DIFFERENTIAL GEOMETRY

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Contents

1. Review of Basic Definitions and Results

1.1. Set Theory.

Definition 1.1.1. *Let $\{A_i\}_{i\in I}$ be a collection of sets. The* ***disjoint union of*** $\{A_i\}_{i\in I}$*, denoted* $\coprod_{i\in I} A_i$*, is defined by*

$$\coprod_{i\in I} A_i = \bigcup_{i\in I} \{i\} \times A_i$$

Define the ***projection map*** $\pi : \coprod_{i\in I} A_i \to I$ *by*

$$\pi(i, A_i) = i$$

Definition 1.1.2. *Let A, B be sets and $\pi : A \to B$ and $\sigma : B \to A$. Then σ is said to be a* ***section of*** π *if* $\pi \circ \sigma = \mathrm{id}_B$.

Exercise 1.1.3. *Let $\{A_i\}_{i\in I}$ be a collection of sets, $\pi : \coprod_{i\in I} A_i \to I$ the projection map and $\sigma : I \to \coprod_{i\in I} A_i$. If σ is a section of π, then for each $i \in I$, $\sigma(i) \in \{i\} \times A_i$.*

Proof. Suppose that σ is a section of π. Let $i \in I$. Then there exists $j \in I$ and $a \in A_j$ such that $\sigma(i) = (j, a)$. Since $\pi \circ \sigma = \mathrm{id}_I$, we have that

$$\begin{aligned} i &= \pi \circ \sigma(i) \\ &= \pi(j, a) \\ &= j \end{aligned}$$

So $i = j$ and $\sigma(i) = (i, a) \in \{i\} \times A_i$ $\square$

1.2. **Differentiation.**

Definition 1.2.1. *Let $n \geq 1$. For $i = 1, \cdots, n$, define $x_i : \mathbb{R}^n \to \mathbb{R}$ by $x_i(a_1, \cdots, a_n) = a_i$. The functions $(x_i)_{i=1}^n$ are called the* ***standard coordinate functions on*** *$\mathbb{R}^n$.*

Definition 1.2.2. *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Thenf is said to be* ***differentiable with respect to*** *x_i* ***at*** *a if*

$$\lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}$$

exists. If f is differentiable with respect to x_i at a, we define the ***partial derivative of*** *f* ***with respect to*** *x_i* ***at*** *a, denoted*

$$\frac{\partial f}{\partial x_i}(a), \ \ \frac{\partial}{\partial x_i}\bigg|_a f, \ \partial_{x_i} f(a) \ \text{or} \ \partial_{x_i}\bigg|_a f$$

to be the limit above.

Definition 1.2.3. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be* ***differentiable with respect to*** *x_i if for each $a \in U$, f is differentiable with respect to x_i at a.*

Exercise 1.2.4. *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x_i x_j}$ and $\frac{\partial^2 f}{\partial x_j x_i}$ exist and are continuous at a. Then*

$$\frac{\partial^2 f}{\partial x_i x_j}(a) = \frac{\partial^2 f}{\partial x_j x_i}(a)$$

Proof. □

Definition 1.2.5. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be* ***smooth*** *if for each $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $\frac{\partial^k f}{\partial i_1 \cdots i_k}$ exists and is continuous on U.*

Definition 1.2.6. *Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$. Then f is said to be* ***smooth*** *if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \to \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.*

Definition 1.2.7. *Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$. Then F is said to be a* ***diffeomorphism*** *if F is a homeomorphism and F, F^{-1} are smooth.*

Exercise 1.2.8. *Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \to F(N)$ is a diffeomorphism*

Proof. content... □

Definition 1.2.9. *Let $U \subset \mathbb{R}^n$ and $p \in U$. Then U is said to be* ***star-shaped*** *if for each $q \in U$, $\{p + t(q - p) : 0 \leq t \leq 1\} \subset U$.*

Theorem 1.2.10. ***(Taylor's Theorem)*** *Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $f \in C^\infty(U)$. Suppose that U is star-shaped with respect to p. Then there exist $g_1, \cdots, g_n \in C^\infty(U)$ such that for each $x \in U$,*

$$f(x) = f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x)$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial f}{\partial x_i}(p)$$

Proof. Since U is star-shaped, $\{p + t(x - p) : 0 \leq t \leq 1\} \subset U$. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(p + t(x - p)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p + t(x - p))(x_i - p_i)$$

Integrating both sides with respect to t from 0 to 1, we obtain

$$f(x) - f(p) = \sum_{i=1}^{n}(x_i - p_i) \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p))dt$$

For $i \in \{1, \cdots, n\}$, define $g_i \in C^\infty(U)$ by

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p))dt$$

Then for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial f}{\partial x_i}(p))$$

□

2. Multilinear Algebra

Note 2.0.1. *For the remainder of this section we let V denote an n-dimensional vector space with basis $\{e_1, \cdots, e_n\}$ with dual space V^* and dual basis $\{\epsilon_1, \cdots, \epsilon_n\}$ defined by $\epsilon_i(e_j) = \delta_{i,j}$.*

2.1. k-Tensors.

Definition 2.1.1. *Let $\alpha : V^k \to \mathbb{R}$. Then α is said to be **multilinear** or a **k-tensor on V** if for $i \in \{1, \cdots, k\}$, $w \in V$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$,*

$$\alpha(v_1, \cdots, v_i + cw, \cdots, v_k) = \alpha(v_1, \cdots, v_i, \cdots, v_k) + c\alpha(v_1, \cdots, w, \cdots, v_k)$$

The set of all k-tensors on V is denoted by $T_k(V)$. Define $L_0(V) = \mathbb{R}$.

