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Introduction to Algebra

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Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Set Theory

1.1 Operations and Relations

Definition 1.1.0.1.

- We define $[0] := \varnothing$ and for $k \in \mathbb{N}$, we define $[k] := \{1, \dots, k\}$.
- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-tupels with entries in** S, denoted S^k, by

$$S^k := \{u : [k] \to S\}$$

- Let S be a set. We define the **set of all tuples with entries in** S, denoted S^*, by

$$S^* := \bigcup_{k \in \mathbb{N}_0} S^k$$

- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary operation on** S, denoted $\mathcal{F}^k(S)$, by $\mathcal{F}^k(S) := S^{(S^k)}$. We define the **set of finitary operations on** S, denoted $\mathcal{F}^*(S)$, by

$$\mathcal{F}^*(S) := \bigcup_{k \in \mathbb{N}_0} \mathcal{F}^k(S)$$

- Let S be a set. We define the **operation arity map**, denoted $\operatorname{arity} : \mathcal{F}^*(S) \to \mathbb{N}_0$, by

$$\operatorname{arity} f := k, \quad f \in \mathcal{F}^k(S)$$

- Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{F}$, denoted $\mathcal{F}_k$, by

$$\mathcal{F}_k := \mathcal{F} \cap \mathcal{F}^k(S)$$

- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary relations on** S, denoted $\mathcal{R}^k(S)$, by $\mathcal{R}^k(S) := \mathcal{P}(S^k)$. We define the **set of finitary relations on** S, denoted $\mathcal{R}^*(S)$, by

$$\mathcal{R}^*(S) := \bigcup_{k \in \mathbb{N}_0} \mathcal{R}^k(S)$$

- Let S be a set. We define the **arity map**, denoted $\operatorname{arity} : \mathcal{R}^*(S) \to \mathbb{N}_0$, by

$$\operatorname{arity} R := k, \quad f \in \mathcal{R}^k(S)$$

- Let S be a set, $\mathcal{R} \subset \mathcal{R}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{R}$, denoted $\mathcal{R}_k$, by

$$\mathcal{R}_k := \mathcal{R} \cap \mathcal{R}^k(S)$$

Definition 1.1.0.2. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $C \subset S$. Then C is said to be $\mathcal{F}$**-closed** if for each $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$ and $a_1, \dots, a_k \in C$, $f(a_1, \dots, a_k) \in C$.

Exercise 1.1.0.3. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $\mathcal{C} \subset \mathcal{P}(S)$. If for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed, then $\bigcap\limits_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed

Proof. Suppose that for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed. Let $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$, $a_1, \dots, a_k \in \bigcap\limits_{C \in \mathcal{C}} C$ and $C_0 \in \mathcal{C}$. Since $C_0 \in \mathcal{C}$, we have that

$$\begin{aligned} a_1, \dots, a_k &\in \bigcap_{C \in \mathcal{C}} C \\ &\subset C_0 \end{aligned}$$

Since C_0 is $\mathcal{F}$-closed, we have that $f(a_1, \dots, a_k) \in C_0$. Since $C_0 \in \mathcal{C}$ is arbitrary, we have that for each $C \in \mathcal{C}$, $f(a_1, \dots, a_k) \in C$. Hence $f(a_1, \dots, a_k) \in \bigcap\limits_{C \in \mathcal{C}} C$. Since $k \in \mathbb{N}_0$ and $a_1, \dots, a_k \in \bigcap\limits_{C \in \mathcal{C}} C$ are arbitrary, we have that $\bigcap\limits_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed. $\square$

Chapter 2

Model Theory

2.1 Introduction

Chapter 3

Lattices

Definition 3.0.0.1. Let L be a set and $\wedge, \vee : L^2 \to L$. Then $(L, \wedge, \vee)$ is said to be a **lattice** if for each $x, y, z \in L$,

1. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
2. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$,
3. $x \vee x = x$ and $x \wedge x = x$,
4. $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$

3.1 Closure Operators

Definition 3.1.0.1. Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Then C is said to be a **closure operator on** A if for each $X, Y \in \mathcal{P}(A)$,

1. $X \subset C(X)$,
2. $C^2(X) = C(X)$,
3. $X \subset Y$ implies that $C(X) \subset C(Y)$.

Exercise 3.1.0.2. Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Suppose that C is a closure operator on A. Then for each $(E_j)_{j\in J} \subset \mathcal{P}(A)$,

1. $C\left(\bigcap\limits_{j\in J} E_j\right) \subset \bigcap\limits_{k\in J} C(E_k)$,
2. $\bigcup\limits_{k\in J} C(E_k) \subset C\left(\bigcup\limits_{j\in J} E_j\right)$.

Proof. Let $(E_j)_{j\in J} \subset \mathcal{P}(A)$.

