1

Introduction to Logic

Carson James

Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Review of Fundamentals

1.1 Set Theory

Definition 1.1.0.1.

- We define $[0] = \varnothing$ and for $k \in \mathbb{N}$, we define $[k] = \{1, \dots, k\}$.
- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-tupels with entries in** S, denoted S^k, by
$$S^k = \{u : [k] \to S\}$$
- Let S be a set. We define the **set of all tuples with entries in** S, denoted S^*, by
$$S^* = \bigcup_{k \in \mathbb{N}_0} S^k$$
- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary operations on** S, denoted $\mathcal{F}^k(S)$, by $\mathcal{F}^k(S) = S^{(S^k)}$. We define the **set of finitary operations on** S, denoted $\mathcal{F}^*(S)$, by
$$\mathcal{F}^*(S) = \bigcup_{k \in \mathbb{N}_0} \mathcal{F}^k(S)$$
- Let S be a set. We define the **arity map**, denoted $\operatorname{arity} : S^* \to \mathbb{N}_0$, by
$$\operatorname{arity} f = k, \quad f \in \mathcal{F}^k(S)$$
- Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{F}$, denoted $\mathcal{F}_k$, by
$$\mathcal{F}_k = \mathcal{F} \cap \mathcal{F}^k(S)$$

Definition 1.1.0.2. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $C \subset S$. Then C is said to be $\mathcal{F}$-closed if for each $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$ and $a_1, \dots, a_k \in C$, $f(a_1, \dots, a_k) \in C$.

Definition 1.1.0.3. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $\mathcal{C} \subset \mathcal{P}(S)$. If for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed, then $\bigcap_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed

Proof. Suppose that for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed. Let $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$, $a_1, \dots, a_k \in \bigcap_{C \in \mathcal{C}} C$ and $C_0 \in \mathcal{C}$. Since $C_0 \in \mathcal{C}$, we have that

$$\begin{aligned} a_1, \dots, a_k &\in \bigcap_{C \in \mathcal{C}} C \\ &\subset C_0 \end{aligned}$$

Since C_0 is $\mathcal{F}$-closed, we have that $f(a_1, \dots, a_k) \in C_0$. Since $C_0 \in \mathcal{C}$ is arbitrary, we have that for each $C \in \mathcal{C}$, $f(a_1, \dots, a_k) \in C$. Hence $f(a_1, \dots, a_k) \in \bigcap\limits_{C \in \mathcal{C}} C$. Since $k \in \mathbb{N}_0$ and $a_1, \dots, a_k \in \bigcap\limits_{C \in \mathcal{C}} C$ are arbitrary, we have that $\bigcap\limits_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed. $\square$

Definition 1.1.0.4. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $B, C \subset S$. Then C is said to be $\mathcal{F}$-inductive over $\mathcal{B}$ if

1. C is $\mathcal{F}$-closed
2. $B \subset C$

Chapter 2

Propositional Logic

Definition 2.0.0.1. Let $\mathcal{A}$ be a set, $\mathcal{V} \subset \mathcal{A}$, $\mathcal{F} \subset \bigcup_{k \in \mathbb{N}_0} A^k$, $\mathcal{C} \subset \bigcup_{k \in \mathbb{N}_0} \mathcal{F}^{(\mathcal{F}^k)}$. Then $(\mathcal{A}, \mathcal{V}, \mathcal{A})$ is said to be a **propositional calculus** if

1. - $\mathcal{V} \subset \mathcal{F}$
   - for each $k \in \mathbb{N}_0$, $p_1, \ldots, p_k \in \mathcal{F}$, and $f \in \mathcal{C}_k$, $f(p_1, \ldots, p_k) \in \mathcal{F}$
   - for each $p \in \mathcal{F}$, there exists some tree $fin\mathcal{T}^k(\mathcal{F})$ such that $\in \mathcal{F}^k$

define the **alphabet** $\mathcal{A}$ define the **variables** $\mathcal{V}$ define the **formulas** $\mathcal{F}$ define the **connectives** $\mathcal{C}$ the **formulas of** $\mathcal{L}$**,**

Exercise 2.0.0.2.