Exercise 2.1.2. *We have that $T_k(V)$ is a vector space.*

Proof. Clear. □

Definition 2.1.3. *For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \to \mathbb{R}$ by*

$$\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

*The map $\alpha \mapsto \sigma\alpha$ is called the **permutation action** of S_k on $T_k(V)$*

Exercise 2.1.4. *The permutation action of S_k on $T_k(V)$ is a group action.*

Proof.

(1) Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V), \sigma\alpha \in T_k(V)$.
(2) Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
(3) Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \cdots, v_k \in V$,

$$\begin{aligned}(\tau\sigma)\alpha(v_1, \cdots, v_k) &= \alpha(v_{\tau\sigma(1)}, \cdots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \cdots, v_k)\end{aligned}$$

□

Exercise 2.1.5. *Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \to T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.*

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$. Then

$$\begin{aligned}\sigma(c\alpha + \beta)(v_1, \cdots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \cdots, v_k) + \sigma\beta(v_1, \cdots, v_k)\end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

Definition 2.1.6. *Let $\alpha \in T_k(V)$. Then α is said to be **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$. and α is said to be **alternating** if for each $\sigma \in S_k$, $\sigma\alpha = \mathrm{sgn}(\sigma)\alpha$. The set of symmetric k-tensors on V is denoted $\Xi_k(V)$ and the set of alternating k-tensors on V is denoted $\Lambda_k(V)$.*

Definition 2.1.7. *Define the **symmetric operator** $S : T_k(V) \to \Xi_k(V)$ by*

$$S(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the ***alternating operator*** $A : T_k(V) \to \Lambda_k(V)$ *by*

$$A(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha$$

Exercise 2.1.8.

(1) *For* $\alpha \in T_k(V)$*,* $S(\alpha)$ *is symmetric.*
(2) *For* $\alpha \in T_k(V)$*,* $A(\alpha)$ *is alternating.*

Proof.

(1) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}\sigma S(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \tau\alpha\right] \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \tau\alpha \\ &= S(\alpha)\end{aligned}$$

(2) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}\sigma A(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right] \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\sigma\tau\alpha \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\ &= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\ &= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \\ &= \operatorname{sgn}(\sigma)A(\alpha)\end{aligned}$$

□

Exercise 2.1.9.

(1) *For* $\alpha \in \Xi_k(V)$*,* $S(\alpha) = \alpha$*.*
(2) *For* $\alpha \in \Lambda_k(V)$*,* $A(\alpha) = \alpha$*.*

Proof.

(1) Let $\alpha \in \Xi_k(V)$. Then

$$\begin{aligned} S(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha \\ &= \frac{1}{k!}\sum_{\sigma \in S_k} \alpha \\ &= \alpha \end{aligned}$$

(2) Let $\alpha \in \Lambda_k(V)$. Then

$$\begin{aligned} A(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha \\ &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2\alpha \\ &= \alpha \end{aligned}$$

□

Exercise 2.1.10. *The symmetric operator $S : T_k(V) \to \Xi_k(V)$ and the alternating operator $A : T_k(V) \to \Lambda_k(V)$ are linear.*

Proof. Clear. □

Definition 2.1.11. *Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. The **tensor product** of α and β is defined to be the map $\alpha \otimes \beta \in T_{k+l}(V)$ given by*

$$\alpha \otimes \beta(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = \alpha(v_1, \cdots, v_k)\beta(v_{k+1}, \cdots, v_{k+l})$$

Thus $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$.

Exercise 2.1.12. *The tensor product $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$ is associative.*

Proof. Clear. □

Exercise 2.1.13. *The tensor product $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$ is bilinear.*

Proof. Clear. □

Definition 2.1.14. *Let $\alpha \in \Lambda_k(V)$ and $\beta \in \Lambda_l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda_{k+l}(V)$ given by*

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!}A(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to \Lambda_{k+l}(V)$.

Exercise 2.1.15. *The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to T_{k+l}(V)$ is bilinear.*

Proof. Clear. □

Exercise 2.1.16. *Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then*

(1) $A(A(\alpha) \otimes \beta) = A(\alpha \otimes \beta)$
(2) $A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \to S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

(1) Then

$$\begin{aligned}
A(A(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \left[A(\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau)(\tau\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau(\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau(\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \operatorname{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \operatorname{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= A(\alpha \otimes \beta)
\end{aligned}$$

(2) Similar to (1).

$\square$

Exercise 2.1.17. *The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to \Lambda_{k+l}(V)$ is associative.*

Proof. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$ and $\gamma \in \Lambda_m(V)$. Then

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!}A(\alpha \otimes \beta)\right] \wedge \gamma \\
&= \frac{(k+l+m)!}{(k+l)!m!}A\left(\left[\frac{(k+l)!}{k!l!}A(\alpha \otimes \beta)\right] \otimes \gamma\right) \\
&= \frac{(k+l+m)!}{(k+l)!m!}\frac{(k+l)!}{k!l!}A(A(\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{m!}\frac{1}{k!l!}A((\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{k!(l+m)!}\frac{(l+m)!}{l!m!}A(\alpha \otimes (\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}\frac{(l+m)!}{l!m!}A(\alpha \otimes A(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}A(\alpha \otimes \frac{(l+m)!}{l!m!}A(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}A(\alpha \otimes (\beta \wedge \gamma)) \\
&= \alpha \wedge (\beta \wedge \gamma))
\end{aligned}$$

