1

Introduction to Logic

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Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Set Theory

1.1 Operations and Relations

Definition 1.1.0.1.

- We define $[0] := \varnothing$ and for $k \in \mathbb{N}$, we define $[k] := \{1, \ldots, k\}$.
- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-tupels with entries in** S, denoted S^k, by

$$S^k := \{u : [k] \to S\}$$

- Let S be a set. We define the **set of all tuples with entries in** S, denoted S^*, by

$$S^* := \bigcup_{k \in \mathbb{N}_0} S^k$$

- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary operation on** S, denoted $\mathcal{F}^k(S)$, by $\mathcal{F}^k(S) := S^{(S^k)}$. We define the **set of finitary operations on** S, denoted $\mathcal{F}^*(S)$, by

$$\mathcal{F}^*(S) := \bigcup_{k \in \mathbb{N}_0} \mathcal{F}^k(S)$$

- Let S be a set. We define the **operation arity map**, denoted $\operatorname{arity} : \mathcal{F}^*(S) \to \mathbb{N}_0$, by

$$\operatorname{arity} f := k, \quad f \in \mathcal{F}^k(S)$$

- Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{F}$, denoted $\mathcal{F}_k$, by

$$\mathcal{F}_k := \mathcal{F} \cap \mathcal{F}^k(S)$$

- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary relations on** S, denoted $\mathcal{R}^k(S)$, by $\mathcal{R}^k(S) := \mathcal{P}(S^k)$. We define the **set of finitary relations on** S, denoted $\mathcal{R}^*(S)$, by

$$\mathcal{R}^*(S) := \bigcup_{k \in \mathbb{N}_0} \mathcal{R}^k(S)$$

- Let S be a set. We define the **arity map**, denoted $\operatorname{arity} : \mathcal{R}^*(S) \to \mathbb{N}_0$, by

$$\operatorname{arity} R := k, \quad f \in \mathcal{R}^k(S)$$

- Let S be a set, $\mathcal{R} \subset \mathcal{R}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{R}$, denoted $\mathcal{R}_k$, by

$$\mathcal{R}_k := \mathcal{R} \cap \mathcal{R}^k(S)$$

Definition 1.1.0.2. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $C \subset S$. Then C is said to be **$\mathcal{F}$-closed** if for each $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$ and $a_1, \dots, a_k \in C$, $f(a_1, \dots, a_k) \in C$.

Exercise 1.1.0.3. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $\mathcal{C} \subset \mathcal{P}(S)$. If for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed, then $\bigcap\limits_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed

Proof. Suppose that for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed. Let $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$, $a_1, \dots, a_k \in \bigcap\limits_{C \in \mathcal{C}} C$ and $C_0 \in \mathcal{C}$. Since $C_0 \in \mathcal{C}$, we have that

$$\begin{aligned} a_1, \dots, a_k &\in \bigcap_{C \in \mathcal{C}} C \\ &\subset C_0 \end{aligned}$$

Since C_0 is $\mathcal{F}$-closed, we have that $f(a_1, \dots, a_k) \in C_0$. Since $C_0 \in \mathcal{C}$ is arbitrary, we have that for each $C \in \mathcal{C}$, $f(a_1, \dots, a_k) \in C$. Hence $f(a_1, \dots, a_k) \in \bigcap\limits_{C \in \mathcal{C}} C$. Since $k \in \mathbb{N}_0$ and $a_1, \dots, a_k \in \bigcap\limits_{C \in \mathcal{C}} C$ are arbitrary, we have that $\bigcap\limits_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed. $\square$

Chapter 2

Propositional Logic

Definition 2.0.0.1. Let $\mathcal{A}$ be a set, $\mathcal{V} \subset \mathcal{A}$, $\mathcal{F} \subset \mathcal{A}^*$, $\mathcal{C} \subset \bigcup_{k \in \mathbb{N}_0} \mathcal{F}^{(\mathcal{F}^k)}$. Then $(\mathcal{A}, \mathcal{V}, \mathcal{A})$ is said to be a **propositional calculus** if

1. - $\mathcal{V} \subset \mathcal{F}$
   - for each $k \in \mathbb{N}_0$, $p_1, \ldots, p_k \in \mathcal{F}$, and $f \in \mathcal{C}_k$, $f(p_1, \ldots, p_k) \in \mathcal{F}$
   - for each $p \in \mathcal{F}$, there exists some tree $f \in \mathcal{T}^k(\mathcal{F})$ such that $\in \mathcal{F}^k$

define the **alphabet** $\mathcal{A}$ define the **variables** $\mathcal{V}$ define the **formulas** $\mathcal{F}$ define the **connectives** $\mathcal{C}$ the **formulas of** $\mathcal{L}$**,**

Exercise 2.0.0.2.

2.1 Language

Definition 2.1.0.1. Let $\mathcal{F}$, $\mathcal{R}$, $\mathcal{C}$ be sets and $(n_f)_{f\in\mathcal{F}}, (n_R)_{R\in\mathcal{R}} \subset \mathbb{N}$. Set $\mathcal{L} := (\mathcal{F}, \mathcal{R}, \mathcal{C}, (n_f)_{f\in\mathcal{F}}, (n_R)_{R\in\mathcal{R}})$. Then $\mathcal{L}$ is said to be a **language**. We define the

- **function symbols of** $\mathcal{L}$, denoted $\mathrm{Fun}(\mathcal{L})$, by $\mathrm{Fun}(\mathcal{L}) := \mathcal{F}$,
- **relation symbols of** $\mathcal{L}$, denoted $\mathrm{Rel}(\mathcal{L})$, by $\mathrm{Rel}(\mathcal{L}) := \mathcal{R}$,
- **constant symbols of** $\mathcal{L}$, denoted $\mathrm{Cons}(\mathcal{L})$, by $\mathrm{Cons}(\mathcal{L}) := \mathcal{C}$.

For each $f \in \mathcal{F}$ and $R \in \mathcal{R}$, we define the **arity** of f and R, denoted $\mathrm{arity}(f)$ and $\mathrm{arity}(R)$ respectively, by $\mathrm{arity}(f) := n_f$ and $\mathrm{arity}(R) := n_R$ respectively.

do we really need constants or can we just use nullary function symbols?

Definition 2.1.0.2. Let $\mathcal{L}$ be a language, M a set, $\phi_{\mathcal{F}} : \mathrm{Fun}(\mathcal{L}) \to \mathcal{F}^*(M)$, $\phi_{\mathcal{R}} : \mathrm{Rel}(\mathcal{L}) \to \mathcal{R}^*(M)$ and $\phi_{\mathcal{C}} : \mathrm{Cons}(\mathcal{L}) \to M$. Set $\mathcal{M} := (M, \phi_{\mathcal{F}}, \phi_{\mathcal{R}}, \phi_{\mathcal{C}})$. Then $\mathcal{M}$ is said to be an **$\mathcal{L}$-structure on** M if

1. for each $f \in \mathrm{Fun}(\mathcal{L})$, $\phi_{\mathcal{F}}(f) \in \mathcal{F}^{n_f}(M)$,
2. for each $R \in \mathrm{Rel}(\mathcal{L})$, $\phi_{\mathcal{R}}(R) \in \mathcal{R}^{n_R}(M)$.

Let $f \in \mathrm{Fun}(\mathcal{L})$, $R \in \mathrm{Rel}(\mathcal{L})$ and $c \in \mathrm{Cons}(\mathcal{L})$. Then $\phi_{\mathcal{F}}(f)$, $\phi_{\mathcal{R}}(R)$ and $\phi_{\mathcal{C}}(c)$ are said to be **interpretations** of f, R and c in M respectively.