INTRODUCTION TO PROBABILITY

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Contents

1. Basic Probability

2. Probability

2.1. Distributions.

Definition 2.1.1. Let Ω be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then $\mathcal{P}$ is said to be a **π-system** on Ω if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Definition 2.1.2. Let Om be a set and $\mathcal{L} \subset \mathcal{P}(\Omega)$. Then $\mathcal{L}$ is said to be a **λ-system** on Ω if

(1) $\mathcal{L} \neq \varnothing$
(2) for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$
(3) for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}$, if $(A_n)_{n\in\mathbb{N}}$ is disjoint, then $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{L}$

Exercise 2.1.3. Let Ω be a set and $\mathcal{L}$ a λ-system on Ω. Then

(1) $\Omega, \varnothing \in \mathcal{L}$

Proof. Straightforward. □

Definition 2.1.4. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. Put

$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(\Omega) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the **λ-system on Ω generated by** $\mathcal{C}$, $\lambda(\mathcal{C})$, to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L}\in\mathcal{S}} \mathcal{L}$$

Exercise 2.1.5. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. If $\mathcal{C}$ is a λ-system and $\mathcal{C}$ is a π-system, then $\mathcal{C}$ is a σ-algebra.

Proof. Suppose that $\mathcal{C}$ is a λ-system and $\mathcal{C}$ is a π-system. Then we need only verify the third axiom in the definition of a σ-algebra. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. Define $B_1 = A_1$ and for $n \geq 2$, define $B_n = A_n \cap \left(\bigcup\limits_{k=1}^{n-1} A_k\right)^c = A_n \cap \left(\bigcap\limits_{k=1}^{n-1} A_k^c\right) \in \mathcal{C}$. Then $(B_n)_{n\in\mathbb{N}}$ is disjoint and therefore $\bigcup\limits_{n\in\mathbb{N}} A_n = \bigcup\limits_{n\in\mathbb{N}} B_n \in \mathcal{C}$. □

Theorem 2.1.6. (Dynkin's Theorem)
Let Ω be a set.

(1) Let $\mathcal{P}$ be a π-system on Ω and $\mathcal{L}$ a λ-system on Ω. If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
(2) Let $\mathcal{P}$ be a π-system on Ω. Then $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

Exercise 2.1.7. Let $(\Omega, \mathcal{F})$ be a measurable space and μ, ν probability measures on $(\Omega, \mathcal{F})$. Put $\mathcal{L}_{\mu,\nu} = \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$. Then $\mathcal{L}_{\mu,\nu}$ is a λ-system on Ω.

Proof.

(1) $\varnothing \in \mathcal{L}_{\mu,\nu}$.
(2) Let $A \in \mathcal{L}_{\mu,\nu}$. Then $\mu(A) = \nu(A)$. Thus

$$\begin{aligned} \mu(A^c) &= 1 - \mu(A) \\ &= 1 - \nu(A) \\ &= \nu(A^c) \end{aligned}$$

So $A^c \in \mathcal{L}_{\mu,\nu}$.

(3) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}_{\mu,\nu}$. So for each $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Then

$$\begin{aligned}\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) &= \sum_{n\in\mathbb{N}} \mu(A_n)\\ &= \sum_{n\in\mathbb{N}} \nu(A_n)\\ &= \nu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)\end{aligned}$$

Hence $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}_{\mu,\nu}$.

□

Exercise 2.1.8. Let $(\Omega, \mathcal{F})$ be a measurable space, μ, ν probability measures on $(\Omega, \mathcal{F})$ and $\mathcal{P} \subset \mathcal{F}$ a π-system on Ω. Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Proof. Using the previous exercise, we see that $\mathcal{P} \subset \mathcal{L}_{\mu,\nu}$. Dynkin's theorem implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu,\nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$. □

Definition 2.1.9. Let $F : \mathbb{R} \to \mathbb{R}$. Then F is said to be a **probability distribution function** if

(1) F is right continuous
(2) F is increasing
(3) $F(-\infty) = 0$ and $F(\infty) = 1$

Definition 2.1.10. Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We define $F_P : \mathbb{R} \to \mathbb{R}$, by

$$F_P(x) = P((-\infty, x])$$

We call F_P the **probability distribution function of** P.

Exercise 2.1.11. Let $(\Omega, \mathcal{F}, P)$ be a probability measure. Then F_P is a probability distribution function.

Proof. (1) Let $x \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}} \subset [x, \infty)$. Suppose that $x_n \to x$. Then $(x, x_n] \to \varnothing$ because $\limsup\limits_{n\to\infty}(x, x_n] = \varnothing$. Thus

$$F(x_n) - F(x) = P((x, x_n]) \to P(\varnothing) = 0$$

This implies that

$$F(x_n) \to F(x)$$

So F is right continuous.

(2) Clearly F_P is increasing.

(3) Continuity from below tells us that

$$F(-\infty) = \lim_{n\to-\infty} F(n) = \lim_{n\to-\infty} P((-\infty, n]) = 0$$

and continuity from above tell us that

$$F(\infty) = \lim_{n\to\infty} F(n) = \lim_{n\to\infty} P((-\infty, n]) = 1$$

□

Exercise 2.1.12. Let μ, ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $F_\mu = F_\nu$ iff $\mu = \nu$.