□

Exercise 2.1.18. *Let $\alpha_i \in \Lambda_{k_i}(V)$ for $i = 1, \cdots, m$. Then*

$$\bigwedge_{i=1}^{m} \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!}A\left(\bigotimes_{i=1}^{m} \alpha_i\right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1+k_2+k_3)!}{k_1!k_2!k_3!}A(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i\right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A\left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A\left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} A\left(\bigotimes_{i=1}^{m_0-1} \alpha_i\right)\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(A\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(\bigotimes_{i=1}^{m_0+1} \alpha_i\right)
\end{aligned}$$

□

Exercise 2.1.19. *Define $\tau \in S_{k+l}$ by*

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl. (Hint: inversion number)

Proof.

$$\begin{aligned}
N(\tau) &= \sum_{i=1}^{l} k \\
&= kl
\end{aligned}$$

Since $\operatorname{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\operatorname{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.1.20. *Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$. Then*

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Note that For $\sigma \in S_{k+l}$ and $v_1, \cdots, v_{k+l} \in V$, we have that

$$\begin{aligned}
\sigma\tau(\beta \otimes \alpha)(v_1, \cdots, v_l, v_{l+1}, \cdots v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}) \alpha(v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \\
&= \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\
&= \alpha \otimes \beta(v_{\sigma(1)}, \cdots v_{\sigma(k)}, v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\
&= \sigma(\alpha \otimes \beta)(v_1, \cdots, v_k, v_{1+k}, \cdots v_{l+k})
\end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Define τ as in the previous exercise. Then

$$\begin{aligned}
\beta \wedge \alpha &= \frac{(k+l)!}{k!l!} A(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma\tau)\sigma\tau(\beta \otimes \alpha) \\
&= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha \otimes \beta) \\
&= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \\
&= \operatorname{sgn}(\tau) \alpha \wedge \beta \\
&= (-1)^{kl} \alpha \wedge \beta
\end{aligned}$$

□

Exercise 2.1.21. *Let* $\alpha_{\in}\Lambda_k(V)$. *If* k *is odd, then* $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned}
\alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\
&= -\alpha \wedge \alpha
\end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.1.22. ***(Fundamental Example)*** *Let* $\alpha_1, \cdots, \alpha_m \in \Lambda_1(V)$ *and* $v_1, \cdots, v_m \in V$. *Then*

$$\left(\bigwedge_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}
\left(\bigwedge_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) &= m! A\left(\bigotimes_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) \\
&= m! \left[\frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\sigma\left(\bigotimes_{i=1}^m \alpha_i \right) \right](v_1, \cdots, v_m) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \left(\bigotimes_{i=1}^m \alpha_i \right)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
&= \det(\alpha_i(v_j))
\end{aligned}$$

□

Definition 2.1.23. *Define* $\mathcal{I}_k = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$. *Each element* $I \in \mathcal{I}_k$ *is called a* ***multi-index***. *Recall that* $\#\mathcal{I}_k = \binom{n}{k}$.

Definition 2.1.24. *Let* $I = \{(i_1, i_2, \cdots, i_k) \in I_k$.
Define $e_I \in V^k$ *by*

$$e_I = (e_{i_1}, \cdots, e_{i_k})$$

Define $\epsilon_I \in \Lambda_k(V)$ *by*

$$\epsilon_I = \epsilon_{i_1} \wedge \cdots, \wedge \epsilon_{i_k}$$

Exercise 2.1.25. *Let* $I = (i_1, \cdots, i_k)$ *and* $J = (j_1, \cdots, j_k) \in \mathcal{I}_k$. *Then* $\epsilon_I(e_J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon_{i_1}(e_{j_1}) & \cdots & \epsilon_{i_1}(e_{j_k}) \\ & \vdots & \\ \epsilon_{i_k}(e_{j_1}) & \cdots & \epsilon_{i_k}(e_{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon_I(e_J) = \det A$. If $I = J$, then $A = I_{k\times k}$ and therefore $\epsilon_I(e_J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the $l_0 th$ row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the $l_0 th$ column of A are 0. □

Exercise 2.1.26. *Let* $\alpha, \beta \in \Lambda_k(V)$. *If for each* $I \in \mathcal{I}_k$, $\alpha(e_I) = \beta(e_I)$, *then* $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e_I) = \beta(e_I)$. Let $v_1, \cdots, v_k \in V$. For $i = 1, \cdots, k$, write $v_i = \sum_{j_i=1}^n a_{i,j_i} e_{j_i}$. Then

$$\begin{aligned}
\alpha(v_1, \cdots, v_k) &= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \alpha(e_{j_1}, \cdots, e_{j_k}) \\
&= \sum_{j_1 \neq \cdots \neq j_k} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \alpha(e_{j_1}, \cdots, e_{j_k}) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{k} a_{i,\sigma(j_i)}\right)\right] \alpha(e_J) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{k} a_{i,\sigma(j_i)}\right)\right] \beta(e_J) \\
&= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \beta(e_{j_1}, \cdots, e_{j_k}) \\
&= \beta(v_1, \cdots, v_k)
\end{aligned}$$

□

Exercise 2.1.27. *The set* $\{\epsilon_I : I \in \mathcal{I}_k\}$ *is a basis for* $\Lambda_k(V)$ *and* $\dim \Lambda_k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I\in\mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I\in\mathcal{I}_k} a_I \epsilon_I$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e_J) = a_J = 0$. Thus $\{e_I : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda_k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e_I)$. define $\mu = \sum_{I\in\mathcal{I}_k} b_I \epsilon_I$. Then for each $J \in \mathcal{I}_k$, $\mu(e_J) = b_J = \beta(e_J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{\epsilon_I : I \in \mathcal{I}_k\}$. □

2.2. (r, s)-Tensors.

