$$

Since $\text{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\text{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.3.2.6. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \cdots, v_{k+l} \in V$, we have that

$$\begin{aligned} \sigma\tau(\beta \otimes \alpha)(v_1, \cdots, v_l, v_{l+1}, \cdots v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)})\alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \\ &= \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)})\beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\ &= \alpha \otimes \beta(v_{\sigma(1)}, \cdots v_{\sigma(k)}, v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\ &= \sigma(\alpha \otimes \beta)(v_1, \cdots, v_k, v_{1+k}, \cdots v_{l+k}) \end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned} \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} \text{Alt}(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)\sigma(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau)\sigma\tau(\beta \otimes \alpha) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)\sigma(\alpha \otimes \beta) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \\ &= \text{sgn}(\tau)\alpha \wedge \beta \\ &= (-1)^{kl} \alpha \wedge \beta \end{aligned}$$

□

Exercise 2.3.2.7. Let $\alpha \in \Lambda^k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned} \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\ &= -\alpha \wedge \alpha \end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.3.2.8. Fundamental Example:
Let $\alpha_1, \cdots, \alpha_m \in \Lambda^1(V)$ and $v_1, \cdots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^{m} \alpha_i\right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}\left(\bigwedge_{i=1}^{m} \alpha_i\right)(v_1, \cdots, v_m) &= m! \operatorname{Alt}\left(\bigotimes_{i=1}^{m} \alpha_i\right)(v_1, \cdots, v_m) \\ &= m!\left[\frac{1}{m!}\sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\sigma\left(\bigotimes_{i=1}^{m} \alpha_i\right)\right](v_1, \cdots, v_m) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\left(\bigotimes_{i=1}^{m} \alpha_i\right)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\prod_{i=1}^{m} \alpha_i(v_{\sigma(i)}) \\ &= \det(\alpha_i(v_j))\end{aligned}$$

□

Note 2.3.2.9. Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\wedge k}$.

Definition 2.3.2.10. Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.
Define $\epsilon^{\wedge I} \in \Lambda^k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$$

Exercise 2.3.2.11. Let $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \cdots & \epsilon^{i_1}(e^{j_k}) \\ & \vdots & \\ \epsilon^{i_k}(e^{j_1}) & \cdots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$. If $I = J$, then $A = I_{k \times k}$ and therefore $\epsilon^I(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the l_0-th row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the l_0-th column of A are 0. □

Exercise 2.3.2.12. Let $\alpha, \beta \in \Lambda^k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \cdots, v_k \in V$. For $i = 1, \cdots, k$, write $v_i =$

$\sum_{j_i=1}^n a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned}
\alpha(v_1, \cdots, v_k) &= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{j_1 \neq \cdots \neq j_k}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{k} a_{i,\sigma(j_i)}\right)\right] \alpha(e^J) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{k} a_{i,\sigma(j_i)}\right)\right] \beta(e^J) \\
&= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \beta(e^{j_1}, \cdots, e^{j_k}) \\
&= \beta(v_1, \cdots, v_k)
\end{aligned}$$

□

Exercise 2.3.2.13. The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(V)$ and $\dim \Lambda^k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$. Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda^k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda^k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$. □

2.3.3 Interior Product

Definition 2.3.3.1. Let V be a finite dimensional vector space and $v \in V$. We define **interior multiplication by** v, denoted $\iota_v : T_k \to T_{k-1}$, by

$$\iota_v \alpha(w_1, \dots, w_{k-1}) = \alpha(v, w_1, \dots, w_{k-1})$$

Exercise 2.3.3.2. Let V be a finite dimensional vector space and $v \in V$. Then $\iota_v|_{\Lambda^k(V)} : \Lambda^k(V) \to \Lambda^{k-1}(V)$.

Proof. Let $\alpha \in \Lambda^k(V)$. Define $\beta \in \Lambda^k(V)$ by $\beta(w_1, \dots, w_k) = \alpha(w_k, w_1, \dots, w_{k-1})$. Let $\sigma \in S_{k-1}$. Define $\tau \in S_k$ by $\tau(j) = \begin{cases} 1 & j = k \\ \sigma(j) & j \neq k \end{cases}$. Let $w_1, \dots, w_{k-1} \in V$. Set $w_k = v$. Then

$$\begin{aligned}
\sigma(\iota_v \alpha)(w_1, \dots, w_{k-1}) &= \iota_v \alpha(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}) \\
&= \alpha(v, w_{\sigma(1)}, \dots, w_{\sigma(k-1)}) \\
&= \beta(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}, v) \\
&= \beta(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}, w_k) \\
&= \beta(w_{\tau(1)}, \dots, w_{\tau(k-1)}, w_{\tau(k)}) \\
&= \operatorname{sgn}(\tau) \beta(w_1, \dots, w_{k-1}, w_k) \\
&= \operatorname{sgn}(\sigma) \beta(w_1, \dots, w_{k-1}, v) \\
&= \operatorname{sgn}(\sigma) \alpha(v, w_1, \dots, w_{k-1}) \\
&= \operatorname{sgn}(\sigma) (\iota_v \alpha)(w_1, \dots, w_{k-1})
\end{aligned}$$

Since $w_1, \dots, w_{k-1} \in V$ are arbitrary, $\sigma(\iota_v \alpha) = \operatorname{sgn}(\sigma) \iota_v \alpha$. Hence $\iota_v \alpha \in \Lambda^{k-1}(V)$. □

2.4 $(0, 2)$-Tensors

Definition 2.4.0.1. Let V be a finite dimensional vector space, $v \in V$ and $\alpha \in T^0_2(V)$. Then α is said to be **degenerate** if there exists $v \in V$ such that $v \neq 0$ and for each $w \in V$, $\alpha(v, w) = 0$.

Definition 2.4.0.2. Let V be a finite dimensional vector space, $\alpha \in T^0_2(V)$. We define $\phi_\alpha : V \to V^*$ by

$$\phi_\alpha(v) = \iota_v \alpha$$

Exercise 2.4.0.3. Let V be a finite dimensional vector space, $\alpha \in T^0_2(V)$. Then $\phi_\alpha \in L(V; V^*)$.

Proof. Let $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$. Then for each $w \in V$,

$$\begin{aligned}\phi_\alpha(v_1 + \lambda v_2)(w) &= (\iota_{v_1+\lambda v_2}\alpha)(w)\\ &= \alpha(v_1 + \lambda v_2, w)\\ &= \alpha(v_1, w) + \lambda\alpha(v_2, w)\\ &= (\iota_{v_1}\alpha)(w) + \lambda(\iota_{v_2}\alpha)(w)\\ &= \phi_\alpha(v_1)(w) + \lambda\phi_\alpha(v_2)(w)\\ &= [\phi_\alpha(v_1) + \lambda\phi_\alpha(v_2)](w)\end{aligned}$$

Therefore, $\phi_\alpha(v_1 + \lambda v_2) = \phi_\alpha(v_1) + \lambda\phi_\alpha(v_2)$. Thus $\phi_\alpha \in L(V; V^*)$. □

Exercise 2.4.0.4. Let V be a finite dimensional vector space and $\alpha \in T^0_2(V)$. Then α is nondegenerate iff ϕ_α is an isomorphism.

Proof.

- $(\implies:)$
  Suppose that α is nondegenerate. Let $v \in \ker\phi_\alpha$. Then for each $w \in V$,

  $$\begin{aligned}\alpha(v, w) &= (\iota_v\alpha)(w)\\ &= \phi_\alpha(v)(w)\\ &= 0\end{aligned}$$

  Since α is nondegenerate, $v = 0$. Since $v \in \ker\phi_\alpha$ is arbitrary, $\ker\phi_\alpha = \{0\}$. Hence ϕ_α is injective. Since $\dim V = \dim V^*$, ϕ_α is surjective. Hence ϕ_α is an isomorphism.

- $(\impliedby:)$
  Suppose that ϕ_α is an isomorphism. Let $v \in V$. Suppose that for each $w \in V$, $\alpha(v, w) = 0$. Then for each $w \in V$,

  $$\begin{aligned}\phi_\alpha(v)(w) &= (\iota_v\alpha)(w)\\ &= \alpha(v, w)\\ &= 0\end{aligned}$$

  Thus $\phi_\alpha(v) = 0$ which implies that $v \in \ker\phi_\alpha$. Since ϕ_α is an isomorphism, $v = 0$. Hence α is nondegenerate.

□

Exercise 2.4.0.5. Let V be a finite dimensional vector space and $\alpha \in T^0_2(V)$. Then

1. $[\phi_\alpha]_{i,j} = \alpha(e_j, e_i)$
2. for each $v, w \in V$,

   $$\alpha(v, w) = [w]^*[\phi_\alpha][v]$$

Proof.

1. Set $A = [\phi_\alpha]$. Let $i, j \in \{1, \dots, n\}$. By definition,

$$\phi_\alpha(e_j) = \sum_{k=1}^{n} A_{k,j} \epsilon^k$$

Then

$$\begin{aligned} \phi_\alpha(e_j)(e_i) &= \sum_{k=1}^{n} A_{k,j} \epsilon^k(e_i) \\ &= \sum_{k=1}^{n} A_{k,j} \delta_{k,i} \\ &= A_{i,j} \end{aligned}$$

2. Let $v, w \in V$. Then there exist $(v^i)_{i=1}^n, (w^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{i=1}^{n} v^i e_i$ and $w = \sum_{j=1}^{n} v^j e_i$. Part (1) implies that

$$\begin{aligned} \alpha(v, w) &= \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j \alpha(e_i, e_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j [\phi_\alpha]_{j,i} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} [v]_i [w]_j [\phi_\alpha]_{j,i} \\ &= [w]^* [\phi_\alpha][v] \end{aligned}$$

□

2.4.1 Scalar Product Spaces

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Definition 2.4.1.1. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$ (define $\Sigma^2(V)$ i.e. symmetric (0,2)-tensors). Then α is said to be

- **positive semidefinite** if for each $v \in V$, $\alpha(v, v) \geq 0$
- **positive definite** if for each $v \in V$, $v \neq 0$ implies that $\alpha(v, v) > 0$
- **negative semidefinite** if $-\alpha$ is positive semidefinite
- **negative definite** if $-\alpha$ is positive definite

Exercise 2.4.1.2. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then

1. α is positive semidefinite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda \geq 0$
2. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda > 0$

Proof.

1. - ($\implies$) :
     Suppose that there exists $\lambda \in \sigma([\phi_\alpha])$ such that $\lambda < 0$. Then there exists $v_\lambda \in \mathbb{R}^n$ $v_\lambda^*[\phi_\alpha]v_\lambda$

- ($\Longleftarrow$) :

Suppose that α is positive semidefinite. Write $\sigma(\phi_\alpha) = \{\lambda_1, \dots, \lambda_n\}$. Define $\Lambda \in \mathbb{R}^{n\times n}$ by $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Since α is symmetric, $[\phi_\alpha]$ is symmetric. There exists $U \in O(n)$ such that $[\phi_\alpha] = U\Lambda U^*$. FINISH!!!

□

Definition 2.4.1.3. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be a **scalar product** if α is nondegenerate. In this case, (V, α) is said to be a **scalar product space**.

Definition 2.4.1.4. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$ a scalar product on V. We define the **index** of α, denoted $\operatorname{ind}\alpha$ by

$$\operatorname{ind}\alpha = \max\{\dim W : W \text{ is a subspace of } V \text{ and } \alpha|_{W\times W} \text{ is negative definite}\}$$

Definition 2.4.1.5. Let (V, α) be a scalar product space.

- Let $v_1, v_2 \in V$. Then v_1 and v_2 are said to be **orthogonal** if $\alpha(v_1, v_2) = 0$.
- Let $U \subset V$ be a subspace. We define the **orthogonal subspace of** U, denoted by $U^\perp$, by

$$U^\perp = \{v \in V : \text{ for each } u \in U,\ \alpha(u, v) = 0\}$$

Exercise 2.4.1.6. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then $U^\perp$ is a subspace of V.

Proof. We note that since $U^\perp = \bigcap_{u\in U} \ker \phi_\alpha(u)$, $U^\perp$ is a subspace of V. □

Exercise 2.4.1.7. Let (V, α) be an n-dimensional scalar product space, $U \subset V$ a k-dimensional subspace and $(e_j)_{j=1}^n \subset V$ a basis for V. Suppose that $(e_j)_{j=1}^k$ is a basis for U. Then for each $v \in V$, $v \in U^\perp$ iff for each $j \in [k]$, $\alpha(v, e_j) = 0$.

Proof. Let $v \in V$.

- ($\Longrightarrow$): Suppose that $v \in U^\perp$. Since $(e_j)_{j=1}^k \subset U$, we have that for each $j \in [k]$, $\alpha(v, e_j) = 0$.
- ($\Longleftarrow$): Suppose that for each $j \in [k]$, $\alpha(v, e_j) = 0$. Let $u \in U$. Then there exist $(a^j)_{j=1}^k \subset \mathbb{R}$ such that $u = \sum\limits_{j=1}^k a^j u_j$. This implies that

$$\begin{aligned}\alpha(v, u) &= \sum_{j=1}^k a^j \alpha(v, u_j) \\ &= 0\end{aligned}$$

Since $u \in U$ is arbitrary, we have that $v \in U^\perp$.

□

Exercise 2.4.1.8. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then

1. $\dim V = \dim U + \dim U^\perp$
2. $(U^\perp)^\perp = U$

Proof.

1. Set $n = \dim V$ and $k = \dim U$. Choose a basis $(e_j)_{j=1}^n$ such that $(e_j)_{j=1}^k$ is a basis for U.

2.

$\square$

Exercise 2.4.1.9. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Set $\sigma([\phi_\alpha])^- = \{\lambda \in \sigma([\phi_\alpha]) : \lambda < 0\}$. Then

$$\operatorname{ind}\alpha = \sum_{\lambda \in \sigma([\phi_\alpha])^-} \mu(\lambda)$$

Proof. Since α is symmetric, there exist $U \in O(n)$ and $\Lambda \in D(n,\mathbb{R})$ such that $[\phi_\alpha] = U\Lambda U^*$. Define $(u_j)_{j=1}^n \subset V$ by $u_j = \sum_{i=1}^n U_{i,j} e_j$. Define $J^- = \{j \in [n] : \Lambda_{j,j} < 0\}$, $n^- = \#J^-$ and $V^- = \operatorname{span}\{u_j : j \in J^-\}$. Let $v \in V^-$. Then there exist $(a^j)_{j \in J^-}$ such that $v = \sum_{j \in J^-} a^j u_j$. We note that

$$\begin{aligned}
U^*[\phi_\alpha]U &= U^*(U\Lambda U^*)U \\
&= (U^*U)\Lambda(U^*U) \\
&= I\Lambda I \\
&= \Lambda
\end{aligned}$$

A previous exercise implies that

$$\begin{aligned}
\alpha(v,v) &= \sum_{j \in J^-}\sum_{k \in J^-} a^j a^k \alpha(u_j,u_k) \\
&= \sum_{j \in J^-}\sum_{k \in J^-} a^j a^k [u_j]^*[\phi_\alpha][u_k] \\
&= \sum_{j \in J^-}\sum_{k \in J^-} a^j a^k ([e_j]^*U^*)[\phi_\alpha](U[e_k]) \\
&= \sum_{j \in J^-}\sum_{k \in J^-} a^j a^k (U^*[\phi_\alpha]U)_{j,k} \\
&= \sum_{j \in J^-}\sum_{k \in J^-} a^j a^k (\Lambda)_{j,k} \\
&= \sum_{j \in J^-} |a^j|^2 \Lambda_{j,j} \\
&< 0
\end{aligned}$$

Since $v \in V^-$ is arbitrary, $\alpha|_{V^- \times V^-}$ is negative definite. Thus

$$\begin{aligned}
\operatorname{ind}\alpha &\geq \dim V^- \\
&= n^-
\end{aligned}$$

Set $J^+ = (J^-)^c$. Let $W \subset V$ be a subspace. Suppose that $\alpha|_{W \times W}$ is negative definite. For the sake of contradiction, suppose that there exists $j_0 \in J^+$ such that $u_{j_0} \in W$. Then

$$\begin{aligned}
\alpha(u_{j_0},u_{j_0}) &= [u_{j_0}]^*[\phi_\alpha][u_{j_0}] \\
&= [u_{j_0}]^*U\Lambda U^*[u_{j_0}] \\
&= \Lambda_{j_0,j_0} \\
&\geq 0
\end{aligned}$$

which is a contradiction since $\alpha|_{W \times W}$ is negative definite. Thus for each $j \in J^+$, $u_j \notin W$. $\square$

Definition 2.4.1.10. Let (V,α) be an n-dimensional scalar product space. We define the **scalar norm associated to** α, denoted $\|\cdot\|_\alpha : V \to \mathbb{R}$ by $\|v\|_\alpha := |\alpha(v,v)|^{1/2}$.

Note 2.4.1.11.

- When the context is clear, we write $\|\cdot\|$ in place of $\|\cdot\|_\alpha$.
- α is not positive definite iff $\|\cdot\|_\alpha$ is not a norm.

alternatively, define GS algorithm in terms of orthogonal projections

Exercise 2.4.1.12. Gram-Schmidt Algorithm:
Let (V, α) be an n-dimensional scalar product space and $(v_j)_{j\in[n]} \subset V$ a basis for V. For $j \in [n]$, define $u_j, e_j \in$ If α is nondegenerate, then there exists $(e_j)_{j=1}^n \subset V$ such that $(e_j)_{j=1}^n$ is an orthonormal basis for V.

Proof. Suppose that α is nondegenerate. Then for each $v \in V$, $\alpha(v, v) \neq 0$. Choose $(v_j)_{j=1}^n \subset V$ such that $(v_j)_{j=1}^n$ is a basis for V. For each $j \in [n]$, we define

$$\begin{aligned} u_j &:= \begin{cases} v_1, & j = 1 \\ v_j - \sum\limits_{k=1}^{j-1} [\alpha(v_j, u_k)/\alpha(u_k, u_k)]u_k, & j \geq 2 \end{cases} \\ e_j &:= u_j/\|u_j\|_\alpha. \end{aligned}$$

Let $j_1, j_2 \in [n]$. Suppose that $j_1 \leq j_2$. Then $\alpha(e_l, e_k)$

- Clearly,

$$\begin{aligned} \alpha(u_1, u_2) &= \alpha(v_1, v_2 - \sum_{k=1}^{j_1} [\alpha(v_2, u_k)/\alpha(u_k, u_k)]u_k) \\ &= \alpha(v_1, v_2 - \frac{\alpha(v_2, u_1)}{\alpha(u_1, u_1)} u_1) \\ &= \alpha(v_1, v_2 - \frac{\alpha(v_2, v_1)}{\alpha(v_1, v_1)} v_1) \\ &= \alpha(v_1, v_2) - \alpha(v_1, \frac{\alpha(v_2, v_1)}{\alpha(v_1, v_1)} v_1) \\ &= \alpha(v_1, v_2) - \frac{\alpha(v_2, v_1)}{\alpha(v_1, v_1)} \alpha(v_1, v_1) \\ &= \alpha(v_1, v_2) - \alpha(v_2, v_1) \end{aligned}$$

-

$$\begin{aligned} \alpha(u_1, u_2) &= \alpha(v_1, v_2 - \sum_{k=1}^{j_1} [\alpha(v_2, u_k)/\alpha(u_k, u_k)]u_k) \\ &= \alpha(v_1, v_2 - \frac{\alpha(v_2, u_1)}{\alpha(u_1, u_1)} u_1) \\ &= \alpha(v_1, v_2 - \frac{\alpha(v_2, v_1)}{\alpha(v_1, v_1)} v_1) \\ &= \alpha(v_1, v_2) - \alpha(v_1, \frac{\alpha(v_2, v_1)}{\alpha(v_1, v_1)} v_1) \\ &= \alpha(v_1, v_2) - \frac{\alpha(v_2, v_1)}{\alpha(v_1, v_1)} \alpha(v_1, v_1) \\ &= \alpha(v_1, v_2) - \alpha(v_2, v_1) \end{aligned}$$

FINISH!!! proof by induction? □

2.4.2 Symplectic Vector Spaces

Definition 2.4.2.1. Let V be a finite dimensional vector space and $\omega \in \Lambda^2(V)$. Then ω is said to be a **symplectic form** if ω is nondegenerate. In this case (V, ω) is said to be a **symplectic space**.

Exercise 2.4.2.2. Let V be a $2n$-dimensional vector space with basis $(a_j, b_j)_{j=1}^n$ and corresponding dual basis $(\alpha^j, \beta^j)_{j=1}^n$. Define $\omega \in \Lambda^2(V)$ by

$$\omega = \sum_{j=1}^n \alpha^j \wedge \beta^j$$

Then

1. for each $j, k \in \{1, \dots, n\}$,
   (a) $\omega(a_j, a_k) = 0$
   (b) $\omega(b_j, b_k) = 0$
   (c) $\omega(a_j, b_k) = \delta_{j,k}$
2. (V, ω) is a symplectic space

Proof.

1. Let $j, k \in \{1, \dots, n\}$.
   (a)
$$\begin{aligned}\omega(a_j, a_k) &= \sum_{l=1}^n \alpha^l \wedge \beta^l(a_j, a_k) \\ &= \sum_{l=1}^n [\alpha^l(a_j)\beta^l(a_k) - \alpha^l(a_k)\beta^l(a_j)] \\ &= 0\end{aligned}$$
   (b) Similar to (a)
   (c)
$$\begin{aligned}\omega(a_j, b_k) &= \sum_{l=1}^n \alpha^l \wedge \beta^l(a_j, b_k) \\ &= \sum_{l=1}^n [\alpha^l(a_j)\beta^l(b_k) - \alpha^l(b_k)\beta^l(a_j)] \\ &= \sum_{l=1}^n \alpha^l(a_j)\beta^l(b_k) \\ &= \sum_{l=1}^n \delta_{j,l}\delta_{l,k} \\ &= \delta_{j,k}\end{aligned}$$
2. Let $v \in V$. Then there exist $(q^j, p^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum\limits_{j=1}^n q^j a_j + p^j b_j$. Suppose that for each

$w \in V$, $\omega(v, w) = 0$. Let $k \in \{1, \dots, n\}$. Then

$$\begin{aligned} 0 &= \omega(v, a_k) \\ &= \sum_{j=1}^{n} q^j \omega(a_j, a_k) + p^j \omega(b_j, a_k) \\ &= \sum_{j=1}^{n} p^j \delta_{j,k} \\ &= p^k \end{aligned}$$

Similarly,

$$\begin{aligned} 0 &= \omega(v, b_k) \\ &= \sum_{j=1}^{n} q^j \omega(a_j, b_k) + p^j \omega(b_j, b_k) \\ &= \sum_{j=1}^{n} q^j \delta_{j,k} \\ &= q^k \end{aligned}$$

Since $k \in \{1, \dots, n\}$ is arbitrary, $v = 0$. Hence ω is nondegenerate. Therefore (V, ω) is symplectic.

□

Exercise 2.4.2.3. Let (V, ω) be a symplectic space. Then $\dim V$ is even.

Proof. Set $n = \dim V$. Let $(e_j)_{j=1}^n$ be a basis for V. Define $[\omega] \in \mathbb{R}^{n \times n}$ by $[\omega]_{i,j} = \omega(e_i, e_j)$. Since $\omega \in \Lambda^2(V)$, $[\omega]^* = -[\omega]$. Therefore

$$\begin{aligned} \det[\omega] &= \det[\omega]^* \\ &= \det(-[\omega]) \\ &= (-1)^n \det[\omega] \end{aligned}$$

For the sake of contradiction, suppose that n is odd. Then $\det[\omega] = -\det[\omega]$ which implies that $\det[\omega] = 0$. Since ω is nondegenerate, $[\omega] \in GL(n, \mathbb{R})$. This is a contradiction. Hence n is even. □

Definition 2.4.2.4. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. We define the **symplectic complement of** V, denoted $S^\perp$, by

$$S^\perp = \{v \in V : \text{ for each } w \in S, \ \omega(v, w) = 0\}$$

Exercise 2.4.2.5. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then $S^\perp$ is a subspace.

Proof. We note that

$$S^\perp = \bigcap_{v \in S} \ker \iota_v \omega$$

Hence $S^\perp$ is a subspace. □

Exercise 2.4.2.6. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then

$$\dim V = \dim S + \dim S^\perp$$

Proof. □

Exercise 2.4.2.7. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then $(S^\perp)^\perp = S$.

Proof. Let $v \in (S^\perp)^\perp$. Then for each $w \in S^\perp$, $\omega(v, w) = 0$. □

Chapter 3

Topological Manifolds

3.1 Introduction

- redo in terms of all charts (U, ϕ) where for some j, $\phi(U) \in \mathcal{T}_{\mathbb{H}^n_j}$ or $\phi(U) \in \mathcal{T}_{\mathbb{R}^n}$ and then make an exercise about equivalently being $\phi(U) \in \mathcal{T}_{\mathbb{H}^n_j}$ and if $\phi(U) \in \mathcal{T}_{\mathbb{R}^n}$ iff interior chart.
- show $\varnothing$ is a top manifold of every dimension

Exercise 3.1.0.1. We have that $\mathbb{R}$ is homeomorphic to $(0, \infty)$

Proof. Define $f : \mathbb{R} \to (0, \infty)$ by $f(x) = e^x$. Then f is a homeomorphism. $\square$

Definition 3.1.0.2. Let $n \in \mathbb{N}$ and $j \in [n]$. We define the **j-th coordinate upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}^n_j$, by

$$\mathbb{H}^n_j = \{(x^1, x^2, \cdots, x^n) \in \mathbb{R}^n : x^j \geq 0\}$$

and we define

$$\partial\mathbb{H}^n_j = \{(x^1, x^2, \cdots, x^n) \in \mathbb{R}^n : x^j = 0\}$$

$$\operatorname{Int}\mathbb{H}^n_j = \{(x^1, x^2, \cdots, x^n) \in \mathbb{R}^n : x^j > 0\}$$

We endow $\mathbb{H}^n_j$, $\partial\mathbb{H}^n_j$ and $\operatorname{Int}\mathbb{H}^n_j$ with the subspace topology inherited from $\mathbb{R}^n$.
We define the projection map $\pi_{\partial\mathbb{H}^n_j} : \partial\mathbb{H}^n_j \to \mathbb{R}^{n-1}$ by

$$\pi_{\partial\mathbb{H}^n_j}(x^1, \dots, x^{j-1}, x^j, x^{j+1}, \dots, x^n) = (x^1, \dots, x^{j-1}, 0, x^{j+1}, \dots, x^{n-1})$$

Definition 3.1.0.3. We define $\mathbb{R}^0 := \{0\}$, $\mathbb{H}^0 := \{0\}$, $\partial\mathbb{H}^0 := \varnothing$, and $\mathbb{H}^{-1}_1 = \varnothing$ endowed with the discrete topology.

Note 3.1.0.4. show in calculus section that $\lambda_{n,k} : \mathbb{H}^n_j \to \mathbb{H}^n_k$ is a diffeo

Exercise 3.1.0.5. Let $n \in \mathbb{N}$ and $j \in [n]$. Then

1. $\partial\mathbb{H}^n_j$ is homeomorphic to $\mathbb{R}^{n-1}$,
2. $\operatorname{Int}\mathbb{H}^n_j$ is homeomorphic to $\mathbb{R}^n$.

Proof.

1. Clearly $\pi_{\partial\mathbb{H}^n_j}$ is a homeomorphism.
2. Define $f_j : \mathbb{R}^n \to \operatorname{Int}\mathbb{H}^n_j$ by $f(x^1, \dots, x^{j-1}, x^j, x^{j+1}, \dots, x^n) = (x^1, \dots, x^{j-1}, e^{x^j}, x^{j+1}, \dots, x^n)$. Then f is a homeomorphism.

$\square$

Exercise 3.1.0.6. Let $A \subset \mathbb{H}^n_j$. Suppose that A is open in $\mathbb{H}^n_j$. Then A is open in $\mathbb{R}^n$ iff $A \cap \partial \mathbb{H}^n_j = \varnothing$.
Hint: simply connected? FINISH!!!

Proof.

- $(\implies)$:
  Suppose that A is open in $\mathbb{R}^n$. For the sake of contradiction, suppose that $A \cap \partial \mathbb{H}^n_j \neq \varnothing$. Then there exists $x \in A$ such that $x \in \partial \mathbb{H}^n_j$. Since A is open in $\mathbb{R}^n$, there exists $B \subset A$ such that B is open in $\mathbb{R}^n$, $x \in B$ and B is simply connected. Set $B' := B \setminus \{x\}$. Then B' is not simply connected. FINISH!!! Just show that you cant get a ball in $\mathbb{R}^n$ around x which is contained in $\mathbb{H}^n_j$.

- $(\impliedby)$:
  Suppose that $A \cap \partial \mathbb{H}^n_j = \varnothing$. Then $A \subset \operatorname{Int} \mathbb{H}^n_j$. Since $\operatorname{Int} \mathbb{H}^n_j$ is open in $\mathbb{R}^n$, we have that

  $$\begin{aligned} \mathcal{T}_{\operatorname{Int} \mathbb{H}^n_j} &= \mathcal{T}_{\mathbb{R}^n} \cap \operatorname{Int} \mathbb{H}^n_j \\ &\subset \mathcal{T}_{\mathbb{R}^n} \end{aligned}$$

  An exercise in the section on subspace topology in the analysis notes implies that

  $$\begin{aligned} \mathcal{T}_{\operatorname{Int} \mathbb{H}^n_j} &= \mathcal{T}_{\mathbb{R}^n} \cap \operatorname{Int} \mathbb{H}^n_j \\ &= (\mathcal{T}_{\mathbb{R}^n} \cap \mathbb{H}^n_j) \cap \operatorname{Int} \mathbb{H}^n_j \\ &= \mathcal{T}_{\mathbb{H}^n_j} \cap \operatorname{Int} \mathbb{H}^n_j \end{aligned}$$

  Since $A \in \mathcal{T}_{\mathbb{H}^n_j}$ and $A \subset \operatorname{Int} \mathbb{H}^n_j$, we have that

  $$\begin{aligned} A &\in \mathcal{T}_{\mathbb{H}^n_j} \cap \operatorname{Int} \mathbb{H}^n_j \\ &= \mathcal{T}_{\operatorname{Int} \mathbb{H}^n_j} \\ &\subset \mathcal{T}_{\mathbb{R}^n} \end{aligned}$$

  Thus A is open in $\mathbb{R}^n$.

□

Definition 3.1.0.7. Let $(M, \mathcal{T})$ be a topological space, $n \in \mathbb{N}$, $j \in [n]$, $U \subset M$, $V \subset \mathbb{R}^n$ and $\phi : U \to V$. Then

- (U, ϕ) is said to be an $\mathbb{R}^n$**-coordinate chart on** $(M, \mathcal{T})$ if
  - $U \in \mathcal{T}$
  - $V \in \mathcal{T}_{\mathbb{R}^n}$
  - ϕ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{R}^n} \cap V)$-homeomorphism
- (U, ϕ) is said to be an $\mathbb{H}^n_j$**-coordinate chart on** $(M, \mathcal{T})$ if
  - $U \in \mathcal{T}$
  - $V \in \mathcal{T}_{\mathbb{H}^n_j}$
  - ϕ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n_j} \cap V)$-homeomorphism
- (U, ϕ) is said to be an n**-coordinate chart on** $(M, \mathcal{T})$ if (U, ϕ) is an $\mathbb{R}^n$-coordinate chart on $(M, \mathcal{T})$ or there exists $j \in [n]$ such that (U, ϕ) is an $\mathbb{H}^n_j$-coordinate chart on $(M, \mathcal{T})$.
- We define

  $$X^{n,j}(M, \mathcal{T}) := \{(U, \phi) : (U, \phi) \text{ is an } \mathbb{H}^n_j\text{-coordinate chart on } (M, \mathcal{T})\}$$

  and

  $$X^n(M, \mathcal{T}) := \{(U, \phi) : (U, \phi) \text{ is an } n\text{-coordinate chart on } (M, \mathcal{T})\}$$

Note 3.1.0.8. From Definition 1.3.3.2, Exercise 1.3.3.3 and Definition 1.3.3.4, we recall

- the definition of the action $S_n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(\sigma, x) \mapsto \sigma \cdot x$,
- for $\sigma \in S_n$, the definition of the map $\Phi_\sigma : \mathbb{R}^n \to \mathbb{R}^n$,
- that Φ_σ is a diffeomorphism,
- for $U \subset \mathbb{R}^n$, the definition of the action $S_n \times (\mathbb{R}^n)^U \to (\mathbb{R}^n)^U$ given by $(\sigma, \phi) \mapsto \sigma \cdot \phi$.

Exercise 3.1.0.9. Let $(M, \mathcal{T})$ be a topological space, $n \in \mathbb{N}$, $j \in [n]$ and $(U, \phi) \in X^{n,j}(M, \mathcal{T})$. For each $\sigma \in S_n$, $\sigma \cdot \phi \in X^{n,\sigma(j)}(M, \mathcal{T})$.

Proof. Let $\sigma \in S_n$. We note the following:

1. By definition, $\sigma \cdot \phi = \Phi_\sigma \circ \phi$. Since $\Phi_\sigma(\mathbb{H}^n_j) = \mathbb{H}^n_{\sigma(j)}$, we have that $(\sigma \cdot \phi)(U) \subset \mathbb{H}^n_{\sigma(j)}$.
2. Since Φ_σ is a diffeomorphism, $\Phi_\sigma|_{\mathbb{H}^n_j}$ is a $(\mathcal{T}_{\mathbb{H}^n_j}, \mathcal{T}_{\mathbb{H}^n_{\sigma(j)}})$-homeomorphism. Since $(U, \phi) \in X^{n,j}(M, \mathcal{T})$, ϕ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n_j} \cap \phi(U))$-homeomorphism. Thus $\sigma \cdot \phi$ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n_{\sigma(j)}} \cap (\sigma \cdot \phi)(U))$-homeomorphism.

Since $(U, \phi) \in X^{n,j}(M, \mathcal{T})$, $U \in \mathcal{T}$. Since $\sigma \cdot \phi$ is a homeomorphism, we have that $(\sigma \cdot \phi)(U) \in \mathcal{T}_{\mathbb{H}^n_{\sigma(j)}}$. Summarizing, we have that

- $U \in \mathcal{T}$,
- $(\sigma \cdot \phi)(U) \in \mathcal{T}_{\mathbb{H}^n_{\sigma(j)}}$,
- $\sigma \cdot \phi$ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n_{\sigma(j)}} \cap \Phi_\sigma(U))$-homeomorphism.

Hence $(U, \sigma \cdot \phi) \in X^{n,\sigma(j)}(M, \mathcal{T})$. □

Exercise 3.1.0.10. Let $(M, \mathcal{T})$ be a topological space, $n \in \mathbb{N}$ and $j, k \in [n]$. For each $p \in M$, there exists $(U, \phi) \in X^{n,j}(M, \mathcal{T})$ such that $p \in U$ iff there exists $(V, \psi) \in X^{n,k}(M, \mathcal{T})$ such that $p \in V$.

Proof. Let $p \in M$.

- $(\implies)$:
  Suppose that there exists $(U, \phi) \in X^{n,j}(M, \mathcal{T})$ such that $p \in U$. Choose $\sigma \in S_n$ such that $\sigma(j) = k$. Define $V := U$ and $\psi := \sigma \cdot \phi$. Then $(V, \psi) \in X^{n,k}(M, \mathcal{T})$ and $p \in V$.
- $(\impliedby)$:
  Suppose that there exists $(V, \psi) \in X^{n,k}(M, \mathcal{T})$ such that $p \in V$. Choose $\tau \in S_n : \tau(k) = j$. Define $U := V$ and $\phi = \tau \cdot \psi$. Then $(U, \phi) \in X^{n,j}(M, \mathcal{T})$ and $p \in U$.

□

Note 3.1.0.11. So if there is at least one coordinate chart to the j-th upper half-space, then there are coordinate charts to all upper half spaces.

need to define $[n] = \{1, \dots, n\}$ if $n \geq 1$ and $[n] = \{1\}$ if $n \in \{-1, 0\}$.

Definition 3.1.0.12. Let $(M, \mathcal{T})$ be a topological space and $n \in \mathbb{N}$. We define

$$X^n(M, \mathcal{T}) := \bigcup_{j=1}^{n} X^{n,j}(M, \mathcal{T})$$

add case $n = 0$.

Note 3.1.0.13. We will write $X^n(M)$ in place of $X^n(M, \mathcal{T})$ when the topology is not ambiguous.

Definition 3.1.0.14. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be **locally Euclidean of dimension** n if for each $p \in M$, there exists $(U, \phi) \in X^n(M)$ such that $p \in U$.

Definition 3.1.0.15. Let M be a topological space and $n \in \mathbb{N}_{-1}$. Then M is said to be an n**-dimensional topological manifold** if

1. M is Hausdorff
2. M is second-countable
3. M is locally Euclidean of dimension n

Exercise 3.1.0.16. Let $n \in \mathbb{N}_{-1}$. Then

1. $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n}) \in X^n(\mathbb{R}^n)$
2. $(\mathbb{H}^n_j, \mathrm{id}_{\mathbb{H}^n_j}) \in X^n(\mathbb{H}^n_j)$. fix

Proof.

1.
2.

$\square$

Exercise 3.1.0.17. Let $n \in \mathbb{N}_0$. Then

1. $\mathbb{R}^n$ is an n-dimensional topological manifold of dimension n,
2. if $n \geq 1$, then $\mathbb{H}^n_j$ is an n-dimensional topological manifold of dimension n. fix

Proof.

1.
2.

$\square$

Theorem 3.1.0.18. Invariance of Domain

Theorem 3.1.0.19. Topological Invariance of Dimension:
Let $n \in \mathbb{N}_0$, M an m-dimensional toplogical manifold and N a n-dimensional toplogical manifold. If M and N are homeomorphic, then $m = n$.

try to prove, first for subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, then the general case, see math stack exchange for short proof https://math.stackexchange.com/questions/1197640/elementary-proof-of-topological-invariance-of-dimension-using-brouwers-fixed-po the idea is that suppose $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are open and $f : U \to V$ is homeo. If $n < m$, then $\iota \circ f$ is a topological embedding onto its image where $\iota : \mathbb{R}^n \to \mathbb{R}^m$ is the inclusion, since $n < m$, no subset of $\iota(\mathbb{R}^n)$ (besides the empty set) is open in $\mathbb{R}^m$. Now use Invariance of domain theorem from algebraic topology.

Note 3.1.0.20. In light of the previous theorem, we write $X(M)$ in place of $X^n(M)$ and refer to n-coordinate charts as coordinate charts when the context is clear.

Exercise 3.1.0.21. Let $n \in \mathbb{N}$, $j, k \in [n]$, $U \in \mathcal{T}_{\mathbb{H}^n_j}$, $V \in \mathcal{T}_{\mathbb{H}^n_k}$ and $\phi : U \to V$. Suppose that ϕ is a $(\mathcal{T}_{\mathbb{H}^n_j} \cap U, \mathcal{T}_{\mathbb{H}^n_k} \cap V)$-homeomorphism. Then for each $p \in U$,

1. $p \in \partial\mathbb{H}^n_j$ iff $\phi(p) \in \partial\mathbb{H}^n_k$,
2. $p \in \operatorname{Int}\mathbb{H}^n_j$ iff $\phi(p) \in \operatorname{Int}\mathbb{H}^n_k$.

Proof. Let $p \in U$.

1. - $(\implies:)$
     For the sake of contradiction, suppose that $p \in \partial\mathbb{H}^n_j$ and $\phi(p) \notin \partial\mathbb{H}^n_k$. Then

$$\begin{aligned}\phi(p) &\in (\partial\mathbb{H}^n_k)^c \\ &= \operatorname{Int}\mathbb{H}^n_k\end{aligned}$$

     Since $\operatorname{Int}\mathbb{H}^n_k \cap V \in \mathcal{T}_{\mathbb{H}^n_k} \cap V$ and $\phi(p) \in \operatorname{Int}\mathbb{H}^n_k \cap V$, there exists $B_V \in \mathcal{T}_{\mathbb{H}^n_k} \cap V$ such that $B_V \subset \operatorname{Int}\mathbb{H}^n_k \cap V$, $\phi(p) \in B_V$ and B_V is simply connected. Define $B_U := \phi^{-1}(B_V)$. Since ϕ is a $(\mathcal{T}_{\mathbb{H}^n_j} \cap U, \mathcal{T}_{\mathbb{H}^n_k} \cap V)$-homeomorphism, $\phi|_{B_U} : B_U \to B_V$ is a $(\mathcal{T}_{\mathbb{H}^n_j} \cap B_U, \mathcal{T}_{\mathbb{H}^n_k} \cap B_V)$-homeomorphism. Therefore $B_U \in \mathcal{T}_{\mathbb{H}^n_j} \cap U$, $p \in B_U$ and B_U is simply connected.

     Define $B'_U \in \mathcal{T}_{\mathbb{H}^n_j} \cap U$ and $B'_V \in \mathcal{T}_{\mathbb{H}^n_k} \cap V$ by $B'_U := B_U \setminus \{p\}$ and $B'_V := B_V \setminus \{\phi(p)\}$. Since $p \in \partial\mathbb{H}^n_j$, B'_U is simply connected. Since ϕ is a $(\mathcal{T}_{\mathbb{H}^n_j} \cap U, \mathcal{T}_{\mathbb{H}^n_k} \cap V)$-homeomorphism, $\phi|_{B'_U} : B'_U \to B'_V$ is a $(\mathcal{T}_{\mathbb{H}^n_j} \cap B'_U, \mathcal{T}_{\mathbb{H}^n_k} \cap B'_V)$-homeomorphism. Therefore B'_V is simply connected.

     Since $\phi(p) \in \operatorname{Int}\mathbb{H}^n_k$, B'_V is not simply connected. This is a contradiction. Hence $p \in \partial\mathbb{H}^n_j$ implies that $\phi(p) \in \partial\mathbb{H}^n_k$.
   - $(\impliedby):$
     Suppose that $\phi(p) \in \partial\mathbb{H}^n_k$. Set $q = \phi(p)$. Then $\phi^{-1} : V \to U$ is a $(\mathcal{T}_{\mathbb{H}^n_k} \cap V, \mathcal{T}_{\mathbb{H}^n_j} \cap U)$-homeomorphism. The previous part implies that

$$\begin{aligned}p &= \phi^{-1}(q) \\ &\in \partial\mathbb{H}^n_j\end{aligned}$$

2. By part (1), we have that

$$\begin{aligned}p \in \operatorname{Int}\mathbb{H}^n_j &\iff p \notin \partial\mathbb{H}^n_j \\ &\iff \phi(p) \notin \partial\mathbb{H}^n_k \\ &\iff \phi(p) \in \operatorname{Int}\mathbb{H}^n_k\end{aligned}$$

□

Definition 3.1.0.22. Let $n \in \mathbb{N}$, $(M, \mathcal{T})$ be an n-dimensional topological manifold and $(U, \phi) \in X^n(M, \mathcal{T})$. Then (U, ϕ) is said to be

- an **interior chart** if there exists $j \in [n]$ such that $(U, \phi) \in X^{n,j}(M, \mathcal{T})$ and $\phi(U) \cap \partial\mathbb{H}^n_j = \varnothing$,
- a **boundary chart** if there exists $j \in [n]$ such that $(U, \phi) \in X^{n,j}(M, \mathcal{T})$ and $\phi(U) \cap \partial\mathbb{H}^n_j \neq \varnothing$.

We set

- $X^n_{\mathrm{Int}}(M, \mathcal{T}) := \{(U, \phi) \in X^n(M, \mathcal{T}) : (U, \phi) \text{ is an interior chart}\}$
- $X^n_{\partial}(M, \mathcal{T}) := \{(U, \phi) \in X^n(M, \mathcal{T}) : (U, \phi) \text{ is a boundary chart}\}$

For $j \in [n]$, we define

- $X^{n,j}_{\mathrm{Int}}(M, \mathcal{T}) := X^n_{\mathrm{Int}}(M, \mathcal{T}) \cap X^{n,j}(M, \mathcal{T})$,
- $X^{n,j}_{\partial}(M, \mathcal{T}) := X^n_{\partial}(M, \mathcal{T}) \cap X^{n,j}(M, \mathcal{T})$.

Exercise 3.1.0.23. Let $n \in \mathbb{N}$, M be an n-dimensional topological manifold, $j \in [n]$ and $(U, \phi) \in X^{n,j}(M, \mathcal{T})$. Then

1. $(U, \phi) \in X^{n,j}_{\mathrm{Int}}(M, \mathcal{T})$ iff for each $k \in [n]$

Proof.

1.

2. for each $p \in M$, there exists $(U, \phi) \in X^{n,j}_{\text{Int}}(M)$ such that $p \in U$ iff there exists $(V, \psi) \in X^{n,k}_{\text{Int}}(M, \mathcal{T})$ such that $p \in V$.
3. for each $p \in M$, there exists $(U, \phi) \in X^{n,j}_{\partial}(M)$ such that $p \in U$ iff there exists $(V, \psi) \in X^{n,k}_{\partial}(M, \mathcal{T})$ such that $p \in V$.

□

Exercise 3.1.0.24. Let $n \in \mathbb{N}$, $(M, \mathcal{T})$ be an n-dimensional topological manifold and $j \in [n]$. Then

1. $X^n(M, \mathcal{T}) = X^n_{\text{Int}}(M, \mathcal{T}) \cup X^n_{\partial}(M, \mathcal{T})$
2. $X^n_{\text{Int}}(M, \mathcal{T}) \cap X^n_{\partial}(M, \mathcal{T}) = \varnothing$

Proof. FIX

1. By definition, $X^n_{\text{Int}}(M, \mathcal{T}) \cup X^n_{\partial}(M, \mathcal{T}) \subset X^n(M, \mathcal{T})$. Let $(U, \phi) \in X^n(M, \mathcal{T})$. By definition, there exists $j \in [n]$ such that $(U, \phi) \in X^{n,j}(M, \mathcal{T})$. If $\phi(U) \cap \partial\mathbb{H}^n_j = \varnothing$, then
$$\begin{aligned}(U, \phi) &\in X^{n,j}_{\text{Int}}(M) \\ &\subset X^{n,j}_{\text{Int}}(M) \cup X^{n,j}_{\partial}(M)\end{aligned}$$
If $\phi(U) \cap \partial\mathbb{H}^n_j \neq \varnothing$, then
$$\begin{aligned}(U, \phi) &\in X^{n,j}_{\partial}(M) \\ &\subset X^{n,j}_{\text{Int}}(M) \cup X^{n,j}_{\partial}(M)\end{aligned}$$
Since $(U, \phi) \in X^n(M, \mathcal{T})$ is arbitrary, $X^n(M, \mathcal{T}) \subset X^n_{\text{Int}}(M) \cup X^n_{\partial}(M)$. Therefore $X^n(M) = X^n_{\text{Int}}(M) \cup X^n_{\partial}(M)$.
2. For the sake of contradiction, suppose that $X^n_{\text{Int}}(M) \cap X^n_{\partial}(M) \neq \varnothing$. Then there exists $(U, \phi) \in X^n(M, \mathcal{T})$ such that $(U, \phi) \in X^n_{\text{Int}}(M, \mathcal{T})$ and $(U, \phi) \in X^n_{\partial}(M, \mathcal{T})$. Therefore
   - there exists $j \in [n]$ such that $(U, \phi) \in X^{n,j}(M, \mathcal{T})$ and $\phi(U) \cap \partial\mathbb{H}^n_j = \varnothing$,
   - there exists $k \in [n]$ such that $(U, \phi) \in X^{n,k}(M, \mathcal{T})$ $\phi(U) \cap \partial\mathbb{H}^n_k \neq \varnothing$.

   Since $(U, \phi) \in X^{n,j}(M, \mathcal{T})$, we have that $\phi(U) \in \mathcal{T}_{\mathbb{H}^n_j}$ and ϕ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n_j} \cap \phi(U))$-homeomorphism. Similarly, since $(U, \phi) \in X^{n,k}(M, \mathcal{T})$, we have that $\phi(U) \in \mathcal{T}_{\mathbb{H}^n_k}$ and ϕ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n_k} \cap \phi(U))$-homeomorphism. Therefore $\text{id}_{\phi(U)} = \phi \circ \phi^{-1}$ is a $(\mathcal{T}_{\mathbb{H}^n_j} \cap \phi(U), \mathcal{T}_{\mathbb{H}^n_k} \cap \phi(U))$-homeomorphism.

   Since $\phi(U) \cap \partial\mathbb{H}^n_k \neq \varnothing$, there exists $p \in U$ such that $\phi(p) \in \partial\mathbb{H}^n_k$. Exercise 3.1.0.21 implies that
$$\begin{aligned}\phi(p) &= \text{id}_{\phi(U)}(\phi(p)) \\ &= \phi \circ \phi^{-1}(\phi(p)) \\ &\in \partial\mathbb{H}^n_j\end{aligned}$$
   This is a contradiction since $\phi(U) \cap \partial\mathbb{H}^n_j = \varnothing$. Hence $X^n_{\text{Int}}(M, \mathcal{T}) \cap X^n_{\partial}(M, \mathcal{T}) = \varnothing$.

□

Definition 3.1.0.25. Let M be an n-dimensional topological manifold. We define the

- **interior** of M, denoted $\text{Int}\, M$, by
$$\text{Int}\, M = \{p \in M : \text{there exists } (U, \phi) \in X_{\text{Int}}(M) \text{ such that } p \in U\}$$
- **boundary** of M, denoted ∂M, by
$$\partial M = \{p \in M : \text{there exists } (V, \psi) \in X_{\partial}(M) \text{ such that } p \in V \text{ and } \psi(p) \in \partial\mathbb{H}^n_j\}$$

  FINISH!!!

Exercise 3.1.0.26. Let M be an n-dimensional topological manifold. Let $(U, \phi) \in X_{\text{Int}}(M)$. Then $U \subset \operatorname{Int} M$.

Proof. Let $p \in U$. Since $(U, \phi) \in X_{\text{Int}}(M)$ and $p \in U$, by definition, $p \in \operatorname{Int} M$. Since $p \in U$ is arbitrary, $U \subset \operatorname{Int} M$. $\square$

Exercise 3.1.0.27. Let M be an n-dimensional topological manifold and $(U, \phi) \in X(M)$. Then $(U, \phi) \in X_{\text{Int}}(M)$ iff $\phi(U)$ is open in $\mathbb{R}^n$.

Proof. Suppose that $(U, \phi) \in X_{\text{Int}}(M)$. Then there exists $j \in [n]$ such that $(U, \phi) \in X^{n,j}(M)$ and $\phi(U) \cap \partial \mathbb{H}^n_j = \varnothing$. Since $\phi(U) \in \mathcal{T}_{\mathbb{H}^n_j}$, Exercise 3.1.0.6 implies that $\phi(U) \in \mathcal{T}_{\mathbb{R}^n}$.
Conversely, suppose that $\phi(U) \in \mathcal{T}_{\mathbb{R}^n}$. Since $(U, \phi) \in X^n(M)$, there exists $j \in [n]$ such that $(U, \phi) \in X^{n,j}(M)$. Therefore $\phi(U) \in \mathcal{T}_{\mathbb{H}^n_j}$. Since $\phi(U) \in \mathcal{T}_{\mathbb{R}^n}$, Exercise 3.1.0.6 implies that $\phi(U) \cap \partial \mathbb{H}^n_j = \varnothing$. Thus $(U, \phi) \in X_{\text{Int}}(M)$. $\square$

Exercise 3.1.0.28. Let M be an n-dimensional topological manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. If $\phi(p) \notin \partial \mathbb{H}^n_j$, then $p \in \operatorname{Int} M$.

Proof. Suppose that $\phi(p) \notin \partial \mathbb{H}^n_j$. Then $\phi(p) \in \operatorname{Int} \mathbb{H}^n_j$. Hence there exists $B' \subset \phi(U)$ such that B' is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then U' is open in M and $\phi' : U' \to B'$ is a homeomorphism. Hence $(U', \phi') \in X_{\text{Int}}(M)$. Since $\phi(p) \in B'$, we have that $p \in U'$. By definition, $p \in \operatorname{Int} M$. $\square$

Exercise 3.1.0.29. Let M be an n-dimensional topological manifold. Then

1. $M = \operatorname{Int} M \cup \partial M$
2. $\operatorname{Int} M \cap \partial M = \varnothing$
   Hint: simply connected

Proof.

1. By definition, $\operatorname{Int} M \cup \partial M \subset M$. Let $p \in M$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. A previous exercise implies that $(U, \phi) \in X_{\text{Int}}(M) \cup X_\partial(M)$. If $(U, \phi) \in X_{\text{Int}}(M)$, then by definition,
$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$
   Suppose that $(U, \phi) \in X_\partial(M)$. If $\phi(p) \in \partial \mathbb{H}^n_j$, then by definition,
$$\begin{aligned} p &\in \partial M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$
   Suppose that $\phi(p) \notin \partial \mathbb{H}^n_j$. The previous exercise implies that $p \in \operatorname{Int} M$. Therefore,
$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$
   Since $p \in M$ is arbitrary, $M \subset \operatorname{Int} M \cup \partial M$. Therefore $M = \operatorname{Int} M \cup \partial M$.
2. For the sake of contradiction, suppose that $\operatorname{Int} M \cap \partial M \neq \varnothing$. Then there exists $p \in M$ such that $p \in \operatorname{Int} M \cap \partial M$. By definition, there exists $(U, \phi) \in X_{\text{Int}}(M)$, $(V, \psi) \in X_\partial(M)$ such that $p \in U \cap V$ and $\psi(p) \in \partial \mathbb{H}^n_j$. Note that $\psi(U \cap V)$ is open in $\mathbb{H}^n_j$, $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.
   Since $\psi(U \cap V)$ is open in $\mathbb{H}^n_j$, there exists an $B_\psi \subset \psi(U \cap V)$ such that B_ψ is open in $\mathbb{H}^n_j$, B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$. Since $\phi(U \cap V)$ is open in $\mathbb{R}^n$, B_ϕ is open in $\mathbb{R}^n$. Since B_ψ is simply connected and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism,

B_ϕ is simply connected.
Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Then $\phi \circ \psi^{-1} : B'_\psi \to B'_\phi$ is a homeomorphism. Since $\psi(p) \in \partial\mathbb{H}^n_j$, B'_ψ is simply connected. Since B_ϕ is open in $\mathbb{R}^n$, B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ. So $\partial M \cap \operatorname{Int} M = \varnothing$.

□

Exercise 3.1.0.30. Let M be an n-dimensional topological manifold. Then

1. $\operatorname{Int} M$ is open
2. ∂M is closed

Proof.

1. Let $p \in \operatorname{Int} M$. Then there exists $(U, \phi) \in X_{\text{Int}}(M)$ such that $p \in U$. By definition, U is open and a previous exercise implies that $U \subset \operatorname{Int} M$. Since $p \in \operatorname{Int} M$ is arbitrary, we have that for each $p \in \operatorname{Int} M$, there exists $U \subset \operatorname{Int} M$ such that U is open. Hence $\operatorname{Int} M$ is open.
2. Since $\partial M = (\operatorname{Int} M)^c$, and $\operatorname{Int} M$ is open, we have that ∂M is closed.

□

Exercise 3.1.0.31. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $p \in U$. If $p \in \partial M$, then $(U, \phi) \in X_\partial(M)$.
Hint: simply connected

Proof. Suppose that $p \in \partial M$. Then there exists a $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial\mathbb{H}^n_j$. Note that $\psi(U \cap V)$ is open in $\mathbb{H}^n_j$, $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi|_{U\cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.
Since $\psi(U \cap V)$ is open in $\mathbb{H}^n_j$, there exists $B_\psi \subset \psi(U \cap V)$ such B_ψ is open in $\mathbb{H}^n_j$, B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$.
For the sake of contradiction, suppose that $(U, \phi) \in X_{\text{Int}}(M)$. Then $\phi(U)$ is open in $\mathbb{R}^n$. Hence $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and B_ϕ is open in $\mathbb{R}^n$. Since $\phi|_{U\cap V} \circ (\psi|_{U\cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism, B_ϕ is simply connected. Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Since $\psi(p) \in \partial\mathbb{H}^n_j$, B'_ψ is simply connected. Since B_ϕ is open in $\mathbb{R}^n$, B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ. So $(U, \phi) \notin X_{\text{Int}}(M)$. Since $(X_{\text{Int}}(M))^c = X_\partial(M)$, we have that $(U, \phi) \in X_\partial(M)$.

□

Exercise 3.1.0.32. Let M be an n-dimensional topological manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. Then

1. $p \in \partial M$ iff $\phi(p) \in \partial\mathbb{H}^n_j$ for some j.
2. $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n_j$

Proof.

1. Suppose that $p \in \partial M$. For the sake of contradiction, suppose that $\phi(p) \notin \partial\mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then $p \in U'$ and $(U', \phi') \in X_{\text{Int}}(M)$. Since $p \in U'$, the previous exercise implies that $(U', \phi') \in X_\partial(M)$. This is a contradiction since $X_{\text{Int}}(M) \cap X_\partial(M) = \varnothing$. So $\phi(p) \in \partial\mathbb{H}^n$.
   Conversely, suppose that $\phi(p) \in \partial\mathbb{H}^n$. By definition, $p \in \partial M$.
2. A previous exercise implies that $\operatorname{Int} M = (\partial M)^c$. Part (1) implies that
$$\begin{aligned} p &\in (\partial M)^c \\ &= \operatorname{Int} M \end{aligned}$$
   if and only if
$$\begin{aligned} \phi(p) &\in (\partial\mathbb{H}^n)^c \\ &= \operatorname{Int} \mathbb{H}^n \end{aligned}$$

□

Exercise 3.1.0.33. Let M be an n-dimensional topological manifold and $p \in M$. Then $p \in \partial M$ iff for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial\mathbb{H}^n$.

Proof. Suppose that $p \in \partial M$. Let $(U, \phi) \in X(M)$. Suppose that $p \in U$. The previous two exercises imply that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial\mathbb{H}^n$.
Conversely, suppose that for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial\mathbb{H}^n$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. By assumption, $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial\mathbb{H}^n$. By definition, $p \in \partial M$. □

Exercise 3.1.0.34. Let M be an n-dimensional topological manifold. Let $(U, \phi) \in X_\partial(M)$. Then

1. $\phi(U \cap \partial M) = \phi(U) \cap \partial\mathbb{H}^n$
2. $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int}\mathbb{H}^n$

Proof.

1. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \partial M$ iff $\phi(p) \in \partial\mathbb{H}^n$. Let $q \in \phi(U \cap \partial M)$. Then there exists $p \in U \cap \partial M$ such that $\phi(p) = q$. Since $p \in \partial M$, $\phi(p) \in \partial\mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \partial\mathbb{H}^n \end{aligned}$$

   Since $q \in \phi(U \cap \partial M)$ is arbitrary, $\phi(U \cap \partial M) \subset \phi(U) \cap \partial\mathbb{H}^n$.
   Let $q \in \phi(U) \cap \partial\mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \partial\mathbb{H}^n$, we have that $p \in \partial M$. Hence $p \in U \cap \partial M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

   Since $q \in \phi(U) \cap \partial\mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial\mathbb{H}^n_j \subset \phi(U \cap \partial M)$. Thus $\phi(U \cap \partial M) = \phi(U) \cap \partial\mathbb{H}^n$.

2. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int}\mathbb{H}^n$. Let $q \in \phi(U \cap \operatorname{Int} M)$. Then there exists $p \in U \cap \operatorname{Int} M$ such that $\phi(p) = q$. Since $p \in \operatorname{Int} M$, $\phi(p) \in \operatorname{Int}\mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \operatorname{Int}\mathbb{H}^n \end{aligned}$$

   Since $q \in \phi(U \cap \operatorname{Int} M)$ is arbitrary, $\phi(U \cap \operatorname{Int} M) \subset \phi(U) \cap \operatorname{Int}\mathbb{H}^n$.
   Let $q \in \phi(U) \cap \operatorname{Int}\mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \operatorname{Int}\mathbb{H}^n$, we have that $p \in \operatorname{Int} M$. Hence $p \in U \cap \operatorname{Int} M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

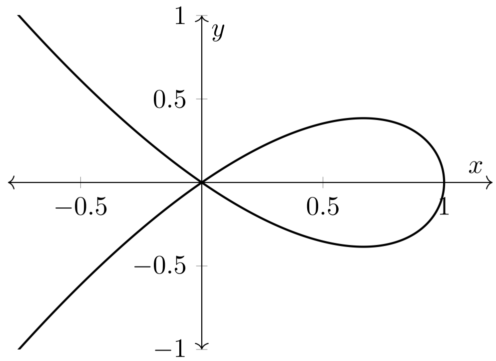
   Since $q \in \phi(U) \cap \partial\mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial\mathbb{H}^n_j \subset \phi(U \cap \operatorname{Int} M)$. Thus $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int}\mathbb{H}^n$.

□

Exercise 3.1.0.35. Graph of Continuous Function:
Let $f \in C(\mathbb{R})$. Set $M = \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$ (i.e. the graph of f). Then M is a 1-dimensional manifold.

Proof. Set $U = \mathbb{R}$ and define $\phi : U \to M$ by $\phi(x) = (x, f(x))$. Then $\phi^{-1} = \pi_1$. Since f is continuous, ϕ is continous. Since π_1 is continiuous, ϕ is a homeomorphism. □

Exercise 3.1.0.36. Nodal Cubic:
Let $M = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2 - x^3\}$. We equip M with the subspace topology.