1. Let $k \in J$. Then $\bigcap\limits_{j\in J} E_j \subset E_k$. So $C\left(\bigcap\limits_{j\in J} E_j\right) \subset C(E_k)$. Since $k \in J$ is arbitrary, we have that
$$C\left(\bigcap_{j\in J} E_j\right) \subset \bigcap_{k\in J} C(E_k).$$
2. Let $k \in J$. Then $E_k \subset \bigcup_{j\in J} E_j$. Hence $C(E_k) \subset C\left(\bigcup\limits_{j\in J} E_j\right)$. Since $k \in J$ is arbitrary, we have that
$$\bigcup_{k\in J} C(E_k) \subset C\left(\bigcup_{j\in J} E_j\right)$$

□

Definition 3.1.0.3. Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ and $X \subset A$. Suppose that C is a closure operator on A. Then X is said to be C-closed if $C(X) = X$.

Definition 3.1.0.4. Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Suppose that C is a closure operator on A. We define the **lattice of C-closed subsets of** A, denoted $L_C(A) \subset \mathcal{P}(A)$, by

$$L_C(A) := \{X \subset A : X \text{ is } C\text{-closed}\}$$

.

Exercise 3.1.0.5. Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Suppose that C is a closure operator on A. Then

1. for each $(E_j)_{j\in J} \subset L_C(A)$, $\bigcap\limits_{j\in J} E_j \in L_C(A)$ and $\bigcup\limits_{j\in J} E_j \in L_C(A)$.

2. $(L_C(A), \subset)$ is a complete lattice define complete lattice

$$C\bigg(\bigcap_{j\in J} E_j\bigg) = \bigcap_{j\in J} E_j$$

and

$$C\bigg(\bigcup_{j\in J} E_j\bigg) = \bigcup_{j\in J} E_j.$$

Proof.

1. Let $(E_j)_{j\in J} \subset L_C(A)$.

   - A previous exercise Exercise B.0.0.3 implies that

   $$\begin{aligned} C\bigg(\bigcap_{j\in J} E_j\bigg) &\subset \bigcap_{k\in J} C(E_k) \\ &= \bigcap_{k\in J} E_k \\ &\subset C\bigg(\bigcap_{k\in J} E_k\bigg). \end{aligned}$$

   Hence $C\bigg(\bigcap\limits_{j\in J} E_j\bigg) = \bigcap\limits_{k\in J} E_k$.

   - A previous exercise Exercise B.0.0.3 implies that

   $$\begin{aligned} \bigcup_{k\in J} E_k &= \bigcup_{k\in J} C(E_k) \\ &\subset C\bigg(\bigcup_{j\in J} E_j\bigg) \\ &\subset \bigcap_{k\in J} C(E_k) \\ &= \bigcap_{k\in J} E_k \\ &\subset C\bigg(\bigcap_{k\in J} E_k\bigg). \end{aligned}$$

   Hence $C\bigg(\bigcap\limits_{j\in J} E_j\bigg) = \bigcap\limits_{k\in J} E_k$.

2.

FINISH!!!, don't need to show second part, $\square$

Definition 3.1.0.6. then is said to be an **algebraic closure operator on** A if

Chapter 4

Universal Algebra

4.1 Introduction

Definition 4.1.0.1. Let $A \in \mathrm{Obj}(\mathbf{Set})$ be a set and J an index set. Suppose that $A \neq \varnothing$. Let $f \in \mathcal{F}^*(A)^J$. Then (A, f) is said to be an **algebra with universe** A **and basic operations** f.

Definition 4.1.0.2. Let (A, f) be an algebra. Set $J := \operatorname{dom} f$. We define the **similarity type of** f, denoted $\rho^f : J \to \mathbb{N}_0$, by $\rho^f(j) := \operatorname{arity} f_j$.

Definition 4.1.0.3. Let (A, f), (B, g) be algebras. Then (A, f) and (B, g) are said to be **type similar** if $\rho^f = \rho^g$.

Note 4.1.0.4. Set $J_f := \operatorname{dom} f$ and $J_g := \operatorname{dom} g$. Then (A, f) and (B, g) are type similar iff $J_f = J_g$ and for each $j \in J_f$, $\operatorname{arity} f_j = \operatorname{arity} g_j$.

maybe define similarity type ρ first and then stipulate algebras belonging to the set of algebras of that type, this way we dont need ρ^f, only ρ.

4.2 Subalgebras

Definition 4.2.0.1. Let (A, f) be an algebra and $B \subset A$. Then B is said to be an f-subuniverse of A if B is f-closed.

Definition 4.2.0.2. Let (A, f) be an algebra and $B \subset A$. Set $\mathcal{S} := \{S \subset A : S \text{ is an } f\text{-subuniverse of } A \text{ and } B \subset S\}$. We define the **$f$-subuniverse of A generated by B**, denoted $\mathrm{Sg}(B, f)$, by

$$\mathrm{Sg}(B, f) := \bigcap_{S \in \mathcal{S}} S$$

Exercise 4.2.0.3. Let (A, f) be an algebra and $B \subset A$. Then

1. $\mathrm{Sg}(B, f)$ is an f-subuniverse of A
2. $B \subset \mathrm{Sg}(B, f)$.