3. Manifolds

3.1. Smooth Manifolds.

Definition 3.1.1. *Define the **upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}_n$, by*

$$\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\partial\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$$
$$(\mathbb{H}^n)^\circ = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

Definition 3.1.2. *Let M be a topological space.*

1. *Let $n \geq 1$, $U \subset M$, $V \subset \mathbb{H}^n$ open and $\phi : U \to V$. Then (U, ϕ) is said to be a **coordinate chart** on M if ϕ is a homeomorphism.*
2. *Let $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$ be a collection of coordinate charts on M. Then $\mathcal{A}$ is said to be an **atlas** on M if $\bigcup\limits_{a \in A} U_a = M$.*
3. *Let $n \geq 1$. Then M is said to be **locally half Euclidean of dimension** n if there exists an atlas $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$ on M such that for each $a \in A$, $\phi_a(U_a) \subset \mathbb{H}^n$.*
4. *The space M is said to be an n**-dimensional manifold** if M is Hausdorff, second countable and locally half Euclidean of dimension n.*

Note 3.1.3. *For the remainder of this section, we assume M is an n-dimensional manifold.*

Definition 3.1.4.

1. *Define the **boundary** of M, denoted ∂M, by*

   $$\partial M = \{p \in M : \text{ there exists a chart } (U, \phi) \text{ on } M \text{ such that } p \in U \text{ and } \phi(p) \in \partial\mathbb{H}^n\}$$
2. *Define the **interior** of M, denoted M°, by*

   $$M^\circ = M \setminus \partial M$$

Exercise 3.1.5. *Let $p \in M$. Then $p \in \partial M$ iff for each chart (U, ϕ) on M, $p \in U$ implies that $\phi(p) \in \partial\mathbb{H}^n$. (Hint: simply connected)*

Proof. Supposet that $p \in \partial M$. Then there exists a coordinate chart (V, ψ) on M such that $\psi(p) \in \partial\mathbb{H}^n$. Let (U, ϕ) be a coordinate chart on M. Suppose that $p \in U$. Note that $\phi \circ \psi : \psi(V \cap U) \to \phi(V \cap U)$ is a homeomorphism. Choose open n-balls B_ϕ, $B_\psi \subset \mathbb{H}^n$ such that $B_\phi \subset \phi(V \cap U)$, $B_\psi \subset \psi(V \cap U)$, $\phi(p) \in B_\phi$ and $\psi(p) \in B_\psi$. For the sake of contradiction, suppose that $\phi(p) \notin \partial\mathbb{H}^n$. Put $U' = B_\phi \setminus \{\phi(p)\}$ and $V' = B_\psi \setminus \{\psi(p)\}$. Define $\lambda : V' \to U'$ by $\lambda = \phi \circ \psi|_{B_\psi}$. Then λ is a homeomorphism. Note that V' is simply connected and U' is not. This is a contradiction. □

Exercise 3.1.6. *If $\partial M \neq \varnothing$, then*

1. *∂M is an $n-1$-dimensional manifold*
2. *$\partial(\partial M) = \varnothing$.*

Proof. (1) Since subspaces of Hausdorff, second countable spaces are Hausdorff and second countable, we need only show that ∂M is locally half euclidean of dimension $n-1$. Let $p \in \partial M$. Then there exists a coordinate chart (U, ϕ) on M such that $p \in U$ and $\phi(p) \in \partial\mathbb{H}^n$.

Put $U' = U \cap \partial M$. Note that U' is open in ∂M and $\phi(U) \cap \partial\mathbb{H}^n$ is open in $\partial\mathbb{H}^n$.

Define $\phi' : U' \to \phi(U) \cap \partial\mathbb{H}^n$ by $\phi' = \phi|_{U'}$. Then ϕ' is a homeomorphism.
Since $\partial\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n-1}$ which is homeomorphic to $(\mathbb{H}^{n-1})^\circ$ there exists $\psi : \partial\mathbb{H}^n \to (\mathbb{H}^{n-1})^\circ$ such that ψ is a homeomorphism.
Define $V' = \psi(\phi(U) \cap \partial\mathbb{H}^n)$ and $\psi' : \phi(U) \cap \partial\mathbb{H}^n \to V'$ by and $\psi' = \psi|_{\phi(U)\cap\partial\mathbb{H}^n}$. Then V' is open in $(\mathbb{H}^{n-1})^\circ$ and ψ' is a homeomrophism.
Define $\lambda : U' \to V'$ by $\lambda = \psi' \circ \phi'$. Then λ is a homeomorhism and (U', λ) is a cooridnate chart on ∂M. So ∂M is locally Euclidean of dimension $n-1$.

(2) Let $p \in \partial M$. Define $(U \cap \partial M, \lambda \circ \psi)$ as in (1). Since $\lambda \circ \psi(p) \in (\mathbb{H}^{n-1})^\circ$, we have that $p \in M^\circ$. Thus $\partial M = (\partial M)^\circ$ and $\partial(\partial M) = \varnothing$.

□

Definition 3.1.7.