Then M is not a 1-dimensional topological manifold.
Hint: connected components

Proof. Suppose that M is a 1-dimensional manifold. Set $p = (0, 0)$. Then there exists $(U, \phi) \in X(M)$ such that $p \in U$. Since $\phi(U)$ is open (in $\mathbb{R}$ or $\mathbb{H}$), there exists a $B \subset \phi(U)$ such that B is open (in $\mathbb{R}$ or $\mathbb{H}$), B is connected and $\phi(p) \in B$. Set $V = \phi^{-1}(B)$, $V' = V \setminus \{p\}$ and $B' = B \setminus \{\phi(p)\}$. Then $\phi : V \to B$ and $\phi' : V' \to B'$ are homeomorphisms. Since B is open (in $\mathbb{R}$ or $\mathbb{H}$) and connected, B' has at most two connected components. Then V' This is a contradiction since V' has four connected components and B' and V' are homeomorphic. □

Exercise 3.1.0.37. Topological Manifold Chart Lemma:
Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

- for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection
- for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous
- there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup\limits_{\alpha \in \Gamma'} U_\alpha$
- for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : V \in \mathcal{T}_{\mathbb{H}^n} \text{ and } \alpha \in \Gamma\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$

Then

1. $\mathcal{B}$ is a basis for $\mathcal{T}_M$
   Hint: For $B_1, B_2 \subset \mathbb{H}^n$, $\phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) = \phi_{\alpha_1}^{-1}(B_1 \cap [\phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}} \circ (\phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}})^{-1}(B_2)])$
2. $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold
3. $\mathcal{T}_M$ is the unique topology $\mathcal{T}$ on M such that $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T})$

Proof.

1. - By assumption, $M \subset \bigcup\limits_{\alpha \in \Gamma} U_\alpha$

- Let $A_1, A_2 \in \mathcal{B}$ and $p \in A_1 \cap A_2$. By definition, there exist $\alpha_1, \alpha_2 \in \Gamma$ and $B_1, B_2 \subset \mathbb{H}^n$ such that B_1, B_2 are open in $\mathbb{H}^n$ and

$$\begin{aligned} A_1 &= \phi_{\alpha_1}^{-1}(B_1) \\ &\subset U_{\alpha_1} \end{aligned} \qquad\qquad \begin{aligned} A_2 &= \phi_{\alpha_2}^{-1}(B_2) \\ &\subset U_{\alpha_2} \end{aligned}$$

Set $\psi_1 = \phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}}$ and $\psi_2 = \phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}}$. We note that

$$\begin{aligned} \psi_1^{-1}(B_1) &= U_{\alpha_2} \cap \phi_{\alpha_1}^{-1}(B_1) \\ &= U_{\alpha_2} \cap A_1 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned} \qquad\qquad \begin{aligned} \psi_2^{-1}(B_2) &= U_{\alpha_1} \cap \phi_{\alpha_2}^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned}$$

Let $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Then $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. Hence $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_1}(q) \in \psi_1 \circ \psi_2^{-1}(B_2)$. This implies that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(B_1) \\ &= A_1 \end{aligned}$$

and since $\psi_2^{-1}(B_2) \subset U_{\alpha_1} \cap U_{\alpha_2}$ and $\phi_{\alpha_1} : U_{\alpha_1} \to \phi_{\alpha_1}(U_{\alpha_1})$ is a bijection, we have that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(\psi_1 \circ \psi_2^{-1}(B_2)) \\ &= \psi_2^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \end{aligned}$$

Thus

$$\begin{aligned} q &\in A_1 \cap (U_{\alpha_1} \cap A_2) \\ &= A_1 \cap A_2 \end{aligned}$$

Since $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$ is arbitrary, we have that $\phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \subset A_1 \cap A_2$. Conversely, let

$$\begin{aligned} q &\in A_1 \cap A_2 \\ &= \phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) \end{aligned}$$

Then $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_2}(q) \in B_2$. Since $A_1 \cap A_2 \subset U_{\alpha_1} \cap U_{\alpha_2}$, we have that

$$\begin{aligned} \psi_2(q) &= \phi_{\alpha_2}(q) \\ &\in B_2 \end{aligned}$$

which implies that $q \in \psi_2^{-1}(B_2)$. Therefore

$$\begin{aligned} \phi_{\alpha_1}(q) &= \psi_1(q) \\ &\in \psi_1(\psi_2^{-1}(B_2)) \\ &= \psi_1 \circ \psi_2^{-1}(B_2) \end{aligned}$$

Hence $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. This implies that $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Since $q \in A_1 \cap A_2$ is arbitrary, we have that $A_1 \cap A_2 \subset \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Thus

$$\begin{aligned} A_1 \cap A_2 &= \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \\ &\in \mathcal{B} \end{aligned}$$

Thus $\mathcal{B}$ is a basis for $\mathcal{T}_M$.

2. (a) **(locally Euclidean of dimension n):**
Let $\alpha \in \Gamma$. By definition, for each $B \subset \mathcal{T}_{\mathbb{H}^n}$,

$$\begin{aligned} \phi_\alpha^{-1}(B) &\in \mathcal{B} \\ &\subset \mathcal{T}_M \end{aligned}$$

Hence ϕ_α is continuous.
Let $A \in \mathcal{T}_{U_\alpha}$. Then there exists $U \subset \mathcal{T}_M$ such that $A = U \cap U_\alpha$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_M$, there exists $\Gamma' \subset \Gamma$, $(V_\beta)_{\beta \in \Gamma'} \subset \mathcal{T}_{\mathbb{H}^n}$ such that $U = \bigcup_{\beta \in \Gamma'} \phi_\beta^{-1}(V_\beta)$. Thus

$$\begin{aligned} A &= U \cap U_\alpha \\ &= \left[\bigcup_{\beta \in \Gamma'} \phi_\beta^{-1}(V_\beta) \right] \cap U_\alpha \\ &= \bigcup_{\beta \in \Gamma'} [\phi_\beta^{-1}(V_\beta) \cap U_\alpha] \end{aligned}$$

Let $\beta \in \Gamma'$. Since $\phi_\alpha(U_\alpha \cap U_\beta) \subset \phi_\alpha(U_\alpha)$ and $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$, we have that

$$\begin{aligned} \phi_\alpha(U_\alpha \cap U_\beta) &= \phi_\alpha(U_\alpha) \cap \phi_\alpha(U_\alpha \cap U_\beta) \\ &\in \mathcal{T}_{\phi_\alpha(U_\alpha)} \end{aligned}$$

Therefore $\mathcal{T}_{\phi_\alpha(U_\alpha \cap U_\beta)} \subset \mathcal{T}_{\phi_\alpha(U_\alpha)}$. Since $(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous, we have that $(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{H}^n$ is continuous and therefore

$$\begin{aligned} [(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1}]^{-1}(V_\beta) &\in \mathcal{T}_{\phi_\alpha(U_\alpha \cap U_\beta)} \\ &\subset \mathcal{T}_{\phi_\alpha(U_\alpha)} \end{aligned}$$

Since $\beta \in \Gamma'$ is arbitrary, we have that

$$\begin{aligned} \phi_\alpha(A) &= \phi_\alpha\left(\bigcup_{\beta \in \Gamma'} [\phi_\beta^{-1}(V_\beta) \cap U_\alpha] \right) \\ &= \bigcup_{\beta \in \Gamma'} \phi_\alpha(\phi_\beta^{-1}(V_\beta) \cap U_\alpha) \\ &= \bigcup_{\beta \in \Gamma'} (\phi_\alpha|_{U_\alpha \cap U_\beta}) \circ (\phi_\beta|_{U_\alpha \cap U_\beta})^{-1}(V_\beta) \\ &= \bigcup_{\beta \in \Gamma'} [(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1}]^{-1}(V_\beta) \\ &\in \mathcal{T}_{\phi_\alpha(U_\alpha)} \end{aligned}$$

Since $A \in \mathcal{T}_{U_\alpha}$ is arbitrary, $\phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to U_\alpha$ is continuous. Hence $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism and $(U_\alpha, \phi_\alpha) \in X^n(M)$. Since $M = \bigcup\limits_{\alpha \in \Gamma} U_\alpha$, we have that M is locally Euclidean of dimension n.

(b) **(Hausdorff):**
Let $p, q \in M$. Suppose that $p \neq q$. Then there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$.

- Suppose that there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$. Since $p \neq q$, $\phi_\alpha(p) \neq \phi_\alpha(q)$. Since $\mathbb{H}^n$ is Hausdorff, there exist $V_p, V_q \subset \phi(U_\alpha)$ such that V_p and V_q are open in $\mathbb{H}^n$, $p \in V_p$, $q \in V_q$ and $V_p \cap V_q = \varnothing$. Set $U_p = \phi_\alpha^{-1}(V_p)$ and $U_q = \phi_\alpha^{-1} V_q$. Then U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$.

- Suppose that there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$. Set $U_p = U_\alpha$ and $U_q = U_\beta$. Then U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$.

Thus for each $p, q \in M$ there exist $U_p, U_q \subset M$ such that U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$. Hence

(c) **(second-countable):**
By assumption, there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$. Let $\alpha \in \Gamma'$. Since $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$ and $\mathbb{H}^n$ is second-countable, we have that $\phi_\alpha(U_\alpha)$ is second-countable. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism, we have that U_α is second-countable. Since $M = \bigcup_{\alpha \in \Gamma'} U_\alpha$, an exercise in topology cite implies that M is second-countable.

3. Let $\mathcal{T}$ be a topology on M. Suppose that $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T})$. Then for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}$ and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism.
Let $U \in \mathcal{B}$. By definition, there exists $\alpha \in \Gamma$ and $V \in \mathcal{T}_{\mathbb{H}^n}$ such that $U = \phi_\alpha^{-1}(V)$. Since $U_\alpha \in \mathcal{T}$, we have that $\mathcal{T} \cap U_\alpha \subset \mathcal{T}$. Since $V \cap \phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$, and ϕ_α is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that

$$\begin{aligned} U &= \phi_\alpha^{-1}(V) \\ &= \phi_\alpha^{-1}(V \cap \phi_\alpha(U_\alpha)) \\ &\in \mathcal{T} \cap U_\alpha \\ &\subset \mathcal{T} \end{aligned}$$

Since $U \in \mathcal{B}$ is arbitrary, $\mathcal{B} \subset \mathcal{T}$. Therefore

$$\begin{aligned} \mathcal{T}_M &= \tau(\mathcal{B}) \\ &\subset \tau(\mathcal{T}) \\ &= \mathcal{T} \end{aligned}$$

Conversely, Let $U \in \mathcal{T}$ and $\alpha \in \Gamma$. Then $U \cap U_\alpha \in \mathcal{T} \cap U_\alpha$. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that $\phi_\alpha(U \cap U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$. Since $U_\alpha \in \mathcal{T}_M$, $\mathcal{T}_M \cap U_\alpha \subset \mathcal{T}_M$. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T}_M \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that

$$\begin{aligned} U \cap U_\alpha &= \phi_\alpha^{-1}(\phi_\alpha(U \cap U_\alpha)) \\ &\in \mathcal{T}_M \cap U_\alpha \\ &\subset \mathcal{T}_M \end{aligned}$$

Then

$$\begin{aligned} U &= U \cap M \\ &= U \cap \left(\bigcup_{\alpha \in \Gamma} U_\alpha \right) \\ &= \bigcup_{\alpha \in \Gamma} (U \cap U_\alpha) \\ &\in \mathcal{T}_M \end{aligned}$$

Since $U \in \mathcal{T}$ is arbitrary, $\mathcal{T} \subset \mathcal{T}_M$. Thus $\mathcal{T} = \mathcal{T}_M$.

□

Exercise 3.1.0.38. Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

- for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$

- for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection
- for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous
- there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$
- for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Then there exists a unique topology $\mathcal{T}_M$ on M such that $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T}_M)$.

Proof. Immediate by previous exercise. $\square$

3.2 Submanifolds

3.2.1 Open Submanifolds

Note 3.2.1.1. Let $(M, \mathcal{T})$ be an n-dimensional topological manifold and $U \subset M$. Suppose that U is open in M. Unless otherwise specified, we equip U with $\mathcal{T} \cap U$.

Exercise 3.2.1.2. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M, then $(U', \phi|_{U'}) \in X^n(M)$.

Proof. Suppose that U' is open in M. Set $\phi' = \phi|_{U'}$.

- By assumption U' is open in M.
- Since U' is open in M, we have that $U' = U' \cap U$ is open in U. Since ϕ is a homeomorphism and U' is open in U, we have that $\phi(U')$ is open in $\phi(U)$. By assumption $\phi(U)$ is open in $\mathbb{R}^n$ or $\phi(U)$ is open in $\mathbb{H}^n$. Therefore $\phi'(U')$ is open in $\mathbb{R}^n$ or $\phi'(U')$ is open in $\mathbb{H}^n$.
- Since $\phi : U \to V$ is a homeomorphism, $\phi' : U' \to \phi'(U')$ is a homeomorphism.

So $(U', \phi') \in X^n(M)$. □

Note 3.2.1.3. Since U is open in M, U' being open in U is equivalent to U' being open in M, so we could have also assumed that U' is open in U.

Exercise 3.2.1.4. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then

$$X^n(U) = \{(V, \psi) \in X^n(M) : V \subset U\}$$

Proof. Suppose that U is open and set $A = \{(V, \psi) \in X^n(M) : V \subset U\}$. Let $(V, \psi) \in X^n(U)$. By definition of $X^n(U)$, V is open in U. Thus, there exists $W \subset M$ such that W is open in M and $V = U \cap W$. Since U is open in M, we have that $V = U \cap W$ is open in M. Hence $(V, \psi) \in X^n(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X^n(U)$ is arbitary, $X^n(U) \subset A$.
Conversely, suppose that $(V, \psi) \in A$. Then $(V, \psi) \in X^n(M)$ and $V \subset U$. By definition of $X^n(M)$, V is open in M. Since $V \subset U$, we have that $V = V \cap U$ is open in U. Hence $(V, \psi) \in X^n(U)$. Since $(V, \psi) \in X^n(U)$ is arbitary, $A \subset X^n(U)$. Hence $X^n(A) = A$. □

Exercise 3.2.1.5. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M, then $(U', \phi|_{U'}) \in X^n(U)$.

Proof. Suppose that U' is open in M. A previous exercise implies that $(U', \phi') \in X^n(M)$. The previous exercise implies that $(U', \phi') \in X^n(U)$. □

Exercise 3.2.1.6. Topological Open Submanifolds:
Let M be an n-dimensional topological manifold and $U \subset M$ open. Then U is an n-dimensional topological manifold.

Proof.

1. Since M is Hausdorff, U is Hausdorff.
2. Since M is second-countable, U is second countable.
3. Let $p \in U$. Since then there exists $(V, \psi) \in X^n(M)$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{U \cap V}$. The previous exercise implies that $(V', \psi') \in X^n(U)$. Therefore U is locally Euclidean of dimension n.

Hence U is an n-dimensional topological manifold. □

Exercise 3.2.1.7. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then

1. $X_{\text{Int}}(U) = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$

2. $X_\partial(U) = \{(V, \psi) \in X_\partial(M) : V \subset U\}$

Proof. Suppose that U is open in M.

1. Set $A = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\text{Int}}(U)$. By definition of $X_{\text{Int}}(U)$, V is open in U and $\phi(V)$ is open in $\mathbb{R}^n$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_{\text{Int}}(M)$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_{\text{Int}}(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X_{\text{Int}}(U)$ is arbitrary, $X_{\text{Int}}(U) \subset A$.
   Conversely, let $(V, \psi) \in A$. Then $(V, \psi) \in X_{\text{Int}}(M)$ and $V \subset U$. By definition of $X_{\text{Int}}(M)$, V is open in M and $\phi(V)$ is open in $\mathbb{R}^n$. Thus $V = V \cap U$ is open in U. So $(V, \psi) \in X_{\text{Int}}(U)$. Since $(V, \psi) \in A$ is arbitrary, $A \subset X_{\text{Int}}(U)$. Thus $X_{\text{Int}}(U) = A$.

2. Set $B = \{(V, \psi) \in X_\partial(M) : V \subset U\}$. Let $(V, \psi) \in X_\partial(U)$. By definition of $X_\partial(U)$, V is open in U, $\phi(V)$ is open in $\mathbb{H}^n$ and $\partial\mathbb{H}^n_j \cap \phi(V) \neq \varnothing$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_\partial(M)$, which implies that $(V, \psi) \in B$. Since $(V, \psi) \in X_\partial(U)$ is arbitrary, $X_\partial(U) \subset B$.
   Conversely, let $(V, \psi) \in B$. Then $(V, \psi) \in X_\partial(M)$ and $V \subset U$. By definition of $X_\partial(M)$, V is open in M, $\phi(V)$ is open in $\mathbb{H}^n$ and $\partial\mathbb{H}^n_j \cap \phi(V) \neq \varnothing$. Thus $V = V \cap U$ is open in U. So $(V, \psi) \in X_\partial(U)$. Since $(V, \psi) \in B$ is arbitrary, $B \subset X_\partial(U)$. Thus $X_\partial(U) = B$.

□

Exercise 3.2.1.8. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then $\partial U = \partial M \cap U$.

Proof. Suppose that U is open. Let $p \in \partial U$. Then there exists $(V, \psi) \in X_\partial(U)$ such that $p \in V$ and $\psi(p) \in \partial\mathbb{H}^n$. Since U is open, the previous exercise implies that $(V, \psi) \in X_\partial(M)$. Thus $p \in \partial M$. Since $p \in \partial U$ is arbitrary, $\partial U \subset \partial M$. Since $\partial U \subset U$, we have that $\partial U \subset \partial M \cap U$.
Conversely, let $p \in \partial M \cap U$. Since $p \in \partial M$, there exists $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial\mathbb{H}^n$. Set $V' = V \cap U$ and $\psi' = \psi|_{V'}$. Then $p \in V'$ since V and U are open in M, V' is open in M. A previous exercise implies that $(V', \psi') \in X(M)$. Since $p \in \partial M$, a previous exercise implies that $(V', \psi') \in X_\partial(M)$. The previous exercise implies that $(V', \psi') \in X_\partial(U)$. Since $\psi'(p) \in \partial\mathbb{H}^n$, $p \in \partial U$. Since $p \in \partial M \cap U$ is arbitrary, $\partial M \cap U \subset \partial U$. Hence $\partial U = \partial M \cap U$.
label exercises and reference them!!! □

3.2.2 Boundary Submanifolds

Note 3.2.2.1. Let $(M, \mathcal{T})$ be an n-dimensional topological manifold. Unless otherwise specified, we equip ∂M with $\mathcal{T} \cap \partial M$.

Definition 3.2.2.2. Let M be an n-dimensional topological manifold and $\pi : \partial\mathbb{H}^n_j \to \mathbb{R}^{n-1}$ the projection map. For $(U, \phi) \in X_\partial(M)$, we define $\bar{U} \subset \partial M$ and $\bar{\phi} : \bar{U} \to \pi(\phi(\bar{U}))$ by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi \circ \phi|_{\bar{U}}$ respectively.

Exercise 3.2.2.3. Let M be an n-dimensional topological manifold, and $\lambda : \partial\mathbb{H}^n_j \to \mathbb{R}^{n-1}$ a homeomorphism. Then $\{(\bar{U}, \bar{\phi}) : (U, \phi) \in X_\partial(M)\} \subset X^{n-1}_{\text{Int}}(\partial M)$.

Proof. Let $(U, \phi) \in X_\partial(M)$.

1. Since U is open in M, $\bar{U} = U \cap \partial M$ is open in ∂M.

2. Since $(U, \phi) \in X_\partial(M)$, $\phi(U)$ is open in $\mathbb{H}^n$. A previous exercise implies that $\phi(\bar{U}) = \phi(U) \cap \partial\mathbb{H}^n$ which is open in $\partial\mathbb{H}^n$. Since $\pi : \partial\mathbb{H}^n_j \to \mathbb{R}^{n-1}$ is a homeomorphism, we have that $\pi(\phi(\bar{U}))$ is open in $\mathbb{R}^{n-1}$.

3. Since $\phi|_{\bar{U}} : \bar{U} \to \phi(U) \cap \partial\mathbb{H}^n$ and $\pi|_{\phi(\bar{U})} : \phi(\bar{U}) \to \lambda(\phi(\bar{U}))$ are homeomorphisms, we have that $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$ is a homeomorphism.

Hence $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$. □

Exercise 3.2.2.4. Topological Boundary Submanifold:
Let M be an n-dimensional topological manifold. Then

1. ∂M is an $(n-1)$-dimensional topological manifold
2. $\partial(\partial M) = \varnothing$

Proof.

1. (a) Since M is Hausdorff, ∂M is Hausdorff.
   (b) Since M is second-countable, ∂M is second countable.
   (c) Let $p \in \partial M$. Then there exists $(U, \phi) \in X_\partial(M)$ such that $\phi(p) \in \partial \mathbb{H}^n$. Then $p \in \bar{U}$ and the previous exercise implies that $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$. Thus ∂M is locally Euclidean of dimension $n-1$.

   Hence ∂M is an $(n-1)$-dimensional topological manifold.
2. Let $p \in \partial M$. Part (1) implies that there exists $(U, \phi) \in X^{n-1}_{\text{Int}}(\partial M)$ such that $p \in U$. Thus $p \in \operatorname{Int} \partial M$. Since $p \in \partial M$ is arbitrary, $\operatorname{Int} \partial M = \partial M$. Hence

$$\begin{aligned}
\partial(\partial M) &= (\operatorname{Int}(\partial M))^c \\
&= (\partial M)^c \\
&= \varnothing
\end{aligned}$$

□

3.3 Product Manifolds

Note 3.3.0.1. Let $(M, \mathcal{T}_M)$ and $(N, \mathcal{T}_N)$ be m-dimensional and n-dimensional topological manifold respectively. Unless otherwise specified, we equip $M \times N$ with $\mathcal{T}_M \otimes \mathcal{T}_N$.

Definition 3.3.0.2. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Define $\lambda_0 : \mathbb{H}^m_j \times \operatorname{Int} \mathbb{H}^n_j \to \mathbb{H}^{m+n}$ by $\lambda((x^1, \dots, x^{m-1}, x^m), (y^1, \dots, y^n)) := (x^1, \dots, x^{m-1}, y^1, \dots, y^{n-1}, \log y^n, x^m)$.

Exercise 3.3.0.3. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then

1. λ_0 is a $(\mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n}, \mathcal{T}_{\mathbb{H}^{m+n}})$-homeomorphism,
2. $\lambda_0(\partial \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n) = \partial \mathbb{H}^{m+n}$,
3. $(\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n, \lambda_0) \in X^{m+n}(\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n, \mathcal{T}_{\mathbb{H}^n} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n})$.

Proof.

1. Clearly λ_0 is a homeomorphism.
2. Clearly $\lambda_0(\partial \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n) = \partial \mathbb{H}^{m+n}$
3. We note that
   - $\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n \in \mathcal{T}_{\mathbb{H}^n} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n}$,
   - $\mathbb{H}^{m+n} \in \mathcal{T}_{\mathbb{H}^{m+n}}$,
   - part (1) implies that λ_0 is a $(\mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n}, \mathcal{T}_{\mathbb{H}^{m+n}})$-homeomorphism.

   Thus $(\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n, \lambda_0) \in X^{m+n}(\mathbb{H}^m \times \mathbb{H}^n, \mathcal{T}_{\mathbb{H}^n} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n})$.

□

Exercise 3.3.0.4. Let $m, n \in \mathbb{N}_0$. Then $\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n$ is an $m+n$-dimensional topological manifold.

Proof.

1. Clearly $\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n$ is Hausdorff.
2. Clearly $\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n$ is second-countable.
3. Since $\lambda_0 \in X^{m+n}(\mathbb{H}^m \times \mathbb{H}^n, \mathcal{T}_{\mathbb{H}^n} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n})$, we have that for each $p \in \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n$, there exists $(U, \phi) \in X^{m+n}(\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n, \mathcal{T}_{\mathbb{H}^n} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n})$ such that $p \in U$. Thus $(\mathbb{H}^m \times \mathbb{H}^n, \mathcal{T}_{\mathbb{H}^n} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n})$ is locally Euclidean of dimension $m+n$.

Thus $(\mathbb{H}^m \times \mathbb{H}^n, \mathcal{T}_{\mathbb{H}^n} \otimes \mathcal{T}_{\operatorname{Int} \mathbb{H}^n})$ is an $m+n$-dimensional topological manifold. □

Exercise 3.3.0.5. Let $(M, \mathcal{T}_M)$, $(N, \mathcal{T}_N)$ be topological manifolds. Set $m = \dim M$ and $n = \dim N$. Suppose that $\partial N = \varnothing$. Then for each $(U, \phi) \in X^m(M, \mathcal{T}_M)$, $(V, \psi) \in X^n(N, \mathcal{T}_N)$,

$$(U \times V, \lambda_0|_{\phi(U) \times \psi(V)} \circ [\phi \times \psi]) \in X^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$$

Proof. Let $(U, \phi) \in X^m(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$.

- Since $U \in \mathcal{T}_M$ and $V \in \mathcal{T}_N$, $U \times V \in \mathcal{T}_M \otimes \mathcal{T}_N$.
- Since $\phi(U) \in \mathcal{T}_{\mathbb{H}^m}$ and $\psi(V) \in \mathcal{T}_{\mathbb{H}^n}$, $\phi(U) \times \psi(V) \in \mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\mathbb{H}^n}$. Since $\partial N = \varnothing$, $(V, \psi) \in X^n_{\operatorname{Int}}(N, \mathcal{T}_N)$ and therefore $\psi(V) \subset \operatorname{Int} \mathbb{H}^n$. Since $\lambda_0 : \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n \to \mathbb{H}^{m+n}$ is a homeomorphism,

$$\begin{aligned} \lambda_0|_{\phi(U) \times \psi(V)} \circ [\phi \times \psi](U \times V) &= \lambda_0(\phi(U) \times \psi(V)) \\ &\in \mathcal{T}_{\mathbb{H}^{m+n}} \end{aligned}$$

- Since $\phi : U \to \phi(U)$ is a $(\mathcal{T}_M \cap U, \mathcal{T}_{\mathbb{H}^m} \cap \phi(U))$-homeomorphism and $\psi : V \to \psi(V)$ is a $(\mathcal{T}_N \cap V, \mathcal{T}_{\mathbb{H}^n} \cap \psi(V))$-homeomorphism, an exercise in the section on product topologies in the analysis notes implies that $\phi \times \psi : U \times V \to \phi(U) \times \phi(V)$ is a $([\mathcal{T}_M \otimes \mathcal{T}_N] \cap [U \times V], [\mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\mathbb{H}^n}] \cap [\phi(U) \times \psi(V)])$-homeomorphism. Since $\lambda_0|_{\phi(U)\times\psi(V)} : \phi(U) \times \psi(V) \to \lambda_0(\phi(U) \times \psi(V))$ is a $([\mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\operatorname{Int}\mathbb{H}^n}] \cap [\phi(U) \times \psi(V)], \mathcal{T}_{\mathbb{H}^{m+n}} \cap \lambda_0(\phi(U) \times \psi(V)))$-homeomorphism, $\lambda_0|_{\phi(U)\times\psi(V)} \circ (\phi \times \psi)$ is a $([\mathcal{T}_M \otimes \mathcal{T}_N] \cap [U \times V], \mathcal{T}_{\mathbb{H}^{m+n}} \cap \lambda_0(U \times V))$-homeomorphism.

Hence $(U \times V, \lambda_0|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$. Since $(U, \phi) \in X^m(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$ are arbitrary, we have that for each $(U, \phi) \in X^m(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$,

$$(U \times V, \lambda_0|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$$

$\square$

Exercise 3.3.0.6. Let M, N be topological manifolds. Set $m = \dim M$ and $n = \dim N$. Suppose that $\partial N = \varnothing$. Then for each $(U, \phi) \in X^m_\partial(M, \mathcal{T}_M)$, $(V, \psi) \in X^n(N, \mathcal{T}_N)$,

$$(U \times V, \lambda_0|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X^{m+n}_\partial(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$$

Proof. Let $(U, \phi) \in X^m_\partial(M)$ and $(V, \psi) \in X^n(N)$. Define $\eta : U \times V \to \lambda_0(\phi(U) \times \psi(V))$ by

$$\eta := \lambda_0|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]$$

Since $(U, \phi) \in X^m_\partial(M)$, $\phi(U) \cap \partial\mathbb{H}^m \neq \varnothing$. Then there exists $p \in U$ such that $\phi(p) \in \partial\mathbb{H}^m$. So $\eta(p, q) \in \partial\mathbb{H}^{m+n}$. Thus $\eta(U \times V) \cap \partial\mathbb{H}^{m+n} \neq \varnothing$ and $(U \times V, \eta) \in X^{m+n}_\partial(M \times N)$. Since $(U, \phi) \in X^m_\partial(M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$ are arbitrary, we have that for each $(U, \phi) \in X^m_p(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$,

$$(U \times V, \lambda_0|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X^{m+n}_\partial(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$$

$\square$

Note 3.3.0.7. The above is still true if $\partial N \neq \varnothing$

Exercise 3.3.0.8. Let M, N be topological manifolds. Suppose that $\partial N = \varnothing$. Then

1. $M \times N$ is a topological manifold
2. $\partial(M \times N) = \partial M \times N$

Proof. Set $m = \dim M$ and $n = \dim N$.

1. - Since M and N are Hausdorff, $M \times N$ is Hausdorff.
   - Since M and N are second-countable, $M \times N$ is second-countable.
   - Let $a \in M \times N$. Then there exist $p \in M$ and $q \in N$ such that $a = (p, q)$. Since M and and N are locally Euclidean, there exist $(U, \phi) \in X^m(M)$ and $(V, \psi) \in X^n(N)$ such that $p \in U$ and $q \in V$. Then $(p, q) \in U \times V$. Exercise 3.3.0.5 implies that $(U \times V, \lambda_0 \circ [\phi \times \psi]) \in X^{m+n}(M \times N)$. Since $a \in M \times N$ is arbitrary, $M \times N$ is locally Euclidean of dimension $m + n$.

   Thus $M \times N$ is an $(m + n)$-dimensional topological manifold.

2. - Let $a \in \partial(M \times N)$. Then there exists $p \in M$ and $q \in N$ such that $a = (p, q)$. Since $(M, \mathcal{T}_M)$ and and (N) are locally Euclidean, there exist $(U, \phi) \in X^m(M)$ and $(V, \psi) \in X^n(N)$ such that $p \in U$ and $q \in V$. Define $\eta : U \times V \to \lambda_0(\phi(U) \times \psi(V))$ by

     $$\eta := \lambda_0|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]$$

     Exercise 3.3.0.5 implies that $\eta \in X^{m+n}(M \times N)$. Since $(p, q) \in \partial(M \times N)$, Exercise 3.3.0.6 implies that $\eta \in X^{m+n}_\partial(M \times N)$ and $\eta(p, q) \in \partial\mathbb{H}^{m+n}$. Therefore

     $$\begin{aligned}\phi \times \psi(p, q) &= \lambda_0|^{-1}_{\phi(U)\times\psi(V)} \circ \eta \\ &\in \partial\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n\end{aligned}$$

Hence $\phi(p) \in \partial\mathbb{H}^m$ and $\psi(q) \in \operatorname{Int}\mathbb{H}^n$. Thus $(U, \phi) \in X_\partial^m(M)$ and $p \in \partial M$. Therefore

$$\begin{aligned} a &= (p, q) \\ &\in \partial M \times N \end{aligned}$$

Since $a \in \partial(M \times N)$ is arbitrary, we have that $\partial(M \times N) \subset \partial M \times N$.

- Let $a \in \partial M \times N$. Then there exists $p \in \partial M$ and $q \in N$ such that $a = (p, q)$. By definition, there exists $(U, \phi) \in X_\partial^m(M)$ and $(V, \psi) \in X^n(N)$ such that $p \in U$, $q \in V$ and $\phi(p) \in \partial\mathbb{H}^m$. Since $\partial N = \varnothing$, $\psi(q) \in \operatorname{Int}\mathbb{H}^n$. Define $\eta : U \times V \to \lambda_0(\phi(U) \times \psi(V))$ by

$$\eta := \lambda_0|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]$$

Exercise 3.3.0.5 implies that $(U \times V, \eta) \in X^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$. Then

$$\begin{aligned} \eta(a) &= \eta(p, q) \\ &= \lambda_0(\phi(p), \psi(q)) \\ &\in \partial\mathbb{H}^{m+n} \end{aligned}$$

Thus $\eta \in X_\partial^{m+n}(M \times N)$ and $a \in \partial(M \times N)$. Since $a \in \partial M \times N$ is arbitrary, $\partial M \times N \subset \partial(M \times N)$. Thus $\partial(M \times N) = \partial M \times N$.

$\square$

3.4 Submanifolds

Definition 3.4.0.1. topological embedding

Definition 3.4.0.2. Let M, N be topological manifolds of dimensions m, n respectively and $F : N \to N$ a topological embedding. Then $\{(F(V), \psi \circ F^{-1}) : (V, \psi) \in X^n(N)\} \subset X^n(F(N))$.

Proof. Since $\square$

Chapter 4

Smooth Manifolds

use smooth manifold chart lemma to show that $\mathbb{H}^n$, $\operatorname{Int}\mathbb{H}^n$ and $\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n$ are smooth manifolds.

4.1 Introduction

Definition 4.1.0.1. Let M be an n-dimensional topological manifold and $(U,\phi),(V,\psi) \in X(M)$. Then (U,ϕ) and (V,ψ) are said to be **smoothly compatible** if

$$\psi|_{U\cap V} \circ (\phi|_{U\cap V})^{-1} : \phi(U\cap V) \to \psi(U \cap V) \text{ is a diffeomorphism}$$

Definition 4.1.0.2. Let $(M,\mathcal{T})$ be an n-dimensional topological manifold.

- Let $\mathcal{A} \subset X(M,\mathcal{T})$. Then $\mathcal{A}$ is said to be an **atlas on** M if $M \subset \bigcup\limits_{(U,\phi)\in\mathcal{A}} U$.
- Let $\mathcal{A}$ be an atlas on M. Then $\mathcal{A}$ is said to be **smooth** if for each $(U,\phi),(V,\psi) \in \mathcal{A}$, (U,ϕ) and (V,ψ) are smoothly compatible.
- Let $\mathcal{A}$ be a smooth atlas on M. Then $\mathcal{A}$ is said to be **maximal** if for each smooth atlas $\mathcal{B}$ on M, $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on** M.
- Let $\mathcal{A}$ be an atlas on M. Then $(M,\mathcal{T},\mathcal{A})$ is said to be an **n-dimensional smooth manifold** if $\mathcal{A}$ is a smooth structure on M.

Note 4.1.0.3. When the context is clear, we write M or $(M,\mathcal{A})$ in place of $(M,\mathcal{T},\mathcal{A})$.

Definition 4.1.0.4. Let M be a topological manifold and $\mathcal{B}$ a smooth atlas on M. We define the **smooth structure on M generated by $\mathcal{B}$**, denoted $\alpha_M(\mathcal{B})$, by

$$\alpha_M(\mathcal{B}) = \{(U,\phi) \in X(M) : \text{ for each } (V,\psi) \in \mathcal{B},\ (U,\phi) \text{ and } (V,\psi) \text{ are smoothly compatible}\}$$

Note 4.1.0.5. When the context is clear, we write $\alpha(\mathcal{B})$ in place of $\alpha_M(\mathcal{B})$.

Exercise 4.1.0.6. Let M be an n-dimensional topological manifold and $\mathcal{B}$ a smooth atlas on M. Then $\alpha(\mathcal{B})$ is the unique smooth structure $\mathcal{A}$ on M such that $\mathcal{B} \subset \mathcal{A}$.

Proof. Clearly $\mathcal{B} \subset \alpha(\mathcal{B})$. Let (U,ϕ) and $(V,\psi) \in \alpha(\mathcal{B})$. Define $F : \phi(U\cap V) \to \psi(U\cap V)$ by

$$F = \psi|_{U\cap V} \circ (\phi|_{U\cap V})^{-1}$$

Let $q \in \phi(U\cap V)$. Set $p = \phi^{-1}(q)$. Since $\mathcal{B}$ is an atlas and $p \in U\cap V \subset M$, there exists $(W,\chi) \in \mathcal{B}$ such that $p \in W$. By definition of $\alpha(\mathcal{B})$, $\psi|_{W\cap V} \circ (\chi|_{W\cap V})^{-1} : \chi(W\cap V) \to \psi(W\cap V)$ and $\chi|_{U\cap W} \circ (\phi|_{U\cap W})^{-1} : \phi(U\cap W) \to \chi(U\cap W)$ are diffeomorphisms. Set $N = U\cap W\cap V$. Then $q \in \phi(N) \subset \phi(U\cap V)$ and

$$\begin{aligned} F|_{\phi(N)} &= \psi|_N \circ (\phi|_N)^{-1} \\ &= [\psi|_N \circ (\chi|_N)^{-1}] \circ [\chi|_N \circ (\phi|_N)^{-1}] \end{aligned}$$

is a diffeomorphism. Thus, for each $q \in \phi(U \cap V)$, there exists $N' \subset \phi(U \cap V)$ such that $F|_{N'}$ is a diffeomorphism. Hence F is a diffeomorphism and (U, ϕ), (V, ψ) are smoothly compatible. Therefore $\alpha(\mathcal{B})$ is a smooth atlas.
To see that $\alpha(\mathcal{B})$ is maximal, let $\mathcal{B}'$ be a smooth atlas on M. Suppose that $\alpha(\mathcal{B}) \subset \mathcal{B}'$ and let $(U, \phi) \in \mathcal{B}'$. By definition, for each chart $(V, \psi) \in \mathcal{B}'$, (U, ϕ) and (V, ψ) are smoothly compatible. Since $\mathcal{B} \subset \alpha(\mathcal{B}) \subset \mathcal{B}'$, we have that $(U, \phi) \in \alpha(\mathcal{B})$. So $\alpha(\mathcal{B}) = \mathcal{B}'$ and $\alpha(\mathcal{B})$ is a maximal smooth atlas on M. □

Exercise 4.1.0.7. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold. Then for each $\sigma \in S_n$, and $(U, \phi) \in \mathcal{A}$, $(U, \sigma \cdot \phi) \in \mathcal{A}$.

Proof. content... □

Definition 4.1.0.8. Let $n \in \mathbb{N}_0$. We define the **standard smooth structure** on $\mathbb{H}^n$, denoted $\mathcal{A}_{\mathbb{H}^n}$, by $\mathcal{A}_{\mathbb{H}^n} = \alpha_{\mathbb{H}^n}(\mathbb{H}^n, \mathrm{id}_{\mathbb{H}^n})$.

Note 4.1.0.9. Unless otherwise specified we equip $\mathbb{H}^n$ with $\mathcal{A}_{\mathbb{H}^n}$.

Note 4.1.0.10. Let $n \in \mathbb{N}$. We recall the definition of $\eta_0 : \mathbb{R}^n \to \operatorname{Int} \mathbb{H}^n$ in Definition **??** given by $\eta_0(a^1, \dots, a^{n-1}, a^n) := (a^1, \dots, a^{n-1}, e^{a^n})$. We know from Exercise **??** that η_0 is a homeomorphism.

Definition 4.1.0.11. Let $n \in \mathbb{N}_0$. Define _0 :We define the **standard smooth structure** on $\mathbb{R}^n$, denoted $\mathcal{A}_{\mathbb{R}^n}$, by $\mathcal{A}_{\mathbb{R}^n} = \alpha_{\mathbb{R}^n}(\mathbb{R}^n, \mathrm{id}_{\mathbb{H}^n})$. finish

Exercise 4.1.0.12. Define $U \subset \mathbb{R}$ and $\phi : U \to \mathbb{R}$ by $U := \mathbb{R}$ and $\phi(x) := x^3$. Then

1. $(U, \phi) \in X^1(\mathbb{R})$
2. $(U, \phi) \notin \mathcal{A}_{\mathbb{R}}$

Proof.

1.
   - Trivially, U is open in $\mathbb{R}$.
   - Trivially, $\mathbb{R}$ is open in $\mathbb{R}$
   - Clearly ϕ is continuous. Also, ϕ is a bijection. and since for each $x \in \mathbb{R}$, $\phi^{-1}(x) = x^{1/3}$, ϕ^{-1} is continuous. Hence ϕ is a homeomorphism.

   So $(U, \phi) \in X^1(\mathbb{R})$.

2. Define $V \subset M$ and $\psi : V \to \mathbb{R}$ by $V := \mathbb{R}$ and $\psi := \mathrm{id}_{\mathbb{R}}$. By defintion, $(V, \psi) \in \mathcal{A}_{\mathbb{R}}$. Since ϕ^{-1} is not differentiable at $x = 0$ and $\psi \circ \phi^{-1} = \phi^{-1}$, we have that $\psi \circ \phi^{-1}$ is not smooth and therefore $\psi \circ \phi^{-1}$ is not a diffeomorphism. Hence (U, ϕ) and (V, ψ) are not smoothly compatible. Thus $(U, \phi) \notin \mathcal{A}_{\mathbb{R}}$.

□

Exercise 4.1.0.13. Let $(M, \mathcal{A})$ be a smooth manifold and $\mathcal{A}_0 \subset \mathcal{A}$. Suppose that $\mathcal{A}_0$ is an atlas on M. Let $(U, \phi) \in X(M)$. Then $(U, \phi) \in \mathcal{A}$ iff for each $(V, \psi) \in \mathcal{A}_0$, (U, ϕ) and (V, ψ) are smoothly compatible.

Proof. Set $n := \dim M$.

- $(\implies)$:
  Suppose that $(U, \phi) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, for each $(V, \psi) \in \mathcal{A}$, (U, ϕ) and (V, ψ) are smoothly compatible. Since $\mathcal{A}_0 \subset \mathcal{A}$, we have that for each $(V, \psi) \in \mathcal{A}_0$, (U, ϕ) and (V, ψ) are smoothly compatible.

- $(\impliedby)$:
  Suppose that for each $(V, \psi) \in \mathcal{A}_0$, (U, ϕ) and (V, ψ) are smoothly compatible. Let $(V, \psi) \in \mathcal{A}$ and $a \in \phi(U \cap V)$. Set $p := \phi^{-1}(a)$. Since $\mathcal{A}_0$ is an atlas on M, there exists $(W_0, \alpha_0) \in \mathcal{A}_0$ such that $p \in W_0$. Define $f : \phi(U \cap W_0) \to \alpha_0(U \cap W_0)$, $g : \alpha_0(W_0 \cap V) \to \psi(W_0 \cap V)$ and $h : \phi(U \cap V) \to \psi(U \cap V)$ by $f := \alpha_0|_{U \cap W_0} \circ \phi|^{-1}_{U \cap W_0}$, $g := \psi|_{W_0 \cap V} \circ \alpha_0|^{-1}_{W_0 \cap V}$ and $h := \psi|_{U \cap V} \circ \phi|^{-1}_{U \cap V}$. By assumption, (U, ϕ) and (W_0, α_0) are smoothly compatible. Thus f is a diffeomorphism and therefore f is smooth.

Since $(W_0, \alpha_0), (V, \psi) \in \mathcal{A}$, we have that (W_0, α_0) and (V, ψ) are smoothly compatible. Thus g is a diffeomorphism and therefore g is smooth. Define $A \subset M$ and $A' \subset \mathbb{R}^n$ by $A := U \cap V \cap W_0$ and $A' = \phi(A)$. Since $p \in A$, $a \in A'$. Since A is open in $U \cap V$ and ϕ is a homeomorphism, A' is open in $\phi(U \cap V)$. Exercise 1.3.2.3 implies that $f|_{A'}$ is smooth. Since $h|_{A'} = g \circ f|_{A'}$, $h|_{A'}$ is smooth. Since $a \in \phi(U \cap V)$ is arbitrary, we have that for each $a \in \phi(U \cap V)$, there exists $A' \subset \phi(U \cap V)$ such that $a \in A'$, A' is open in $\phi(U \cap V)$ and $h|_{A'}$ is smooth. Exercise 1.3.2.4 implies that h is smooth. Thus (U, ϕ) and (V, ψ) are smoothly compatible. Since $(V, \psi) \in \mathcal{A}$ is arbitrary, we have that $\mathcal{A} \cup \{(U, \phi)\}$ is a smooth atlas on M. Since $\mathcal{A}$ is maximal, $\mathcal{A} \cup \{(U, \phi)\} = \mathcal{A}$. Thus $(U, \phi) \in \mathcal{A}$.

□

Exercise 4.1.0.14. Smooth Manifold Chart Lemma:
Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

(a) for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$

(b) for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$

(c) for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection

(d) for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

(e) there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup\limits_{\alpha \in \Gamma'} U_\alpha$

(f) for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Then there exists a unique topology $\mathcal{T}_M$ on M and smooth structure $\mathcal{A}_M$ on $(M, \mathcal{T}_M)$ such that $(M, \mathcal{T}_M, \mathcal{A}_M)$ is an n-dimensional smooth manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset \mathcal{A}_M$.

Proof. Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : \alpha \in \Gamma \text{ and } V \in \mathcal{T}_{\mathbb{H}^n}\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$
- $\mathcal{A}' = \{(U_\alpha, \phi_\alpha) : \alpha \in \Gamma\}$.

Exercise 3.1.0.37 (the topological manifold chart lemma) implies that $\mathcal{T}_M$ is the unique topology on M such that $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold and $\mathcal{A}' \subset X^n(M, \mathcal{T}_M)$. Since $M = \bigcup_{\alpha \in \Gamma} U_\alpha$, $\mathcal{A}'$ is an atlas on M. Since for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth, we have that $\mathcal{A}'$ is smooth. Set $\mathcal{A}_M = \alpha(\mathcal{A}')$. A previous exercise implies that $\mathcal{A}_M$ is the unique smooth structure $\mathcal{A}$ on M such that $\mathcal{A}' \subset \mathcal{A}$. Hence $(M, \mathcal{A}_M)$ is an n-dimensional smooth manifold and $\mathcal{A}' \subset \mathcal{A}_M$.
link exercises □

4.2 Open and Boundary Submanifolds

4.2.1 Open Submanifolds

Exercise 4.2.1.1. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold, $(U, \phi) \in \mathcal{A}$ and $U' \subset U$. If U' is open, then $(U', \phi|_{U'}) \in \mathcal{A}$.

Proof. Set $\phi' = \phi|_{U'}$. A previous exercise implies that $(U', \phi') \in X(U)$. Define $\mathcal{B} = \mathcal{A} \cup \{(U', \phi')\}$. Let $(V, \psi) \in \mathcal{B}$. If $(V, \psi) = (U', \phi')$, then

$$\phi' \circ \psi^{-1} = \mathrm{id}_{U'}$$

which is a diffeomorphism. Thus (U', ϕ'), (V, ψ) are smoothly compatible. Suppose that $(V, \psi) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism. Therefore $\psi|_{U' \cap V} \circ (\phi'|_{U' \cap V})^{-1} : \phi'(U' \cap V) \to \psi(U' \cap V)$ is a diffeomorphism and (U', ϕ'), (V, ψ) are smoothly compatible. Since $(V, \psi) \in \mathcal{B}$ is arbitrary, $\mathcal{B}$ is smooth. Since $\mathcal{A}$ is maximal and $\mathcal{A} \subset \mathcal{B}$, we have that $\mathcal{A} = \mathcal{B}$ and $(U', \phi') \in \mathcal{A}$. □

Exercise 4.2.1.2. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold and $U \subset M$ open. Set $\mathcal{B} = \{(V, \psi) \in \mathcal{A} : V \subset U\}$. Then $\mathcal{B}$ is a smooth atlas on U.

Proof.

- Some previous exercises imply that U is an n-dimensional topological manifold and $X(U) = \{(V, \psi) \in X(M) : V \subset U\}$. Since

$$\begin{aligned} \mathcal{B} &\subset \mathcal{A} \\ &\subset X(M) \end{aligned}$$

  we have that $\mathcal{B} \subset X(U)$. Let $p \in U$. Then there exists $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{V'}$. The previous exercise implies that $(V', \psi') \in \mathcal{A}$. By definition, $(V', \psi') \in \mathcal{B}$. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $(V', \psi') \in \mathcal{B}$ such that $p \in V'$. Hence $\mathcal{B}$ is an atlas on U.

- Let $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$. Then $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, (V_1, ψ_1) and (V_2, ψ_2) are smoothly compatible. Since $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ are arbitrary, $\mathcal{B}$ is smooth.

□

Definition 4.2.1.3. Smooth Open Submanifold:
Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $U \subset M$ open. A previous exercise implies that U is an n-dimensional topological manifold. We define the **induced smooth structure on** U, denoted $\mathcal{A}|_U \subset X(U)$, by

$$\mathcal{A}|_U = \alpha_U(\{(V, \psi) \in \mathcal{A} : V \subset U\})$$

Then $(U, \mathcal{A}|_U)$ is said to be a **smooth open submanifold of** $(M, \mathcal{A})$.

Exercise 4.2.1.4. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $U \subset M$ open. Then

1. $\mathcal{A}|_U \subset \mathcal{A}$,
2. $\mathcal{A}|_U = \{(V, \psi) \in \mathcal{A} : V \subset U\}$.

Proof.

1. Set $\mathcal{B} = \{(V, \psi) \in \mathcal{A} : V \subset U\}$. Let $(U', \phi) \in \mathcal{A}|_U$, $(V, \psi) \in \mathcal{A}$ and $a \in \phi(U' \cap V)$. Set $p = \phi^{-1}(a)$. Exercise 4.2.1.2 implies that $\mathcal{B}$ is a smooth atlas on U. Thus there exists $(W, \alpha) \in \mathcal{B}$ such that $p \in W$. Set $A := W \cap U' \cap V$ and $A_0 := \phi(A)$. Then $p \in A$, $a \in A_0$, A is open in M, A_0 is open in $\phi(U' \cap V)$ and A_0 is open in $\phi(W \cap U')$. Define $f : \phi(W \cap U') \to \alpha(W \cap U')$, $g : \alpha(W \cap V) \to \psi(W \cap V)$ and $h : \phi(U' \cap V) \to \psi(U' \cap V)$ by $f := \alpha|_{W \cap U'} \circ \phi|_{W \cap U'}^{-1}$, $g := \psi|_{W \cap V} \circ \alpha|_{W \cap V}^{-1}$ and $h := \psi_{U' \cap V} \circ \phi|_{U' \cap V}^{-1}$. Since $\mathcal{B} \subset \mathcal{A}$, g is smooth. Since $\mathcal{B} \subset \mathcal{A}|_U$, f is smooth. Exercise 1.3.2.3 implies that $f|_{A_0}$ is smooth. Since $h|_{A_0} = g \circ f|_{A_0}$, Exercise 1.3.2.5 implies that $h|_{A_0}$ is smooth. Since $a \in \phi(U' \cap V)$ is arbitrary,

we have that for each $a \in \phi(U' \cap V)$, there exists $A_0 \subset \phi(U' \cap V)$ such that $a \in A_0$, A_0 is open in $\phi(U' \cap V)$ and $h|_{A_0}$ is smooth. Exercise 1.3.2.4 implies that h is smooth. Similarly h^{-1} is smooth. Thus h is a diffeomorphism. Therefore (V, ψ) and (U', ϕ) are smoothly compatible. Since $(V, \psi) \in \mathcal{A}$ is arbitrary, we have that $\{(U', \phi)\} \cup \mathcal{A}$ is a smooth atlas. Since $\mathcal{A}$ is maximal, $\{(U', \phi)\} \cup \mathcal{A} = \mathcal{A}$. Thus $(U', \phi) \in \mathcal{A}$. Since $(U', \phi) \in \mathcal{A}|_U$ is arbitrary, we have that $\mathcal{A}|_U \subset \mathcal{A}$.

2. By definition,

$$\begin{aligned} \mathcal{B} &\subset \alpha_U(\mathcal{B}) \\ &= \mathcal{A}|_U \end{aligned}$$

Since $\mathcal{A}|_U \subset \mathcal{A}$, the definition of $\mathcal{B}$ implies that $\mathcal{A}|_U \subset \mathcal{B}$. Hence $\mathcal{A}|_U = \mathcal{B}$.

□

Note 4.2.1.5. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $U \subset M$. Suppose that U is open in M. Unless otherwise specified, we equip U with $\mathcal{A}|_U$.

4.2.2 Boundary Submanifolds

Exercise 4.2.2.1. Let $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ be the projection map given by $\pi(x^1, \ldots, x^{n-1}, 0) = (x^1, \ldots, x^{n-1})$. Then π is a diffeomorphism.

Proof. Define projection map $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by $\pi'(x^1, \ldots, x^{n-1}, x^n) = (x^1, \ldots, x^{n-1})$. Then $\mathbb{R}^n$ is an open neighborhood of $\partial\mathbb{H}^n$, $\pi'|_{\partial H^n} = \pi$ and π' is smooth. Then by definition, π is smooth. Clearly, π^{-1} is smooth. So π is a diffeomorphism. □

Definition 4.2.2.2. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold and $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ the projection map. Recall that for $(U, \phi) \in X^n_\partial(M)$, the $(n-1)$-coordinate chart $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$ is defined by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$.
We define

$$\bar{\mathcal{A}} = \{(\bar{U}, \bar{\phi}) \in X^{n-1}_\partial(M) : (U, \phi) \in \mathcal{A}\}$$

Exercise 4.2.2.3. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold. Then $\bar{\mathcal{A}}$ is a smooth atlas on ∂M.

Proof.

- A previous exercise implies that ∂M is an $(n-1)$-dimensional topological manifold. Let $p \in \partial M$. Then there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Since $\mathcal{A} \subset X^n(M)$ and $p \in \partial M$, we have that $p \in \bar{U}$ and a previous exercise implies that $(U, \phi) \in X^n_\partial(M)$. By definition of $\bar{\mathcal{A}}$, $(\bar{U}, \bar{\phi}) \in \bar{\mathcal{A}}$. Since $p \in \partial M$ is arbitrary, $\bar{\mathcal{A}}$ is an atlas on ∂M.

- Let $(\bar{U}, \bar{\phi})$, $(\bar{V}, \bar{\psi}) \in \bar{\mathcal{A}}$. Since (U, ϕ) and (V, ψ) are smoothly compatible, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$ is a diffeomorphism. Thus $\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}$ is a diffeomorphism. Since $\pi|_{\phi(U \cap V)}$ and $\pi|_{\psi(U \cap V)}$ are diffeomorphisms, $\pi|_{\phi(\bar{U} \cap \bar{V})}$ and $\pi|_{\psi(\bar{U} \cap \bar{V})}$ are diffeomorphisms. Then

$$\begin{aligned} \bar{\psi}|_{\bar{U} \cap \bar{V}} \circ (\bar{\phi}|_{\bar{U} \cap \bar{V}})^{-1} &= \left[\pi|_{\psi(\bar{U} \cap \bar{V})} \circ \psi|_{\bar{U} \cap \bar{V}}\right] \circ \left[(\phi|_{\bar{U} \cap \bar{V}})^{-1} \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1}\right] \\ &= \pi|_{\psi(\bar{U} \cap \bar{V})} \circ \left[\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}\right] \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \end{aligned}$$

is a diffeomorphism. Therefore $(\bar{U}, \bar{\phi})$ and $(\bar{V}, \bar{\psi})$ are smoothly compatible. Since $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \bar{\mathcal{A}}$ are arbitrary, $\mathcal{A}$ is smooth.

□

Definition 4.2.2.4. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold. We define the **induced smooth structure on the boundary**, denoted $\mathcal{A}|_{\partial M}$, by

$$\mathcal{A}|_{\partial M} = \alpha(\bar{\mathcal{A}})$$

We define the **smooth boundary submanifold of** M to be $(\partial M, \mathcal{A}|_{\partial M})$.

Note 4.2.2.5. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold. Unless otherwise specified, we equip ∂M with $\mathcal{A}|_{\partial M}$.

4.3 Product Manifolds

Note 4.3.0.1. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. We recall the definition of $\lambda_0 : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^{m+n}$ in Definition 3.3.0.2 by $\lambda((x^1, \dots, x^{m-1}, x^m), (y^1, \dots, y^n)) := (x^1, \dots, x^{m-1}, y^1, \dots, y^{n-1}, \log y^n, x^m)$ and from Exercise 3.3.0.3, we know that

- $\lambda_0(\partial\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n) = \partial\mathbb{H}^{m+n}$,
- $(\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n, \lambda_0) \in X^{m+n}(\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n)$.

Definition 4.3.0.2. Let M, N be topological manifolds of dimension m and n respectively, $\mathcal{A} \subset X^m(M)$ and $\mathcal{B} \subset X^n(N)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are smooth atlases on M and N respectively and $\partial N = \varnothing$. We define the **product atlas of $\mathcal{A}$ and $\mathcal{B}$ on** $M \times N$, denoted $\mathcal{A} \otimes_0 \mathcal{B}$, by

$$\mathcal{A} \otimes_0 \mathcal{B} = \{(U \times V, \lambda_0|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) : (U, \phi) \in \mathcal{A} \text{ and } (V, \psi) \in \mathcal{B}\}$$

Exercise 4.3.0.3. Let M, N be topological manifolds of dimension m and n respectively, $\mathcal{A} \subset X^m(M)$ and $\mathcal{B} \subset X^n(N)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are smooth atlases on M and N respectively and $\partial N = \varnothing$. Then $\mathcal{A} \otimes_0 \mathcal{B}$ is a smooth atlas on $M \times N$.

Proof.

- Exercise 3.3.0.5 and the proof of Exercise 3.3.0.6 implies that $\mathcal{A} \otimes_0 \mathcal{B}$ is an atlas on $M \times N$.
- Let $(W_1, \eta_1), (W_2, \eta_2) \in \mathcal{A} \otimes_0 \mathcal{B}$. Then there exist $(U_1, \phi_1), (U_2, \phi_2) \in \mathcal{A}$, $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ such that $W_1 = U_1 \times V_1$, $W_2 = U_2 \times V_2$, $\eta_1 = \lambda_0|_{\phi_1(U_1)\times\psi_1(V_1)} \circ [\phi_1 \times \psi_1]$ and $\eta_2 = \lambda_0|_{\phi_2(U_2)\times\psi_2(V_2)} \circ [\phi_2 \times \psi_2]$. For notational convenience, set $U := U_1 \cap U_2$ and $V := V_1 \cap V_2$. Then $W_1 \cap W_2 = U \cap V$ and

$$\begin{aligned}
\eta_2|_{W_1\cap W_2} \circ \eta_1|_{W_1\cap W_2}^{-1} &= \eta_2|_{U\cap V} \circ \eta_1|_{U\cap V}^{-1} \\
&= \lambda_0|_{\phi_2(U)\times\psi_2(V)} \circ [\phi_2 \times \psi_2]|_{U\times V} \circ [\phi_1 \times \psi_1]|_{U\times V}^{-1} \circ \lambda_0|_{\phi_1(U)\times\psi_1(V)}^{-1} \\
&= \lambda_0|_{\phi_2(U)\times\psi_2(V)} \circ [\phi_2|_U \times \psi_2|_V] \circ [\phi_1|_U^{-1} \times \psi_1|_V^{-1}] \circ \lambda_0|_{\phi_1(U)\times\psi_1(V)}^{-1} \\
&= \lambda_0|_{\phi_2(U)\times\psi_2(V)} \circ [(\phi_2|_U \circ \phi_1|_U^{-1}) \times (\psi_2|_V \circ \psi_1|_V^{-1})] \circ \lambda_0|_{\phi_1(U)\times\psi_1(V)}^{-1}
\end{aligned}$$

  Write $\phi_2 = (x_2^1, \dots, x_2^m)$ and $\psi_2 = (y_2^1, \dots, y_2^n)$. Since $\phi_2|_U \circ \phi_1|_U^{-1}$ and $\psi_2|_V \circ \psi_1|_V^{-1}$ are smooth, reference components of smooth tuples are smooth implies that for each $j \in [m]$ and $k \in [n]$, $x_2^j \circ \phi_1|_U^{-1}$ and $y_2^k \circ \psi_1|_V^{-1}$ are smooth. Let $(a^1, \dots, a^{m-1}, b^1, \dots, b^n, a^m) \in \eta_1(W_1 \cap W_2)$. Then

$$\begin{aligned}
\eta_2|_{W_1\cap W_2} \circ \eta_1|_{W_1\cap W_2}^{-1}(a^1, \dots, a^{m-1}, b^1, \dots, b^n, a^m) = (&x_2^1 \circ \phi_1^{-1}(a^1, \dots, a^m), \dots, x_2^{m-1} \circ \phi_1^{-1}(a^1, \dots, a^m), \\
&y_2^1 \circ \psi_1^{-1}(b^1, \dots, b^{n-1}, e^{b^n}), \dots, y_2^{n-1} \circ \psi_1^{-1}(b^1, \dots, b^{n-1}, e^{b^n}), \\
&\log y_2^n \circ \psi_1^{-1}(b^1, \dots, b^{n-1}, e^{b^n}), x_2^m \circ \phi_1^{-1}(a^1, \dots, a^m))
\end{aligned}$$

  Hence reference tuples of smooth maps are smooth $\eta_2|_{W_1\cap W_2} \circ \eta_1|_{W_1\cap W_2}^{-1}$ is smooth. Since $(W_1, \eta_1), (W_2, \eta_2) \in \mathcal{A} \otimes_0 \mathcal{B}$ are arbitrary, we have that $\mathcal{A} \otimes_0 \mathcal{B}$ is smooth.