Proof. Clearly if $\mu = \nu$, then $F_\mu = F_\nu$. Conversely, suppose that $F_\mu = F_\nu$. Then for each $x \in \mathbb{R}$,

$$\begin{aligned} \mu((-\infty, x]) &= F_\mu(x) \\ &= F_\nu(x) \\ &= \nu((-\infty, x]) \end{aligned}$$

Put $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then $\mathcal{C}$ is a π-system and for each $A \in \mathcal{C}$, $\mu(A) = \nu(A)$. Hence for each $A \in \sigma(C) = \mathcal{B}(\mathbb{R})$, $\mu(A) = \nu(A)$. So $\mu = \nu$. $\square$

Definition 2.1.13. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}^n$. Then X is said to be a **random vector** on $(\Omega, \mathcal{F})$ if X is $\mathcal{F}$-$\mathcal{B}(\mathbb{R}^n)$ measurable. If $n = 1$, then X is said to be a **random variable**. We define

$$L_n^0(\Omega, \mathcal{F}, P) = \{X : \Omega \to \mathbb{R}^n : X \text{ is a random vector}\}$$

and

$$L_n^p(\Omega, \mathcal{F}, P) = \left\{ X \in L_n^0 : \int \|X\|^p dP < \infty \right\}$$

Definition 2.1.14. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable on $(\Omega, \mathcal{F})$. We define the **probability distribution** of X, $P_X : \mathcal{B}(R) \to [0, 1]$, to be the measure

$$P_X = X_* P$$

That is, for each $A \in \mathcal{B}(\mathbb{R})$,

$$P_X(A) = P(X^{-1}(F))$$

We define the **probability distribution function** of X, $F_X : \mathbb{R} \to [0, 1]$, to be

$$F_X = F_{P_X}$$

Definition 2.1.15. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable on $(\Omega, \mathcal{F})$. If $P_X \ll m$, we define the **probability density** of X, $f_X : \mathbb{R} \to \mathbb{R}$, by

$$f_X = \frac{dP_X}{dm}$$

Exercise 2.1.16. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{F})$. Then for each $x \in \mathbb{R}$,

$$\mathbb{P}\left(\liminf_{n \to \infty} X_n > x \right) \leq \liminf_{n \to \infty} P(X_n > x)$$

Proof. Let $\omega \in \left\{ \liminf_{n \to \infty} X_n > x \right\}$. Then $x < \liminf_{n \to \infty} X_n(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} X_k(\omega) \right)$. So there exists $n^* \in \mathbb{N}$ such that $x < \inf_{k \geq n^*} X_k(\omega)$. Then for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x < X_k(\omega)$. So there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Hence $\inf_{k \geq n^*} \mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Thus $\liminf_{n \to \infty} \mathbf{1}_{\{X_k > x\}}(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \mathbf{1}_{\{X_k > x\}}(\omega) \right) = 1$. Therefore

$\omega \in \liminf_{n\to\infty}\{X_k > x\}$ and we have shown that

$$\left\{\liminf_{n\to\infty} X_n > x\right\} \subset \liminf_{n\to\infty}\{X_k > x\}$$

Then

$$\begin{aligned} P\left(\liminf_{n\to\infty} X_n > x\right) &\leq P\left(\liminf_{n\to\infty}\{X_k > x\}\right) \\ &\leq \liminf_{n\to\infty} P(\{X_k > x\}) \end{aligned}$$

□

Definition 2.1.17. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^+(\Omega) \cup L^1$. Define the **expectation of X**, $E[X]$, to be

$$\mathrm{E}[X] = \int X dP$$

.

2.2. **Independence.**

Definition 2.2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C} \subset \mathcal{F}$. Then $\mathcal{C}$ is said to be **independent** if for each $(A_i)_{i=1}^n \subset \mathcal{C}$,

$$P\left(\bigcap_{k=1}^n A_k\right) = \prod_{k=1}^n P(A_k)$$

Definition 2.2.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C}_1, \cdots, \mathcal{C}_n \subset \mathcal{F}$. Then $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are said to be **independent** if for each $A_1 \in \mathcal{C}_1, \cdots, A_n \in \mathcal{C}_n$, $A_1, \cdots, A_n$ are independent.

Note 2.2.3. We will explicitely say that for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is independent when talking about the independence of the elements of $\mathcal{C}_i$ to avoid ambiguity.

Definition 2.2.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_2$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are said to be **independent** if for each $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R})$, $X_1^{-1}B_1, \cdots, X_n^{-1}B_n$ are independent.

Exercise 2.2.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are independent iff $\sigma(X_1), \cdots, \sigma(X_n)$ are independent.

Proof. Suppose that $X_1, \cdots, X_n$ are independent. Let $A_1, \in \sigma(X_1), \cdots, A_n \in \sigma(A_n)$. Then for each $i = 1, \cdots, n$, there exists $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X_i^{-1}(B_i)$. Then $A_1, \cdots, A_n$ are independent. Hence $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Conversely, suppose that $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Let $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R})$. Then for each $i = 1, \cdots, n$, $X_i^{-1}B_i \in \sigma(X_i)$. Then $X_1^{-1}B_1, \cdots, X_n^{-1}B_n$ are independent. Hence $X_1, \cdots, X_n$ are independent. □

Exercise 2.2.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$ and $\mathcal{F}_1, \cdots, \mathcal{F}_n \subset \mathcal{F}$ a collection of σ-algebras on Ω. Suppose that for each $i = 1, \cdots, n$, X_i is $\mathcal{F}_i$-measurable. If $\mathcal{F}_1, \cdots, \mathcal{F}_n$ are independent, then $X_1, \cdots, X_n$ are independent.

Proof. For each $i = 1, \cdots, n$, $\sigma(X_i) \subset \mathcal{F}_i$. So $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Hence $X_1, \cdots, X_n$ are independent. □

Exercise 2.2.7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C}_1, \cdots, \mathcal{C}_n \subset \mathcal{F}$. Suppose that for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is a π-system and $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are independent, then $\sigma(\mathcal{C}_1), \cdots, \sigma(\mathcal{C}_n)$ are independent.