(1) *Let $(U, \phi), (V, \psi)$ be coordinate charts on M. Then (U, ϕ) and (V, ψ) are said to be* ***smoothly compatible*** *if*

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \text{ is a diffeomorphism}$$

(2) *Let $\mathcal{A} = ((U_a, \phi_a))_{a\in A}$ be an atlas on M. Then $\mathcal{A}$ is said to be* ***smooth*** *if for each $a, b \in A$, (U_a, ϕ_a) and (U_b, ϕ_b) are smoothly compatible.*

(3) *Let $\mathcal{A}$ be a smooth atlas on M. Then $\mathcal{A}$ is said to be* ***maximal*** *if for each smooth atlas $\mathcal{B}$ on M, $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a* ***smooth structure on*** *M.*

(4) *Let $\mathcal{A}$ be a smooth structure on M. Then $(M, \mathcal{A})$ is said to be a* ***smooth n-dimensional manifold****.*

Exercise 3.1.8. *Let $\mathcal{B}$ be a smooth atlas on M. Then there exists a unique smooth structure $\mathcal{A}$ on M such that $\mathcal{B} \subset \mathcal{A}$.*

Proof. Define $\mathcal{A}$ to be the set of all coordinate charts (U, ϕ) on M such that for each coordinate chart $(V, \psi) \in \mathcal{B}$, (U, ϕ) and (V, ψ) are smoothly compatible.
Clearly $\mathcal{B} \subset \mathcal{A}$.
Let $(U, \phi), (V, \psi) \in \mathcal{A}$ and $p \in U \cap V$. Then there exists $(W, \chi) \in \mathcal{B}$ such that $p \in W$. By assumption, $\phi \circ \chi^{-1} : \chi(U \cap W) \to \phi(U \cap W)$ and $\chi \circ \psi^{-1} : \psi(W \cap V) \to \chi(W \cap V)$ are diffeomorphisms. Then $(\phi\circ\chi^{-1})\circ(\chi\circ\psi^{-1}) = \phi\circ\psi^{-1} : \psi(U\cap W\cap V) \to \phi(U\cap W\cap V)$ is a diffeomorphism. Since for each $q \in \psi(U \cap V)$, there exits an open neighborhood $N \subset \psi(U \cap V)$ of q on which $\phi\circ\psi^{-1}$ are diffeomorphic, we have that $\phi\circ\psi^{-1}$ is a diffeomorphism on $\psi(U\cap V)$ and therefore (U, ϕ) and (V, ψ) are smoothly compatible. Hence $\mathcal{A}$ is a smooth atlas.
To see that $\mathcal{A}$ is maximal, let $\mathcal{B}'$ be a smooth atlas on M. Suppose that $\mathcal{A} \subset \mathcal{B}'$ and let $(U, \phi) \in \mathcal{B}'$. By definition, for each chart $(V, \psi) \in \mathcal{B}'$, (U, ϕ) and (V, ψ) are smoothly compatible. Since $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$, we have that $(U, \phi) \in \mathcal{A}$. So $\mathcal{A} = \mathcal{B}'$ and $\mathcal{A}$ is a maximal smooth atlas on M. □

Exercise 3.1.9. *Let $\mathcal{A}$ be a smooth atlas on M. Define $\lambda : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ by $\lambda(x_1, \cdots, x_{n-1,0}) = (x_1, \cdots, x_{n-1})$. Put $\mathcal{A}|_{\partial M} = \{(U \cap \partial M, \lambda \circ \phi_{U\cap\partial M}) : (U, \phi) \in \mathcal{A}\}$. Then*

(1) *$\mathcal{A}|_{\partial M}$ is a smooth atlas on ∂M.*
(2) *if $\mathcal{A}$ is maximal, then $\mathcal{A}|_{\partial M}$ is maximal.*

Proof. □

Note 3.1.10. *For the rest of this section, we assume that $(M, \mathcal{A})$ is a smooth n-dimensional manifold and we denote the standard coordinate functions on $\mathbb{R}^n$ by $u_1, \cdots, u_n$. For a coordinate chart $(U, \phi) \in \mathcal{A}$ and $i \in \{1, \cdots, n\}$, we will typically denote the ith coordinate of ϕ by x_i, that is, $x_i = u_i(\phi)$.*

Definition 3.1.11. *Let $f : M \to \mathbb{R}$. Then f is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M)$.*

Exercise 3.1.12. *We have that $C^\infty(M)$ is a vector space.*

Proof. Clear. □

Definition 3.1.13. *Let $p \in M$. Define the relation $\sim_p$ on $C^\infty(M)$ by $f \sim_p g$ iff there exists an open $U \subset M$ such that $f|_U = g|_U$. Clearly $\sim_p$ is an equivalence relation on $C^\infty(M)$. We denote $C^\infty(M)/\sim_p$ by $C^\infty_p(M)$. For $f \in C^\infty(M)$, we define the* ***germ of*** *f* ***at*** *p to be the equivalence class of f under $\sim_p$.*

Exercise 3.1.14. *Let $p \in$We have that $C^\infty_p(M)$ is a vector space.*

Proof. Clear. □

Definition 3.1.15. *Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$, $p \in U$ and $f \in C^\infty_p(M)$. For $i \in \{1, \cdots, n\}$, define the partial derivative of f with respect to x_i at p, denoted*

$$\frac{\partial f}{\partial x_i}(p), \quad \frac{\partial}{\partial x_i}\bigg|_p f, \ \partial_{x_i} f(p) \ \text{or} \ \partial_{x_i}\bigg|_p f$$

by

$$\frac{\partial}{\partial x_i}\bigg|_p f = \frac{\partial}{\partial u_i}\bigg|_{\phi(p)} f \circ \phi^{-1}$$