□

Definition 4.3.0.4. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ be smooth manifolds. Suppose that $\partial N = \varnothing$. We define the **product smooth structure**, denoted $\mathcal{A} \otimes \mathcal{B}$, by

$$\mathcal{A} \otimes \mathcal{B} = \alpha_{M\times N}(\mathcal{A} \otimes_0 \mathcal{B})$$

We define the **smooth product manifold of** $(M, \mathcal{A})$ **and** $(N, \mathcal{B})$ to be $(M \times N, \mathcal{A} \otimes \mathcal{B})$.

Note 4.3.0.5. Let $(M, \mathcal{A})$ and $(M, \mathcal{B})$ be an n-dimensional smooth manifolds. Unless otherwise specified, we equip $M \times N$ with $\mathcal{A} \otimes \mathcal{B}$.

Exercise 4.3.0.6. Show that if $U \subset M$ is open, $V \subset N$ open, then $(\mathcal{A} \otimes \mathcal{B})|_{U\times V} = \mathcal{A}|_U \otimes \mathcal{B}|_V$.

Proof. FINISH!!! □

Chapter 5

Smooth Maps

5.1 Smooth Maps between Manifolds

Note 5.1.0.1. it might be better to phrase smoothness as F is smooth if there exists $\mathcal{A}_0 \subset \mathcal{A}$ ... such that for each $(U, \phi) \in \mathcal{A}_0$, $\psi \circ F \circ \phi|_{U \cap F^{-1}(V)}^{-1}$

Definition 5.1.0.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then F is said to be

- $(\mathcal{A}, \mathcal{B})$**-smooth** if for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is smooth.
- a $(\mathcal{A}, \mathcal{B})$**-diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Note 5.1.0.3. When the context is clear, we write "smooth" in place of "$(\mathcal{A}, \mathcal{B})$-smooth".

Exercise 5.1.0.4. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifold and $F : M \to N$. If F is smooth, then F is continuous.

Proof. Suppose that F is smooth. Let $p \in M$. By defintion, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is smooth. Define $F_0 : \phi(U) \to \psi(V)$ by

$$F_0 = \psi \circ F \circ \phi^{-1}$$

By definition, F_0 is smooth. Exercise 1.3.2.2 implies that F_0 is continuous. Since ϕ and ψ are homeomorphisms and $F|_U = \psi^{-1} \circ F_0 \circ \phi$, we have that $F|_U$ is continuous. In particular, F is continuous at p. Since $p \in M$ is arbitrary, F is continuous. $\square$

Exercise 5.1.0.5. Equivalence of Smoothness:
Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then the following are equivalent:

1. $F : M \to N$ is smooth
2. for each $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, if $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N, then for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|_{U \cap F^{-1}(V)}^{-1}$ is smooth.
3. for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|_{U \cap F^{-1}(V)}^{-1}$ is smooth.
4. F is continuous and there exist $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ such that $\mathcal{A}_0$ is an atlas on $\mathcal{A}$, $\mathcal{B}_0$ is an atlas on N and for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $\psi \circ F \circ \phi|_{U \cap F^{-1}(V)}^{-1}$ is smooth

Proof. Set $m := \dim M$ and $n := \dim N$.

1. (1) $\implies$ (2):
Suppose that F is smooth. Let $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$. Suppose that $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N. Let $(U_0, \phi_0) \in \mathcal{A}_0$ and $(V_0, \psi_0) \in \mathcal{B}_0$. Since $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, we have that $(U_0, \phi_0) \in \mathcal{A}$ and $(V_0, \psi_0) \in \mathcal{B}$. Since F is smooth, Exercise 5.1.0.4 implies that F is continuous and therefore $U_0 \cap F^{-1}(V_0)$ is open in M. Define $F_0 : \phi_0(U_0 \cap F^{-1}(V_0)) \to \psi_0(V_0)$ by $F_0 := \psi_0 \circ F \circ \phi_0|^{-1}_{U_0 \cap F^{-1}(V_0)}$. Let $a \in \phi_0(U_0 \cap F^{-1}(V_0))$. Define $p \in M$ by $p := \phi_0^{-1}(a)$. Since F is smooth, by definition there exists $(U_1, \phi_1) \in \mathcal{A}$ and $(V_1, \psi_1) \in \mathcal{B}$ such that $p \in U_1$, $F(p) \in V_1$, $F(U_1) \subset V_1$ and $\psi_1 \circ F \circ \phi_1^{-1}$ is smooth. Define $U \subset M$, $\alpha : \phi_1(U_0 \cap U_1) \to \phi_0(U_0 \cap U_1)$, $\beta : \psi_1(V_0 \cap V_1) \to \psi_0(V_0 \cap V_1)$ and $F_1 : \phi_1(U_1) \to \psi_1(V_1)$ by $U := U_0 \cap U_1 \cap F^{-1}(V_0 \cap V_1)$, $\alpha := \phi_0|_{U_0 \cap U_1} \circ \phi_1|^{-1}_{U_0 \cap U_1}$, $\beta := \psi_0|_{V_0 \cap V_1} \circ \psi_1|^{-1}_{V_0 \cap V_1}$ and $F_1 := \psi_1 \circ F \circ \phi_1^{-1}$. We note the following:

   - since $p \in U$ and $a = \phi_0(p)$, we have that $a \in \phi_0(U)$
   - $\phi_0(U)$ is open in $\phi_0(U_0 \cap F^{-1}(V_0))$
   - since $(U_0, \phi_0), (U_1, \phi_1) \in \mathcal{A}$, (U_0, ϕ_0) and (U_1, ϕ_1) are smoothly compatible and α is a diffeomorphism
   - since $(V_0, \psi_0), (V_1, \psi_1) \in \mathcal{B}$, (V_0, ψ_0) and (V_1, ψ_1) are smoothly compatible and β is a diffeomorphism
   - since $F_1 = \psi_1 \circ F \circ \phi_1^{-1}$, F_1 is smooth
   - since α^{-1} is smooth, Exercise 1.3.2.3 implies that $\alpha|^{-1}_{\phi_1(U)}$ is smooth
   - since $F_0|_{\phi_0(U)} = \beta \circ F_1 \circ \alpha|^{-1}_{\phi_1(U)}$, Exercise 1.3.2.5 implies that that $F_0|_{\phi_0(U)}$ is smooth

   Since $a \in \phi_0(U_0 \cap F^{-1}(V_0))$ is arbitrary, we have that for each $a \in \phi_0(U_0 \cap F^{-1}(V_0))$, there exists $A \subset \phi_0(U_0 \cap F^{-1}(V_0))$ such that $a \in A$, A is open in $\phi_0(U_0 \cap F^{-1}(V_0))$ and $F_0|_A$ is smooth. Exercise 1.3.2.4 implies that F_0 is smooth.

   Since $(U_0, \phi_0) \in \mathcal{A}_0$ and $(V_0, \psi_0) \in \mathcal{B}_0$ are arbitrary, we have that for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

   Since $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ such that $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N are arbitrary, we have that for each $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, if $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N, then for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

2. (2) $\implies$ (3):
Suppose that for each $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, if $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N, then for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth. Let $p \in M$. Since $\mathcal{A}$ is an atlas on M and $\mathcal{B}$ is an atlas on N, there exists $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $F(p) \in V$. By assumption, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

3. (3) $\implies$ (4):
Suppose that for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

   - Let $p \in M$. By assumption, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth. Define $A \subset M$, $A_1 \subset \mathbb{H}^m$ and $F_1 : A_1 \to \mathbb{R}^n$ by $A := U \cap F^{-1}(V)$, $A_1 := \phi(A)$ and $F_1 := \psi \circ F \circ \phi|_A^{-1}$. Since F_1 is smooth, Exercise 1.3.2.2 implies that $F_1 : A_1 \to \mathbb{R}^n$ is continuous. Since $\phi|_A$ and ψ are homeomorphisms,
$$\begin{aligned} F|_A &= \psi^{-1} \circ (\psi \circ F \circ \phi|_A) \circ \phi|_A^{-1} \\ &= \psi^{-1} \circ F_1 \circ \phi_A^{-1} \end{aligned}$$
which is continuous. We note that $p \in A$ and A is open in M. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exists $A \subset M$ such that $p \in A$, A is open in M and $F|_A$ is continuous. Thus F is continuous.

- – By assumption, for each $p \in M$, there exists $(U_p, \phi_p) \in \mathcal{A}$ and $(V_p, \psi_p) \in \mathcal{B}$ such that $p \in U_p$, $F(p) \in V_p$, $U_p \cap F^{-1}(V_p)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(p)}$ is smooth. The axiom of choice implies that there exist $(U_p, \phi_p)_{p \in M} \subset \mathcal{A}$ and $(V_p, \psi_p)_{p \in M} \subset \mathcal{B}$ such that for each $p \in M$, $p \in U_p$, $F(p) \in V_p$, $U_p \cap F^{-1}(V_p)$ is open in M and $\psi_p \circ F \circ \phi_p|^{-1}_{U_p \cap F^{-1}(V_p)}$ is smooth. Define $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ by $\mathcal{A}_0 := (U_p, \phi_p)_{p \in M}$ and $\mathcal{B}_0 := (B_p, \psi_p)_{p \in M}$ respectively. By construction, $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N.
  – Let $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$. Define $\tilde{A} \subset \mathbb{H}^m$ and $\tilde{F} : \tilde{A} \to \mathbb{R}^n$ by $\tilde{A} = \phi(U \cap F^{-1}(V))$ and $\tilde{F} = \psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$. Since F is continuous, $U \cap F^{-1}(V)$ is open in M. Since ϕ is a homeomorphism, $\tilde{A}$ is open in $\mathbb{H}^n$. Let $a \in \tilde{A}$. Set $p := \phi^{-1}(a)$. Define $A \subset M$ by $A := U \cap U_p \cap F^{-1}(V \cap V_p)$. We note that $p \in A$ and since F is continuous, A is open in M. Define $A_0 \subset \mathbb{H}^m$ and $F_0 : A_0 \to \mathbb{R}^n$ by $A_0 = \phi_p(A)$ and $F_0 = \psi_p \circ F \circ \phi_p|^{-1}_A$. By construction, $\psi_p \circ F \circ \phi_p|^{-1}_{U_p \cap F^{-1}(V_p)}$ is smooth. An exercise about restriction in the section on differentation on subspaces implies that F_0 is smooth. We define $\alpha : \phi_p(U \cap U_p) \to \phi(U \cap U_p)$ and $\beta : \psi_p(V \cap V_p) \to \psi(V \cap V_p)$ by

$$\alpha := \phi|_{U \cap U_p} \circ \phi_p|^{-1}_{U \cap U_p}, \quad \beta := \psi|_{V \cap V_p} \circ \psi_p|^{-1}_{V \cap V_p}$$

  Since $\phi, \phi_p \in \mathcal{A}$, we know that ϕ and ϕ_p are smoothly compatible. Therefore α is a diffeomorphism. Similarly, β is a diffeomorphism. the restriction exercise again implies that $\alpha|_{A_0}$ is a diffeomorphism. Since $\tilde{F}|_{\phi(A)} = \beta \circ F_0 \circ \alpha|^{-1}_{A_0}$, we have that $\tilde{F}|_{\phi(A)}$ is smooth. We note that $a \in \phi(A)$, $\phi(A)$ is open in $\tilde{A}$. Since $a \in \tilde{A}$ is arbitrary, we have that for each $a \in \tilde{A}$, there exists $E \subset \tilde{A}$ such that $a \in E$, E is open in $\tilde{A}$ and $\tilde{F}|_E$ is smooth. An exercise in the section on differentiation on subspaces implies that $\tilde{F}$ is smooth. Since $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$ are arbitrary, we have that for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

4. (4) $\implies$ (1):
Suppose that F is continuous and there exist $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ such that $\mathcal{A}_0$ is an atlas on $\mathcal{A}$, $\mathcal{B}_0$ is an atlas on N and for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth. Let $p \in M$. Since $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N, there exists $(U', \phi') \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$ such that $p \in U'$ and $F(p) \in V$. Define $A_0 \subset \mathbb{H}^m$ and $F_0 : A_0 \to \mathbb{R}^n$ by $A_0 = \phi'(U' \cap F^{-1}(V))$ and $F_0 = \psi \circ F \circ \phi'|^{-1}_{U' \cap F^{-1}(V)}$. By assumption F_0 is smooth. Since F is continuous, $F(p) \in V$ and V is open in N, we have that there exists $U_0 \subset M$ such that $p \in U_0$, U_0 is open in M and $F(U_0) \subset V$. Define $U \subset M$ and $\phi : U \to \phi'(U)$ by $U := U' \cap U_0$ and $\phi = \phi'|_U$. Then $p \in U$, U is open in M and

$$\begin{aligned} F(U) &= F(U' \cap U_0) \\ &\subset F(U_0) \\ &\subset V \end{aligned}$$

An exercise in the section on smooth manifolds implies that $(U, \phi) \in \mathcal{A}$. Since F_0 is smooth, an exercise in the section on subspace differentiation implies that $F_0|_{\phi(U)}$ is smooth. Since $\psi \circ F \circ \phi^{-1} = F_0|_{\phi(U)}$, we have that $\psi \circ F \circ \phi^{-1}$. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is smooth. Hence F is smooth.

$\square$

Exercise 5.1.0.6. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ $(E, \mathcal{C})$be smooth manifolds and $F : M \to N$, $G : N \to E$. If F and G are smooth, then $G \circ F : M \to E$ is smooth.

Proof. Set $m = \dim M$, $n = \dim N$ and $e = \dim E$. Suppose that F and G are smooth. Let $p_0 \in M$. Since F is smooth, there exists $(U_0, \phi_0) \in \mathcal{A}$ and $(V_0, \psi_0) \in \mathcal{B}$ such that $p_0 \in U_0$, $F(p_0) \in V_0$, $F(U_0) \subset V_0$ and $\psi_0 \circ F \circ \phi_0^{-1}$ is smooth. Set $p_1 = F(p_0)$. Since G is smooth, there exists $(U_1, \phi_1) \in \mathcal{B}$ and $(V_1, \psi_1) \in \mathcal{C}$ such that $p_1 \in U_1$, $G(p_1) \in V_1$, $G(U_1) \subset V_1$ and $\psi_1 \circ F \circ \phi_1^{-1}$ is smooth. Define $f : \phi_0(U_0) \to \mathbb{H}^n$ and $g : \phi_1(U_1) \to \mathbb{H}^e$ by $f = \psi_0 \circ F \circ \phi_0^{-1}$ and $g = \psi_1 \circ G \circ \phi_1^{-1}$ respectively. Set $W_1 = U_1 \cap V_0$ and $W_0 = F^{-1}(W_1)$. Since W_1 is open in N and F is continuous, W_0 is open in M. An exercise in the section on open submanifolds implies that

$$\begin{aligned}(W_0, \phi_0|_{W_0}) &\in \mathcal{A}|_{W_0} \\ &\subset \mathcal{A}\end{aligned}$$

Since $p_1 \in W_1$, $p_0 \in W_0$. Furthermore,

$$\begin{aligned}G \circ F(p_0) &= G(p_1) \\ &\in V_1\end{aligned}$$

and

$$\begin{aligned}G \circ F(W_0) &= G(F(W_0)) \\ &\subset G(W_1) \\ &\subset G(U_1) \\ &\subset V_1\end{aligned}$$

Since $(U_1, \phi_1), (V_0, \psi_0) \in \mathcal{B}$, (U_1, ϕ_1) and (V_0, ψ_0) are smoothly-compatible. Thus $\phi_1|_{W_1} \circ \psi_0|_{W_1}^{-1} : \psi_0(W_1) \to \phi_1(W_1)$ is smooth. Since f and g are smooth, we have that $f|_{\phi_0(W_0)}$ is smooth and therefore

$$\begin{aligned}\psi_1 \circ (G \circ F) \circ \phi_0|_{W_0}^{-1} &= (\psi_1 \circ G \circ \phi_1|_{W_1}^{-1}) \circ (\phi_1|_{W_1} \circ \psi_0|_{W_1}^{-1}) \circ (\psi_0 \circ F \circ \phi_0|_{W_0}^{-1}) \\ &= g \circ (\phi_1|_{W_1} \circ \psi_0|_{W_1}^{-1}) \circ f|_{\phi_0(W_0)}\end{aligned}$$

is smooth. Since $p_0 \in M$ is arbitrary, we have that for each $p_0 \in M$, there exists $(W_0, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{C}$ such that $p_0 \in W_0$, $G \circ F(p_0) \in V$, $G \circ F(W_0) \subset V$ and $\psi \circ (G \circ F) \circ \phi^{-1}$ is smooth. Thus $G \circ F$ is smooth. $\square$

5.2 Smooth Maps on Open and Boundary Submanifolds

Exercise 5.2.0.1. Locality of Smoothness:
Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then the following are equivalent:

1. F is smooth
2. for each $U \subset M$, if U is open in M, then $F|_U : U \to N$ is smooth.
3. for each $p \in M$, there exists $U \subset M$ such that $p \in U$, U is open in M and $F|_U : U \to N$ is smooth.

Proof.

- (1) $\implies$ (2):
  Suppose that F is smooth. Let $U \subset M$. Suppose that U is open in M. Let $p \in U$. Since $\mathcal{A}|_U$ is an atlas on U and $\mathcal{B}$ is an atlas on N, there exist $(U_0, \phi_0) \in \mathcal{A}|_U$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U_0$ and $F(p) \in V$. Since $p \in U$, we have that

  $$\begin{aligned} F|_U(p) &= F(p) \\ &\in V \end{aligned}$$

  An exercise in the section on open submanifolds implies that $\mathcal{A}|_U \subset \mathcal{A}$. Thus $(U_0, \phi_0) \in \mathcal{A}$. Since F is smooth a previous exercise implies that $U_0 \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi_0|_{U_0 \cap F^{-1}(V)}$ is smooth. Since $U_0 \subset U$, we have that

  $$\begin{aligned} U_0 \cap F|_U^{-1}(V) &= U_0 \cap (U \cap F^{-1}(V)) \\ &= U_0 \cap F^{-1}(V) \end{aligned}$$

  and $\psi \circ F|_U \circ \phi_0|^{-1}_{U_0 \cap F|_U^{-1}(V)} = \psi \circ F \circ \phi_0|^{-1}_{U_0 \cap F^{-1}(V)}$. Thus $U_0 \cap F|_U^{-1}(V)$ is open in U and $\psi \circ F|_U \circ \phi_0|^{-1}_{U_0 \cap F|_U^{-1}(V)}$ is smooth. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $(U_0, \phi_0) \in \mathcal{A}|_U$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U_0$, $F|_U(p) \in V$, $U_0 \cap F|_U^{-1}(V)$ is open in U and $\psi \circ F|_U \circ \phi_0|^{-1}_{U_0 \cap F|_U^{-1}(V)}$ is smooth. (3) in smooth equivalence implies that $F|_U$ is smooth. Since $U \subset M$ with U open in M is arbitrary, we have that for each $U \subset M$, if U is open in M, then $F|_U : U \to N$ is smooth.

- (2) $\implies$ (3):
  Suppose that for each $U \subset M$, if U is open in M, then $F|_U : U \to N$ is smooth. Let $p \in M$. Since $\mathcal{A}$ is an atlas on M, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Since $(U, \phi) \in X(M)$, U is open in M. By assumption, $F|_U : U \to N$ is smooth. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exists $U \subset M$ such that $p \in U$, U is open in M and $F|_U : U \to N$ is smooth.

- (3) $\implies$ (1):
  Suppose that for each $p \in M$, there exists $U \subset M$ such that $p \in U$, U is open in M and $F|_U : U \to N$ is smooth. Let $p \in M$. Let $p \in M$. By assumption, there exists $U \subset M$ such that $p \in U$, U is open in M and $F|_U : U \to N$ is smooth. Since $F|_U$ is smooth, there exist $(U', \phi) \in \mathcal{A}|_U$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U'$, $F(p) \in V$, $F|_U(U') \subset V$ and $\psi \circ F|_U \circ \phi^{-1}$ is smooth. An exercise in the section on open submanifolds implies that $\mathcal{A}|_U \subset \mathcal{A}$. Thus $(U', \phi) \in \mathcal{A}$. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exists $(U', \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U'$, $F(p) \in V$, $F(U') \subset V$ and $\psi \circ F \circ \phi^{-1}$ is smooth. Thus F is smooth.

□

Exercise 5.2.0.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $U \subset M$ and $F : M \to N$. Suppose that U is open in M. If F is a diffeomorphism, then $F|_U : U \to F(U)$ is a diffeomorphism.

Proof. Suppose that F is a diffeomorphism. Then F and F^{-1} are smooth. Hence F is a homeomorphism and $F(U)$ is open in N., By definition, F and F^{-1} are smooth. A previous exercise about locality of smoothness implies that $F|_U$ and $F^{-1}|_{F(U)}$ are smooth. Since $F|_U^{-1} = F^{-1}|_{F(U)}$, $F|_U$ is a diffeomorphism. □

Exercise 5.2.0.3. Let $(M, \mathcal{A})$ be a smooth manifold and $(U, \phi) \in \mathcal{A}$. Then $\phi : U \to \phi(U)$ is a diffeomorphism.

Proof. Set $n := \dim M$. Let $(V, \psi) \in \mathcal{A}$. By definition, ϕ is continuous. Since $(U, \phi), (V, \psi) \in \mathcal{A}$, we have that (U, ϕ) and (V, ψ) are smoothly compatible. Hence $\phi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}$ is a diffeomorphism. Define $\alpha : \psi(U\cap V) \to \phi(U \cap V)$ by $\alpha = \phi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}$. Since $V \cap \phi^{-1}(\phi(U)) = U \cap V$ and $\phi(U) \cap (\phi^{-1})^{-1}(V) = \phi(U \cap V)$, we have that $V \cap \phi^{-1}(\phi(U))$ and $\phi(U) \cap (\phi^{-1})^{-1}(V)$ are open. Furthermore,

$$\begin{aligned}
\mathrm{id}_{\phi(U)} \circ \phi \circ \psi|_{V\cap\phi^{-1}(\phi(U))}^{-1} &= \mathrm{id}_{\phi(U)} \circ \phi \circ \psi|_{V\cap U}^{-1} \\
&= \mathrm{id}_{\phi(U)} \circ \alpha \\
&= \alpha
\end{aligned}$$

and

$$\begin{aligned}
\psi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} |_{\phi(U)\cap(\phi^{-1})^{-1}(V)} &= \psi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} |_{\phi(U\cap V)} \\
&= \alpha^{-1} \circ \mathrm{id}_{\phi(U\cap V)} \\
&= \alpha^{-1}
\end{aligned}$$

Since α is a diffeomorphism, we have that $\mathrm{id}_{\phi(U)} \circ \phi \circ \psi|_{V\cap\phi^{-1}(\phi(U))}^{-1}$ and $\psi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} |_{\phi(U)\cap(\phi^{-1})^{-1}(V)}$ are smooth. Since $(\mathcal{A}|_{\mathbb{H}^n})_{\phi(U)} = \alpha(\mathrm{id}_{\phi(U)})$, $\mathcal{A} = \alpha(\mathcal{A})$ and $(V, \psi) \in \mathcal{A}$ is arbitrary, a previous exercise about smoothness depending on a smooth atlas implies that ϕ and ϕ^{-1} are smooth. Hence ϕ is a diffeomorphism. □

Exercise 5.2.0.4. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$ a diffeomorphism. Then

1. for each $(V, \psi) \in \mathcal{B}$, $(F^{-1}(V), \psi \circ F|_{F^{-1}(V)}) \in \mathcal{A}$
2. for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F|_{F(U)}^{-1}) \in \mathcal{B}$

Proof. Set $n := \dim M$.

1. Let $(V, \psi) \in \mathcal{B}$. Since $F^{-1}(V)$ is open in M, a previous exercise implies that $F|_{F^{-1}(V)}^{-1}$ is a diffeomorphism. A previous exercise implies that ψ is a diffeomorphism. Therefore $\psi \circ F|_{F^{-1}(V)}^{-1}$ is a diffeomorphism.
   (a) Since $(V, \psi) \in \mathcal{B}$ and $F|_{F^{-1}(V)}^{-1}$ is a homeomorphism, we have that
      - $F^{-1}(V)$ is open in M.
      - $\psi(V)$ is open in $\mathbb{H}^n$
      - $\psi \circ F|_{F^{-1}(V)} : F^{-1}(V) \to \psi(V)$ is a homeomorphism

      So $(F^{-1}(V), \psi \circ F|_{F^{-1}(V)}) \in X^n(M)$.
   (b) Let $(U, \phi) \in \mathcal{A}$. A previous exercise implies that ψ is a diffeomorphism. A previous exercise implies that $\phi|_{U\cap F^{-1}(V)}$ and $\psi \circ F|_{U\cap F^1(V)}$ are diffeomorphisms. Hence $(\psi \circ F|_F^{-1}(V))|_{U\cap F^{-1}(V)} \circ \phi|_{U\cap F^{-1}(V)}^{-1}$ is a diffeomorphism. Therefore $(F(U), \psi \circ F|_{F^{-1}(V)}^{-1})$ and (V, ψ) are smoothly compatible. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{A}$, (U, ϕ) and $(F^{-1}(V), \psi \circ F|_{F^{-1}(V)})$ are smoothly compatible. Since $\mathcal{A}$ is maximal, $(F^{-1}(V), \psi \circ F^{-1}) \in \mathcal{A}$.
2. Similar to (1).

□

Exercise 5.2.0.5. Let $M \in \mathrm{Obj}(\mathbf{Man}^0)$ and $\mathcal{A}_1, \mathcal{A}_2$ smooth structures on M. Define $\iota : M \to M$ by $\iota(p) = p$. If $\iota \in \mathrm{Iso}_{\mathbf{ManBnd}^\infty}[(M, \mathcal{A}_1), (M, \mathcal{A}_2)]$, then $\mathcal{A}_1 = \mathcal{A}_2$.

Proof. Set $n := \dim M$. Suppose that ι is a $(\mathcal{A}_1, \mathcal{A}_2)$-diffeomorphism. Exercise 5.2.0.4 implies that $\mathcal{A}_1 = \mathcal{A}_2$. maybe give more details. □

Exercise 5.2.0.6. Let $(M,\mathcal{A})$ and $(N,\mathcal{B})$ be smooth manifolds and $F : M \to N$. Then F is smooth iff for each $(V,\psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$, $y^i \circ F$ is smooth.

Proof. Suppose that F is smooth. Let $(V,\psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. Then for each $i \in \{1, \dots, n\}$, F^i is smooth.
Conversely, suppose that for each $(V,\psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$ and $i \in \{1, \dots, n\}$, $y^i \circ F$ is smooth. $\square$

Definition 5.2.0.7. Let $(N,\mathcal{B})$ be a smooth n-dimensional manifold, $F : M \to N$ smooth and $(V,\psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. For $i \in \{1, \dots, n\}$, We define the **i-th component of F with respect to** (V,ψ), denoted $F^i : V \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

Exercise 5.2.0.8. Let $(M,\mathcal{A})$ be a smooth manifold, $(U,\phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $p \in U$ and $f \in C^\infty(M,\mathcal{A})$. Then $f|_U \in C^\infty(U,\mathcal{A}|_U)$.

Proof. Let $\square$

5.3 Smooth Maps and Product Manifolds

Note 5.3.0.1. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. We recall the definition of $\lambda_0 : \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n \to \mathbb{H}^{m+n}$ in Definition 3.3.0.2 by $\lambda((x^1, \dots, x^{m-1}, x^m), (y^1, \dots, y^n)) := (x^1, \dots, x^{m-1}, y^1, \dots, y^{n-1}, \log y^n, x^m)$.

Exercise 5.3.0.2. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$, $(E, \mathcal{C})$ be smooth manifolds and $F : M \times N \to E$. Suppose that $\partial N = \varnothing$. Then the following are equivalent:

1. F is smooth
2. there exist $\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{C}_0 \subset \mathcal{C}$, such that $\mathcal{A}_0$ is an atlas on M, $\mathcal{B}_0$ is an atlas on N, $\mathcal{C}_0$ is an atlas on E and for each $(U, \phi) \in \mathcal{A}_0$, $(V, \psi) \in \mathcal{B}_0$, $(W, \chi) \in \mathcal{C}_0$, $(U \times V) \cap F^{-1}(W)$ is open and $\chi \circ F \circ [\lambda_0 \circ (\phi \times \psi)|_{(U \times V) \cap F^{-1}(W)}]^{-1}$ is smooth.
3. for each $(p, q) \in M \times N$, there exist $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ and $(W, \chi) \in \mathcal{C}$ such that $(p, q) \in U \times V$, $F(p, q) \in W$, $(U \times V) \cap F^{-1}(W)$ is open in $M \times N$ and $\circ F \circ \chi|^{-1}_{W \cap G^{-1}(U \times V)}[\lambda_0 \circ (\phi \times \psi)|_{(U \times V) \cap F^{-1}(W)}]$ is smooth.

Proof. Set $m := \dim M$, $n = \dim N$ and $e = \dim E$.

1.
   - ($\Longrightarrow$):
     Suppose that F is smooth. Let $(U, \phi) \in \mathcal{A}_0$, $(V, \psi) \in \mathcal{B}_0$ and $(W, \chi) \in \mathcal{C}_0$. Set $\eta := \lambda_0|_{\phi(U) \times \psi(V)} \circ (\phi \times \psi)$. By Definition 4.3.0.2 and Definition 4.3.0.4, $\eta \in \mathcal{A} \otimes \mathcal{B}$. Since F is smooth the second characterization in Exercise 5.1.0.5 implies that $(U \times V) \cap F^{-1}(W)$ is open in $M \times N$ and $\chi \circ F \circ \eta|^{-1}_{(U \times V) \cap F^{-1}(W)}$ is smooth.
     Since $(U, \phi) \in \mathcal{A}_0$, $(V, \psi) \in \mathcal{B}_0$ and $(W, \chi) \in \mathcal{C}_0$ are arbitrary, we have that for each $(U, \phi) \in \mathcal{A}_0$, $(V, \psi) \in \mathcal{B}_0$, $(W, \chi) \in \mathcal{C}_0$, $(U \times V) \cap F^{-1}(W)$ is open in $M \times N$ and $\chi \circ F \circ [\lambda_0 \circ (\phi \times \psi)|_{(U \times V) \cap F^{-1}(W)}]^{-1}$ is smooth.
   - ($\Longleftarrow$):
     Suppose that for each $(U, \phi) \in \mathcal{A}_0$, $(V, \psi) \in \mathcal{B}_0$, $(W, \chi) \in \mathcal{C}_0$, $(U \times V) \cap F^{-1}(W)$ is open and $\chi \circ F \circ [\lambda_0 \circ (\phi \times \psi)|_{(U \times V) \cap F^{-1}(W)}]^{-1}$ is smooth. Let $(p, q) \in M \times N$. Since $\mathcal{A}_0$ is an atlas on M, $\mathcal{B}_0$ is an atlas on N and $\mathcal{C}_0$ is an atlas on E, there exist $(U, \phi) \in \mathcal{A}_0$, $(V, \psi) \in \mathcal{B}_0$, $(W, \chi) \in \mathcal{C}_0$ such that $p \in U$, $q \in V$ and $F(p, q) \in W$. Define $\eta := \lambda_0 \circ (\phi \times \psi)|_{(U \times V) \cap F^{-1}(W)}$. Definition 4.3.0.2 and Definition 4.3.0.4 imply that and $\eta \in \mathcal{A} \otimes \mathcal{B}$. Set $F_0 := \chi \circ F \circ \eta|^{-1}_{(U \times V) \cap F^{-1}(W)}$. By assumption, $(U \times V) \cap F^{-1}(W)$ is open and F_0 is smooth.
     Since $(p, q) \in M \times N$ is arbitrary, the third characterization in Exercise 5.1.0.5 implies that F is smooth. FINISH!!!
2. Similar to (1).

□

Exercise 5.3.0.3. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$, $(E, \mathcal{C})$ be smooth manifolds, $G : E \to M \times N$. Suppose that $\partial N = \varnothing$. Then the following are equivalent:

1. G is smooth iff
2. there exist $\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{C}_0 \subset \mathcal{C}$ such that $\mathcal{A}_0$ is an atlas on M, $\mathcal{B}_0$ is an atlas on N, $\mathcal{C}_0$ is an atlas on E and for each $(U, \phi) \in \mathcal{A}_0$, $(V, \psi) \in \mathcal{B}_0$, $(W, \chi) \in \mathcal{C}_0$, $[\lambda_0 \circ (\phi \times \psi)] \circ G \circ \chi|^{-1}_{W \cap G^{-1}(U \times V)}$ is smooth.
3. for each $p \in E$, there exist $(W, \chi) \in \mathcal{C}$, $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in W$, $G(p) \in U \times V$, $W \cap F^{-1}(U \times V)$ is open in E and $[\lambda_0 \circ (\phi \times \psi)] \circ G \circ \chi|^{-1}_{W \cap G^{-1}(U \times V)}$ is smooth.

Proof.

1. FINISH!!!
2.

□

Exercise 5.3.0.4. We have that $\lambda_0 : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^{m+n}$ is a diffeomorphism.

Proof. Define $(U,\phi) \in \mathcal{A}$, $(V,\psi) \in \mathcal{A}_{\mathbb{H}^n}|_{\operatorname{Int}\mathbb{H}^n}$ and $(W,\chi) \in \mathcal{A}_{\mathbb{H}^{m+n}}$ by $(U,\phi) := (\mathbb{H}^m, \mathrm{id}_{\mathbb{H}^m})$, $(V,\psi) := (\operatorname{Int}\mathbb{H}^n, \mathrm{id}_{\operatorname{Int}\mathbb{H}^n})$ and $(W,\chi) := (\mathbb{H}^{m+n}, \mathrm{id}_{\mathbb{H}^{m+n}})$. Set $\mathcal{A}_0 = \{(U,\phi)\}$, $\mathcal{B}_0 = \{(V,\psi)\}$ and $\mathcal{C}_0 := \{(W,\chi)\}$. Then $\mathcal{A}_0$ is a smooth atlas on $\mathbb{H}^m$, $\mathcal{B}_0$ is a smooth atlas on $\operatorname{Int}\mathbb{H}^n$ and $\mathcal{C}_0$ is a smooth atlas on $\mathbb{H}^m$.

Define $F := \lambda_0$, $\eta := \lambda_0 \circ (\phi \times \psi)$ and $F_0 := \chi \circ F \circ \eta|^{-1}_{(U\times V)\cap F^{-1}(W)}$. We note that for each $(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) \in \lambda_0[\phi\times\psi(U\times V \cap F^{-1}(W))]$,

$$\begin{aligned}
F_0(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) &= \chi \circ F \circ \eta|^{-1}_{(U\times V)\cap \mathrm{proj}_1^{-1}(W)}(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) \\
&= \mathrm{id}_{\mathbb{H}^m} \circ \lambda_0 \circ \lambda_0^{-1}(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) \\
&= (a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) \\
&= \mathrm{id}_{\mathbb{H}^{m+n}}(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m)
\end{aligned}$$

Hence F_0 is smooth. Exercise 5.2.0.1 implies that λ_0 is smooth. Similarly, λ_0^{-1} is smooth. Thus λ_0 is a diffeomorphism. □

Exercise 5.3.0.5. Let $m, n \in \mathbb{N}$. Then

1. $\mathrm{proj}_1 : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^m$ is smooth
2. $\mathrm{proj}_2 : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^n$ is smooth

Proof.

1. Define $(U,\phi) \in \mathcal{A}$, $(V,\psi) \in \mathcal{A}_{\mathbb{H}^n}|_{\operatorname{Int}\mathbb{H}^n}$ and $(W,\chi) \in \mathcal{A}_{\mathbb{H}^m}$ by $(U,\phi) := (\mathbb{H}^m, \mathrm{id}_{\mathbb{H}^m})$, $(V,\psi) := (\operatorname{Int}\mathbb{H}^n, \mathrm{id}_{\operatorname{Int}\mathbb{H}^n})$ and $(W,\chi) := (\mathbb{H}^m, \mathrm{id}_{\mathbb{H}^m})$. Set $\mathcal{A}_0 = \{(U,\phi)\}$, $\mathcal{B}_0 = \{(V,\psi)\}$ and $\mathcal{C}_0 := \{(W,\chi)\}$. Then $\mathcal{A}_0$ is a smooth atlas on $\mathbb{H}^m$, $\mathcal{B}_0$ is a smooth atlas on $\operatorname{Int}\mathbb{H}^n$ and $\mathcal{C}_0$ is a smooth atlas on $\mathbb{H}^m$.

   Define $F := \mathrm{proj}_1$, $\eta := \lambda_0 \circ (\phi \times \psi)$ and $F_0 := \chi \circ F \circ \eta|^{-1}_{(U\times V)\cap F^{-1}(W)}$. We note that for each $(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) \in \lambda_0[\phi\times\psi(U\times V \cap F^{-1}(W))]$,

$$\begin{aligned}
F_0(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) &= \chi \circ F \circ \eta|^{-1}_{(U\times V)\cap \mathrm{proj}_1^{-1}(W)}(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) \\
&= \mathrm{id}_{\mathbb{H}^m} \circ \mathrm{proj}_1 \circ \lambda_0^{-1}(a^1,\dots,a^{m-1},b^1,\dots,b^n,a^m) \\
&= \mathrm{proj}_1(a^1,\dots,a^m,e^{b^1},\dots,e^{b^n}) \\
&= (a^1,\dots,a^m)
\end{aligned}$$

   Hence F_0 is smooth. Exercise 5.2.0.1 implies that proj_1 is smooth.

2. Similar to (1).

□

Definition 5.3.0.6. Let $(M,\mathcal{A})$ and $(N,\mathcal{B})$ be smooth manifolds. We define the **projection maps onto M and N**, denoted by $\pi_M : M\times N \to M$ and $\pi_N : M \times N \to N$ respectively, by

- $\pi_M(p,q) = p$
- $\pi_N(p,q) = q$

Exercise 5.3.0.7. Let M and N be smooth manifolds. Suppose that $\partial N = \varnothing$. Then

1. $\pi_M : M \times N \to M$ is smooth,
2. $\pi_N : M \times N \to N$ is smooth.

Proof.

1. Set $m = \dim M$ and $n = \dim N$.
   Let $(p, q) \in M \times N$. Then there exists $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $q \in V$.
   Define $F := \pi_M$, $\eta := \lambda_0 \circ (\phi \times \psi)$ and $F_0 := \phi \circ F \circ \eta|^{-1}_{(U \times V) \cap F^{-1}(W)}$. We note that for each $(a^1, \dots, a^{m-1}, b^1, \dots, b^n, a^m) \in \lambda_0[\phi \times \psi(U \times V \cap F^{-1}(W))]$,
   $$\begin{aligned} F_0(a^1, \dots, a^{m-1}, b^1, \dots, b^n, a^m) &= \chi \circ F \circ \eta|^{-1}_{(U \times V) \cap \mathrm{proj}_1^{-1}(W)}(a^1, \dots, a^{m-1}, b^1, \dots, b^n, a^m) \\ &= \mathrm{id}_{\mathbb{H}^m} \circ \pi_M \circ \lambda_0^{-1} \\ &= (a^1, \dots, a^{m-1}, b^1, \dots, b^n, a^m) \\ &= \mathrm{id}_{\mathbb{H}^{m+n}}(a^1, \dots, a^{m-1}, b^1, \dots, b^n, a^m) \end{aligned}$$
   Hence F_0 is smooth. Exercise 5.2.0.1 implies that λ_0 is smooth. Similarly, λ_0^{-1} is smooth. Thus λ_0 is a diffeomorphism.
   Let (U, ϕ), $(U', \phi') \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$. Then for each $(a, b) \in \phi(U) \times \psi(V)$
   $$\begin{aligned} \phi'|_{U' \cap U} \circ \pi_M \circ [\phi \times \psi]^{-1}|_{\phi(U) \times \psi(V)}(a, b) &= \phi'|_{U' \cap U} \circ \pi_M \circ [\phi|^{-1}_{\phi(U)} \times \psi|^{-1}_{\psi(V)}](a, b) \\ &= \phi' \circ \phi^{-1}(a) \\ &= (\phi' \circ \phi^{-1}) \circ \mathrm{proj}_1(a, b) \end{aligned}$$
   Since $(a, b) \in \phi(U) \times \psi(V)$ is arbitrary,
   $$\phi'|_{U' \cap U} \circ \pi_M \circ [\phi \times \psi]^{-1}|_{\phi(U \cap U') \times \psi(V)} = \phi'|_{U' \cap U} \circ \phi|^{-1}_{U' \cap U} \circ \mathrm{proj}_1|_{\phi(U \cap U') \times \psi(V)}$$
   where $\mathrm{proj}_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the usual projection map. Since (U, ϕ), $(U', \phi') \in \mathcal{A}_M$, (U, ϕ) and (U', ϕ') are smoothly compatible. Hence $\phi'|_{U \cap U'} \circ \phi|^{-1}_{U \cap U'}$ is smooth. Since proj_1 is smooth need to show smooth functions in the calculus sense are smooth in the manifold sense, what does it mean for a projection to be smooth?, BIG ISSSUE, may need to define differentiation on product spaces in calculus section and redo product manifold stuff, therefore $\phi'|_{U' \cap U} \circ \pi_M \circ [\phi \times \psi]^{-1}|_{\phi(U) \times \psi(V)}$ is smooth. Since fix here and $(V, \psi) \in \mathcal{A}_N$ are arbitrary, we have that $\pi_M : M \times N \to M$ is smooth. we have that (U, ϕ) and (U', ϕ') are smoothly compatible. Thus $\phi'|_{U \cap U'} \circ \phi^{-1}|^{-1}_{U \cap U'}$ is smooth. FINISH!!!

2. Similar to (1).

□

Exercise 5.3.0.8. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ and $(E, \mathcal{C})$ be smooth manifolds and $F : E \to M \times N$. Then F is smooth iff $\pi_M \circ F$ is smooth and $\pi_N \circ F$ is smooth.

Proof.

- ($\Longrightarrow$):
  Suppose that F is smooth.
- ($\Longleftarrow$):

□

Definition 5.3.0.9. Let M and N be smooth manifolds and $(p, q) \in M \times N$. We define the **slice maps at** q **and** p, denoted by $\iota_q^M : M \to M \times N$ and $\iota_p^N : N \to M \times N$ respectively, by

- $\iota_q^M(a) = (a, q)$
- $\iota_p^N(b) = (p, b)$

Exercise 5.3.0.10. Let M and N be smooth manifolds and $(p, q) \in M \times N$. Then

1. $\iota_q^M : M \to M \times N$ is smooth,
2. $\iota_p^N : N \to M \times N$ is smooth.

Proof. Let () □

5.4 Partitions of Unity

Definition 5.4.0.1. Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then ρ is said to be a **bump function at p supported in** U if

1. $\rho \geq 0$
2. there exists $V \in \mathcal{N}_p$ such that V is open and $\rho|_V = 1$
3. $\operatorname{supp} \rho \subset U$

Exercise 5.4.0.2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

Proof. □

5.5 Smooth Functions on Manifolds

Definition 5.5.0.1. Let $(M, \mathcal{A})$ be a smooth manifold and $f : M \to \mathbb{R}$. Then f is said to be **smooth** if for each $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M, \mathcal{A})$.

Note 5.5.0.2. When the context is clear, we write $C^\infty(M)$ in place of $C^\infty(M, \mathcal{A})$.

Exercise 5.5.0.3. Let $(M, \mathcal{A})$ be a smooth manifold and $f : M \to \mathbb{R}$. Then f is smooth iff f is $(\mathcal{A}, \mathcal{A}_\mathbb{R})$-smooth.

Proof.

- $(\implies)$:
  Suppose that f is smooth. Let $(U, \phi) \in \mathcal{A}$. Since $\mathrm{id}_\mathbb{R} \circ f \circ \phi^{-1} = f \circ \phi^{-1}$ and $f \circ \phi^{-1}$ is smooth, we have that $\mathrm{id}_\mathbb{R} \circ f \circ \phi^{-1}$ is smooth. Since $\mathcal{A} = \alpha(\mathcal{A})$ and $\mathcal{A}_\mathbb{R} = \alpha((\mathbb{R}, \mathrm{id}_\mathbb{R}))$, an exercise in the section on smooth maps implies that f is $(\mathcal{A}, \mathcal{A}_\mathbb{R})$-smooth.

- $(\impliedby)$:
  Suppose that f is $(\mathcal{A}, \mathcal{A}_\mathbb{R})$-smooth. Let $(U, \phi) \in \mathcal{A}$. Since $(\mathbb{R}, \mathrm{id}_\mathbb{R}) \in \mathcal{A}_\mathbb{R}$ and $f \circ \phi^{-1} = \mathrm{id}_\mathbb{R} \circ f \circ \phi^{-1}$, we have that $f \circ \phi^{-1}$ is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, we have that f is smooth.

□

Note 5.5.0.4. When the context is clear, we write $C^\infty(M, \mathcal{A})$ in place of $C^\infty(M)$.

Exercise 5.5.0.5. Let $(M, \mathcal{A})$ be a smooth manifold, $\mathcal{A}_0 \subset \mathcal{A}$. Suppose that $\mathcal{A}_0$ is an atlas on M and $f : M \to \mathbb{R}$. Then f is smooth iff for each $(U, \phi) \in \mathcal{A}_0$, $f \circ \phi^{-1}$ is smooth.

Proof.

- $(\implies)$:
  Suppose that f is smooth. Let $(U, \phi) \in \mathcal{A}_0$. Since $\mathcal{A}_0 \subset \mathcal{A}$, $(U, \phi) \in \mathcal{A}$. Since f is smooth, $f \circ \phi^{-1}$ is smooth. Since $(U, \phi) \in \mathcal{A}_0$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{A}_0$, $f \circ \phi^{-1}$ is smooth.

- $(\impliedby)$:
  Suppose that for each $(U, \phi) \in \mathcal{A}_0$, $f \circ \phi^{-1}$ is smooth. Then for each $(U, \phi) \in \mathcal{A}_0$, $\mathrm{id}_\mathbb{R} \circ f \circ \phi^{-1}$ is smooth. Since $\mathcal{A} = \alpha(\mathcal{A}_0)$ and $\mathcal{A}_\mathbb{R} = \alpha(\mathbb{R}, \mathrm{id}_\mathbb{R})$, an exercise in the section on smooth maps implies that f is $(\mathcal{A}, \mathcal{A}_\mathbb{R})$-smooth. A previous exercise implies that f is smooth.

□

Exercise 5.5.0.6. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then F is smooth iff F is continuous and for each $g \in C^\infty(N)$, $g \circ F$ is smooth.

Proof.

- $(\implies)$:
  Suppose that F is smooth. Then F is continuous. Let $g \in C^\infty(N)$. Then $g \circ F$ is smooth. Since $g \in C^\infty(N)$ is arbitrary, we have that for each $g \in C^\infty(N)$, $g \circ F$ is smooth.

- $(\impliedby)$:
  Suppose that F is continuous and for each $g \in C^\infty(N)$, $g \circ F$ is smooth. Let $p \in U$.

  Let $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$. Set $W = U \cap F^{-1}(V)$. Since F is continuous, W is open in M. Define $G : W \to V$ by $G := F|_W$. FINISH!!!, maybe use bump functions to go from a smooth g on V to N

□

Exercise 5.5.0.7. Let M be a smooth manifold. Then $C^\infty(M)$ is a vector space.

Proof. Let $f, g \in C^\infty(M)$, $\lambda \in \mathbb{R}$ and $(U, \phi) \in \mathcal{A}$. By assumption, $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are smooth. Hence

$$(f + \lambda g) \circ \phi^{-1} = f \circ \phi^{-1} + \lambda g \circ \phi^{-1}$$

is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, $f + \lambda g \in C^\infty(M)$. Since $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$ are arbitrary, $C^\infty(M)$ is a vector space. $\square$

Definition 5.5.0.8. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. We define the **partial derivative of f with respect to x^i**, denoted

$$\partial f / \partial x^i : U \to \mathbb{R} \ \text{ or } \ \partial_i f : U \to \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left(\frac{\partial}{\partial u^i}[f \circ \phi^{-1}] \right) \circ \phi$$

Exercise 5.5.0.9. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. Then $\partial / \partial x^i : C^\infty(U) \to C^\infty(U)$ is linear.

Proof. **FINISH!!!** $\square$

Exercise 5.5.0.10. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j}[f \circ \phi^{-1}] \right) \circ \phi$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} f \right) \\
&= \frac{\partial}{\partial x^i} \left(\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}] \right] \circ \phi \right) \\
&= \left(\frac{\partial}{\partial u^i} \left[\left(\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}] \right] \circ \phi \right) \circ \phi^{-1} \right] \right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i} \left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}] \right] \right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j}[f \circ \phi^{-1}] \right) \circ \phi
\end{aligned}$$

$\square$

Exercise 5.5.0.11. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i}$$

Proof. Let $f \in C^\infty(U)$. Since $f \circ \phi^{-1}$ is smooth,

$$\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j}[f \circ \phi^{-1}] = \frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i}[f \circ \phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}f &= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f\circ\phi^{-1}]\right)\circ\phi \\ &= \left(\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f\circ\phi^{-1}]\right)\circ\phi \\ &= \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}f\end{aligned}$$

□

Exercise 5.5.0.12. Let $(M,\mathcal{A})$ be a smooth manifold, $(U,\phi)\in\mathcal{A}$ with $\phi=(x^1,\cdots,x^n)$ and $f\in C^\infty(U)$. Then for each $\alpha\in\mathbb{N}_0^n$,

$$\partial^\alpha f = (\partial^\alpha[f\circ\phi^{-1}])\circ\phi$$

Proof. The claim is clearly true when $|\alpha|=0$ or by definition if $|\alpha|=1$. Let $n\in\mathbb{N}$ and suppose the claim is true for each $|\alpha|\in\{1,\dots,n-1\}$. Then there exists $i\in\{1,\dots,n\}$ such that $\alpha_i\geq 1$. Hence

$$\begin{aligned}\partial^\alpha f &= \partial^{e^i}(\partial^{\alpha-e^i}f) \\ &= \partial^{e^i}(\partial^{\alpha-e^i}[f\circ\phi^{-1}]\circ\phi) \\ &= (\partial^{e^i}[(\partial^{\alpha-e^i}[f\circ\phi^{-1}]\circ\phi)\circ\phi^{-1}])\circ\phi \\ &= (\partial^{e^i}[\partial^{\alpha-e^i}[f\circ\phi^{-1}]])\circ\phi \\ &= (\partial^\alpha[f\circ\phi^{-1}])\circ\phi\end{aligned}$$

□

Exercise 5.5.0.13. Taylor's Theorem:
Let $(M,\mathcal{A})$ be a smooth manifold, $(U,\phi)\in\mathcal{A}$ with $\phi=(x^1,\dots,x^n)$ and $\phi(U)$ convex, $p\in U$, $f\in C^\infty(U)$ and $T\in\mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1}\subset C^\infty(U)$ such that

$$f=\sum_{k=0}^{T}\left[\sum_{|\alpha|=k}(x-p)^\alpha\partial^\alpha f(x_0)\right]+\sum_{|\alpha|=T+1}(x^i-x^i(p))^\alpha g_\alpha$$

and for each $|\alpha|=T+1$,

$$g_\alpha(p)=\frac{1}{(T+1)!}\partial^\alpha f(p)$$

Proof. Since $\phi(U)$ is open and convex and $f\circ\phi^{-1}\in C^\infty(\phi(U))$, Taylors thorem in section 2.1 implies that there exist $(\tilde{g}_\alpha)_{|\alpha|=T+1}\subset C^\infty(\phi(U))$ such that for each $q\in U$,

$$f\circ\phi^{-1}(\phi(q))=\sum_{k=0}^{T}\left[\sum_{|\alpha|=k}(x^i(q)-x^i(p))^\alpha\partial^\alpha[f\circ\phi^{-1}](\phi(p))\right]+\sum_{|\alpha|=T+1}(x^i(q)-x^i(p))^\alpha\tilde{g}_\alpha(\phi(q))$$

and for each $|\alpha|=T+1$,

$$\begin{aligned}\tilde{g}_\alpha(\phi(p)) &= \frac{1}{(T+1)!}\partial^\alpha[f\circ\phi^{-1}](\phi(p)) \\ &= \frac{1}{(T+1)!}\partial^\alpha f(p)\end{aligned}$$

For $|\alpha| = T+1$, set $g_\alpha = \tilde{g} \circ \phi$. Then

$$\begin{aligned} f(q) &= f \circ \phi^{-1}(\phi(q)) \\ &= \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\ &= \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q) \end{aligned}$$

$\square$

Chapter 6

The Tangent and Cotangent Spaces

6.1 The Tangent Space

6.1.1 Introduction

Definition 6.1.1.1. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. For $i \in \{1, \cdots, n\}$, define the partial derivative with respect to x^i at p, denoted

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(M) \to \mathbb{R}, \text{ or } \partial_i|_p : C^\infty(M) \to \mathbb{R}$$

by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p)$$

Exercise 6.1.1.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$, we have that

$$\frac{\partial}{\partial x^i} x^j(p) = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \cdots, n\}$. Then

$$\begin{aligned}
\left.\frac{\partial}{\partial x^i}\right|_p x^i &= \left.\frac{\partial}{\partial u^i}\right|_{\phi(p)} x^i \circ \phi^{-1} \\
&= \left.\frac{\partial}{\partial u^i}\right|_{\phi(p)} u^i \circ \phi \circ \phi^{-1} \\
&= \left.\frac{\partial}{\partial u^i}\right|_{\phi(p)} u^i \\
&= \delta_{i,j}
\end{aligned}$$

□

Exercise 6.1.1.3. Change of Coordinates:
Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $\psi = (y^1, \cdots, y^n)$, $p \in U \cap V$ and $f \in C^\infty(M)$. Then for each $i \in \{1, \cdots, n\}$,

$$\left.\frac{\partial}{\partial y^i}\right|_p = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i}(p) \left.\frac{\partial}{\partial x^j}\right|_p.$$

Proof. Let $f \in C^\infty(M)$. Set $h := \phi \circ \psi^{-1}$ and write $h = (h^1, \cdots, h^n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned}
\frac{\partial}{\partial y^i}\bigg|_p f &= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \psi^{-1} \\
&= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \phi^{-1} \circ h \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial u^j}\bigg|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left(\frac{\partial}{\partial u^i}\bigg|_{\psi(p)} h^j \right) \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial u^j}\bigg|_{\phi(p)} f \circ \phi^{-1} \right) \left(\frac{\partial}{\partial u^i}\bigg|_{\psi(p)} x^j \circ \psi^{-1} \right) \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial x^j}\bigg|_p f \right) \left(\frac{\partial}{\partial y^i}\bigg|_p x^j \right) \\
&= \left[\sum_{j=1}^n \frac{\partial x^j}{\partial y^i}(p) \frac{\partial}{\partial x^j}\bigg|_p \right] f
\end{aligned}$$

Since $f \in C^\infty(M)$ is arbitrary, we have that

$$\frac{\partial}{\partial y^i}\bigg|_p = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i}(p) \frac{\partial}{\partial x^j}\bigg|_p .$$

□

Definition 6.1.1.4. Let $p \in M$ and $v : C^\infty(M) \to \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation on** $C^\infty(M)$ **at** p if for each $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,

1. v is linear
2. v is Leibnizian

We define the **tangent space of** M **at** p, denoted T_pM, by

$$T_pM = \{v : C^\infty(M) \to \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 6.1.1.5. T_pM is a vector space

Proof. content... □

Exercise 6.1.1.6. Let $f \in C^\infty(M)$ and $v \in T_pM$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. □

Exercise 6.1.1.7. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

is a basis for T_pM and $\dim T_pM = n$.

Proof. Clearly $\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \in T_pM$. Let $a_1, \cdots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p x^j \\ &= a_j \end{aligned}$$

Hence $\left\{\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ is independent.
Now, let $v \in T_pM$ and $f \in \mathbb{C}^\infty(M)$. By Taylor's theorem, there exist $g_1, \cdots g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x^i}\bigg|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p f \\ &= \left[\sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p\right] f \end{aligned}$$

So

$$v = \sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p$$

and

$$v \in \text{span}\left\{\frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p\right\}$$

□

Definition 6.1.1.8. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. We define the **derivative of F at p**, denoted $DF_p : T_pM \to T_{F(p)}N$, by

$$\Big[DF_p(v)\Big](f) = v(f \circ F)$$

for $v \in T_pM$ and $f \in C^\infty(N)$.

Exercise 6.1.1.9. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. Then for each $v \in T_pM$, $DF_p(v)$ is a derivation.

Proof. Let $v \in T_pM$, $f, g \in C^\infty_{F(p)}(N)$ and $c \in \mathbb{R}$. Then

1. $$\begin{aligned} DF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= DF_p(v)(f) + cDF_p(v)(g) \end{aligned}$$

   So $DF_p(v)$ is linear.

2. $$\begin{aligned} DF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= DF_p(v)(f) * g(F(p)) + f(F(p)) * DF_p(v)(g) \end{aligned}$$

So $DF_p(v)$ is Leibnizian and hence $DF_p(v) \in T_{F(p)}N$ □

Exercise 6.1.1.10. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. If F is a diffeomorphism, then DF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \cdots, x^n)$ and $\phi \circ F^{-1} = (y^1, \cdots, y^n)$. Let $f \in C^\infty(N)$ Then

$$\begin{aligned} \frac{\partial}{\partial y^i}\bigg|_{F(p)} f &= \frac{\partial}{\partial u^i}\bigg|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \frac{\partial}{\partial x^i}\bigg|_{p} f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[DF(p)\left(\frac{\partial}{\partial x^i}\bigg|_p\right)\right](f) &= \frac{\partial}{\partial x^i}\bigg|_p f \circ F \\ &= \frac{\partial}{\partial y^i}\bigg|_{F(p)} f \end{aligned}$$

Hence

$$DF(p)\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial y^i}\bigg|_{F(p)}$$

Since $\left\{\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ is a basis for T_pM and $\left\{\frac{\partial}{\partial y^1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}\Big|_{F(p)}\right\}$ is a basis for $T_{F(p)}N$, $DF(p)$ is an isomorphism. □

Exercise 6.1.1.11. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $(U,\phi) \in \mathcal{A}_M$ and $p \in U$. Write $\phi = (x^1, \dots, x^n)$. Then for each $j \in [n]$,

$$D\phi(p)\bigg(\frac{\partial}{\partial x^j}\bigg|_p\bigg) = \frac{\partial}{\partial u^j}\bigg|_{\phi(p)}$$

Proof. Let $j \in [n]$, $f \in C^\infty_{\phi(p)}(\phi(U))$. Then

$$\begin{aligned} D\phi(p)\bigg(\frac{\partial}{\partial x^j}\bigg|_p\bigg)(f) &= \frac{\partial}{\partial x^j}\bigg|_p\bigg[f \circ \phi\bigg] \\ &= \frac{\partial}{\partial u^j}\bigg|_{\phi(p)}\bigg[f \circ \phi \circ \phi^{-1}\bigg] \\ &= \frac{\partial}{\partial u^j}\bigg|_{\phi(p)}(f). \end{aligned}$$

Since $f \in C^\infty_{\phi(p)}(\phi(U))$ is arbitrary, we have that for each $f \in C^\infty_{\phi(p)}(\phi(U))$,

$$D\phi(p)\bigg(\frac{\partial}{\partial x^j}\bigg|_p\bigg)(f) = \frac{\partial}{\partial u^j}\bigg|_{\phi(p)}(f).$$

Thus

$$D\phi(p)\bigg(\frac{\partial}{\partial x^j}\bigg|_p\bigg) = \frac{\partial}{\partial u^j}\bigg|_{\phi(p)}.$$

□

Exercise 6.1.1.12. Let $(M,\mathcal{A})$ be a smooth m-dimensional manifold, $(N,\mathcal{B})$ a n-dimensional smooth manifold, $F : M \to N$ smooth, $(U,\phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^m)$ and $(V,\psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. Suppose that $p \in U$ and $F(p) \in V$. Define the ordered bases $B_\phi = \left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^m}\Big|_p \right\}$ and $B_\psi = \left\{ \frac{\partial}{\partial y^1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}\Big|_{F(p)} \right\}$. Then the matrix representation of DF_p with respect to the bases B_ϕ and B_ψ is

$$([DF(p)]_{\phi,\psi})_{j,k} = \frac{\partial (y^j \circ F)}{\partial x^k}(p)$$

Proof. Let $[DF(p)]_{\phi,\psi} = (a_{j,k})_{j,k} \in \mathbb{R}^{n \times m}$. Then for each $k \in [n]$,

$$DF(p)\bigg(\frac{\partial}{\partial x^k}\bigg|_p\bigg) = \sum_{j=1}^n a_{j,k}\frac{\partial}{\partial y^j}\bigg|_{F(p)}$$

This implies that for each $k, l \in [n]$,

$$\begin{aligned} DF(p)\bigg(\frac{\partial}{\partial x^k}\bigg|_p\bigg)(y^l) &= \sum_{j=1}^n a_{j,k}\frac{\partial}{\partial y^j}\bigg|_{F(p)}(y^l) \\ &= \sum_{j=1}^n a_{j,k}\delta_{j,l} \\ &= a_{l,k} \end{aligned}$$

By definition,

$$\begin{aligned} a_{j,k} &= DF_p\bigg(\frac{\partial}{\partial x^k}\bigg|_p\bigg)(y^j) \\ &= \frac{\partial}{\partial x^k}\bigg|_p(y^j \circ F) \\ &= \frac{\partial (y^j \circ F)}{\partial x^k}(p). \end{aligned}$$

□

Note 6.1.1.13. Since rank DF_p is independent of basis, it is independent of coordinate charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$.

