1

Introduction to Dynamical Systems

Carson James

Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Basic Concepts

1.1 Measure Preserving Transformations

Definition 1.1.0.1. We define **Meas** by

- $\text{Obj}(\mathbf{Meas}) := \{(X, \mathcal{A}) : (X, \mathcal{A}) \text{ is a measurable space}\}$.
- for $(X, \mathcal{A}), (Y, \mathcal{B}) \in \text{Obj}(\mathbf{Meas})$,
$$\text{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B})) := \{f : X \to Y : f \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable}\}$$
- for $(X, \mathcal{A}), (Y, \mathcal{B}), (Z, \mathcal{C}) \in \text{Obj}(\mathbf{Meas})$, $f \in \text{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B}))$ and $g \in \text{Hom}_{\mathbf{Meas}}((Y, \mathcal{B}), (Z, \mathcal{C}))$,
$$g \circ_{\mathbf{Meas}} f := g \circ_{\mathbf{Set}} f$$

Exercise 1.1.0.2. We have that **Meas** is a category.

Proof. □

Exercise 1.1.0.3. We have that **Meas** is a Cartesian monoidal category.

Definition 1.1.0.4. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be probability spaces and $f \in \text{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B}))$. Then T is said to be **measure preserving** if $f_*\mu = \nu$.

Exercise 1.1.0.5. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be probability spaces and $f \in \text{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B}))$. Then f is measure preserving iff for each $\phi \in L^1(Y, \mathcal{B}, \nu)$, $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and

$$\int_Y \phi \, d\nu = \int_X \phi \circ f \, d\mu$$

Proof.

- ($\Longrightarrow$):
Suppose that f is measure preserving. $\phi \in L^1(Y, \mathcal{B}, \nu)$. Then the a basic result on the change of variables implies that $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and
$$\begin{aligned}\int_Y \phi \, d\nu &= \int_Y \phi d \, f_* mu \\ &= \int_X \phi \, d\mu\end{aligned}$$

- ($\Longleftarrow$):
  Suppose that for each $\phi \in L^1(Y, \mathcal{B}, \nu)$, $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and

$$\int_Y \phi \, d\nu = \int_X \phi \circ f \, d\mu$$

  Let $B \in \mathcal{B}$. Since ν is a probability measure, $\chi_B \in L^1(Y, \mathcal{B}, \nu)$. Thus

$$\begin{aligned}\nu(B) &= \int_Y \chi_B \, d\nu \\ &= \int_X \chi_B \circ f \, d\mu \\ &= \int_X \chi_{f^{-1}(B)} \, d\mu \\ &= \mu(f^{-1}(B)) \\ &= f_*\mu(B)\end{aligned}$$

  Since $B \in \mathcal{B}$ is arbitrary, $f_*\mu = \nu$.

□

Definition 1.1.0.6. We define **Prob** by

- $\text{Obj}(\textbf{Prob}) = \{(X, \mathcal{A}, \mu) : (X, \mathcal{A}, \mu) \text{ is a probability space}\}$.
- for $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu) \in \text{Obj}(\textbf{Prob})$,

$$\text{Hom}_{\textbf{Prob}}((X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)) = \{f \in \text{Hom}_{\textbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B})) : f \text{ is measure preserving}\}$$

- for $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu), (Z, \mathcal{C}, \lambda) \in \text{Obj}(\textbf{Prob})$, $f \in \text{Hom}_{\textbf{Prob}}((X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu))$ and $g \in \text{Hom}_{\textbf{Prob}}((Y, \mathcal{B}, \nu), (Z, \mathcal{C}, \lambda))$,

$$g \circ_{\textbf{Prob}} f := g \circ_{\textbf{Set}} f$$

Exercise 1.1.0.7. We have that **Prob** is a category.

Proof. □

Exercise 1.1.0.8. We have that **Prob** is not a Cartesian monoidal category.

Proof. content... □

Even though **Prob** does not have products, when applying the forgetful functor $U : \textbf{Prob} \to \textbf{Meas}$, we get a category with products **Meas**, so in some sense, an object in **Meas** is an equivalence class of objects in **Prob** where we ignore our notions of size/interaction of sub-objects. After applying the U to a potential product $(Z, \mathcal{C}, \lambda) \in \text{Obj}(\textbf{Prob})$ (i.e. there are associated measure preserving maps $f_X : Z \to X$ and $f_Y : Z \to Y$) to get $(Z, \mathcal{C}) \in \text{Obj}(\textbf{Meas})$, then $(Z, \mathcal{C}) \in \text{Obj}(\textbf{Meas})$ is a potential product with the same associated maps and we get the unique map $h : Z \to X \times Y$ in **Meas** yielding the typical commutative diagram for products in **Meas** (i.e. $h = f_X, f_Y$). In general h does not preserve measure unless λ can be written as a tensor product. We can quantify how far off a potential product $(Z, \mathcal{C}, \lambda) \in \text{Obj}(\textbf{Prob})$ (i.e. an element of the equivalence class) is from being a product by looking at the information loss (relative entropy) across h

1.2 Measure Preserving Systems

Definition 1.2.0.1. Let $(X, \mathcal{A}) \in \mathrm{Obj}(\mathbf{Meas})$, $f \in \mathrm{End}_{\mathbf{Meas}}(X, \mathcal{A})$ and $\mu \in \mathcal{M}(X, \mathcal{A})$. Then μ is said to be f**-invariant** if $f_*\mu = \mu$.

Exercise 1.2.0.2. Let X be a compact metric space and $f \in \mathrm{End}_{\mathbf{Top}}(X)$. Then there exists $\mu \in \mathcal{P}(X, \mathcal{A})$ such that μ is f-invariant.
Hint:

Proof. □

Definition 1.2.0.3. Let $(X, \mathcal{A}, \mu) \in \mathbf{Prob}$ and $f \in \mathrm{End}_{\mathbf{Prob}}(X, \mathcal{A}, \mu)$. Then $(X, \mathcal{A}, \mu, f)$ is said to be a **measure-preserving dynamical system**.

Exercise 1.2.0.4.

Appendix A

App

A.1 Reading Diagrams and associated digraphs of diagrams

Definition A.1.0.1. Let

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "h"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \implies \begin{tikzcd} C \arrow[rr, bend left, "g"] \arrow[rr, bend right, "h"'] & & A \end{tikzcd}$$

see an intro to the language of category theory by roman for description

Definition A.1.0.2. A diagram is said to be **commutative** if for each path of length ≥ 2, in the associated digraph gives the same morphism.