Exercise 3.1.16. *Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$, we have that*

$$\frac{\partial x_i}{\partial x_j}(p) = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \cdots, n\}$. Then

$$\begin{aligned} \frac{\partial}{\partial x_j}\bigg|_p x_i &= \frac{\partial}{\partial u_j}\bigg|_{\phi(p)} x_i \circ \phi^{-1} \\ &= \frac{\partial}{\partial u_j}\bigg|_{\phi(p)} u_i \circ \phi \circ \phi^{-1} \\ &= \frac{\partial}{\partial u_j}\bigg|_{\phi(p)} u_i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 3.1.17. ***(Taylor's Theorem)*** *Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$, $p \in U$ and $f \in C^\infty_p(M)$. Then there exist $g_1, \cdots g_n \in C^\infty_p(M)$ such that*

$$f = f(p) + \sum_{i=1}^{n} (x_i - x_i(p)) g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x_i}\bigg|_p f$$

Proof. Since we are interested in the germ of f at p, we may assume that $\phi(U)$ is star-shaped with respect to $\phi(p)$. Let $q \in U$. From Taylor's theorem in section 1, we know that there exist $g'_1, \cdots, g'_n \in C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = f \circ \phi^{-1}(\phi(p)) + \sum_{i=1}^{n} [u_i \circ \phi(q) - u_i \circ \phi(p)] g'_i(\phi(q))$$

and for each $i \in \{1, \cdots, n\}$,

$$g'_i(\phi(p)) = \frac{\partial}{\partial u_i}\bigg|_{\phi(p)} f \circ \phi^{-1}$$

For each $i \in \{1, \cdots, n\}$, define $g_i = g'_i \circ \phi$. Then for each $q \in U$,

$$f(q) = f(p) + \sum_{i=1}^{n} [x_i(q) - x_i(p)] g_i(q)$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x_i}\bigg|_p f$$

$\square$

Definition 3.1.18. *Let $D : C_p^\infty(M) \to \mathbb{R}$ and $p \in M$. Then D is said to be a **derivation at** p if for each $f, g \in C_p^\infty(M)$ and $a \in \mathbb{R}$,*

1. *$D(f + cg) = D(f) + cD(g)$ (D is linear)*
2. *$D(fg) = D(f)g(p) + f(p)D(g)$ (D is Leibnizian)*

Definition 3.1.19. *Let $p \in M$. The set of derivations at p, denoted T_pM is called the **tangent space of** M **at** p.*

Exercise 3.1.20. *Let $f \in C_p^\infty(M)$ and $D \in T_pM)$. If f is constant, then $Df = 0$.*

Proof. Suppose that $f \equiv 1$. Then $f^2 = f$ and $D(f^2) = 2D(f)$. So $D(f) = 2D(f)$ which implies that $D(f) = 0$. If $f \not\equiv 1$, then there exists $c \in \mathbb{R}$ such that $f \equiv c$. Since D is linear, $D(f) = cD(1) = 0$. $\square$

Exercise 3.1.21. *Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$ and $p \in U$. Then*

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \cdots, \frac{\partial}{\partial x_n}\bigg|_p \right\}$$

is a basis for T_pM and $\dim T_pM = n$.

Proof. Clearly $\frac{\partial}{\partial x_1}\big|_p, \cdots, \frac{\partial}{\partial x_n}\big|_p \in T_pM$. Let $a_1, \cdots, a_n \in \mathbb{R}$. Suppose that

$$D = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_p = 0$$

Then

$$\begin{aligned} 0 &= Dx_j \\ &= \sum_{i=1}^n a_i \left.\frac{\partial}{\partial x_i}\right|_p x_j \\ &= a_j \end{aligned}$$

Hence $\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p \right\}$ is independent.
Now, let $D \in T_pM$ and $f \in \mathbb{C}_p^\infty(M)$. By Taylor's theorem, there exist $g_1, \cdots g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x_i - x_i(p))g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \left.\frac{\partial}{\partial x_i}\right|_p f$$

Then

$$\begin{aligned} D(f) &= \sum_{i=1}^n D(x_i - x_i(p))g_i(p) + \sum_{i=1}^n (x_i(p) - x_i(p))D(g_i) \\ &= \sum_{i=1}^n D(x_i)g_i(p) \\ &= \sum_{i=1}^n D(x_i) \left.\frac{\partial}{\partial x_i}\right|_p f \\ &= \left[\sum_{i=1}^n D(x_i) \left.\frac{\partial}{\partial x_i}\right|_p \right] f \end{aligned}$$

So

$$D = \sum_{i=1}^n D(x_i) \left.\frac{\partial}{\partial x_i}\right|_p$$

and

$$D \in \operatorname{span}\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p \right\}$$

□

Definition 3.1.22. *We define the* ***tangent bundle of*** M, *denoted* TM, *by*

$$TM = \bigcup_{p \in M} T_pM$$

and we define the ***cotangent bundle of*** M, *denoted* T^*M, *by*

$$T^*M = \bigcup_{p \in M} T_p^*M$$

Definition 3.1.23. *Let* $(U, \phi) \in \mathcal{A}$ *with* $\phi = (x_1, \cdots, x_n)$ *and* $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$. *For* $p \in U$*, define*

$$\frac{\partial}{\partial x_I}\bigg|_p = \left(\frac{\partial}{\partial x_{i_1}}\bigg|_p, \cdots, \frac{\partial}{\partial x_{i_k}}\bigg|_p \right)$$

and define

$$\frac{\partial}{\partial x_I} : U \to TM$$

by

$$p \mapsto \frac{\partial}{\partial x_I}\bigg|_p$$

Definition 3.1.24. *Let* $(N, \mathcal{B})$ *be a smooth manifold and* $F : M \to N$. *Then* F *is said to be* ***smooth*** *if for each* $(U, \phi) \in \mathcal{A}$ *and* $(V, \psi) \in \mathcal{B}$*,*