.

Proof.

1. Set $\mathcal{S} := \{S \subset A : S \text{ is an } f\text{-subuniverse of } A\}$. By construction, for each $S \in S$, S is f-closed. Since $\mathrm{Sg}(B, f) = \bigcap_{S \in \mathcal{S}} S$, Exercise B.0.0.3 A previous exercise in the set theory section implies that $\mathrm{Sg}(B, f)$ is f-closed. Hence $\mathrm{Sg}(B, f)$ is an f-subuniverse of A.
2. By construction, for each $S \in S$, $B \subset S$. Thus

$$\begin{aligned} B &\subset \bigcap_{S \in \mathcal{S}} S \\ &= \mathrm{Sg}(B, f). \end{aligned}$$

□

Exercise 4.2.0.4. Let (A, f) be an algebra. Then $\mathrm{Sg}(\cdot, f)$ is an algebraic closure operator on A.

Proof. □

Definition 4.2.0.5. Let (A, f), (B, g) be algebras. Suppose that (A, f) and (B, g) are type similar. Set $J := \operatorname{dom} f$ and $\rho := \rho^f$. Then (B, g) is said to be a **subalgebra** of (A, f) if

1. $A \subset B$
2. for each $j \in J$, $f_j|_{B^{\rho(j)}} = g_j$.

Exercise 4.2.0.6. Let (A, f), (B, g) be algebras. Suppose that (A, f) and (B, g) are type similar. If (B, g) is a sub algebra of (A, f), then B is a subuniverse of A.

Proof. Set $J := \operatorname{dom} f$. Suppose that (B, g) is a sub algebra of (A, f). Let $j \in J$. Then for each $a_1, \ldots, a_{\rho^f(j)} \in B$,

$$\begin{aligned} f_j(a_1, \ldots, a_{\rho^f(j)}) &= f_j|_{B^{\rho^f(j)}}(a_1, \ldots, a_{\rho^f(j)}) \\ &= g_j((a_1, \ldots, a_{\rho^f(j)})) \\ &\in B. \end{aligned}$$

Since $j \in J$ is arbitrary, we have that B is f-closed. Thus B is a subuniverse of A. □

4.3 Homomorphisms

Definition 4.3.0.1. Let (A, f), (B, g) be algebras and $h : A \to B$. Suppose that (A, f) and (B, g) are type similar and set $J := \operatorname{dom} f$, $\rho := \rho^f$. Then h is said to be a **homomorphism** if for each $j \in J$, and $a_1, \ldots, a_{\rho(j)}$,

$$h(f_j(a_1, \ldots, a_{\rho(j)})) = g_j(h(a_1), \ldots, h(a_{\rho(j)})).$$

Chapter 5

Groups

5.0.1 Direct Products

Definition 5.0.1.1. Let G, H be groups. Define a product $* : (G \times H) \times (G \times H) \to G \times H$ by

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

Then $(G \times H, *)$ is called the **direct product of** G **and** H.

Exercise 5.0.1.2. Let G, H be groups. Then the direct product $G \times H$ is a group.

Proof. Clear. $\square$

Definition 5.0.1.3. Let G, H be groups. Define $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ by $\pi_G(x, y) = x$ and $\pi_H(x, y) = y$. Then π_G and π_H are respectively called the **projection maps onto** G **and** H.

Exercise 5.0.1.4. Let G, H be groups. Then

1. $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ are homomorphisms
2. $\ker \pi_G \cong H$ and $\ker \pi_H \cong G$

Proof.

1. Clear
2. Define $\iota_G : G \to \ker \pi_H$ by
$$\iota_G(x) = (x, e_H)$$
Then ι_G is an isomorphism. Similarly, we can define $\iota_H : H \to \ker \pi_G$ and show that it is an isomorphism.

$\square$

Definition 5.0.1.5. Let G, H, K be groups, $\phi \in \mathrm{Hom}(G, K)$ and $\psi \in \mathrm{Hom}(H, K)$. We define $\phi \times \psi : G \times H \to K$ by $\phi \times \psi(x, y) = \phi(x)\psi(y)$

Exercise 5.0.1.6. Let G, H, K be groups, $\phi \in \mathrm{Hom}(G, K)$ and $\psi \in \mathrm{Hom}(H, K)$. If K is abelian, then $\phi \times \psi \in Hom(G \times H, K)$.

Proof. Let $x_1, x_2 \in G$ and $y_1, y_2 \in H$. Then

$$\begin{aligned}
\phi \times \psi[(x_1, y_1)(x_2, y_2)] &= \phi \times \psi(x_1 x_2, y_1 y_2) \\
&= \phi(x_1 x_2)\psi(y_1 y_2) \\
&= \phi(x_1)\phi(x_2)\psi(y_1)\psi(y_2) \\
&= \phi(x_1)\psi(y_1)\phi(x_2)\psi(y_2) \\
&= [\phi \times \psi(x_1, y_1)][\phi \times \psi(x_2, y_2)]
\end{aligned}$$

$\square$

Exercise 5.0.1.7. Let G, H, K be groups and $\phi \in \text{Hom}(G \times H, K)$. Then there exist $\phi_G \in \text{Hom}(G, K)$, $\phi_H \in \text{Hom}(H, K)$ such that $\phi_G \times \phi_H = \phi$.

Proof. Suppose that K is abelian. Define $\iota_G \in \text{Hom}(G, \ker \pi_H)$ and $\iota_H \in \text{Hom}(H, \ker \pi_G)$ as in part (2) of Exercise 5.0.1.4 Define $\phi_G \in \text{Hom}(G, K)$ and $\phi_H \in \text{Hom}(H, K)$ by $\phi_G = \phi \circ \iota_G$ and $\phi_H = \phi \circ \iota_H$. Let $(x, y) \in G \times H$. Then

$$\begin{aligned}
\phi_G \times \phi_H(x, y) &= \phi_G(x)\phi_H(y) \\
&= \phi \circ \iota_G(x)\phi \circ \iota_H(y) \\
&= \phi(x, e_H)\phi(e_G, y) \\
&= \phi(x, y)
\end{aligned}$$

So $\phi = \phi_G \times \phi_H$ $\square$

5.1 Rings

Definition 5.1.0.1. Let R be a set and $+, * : R \times R \to R$ (we write $a+b$ and ab in place of $+(a,b)$ and $*(a,b)$ respectively). Then R is said to be a **ring** if for each $a, b, c \in R$,

1. R is an abelian group with respect to $+$. The identity element with respect to $+$ is denoted by 0.
2. R is a monoid with respect to $*$. The identity element of R with respect to $*$ is denoted 1.
3. R is commutative with respect to $*$.
4. $*$ distributes over $+$.

Definition 5.1.0.2. Let R be a ring and $I \subset R$. Then I is said to be an **ideal** of R if for each $a \in R$ and $x, y \in I$,

1. $x + y \in I$
2. $ax \in I$

Definition 5.1.0.3. Let R be a ring and $A, B \subset R$. We define the **product** of A and B, denoted AB, to be

$$AB = \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in A, b_i \in B, n \in \mathbb{N} \right\}$$

Exercise 5.1.0.4. Let R be a ring and $I \subset R$. Then I is an ideal of R iff $RI \subset I$.

Proof. Suppose that $RI \subset I$. Let $a \in R$ and $x, y \in I$. Then by assumption $x + y = 1x + 1y \in I$ and $ax \in I$. So I is an ideal of R
Conversely, suppose that I is an ideal of R. Let $a_1, \cdots, a_n \in R$ and $x_1, \cdots, x_n \in I$. Then by assumption, for each $i = 1, \cdots, n$, $a_i x_i \in I$ and therefore $\sum\limits_{i=1}^{n} a_i b_i \in I$. Hence $RI \subset I$. $\square$

5.2 Modules

5.2.1 Introduction

Definition 5.2.1.1. Let R be a ring, M a set, $+ : M \times M \to M$ and $* : R \times M \to M$ (we write rx in place of $*(r, x)$). Then M is said to be an R**-module** if

1. M is an abelian group with respect to $+$. The identity element of M with respect to $+$ is denoted by 0.
2. for each $r \in R$, $*(r, \cdot)$ is a group endomorphism of M
3. for each $x \in M$, $*(\cdot, x)$ is a group homomorphism from R to M
4. $*$ is a monoid action of R on M

Note 5.2.1.2. For the remainder of this section, we assume that R is a commutative ring.

Exercise 5.2.1.3. Let M be an R-module. Then for each $r \in R$ and $x \in M$,

1. $r0 = 0$
2. $0x = 0$
3. $(-1)x = -x$

Proof. Let $r \in R$ and $x \in M$. Then

1.
$$\begin{aligned} r0 &= r(0+0) \\ &= r0 + r0 \end{aligned}$$
which implies that $r0 = 0$.

2.
$$\begin{aligned} 0x &= (0+0)x \\ &= 0x + 0x \end{aligned}$$
which implies that $0x = 0$.

3.
$$\begin{aligned} (-1)x + x &= (-1)x + 1x \\ &= (-1+1)x \\ &= 0x \\ &= 0 \end{aligned}$$
which implies that $(-1)x = -x$.

□

Definition 5.2.1.4. Let M an R-module and $N \subset M$. Then N is said to be a **submodule** of M if for each $r \in R$ and $x, y \in N$, we have that $rx \in N$ and $x + y \in N$.

Definition 5.2.1.5. Let M be an R-module. We define $\mathcal{S}(M) = \{N \subset M : N \text{ is a submodule of } M\}$.

Exercise 5.2.1.6. Let M be an R-module and $N \in \mathcal{S}(M)$. Then N is a subgroup of M.

Proof. Let $x, y \in M$. Then $x - y = 1x + (-1)y \in N$. So N is a subgroup of M. □

Definition 5.2.1.7. Let M be an R-module and $N \in \mathcal{S}(M)$. We define

1. $M/N = \{x + N : x \in M\}$
2. $+ : M/N \times M/N \to M/N$ by
$$(x + N) + (y + N) = (x + y) + N$$
3. $* : R \times M/N \to M/N$ by
$$r(x + N) = (rx) + N$$

Under these operations (see next exercise), M/N is an R-module known as the **quotient module** of M by N.

Exercise 5.2.1.8. Let M be an R-module and $N \in \mathcal{S}(M)$. Then

1. the monoid action defined above is well defined
2. the quotient M/N is an R-module

Proof.

1. Let $r \in R$ and $x + N, y + N \in M/N$. Recall from group theory that $x + N = y + N$ iff $x - y \in N$. Suppose that $x + N = y + N$. Then $x - y \in N$ and there exists $n \in N$ such that $x - y = n$. Therefore
$$\begin{aligned} rx - ry &= r(x - y) \\ &= rn \\ &\in N \end{aligned}$$
So $rx + N = ry + N$.
2. Properties (1) - (4) in the definition of a module are easily shown to be satisfied for M/N since they are true for M.

$\square$

Definition 5.2.1.9. Let M and N be R-modules and $\phi : M \to N$. Then ϕ is said to be a **module homomorphism** if for each $r \in R$ and $x, y \in M$

1. $\phi(rx) = r\phi(x)$
2. $\phi(x + y) = \phi(x) + \phi(y)$

Exercise 5.2.1.10. Let M and N be R-modules and $\phi : M \to N$. Then ϕ is a iff for each $r \in R$ and $x, y \in M$, $\phi(x + ry) = \phi(x) + r\phi(y)$.

Proof. Clear. $\square$

Exercise 5.2.1.11. Let M and N be R-modules and $\phi : M \to N$ a homomorphism. Then

1. $\ker \phi$ is a submodule of M
2. $\operatorname{Im} \phi$ is a submodule of N

Proof. Let $r \in R$, $x, y \in \ker \phi$ and $w, z \in \operatorname{Im} \phi$. Then

1.
$$\begin{aligned} \phi(rx) &= r\phi(x) \\ &= r0 \\ &= 0 \end{aligned}$$
So $rx \in \ker \phi$. Group theory tells us that $\ker \phi$ is a subgroup of M, so $x + y \in \ker \phi$. Hence $\ker \phi$ is a submodule of M.

2. Similar.

□

Definition 5.2.1.12. Let M be an R-module and $A \subset M$. We define the **submodule of** M **generated by** A, denoted $\text{span}(A)$, to be

$$\text{span}(A) = \bigcap_{N \in \mathcal{S}(M)} N$$

Exercise 5.2.1.13. Let M be an R-module and $A \subset M$. Then $\text{span}(A) \in \mathcal{S}(M)$

Proof. Let $r \in R$ and $x, y \in \text{span}(A)$. Basic group theory tells us that $\text{span}(A)$ is a subgroup of M. So $x + y \in \text{span}(A)$. For $N \in \mathcal{S}(M)$, by definition we have $x \in N$ and therefore $rx \in N$. So $rx \in \text{span}(A)$. Hence $\text{span}(A)$ is a submodule of M. □

Exercise 5.2.1.14. Let M be an R-module and $A \subset M$. If $A \neq \varnothing$, then

$$\text{span}(A) = \left\{ \sum_{i=1}^{n} r_i a_i : r_i \in R, a_i \in A, n \in \mathbb{N} \right\}$$

Proof. Clearly □

Definition 5.2.1.15. Let M

5.3 Fields

5.4 Vector Spaces

5.5 Appendix

5.5.1 Monoids

Definition 5.5.1.1. Let G be a set and $* : G \times G \to G$ (we write ab in place of $*(a,b)$). Then

1. $*$ is called a **binary operation** on G
2. $*$ is said to be **associative** if for each $x, y, z \in G$, $(xy)z = x(yz)$
3. $*$ is said to be **commutative** if for each $x, y \in G$, $xy = yx$

Definition 5.5.1.2. Let G be a set, $* : G \times G \to G$, $e, x, y \in G$. Then e is said to be an **identity element** if for each $x \in G$, $ex = xe = x$.

Definition 5.5.1.3. Let G be a set and $* : G \times G \to G$. Then G is said to be a **monoid** if

1. $*$ is associative
2. there exits $e \in G$ such that e is an identity element.

Exercise 5.5.1.4. Let G be a monoid. Then the identity element is unique.

Proof. Let $e, f \in G$. Suppose that e and f are identity elements. Then $e = ef = f$. □

Note 5.5.1.5. Unless otherwise specified, we will denote the identity element of a monoid by e.

Definition 5.5.1.6. Let G be a monoid, X a set and $* : G \times X \to X$ (we write gx in place of $*(g,x)$). Then $*$ is said to be a **monoid action** of G on X if for each $g, h \in G$ and $x \in X$,

1. $(gh)x = g(hx)$
2. $ex = x$

Appendix A

Summation

Definition A.0.0.1. Let $f : X \to [0, \infty)$, Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note A.0.0.2. Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.

Appendix B

Asymptotic Notation

Definition B.0.0.1. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = o(g) \qquad \text{as } x \to x_0$$

if for each $\epsilon > 0$, there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U$,

$$\|f(x)\| \leq \epsilon \|g(x)\|$$

Exercise B.0.0.2. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. If there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U \setminus \{x_0\}$, $g(x) > 0$, then

$$f = o(g) \text{ as } x \to x_0 \quad \text{iff} \quad \lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = 0$$

Exercise B.0.0.3. Let X and Y a be normed vector spaces, $A \subset X$ open and $f : A \to Y$. Suppose that $0 \in A$. If $f(h) = o(\|h\|)$ as $h \to 0$, then for each $h \in X$, $f(th) = o(|t|)$ as $t \to 0$.

Proof. Suppose that $f(h) = o(\|h\|)$ as $h \to 0$. Let $h \in X$ and $\epsilon > 0$. Choose $\delta' > 0$ such that for each $h' \in B(0, \delta')$, $h' \in A$ and

$$\|f(h')\| \leq \frac{\epsilon}{\|h\| + 1} \|h'\|$$

Choose $\delta > 0$ such that for each $t \in B(0, \delta)$, $th \in B(0, \delta')$. Let $t \in B(0, \delta)$. Then

$$\begin{aligned} \|f(th)\| &\leq \frac{\epsilon}{\|h\| + 1} |t| \|h\| \\ &< \epsilon |t| \end{aligned}$$

So $f(th) = o(|t|)$ as $t \to 0$. $\square$

Definition B.0.0.4. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = O(g) \qquad \text{as } x \to x_0$$

if there exists $U \in \mathcal{N}(x_0)$ and $M \geq 0$ such that for each $x \in U$,

$$\|f(x)\| \leq M \|g(x)\|$$

Appendix C

Categories

move to notation?

Definition C.0.0.1. We define the category of topological measure spaces, denoted $\mathbf{TopMsr}_+$, by

- $\mathrm{Obj}(\mathbf{TopMsr}_+) := \{(X, \mu) : X \in \mathrm{Obj}(\mathbf{Top}) \text{ and } \mu \in M(X)\}$
- $\mathrm{Hom}_{\mathbf{TopMsr}_+}((X, \mu), (Y, \nu)) := \mathrm{Hom}_{\mathbf{Top}}(X, Y) \cap \mathrm{Hom}_{\mathbf{Msr}_+}((X, \mathcal{B}(X), \mu), (Y, \mathcal{B}(Y), \nu))$

Appendix D

Vector Spaces

it might be better to cover some category theory and write everything in terms of $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}$ and $\mathrm{Obj}(\mathbf{Vect}_{\mathbb{K}})$

D.1 Introduction

Definition D.1.0.1. Let X be a set, $\mathbb{K}$ a field, $+ : X \times X \to X$ and $\cdot : \mathbb{K} \times X \to X$. Then $(X, +, \cdot)$ is said to be a **$\mathbb{K}$-vector space** if

1. $(X, +)$ is an abelian group
2.

Definition D.1.0.2. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$. Then $(E, +_E, \cdot_E)$ is said to be a subspace of X if

1. $+_E = +_X|_{E \times E}$
2. $\cdot_E = \cdot_X|_{\mathbb{K} \times E}$

Exercise D.1.0.3. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$.

Exercise D.1.0.4. Let $(X, +, \cdot)$ be a vector space and $E \subset X$. Then E is a subspace of X

Definition D.1.0.5. Let X be a vector space and $(E_j)_{j \in J}$ a collection of subspaces of X. Then $\bigcap\limits_{j \in J} E_j$ is a subspace of X.

Proof. Set $E := \bigcap\limits_{j \in J} E_j$. Let $x, y \in E$ and $\lambda \in \mathbb{K}$. Then for each $j \in J$, $x, y \in E_j$. Since for each $j \in J$, E_j is a subspace of X, we have that for each $j \in J$, $x + \lambda y \in E_j$. Thus $x + \lambda y \in E$. Since $x, y \in E$ and $\lambda \in \mathbb{K}$ are arbitrary, (cite exercise here) we have that E is a subspace of X. $\square$

Definition D.1.0.6. Let X, Y be vector spaces and $T : X \to Y$. Then T is said to be **linear** if for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

1. $T(x_1 + x_2) = T(x_1) + T(x_2)$,
2. $T(\lambda x_1) = \lambda T(x_1)$.

We define $L(X; Y) := \{T : X \to Y : T \text{ is linear}\}$.

Exercise D.1.0.7. Let X, Y be vector spaces and $T : X \to Y$. Then T is linear iff for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

$$T(x_1 + \lambda x_2) = T(x_1) + \lambda T(x_2)$$

Proof. Clear. (add details) $\square$

Definition D.1.0.8. define addition/scalar multiplication of linear maps

Exercise D.1.0.9. Let X, Y be vector spaces. Then $L(X;Y)$ is a $\mathbb{K}$-vector space.

Proof. Clear $\square$

Definition D.1.0.10. Let X be a vector space over $\mathbb{K}$ and $T : X \to \mathbb{K}$. Then T is said to be a **linear functional on** X if T is linear. We define the **dual space of** X, denoted X^*, by $X^* := \{T : X \to \mathbb{K} : T \text{ is linear}\}$.

Exercise D.1.0.11. Let X be a vector space. Then X^* is a vector space.

Proof. Clear. $\square$

D.2 Bases

Definition D.2.0.1. Let X be a vector space and $(e_\alpha)_{\alpha \in A} \subset X$. Then $(e_\alpha)_{\alpha \in A}$ is said to be

- **linearly independent** if for each $(\alpha_j)_{j=1}^n \subset A$, $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $\sum\limits_{j=1}^n \lambda_j e_{\alpha_j} = 0$ implies that for each $j \in [n]$, $\lambda_j = 0$.
- a **Hamel basis for** X if $(e_\alpha)_{\alpha \in A}$ is linearly independent and $\text{span}(e_\alpha)_{\alpha \in A} = X$.

Exercise D.2.0.2. every vector space has a Hamel basis

Proof. $\square$

Exercise D.2.0.3.

Exercise D.2.0.4. Let X be a $\mathbb{K}$-vector space and $x \in X$. Then $x = 0$ iff for each $\phi \in X^*$, $\phi(x) = 0$.

Proof.

- $(\implies)$:
  Suppose that $x = 0$. Linearity implies that for each $\phi \in X^*$ $\phi(x) = 0$.
- $(\impliedby)$:
  Conversely, suppose that $x \neq 0$. Define $\epsilon_x : \text{span}(x) \to \mathbb{K}$ by $\epsilon_x(\lambda x) := \lambda$. Let $u, v \in \text{span}(x)$. Then there exists $\lambda_u, \lambda_v \in \mathbb{K}$ such that $u = \lambda_u x$ and $v = \lambda_v x$. Suppose that $u = v$. Then

  $$\begin{aligned}(\lambda_u - \lambda_v)x &= \lambda_u x - \lambda_v x \\ &= u - v \\ &= 0\end{aligned}$$

  Since $x \neq 0$, we have that $\lambda_u - \lambda_v = 0$ and therefore $\lambda_u = \lambda_v$. Hence

  $$\begin{aligned}\lambda_u &= \epsilon_x(u) \\ &= \epsilon_x(v) \\ &= \lambda_v.\end{aligned}$$

  Thus ϵ_x is well defined.

$\square$

D.3 Multilinear Maps

Definition D.3.0.1. Let $X_1, \cdots, X_n, Y$ be vector spaces and $T : \prod_{j=1}^{n} X_j \to \mathbb{K}$. Then T is said to be **multilinear** if for each $j_0 \in [n]$ and $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$, $T(x_1, \ldots, x_{j_0-1}, \cdot, x_{j_0+1})$ is linear.

$$L^n(X_1, \ldots, X_n; Y) = \left\{ T : \prod_{j=1}^{n} X_j \to Y : T \text{ is multilinear} \right\}$$

If $X_1 = \cdots = X_n = X$, we write $L^n(X; Y)$ in place of $L^n(X, \ldots, X; Y)$.

Definition D.3.0.2. define addition and scalar mult of multilinear maps

Exercise D.3.0.3. Let $X_1, \cdots, X_n, Y$ be vector spaces. Then $L^n(X_1, \ldots, X_n; Y)$ is a $\mathbb{K}$-vector space.

Proof. content... □

Exercise D.3.0.4. Let $X_1, \cdots, X_n, Y, Z$ be $\mathbb{K}$-vector spaces, $\alpha \in L^n(X_1, \ldots, X_n; Y)$ and $\phi \in L^1(Y; Z)$. Then $\phi \circ \alpha \in L^n(X_1, \ldots, X_n; Z)$.

Proof. Let $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$. Define $f : X_{j_0} \to Y$ by

$$f(a) := \alpha(x_1, \ldots, x_{j_0-1}, a, x_{j_0+1}, \ldots, x_n)$$

Since $\alpha \in L^n(X_1, \ldots, X_n; Y)$, f is linear. Since ϕ is linear, and $\phi \circ f$ is linear. Since $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$ are arbitrary, we have that $\phi \circ \alpha \in L^n(X_1, \ldots, X_n; Y)$. □

D.4 Tensor Products

Definition D.4.0.1. Let X, Y and T be vector spaces over $\mathbb{K}$ and $\alpha \in L^2(X, Y; T)$. Then (T, α) is said to be a **tensor product of** X **and** Y if for each vector space Z and $\beta \in L^2(X, Y; Z)$, there exists a unique $\phi \in L^1(T; Z)$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & T \arrow[d, dashed, "\phi"] \\ & Z \end{tikzcd}$$

Exercise D.4.0.2. Let X, Y, S, T be vector spaces, $\alpha \in L^2(X, Y; S)$ and $\beta \in L^2(X, Y; T)$. Suppose that (S, α) and (T, β) are tensor products of X and Y. Then S and T are isomorphic.

Proof. Since (T, β) is a tensor product of X and Y, $\beta \in L^2(X, Y; T)$ there exists a unique $f \in L^1(T; T)$ such that $f\ circ \beta = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dd, dashed, "f"] \\ X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T \end{tikzcd}$$

Since $\mathrm{id}_T \in L^1(T; T)$ and $\mathrm{id}_T \circ \beta = \beta$, we have that $f = \mathrm{id}_T$. Since (S, α) is a tensor product of X and Y, there exists a unique $\phi : S \to T$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] \\ & T \end{tikzcd}$$

Similarly, since (T, β) is a tensor product of X and Y, there exists a unique $\psi : T \to S$ such that $\psi \circ \beta = \alpha$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & T \arrow[d, dashed, "\psi"] \\ & S \end{tikzcd}$$

Therefore

$$\begin{aligned} (\phi \circ \psi) \circ \beta &= \phi \circ (\psi \circ \beta) \\ &= \phi \circ \alpha \\ &= \beta, \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, dashed, "\psi"] & & & T \arrow[dd, dashed, "\phi\circ\psi"] \\ X \times Y \arrow[ur, "\beta"] \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] & \Longrightarrow & X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T & & & T \end{tikzcd}$$

By uniqueness of $f \in L^1(T; T)$, we have that

$$\begin{aligned} \mathrm{id}_T &= f \\ &= \phi \circ \psi \end{aligned}$$

A similar argument implies that $\psi \circ \phi = \mathrm{id}_S$. Hence ϕ and ψ are isomorphisms with $\phi^{-1} = \psi$. Hence S and T are isomorphic. $\square$

Definition D.4.0.3. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. We define $x \otimes y : X^* \times Y^* \to \mathbb{K}$ by $x \otimes y(\phi, \psi) := \phi(x)\psi(y)$.

Exercise D.4.0.4. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. Then $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$.

Proof. Let $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} x \otimes y(\phi_1 + \lambda\phi_2, \psi) &= [\phi_1 + \lambda\phi_2(x)]\psi(y) \\ &= \phi_1(x)\psi(y) + \lambda\phi_2(x)\psi(y) \\ &= x \otimes y(\phi_1, \psi) + \lambda x \otimes y(\phi_2, \psi) \end{aligned}$$

Since $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$ are arbitrary, we have that for each $\psi \in Y^*$, $x \otimes y(\cdot, \psi)$ is linear. Similarly for each $\phi \in X^*$, $x \otimes y(\phi, \cdot)$ is linear. Hence $x \otimes y$ is bilinear and $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$. $\square$

Definition D.4.0.5. Let X, Y be vector spaces. We define

- the **tensor product of** X **and** Y, denoted $X \otimes Y \subset L^2(X^*, Y^*; \mathbb{K})$, by

$$X \otimes Y := \mathrm{span}(x \otimes y : x \in X \text{ and } y \in Y),$$

- the **tensor map**, denoted $\otimes : X \times Y \to X \otimes Y$, by $\otimes(x, y) := x \otimes y$.

Exercise D.4.0.6. Let X, Y be vector spaces, $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$. The following are equivalent:

1. $\sum\limits_{j=1}^n x_j \otimes y_j = 0$
2. for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum\limits_{j=1}^n \phi(x_j)\psi(y_j) = 0$
3. for each $\phi \in X^*$, $\sum\limits_{j=1}^n \phi(x_j)y_j = 0$
4. for each $\psi \in Y^*$, $\sum\limits_{j=1}^n \psi(y_j)x_j = 0$

Proof.

1. (1) $\implies$ (2) :
   Suppose that $\sum\limits_{j=1}^n x_j \otimes y_j = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Then

$$\begin{aligned} \sum_{j=1}^n \phi(x_j)\psi(y_j) &= \phi\bigg(\sum_{j=1}^n \psi(y_j)x_j\bigg) \\ &= \end{aligned}$$

2.
3.

$\square$

Exercise D.4.0.7. Let X, Y be vector spaces. Then $(X \otimes Y, \otimes)$ is a tensor product of X and Y.

Proof. Let Z be a vector space and $\alpha \in L^2(X, Y; Z)$. Define $\phi : X \otimes Y \to Z$ by $\phi\Big(\sum_{j=1}^{n} \lambda_j x_j \otimes y_j) := \sum_{j=1}^{n} \lambda_j \alpha(x_j, y_j)$.

- **(well defined):**
  Let $u \in X \otimes Y$. Then there exist $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $(x_j)_{j=1}^n \subset X$, $(y_j)_{j=1}^n \subset Y$ such that $u = \sum_{j=1}^{n} \lambda_j x_j \otimes y_j$.
  Suppose that $u = 0$. Let $\phi \in Z^*$. Then $\phi \circ \alpha \in L^2(X, Y; Z)$.

$\square$

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