INTRODUCTION TO RANDOM FIELDS

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1. Random Fields

1.1. Introduction.

Definition 1.1.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(X, \mathcal{A})$ a measureable space, Y a Banach space and $f : X \to L^2_Y(\Omega, \mathcal{F})$. Then f is said to be a **random field** if for each $\omega \in \Omega$, $f(\cdot)(\omega) \in L^0_Y(X, \mathcal{A})$. We define

$$F_Y(X) = \{f : X \to L^2_Y(\Omega, \mathcal{F}) : f \text{ is a random field}\}$$

Definition 1.1.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(X, \mathcal{A})$ a measureable space, Y a Banach space and $f \in F_Y(X)$. For $\omega \in \Omega$, we define the **sample of f at ω**, denoted $f_\omega \in L^0_Y(X, \mathcal{A})$, by

$$f_\omega(x) = f(x)(\omega)$$

We define $f_\Omega = \{f_\omega : \omega \in \Omega\} \subset L^0_Y(X, \mathcal{A})$. Let p be a property on $L^0_Y(X, \mathcal{A})$. Then f is said to have **samples with property** p if for each $\omega \in \Omega$, f_ω has property p.

Definition 1.1.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(X, \mathcal{A})$ a measureable space, Y a Banach space and $f \in F_Y(X)$. We define the **mean of** f, denoted $\mu_f : X \to Y$, by

$$\mu_f(x) = E(f(x))$$

Definition 1.1.4. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(X, \mathcal{A})$ a measureable space, Y a Hilbert space and $f \in F_Y(X)$. We define the **covariance of** f, denoted $c_f : X \times X \to Y$, by

$$c_f(x, y) = E[\langle f(x) - \mu(x), f(y) - \mu(y)\rangle]$$

1.2. Differentiability.

Note 1.2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, a X a Banach space, Y a Banach space and $f \in F_Y(X)$. Let $x_0 \in X$. Many sources define mean square differentiablilty of f. However, this is just the Frechet derivative of f.

Exercise 1.2.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, a X a Banach space, Y a separable Banach space, $f \in F_Y(X)$ and $x_0 \in X$. If f is Frechet differentiable at x_0, then μ_f is Frechet differentiable at x_0 and $E(Df(x)) = D\mu_f(x)$.

Proof. Suppose that f is Frechet differentiable at x_0. Then

$$f(x_0 + h) = f(x_0) + Df(x_0)(h) + o(\|h\|) \quad \text{as } h \to 0$$

$\square$