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(F(U) \cap V)$$

is smooth and F *is said to be a* ***diffeomorphism*** *if* F *is a homeomorphism and* F, F^{-1} *are smooth.*

Exercise 3.1.25. *Let* $(N, \mathcal{B})$ *be a smooth manifold and* $F : M \to N$ *a diffeomorphism. Then for each* $(U, \phi) \in \mathcal{A}$*,* $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

Proof. Let $(V, \psi) \in \mathcal{B}$. Since F is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \to \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \psi(V \cap F(U))$$

are smooth. Therefore $(F(U), \phi \circ F^{-1})$ and (V, ψ) are smoothly compatible. Since $\mathcal{B}$ is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. □

Definition 3.1.26. *Let* $(N, \mathcal{B})$ *be a smooth manifold,* $F : M \to N$ *a diffeomorphism and* $p \in M$. *We define the* ***push forward of*** F ***at*** p*, denoted* $(F_*)_p : T_pM \to T_{F(p)}N$*, by*

$$\left[(F_*)_p(D)\right](f) = D(f \circ F)$$

for $D \in T_pM$ *and* $f \in C^\infty_{F(p)}(N)$.

Exercise 3.1.27. *Let* $(N, \mathcal{B})$ *be a smooth manifold,* $F : M \to N$ *a diffeomorphism and* $p \in M$. *Then* $(F_*)_p$ *is an isomorphism.*

Proof. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x_!, \cdots, x_n)$ and $\phi \circ F^{-1} = (y_1, \cdots, y_n)$. Let $f \in C^\infty_{F(p)}(N)$ Then

$$\begin{aligned}
\frac{\partial}{\partial y_i}\bigg|_{F(p)} f &= \frac{\partial}{\partial u_i}\bigg|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\
&= \frac{\partial}{\partial u_i}\bigg|_{\phi(p)} f \circ F \circ \phi^{-1} \\
&= \frac{\partial}{\partial x_i}\bigg|_p f \circ F
\end{aligned}$$

Therefore

$$\left[(F_*)_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)\right](f) = \left.\frac{\partial}{\partial x_i}\right|_p f \circ F$$
$$= \left.\frac{\partial}{\partial y_i}\right|_{F(p)} f$$

Hence

$$(F_*)_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right) = \left.\frac{\partial}{\partial y_i}\right|_{F(p)}$$

Since $\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p \right\}$ is a basis for T_pM and $\left\{ \left.\frac{\partial}{\partial y_1}\right|_{F(p)}, \cdots, \left.\frac{\partial}{\partial y_n}\right|_{F(p)} \right\}$ is a basis for $T_{F(p)}N$, $(F_*)_p$ is an isomorphism. $\square$

Definition 3.1.28. *Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ a diffeomorphism. Define the **push forward of** F, denoted*

$$F_* : M \to \bigcup_{p \in M} \mathrm{Iso}(T_pM, T_{F(p)}N)$$

by

$$p \mapsto (F_*)_p$$

.

3.2. **Integration on Manifolds.**

Definition 3.2.1. *We define*

$$\Lambda_k(TM) = \bigcup_{p \in M} \Lambda_k(T_pM)$$

Definition 3.2.2. *Let* $(U, \phi) \in \mathcal{A}$ *with* $\phi = (x_1, \cdots, x_n)$. *Then we define* $dx_1, \cdots, dx_n : U \to \Lambda_1(TU)$, *by*

$$(dx_i)_p\bigg(\frac{\partial}{\partial x_j}\bigg|_p\bigg) = \delta_{i,j}$$

That is, for each $p \in U$, $\{(dx_1)_p, \cdots, (dx_n)_p\}$ *is the dual basis of* $\left\{\frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p\right\}$ *for* T_p^*M

Definition 3.2.3. *Let* $\omega : M \to \Lambda_k(TM)$. *Then* ω *is said to be a **k-form on** M if for each* $p \in M$, $\omega_p \in \Lambda_k(T_pM)$.
Let (U, ϕ) *be a coordinate chart on* M *with* $\phi = (x_1, \cdots, x_n)$. *Then* ω *is said to be a **smooth on** U if for each* $I \in \mathcal{I}_k$, *there exist* $f_I \in C^\infty(U)$ *such that*

$$\omega\bigg(\frac{\partial}{\partial x_I}\bigg) : U \to \mathbb{R}$$

is smooth.

$$\omega|_U = \sum_{I \in \mathcal{I}_k} f_I dx_I$$

where for $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$, $(dx_I)_p = (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p$.
If for each $(U, \phi) \in \mathcal{A}$, ω *is smooth on* U, *then* ω *is said to be a **differential k-form on** M. The set of differential k-forms on* M *is denoted* $\Psi_k(M)$.

Note 3.2.4. *Observe that* $\Omega_0(M) = C^\infty(M)$.