Proof. Let $A_2 \in \mathcal{C}_2$. Define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap A_2) = P(A)P(A_2)\}$. Then

(1) $\Omega \in \mathcal{L}$

(2) If $A \in \mathcal{L}$, then

$$\begin{aligned} P(A^c \cap A_2) &= P(A_2) - P(A_2 \cap A) \\ &= P(A_2) - P(A_2)P(A) \\ &= (1 - P(A))P(A_2) \\ &= P(A^c)P(A_2) \end{aligned}$$

So $A^c \in \mathcal{L}$

(3) If $(B_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ is disjoint, then

$$\begin{aligned} P\bigg(\bigg[\bigcup_{n\in\mathbb{N}} B_n\bigg] \cap A_2\bigg) &= P\bigg(\bigcup_{n\in\mathbb{N}} B_n \cap A_2\bigg) \\ &= \sum_{n\in\mathbb{N}} P(B_n \cap A_2) \\ &= \sum_{n\in\mathbb{N}} P(B_n)P(A_2) \\ &= \bigg[\sum_{n\in\mathbb{N}} P(B_n)\bigg] P(A_2) \\ &= P\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)P(A_2) \end{aligned}$$

So $\bigcup\limits_{n\in\mathbb{N}} B_n \in \mathcal{L}$.

Thus $\mathcal{L}$ is a λ-system. Since $\mathcal{C}_1 \subset \mathcal{L}$ is a π-system, Dynkin's theorem tells us that $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Since $A_2 \in \mathcal{C}_2$ is arbitrary $\sigma(\mathcal{C}_1)$ and $\mathcal{C}_2$ are independent. The same reasoning implies that $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent. Let $A_2 \in \mathcal{C}_1, \cdots, A_n \in \mathcal{C}_n$ We may do the same process with

$$\mathcal{L} = \bigg\{A \in \mathcal{F} : P\bigg(A \cap \bigg(\bigcap_{i=2}^{n} A_i\bigg)\bigg) = P(A)\prod_{i=2}^{n} P(A_i)\bigg\}$$

and conclude that $\sigma(\mathcal{C}_1), \mathcal{C}_2, \cdots, \mathcal{C}_n$ are independent. Which, using the same reasoning would imply that $\sigma(\mathcal{C}_1), \cdots, \sigma(\mathcal{C}_n)$ are independent. $\square$

Exercise 2.2.8. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are independent iff for each $x_1, \cdots, x_n \in \mathbb{R}$,

$$P(X_1 \le x_1, \cdots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i)$$

Proof. Suppose that $X_1, \cdots, X_n$ are independent. Then $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Let $x_1, \cdots, x_n \in \mathbb{R}$. Then for each $i = 1, \cdots, n$, $\{X_i \le x_i\} \in \sigma(X_i)$. Hence

$P(X_1 \leq x_1, \cdots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Conversely, suppose that for each $x_1, \cdots, x_n \in \mathbb{R}$, $P(X_1 \leq x_1, \cdots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Define $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$. For each $i = 1, \cdots, n$, define $\mathcal{C}_i = X_i^{-1}\mathcal{C}$. Then for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is a π-system and

$$\begin{aligned}\sigma(\mathcal{C}_i) &= \sigma(X^{-1}(\mathcal{C})) \\ &= X_i^{-1}(\sigma(\mathcal{C})) \\ &= X_i^{-1}(\mathcal{B}(\mathbb{R})) \\ &= \sigma(X_i)\end{aligned}$$

By assumption, $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are independent. The previous exercies tells us that $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Then $X_1, \cdots, X_n$ are independent. □

Exercise 2.2.9. Let Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Define $X = (X_1, \cdots, X_n)$. If $X_1, \cdots, X_n$ are independent, then

$$P_X = \prod_{i=1}^n P_{X_i}$$

.

Proof. Let $A_1, \cdots, A_n \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}P_X(A_1 \times \cdots \times A_n) &= P(X \in A_1 \times \cdots \times \in A_n) \\ &= P(X_1 \in A_1, \cdots, X_n \in A_n) \\ &= P(X_1 \in A_1) \cdots P(X_n \in A_n) \\ &= P_{X_1}(A_1) \cdots P_{X_n}(A_n) \\ &= \prod_{i=1}^n P_{X_i}(A_1 \times \cdots \times A_n)\end{aligned}$$

Put

$$\mathcal{P} = \{A_1 \times \cdots \times A_n : A_1 \in \mathcal{B}(R), \cdots, A_n \in \mathcal{B}(R)\}$$

Then $\mathcal{P}$ is a π-system and

$$\sigma(\mathcal{P}) = \mathcal{B}(R) \otimes \cdots \otimes \mathcal{B}(R) = \mathcal{B}(\mathbb{R}^n)$$

A previous exercise then tells us that $P_X = \prod_{i=1}^n P_{X_i}$ □

Exercise 2.2.10. Let Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$ and $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R} \in L^0$. Suppose that $f_1 \circ X_1, \cdots, f_n \circ X_n \in L^+(\Omega)$ or $f_1 \circ X_1, \cdots, f_n \circ X_n \in L^1(\Omega)$. If $X_1, \cdots, X_n$ are independent, then

$$E[f_1(X_1) \cdots f_n(X_n)] = \prod_{i=1}^n E[f_i(X_i)]$$

Proof. Define the random vector $X : \Omega \to \mathbb{R}^n$ by $X = (X_1, \cdots, X_n)$ and $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x_1, \cdots, x_n) = f_1(x_1) \cdots f_n(x_n)$. Suppose that for each $i = 1, \cdots, n$, $f_i \in L^+(\mathbb{R})$. Then $g \in L^+(\mathbb{R}^n)$ and by change of variables,

$$\begin{aligned} E[f_1(X_1) \cdots f_n(X_n)] &= E[g(X)] \\ &= \int_\Omega g \circ X dP \\ &= \int_{\mathbb{R}^n} g(x) dP_X(x) \\ &= \int_{R^n} g(x) d \prod_{i=1}^n P_{X_i}(x) \\ &= \prod_{i=1}^n \int_{\mathbb{R}} f_i(x) dP_{X_i}(x) \\ &= \prod_{i=1}^n \int_\Omega f_i \circ X dP \\ &= \prod_{i=1}^n E[f_i(X_i)] \end{aligned}$$

If for each $i = 1, \cdots, n$, $f_i \in L^1(\mathbb{R}, P_{X_i})$, then following the above reasoning with $|g|$ tells us that $g \in L^1(\mathbb{R}^n, P_X)$ and we use change of variables and Fubini's theorem to get the same result. $\square$

2.3. L^p Spaces for Probability.

Note 2.3.1. Recall that for a probability space $(\Omega, \mathcal{F}, P)$ and $1 \leq p \leq q \leq \infty$ we have $L^q \subset L^p$ and for each $X \in L^q$, $\|X\|_p \leq \|X\|_q$. Also recall that for $X, Y \in Ł^2$, we have that $\|XY\|_1 \leq \|X\|_2 \|X\|_2$.