Exercise 6.1.1.14. need exercise giving $\sigma\phi$ has derivative $P_\sigma D\phi$.

Exercise 6.1.1.15.

6.1.2 Tangent Space and Product Manifolds

Exercise 6.1.2.1. Let $M, N \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. Set $m := \dim M$ and $n := \dim N$. Let $(U_M, \phi_M) \in \mathcal{A}_M$ and $(U_N \phi_N) \in \mathcal{A}_N$. Write $\phi_M = (x^1, \ldots, x^m)$ and $\phi_N = (y^1, \ldots, y^n)$. Define $\phi \in \mathcal{A}_M \otimes \mathcal{A}_N$ by $\phi := \phi_M \times \phi_N$. Write $\phi = (\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{y}^1, \ldots, \tilde{y}^n)$. Then

1. for each $j \in [m]$, $k \in [n]$ and $(p, q) \in M \times N$,

$$\frac{\partial}{\partial \tilde{x}^k}\bigg|_{(p,q)} (x^j \circ \pi_M) = \frac{\partial}{\partial x^k}\bigg|_p (x^j), \qquad \frac{\partial}{\partial \tilde{y}^k}\bigg|_{(p,q)} (x^j \circ \pi_M) = 0,$$
$$\frac{\partial}{\partial \tilde{x}^k}\bigg|_{(p,q)} (y^j \circ \pi_N) = 0, \qquad \frac{\partial}{\partial \tilde{y}^k}\bigg|_{(p,q)} (y^j \circ \pi_N) = \frac{\partial}{\partial y^k}\bigg|_q (y^j).$$

2. $[D\pi_M(p,q)]_{\phi_M, \phi} = \begin{pmatrix} I_m & 0 \end{pmatrix}$ and $[D\pi_N(p,q)]_{\phi_N, \phi} = \begin{pmatrix} 0 & I_n \end{pmatrix}$

Proof.

1. Let $j \in [m]$, $k \in [n]$ and $(p, q) \in M \times N$. Let $(u^i, v^j) \in \mathbb{R}^{m+n}$ denote the usual coordinates with $(e^j)_j, (f^k)_k$ the standard bases (use wording used elsewhere). Then Exercise **??** implies that

$$\begin{aligned}
\frac{\partial}{\partial \tilde{x}^k}\bigg|_{(p,q)} (x^j \circ \pi_M) &= \frac{\partial}{\partial u^k}\bigg|_{\phi(p,q)} (x^j \circ \pi_M \circ \phi^{-1}) \\
&= \frac{\partial}{\partial u^k}\bigg|_{\phi(p,q)} (x^j \circ \phi_M^{-1} \circ \mathrm{proj}_{[m]}) \\
&= \sum_{l=1}^m \frac{\partial (x^j \circ \phi_M^{-1})}{\partial u^l}(\phi_M(p)) \frac{\partial (u^l \circ \mathrm{proj}_{[m]})}{\partial u^k}(\phi(p,q)) \\
&= \sum_{l=1}^m \frac{\partial (x^j \circ \phi_M^{-1})}{\partial u^l}(\phi_M(p)) \delta_{l,k} \\
&= \frac{\partial (x^j \circ \phi_M^{-1})}{\partial u^k}(\phi_M(p)) \\
&= \frac{\partial}{\partial u^k}\bigg|_{\phi_M(p)} x^j \circ \phi_M^{-1} \\
&= \frac{\partial}{\partial x^k}\bigg|_p x^j
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \tilde{y}^k}\bigg|_{(p,q)}(x^j \circ \pi_M) &= \frac{\partial}{\partial v^k}\bigg|_{\phi(p,q)}(x^j \circ \pi_M \circ \phi^{-1}) \\
&= \frac{\partial}{\partial v^k}\bigg|_{\phi(p,q)}(x^j \circ \phi_M^{-1} \circ \mathrm{proj}_{[m]}) \\
&= \sum_{l=1}^{m} \frac{\partial(x^j \circ \phi_M^{-1})}{\partial u^l}(\phi_M(p)) \frac{\partial(u^l \circ \mathrm{proj}_{[m]})}{\partial v^k}(\phi(p,q)) \\
&= \sum_{l=1}^{m} \frac{\partial(x^j \circ \phi_M^{-1})}{\partial u^l}(\phi_M(p))0 \\
&= 0
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial \tilde{x}^k}\bigg|_{(p,q)}(y^j \circ \pi_N) = 0, \quad \text{and} \quad \frac{\partial}{\partial \tilde{y}^k}\bigg|_{(p,q)}(y^j \circ \pi_N) = \frac{\partial}{\partial y^k}\bigg|_{q}(y^j)$$

2. The previous part implies that

$$\begin{aligned}
([D\pi_M(p,q)]_{\phi_M,\phi})_{j,k} &= \left(\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_{(p,q)}(x^i \circ \pi_M)\right)_{i,j} \quad \left(\frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,q)}(x^i \circ \pi_M)\right)_{i,j}\right) \\
&= \begin{pmatrix}
\frac{\partial}{\partial x^1}\big|_p (x^1) & \cdots & \frac{\partial}{\partial x^m}\big|_p (x^1) & 0 & \cdots & 0 \\
 & & \vdots & & & \\
\frac{\partial}{\partial x^1}\big|_p (x^m) & \cdots & \frac{\partial}{\partial x^m}\big|_p (x^m) & 0 & \cdots & 0
\end{pmatrix} \\
&= \begin{pmatrix} I_m & 0 \end{pmatrix}.
\end{aligned}$$

Similarly, $([D\pi_N(p,q)]_{\phi_N,\phi})_{j,k} = \begin{pmatrix} 0 & I_n \end{pmatrix}$.

$\square$

Exercise 6.1.2.2. Let $M, N \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $p \in M$ and $q \in N$. Set $m := \dim M$ and $n := \dim N$. Define $\alpha \in \mathrm{Hom}_{\mathbf{Vect}_\mathbb{R}}(T_{(p,q)}(M \times N), T_pM \times T_qN)$ by $\alpha := (D\pi_M(p,q), D\pi_N(p,q))$. Then

1. Let $(U_M, \phi_M) \in \mathcal{A}_M$ and $(U_N\phi_N) \in \mathcal{A}_N$. Write $\phi_M = (x^1, \dots, x^m)$ and $\phi_N = (y^1, \dots, y^n)$. Define $(U, \phi) \in \mathcal{A}_M \otimes \mathcal{A}_N$ by $U := U_M \times U_N$ and $\phi := \phi_M \times \phi_N$. Write $\phi = (\tilde{x}^1, \dots, \tilde{x}^m, \tilde{y}^1, \dots, \tilde{y}^n)$. Then for each $j \in [m]$ and $k \in [n]$,

$$\alpha\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_{(p,q)}\right) = \left(\frac{\partial}{\partial x^j}\bigg|_p, 0\right), \qquad \alpha\left(\frac{\partial}{\partial \tilde{y}^k}\bigg|_{(p,q)}\right) = \left(0, \frac{\partial}{\partial y^j}\bigg|_p\right)$$

2. $\alpha \in \mathrm{Iso}_{\mathbf{Vect}_\mathbb{R}}(T_{p,q}(M \times N), T_pM \times T_qN)$.

Proof.

1. Clear by previous exercise

2. The previous part implies that $\mathrm{Im}\,\alpha = T_pM \oplus T_qN$ and α is surjective. Since

$$\begin{aligned}
\dim T_{(p,q)}(M \times N) &= m + n \\
&= \dim(T_pM \oplus T_qN),
\end{aligned}$$

we have that α is surjective and therefore α is an isomorphism and $\alpha \in \mathrm{Iso}_{\mathbf{Vect}_\mathbb{R}}(T_{p,q}(M \times N), T_pM \times T_qN)$.

$\square$

Exercise 6.1.2.3. there exists $\alpha \in \mathrm{Iso}_{\mathbf{Vect}_{\mathbb{R}}}(T_{p,q}(M \times N), T_pM \times T_qN)$ such that $\alpha\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_{(p,q)}\right) = \left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_{(p,q)}\right)$ i.e. the following diagram commutes:

6.2 The Cotangent Space

Definition 6.2.0.1. Let $p \in M$. We define the **cotangent space of** M **at** p, denoted T_p^*M, by

$$T_p^*M := (T_pM)^*$$

Definition 6.2.0.2. Let $f \in C^\infty(M)$. We define the **differential of** f **at** p, denoted $df_p : T_pM \to \mathbb{R}$, by

$$df_p(v) = v(f)$$

Exercise 6.2.0.3. Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

Proof. Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1 f + \lambda v_2 f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that df_p is linear and hence $df_p \in T_p^*M$. □

Exercise 6.2.0.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$,

$$dx_p^i \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta_{i,j}$$

In particular, $\{dx_p^1, \cdots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ and $T_p^*M = \operatorname{span}\{dx_p^1, \cdots, dx_p^n\}$.

Proof. Let $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned} \left[dx_p^i \left(\frac{\partial}{\partial x^i}\bigg|_p \right) \right]_p &= \frac{\partial}{\partial x^i}\bigg|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 6.2.0.5. Let $f \in C^\infty(M)$, (U, ϕ) a chart on M with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i{}_p$$

Proof. Since $\{dx_p^1, \cdots, dx_p^n\}$ is a basis for T_p^*M, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left(\frac{\partial}{\partial x^i}\bigg|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= \frac{\partial}{\partial x^i}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p) dx_p^i$$

$\square$

Chapter 7

Immersions and Submersions

7.1 Maps of Constant Rank

Do this section assuming $\partial M, \partial N = \varnothing$

Definition 7.1.0.1. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. We define the **rank map of** F, denoted $\operatorname{rank} F : M \to \mathbb{N}_0$ by

$$\operatorname{rank}_p F = \dim \operatorname{Im} DF(p)$$

and F is said to have **constant rank** if for each $p, q \in M$, $\operatorname{rank}_p F = \operatorname{rank}_q F$. If F has constant rank, we define the **rank of** F, denoted $\operatorname{rank} F$, by $\operatorname{rank} F = \operatorname{rank}_p F$ for $p \in M$.

Exercise 7.1.0.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$ and $p \in M$. Suppose that $\partial N = \varnothing$ and $\operatorname{rank}_p F = k$. Then there exist $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi,\psi})_{i,j} = A_{i,j}$$

Does the boundary need to be empty?

Proof. Define $q \in V$ by $q = F(p)$. Choose $(U, \phi') \in \mathcal{A}$ and $(V, \psi') \in \mathcal{B}$ such that $p \in U$, $q \in V$. Since $\partial N = \varnothing$, $\phi'(U) \subset \operatorname{Int} \mathbb{H}^m_j$ and $\psi'(V) \subset \operatorname{Int} \mathbb{H}^n_k$. Set $Z = [DF(p)]_{\phi',\psi'}$. By assumption, $\operatorname{rank} Z = k$. Exercise 1.2.0.9 implies that there exist $\sigma \in S_m$, $\tau \in S_n$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P^*_\sigma)_{i,j} = A_{i,j}$$

Define $\phi : U \to (\sigma \cdot \phi')(U)$ and $\psi : V \to (\tau \cdot \psi')(V)$ by

$$\phi = \sigma \cdot \phi', \quad \psi = \tau \cdot \psi'$$

Exercise 4.1.0.7 implies that $(U, \phi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$ and Exercise 1.3.3.3 implies that

$$[DF(p)]_{\phi,\psi} = P_\tau Z P^*_\sigma$$

$\square$

Exercise 7.1.0.3. Local Rank Theorem:
rework for $\mathbb{H}^m$ instead of $\mathbb{R}^m$ Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$. Suppose that $\partial M, \partial N = \varnothing$, F has constant rank and $\operatorname{rank} F = k$. Then for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(U) \subset V$ and

$$\psi \circ F \circ \phi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

Hint: Needs a hint

Proof. Let $p \in M$. The previous exercise implies that there exist $(U_0, \phi_0) \in \mathcal{A}$, $(V_0, \psi_0) \in \mathcal{B}$ and $L \in GL(k, \mathbb{R})$ such that $p \in U$, $F(p) \in V_0$ and for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi_0, \psi_0})_{i,j} = L_{i,j}$$

Define $\hat{M} \subset \mathbb{R}^m$, $\hat{N} \subset \mathbb{R}^n$ and $\hat{F} : \hat{M} \to \hat{N}$ by $\hat{M} := \phi_0(U_0)$, $\hat{N} := \psi_0(V_0)$ and $\hat{F} := \psi_0 \circ F \circ \phi_0^{-1}$. Set $\hat{p} := \phi_0(p)$. Let (x, y) be the standard coordinates on $\mathbb{R}^m$, with $\pi_x : \mathbb{R}^m \to \mathbb{R}^k$ and $\pi_y : \mathbb{R}^m \to \mathbb{R}^{m-k}$ the standard projection maps. Write $\hat{p} = (x_0, y_0)$. There exist $Q : \hat{M} \to \mathbb{R}^k$ and $R : \hat{M} \to \mathbb{R}^{n-k}$ such that $\hat{F} = (Q, R)$. By construction, $[D_x Q(x_0, y_0)] = L$. Define $G : \hat{M} \to \mathbb{R}^m$ by $G(x, y) := (Q(x, y), y)$. Then

$$\begin{aligned}
[DG(x_0, y_0)] &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_x Q(x_0, y_0)] \\ [D_x \pi_y(x_0, y_0)] & [D_y \pi_y(x_0, y_0)] \end{pmatrix} \\
&= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} L & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\det([DG(x_0, y_0)]) &= \det(L)\det(I) \\
&= \det(L) \\
&\neq 0
\end{aligned}$$

The inverse function theorem implies that there exist $\hat{U} \subset \hat{M}$ such that $\hat{U}$ is open, $\hat{p} \in \hat{U}$ and $G|_{\hat{U}} : \hat{U} \to G(\hat{U})$ is a diffeomorphism. Since

$$\{U_1 \times U_2 : U_1 \subset \mathbb{R}^k, U_2 \subset \mathbb{R}^{m-k} \text{ and } U_1,\, U_2 \text{ are open}\}$$

is a basis for the topology on $\mathbb{R}^m$, there exist $\hat{U}_1 \subset \mathbb{R}^k$ and $\hat{U}_2 \subset \mathbb{R}^{m-k}$ such that $\hat{U}_1$, $\hat{U}_2$ are open, $\hat{p} \in \hat{U}_1 \times \hat{U}_2$ and $\hat{U}_1 \times \hat{U}_2 \subset \hat{U}$. Set $\hat{U}_{12} := \hat{U}_1 \times \hat{U}_2$ and define $G_{12} : \hat{U}_{12} \to Q(\hat{U}_{12}) \times \hat{U}_2$ by $G_{12} := G|_{\hat{U}_{12}}$. Since $G|_{\hat{U}} : \hat{U} \to G(\hat{U})$ is a diffeomorphism, $\hat{U}_{12} \subset \hat{U}$ and

$$\begin{aligned}
G(\hat{U}_{12}) &= G(\hat{U}_1 \times \hat{U}_2) \\
&= Q(\hat{U}_{12}) \times \hat{U}_2
\end{aligned}$$

we have that $G_{12} : \hat{U}_{12} \to Q(\hat{U}_{12}) \times \hat{U}_2$ is a diffeomorphism. Since G_{12} is a homeomorphism and π_x is open, $Q(\hat{U}_{12})$ is open. Since $G_{12}^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_{12}$, there exist $A : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_1$ and $B : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_2$ such that A, B are smooth and $G_{12}^{-1} = (A, B)$. Define $\tilde{R} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \mathbb{R}^{n-k}$ by $\tilde{R}(x, y) := R(A(x, y), y)$. Then $\tilde{R}$ is smooth. Let $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$. Then

$$\begin{aligned}
(x, y) &= G_{12} \circ G_{12}^{-1}(x, y) \\
&= G(A(x, y), B(x, y)) \\
&= (Q(A(x, y), B(x, y)), B(x, y))
\end{aligned}$$

This implies that $B(x, y) = y$,

$$\begin{aligned}
x &= Q(A(x, y), B(x, y)) \\
&= Q(A(x, y), y)
\end{aligned}$$

and

$$\begin{aligned}
G_{12}^{-1}(x, y) &= (A(x, y), B(x, y)) \\
&= (A(x, y), y)
\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{F}\circ G_{12}^{-1}(x,y) &= \hat{F}(A(x,y),y)\\ &= (Q(A(x,y),y),R(A(x,y),y))\\ &= (x,R(A(x,y),y))\\ &= (x,\tilde{R}(x,y))\end{aligned}$$

We note that

$$\begin{aligned}[D(\hat{F}\circ G_{12}^{-1})(x,y)] &= \begin{pmatrix}[D_x\pi_x(x,y)] & [D_y\pi_x(x,y)]\\ [D_x\tilde{R}(x,y)] & [D_y\tilde{R}(x,y)]\end{pmatrix}\\ &= \begin{pmatrix} I & 0\\ [D_x\tilde{R}(x,y)] & [D_y\tilde{R}(x,y)]\end{pmatrix}\end{aligned}$$

Since $G_{12}^{-1}:Q(\hat{U}_{12})\times\hat{U}_2\to\hat{U}_{12}$ is a diffeomorphism, we have that $[DG^{-1}(x,y)]\in GL(m,\mathbb{R})$. Since $\hat{F}$ has constant rank and $\operatorname{rank}\hat{F}=k$, we have that

$$\begin{aligned}\operatorname{rank}[D(\hat{F}\circ G_{12}^{-1})(x,y)] &= \operatorname{rank}([D\hat{F}(G_{12}^{-1}(x,y))][DG_{12}^{-1}(x,y)])\\ &= \operatorname{rank}[D\hat{F}(G_{12}^{-1}(x,y))]\\ &= k\end{aligned}$$

Since $\operatorname{rank}\begin{pmatrix} I\\ [D_x\tilde{R}(x,y)]\end{pmatrix}=k$, we have that $\operatorname{rank}\begin{pmatrix} 0\\ [D_y\tilde{R}(x,y)]\end{pmatrix}=0$. Thus $[D_y\tilde{R}(x,y)]=0$. Since $(x,y)\in Q(\hat{U}_{12})\times\hat{U}_2$ is arbitrary, for each $(x,y)\in Q(\hat{U}_{12})\times\hat{U}_2$,

$$\tilde{R}(x,y)=\tilde{R}(x,y_0)$$

Define $\tilde{S}:Q(\hat{U}_{12})\to\mathbb{R}^{n-k}$ by $\tilde{S}(x):=\tilde{R}(x,y_0)$. Then $\tilde{S}$ is smooth and for each $(x,y)\in Q(\hat{U}_{12})\times\hat{U}_2$,

$$\hat{F}\circ G_{12}^{-1}(x,y)=(x,\tilde{S}(x))$$

Let (a,b) be the standard coordinates on $\mathbb{R}^n$, with $\pi_a:\mathbb{R}^n\to\mathbb{R}^k$ and $\pi_b:\mathbb{R}^n\to\mathbb{R}^{n-k}$ the standard projection maps. Write $\hat{F}(\hat{p})=(a_0,b_0)$. Set

$$\begin{aligned}\hat{V}_{12} &:= \pi_a|_{\hat{N}}^{-1}(Q(\hat{U}_{12}))\\ &= \pi_a^{-1}(Q(\hat{U}_{12}))\cap\hat{N}\end{aligned}$$

Since $Q(\hat{U}_{12})$ is open, $\hat{N}$ is open and π_a is continuous, we have that $\hat{V}_{12}$ is open. Since

$$\begin{aligned}Q(\hat{U}_{12}) &= \pi_a|_{\hat{N}}\circ\hat{F}\circ G^{-1}(Q(\hat{U}_{12})\times\hat{U}_2)\\ &= \pi_a|_{\hat{N}}\circ\hat{F}(\hat{U}_{12})\end{aligned}$$

we have that

$$\begin{aligned}\hat{F}(\hat{U}_{12}) &\subset \pi_a|_{\hat{N}}^{-1}(Q(\hat{U}_{12}))\\ &\subset \hat{V}_{12}\end{aligned}$$

In particular, $\hat{F}(\hat{p})\in\hat{V}_{12}$. Define $H:Q(\hat{U}_{12})\times\mathbb{R}^{n-k}\to Q(\hat{U}_{12})\times\mathbb{R}^{n-k}$ by $H:=(\pi_a,\pi_b-\tilde{S}\circ\pi_a)$, i.e. for each $(a,b)\in Q(\hat{U}_{12})\times\mathbb{R}^{n-k}$, $H(a,b)=(a,b-\tilde{S}(a))$. Then H is a bijection and $H^{-1}(a,b)=(\pi_a,\pi_b+\tilde{S}\circ\pi_a)$. Thus H and H^{-1} are smooth and therefore H is a diffeomorphism. Define $H_{12}:\hat{V}_{12}\to H(\hat{V}_{12})$ by $H_{12}=H|_{\hat{V}_{12}}$. Then H_{12} is a diffeomorphism and for each $x,y\in Q(\hat{U}_{12}\times\hat{U}_2)$, $H_{12}\circ\hat{F}\circ G_{12}^{-1}(x,y)=(x,0)$. Define $(U,\phi)\in\mathcal{A}$

and $(V,\psi) \in \mathcal{B}$ by $U := \phi_0^{-1}(\hat{U}_{12})$, $V := \psi_0^{-1}(\hat{V}_{12})$, $\phi := G_{12} \circ \phi_0|_U$ and $\psi := H_{12} \circ \psi_0|_V$. Show that $F(U) \subset V$. Then for each $(x,y) \in \phi(U)$,

$$\begin{aligned}\psi \circ F \circ \phi^{-1}(x,y) &= H_{12} \circ \psi_0|_V \circ F \circ \phi_0|_U^{-1} \circ G_{12}^{-1}(x,y) \\ &= H_{12} \circ \hat{F} \circ G_{12}^{-1}(x,y) \\ &= (x,0)\end{aligned}$$

need to start with compact chart domain and add constant so we stay in $\mathbb{H}^n$, i.e. need U to be compact, so set U_1 and U_2 to be compact, then U_{12} will be and thus U. □

Exercise 7.1.0.4. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M,N)$. Suppose that $\dim M = m$ and $\dim N = n$, F has constant rank and $\operatorname{rank} F = r$. Then for each $p \in M$, there exist $(U,\phi) \in \mathcal{A}$ and $(V,\psi) \in \mathcal{B}$ such that $p \in U$, $F(\operatorname{cl} U) \subset V$, $\operatorname{cl} U$ is compact and

$$\psi \circ F \circ \phi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

Proof. content... □

Exercise 7.1.0.5. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M,N)$. Suppose that F has constant rank.

1.
2.
3.

Proof. Set $m := \dim M$, $n := \dim N$ and $r := \operatorname{rank} F$.

1. Let $p \in M$. The local rank theorem (Exercise 7.1.0.3) implies that there exists $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V,\psi) \in \mathcal{A}_N$ such that $p \in U$, $F(U) \subset V$ and $\psi \circ F \circ \phi_0^{-1} = (\mathrm{proj}^n_{[r]}, 0)$. Choose $\epsilon > 0$ such that $\bar{B}(\phi_0(p), \epsilon) \subset \phi(U)$. Set $U := \phi_0^{-1}(B(\phi_0(p), \epsilon))$. Since $\bar{B}(\phi_0(p), \epsilon)$ is compact, ϕ_0 is a homeomorphism and $\operatorname{cl} U = \phi_0^{-1}(\bar{B}(\phi_0(p), \epsilon))$, we have that $\operatorname{cl} U$ is compact and $\operatorname{cl} U \subset U_0$.
2.
3.

□

Exercise 7.1.0.6. Global Rank Theorem:
Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M,N)$. Suppose that F has constant rank.

1.
2.
3.

If F is surjective, then F is a $\mathbf{Man}^\infty$-submersion,

Proof. Set $m := \dim M$, $n := \dim N$ and $r := \operatorname{rank} F$. Suppose that F is surjective. For the sake of contradiction, suppose that F is not a $\mathbf{Man}^\infty$ submersion. Then $r < n$.

Let $p \in M$. The local rank theorem (Exercise 7.1.0.3) implies that there exists $(U,\phi) \in \mathcal{A}_M$ and $(V,\psi) \in \mathcal{A}_N$ such that $p \in U$, $F(U) \subset V$ and $\psi \circ F \circ \phi = (\mathrm{proj}^n_{[r]}, 0)$.

□

Proof. Set $m := \dim M$, $n := \dim N$ and $r := \operatorname{rank} F$.

1. Suppose that F is surjective. For the sake of contradiction, suppose that F is not a $\mathbf{Man}^\infty$-submersion. Then $r < n$.

2.

3.

□

Definition 7.1.0.7. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. Then F is said to be

- a **smooth immersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is injective
- a **smooth submersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is surjective

Exercise 7.1.0.8. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. Let $p \in M$.

1. If that $DF(p)$ is injective, then there exists $U \subset M$ such that U is open and $F|_U$ is a smooth immersion.
2. If $DF(p)$ is surjective, then there exists $U \subset M$ such that U is open and $F|_U$ is a smooth submersion.

Proof.

1. Suppose that $DF(p)$ is injective. Exercise 7.1.0.3 implies that there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$ and $([DF(p)]_{\phi,\psi})_{i,j}$
2. Similar to (1).

□

7.2 Immersions

Definition 7.2.0.1. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$ and $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$. Then F is said to be a **$\mathbf{ManBnd}^\infty$-immersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is injective.

Exercise 7.2.0.2. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$, $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$ and $p \in M$. If $DF(p)$ is injective, then there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $F|_U$ is a smooth immersion.

Proof. content... □

Exercise 7.2.0.3. Let $M, N \in \text{Obj}(\mathbf{Man}^\infty)$ and $F \in \text{Hom}_{\mathbf{Man}^\infty}(M, N)$. Define $U \subset M$ by $U := \{p \in M : \text{rank}\, DF(p) = \dim M\}$. Then

1. $U \in \mathcal{T}_M$,
2. $F|_U$ is a submersion.

Proof.
1. Let $p \in U$. Then $\text{rank}\, DF(p) = M$. Hence Exercise 7.2.0.2 implies that there exists $V \in \mathcal{T}_M$ such that $p \in V$ and $F|_V$ is an immersion. Since $F|_V$ is a immersion, for each $x \in V$, $\text{rank}\, DF(x) = \dim M$. Hence $V \subset U$. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $V \in \mathcal{T}_M$ such that $p \in V$ and $V \subset U$. Hence $U \in \mathcal{T}_M$.
2. Let $p \in U$. By construction

$$\begin{aligned}\text{rank}\, DF|_U(p) &= \text{rank}\, DF(p) \\ &= \dim M.\end{aligned}$$

Hence $DF|_U(p)$ is injective. Since $p \in U$ is arbitrary, we have that for each $p \in U$, $DF(p)$ is injective. Hence $F|_U$ is an immersion.

□

Definition 7.2.0.4. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$ and $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$. Then F is said to be a **$\mathbf{ManBnd}^\infty$-embedding** if

1. F is a $\mathbf{ManBnd}^\infty$-immersion,
2. $F \in \text{Iso}_{\mathbf{Top}}[(M, \mathcal{T}_M), (F(M), \mathcal{T}_N \cap F(M))]$.

Note 7.2.0.5. Here the topology on $F(M)$ is the subspace topology.

Exercise 7.2.0.6. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$, $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$. Suppose that F is an immersion. Then for each $U \in \mathcal{T}_M$, $F|_U$ is an immersion.

Proof. Let $p \in U$. Since $p \in M$ and F is an immersion, $\text{rank}\, DF(p) = \dim M$. Let $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V', \psi') \in \mathcal{A}_N$. Define $(U', \phi') \in \mathcal{A}_M|_U$ by $U' := U \cap U_0$ and $(\phi' := \phi_0|_{U'})$. Since $\mathcal{A}_M|_U \subset \mathcal{A}_M$, we have that

$$\begin{aligned}\text{rank}\, D(F|_U)(p) &= \text{rank}[D(F|_U)(p)]_{\phi', \psi} \\ &= \text{rank}[DF(p)]_{\phi', \psi} \\ &= \text{rank}\, DF(p) \\ &= m\end{aligned}$$

Since $p \in U$ is arbitrary, we have that for each $p \in U$, $D(F|_U)(p)$ is injective. Hence $F|_U$ is an immersion. □

Exercise 7.2.0.7. Local Embedding Theorem:
Let $M, N \in \text{Obj}(\mathbf{Man}^\infty)$ and $F \in \text{Hom}_{\mathbf{Man}^\infty}(M, N)$. Then F is an immersion iff for each $p \in M$, there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $F|_U : U \to N$ is a $\mathbf{Man}^\infty$-embedding. generalize to $\mathbf{ManBnd}^\infty$ with local embedding theorem for manifolds with boundary with Lee pg 87

Proof. Set $\dim M = m$ and $\dim N = n$.

- $(\implies)$:
  Suppose that F is an immersion. Let $p \in M$.
  - Let $p \in M$. Exercise 7.1.0.3 implies that there exists $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ such that $p \in U$, $F(U_0) \subset V$, and $\psi \circ F \circ \phi^{-1} = (\mathrm{id}_{\phi(U_0)}, 0)$. Thus $\psi \circ F \circ \phi^{-1}$ is injective. Since ϕ, ψ are bijections and $F|_{U_0} = \psi^{-1} \circ (\psi \circ F \circ \phi^{-1}) \circ \phi$, we have that $F|_{U_0}$ is injective. Choose $K \subset U_0$ such that K is compact and $p \in \operatorname{Int} K$. Since $F|_{U_0}$ is injective and continuous, $F|_K$ is injective and continuous. Since K is compact and N is Hausdorff, the closed map lemma in the analysis notes section on compact spaces and continuity implies that $F|_K : K \to F(K)$ is a homeomorphism. Set $U := \operatorname{Int} K$. Then $F|_U : U \to F(U)$ is a homeomorphism. Since F is an immersion, $F|_U$ is an immersion. Hence $F|_U$ is a $\mathbf{Man}^\infty$-embedding. generalize to boundary using Lee pg 87
- $(\impliedby)$:
  Suppose that for each $p \in M$, there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $F|_U : U \to N$ is a $\mathbf{Man}^\infty$-embedding. Let $p \in M$. Then there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $F|_U : U \to N$ is a $\mathbf{Man}^\infty$-embedding. Since $F|_U$ is a $\mathbf{Man}^\infty$-embedding, $F|_U$ is a $\mathbf{Man}^\infty$-immersion. Thus $DF|_U(p) : T_pU \to T_pN$ is injective. Since $DF(p) = DF|_U(p)$, $DF(p) : T_pM \to T_pN$ is injective. Since $p \in M$ is arbitrary, we have that for each $p \in M$, $DF(p)$ is injective. Hence F is a $\mathbf{Man}^\infty$-immersion.

□

Exercise 7.2.0.8. Let $(M, \mathcal{A})$ be a smooth manifold and $U \subset M$ open. Then the inclusion map $\iota_U : U \to M$ is a smooth embedding.

Proof. content... □

Exercise 7.2.0.9. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $p \in M$ and $q \in N$. Suppose that $\partial N = \varnothing$. Then

1. $\iota_q^M : M \to M \times N$ is a smooth embedding,
2. $\iota_p^N : N \to M \times N$ is a smooth embedding.

Proof.

1. Exercise 5.3.0.10 implies that ι_q^M is smooth. Let $p \in M$. Then

□

Exercise 7.2.0.10. Local Representation of Immersions:
Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. Then F is am immersion iff for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ such that $p \in U$, $\phi(U) = V$, and $\psi \circ F \circ \phi^{-1} = (\mathrm{id}_{\phi(U)}, 0)$.

Proof. FINISH!!! □

Exercise 7.2.0.11. Discuss Lemniscate (pg 86 Lee)

7.3 Submersions

give boundary assumptions being empty

Definition 7.3.0.1. Let $M, N \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$. Then F is said to be a **submersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is surjective.

Exercise 7.3.0.2. Let $M, N \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $F \in \mathrm{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$ and $p \in M$. If $DF(p)$ is surjective, then there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $F|_U$ is a smooth submersion.

Proof. content... $\square$

Exercise 7.3.0.3. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. Define $U \subset M$ by $U := \{p \in M : \mathrm{rank}\, DF(p) = \dim N\}$. Then

1. $U \in \mathcal{T}_M$,
2. $F|_U$ is a submersion.

Proof.

1. Let $p \in U$. Then $\mathrm{rank}\, DF(p) = N$. Hence Exercise 7.3.0.2 implies that there exists $V \in \mathcal{T}_M$ such that $p \in V$ and $F|_V$ is a submersion. Since $F|_V$ is a submersion, for each $x \in V$, $\mathrm{rank}\, DF(x) = \dim N$. Hence $V \subset U$. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $V \in \mathcal{T}_M$ such that $p \in V$ and $V \subset U$. Hence $U \in \mathcal{T}_M$.
2. Let $p \in U$. By construction
$$\begin{aligned}\mathrm{rank}\, DF|_U(p) &= \mathrm{rank}\, DF(p)\\ &= \dim N.\end{aligned}$$
Hence $DF|_U(p)$ is surjective. Since $p \in U$ is arbitrary, we have that for each $p \in U$, $DF(p)$ is surjective. Hence $F|_U$ is a submersion.

$\square$

Exercise 7.3.0.4. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Then $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are submersions.

Proof. Exercise 6.1.2.1 implies that $[D\pi_M(p,q)]_{\phi,\phi_M} = [I_m, 0]$. Hence $\mathrm{rank}[D\pi_M(p,q)]_{\phi,\phi_M} = m$. Since $\dim T_pM = m$, $D\pi_M(p,q) : M \times N \to T_pM$ is surjective. Since $(p,q) \in M \times N$ is arbtrary, we have that for each $(p,q) \in M \times N$, $D\pi_M(p,q)$ is surjective. Hence π_M is a submersion. $\square$

Exercise 7.3.0.5. Let $E, M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$, $G \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. If F, G are submersions, then $G \circ F$ is a submersion.

Proof. Suppose that F, G are submersions. Let $a \in E$. Then $DF(a)$ and $DG(F(a))$ are surjective. Since $D(G \circ F)(a) = DG(F(a)) \circ DF(a)$, we have that $D(G \circ F)(a)$ is surjective. Since $a \in E$ is arbitrary, we have that for each $a \in E$, $D(G \circ F)(a)$ is surjective. Hence $G \circ F$ is a submersion. $\square$

Exercise 7.3.0.6. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. Then F is a submersion iff for each $p \in M$, there exists $U \in \mathcal{T}_M$ such that $p \in M$ and $F|_U$ is a submersion.

Proof. FINISH!!! $\square$

Exercise 7.3.0.7. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ be smooth manifolds, $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$ a smooth map and $p \in M$.

1. If that $DF(p)$ is injective, then there exists $U \subset M$ such that U is open and $F|_U$ is a smooth immersion.
2. If $DF(p)$ is surjective, then there exists $U \subset M$ such that U is open and $F|_U$ is a smooth submersion.

Proof. FINISH!!! $\square$

Note 7.3.0.8. We define $\text{proj}^{n+k}_{[n]} : \mathbb{R}^{n+k} \to \mathbb{R}^n$ by $\text{proj}^{n+k}_{[n]}(a^1, \dots, a^{n+k}) = (a^1, \dots, a^n)$.

Exercise 7.3.0.9. Local Representation of Submersions:
Let $E, M \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$. Then π is a submersion iff for each $a \in E$, there exist $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_E$ such that $a \in V$, $U = \pi(V)$, and $\phi \circ \pi \circ \psi^{-1} = \text{proj}^{n+k}_{[n]} |_{\psi(V)}$.

Proof.

- $(\implies)$:
  Suppose that π is a submersion. Set $n := \dim M$, $k := \dim E - n$. Let $a \in E$. Set $p := \pi(a)$. Since $\pi : E \to M$ is a submersion, π has constant rank and $\text{rank}\, \pi = n$. Exercise 7.1.0.3 implies that there exist $(V, \psi) \in \mathcal{A}_E$, $(U_0, \phi_0) \in \mathcal{A}_M$ such that $a \in V$, $\pi(V) \subset U_0$ and $\phi_0 \circ \pi \circ \psi^{-1} = \text{proj}^{n+k}_{[n]} |_{\psi(V)}$. Define $U := \phi_0^{-1}(\text{proj}^{n+k}_{[n]}(\psi(V)))$. Since $\text{proj}^{n+k}_{[n]}$ is open and $\psi(V)$ is open in $\mathbb{R}^{n+k}$, we have that $\text{proj}^{n+k}_{[n]}(\psi(V))$ is open in $\mathbb{R}^n$. Since ϕ_0 is a homeomorphism, U is open in M. Set $\phi := \phi_0|_U$. a previous exercise in the section on smooth atlases implies that $(U, \phi) \in \mathcal{A}_M$. By construction,

  –
  $$\begin{aligned} \pi(V) &= [\phi_0^{-1} \circ (\phi_0 \circ \pi \circ \psi^{-1}) \circ \psi](V) \\ &= \phi_0^{-1} \circ \text{proj}^{n+k}_{[n]} \circ \psi(V) \\ &= U. \end{aligned}$$

  –
  $$\begin{aligned} \phi \circ \pi \circ \psi^{-1} &= \phi_0|_U \circ \pi \circ \psi^{-1} \\ &= \phi_0 \circ \pi \circ \psi^{-1} \\ &= \text{proj}^{n+k}_{[n]}. \end{aligned}$$

  Since $a \in E$ is arbitrary, we have that for each $a \in E$, there exist $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_E$ such that $a \in V$, $U = \pi(V)$, and $\phi \circ \pi \circ \psi^{-1} = \text{proj}^{n+k}_{[n]} |_{\psi(V)}$.

- $(\impliedby)$:
  Conversely, suppose that for each $a \in E$, there exist $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_E$ such that $a \in V$, $U = \pi(V)$, and $\phi \circ \pi \circ \psi^{-1} = \text{proj}^{n+k}_{[n]} |_{\psi(V)}$. Let $a \in E$. By assumption, there exists $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_E$ such that $a \in V$, $U = \pi(V)$, and $\phi \circ \pi \circ \psi^{-1} = \text{proj}^{n+k}_{[n]} |_{\psi(V)}$. Since ϕ and ψ are diffeomorphisms, we have that

  $$\begin{aligned} \text{rank}\, D\pi(a) &= \text{rank}[D\phi(\pi(a)) \circ D\pi(a) \circ D\psi^{-1}(\psi(a))] \\ &= \text{rank}\, D(\phi \circ \pi \circ \psi^{-1})(\psi(a)) \\ &= \text{rank}\, D\, \text{proj}^{n+k}_{[n]} |_{\psi(V)}(\psi(a)) \\ &= n \\ &= \dim T_{\pi(a)} M. \end{aligned}$$

  Thus $D\pi(a) : T_a E \to T_{\pi(a)} M$ is surjective. Since $a \in E$ is arbitrary, we have that for each $a \in E$, $D\pi(a)$ is surjective. Hence π is a submersion.

$\square$

Exercise 7.3.0.10. Let $E, M \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$.

1. If π is a submersion, then π is open.

2. If π is a surjective submersion, then π is a quotient map.

Proof.

1. Suppose that π is a submersion. Let $a \in E$. Exercise 7.3.0.9 implies that there exist $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_E$ such that

    - $a \in V$ and $U = \pi(V)$,
    - $\phi \circ \pi \circ \psi^{-1} = \mathrm{proj}_{[n]}^{n+k}|_{\psi(V)}$.

    Since $\mathrm{proj}_{[n]}^{n+k}$ is open and $\psi(V)$ is open in $\mathbb{R}^{n+k}$, we have that $\mathrm{proj}_{[n]}^{n+k}|_{\psi(V)}$ is open. Since ϕ, ψ are homeomorphisms and $\pi|_V = \phi^{-1} \circ \mathrm{proj}_{[n]}^{n+k}|_{\psi(V)} \circ \psi$, we have that $\pi|_V$ is open. Since $a \in E$ is arbitrary, we have that for each $a \in E$, there exists $V \subset E$ such that V is open in E and $\pi|_E$ is open. An exercise in the analysis notes section on subspace topology implies that π is open.

2. Suppose that π is a surjective submersion. Part (1) implies that π is open. Since π is surjective, open and continuous, an exercise in the analysis notes section on quotient maps implies that π is a quotient map.

$\square$

Definition 7.3.0.11. Let $E, M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $\pi \in \mathrm{Hom}_{\mathbf{ManBnd}^\infty}(E, M)$ a surjection and $\sigma : M \to E$. Then σ is said to be a smooth section of π if

1. $\sigma \in \mathrm{Hom}_{\mathbf{ManBnd}^\infty}(M, E)$
2. σ is a section of π

We define
$$\Gamma(\pi) := \{\sigma \in \mathrm{Hom}_{\mathbf{ManBnd}^\infty}(M, E) : \sigma \text{ is a smooth section of } \pi.\}$$

Definition 7.3.0.12. Let $E, M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $\pi \in \mathrm{Hom}_{\mathbf{ManBnd}^\infty}(E, M)$, $U \in \mathcal{T}_M$ and $\sigma : U \to E$. Then

- (U, σ) is said to be a **smooth local section of** π if $\sigma \in \Gamma(\pi|_{\pi^{-1}(U)})$,
- for each $p \in M$, we define
$$\Gamma_p(\pi) := \{(U, \sigma) : (U, \sigma) \text{ is a smooth local section of } \pi \text{ and } p \in U\}$$

Exercise 7.3.0.13. Let $E, M \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$. Suppose that π is a surjective submersion. Then π admits local sections. define this, maybe each $a \in E$ is in the image of a smooth section, or for each $p \in M$, there is a local section around p, or both

Proof. Set $n := \dim M$ and $k := \dim E - n$. Let $p \in M$. Since π is surjective, there exists $a \in E$ such that $\pi(a) = p$. Exercise 7.3.0.9 implies that there exist $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_E$ such that

- $a \in V$ and $U = \pi(V)$,
- $\phi \circ \pi \circ \psi^{-1} = \mathrm{proj}_{[n]}^{n+k}|_{\psi(V)}$.

Set $\hat{x} := \mathrm{proj}_{[n]}^{n+k}(\psi(a))$ and $\hat{y} := \mathrm{proj}_{[-k]}^{n+k}(\psi(a))$ so that $\psi(a) = (\hat{x}, \hat{y})$. An exercise in the analysis notes from the section on the product topology implies that there exist $A \in \mathcal{T}_{\mathbb{R}^n}$ and $B \in \mathcal{T}_{\mathbb{R}^k}$ such that $(\hat{x}, \hat{y}) \in A \times B$ and $A \times B \subset \psi(V)$. We note that $\hat{x} = \phi(p)$, $A \subset \phi(U)$ and for each $(x^1, \dots, x^n) \in A$, $(x^1, \dots, x^n, \hat{y}) \in \psi(V)$. Define $\hat{\sigma} : A \to \psi(V)$ by $\hat{\sigma}(x^1, \dots, x^n) := (x^1, \dots, x^n, \hat{y})$. Then $\hat{\sigma}$ is smooth. Define $\sigma : \phi^{-1}(A) \to V$ by

$\sigma := \psi^{-1} \circ \hat{\sigma} \circ \phi|_{\phi^{-1}(A)}$. Then σ is smooth. Let $q \in \phi^{-1}(A)$. Set $x := \phi(q)$. Then

$$\begin{aligned}
\pi \circ \sigma(q) &= [\pi \circ (\psi^{-1} \circ \hat{\sigma} \circ \phi|_{\phi^{-1}(A)})](q) \\
&= [\pi \circ (\psi^{-1} \circ \hat{\sigma} \circ \phi|_{\phi^{-1}(A)})](\phi^{-1}(x)) \\
&= [\pi \circ (\psi^{-1} \circ \hat{\sigma})](x) \\
&= [(\pi \circ \psi^{-1}) \circ \hat{\sigma}](x) \\
&= (\phi^{-1} \circ \mathrm{proj}^{n+k}_{[n]})(x, \hat{y}) \\
&= \phi^{-1}(x) \\
&= q
\end{aligned}$$

Since $q \in \phi^{-1}(A)$ is arbitrary, we have that $\pi \circ \sigma = \mathrm{id}_{\phi^{-1}(A)}$ and therefore $(\phi^{-1}(A), \sigma) \in \Gamma_p(\pi)$. □

Exercise 7.3.0.14. Let $E, M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ and $F : M \to N$. Suppose that π is a surjective submersion. Then $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$ iff $F \circ \pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, N)$, in which case the following diagram commutes in $\mathbf{Man}^\infty$:

$$\begin{tikzcd}
E \arrow[d, "\pi"'] \arrow[dr, "F \circ \pi"] & \\
M \arrow[r, "F"'] & N
\end{tikzcd}$$

Proof.

- $(\implies)$:
  Suppose that F is smooth. Then clearly $F \circ \pi$ is smooth.
- $(\impliedby)$:
  Suppose that $F \circ \pi$ is smooth. Let $p \in M$. Then there exists a local section $(U, \sigma) \in \Gamma_p(\pi)$ such that $p \in U$. Since $F \circ \pi$ are smooth and σ is smooth, we have that

  $$\begin{aligned}
  (F \circ \pi) \circ \sigma &= F \circ (\pi \circ \sigma) \\
  &= F \circ \mathrm{id}_U \\
  &= F|_U
  \end{aligned}$$

  is smooth. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exists $U \subset M$ such that U is open in M, $p \in U$ and $F|_U$ is smooth. Thus F is smooth.

□

Exercise 7.3.0.15. Let $(E, \mathcal{C})$ be a smooth manifold, M a topological manifold, $\mathcal{A}_1$ and $\mathcal{A}_2$ smooth structures on M and $\pi : E \to M$. Suppose that π is a surjective. If π is a $(\mathcal{C}, \mathcal{A}_1)$-smooth subsmersion and π is a $(\mathcal{C}, \mathcal{A}_2)$-smooth subsmersion, then $\mathcal{A}_1 = \mathcal{A}_2$. clean up notation with $\mathcal{A}_E$ instead of $\mathcal{C}$

Proof. Suppose that π is a $(\mathcal{C}, \mathcal{A}_1)$-smooth subsmersion and π is a $(\mathcal{C}, \mathcal{A}_2)$-smooth subsmersion. Since $\mathrm{id}_M \circ \pi = \pi$ and π is $(\mathcal{C}, \mathcal{A}_2)$-smooth, Exercise 7.3.0.14 implies that id_M is $(\mathcal{A}_1, \mathcal{A}_2)$-smooth. Similarly, Since π is $(\mathcal{C}, \mathcal{A}_1)$-smooth Exercise 7.3.0.14 implies that id_M is $(\mathcal{A}_2, \mathcal{A}_1)$-smooth. Thus id_M is a $(\mathcal{A}_1, \mathcal{A}_2)$ diffeomorphism. Exercise 5.2.0.5 implies that $\mathcal{A}_1 = \mathcal{A}_2$. □

Exercise 7.3.0.16. Let $E, M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, N)$. Suppose that π is a surjective submersion. If for each $a, b \in E$, $\pi(a) = \pi(b)$ implies that $F(a) = F(b)$, then there exists a unique $\tilde{F} \in \mathrm{Hom}(\mathbf{Man}^\infty)(M, N)$ such that $\tilde{F} \circ \pi = F$, i.e. the following diagram commutes:

$$\begin{tikzcd}
E \arrow[d, "\pi"'] \arrow[dr, "F"] & \\
M \arrow[r, dashed, "\tilde{F}"'] & N
\end{tikzcd}$$

Proof. Exercise 7.3.0.10 implies that π is a quotient space. We define the relation $\sim_\pi$ on E by $a \sim_\pi b$ iff $\pi(a) = \pi(b)$. Let $p_\pi : E \to E/\sim_\pi$ be the projection map. An exercise in the analysis notes section on quotient spaces implies that there exists $h : E/\sim_\pi \to M$ such that h is a homeomorphism and $h \circ p_\pi = \pi$. Thus $p_\pi = h^{-1} \circ \pi$. By assumption, F is $\sim_\pi$-invariant. Another exercise in the analysis notes section on quotient spaces implies that there exists a unique $\bar{F} : E/\sim_\pi \to N$ such that $\bar{F}$ is continuous and $\bar{F} \circ p_\pi = F$. Set $\tilde{F} := \bar{F} \circ h^{-1}$. Therefore,

$$\begin{aligned}
\tilde{F} \circ \pi &= (\bar{F} \circ h^{-1}) \circ \pi \\
&= \bar{F} \circ (h^{-1} \circ \pi) \\
&= \bar{F} \circ p_\pi \\
&= F,
\end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd}
& E \arrow[dl, "F"'] \arrow[d, "p_\pi"] \arrow[dr, "\pi"] & \\
N & E/\sim_\pi \arrow[l, "\bar{F}"] & M \arrow[l, "h^{-1}"]
\end{tikzcd}$$

Since F is smooth and $\tilde{F} \circ \pi = F$, we have that $\tilde{F} \circ \pi$ is smooth, i.e. the following diagram commutes:

$$\begin{tikzcd}
E \arrow[d, "\pi"'] \arrow[dr, "\tilde{F}\circ\pi"] & \\
M \arrow[r, "\tilde{F}"'] & N
\end{tikzcd}$$

Exercise 7.3.0.14 then implies that $\tilde{F}$ is smooth. $\square$

Chapter 8

Submanifolds

8.1 Introduction

Definition 8.1.0.1. Let $M, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$.

- Then S is said to be an **immersed submanifold** of M if the inclusion map $\iota_S : S \to M$ is an immersion.
- If S is an immersed submanifold of M, then M is said to be the **ambient manifold of** S.
- If S is an immersed submanifold of M, we define the **codimension of** S **with respect to** M, denoted $\mathrm{codim}_M(S)$, by $\mathrm{codim}_M(S) = \dim M - \dim S$.

Exercise 8.1.0.2. Let $M, N, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. Suppose that S is an immersed submanifold of M. Then $F|_S \in \mathrm{Hom}_{\mathbf{Man}^\infty}(S, N)$.

Proof. Since S is an immersed submanifold of M, the inclusion $\iota_S \in \mathrm{Hom}_{\mathbf{Man}^\infty}(S, M)$. Therefore

$$\begin{aligned} F|_S &= F \circ \iota \\ &\in \mathrm{Hom}_{\mathbf{Man}^\infty}(S, N). \end{aligned}$$

□

Definition 8.1.0.3. Let $(M, \mathcal{T}_M, \mathcal{A}_M), (S, \mathcal{T}_S, \mathcal{A}_S) \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$. Then S is said to be an **embedded submanifold** of M if the inclusion map $\iota_S : (S, \mathcal{T}_S, \mathcal{A}_S) \to (M, \mathcal{T}_M, \mathcal{A}_M)$ is a $\mathbf{Man}^\infty$-embedding.

Exercise 8.1.0.4. Let $M, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$. If S is an embedded submanifold of M, then S is an immersed submanifold of M.

Proof. Clear. □

Exercise 8.1.0.5. Immersed Implies Locally Embedded:
Let $M, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$. Then S is an immersed submanifold fo M iff for each $p \in S$, there exists $U \in \mathcal{T}_S$ such that $p \in U$ and U is an embedded submanifold of M.

Proof.

- ($\implies$) :
  Suppose that S is an immersed submanifold fo M. Then $\iota_S : S \to M$ is an immersion. Let $p \in S$. Since ι_S is an immersion, Exercise 7.2.0.7 implies that there exists $U \in \mathcal{T}_S$ such that $p \in U$ and $\iota_S|_U$ is a $\mathbf{Man}^\infty$-embedding. Since $\iota_S|_U = \iota_U$, we have that ι_U is a $\mathbf{Man}^\infty$-embedding and U is an embedded submanifold of M.

- $(\Longleftarrow)$:
Suppose that for each $p \in S$, there exists $U \in \mathcal{T}_S$ such that $p \in U$ and U is an embedded submanifold of M. Let $p \in S$. By assumption, there exists $U \in \mathcal{T}_S$ such that $p \in U$ and U is an embedded submanifold of M. Thus ι_U is a $\mathbf{Man}^\infty$-embedding. Since $\iota_U = \iota_S|_U$, we have that $\iota_S|_U$ is a $\mathbf{Man}^\infty$-embedding. Since $p \in S$ is arbitrary, we have that for each $p \in S$, there exists $U \in \mathcal{T}_S$ such that $p \in U$ and $\iota_S|_U$ is a $\mathbf{Man}^\infty$-embedding. Exercise 7.2.0.7 implies that ι_S is an immersion. Thus S is an immersed submanifold of M.

□

Exercise 8.1.0.6. Uniqueness of Topology for Embedded Submanifolds Let $(M, \mathcal{T}_M, \mathcal{A}_M), (S, \mathcal{T}_S, \mathcal{A}_S) \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$ and $(S, \mathcal{T}_S, \mathcal{A}_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$. Then $\mathcal{T}_S = \mathcal{T}_M \cap S$.

Proof. Since $(S, \mathcal{T}_S, \mathcal{A}_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$, $\iota_S \in \mathrm{Iso}_{\mathbf{Top}}[(S, \mathcal{T}_S), (S, \mathcal{T}_M \cap S)]$. An exercise in the analysis notes section on subspaces implies that $\mathcal{T}_S = \mathcal{T}_M \cap S$. get rid of the following:

- Let $U \in \mathcal{T}_S$. Since $\iota_S(U) = U$ and ι_S is $(\mathcal{T}_S, \mathcal{T}_M \cap S)$-open, we have that
$$\begin{aligned} U &= \iota_S(U) \\ &\in \mathcal{T}_M \cap S. \end{aligned}$$
Since $U \in \mathcal{T}_S$ is arbitrary, we have that $\mathcal{T}_S \subset \mathcal{T}_M \cap S$.

- Let $U \in \mathcal{T}_M \cap S$. Since ι_S is $(\mathcal{T}_S, \mathcal{T}_M \cap S)$-continuous and $U \subset S$, we have that we have that
$$\begin{aligned} U &= \iota_S^{-1}(U) \\ &= \in \mathcal{T}_S. \end{aligned}$$
Since $U \in \mathcal{T}_M \cap S$ is arbitrary, we have that $\mathcal{T}_M \cap S \subset \mathcal{T}_S$.

Hence $\mathcal{T}_S = \mathcal{T}_M \cap S$. Make this an exercise in the analysis notes section on topology and subspaces, then just cite that exercise here in the context of smooth manifolds. □

Exercise 8.1.0.7. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $p \in M$ and $q \in N$. Then $M \times \{q\}$ and $N \times \{p\}$ are embedded submanifold of $M \times N$.

Proof. FINISH!!! □

Exercise 8.1.0.8. Let M, U be a smooth manifolds. Suppose that $U \subset M$. Then U is an embedded submanifold of M and $\mathrm{codim}_M(U) = 0$ iff U is an open submanifold of M.

Proof.

- $(\Longrightarrow)$:
Suppose that U is an embedded submanifold of M and $\mathrm{codim}_M(U) = 0$. FINISH!!!

- $(\Longleftarrow)$:
Suppose that U is an open submanifold of M. need to say why U is embedded Exercise 3.2.1.6 and Definition 4.2.1.3 implies that $\dim U = n$, so that $\mathrm{codim}_M(U) = 0$.

□

Definition 8.1.0.9. Let $(M, \mathcal{A}), (S, \mathcal{B}) \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$ and $(S, \mathcal{B})$ is an embedded submanifold of $(M, \mathcal{A})$. Then $(S, \mathcal{B})$ is said to be **properly embedded** if $\iota_S : S \to M$ is proper.

Exercise 8.1.0.10. Let $(M, \mathcal{A}), (S, \mathcal{B}) \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$ and $(S, \mathcal{B})$ is an embedded submanifold of $(M, \mathcal{A})$. Then $(S, \mathcal{B})$ is properly embedded iff S is closed in M.

Proof.

- ($\Longrightarrow$) :
  Suppose that $(S, \mathcal{B})$ is properly embedded. Then $\iota_S : S \to M$ is proper. An exercise in the analysis notes section on locally compact Hausdorff spaces implies that ι_S is closed. Since S is closed in S and ι_S is closed, we have that $\iota_S(S)$ is closed in M. Since $\iota_S(S) = S$, we have that S is closed in M.

- ($\Longleftarrow$) :
  Conversely, suppose that S is closed in M. Let $K \subset M$. Suppose that K is compact in M. Since M is Hausdorff and S is closed in M, an exercise in the analysis notes section on compactness implies that $K \cap S$ is compact in M. An exercise in the analysis notes section on compactness implies that $K \cap S$ is compact in S. Since $\iota_S^{-1}(K) = K \cap S$, $\iota_S^{-1}(K)$ is compact in S. Since $K \subset M$ with K compact in M is arbitrary, we have that for each $K \subset M$, K is compact implies that $\iota_S^{-1}(K)$ is compact in S. Thus ι_S is proper.

$\square$

Definition 8.1.0.11. Let $n \in \mathbb{N}$ and $k \in [n]$. We define the ***k*-slice of** $\mathbb{R}^n$, denoted $\mathbb{S}^{n,k}$, by $\mathbb{S}^{n,k} := \{a \in \mathbb{R}^n : a^{k+1}, \dots, a^n = 0\}$.

Definition 8.1.0.12. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Then S is said to be a ***k*-slice** of U if $S = U \cap \mathbb{S}^{n,k}$.

Exercise 8.1.0.13. show $\mathbb{S}^{n,k}$ is a k-slice of $\mathbb{R}^n$.

Proof. Clear. $\square$

Definition 8.1.0.14. Let M be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}_M$. Then (U, ϕ) is said to be a ***k*-slice chart on** S if $\phi(U \cap S)$ is a k-slice of $\phi(U)$. We define

$$\mathbb{S}^k(M; S) := \{(U, \phi) \in \mathcal{A}_M : (U, \phi) \text{ is a } k\text{-slice chart on } S\}$$

Exercise 8.1.0.15. Let M be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}_M$ with $\phi = (x^1, \dots, x^n)$. If (U, ϕ) is a k-slice chart on S, then $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$.

Proof. Clear. $\square$

Definition 8.1.0.16. Let M be a smooth manifold and $S \subset M$. Then S is said to **satisfy the local *k*-slice condition with respect to** M if for each $p \in S$, there exists $(U, \phi) \in \mathbb{S}^k(M; S)$ such that $p \in U$.

Exercise 8.1.0.17. Let M, N be smooth manifolds and $S \subset M$. Suppose that $\dim M = m$, $\dim N = n$ and $M \subset N$. Then

1. $S^k(M; S) \subset S^k(N; S)$
2.

Proof. FINISH!!! $\square$

Exercise 8.1.0.18. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Suppose that S is a k-slice of U. Define $\pi^n_{[k]} : \mathbb{R}^n \to \mathbb{R}^k$ by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then $\pi^n_{[k]}|_S \to \pi(S)$ is a diffeomorphism.

Proof. Clear. FINISH!!! $\square$

Exercise 8.1.0.19. Let $M, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$. If S is a k-dimensional embedded submanifold of M, then S satisfies the local k-slice condition with respect to M.
Hint: Draw a picture

Proof. Set $n := \dim M$. Suppose that S is a k-dimensional embedded submanifold of M. Let $p \in S$. Since S is an emebedded submanifold of M, the inclusion map $\iota : S \to M$ is an immersion. The local rank theorem (Exercise 7.1.0.3) implies that Then there exists $(U_0, \phi_0) \in \mathcal{A}_S$, $(V_0, \psi_0) \in \mathcal{A}_M$ such that $p \in U_0$, $\iota(p) \in V_0$, $\iota(U_0) \subset V_0$ and $\psi_0 \circ \iota \circ \phi_0^{-1} = (\mathrm{id}_{\phi_0(U_0)}, 0)$. Since for each $q \in U_0$, $\iota(q) = q$, we have that $U_0 \subset V_0$ and $\psi_0 \circ \iota \circ \phi_0^{-1} = \psi_0 \circ \phi_0^{-1}$. Therefore for each $q \in U_0$,

$$\begin{aligned}\psi_0(q) &= \psi_0 \circ \phi_0^{-1}(\phi_0(q)) \\ &= \psi_0 \circ \iota \circ \phi_0^{-1}(\phi_0(q)) \\ &= (\mathrm{id}_{\mathbb{R}^k}(\phi_0(q)), 0) \\ &= (\phi_0(q), 0)\end{aligned}$$

and in particular, $\psi_0(p) = (\phi_0(p), 0)$. Since $U_0 \in \mathcal{T}_S$ and $\mathcal{T}_S = \mathcal{T}_M \cap S$, there exists $U' \in \mathcal{T}_M$ such that $U_0 = U' \cap S$. An exercise in the analysis notes in the section on product topology implies that there exist $A_0 \in \mathcal{T}_{\mathbb{R}^k}$ and $B_0 \in \mathcal{T}_{\mathbb{R}^{n-k}}$ such that $(\phi(p), 0) \in A_0 \times B_0$ and $A_0 \times B_0 \subset \psi_0(V_0 \cap U') \cap [\phi_0(U_0) \times \mathbb{R}^{n-k}]$. Define $(V, \psi) \in \mathcal{A}_M$ by $V := \psi_0^{-1}(A_0 \times B_0)$ and $\psi := \psi_0|_V$. A previous exercise in the subsection about smooth maps on subspaces implies that $(V, \psi) \in \mathcal{A}_M$. Then $p \in V$.