Definition 3.2.5. *Define the **exterior product***

$$\wedge : \Omega_k(M) \times \Omega_l(M) \to \Omega_{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

*Define the **permutation action of** S_k **on** $\Omega_k(M)$ by*

$$(\sigma\omega)_p = \sigma\omega_p$$

Note 3.2.6. *All of the results from multilinear algebra apply here.*

Definition 3.2.7. *Let* $(N, \mathcal{B})$ *be a smooth manifold and* $F : M \to N$ *be a diffeomorphism. Define the **pullback of** F, denoted* $F^* : \Omega_k(N) \to \Omega_k(M)$ *by*

$$(F^*\omega)_p(D_1, \cdots, D_k) = \omega_{F(p)}((F_*)_p(D_1), \cdots, (F_*)_p(D_k))$$

for $p \in M$ *and* $D_1, \cdots, D_k \in T_pM$

.

Definition 3.2.8. *When working in $\mathbb{R}^n$, we introduce the formal objects $dx_1, dx_2, \cdots, dx_n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.*

Definition 3.2.9. *Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be*

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \operatorname{span}\{dx_I : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their ***exterior product****, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:*

(1) *the exterior product is bilinear*
(2) *for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$*
(3) *for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$*
(4) *for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$*

We call $\Phi_k(\mathbb{R}^n)$ the differential k-forms on $\mathbb{R}^n$. Let ω be a k-form on $\mathbb{R}^n$. If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$

Note 3.2.10. *The terms $dx_1, dx_2, \cdots, dx_n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du_1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx_1, dx_2, \cdots, dx_n\}$ when referencing the coordinates on $\mathbb{R}^n$.*

Exercise 3.2.11. *Let $B_{n \times n} = (b_{i,j}) \in [C^\infty(M)]^{n \times n}$ be an $n \times n$ matrix. Then*

$$\bigwedge_{i=1}^{n} \left(\sum_{j=1}^{n} b_{i,j} dx_j \right) = (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned}
\bigwedge_{i=1}^{n} \left(\sum_{j=1}^{n} b_{i,j} dx_j \right) &= \left(\sum_{j=1}^{n} b_{1,j} dx_j \right) \wedge \left(\sum_{j=1}^{n} b_{2,j} dx_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} b_{n,j} dx_j \right) \\
&= \sum_{j_1, \cdots, j_n = 1}^{n} \left(\prod_{i=1}^{n} b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\
&= \sum_{j_1 \neq \cdots \neq j_n} \left(\prod_{i=1}^{n} b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\
&= \left[\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{n} b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\
&= (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n
\end{aligned}$$

$\square$

Definition 3.2.12. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted df, on $\mathbb{R}^n$ by*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Let $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx_I$ be a k-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx_I$$

Exercise 3.2.13. *On $\mathbb{R}^3$, put*

(1) $\omega_0 = f_0$,
(2) $\omega_1 = f_1 dx_1 + f2 dx_2 + f_2 dx_3$,
(3) $\omega_2 = f_1 dx_2 \wedge dx_3 - f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2$

Show that

(1) $d\omega_0 = \frac{\partial f_0}{\partial x_1} dx_1 + \frac{\partial f_0}{\partial x_2} dx_2 + \frac{\partial f_0}{\partial x_3} dx_3$
(2) $d\omega_1 = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 \wedge dx_3 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2$
(3) $d\omega_2 = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$

Proof. Straightforward. □

Exercise 3.2.14. *Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx_I \wedge dx_{I_*} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.*

Definition 3.2.15. *We define a linear map $* : \Phi_k(\mathbb{R}^n) \to \Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge $*$-operator** by*

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 3.2.16. *Let $\phi : \mathbb{R}^k \to \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We define $\phi^* : \Phi_k(\mathbb{R}^n) \to \Phi_k(\mathbb{R}^k)$ via the following properties:*

(1) *for each 0-form f on $\mathbb{R}^n$, $\phi^* f = f \circ \phi$*
(2) *for $i = 1, \cdots, n$, $\phi^* dx_i = d\phi_i$*
(3) *for an s-form ω, and a t-form χ on $\mathbb{R}^n$, $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$*
(4) *for l-forms ω, χ on $\mathbb{R}^n$, $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$*

Exercise 3.2.17. *Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold of $\mathbb{R}^n$, $\phi : U \to V$ a smooth parametrization of M, $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$ an k-form on $\mathbb{R}^n$. Then*

$$\phi^*\omega = \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det D\phi_I)\right) du_1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof. Using the definitions, we see that

$$\begin{aligned}\phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I)\phi^* dx_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\ &= (\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u_j} du_j) \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u_j} du_j) \wedge \cdots \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u_j} du_j) \\ &= (\det D\phi_I) du_1 \wedge du_2 \wedge \cdots \wedge du_k\end{aligned}$$

Therefore

$$\begin{aligned}\phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det D\phi_I) du_1 \wedge du_2 \wedge \cdots \wedge du_k \\ &= \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det D\phi_I) \right) du_1 \wedge du_2 \wedge \cdots \wedge du_k\end{aligned}$$

□

3.3. Integration of Differential Forms.

Definition 3.3.1. *Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$ a k-form on $\mathbb{R}^k$. Define*

$$\int_U \omega = \int_U f dx$$

Definition 3.3.2. *Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$, ω a k-form on $\mathbb{R}^n$ and $\phi : U \to V$ a local smooth, orientation-preserving parametrization of M. Define*

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 3.3.3.

Theorem 3.3.4. ***(Stokes Theorem)*** *Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$ and ω a $k-1$-form on $\mathbb{R}^n$. Then*

$$\int_{\partial M} \omega = \int_M d\omega$$