Definition 2.3.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Define the **variance of X**, $Var(X)$, to be

$$Var(X) = \mathrm{E}\big[\big(X - E[X]\big)^2\big]$$

.

Definition 2.3.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Define the

Definition 2.3.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Define the **covariance of X and Y**, $Cov(X, Y)$, to be

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Exercise 2.3.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Then the covariance is well defined and $Cov(X, Y)^2 \leq Var(X)Var(Y)$

Proof. By Holder's inequality,

$$\begin{aligned}
|Cov(X,Y)| &= \left| \int (X - E[X])(Y - E[Y]) dP \right| \\
&\leq \int |(X - E[X])(Y - E[Y])| dP \\
&= \|(X - E[X])(Y - E[Y])\|_1 \\
&\leq \|X - E[X]\|_2 \|(Y - E[Y]\|_2 \\
&= \left(\int |X - E[X]|^2 dP \right)^{\frac{1}{2}} \left(|Y - E[Y]|^2 \right)^{\frac{1}{2}} \\
&= Var(X)^{\frac{1}{2}} Var(Y)^{\frac{1}{2}}
\end{aligned}$$

So $Cov(X,Y)^2 \leq Var(X)Var(Y)$. □

Exercise 2.3.6. Let $(\Omega, \mathcal{F}, P)$ be a measure space and $X, Y \in L^2$. Then

(1) $Cov(X,Y) = E[XY] - E[X]E[Y]$
(2) If X, Y are independent, then $Cov(X,Y) = 0$
(3) $Var(X) = E[X^2] - E[X]^2$
(4) for each $a, b \in \mathbb{R}$, $Var(aX + b) = a^2 Var(X)$.
(5) $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

Proof.

(1) We have that

$$\begin{aligned}
Cov(X,Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY - E[Y]X - E[X]Y + E[X]E[Y]] \\
&= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

(2) Suppose that X, Y are independent. Then $E[XY] = E[X]E[Y]$. Hence

$$\begin{aligned}
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= E[X]E[Y] - E[X]E[Y] \\
&= 0
\end{aligned}$$

(3) Part (1) implies that

$$\begin{aligned}
Var(X) &= Cov(X,X) \\
&= E[X^2] - E[X]^2
\end{aligned}$$

(4) Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
Var(aX + b) &= E[(aX + b)^2] - E[aX + b]^2 \\
&= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - (a^2E[X]^2 + 2abE[X] + b^2) \\
&= a^2(E[X^2] - E[X]^2) \\
&= a^2 Var(X)
\end{aligned}$$

(5) We have that

$$\begin{aligned}
Var(X+Y) &= E[(X+Y)^2] - E[X+Y]^2 \\
&= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
&= E[X^2] + 2E[XY] + E[Y^2] - (E[X]^2 + 2E[X]E[Y] + E[Y]^2) \\
&= (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2) + 2(E[XY] - E[X]E[Y]) \\
&= Var(X) + Var(Y) + 2Cov(X,Y)
\end{aligned}$$

□

Definition 2.3.7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. The **correlation of X and Y**, $Cor(X,Y)$, is defined to be

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Exercise 2.3.8.

Exercise 2.3.9. Jensen's Inequality Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X \in L^1$ and $\phi : \mathbb{R} \to \mathbb{R}$. If ϕ is convex, then

$$\phi(E[X]) \leq E[\phi(X)]$$

Proof. Put $x_0 = E[X]$. Since ϕ is convex, there exist $a, b \in \mathbb{R}$ such that $\phi(x_0) = ax_0 + b$ and for each $x \in \mathbb{R}$, $\phi(x) \geq ax + b$. Then

$$\begin{aligned}
E[\phi(X)] &= \int \phi(X) dP \\
&\geq \int [aX + b] dP \\
&= a \int X dP + b \\
&= aE[X] + b \\
&= ax_0 + b \\
&= \phi(x_0) \\
&= \phi(E[X])
\end{aligned}$$

□

Exercise 2.3.10. Markov's Inequality: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^+$. Then for each $a \in (0, \infty)$,

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Proof. Let $a \in (0, \infty)$. Then $a\mathbf{1}_{\{X \geq a\}} \leq X\mathbf{1}_{\{X \geq a\}}$. Thus

$$\begin{aligned} aP(X \geq a) &= \int a\mathbf{1}_{\{X \geq a\}} dP \\ &= \int X\mathbf{1}_{\{X \geq a\}} dP \\ &\leq \int X dP \\ &= E[X] \end{aligned}$$

Therefore

$$P(X \geq a) \leq \frac{E[X]}{a}$$

. □

Exercise 2.3.11. Chebychev's Inequality: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Then for each $a \in (0, \infty)$,

$$P(|X - E[X]| \geq a) \leq \frac{Var(X)}{a^2}$$

Proof. Let $a \in (0, \infty)$. Then

$$\begin{aligned} P(|X - E[X]| \geq a) &= P((X - E[X])^2 \geq a^2) \\ &\leq \frac{E[(X - E[X])^2]}{a^2} \\ &= \frac{Var(X)}{a^2} \end{aligned}$$