- Let $y \in A_0 \times \{0\}$. Then there exists $a \in A_0$ such that $y = (a, 0)$. Since $A_0 \times B_0 \subset \phi_0(U_0) \times \mathbb{R}^{n-k}$, we have that $A_0 \subset \phi_0(U_0)$. In particular, $a \in \phi_0(U_0)$ and $\phi_0^{-1}(a) \in U_0$. Hence

  $$\begin{aligned}y &= (a, 0) \\ &= \psi_0 \circ \phi_0^{-1}(a) \\ &\in \psi_0(U_0).\end{aligned}$$

  By construction,

  $$\begin{aligned}y &= (a, 0) \\ &= \psi_0(\psi_0^{-1}(a, 0)) \\ &\in \psi_0[\psi_0^{-1}(A_0 \times \{0\})] \\ &\subset \psi_0[\psi_0^{-1}(A_0 \times B_0)] \\ &= \psi_0(V).\end{aligned}$$

  Therefore

  $$\begin{aligned}y &\in \psi_0(U_0) \cap \psi_0(V) \\ &= \psi_0[(U_0) \cap V] \\ &= \psi_0([(U' \cap S) \cap V_0] \cap V) \\ &= \psi_0(V \cap S).\end{aligned}$$

  Since $y \in A_0 \times \{0\}$ is arbitrary, we have that $A_0 \times \{0\} \subset \psi_0(V \cap S)$.

- Conversely, we note that for each $q \in V \cap S$,

  $$\begin{aligned}(\phi_0(q), 0) &= \psi_0(q) \\ &\in \psi_0(V \cap S) \\ &\subset \psi_0(V) \\ &= A_0 \times B_0,\end{aligned}$$

  and therefore $\phi_0(V \cap S) \subset A_0$. Hence

  $$\begin{aligned}\psi_0(V \cap S) &= \phi_0(V \cap S) \times \{0\} \\ &\subset A_0 \times \{0\}.\end{aligned}$$

Thus $A_0 \times \{0\} = \psi_0(V \cap S)$ and

$$\begin{aligned}\psi(V \cap S) &= \psi_0(V \cap S)\\ &= A_0 \times \{0\}\\ &= (A_0 \times B_0) \cap \mathbb{S}^{n,k}\\ &= \psi(V) \cap \mathbb{S}^{n,k}.\end{aligned}$$

Hence $\psi(V \cap S)$ is a k-slice of $\psi(V)$ and therefore $(V, \psi) \in \mathbb{S}^k(M; S)$. Since $p \in S$ is arbitrary, we have that for each $p \in S$, there exists $(V, \psi) \in \mathbb{S}^k(M; S)$ such that $p \in V$. Therefore S satisfies the local k-slice condition with respect to M. □

Exercise 8.1.0.20. Let $(M, \mathcal{A}) \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $S \subset M$. Suppose that $\dim M = n$ and S satisfies the local k-slice condition with respect to M. Then

1. for each $(U, \phi) \in \mathbb{S}^k(M; S)$, if $U \cap S \neq \varnothing$, then $(U \cap S, \pi_{n,k} \circ \phi|_{U \cap S}) \in X^k(S)$,
2. $(S, \mathcal{T}_M \cap S) \in \mathrm{Obj}(\mathbf{Man}^0)$ and $\dim S = k$.

Proof.

1. Let $(U_0, \phi_0) \in \mathbb{S}^k(M; S)$. Suppose that $U_0 \cap S \neq \varnothing$. Set $U := U_0 \cap S$ and $\phi := \phi_0|_U$. Since $(U_0, \phi_0) \in \mathbb{S}^k(M; S)$, we have that

$$\begin{aligned}\phi_0(U) &= \phi_0(U_0 \cap S)\\ &= \phi_0(U_0) \cap \mathbb{S}^{n,k}\\ &\in \mathcal{T}_{\mathbb{R}^n} \cap \mathbb{S}^{n,k}\end{aligned}$$

   (a) By assumption, $U_0 \in \mathcal{T}_M$. Therefore $U \in \mathcal{T}_M \cap S$.

   (b) Since $(U_0, \phi_0) \in X^n(M, \mathcal{T}_M)$, $\phi_0(U_0) \in \mathcal{T}_{\mathbb{R}^n}$. Since $(U_0, \phi_0) \in \mathbb{S}^k(M; S)$, we have that

$$\begin{aligned}\phi_0(U_0 \cap S) &= \phi_0(U_0) \cap \mathbb{S}^{n,k}\\ &\in \mathcal{T}_{\mathbb{R}^n} \cap \mathbb{S}^{n,k}\\ &= \mathcal{T}_{\mathbb{S}^{n,k}}\end{aligned}$$

   By a previous exercise, $\pi^n_{[k]}|_{\mathbb{S}^k}$ is a $(\mathcal{T}_{\mathbb{S}^{n,k}}, \mathcal{T}_{\mathbb{R}^k})$-homeomorphism. Hence

$$\begin{aligned}\phi(U) &= \pi^n_{[k]} \circ \phi_0(U_0 \cap S)\\ &\in \mathcal{T}_{\mathbb{R}^k}\end{aligned}$$

   (c) Since $\phi_0|_U$ is a $(\mathcal{T}_M \cap U, \mathcal{T}_{\mathbb{S}^{n,k}} \cap \phi_0(U_0))$-homeomorphism and $\pi^n_{[k]}|_{\phi(U)}$ is a $(\mathcal{T}_{\mathbb{S}^{n,k}} \cap \phi_0(U_0), \mathcal{T}_{\mathbb{R}^k} \cap \phi(U))$-homeomorphism, we have that ϕ is a $(\mathcal{T}_M \cap U, \mathcal{T}_{\mathbb{R}^k} \cap \phi(U))$-homeomorphism.

   Hence $(U, \phi) \in X^k(S)$.

2. (a) Since $(M, \mathcal{T}_M)$ is Hausdorff, $(S, \mathcal{T}_M \cap S)$ is Hausdorff.

   (b) Since $(M, \mathcal{T}_M)$ is second-countable, $(S, \mathcal{T}_M \cap S)$ is second-countable.

   (c) Let $p \in S$. Since S satisfies the local k-slice condition with respect to M, there exists $(U_0, \phi_0) \in \mathcal{A}$ such that $p \in U_0$ and $(U_0, \phi_0) \in \mathbb{S}^k(M; S)$. Set $U := U_0 \cap S$ and $\phi := \pi^n_{[k]} \circ \phi_0|_U$. Then $p \in U$ and the prevous part implies that $(U, \phi) \in X^k(S, \mathcal{T}_M \cap S)$. Since $p \in S$ is arbitrary, we have that for each $p \in S$, there exists $(U, \phi) \in X^k(S, \mathcal{T}_M \cap S)$ such that $p \in U$. Hence S is locally Euclidean of dimension k.

   Thus $(S, \mathcal{T}_M \cap S) \in \mathrm{Obj}(\mathbf{Man}^0)$ and $\dim S = k$.

□

Definition 8.1.0.21. Let $(M,\mathcal{A}) \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $S \subset M$. Suppose that $\dim M = n$ and S satisfies the local k-slice condition with respect to M. We define

$$\mathcal{A}|_S^0 := \{(U \cap S, \pi^n_{[k]} \circ \phi_{U\cap S}) : (U,\phi) \in \mathbb{S}^k(M;S)\}.$$

Exercise 8.1.0.22. Let $(M,\mathcal{A}) \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $S \subset M$. Suppose that S satisfies the local k-slice condition with respect to M. Then

1. $\mathcal{A}|_S^0$ is an atlas on S,
2. $\mathcal{A}|_S^0$ is smooth.

Proof.

1. The previous exercise implies that $\mathcal{A}|_S^0 \subset X^k(M, \mathcal{T}_M \cap S)$. Let $p \in S$. Since S satisifes the local k-slice condition with respect to M, there exists $(U_0,\phi_0) \in \mathbb{S}^k(M;S)$ such that $p \in U_0$. Set $U := U_0 \cap S$ and $\phi := \phi_0|_U$. By definition, $(U,\phi) \in \mathcal{A}|_S^0$. By construction, $p \in U$. Since $p \in S$ is arbitrary, we have that for each $p \in S$, there exists $(U,\phi) \in \mathcal{A}|_S^0$ such that $p \in U$. Hence $\mathcal{A}|_S^0$ is an atlas on S.

2. Let $(U,\phi),(V,\psi) \in \mathcal{A}|_S^0$. Then there exist $(U_0,\phi_0),(V_0,\psi_0) \in \mathbb{S}^k(M;S)$ such that $U = U_0 \cap S$, $V = V_0 \cap S$, $\phi = \pi^n_{[k]} \circ \phi_0|_U$ and $\psi = \pi^n_{[k]} \circ \psi_0|_V$.

$$\begin{aligned}
\psi|_{U\cap V} \circ \phi|_{U\cap V}^{-1} &= (\pi^n_{[k]}|_{\psi_0(S\cap U_0\cap V_0)} \circ \psi_0|_{S\cap(U_0\cap V_0)}) \circ (\pi^n_{[k]}|_{\phi_0(S\cap U_0\cap V_0)} \circ \phi_0|_{S\cap(U_0\cap V_0)})^{-1} \\
&= (\pi^n_{[k]}|_{\psi_0(S\cap U_0\cap V_0)} \circ \psi_0|_{S\cap(U_0\cap V_0)}) \circ (\phi_0|_{S\cap(U_0\cap V_0)}^{-1} \circ \pi^n_{[k]}|_{\phi_0(S\cap U_0\cap V_0)}^{-1}) \\
&= \pi^n_{[k]}|_{\psi_0(S\cap U_0\cap V_0)} \circ [\psi_0|_{S\cap(U_0\cap V_0)} \circ \phi_0|_{S\cap(U_0\cap V_0)}^{-1}] \circ \pi^n_{[k]}|_{\phi_0(S\cap U_0\cap V_0)}^{-1} \\
&= \pi^n_{[k]}|_{\psi_0(S\cap U_0\cap V_0)} \circ [\psi_0|_{U_0\cap V_0} \circ \phi_0|_{U_0\cap V_0}^{-1}]|_{\phi_0(S\cap(U_0\cap V_0))} \circ \pi^n_{[k]}|_{\phi_0(S\cap U_0\cap V_0)}^{-1} \\
&= \pi^n_{[k]}|_{\psi_0(U\cap V)} \circ [\psi_0|_{U_0\cap V_0} \circ \phi_0|_{U_0\cap V_0}^{-1}]|_{\phi_0(U\cap V)} \circ \pi^n_{[k]}|_{\phi_0(U\cap V)}^{-1}
\end{aligned}$$

Since $\mathcal{A}$ is smooth, we have that $\psi_0|_{U_0\cap V_0} \circ \phi_0|_{U_0\cap V_0}^{-1}$ is smooth. Thus $(\psi_0|_{U_0\cap V_0} \circ \phi_0|_{U_0\cap V_0}^{-1})|_{\phi_0(U\cap V)}$ is smooth. A previous exercise implies that $\pi^n_{[k]}|_{\phi_0(U\cap V)}$ and $\pi^n_{[k]}|_{\psi_0(U\cap V)}$ are smooth. Thus $\psi|_{U\cap V} \circ \phi|_{U\cap V}^{-1}$ is smooth. Similarly, $\phi|_{U\cap V} \circ \psi|_{U\cap V}^{-1}$ is smooth. Hence$\psi|_{U\cap V} \circ \phi|_{U\cap V}^{-1}$ is a diffeomorphism and (U,ϕ), (V,ψ) are smoothly compatible. Since $(U,\phi),(V,\psi) \in \mathcal{A}|_S^0$ are arbitrary, we have that for each $(U,\phi),(V,\psi) \in \mathcal{A}|_S^0$, (U,ϕ) and (V,ψ) are smoothly compatible. Therefore $\mathcal{A}|_S^0$ is smooth.

$\square$

Definition 8.1.0.23. Let $(M,\mathcal{A}) \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $S \subset M$. Suppose that S satisfies the local k-slice condition with respect to M. We define the **embedded smooth structure on S induced by $\mathcal{A}$**, denoted $\mathcal{A}|_S$, by

$$\mathcal{A}|_S := \alpha(\mathcal{A}|_S^0).$$

Exercise 8.1.0.24. Let $(M,\mathcal{A}) \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $S \subset M$. Suppose that S satisfies the local k-slice condition with respect to M. Then $(S, \mathcal{T}_M \cap S, \mathcal{A}|_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A})$,

Proof. By definition, ι_S is a topological embedding (check this). Let $p \in S$. Since S atisfies the local k-slice condition with respect to M, there exists $(V_0,\psi_0) \in \mathbb{S}^k(M;S)$ such that $p \in V_0$. Set $V := V_0 \cap S$ and $\psi := \pi^n_{[k]} \circ \psi_0|_V$. By definition,

$$\begin{aligned}
(V,\psi) &\in \mathcal{A}|_S^0 \\
&\subset \mathcal{A}|_S.
\end{aligned}$$

Hence

$$\begin{aligned}\psi_0 \circ \iota \circ \psi^{-1} \\ &= \psi_0 \circ \psi^{-1} \\ &= \psi_0 \circ (\pi^n_{[k]}|_{\psi_0(V)} \circ \psi_0|_V)^{-1} \\ &= \psi_0 \circ \psi_0|_V^{-1} \circ \pi^n_{[k]}|^{-1}_{\psi_0(V)} \\ &= \pi^n_{[k]}|^{-1}_{\psi_0(V)}\end{aligned}$$

A previous exercise in the section on immersions implies that $\pi^n_{[k]}|^{-1}_{\psi_0(V)}$ is an immersion and $\operatorname{rank} \pi^n_{[k]}|^{-1}_{\psi_0(V)} = k$. Since $(V, \psi) \in \mathcal{A}$ and $(V_0, \psi_0) \in \mathcal{A}|_S$, an exercise in the section on smooth maps on submaifolds implies that ψ and ψ_0 are diffeomorphisms. Therefore

$$\begin{aligned}\operatorname{rank} D\iota(p) &= \operatorname{rank} D(\psi_0 \circ \iota \circ \psi^{-1})(\psi(p)) \\ &= \operatorname{rank} D(\psi_0 \circ \psi^{-1})(\psi(p)) \\ &= \operatorname{rank} D(\pi^n_{[k]}|^{-1}_{\psi_0(V)})(\psi(p)) \\ &= k\end{aligned}$$

Since $p \in S$ is arbitrary, we have that for each $p \in S$, $\operatorname{rank} D\iota(p) = k$. Thus ι has constant rank and $\operatorname{rank} \iota = k$. Since $\dim S = k$, an exercise in the section on maps of constant rank implies that ι is an immersion. Thus $(S, \mathcal{A}|_S)$ is an embedded submanifold of $(M, \mathcal{A})$. □

Note 8.1.0.25. Let $(M, \mathcal{A}) \in \operatorname{Obj}(\mathbf{Man}^\infty)$ and $S \subset M$. Suppose that S satisfies the local k-slice condition with respect to M. Unless otherwise specified, we equip S with $\mathcal{A}|_S$.

Exercise 8.1.0.26. Let $M, N, S \in \operatorname{Obj}(\mathbf{Man}^\infty)$, $F \in \operatorname{Hom}_{\mathbf{Man}^\infty}(N, M)$. Suppose that $S \subset M$ and S is an immersed submanifold of M, $F(N) \subset S$ and $F \in \operatorname{Hom}_{\mathbf{Top}}(N, S)$. Then $F \in \operatorname{Hom}_{\mathbf{Man}^\infty}(N, S)$.
Hint: Define $F_0 : N \to S$ by $F_0(p) = F(p)$. Then $F = \iota_S \circ F_0$.

Proof. Set $m := \dim M$, $k := \dim S$ and $n := \dim N$. Define $F_0 : N \to S$ by $F_0(p) := F(p)$. We note that $\iota_S \circ F_0 = F$. Since S is an immersed submanifold of M, ι_S is an immersion. Let $p \in N$. Define $q \in S$ by $q := F(p)$. Exercise 7.2.0.7 implies that there exist $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_S$ such that $q \in V$, $\iota_S(V) \subset U$ and $\phi \circ \iota_S \circ \psi^{-1} = (\mathrm{id}_{\psi(V)}, 0)$. Since F_0 is $(\mathcal{T}_N, \mathcal{T}_S)$-continous, $F_0^{-1}(V) \in \mathcal{T}_N$. Then there exists $(W, \eta) \in \mathcal{A}_N$ such that $p \in W$ and $W \subset F_0^{-1}(V)$. Define $\widehat{F} : \eta(W) \to \phi(U)$ and $\widehat{F}_0 : \eta(W) \to \psi(V)$ by $\widehat{F} := \phi \circ F \circ \eta^{-1}$ and $\widehat{F}_0 := \psi \circ F_0 \circ \eta^{-1}$. Since F is smooth, $\widehat{F}$ is smooth. Then

$$\begin{aligned}(\widehat{F}_0, 0) &= (\mathrm{id}_{\psi(V)} \circ \widehat{F}_0, 0) \\ &= (\mathrm{id}_{\psi(V)}, 0) \circ \widehat{F}_0 \\ &= (\phi \circ \iota_S \circ \psi^{-1}) \circ (\psi \circ F_0 \circ \eta^{-1}) \\ &= \phi \circ \iota_S \circ F_0 \circ \eta^{-1} \\ &= \phi \circ F \circ \eta^{-1} \\ &= \widehat{F}\end{aligned}$$

Since $\widehat{F}$ is smooth, we have that $\widehat{F}_0$ is smooth. Since $p \in N$ is arbitrary, we have that for each $p \in N$, there exists $(W, \eta) \in \mathcal{A}_N$ and $(V, \psi) \in \mathcal{A}_S$ such that $p \in W$, $F_0(p) \in V$, $W \cap F_0^{-1}(V) \in \mathcal{T}_N$ and $\psi \circ F_0 \circ \eta^{-1}|_{W \cap F_0^{-1}(V)}$ is smooth. Exercise 5.1.0.5 implies that F_0 is smooth. □

Exercise 8.1.0.27. Let $M, N \in \operatorname{Obj}(\mathbf{Man}^\infty)$, $F \in \operatorname{Hom}_{\mathbf{Man}^\infty}(N, M)$ and $S \subset M$. Suppose that S is an embedded submanifold of M and $F(N) \subset S$. Then $F \in \operatorname{Hom}_{\mathbf{Man}^\infty}(N, S)$.

Proof. Since S is an embedded submanifold of M, ι_S is a $\mathbf{Man}^\infty$-embedding. Let $V \in \mathcal{T}_S$. Then

$$\begin{aligned}V &= \iota_S(V) \\ &\in \mathcal{T}_M \cap S.\end{aligned}$$

Therefore there exists $U \in \mathcal{T}_M$ such that $V = U \cap S$. Since F is $(\mathcal{T}_N, \mathcal{T}_M)$-continuous, $F^{-1}(U) \in \mathcal{T}_N$. Hence

$$\begin{aligned} F^{-1}(V) &= F^{-1}(U \cap S) \\ &= F^{-1}(U) \cap F^{-1}(S) \\ &= F^{-1}(U) \cap N \\ &= F^{-1}(U) \\ &\in \mathcal{T}_N. \end{aligned}$$

Since $V \in \mathcal{T}_S$ is arbitrary, we have that for each $V \in \mathcal{T}_S$, $F^{-1}(V) \in \mathcal{T}_N$. Hence F is $(\mathcal{T}_N, \mathcal{T}_S)$-continuous. Since S is an embedded submanifold of M, S is an immersed submanifold of M. Exercise **??** (reference previous exercise here) implies that $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(N, S)$. □

Exercise 8.1.0.28. Uniqueness of Topological and Smooth Structure for Embedded Submanifolds
Let $(M, \mathcal{T}_M, \mathcal{A}_M), (S, \mathcal{T}_S, \mathcal{A}_S) \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$. If $(S, \mathcal{T}_S, \mathcal{A}_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$, then

1. $\mathcal{T}_S = \mathcal{T}_M \cap S$,
2. $\mathcal{A}_S = \mathcal{A}_M|_S$.

Proof. Suppose that $(S, \mathcal{T}_S, \mathcal{A}_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$.

1. Since $(S, \mathcal{T}_S, \mathcal{A}_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$, $\iota_S \in \mathrm{Iso}_{\mathbf{Top}}[(S, \mathcal{T}_S), (S, \mathcal{T}_M \cap S)]$. An exercise in the analysis notes section on subspaces implies that $\mathcal{T}_S = \mathcal{T}_M \cap S$.

2. Define $\iota : S \to S$ by $\iota(p) := p$. Clearly, ι is a bijection. Since $(S, \mathcal{T}_S, \mathcal{A}_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$, Exercise **??** (reference a previous exercise here) implies that S satisfies the local k-slice condition with respect to M. arg1 Exercise **??** (reference a previous exercise here) then implies that $((S, \mathcal{T}_M \cap S, \mathcal{A}_M|_S))$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$.

   - Since $(S, \mathcal{T}_S, \mathcal{A}_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$, $\iota \in \mathrm{Hom}_{\mathbf{Man}^\infty}[(S, \mathcal{T}_S, \mathcal{A}_S), (M, \mathcal{T}_M, \mathcal{A}_M)]$. Since $\iota(S) = S$, Exercise **??** the previous exercise implies that $\iota \in \mathrm{Hom}_{\mathbf{Man}^\infty}[(S, \mathcal{T}_S, \mathcal{A}_S), (S, \mathcal{T}_M \cap S, \mathcal{A}_M|_S)]$.
   - Since $(S, \mathcal{T}_M \cap S, \mathcal{A}_M|_S)$ is an embedded submanifold of $(M, \mathcal{T}_M, \mathcal{A}_M)$, $\iota^{-1} \in \mathrm{Hom}_{\mathbf{Man}^\infty}[(S, \mathcal{T}_M \cap S, \mathcal{A}_M|_S), (M, \mathcal{T}_M, \mathcal{A}_M)]$. Since $\iota^{-1}(S) = S$, Exercise **??** the previous exercise implies that $\iota^{-1} \in \mathrm{Hom}_{\mathbf{Man}^\infty}[(S, \mathcal{T}_M \cap S, \mathcal{A}_M|_S), (S, \mathcal{T}_S, \mathcal{A}_S)]$.

   Exercise 5.2.0.5 then implies that ι is a diffeomorphism and $\mathcal{A}_S = \mathcal{A}_M|_S$.

□

Exercise 8.1.0.29. Uniqueness of Smooth Structure for Immersed Submanifolds Let $(M, \mathcal{T}_M, \mathcal{A}_M) \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $(S, \mathcal{T}_S) \in \mathrm{Obj}(\mathbf{Man}^0)$ and $\mathcal{A}_1\mathcal{A}_2$ smooth structures on $(S, \mathcal{T}_S)$. Suppose that $S \subset M$. If $(S, \mathcal{T}_S, \mathcal{A}_1)$ and $(S, \mathcal{T}_S, \mathcal{A}_2)$ are immersed submanifolds of $(M, \mathcal{T}_M, \mathcal{A}_M)$, then $\mathcal{A}_1 = \mathcal{A}_2$.

Proof. Let $p \in S$. Since $(S, \mathcal{T}_S, \mathcal{A}_1)$, $(S, \mathcal{T}_S, \mathcal{A}_2)$ are immersed submanifolds of $(M, \mathcal{T}_M, \mathcal{A}_M)$, there exists $W_1, W_2 \in \mathcal{T}_S$ such that $p \in W_1 \cap W_2$ and $(W_1, \mathcal{T}_S \cap W_1, \mathcal{A}_1|_{W_1})$, $(W_2, \mathcal{T}_S \cap W_2, \mathcal{A}_2|_{W_2})$ are embedded submanifolds of $(M, \mathcal{T}_M, \mathcal{A}_M)$. Define $W \in \mathcal{T}_S$ by $W := W_1 \cap W_2$. Exercise **??** (reference previous exercise about open submanifolds here) implies that $(W, \mathcal{T}_S \cap W, \mathcal{A}_1|_W)$, $(W, \mathcal{T}_S \cap W, \mathcal{A}_2|_W)$ are embedded submanifolds of $(M, \mathcal{T}_M, \mathcal{A}_M)$. Exercise **??** (reference previous exercise here) implies that $\mathcal{T}_S \cap W = \mathcal{T}_M \cap W$ and

$$\begin{aligned} \mathcal{A}_1|_W &= \mathcal{A}_M|_W \\ &= \mathcal{A}_2|_W. \end{aligned}$$

Since $\mathcal{A}_1|_W \subset \mathcal{A}_1$ and $\mathcal{A}_2|_W \subset \mathcal{A}_2$, we have that $\mathcal{A}_1|_W, \mathcal{A}_2|_W \subset \mathcal{A}_1 \cap \mathcal{A}_2$. Since $\mathcal{A}_1$ is an atlas on $(S, \mathcal{T}_S)$, there exists $(V', \psi') \in \mathcal{A}_1$ such that $p \in V'$. Define $(V, \psi) \in \mathcal{A}_1|_W$ by $V := V' \cap W$ and $\psi := \psi'|_{V' \cap W}$. Then $p \in V$ and

$$\begin{aligned}(V, \psi) &\in \mathcal{A}_1|_W \\ &\subset \mathcal{A}_1 \cap \mathcal{A}_2.\end{aligned}$$

Since $p \in S$ is arbitrary, we have that for each $p \in S$, there exists $(V, \psi) \in \mathcal{A}_1 \cap \mathcal{A}_2$ such that $p \in V$. The axiom of choice implies that there exists $\mathcal{A} \subset \mathcal{A}_1 \cap \mathcal{A}_2$ such that for each $p \in S$, there exists $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Then $\mathcal{A}$ is a smooth atlas on $(S, \mathcal{T}_S)$. Since $\mathcal{A} \subset \mathcal{A}_1 \cap \mathcal{A}_2$, we have that

$$\begin{aligned}\mathcal{A}_1 &= \alpha(\mathcal{A}) \\ &= \mathcal{A}_2.\end{aligned}$$

□

Exercise 8.1.0.30. Let $M, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$ and S is an immersed submanifold of M. If for each $p \in S$, there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $S \cap U$ is an embedded submanifold of U, then S is an embedded submanifold of M.

Proof. Suppose that for each $p \in S$, there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $S \cap U$ is an embedded submanifold of U. Let $p \in S$. By assumption, there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $S \cap U$ is an embedded submanifold of U. Since U is an embedded submaifold of M, we have that $S \cap U$ is an embedded submanifold of M (need exercise showing composition of embeddings is embedding?). Then $S \cap U$ satisfies the local k-slice condition with respect to M. Thus there exists $(V, \psi) \in \mathbb{S}^k(M; S \cap U)$ such that $p \in V$ and $V \subset U$. By definition of $\mathbb{S}^k(M; S \cap U)$, we have that

$$\begin{aligned}\psi(S \cap V) &= \psi(V \cap (S \cap U)) \\ &= \psi(V) \cap \mathbb{S}^{n,k}.\end{aligned}$$

Hence $(V, \psi) \in \mathbb{S}^k(M; S)$. Since $p \in S$ is arbitrary, we have that for each $p \in S$, there exists $(V, \psi) \in \mathbb{S}^k(M; S)$ such that $p \in V$. Hence S satisfies the local k-slice condition with respect to M. Thus S is an embedded submanifold of M. □

8.2 Embedded Submanifolds

Definition 8.2.0.1. Let $(M, \mathcal{A}), (N, \mathcal{B})$ be smooth manifolds and $F : N \to M$ a smooth embedding. We define the restriction of $\mathcal{A}$ to $F(N)$, denoted $\mathcal{A}|^0_{F(N)}$, by

$$\mathcal{A}|^0_{F(N)} := \alpha(\{(F(V), \psi \circ F^{-1}) : (V, \psi) \in \mathcal{B}\})$$

Exercise 8.2.0.2. Let $(M, \mathcal{A}), (N, \mathcal{B})$ be smooth manifolds and $F : N \to M$ a smooth embedding. Then $\mathcal{A}|^0_{F(N)}$ is a smooth atlas on $F(N)$.

Proof. exercise in topological manifold section implies that $\mathcal{A}_0 \subset X^n(F(N))$ □

Definition 8.2.0.3. Let $(M, \mathcal{A}), (N, \mathcal{B})$ be smooth manifolds and $F : N \to M$ a smooth embedding. We define the smooth structure on $F(N)$ induced by F, denoted $\mathcal{A}|_{F(N)}$, by

$$\mathcal{A}|_{F(N)} := \alpha(\mathcal{A}|^0_{F(N)})$$

Exercise 8.2.0.4. Let $(M, \mathcal{A}), (N, \mathcal{B})$ be smooth manifolds and $F : N \to M$ a smooth embedding. Suppose that $\partial N = \varnothing$. Then $\mathcal{A}|_{F(M)}$ is the unique smooth structure on $F(M)$ such that $F : M \to F(M)$ is a diffeomorphism and $(F(M), \mathcal{A}_{F(M)})$ is an embedded submanifold of N.

Proof.

- Since $F : N \to M$ is a smooth embedding, $F : N \to F(M)$ is a bijection. F is a local diffeo. make exercise about local diffeo and bijecition imply diffeo. So F is a diffeomorphism
- Show $\iota : F(N) \to M$ is smooth embedding
- Let $\mathcal{A}'$ be a smooth structure on $F(N)$. Then cite exercise in section on smooth maps implies that $F^*\mathcal{A}' = \mathcal{N}$.

Question: can I define product and boundary submanifolds while discussing embedded submanifolds in an easier way than currently?

□

Exercise 8.2.0.5. Let M, S be smooth manifolds. Suppose that $S \subset M$. Then S is an embedded submanifold of M iff there exists smooth manifold N and smooth embedding $F : N \to M$ such that $F(N) = S$.

Proof. content... □

Exercise 8.2.0.6. talk about the boundary as an embedded submanifold. In particular if $\dim M = n$, then ∂M satisfies the local $n-1$-slice condition Let $M \in \text{Obj}(\mathbf{ManBnd}^\infty)$. Then ∂M is an embedded submanifold of M.

Proof. content... □

Exercise 8.2.0.7. Constant Rank Level Set Theorem:
Let $M, N \in \text{Obj}(\mathbf{Man}^\infty)$, $F \in \text{Hom}_{\mathbf{Man}^\infty}(M, N)$ and $q_0 \in F(M)$. Suppose F has constant rank and $\operatorname{rank} F = r$. Then

1. $F^{-1}(\{q_0\})$ satisfies the local $(m-r)$-slice condition with respect to M.
2. $(F^{-1}(\{q_0\}), \mathcal{T}_M \cap F^{-1}(\{q_0\}), \mathcal{A}_M|_{F^{-1}(\{q_0\})})$ is a properly embedded submanifold of M.

Proof.

1. Set $S := F^{-1}(\{q_0\})$. Let $p \in S$. Define $\text{proj}_{-r} : \mathbb{R}^m \to \mathbb{R}^r$ by $\text{proj}_{-r}(x^1, \dots, x^m) = (x^{m-r+1}, x^m)$. Since F has constant rank and $\operatorname{rank} F = r$, Exercise 7.1.0.3 (the local rank theorem) (add exercise about permutations on charts to get the 0's at the beginning) implies that there exist $(U_0, \phi_0) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ such that $p \in U$, $F(U) \subset V$, $\psi(q_0) = 0$ and $\psi \circ F \circ \phi_0^{-1} = (0, \text{proj}_{-r}|_{\phi_0(U_0)})$. Since $\phi(U_0) \in \mathcal{T}_{\mathbb{R}^m}$, an exercise about bases of the product topology in the analysis notes implies that there exists $A_0 \in \mathcal{T}_{\mathbb{R}^{m-r}}$ and $B_0 \in \mathcal{T}_{\mathbb{R}^r}$ such that $\phi_0(p) \in A_0 \times B_0$ and $A_0 \times B_0 \subset \phi(U_0)$. Set $U := \phi_0^{-1}(A_0 \times B_0)$ and $\phi := \phi_0|_U$. Then $(U, \phi) \in \mathcal{A}_M$, $p \in U$.

   - By definition, $\phi(U) = A_0 \times B_0$. Hence $\text{proj}_{m-r}(\phi(U)) = A_0$. Since $U \subset U_0$, for each $p' \in U \cap S$,

     $$\begin{aligned} 0 &= \psi(q_0) \\ &= \psi(F(p')) \\ &= \psi \circ F \circ \phi_0^{-1}(\phi_0(p')) \\ &= (0, \text{proj}^{-r}(\phi(p'))) \end{aligned}$$

     Thus for each $p' \in U \cap S$, $\text{proj}^{-r}(\phi(p')) = 0$ and therefore

     $$\begin{aligned} \phi(U \cap S) &\subset A_0 \times \{0\} \\ &= (A_0 \times B_0) \cap \mathbb{S}^{m,m-r} \\ &= \phi(U) \cap \mathbb{S}^{m,m-r}. \end{aligned}$$

   - Let $y \in \phi(U) \cap \mathbb{S}^{m,m-r}$. Then here exists $p' \in U$ such that $\phi(p') = y$. Since $\phi(U) \cap \mathbb{S}^{m,m-r} = A_0 \times \{0\}$, there exists $a \in A_0$ such that $y = (a, 0)$. Let $p' \in (U \cap S)^c$. Since $p' \in U$, we have that $p' \in S^c$. Thus $F^{-1}(p') \neq q_0$. Since ϕ is injective,

     $$\begin{aligned} 0 &= \psi(q_0) \\ &\neq \psi \circ F \circ \phi_0^{-1}(\phi_0(p')) \\ &= (0, \text{proj}_{-r}(\phi(p'))). \end{aligned}$$

     Therefore $\text{proj}_{-r}(\phi(p')) \neq 0$. Hence $\phi(p') \in (\mathbb{S}^{m,m-r})^c$. Since $p' \in (U \cap S)^c$ is arbitrary, we have that

     $$\begin{aligned} \phi(U \cap S)^c &= \phi((U \cap S)^c) \\ &\subset (\mathbb{S}^{m,m-r})^c \\ &\subset (\phi(U) \cap \mathbb{S}^{m,m-r})^c \end{aligned}$$

     Thus $\phi(U) \cap \mathbb{S}^{m,m-r} \subset \phi(U \cap S)$.

   Therefore $\phi(U \cap S) = \phi(U) \cap \mathbb{S}^{m,m-r}$ and $\phi(U \cap S)$ is a $(m-r)$-slice of $\phi(U)$. Hence (U, ϕ) is an $(m-r)$-slice chart on S. Since $p \in S$ is arbitrary, we have that for each $p \in S$, there exists $(U, \phi) \in \mathcal{A}_M$ such that $p \in U$ and (U, ϕ) is an $(m-r)$-slice chart on S. So S satisfies the local $(m-r)$-slice condition with respect to M.

2. Since F is $(\mathcal{T}_M, \mathcal{T}_N)$-continuous and $\{q_0\}$ is closed in $(N, \mathcal{T}_N)$, we have that S is closed in $(M, \mathcal{T}_M)$. Exercise **??** (a previous exercise) implies that S is properly embedded.

$\square$

Exercise 8.2.0.8. Submersion Level Set Theorem:
Let $M, N \in \text{Obj}(\mathbf{Man}^\infty)$, $F \in \text{Hom}_{\mathbf{Man}^\infty}(M, N)$. Set $m := \dim M$ and $n := \dim N$. Suppose F is a submersion. Then for each $q \in N$,

1. $F^{-1}(\{q\})$ satisfies the local $(m-n)$-slice condition with respect to M,
2. $(F^{-1}(\{q\}), \mathcal{T}_M \cap F^{-1}(\{q\}), \mathcal{A}_M|_{F^{-1}(\{q\})})$ is a properly embedded submanifold of M.

Proof. Since F is a submersion, F has constant rank and $\operatorname{rank} F = n$. Let $q \in N$. Exercise **??** (the previous exercise) implies that

1. $F^{-1}(\{q\})$ satisfies the local $(m-n)$-slice condition with respect to M,
2. $(F^{-1}(\{q\}), \mathcal{T}_M \cap F^{-1}(\{q\}), \mathcal{A}_M|_{F^{-1}(\{q\})})$ is a properly embedded submanifold of M.

□

Definition 8.2.0.9. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$ and $p \in M$ and $q \in N$. Then p is said to be a

- **regular point of** F if $DF(p) : T_pM \to T_{F(p)}N$ is surjective,
- **critical point of** F if p is not a regular point of F

and q is said to be a

- **regular value of** F if for each $x \in F^{-1}(\{q\})$, x is a regular point of F,
- **critical value of** F if q is not a regular value of F.

Note 8.2.0.10. In particular, if $\dim M < \dim N$, then for each $p \in M$, p is a critical point of F and for each $q \in N$, if $F^{-1}(\{q\}) = \varnothing$, then q is a regular value of F.

Exercise 8.2.0.11. Let $M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. If F is a submersion, then for each $q \in N$, q is a regular value of F.

Proof. Suppose that F is a submersion. Let $q \in N$ and $p \in F^{-1}(\{q\})$. Since F is a submersion, $DF(p)$ is surjective. Hence p is a regular point of F. Since $p \in F^{-1}(\{q\})$ is arbitrary, we have that for each $p \in F^{-1}(\{q\})$, p is a regular point of F. Hence q is a regular value of F. Since $q \in N$ is arbitrary, we have that for each $q \in N$, q is a regular value of F. □

Definition 8.2.0.12. Let $M, N, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. Suppose that $S \subset M$. Then S is said to be a **regular level set of** F if there exists $q \in N$ such that q is a regular value of F and $S = F^{-1}(\{q\})$.

Exercise 8.2.0.13. Regular Level Set Theorem:
Let $M, N, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. Set $m := \dim M$, $n := \dim N$ and $k := \dim S$. Suppose that $S \subset M$ and S is a regular level set of F. Then

1. S satisfies the local $(m-n)$-slice condition with respect to M,
2. $(S, \mathcal{T}_M \cap S, \mathcal{A}_M|_S)$ is a properly embedded submanifold of M.

Hint:
Define $U \subset M$ by $U := \{p \in M : \operatorname{rank} DF(p) = \dim N\}$ and consider Exercise 7.3.0.3.

Proof. Define $U \subset M$ by $U := \{p \in M : \operatorname{rank} DF(p) = \dim N\}$. Exercise 7.3.0.3 implies that $U \in \mathcal{T}_M$ and $F|_U$ is a submersion. Let $S \subset M$. Suppose that S is a regular level set of F. Then there exists $q \in N$ such that q is a regular value of F and $S = F^{-1}(\{q\})$. Since q is a regular value of F, for each $x \in S$, x is a regular point of F. Thus for each $x \in S$, $DF(x)$ is surjective. Thus $S \subset U$. Since $F|_U$ is a submersion and

$$\begin{aligned} S &= F^{-1}(\{q\}) \\ &= F|_U^{-1}(\{q\}), \end{aligned}$$

Exercise **??** (the previous exercise) implies that S is a properly embedded submanifold of U. Since $U \in \mathcal{T}_M$, U is a properly embedded submanifold of M. Hence $F^{-1}(\{q\})$ is a properly embedded submanifold of M. (flesh out some of the last details here, like composition of proper maps is proper, composition of $\mathbf{Man}^\infty$-embeddings is a $\mathbf{Man}^\infty$-embedding, etc)

1.

2.

FINISH!!! □

Exercise 8.2.0.14. Let $M, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Set $m := \dim M$ and $k := \dim S$. Suppose that $S \subset M$. Then S is an embedded submanifold of M iff for each $p \in S$, there exists $U \in \mathcal{T}_M$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(U, \mathbb{R}^{n-k})$ such that $p \in U$, F is a smooth submersion and $S \cap U$ is a regular level set of F.

Proof.

- $(\implies)$:

  - Suppose that S is an embedded submanifold of M. Let $p \in S$. Since S is an embedded submanifold of M, there exists $(U_0, \phi_0) \in \mathcal{A}_M|_S^0$ such that $p \in U$. Thus there exists $(U, \phi) \in \mathbb{S}^k(M; S)$ such that $U_0 = U \cap S$ and $\phi_0 = \pi^m_{[k]} \circ \phi$. Set $r := m - k$ and define $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(U, \mathbb{R}^r)$ by $F \circ \phi$. Then $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(U, \mathbb{R}^r)$ and $p \in U$. By definition of $\mathbb{S}^k(M; S)$, $\phi(S \cap U) = \phi(U) \cap \mathbb{S}^{m,k}$. Hence

$$\begin{aligned} F(S \cap U) &= \pi^m_{[-r]} \circ \phi(S \cap U) \\ &= \pi^m_{[-r]}(\phi(U) \cap \mathbb{S}^{m,k}) \\ &= \{0\} \end{aligned}$$

    Hence $S \cap U \subset F^{-1}(\{0\})$.
  - Let $q \in F^{-1}(\{0\})$. Then $q \in U$ and $F(q) = 0$. Since

$$\begin{aligned} \phi(q) &= (\pi^m_{[k]} \circ \phi(q), F(q)) \\ &= (\pi^m_{[k]} \circ \phi(q), 0) \\ &\in \mathbb{S}^{m,k}, \end{aligned}$$

    we have that

$$\begin{aligned} \phi(q) &\in \phi(U) \cap \mathbb{S}^{m,k} \\ &= \phi(S \cap U). \end{aligned}$$

    Since ϕ is a bijection, $q \in S \cap U$. Since $q \in F^{-1}(\{0\})$ is arbitrary, we have that for each $q \in F^{-1}(\{0\})$, $q \in S \cap U$. Thus $F^{-1}(\{0\}) \subset S \cap U$.

  Hence $F^{-1}(\{0\}) = S \cap U$. Let $q \in U$. Since $[D\phi(q)]_{\phi, \mathrm{id}_{\mathbb{R}^m}} = \begin{pmatrix} [D\pi^m_{[k]} \circ \phi(q)]_{\phi, \mathrm{id}_{\mathbb{R}^k}} \\ [DF(q)]_{\phi, \mathrm{id}_{\mathbb{R}^r}} \end{pmatrix}$ and $[D\phi(q)]_{\phi, \mathrm{id}_{\mathbb{R}^m}}$ is a bijection, we have that $\mathrm{rank}[DF(q)]_{\phi, \mathrm{id}_{\mathbb{R}^r}} = r$. Thus $DF(q)$ is surjective. Since $q \in U$ is arbitrary, we have that for each $q \in U$, $DF(q)$ is surjective. Thus F is a submersion. Since F is a submersion, Exercise **??** a previous exercise implies that 0 is a regular value of F. Since $F^{-1}(0) = S \cap U$, $S \cap U$ is a regular level set of F.

- $(\impliedby)$:
  Suppose that for each $p \in S$, there exists $U \in \mathcal{T}_M$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(U, \mathbb{R}^{m-k})$ such that $p \in U$, F is a smooth submersion and $S \cap U$ is a regular level set of F. Let $p \in S$. By assumption, there exists $U \in \mathcal{T}_M$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(U, \mathbb{R}^{m-k})$ such that $p \in U$, F is a smooth submersion and $S \cap U$ is a regular level set of F. Exercise **??** a previous exercise implies that $S \cap U$ is an embedded submanifold of U. Since $p \in S$ is arbitrary, we have that for each $p \in S$, there exists $U \in \mathcal{T}_M$ such that $p \in U$ and $S \cap U$ is an embedded submanifold of U. Exercise **??** (an exercise in the previous section) implies that S is an embedded submanifold of M.

□

Definition 8.2.0.15. Let $M, N, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $U \in \mathcal{T}_M$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(U, N)$. Suppose that $S \subset M$. Then F is said to be a

- **local defining map for** S if $S \cap U$ is a regular level set of F,
- **defining map for** S if F is a local defining map for S and $U = M$.

Exercise 8.2.0.16. Let $M, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Set $m := \dim M$ and $k := \dim S$. Suppose that $S \subset M$. Then S is an embedded submanifold of M iff for each $p \in S$, there exists $U \in \mathcal{T}_M$ and $F \in \mathrm{Hom}_{\mathbf{Man}^\infty}(U, \mathbb{R}^{m-k})$ such that $p \in U$ and F is a local defining map for $S \cap U$.

Proof. FINISH!!!, basically previous exercise □

8.3 Immersed Submanifolds

8.4 The Tangent Space of Submanifolds

Exercise 8.4.0.1. Let $M, S \in \text{Obj}(\mathbf{Man}^\infty)$. Suppose that $S \subset M$ and S is an embedded submanifold of M. Set $n := \dim M$ and $k := \dim S$. Let $(U_0, \phi_0) \in \mathbb{S}^k(M;S)$ with $\phi_0 = (x^1, \dots, x^n)$. Set $U := U_0 \cap S$ and $\phi := \pi^n_k \circ \phi_0|_U$ so that $(U, \phi) \in \mathcal{A}_M|^0_S$. Let $p \in U$. Then for each $j \in [k]$,

$$D(\iota_S)(p)\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right) = \frac{\partial}{\partial x^j}\bigg|_p$$

Proof. Let $j \in [k]$ and $f \in C^\infty_p(M)$. By construction, $f \circ \phi_0^{-1} = f \circ \phi^{-1} \circ \pi^n_k$. Thus

$$\begin{aligned}
D(\iota_S)(p)\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right)(f) &= \frac{\partial}{\partial \tilde{x}^j}\bigg|_p (f \circ \iota_S) \\
&= \frac{\partial}{\partial u^j}\bigg|_{\phi(p)} (f \circ \iota_S \circ \phi^{-1}) \\
&= \frac{\partial}{\partial u^j}\bigg|_{\phi(p)} (f \circ \phi^{-1}) \\
&= \lim_{\epsilon \to 0} \frac{f \circ \phi^{-1}(\phi(p) + \epsilon e^j) - f \circ \phi^{-1}(\phi(p))}{\epsilon}, \quad (\text{in } \mathbb{R}^k) \\
&= \lim_{\epsilon \to 0} \frac{f \circ \phi_0^{-1}(\phi_0(p) + \epsilon e^j) - f \circ \phi_0^{-1}(\phi_0(p))}{\epsilon}, \quad (\text{in } \mathbb{R}^n) \\
&= \frac{\partial}{\partial x^j}\bigg|_p f
\end{aligned}$$

Since $f \in C^\infty_p(M)$ is arbitrary, we have that

$$D(\iota_S)(p)\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right) = \frac{\partial}{\partial x^j}\bigg|_p.$$

□

discuss how to identify T_pM and T_pU where $U \in \mathcal{T}_M$. Can use germs since derivations at a point are determined locally around that point. So in some sense even though T_pM and T_pU are ismorphic, they are isomorphic in a strong sense where we can define derivations on the germ at a point and discarding any nonlocal information about the functions at the point.

Need to define T_pM in terms of germs, then explain how

Definition 8.4.0.2. Let $M, S \in \text{Obj}(\mathbf{Man}^\infty)$ and $p \in S$. Suppose that $S \subset M$ and S is an immersed submanifold of M. We identify T_pS with $\operatorname{Im} D\iota_S(p)$.

Exercise 8.4.0.3. Let $M, N, S \in \text{Obj}(\mathbf{Man}^\infty)$, $U \in \mathcal{T}_M$ and $F \in \text{Hom}_{\mathbf{Man}^\infty}(U, N)$. Suppose that $S \subset M$, S is an embedded submanifold of M and F is a local defining map for S. Then for each $p \in S \cap U$, $T_pS = \ker DF(p)$.

Proof. Let $p \in S \cap U$.

- Since F is a local defining map for S, $S \cap U$ is a regular level set of F. Hence there exists $q \in N$ such that q is a regular value of F and $S \cap U = F^{-1}(\{q\})$. Thus $F|_{S\cap U}$ is constant. Hence

$$\begin{aligned}
0 &= D(F|_{S\cap U})(p) \\
&= D(F \circ \iota_{S\cap U})(p) \\
&= DF(p) \circ D\iota_{S\cap U}(p).
\end{aligned}$$

Since S is an embedded submanifold of M, $\mathcal{T}_S = \mathcal{T}_M \cap S$ and $S \cap U \in \mathcal{T}_S$. Then

$$\begin{aligned}
T_pS &= T_pS \cap U \\
&= \operatorname{Im} D\iota_{S\cap U}(p) \\
&\subset \ker DF(p).
\end{aligned}$$

- Set $m := \dim M$, $n := \dim N$ and $k := \dim S$. Since q is a regular value of F, $DF(p)$ is surjective. Exercise **??** (an exercise in the previous section on regular level sets dimension) implies that

$$\begin{aligned}
\dim \ker DF(p) &= \dim T_pM - \dim \operatorname{Im} DF(p) \\
&= \dim T_pM - \dim T_{F(p)}N \\
&= m - n \\
&= \dim T_pS \cap U \\
&= \dim T_pS.
\end{aligned}$$

Since $T_pS \subset \ker DF(p)$ and $\dim T_pS = \dim \ker DF(p)$, we have that $T_pS = \ker DF(p)$. □

8.5 Transverse Submanifolds

Definition 8.5.0.1. Let $M, S_1, S_2 \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that $S_1, S_2 \subset M$, S_1, S_2 are immersed submanifolds of M. Then S_1 and S_2 are said to be **transverse** if for each $p \in S_1 \cap S_2$, $T_pM = T_pS_1 + T_pS_2$.

Exercise 8.5.0.2. Define $S_1, S_2 \subset \mathbb{R}^n$ by $S_1 := \{(a, 0) \in \mathbb{R}^n : a \in \mathbb{R}^k\}$ and $S_2 := \{(0, b) \in \mathbb{R}^n : b \in \mathbb{R}^{n-k}\}$. Then S_1 and S_2 are transverse.

Proof. Define $\phi_0, \psi_0 : \mathbb{R}^n \to \mathbb{R}^n$ by $\phi_0(a^1, \dots, a^n) := (a^1, \dots, a^n)$ and $\phi_0(a^1, \dots, a^k, a^{k+1}, \dots, a^n) := (a^{k+1}, \dots, a^n, a^1, \dots, a^k)$. Write $\phi_0 = (x^1, \dots, x^n)$ and $\psi_0 = (y^1, \dots, y^n)$. Then $(\mathbb{R}^n, \phi_0) \in \mathbb{S}^k(\mathbb{R}^n, S_1)$ and $(\mathbb{R}^n, \psi_0) \in \mathbb{S}^k(\mathbb{R}^n, S_2)$. Set $\phi := \pi^n_{[k]} \circ \phi_0|_{S_1}$ and $\psi := \pi^n_{[n-k]} \circ \psi_0|_{S_2}$. Write $\phi = (\tilde{x}^1, \dots, \tilde{x}^k)$ and $\psi = (\tilde{y}^1, \dots, \tilde{y}^{n-k})$. An exercise in the section on tangent space of submanifolds implies that for each $j \in [k]$,

$$\begin{aligned} D\iota_{S_1}(0)\bigg(\frac{\partial}{\partial \tilde{x}^j}\bigg|_0\bigg) &= \frac{\partial}{\partial x^j}\bigg|_0 \\ &= \frac{\partial}{\partial u^j}\bigg|_0 \end{aligned}$$

and for each $j \in [n-k]$

$$\begin{aligned} D\iota_{S_2}(0)\bigg(\frac{\partial}{\partial \tilde{y}^j}\bigg|_0\bigg) &= \frac{\partial}{\partial y^j}\bigg|_0 \\ &= \frac{\partial}{\partial u^{k+j}}\bigg|_0. \end{aligned}$$

Hence

$$\begin{aligned} T_0(\mathbb{R}^n) &= \operatorname{span}\bigg\{\frac{\partial}{\partial u^j}\bigg|_0 : j \in [k]\bigg\} \oplus \operatorname{span}\bigg\{\frac{\partial}{\partial u^{k+j}}\bigg|_0 : j \in [n-k]\bigg\} \\ &= \operatorname{Im} D\iota_{S_1}(0) \oplus \operatorname{Im} D\iota_{S_2}(0) \\ &= T_0S_1 \oplus T_0S_2. \end{aligned}$$

Since $S_1 \cap S_2 = \{0\}$, we have that for each $p \in S_1 \cap S_2$, $T_p(\mathbb{R}^n) = T_pS_1 \oplus T_pS_2$. Hence S_1 and S_2 are transverse. □

Exercise 8.5.0.3. Let $M, S \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $p \in S$. Suppose that $S \subset M$, S is an embedded submanifold of M and $\dim S < \dim M$. Then there exists $S' \in \mathrm{Obj}(\mathbf{Man}^\infty)$ such that $S' \subset M$, S' is an immersed submanifold of M, $p \in S'$ and S, S' are transverse.

Proof. Set $n := \dim M$ and $k := \dim S$. Then there exists $(U, \phi) \in \mathcal{A}_M|^0_S$ such that $p \in U$ and $\phi(p) = 0$. Then there exists $(U_0, \phi_0) \in \mathbb{S}^k(M; S)$ such that $U = U_0 \cap S$ and $\phi = \pi^n_k \circ \phi_0|_U$. Thus $\phi_0(p) = 0$. Write $\phi_0 = (x^1, \dots, x^n)$ and $\phi = (\tilde{x}^1, \dots, \tilde{x}^k)$. Define $B, B' \subset \mathbb{R}^n$ by $B := \{(a, 0) \in \mathbb{R}^n : a \in \mathbb{R}^k\} \cap \phi_0(U_0)$ and $B' := \{(0, b) \in \mathbb{R}^n : b \in \mathbb{R}^{n-k}\} \cap \phi_0(U_0)$. Then

$$\begin{aligned} B &= \phi_0(U_0) \cap \mathbb{S}^{n,k} \\ &= \phi_0(U_0 \cap V) \\ &= \phi_0(U) \end{aligned}$$

Define $U' \subset M$, $\sigma \in S_n$ and $\psi_0 : U_0 \to \sigma \cdot \phi_0(U_0)$ by $U' := \phi_0^{-1}(B')$, $\sigma := \begin{pmatrix} 1 & \dots k & k+1 & \dots & n \\ k+1 & \dots n & 1 & \dots & k \end{pmatrix}$ and $\psi_0 := \sigma \cdot \phi_0$. Then need exercise saying U' is embedded submanifold of M, $(U_0, \psi_0) \in \mathcal{A}_M$ and

$$\begin{aligned} \psi_0(U_0 \cap U') &= \psi_0(U') \\ &= \sigma \cdot \phi_0(U') \\ &= \sigma \cdot B' \\ &= \sigma \cdot [\phi_0(U_0) \cap \{(0, b) \in \mathbb{R}^n : b \in \mathbb{R}^{n-k}\}] \\ &= \sigma \cdot \phi_0(U_0) \cap \sigma \cdot \{(0, b) \in \mathbb{R}^n : b \in \mathbb{R}^{n-k}\} \\ &= \psi_0(U_0) \cap \mathbb{S}^{n,n-k}. \end{aligned}$$

Thus $(U_0, \psi_0) \in \mathbb{S}^{n-k}(M, U')$. Write $\psi_0 = (y^1, \dots, y^n)$. Define $(U', \psi') \in \mathcal{A}_M|_{U'}$ by $\psi' := \pi^n_{n-k} \circ \psi_0|_{U'}$. Write $\psi' = (\tilde{y}^1, \dots, \tilde{y}^{n-k})$. Since $B \cap B' = \{0\}$,

$$\begin{aligned} U \cap U' &= \phi_0^{-1}(B) \cap \phi_0^{-1}(B') \\ &= \phi_0^{-1}(B \cap B') \\ &= \phi_0^{-1}(\{0\}) \\ &= p. \end{aligned}$$

An exercise in the section on tangent spaces of submanifolds implies that for each $j \in [k]$

$$D\iota_U(p)\left(\frac{\partial}{\partial \tilde{x}^j}\bigg|_p\right) = \frac{\partial}{\partial x^j}\bigg|_p$$

and for each $j \in [n-k]$

$$\begin{aligned} D\iota_{U'}(p)\left(\frac{\partial}{\partial \tilde{y}^j}\bigg|_p\right) &= \frac{\partial}{\partial y^j}\bigg|_p \\ &= \frac{\partial}{\partial x^{k+j}}\bigg|_p. \end{aligned}$$

Therefore

$$\begin{aligned} T_pM &= \operatorname{span}\left\{\frac{\partial}{\partial x^j}\bigg|_p : j \in [k]\right\} \oplus \operatorname{span}\left\{\frac{\partial}{\partial x^{k+j}}\bigg|_p : j \in [n-k]\right\} \\ &= \operatorname{Im} D\iota_U(p) \oplus \operatorname{Im} D\iota_{U'}(p) \\ &= T_pU \oplus T_pU'. \end{aligned}$$

Set $S' := U'$. Since $U \in \mathcal{T}_V$ and $V \in \mathcal{T}_S$, we have that $U \in \mathcal{T}_S$. Thus

$$\begin{aligned} T_pM &= T_pU \oplus T_pU' \\ &= T_pS \oplus T_pS'. \end{aligned}$$

Let $q \in S \cap S'$. Then

$$\begin{aligned} q &\in S' \\ &= U' \\ &\subset U_0. \end{aligned}$$

and therefore

$$\begin{aligned} q &\in U_0 \cap S \\ &= U. \end{aligned}$$

Hence

$$\begin{aligned} q &\in U \cap U' \\ &= \{p\}. \end{aligned}$$

Since $q \in S \cap S'$ is arbitrary, we have that $S \cap S' \subset \{p\}$. Since $\{p\} \subset S \cap S'$, we have that $S \cap S' = \{p\}$. Thus for each $q \in S \cap S'$, $T_pM = T_pS \oplus T_pS'$ and S, S' are transverse. □

Exercise 8.5.0.4. Let $M, N, S_1, S_2, E_1, E_2 \in \operatorname{Obj}(\mathbf{Man}^\infty)$. Suppose that $S_1, S_2 \subset M$, S_1, S_2 are immersed submanifolds of M, $E_1, E_2 \subset N$, E_1, E_2 are immersed submanifolds of N. If $S_1, 2_1$ are transverse and E_1, E_2 are transverse, then $S_1 \times E_1$ and $S_2 \times E_2$ are transverse.

Proof. □

need exercise about $S \subset M, E \subset N$ are embedded submanifolds, then $S \times E \subset M \times N$ is embedded submanifold

Exercise 8.5.0.5. generalize the preimage submanifold result using transversality

Chapter 9

Quotient Manifolds

the surjective submersion assumption is not necessary

Exercise 9.0.0.1. Let $M, R \in \text{Obj}(\mathbf{Man}^\infty)$. Suppose that R is a properly embedded submanifold of $M \times M$, R is an equivlance relation on M, and $\text{proj}_1|_R : R \to M$ the projection map. Then

1. for each $U \in \mathcal{T}_M$, $\pi^{-1}(\pi(U)) = \text{proj}_1((M \times U) \cap R)$,
2. $\pi : M \to M/R$ is open,
3. M/R is Hausdorff.

Proof.

1. Let $U \in \mathcal{T}_M$ and $x \in M$. Then

$$\begin{aligned} x \in \pi^{-1}(\pi(U)) &\iff \pi(x) \in \pi(U) \\ &\iff \text{there exists } u \in U \text{ such that } \pi(x) = \pi(u) \\ &\iff \text{there exists } u \in U \text{ such that } (x, u) \in R \\ &\iff \text{there exists } u \in U \text{ such that } (x, u) \in (M \times U) \cap R \\ &\iff x \in \text{proj}_1((M \times U) \cap R) \end{aligned}$$

   Hence $\pi^{-1}(\pi(U)) = \text{proj}_1((M \times U) \cap R)$. Since $U \in \mathcal{T}_M$ is arbitrary, we have that for each $U \in \mathcal{T}_M$, $\pi^{-1}(\pi(U)) = \text{proj}_1((M \times U) \cap R)$.

2. Let $U \in \mathcal{T}_M$. Then $(M \times U) \cap R \in \mathcal{T}_R$. Since $\text{proj}_1|_R$ is a surjective submersion, Exercise 7.3.0.10 implies that $\text{proj}_1|_R$ is open. Part (1) implies that for each $U \in \mathcal{T}_M$,

$$\begin{aligned} \pi^{-1}(\pi(U)) &= \text{proj}_1((M \times U) \cap R) \\ &= \text{proj}_1|_R((M \times U) \cap R) \\ &\in \mathcal{T}_M \end{aligned}$$

   Since π is a quotient map, an exercise in the analysis notes section on the quotient topology implies that π is open.

3. Since R is properly embedded an exercise in the section on embedded submanifolds implies that R is closed in $M \times M$. An exercise in the analysis notes section on separation axioms on quotient spaces implies that M/R is Hausdorff.

□

Exercise 9.0.0.2. Let $M, R \in \text{Obj}(\mathbf{Man}^\infty)$. Suppose that R is a properly embedded submanifold of $M \times M$, R is an equivlance relation on M, and $\text{proj}_1|_R, \text{proj}_2|_R : R \to M$ the projection maps. Then for each $p \in M$, $\pi(p)$ is a properly embedded submanifold of M and $\dim \pi(p) = \dim R - \dim M$.
Hint: For each $p \in M$, $\pi(p) = \text{proj}_1|_R(\text{proj}_2|_R^{-1}(\{p\}))$ and $\text{proj}_1|_{M \times \{p\}}$ is a diffeomorphism.

