INTRODUCTION TO CATEGORY THEORY

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Contents

Preface

1. Categories and Functors

1.1. Categories.

Definition 1.1.1. Let $\mathcal{C}_0$, $\mathcal{C}_1$ be classes and $\text{dom}, \text{cod} : \mathcal{C}_1 \to \mathcal{C}_0$. Set $\mathcal{C} = (C_0, C_1, \text{dom}, \text{cod})$. Then $\mathcal{C}$ is said to be a **category** if

(1) (composition): for each $f, g \in C_1$, if $\text{cod}(f) = \text{dom}(g)$, then there exists $g \circ f \in C_1$ such that $\text{dom}(g \circ f) = \text{dom}(f)$ and $\text{cod}(g \circ f) = \text{cod}(g)$
(2) (associativity): for each $f, g, h \in C_1$, if $\text{cod}(f) = \text{dom}(g)$ and $\text{cod}(g) = \text{dom}(h)$, then
$$(h \circ g) \circ f = h \circ (g \circ f)$$
(3) (identity): for each $X \in \mathcal{C}_0$, there exists $1_X \in C_1$ such that $\text{dom}(1_X) = \text{cod}(1_X) = X$ and for each $f, g \in C_1$, if $\text{dom}(f) = X$ and $\text{cod}(g) = X$, then
$$f \circ 1_X = f \text{ and } 1_X \circ g = g$$

We define the

- **objects of** $\mathcal{C}$, denoted $\text{Obj}(\mathcal{C})$, by $\text{Obj}(\mathcal{C}) = C_0$
- **morphisms of** $\mathcal{C}$, denoted $\text{Hom}_{\mathcal{C}}$, by $\text{Hom}_{\mathcal{C}} = C_1$

For $X, Y \in \text{Obj}(\mathcal{C})$, we define the **morphisms from** X **to** Y, denoted $\text{Hom}_{\mathcal{C}}(X, Y)$, by $\text{Hom}_{\mathcal{C}}(X, Y) = \{f \in \text{Hom}(\mathcal{C}) : \text{dom}(f) = X \text{ and } \text{cod}(f) = Y\}$.

Note 1.1.2. We typically define a category $\mathcal{C}$ by specifying

- $\text{Obj}(\mathcal{C})$
- for $X, Y \in \text{Obj}(\mathcal{C})$, the class $\text{Hom}_{\mathcal{C}}(X, Y)$
- for $X, Y, Z \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$, the composite morphism $g \circ f \in \text{Hom}_{\mathcal{C}}(X, Z)$.

and then show

- composition is well defined
- associativity of composition
- existence of identities

Definition 1.1.3. Let $\mathcal{C}$ be a category, we define the dual of $\mathcal{C}$ or the **opposite of** $\mathcal{C}$, denoted $\mathcal{C}^{\text{op}}$, by

- $\text{Obj}(\mathcal{C}^{\text{op}}) = \text{Obj}(\mathcal{C})$
- for $X, Y \in \text{Obj}(\mathcal{C}^{\text{op}})$, $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$
- for $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y), g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$, $g \circ_{\mathcal{C}^{\text{op}}} f = f \circ_{\mathcal{C}} g$

Exercise 1.1.4. Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\text{op}}$ is a category.

Proof.

- for $W, X, Y, Z \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(W, X)$ and $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$ and $h \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$. Then
$$\begin{aligned}
(h \circ_{\mathcal{C}^{\text{op}}} g) \circ_{\mathcal{C}^{\text{op}}} f &= f \circ_{\mathcal{C}} (h \circ_{\mathcal{C}^{\text{op}}} g) \\
&= f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h) \\
&= (f \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} h \\
&= h \circ_{\mathcal{C}^{\text{op}}} (f \circ_{\mathcal{C}} g) \\
&= h \circ_{\mathcal{C}^{\text{op}}} (g \circ_{\mathcal{C}^{\text{op}}} f)
\end{aligned}$$

So composition is associative.
- Let $X \in \mathrm{Obj}(\mathcal{C})$ and $f, g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}$. Suppose that $\mathrm{dom}(f) = X$ and $\mathrm{cod}(g) = X$ Then

$$\begin{aligned} f \circ_{\mathcal{C}^{\mathrm{op}}} 1_X &= 1_X \circ_{\mathcal{C}} f \\ &= f \end{aligned}$$

and

$$\begin{aligned} 1_X \circ_{\mathcal{C}^{\mathrm{op}}} g &= g \circ_{\mathcal{C}} 1_X \\ &= g \end{aligned}$$

So $(1_X)_{\mathcal{C}^{\mathrm{op}}} = (1_X)_{\mathcal{C}}$.

□

Definition 1.1.5. Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. We define the **slice category of $\mathcal{C}$ over X**, denoted $\mathcal{C}/X$, by
- $\mathrm{Obj}(\mathcal{C}/X) = \{f \in \mathrm{Hom}_{\mathcal{C}} : \mathrm{cod}(f) = X\}$
- for $f, g \in \mathrm{Obj}(\mathcal{C}/X)$,

$$\mathrm{Hom}_{\mathcal{C}/X}(f, g) = \{\alpha \in \mathrm{Hom}_{\mathcal{C}} : \mathrm{dom}(\alpha) = \mathrm{dom}(f),\ \mathrm{cod}(\alpha) = \mathrm{dom}(g) \text{ and } f = g \circ \alpha\}$$

i.e. for $f \in \mathrm{Hom}_{\mathcal{C}}(A, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f, g)$ iff the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

- for $f, g, h \in \mathrm{Obj}(\mathcal{C}/X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}/X}(g, h)$, $\beta \circ_{\mathcal{C}/X} \alpha = \beta \circ_{\mathcal{C}} \alpha$

Exercise 1.1.6. Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. Then $\mathcal{C}/X$ is a category.

Proof.
- $f, g, h \in \mathrm{Obj}(\mathcal{C}/X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}/X}(g, h)$. Then $f = g \circ_{\mathcal{C}} \alpha$ and $g = h \circ_{\mathcal{C}} \beta$, i.e. the following diagrams commute:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & \mathrm{dom}(g) \arrow[dl, "g"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{dom}(g) \arrow[rr, "\beta"] \arrow[dr, "g"'] & & \mathrm{dom}(h) \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

Therefore, we have that

$$\begin{aligned} f &= g \circ_{\mathcal{C}} \alpha \\ &= (h \circ_{\mathcal{C}} \beta) \circ_{\mathcal{C}} \alpha \\ &= h \circ_{\mathcal{C}} (\beta \circ_{\mathcal{C}} \alpha) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\beta \circ_{\mathcal{C}} \alpha"] \arrow[dr, "f"'] & & \mathrm{dom}(h) \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

which implies that

$$\begin{aligned}\beta \circ_{\mathcal{C}/X} \alpha &= \beta \circ_{\mathcal{C}} \alpha \\ &\in \mathrm{Hom}_{\mathcal{C}/X}(f, h)\end{aligned}$$

and composition is well defined.

- Associativity of $\circ_{\mathcal{C}/X}$ follows from associativity of $\circ_{\mathcal{C}}$.
- Let $f \in \mathrm{Obj}(\mathcal{C}/X)$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}/X}$. Since $f \circ 1_{\mathrm{dom}_{\mathcal{C}}(f)} = f$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "1_{\mathrm{dom}(f)}"] \arrow[dr, "f"'] & & \mathrm{dom}(f) \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

we have that $1_{\mathrm{dom}(f)} \in \mathrm{Hom}_{\mathcal{C}/X}(f, f)$. Suppose that $\mathrm{dom}_{\mathcal{C}/X}(\alpha) = f$ and $\mathrm{cod}_{\mathcal{C}/X}(\beta) = f$. Then

$$\begin{aligned}\alpha \circ_{\mathcal{C}/X} 1_{\mathrm{dom}(f)} &= \alpha \circ_{C} 1_{\mathrm{dom}(f)} \\ &= \alpha\end{aligned}$$

and

$$\begin{aligned}1_{\mathrm{dom}(f)} \circ_{\mathcal{C}/X} \beta &= 1_{\mathrm{dom}(f)} \circ_{\mathcal{C}} \beta \\ &= \beta\end{aligned}$$

So $(1_f)_{\mathcal{C}/X} = (1_{\mathrm{dom}(f)})_{\mathcal{C}}$.

$\square$

1.2. **Functors.**

Definition 1.2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $F_1 : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}$. Set $F = (F_0, F_1)$. Then F is said to be a functor from $\mathcal{C}$ to $\mathcal{D}$ if

(1) for each $A, B \in \mathrm{Obj}(C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F_1(f) \in \mathrm{Hom}_{\mathcal{D}}(F_0(A), F_0(B))$
(2) for each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$
(3) for each $A \in \mathrm{Obj}(\mathcal{C})$, $F_1(\mathrm{id}_A) = \mathrm{id}_{F_0(A)}$

Note 1.2.2. For $A \in \mathrm{Obj}(C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}$, we typically write $F(A)$ and $F(f)$ instead of $F_0(A)$ and $F_1(f)$ respectively.

1.3. **Natural Transformations.**