□

Exercise 2.3.12. Chernoff's Bound: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Then for each $a, t \in (0, \infty)$,

$$P(X \geq a) \leq e^{-ta} E[e^{tX}]$$

Proof. Let $a, t \in (0, \infty)$. Then

$$\begin{aligned} P(X \geq a) &= P(tX \geq ta) \\ &= P(e^{tX} \geq e^{ta}) \\ &\leq e^{-ta} E[e^{tX}] \end{aligned}$$

□

Exercise 2.3.13. Weak Law of Large Numbers: Let $(\Omega, \mathcal{F}, P)$ be a probability space $(X_i)_{i \in \mathbb{N}} \subset L^2$. Suppose that $(X_i)_{i \in \mathbb{N}}$ are iid. Then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} E[X_1]$$

Proof. Put $\mu = E[X_1]$ and $\sigma^2 = Var(X_1)$. Then

$$\begin{aligned} E[\frac{1}{n}\sum_{i=1}^n X_i] &= \frac{1}{n}\sum_{i=1}^n E[X_i] \\ &= \frac{1}{n}\sum_{i=1}^n \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} Var(\frac{1}{n}\sum_{i=1}^n X_i) &= \frac{1}{n^2}Var(\sum_{i=1}^n X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^n Var(X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^n \sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Let $\epsilon > 0$. Then

$$\begin{aligned} P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - E[X_1]\right| \geq \epsilon\right) &= P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) \\ &= P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^n X_i\right]\right| \geq \epsilon\right) \\ &\leq \frac{Var(\frac{1}{n}\sum_{i=1}^n X_i)}{\epsilon^2} \\ &= \frac{\sigma^2/n}{\epsilon^2} \\ &= \frac{\sigma^2}{n\epsilon^2} \to 0 \end{aligned}$$

So

$$\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

$\square$

2.4. **Borel Cantelli Lemma.**

Exercise 2.4.1. Borel Cantelli Lemma:
Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$.

(1) If $\sum\limits_{n\in\mathbb{N}} P(A_n) < \infty$, then $P(\limsup\limits_{n\to\infty} A_n) = 0$.
(2) If $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum\limits_{n\in\mathbb{N}} P(A_n) = \infty$, then $P(\limsup\limits_{n\to\infty} A_n) = 1$

Proof.

(1) Suppose that $\sum\limits_{n\in\mathbb{N}} P(A_n) < \infty$. Recall that

$$\limsup_{n\to\infty} A_n = \left\{\omega \in \Omega : \sum_{n\in\mathbb{N}} 1_{A_n}(\omega) = \infty\right\}$$

Then

$$\begin{aligned}\infty &> \sum_{n\in\mathbb{N}} P(A_n) \\ &= \sum_{n\in\mathbb{N}} \int 1_{A_n} dP \\ &= \int \sum_{n\in\mathbb{N}} 1_{A_n} dP\end{aligned}$$

Thus $\sum\limits_{n\in\mathbb{N}} 1_{A_n} < \infty$ a.e. and $P(\limsup\limits_{n\to\infty} A_n) = 0$.

(2) Suppose that $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum\limits_{n\in\mathbb{N}} P(A_n) = \infty$.

□

Exercise 2.4.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n)_{n\in\mathbb{N}} \subset L^0$ and $X \in L^0$.

(1) If for each $\epsilon > 0$, $\sum\limits_{n\in\mathbb{N}} P(|X_n - X| \geq \epsilon) < \infty$, then $X_n \to X$ a.e.
(2) If $(X_n)_{n\in\mathbb{N}}$ are independent and there exists $\epsilon > 0$ such that $\sum\limits_{n\in\mathbb{N}} P(|X_n - X| \geq \epsilon) = \infty$, then $X_n \not\to X$ a.e.

Proof.

(1) For $\epsilon > 0$ and $n \in \mathbb{N}$, set $A_n(\epsilon) = \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}$. Suppose that for each $\epsilon > 0$, $\sum\limits_{n\in\mathbb{N}} P(|X_n - X| \geq \epsilon) < \infty$. The Borel-Cantelli lemma implies that for each $m \in \mathbb{N}$,

$$P(\limsup_{n\to\infty} A_n(1/m)) = 0$$

Let $\omega \in \Omega$. Then $X_n(\omega) \not\to X(\omega)$ iff

$$\omega \in \bigcup_{m\in\mathbb{N}} \limsup_{n\to\infty} A_n(1/m)$$

So

$$\begin{aligned} P(X_n \not\to X) &= P\bigg(\bigcup_{m\in\mathbb{N}} \limsup_{n\to\infty} A_n(1/m)\bigg) \\ &\leq \sum_{m\in\mathbb{N}} P(\limsup_{n\to\infty} A_n(1/m)) \\ &= 0 \end{aligned}$$

Hence $X_n \to X$ a.e.

(2)

$\square$

3. Probability on locally compact Groups

Note 3.0.1. In this section, familiarity with Haar measure will be assumed. This section is intended as a continuation of section 7 of [3].

3.1. Action on Probability Measures.

Note 3.1.1. We recall some notation from section 7.1 of [3].

- $l_g \in \text{Homeo}(G)$, $l_g(x) = gc$
- $L_g \in \text{Sym}(L_0(G))$, $L_g f = f \circ l_g^{-1}$ We continue from section 7

Note 3.1.2. The next exercise generalizes the notion of a scale-family.

Exercise 3.1.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, G a locally compact group, μ a left Haar measure on G, $X \in L^0_G$ and $g \in G$. If $P_X \ll \mu$, then $f_{gX} = L_g f_X$.