Proof. Let $p \in M$. Exercise **??** implies that $\mathrm{proj}_1 : M \times M \to M$ is a submersion. Exercise **??** implies that $M \times \{p\}$ is an embedded submanifold of $M \times M$. Exercise **??** implies that $\mathrm{proj}_2|_R$ is a submersion. Since $\mathrm{proj}_2|_R$ is a surjective submersion, Exercise **??** implies that $\mathrm{proj}_2|_R^{-1}(\{p\})$ is a properly embedded submanifold of R and $\dim \mathrm{proj}_2|_R^{-1}(\{p\}) = \dim R - \dim M$. need to show $\mathrm{proj}_2|_R^{-1}(\{p\})$ is an embedded submanifold of $M \times \{p\}$. Since $\mathrm{proj}_1|_{M\times\{p\}}$ is a diffeomorphism and $\pi(p) = \mathrm{proj}_1|_{M\times\{p\}}(\mathrm{proj}_2|_R^{-1}(\{p\}))$, Exercise **??** make exercise in the section on embedded submanifolds implies that $\pi(p)$ is an embedded submanifold of M and $\dim \pi(p) = \dim R - \dim M$. □

Exercise 9.0.0.3. Let M, N, E be smooth manifolds with $\dim M = m$, $\dim N = n$ and $\dim E = e$. Suppose that N is an embedded submanifold of E. Then M is an embedded submanifold of N iff M is an embedded submanifold of E.

Proof. Exercise **??** implies that N satisfies the local n-slice condition with respect to E.

- $(\implies)$:
  Suppose that M is an embedded submanifold of N. Exercise **??** implies that M satisfies the local m-slice condition with respect to N. Let $p \in M$. Then there exists $(U_N, \phi_N) \in \mathbb{S}^m(N; M)$ and $(U_E, \phi_E) \in \mathbb{S}^n(E; N)$ such that $p \in U_N \cap U_E$.
- $(\impliedby)$:

□

Definition 9.0.0.4. content...

Chapter 10

The Tangent and Cotangent Bundles

10.1 Introduction

Definition 10.1.0.1. Let $M \in \operatorname{Obj}(\mathbf{ManBnd}^\infty)$. Set $n := \dim M$. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p \in M} T_p M$$

and we define the **tangent bundle projection**, denoted $\pi_{TM} : TM \to M$, by

$$\pi_{TM}(p, v) := p$$

Let $(U, \phi) \in \mathcal{A}_M$ with $\phi = (x^1, \dots, x^n)$. We define $\tilde{\phi} : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n$ by

$$\tilde{\phi}\bigg(p, \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j}\bigg|_p\bigg) := (\phi(p), \xi^1, \dots, \xi^n)$$

Note 10.1.0.2. When the context is clear, we write π in place of π_{TM}.

Exercise 10.1.0.3. Let $M \in \operatorname{Obj}(\mathbf{ManBnd}^\infty)$ and $(U, \phi) \in \mathcal{A}_M$. Set $n := \dim M$. Then

- π is surjective,
- for each $A \subset U$, $\tilde{\phi}(\pi^{-1}(A)) = \phi(A) \times \mathbb{R}^n$.

Proof. FINISH!!! □

Exercise 10.1.0.4. Let $M \in \operatorname{Obj}(\mathbf{ManBnd}^\infty)$. Then there exists a unique topology $\mathcal{T}_{TM}$ on TM and smooth structure $\mathcal{A}_{TM}$ on $(TM, \mathcal{T}_{TM})$ such that $(TM, \mathcal{T}_{TM}, \mathcal{A}_{TM}) \in \operatorname{Obj}(\mathbf{ManBnd}^\infty)$, $(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)_{\alpha \in \Gamma} \subset \mathcal{A}_{TM}$ and $\pi \in \operatorname{Hom}_{\mathbf{ManBnd}^\infty}(TM, M)$.

Proof. Write $\mathcal{A}_M = (U_\alpha, \phi_\alpha)_{\alpha \in \Gamma}$.

(a) Let $\alpha \in \Gamma$. Since $U_\alpha \in \mathcal{T}_M$ and ϕ_α is a homeomorphism, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n_n}$. Hence

$$\begin{aligned}\tilde{\phi}_\alpha(\pi^{-1}(U_\alpha)) &= \phi_\alpha(U_\alpha) \times \mathbb{R}^n \\ &\in \mathbb{H}^{2n}_n.\end{aligned}$$

(b) Let $\alpha, \beta \in \Gamma$. Since $U_\alpha, U_\beta \in \mathcal{T}_M$, we have that $U_\alpha \cap U_\beta \in \mathcal{T}_M$. Since ϕ_α is a homeomorphism, and $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n_n}$. Therefore

$$\begin{aligned}\tilde{\phi}_\alpha(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)) &= \tilde{\phi}_\alpha(\pi^{-1}(U_\alpha \cap U_\beta)) \\ &= \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n \\ &\in \mathcal{T}_{\mathbb{H}^{2n}_n}.\end{aligned}$$

(c) Let $\alpha, \beta \in \Gamma$. Write $\phi_\alpha = (x^1, \dots, x^n)$. Then $\tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^n$ is a bijection with

$$\tilde{\phi}_\alpha^{-1}(a, \xi^1, \dots, \xi^n) = \bigg(\phi_\alpha^{-1}(a), \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_{\phi^{-1}(a)}\bigg).$$

(d) Let $\alpha, \beta \in \Gamma$. Write $\phi_\alpha = (x^1, \dots, x^n)$ and $\phi_\beta = (y^1, \dots, y^n)$. Set $f_\alpha := \tilde{\phi}_\alpha|_{\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta)}$ and $f_\beta := \tilde{\phi}_\beta|_{\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta)}$. Let $(a, \xi^1, \dots, \xi^n) \in \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$. Then

$$\begin{aligned}
f_\beta \circ f_\alpha^{-1}(a, \xi^1, \dots, \xi^n) &= \tilde{\phi}_\beta\bigg(\phi_\alpha^{-1}(a), \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_{\phi_\alpha^{-1}(a)}\bigg) \\
&= \tilde{\phi}_\beta\bigg(\phi_\alpha^{-1}(a), \sum_{k=1}^n \bigg[\sum_{j=1}^n \xi^j \frac{\partial y^k}{\partial x^j}(\phi_\alpha^{-1}(a))\bigg] \frac{\partial}{\partial y^k}\bigg|_{\phi_\alpha^{-1}(a)}\bigg) \\
&= \bigg(\phi_\beta(\phi_\alpha^{-1}(a)), \sum_{j=1}^n \xi^j \frac{\partial y^1}{\partial x^j}(\phi_\alpha^{-1}(a)), \dots, \sum_{j=1}^n \xi^j \frac{\partial y^n}{\partial x^j}(\phi_\alpha^{-1}(a))\bigg).
\end{aligned}$$

Since $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta) \in \mathcal{A}_M$, we have that $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ are smoothly compatible. Hence $\phi_\beta \circ \phi_\alpha|_{U_\alpha\cap U_\beta}^{-1}$ is smooth. In particular, for each $k \in [n]$, $y^k \circ \phi|_{U_\alpha\cap U_\beta}^{-1}$ is smooth. By definition, for each $a \in \phi_\alpha(U_\alpha \cap U_\beta)$ and $j, k \in [n]$, we have that $\frac{\partial y^k}{\partial x^j}(\phi_\alpha^{-1}(a)) = \frac{\partial}{\partial u^j}[y^k \circ \phi_\alpha|_{U_\alpha\cap U_\beta}^{-1}](a)$. Hence for each $j, k \in [n]$, $\frac{\partial y^k}{\partial x^j} \circ \phi_\alpha|_{U_\alpha\cap U_\beta}^{-1}$ is smooth. Thus $\tilde{\phi}_\beta|_{\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta)} \circ \tilde{\phi}_\alpha|_{\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta)}^{-1}$ is smooth.

(e) Since $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, M is second-countable. Thus M is Lindelof. Since $(U_\alpha, \phi_\alpha)_{\alpha\in A}$ is an atlas on M, $(U_\alpha)_{\alpha\in\Gamma}$ is an open cover of M. Hence there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha\in\Gamma'} U_\alpha$. Hence

$$\begin{aligned}
TM &= \pi^{-1}(M) \\
&\subset \pi^{-1}\bigg(\bigcup_{\alpha\in\Gamma'} U_\alpha\bigg) \\
&= \bigcup_{\alpha\in\Gamma'} \pi^{-1}(U_\alpha).
\end{aligned}$$

(f) Let $(p_1, v_1), (p_2, v_2) \in TM$.

- Suppose that $p_1 \neq p_2$. Since $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, M is Hausdorff. Thus there exist $U_1', U_2' \in \mathcal{T}_M$ such that $p_1 \in U_1'$, $p_2 \in U_2'$ and $U_1' \cap U_2' = \varnothing$. Since $(U_\alpha)_{\alpha\in\Gamma}$ is an open cover of M, there exist $\alpha_1', \alpha_2' \in \Gamma$ such that $p_1 \in U_{\alpha_1'}$ and $p_2 \in U_{\alpha_2'}$. Set $U_1 := U_1' \cap U_{\alpha_1'}$, $U_2 := U_2' \cap U_{\alpha_2'}$, $\phi_1 := \phi_{\alpha_1'}|_{U_1}$ and $\phi_2 := \phi_{\alpha_2'}|_{U_2}$. Exercise **??** (reference ex here) implies that $(U_1, \phi_1), (U_2, \phi_2) \in \mathcal{A}_M$. Hence there exists $\alpha_1, \alpha_2 \in \Gamma$ such that $(U_1, \phi_1) = (U_{\alpha_1}, \phi_{\alpha_1})$ and $(U_2, \phi_2) = (U_{\alpha_2}, \phi_{\alpha_2})$. By construction, $p_1 \in U_{\alpha_1}$, $p_2 \in U_{\alpha_2}$ and $U_{\alpha_1} \cap U_{\alpha_2} = \varnothing$. Therefore $(p_1, v_1) \in \pi^{-1}(U_{\alpha_1})$, $(p_2, v_2) \in \pi^{-1}(U_{\alpha_2})$ and

$$\begin{aligned}
\pi^{-1}(U_{\alpha_1}) \cap \pi^{-1}(U_{\alpha_2}) &= \pi^{-1}(U_{\alpha_1} \cap U_{\alpha_2}) \\
&= \pi^{-1}(\varnothing) \\
&= \varnothing.
\end{aligned}$$

- Suppose that $p_1 = p_2$. Since $\mathcal{A}_M$ is an atlas on M, there exists $\alpha \in \Gamma$ such that $p_1 \in U_\alpha$. Since $p_1 = p_2$, we have that $(p_1, v_1), (p_2, v_2) \in \pi^{-1}(U_\alpha)$.

Exercise 4.1.0.14 implies that there exists a unique topology $\mathcal{T}_{TM}$ on TM and smooth structure $\mathcal{A}_{TM}$ on $(TM, \mathcal{T}_{TM})$ such that $(TM, \mathcal{T}_{TM}, \mathcal{A}_{TM}) \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)_{\alpha\in\Gamma} \subset \mathcal{A}_{TM}$.

Let $(p, v) \in TM$. Since $(\pi^{-1}(U_\alpha), \tilde{\phi}_\alpha)_{\alpha\in\Gamma} \subset \mathcal{A}_{TM}$ is an atlas on TM, there exists $\alpha \in \Gamma$ such that

$(p,v) \in \pi^{-1}(U_\alpha)$. Set $U := \pi^{-1}(U_\alpha)$, $V := U_\alpha$, $\phi := \tilde{\phi}_\alpha$ and $\psi := \phi_\alpha$. $(U,\phi) \in \mathcal{A}_{TM}$, $(V,\psi) \in \mathcal{A}_M$, $(p,v) \in U$, $\pi(p,v) \in V$ and

$$\begin{aligned} U \cap \pi^{-1}(V) &= \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\alpha) \\ &= \pi^{-1}(U_\alpha) \\ &\in \mathcal{T}_{TM}. \end{aligned}$$

Write $\phi_\alpha = (x^1, \ldots, x^n)$. Then for each $(a, \xi^1, \ldots, \xi^n) \in \tilde{\phi}_\alpha(\pi^{-1}(U_\alpha))$,

$$\begin{aligned} \psi \circ \pi \circ \phi|_{U \cap \pi^{-1}(V)}^{-1}(a, \xi^1, \ldots, \xi^n) &= \phi_\alpha \circ \pi \circ \tilde{\phi}_\alpha|_{\pi^{-1}(U_\alpha)}^{-1}(a, \xi^1, \ldots, \xi^n) \\ &= \phi_\alpha \circ \pi \bigg(\phi_\alpha^{-1}(a), \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_{\phi_\alpha^{-1}(a)} \bigg) \\ &= \phi_\alpha(\phi_\alpha^{-1}(a)) \\ &= \mathrm{id}_{\phi_\alpha(U_\alpha)}(a) \end{aligned}$$

Hence $\psi \circ \pi \circ \phi|_{U \cap \pi^{-1}(V)}^{-1} = \mathrm{id}_{\phi_\alpha(U_\alpha)}$ which is smooth. Exercise 5.1.0.5 implies that π is smooth. $\square$

Exercise 10.1.0.5. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. Then $\pi : TM \to M$ is a submersion.

Proof. Let $(p,v) \in TM$. Choose $(U,\phi) \in \mathcal{A}_M$ such that $p \in U$. Set $V := \pi^{-1}(U)$ and $\psi := \tilde{\phi}$. Then $(V,\psi) \in \mathcal{A}_{TM}$, $(p,v) \in V$, $U = \pi(V)$,

$$\begin{aligned} \psi(V) &= \tilde{\phi}(\pi^{-1}(U)) \\ &= \phi(U) \times \mathbb{R}^n, \end{aligned}$$

and since π is surjective,

$$\begin{aligned} \pi(V) &= \pi(\pi^{-1}(U)) \\ &= U. \end{aligned}$$

Since for each $(a, \xi^1, \ldots, \xi^n) \in \psi(V)$,

$$\begin{aligned} \phi \circ \pi \circ \psi^{-1}(a, \xi^1, \ldots, \xi^n) &= \phi \circ \pi \bigg(\phi^{-1}(a), \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_{\phi^{-1}(a)} \bigg) \\ &= \phi(\phi^{-1}(a)) \\ &= a \\ &= \mathrm{proj}_{[n]}^{2n}(a), \end{aligned}$$

we have that $\phi \circ \pi \circ \psi^{-1} = \mathrm{proj}_{[n]}^{2n}(a)|_{\psi(V)}$. Since $(p,v) \in TM$ is arbitrary, we have that for each $(p,v) \in TM$, there exists $(U,\phi) \in \mathcal{T}_M, (V,\psi) \in \mathcal{T}_{TM}$ such that $(p,v) \in V$, $U = \pi(V)$ and $\phi \circ \pi \circ \psi^{-1} = \mathrm{proj}_{[n]}^{2n}|_{\psi(V)}$. Exercise 7.3.0.9 implies that π is a submersion. $\square$

Exercise 10.1.0.6. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $(U,\phi) \in \mathcal{A}_M$. Set $n := \dim M$ and write $\phi = (x^1, \ldots, x^n)$ and $\tilde{\phi} = (\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}^1, \ldots, \tilde{y}^n)$. Then for each $(p,v) \in \pi^{-1}(U)$,

1. $[D\pi(p,v)]_{\tilde{\phi},\phi} = \begin{pmatrix} I_n & 0_n \end{pmatrix}$
2. $\ker D\pi(p,v) = \mathrm{span}\left\{ \frac{\partial}{\partial \tilde{y}^j}\Big|_{(p,v)} : j \in [n] \right\}$

Proof. 1. The previous exercise Exercise **??** implies that for each $(p,v) \in \pi^{-1}(U)$, $\phi \circ \pi \circ \tilde{\phi}^{-1} = \mathrm{proj}_{[n]}^{2n}|_{\phi(U) \times \mathbb{R}^n}$. Hence

$$\begin{aligned} [D\pi(p,v)]_{\tilde{\phi},\phi} &= [D\,\mathrm{proj}_{[n]}^{2n}(p,v)] \\ &= \begin{pmatrix} I_n & 0_n \end{pmatrix}. \end{aligned}$$

2. Clear from previous part.

□

Definition 10.1.0.7. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$ and $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$. We define the **push-forward of** F, denoted by $F_* : TM \to TN$ by

$$F_*(p, v) := (F(p), DF(p)(v))$$

Note 10.1.0.8. Other common notations for F_* are DF and TF.

Exercise 10.1.0.9. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$ and $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$. Then

1. $\pi_{TN} \circ F_* = F \circ \pi_{TM}$, i.e. the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "F_*"] \arrow[d, "\pi_{TM}"'] & TN \arrow[d, "\pi_{TN}"] \\ M \arrow[r, "F"'] & N \end{tikzcd}$$

2. for each $V \in \mathcal{T}_N$, $F_*^{-1}(\pi_{T_N}^{-1}(V)) = \pi_{TM}^{-1}(F^{-1}(V))$

Proof.

1. We note that for each $(p, v) \in TM$,

$$\begin{aligned} \pi_{TN} \circ F_*(p, v) &= \pi_{TN}(F(p), DF(p)(v)) \\ &= F(p) \\ &= F \circ \pi_{TM}(p, v). \end{aligned}$$

   Thus $\pi_{TN} \circ F_* = F \circ \pi_{TM}$.

2. Let $V \in \mathcal{T}_N$. Then

$$\begin{aligned} F_*^{-1}(\pi_{T_N}^{-1}(V)) &= (\pi_{TN} \circ F_*)^{-1}(V) \\ &= (F \circ \pi_{TM})^{-1}(V) \\ &= \pi_{TM}^{-1}(F^{-1}(V)). \end{aligned}$$

□

Exercise 10.1.0.10. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$ and $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$. Then $F_* \in \text{Hom}_{\mathbf{ManBnd}^\infty}(TM, TN)$.

Proof. Let $(p, v) \in TM$. Since $\mathcal{A}_M$ is an atlas on M and $\mathcal{A}_N$ is an atlas on N, there exist $(U, \phi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ such that $p \in U$ and $F(p) \in V$. Since $p \in U$, $(p, v) \in \pi_{TM}^{-1}(U)$. The previous exercise implies that $F_*^{-1}(\pi_{TN}^{-1}(V)) = \pi_{TM}^{-1}(F^{-1}(V))$. Since F is smooth, $U \cap F^{-1}(V) \in \mathcal{T}_M$. Since π_{TM} is smooth, we have that

$$\begin{aligned} \pi_{TM}^{-1}(U) \cap F_*^{-1}(\pi_{TN}^{-1}(V)) &= \pi_{TM}^{-1}(U) \cap \pi_{TM}^{-1}(F^{-1}(V)) \\ &= \pi_{TM}^{-1}(U \cap F^{-1}(V)) \\ &\in \mathcal{T}_{TM}. \end{aligned}$$

Set $m := \dim M$, $n := \dim N$ and write $\phi = (x^1, \dots, x^m)$ and $\psi = (y^1, \dots, y^n)$. Then for each $(a, \xi^1, \dots, \xi^m) \in \tilde{\phi}[\pi_{TM}^{-1}(U) \cap F_*^{-1}(\pi_{TN}^{-1}(V))]$, we have that

$$\begin{aligned}
\tilde{\psi} \circ F_* \circ \tilde{\phi}^{-1}(a, \xi^1, \dots, \xi^m) &= \tilde{\psi} \circ F_* \left(\phi^{-1}(a), \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_{\phi^{-1}(a)} \right) \\
&= \tilde{\psi} \left(F \circ \phi^{-1}(a), \sum_{j=1}^n \xi^j DF(\phi^{-1}(a)) \left(\frac{\partial}{\partial x^j}\bigg|_{\phi^{-1}(a)} \right) \right) \\
&= \tilde{\psi} \left(F \circ \phi^{-1}(a), \sum_{j=1}^n \xi^j \left[\sum_{k=1}^n \frac{\partial (y^k \circ F)}{\partial x^j} (\phi^{-1}(a)) \frac{\partial}{\partial y^k}\bigg|_{F \circ \phi^{-1}(a)} \right] \right) \\
&= \tilde{\psi} \left(F \circ \phi^{-1}(a), \sum_{k=1}^n \left[\sum_{j=1}^n \xi^j \frac{\partial (y^k \circ F)}{\partial x^j} (\phi^{-1}(a)) \right] \frac{\partial}{\partial y^k}\bigg|_{F \circ \phi^{-1}(a)} \right) \\
&= \left(\psi \circ F \circ \phi^{-1}(a), \sum_{j=1}^n \xi^j \frac{\partial (y^1 \circ F)}{\partial x^j} (\phi^{-1}(a)), \dots, \sum_{j=1}^n \xi^j \frac{\partial (y^n \circ F)}{\partial x^j} (\phi^{-1}(a)) \right).
\end{aligned}$$

Thus $\tilde{\psi} \circ F_* \circ \tilde{\phi}|^{-1}_{\pi_{TM}^{-1}(U) \cap F_*^{-1}(\pi_{TN}^{-1}(V))}$ is smooth. Exercise 5.1.0.5 implies that F_* is smooth. (maybe add more details here). □

Exercise 10.1.0.11. Let $M, N, E \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $F \in \mathrm{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$ and $G \in \mathrm{Hom}_{\mathbf{ManBnd}^\infty}(N, E)$. Then

1. for each $p \in M$, $DF|_{\{p\} \times T_p M} = \mathrm{id}_{\{p\}} \times DF(p)$.
2. $D(G \circ F) = DG \circ DF$
3. $D(\mathrm{id}_M) = \mathrm{id}_{TM}$
4. $F \in \mathrm{Iso}_{\mathbf{ManBnd}^\infty}(M, N)$ implies that $DF \in \mathrm{Iso}_{\mathbf{ManBnd}^\infty}(TM, TN)$ and $D(F^{-1}) = DF^{-1}$.

Proof.

1.
2.
3.
4.

FINISH!!! □

10.2 Cotangent Bundle

Chapter 11

Vector and Covector Fields

11.1 Vector Fields

Definition 11.1.0.1. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. We define the **vector fields on** M, denoted $\mathfrak{X}(M)$, by $\mathfrak{X}(M) := \Gamma(\pi_{TM})$.

Exercise 11.1.0.2. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X : M \to TM$. If X is a section of π_{TM}, then for each $p \in M$, $X(p) \in \{p\} \times T_pM$.

Proof. Suppose that X is a section of π_{TM}. Let $p \in M$. Since $X(p) \in TM$, there exists $q \in M$ and $v \in T_qM$ such that $X(p) = (q, v)$. Since X is a section of π_{TM},

$$\begin{aligned} p &= \mathrm{id}_M(p) \\ &= \pi_{TM} \circ X(p) \\ &= \pi_{TM}(q, v) \\ &= q. \end{aligned}$$

Hence

$$\begin{aligned} X(p) &= (p, v) \\ &\in \{p\} \times T_pM. \end{aligned}$$

actually just reference exercise in set theory section □

Note 11.1.0.3. When the context is clear, we write X_p in place of $X(p)$ and if $X_p = (p, v)$, we write X_p to refer to both $X_p \in TM$ and to $v \in T_pM$.

Definition 11.1.0.4. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $(U, \phi) \in \mathcal{A}_M$ and $X : M \to TM$. Suppose that X is a section of π_{TM}. Set $n := \dim M$ and write $\phi = (x^1, \dots, x^n)$. We define the **component functions of** X **with respect to** (U, ϕ), denoted $X^1, \dots, X^n : U \to TM$ by $X^j(p) := dx^j_p(X_p)$. In particular, for each $p \in U$,

$$X_p = \sum_{j=1}^n X^j(p) \frac{\partial}{\partial x^j}\bigg|_p.$$

Note 11.1.0.5. In particular, for $(U, \phi) \in \mathcal{A}_M$ with $\phi = (x^1, \dots, x^n)$, we have that for each $p \in U$, $[\tilde{\phi} \circ X](p) = (\phi(p), X^1_p, \dots, X^n_p)$.

Exercise 11.1.0.6. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $(U, \phi) \in \mathcal{A}_M$ and $X : M \to TM$. Suppose that X is a section of π_{TM}. Set $n := \dim M$ and write $\phi = (x^1, \dots, x^n)$. Then $X|_U \in \mathfrak{X}(U)$ iff for each $j \in [n]$, $X^j \in C^\infty(U)$.

Proof.

- ($\implies$):
Suppose that X is smooth. Then $\tilde{\phi} \circ X \circ \phi^{-1}$ is smooth. Since $\tilde{\phi} \circ X \circ \phi^{-1} = (\mathrm{id}_{\phi(U)}, X^1 \circ \phi^{-1}, \dots, X^n \circ \phi^{-1})$, we have that for each $j \in [n]$, $X^j \circ \phi^{-1}$ is smooth. Hence for each $j \in [n]$, X^j is smooth.

- ($\impliedby$):
Suppose that for each $j \in [n]$, X^j is smooth. Then for each $j \in [n]$, $X^j \circ \phi^{-1}$ is smooth. Since $\tilde{\phi} \circ X \circ \phi^{-1} = (\mathrm{id}_{\phi(U)}, X^1 \circ \phi^{-1}, \dots, X^n \circ \phi^{-1})$, we have that $\tilde{\phi} \circ X \circ \phi^{-1}$ is smooth. Since $X|_U = \tilde{\phi}^{-1} \circ [\tilde{\phi} \circ X \circ \phi^{-1}] \circ \phi$, we have that $X|_U$ is smooth.

□

Exercise 11.1.0.7. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X : M \to TM$. Set $n := \dim M$. Suppose that X is a section of π_{TM}. Then $X \in \mathfrak{X}(M)$ iff for each $(U, \phi) \in \mathcal{A}_M$, $X^1, \dots, X^n \in C^\infty(U)$.

Proof. Since X is smooth iff for each $(U, \phi) \in \mathcal{A}_M$, $X|_U$ is smooth, the previous exercise implies that $X \in \mathfrak{X}(M)$ iff for each $(U, \phi) \in \mathcal{A}_M$, $X^1, \dots, X^n \in C^\infty(U)$. reword □

Exercise 11.1.0.8. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $(U, \phi) \in \mathcal{A}_M$. Set $n := \dim M$ and write $\phi = (x^1, \dots, x^n)$. Then for each $j \in [n]$, $\frac{\partial}{\partial x^j} \in \mathfrak{X}(U)$.

Proof. Let $j \in [n]$. Define $X : U \to TM$ by $X_p := \left.\frac{\partial}{\partial x^j}\right|_p$. Clearly, X is a section of π_{TU}. Since for each $k \in [n]$, $X^k = \delta_{j,k}$, the previous exercise implies that $X \in \mathfrak{X}(U)$. □

Definition 11.1.0.9. Let $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. We define

- $fX : M \to TM$ by
$$(fX)_p = f(p)X_p$$

- $X + Y : M \to TM$ by
$$(X + Y)_p = X_p + Y_p$$

Exercise 11.1.0.10. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. Then

1. for each $f \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$,
   (a) $fX \in \mathfrak{X}(M)$
   (b) $X + Y \in \mathfrak{X}(M)$
2. $\mathfrak{X}(M) \in \mathrm{Obj}(\mathbf{Mod}_{C^\infty(M)})$.

Proof.

1. Let $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$ and $(U, \phi) \in \mathcal{A}_M$. Set $n := \dim M$ and write $\phi = (x^1, \dots, x^n)$.
   (a) Clearly fX is a section of π_{TM}. Since
$$\begin{aligned}(fX)|_U &= f|_U \sum_{j=1}^n X^j \frac{\partial}{\partial x^j} \\ &= \sum_{j=1}^n f|_U X^j \frac{\partial}{\partial x^j},\end{aligned}$$
   we have that for each $j \in [n]$, $(fX)^j = f|_U X^j$. Since $f|_U, X^j \in C^\infty(U)$, $f|_U X^j \in C^\infty(U)$. a previous exercise implies that $(fX)|_U$ is smooth. Since $(U, \phi) \in \mathcal{A}_M$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{A}_M$, $(fX)|_U$ is smooth. Hence fX is smooth and $fX \in \mathfrak{X}(M)$.

(b) Clearly $X+Y$ is a section of π_{TM}. Since

$$\begin{aligned}(X+Y)|_U &= \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j} + \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j} \\ &= \sum_{j=1}^{n} (X^j + Y^j) \frac{\partial}{\partial x^j}\end{aligned}$$

we have that for each $j \in [n]$, $(X+Y)^j = X^j + Y^j$. Since $X^j, Y^j \in C^\infty(U)$, $X^j + Y^j \in C^\infty(U)$. a previous exercise implies that $(X+Y)|_U$ is smooth. Since $(U,\phi) \in \mathcal{A}_M$ is arbitrary, we have that for each $(U,\phi) \in \mathcal{A}_M$, $(X+Y)|_U$ is smooth. Hence $X+Y$ is smooth and $X+Y \in \mathfrak{X}(M)$.

2. Clearly by previous part.

$\square$

11.2 Vector Fields as Derivations on $C^\infty(M)$

Definition 11.2.0.1. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $D : C^\infty(M) \to C^\infty(M)$. Then D is said to be a **derivation on** $C^\infty(M)$ if

- (linearity):
  for each $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$, $D(f + \lambda g) = D(f) + \lambda D(g)$,
- (Leibnizianity):
  for each $f, g \in C^\infty(M)$, $D(fg) = fD(g) + D(f)g$.

We define
$$\mathrm{Deriv}^\infty(M) := \{D : C^\infty(M) \to C^\infty(M) : D \text{ is a derivation on } C^\infty(M)\}.$$

Exercise 11.2.0.2. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $D \in \mathrm{Deriv}^\infty(M)$.

Definition 11.2.0.3. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $D_1, D_2 \in \mathrm{Deriv}^\infty(M)$ and $f \in C^\infty(M)$. For each $g \in C^\infty(M)$, we define

- $[D_1 + D_2](g) := D_1(g) + D_2(g)$
- $fD_1(g) := fD_1(g)$

Exercise 11.2.0.4. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. Then

1. for each $D_1, D_2 \in \mathrm{Deriv}^\infty(M)$ and $f \in C^\infty(M)$,
   (a) $D_1 + D_2 \in \mathrm{Deriv}^\infty(M)$
   (b) $fD_1 \in \mathrm{Deriv}^\infty(M)$
2. $\mathrm{Deriv}^\infty(M) \in \mathrm{Obj}(\mathbf{Mod}_{C^\infty(M)})$.

Proof. FINISH!!! □

Definition 11.2.0.5. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X : M \to TM$. Suppose that X is a section of π_{TM}. For each $f \in C^\infty(M)$, we define $Xf : M \to \mathbb{R}$ by
$$(Xf)_p := X_p(f).$$

Exercise 11.2.0.6. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $X : M \to TM$ and $(U, \phi) \in \mathcal{A}_M$. Suppose that X is a section of π_{TM}. Set $n := \dim M$ and write $\phi = (x^1, \cdots, x^n)$. Then
$$X|_U = \sum_{j=1}^n (X|_U(x^j)) \frac{\partial}{\partial x^j}$$

Proof. We have that for each $k \in [n]$,
$$\begin{aligned} X|_U(x^k) &= \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}(x^k) \\ &= \sum_{j=1}^n X^j \delta_{j,k} \\ &= X^k. \end{aligned}$$

Hence
$$X|_U = \sum_{j=1}^n (X|_U(x^j)) \frac{\partial}{\partial x^j}.$$

□

Exercise 11.2.0.7. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X \in \mathfrak{X}(M)$. Then for each $f \in C^\infty(M)$, $Xf \in C^\infty(M)$.

Proof. Let $(U, \phi) \in \mathcal{A}_M$. Set $n := \dim M$ and write $\phi = (x^1, \dots, x^n)$. Then need exercise about how Xf only depends on neighborhood of p, maybe already exists in tangent space section, need reference implies that for each $p \in U$,

$$\begin{aligned}
[X|_U f|_U](p) &= X_p(f) \\
&= \left[\sum_{j=1}^n X^j(p) \frac{\partial}{\partial x^j}\bigg|_p\right] f \\
&= \sum_{j=1}^n X^j(p) \frac{\partial f}{\partial x^j}(p) \\
&= \left[\sum_{j=1}^n X^j \frac{\partial f}{\partial x^j}\right](p).
\end{aligned}$$

Since $X|_U \in \mathfrak{X}(U)$, and $f|_U \in C^\infty(U)$, we have that for each $j \in [n]$, $X^j \frac{\partial f}{\partial x^j} \in C^\infty(U)$. Thus $\sum\limits_{j=1}^n X^j \frac{\partial f}{\partial x^j} \in C^\infty(U)$. Hence $X|_U f|_U \in C^\infty(U)$. Since $(Xf)|_U = X|_U f|_U$, we have that $(Xf)|_U \in C^\infty(U)$. Since $(U, \phi) \in \mathcal{A}_M$ is arbitrary, we have that for each $U \in \mathcal{T}_M$, $(Xf)|_U \in C^\infty(U)$. Thus $Xf \in C^\infty(M)$. □

Definition 11.2.0.8. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X \in \mathfrak{X}(M)$. We define $D^X : C^\infty(M) \to C^\infty(M)$ by $D^X(f) := Xf$.

Exercise 11.2.0.9. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X \in \mathfrak{X}(M)$. Then $D^X \in \mathrm{Deriv}^\infty(M)$.

Proof.

- Let $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$. Then for each $p \in M$,

$$\begin{aligned}
D^X(f + \lambda g) &= X(f + \lambda g)(p) \\
&= X_p(f + \lambda g) \\
&= X_p f + \lambda X_p g \\
&= (Xf)(p) + \lambda (Xg)(p) \\
&= [Xf + \lambda Xg](p) \\
&= [D^X(f) + \lambda D^X(g)](p)
\end{aligned}$$

  Hence $D^X(f + \lambda g) = D^X(f) + \lambda D^X(g)$ and $D^X : C^\infty(M) \to C^\infty(M)$ is linear.

- Let $f, g \in C^\infty(M)$. Then for each $p \in M$,

$$\begin{aligned}
[D^X(fg)](p) &= [X(fg)](p) \\
&= X_p(fg) \\
&= (X_p f) g(p) + f(p) X_p(g) \\
&= (Xf)(p) g(p) + f(p)(Xg)(p) \\
&= [(Xf)g + f(Xg)](p) \\
&= D^X(f)g + fD^X(g).
\end{aligned}$$

  Hence $D^X(fg) = D^X(f)g + fD^X(g)$ and $D^X : C^\infty(M) \to C^\infty(M)$ is Leibnizian.

Thus $D^X \in \mathrm{Deriv}^\infty(M)$. □

Definition 11.2.0.10. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. We define the **Derivation map**, denoted $\mathrm{Der} : \mathfrak{X}(M) \to \mathrm{Deriv}^\infty(M)$, by $\mathrm{Der}(X) := D^X$.

Exercise 11.2.0.11. Let $M \in \text{Obj}(\mathbf{ManBnd}^\infty)$. Then $\text{Der} \in \text{Hom}_{\mathbf{Mod}_{C^\infty(M)}}(\mathfrak{X}(M), \text{Deriv}^\infty(M))$.

Proof. Let $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Then for each $p \in M$,

$$\begin{aligned}
[D^{X+fY}(g)](p) &= ([X + fY]g)(p) \\
&= [X + fY]_p(g) \\
&= [X_p + f(p)Y_p](g) \\
&= X_p(g) + f(p)Y_p(g) \\
&= (Xg)(p) + [f(Yg)](p) \\
&= [Xg + f(Yg)](p) \\
&= [D^X(g) + fD^Y(g)](p).
\end{aligned}$$

Hence $D^{X+fY}(g) = D^X(g) + fD^Y(g)$. Since $g \in C^\infty(M)$ is arbitrary, we have that

$$\begin{aligned}
\text{Der}(X + fY) &= D^{X+fY} \\
&= D^X + fD^Y \\
&= \text{Der}(X) + f\text{Der}(Y).
\end{aligned}$$

Thus Der is $C^\infty(M)$-linear. □

Exercise 11.2.0.12. Let $M \in \text{Obj}(\mathbf{ManBnd}^\infty)$, $X : M \to TM$. Suppose that X is a section of π_{TM}. Then the following are equivalent:

1. X is smooth
2. for each $f \in C^\infty(M)$, $Xf \in C^\infty(M)$
3. for each $U \in \mathcal{T}_M$ $f \in C^\infty(U)$, $X|_U(f) \in C^\infty(U)$

Proof.

- (1) $\implies$ (2):
  Suppose that X is smooth. Let $f \in C^\infty$ and $(U, \phi) \in \mathcal{A}_M$. Then

$$\begin{aligned}
X|_U f|_U &= \left[\sum_{j=1}^n X^j \frac{\partial}{\partial x^j}\right] f|_U \\
&= \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}(f|_U) \\
&= \sum_{j=1}^n X^j \frac{\partial f|_U}{\partial x^j}.
\end{aligned}$$

  Since X and f are smooth, for each $j \in [n]$, $X^j, \frac{\partial f|_U}{\partial x^j} \in C^\infty(U)$. Hence $X|_U f|_U$ is smooth. Since $X|_U f|_U = (Xf)|_U$, we have that $(Xf)|_U$ is smooth. Since $U \in \mathcal{T}_M$ is arbitrary, we have that for each $U \in \mathcal{T}_M$, $(Xf)|_U$ is smooth. Exercise **??** A previous exercise implies that Xf is smooth.
- (2) $\implies$ (3):
  Clear. maybe add details, maybe bump function.
- (3) $\implies$ (1):
  FINISH!!!

□

Definition 11.2.0.13. Let $M \in \text{Obj}(\mathbf{ManBnd}^\infty)$ and $D \in \text{Deriv}^\infty(M)$. For each $p \in M$, we define $X^D_p : C^\infty(M) \to \mathbb{R}$ by $X^D_p(f) := D(f)(p)$.

Exercise 11.2.0.14. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $D \in \mathrm{Deriv}^\infty(M)$. Then or each $p \in M$, $X_p^D \in T_pM$.

Proof. Let $p \in M$.

- (linearity):
  Let $f, g \in C^\infty$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} X_p^D(f + \lambda g) &= D(f + \lambda g)(p) \\ &= [D(f) + \lambda D(g)](p) \\ &= D(f)(p) + \lambda D(g)(p) \\ &= X_p^D(f) + \lambda X_p^D(g). \end{aligned}$$

- (Leibnizianity):
  Let $f, g \in C^\infty(M)$. Then

$$\begin{aligned} X_p^D(fg) &= D(fg)(p) \\ &= [(Df)g + f(Dg)](p) \\ &= Df(p)g(p) + f(p)Dg(p) \\ &= X_p^D(f)g(p) + f(p)X_p^D(g). \end{aligned}$$

Thus $X_p^D \in T_pM$. □

Definition 11.2.0.15. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $D \in \mathrm{Deriv}^\infty(M)$. We define $X^D : M \to TM$ by $X^D(p) := (p, X_p^D)$.

Exercise 11.2.0.16. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $D \in \mathrm{Deriv}^\infty(M)$. Then $X^D \in \mathfrak{X}(M)$.

Proof. By construction X^D is a section of π_{TM}. Let $(U, \phi) \in \mathcal{A}_M$. Set $n := M$ and write $\phi = (x^1, \ldots, x^n)$. Then for each $j \in [n]$,

$$\begin{aligned} (X^D)^j &= X^D|_U(x^j) \\ &= D(x^j) \\ &\in C^\infty(U) \end{aligned}$$

(maybe need to make more rigorous with a bump function or maybe talk about restrictions of derivations, doesnt feel clean here). □

Exercise 11.2.0.17. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. Then $\mathrm{Der} \in \mathrm{Iso}_{\mathbf{Mod}_{C^\infty(M)}}(\mathfrak{X}(M), \mathrm{Deriv}^\infty(M))$.

Proof.

- **(injectivity)**:
  Let $X, Y \in \mathfrak{X}(M)$. Suppose that $\mathrm{Der}(X) = \mathrm{Der}(Y)$. Let $(U, \phi) \in \mathcal{A}_M$. Set $n := \dim M$ and write $\phi = (x^1, \ldots, x^n)$. Then for each $j \in [n]$,

$$\begin{aligned} X^j &= X|U(x^j) \\ &= D^{X|_U}(x^j) \\ &= D^{Y|_U}(x^j) \\ &= Y|_U(x^j) \\ &= Y^j. \end{aligned}$$

  Hence $X|_U = Y|_U$. Since $(U, \phi) \in \mathcal{A}_M$ is arbitrary, for each $U \in \mathcal{T}_M$, $X|_U = Y|_U$. Thus $X = Y$. Since $X, Y \in \mathfrak{X}(M)$ are arbitrary, we have that Der is injective

- **(sujectivity)**:
  Let $D \in \operatorname{Deriv}^{\infty}(M)$. Define $X \in \mathfrak{X}(M)$ by $X := X^D$. Then for each $f \in C^{\infty}(M)$,

$$\begin{aligned}\operatorname{Der}(X)(f) &= D^X(f)\\ &= Xf\\ &= X^D(f)\\ &= D(f).\end{aligned}$$

  Hence $\operatorname{Der}(X) = D$. Thus for each $D \in \operatorname{Deriv}^{\infty}(M)$, there exists $X \in \mathfrak{X}(M)$ such that $\operatorname{Der}(X) = D$. Thus Der is surjective.

Thus $\operatorname{Der} \in \operatorname{Iso}_{\mathbf{Mod}_{C^{\infty}(M)}}(\mathfrak{X}(M), \operatorname{Deriv}^{\infty}(M))$. $\square$

11.3 The Commutator

Definition 11.3.0.1. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X, Y \in \mathfrak{X}(M)$. We define $XY : C^\infty(M) \to C^\infty(M)$ by $XY(f) := X(Yf)$.

Exercise 11.3.0.2. There exist $X, Y \in \mathfrak{X}(\mathbb{R}^2)$ such that $XY \notin \mathrm{Deriv}^\infty(\mathbb{R}^2)$.

Proof. Set $X := \frac{\partial}{\partial x^1}$ and $Y := \frac{\partial}{\partial x^2}$. Then $XY = \frac{\partial^2}{\partial x^1 \partial x^2}$. Define $f, g \in C^\infty(\mathbb{R}^2)$ by $f(x^1, x^2) := x^1$ and $g(x^1, x^2) := x^2$. Then

$$\begin{aligned} XY(fg) &= \frac{\partial^2}{\partial x^1 \partial x^2}(fg) \\ &= \frac{\partial}{\partial x^1}\left[\frac{\partial(fg)}{\partial x^2}\right] \\ &= \frac{\partial}{\partial x^1}\left[\frac{\partial f}{\partial x^2} g + f \frac{\partial g}{\partial x^2}\right] \\ &= \frac{\partial^2 f}{\partial x^1 \partial x^2} g + \frac{\partial f}{\partial x^2}\frac{\partial g}{\partial x^1} + \frac{\partial f}{\partial x^1}\frac{\partial g}{\partial x^2} + f \frac{\partial^2 g}{\partial x^1 \partial x^2} \\ &= \frac{\partial^2 f}{\partial x^1 \partial x^2} g + f \frac{\partial^2 g}{\partial x^1 \partial x^2} + 1 \\ &= [XY(f)]g + fXY(g) + 1 \\ &\neq [XY(f)]g + fXY(g). \end{aligned}$$

Thus XY is not Leibnizian and therefore $XY \notin \mathrm{Deriv}^\infty(M)$. □

Definition 11.3.0.3. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X, Y \in \mathfrak{X}(M)$. We define the **derivation commutator of X and Y**, denoted $[X, Y]_D : C^\infty(M) \to C^\infty(M)$, by

$$[X, Y] := XY - YX$$

Exercise 11.3.0.4. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X, Y \in \mathfrak{X}(M)$. Then $[X, Y]_D \in \mathrm{Deriv}^\infty(M)$.

Proof. Let $f, g \in C^\infty(M)$. Then

- (linearity):
  Let $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} [X, Y](f + \lambda g) &= (XY - YX)(f + \lambda g) \\ &= XY(f + \lambda g) - YX(f + \lambda g) \\ &= X(Yf + \lambda Yg) - Y(Xf + \lambda Xg) \\ &= XY(f) + \lambda XY(g) - (YX(f) + \lambda YX(g)) \\ &= XY(f) - YX(f) + \lambda(XY(g) - YX(g)) \\ &= (XY - YX)(f) + \lambda(XY - YX)(g) \\ &= [X, Y]_D(f) + \lambda[X, Y]_D(g). \end{aligned}$$

  Thus $[X, Y]$ is $\mathbb{R}$-linear.

- (Leibnizianity):

$$\begin{aligned} (XY)(fg) &= X(Y(fg)) \\ &= X((Yf)g + f(Yg)) \\ &= X((Yf)g) + X(f(Yg)) \\ &= [X(Yf)]g + (Yf)(Xg) + (Xf)(Yg) + f[X(Yg)] \\ &= [(XY)(f)]g + (Yf)(Xg) + (Xf)(Yg) + f[(XY)(g)]. \end{aligned}$$

Similarly, $(YX)(fg) = [(YX)(f)]g + (Xf)(Yg) + (Yf)(Xg) + f[(YX)(g)]$. Hence

$$\begin{aligned}
[X,Y]_D(fg) &= (XY - YX)(fg) \\
&= XY(fg) - YX(fg) \\
&= [(XY)(f)]g + (Yf)(Xg) + (Xf)(Yg) + f[(XY)(g)] - ([(YX)(f)]g + (Xf)(Yg) + (Yf)(Xg) + f[(YX)(\\
&= [(XY)(f)]g - [(YX)(f)]g + f[(XY)(g)] - f[(YX)(g)] \\
&= [(XY)(f) - (YX)(f)](g) + f[(XY)(g) - (YX)(g)] \\
&= [(XY - YX)(f)]g + f[(XY - YX](g)) \\
&= ([X,Y]_D(f))g + f([X,Y]_D(g)).
\end{aligned}$$

Thus $[X,Y]_D$ is Leibnizian.

Hence $[X,Y]_D \in \mathrm{Deriv}^\infty(M)$. $\square$

Definition 11.3.0.5. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X, Y \in \mathfrak{X}(M)$. We define the **vector field commutator of X and Y**, denoted $[X,Y] \in \mathfrak{X}(M)$, by $[X,Y] := \mathrm{Der}^{-1}([X,Y]_D)$.

Exercise 11.3.0.6. Jacobi Identity:
Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $X, Y, Z \in \mathfrak{X}(M)$. Then

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Proof. Let FINISH!!! $\square$

11.4 Vector Fields and Smooth Maps

Definition 11.4.0.1. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$, $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X is said to be ***F*-related** to Y if for each $p \in M$, $Y_{F(p)} = F_* X_p$.

Exercise 11.4.0.2. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$, $F \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, N)$, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X is F-related to Y iff for each $V \in \mathcal{T}_N$ and $f \in C^\infty(V)$, $X|_V(f \circ F|_{F^{-1}(V)}) = Y|_V(f) \circ F|_{F^{-1}(V)}$.

Proof. FINISH!!! □

Exercise 11.4.0.3. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$, $F \in \text{Iso}_{\mathbf{ManBnd}^\infty}(M, N)$. Then for each $X \in \mathfrak{X}(M)$, there exists a unique $Y \in \mathfrak{X}(N)$ such that X is F-related to Y.

Proof. Let $X \in \mathfrak{X}(M)$. Define $Y : N \to TN$ by $Y := F_* \circ X \circ F^{-1}$.

- Since $F_* \in \text{Hom}_{\mathbf{ManBnd}^\infty}(TM, TN)$, $X \in \text{Hom}_{\mathbf{ManBnd}^\infty}(M, TM)$ and $F^{-1} \in \text{Hom}_{\mathbf{ManBnd}^\infty}(N, M)$, we have that

$$\begin{aligned} Y &= F_* \circ X \circ F^{-1} \\ &\in \text{Hom}_{\mathbf{ManBnd}^\infty}(N, TN). \end{aligned}$$

- Let $q \in N$. Define $p \in M$ by $p := F^{-1}(q)$. Since $X \in \mathfrak{X}(M)$, there exists $v \in T_p M$ such that $X(p) = (p, v)$. Then

$$\begin{aligned} &= \pi_{TN} \circ Y(q) \\ &= \pi_{T_N}(F_* X_{F^{-1}(q)}) \\ &= \pi_{T_N}(F_* X_p) \\ &= \pi_{TM}(F_*(p, v)) \\ &= \pi_{TM}(F(p), DF(p)(v)) \\ &= F(p) \\ &= q \\ &= \text{id}_N(q). \end{aligned}$$

  Since $q \in N$ is arbitrary, we have theat $\pi_{TN} \circ Y = \text{id}_N$. Hence Y is a section of π_{TN}.

Since Y is smooth and Y is a section of π_{TN}, we have that $Y \in \mathfrak{X}(N)$. □

Definition 11.4.0.4. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$ and $F \in \text{Iso}_{\mathbf{ManBnd}^\infty}(M, N)$. For each $X \in \mathfrak{X}(M)$, we define the **pushforward of** X **by** F, denoted $F_* X \in \mathfrak{X}(N)$ by $F_* X := DF \circ X \circ F^{-1}$.

Exercise 11.4.0.5. Let $M, N \in \text{Obj}(\mathbf{ManBnd}^\infty)$, $F \in \text{Iso}_{\mathbf{ManBnd}^\infty}(M, N)$. Then for each $X, Y \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$, $F_*(X + fY) = F_* X + \lambda F_* Y$.

Proof. Let $X, Y \in \mathfrak{X}(M)$, $\lambda \in \mathbb{R}$ and $q \in N$. Set $p := F^{-1}(q)$. Since $DF|_{\{p\} \times T_p M} = \text{id}_{\{p\}} \times DF(p)$, and $DF(p) : T_p M \to T_q N$ is $\mathbb{R}$-linear, we have that $DF|_{\{p\} \times T_p M}$ is $\mathbb{R}$-linear and

$$\begin{aligned} [F_*(X + \lambda Y)](q) &= F_*([X + \lambda Y]_p) \\ &= F_*([X_p + \lambda Y_p]) \\ &= F_*(X_p) + \lambda F_*(Y_p) \\ &= F_* \circ X \circ F^{-1}(q) + \lambda F_* \circ Y \circ F^{-1}(q) \\ &= F_* X(q) + \lambda F_* Y(q) \\ &= [F_* X + \lambda F_* Y](q). \end{aligned}$$

Since $q \in N$ is arbitrary, we have that $F_*(X + \lambda Y) = F_* X + \lambda F_* Y$. □

11.5 1-Forms

Definition 11.5.0.1. Let $\omega : M \to T^*M$. Then ω is said to be a **1-form on** M if for each $p \in M$, $\omega_p \in T^*_pM$.
For each $X \in \mathfrak{X}(M)(M)$, we define $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \mathfrak{X}(M)(M)$, $\omega(X)$ is smooth.
The set of smooth 1-forms on M is denoted $\Gamma_1(M)$.

Definition 11.5.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \mathfrak{X}(M)(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta \in \mathfrak{X}(M)(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 11.5.0.3. The set $\Gamma_1(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Chapter 12

Lie Groups

12.1 Introduction

Definition 12.1.0.1. Let $G \in \text{Obj}(\mathbf{ManBnd}^\infty)$. For each $g \in G$, we define $\iota^l_g : G \to G \times G$ and $\iota^r_g : G \to G \times G$ by $\iota^l_g(x) = (g, x)$ and $\iota^r_g(x) = (x, g)$ respectively.

Note 12.1.0.2. Exercise 5.3.0.10 implies that for each $g \in G$, $\iota^l_g, \iota^r_h \in \text{Hom}_{\mathbf{ManBnd}^\infty}(G, G \times G)$.

Definition 12.1.0.3. Let G be a set and $\text{mult} : G \times G \to G$. Suppose that (G, mult) is a group. We define the **inversion map**, denoted $\text{inv} : G \to G$, by $\text{inv}(g) = g^{-1}$.

Note 12.1.0.4. When the context is clear, we write gh in place of $\text{mult}(g, h)$.

Definition 12.1.0.5. Let $G \in \text{Obj}(\mathbf{ManBnd}^\infty)$ and $\text{mult} : G \times G \to G$. Suppose that (G, mult) is a group. Then (G, mult) is said to be a **Lie group** if

1. $\text{mult} \in \text{Hom}_{\mathbf{ManBnd}^\infty}(G \times G, G)$,
2. $\text{inv} \in \text{Hom}_{\mathbf{ManBnd}^\infty}(G, G)$.

Note 12.1.0.6. When the context is clear, we write G in place of (G, mult).

Definition 12.1.0.7. Let G be a Lie group and $g \in G$. We define the **left and right translation maps**, denoted $l_g : G \to G$ and $r_g : G \to G$ respectively, by $l_g(x) = gx$ and $r_g(x) = xg^{-1}$.

Exercise 12.1.0.8. Let G be a Lie group. Then for each $g \in G$,

1. $l_g^{-1} = l_{g^{-1}}$ and $r_g^{-1} = r_{g^{-1}}$,
2. $l_g, r_g \in \text{Hom}_{\mathbf{ManBnd}^\infty}(G, G)$,
3. $l_g, r_g \in \text{Aut}_{\mathbf{ManBnd}^\infty}(G)$.

Proof. Let $g \in G$.

1. Clear
2. Since G is a Lie group, mult is smooth. Since $l_g = \text{mult} \circ \iota^l_g$ and $r_g = \text{mult} \circ \iota^r_{g^{-1}}$, we have that l_g and r_g are smooth.
3. Since $l_g \in \text{Hom}_{\mathbf{ManBnd}^\infty}(G, G)$ and

$$\begin{aligned} l_g^{-1} &= l_{g^{-1}} \\ &\in \text{Hom}_{\mathbf{ManBnd}^\infty}(G, G), \end{aligned}$$

we have that $l_g \in \text{Aut}_{\mathbf{ManBnd}^\infty}(G)$. Similarly, $r_g \in \text{Aut}_{\mathbf{ManBnd}^\infty}(G)$.

□

Exercise 12.1.0.9. Let $G \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. Suppose that G is a Lie Group. Then $\partial G = \varnothing$.

Proof. Let $g \in G$. Since $\mathcal{A}_G$ is a smooth atlas, there exists $(U_0, \phi_0) \in \mathcal{A}_G$ such that $e \in U_0$. There exists $x \in U_0$ such that $x \in \mathrm{Int}\, G$ (add details). Set $U := U_0 \cap \mathrm{Int}\, G$. Since $U_0, \mathrm{Int}\, G \in \mathcal{T}_G$, $x \in U_0$ and $x \in \mathrm{Int}\, G$, we have that $U \in \mathcal{T}_G$ and $x \in U$. Set $\phi := \phi_0|_U$. Exercise **??** (exercise in section on open submanifolds) implies that $(U, \phi) \in \mathcal{A}_G$. Since $l_{gx^{-1}}$ is a diffeomorphism, $l_{gx^{-1}}$ is a homeomorphism. Hence

$$\begin{aligned} g &= l_{gx^{-1}}(x) \\ &\in l_{gx^{-1}}(U) \\ &\subset \mathrm{Int}\, G \end{aligned}$$

Since $g \in G$ is arbitrary, we have that for each $g \in G$, $g \in \mathrm{Int}\, G$. Thus $\mathrm{Int}\, G = G$ and Exercise **??** (ref ex from intro to topological manifolds) implies that

$$\begin{aligned} \partial G &= (\mathrm{Int}\, G)^c \\ &= \varnothing. \end{aligned}$$

□

Exercise 12.1.0.10. Let $G \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Suppose that G is a group. Define $f : G \times G \to G$ by $f(g, h) = gh^{-1}$. Then G is a Lie group iff f is smooth.

Proof.

- ($\implies$):
  Suppose that G is a Lie group. Then mult is smooth and inv is smooth. Thus $\mathrm{id}_G \times \mathrm{inv}$ is smooth. Since $f = \mathrm{mult} \circ (\mathrm{id}_G \times \mathrm{inv})$, we have that f is smooth.
- ($\impliedby$):
  Suppose that f is smooth. Since $\mathrm{inv} = f \circ \iota^l_e$, inv is smooth. Therefore $id_G \times \mathrm{inv}$ is smooth and since $\mathrm{mult} = f \circ (\mathrm{id}_G \times \mathrm{inv})$, mult is smooth. Since mult and inv are smooth, G is a Lie group.

□

Exercise 12.1.0.11. Let $G, H \in \mathrm{Obj}(Maninf)$ and $\phi : G \to H$. Suppose that G, H are Lie groups. Then ϕ is said to be a **Lie group homomorphism** if $\phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(G, H) \cap \mathrm{Hom}_{\mathbf{Grp}}(G, H)$.

Definition 12.1.0.12. We define the category of Lie groups, denoted **LieGrp**, by

- $\mathrm{Obj}(\mathbf{LieGrp}) = \{G : G \text{ is a Lie group}\}$
- For $G_1, G_2 \in \mathrm{Obj}(\mathbf{LieGrp})$,

$$\mathrm{Hom}_{\mathbf{LieGrp}}(G_1, G_2) = \mathrm{Hom}_{\mathbf{Man}^\infty}(G, H) \cap \mathrm{Hom}_{\mathbf{Grp}}(G, H)$$

- For
  - $G_1, G_2, G_3 \in \mathrm{Obj}(\mathbf{LieGrp})$
  - $\phi_{12} \in \mathrm{Hom}_{\mathbf{LieGrp}}(G_1, G_2)$
  - $\phi_{23} \in \mathrm{Hom}_{\mathbf{LieGrp}}(G_2, G_3)$

  we define $\phi_{23} \circ_{\mathbf{LieGrp}} \phi_{12} \in \mathrm{Hom}_{\mathbf{LieGrp}}(G_1, G_3)$ by

$$\phi_{23} \circ_{\mathbf{LieGrp}} \phi_{12} = \phi_{23} \circ_{\mathbf{Set}} \phi_{12}$$

Exercise 12.1.0.13. We have that **LieGrp** is a subcategory of **Grp** and $\mathbf{Man}^\infty$.

Proof. FINISH!!! □

Exercise 12.1.0.14. Let $G, H \in \mathrm{Obj}(\mathbf{LieGrp})$ and $\phi \in \mathrm{Hom}_{\mathbf{LieGrp}}(G, H)$. Then ϕ has constant rank.

Proof. Let $g \in G$. Since ϕ is a homomorphism, we have that for each $x \in G$, $\phi(gx) = \phi(g)\phi(x)$. Thus $\phi \circ l_g = l_{\phi(g)} \circ \phi$, i.e. the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "l_g"'] & H \arrow[d, "l_{\phi(g)}"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

Let $x \in G$. Then

$$\begin{aligned} D\phi(gx) \circ Dl_g(x) &= D(\phi \circ l_g)(x) \\ &= D(l_{\phi(g)} \circ \phi) \\ &= Dl_{\phi(g)}(\phi(x)) \circ D\phi(x) \end{aligned}$$

Since $l_g \in \mathrm{Aut}_{\mathbf{Man}^\infty}(G), l_{\phi(g)} \in \mathrm{Aut}_{\mathbf{Man}^\infty}(H)$, we have that $Dl_g(x) \in \mathrm{Iso}_{\mathbf{Vect}_\mathbb{R}}(T_xG, T_{gx}G)$ and $Dl_{\phi(g)}(\phi(x)) \in \mathrm{Iso}_{\mathbf{Vect}_\mathbb{R}}(T_{\phi(x)}H, T_{\phi(g)\phi(x)}H)$. Hence

$$\begin{aligned} \mathrm{rank}\, D\phi(gx) &= \mathrm{rank}\, D\phi(gx) \circ Dl_g(x) \\ &= \mathrm{rank}\, Dl_{\phi(g)}(\phi(x)) \circ D\phi(x) \\ &= \mathrm{rank}\, D\phi(x) \end{aligned}$$

Since $x \in G$ is arbitrary, for each $x \in G$, $\mathrm{rank}\, D\phi(gx) = \mathrm{rank}\, D\phi(x)$. In particular, $\mathrm{rank}\, D\phi(g) = \mathrm{rank}\, D\phi(e)$. Since $g \in G$ is arbitrary, for each $g \in G$, $\mathrm{rank}\, D\phi(g) = \mathrm{rank}\, D\phi(e)$ and ϕ has constant rank. □

Exercise 12.1.0.15. Let $G, H \in \mathrm{Obj}(\mathbf{LieGrp})$ and $\phi \in \mathrm{Hom}_{\mathbf{LieGrp}}(G, H)$. Then $\phi \in \mathrm{Iso}_{\mathbf{LieGrp}}(G, H)$ iff ϕ is a bijection.

Proof. global rank theorem FINISH!!! □

Definition 12.1.0.16. Let $G, H \in \mathrm{Obj}(\mathbf{LieGrp})$ and $\phi \in \mathrm{Hom}_{\mathbf{LieGrp}}(G, H)$. Then ϕ is said to be a

- **LieGrp-immersion** if ϕ is a $\mathbf{Man}^\infty$-immersion
- **LieGrp-embedding** if ϕ is a $\mathbf{Man}^\infty$-embedding

Exercise 12.1.0.17. Let $G, H \in \mathrm{Obj}(\mathbf{LieGrp})$ and $\phi \in \mathrm{Hom}_{\mathbf{LieGrp}}(G, H)$. Suppose that ϕ is a **LieGrp**-immersion. If G is compact, then ϕ is a **LieGrp**-embedding.

12.2 Lie Subgroups

Definition 12.2.0.1. Let $G, H \in \mathrm{Obj}(\mathbf{LieGrp})$. Suppose that $G \leqslant H$. Then H is said to be an

- **immersed Lie subgroup** of G if G is an immersed submanifold of H,
- **embedded Lie subgroup** of G if G is an embedded submanifold of H.

Definition 12.2.0.2. content...

Exercise 12.2.0.3. Let $G, H \in \mathrm{Obj}(\mathbf{LieGrp})$. Suppose that $G \leqslant H$.

12.3 Product Lie Groups

Definition 12.3.0.1. Let $G, H \in \mathrm{Obj}(\mathbf{LieGrp})$. Suppose that $G \subset H$. Then G is said to be a **Lie subgroup** of H if

1. $G \leqslant H$
2. G is an immersed submanifold of H. FIX!!!

12.4 Representations of Lie Groups

12.5 Lie Algebras

12.5.1 Introduction

Definition 12.5.1.1. Let $V \in \mathrm{Obj}(\mathbf{Vect}_{\mathbb{K}})$ and $[\cdot,\cdot] : V \times V \to V$. Then $[\cdot,\cdot]$ is said to be a **Lie bracket on** V if

1. (bilinearity): for each $x, y, z \in V$ and $\lambda \in \mathbb{K}$, $[x + \lambda y, z] = [x, z] + \lambda[y, z]$
2. (antisymmetry): for each $x, y \in V$, $[x, y] = -[y, x]$
3. (Jacobi identity): for each $x, y, z \in V$, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

and $(V, [\cdot,\cdot])$ is said to be a $\mathbb{K}$**-Lie Algebra** if $[\cdot,\cdot]$ is a Lie bracket on V.

12.5.2 Lie Subalgebras

Definition 12.5.2.1. Let $(V, [\cdot,\cdot])$ be a $\mathbb{K}$-Lie algebra and $W \subset V$ a subsapce. Then $(W, [\cdot,\cdot]|_{W\times W})$ is said to be a **Lie subalgebra of** $(V, [\cdot,\cdot])$ if for each $x, y \in W$, $[x, y] \in W$.

Note 12.5.2.2. When the context is clear, we will typically suppress the Lie bracket $[\cdot,\cdot]$.

Exercise 12.5.2.3. exercise about intersection of two lie subalgebras is a lie subalgebra

Proof. FINISH!!! □

12.6 Lie Algebras from Lie Groups

Exercise 12.6.0.1. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$. Then $(\mathfrak{X}(M), [\cdot,\cdot])$ is an $\mathbb{R}$-Lie Algebra.

Proof. Clear by **??** (make exercise in section on vector fields about $[\cdot,\cdot]$). □

Definition 12.6.0.2. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$, $\Gamma \subset \mathrm{Aut}_{\mathbf{ManBnd}^\infty}(M)$ and $X \in \mathfrak{X}(M)$. Then X is said to be Γ-invariant if for each $\phi \in \Gamma$, $\phi_* X = X$. We define the Γ**-invariant vector fields on** M, denoted $\mathfrak{X}^\Gamma(M)$, by $\mathfrak{X}^\Gamma(M) := \{X \in \mathfrak{X}(M) : X \text{ is } \Gamma\text{-invariant}\}$.

Exercise 12.6.0.3. Let $M \in \mathrm{Obj}(\mathbf{ManBnd}^\infty)$ and $\Gamma \subset \mathrm{Aut}_{\mathbf{ManBnd}^\infty}(M)$. Then

1. $\mathfrak{X}^\Gamma(M)$ is a subspace of $\mathfrak{X}(M)$,
2. $\mathfrak{X}^\Gamma(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Proof. 1. Let $X, Y \in \mathfrak{X}^\Gamma(M)$, $\lambda \in \mathbb{R}$ and $\phi \in \Gamma$. Then Exercise **??** an exercise in the section on vector fields and smooth maps implies that

$$\begin{aligned}\phi_*(X + \lambda Y) &= \phi_* X + \lambda \phi_* Y \\ &= X + \lambda Y.\end{aligned}$$

Hence $X + \lambda Y \in \mathfrak{X}^\Gamma(M)$. Thus $\mathfrak{X}^\Gamma(M)$ is a subsapce of $\mathfrak{X}(M)$.

2. Let $X, Y \in \mathfrak{X}^{\Gamma}(M)$. Then

$$\begin{aligned}
\phi_*[X,Y] &= \phi_*(XY - YX) \\
&= \phi_*(XY) - \phi_*(YX) \\
&= (\phi_* X)(\phi_* Y) - (\phi_* Y)(\phi_* X)\text{prove this} \\
&= XY - YX \\
&= [X,Y].
\end{aligned}$$

Hence $[X, Y] \in \mathfrak{X}^{\Gamma}(M)$. Thus $\mathfrak{X}^{\Gamma}(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$.

$\square$

Chapter 13

Fiber Bundles

13.1 Introduction

13.1.1 Local Trivializations

Note 13.1.1.1. Let M, F be sets, we write $\mathrm{proj}_1 : M \times F \to M$ to denote the projection onto M.

Definition 13.1.1.2. Let $E, M, F \in \mathrm{Obj}(\mathbf{Set})$, $\pi \in \mathrm{Hom}_{\mathbf{Set}}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **local trivialization with respect to** π **of** E **over** U **with fiber** F if

1. Φ is a bijection
2. $\mathrm{proj}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$, i.e. the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\mathrm{proj}_1"] \\ & U \end{tikzcd}$$

Exercise 13.1.1.3. Let $E, M, F \in \mathrm{Obj}(\mathbf{Set})$ and $\pi \in \mathrm{Hom}_{\mathbf{Set}}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$ a local trivialization with respect to π of E over U with fiber F. Then for each $A \subset U$,

$$\Phi(\pi^{-1}(A)) = A \times F$$

Hint: consider $\Phi^{-1}(A \times F)$

Proof. Let $A \subset U$. Since $\mathrm{proj}_1^{-1}(A) = A \times F$, we have that

$$\begin{aligned} \Phi^{-1}(A \times F) &= \Phi^{-1}(\mathrm{proj}_1^{-1}(A)) \\ &= (\mathrm{proj}_1 \circ \Phi)^{-1}(A) \\ &= (\pi|_{\pi^{-1}(U)})^{-1}(A) \\ &= \pi^{-1}(A) \cap \pi^{-1}(U) \\ &\pi^{-1}(A \cap U) \\ &= \pi^{-1}(A) \end{aligned}$$

Since Φ is a bijection, we have that

$$\begin{aligned} \Phi(\pi^{-1}(A)) &= \Phi \circ \Phi^{-1}(A \times F) \\ &= A \times F \end{aligned}$$

$\square$

13.1.2 $\mathbf{Man}^0$ Fiber Bundles

Definition 13.1.2.1. Let $E, M, F \in \mathrm{Obj}(\mathbf{Man}^0)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^0}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **continuous fiber bundle local trivialization with respect to π of E over U with fiber F** if

1. U is open in M
2. (U, Φ) is a local trivialization with respect to π of E over U with fiber F
3. Φ is a homeomorphism

Definition 13.1.2.2. Let $E, M, F \in \mathrm{Obj}(\mathbf{Man}^0)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^0}(E, M)$ a surjection. Then (E, M, π, F) is said to be a **$\mathbf{Man}^0$ fiber bundle with total space E, base space M, fiber F and projection π** if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that (U, Φ) is a continuous local trivialization with respect to π of E over U with fiber F. For $p \in M$, we define the **fiber over** p, denoted E_p, by $E_p = \pi^{-1}(\{p\})$.

Exercise 13.1.2.3. $\mathbf{Man}^0$ Fiber Bundle Chart Lemma:
Let $E \in \mathrm{Obj}(\mathbf{Set})$, $M, F \in \mathrm{Obj}(\mathbf{Man}^0)$, $\pi : E \to M$ a surjection, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. Set $n = \dim M$ and $k = \dim F$. Suppose that

- for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}_M$
- $M \subset \bigcup\limits_{\alpha \in \Gamma} U_\alpha$
- for each $\alpha \in \Gamma$, (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
- for each $\alpha, \beta \in \Gamma$, $\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is continuous.

Then there exist a unique topology, $\mathcal{T}_E$, on E such that

1. $(E, \mathcal{T}_E)$ is a $n + k$-dimensional topological manifold
2. for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism
3. $\pi : E \to M$ is continuous
4. (E, M, π, F) is an $\mathbf{Man}^0$ fiber bundle

Proof.

1. For $\alpha \in \Gamma$, we define $X^n_\alpha(M, \mathcal{T}_M) \subset X^n(M, \mathcal{T}_M)$ by
$$X^n_\alpha(M, \mathcal{T}_M) = \{(V^M, \psi^M) \in X^n(M, \mathcal{T}_M) : V^M \subset U_\alpha\}$$
Choose index sets $(\Pi^M_\alpha)_{\alpha \in \Gamma}$ and Π^F such that for each $\alpha \in \Gamma$, $X^n_\alpha(M, \mathcal{T}_M) = (V^M_{\alpha,\mu}, \psi^M_{\alpha,\mu})_{\mu \in \Pi^M_\alpha}$ and $X^k(F, \mathcal{T}_F) = (V^F_\nu, \psi^F_\nu)_{\nu \in \Pi^F}$. Set $\Pi^M = \coprod_{\alpha \in \Gamma} \Pi^M_\alpha$ and $\Pi^E = \Pi^M \times \Pi^F$. For $(\alpha, \mu, \nu) \in \Pi^E$, we define $V^E_{\alpha,\mu,\nu} \subset E$ and $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ by
    - $V^E_{\alpha,\mu,\nu} = \Phi^{-1}_\alpha(V^M_{\alpha,\mu} \times V^F_\nu)$
    - $\psi^E_{\alpha,\mu,\nu} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu) \circ \Phi_\alpha|_{V^E_{\alpha,\mu,\nu}}$

    We have the following:
    - For each $(\alpha, \mu, \nu) \in \Pi^E$, $\psi^E_{\alpha,\mu,\nu}(V^E_{\alpha,\mu,\nu}) = \psi^M_\mu(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ and thus $\psi^E_{\alpha,\mu,\nu}(V^E_{\alpha,\mu,\nu}) \in \mathcal{T}_{\mathbb{H}^{n+k}}$

- For each $(\alpha_1,\mu_1,\nu_1),(\alpha_2,\mu_2,\nu_2)\in\Pi^E$,

$$\begin{aligned}\psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1}\cap V^E_{\alpha_2,\mu_2,\nu_2}) &= (\psi^M_{\alpha_1,\mu_1}\times\psi^F_{\nu_1})\circ\Phi_{\alpha_1}|_{V^E_{\alpha_1,\mu_1,\nu_1}}(\Phi^{-1}_{\alpha_1}([V^M_{\alpha_1,\mu_1}\times V^F_{\nu_1}]\cap[V^M_{\alpha_2,\mu_2}\times V^F_{\nu_2}]))\\ &= (\psi^M_{\alpha_1,\mu_1}\times\psi^F_{\nu_1})([V^M_{\alpha_1,\mu_1}\times V^F_{\nu_1}]\cap[V^M_{\alpha_2,\mu_2}\times V^F_{\nu_2}])\\ &= (\psi^M_{\alpha_1,\mu_1}\times\psi^F_{\nu_1})([V^M_{\alpha_1,\mu_1}\cap V^M_{\alpha_2,\mu_2}]\times[V^F_{\nu_1}\cap V^F_{q_2}])\\ &= \psi^M_{\alpha_1,\mu_1}(V^M_{\alpha_1,\mu_1}\cap V^M_{\alpha_2,\mu_2})\times\psi^F_{\nu_1}(V^F_{\nu_1}\cap V^F_{\nu_2})\\ &\in \mathcal{T}_{\mathbb{H}^{n+k}}\end{aligned}$$

- For each $(\alpha,\mu,\nu)\in\Pi^E$, $\psi^E_{\alpha,\mu,\nu}:V^E_{\alpha,\mu,\nu}\to\psi^M_{\alpha,\mu}(V^M_{\alpha,\mu})\times\psi^F_\nu(V^F_\nu)$ is a bijection
- Let $(\alpha_1,\mu_1,\nu_1),(\alpha_2,\mu_2,\nu_2)\in\Pi^E$. For notational convenience, set $\psi^E_1=\psi^E_{\alpha_1,\mu_1,\nu_1}$, $\psi^E_2=\psi_{\alpha_2,\mu_2,\nu_2}$, $V^E=V^E_{\alpha_1,\mu_1,\nu_1}\cap V^E_{\alpha_2,\mu_2,\nu_2}$, $V^M=V^M_{\alpha_1,\mu_1}\cap V^M_{\alpha_2,\mu_2}$ and $V^F=V^F_{\nu_1}\cap V^F_{\nu_2}$. Then $\psi_2|_{V^E}\circ(\psi_1|_{V^E})^{-1}:\psi_1(V^E)\to\psi_2(V^E)$ is given by

$$\begin{aligned}\psi^E_2|_{V^E}\circ(\psi^E_1|_{V^E})^{-1} &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M}\times\psi^F_{\nu_2}|_{V^F})\circ\Phi_{\alpha_2}|_{V^E}]\circ[(\psi^M_{\alpha_1,\mu_1}|_{V^M}\times\psi^F_{\nu_1}|_{V^F})\circ\Phi_{\alpha_1}|_{V^E}]^{-1}\\ &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M}\times\psi^F_{\nu_2}|_{V^F})\circ\Phi_{\alpha_2}|_{V^E}]\circ[(\Phi_{\alpha_1}|_{V^E})^{-1}\circ(\psi^M_{\alpha_1,\mu_1}|_{V^M}\times\psi^F_{\nu_1}|_{V^F})^{-1}]\\ &= (\psi^M_{\alpha_2,\mu_2}|_{V^M}\times\psi^F_{\nu_2}|_{V^F})\circ[\Phi_{\alpha_2}|_{V^E}\circ(\Phi_{\alpha_1}|_{V^E})^{-1}]\circ(\psi^M_{\alpha_1,\mu_1}|_{V^M}\times\psi^F_{\nu_1}|_{V^F})^{-1}\end{aligned}$$

Since $\Phi_{\alpha_2}|_{V^E}\circ(\Phi_{\alpha_1}|_{V^E})^{-1}$ is continuous, we have that $\psi^E_{\alpha_2,\mu_2,\nu_2}|_{V^E_{\alpha_1,\mu_1,\nu_1}\cap V^E_{\alpha_2,\mu_2,\nu_2}}\circ(\psi^E_{\alpha_1,\mu_1,\nu_1}|_{V^E_{\alpha_1,\mu_1,\nu_1}\cap V^E_{\alpha_2,\mu_2,\nu_2}})^{-1}:\psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1}\cap V^E_{\alpha_2,\mu_2,\nu_2})\to\psi^E_{\alpha_2,\mu_2,\nu_2}(V^E_{\alpha_1,\mu_1,\nu_1}\cap V^E_{\alpha_2,\mu_2,\nu_2})$ is continuous.
- A previous exercise in the section on topological manifolds implies that $(V^M_{\alpha,\mu})_{(\alpha,\mu)\in\Pi^M}$ is an open cover of M and $(V^F_\nu)_{\nu\in\Pi^F}$ is an open cover of F. Since M,F are second-countable M,F are Lindelöf and there exists $S^M\subset\Pi^M$, $S^F\subset\Pi^F$ such that S^M,S^F are countable, $(V^M_{\alpha,\mu})_{(\alpha,\mu)\in S^M}$ is an open cover of M and $(V^F_\nu)_{\nu\in\Pi^F}$ is an open cover of F. Then $S^M\times S^F$ is countable and $(V^M_{\alpha,\mu}\times V^F_\nu)_{(\alpha,\mu,\nu)\in S^M\times S^F}$ is an open cover of $M\times F$.
Let $a\in E$. Set $p=\pi(a)$. Choose $(\alpha,\mu)\in S^M$ such that $p\in V^M_{\alpha,\mu}$. Since $V^M_{\alpha,\mu}\subset U_\alpha$, $a\in\pi^{-1}(U_\alpha)$ which implies that

$$\begin{aligned}p&=\pi(a)\\ &=\mathrm{proj}_1\circ\Phi_\alpha(a)\end{aligned}$$

Set $q=\mathrm{proj}_2\circ\Phi_\alpha(a)$. Choose $\nu\in S^F$ such that $q\in V^F_\nu$. Then

$$\begin{aligned}\Phi_\alpha(a)&=(\mathrm{proj}_1\circ\Phi_\alpha(a),\mathrm{proj}_2\circ\Phi_\alpha(a))\\ &=(p,q)\\ &\in V^M_{\alpha,\mu}\times V^F_\nu\end{aligned}$$

Thus

$$\begin{aligned}a&\in\Phi^{-1}_\alpha(V^M_{\alpha,\mu}\times V^F_\nu)\\ &=V^E_{\alpha,\mu,\nu}\end{aligned}$$

Since $a\in E$ is arbitrary, we have that for each $a\in E$, there exists $(\alpha,\mu,\nu)\in S^M\times S^F\subset\Pi^E$ such that $a\in V^E_{\alpha,\mu,\nu}$. Thus

$$E\subset\bigcup_{(\alpha,\mu,\nu)\in S^M\times S^F}V_{\alpha,\mu,\nu}$$

- Let $a_1,a_2\in E$.
For now, suppose that $\pi(a_1)\neq\pi(a_2)$. Set $p_1=\pi(a_1)$ and $p_2=\pi(a_2)$. Since M is Hausdorff, there exist $(\alpha_1,\mu_1),(\alpha_2,\mu_2)\in\Pi^M$ such that $p_1\in V^M_{\alpha_1,\mu_1}$, $p_2\in V^M_{\alpha_2,\mu_2}$ and $V^M_{\alpha_1,\mu_1}\cap V^M_{\alpha_2,\mu_2}=\varnothing$.

Set $q_1 = \text{proj}_2 \circ \Phi_{\alpha_1}(a_1)$ and $q_2 = \text{proj}_2 \circ \Phi_{\alpha_2}(a_2)$. Choose $\nu_1, \nu_2 \in \Pi^F$ such that $q_1 \in V^F_{\nu_1}$ and $q_2 \in V^F_{\nu_2}$. Then similarly to the previous part, $a_1 \in V^E_{\alpha_1,\mu_1,\nu_1}$ and $a_2 \in V^E_{\alpha_2,\mu_2,\nu_2}$ and therefore

$$\begin{aligned} V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2} &= \Phi^{-1}_{\alpha_1}(V^M_{\alpha_1,\mu_1} \times V^F_{\nu_1}) \cap \Phi^{-1}_{\alpha_2}(V^M_{\alpha_2,\mu_2} \times V^F_{\nu_2}) \\ &\subset \pi^{-1}(V^M_{\alpha_1,\mu_1}) \cap \pi^{-1}(V^M_{\alpha_2,\mu_2}) \\ &= \pi^{-1}(V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}) \\ &= \pi^{-1}(\varnothing) \\ &= \varnothing \end{aligned}$$

Now suppose that $\pi(a_1) = \pi(a_2)$. Set $p = \pi(a_1)$. Then there exists $(\alpha, \mu) \in \Pi^M$ such that $p \in V^M_{\alpha,\mu} \subset U_\alpha$.
For now, suppose that $\text{proj}_2 \circ \Phi_\alpha(a_1) \neq \text{proj}_2 \circ \Phi_\alpha(a_2)$. Set $q_1 = \text{proj}_2 \circ \Phi_\alpha(a_1)$ and $q_2 = \text{proj}_2 \circ \Phi_\alpha(a_2)$. Since F is Hausdorff, there exist $\nu_1, \nu_2 \in \Pi^F$ such that $q_1 \in V^F_{\nu_1}$ and $q_2 \in V^F_{\nu_2}$ and $V^F_{\nu_1} \cap V^F_{\nu_2} = \varnothing$. Then $a_1 \in V^E_{\alpha,\mu,\nu_1}$, $a_2 \in V^E_{\alpha,\mu,\nu_2}$ and

$$\begin{aligned} V^E_{\alpha,\mu,\nu_1} \cap V^E_{\alpha,\mu,\nu_2} &= \Phi^{-1}_{\alpha}(V^M_{\alpha,\mu} \times V^F_{\nu_1}) \cap \Phi^{-1}_{\alpha}(V^M_{\alpha,\mu} \times V^F_{\nu_2}) \\ &= \Phi^{-1}_{\alpha}([V^M_{\alpha,\mu} \times V^F_{\nu_1}] \cap [V^M_{\alpha,\mu} \times V^F_{\nu_2}]) \\ &= \Phi^{-1}_{\alpha}([V^M_{\alpha,\mu} \cap V^M_{\alpha,\mu}] \times [V^F_{\nu_1} \cap V^F_{\nu_2}]) \\ &= \Phi^{-1}_{\alpha}(V^M_{\alpha,\mu} \times [V^F_{\nu_1} \cap V^F_{\nu_2}]) \\ &= \Phi^{-1}_{\alpha}(V^M_{\alpha,\mu} \times \varnothing) \\ &= \Phi^{-1}_{\alpha}(\varnothing) \\ &= \varnothing \end{aligned}$$

Now, suppose that $\text{proj}_2 \circ \Phi_\alpha(a_1) = \text{proj}_2 \circ \Phi_\alpha(a_2)$. Set $q = \text{proj}_2 \circ \Phi_\alpha(a_1)$. Choose $\nu \in \Pi^F$ such that $q \in V^F_\nu$. Since

$$\begin{aligned} \Phi_\alpha(a_1) &= (\text{proj}_1 \circ \Phi_\alpha(a_1), \text{proj}_2 \circ \Phi_\alpha(a_1)) \\ &= (p, q) \\ &= (\text{proj}_1 \circ \Phi_\alpha(a_2), \text{proj}_2 \circ \Phi_\alpha(a_2)) \\ &= \Phi_\alpha(a_2) \end{aligned}$$

we have that $a_1 = a_2$ and $a_1, a_2 \in V^E_{\alpha,\mu,\nu}$. Therefore, for each $a_1, a_2 \in E$, there exists $(\alpha, \mu, \nu) \in \Pi^E$ such that $p, q \in V^E_{\alpha,\mu,\nu}$ or there exist $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$ such that $a_1 \in V^E_{\alpha_1,\mu_1,\nu_1}$, $a_2 \in V^E_{\alpha_2,\mu_2,\nu_2}$ and $V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2} = \varnothing$.

The topological manifold chart lemma implies that there exists a unique topology $\mathcal{T}_E$ on E such that $(E, \mathcal{T}_E)$ is an $n+k$-dimensional topological manifold and $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E} \subset X^{n+k}(E, \mathcal{T}_E)$.

2. Let $\alpha \in \Gamma$. By assumption $U_\alpha \in \mathcal{T}_M$. Let $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$. Then $(\alpha, \mu, \nu) \in \Pi^E$. Since

- $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a homeomorphism
- $\psi^M_{\alpha,\mu} \times \psi^F_\nu : V^M_{\alpha,\mu} \times V^F_\nu \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a homeomorphism
- $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is given by $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu)^{-1} \circ \psi^E_{\alpha,\mu,\nu}$,

we have that $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a homeomorphism. Since $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$ are arbitrary we have that for each $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$, $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a homeomorphism. Since $(V^M_{\alpha,\mu})_{\mu\in\Pi^M_\alpha}$ is an open cover of U_α and $(V^M_{\alpha,\mu} \times V^F_\nu)_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F}$ is an open cover of $U_\alpha \times F$, we have that

$$\begin{aligned}
\pi^{-1}(U_\alpha) &= \pi^{-1}\bigg(\bigcup_{\mu\in\Pi^M_\alpha} V^M_{\alpha,\mu}\bigg)\\
&= \bigcup_{\mu\in\Pi^M_\alpha} \pi^{-1}(V^M_{\alpha,\mu})\\
&= \bigcup_{\mu\in\Pi^M_\alpha} \Phi^{-1}_\alpha(V^M_{\alpha,\mu}\times F)\\
&= \bigcup_{\mu\in\Pi^M_\alpha} \Phi^{-1}_\alpha\bigg(V^M_{\alpha,\mu}\times \bigg[\bigcup_{\nu\in\Pi^F} V^F_\nu\bigg]\bigg)\\
&= \bigcup_{\mu\in\Pi^M_\alpha} \Phi^{-1}_\alpha\bigg(\bigcup_{\nu\in\Pi^F} [V^M_{\alpha,\mu}\times V^F_\nu]\bigg)\\
&= \bigcup_{\mu\in\Pi^M_\alpha} \bigg[\bigcup_{\nu\in\Pi^F} \Phi^{-1}_\alpha(V^M_{\alpha,\mu}\times V^F_\nu)\bigg]\\
&= \bigcup_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F} V^E_{\alpha,\mu,\nu}
\end{aligned}$$

Hence $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$, $(V^E_{\alpha,\mu,\nu})_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F}$ is an open cover of $\pi^{-1}(U_\alpha)$ and Φ_α is a local homeomorphism. Since Φ_α is a bijection, Φ_α is a homeomorphism. Since $\alpha \in \Gamma$ is arbitrary, we have that for each $\alpha \in \Gamma$, $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism.

3. Let $(\alpha,\mu,\nu) \in \Pi^E$. Since

   - $V^E_{\alpha,\mu,\nu} \subset \pi^{-1}(U_\alpha)$
   - $\mathrm{proj}_1 : M \times F \to M$ is continuous
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is continuous
   - $\pi|_{V^E_{\alpha,\mu,\nu}} = \mathrm{proj}_1 \circ \Phi|_{V^E_{\alpha,\mu,\nu}}$

   we have that $\pi|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu}$ is continuous. Since $(\alpha,\mu,\nu) \in \Pi^E$ is arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is an open cover of E, we have that $\pi : E \to M$ is continuous.

4. Let $p \in M$. By assumption, there exists $\alpha \in \Gamma$ such that $p \in U_\alpha$, $U_\alpha \in \mathcal{T}_M$. Since $E, M, F \in \mathrm{Obj}(\mathbf{Man}^0)$, $\pi \in \mathrm{Hom}_{\mathbf{Man}^0}(E, M)$ is a surjection, and

   - U_α is open
   - (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism

   we have that (U_α, Φ_α) is a continuous local trivialization with respect to π of E over U_α with fiber F. Since $p \in M$ is arbitrary, (E, M, π, F) is a $\mathbf{Man}^0$ fiber bundle.

□

13.1.3 Man$^\infty$ Fiber Bundles

Definition 13.1.3.1. Let $E, M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **smooth fiber bundle local trivialization of** E **over** U **with fiber** F if

1. U is open in M

2. (U, Φ) is a local trivialization of E over U with fiber F with respect to π

3. Φ is a diffeomorphism

Definition 13.1.3.2. Let $E, M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then (E, M, π, F) is said to be a **$\mathbf{Man}^\infty$ fiber bundle with total space E, base space M, fiber F and projection π** if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that U is open and (U, Φ) is a smooth local trivialization of E over U with fiber F. For $p \in M$, we define the **fiber over** p, denoted E_p, by $E_p = \pi^{-1}(\{p\})$.

Exercise 13.1.3.3. $\mathbf{Man}^\infty$ Fiber Bundle Chart Lemma:
Let $E \in \mathrm{Obj}(\mathbf{Set})$, $M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $\pi : E \to M$ a surjection, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. Set $n := \dim M$ and $k := \dim F$. Suppose that

- for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}_M$
- $M \subset \bigcup\limits_{\alpha \in \Gamma} U_\alpha$
- for each $\alpha \in \Gamma$, (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
- for each $\alpha, \beta \in \Gamma$, $\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is smooth.

Then there exist a unique topology $\mathcal{T}_E$ on E and smooth structure $\mathcal{A}_E \subset X^{n+k}(M, \mathcal{T}_E)$ on E such that

1. $(E, \mathcal{T}_E)$ is an $n+k$-dimensional topologocal manifold and $(E, \mathcal{T}_E, \mathcal{A}_E)$ is a smooth manifold,
2. for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism
3. $\pi : E \to M$ is smooth
4. (E, M, π, F) is an $\mathbf{Man}^\infty$ fiber bundle

Proof. Exercise 13.1.2.3 implies that there exists a unique topology $\mathcal{T}_E$ on E such that

- $(E, \mathcal{T}_E)$ is a $n+k$-dimensional topological manifold
- for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism
- $\pi : E \to M$ is continuous
- (E, M, π, F) is an $\mathbf{Man}^0$ fiber bundle

1. Define $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E} \subset X^{n+k}(E, \mathcal{T}_E)$ as in the proof of the $\mathbf{Man}^0$ fiber bundle chart lemma. Let $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$. For notational convenience, set $\psi^E_1 = \psi^E_{\alpha_1,\mu_1,\nu_1}$, $\psi^E_2 = \psi_{\alpha_2,\mu_2,\nu_2}$, $V^E = V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}$, $V^M = V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}$ and $V^F = V^F_{\nu_1} \cap V^F_{\nu_2}$. Then $\psi_2|_{V^E} \circ (\psi_1|_{V^E})^{-1} : \psi_1(V^E) \to \psi_2(V^E)$ is given by
$$\begin{aligned}
\psi^E_2|_{V^E} \circ (\psi^E_1|_{V^E})^{-1} &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F}) \circ \Phi_{\alpha_1}|_{V^E}]^{-1} \\
&= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\Phi_{\alpha_1}|_{V^E})^{-1} \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}] \\
&= (\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ [\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}] \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}
\end{aligned}$$
Since $\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}$ is smooth, we have that $\psi^E_{\alpha_2,\mu_2,\nu_2}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}} \circ (\psi^E_{\alpha_1,\mu_1,\nu_1}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}})^{-1} : \psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}) \to \psi^E_{\alpha_2,\mu_2,\nu_2}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2})$ is smooth. Since $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$ are arbitrary, we have that $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is a smooth atlas on E. An exercise in the section on smooth manifolds implies that there exists a unique smooth structure $\mathcal{A}_E$ on E such that $(E, \mathcal{A}_E)$ is an $n+k$-dimensional smooth manifold.
2. Let $\alpha \in \Gamma$. By assumption $U_\alpha \in \mathcal{T}_M$. Let $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$. Then $(\alpha, \mu, \nu) \in \Pi^E$. Since
   - $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a diffeomorphism
   - $\psi^M_{\alpha,\mu} \times \psi^F_\nu : V^M_{\alpha,\mu} \times V^F_\nu \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a diffeomorphism

- $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is given by $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu)^{-1} \circ \psi^E_{\alpha,\mu,\nu}$,

we have that $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a diffeomorphism. Since $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$ are arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\mu,\nu)\in\Pi^M_\alpha \times \Pi^F}$ is an open cover of $\pi^{-1}(U_\alpha)$, we have that $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a local diffeomorphism. Since Φ_α is a bijection, Φ_α is a diffeomorphism. Since $\alpha \in \Gamma$ is arbitrary, we have that for each $\alpha \in \Gamma$, $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism.

3. Let $(\alpha, \mu, \nu) \in \Pi^E$. Since
   - $V^E_{\alpha,\mu,\nu} \subset \pi^{-1}(U_\alpha)$
   - $\mathrm{proj}_1 : M \times F \to M$ is smooth
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is smooth
   - $\pi|_{V^E_{\alpha,\mu,\nu}} = \mathrm{proj}_1 \circ \Phi|_{V^E_{\alpha,\mu,\nu}}$

   we have that $\pi|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu}$ is smooth. Since $(\alpha, \mu, \nu) \in \Pi^E$ is arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is an open cover of E, we have that $\pi : E \to M$ is smooth.

4. Let $p \in M$. By assumption, there exists $\alpha \in \Gamma$ such that $p \in U_\alpha$, $U_\alpha \in \mathcal{T}_M$. Since $E, M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ is a surjection, and
   - U_α is open
   - (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism

   we have that (U_α, Φ_α) is a smooth local trivialization with respect to π of E over U_α with fiber F. Since $p \in M$ is arbitrary, (E, M, π, F) is a $\mathbf{Man}^\infty$ fiber bundle.

□

Definition 13.1.3.4. Let (E_1, M_1, π_1, F_1) and (E_2, M_2, π_2, F_2) be $\mathbf{Man}^\infty$ fiber bundles, $\Phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E_1, E_2)$ and $\phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M_1, M_2)$. Then (Φ, ϕ) is said to be a **smooth bundle morphism** from (E_1, M_1, π_1, F_1) to (E_2, M_2, π_2, F_2) if $\pi_2 \circ \Phi = \phi \circ \pi_1$, i.e. the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\phi"'] & M_2 \end{tikzcd}$$

Exercise 13.1.3.5. Let (E_1, M_1, π_1, F_1) and (E_2, M_2, π_2, F_2) be $\mathbf{Man}^\infty$ fiber bundles, $\Phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E_1, E_2)$ and $\phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M_1, M_2)$. If (Φ, ϕ) is a smooth bundle morphism from (E_1, M_1, π_1, F_1) to (E_2, M_2, π_2, F_2), then for each $p \in M_1$, $\Phi((E_1)_p) \subset (E_2)_{\phi(p)}$.

Proof. Suppose that (Φ, ϕ) is a smooth bundle morphism from (E_1, M_1, π_1, F_1) to (E_2, M_2, π_2, F_2). Let $p \in M_1$ and $y \in \Phi((E_1)_p)$. Then there exists $x \in (E_1)_p$ such that $y = \Phi(x)$. Since $x \in (E_1)_p$, we have that $\pi_1(x) = p$. Since (Φ, ϕ) is a smooth bundle morphism from (E_1, M_1, π_1, F_1) to (E_2, M_2, π_2, F_2), we have that $\pi_2 \circ \Phi = \phi \circ \pi_1$. Therefore

$$\begin{aligned} \pi_2(y) &= \pi_2(\Phi(x)) \\ &= \pi_2 \circ \Phi(x) \\ &= \phi \circ \pi_1(x) \\ &= \phi(p) \end{aligned}$$

Thus

$$\begin{aligned} y &\in \pi_2^{-1}(\phi(p)) \\ &= (E_2)_{\phi(p)} \end{aligned}$$

Since $y \in \Phi((E_1)_p)$ is arbitrary, we have that $\Phi((E_1)_p) \subset (E_2)_{\phi(p)}$. □

Definition 13.1.3.6. We define the category of $\mathbf{Man}^\infty$ fiber bundles, denoted $\mathbf{Bun}^\infty$, by

- $\mathrm{Obj}(\mathbf{Bun}^\infty) := \{(E,M,\pi,F) : (E,M,\pi,F) \text{ is a } \mathbf{Man}^\infty \text{ fiber bundle}\}$
- For $(E_1,M_1,\pi_1,F_1),(E_2,M_2,\pi_2,F_2) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$,

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1,M_1,\pi_1,F_1),(E_2,M_2,\pi_2,F_2)) := \\ &\{(\Phi,\phi) : (\Phi,\phi) \text{ is a smooth bundle morphism from } (E_1,M_1,\pi_1,F_1) \text{ to } (E_2,M_2,\pi_2,F_2)\}\end{aligned}$$

- For
  - $(E_1,M_1,\pi_1,F_1),(E_2,M_2,\pi_2,F_2),(E_3,M_3,\pi_3) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$
  - $(\Phi_{12},\phi_{12}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1,M_1,\pi_1,F_1),(E_2,M_2,\pi_2,F_2))$
  - $(\Phi_{23},\phi_{23}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_2,M_2,\pi_2,F_2),(E_3,M_3,\pi_3))$

  we define $(\Phi_{23},\phi_{23}) \circ (\Phi_{12},\phi_{12}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1,M_1,\pi_1,F_1),(E_3,M_3,\pi_3))$ by

$$(\Phi_{23},\phi_{23}) \circ (\Phi_{12},\phi_{12}) := (\Phi_{23} \circ \Phi_{12}, \phi_{23} \circ \phi_{12})$$

Exercise 13.1.3.7. We have that $\mathbf{Bun}^\infty$ is a full subcategory of $(\mathrm{id}_{\mathbf{Man}^\infty} \downarrow \mathrm{id}_{\mathbf{Man}^\infty})$.

Proof. Set $\mathcal{C} = (\mathrm{id}_{\mathbf{Man}^\infty} \downarrow \mathrm{id}_{\mathbf{Man}^\infty})$. We note that

- $\mathrm{Obj}(\mathbf{Bun}^\infty) \subset \mathrm{Obj}(\mathcal{C})$
- for each $(E_1,M_1,\pi_1,F_1),(E_2,M_2,\pi_2,F_2) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$,

$$\mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1,M_1,\pi_1,F_1),(E_2,M_2,\pi_2,F_2)) = \mathrm{Hom}_{\mathcal{C}}((E_1,M_1,\pi_1,F_1),(E_2,M_2,\pi_2,F_2))$$

So $\mathbf{Bun}^\infty$ is a full subcategory of $\mathcal{C}$. □

Exercise 13.1.3.8. Let $(E,M,\pi,F) \in \mathbf{Bun}^\infty$. Then π is a submersion.

Proof. Let $a \in E$. Set $p := \pi(a)$. Since $(E,M,\pi,F) \in \mathbf{Bun}^\infty$, there exists $U \in \mathcal{T}_M$ and $\Phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(\pi^{-1}(U), U \times F)$ such that $p \in U$ and (U,Φ) is a smooth fiber bundle local trivialization of E over U with fiber F with respect to π. Then Φ is a diffeomorphism and $\mathrm{proj}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$. Exercise 7.3.0.4 implies that $\mathrm{proj}_1 : U \times F \to U$ is a submersion. Since Φ is a diffeomorphism, Φ is a submersion. Exercise 7.3.0.5 then implies that $\pi|_{\pi^{-1}(U)}$ is a submersion. Since $a \in E$ is arbitrary, we have that for each $a \in E$, there exists $V \in \mathcal{T}_E$ such that $a \in V$ and $\pi|_V$ is a submersion. (cite exercise) Exercise **??** implies that π is a submersion. □

Exercise 13.1.3.9. Let $(E,M,\pi,F) \in \mathbf{Bun}^\infty$ and (U,Φ) a local trivialization of E over U. For each $p \in M$,

1. E_p is an embedded submanifold of E,
2. $\Phi|_{E_p} : E_p \to \{p\} \times F$ is a diffeomorphism.

Proof. Let $p \in M$.

1. Since $E_p = \pi^{-1}(\{p\})$ and π is a surjective submersion Exercise **??** ref exercise in section on submersion implies that E_p is an embedded submanifold of E.
2. Exercise **??** ref exercise in section on immersed submanifolds implies that $\Phi|_{E_p}$ is a diffeomorphism.

□

Exercise 13.1.3.10. Let $(E,M,\pi,F) \in \mathbf{Bun}^\infty$, (U,Φ) a local trivialization of E over U and (V,Ψ) a local trivialization of E over V. Then

1. $\mathrm{proj}_1 \circ \Psi|_{\pi^{-1}(U\cap V)} \circ \Phi|^{-1}_{\pi^{-1}(U\cap V)} = \mathrm{proj}_1$

2. there exists $\sigma \in \mathrm{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that $\Psi|_{\pi^{-1}(U\cap V)} \circ \Phi|^{-1}_{\pi^{-1}(U\cap V)} = (\mathrm{proj}_1, \sigma)$ and for each $p \in U \cap V$, $\sigma(p, \cdot) \in \mathrm{Aut}_{\mathbf{Man}^\infty}(F)$.

Proof.

1. By definition and Exercise 13.1.1.3, the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times F \arrow[dr, "\mathrm{proj}_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\Phi"'] \arrow[r, "\Psi"] \arrow[d, "\pi"] & (U \cap V) \times F \arrow[dl, "\mathrm{proj}_1"] \\
& U \cap V &
\end{tikzcd}$$

Therefore $\mathrm{proj}_1 \circ \Psi|_{\pi^{-1}(U\cap V)} \circ (\Phi|_{\pi^{-1}(U\cap V)})^{-1} = \mathrm{proj}_1$.

2. Define $\sigma, \tau \in \mathrm{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ by $\sigma := \mathrm{proj}_2 \circ \Psi|_{\pi^{-1}(U\cap V)} \circ (\Phi|_{\pi^{-1}(U\cap V)})^{-1}$ and $\tau := \mathrm{proj}_2 \circ \Phi|_{\pi^{-1}(U\cap V)} \circ (\Psi|_{\pi^{-1}(U\cap V)})^{-1}$. Part (1) implies that for each $(p, x) \in (U \cap V) \times F$,

$$\begin{aligned}
\Psi|_{\pi^{-1}(U\cap V)} \circ (\Phi|_{\pi^{-1}(U\cap V)})^{-1}(p, x) &= (\mathrm{proj}_1(p, x), \sigma(p, x)) \\
&= (p, \sigma(p, x)).
\end{aligned}$$

Similarly, for each $(p, x) \in (U \cap V) \times F$, $\Phi|_{\pi^{-1}(U\cap V)} \circ (\Psi|_{\pi^{-1}(U\cap V)})^{-1}(p, x) = (p, \tau(x))$. Let $p \in U \cap V$ and $x \in F$. Set $\sigma_p := \sigma \circ \iota_p^F$ and $\tau_p := \tau \circ \iota_p^F$. Exercise 7.2.0.10 implies that σ_p and τ_p are smooth (clean up a bit here). Then

$$\begin{aligned}
(p, x) &= \mathrm{id}_{(U\cap V)\times F}(p, x) \\
&= [\Psi|_{\pi^{-1}(U\cap V)} \circ (\Phi|_{\pi^{-1}(U\cap V)})^{-1}] \circ [\Phi|_{\pi^{-1}(U\cap V)} \circ (\Psi|_{\pi^{-1}(U\cap V)})^{-1}](p, x) \\
&= (p, \sigma(\Phi|_{\pi^{-1}(U\cap V)} \circ (\Psi|_{\pi^{-1}(U\cap V)})^{-1}(p, x))) \\
&= (p, \sigma(p, \tau(p, x))) \\
&= (p, \sigma_p \circ \tau_p(x))
\end{aligned}$$

Since $x \in F$ is arbitary, we have that for each $x \in F$, $\mathrm{id}_F(x) = \sigma_p \circ \tau_p(x)$. Thus $\sigma_p \circ \tau_p = \mathrm{id}_F$. Similarly, $\tau_p \circ \sigma_p = \mathrm{id}_F$. Thus σ_p is a bijection and $\sigma_p^{-1} = \tau_p$. Therefore $\sigma_p \in \mathrm{Aut}_{\mathbf{Man}^\infty}(F)$. Since $p \in U \cap V$ is arbitrary, we have that for each $p \in U \cap V$, $\sigma(p, \cdot) \in \mathrm{Aut}_{\mathbf{Man}^\infty}(F)$.

$\square$

13.1.4 cocycles

Definition 13.1.4.1. Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$, A an index set and for each $\alpha \in A$, (U_α, Φ_α) a smooth local trivializations of E. Then $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ is said to be a **smooth fiber bundle atlas on** (E, M, π, F) if for each $p \in M$, there exists $\alpha \in A$ such that $p \in U_\alpha$.

Definition 13.1.4.2. Let $(E, M, \pi, F) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$, A an index set and $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ a smooth fiber bundle atlas on (E, M, π, F). For each $\alpha, \beta \in A$, we define $U_{\alpha,\beta} \subset M$ and $\Phi_{\alpha,\beta} : U_{\alpha,\beta} \times F \to U_{\alpha,\beta} \times F$ by

- $U_{\alpha,\beta} = U_\alpha \cap U_\beta$
- $\Phi_{\alpha,\beta} = \Phi_\alpha|_{U_{\alpha,\beta}} \circ \Phi_\beta|^{-1}_{U_{\alpha,\beta}}$

Exercise 13.1.4.3. Let $(E, M, \pi, F) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$, A an index set and $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ a smooth fiber bundle atlas on (E, M, π, F). Then for each $\alpha, \beta \in A$ and $p \in U_{\alpha,\beta}$, $\Phi_{\alpha,\beta}(p, \cdot) \in \mathrm{Aut}_{\mathbf{Man}^\infty}(F)$.

Proof. Let $\alpha, \beta \in \Gamma$ and $p \in U_{\alpha,\beta}$. Since FINISH, basically reference the previous exercise $\square$

13.2 Product Bundles

Definition 13.2.0.1.

13.3 Vertical and Horizontal Subbundles

Definition 13.3.0.1. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$. We define the **vertical bundle associated to** (E, M, π) , denoted $(VE, M, \pi_V) \in \mathbf{Bun}^\infty$, by

$$VE = \coprod_{q \in E} \ker D\pi(q)$$

relocate this to after tangent bundle is introduced

Exercise 13.3.0.2. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $(\pi^{-1}(U), \Phi_\phi) \in \mathcal{A}_{TM}$ the induced chart on TM with $\Phi_\phi = (\tilde{x}^1, \dots, \tilde{x}^n, \tilde{y}^1, \dots, \tilde{y}^n)$. Then

$$V(TM)|_{\pi^{-1}(U)} = \coprod_{(p,\xi) \in \pi^{-1}(U)} \mathrm{span}\left\{ \frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \dots, n\} \right\}$$

Split into smaller exercises

Proof. Let $f \in C^\infty(M)$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$ the standard coordinates on $\mathbb{R}^n \times \mathbb{R}^n$. We note that by definition, $\Phi_\phi(p, \xi) = (\phi(p), \psi(\xi))$ where $\psi : \bigcup_{p \in U} T_pM \to \mathbb{R}^n$ is given by

$$\psi\left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{(\phi(p),\psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial v^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i}\bigg|_{(\phi(p),\psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= 0 \end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned} D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} f \circ \pi \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i}(p,\xi) \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)\delta_{i,k} \\ &= \frac{\partial f}{\partial x^i}(p) \end{aligned}$$

and

$$\begin{aligned} D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} f \circ \pi \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i}(p,\xi) \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)0 \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \ker D\pi(p,\xi) \\ &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \operatorname{span}\left\{ \frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \dots, n\} \right\} \end{aligned}$$

$\square$

Chapter 14

Vector Bundles

14.1 Introduction

14.1.1 $\mathbf{Man}^\infty$ Vector Bundles

Note 14.1.1.1. Let M be a set and $p \in M$. We endow $\{p\} \times \mathbb{R}^n$ with the natural vector space structure such that $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

Definition 14.1.1.2. Let $E, M \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$. Then (U, Φ) is said to be a **smooth vector bundle local trivialization of E over U** if

1. U is open in M
2. (U, ϕ) is a smooth local trivialization of E over U with fiber $\mathbb{R}^k$ (Definition 13.1.3.1)
3. for each $q \in U$, $\Phi|_{E_q} : E_q \to \{p\} \times \mathbb{R}^k$ is a vector space

Definition 14.1.1.3. Let $E, M \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then (E, M, π) is said to be a **rank-k smooth vector bundle** if

1. $(E, M, \pi, \mathbb{R}^k) \in \text{Obj}(\mathbf{Bun}^\infty)$
2. for each $p \in M$, E_p is a k-dimensional real vector space and there exists $U \in \mathcal{T}_M$, $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that
   (a) $p \in U$
   (b) (U, ϕ) is a smooth vector bundle local trivialization of E over U

In this case we define the **rank of** (E, M, π), denoted $\text{rank}(E, M, \pi)$, by $\text{rank}(E, M, \pi) = k$.

Exercise 14.1.1.4. Let (E, M, π) be a rank-k smooth vector bundle, (U, Φ) a local trivialization of E over U and (V, Ψ) a smooth vector bundle local trivialization of E over V. Then

1. $\text{proj}_1 \circ \Psi|_{\pi^{-1}(U \cap V)} \circ \Phi|^{-1}_{\pi^{-1}(U \cap V)} = \text{proj}_1$
2. there exists $\tau \in \text{Hom}_{\mathbf{Man}^\infty}(U \cap V, GL(k, \mathbb{R}))$ such that for each $(p, v) \in (U \cap V) \times \mathbb{R}^k$, $\Psi|_{\pi^{-1}(U \cap V)} \circ \Phi|^{-1}_{\pi^{-1}(U \cap V)}(p, v) = (p, \tau(p)(v))$.

Proof. Exercise 13.1.3.10 implies that there exists $\sigma \in \text{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times \mathbb{R}^k, \mathbb{R}^k)$ such that $\Psi|_{\pi^{-1}(U \cap V)} \circ \Phi|^{-1}_{\pi^{-1}(U \cap V)} = (\text{proj}_1, \sigma)$ and for each $p \in U \cap V$, $\sigma(p, \cdot) \in \text{Aut}_{\mathbf{Man}^\infty}(\mathbb{R}^k)$. Define $\tau : U \cap V \to \text{Aut}_{\mathbf{Man}^\infty}(\mathbb{R}^k)$ by $\tau(p) = \sigma(p, \cdot)$. Since (U, Φ), (V, Ψ) are smooth vector bundle local trivializations, for each $q \in U \cap V$,

$\Phi|_{E_q} \to \{q\} \times \mathbb{R}^k$ and $\Psi|_{E_q} \to \{q\} \times \mathbb{R}^k$ are linear isomorphism. Let $q \in U \cap V$. Since $\Psi|_{E_q} \circ \Phi|_{E_q}^{-1} : \{q\} \times \mathbb{R}^k \to \{q\} \times \mathbb{R}^k$, is a vector space isomorphism and for each $v \in \mathbb{R}^k$,

$$\begin{aligned}\Psi|_{E_q} \circ \Phi|_{E_q}^{-1}(q, v) &= (q, \sigma(q, v)) \\ &= (q, \tau(q)(v)),\end{aligned}$$

we have that $\tau(q) \in GL(k, \mathbb{R})$. need to show τ is smooth, use hint in book, make exercise in a previous section about actions □

the fiber bundle construction theorems dont actually construct a fiber bundle, they just show that a given set is one and characterize the topology and smooth structure under some assumptions, maybe go back and rename them to "characterization theorem" and then actually have a construction theorem. then here, introduce a characterization theorem and then have a separate short construction theorem.

Exercise 14.1.1.5. Smooth Vector Bundle Chart Lemma:
Let $M \in \text{Obj}(\mathbf{Man}^\infty)$ and $(E_p)_{p \in M} \subset \text{Obj}(\mathbf{Vect}_{\mathbb{R}})$. Set $n := \dim M$. Suppose that for each $p \in M$, $\dim E_p = k$. We define $E \in \text{Obj}(\mathbf{Set})$ and $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$ by

$$E = \coprod_{p \in M} E_p$$

and $\pi(p, v) = p$. Let Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$. Set $n := \dim M$ and $k := \dim F$. Suppose that

1. for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}_M$
2. $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
3. for each $\alpha \in \Gamma$, there exists $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ is a bijection
   - for each $q \in U_\alpha$, $\Phi_\alpha|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$ is a vector space isomorphism
4. for each $\alpha, \beta \in \Gamma$, there exists $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that
   - $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ is smooth
   - $\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is given by $\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1}(p, v) = (p, \tau_{\alpha,\beta}(p)(v))$.

Then there exists a unique topology $\mathcal{T}_E$ on E and smooth structure $\mathcal{A}_E$ on $(E, \mathcal{T}_E)$ such that

1. $(E, \mathcal{T}_E)$ is an $(n + k)$-dimensional topological manofold and $(E, \mathcal{T}_E, \mathcal{A}_E)$ is a smooth manifold
2. for each $\alpha \in \Gamma$, (U_α, Φ_α) is a diffeomorphism
3. $\pi : E \to M$ is smooth
4. (E, M, π) is a rank-k $\mathbf{Man}^\infty$ vector bundle.

Proof. Let $\alpha \in \Gamma$ and $a \in \pi^{-1}(U_\alpha)$. By definition, there exists $q \in U_\alpha$ and $v_0 \in E_q$ such that $a = (q, v_0)$. Since $\Phi_\alpha|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$ is a vector space isomorphism, there exists $v \in \mathbb{R}^k$ such that $\Phi_\alpha(q, v_0) = (q, v)$. Then

$$\begin{aligned}\text{proj}_1 \circ \Phi_\alpha(a) &= \text{proj}_1 \circ \Phi_\alpha(q, v_0) \\ &= \text{proj}_1(q, v) \\ &= q \\ &= \pi(q, v_0) \\ &= \pi(a).\end{aligned}$$

Since $a \in \pi^{-1}(U_\alpha)$ is arbitrary, we have that $\text{proj}_1 \circ \Phi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$. Therefore (U_α, Φ_α) is a local trivialization of E over U_α with fiber $\mathbb{R}^k$ with respect to π.

such that need to show that (U_α, Φ_α) smooth vector bundle local trivialization of E over U with fiber $\mathbb{R}^k$ with respect to π here using the cocycle condition. Let $\alpha \in A$.

1. By assumption, Φ_α is a bijection
2. $\text{proj}_1 \circ \Phi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$, i.e. the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times \mathbb{R}^k \arrow[d, "\text{proj}_1"] \\ & U_\alpha \end{tikzcd}$$

then Exercise 13.1.3.3 implies that there exist a unique topology $\mathcal{T}_E$ on E and smooth structure $\mathcal{A}_E \subset X^{n+k}(M, \mathcal{T}_E)$ on E such that

1. $(E, \mathcal{T}_E)$ is an $n+k$-dimensional topologocal manifold and $(E, \mathcal{T}_E, \mathcal{A}_E)$ is a smooth manifold,
2. for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ is a diffeomorphism,
3. $\pi : E \to M$ is smooth,
4. $(E, M, \pi, \mathbb{R}^k)$ is an $\mathbf{Man}^\infty$ fiber bundle.
   - As noted above, $(E, M, \pi, \mathbb{R}^k) \in \text{Obj}(\mathbf{Bun}^\infty)$.
   - Let $p \in M$, Clearly E_p is a k-dimensional real vector space. By assumption, there exists $\alpha \in \Gamma$ such that
     (a) $p \in U_\alpha$.
     (b) As noted above, (U_α, Φ_α) is a smooth local trivialization of E over U with fiber $\mathbb{R}^k$ with respect to π.
     (c) Let $q \in U_\alpha$. By assumption, $\Phi|_{E_q} : E_q \to \{p\} \times \mathbb{R}^k$ is a vector space isomorphism.

   FINISH!!!

□

Definition 14.1.1.6. Let (E_1, M_1, π_1) and (E_2, M_2, π_2) be rank-k_1 and rank-k_2 smooth vector bundles respectively, $(\Phi, \phi) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$. Then (Φ, ϕ) is said to be a **smooth vector bundle morphism** from (E_1, M_1, π_1) to (E_2, M_2, π_2) if for each $p \in M_1$, $\Phi|_{(E_1)_p} : (E_1)_p \to (E_2)_{\phi(p)}$ is linear.

Definition 14.1.1.7. We define the category of smooth vector bundles, denoted $\mathbf{VecBun}^\infty$, by

- $\text{Obj}(\mathbf{VecBun}^\infty) := \{(E, M, \pi) : (E, M, \pi) \text{ is a smooth vector bundle}\}$
- For $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{VecBun}^\infty)$ with $\text{rank}(E_1, M_1, \pi_1) = k_1$ and $\text{rank}(E_2, M_2, \pi_2) = k_2$,

$$\begin{aligned} \text{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) := \{&(\Phi, \phi) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2})) : \\ &(\Phi, \phi) \text{ is a smooth vector bundle morphism from} \\ &(E_1, M_1, \pi_1) \text{ to } (E_2, M_2, \pi_2)\} \end{aligned}$$

Exercise 14.1.1.8. We have that $\mathbf{VecBun}^\infty$ is a subcategory of $\mathbf{Bun}^\infty$.

Proof. We note that

- $\text{Obj}(\mathbf{VecBun}^\infty) \subset \text{Obj}(\mathbf{Bun}^\infty)$

- for each $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ with $\mathrm{rank}(E_1, M_1, \pi_1) = k_1$ and $\mathrm{rank}(E_2, M_2, \pi_2) = k_2$,

$$\mathrm{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

FINISH!!!
So $\mathbf{Bun}^\infty$ is a subcategory of $\mathcal{C}$. □

Exercise 14.1.1.9. Let $M \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Set $n := \dim M$, $E := M \times \mathbb{R}^k$ and define $\pi : E \to M$ by $\pi(p, x) := p$. Then (E, M, π) is a rank-k smooth vector bundle.

Proof.

1. For each $p \in M$, $E_p = \{p\} \times \mathbb{R}^k$ is an n-dimensional real vector space.
2. Let $p \in M$. Set $U = M$. Then $\pi^{-1}(U) = E$. Define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ by $\Phi = \mathrm{id}_E$. Then (U, Φ) is a smooth local trivialization of E over U.
3. Let $p \in M$. Then $\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is clearly an isomorphism.

□

14.1.2 Subbundles

Definition 14.1.2.1. Let $(E, M, \pi_E), (D, M, \pi_D) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$. Then (D, M, π_D) is said to be a **subbundle of** (E, M, π_E) if

1. D is an embedded submanifold of E
2. $\pi_E|_D = \pi_D$
3. for each $p \in M$, D_p is a subspace of E_p.

Exercise 14.1.2.2. Local Frame Criterion:
FINISH!!!

14.1.3 Direct Sum Bundles

Definition 14.1.3.1. Let $(E_1, M, \pi_1), (E_2, M, \pi_2) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$. We define the **tensor product of** (E_1, M, π_1) **and** (E_2, M, π_2), denoted $(E_1 \otimes E_2, M, \pi)$, by

14.1.4 Tensor Product Bundles

Definition 14.1.4.1. Let $(E_1, M, \pi_1), (E_2, M, \pi_2) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$. Set

- $$E_1 \otimes E_2 := \coprod_{p \in M} (E_1)_p \otimes (E_2)_p$$
- $\pi : E_1 \otimes E_2 \to M$ by
$$\pi(p, v) = p$$

We define the **tensor product bundle of** (E_1, M, π_1) **and** (E_2, M, π_2), denoted $(E_1 \otimes E_2, M, \pi)$.

14.1.5 Hom Bundles

Definition 14.1.5.1. Let $(E_1, M, \pi_1), (E_2, M, \pi_2) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$. Set

- $$\mathrm{Hom}(E_1, E_2) := \coprod_{p \in M} L((E_1)_p, (E_2)_p)$$

- $\pi : E_1 \otimes E_2 \to M$ by
$$\pi(p, v) = p$$

We define the **Hom bundle of** (E_1, M, π_1) **and** (E_2, M, π_2), denoted $(\mathrm{Hom}(E_1, E_2), M, \pi)$, by $\mathrm{Hom}(E_1, E_2)$.

need to show the hom and tensor bundles are bundle isomorphic, then use that to define a covariant derivative from a connection

Chapter 15

The Tangent and Cotangent Bundle

15.1 The Tangent Bundle

Definition 15.1.0.1. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p \in M} T_pM$$

We denote the natrual projection map by $\pi : TM \to M$.

Definition 15.1.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \ldots, x^n)$. Define $\tilde{U} \subset TM$ and $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}^n$ by

- $\tilde{U} = \pi^{-1}(U)$
- $$\begin{aligned} \tilde{\phi}\bigg(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\bigg|_p\bigg) &= (\phi(p), v) \\ &= (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n) \end{aligned}$$
  ,

Exercise 15.1.0.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \ldots, x^n)$. Then $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}$ is a bijection.

15.2 The cotangent Bundle

Definition 15.2.0.1. We define the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p \in M} T_p^*M$$

15.3 The (r,s)-Tensor Bundle

Definition 15.3.0.1.

1. the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p \in M} T_p^*M$$

2. the (r,s)**-tensor bundle of** M, denoted T^r_sM, by

$$T^r_sM = \coprod_{p \in M} T^r_s(T_pM)$$

3. the k**-alternating tensor bundle of** M, denoted $\Lambda^k(M)$, by

$$\Lambda^kM = \coprod_{p \in M} \Lambda^k(T_pM)$$

15.4 Vector Fields

Definition 15.4.0.1. Let $X : M \to TM$. Then X is said to be a **vector field on** M if for each $p \in M$, $X_p \in T_pM$.
For $f \in \mathbb{C}^\infty(M)$, we define $Xf : M \to \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in \mathbb{C}^\infty(M)$, Xf is smooth.
We denote the set of smooth vector fields on M by $\Gamma^1(M)$.

Exercise 15.4.0.2.

15.5 (r, s)-Tensor Fields

Definition 15.5.0.1. Let $\alpha : M \to T^r_s M$. Then α is said to be an **(r, s)-tensor field on M** if for each $p \in M$, $\alpha_p \in T^r_s(T_p M)$.
For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \to \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and α is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth.
The set of smooth (r, s)-tensor fields on M is denoted $T^r_s(M)$.

Definition 15.5.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T^r_s(M)$. We define

- $f\alpha : M \to T^r_s M$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta : M \to T^r_s M$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 15.5.0.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T^r_s(M)$. Then

1. $f\alpha \in T^r_s(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
2. $\alpha + \beta \in T^r_s(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Proof. Clear. □

Exercise 15.5.0.4. The set $T^r_s(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Definition 15.5.0.5. Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. We define the **tensor product of α with β**, denoted $\alpha \otimes \beta : M \to T^{r_1+r_2}_{s_1+s_2} M$, by

$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

Exercise 15.5.0.6. Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma^{r_1+r_2}_{s_1+s_2}(M)$

Proof. Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since α_1 and α_2 are smooth by assumption. □

Definition 15.5.0.7. We define the **tensor product**, denoted $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

Exercise 15.5.0.8. The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is associative.

Proof. Clear. □

Exercise 15.5.0.9. The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is $C^\infty(M)$-bilinear.