Proof. Suppose that $P_X \ll \mu$. Let $A \in \mathcal{B}(G)$. Then

$$\begin{aligned} P_{gX}(A) &= P(gX \in A) \\ &= P(X \in g^{-1}A) \\ &= P_X(g^{-1}A) \\ &= P_X(l_g^{-1}(A)) \\ &= l_{g*}P_X(A) \\ &= g \cdot P_X(A) \end{aligned}$$

The previous exercise tells us that $f_{gX} = L_g f_X$. □

4. Weak Convergence of Measures

5. Concentration Inequalities

5.1. Sub α-Exponential Random Variables.

Definition 5.1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then X is said to be **sub α-exponential** if there exist $M, K > 0$ such that for each $t \geq 0$,

$$P(|X| \geq t) \leq Me^{-Kt^\alpha}$$

Exercise 5.1.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then the following are equivalent:

(1) X is sub α-exponential
(2) there exists $K > 0$ such that for each $p \geq 1$, $\|X\|_p \leq Kp^{1/\alpha}$
(3)
(4)

Proof.

- (1) $\implies$ (2):
  Choose $C_\alpha > 0$ such that for each $x \geq \alpha^{-1}$, $\Gamma(x) \leq C_\alpha x^x$. Since X is sub α-exponential, there exist $M, K_0 > 0$ such that for each $t \geq 0$, $P(|X| \geq t) \leq Me^{-Kt^\alpha}$. Set $K = \max(M\alpha^{-1}C_\alpha, 1)2K_0^{-1/\alpha}\alpha^{-1/\alpha}$. Let $p \geq 1$. Then $p\alpha^{-1} \geq \alpha^{-1}$

$$\begin{aligned}
\|X\|_p^p &= E(|X|^p) \\
&= \int_0^\infty P(|X|^p \geq t)dt \\
&= \int_0^\infty P(|X| \geq t^{1/p})dt \\
&\leq \int_0^\infty Me^{-K_0 t^{\alpha/p}}dt \\
&= Mp\alpha^{-1}\int_0^\infty u^{p/\alpha-1}e^{-K_0 u}du \\
&= Mp\alpha^{-1}\Gamma(p/\alpha)K_0^{-p/\alpha} \\
&\leq Mp\alpha^{-1}C_\alpha(p\alpha^{-1})^{p/\alpha}K_0^{-p/\alpha}
\end{aligned}$$

  Therefore

$$\begin{aligned}
\|X\|_p &\leq (M\alpha^{-1}C_\alpha)^{1/p}p^{1/p}K_0^{-1/\alpha}\alpha^{-1/\alpha}p^{1/\alpha} \\
&\leq \max(M\alpha^{-1}C_\alpha, 1)2K_0^{-1/\alpha}\alpha^{-1/\alpha}p^{1/\alpha} \\
&= Kp^{1/\alpha}
\end{aligned}$$

- (2) $\implies$ (3):

$\square$

Definition 5.1.3. Let $\psi : [0, \infty) \to [0, \infty)$. Then ψ is said to be an **Orlicz function** if

(1) ψ is convex
(2) ψ is increasing

(3) $\psi(x) \to \infty$ as $x \to \infty$
(4) $\psi(0) = 0$

Definition 5.1.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\psi : [0, \infty) \to [0, \infty)$ and Orlicz function. We define the **Orlicz ψ-norm**, denoted $\| \cdot \|_\psi : L^0(\Omega, \mathcal{F}, P) \to [0, \infty]$, by

$$\|X\|_\psi = \inf\{t > 0 : E[\psi(|X|/t)] \leq 1\}$$

We define the **Orlicz ψ-space**, denoted $L_\psi(\Omega, \mathcal{F}, P)$, by

$$L_\psi(\Omega, \mathcal{F}, P) = \{X \in L^0(\Omega, \mathcal{F}, P) : \|X\|_\psi < \infty\}$$

Exercise 5.1.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\psi : [0, \infty) \to [0, \infty)$ and Orlicz function. Then L_ψ is a vector space and $\| \cdot \|_\psi : L_\psi \to [0, \infty)$ is a norm.
Hint: Orlicz functions are super-additive: for each $x, y \in [0, \infty)$, $\psi(x) + \psi(y) \leq \psi(x + y)$

Proof. □

6. Conditional Expectation and Probability

6.1. Conditional Expectation.

Definition 6.1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub σ-alg of $\mathcal{F}$ and $X, Y \in L^1(\Omega, \mathcal{F}, P)$. Then Y is said to be a **conditional expectation of X given $\mathcal{G}$** if

(1) Y is $\mathcal{G}$-measurable
(2) for each $G \in \mathcal{G}$,

$$\int_G Y\,dP = \int_G X\,dP$$

To denote this, we write $Y = E[X|\mathcal{G}]$

Note 6.1.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(S, \mathcal{S})$ a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. We typically write $E[X|Y]$ instead of $E[X|Y^*\mathcal{S}]$.

Exercise 6.1.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub σ-alg of $\mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F}, P)$. Define $P_{\mathcal{G}} = P|_{\mathcal{G}}$ and $Q : \mathcal{G} \to [0, \infty)$ by $Q(G) = \int_G X dP$. Then $Q \ll P_{\mathcal{G}}$.

Proof. Let $G \in \mathcal{G}$. Suppose that $P_{\mathcal{G}}(G) = 0$. By definition, $P(G) = 0$. So $Q(G) = 0$ and $Q \ll P_{\mathcal{G}}$. □

Exercise 6.1.4. Existence of Conditional Expectation:
Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub σ-alg of $\mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F}, P)$. Define Q and $P_{\mathcal{G}}$ as in the previous exercise. Define $Y = dQ/dP_{\mathcal{G}}$. Then Y is a conditional expectation of X given $\mathcal{G}$.

Proof. The Radon-Nikodym theorem implies that Y is $\mathcal{G}$-measurable. Since Q is finite, so is $|Q|$. Since $d|Q| = |Y|\,dP$, we have that $Y \in L^1(\Omega, \mathcal{G}, P_{\mathcal{G}})$. An exercise in section 3.3 of [3], implies that for each $G \in \mathcal{G}$

$$\begin{aligned}\int_G Y\,dP &= \int_G Y\,dP_{\mathcal{G}} \\ &= Q(G) \\ &= \int_G X\,dP\end{aligned}$$

□

Definition 6.1.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(S, \mathcal{S})$ a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. Let $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$. Then ϕ is said to be a **conditional expectation function of X given Y** if for each $B \in \mathcal{S} \cap Y(\Omega)$,

$$\int_{Y^{-1}(B)} X\,dP = \int_B \phi\,dP_Y$$

To denote this, we write $\phi(y) = E[X|Y = y]$.

Exercise 6.1.6. Existence of Conditional Expectation Function:
Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(S, \mathcal{S})$ a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. Suppose that for each $y \in S$, $\{y\} \in \mathcal{S}$. Then there exists $\phi \in L^0(Y(S), \mathcal{S} \cap Y(\Omega))$ such that ϕ is a conditional expectation function of X given Y.
Hint: Doob-Dynkin lemma

Proof. Since $E[X|Y] \in L^0(\Omega, Y^*\mathcal{S})$, the Doob-Dynkin lemma implies that there exists $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$ such that $\phi \circ Y = E[X|Y]$. Let $B \in \mathcal{S} \cap Y(\Omega)$. Then

$$\begin{aligned}\int_B \phi \, dP_Y &= \int_{Y^{-1}(B)} \phi \circ Y \, dP \\ &= \int_{Y^{-1}(B)} E[X|Y] \, dP \\ &= \int_{Y^{-1}(B)} X \, dP\end{aligned}$$

□

6.2. **Conditional Probability.**

Definition 6.2.1. Let $(A, \mathcal{A})$ be a measurable space, $(B, \mathcal{B}, P_Y)$ a probability space and $Q : B \times \mathcal{A} \to [0, 1]$. Then Q is said to be a **stochastic transition kernel from** $(B, \mathcal{B}, P)$ **to** $(A, \mathcal{A})$ if

(1) for each $E \in \mathcal{A}$, $Q(\cdot, E)$ is $\mathcal{B}$-measurable
(2) for P-a.e. $b \in B$, $Q(b, \cdot)$ is a probability measure on $(A, \mathcal{A})$

Definition 6.2.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L^0_n(\Omega, \mathcal{F}, P)$ and $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$. Then Q is said to be a **conditional probability distribution of** X **given** Y if

(1) Q is a stochastic transition kernel from $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_Y)$ to $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$
(2) for each $A, B \in \mathcal{F}$,

$$\int_B Q(y, A) dP_Y(y) = P(X \in A, Y \in B)$$

Note 6.2.3. It is helpful to connect this notion of conditional probability with the elementary one by writing $Q(y, A) = P(X \in A|Y = y)$. If $P_Y \ll \mu$, then property (2) in the definition becomes

$$\begin{aligned}P(X \in A, Y \in B) &= \int_B Q(y, A) dP_Y(y) \\ &= \int_B P(X \in A|Y = y) f_Y(y) d\mu(y)\end{aligned}$$

as in a first course on probability.

Exercise 6.2.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L^0_n$ and $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$. Suppose that for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$-measurable, for P_Y-a.e. $y \in \mathbb{R}^n$, $P_{X|Y}(y, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$ and $Q(Y, A) = P(X \in A|Y)$ a.e. Then Q is a conditional probability of X given Y.

Proof. By assumption, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$-measurable and for P_Y-a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$. Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned}
\int_B Q(y, A) dP_Y(y) &= \int_{Y^{-1}(B)} Q(Y(\omega), A) dP(\omega) \\
&= \int_{Y^{-1}(B)} P(X \in A | Y) dP \\
&= \int_{Y^{-1}(B)} E[1_{X^{-1}(A)} | Y] dP \\
&= \int_{Y^{-1}(B)} 1_{X^{-1}(A)} dP \\
&= \int 1_{X^{-1}(A)} 1_{Y^{-1}(B)} dP \\
&= \int 1_{X^{-1}(A) \cap Y^{-1}(B)} dP \\
&= P(X \in A, Y \in B)
\end{aligned}$$

So Q is a conditional probability distribution of X given Y. □

Definition 6.2.5. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L_n^0$ and μ a σ-finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Then $P_{X,Y} \ll \mu^2$. Let $f_X = dP_X/d\mu$, $f_Y = dP_Y/d\mu$ and $f_{X,Y} = dP_{X,Y}/d\mu^2$. Define $f_{X|Y} : \mathbb{R}^n \times \mathbb{R}^n$ by

$$f_{X|Y}(x, y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)}, & y \in \operatorname{supp} Y \\ 0, & y \notin \operatorname{supp} Y \end{cases}$$

Then $f_{X|Y}$ is called the **conditional probability density of X given Y**.

Exercise 6.2.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L_n^0$ and μ a σ-finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Define $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$ by

$$Q(y, A) = \int_A f_{X|Y}(x, y) d\mu(x)$$

Then Q is a conditional probability distribution of X given Y.

Proof. By the Fubini-Tonelli Theorem, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$-measurable and for P_Y-a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$. Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned}
\int_B Q(y, A) dP_Y(y) &= \int_B \left[\int_A f_{X|Y}(x, y) d\mu(x) \right] dP_Y(y) \\
&= \int_{B \cap \operatorname{supp} Y} \left[\int_{A \cap \operatorname{supp} Y} \frac{f_{X,Y}(x, y)}{f_Y(y)} d\mu(x) f_Y(y) \right] d\mu(y) \\
&= \int_{B \cap \operatorname{supp} Y} \left[\int_A f_{X,Y}(x, y) d\mu(x) \right] d\mu(y) \\
&= P(X \in A, Y \in B \cap \operatorname{supp} Y) \\
&= P(X \in A, Y \in B)
\end{aligned}$$

$\square$

Theorem 6.2.7. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X, Y \in L^1_n(\Omega, \mathcal{F}, P)$. Suppose that $\operatorname{Im} X \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a conditional probability distribution of Y given X.

7. Markov Chains

Definition 7.0.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n)_{n\in\mathbb{N}_0} \in L_n^0$. Then $(X_n)_{n\in\mathbb{N}_0}$ is said to be a **homogeneous Markov chain** if for each $A \in \mathcal{F}$ and $n \in \mathbb{N}$, $P(X_n \in A|X_1, \cdots, X_{n-1}) = P(X_1 \in A|X_0)$ a.e.

8. Probabilities Induced by Isometric Group Actions

8.1. Applications to Bayesian Statistics.

Exercise 8.1.1. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \to \Theta$ an isometric group action. Suppose that $\bar{d}$ is a metric on Θ/G. Let

- H_p^Θ be the Hausdorff measure on Θ, $\mu_{\mathcal{X}}$ a measure on $\mathcal{X}$,
- p a denisty on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on $\mathcal{X}$.
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose that μ_Θ is G-invariant, p is G-invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G-invariant and continuous on Θ. For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \ldots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \to [0,1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)})\, dH_p^\Theta(\theta)$$

Then there exists a density $\bar{p}(\cdot|X^{(n)})$ on Θ/G such that

$$d\bar{P}_{\Theta|X^{(n)}}(\theta) = \bar{p}(\theta|X^{(n)})\, d\bar{H}^\Theta(\theta)$$

Proof. Clear from previous work. □

Exercise 8.1.2. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \to \Theta$ an isometric group action. Suppose that $\bar{d}$ is a metric on Θ/G. Let

- H_p^Θ be the Hausdorff measure on Θ, $\mu_{\mathcal{X}}$ a measure on $\mathcal{X}$,
- p a denisty on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on $\mathcal{X}$.
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose p is G-invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G-invariant and continuous on Θ. For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \ldots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \to [0,1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)})\, dH_p^\Theta(\theta)$$

Suppose that $(P_{\Theta|X^{(n)}})_{n\in\mathbb{N}}$ concentrates on $\bar{\theta}_0 \subset \Theta$ a.s. or in probability. Then $(\bar{P}_{\Theta|X^{(n)}})_{n\in\mathbb{N}}$ concentrates a.s. or in probability on $\{\bar{\theta}_0\} \subset \Theta/G$ (i.e. is consistent a.s. or in probability)

Proof. Let $V \in \mathcal{N}_{\bar{\theta}_0}$. Then $\pi^{-1}(V) \in \mathcal{N}_{\bar{\theta}_0}$. By definition,

$$\begin{aligned}
\bar{P}_{\Theta|X^{(n)}}(V^c) &= P_{\Theta|X^{(n)}}(\pi^{-1}(V^c)) \\
&= P_{\Theta|X^{(n)}}(\pi^{-1}(V)^c) \\
&\xrightarrow{\text{a.s.}/P} 0
\end{aligned}$$

□

Note 8.1.3. Some examples of G-invariant priors would be the uniform distribution, or $N_n(0, \sigma^2 I)$ on $\mathbb{R}^n$ when acted on by $O(n)$. An example of a G-invariant likelihood would be $f(A|Z) \sim \mathrm{Ber}(ZZ^T)$ as in a latent position random graph model where $Z \in \mathbb{R}^{n\times d}$ is the parameter is invariant under right multiplication by $U \in O_d$.

9. Stochastic Integration

Exercise 9.0.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, X a set $\mathcal{A}_0$ an algebra, $\mu_0 : \mathcal{A}_0 \to \mathbb{C}$ and $B : \mathcal{A}_0 \to L^2(\Omega, \mathcal{F}, P)$. Suppose that

1. $B(\varnothing) = 0$
2. for each $E, F \in \mathcal{A}_0$, if $E \cap F = \varnothing$, then $B(E \cup F) = B(E) + B(F)$
3. $\mathrm{E}[B(E)B(F)^*] = \mu_0(E \cap F)$

Then

1. for each $E \in \mathcal{A}_0$, $\mu_0(E) = \mathrm{E}[|B(E)|^2]$.
2. for each $E \in \mathcal{A}_0$, $0 \leq \mu_0(E) < \infty$
3. for each $E, F \in \mathcal{A}_0$, if $E \cap F = \varnothing$, then $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$

Proof.

1. Clear
2. Clear
3. Let $E, F \in \mathcal{A}_0$. Suppose that $E \cap F = \varnothing$. Then
$$\begin{aligned} \mathrm{E}[B(E)B(F)^*] &= \mu_0(E \cap F) \\ &= \mu_0(\varnothing) \\ &= \mathrm{E}[|B(\varnothing)|^2] \\ &= \mathrm{E}[0] \\ &= 0 \end{aligned}$$
This implies that
$$\begin{aligned} \mu_0(E \cup F) &= \mathrm{E}[|B(E \cup F)|^2] \\ &= \mathrm{E}[|B(E) + B(F)|^2] \\ &= \mathrm{E}[|B(E)|^2] + \mathrm{E}[|B(F)|^2] + 2\mathrm{Re}\mathrm{E}[B(E)B(F)^*] \\ &= \mu_0(E) + \mu_0(F) + 0 \\ &= \mu_0(E) + \mu_0(F) \end{aligned}$$

□

Definition 9.0.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, X a set $\mathcal{A}_0$ an algebra, $\mu_0 : \mathcal{A}_0 \to [0, \infty)$ a premeasure and $B : \mathcal{A}_0 \to L^2(\Omega, \mathcal{F}, P)$. Suppose that

1. $B(\varnothing) = 0$
2. for each $E, F \in \mathcal{A}_0$, if $E \cap F = \varnothing$, then $B(E \cup F) = B(E) + B(F)$
3. $\mathrm{E}[B(E)B(F)^*] = \mu_0(E \cap F)$

Then B is said to be a **stochastic premeasure with sturcture** μ_0

References

[1] Introduction to Analysis
[2] Introduction to Group Theory
[3] Introduction to Measure and Integration