Proof. Clear. □

Definition 15.5.0.10. Let $(N,\mathcal{B})$ be a smooth manifold, $F: M \to N$ a smooth map and $\alpha \in \Gamma^0_k(N)$. We define the **pullback of** α **by** F, denoted $F^*\alpha \in \Gamma^0_k(M)$, by

$$(F^*\alpha)_p(v_1,\dots,v_k) = \alpha_{F(p)}(DF_p(v_1),\dots,DF_p(v_k))$$

for $p \in M$ and $v_1,\dots,v_k \in T_pM$

Exercise 15.5.0.11. Let $(M,\mathcal{A})$, $(N,\mathcal{B})$ and $(L,\mathcal{C})$ be smooth manifolds, $F: M \to N$ and $G: N \to L$ smooth maps, $\alpha \in \Gamma^0_k(N)$, $\beta \in \Gamma^0_l(N)$, $\gamma \in \Gamma^0_k(L)$ and $f \in C^\infty(N)$. Then

1. $F^*(f\alpha) = (f \circ F)F^*\alpha$
2. $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
3. $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
4. $(G \circ F)^*\gamma = F^*(G^*\gamma)$
5. $id_N^*\alpha = \alpha$

Proof.

1.
$$\begin{aligned}[F^*(f\alpha)]_p(v_1,\dots,v_k) &= (f\alpha)_{F(p)}(DF_p(v_1),\dots,DF_p(v_k)) \\ &= f(F(p))\alpha_{F(p)}(DF_p(v_1),\dots,DF_p(v_k)) \\ &= (f\circ F)(p)(F^*\alpha)_p(v_1,\dots,v_k)\end{aligned}$$

   So that $F^*(f\alpha) = (f \circ F)F^*\alpha$

2.
$$F^*$$

□

Definition 15.5.0.12.

Exercise 15.5.0.13.

Proof. □

Exercise 15.5.0.14. Let $\alpha \in T^r_s(M)$ and $(U,\phi) \in \mathcal{A}$ with $\phi = (x^1,\cdots,x^n)$. Then there exist $(f^I_J)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I,J)\in\mathcal{I}_r\times\mathcal{I}_s} f^I_J \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

Proof. Let $p \in M$. Then $\omega_p \in T^r_s(T_pM)$ and $\left\{\partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}\right\}$ is a basis of $T^r_s(T_pM)$. So there exist $(f^I_J(p))_{I\in\mathcal{I}_r,J\in\mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I,J)\in\mathcal{I}_r\times\mathcal{I}_s} f^I_J(p)\partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let $(K,L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned}\alpha_p(dx^K_p, \partial_{x^L}|_p) &= \sum_{(I,J)\in\mathcal{I}_r\times\mathcal{I}_s} f^I_J(p)\partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}(dx^K_p, \partial_{x^L}|_p) \\ &= \sum_{(I,J)\in\mathcal{I}_r\times\mathcal{I}_s} f^I_J(p)\partial_{x^{\otimes I}}|_p(dx^K_p)dx_p^{\otimes J}(\partial_{x^L}|_p) \\ &= f^K_L(p)\end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx^K, \partial_{x^L})_p$ is smooth, so that $f^K_L \in C^\infty(U)$.

□

Definition 15.5.0.15.

15.6 Differential Forms

Definition 15.6.0.1. We define
$$\Lambda^k(TM) = \coprod_{p\in M} \Lambda^k(T_pM)$$

Definition 15.6.0.2. Let $\omega : M \to \Lambda^k(TM)$. Then ω is said to be a **k-form on** M if for each $p \in M$, $\omega_p \in \Lambda^k(T_pM)$.
For each $X \in \Gamma^1(M)^k$, we define $\omega(X) : M \to \mathbb{R}$ by
$$\omega(X)_p = \omega_p(X_p)$$
and ω is said to be **smooth** if for each $X \in \Gamma^1(M)^k$, $\omega(X)$ is smooth.
The set of smooth k-forms on M is denoted $\Omega^k(M)$.

Note 15.6.0.3. Observe that

1. $\Omega^k(M) \subset \Gamma^0_k(M)$
2. $\Omega^0(M) = C^\infty(M)$

Exercise 15.6.0.4. The set $\Omega^k(M)$ is a $C^\infty(M)$-submodule of $\Gamma^0_k(M)$.

Proof. Clear. □

Definition 15.6.0.5. Define the **exterior product**
$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$
by
$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Note 15.6.0.6. For $f \in \Omega^0(M)$ and $\alpha \in \Omega^k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 15.6.0.7. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is well defined.

Proof. Let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{i=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then
$$\begin{aligned}
\alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)(\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \cdots, X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p))
\end{aligned}$$

□

Exercise 15.6.0.8. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is $C^\infty(M)$-bilinear.

Proof.

1. $C^\infty(M)$-linearity in the first argument:
   Let $\alpha \in \Omega^k(M)$, $\beta, \gamma \in \Omega^l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda^k(T_pM) \times \Lambda^l(T_pM) \to \Lambda^{k+l}(T_pM)$ implies that

$$\begin{aligned}
[(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\
&= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\
&= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\
&= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p
\end{aligned}$$

   So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

   and $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is $C^\infty(M)$-linear in the first argument.

2. $C^\infty(M)$-linearity in the second argument:
   Similar to (1).

□

Note 15.6.0.9. All of the results from multilinear algebra apply here.

Definition 15.6.0.10. We define the **exterior derivative** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ inductively by

1. $d(d\alpha) = 0$ for $\alpha \in \Omega^p(M)$
2. $df(X) = Xf$ for $f \in \Omega^0(M)$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$
4. extending linearly

Exercise 15.6.0.11. Let (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then on U, for each $i, j \in \{1, \cdots, n\}$,

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx^1_p, \cdots, dx^n_p\}$ is the dual basis to $\left\{\frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p\right\}$ and $T_p^*M = \operatorname{span}\{dx^1_p, \cdots, dx^n_p\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned}
\left[dx^i\left(\frac{\partial}{\partial x^j}\right)\right]_p &= \left(\frac{\partial}{\partial x^j} x^i\right)_p \\
&= \frac{\partial}{\partial x^i}\bigg|_p x^i \\
&= \delta_{i,j}
\end{aligned}$$

□

Exercise 15.6.0.12. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Proof. Let $p \in U$. Since $\{dx^1, \cdots, dx^n\}$ is a basis for $\Lambda(T_pM)$, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum\limits_{i=1}^{n} a^i(p)dx^i_p$. Therefore, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) &= \sum_{i=1}^{n} a^i(p)dx^i_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) &= \frac{\partial}{\partial x^i}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p)dx^i_p$$

Therefore

$$df|_U = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}dx^i$$

□

Exercise 15.6.0.13. Let $f \in \Omega^0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \dots, x_n)$. Then for each $i \in \{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i}\bigg|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p)dx^i_p \\ &= 0 \end{aligned}$$

□

Exercise 15.6.0.14.

Definition 15.6.0.15. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$. We define

$$dx^i = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_k}}\right)$$

Note 15.6.0.16. We have that

1. $$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{I,J}$$

2. Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega^k(U)$,

$$\omega\left(\frac{\partial}{\partial x^i}\right) \in C^\infty(U)$$

Exercise 15.6.0.17. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega\left(\frac{\partial}{\partial x^i}\right) dx^i$$

Proof. Let $p \in U$. Since $\{dx^i_p : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(T_pM)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p) dx^i_p$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned}\omega\left(\frac{\partial}{\partial x^j}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j}\right) \\ &= f_J\end{aligned}$$

□

Exercise 15.6.0.18. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. If $\omega = \sum_{I \in \mathcal{I}_k} f_I dx^i$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

.

Proof. First we note that

$$\begin{aligned}d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f d(dx^i) \\ &= df_I \wedge dx^i \\ &= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i\right) \wedge dx^i \\ &= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i\end{aligned}$$

Then we extend linearly. □

Definition 15.6.0.19. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ be a diffeomorphism. Define the **pullback of** F, denoted $F^* : \Omega^k(N) \to \Omega^k(M)$ by

$$(F^*\omega)_p(v_1, \cdots, v_k) = \omega_{F(p)}(DF_p(v_1), \cdots, DF_p(v_k))$$

for $\omega \in \Omega^k(N)$, $p \in M$ and $v_1, \cdots, v_k \in T_pM$

15.7 Vector Bundle Valued Differential Forms

change notation in earlier sections so that $\Lambda^k(V^*)$ is k-forms instead of $\Lambda^k(V)$

Definition 15.7.0.1. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$. For each $k \in \mathbb{N}_0$, we define the **E-valued k-forms on** M, denoted $\Omega^k(M; E)$ by $\Omega^k(M; E) := \Gamma(\Lambda^k T^*M \otimes E)$.

Note 15.7.0.2. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ and $V \in \mathrm{Obj}(\mathbf{Vect}_{\mathbb{R}})$. Then we write $\Omega^k(M; V)$ in place of $\Omega^k(M; M \times V)$.

Chapter 16

The Tangent Bundle

16.1 The Tangent Bundle

Definition 16.1.0.1. Let $(M, \mathcal{A}_M)$ be an n-dimensional smooth manifold. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p \in M} T_pM$$

and we define the **tangent bundle projection**, denoted $\pi : TM \to M$, by

$$\pi(p, v) = p$$

Let $(U, \phi) \in \mathcal{A}_M$ with $\phi = (x^1, \dots, x^n)$. We define $\Phi_\phi : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\Phi_\phi\left(p, \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p\right) = (\phi(p), \xi^1, \dots, \xi^n)$$

We define $\mathcal{T}_{TM} = \tau_{TM}(\iota_p : p \in M)$.

Exercise 16.1.0.2. $\psi : \bigcup_{p \in U} T_pM \to \mathbb{R}^n$ is given by

$$\psi\left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p\right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}[x^k \circ \pi] &= \frac{\partial}{\partial v^i}\bigg|_{\Phi_\phi(p,\xi)}[x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \frac{\partial}{\partial v^i}\bigg|_{(\phi(p),\psi(\xi))}[x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \frac{\partial}{\partial v^i}\bigg|_{\phi(p)}[x^k \circ \phi^{-1}] \\
&= 0
\end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(\pi(p,\xi))\frac{\partial x^k \circ \pi}{\partial \tilde{x}^i}(p,\xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(p)\delta_{i,k} \\
&= \frac{\partial f}{\partial x^i}(p)
\end{aligned}$$

and

$$\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(\pi(p,\xi))\frac{\partial x^k \circ \pi}{\partial \tilde{y}^i}(p,\xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(p)0 \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \ker D\pi(p,\xi) \\
&= \coprod_{(p,\xi)\in\pi^{-1}(U)} \operatorname{span}\left\{\frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1,\dots,n\}\right\}
\end{aligned}$$

Definition 16.1.0.3. Let $M, N \in \text{Obj}(\mathbf{Man}^\infty)$ and $F \in \text{Hom}_{\mathbf{Man}^\infty}(M, N)$. We define the **push-forward of** F, denoted $F_* : TM \to TN$, by $F_*(p, v) = (F(p), DF(p)(v))$.

Exercise 16.1.0.4. Let $M, N \in \text{Obj}(\mathbf{Man}^\infty)$ and $F \in \text{Hom}_{\mathbf{Man}^\infty}(M, N)$. Then $F_* \in \text{Hom}_{\mathbf{Man}^\infty}(TM, TN)$.

Proof. □

Definition 16.1.0.5. Let $M, N \in \text{Obj}(\mathbf{Man}^\infty)$ and $F \in \text{Hom}_{\mathbf{Man}^\infty}(M, N)$. We define the **tangent functor**, denoted $T : \mathbf{Man}^\infty \to \mathbf{Man}^\infty$, by

- $T(M) = TM$
- $TF = F_*$

Exercise 16.1.0.6. Let $M, N \in \text{Obj}(\mathbf{Man}^\infty)$ and $F \in \text{Hom}_{\mathbf{Man}^\infty}(M, N)$. Then $T : \mathbf{Man}^\infty \to \mathbf{Man}^\infty$ is a functor.

Proof. content... □

16.2 Vector Fields

Exercise 16.2.0.1.

Chapter 17

Lie Algebras

17.1 Introduction

Definition 17.1.0.1. Let $\mathfrak{g}$ be a vector space and $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $[\cdot,\cdot]$ is said to be a **Lie bracket** on $\mathfrak{g}$ if

1. $[\cdot,\cdot]$ is bilinear
2. $[\cdot,\cdot]$ is antisymmetric
3. $[\cdot,\cdot]$ satisfies the Jacobi identity:
   for each $x, y, z \in \mathfrak{g}$,
   $$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

In this case, $(\mathfrak{g}, [\cdot,\cdot])$ is said to be a **Lie algebra**.

Definition 17.1.0.2. Let $G \in \mathrm{Obj}(\mathbf{LieGrp})$ and $X \in \mathfrak{X}(G)$. Then X is said to be **left G-invariant** if for

Exercise 17.1.0.3. Let $G \in \mathrm{Obj}(\mathbf{LieGrp})$ and $X \in \mathfrak{X}(G)$. Then

Chapter 18

Principle Bundles

18.1 Introduction

define $\triangleleft$-invariance and $(\triangleleft_1, \triangleleft_2)$-equivariance

Definition 18.1.0.1. Let X be a set and G a group. We define the **trivial right action on** $X \times G$, denoted $\triangleleft^{\text{Triv}}_{X\times G} : (X \times G) \times G \to X \times G$, by

$$(x, g) \triangleleft^{\text{Triv}}_{X\times G} h = (x, gh)$$

Exercise 18.1.0.2. Let $(P, X, \pi, G) \in \text{Obj}(\mathbf{Bun}^\infty)$ and $\triangleleft \in \text{Hom}_{\mathbf{Man}^\infty}(P \times G, P)$. Suppose that $\triangleleft$ is a right group action. Then π is $\triangleleft$-invariant iff for each $x \in X$, $P_x \triangleleft G = P_x$.

Proof.

- $(\implies)$:
  Suppose that π is $\triangleleft$-invariant. Let $x \in X$, $p \in P_x$ and $g \in G$. Then

$$\begin{aligned}\pi(p \triangleleft g) &= \pi(p) \\ &= x.\end{aligned}$$

  Hence $p \triangleleft g \in P_x$. Since $p \in P_x$ and $g \in G$ are arbitrary, we have that $P_x \triangleleft G \subset P_x$. Let $p \in P_x$. Then

$$\begin{aligned}p &= p \triangleleft e \\ &\in P_x \triangleleft G.\end{aligned}$$

  Since $p \in P_x$ is arbitrary, we have that $P_x \subset P_x \triangleleft G$. Hence $P_x \triangleleft G = P_x$. Since $x \in X$ is arbitrary, we have that for each $x \in X$, $P_x \triangleleft G = P_x$.

- $(\impliedby)$:
  Conversely, suppose that for each $x \in X$, $P_x \triangleleft G = P_x$. Let $p \in P$ and $g \in G$. Set $x := \pi(p)$. Since $p \in P_x$, by assumption, we have that

$$\begin{aligned}p \triangleleft g &\in P_x \triangleleft G \\ &= P_x.\end{aligned}$$

  Therefore

$$\begin{aligned}\pi(p \triangleleft g) &= x \\ &= \pi(p).\end{aligned}$$

  Since $p \in P$ and $g \in G$ are arbitrary, we have that for each $p \in P$ and $g \in G$, $\pi(p \triangleleft g) = \pi(p)$. Hence π is $\triangleleft$-invariant.

□

Definition 18.1.0.3. Let $(P, X, \pi, G) \in \text{Obj}(\mathbf{Bun}^\infty)$ and $\triangleleft \in \text{Hom}_{\mathbf{Man}^\infty}(P \times G, P)$. Suppose that

- G is a Lie group
- $\triangleleft$ a right group action
- π is $\triangleleft$-invariant.

For each $x \in X$, we define the **right action of G on P_x induced by** $\triangleleft$, denoted $\triangleleft_x$, by $\triangleleft_x := \triangleleft|_{P_x \times G}$.

Exercise 18.1.0.4. Let Let $(P, X, \pi, G) \in \text{Obj}(\mathbf{Bun}^\infty)$ and $\triangleleft \in \text{Hom}_{\mathbf{Man}^\infty}(P \times G, P)$. Suppose that

- G is a Lie group
- $\triangleleft$ a right group action
- π is $\triangleleft$-invariant.

Then for each $x \in X$, $\triangleleft_x : P_x \times G \to P_x$ is a smooth group action.

Proof. Let $x \in X$, $g, h \in G$ and $p \in P_x$.

- Then
$$\begin{aligned} p \triangleleft_x (gh) &= p \triangleleft (gh) \\ &= (p \triangleleft g) \triangleleft h \\ &= (p \triangleleft_x g) \triangleleft_x h \end{aligned}$$
and
$$\begin{aligned} p \triangleleft_x e &= p \triangleleft e \\ &= p. \end{aligned}$$
Since $g, h \in G$ and $p \in P_x$ is arbitrary, we have that $\triangleleft_x$ is a group action.
- Since π is a surjective submersion,
FINISH!!!, need previous exercise showing P_x is a smooth embedded submanifold of P in a fiber bundle and therefore the restriction of a smooth map to a smooth embedded submanifold is smooth.

□

Definition 18.1.0.5. Let $P, X, G \in \text{Obj}(\mathbf{Man}^\infty)$, $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(P, X)$ a surjection, $\triangleleft \in \text{Hom}_{\mathbf{Man}^\infty}(P \times G, P)$, $U \in \mathcal{T}_X$ and $\Phi \in \text{Hom}_{\mathbf{Man}^\infty}(\pi^{-1}(U), U \times G)$. Suppose that

- G is a Lie Group,
- $\triangleleft$ is a right group action,
- π is $\triangleleft$-invariant.

Then (U, Φ) is said to be a **smooth principle bundle local trivialization of P over U with respect to π and** $\triangleleft$ if

1. (U, Φ) is a smooth fiber bundle local trivialization of P over U with fiber G with respect to π
2. Φ is $(\triangleleft, \triangleleft^{\text{Triv}}_{U \times G})$-equivariant

Definition 18.1.0.6. Let $P, X, G \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(P, X)$ a surjection and $\triangleleft \in \text{Hom}_{\mathbf{Man}^\infty}(P \times G, P)$. Suppose that

- G is a Lie Group,
- $\triangleleft$ is a right group action.

Then $(P, X, \pi, G, \triangleleft)$ is said to be a **$\mathbf{Man}^\infty$ principle bundle with total space P, base space X, structure group G, projection π and action $\triangleleft$** if

1. $(P, X, \pi, G) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$,
2. π is $\triangleleft$-invariant,
3. for each $x \in X$,
   (a) $\triangleleft_x : P_x \times G \to P_x$ is transitive and free,
   (b) there exists $U \in \mathcal{T}_X$ and $\Phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(\pi^{-1}(U), U \times G)$ such that (U, Φ) is a smooth principle bundle local trivialization of P over U with respect to π and $\triangleleft$.

Exercise 18.1.0.7. Exercise 13.1.3.10
FINISH!!!

Definition 18.1.0.8. We define the category of $\mathbf{Man}^\infty$-principle bundles, denoted $\mathbf{PrinBun}^\infty$, by

- $\mathrm{Obj}(\mathbf{PrinBun}^\infty) := \{(P, X, \pi, G, \triangleleft) : (P, X, \pi, G) \text{ is a } \mathbf{Man}^\infty\text{-principal bundle}\}$
- For $(P_1, X_1, \pi_1, G_1, \triangleleft_1), (P_2, X_2, \pi_2, G_2, \triangleleft_2) \in \mathrm{Obj}(\mathbf{PrinBun}^\infty)$,

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Bun}^\infty}((P_1, X_1, \pi_1, G_1, \triangleleft_1), (P_2, X_2, \pi_2, G_2, \triangleleft_2)) :=\\ &\{(\Phi, \phi) : (\Phi, \phi) \text{ is a smooth bundle morphism from } (E_1, M_1, \pi_1, F_1) \text{ to } (E_2, M_2, \pi_2, F_2)\}\end{aligned}$$

- For
  - $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$
  - $(\Phi_{12}, \phi_{12}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$
  - $(\Phi_{23}, \phi_{23}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3))$

  we define $(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_3, M_3, \pi_3))$ by

$$(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) := (\Phi_{23} \circ \Phi_{12}, \phi_{23} \circ \phi_{12})$$

FINISH!!!

Chapter 19

de Rham Cohomology

19.1 TO DO

1. de Rham cohomology
2. de Rham homology
3. in de Rham homology, measures on the manifold can be identified with the 0th Homology, group
4. think about how the other homology groups can be used in statistics

19.2 Introduction

Note 19.2.0.1. We recall that $d : \Omega^*(M) \to \Omega^*(M)$ satisfies the properties:

1. $d^2 = 0$
2.
3.

Definition 19.2.0.2. Let M be an n-dimensional smooth manifold. For $k \in \{1, \ldots, n\}$, we define the

- **k-th coboundary operator**, denoted $d^k : \Omega^k(M) \to \Omega^{k+1}(M)$, by $d^k = d|_{\Omega^k(M)}$
-
-

Chapter 20

Jet Bundles

20.1 Fibered Manifolds

Definition 20.1.0.1. Let $E, M \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$. Then (E, M, π) is said to be a **fibered manifold** if π is a surjective submersion.

Definition 20.1.0.2. Let $E, F, M, N \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$, $\tau \in \mathrm{Hom}_{\mathbf{Man}^\infty}(F, N)$, $\Phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, F)$ and $\phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M, N)$. Suppose that (E, M, π) and (F, N, τ) are fibered manifolds. Then (Φ, ϕ) is said to be a **fibered manifold morphism** if $\tau \circ \Phi = \phi \circ \pi$, i.e. the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\pi"'] & F \arrow[d, "\tau"] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

Note 20.1.0.3. We write $\mathrm{proj}_1^n : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ to denote the projection onto M.

- Define fibered manifold morphism and category
- Define set of atlas charts which are fibered
- define jet bundles

Definition 20.1.0.4. Let (E, M, π) be a fibered manifold and $(V, \psi) \in \mathcal{A}_E$. Set $n := \dim M$ and $k := \dim E - n$. Then (V, ψ) is said to be a **π-fibered chart on** E if there exists $(U, \phi) \in \mathcal{A}_M$ such that

1. $U = \pi(V)$
2. $\phi \circ \pi|_V = \pi^{n+k}_{[n]} \circ \psi$,
   i.e. if $\psi = (y^1, \dots, y^{n+k})$ and $\phi = (x^1, \dots, x^n)$, then $\psi = (x^1 \circ \pi|_V, \dots, x^n \circ \pi|_V, y^{n+1}, \dots, y^{n+k})$.

We define $\mathbb{F}(\pi) := \{(U, \phi) \in \mathcal{A}_E : (U, \phi) \text{ is } \pi\text{-fibered}\}$.

Exercise 20.1.0.5. Let (E, M, π) be a smooth fibered manifold. Suppose that $\partial E, \partial M = \varnothing$. Then for each $a \in E$, there exists $(V, \psi) \in \mathbb{F}(\pi)$ such that $a \in V$.
Hint: local rank theorem reference ex from submersions section

Proof. Set $n := \dim M$, $k := \dim E - n$. Let $a \in E$. Set $p := \pi(a)$. Since $\pi : E \to M$ is a submersion, π has constant rank and $\operatorname{rank} \pi = n$. Exercise 7.1.0.3 implies that there exist $(V, \psi) \in \mathcal{A}_E$, $(U_0, \phi_0) \in \mathcal{A}_M$ such that $a \in V$, $p \in U_0$, $\pi(V) \subset U_0$ and $\phi_0 \circ \pi \circ \psi^{-1} = \mathrm{proj}^{n+k}_n|_{\psi(V)}$. Hence $\phi_0 \circ \pi = \mathrm{proj}^{n+k}_n \circ \psi$. Define $U := \pi(V)$ and $\phi := \phi_0|_U$. An exercise in the section on submersions implies that π is open. Hence $U \in \mathcal{T}_M$ and $(U, \phi) \in \mathcal{A}_M$. By construction,

1. $U = \pi(V)$

2. $\phi \circ \pi|_V = \text{proj}_n^{n+k} \circ \psi$

Hence (V, ψ) is a π-fibered chart on E. $\square$

Exercise 20.1.0.6. Let (E, M, π) be a smooth fibered manifold and $a \in E$ and $(U_0, \phi_0) \in \mathbb{F}(\pi)$. Set $n := \dim M$ and $k := \dim E - n$. Since $(U, \phi) \in \mathbb{F}(\pi)$, there exists $(U, \phi) \in \mathcal{A}_M$ such that $\pi(U_0) = U$ and $\phi \circ \pi = \pi_{[n]}^{n+k} \circ \phi_0$. Suppose that $\partial E, \partial M = \varnothing$ and $a \in U_0$. Write $\phi_0 = (x^1, \dots, x^n, \upsilon^1, \dots, \upsilon^k)$ and $\phi = (\tilde{x}^1, \dots, \tilde{x}^1)$. Then for each $j \in [n]$ and $l \in [k]$,

$$D\pi(a)\bigg(\frac{\partial}{\partial x^j}\bigg|_a\bigg) = \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\pi(a)}, \qquad D\pi(a)\bigg(\frac{\partial}{\partial \upsilon^l}\bigg|_a\bigg) = 0.$$

Proof. Let $j \in [n]$, $l \in [k]$ and $f \in C^\infty(M)$. Set $p := \pi(a)$. Then

$$\begin{aligned}
D\pi(a)\bigg(\frac{\partial}{\partial x^j}\bigg|_a\bigg)(f) &= \frac{\partial}{\partial x^j}\bigg|_a (f \circ \pi) \\
&= \frac{\partial}{\partial x^j}\bigg|_a (f \circ \phi^{-1} \circ \phi \circ \pi) \\
&= \frac{\partial}{\partial x^j}\bigg|_a (f \circ \phi^{-1} \circ \pi_{[n]}^{n+k} \circ \phi_0) \\
&= \frac{\partial}{\partial u^j}\bigg|_{\phi_0(a)} (f \circ \phi^{-1} \circ \pi_{[n]}^{n+k} \circ \phi_0 \circ \phi_0^{-1}) \\
&= \frac{\partial}{\partial u^j}\bigg|_{\phi_0(a)} (f \circ \phi^{-1} \circ \pi_{[n]}^{n+k}) \\
&= \sum_{l=1}^n \frac{\partial(f \circ \phi^{-1})}{\partial u^l}(\pi_{[n]}^{n+k}(\phi_0(a)))\frac{\partial(\pi_l^n \circ \pi_{[n]}^{n+k})}{\partial u^j}(\phi_0(a)) \\
&= \sum_{l=1}^n \frac{\partial(f \circ \phi^{-1})}{\partial u^l}(\phi \circ \pi(a))\frac{\partial(\pi_l^{n+k})}{\partial u^j}(\phi_0(a)) \\
&= \sum_{l=1}^n \frac{\partial(f \circ \phi^{-1})}{\partial u^l}(\phi(p))\delta_{l,j} \\
&= \frac{\partial(f \circ \phi^{-1})}{\partial u^j}(\phi(p)) \\
&= \frac{\partial}{\partial \tilde{x}^j}\bigg|_p f
\end{aligned}$$

and similarly,

$$\begin{aligned}
D\pi(a)\bigg(\frac{\partial}{\partial \upsilon^l}\bigg|_a\bigg)(f) &= \frac{\partial}{\partial \upsilon^l}\bigg|_a (f \circ \pi) \\
&= \frac{\partial}{\partial u^{n+l}}\bigg|_{\phi_0(a)} (f \circ \phi^{-1} \circ \pi_{[n]}^{n+k}) \\
&= \sum_{j=1}^n \frac{\partial(f \circ \phi^{-1})}{\partial u^j}(\phi \circ \pi(a))\frac{\partial(\pi_j^{n+k})}{\partial u^{n+l}}(\phi_0(a)) \\
&= 0
\end{aligned}$$

Since $f \in C^\infty(M)$ is arbitrary, we have that

$$D\pi(a)\bigg(\frac{\partial}{\partial x^j}\bigg|_a\bigg) = \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\pi(a)}, \qquad D\pi(a)\bigg(\frac{\partial}{\partial v^l}\bigg|_a\bigg) = 0.$$

FINISH!!! (math scribbles) □

Exercise 20.1.0.7. Let $s_1, s_2 \in \Gamma_p(\pi)$. Write $\phi_0 = (x^1, \dots, x^n, v^1, \dots, v^k)$ and $\psi_0 = (y^1, \dots, y^n, \omega^1, \dots, \omega^k)$, $\phi = (\tilde{x}^1, \dots, \tilde{x}^n)$ and $\psi = (\tilde{y}^1, \dots, \tilde{y}^n)$. Then for each $j \in [n]$ and $l \in [k]$,

$$\frac{\partial}{\partial \tilde{x}^j}\bigg|_{\pi(a)}(v^l \circ s_1) = \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\pi(a)}(v^l \circ s_2)$$

iff for each $j' \in [n]$ and $l' \in [k]$,

$$\frac{\partial}{\partial \tilde{y}^{j'}}\bigg|_{\pi(a)}(\omega^{l'} \circ s_1) = \frac{\partial}{\partial \tilde{y}^{j'}}\bigg|_{\pi(a)}(\omega^{l'} \circ s_2).$$

Proof. Set $p := \pi(a)$.

- $(\implies:)$
  Suppose that for each $j \in [n]$ and $l \in [k]$,

  $$\frac{\partial}{\partial \tilde{x}^j}\bigg|_p(v^l \circ s_1) = \frac{\partial}{\partial \tilde{x}^j}\bigg|_p(v^l \circ s_2).$$

  Let $j' \in [j]$ and $l' \in [k]$. Then

  $$\begin{aligned}\frac{\partial}{\partial \tilde{y}^{j'}}\bigg|_p(\omega^{l'} \circ s_1) &= \sum_{m=1}^n \frac{\partial \tilde{x}^m}{\partial \tilde{y}^{j'}}(a)\frac{\partial}{\partial \tilde{x}^m}\bigg|_p(\omega^{l'} \circ s_1) \\ &= \sum_{m=1}^n \frac{\partial \tilde{x}^m}{\partial \tilde{y}^{j'}}(a)\bigg[\sum_{j=1}^n \frac{\partial \omega^{l'}}{\partial x^j}(s_1(p))\frac{\partial}{\partial \tilde{x}^m}\bigg|_p(x^j \circ s_1) + \sum_{l=1}^k \frac{\partial \omega^{l'}}{\partial v^l}(s_1(p))\frac{\partial}{\partial \tilde{x}^m}\bigg|_p(v^l \circ s_1)\bigg] \\ &= \sum_{m=1}^n \frac{\partial \tilde{x}^m}{\partial \tilde{y}^{j'}}(a)\bigg[\sum_{j=1}^n \frac{\partial \omega^{l'}}{\partial x^j}(s_1(p))\frac{\partial}{\partial \tilde{x}^m}\bigg|_p(x^j \circ s_1) + \sum_{l=1}^k \frac{\partial \omega^{l'}}{\partial v^l}(s_1(p))\frac{\partial}{\partial \tilde{x}^m}\bigg|_p(v^l \circ s_2)\bigg]\end{aligned}$$

  FINISH!!!, need to get rid of fibered charts, contact order is defined more generally, should move this exercise to the smooth maps section

- $(\impliedby:)$

□

Exercise 20.1.0.8. Let $(E, M, \pi, F) \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Then (E, M, π) is a smooth fibered manifold.

Proof. Since $(E, M, \pi, F) \in \mathrm{Obj}(\mathbf{Man}^\infty)$, π is surjective. An exercise in the section on smooth fiber bundles implies that π is a submersion. Since π is a surjective submersion, (E, M, π) is a smooth fibered manifold. □

need to go over multi index notation for partial derivatives

Definition 20.1.0.9. Let (E, M, π) be a smooth fibered manifold.

Exercise 20.1.0.10.

Chapter 21

Connections

21.1 Ehresmann Connections

Definition 21.1.0.1. Let $(P, X, \pi, G, \triangleleft) \in \mathrm{Obj}(\mathbf{PrinBun}^\infty)$ and $p \in P$. Set $x := \pi(p)$. We define the **verticle tangent space of P at p**, denoted V_p, by $V_p := T_p(P_x)$.

Exercise 21.1.0.2. Let $(P, X, \pi, G, \triangleleft) \in \mathrm{Obj}(\mathbf{PrinBun}^\infty)$. For each $p \in P$, $V_p = \ker D\pi(p)$.

Proof. Let $p \in P$. Set $x := \pi(p)$. ref ex about tangent space of subamnifold being the kernel of derivative $\square$

21.2 Koszul Connections

Definition 21.2.0.1.

- Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ and $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$. Then ∇ is said to be a **Koszul connection on** E if for each $f \in C^\infty(M)$ and $s \in \Gamma(E)$, $\nabla(fs) = df \otimes s + f \nabla s$.
- We define $\mathrm{Con}_{\mathrm{Kos}}(E) := \{\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) : \nabla \text{ is a Koszul connection}\}$.

Exercise 21.2.0.2. content...

Definition 21.2.0.3. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ and $\nabla \in \mathrm{Con}_{\mathrm{Kos}}$. We define the **covariant derivative induced by** ∇, denoted $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, by $\nabla(X, s) := \nabla(s)$

Definition 21.2.0.4. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$, $\nabla_1 : \Gamma(E) \to \Gamma(T^*M \otimes E)$ and $\nabla_2 : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then

- ∇_1 is said to be a **type-1 Koszul connection on** E if for each $f \in C^\infty(M)$ and $s \in \Gamma(E)$, $\nabla_1(fs) = df \otimes s + f \nabla_1 s$.
- ∇_2 is said to be a **type-2 Koszul connection on** E if
  1. for each $s \in \Gamma(E)$, $\nabla(\cdot, s)$ is $C^\infty(M)$-linear
  2. for each $X \in \mathfrak{X}(M)$, $\nabla(X, \cdot)$ is $\mathbb{R}$-linear
  3. for each $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$,
  $$\nabla(X, fs) = f \nabla(X, s) + X(f)s$$
- We define
  - $\mathrm{Con}_1(E) := \{\nabla_1 : \Gamma(E) \to \Gamma(T^*M \otimes E) : \nabla \text{ is a type-1 Koszul connection}\}$
  - $\mathrm{Con}_2(E) := \{\nabla_2 : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) : \nabla \text{ is a type-2 Koszul connection}\}$

  .

Exercise 21.2.0.5. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$. There exists $\phi : \mathrm{Con}_1 \to \mathrm{Con}_2$ such that ϕ is a bijection.

Proof.

- Let $\nabla_1 \in \mathrm{Con}_1$, $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$. Set $\nabla_2(X, s) := \nabla_1(s)(X)$.

□

Exercise 21.2.0.6. We define $\mathrm{Con}_1(E) := \{\nabla_1 : \Gamma(E) \to \Gamma(T^*M \otimes E) : \nabla \text{ is a Koszul connection}\}$.

Proof. content... □

Note 21.2.0.7. We identify type-1 and type-2 Koszul connections.

Definition 21.2.0.8. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ be a smooth vector bundle and $\nabla : \Gamma(E) \to T^*M \otimes \Gamma(E)$. Then ∇ is said to be a **Koszul connection on** E **in the second representation** if

1. ∇ is $\mathbb{R}$-linear
2. for each $s \in \Gamma(E)$ and $f \in C^\infty(M)$,
$$\nabla(fs) = f \nabla s + df \otimes s$$

Exercise 21.2.0.9. There exists a bijection $\phi : \mathrm{Con}_1 \to \mathrm{Con}_2$.

Proof. Let $\nabla \in \mathrm{Con}_1$. We define $\phi(\nabla) : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ by
$$\phi(\nabla)(X, s) = (\nabla s)(X)$$

FINISH!!! □

Note 21.2.0.10. When the context is clear, we will write $\nabla_X Y$ in place of $\nabla(X, Y)$ and we will refer to ∇ as a connection.

Exercise 21.2.0.11. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$, ∇ a connection on E, $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$. If $X = 0$ or $Y = 0$, then $\nabla_X Y = 0$.

Proof.

- If $X = 0$, then

$$\begin{aligned}\nabla_X Y &= \nabla_{0X} Y \\ &= 0 \nabla_X Y \\ &= 0\end{aligned}$$

- Similarly, if $Y = 0$, then $\nabla_X Y = 0$.

□

Exercise 21.2.0.12. Let (E, M, π) be a smooth vector bundle, ∇ a connection on E, $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$ and $p \in M$. If $X \sim_p 0$ or $Y \sim_p 0$, then $[\nabla_X Y]_p = 0$.

Proof.

- Suppose that $X \sim_p 0$. Then there exists $U \subset M$ such that U is open and $X|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\operatorname{supp} \phi \subset U$ and $\phi \sim_p 1$. Then $\phi X = 0$. The previous exercise implies that $\nabla_{\phi X} Y = 0$. Therefore

$$\begin{aligned}\nabla_X Y &= \nabla_{\phi X + (1-\phi) X} Y \\ &= \nabla_{\phi X} Y + \nabla_{(1-\phi) X} Y \\ &= 0 + (1 - \phi) \nabla_X Y \\ &= (1 - \phi) \nabla_X Y\end{aligned}$$

  Hence

$$\begin{aligned}[\nabla_X Y]_p &= [(1 - \phi) \nabla_X Y]_p \\ &= (1 - \phi(p))[\nabla_X Y]_p \\ &= 0\end{aligned}$$

- Suppose that $Y \sim_p 0$. Then there exists $U \subset M$ such that U is open and $Y|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\operatorname{supp} \phi \subset U$ and $\phi \sim_p = 1$. Then $\phi Y = 0$. The previous exercise implies that $\nabla_X \phi Y = 0$. Since $\phi \sim_p 1$, we have that $1 - \phi \sim_p 0$. Thus $X(1 - \phi) \sim_p 0$ and

$$\begin{aligned}\nabla_X Y &= \nabla_X[\phi Y + (1 - \phi)Y] \\ &= \nabla_X[\phi Y] + \nabla_X[(1 - \phi)Y] \\ &= \nabla_X[(1 - \phi)Y] \\ &= (1 - \phi) \nabla_X Y + [X(1 - \phi)] \nabla_X Y\end{aligned}$$

  Hence

$$\begin{aligned}[\nabla_X Y]_p &= (1 - \phi(p))[\nabla_X Y]_p + [X(1 - \phi)](p)[\nabla_X Y]_p \\ &= 0\end{aligned}$$

□

Exercise 21.2.0.13. Let (E, M, π) be a smooth vector bundle and ∇ a connection on E. Then for each $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$, $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$ implies that $[\nabla_{X_1} Y_1]_p = [\nabla_{X_2} Y_2]_p$.

Proof. Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$. Suppose that $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$. Define $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$ by $X = X_2 - X_1$ and $Y = Y_2 - Y_1$. Then $X \sim_p 0$ and $Y \sim_p 0$. The previous exercise implies that $[\nabla_X Y_1]_p = 0$ and $[\nabla_{X_2} Y]_p = 0$. Therefore

$$\begin{aligned}
[\nabla_{X_1} Y_1]_p &= [\nabla_{X_1} Y_1]_p + [\nabla_X Y_1]_p \\
&= [\nabla_{X_1} Y_1 + \nabla_X Y_1]_p \\
&= [\nabla_{X_1 + X} Y_1]_p \\
&= [\nabla_{X_2} Y_1]_p \\
&= [\nabla_{X_2} Y_1]_p + [\nabla_{X_2} Y]_p \\
&= [\nabla_{X_2} Y_1 + \nabla_{X_2} Y]_p \\
&= [\nabla_{X_2} (Y_1 + Y)]_p \\
&= [\nabla_{X_2} Y_2]_p
\end{aligned}$$

□

Exercise 21.2.0.14. Let (E, M, π) be a smooth vector bundle, ∇ a connection on E and $U \subset M$. If U is open, then there exists a unique connection $\nabla^U : \mathfrak{X}(U) \times \Gamma(E|_U) \to \Gamma(E|_U)$ such that for each $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$,

$$\nabla^U_{X|_U} Y|_U = (\nabla_X Y)|_U$$

Chapter 22

Semi-Riemannian Geometry

22.1 Metric Tensors

Definition 22.1.0.1. Let M be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then g is said to be nondegenerate if for each $p \in M$, g_p is nondegenerate.

Definition 22.1.0.2. Let M be a manifold and $g \in \Gamma(\Sigma^2 M)$.

- Then g is said to be a **metric tensor field** on M if
  1. g is nondegenerate,
  2. g has constant index.
- If g is a metric tensor field on M, then (M, g) is said to be a **semi-Riemannian manifold**.

Definition 22.1.0.3.

22.2 Curvature

Definition 22.2.0.1. Define Interval
FINISH!!!

Definition 22.2.0.2. Let $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ and $\gamma \in \text{Hom}_{\mathbf{Man}^\infty}(I, E)$. Then γ is said to be a **section of E over α** if $\pi \circ \gamma = \alpha$. We denote the set of sections of E over α by $\Gamma(E, \alpha)$.

Definition 22.2.0.3. Let $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ and $\gamma \in \Gamma(E, \alpha)$. Then γ is said to be said to be **extendible** if there exists $U \in \mathcal{N}_{\alpha(I)}$ and $\tilde{\gamma} \in \Gamma(E|_U)$ such that U is open and $\tilde{\gamma} \circ \alpha = \gamma$.

Exercise 22.2.0.4. figure 8 not extendible FINISH!!!

Exercise 22.2.0.5. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$, ∇ a connection on E, $I \subset \mathbb{R}$ an interval and $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$. There exists a unique $D_\alpha : \Gamma(E, \alpha) \to \Gamma(E, \alpha)$ such that

1. for each $\lambda \in \mathbb{R}$ and $\gamma, \sigma \in \Gamma(E, \alpha)$,
$$D_\alpha(\gamma + \lambda\sigma) = D_\alpha\gamma + \lambda D_\alpha\sigma$$
2. for each $f \in C^\infty(I)$ and $\gamma \in \Gamma(E, \alpha)$,
$$D_\alpha(f\gamma) = f'\gamma + f D_\alpha\gamma$$
3. for each $\gamma \in \Gamma(E)$, if $\tilde{\gamma}$ extends γ, then
$$D_\alpha\gamma = \nabla_{\alpha'}\gamma$$

Proof. □

Chapter 23

Riemannian Geometry

Definition 23.0.0.1. Let M be a smooth manifold and $g \in T^0_2(M)$ a metric tensor on M. We define $\hat{g} \in T^2_0(M)$ by $\hat{g}(\omega, \eta) = g(\phi_g^{-1}(\omega), \phi_g^{-1}(\eta))$.

Exercise 23.0.0.2. content...

Exercise 23.0.0.3. Let (M, g) be a semi-Riemannian manifold and $(U, \phi) \in \mathcal{A}$. Then the induced metric $\langle\rangle_{T^*M \otimes TM}$ on $T^*M \otimes TM$ is given by

$$\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} = g^{i,j} g_{kl}$$

Proof. We have that

$$\begin{aligned}\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} &= \langle dx^i, dx^j \rangle_{T^*M} \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle_{TM} \\ &= g^{i,j} g_{k,l}\end{aligned}$$

□

Exercise 23.0.0.4. Let (M, g) be an n-dimensional Riemannian manifold.

1. There exists $\lambda \in \Omega^n(M)$ such that for each orthonormal frame $e_1, \ldots, e_n$,

$$\lambda(e_1, \ldots, e_n) = 1$$

   Hint: Choose a frame $z_1, \ldots, z_n$ on M with corresponding dual frame $\zeta^1, \ldots, \zeta^n$. Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n$$

2. Let $N \in \mathfrak{X}(M)$ be the outward pointing normal to ∂M and $X \in \mathfrak{X}(M)$. Then

$$\int_M \operatorname{div} X \lambda = \int_{\partial M} g(X, N) \tilde{\lambda}$$

3. For each $u \in \mathbb{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, we have that

$$\operatorname{div}(uX) = u \operatorname{div}(X) + du(X)$$

   and therefore

$$\int_M du(X) \lambda = \int_{\partial M} u g(X, N) \tilde{\lambda} - \int_M u \operatorname{div}(X) \lambda$$

Proof.

1. Let $z_1, \dots, z_n$ be a frame on M and $\zeta^1, \dots, \zeta^n$ with corresponding dual frame $\zeta^1, \dots, \zeta^n$. Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n$$

Let $e_1, \dots, e_n$, be an orthonormal frame on M with corresponding dual coframe $\epsilon^1, \dots, \epsilon^n$. Let $i, j \in \{1, \dots, n\}$. Then there exist $(a_{k,i}) \subset \mathbb{R}$ such that $\zeta^i = \sum_{k=1}^n a_{k,i}\epsilon^k$. Then

$$\begin{aligned}
\hat{g}(\epsilon^j, \zeta^i) &= \sum_{k=1}^n a_{k,i}\hat{g}(\epsilon^j, \epsilon^k) \\
&= \sum_{k=1}^n a_{k,i} g(\phi_g^{-1}(\epsilon^j), \phi_g^{-1}(\epsilon^k)) \\
&= \sum_{k=1}^n a_{k,i} g(e_j, e_k) \\
&= \sum_{k=1}^n a_{k,i}\delta_{j,k} \\
&= a_{j,i}
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta_{i,j} &= \zeta^i(z_j) \\
&= \sum_{k=1}^n a_{k,i}\epsilon^k(z_j) \\
&= \sum_{k=1}^n a_{k,i} g(e_k, z_j) \\
&= \sum_{k=1}^n \hat{g}(\epsilon^k, \zeta^i) g(e_k, z_j)
\end{aligned}$$

Define $U, V \in \mathbb{R}^{n\times n}$ by $U_{i,k} = \hat{g}(\zeta^i, \epsilon^k)$ and $V_{k,j} = g(e_k, z_j)$. Then from above, we have that $UV = I$. Since $U, V \in \mathbb{R}^{n\times n}$, $VU = I$. Hence $U = V^{-1}$. Since

$$\begin{aligned}
\zeta^i(e_j) &= \sum_{k=1}^n a_{k,i}\epsilon^k(e_j) \\
&= \sum_{k=1}^n a_{k,i}\delta_{k,j} \\
&= a_{j,i} \\
&= \hat{g}(\epsilon^j, \zeta^i) \\
&= U_{i,j}
\end{aligned}$$

and

$$\begin{aligned}
g(z_i, z_j) &= \left(\sum_{k=1}^{n} g(e_k, z_i) e_k, \sum_{l=1}^{n} g(e_l, z_j) e_l \right) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} g(e_k, z_i) g(e_l, z_j) g(e_k, e_l) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} g(e_k, z_i) g(e_l, z_j) \delta_{k,l} \\
&= \sum_{k=1}^{n} g(e_k, z_i) g(e_k, z_j) \\
&= (V^* V)_{i,j}
\end{aligned}$$

we have that

$$\begin{aligned}
\lambda(e_1, \ldots, e_n) &= \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n (e_1, \ldots, e_n) \\
&= \det[g(z_i, z_j)]^{1/2} \det[\zeta^i(e_j)] \\
&= \det(V^* V)^{1/2} \det U \\
&= \det V (\det V)^{-1} \\
&= 1
\end{aligned}$$

2. Choose an orthonormal frame $e_1, \ldots, e_{n-1} \in \mathfrak{X}(\partial M)$ with dual coframe $\epsilon^1, \ldots, \epsilon^{n-1}$. Define $\nu \in \Omega^1(M)$ to be the dual covector to N. We note that $N, e_1, \ldots, e_{n-1}$ is an orthonormal frame on $\mathfrak{X}(M)$. Let $X_1, \ldots, X_{n-1} \in \mathfrak{X}(\partial M)$. Since for each $j \in \{1, \ldots, n-1\}$, $X_j \in \mathfrak{X}(\partial M)$ and for each $p \in \partial M$, $N_p \in (T_p \partial M)^\perp$, we have that for each $j \in \{1, \ldots, n-1\}$, $g(X_j, N) = 0$. This implies that

$$\begin{aligned}
\iota^* \iota_X \lambda(X_1, \ldots, X_{n-1}) &= \lambda(X, X_1, \ldots, X_{n-1}) \\
&= \nu \wedge \epsilon^1 \wedge \cdots \wedge \epsilon^{n-1}(X, X_1, \ldots, X_{n-1}) \\
&= \det \begin{pmatrix} \nu(X) & \nu(X_1) & \cdots & \nu(X_{n-1}) \\ \epsilon^1(X) & \epsilon^1(X_1) & \cdots & \epsilon^1(X_{n-1}) \\ \vdots & & & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \cdots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= \det \begin{pmatrix} g(X, N) & g(X_1, N) & \cdots & g(X_{n-1}, N) \\ \epsilon^1(X) & \epsilon^1(X_1) & \cdots & \epsilon^1(X_{n-1}) \\ \vdots & & & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \cdots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= g(X, N) \det(\epsilon^i(X_j)) \\
&= g(X, N) \epsilon^1 \wedge \ldots \wedge \epsilon^{n-1}(X_1, \ldots, X_{n-1}) \\
&= g(X, N) \tilde{\lambda}(X_1, \ldots, X_n)
\end{aligned}$$

Therefore $\iota^* \iota_X \lambda = g(X, N) \tilde{\lambda}$ and

$$\begin{aligned}
\int_M \operatorname{div} X \lambda &= \int_M d(\iota_X \lambda) \\
&= \int_{\partial M} \iota^*(\iota_X \lambda) \\
&= \int_{\partial M} g(X, N) \tilde{\lambda}
\end{aligned}$$

3. We note that

$$\begin{aligned} 0 &= \iota_X(du \wedge \lambda) \\ &= \iota_X(du) \wedge \lambda - du \wedge (\iota_X \lambda) \\ &= du(X)\lambda - du \wedge (\iota_X \lambda) \end{aligned}$$

which implies that

$$\begin{aligned} \mathrm{div}(uX)\lambda &= d(\iota_{uX}\lambda) \\ &= d(u\iota_X \lambda) \\ &= du \wedge (\iota_X \lambda) + ud(\iota_X \lambda) \\ &= du(X)\lambda + u\mathrm{div}(X)\lambda \\ &= [du(X) + u\mathrm{div}(X)]\lambda \end{aligned}$$

This implies that $\mathrm{div}(uX) = du(X) + u\mathrm{div}(X)$. From before, we have that

$$\begin{aligned} \int_M du(X)\lambda &= \int_M \mathrm{div}(uX)\lambda - \int_M u\mathrm{div}(X)\lambda \\ &= \int_{\partial M} g(uX, N)\tilde{\lambda} - \int_M u\mathrm{div}(X)\lambda \\ &= \int_{\partial M} ug(X, N)\tilde{\lambda} - \int_M u\mathrm{div}(X)\lambda \end{aligned}$$

$\square$

Exercise 23.0.0.5.

$$\mathrm{div}(X) = \sum_{j=1}^{n} (\nabla_{\partial_j} X)^j$$

Proof. We have that

$$\begin{aligned} \nabla_{\partial_i}(X) &= \sum_{j=1}^{n} \nabla_{\partial_i}(X^j \partial_j) \\ &= \sum_{j=1}^{n} \left[X^j \, \nabla_{\partial_i} \partial_j + \partial_i(X^j)\partial_j \right] \\ &= \sum_{j=1}^{n} \left[X^j (\sum_{k=1}^{n} \Gamma_{i,j}^k \partial_k) + \partial_i(X^j)\partial_j \right] \\ &= \sum_{j=1}^{n} \left[X^j \left(\sum_{k=1}^{n} \Gamma_{i,j}^k \partial_k \right) + \partial_i(X^j)\partial_j \right] \\ &= \sum_{j=1}^{n} X^j \left(\sum_{k=1}^{n} \Gamma_{i,j}^k \partial_k \right) + \sum_{j=1}^{n} \partial_i(X^j)\partial_j \\ &= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} X^j \Gamma_{i,j}^k \right) \partial_k + \sum_{k=1}^{n} \partial_i(X^k)\partial_k \\ &= \sum_{k=1}^{n} \left[\left(\sum_{j=1}^{n} X^j \Gamma_{i,j}^k \right) + \partial_i(X^k) \right] \partial_k \end{aligned}$$

so that $(\nabla_{\partial_i}(X))^i = \left(\sum_{j=1}^n X^j \Gamma^i_{i,j}\right) + \partial_i(X^i)$. We note that

$$\begin{aligned}\operatorname{div}(X) &= \sum_{i=1}^n \operatorname{div}(X^i \partial_i) \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + dx^i(\partial_i)] \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + 1]\end{aligned}$$

Since $\lambda = [\det g(\partial_i, \partial_j)]^{1/2} dx^1 \wedge \cdots \wedge dx^n = (\det g)^{1/2} dx$, we have that

$$\begin{aligned}d(\iota_{\partial_i}\lambda) &= d((\det g)^{1/2} \iota_{\partial_i} dx) \\ &= d[(\det g)^{1/2}]\iota_{\partial_i} dx + (\det g)^{1/2} d(\iota_{\partial_i} dx) \\ &= d[(\det(g)^{1/2}] \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots dx^n + (\det g)^{1/2} \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots dx^n)\end{aligned}$$

FINISH!!! □

Exercise 23.0.0.6. Let (M, g) be a Riemannian manifold.

1. For each $u, v \in C^\infty(M)$. Then
   (a)
   $$\int_M u\Delta v \lambda + \int_M g(\nabla u, \nabla v)\lambda = \int_{\partial M} uN(v)\tilde{\lambda}$$
   (b)
   $$\int_M [u\Delta v - v\Delta u]\lambda = \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}$$
2. (a) If $\partial M \neq \varnothing$, then for each $u, v \in C^{\infty(M)}$, u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$ implies that $u = v$.
   (b) If $\partial M = \varnothing$, then for each $u \in C^\infty(M)$, u is harmonic implies that u is constant.

Proof.

1. Let $u, v \in C^\infty(M)$. Then
   (a)
   $$\begin{aligned}\int_M u\Delta v\lambda &= \int_M u \operatorname{div}(\nabla v)\lambda \\ &= \int_{\partial M} ug(\nabla v, N)\tilde{\lambda} - \int_M du(\nabla v)\lambda \\ &= \int_{\partial M} udv(N)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda \\ &= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda\end{aligned}$$

(b) From above, we have that

$$\begin{aligned}
\int_M [u\Delta v - v\Delta u]\lambda &= \int_M u\Delta v\lambda - \int_M v\Delta u\lambda \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda - \left(\int_{\partial M} vN(u)\tilde{\lambda} - \int_M g(\nabla v, \nabla u)\lambda\right) \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_{\partial M} vN(u)\tilde{\lambda} \\
&= \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}
\end{aligned}$$

2. (a) Suppose that $\partial M \neq \varnothing$. Let $u, v \in C^{\infty(M)}$. Suppose that u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$. Then $u - v$ is harmonic and

$$\begin{aligned}
\int_M \| \nabla(u-v)\|_g^2\lambda &= \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= 0 + \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= \int_M (u-v)\Delta(u-v)\lambda + \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= \int_{\partial M} (u-v)N(u-v)\tilde{\lambda} \\
&= 0
\end{aligned}$$

Thus $\nabla(u-v) = 0$ and $u - v$ is constant. Since $u|_{\partial M} = v|_{\partial M}$, we have that $u - v = 0$ and thus $u = v$.

(b) Suppose that $\partial M = \varnothing$. Let $u \in C^\infty(M)$. Suppose that u is harmonic. Then

$$\begin{aligned}
\int_M \| \nabla u\|_g^2\lambda &= \int_M g(\nabla u, \nabla u)\lambda \\
&= 0 + \int_M g(\nabla u, \nabla u)\lambda \\
&= \int_M u\Delta u\lambda + \int_M g(\nabla u, \nabla u)\lambda \\
&= \int_{\partial M} (u-v)g(\nabla(u-v), N)\tilde{\lambda} \\
&= 0
\end{aligned}$$

Therefore $\nabla u - 0$ and u is constant.

$\square$

Chapter 24

Symplectic Geometry

24.1 Symplectic Manifolds

Definition 24.1.0.1. Let $M \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\omega \in \Omega^2(M)$. Then ω is said to be **symplectic** if

1. ω is nondegenerate
2. ω is closed

Chapter 25

Extra

Definition 25.0.0.1. When working in $\mathbb{R}^n$, we introduce the formal objects $dx^1, dx_2, \cdots, dx^n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx^i = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.

Definition 25.0.0.2. Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx^i : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

1. the exterior product is bilinear

2. for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$

3. for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$

4. for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k-forms on $\mathbb{R}^n$. Let ω be a k-form on $\mathbb{R}^n$. If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx^i$

Note 25.0.0.3. The terms $dx^1, dx_2, \cdots, dx^n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx^1, dx_2, \cdots, dx^n\}$ when referencing the coordinates on $\mathbb{R}^n$.

Exercise 25.0.0.4. Let $B_{n \times n} = (b_{i,j}) \in [C^\infty(M)]^{n \times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^{n} \left(\sum_{j=1}^{n} b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned}\bigwedge_{i=1}^{n}\left(\sum_{j=1}^{n} b_{i,j}dx^j\right) &= \left(\sum_{j=1}^{n} b_{1,j}dx^j\right)\wedge\left(\sum_{j=1}^{n} b_{2,j}dx^j\right)\wedge\cdots\wedge\left(\sum_{j=1}^{n} b_{n,j}dx^j\right)\\ &= \sum_{j_1,\cdots,j_n=1}^{n}\left(\prod_{i=1}^{n} b_{i,j_i}\right)dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}\\ &= \sum_{j_1\neq\cdots\neq j_n}\left(\prod_{i=1}^{n} b_{i,j_i}\right)dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}\\ &= \left[\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\left(\prod_{i=1}^{n} b_{i,\sigma(i)}\right)\right]dx_1\wedge dx_2\wedge\cdots\wedge dx_n\\ &= (\det B)dx_1\wedge dx_2\wedge\cdots\wedge dx_n\end{aligned}$$

□

Definition 25.0.0.5. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted df, on $\mathbb{R}^n$ by

$$df=\sum_{i=1}^{n}\frac{\partial f}{\partial x^i}dx^i$$

Let $\omega=\sum_{I\in\mathcal{I}_{k,n}} f_I dx^i$ be a k-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega=\sum_{I\in\mathcal{I}_{k,n}} df_I\wedge dx^i$$

Exercise 25.0.0.6. On $\mathbb{R}^3$, put

1. $\omega_0=f_0$,
2. $\omega_1=f_1dx^1+f2dx_2+f_2dx_3$,
3. $\omega_2=f_1dx_2\wedge dx_3-f_2dx^1\wedge dx_3+f_3dx^1\wedge dx_2$

Show that

1. $d\omega_0=\frac{\partial f_0}{\partial x^1}dx^1+\frac{\partial f_0}{\partial x^2}dx_2+\frac{\partial f_0}{\partial x^3}dx_3$
2. $d\omega_1=\left(\frac{\partial f_3}{\partial x^2}-\frac{\partial f_2}{\partial x^3}\right)dx_2\wedge dx_3+\left(\frac{\partial f_3}{\partial x^1}-\frac{\partial f_1}{\partial x^3}\right)dx^1\wedge dx_3+\left(\frac{\partial f_2}{\partial x^1}-\frac{\partial f_1}{\partial x^2}\right)dx^1\wedge dx_2$
3. $d\omega_2=\left(\frac{\partial f_1}{\partial x^1}+\frac{\partial f_2}{\partial x^2}+\frac{\partial f_3}{\partial x^3}\right)dx^1\wedge dx_2\wedge dx_3$

Proof. Straightforward. □

Exercise 25.0.0.7. Let $I\in\mathcal{I}_{k,n}$. Then there is a unique $I_*\in\mathcal{I}_{n-k,n}$ such that $dx^i\wedge dx_{I_*}=dx^1\wedge dx_2\wedge\cdots\wedge dx^n$.

Definition 25.0.0.8. We define a linear map $*:\Phi_k(\mathbb{R}^n)\to\Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge $*$-operator** by

$$*\sum_{I\in\mathcal{I}_{k,n}} f_I dx^i=\sum_{I\in\mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 25.0.0.9. Let $\phi:\mathbb{R}^k\to\mathbb{R}^n$ be smooth. Write $\phi=(\phi_1,\phi_2,\cdots,\phi_n)$. We define $\phi^*:\Phi_k(\mathbb{R}^n)\to\Phi_k(\mathbb{R}^k)$ via the following properties:

1. for each 0-form f on $\mathbb{R}^n$, $\phi^* f = f \circ \phi$
2. for $i = 1, \cdots, n$, $\phi^* dx^i = d\phi_i$
3. for an s-form ω, and a t-form χ on $\mathbb{R}^n$, $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
4. for l-forms ω, χ on $\mathbb{R}^n$, $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 25.0.0.10. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold of $\mathbb{R}^n$, $\phi : U \to V$ a smooth parametrization of M, $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$ an k-form on $\mathbb{R}^n$. Then

$$\phi^*\omega = \bigg(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \bigg) du^1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof. By definition,

$$\begin{aligned}
\phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i \\
&= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I)\phi^* dx^i \\
&= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I
\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}
d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\
&= (\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u^j} du^j) \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u^j} du^j) \wedge \cdots \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u^j} du^j) \\
&= (\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\
&= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k \\
&= \bigg(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \bigg) du^1 \wedge du_2 \wedge \cdots \wedge du_k
\end{aligned}$$

□

25.1 Integration of Differential Forms

Definition 25.1.0.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \cdots \wedge dx_k$ a k-form on $\mathbb{R}^k$. Define

$$\int_U \omega = \int_U f dx$$

Definition 25.1.0.2. Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$, ω a k-form on $\mathbb{R}^n$ and $\phi : U \to V$ a local smooth, orientation-preserving parametrization of M. Define

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 25.1.0.3.

Theorem 25.1.0.4. Stokes Theorem:
Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$ and ω a $k-1$-form on $\mathbb{R}^n$. Then

$$\int_{\partial M} \omega = \int_M d\omega$$

Appendix A

Summation

Appendix B

Asymptotic Notation

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration