1

Introduction to Differential Geometry

Carson James

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Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Review of Fundamentals

1.1 Set Theory

Definition 1.1.0.1. Let $\{A_i\}_{i\in I}$ be a collection of sets. The **disjoint union of** $\{A_i\}_{i\in I}$, denoted $\coprod\limits_{i\in I} A_i$, is defined by

$$\coprod_{i\in I} A_i = \bigcup_{i\in I}\{i\} \times A_i$$

We define the **natural projection map**, denoted $\pi : \coprod\limits_{i\in I} A_i \to I$, by $\pi(i,a) = i$.

Definition 1.1.0.2. Let E and M be sets, $\pi : E \to B$ a surjection and $\sigma : B \to E$. Then σ is said to be a section of (E, M, π) if $\pi \circ \sigma = \mathrm{id}_M$.

Note 1.1.0.3. Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod\limits_{i\in I} A_i$. We will typically be interested in sections σ of $\left(\coprod\limits_{i\in I} A_i, I, \pi\right)$.

Exercise 1.1.0.4. Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod\limits_{i\in I} A_i$. Then σ is a section of $\coprod\limits_{i\in I} A_i$ iff for each $i \in I$, $\sigma(i) \in A_i$

Proof. Clear. $\square$

1.2 Linear Algebra

Note 1.2.0.1. We denote the standard basis on $\mathbb{R}^n$ by $(e_1, \dots, e_n)$.

Definition 1.2.0.2. Let $A \in \mathbb{R}^{n \times n}$. Then A is said to be **invertible** if $\det(A) \neq 0$. We denote the set of $n \times n$ invertible matrices by $GL(n, \mathbb{R})$.

$$O(n)$$

Exercise 1.2.0.3. Let $A, B \in \mathbb{R}^{n \times n}$. Then $AB = I$ iff $BA = I$.

Proof.

- ($\implies$):
  Suppose that $AB = I$. Then

  $$\begin{aligned} \ker B &\subset \ker AB \\ &= \ker I \\ &= \{0\} \end{aligned}$$

  so that $\ker B = \{0\}$. Hence $\operatorname{Im} B = \mathbb{R}^n$ and B is surjective. Then

  $$\begin{aligned} IB &= BI \\ &= B(AB) \\ &= (BA)B \end{aligned}$$

  Since B is surjective, $I = BA$.

- ($\impliedby$):
  Immediate by the previous part.

□

Definition 1.2.0.4. Let $A \in \mathbb{R}^{n \times p}$. Then A is said to be an **orthogonal matrix** if $A^* A = I$. We denote the set of $n \times p$ orthogonal matrices by $O(n, p)$. We write $O(n)$ in place of $O(n, n)$.

$$O(n)$$

Exercise 1.2.0.5. Define $\phi : S_n \to GL(n, \mathbb{R})$ by

$$\phi(\sigma) = \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix}$$

Then

1. for each $A \in \mathbb{R}^{n \times p}$,

   $$(\phi(\sigma)A)_{i,j} = A_{\sigma(i),j}$$

   i.e. left multiplying A by $\phi(\sigma)$ the the same as permuting the rows of A by σ
2. ϕ is a group homomorphism

Proof. 1. Let $A \in \mathbb{R}^{n \times p}$. Then

$$\begin{aligned} (\phi(\sigma)A)_{i,j} &= \langle e^*_{\sigma(i)}, Ae_j \rangle \\ &= A_{\sigma(i),j} \end{aligned}$$

2. Let $\sigma, \tau \in S_n$. Part (1) implies that

$$\begin{aligned}\phi(\sigma\tau) &= \begin{pmatrix} e^*_{\sigma\tau(1)} \\ \vdots \\ e^*_{\sigma\tau(n)} \end{pmatrix} \\ &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e^*_{\tau(1)} \\ \vdots \\ e^*_{\tau(n)} \end{pmatrix} \\ &= \phi(\sigma)\phi(\tau)\end{aligned}$$

Since $\sigma, \tau \in S_n$ are arbitrary, ϕ is a group homomorphism.

□

Definition 1.2.0.6. Define $\phi : S_n \to GL(n, \mathbb{R})$ as in the previous exercise. Let $P \in GL(n, \mathbb{R})$. Then P is said to be a **permutation matrix** if there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. We denote the set of $n \times n$ permutation matrices by $\text{Perm}(n)$.

Exercise 1.2.0.7. We have that

1. $\text{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$
2. $\text{Perm}(n)$ is a subgroup of $O(n)$

Proof.

1. By definition, $\text{Perm}(n) = \operatorname{Im} \phi$. Since $\phi : S_n \to GL(n, \mathbb{R})$ is a group homomorphism, $\operatorname{Im} \phi$ is a subgroup of $GL(n, \mathbb{R})$. Hence $\text{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$.
2. Let $P \in \text{Perm}(n)$. Then there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. Then

$$\begin{aligned}PP^* &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix}^* \\ &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e_{\sigma(1)} & \cdots & e_{\sigma(n)} \end{pmatrix} \\ &= (\langle e_{\sigma(i)}, e_{\sigma(j)} \rangle)_{i,j} \\ &= I\end{aligned}$$

A previous exercise implies that $P^*P = I$. Hence $P \in O(n)$. Since $P \in \text{Perm}(n)$ is arbitrary, $\text{Perm}(n) \subset O(n)$. Part (1) implies that $\text{Perm}(n)$ is a group. Hence $\text{Perm}(n)$ is a subgroup of $O(n)$

□

Note 1.2.0.8. We will write P_σ in place of $\phi(\sigma)$.

Exercise 1.2.0.9. Let $Z \in \mathbb{R}^{p \times n}$. If $\operatorname{rank} Z = k$, then there exist $\sigma \in S_n$, $\tau \in S_p$ and $A \in GL(k, \mathbb{R})$, such that for each $i, j \in \{1, \ldots, k\}$,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Proof. Suppose that $\operatorname{rank} Z - k$. Then there exist $i_1, \ldots, i_k \in \{1, \ldots, p\}$ such that $i_1 < \cdots < i_k$ and $\{e^*_{i_1} Z, \ldots, e^*_{i_k} Z\}$ is linearly independent. Set

$$Z' = \begin{pmatrix} e^*_{i_1} Z \\ \vdots \\ e^*_{i_k} Z \end{pmatrix}$$

Then $\operatorname{rank} Z' = k$. Hence there exist $j_1, \dots, j_k \in \{1, \dots, n\}$ such that $j_1 < \cdots < j_k$, and $\{Z'e_{i_1}, \dots, Z'e_{i_k}\}$ is linearly independent. Set

$$A = \begin{pmatrix} Z'e_{i_1} & \cdots & Z'e_{i_k} \end{pmatrix}$$

Then $A \in \mathbb{R}^{k \times k}$ and $\operatorname{rank} A = k$. Thus $A \in GL(k, \mathbb{R})$. Choose $\sigma \in S_n$ and $\tau \in S_p$ such that $\sigma(1) = j_1, \dots, \sigma(k) = j_k$ and $\tau(1) = i_1, \dots, \tau(k) = i_k$. Let $a, b \in \{1, \dots, k\}$. By construction,

$$\begin{aligned} (P_\tau Z P_\sigma^*)_{a,b} &= Z_{\tau(a),\sigma(b)} \\ &= Z_{i_a, j_b} \\ &= A_{a,b} \end{aligned}$$

□

Definition 1.2.0.10. Let $A \in \mathbb{R}^{n \times p}$. Then A is said to be a **diagonal matrix** if for each $i \in [n]$ and $j \in [p]$, $i \neq j$ implies that $A_{i,j} = 0$. We denote the set of $n \times p$ diagonal matrices by $D(n, p, \mathbb{R})$. We write $D(n, \mathbb{R})$ in place of $D(n, n, \mathbb{R})$.

Definition 1.2.0.11. For (n, k), (m, l) $\operatorname{diag}_{p,(n \times p)} : \mathbb{R}^p \to \mathbb{R}^{n \times p}$ and $\operatorname{diag}_{n,(n \times p)} : \mathbb{R}^p \to \mathbb{R}^{n \times p}$ by $\operatorname{diag}(v)$ FINISH!!!

Definition 1.2.0.12. Let $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \sigma(A)$. Suppose that A is symmetric. We define the **geometric multiplicity** of λ, denoted $\mu(\lambda)$, by

$$\mu(\lambda) = \dim \ker([\phi_\alpha] - \lambda I)$$

Definition 1.2.0.13. Let V be an n-dimensional vector space, $U \subset V$ a k-dimensional subspace and $(e_j)_{j=1}^n \subset V$ a be a basis. Then $(e_j)_{j=1}^n$ is said to be **adapted to** U if $(e_j)_{j=1}^k$ is a basis for U.

1.3 Calculus

1.3.1 Differentiation

Definition 1.3.1.1. Let $n \geq 1$. For $i = 1, \cdots, n$, define $x^i : \mathbb{R}^n \to \mathbb{R}$ by $x^i(a^1, \cdots, a^n) = a^i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on** $\mathbb{R}^n$.

Definition 1.3.1.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to** x^i **at** a if

$$\lim_{h \to 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If f is differentiable with respect to x^i at a, we define the **partial derivative of** f **with respect to** x^i **at** a, denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \frac{\partial}{\partial x^i} f$$

to be the limit above.

Definition 1.3.1.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **differentiable with respect to** x^i if for each $a \in U$, f is differentiable with respect to x^i at a.

Exercise 1.3.1.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i x^j}$ and $\frac{\partial^2 f}{\partial x^j x^i}$ exist and are continuous at a. Then

$$\frac{\partial^2 f}{\partial x^i x^j}(a) = \frac{\partial^2 f}{\partial x^j x^i}(a)$$

Proof. □

Definition 1.3.1.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $\frac{\partial^k f}{\partial i_1 \cdots i_k}$ exists and is continuous on U.

Definition 1.3.1.6. Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \to \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Theorem 1.3.1.7. Taylor's Theorem:
Let $U \subset \mathbb{R}^n$ be open and convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that for each $x \in U$,

$$f(x) = \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each $|\alpha| = T + 1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. See analysis notes □

Definition 1.3.1.8. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Let $x^1, \cdots, x^n$ be the standard coordinate functions on $\mathbb{R}^n$ and $y_1, \cdots, y_m$ be the standard coordinate functions on $\mathbb{R}^m$. For $i \in \{1, \cdots, m\}$, we define the i**th component of** F, denoted $F^i : U \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \cdots, F_m)$

Definition 1.3.1.9. Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \cdots, m\}$, the ith component of F, $F^i : U \to \mathbb{R}$, is smooth.

Definition 1.3.1.10. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Then F is said to be **smooth** if for each $x \in U$, there exists $U_x \in \mathcal{N}_x$ and $\tilde{F} : U_x \to \mathbb{R}^m$ such that U_x is open, $\tilde{F}$ is smooth and $\tilde{F}|_{U \cap U_x} = F|_{U \cap U_x}$.

Definition 1.3.1.11. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. Then F is said to be a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 1.3.1.12. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F is a bijection and F and F^{-1} are smooth. Thus, F and F^{-1} are continuous and F is a homeomorphism. □

Definition 1.3.1.13. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \to \mathbb{R}^m$. We define the **Jacobian of F at p**, denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p)\right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Exercise 1.3.1.14. Inverse Function Theorem:
Let $U, V \subset \mathbb{R}^n$ be open and $F : U \to V$.

Exercise 1.3.1.15. Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \to F(N)$ is a diffeomorphism

Proof. content... □

Exercise 1.3.1.16. Let $\sigma \in S_n$. Define $\phi_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by $\phi(x^1, \ldots, x^n) = \phi(x^{\sigma(1), \ldots, x^{\sigma(n)}})$. Then $D\phi = P_\sigma$

Definition 1.3.1.17. Let $\sigma \in S_n$ and $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$. We define $\sigma x \in \mathbb{R}^n$ by

$$\sigma x = (x^{\sigma(1)}, \ldots, x^{\sigma(n)})$$

We define the **permutation action** of S_n on $\mathbb{R}^n$ to be the map $S_n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(\sigma, x) \mapsto \sigma x$

Definition 1.3.1.18. Let $\sigma \in S_n$, U a set, $V \subset \mathbb{R}^n$ and $\phi : U \to \mathbb{R}^n$ with $\phi = (x^1, \ldots, x^m)$. We define $\sigma\phi : U \to \mathbb{R}^n$ by

$$\sigma\phi = (x^{\sigma(1)}, \ldots, x^{\sigma(n)})$$

We define the **permutation action** of S_n on $(\mathbb{R}^n)^U$ to be the map $S_n \times (\mathbb{R}^n)^U \to \mathbb{R}^n$ given by $(\sigma, \phi) \mapsto \sigma\phi$.

Exercise 1.3.1.19. Let $\sigma \in S_m$. Then for each $p \in \mathbb{R}^n$, $D(\sigma \operatorname{id}_{\mathbb{R}^n})(p) = P_\sigma$.

Proof. Note that since $\operatorname{id}_{\mathbb{R}^n} = (\pi_1, \ldots, \pi_n)$, we have that $\sigma \operatorname{id}_{\mathbb{R}^n} = (\pi_{\sigma(1)}, \ldots, \pi_{\sigma(n)})$. Let $p \in \mathbb{R}^n$. Then

$$\begin{aligned}
D(\sigma \operatorname{id}_{\mathbb{R}^n})(p) &= \left(\frac{\partial \pi_i \circ \sigma \operatorname{id}_{\mathbb{R}^n}}{\partial x^j}(p)\right)_{i,j} \\
&= \left(\frac{\partial \pi_{\sigma(i)}}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma \left(\frac{\partial \pi_i}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma \left(\frac{\partial \pi_i \circ \operatorname{id}_{\mathbb{R}^n}}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma D \operatorname{id}_{\mathbb{R}^n}(p) \\
&= P_\sigma I \\
&= P_\sigma
\end{aligned}$$

□

1.3.2 Integration

1.4 Topology

Definition 1.4.0.1. Let $(X, \mathbb{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 1.4.0.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Definition 1.4.0.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \to Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 1.4.0.4. Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \ncong \mathbb{R}^n$

Chapter 2

Multilinear Algebra

2.1 Tensor Products

Let V and W be vector spaces.

2.2 (r, s)-Tensors

Definition 2.2.0.1. Let $V_1, \dots, V_k, W$ be vector spaces and $\alpha : \prod_{i=1}^n V_i \to W$. Then α is said to be **multilinear** if for each $i \in \{1, \cdots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$,

$$\alpha(v_1, \cdots, v_i + cv, \cdots, v_k) = \alpha(v_1, \cdots, v_i, \cdots, v_k) + c\alpha(v_1, \cdots, v, \cdots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{ \alpha : \prod_{i=1}^n V_i \to W : \alpha \text{ is multilinear} \right\}$$

Note 2.2.0.2. For the remainder of this section we let V denote an n-dimensional vector space with basis $\{e^1, \cdots, e^n\}$ with dual space V^* and dual basis $\{\epsilon_1, \cdots, \epsilon_n\}$ defined by $\epsilon^i(e^j) = \delta_{i,j}$. We identify V with V^{**} by the isomorphism $V \to V^{**}$ defined by $v \mapsto \hat{v}$ where $\hat{v}(\alpha) = \alpha(v)$ for each $\alpha \in V^*$.

Definition 2.2.0.3. Let $\alpha : (V^*)^r \times V^s \to \mathbb{R}$. Then α is said to be an (r, s)-tensor on V if $\alpha \in L(\underbrace{V^*, \dots, V^*}_{r}, \underbrace{V, \dots, V}_{s}; \mathbb{R})$. The set of all (r, s)-tensors on V is denoted $T^r_s(V)$.
When $r = s = 0$, we set $T^r_s = \mathbb{R}$.

Exercise 2.2.0.4. We have that $T^r_s(V)$ is a vector space.

Proof. Clear. $\square$

Exercise 2.2.0.5. Under the identification of V with V^{**} as noted above, we have that $V = T^1_0(V)$.

Proof. By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T^1_0(V) \end{aligned}$$

$\square$

Definition 2.2.0.6. Let $\alpha \in T^{r_1}_{s_1}(V)$ and $\beta \in T^{r_2}_{s_2}(V)$. We define the **tensor product of α with β**, denoted $\alpha \otimes \beta \in T^{r_1+r_2}_{s_1+s_2}(V)$, by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$.
When $r_1 = s_1 = r_2 = s_2 = 0$ (so that $\alpha, \beta \in \mathbb{R}$), we set $\alpha \otimes \beta = \alpha\beta$.

Definition 2.2.0.7. We define the **tensor product**, denoted $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

Exercise 2.2.0.8. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is well defined.

Proof. Tedious but straightforward. □

Exercise 2.2.0.9. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is associative.

Proof. Let $\alpha \in T^{r_1}_{s_1}(V)$, $\beta \in T^{r_2}_{s_2}(V)$ and $\gamma \in T^{r_3}_{s_3}(V)$. Then for each $u^* \in (V^*)^{r_1}, v^* \in (V^*)^{r_2}, w^* \in (V^*)^{r_3}, u \in V^{s_1}, v \in V^{s_2}, w \in V^{s_3}$,

$$\begin{aligned}
(\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v)\gamma(w^*, w) \\
&= [\alpha(u^*, u)\beta(v^*, v)]\gamma(w^*, w) \\
&= \alpha(u^*, u)[\beta(v^*, v)\gamma(w^*, w)] \\
&= \alpha(u^*, u)(\beta \otimes \gamma)(v^*, w^*, v, w) \\
&= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w)
\end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

Exercise 2.2.0.10. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is bilinear.

Proof.

1. Linearity in the first argument:
   Let $\alpha, \beta \in T^{r_1}_{s_1}(V)$, $\gamma \in T^{r_2}_{s_2}(V), \lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $vinV^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned}
[(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v)\gamma(w^*, w) \\
&= [\alpha(v^*, v) + \lambda\beta(v^*, v)]\gamma(w^*, w) \\
&= \alpha(v^*, v)\gamma(w^*, w) + \lambda\beta(v^*, v)\gamma(w^*, w) \\
&= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\
&= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w)
\end{aligned}$$

   So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

2. Linearity in the second argument:
   Similar to (1).

□

Definition 2.2.0.11.

1. Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1, \cdots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length** k. Recall that $\#\mathcal{I}_{\otimes k} = n^k$.

2. Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length** k. Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

Note 2.2.0.12. For the remainder of this section we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\otimes k}$.

Definition 2.2.0.13. Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.

1. Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by
$$\epsilon^I = (\epsilon^{i_1}, \cdots, \epsilon^{i_k})$$
and
$$e^I = (e^{i_1}, \cdots, e^{i_k})$$
2. Define $e^{\otimes I} \in T^k_0(V)$ and $\epsilon^{\otimes I} \in T^0_k(V)$ by
$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$
and
$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

Exercise 2.2.0.14. Let $\alpha, \beta \in T^r_s(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v^*_1, \dots, v^*_r \in V^*$ and $v_1, \dots, v_s \in V$. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, write
$$v^*_i = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$
and
$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$
Then
$$\begin{aligned}
\alpha(v^*_1, \dots, v^*_r, v_1, \dots, v_s) &= \sum_{k_1,\dots,k_r=1}^n \sum_{l_1,\dots,l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \sum_{k_1,\dots,k_r=1}^n \sum_{l_1,\dots,l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \beta(v^*_1, \dots, v^*_r, v_1, \dots, v_s)
\end{aligned}$$
So that $\alpha = \beta$. □

Exercise 2.2.0.15. Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K}\delta_{J,L}$.

Proof. Write $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$ and $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$. Then
$$\begin{aligned}
e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K)\epsilon^{\otimes J}(e^L) \\
&= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r})\epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\
&= \left[\prod_{m=1}^r e^{i_m}(\epsilon^{k_m})\right]\left[\prod_{n=1}^s \epsilon^{j_n}(e^{l_n})\right] \\
&= \left[\prod_{m=1}^r \delta_{i_m,k_m}\right]\left[\prod_{n=1}^s \delta_{j_n,l_n}\right] \\
&= \delta_{I,K}\delta_{J,L}
\end{aligned}$$
□

Exercise 2.2.0.16. The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T^r_s(V)$ and $\dim T^r_s(V) = n^{r+s}$.

Proof. Let $(a^I_J)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a^I_J e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a^I_J = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T^r_s(V)$. For $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b^I_J = \beta(\epsilon^J, e^I)$. Define $\mu = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b^I_J e^{\otimes I} \otimes \epsilon^{\otimes J} \in T^r_s(V)$. Then for each $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b^I_J = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. □

2.3 Covariant k-Tensors

2.3.1 Symmetric and Alternating Covariant k-Tensors

Definition 2.3.1.1. Let $\alpha : V^k \to \mathbb{R}$. Then α is said to be a **covariant k-tensor on V** if $\alpha \in T^0_k(V)$. We denote the set of covariant k-tensors by $T_k(V)$.

Definition 2.3.1.2. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \to \mathbb{R}$ by

$$\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

We define the **permutation action** of of S_k on $T_k(V)$ to be the map $S_k \times T_k(V) \to T_k(V)$ given by $(\sigma, \alpha) \mapsto \sigma\alpha$

Exercise 2.3.1.3. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

1. Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V), \sigma\alpha \in T_k(V)$.
2. Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
3. Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \cdots, v_k \in V$,

$$\begin{aligned}(\tau\sigma)\alpha(v_1, \cdots, v_k) &= \alpha(v_{\tau\sigma(1)}, \cdots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \cdots, v_k)\end{aligned}$$

□

Exercise 2.3.1.4. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \to T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$. Then

$$\begin{aligned}\sigma(c\alpha + \beta)(v_1, \cdots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \cdots, v_k) + \sigma\beta(v_1, \cdots, v_k)\end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

Definition 2.3.1.5. Let $\alpha \in T_k(V)$. Then α is said to be

- **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$
- **antisymmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \operatorname{sgn}(\sigma)\alpha$
- **alternating** if for each $v_1, \ldots, v_k \in V$, if there exists $i, j \in \{1, \ldots, k\}$ such that $v_i = v_j$, then $\alpha(v_1, \cdots, v_k) = 0$.

We denote the set of symmetric k-tensors on V by $\Sigma^k(V)$. We denote the set of alternating k-tensors on V by $\Lambda^k(V)$.

Exercise 2.3.1.6. Let $\alpha \in T_k(V)$. Then α is antisymmetric iff α is alternating.

Proof. Suppose that α is antisymmetric. Let $v_1, \ldots, v_k \in V$. Suppose that there exists $i, j \in \{1, \ldots, k\}$ such that $v_i = v_j$. Define $\sigma \in S_k$ by $\sigma = (i, j)$. Then

$$\begin{aligned}\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) &= \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \\ &= \sigma(\alpha)(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &= \operatorname{sgn}(\sigma)\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &= -\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k)\end{aligned}$$

Therefore $2\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = 0$ which implies that $\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = 0$. Hence α is alternating.
Conversely, suppose that α is alternating. Let $i, j \in \{1, \dots, k\}$ and $v_1, \dots, v_k \in V$. Then

$$\begin{aligned} 0 &= \alpha(v_1, \cdots, v_i + v_j, \cdots, v_i + v_j, \cdots, v_k) \\ &= \alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) + \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \end{aligned}$$

Since $i, j \in \{1, \dots, k\}$ and $v_1, \dots, v_k \in V$ are arbitrary, we have that for each $\tau \in S_k$, τ is a transposition implies that

$$\begin{aligned} \tau\alpha &= -\alpha \\ &= \operatorname{sgn}(\tau)\alpha \end{aligned}$$

Let $n \in \mathbb{N}$. Suppose that for each $\tau_1, \dots, \tau_{n-1} \in S_k$ if for each $j \in \{1, \dots, n-1\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_{n-1})\alpha = \sigma(\tau_1 \cdots \tau_{n-1})\alpha$. Let $\tau_1, \dots, \tau_n \in S_k$. Suppose that for each $j \in \{1, \dots, n\}$, τ_j is a transposition. Then

$$\begin{aligned} (\tau_1 \cdots \tau_n)\alpha &= (\tau_1 \cdots \tau_{n-1})(\tau_n \alpha) \\ &= (\tau_1 \cdots \tau_{n-1})(\operatorname{sgn}(\tau_n)\alpha) \\ &= (\operatorname{sgn}(\tau_n)(\tau_1 \cdots \tau_{n-1})\alpha \\ &= (\operatorname{sgn}(\tau_n) \operatorname{sgn}((\tau_1 \cdots \tau_{n-1})\alpha \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha \end{aligned}$$

By induction, for each $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in S_k$, if for each $j \in \{1, \dots, n\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_n)\alpha = \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha$. Now let $\sigma \in S_k$. Then there exist $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in S_k$ such that $\sigma = \tau_1 \cdots \tau_n$ and for each $j \in \{1, \dots n\}$, τ_j is a transposition. Hence

$$\begin{aligned} \sigma\alpha &= (\tau_1 \cdots \tau_n)\alpha \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha \\ &= \operatorname{sgn}(\sigma)\alpha \end{aligned}$$

Therefore α is antisymmetric. □

Definition 2.3.1.7. Define the **symmetric operator** $S : T_k(V) \to \Sigma^k(V)$ by

$$\operatorname{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \to \Lambda^k(V)$ by

$$\operatorname{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha$$

Exercise 2.3.1.8.

1. For $\alpha \in T_k(V)$, $\operatorname{Sym}(\alpha)$ is symmetric.
2. For $\alpha \in T_k(V)$, $\operatorname{Alt}(\alpha)$ is alternating.

Proof.

1. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned} \sigma \operatorname{Sym}(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \\ &= \operatorname{Sym}(\alpha) \end{aligned}$$

2. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
\sigma \operatorname{Alt}(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right] \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\sigma\tau\alpha \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \\
&= \operatorname{sgn}(\sigma)\operatorname{Alt}(\alpha)
\end{aligned}$$

□

Exercise 2.3.1.9.

1. For $\alpha \in \Sigma^k(V)$, $\operatorname{Sym}(\alpha) = \alpha$.
2. For $\alpha \in \Lambda^k(V)$, $\operatorname{Alt}(\alpha) = \alpha$.

Proof.

1. Let $\alpha \in \Sigma^k(V)$. Then

$$\begin{aligned}
\operatorname{Sym}(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} \alpha \\
&= \alpha
\end{aligned}$$

2. Let $\alpha \in \Lambda^k(V)$. Then

$$\begin{aligned}
\operatorname{Alt}(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2\alpha \\
&= \alpha
\end{aligned}$$

□

Exercise 2.3.1.10. The symmetric operator $S : T_k(V) \to \Sigma^k(V)$ and the alternating operator $A : T_k(V) \to \Lambda^k(V)$ are linear.

Proof. Clear. □

Exercise 2.3.1.11. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

1. $\operatorname{Alt}(\operatorname{Alt}(\alpha) \otimes \beta) = \operatorname{Alt}(\alpha \otimes \beta)$
2. $\operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta)) = \operatorname{Alt}(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \to S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma,\tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

1. Then

$$\begin{aligned}
\mathrm{Alt}(\mathrm{Alt}(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\mathrm{Alt}(\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)\tau\alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)(\tau\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)\tau(\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \mathrm{sgn}(\sigma\tau)\sigma\tau(\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \mathrm{Alt}(\alpha \otimes \beta)
\end{aligned}$$

2. Similar to (1).

□

2.3.2 Exterior Product

Definition 2.3.2.1. Let $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$.

Exercise 2.3.2.2. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is bilinear.

Proof. Clear. □

Exercise 2.3.2.3. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$ and $\gamma \in \Lambda^m(V)$. Then

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)\right] \wedge \gamma \\
&= \frac{(k+l+m)!}{(k+l)!m!} \operatorname{Alt}\left(\left[\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)\right] \otimes \gamma\right) \\
&= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} \operatorname{Alt}(\operatorname{Alt}(\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} \operatorname{Alt}((\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \operatorname{Alt}(\alpha \otimes (\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \operatorname{Alt}(\alpha \otimes \frac{(l+m)!}{l!m!} \operatorname{Alt}(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \operatorname{Alt}(\alpha \otimes (\beta \wedge \gamma)) \\
&= \alpha \wedge (\beta \wedge \gamma))
\end{aligned}$$

□

Exercise 2.3.2.4. Let $\alpha_i \in \Lambda^{k_i}(V)$ for $i = 1, \cdots, m$. Then

$$\bigwedge_{i=1}^{m} \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m} \alpha_i\right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} \operatorname{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i\right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \operatorname{Alt}\left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \operatorname{Alt}\left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m_0-1} \alpha_i\right)\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\operatorname{Alt}\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m_0+1} \alpha_i\right)
\end{aligned}$$

□

Exercise 2.3.2.5. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl. (Hint: inversion number)

Proof.

$$\begin{aligned} N(\tau) &= \sum_{i=1}^{l} k \\ &= kl \end{aligned}$$

Since $\mathrm{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\mathrm{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.3.2.6. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \cdots, v_{k+l} \in V$, we have that

$$\begin{aligned} \sigma\tau(\beta \otimes \alpha)(v_1, \cdots, v_l, v_{l+1}, \cdots v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)})\alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \\ &= \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)})\beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\ &= \alpha \otimes \beta(v_{\sigma(1)}, \cdots v_{\sigma(k)}, v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\ &= \sigma(\alpha \otimes \beta)(v_1, \cdots, v_k, v_{1+k}, \cdots v_{l+k}) \end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned} \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} \mathrm{Alt}(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma\tau)\sigma\tau(\beta \otimes \alpha) \\ &= \mathrm{sgn}(\tau)\frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma(\alpha \otimes \beta) \\ &= \mathrm{sgn}(\tau)\frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha \otimes \beta) \\ &= \mathrm{sgn}(\tau)\alpha \wedge \beta \\ &= (-1)^{kl}\alpha \wedge \beta \end{aligned}$$

□

Exercise 2.3.2.7. Let $\alpha \in \Lambda^k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned} \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\ &= -\alpha \wedge \alpha \end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.3.2.8. Fundamental Example:
Let $\alpha_1, \cdots, \alpha_m \in \Lambda^1(V)$ and $v_1, \cdots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}
\left(\bigwedge_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) &= m! \operatorname{Alt}\left(\bigotimes_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) \\
&= m!\left[\frac{1}{m!}\sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\sigma\left(\bigotimes_{i=1}^m \alpha_i\right)\right](v_1, \cdots, v_m) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\left(\bigotimes_{i=1}^m \alpha_i\right)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
&= \det(\alpha_i(v_j))
\end{aligned}$$

□

Note 2.3.2.9. Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\wedge k}$.

Definition 2.3.2.10. Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.
Define $\epsilon^{\wedge I} \in \Lambda^k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$$

Exercise 2.3.2.11. Let $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \cdots & \epsilon^{i_1}(e^{j_k}) \\ & \vdots & \\ \epsilon^{i_k}(e^{j_1}) & \cdots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$. If $I = J$, then $A = I_{k\times k}$ and therefore $\epsilon^I(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the l_0-th row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the l_0-th column of A are 0. □

Exercise 2.3.2.12. Let $\alpha, \beta \in \Lambda^k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \cdots, v_k \in V$. For $i = 1, \cdots, k$, write $v_i =$

$\sum_{j_i=1}^{n} a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned}
\alpha(v_1, \cdots, v_k) &= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{j_1 \neq \cdots \neq j_k}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{k} a_{i,\sigma(j_i)}\right)\right] \alpha(e^J) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{k} a_{i,\sigma(j_i)}\right)\right] \beta(e^J) \\
&= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right) \beta(e^{j_1}, \cdots, e^{j_k}) \\
&= \beta(v_1, \cdots, v_k)
\end{aligned}$$

□

Exercise 2.3.2.13. The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(V)$ and $\dim \Lambda^k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$. Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda^k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda^k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$. □

2.3.3 Interior Product

Definition 2.3.3.1. Let V be a finite dimensional vector space and $v \in V$. We define **interior multiplication by** v, denoted $\iota_v : T_k \to T_{k-1}$, by

$$\iota_v \alpha(w_1, \ldots, w_{k-1}) = \alpha(v, w_1, \ldots, w_{k-1})$$

Exercise 2.3.3.2. Let V be a finite dimensional vector space and $v \in V$. Then $\iota_v|_{\Lambda^k(V)} : \Lambda^k(V) \to \Lambda^{k-1}(V)$.

Proof. Let $\alpha \in \Lambda^k(V)$. Define $\beta \in \Lambda^k(V)$ by $\beta(w_1, \ldots, w_k) = \alpha(w_k, w_1, \ldots, w_{k-1})$. Let $\sigma \in S_{k-1}$. Define $\tau \in S_k$ by $\tau(j) = \begin{cases} 1 & j = k \\ \sigma(j) & j \neq k \end{cases}$. Let $w_1, \ldots, w_{k-1} \in V$. Set $w_k = v$. Then

$$\begin{aligned}
\sigma(\iota_v \alpha)(w_1, \ldots, w_{k-1}) &= \iota_v \alpha(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}) \\
&= \alpha(v, w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}) \\
&= \beta(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}, v) \\
&= \beta(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}, w_k) \\
&= \beta(w_{\tau(1)}, \ldots, w_{\tau(k-1)}, w_{\tau(k)}) \\
&= \operatorname{sgn}(\tau) \beta(w_1, \ldots, w_{k-1}, w_k) \\
&= \operatorname{sgn}(\sigma) \beta(w_1, \ldots, w_{k-1}, v) \\
&= \operatorname{sgn}(\sigma) \alpha(v, w_1, \ldots, w_{k-1}) \\
&= \operatorname{sgn}(\sigma) (\iota_v \alpha)(w_1, \ldots, w_{k-1})
\end{aligned}$$

Since $w_1, \ldots, w_{k-1} \in V$ are arbitrary, $\sigma(\iota_v \alpha) = \operatorname{sgn}(\sigma) \iota_v \alpha$. Hence $\iota_v \alpha \in \Lambda^{k-1}(V)$. □

2.4 $(0, 2)$-Tensors

Definition 2.4.0.1. Let V be a finite dimensional vector space, $v \in V$ and $\alpha \in T^0_2(V)$. Then α is said to be **degenerate** if there exists $v \in V$ such that for each $w \in V$, $\alpha(v, w) = 0$ and $v \neq 0$.

Definition 2.4.0.2. Let V be a finite dimensional vector space, $\alpha \in T^0_2(V)$. We define $\phi_\alpha : V \to V^*$ by

$$\phi_\alpha(v) = \iota_v \alpha$$

Exercise 2.4.0.3. Let V be a finite dimensional vector space, $\alpha \in T^0_2(V)$. Then $\phi_\alpha \in L(V; V^*)$.

Proof. Let $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$. Then for each $w \in V$,

$$\begin{aligned}
\phi_\alpha(v_1 + \lambda v_2)(w) &= (\iota_{v_1+\lambda v_2}\alpha)(w) \\
&= \alpha(v_1 + \lambda v_2, w) \\
&= \alpha(v_1, w) + \lambda\alpha(v_2, w) \\
&= (\iota_{v_1}\alpha)(w) + \lambda(\iota_{v_2}\alpha)(w) \\
&= \phi_\alpha(v_1)(w) + \lambda\phi_\alpha(v_2)(w) \\
&= [\phi_\alpha(v_1) + \lambda\phi_\alpha(v_2)](w)
\end{aligned}$$

Therefore, $\phi_\alpha(v_1 + \lambda v_2) = \phi_\alpha(v_1) + \lambda\phi_\alpha(v_2)$. Thus $\phi_\alpha \in L(V; V^*)$. □

Exercise 2.4.0.4. Let V be a finite dimensional vector space and $\alpha \in T^0_2(V)$. Then α is nondegenerate iff ϕ_α is an isomorphism.

Proof.

- $(\implies:)$
  Suppose that α is nondegenerate. Let $v \in \ker\phi_\alpha$. Then for each $w \in V$,

$$\begin{aligned}
\alpha(v, w) &= (\iota_v\alpha)(w) \\
&= \phi_\alpha(v)(w) \\
&= 0
\end{aligned}$$

  Since α is nondegenerate, $v = 0$. Since $v \in \ker\phi_\alpha$ is arbitrary, $\ker\phi_\alpha = \{0\}$. Hence ϕ_α is injective. Since $\dim V = \dim V^*$, ϕ_α is surjective. Hence ϕ_α is an isomorphism.

- $(\impliedby:)$
  Suppose that ϕ_α is an isomorphism. Let $v \in V$. Suppose that for each $w \in V$, $\alpha(v, w) = 0$. Then for each $w \in V$,

$$\begin{aligned}
\phi_\alpha(v)(w) &= (\iota_v\alpha)(w) \\
&= \alpha(v, w) \\
&= 0
\end{aligned}$$

  Thus $\phi_\alpha(v) = 0$ which implies that $v \in \ker\phi_\alpha$. Since ϕ_α is an isomorphism, $v = 0$. Hence α is nondegenerate.

□

Exercise 2.4.0.5. Let V be a finite dimensional vector space and $\alpha \in T^0_2(V)$. Then

1. $[\phi_\alpha]_{i,j} = \alpha(e_j, e_i)$
2. for each $v, w \in V$,

$$\alpha(v, w) = [w]^*[\phi_\alpha][v]$$

Proof. 1. Set $A = [\phi_\alpha]$. Let $i, j \in \{1, \dots, n\}$. By definition,

$$\phi_\alpha(e_j) = \sum_{k=1}^n A_{k,j}\epsilon^k$$

Then

$$\begin{aligned}\phi_\alpha(e_j)(e_i) &= \sum_{k=1}^n A_{k,j}\epsilon^k(e_i)\\ &= \sum_{k=1}^n A_{k,j}\delta_{k,i}\\ &= A_{i,j}\end{aligned}$$

2. Let $v, w \in V$. Then there exist $(v^i)_{i=1}^n, (w^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum\limits_{i=1}^n v^i e_i$ and $w = \sum\limits_{j=1}^n v^j e_i$. Part (1) implies that

$$\begin{aligned}\alpha(v, w) &= \sum_{i=1}^n \sum_{j=1}^n v^i w^j \alpha(e_i, e_j)\\ &= \sum_{i=1}^n \sum_{j=1}^n v^i w^j [\phi_\alpha]_{j,i}\\ &= \sum_{i=1}^n \sum_{j=1}^n [v]_i [w]_j [\phi_\alpha]_{j,i}\\ &= [w]^*[\phi_\alpha][v]\end{aligned}$$

□

2.4.1 Scalar Product Spaces

Definition 2.4.1.1. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be

- **positive semidefinite** if for each $v \in V$, $\alpha(v, v) \geq 0$
- **positive definite** if for each $v \in V$, $v \neq 0$ implies that $\alpha(v, v) > 0$
- **negative semidefinite** if $-\alpha$ is positive semidefinite
- **negative definite** if $-\alpha$ is positive definite

Exercise 2.4.1.2. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then

1. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda > 0$
2. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda \geq 0$

Proof.

1. Suppose that α is positive definite. Write $\sigma(\phi_\alpha) = \{\lambda_1, \dots, \lambda_n\}$. Define $\Lambda \in \mathbb{R}^{n\times n}$ by $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Since α is symmetric, $[\phi_\alpha]$ is symmetric. There exists $U \in O(n)$ such that $[\phi_\alpha] = U\Lambda U^*$. FINISH!!!

□

Definition 2.4.1.3. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be a **scalar product** if α is nondegenerate. In this case, (V, α) is said to be a **scalar product space**.

Definition 2.4.1.4. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$ a scalar product on V. We define the **index** of α, denoted $\operatorname{ind} \alpha$ by

$$\operatorname{ind} \alpha = \max\{\dim W : W \text{ is a subspace of } V \text{ and } \alpha|_{W \times W} \text{ is negative definite}\}$$

Definition 2.4.1.5. Let (V, α) be a scalar product space.

- Let $v_1, v_2 \in V$. Then v_1 and v_2 are said to be **orthogonal** if $\alpha(v_1, v_2) = 0$.
- Let $U \subset V$ be a subspace. We define the **orthogonal subspace of** U, denoted by $U^\perp$, by

$$U^\perp = \{v \in V : \text{ for each } u \in U, \, \alpha(u, v) = 0\}$$

Exercise 2.4.1.6. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then $U^\perp$ is a subspace of V.

Proof. We note that since $U^\perp = \bigcap_{u \in U} \ker \phi_\alpha(u)$, $U^\perp$ is a subspace of V. □

Exercise 2.4.1.7. Let (V, α) be an n-dimensional scalar product space, $U \subset V$ a k-dimensional subspace and $(e_j)_{j=1}^n \subset V$ a basis for V. Suppose that $(e_j)_{j=1}^k$ is a basis for U. Then for each $v \in V$, $v \in U^\perp$ iff for each $j \in [k]$, $\alpha(v, e_j) = 0$.

Proof. Let $v \in V$.

- ($\Longrightarrow$): Suppose that $v \in U^\perp$. Since $(e_j)_{j=1}^k \subset U$, we have that for each $j \in [k]$, $\alpha(v, e_j) = 0$.
- ($\Longleftarrow$): Suppose that for each $j \in [k]$, $\alpha(v, e_j) = 0$. Let $u \in U$. Then there exist $(a^j)_{j=1}^k \subset \mathbb{R}$ such that $u = \sum_{j=1}^k a^j u_j$. This implies that

$$\begin{aligned} \alpha(v, u) &= \sum_{j=1}^k a^j \alpha(v, u_j) \\ &= 0 \end{aligned}$$

  Since $u \in U$ is arbitrary, we have that $v \in U^\perp$.

□

Exercise 2.4.1.8. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then

1. $\dim V = \dim U + \dim U^\perp$
2. $(U^\perp)^\perp = U$

Proof.

1. Set $n = \dim V$ and $k = \dim U$. Choose a basis $(e_j)_{j=1}^n$ such that $(e_j)_{j=1}^k$ is a basis for U.
2.

□

Exercise 2.4.1.9. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Set $\sigma([\phi_\alpha])^- = \{\lambda \in \sigma([\phi_\alpha]) : \lambda < 0\}$. Then

$$\operatorname{ind} \alpha = \sum_{\lambda \in \sigma([\phi_\alpha])^-} \mu(\lambda)$$

Proof. Since α is symmetric, there exist $U \in O(n)$ and $\Lambda \in D(n, \mathbb{R})$ such that $[\phi_\alpha] = U\Lambda U^*$. Define $(u_j)_{j=1}^n \subset V$ by $u_j = \sum_{i=1}^n U_{i,j} e_j$. Define $J^- = \{j \in [n] : \Lambda_{j,j} < 0\}$, $n^- = \# J^-$ and $V^- = \operatorname{span}\{u_j : j \in J^-\}$. Let $v \in V^-$. Then there exist $(a^j)_{j \in J^-}$ such that $v = \sum_{j \in J^-} a^j u_j$. We note that

$$\begin{aligned} U^*[\phi_\alpha]U &= U^*(U\Lambda U^*)U \\ &= (U^*U)\Lambda(U^*U) \\ &= I\Lambda I \\ &= \Lambda \end{aligned}$$

A previous exercise implies that

$$\begin{aligned} \alpha(v, v) &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k \alpha(u_j, u_k) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k [u_j]^* [\phi_\alpha][u_k] \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k ([e_j]^* U^*)[\phi_\alpha](U[e_k]) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (U^*[\phi_\alpha]U)_{j,k} \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (\Lambda)_{j,k} \\ &= \sum_{j \in J^-} |a^j|^2 \Lambda_{j,j} \\ &< 0 \end{aligned}$$

Since $v \in V^-$ is arbitrary, $\alpha|_{V^- \times V^-}$ is negative definite. Thus

$$\begin{aligned} \operatorname{ind} \alpha &\geq \dim V^- \\ &= n^- \end{aligned}$$

Set $J^+ = (J^-)^c$. Let $W \subset V$ be a subspace. Suppose that $\alpha|_{W \times W}$ is negative definite. For the sake of contradiction, suppose that there exists $j_0 \in J^+$ such that $u_{j_0} \in W$. Then

$$\begin{aligned} \alpha(u_{j_0}, u_{j_0}) &= [u_{j_0}]^* [\phi_\alpha][u_{j_0}] \\ &= [u_{j_0}]^* U\Lambda U^* [u_{j_0}] \\ &= \Lambda_{j_0, j_0} \\ &\geq 0 \end{aligned}$$

which is a contradiction since $\alpha|_{W \times W}$ is negative definite. Thus for each $j \in J^+$, $u_j \notin W$. $\square$

2.4.2 Symplectic Vector Spaces

Definition 2.4.2.1. Let V be a finite dimensional vector space and $\omega \in \Lambda^2(V)$. Then ω is said to be a **symplectic form** if ω is nondegenerate. In this case (V, ω) is said to be a **symplectic space**.

Exercise 2.4.2.2. Let V be a $2n$-dimensional vector space with basis $(a_j, b_j)_{j=1}^n$ and corresponding dual basis $(\alpha^j, \beta^j)_{j=1}^n$. Define $\omega \in \Lambda^2(V)$ by

$$\omega = \sum_{j=1}^n \alpha^j \wedge \beta^j$$

Then

1. for each $j, k \in \{1, \dots, n\}$,
   (a) $\omega(a_j, a_k) = 0$
   (b) $\omega(b_j, b_k) = 0$
   (c) $\omega(a_j, b_k) = \delta_{j,k}$
2. (V, ω) is a symplectic space

Proof.

1. Let $j, k \in \{1, \dots, n\}$.
   (a)
$$\begin{aligned}\omega(a_j, a_k) &= \sum_{l=1}^{n} \alpha^l \wedge \beta^l(a_j, a_k) \\ &= \sum_{l=1}^{n}[\alpha^l(a_j)\beta^l(a_k) - \alpha^l(a_k)\beta^l(a_j)] \\ &= 0\end{aligned}$$
   (b) Similar to (a)
   (c)
$$\begin{aligned}\omega(a_j, b_k) &= \sum_{l=1}^{n} \alpha^l \wedge \beta^l(a_j, b_k) \\ &= \sum_{l=1}^{n}[\alpha^l(a_j)\beta^l(b_k) - \alpha^l(b_k)\beta^l(a_j)] \\ &= \sum_{l=1}^{n} \alpha^l(a_j)\beta^l(b_k) \\ &= \sum_{l=1}^{n} \delta_{j,l}\delta_{l,k} \\ &= \delta_{j,k}\end{aligned}$$

2. Let $v \in V$. Then there exist $(q^j, p^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{j=1}^{n} q^j a_j + p^j b_j$. Suppose that for each $w \in V$, $\omega(v, w) = 0$. Let $k \in \{1, \dots, n\}$. Then
$$\begin{aligned}0 &= \omega(v, a_k) \\ &= \sum_{j=1}^{n} q^j \omega(a_j, a_k) + p^j \omega(b_j, a_k) \\ &= \sum_{j=1}^{n} p^j \delta_{j,k} \\ &= p^k\end{aligned}$$

Similarly,

$$\begin{aligned}
0 &= \omega(v, b_k) \\
&= \sum_{j=1}^{n} q^j \omega(a_j, b_k) + p^j \omega(b_j, b_k) \\
&= \sum_{j=1}^{n} q^j \delta_{j,k} \\
&= q^k
\end{aligned}$$

Since $k \in \{1, \dots, n\}$ is arbitrary, $v = 0$. Hence ω is nondegenerate. Therefore (V, ω) is symplectic.

$\square$

Exercise 2.4.2.3. Let (V, ω) be a symplectic space. Then $\dim V$ is even.

Proof. Set $n = \dim V$. Let $(e_j)_{j=1}^n$ be a basis for V. Define $[\omega] \in \mathbb{R}^{n \times n}$ by $[\omega]_{i,j} = \omega(e_i, e_j)$. Since $\omega \in \Lambda^2(V)$, $[\omega]^* = -[\omega]$. Therefore

$$\begin{aligned}
\det[\omega] &= \det[\omega]^* \\
&= \det(-[\omega]) \\
&= (-1)^n \det[\omega]
\end{aligned}$$

For the sake of contradiction, suppose that n is odd. Then $\det[\omega] = -\det[\omega]$ which implies that $\det[\omega] = 0$. Since ω is nondegenerate, $[\omega] \in GL(n, \mathbb{R})$. This is a contradiction. Hence n is even. $\square$

Definition 2.4.2.4. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. We define the **symplectic complement of** V, denoted $S^\perp$, by

$$S^\perp = \{v \in V : \text{ for each } w \in S, \, \omega(v, w) = 0\}$$

Exercise 2.4.2.5. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then $S^\perp$ is a subspace.

Proof. We note that

$$S^\perp = \bigcap_{v \in S} \ker \iota_v \omega$$

Hence $S^\perp$ is a subspace. $\square$

Exercise 2.4.2.6. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then

$$\dim V = \dim S + \dim S^\perp$$

Proof. $\square$

Exercise 2.4.2.7. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then $(S^\perp)^\perp = S$.

Proof. Let $v \in (S^\perp)^\perp$. Then for each $w \in S^\perp$, $\omega(v, w) = 0$. $\square$

2.5 Vector-Valued Covariant k-Tensors

Chapter 3

Smooth Manifolds

3.1 Topological Manifolds

Exercise 3.1.0.1. We have that $\mathbb{R}$ is homeomorphic to $(0, \infty)$

Proof. Define $f : \mathbb{R} \to (0, \infty)$ by $f(x) = e^x$. Then f is a homeomorphism. $\square$

Definition 3.1.0.2. Let $n \in \mathbb{N}$. We define the **upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}^n$, by

$$\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and we define

$$\partial\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$$
$$\operatorname{Int}\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

We endow $\mathbb{H}^n$, $\partial\mathbb{H}^n$ and $\operatorname{Int}\mathbb{H}^n$ with the subspace topology inherited from $\mathbb{R}^n$.
We define the projection map $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ by

$$\pi(x_1, \ldots, x_{n-1}, 0) = (x_1, \ldots, x_{n-1})$$

Definition 3.1.0.3. We define $\mathbb{R}^0 = \{0\}$ and $\mathbb{H}^0 = \varnothing$ endowed with the discrete topology.

Exercise 3.1.0.4. Let $n \in \mathbb{N}$.

1. $\partial\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n-1}$
2. $\operatorname{Int}\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^n$

Proof.

1. Let $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ be the projection map given by

$$\pi(x_1, \ldots, x_{n-1}, 0) = (x_1, \ldots, x_{n-1})$$

   Then π is a homeomorphism.

2. Define $f : \mathbb{R}^n \to \operatorname{Int}\mathbb{H}^n$ by $f(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, e^{x_n})$. Then f is a homeomorphism.

$\square$

Definition 3.1.0.5. Let $(M, \mathcal{T})$ be a topological space and $n \in \mathbb{N}_0$. Let $U \subset M$ and $V \subset \mathbb{H}^n$ and $\phi : U \to V$. Then (U, ϕ) is said to be a n**-coordinate chart on** $(M, \mathcal{T})$ if

- $U \in \mathcal{T}$
- $V \in \mathcal{T}_{\mathbb{H}^n}$

- ϕ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n} \cap V)$-homeomorphism

We denote the set of all n-coordinate charts on M by $X^n(M, \mathcal{T})$.

Note 3.1.0.6. We will write $X^n(M)$ in place of $X^n(M, \mathcal{T})$ when the topology is not ambiguous.

Definition 3.1.0.7. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be **locally Euclidean of dimension** n if for each $p \in M$, there exists $(U, \phi) \in X^n(M)$ such that $p \in U$.

Definition 3.1.0.8. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be an n**-dimensional topological manifold** if

1. M is Hausdorff
2. M is second-countable
3. M is locally Euclidean of dimension n

Theorem 3.1.0.9. Topological Invariance of Dimension:
Let M be an n-dimensional toplogical manifold and N a p-dimensional toplogical manifold. If M and N are homeomorphic, then $n = p$.

Note 3.1.0.10. In light of the previous theorem, we write $X(M)$ in place of $X^n(M)$ and refer to n-coordinate charts as coordinate charts when the context is clear.

Definition 3.1.0.11. Let M be an n-dimensional topological manifold and $(U, \phi) \in X(M)$. Then (U, ϕ) is said to be an

- **interior chart** if $\phi(U)$ is open in $\mathbb{R}^n$
- **boundary chart** if $\phi(U)$ is open in $\mathbb{H}^n$ and $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$

We denote the set of all interior charts on M and the set of all boundary charts on M by $X_{\text{Int}}(M)$ and $X_\partial(M)$ respectively.

Exercise 3.1.0.12. Let M be an n-dimensional topological manifold. Then

1. $X(M) = X_{\text{Int}}(M) \cup X_\partial(M)$
2. $X_{\text{Int}}(M) \cap X_\partial(M) = \varnothing$

Proof.

1. By definition, $X_{\text{Int}}(M) \cup X_\partial(M) \subset X(M)$. Let $(U, \phi) \in X(M)$. Since (U, ϕ) is a coordinate chart on M, $\phi(U)$ is open in $\mathbb{R}^n$ or $\phi(U)$ is open in $\mathbb{H}^n$. If $\phi(U)$ is open in $\mathbb{R}^n$, then

$$\begin{aligned} (U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M) \end{aligned}$$

Suppose that $\phi(U)$ is open in $\mathbb{H}^n$. If $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$, then $\phi(U)$ is open in $\mathbb{R}^n$ and

$$\begin{aligned} (U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M) \end{aligned}$$

Suppose that $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$. Then

$$\begin{aligned} (U, \phi) &\in X_\partial(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M) \end{aligned}$$

Since $(U, \phi) \in X(M)$ is arbitrary, $X(M) \subset X_{\text{Int}}(M) \cup X_\partial(M)$. Therefore $X(M) = X_{\text{Int}}(M) \cup X_\partial(M)$.

2. For the sake of contradiction, suppose that $X_{\text{Int}}(M) \cup X_\partial(M) \neq \varnothing$. Then there exists $(U, \phi) \in X(M)$ such that $(U, \phi) \in X_{\text{Int}}(M)$ and $(U, \phi) \in X_\partial(M)$. Therefore $\phi(U)$ is open in $\mathbb{R}^n$, $\phi(U)$ is open in $\mathbb{H}^n$ and $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$. Since $\phi(U)$ is open in $\mathbb{R}^n$ and $\phi(U) \subset \mathbb{H}^n$, $\phi(U) \subset \operatorname{Int}\mathbb{H}^n$ and therefore $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$ which is a contradiction.

□

Definition 3.1.0.13. Let M be an n-dimensional topological manifold. We define the

- **interior** of M, denoted $\operatorname{Int} M$, by

$$\operatorname{Int} M = \{p \in M : \text{there exists } (U, \phi) \in X_{\text{Int}}(M) \text{ such that } p \in U\}$$

- **boundary** of M, denoted ∂M, by

$$\partial M = \{p \in M : \text{there exists } (V, \psi) \in X_\partial(M) \text{ such that } p \in V \text{ and } \psi(p) \in \partial\mathbb{H}^n\}$$

Exercise 3.1.0.14. Let M be an n-dimensional topological manifold. Let $(U, \phi) \in X_{\text{Int}}(M)$. Then $U \subset \operatorname{Int} M$.

Proof. Let $p \in U$. Since $(U, \phi) \in X_{\text{Int}}(M)$ and $p \in U$, by definition, $p \in \operatorname{Int} M$. Since $p \in U$ is arbitrary, $U \subset \operatorname{Int} M$. □

Exercise 3.1.0.15. Let M be an n-dimensional topological manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. If $\phi(p) \notin \partial\mathbb{H}^n$, then $p \in \operatorname{Int} M$.

Proof. Suppose that $\phi(p) \notin \partial\mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int}\mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then U' is open in M and $\phi' : U' \to B'$ is a homeomorphism. Hence $(U', \phi') \in X_{\text{Int}}(M)$. Since $\phi(p) \in B'$, we have that $p \in U'$. By definition, $p \in \operatorname{Int} M$. □

Exercise 3.1.0.16. Let M be an n-dimensional topological manifold. Then

1. $M = \operatorname{Int} M \cup \partial M$
2. $\operatorname{Int} M \cap \partial M = \varnothing$
   Hint: simply connected

Proof.

1. By definition, $\operatorname{Int} M \cup \partial M \subset M$. Let $p \in M$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. A previous exercise implies that $(U, \phi) \in X_{\text{Int}}(M) \cup X_\partial(M)$. If $(U, \phi) \in X_{\text{Int}}(M)$, then by definition,

$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Suppose that $(U, \phi) \in X_\partial(M)$. If $\phi(p) \in \partial\mathbb{H}^n$, then by definition,

$$\begin{aligned} p &\in \partial M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Suppose that $\phi(p) \notin \partial\mathbb{H}^n$. The previous exercise implies that $p \in \operatorname{Int} M$. Therefore,

$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Since $p \in M$ is arbitrary, $M \subset \operatorname{Int} M \cup \partial M$. Therefore $M = \operatorname{Int} M \cup \partial M$.

2. For the sake of contradiction, suppose that $\operatorname{Int} M \cap \partial M \neq \varnothing$. Then there exists $p \in M$ such that $p \in \operatorname{Int} M \cap \partial M$. By definition, there exists $(U, \phi) \in X_{\text{Int}}(M)$, $(V, \psi) \in X_{\partial}(M)$ such that $p \in U \cap V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in $\mathbb{H}^n$, $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi|_{U\cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.
   Since $\psi(U \cap V)$ is open in $\mathbb{H}^n$, there exists an $B_\psi \subset \psi(U \cap V)$ such that B_ψ is open in $\mathbb{H}^n$, B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$. Since $\phi(U \cap V)$ is open in $\mathbb{R}^n$, B_ϕ is open in $\mathbb{R}^n$. Since B_ψ is simply connected and $\phi|_{U\cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism, B_ϕ is simply connected.
   Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Then $\phi \circ \psi^{-1} : B'_\psi \to B'_\phi$ is a homeomorphism. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_ψ is simply connected. Since B_ϕ is open in $\mathbb{R}^n$, B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ. So $\partial M \cap \operatorname{Int} M = \varnothing$.

□

Exercise 3.1.0.17. Let M be an n-dimensional topological manifold. Then

1. $\operatorname{Int} M$ is open
2. ∂M is closed

Proof.

1. Let $p \in \operatorname{Int} M$. Then there exists $(U, \phi) \in X_{\text{Int}}(M)$ such that $p \in U$. By definition, U is open and a previous exercise implies that $U \subset \operatorname{Int} M$. Since $p \in \operatorname{Int} M$ is arbitrary, we have that for each $p \in \operatorname{Int} M$, there exists $U \subset \operatorname{Int} M$ such that U is open. Hence $\operatorname{Int} M$ is open.
2. Since $\partial M = (\operatorname{Int} M)^c$, and $\operatorname{Int} M$ is open, we have that ∂M is closed.

□

Exercise 3.1.0.18. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $p \in U$. If $p \in \partial M$, then $(U, \phi) \in X_\partial(M)$.
Hint: simply connected

Proof. Suppose that $p \in \partial M$. Then there exists a $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in $\mathbb{H}^n$, $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi|_{U\cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.
Since $\psi(U \cap V)$ is open in $\mathbb{H}^n$, there exists $B_\psi \subset \psi(U \cap V)$ such B_ψ is open in $\mathbb{H}^n$, B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$.
For the sake of contradiction, suppose that $(U, \phi) \in X_{\text{Int}}(M)$. Then $\phi(U)$ is open in $\mathbb{R}^n$. Hence $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and B_ϕ is open in $\mathbb{R}^n$. Since $\phi|_{U\cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism, B_ϕ is simply connected. Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_ψ is simply connected. Since B_ϕ is open in $\mathbb{R}^n$, B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ. So $(U, \phi) \notin X_{\text{Int}}(M)$. Since $(X_{\text{Int}}(M))^c = X_\partial(M)$, we have that $(U, \phi) \in X_\partial(M)$.

□

Exercise 3.1.0.19. Let M be an n-dimensional topological manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. Then

1. $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$
2. $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n$

Proof.

1. Suppose that $p \in \partial M$. For the sake of contradiction, suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then $p \in U'$ and $(U', \phi') \in X_{\text{Int}}(M)$. Since $p \in U'$, the previous exercise implies that $(U', \phi') \in X_\partial(M)$. This is a contradiction since $X_{\text{Int}}(M) \cap X_\partial(M) = \varnothing$. So $\phi(p) \in \partial \mathbb{H}^n$.
   Conversely, suppose that $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$.

2. A previous exercise implies that $\operatorname{Int} M = (\partial M)^c$. Part (1) implies that

$$\begin{aligned} p &\in (\partial M)^c \\ &= \operatorname{Int} M \end{aligned}$$

if and only if

$$\begin{aligned} \phi(p) &\in (\partial \mathbb{H}^n)^c \\ &= \operatorname{Int} \mathbb{H}^n \end{aligned}$$

□

Exercise 3.1.0.20. Let M be an n-dimensional topological manifold and $p \in M$. Then $p \in \partial M$ iff for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.

Proof. Suppose that $p \in \partial M$. Let $(U, \phi) \in X(M)$. Suppose that $p \in U$. The previous two exercises imply that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.
Conversely, suppose that for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. By assumption, $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$. □

Exercise 3.1.0.21. Let M be an n-dimensional topological manifold. Let $(U, \phi) \in X_\partial(M)$. Then

1. $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$
2. $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int} \mathbb{H}^n$

Proof.

1. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$. Let $q \in \phi(U \cap \partial M)$. Then there exists $p \in U \cap \partial M$ such that $\phi(p) = q$. Since $p \in \partial M$, $\phi(p) \in \partial \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \partial \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \partial M)$ is arbitrary, $\phi(U \cap \partial M) \subset \phi(U) \cap \partial \mathbb{H}^n$.
Let $q \in \phi(U) \cap \partial \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \partial \mathbb{H}^n$, we have that $p \in \partial M$. Hence $p \in U \cap \partial M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \partial M)$. Thus $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$.

2. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Let $q \in \phi(U \cap \operatorname{Int} M)$. Then there exists $p \in U \cap \operatorname{Int} M$ such that $\phi(p) = q$. Since $p \in \operatorname{Int} M$, $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \operatorname{Int} \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \operatorname{Int} M)$ is arbitrary, $\phi(U \cap \operatorname{Int} M) \subset \phi(U) \cap \operatorname{Int} \mathbb{H}^n$.
Let $q \in \phi(U) \cap \operatorname{Int} \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \operatorname{Int} \mathbb{H}^n$, we have that $p \in \operatorname{Int} M$. Hence $p \in U \cap \operatorname{Int} M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \operatorname{Int} M)$. Thus $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int} \mathbb{H}^n$.

□

Definition 3.1.0.22. Let M be an n-dimensional topological manifold and $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ the projection map. For $(U, \phi) \in X_\partial(M)$, we define $\bar{U} \subset \partial M$ and $\bar{\phi} : \bar{U} \to \pi(\phi(\bar{U}))$ by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi \circ \phi|_{\bar{U}}$ respectively.

Exercise 3.1.0.23. Let M be an n-dimensional topological manifold, and $\lambda : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ a homeomorphism. Then $\{(\bar{U}, \bar{\phi}) : (U, \phi) \in X_\partial(M)\} \subset X^{n-1}_{\text{Int}}(\partial M)$.

Proof. Let $(U, \phi) \in X_\partial(M)$.

1. Since U is open in M, $\bar{U} = U \cap \partial M$ is open in ∂M.
2. Since $(U, \phi) \in X_\partial(M)$, $\phi(U)$ is open in $\mathbb{H}^n$. A previous exercise implies that $\phi(\bar{U}) = \phi(U) \cap \partial\mathbb{H}^n$ which is open in $\partial\mathbb{H}^n$. Since $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ is a homeomorphism, we have that $\pi(\phi(\bar{U}))$ is open in $\mathbb{R}^{n-1}$.
3. Since $\phi|_{\bar{U}} : \bar{U} \to \phi(U) \cap \partial\mathbb{H}^n$ and $\pi|_{\phi(\bar{U})} : \phi(\bar{U}) \to \lambda(\phi(\bar{U}))$ are homeomorphisms, we have that $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$ is a homeomorphism.

Hence $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$. □

Exercise 3.1.0.24. Let M be an n-dimensional topological manifold. Then

1. ∂M is an $(n-1)$-dimensional topological manifold
2. $\partial(\partial M) = \varnothing$

Proof.

1. (a) Since M is Hausdorff, ∂M is Hausdorff.
   (b) Since M is second-countable, ∂M is second countable.
   (c) Let $p \in \partial M$. Then there exists $(U, \phi) \in X_\partial(M)$ such that $\phi(p) \in \partial\mathbb{H}^n$. Then $p \in \bar{U}$ and the previous exercise implies that $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$. Thus ∂M is locally Euclidean of dimension $n-1$.

   Hence ∂M is an $(n-1)$-dimensional topological manifold.
2. Let $p \in \partial M$. Part (1) implies that there exists $(U, \phi) \in X^{n-1}_{\text{Int}}(\partial M)$ such that $p \in U$. Thus $p \in \text{Int}\,\partial M$. Since $p \in \partial M$ is arbitrary, $\text{Int}\,\partial M = \partial M$. Hence

$$\begin{aligned}\partial(\partial M) &= (\text{Int}(\partial M))^c \\ &= (\partial M)^c \\ &= \varnothing\end{aligned}$$

□

Exercise 3.1.0.25. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M, then $(U', \phi|_{U'}) \in X^n(M)$.

Proof. Suppose that U' is open in M. Set $\phi' = \phi|_{U'}$.

- By assumption U' is open in M.
- Since U' is open in M, we have that $U' = U' \cap U$ is open in U. Since ϕ is a homeomorphism and U' is open in U, we have that $\phi(U')$ is open in $\phi(U)$. By assumption $\phi(U)$ is open in $\mathbb{R}^n$ or $\phi(U)$ is open in $\mathbb{H}^n$. Therefore $\phi'(U')$ is open in $\mathbb{R}^n$ or $\phi'(U')$ is open in $\mathbb{H}^n$.
- Since $\phi : U \to V$ is a homeomorphism, $\phi' : U' \to \phi'(U')$ is a homeomorphism.

So $(U', \phi') \in X^n(M)$. □

Note 3.1.0.26. Since U is open in M, U' being open in U is equivalent to U' being open in M, so we could have also assumed that U' is open in U.

Exercise 3.1.0.27. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then

$$X^n(U) = \{(V, \psi) \in X^n(M) : V \subset U\}$$

Proof. Suppose that U is open and set $A = \{(V, \psi) \in X^n(M) : V \subset U\}$. Let $(V, \psi) \in X^n(U)$. By definition of $X^n(U)$, V is open in U. Thus, there exists $W \subset M$ such that W is open in M and $V = U \cap W$. Since U is open in M, we have that $V = U \cap W$ is open in M. Hence $(V, \psi) \in X^n(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X^n(U)$ is arbitary, $X^n(U) \subset A$.
Conversely, suppose that $(V, \psi) \in A$. Then $(V, \psi) \in X^n(M)$ and $V \subset U$. By definition of $X^n(M)$, V is open in M. Since $V \subset U$, we have that $V = V \cap U$ is open in U. Hence $(V, \psi) \in X^n(U)$. Since $(V, \psi) \in X^n(U)$ is arbitary, $A \subset X^n(U)$. Hence $X^n(A) = A$. $\square$

Exercise 3.1.0.28. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M, then $(U', \phi|_{U'}) \in X^n(U)$.

Proof. Suppose that U' is open in M. A previous exercise implies that $(U', \phi') \in X^n(M)$. The previous exercise implies that $(U', \phi') \in X^n(U)$. $\square$

Exercise 3.1.0.29. Topological Open Submanifolds:
Let M be an n-dimensional topological manifold and $U \subset M$ open. Then U is an n-dimensional topological manifold.

Proof.

1. Since M is Hausdorff, U is Hausdorff.

2. M is second-countable, U is second countable.

3. Let $p \in U$. Since then there exists $(V, \psi) \in X^n(M)$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{U \cap V}$. The previous exercise implies that $(V', \psi') \in X^n(U)$. Therefore U is locally Euclidean of dimension n.

Hence U is an n-dimensional topological manifold. $\square$

Exercise 3.1.0.30. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then

1. $X_{\text{Int}}(U) = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$

2. $X_{\partial}(U) = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$

Proof. Suppose that U is open in M.

1. Set $A = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\text{Int}}(U)$. By definition of $X_{\text{Int}}(U)$, V is open in U and $\phi(V)$ is open in $\mathbb{R}^n$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_{\text{Int}}(M)$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_{\text{Int}}(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X_{\text{Int}}(U)$ is arbitrary, $X_{\text{Int}}(U) \subset A$.
   Conversely, let $(V, \psi) \in A$. Then $(V, \psi) \in X_{\text{Int}}(M)$ and $V \subset U$. By definition of $X_{\text{Int}}(M)$, V is open in M and $\phi(V)$ is open in $\mathbb{R}^n$. Thus $V = V \cap U$ is open in U. So $(V, \psi) \in X_{\text{Int}}(U)$. Since $(V, \psi) \in A$ is arbitrary, $A \subset X_{\text{Int}}(U)$. Thus $X_{\text{Int}}(U) = A$.

2. Set $B = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\partial}(U)$. By definition of $X_{\partial}(U)$, V is open in U, $\phi(V)$ is open in $\mathbb{H}^n$ and $\partial \mathbb{H}^n \cap \phi(V) \neq \varnothing$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_{\partial}(M)$, which implies that $(V, \psi) \in B$. Since $(V, \psi) \in X_{\partial}(U)$ is arbitrary, $X_{\partial}(U) \subset B$.
   Conversely, let $(V, \psi) \in B$. Then $(V, \psi) \in X_{\partial}(M)$ and $V \subset U$. By definition of $X_{\partial}(M)$, V is open in M, $\phi(V)$ is open in $\mathbb{H}^n$ and $\partial \mathbb{H}^n \cap \phi(V) \neq \varnothing$. Thus $V = V \cap U$ is open in U. So $(V, \psi) \in X_{\partial}(U)$. Since $(V, \psi) \in B$ is arbitrary, $B \subset X_{\partial}(U)$. Thus $X_{\partial}(U) = B$.

$\square$

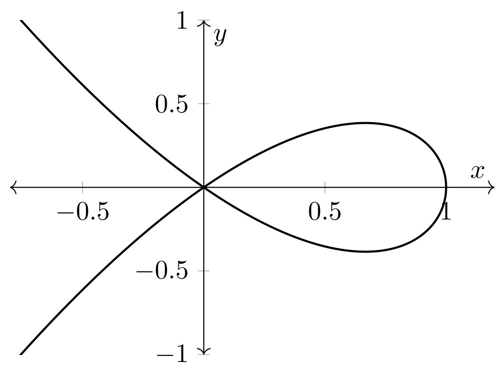
Exercise 3.1.0.31. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then $\partial U = \partial M \cap U$.

Proof. Suppose that U is open. Let $p \in \partial U$. Then there exists $(V, \psi) \in X_\partial(U)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Since U is open, the previous exercise implies that $(V, \psi) \in X_\partial(M)$. Thus $p \in \partial M$. Since $p \in \partial U$ is arbitrary, $\partial U \subset \partial M$. Since $\partial U \subset U$, we have that $\partial U \subset \partial M \cap U$.
Conversely, let $p \in \partial M \cap U$. Since $p \in \partial M$, there exists $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Set $V' = V \cap U$ and $\psi' = \psi|_{V'}$. Then $p \in V'$ since V and U are open in M, V' is open in M. A previous exercise implies that $(V', \psi') \in X(M)$. Since $p \in \partial M$, a previous exercise implies that $(V', \psi') \in X_\partial(M)$. The previous exercise implies that $(V', \psi') \in X_\partial(U)$. Since $\psi'(p) \in \partial \mathbb{H}^n$, $p \in \partial U$. Since $p \in \partial M \cap U$ is arbitrary, $\partial M \cap U \subset \partial U$. Hence $\partial U = \partial M \cap U$.
label exercises and reference them!!! □

Exercise 3.1.0.32. Graph of Continuous Function:
Let $f \in C(\mathbb{R})$. Set $M = \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$ (i.e. the graph of f). Then M is a 1-dimensional manifold.

Proof. Set $U = \mathbb{R}$ and define $\phi : U \to M$ by $\phi(x) = (x, f(x))$. Then $\phi^{-1} = \pi_1$. Since f is continuous, ϕ is continous. Since π_1 is continiuous, ϕ is a homeomorphism. □

Exercise 3.1.0.33. Nodal Cubic:
Let $M = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2 - x^3\}$. We equip M with the subspace topology.



Then M is not a 1-dimensional topological manifold.
Hint: connected components

Proof. Suppose that M is a 1-dimensional manifold. Set $p = (0, 0)$. Then there exists $(U, \phi) \in X(M)$ such that $p \in U$. Since $\phi(U)$ is open (in $\mathbb{R}$ or $\mathbb{H}$), there exists a $B \subset \phi(U)$ such that B is open (in $\mathbb{R}$ or $\mathbb{H}$), B is connected and $\phi(p) \in B$. Set $V = \phi^{-1}(B)$, $V' = V \setminus \{p\}$ and $B' = B \setminus \{\phi(p)\}$. Then $\phi : V \to B$ and $\phi' : V' \to B'$ are homeomorphisms. Since B is open (in $\mathbb{R}$ or $\mathbb{H}$) and connected, B' has at most two connected components. Then V' This is a contradiction since V' has four connected components and B' and V' are homeomorphic. □

Exercise 3.1.0.34. Topological Manifold Chart Lemma:
Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

- for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection
- for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous
- there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$

- for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : V \in \mathcal{T}_{\mathbb{H}^n} \text{ and } \alpha \in \Gamma\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$

Then

1. $\mathcal{B}$ is a basis for $\mathcal{T}_M$
   Hint: For $B_1, B_2 \subset \mathbb{H}^n$, $\phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) = \phi_{\alpha_1}^{-1}(B_1 \cap [\phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}} \circ (\phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}})^{-1}(B_2)])$
2. $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold
3. $\mathcal{T}_M$ is the unique topology $\mathcal{T}$ on M such that $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T})$

Proof.

1.
   - By assumption, $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
   - Let $A_1, A_2 \in \mathcal{B}$ and $p \in A_1 \cap A_2$. By definition, there exist $\alpha_1, \alpha_2 \in \Gamma$ and $B_1, B_2 \subset \mathbb{H}^n$ such that B_1, B_2 are open in $\mathbb{H}^n$ and

$$\begin{aligned} A_1 &= \phi_{\alpha_1}^{-1}(B_1) \\ &\subset U_{\alpha_1} \end{aligned} \qquad\qquad \begin{aligned} A_2 &= \phi_{\alpha_2}^{-1}(B_2) \\ &\subset U_{\alpha_2} \end{aligned}$$

Set $\psi_1 = \phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}}$ and $\psi_2 = \phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}}$. We note that

$$\begin{aligned} \psi_1^{-1}(B_1) &= U_{\alpha_2} \cap \phi_{\alpha_1}^{-1}(B_1) \\ &= U_{\alpha_2} \cap A_1 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned} \qquad\qquad \begin{aligned} \psi_2^{-1}(B_2) &= U_{\alpha_1} \cap \phi_{\alpha_2}^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned}$$

Let $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Then $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. Hence $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_1}(q) \in \psi_1 \circ \psi_2^{-1}(B_2)$. This implies that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(B_1) \\ &= A_1 \end{aligned}$$

and since $\psi_2^{-1}(B_2) \subset U_{\alpha_1} \cap U_{\alpha_2}$ and $\phi_{\alpha_1} : U_{\alpha_1} \to \phi_{\alpha_1}(U_{\alpha_1})$ is a bijection, we have that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(\psi_1 \circ \psi_2^{-1}(B_2)) \\ &= \psi_2^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \end{aligned}$$

Thus

$$\begin{aligned} q &\in A_1 \cap (U_{\alpha_1} \cap A_2) \\ &= A_1 \cap A_2 \end{aligned}$$

Since $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$ is arbitrary, we have that $\phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \subset A_1 \cap A_2$. Conversely, let

$$\begin{aligned} q &\in A_1 \cap A_2 \\ &= \phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) \end{aligned}$$

Then $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_2}(q) \in B_2$. Since $A_1 \cap A_2 \subset U_{\alpha_1} \cap U_{\alpha_2}$, we have that

$$\begin{aligned}\psi_2(q) &= \phi_{\alpha_2}(q)\\ &\in B_2\end{aligned}$$

which implies that $q \in \psi_2^{-1}(B_2)$. Therefore

$$\begin{aligned}\phi_{\alpha_1}(q) &= \psi_1(q)\\ &\in \psi_1(\psi_2^{-1}(B_2))\\ &= \psi_1 \circ \psi_2^{-1}(B_2)\end{aligned}$$

Hence $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. This implies that $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Since $q \in A_1 \cap A_2$ is arbitrary, we have that $A_1 \cap A_2 \subset \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Thus

$$\begin{aligned}A_1 \cap A_2 &= \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])\\ &\in \mathcal{B}\end{aligned}$$

Thus $\mathcal{B}$ is a basis for $\mathcal{T}_M$.

2. (a) **(locally Euclidean of dimension n):**
Let $\alpha \in \Gamma$. By definition, for each $B \subset \mathcal{T}_{\mathbb{H}^n}$,

$$\begin{aligned}\phi_\alpha^{-1}(B) &\in \mathcal{B}\\ &\subset \mathcal{T}_M\end{aligned}$$

Hence ϕ_α is continuous.
Let $A \in \mathcal{T}_{U_\alpha}$. Then there exists $U \subset \mathcal{T}_M$ such that $A = U \cap U_\alpha$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_M$, there exists $\Gamma' \subset \Gamma$, $(V_\beta)_{\beta\in\Gamma'} \subset \mathcal{T}_{\mathbb{H}^n}$ such that $U = \bigcup_{\beta\in\Gamma'} \phi_\beta^{-1}(V_\beta)$. Thus

$$\begin{aligned}A &= U \cap U_\alpha\\ &= \left[\bigcup_{\beta\in\Gamma'} \phi_\beta^{-1}(V_\beta)\right] \cap U_\alpha\\ &= \bigcup_{\beta\in\Gamma'} [\phi_\beta^{-1}(V_\beta) \cap U_\alpha]\end{aligned}$$

Let $\beta \in \Gamma'$. Since $\phi_\alpha(U_\alpha \cap U_\beta) \subset \phi_\alpha(U_\alpha)$ and $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$, we have that

$$\begin{aligned}\phi_\alpha(U_\alpha \cap U_\beta) &= \phi_\alpha(U_\alpha) \cap \phi_\alpha(U_\alpha \cap U_\beta)\\ &\in \mathcal{T}_{\phi_\alpha(U_\alpha)}\end{aligned}$$

Therefore $\mathcal{T}_{\phi_\alpha(U_\alpha\cap U_\beta)} \subset \mathcal{T}_{\phi_\alpha(U_\alpha)}$. Since $(\phi_\beta|_{U_\alpha\cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha\cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous, we have that $(\phi_\beta|_{U_\alpha\cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha\cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{H}^n$ is continuous and therefore

$$\begin{aligned}[(\phi_\beta|_{U_\alpha\cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha\cap U_\beta})^{-1}]^{-1}(V_\beta) &\in \mathcal{T}_{\phi_\alpha(U_\alpha\cap U_\beta)}\\ &\subset \mathcal{T}_{\phi_\alpha(U_\alpha)}\end{aligned}$$

Since $\beta \in \Gamma'$ is arbitrary, we have that

$$\begin{aligned}\phi_\alpha(A) &= \phi_\alpha\bigg(\bigcup_{\beta\in\Gamma'}[\phi_\beta^{-1}(V_\beta)\cap U_\alpha]\bigg)\\ &= \bigcup_{\beta\in\Gamma'}\phi_\alpha(\phi_\beta^{-1}(V_\beta)\cap U_\alpha)\\ &= \bigcup_{\beta\in\Gamma'}(\phi_\alpha|_{U_\alpha\cap U_\beta})\circ(\phi_\beta|_{U_\alpha\cap U_\beta})^{-1}(V_\beta)\\ &= \bigcup_{\beta\in\Gamma'}[(\phi_\beta|_{U_\alpha\cap U_\beta})\circ(\phi_\alpha|_{U_\alpha\cap U_\beta})^{-1}]^{-1}(V_\beta)\\ &\in \mathcal{T}_{\phi_\alpha(U_\alpha)}\end{aligned}$$

Since $A \in \mathcal{T}_{U_\alpha}$ is arbitrary, $\phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to U_\alpha$ is continuous. Hence $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism and $(U_\alpha, \phi_\alpha) \in X^n(M)$. Since $M = \bigcup_{\alpha\in\Gamma} U_\alpha$, we have that M is locally Euclidean of dimension n.

(b) **(Hausdorff):**
Let $p, q \in M$. Suppose that $p \neq q$. Then there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$.

- Suppose that there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$. Since $p \neq q$, $\phi_\alpha(p) \neq \phi_\alpha(q)$. Since $\mathbb{H}^n$ is Hausdorff, there exist $V_p, V_q \subset \phi(U_\alpha)$ such that V_p and V_q are open in $\mathbb{H}^n$, $p \in V_p$, $q \in V_q$ and $V_p \cap V_q = \varnothing$. Set $U_p = \phi_\alpha^{-1}(V_p)$ and $U_q = \phi_\alpha^{-1}V_q$. Then U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$.
- Suppose that there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$. Set $U_p = U_\alpha$ and $U_q = U_\beta$. Then U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$.

Thus for each $p, q \in M$ there exist $U_p, U_q \subset M$ such that U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$. Hence

(c) **(second-countable):**
By assumption, there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha\in\Gamma'} U_\alpha$. Let $\alpha \in \Gamma'$. Since $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$ and $\mathbb{H}^n$ is second-countable, we have that $\phi_\alpha(U_\alpha)$ is second-countable. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism, we have that U_α is second-countable. Since $M = \bigcup_{\alpha\in\Gamma'} U_\alpha$, an exercise in topology cite implies that M is second-countable.

3. Let $\mathcal{T}$ be a topology on M. Suppose that $(U_\alpha, \phi_\alpha)_{\alpha\in\Gamma} \subset X^n(M, \mathcal{T})$. Then for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}$ and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism.
Let $U \in \mathcal{B}$. By definition, there exists $\alpha \in \Gamma$ and $V \in \mathcal{T}_{\mathbb{H}^n}$ such that $U = \phi_\alpha^{-1}(V)$. Since $U_\alpha \in \mathcal{T}$, we have that $\mathcal{T} \cap U_\alpha \subset \mathcal{T}$. Since $V \cap \phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$, and ϕ_α is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that

$$\begin{aligned}U &= \phi_\alpha^{-1}(V)\\ &= \phi_\alpha^{-1}(V \cap \phi_\alpha(U_\alpha))\\ &\in \mathcal{T} \cap U_\alpha\\ &\subset \mathcal{T}\end{aligned}$$

Since $U \in \mathcal{B}$ is arbitrary, $\mathcal{B} \subset \mathcal{T}$. Therefore

$$\begin{aligned}\mathcal{T}_M &= \tau(\mathcal{B})\\ &\subset \tau(\mathcal{T})\\ &= \mathcal{T}\end{aligned}$$

Conversely, Let $U \in \mathcal{T}$ and $\alpha \in \Gamma$. Then $U \cap U_\alpha \in \mathcal{T} \cap U_\alpha$. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that $\phi_\alpha(U \cap U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$. Since $U_\alpha \in \mathcal{T}_M$, $\mathcal{T}_M \cap U_\alpha \subset \mathcal{T}_M$. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T}_M \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that

$$\begin{aligned} U \cap U_\alpha &= \phi_\alpha^{-1}(\phi_\alpha(U \cap U_\alpha)) \\ &\in \mathcal{T}_M \cap U_\alpha \\ &\subset \mathcal{T}_M \end{aligned}$$

Then

$$\begin{aligned} U &= U \cap M \\ &= U \cap \left(\bigcup_{\alpha \in \Gamma} U_\alpha \right) \\ &= \bigcup_{\alpha \in \Gamma} (U \cap U_\alpha) \\ &\in \mathcal{T}_M \end{aligned}$$

Since $U \in \mathcal{T}$ is arbitrary, $\mathcal{T} \subset \mathcal{T}_M$. Thus $\mathcal{T} = \mathcal{T}_M$.

$\square$

Exercise 3.1.0.35. Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

- for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection
- for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous
- there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$
- for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Then there exists a unique topology $\mathcal{T}_M$ on M such that $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T}_M)$.

Proof. Immediate by previous exercise. $\square$

3.2 Smooth Manifolds

Definition 3.2.0.1. Let M be an n-dimensional topological manifold and $(U,\phi),(V,\psi)\in X(M)$. Then (U,ϕ) and (V,ψ) are said to be **smoothly compatible** if

$$\psi|_{U\cap V}\circ(\phi|_{U\cap V})^{-1}:\phi(U\cap V)\to\psi(U\cap V) \text{ is a diffeomorphism}$$

Definition 3.2.0.2. Let M be an n-dimensional topological manifold.

- Let $\mathcal{A}\subset X(M)$. Then $\mathcal{A}$ is said to be an **atlas on** M if $M\subset \bigcup\limits_{(U,\phi)\in\mathcal{A}} U$.
- Let $\mathcal{A}$ be an atlas on M. Then $\mathcal{A}$ is said to be **smooth** if for each $(U,\phi),(V,\psi)\in\mathcal{A}$, (U,ϕ) and (V,ψ) are smoothly compatible.
- Let $\mathcal{A}$ be a smooth atlas on M. Then $\mathcal{A}$ is said to be **maximal** if for each smooth atlas $\mathcal{B}$ on M, $\mathcal{A}\subset\mathcal{B}$ implies that $\mathcal{A}=\mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on** M.
- Let $\mathcal{A}$ be an atlas on M. Then $(M,\mathcal{A})$ is said to be an **n-dimensional smooth manifold** if $\mathcal{A}$ is a smooth structure on M.

Exercise 3.2.0.3. Let M be an n-dimensional topological manifold and $\mathcal{B}$ a smooth atlas on M. Then there exists a unique smooth structure $\mathcal{A}$ on M such that $\mathcal{B}\subset\mathcal{A}$.

Proof. Define

$$\mathcal{A}=\{(U,\phi)\in X(M): \text{ for each } (V,\psi)\in\mathcal{B},\ (U,\phi) \text{ and } (V,\psi) \text{ are smoothly compatible}\}$$

Clearly $\mathcal{B}\subset\mathcal{A}$. Let (U,ϕ) and $(V,\psi)\in\mathcal{A}$. Define $F:\phi(U\cap V)\to\psi(U\cap V)$ by

$$F=\psi|_{U\cap V}\circ(\phi|_{U\cap V})^{-1}$$

Let $q\in\phi(U\cap V)$. Set $p=\phi^{-1}(q)$. Since $p\in U\cap V\subset M$, there exists $(W,\chi)\in\mathcal{B}$ such that $p\in W$. By definition of $\mathcal{A}$, $\psi|_{W\cap V}\circ(\chi|_{W\cap V})^{-1}:\chi(W\cap V)\to\psi(W\cap V)$ and $\chi|_{U\cap W}\circ(\phi|_{U\cap W})^{-1}:\phi(U\cap W)\to\chi(U\cap W)$ are diffeomorphisms. Set $N=U\cap W\cap V$. Then $q\in\phi(N)\subset\phi(U\cap V)$ and

$$\begin{aligned}F|_{\phi(N)}&=\psi|_N\circ(\phi|_N)^{-1}\\&=[\psi|_N\circ(\chi|_N)^{-1}]\circ[\chi|_N\circ(\phi|_N)^{-1}]\end{aligned}$$

is a diffeomorphism. Thus, for each $q\in\phi(U\cap V)$, there exists $N'\subset\phi(U\cap V)$ such that $F|_{N'}$ is a diffeomorphism. Hence F is a diffeomorphism and (U,ϕ), (V,ψ) are smoothly compatible. Therefore $\mathcal{A}$ is a smooth atlas.
To see that $\mathcal{A}$ is maximal, let $\mathcal{B}'$ be a smooth atlas on M. Suppose that $\mathcal{A}\subset\mathcal{B}'$ and let $(U,\phi)\in\mathcal{B}'$. By definition, for each chart $(V,\psi)\in\mathcal{B}'$, (U,ϕ) and (V,ψ) are smoothly compatible. Since $\mathcal{B}\subset\mathcal{A}\subset\mathcal{B}'$, we have that $(U,\phi)\in\mathcal{A}$. So $\mathcal{A}=\mathcal{B}'$ and $\mathcal{A}$ is a maximal smooth atlas on M. □

Exercise 3.2.0.4. Let $(M,\mathcal{A})$ be an n-dimensional smooth manifold, $(U,\phi)\in\mathcal{A}$ and $U'\subset U$. If U' is open, then $(U',\phi|_{U'})\in\mathcal{A}$.

Proof. Set $\phi'=\phi|_{U'}$. A previous exercise implies that $(U',\phi')\in X(U)$. Define $\mathcal{B}=\mathcal{A}\cup\{(U',\phi')\}$. Let $(V,\psi)\in\mathcal{B}$. If $(V,\psi)=(U',\phi')$, then

$$\phi'\circ\psi^{-1}=\mathrm{id}_{U'}$$

which is a diffeomorphism. Thus (U',ϕ'), (V,ψ) are smoothly compatible. Suppose that $(V,\psi)\in\mathcal{A}$. Since $\mathcal{A}$ is smooth, $\psi|_{U\cap V}\circ(\phi|_{U\cap V})^{-1}:\phi(U\cap V)\to\psi(U\cap V)$ is a diffeomorphism. Therefore $\psi|_{U'\cap V}\circ(\phi'|_{U'\cap V})^{-1}:\phi'(U'\cap V)\to\psi(U'\cap V)$ is a diffeomorphism and (U',ϕ'), (V,ψ) are smoothly compatible. Since $(V,\psi)\in\mathcal{A}$ is arbitrary, $\mathcal{B}$ is smooth. Since $\mathcal{A}$ is maximal and $\mathcal{A}\subset\mathcal{B}$, we have that $\mathcal{A}=\mathcal{B}$ and $(U',\phi')\in\mathcal{A}$. □

Exercise 3.2.0.5. Let $(M,\mathcal{A})$ be a n-dimensional smooth manifold and $U\subset M$ open. Set $\mathcal{B}=\{(V,\psi)\in\mathcal{A}:V\subset U\}$. Then $\mathcal{B}$ is a smooth atlas on U.

Proof.

- Some previous exercises imply that U is an n-dimensional topological manifold and $X(U) = \{(V, \psi) \in X(M) : V \subset U\}$. Since

$$\begin{aligned} \mathcal{B} &\subset \mathcal{A} \\ &\subset X(M) \end{aligned}$$

  we have that $\mathcal{B} \subset X(U)$. Let $p \in U$. Then there exists $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{V'}$. The previous exercise implies that $(V', \psi') \in \mathcal{A}$. By definition, $(V', \psi') \in \mathcal{B}$. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $(V', \psi') \in \mathcal{B}$ such that $p \in V'$. Hence $\mathcal{B}$ is an atlas on U.

- Let $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$. Then $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, (V_1, ψ_1) and (V_2, ψ_2) are smoothly compatible. Since $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ are arbitrary, $\mathcal{B}$ is smooth.

□

Definition 3.2.0.6. Smooth Open Submanifold:
Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $U \subset M$ open. A previous exercise implies that U is an n-dimensional topological manifold. We define $\mathcal{A}|_U \subset X(U)$ to be the unique smooth structure on U such that $\{(V, \psi) \in \mathcal{A} : V \subset U\} \subset \mathcal{A}|_{\mathcal{U}}$. Then $(U, \mathcal{A}|_U)$ is said to be a **smooth open submanifold of** $(M, \mathcal{A})$.

Exercise 3.2.0.7. Let $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ be the projection map given by $\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$. Then π is a diffeomorphism.

Proof. Define projection map $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by $\pi'(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1})$. Then $\mathbb{R}^n$ is an open neighborhood of ∂H^n, $\pi'|_{\partial H^n} = \pi$ and π' is smooth. Then by definition, π is smooth. Clearly, π^{-1} is smooth. So π is a diffeomorphism. □

Definition 3.2.0.8. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold and $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ the projection map. Recall that for $(U, \phi) \in X^n_\partial(M)$, the $(n-1)$-coordinate chart $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$ is defined by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$.
We define

$$\overline{\mathcal{A}} = \{(\bar{U}, \bar{\phi}) : (U, \phi) \in \mathcal{A} \cap X^n_\partial(M)\}$$

Exercise 3.2.0.9. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold. Then $\overline{\mathcal{A}}$ is a smooth atlas on ∂M.

Proof.

- A previous exercise implies that ∂M is an $(n-1)$-dimensional topological manifold. Let $p \in \partial M$. Then there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Since $\mathcal{A} \subset X^n(M)$ and $p \in \partial M$, we have that $p \in \bar{U}$ and a previous exercise implies that $(U, \phi) \in X^n_\partial(M)$. By definition of $\overline{\mathcal{A}}$, $(\bar{U}, \bar{\phi}) \in \overline{\mathcal{A}}$. Since $p \in \partial M$ is arbitrary, $\overline{\mathcal{A}}$ is an atlas on ∂M.

- Let $(\bar{U}, \bar{\phi})$, $(\bar{V}, \bar{\psi}) \in \overline{\mathcal{A}}$. Since (U, ϕ) and (V, ψ) are smoothly compatible, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$ is a diffeomorphism. Thus $\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}$ is a diffeomorphism. Since $\pi|_{\phi(U \cap V)}$ and $\pi|_{\psi(U \cap V)}$ are diffeomorphisms, $\pi|_{\phi(\bar{U} \cap \bar{V})}$ and $\pi|_{\psi(\bar{U} \cap \bar{V})}$ are diffeomorphisms. Then

$$\begin{aligned} \bar{\psi}|_{\bar{U} \cap \bar{V}} \circ (\bar{\phi}|_{\bar{U} \cap \bar{V}})^{-1} &= \left[\pi|_{\psi(\bar{U} \cap \bar{V})} \circ \psi|_{\bar{U} \cap \bar{V}}\right] \circ \left[(\phi|_{\bar{U} \cap \bar{V}})^{-1} \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1}\right] \\ &= \pi|_{\psi(\bar{U} \cap \bar{V})} \circ \left[\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}\right] \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \end{aligned}$$

  is a diffeomorphism. Therefore $(\bar{U}, \bar{\phi})$ and $(\bar{V}, \bar{\psi})$ are smoothly compatible. Since $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \overline{\mathcal{A}}$ are arbitrary, $\mathcal{A}$ is smooth.

□

Definition 3.2.0.10. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold. We define $\mathcal{A}|_{\partial M}$ to be the unique smooth structure on ∂M such that $\overline{\mathcal{A}} \subset \mathcal{A}|_{\partial M}$. We define the **smooth boundary submanifold of** M to be $(\partial M, \mathcal{A}|_{\partial M})$.

Exercise 3.2.0.11. Smooth Manifold Chart Lemma:
Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

(a) for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$

(b) for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$

(c) for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection

(d) for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

(e) there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$

(f) for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Then there exists a unique smooth structure $\mathcal{A}_M$ on M such that $(M, \mathcal{A}_M)$ is an n-dimensional smooth manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset \mathcal{A}_M$.

Proof. Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : \alpha \in \Gamma \text{ and } V \in \mathcal{T}_{\mathbb{H}^n}\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$
- $\mathcal{A}' = \{(U_\alpha, \phi_\alpha) : \alpha \in \Gamma\}$.

The topological manifold chart lemma implies that $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold and $\mathcal{A}' \subset X^n(M, \mathcal{T}_M)$. Since $M = \bigcup_{\alpha \in \Gamma} U_\alpha$, $\mathcal{A}'$ is an atlas on M. Since for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth, we have that $\mathcal{A}'$ is smooth. A previous exercise implies that there exists a unique smooth structure $\mathcal{A}_M$ on M such that $\mathcal{A}' \subset \mathcal{A}_M$. $\square$

3.3 Smooth Maps

Definition 3.3.0.1. Let $(M, \mathcal{A})$ be a smooth manifold and $f : M \to \mathbb{R}$. Then f is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M)$.

Exercise 3.3.0.2. Let $(M, \mathcal{A})$ be a smooth manifold. Then $C^\infty(M)$ is a vector space.

Proof. Let $f, g \in C^\infty(M)$, $\lambda \in \mathbb{R}$ and $(U, \phi) \in \mathcal{A}$. By assumption, $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are smooth. Hence

$$(f + \lambda g) \circ \phi^{-1} = f \circ \phi^{-1} + \lambda g \circ \phi^{-1}$$

is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, $f + \lambda g \in C^\infty(M)$. Since $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$ are arbitrary, $C^\infty(M)$ is a vector space. □

Exercise 3.3.0.3. Let $(M, \mathcal{A})$ be a smooth manifold, $\mathcal{B}$ an atlas on M and $f : M \to \mathbb{R}$. Suppose that $\mathcal{B} \subset \mathcal{A}$. Then f is smooth iff for each $(U, \phi) \in \mathcal{B}$, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth.

Proof.

- ($\implies$):
  Suppose that f is smooth. Let $(U, \phi) \in \mathcal{B}$. Since $\mathcal{B} \subset \mathcal{A}$, $(U, \phi) \in \mathcal{A}$. Since f is smooth, $f \circ \phi^{-1}$ is smooth. Since $(U, \phi) \in \mathcal{B}$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{B}$, $f \circ \phi^{-1}$ is smooth.

- ($\impliedby$):
  Suppose that for each $(V, \psi) \in \mathcal{B}$, $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}$ is smooth. Let $(U, \phi) \in \mathcal{A}$ and $q \in \phi(U)$. Set $p = \phi^{-1}(q)$. Since $\mathcal{B}$ is an atlas, there exists $(V, \psi) \in \mathcal{B}$ such that $p \in V$. Since $\mathcal{B} \subset \mathcal{A}$, $(V, \psi) \in \mathcal{A}$. Set $W = U \cap V$ and $\tilde{\phi} = \phi|_W$ and $\tilde{\psi} = \psi|_W$. We note that $\phi(W) \in \mathcal{N}_q$ and $\phi(W)$ is open. An exercise in the section on smooth manifolds implies that $(W, \tilde{\phi}), (W, \tilde{\psi}) \in \mathcal{A}$. Therefore $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(W) \to \psi(W)$ is smooth. By assumption, $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}$ is smooth. This implies that $(f \circ \psi^{-1})|_{\psi(W)} : \psi(W) \to \mathbb{R}$ is smooth. Hence

  $$\begin{aligned}(f \circ \phi^{-1})|_{\phi(W)} &= f \circ \tilde{\phi}^{-1} \\ &= f \circ (\tilde{\psi}^{-1} \circ \tilde{\psi}) \circ \tilde{\phi}^{-1} \\ &= (f \circ \tilde{\psi}^{-1}) \circ (\tilde{\psi} \circ \tilde{\phi}^{-1})\end{aligned}$$

  is smooth. Since $q \in \phi(U)$ is arbitrary, for each $q \in \phi(U)$, there exists $A \in \mathcal{N}_q$ such that A is open and $(f \circ \phi^{-1})|_A : A \to \mathbb{R}$ is smooth. This implies that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, f is smooth.

□

Exercise 3.3.0.4. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $p \in U$ and $f \in C^\infty(M)$. Then $f|_U \in C^\infty(U)$.

Proof. Let □

Definition 3.3.0.5. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. We define the **partial derivative of f with respect to x^i**, denoted

$$\partial f / \partial x^i : U \to \mathbb{R} \quad \text{or} \quad \partial_i f : U \to \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left(\frac{\partial}{\partial u^i}[f \circ \phi^{-1}] \right) \circ \phi$$

Exercise 3.3.0.6. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. Then $\partial/\partial x^i : C^\infty(U) \to C^\infty(U)$ is linear.

Proof. **FINISH!!!** □

Exercise 3.3.0.7. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f = \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f &= \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j} f\right) \\
&= \frac{\partial}{\partial x^i}\left(\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right] \circ \phi\right) \\
&= \left(\frac{\partial}{\partial u^i}\left[\left(\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right] \circ \phi\right) \circ \phi^{-1}\right]\right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i}\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right]\right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi
\end{aligned}$$

□

Exercise 3.3.0.8. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}$$

Proof. Let $f \in C^\infty(U)$. Since $f \circ \phi^{-1}$ is smooth,

$$\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}] = \frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f \circ \phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned}
\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f &= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f \circ \phi^{-1}]\right) \circ \phi \\
&= \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} f
\end{aligned}$$

□

Exercise 3.3.0.9. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $f \in C^\infty(U)$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha f = (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi$$

Proof. The claim is clearly true when $|\alpha| = 0$ or by definition if $|\alpha| = 1$. Let $n \in \mathbb{N}$ and suppose the claim is true for each $|\alpha| \in \{1, \dots, n-1\}$. Then there exists $i \in \{1, \dots, n\}$ such that $\alpha_i \geq 1$. Hence

$$\begin{aligned}
\partial^\alpha f &= \partial^{e^i}(\partial^{\alpha - e^i} f) \\
&= \partial^{e^i}(\partial^{\alpha - e^i}[f \circ \phi^{-1}] \circ \phi) \\
&= (\partial^{e^i}[(\partial^{\alpha - e^i}[f \circ \phi^{-1}] \circ \phi) \circ \phi^{-1}]) \circ \phi \\
&= (\partial^{e^i}[\partial^{\alpha - e^i}[f \circ \phi^{-1}]]) \circ \phi \\
&= (\partial^\alpha[f \circ \phi^{-1}]) \circ \phi
\end{aligned}$$

□

Exercise 3.3.0.10. Taylor's Theorem:
Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\phi(U)$ convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha| = T+1} \subset C^\infty(U)$ such that

$$f = \sum_{k=0}^{T} \left[\sum_{|\alpha| = k} (x - p)^\alpha \partial^\alpha f(x_0) \right] + \sum_{|\alpha| = T+1} (x^i - x^i(p))^\alpha g_\alpha$$

and for each $|\alpha| = T + 1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. Since $\phi(U)$ is open and convex and $f \circ \phi^{-1} \in C^\infty(\phi(U))$, Taylors thorem in section 2.1 implies that there exist $(\tilde{g}_\alpha)_{|\alpha| = T+1} \subset C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = \sum_{k=0}^{T} \left[\sum_{|\alpha| = k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha| = T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q))$$

and for each $|\alpha| = T + 1$,

$$\begin{aligned}
\tilde{g}_\alpha(\phi(p)) &= \frac{1}{(T+1)!} \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \\
&= \frac{1}{(T+1)!} \partial^\alpha f(p)
\end{aligned}$$

For $|\alpha| = T + 1$, set $g_\alpha = \tilde{g} \circ \phi$. Then

$$\begin{aligned}
f(q) &= f \circ \phi^{-1}(\phi(q)) \\
&= \sum_{k=0}^{T} \left[\sum_{|\alpha| = k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha| = T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\
&= \sum_{k=0}^{T} \left[\sum_{|\alpha| = k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha| = T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q)
\end{aligned}$$

□

Definition 3.3.0.11. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$. Then F is said to be

- **smooth** if for each $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(F(U) \cap V)$$

  is smooth

- a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 3.3.0.12. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifold and $F : M \to N$. If F is smooth, then F is continuous.

Proof. Suppose that F is smooth. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $F(p) \in V$. Put $\tilde{U} = U \cap F^{-1}(V)$ and $\tilde{V} = F(U) \cap V$.
Define $\tilde{\phi} : \tilde{U} \to \phi(\tilde{U})$ and $\tilde{\psi} : \tilde{V} \to \psi(\tilde{V})$ by

$$\tilde{\phi} = \phi|_{\tilde{U}}, \quad \tilde{\phi} = \psi|_{\tilde{V}}$$

Then $\tilde{\phi}$ and $\tilde{\psi}$ are homeomorphisms, $p \in \tilde{U}$ and $F(\tilde{U}) \subset \tilde{V}$. Define $\tilde{F} : \phi(\tilde{U}) \to \psi(\tilde{V})$ by

$$\tilde{F} = \tilde{\psi} \circ F \circ \tilde{\phi}^{-1}$$

By definition, $\tilde{F}$ is smooth and therefore continuous. Since ϕ and ψ are homeomorphisms and $F|_{\tilde{U}} = \tilde{\psi}^{-1} \circ \tilde{F} \circ \tilde{\phi}$, we have that $F|_{\tilde{U}}$ is continuous. In particular, F is continuous at p and since $p \in M$ is arbitrary, F is continuous. □

Exercise 3.3.0.13. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifold and $F : M \to N$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F and F^{-1} are smooth. The previous exercise implies that F and F^{-1} are continuous. Hence F is a homeomorphism. □

Exercise 3.3.0.14. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ a diffeomorphism. Then for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

Proof. Let $(V, \psi) \in \mathcal{B}$.

1. Since ϕ and F^{-1} are homeomorphisms, $\phi \circ F^{-1} : F(U) \cap V \to \phi(U \cap F^{-1}(V))$ is a homeomorphism
2. Since F is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \to \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \psi(V \cap F(U))$$

are smooth.

Therefore $(F(U), \phi \circ F^{-1})$ and (V, ψ) are smoothly compatible. Since $\mathcal{B}$ is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. □

Definition 3.3.0.15. Let $(N, \mathcal{B})$ be a smooth n-dimensional manifold, $F : M \to N$ smooth and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. For $i \in \{1, \dots, n\}$, We define the ***i*-th component of F with respect to** (V, ψ), denoted $F^i : V \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

3.4 Partitions of Unity

Definition 3.4.0.1. Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then ρ is said to be a **bump function at p supported in** U if

1. $\rho \geq 0$
2. there exists $V \in \mathcal{N}_p$ such that V is open and $\rho|_V = 1$
3. $\operatorname{supp} \rho \subset U$

Exercise 3.4.0.2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

Proof. □

3.5 The Tangent Space

Definition 3.5.0.1. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. For $i \in \{1, \cdots, n\}$, define the partial derivative with respect to x^i at p, denoted

$$\frac{\partial}{\partial x^i}\bigg|_p : C^\infty(M) \to \mathbb{R}, \text{ or } \partial_i|_p : C^\infty(M) \to \mathbb{R}$$

by

$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}(p)$$

Exercise 3.5.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$, we have that

$$\frac{\partial}{\partial x^i} x^j(p) = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \cdots, n\}$. Then

$$\begin{aligned}
\frac{\partial}{\partial x^i}\bigg|_p x^i &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} x^i \circ \phi^{-1} \\
&= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} u^i \circ \phi \circ \phi^{-1} \\
&= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} u^i \\
&= \delta_{i,j}
\end{aligned}$$

□

Exercise 3.5.0.3. Change of Coordinates:
Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $\psi = (y^1, \cdots, y^n)$, $p \in U \cap V$ and $f \in C^\infty(M)$. Then for each $i \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial y^i}\bigg|_p = \sum_{j=1}^n \frac{\partial}{\partial x^j} y^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

Proof. Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \cdots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned}
\frac{\partial}{\partial y^i}\bigg|_p f &= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \psi^{-1} \\
&= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \phi^{-1} \circ h \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial u^j}\bigg|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left(\frac{\partial}{\partial u^i}\bigg|_{\psi(p)} h_j \right) \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial u^j}\bigg|_{\phi(p)} f \circ \phi^{-1} \right) \left(\frac{\partial}{\partial u^i}\bigg|_{\psi(p)} x^j \circ \psi^{-1} \right) \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial x^i}\bigg|_p f \right) \left(\frac{\partial}{\partial y^i}\bigg|_p x^j \right)
\end{aligned}$$

□

Definition 3.5.0.4. Let $p \in M$ and $v : C^\infty(M) \to \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation at** p if for each $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,

1. v is linear
2. v is Leibnizian

We define the **tangent space of** M **at** p, denoted T_pM, by

$$T_pM = \{v : C^\infty(M) \to \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 3.5.0.5. Let $f \in C^\infty(M)$ and $v \in T_pM$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. □

Exercise 3.5.0.6. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

is a basis for T_pM and $\dim T_pM = n$.

Proof. Clearly $\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \in T_pM$. Let $a_1, \cdots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p x^j \\ &= a_j \end{aligned}$$

Hence $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is independent.

Now, let $v \in T_pM$ and $f \in \mathbb{C}^\infty(M)$. By Taylor's theorem, there exist $g_1, \cdots g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x^i}\bigg|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p f \\ &= \left[\sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p\right] f \end{aligned}$$

So

$$v = \sum_{i=1}^{n} v(x^i) \frac{\partial}{\partial x^i}\bigg|_p$$

and

$$v \in \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

□

Definition 3.5.0.7. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. We define the **differential of** F **at** p, denoted $DF_p : T_pM \to T_{F(p)}N$, by

$$\Big[DF_p(v)\Big](f) = v(f \circ F)$$

for $v \in T_pM$ and $f \in C^\infty(N)$.

Exercise 3.5.0.8. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. Then for each $v \in T_pM$, $DF_p(v)$ is a derivation.

Proof. Let $v \in T_pM$, $f, g \in C^\infty_{F(p)}(N)$ and $c \in \mathbb{R}$. Then

1.
$$\begin{aligned} DF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= DF_p(v)(f) + cDF_p(v)(g) \end{aligned}$$

   So $DF_p(v)$ is linear.

2.
$$\begin{aligned} DF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= DF_p(v)(f) * g(F(p)) + f(F(p)) * DF_p(v)(g) \end{aligned}$$

So $DF_p(v)$ is Leibnizian and hence $DF_p(v) \in T_{F(p)}N$ □

Exercise 3.5.0.9. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. If F is a diffeomorphism, then DF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \cdots, x^n)$ and $\phi \circ F^{-1} = (y^1, \cdots, y^n)$. Let $f \in C^\infty(N)$ Then

$$\begin{aligned} \frac{\partial}{\partial y^i}\bigg|_{F(p)} f &= \frac{\partial}{\partial u^i}\bigg|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \frac{\partial}{\partial x^i}\bigg|_p f \circ F \end{aligned}$$

Therefore

$$\begin{aligned}\left[DF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right)\right](f) &= \frac{\partial}{\partial x^i}\bigg|_p f\circ F \\ &= \frac{\partial}{\partial y^i}\bigg|_{F(p)} f\end{aligned}$$

Hence

$$DF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial y^i}\bigg|_{F(p)}$$

Since $\left\{\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ is a basis for T_pM and $\left\{\frac{\partial}{\partial y^1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}\Big|_{F(p)}\right\}$ is a basis for $T_{F(p)}N$, DF_p is an isomorphism. $\square$

Exercise 3.5.0.10. Let $(M, \mathcal{A})$ be a smooth m-dimensional manifold, $(N, \mathcal{B})$ a n-dimensional smooth manifold, $F : M \to N$ smooth, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^m)$ and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. Suppose that $p \in U$ and $F(p) \in V$. Define the ordered bases $B_\phi = \left\{\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^m}\Big|_p\right\}$ and $B_\psi = \left\{\frac{\partial}{\partial y^1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}\Big|_{F(p)}\right\}$. Then the matrix representation of DF_p with respect to the bases B_ϕ and B_ψ is

$$DF_p^{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Proof. Let $(DF_p)_{B_\phi, B_\psi} = (a_{i,j})_{i,j} \in \mathbb{R}^{n\times m}$. Then for each $j \in \{1, \dots, m\}$,

$$DF_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{i=1}^n a_{i,j}\frac{\partial}{\partial y^i}\bigg|_{F(p)}$$

This implies that

$$\begin{aligned}DF_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right)(y^k) &= \sum_{i=1}^n a_{i,j}\frac{\partial}{\partial y^i}\bigg|_{F(p)}(y^k) \\ &= \sum_{i=1}^n a_{i,j}\delta_{i,k} \\ &= a_{k,j}\end{aligned}$$

By definition,

$$\begin{aligned}DF_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right)(y^k) &= \frac{\partial}{\partial x^j}\bigg|_p y^k\circ F \\ &= \frac{\partial}{\partial x^j}\bigg|_p F^k \\ &= \frac{\partial F^k}{\partial x^j}(p)\end{aligned}$$

$\square$

Note 3.5.0.11. Since $\operatorname{rank} DF_p$ is independent of basis, it is independent of coordinate charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$.

3.6 The Cotangent Space

Definition 3.6.0.1. Let $p \in M$. We define the **cotangent space of** M **at** p, denoted T_p^*M, by

$$T_p^*M = (T_pM)^*$$

Definition 3.6.0.2. Let $f \in C^\infty(M)$. We define the **differential of** f **at** p, denoted $df_p : T_pM \to \mathbb{R}$, by

$$df_p(v) = vf$$

Exercise 3.6.0.3. Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

Proof. Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1 f + \lambda v_2 f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that df_p is linear and hence $df_p \in T_p^*M$. $\square$

Exercise 3.6.0.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$,

$$dx^i_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta_{i,j}$$

In particular, $\{dx^1_p, \cdots, dx^n_p\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ and $T_p^*M = \operatorname{span}\{dx^1_p, \cdots, dx^n_p\}$.

Proof. Let $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned} \left[dx^i_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) \right]_p &= \frac{\partial}{\partial x^i}\bigg|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

$\square$

Exercise 3.6.0.5. Let $f \in C^\infty(M)$, (U, ϕ) a chart on M with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i{}_p$$

Proof. Since $\{dx^1_p, \cdots, dx^n_p\}$ is a basis for T_p^*M, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx^i_p$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= \sum_{i=1}^n a_i(p) dx^i_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= \frac{\partial}{\partial x^i}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p) dx_p^i$$

$\square$

Chapter 4

Submersions and Immersions

4.1 Maps of Constant Rank

Definition 4.1.0.1. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. We define the **rank map of** F, denoted $\operatorname{rank} F : M \to \mathbb{N}_0$ by

$$\operatorname{rank}_p F = \dim \operatorname{Im} DF(p)$$

and F is said to have **constant rank** if for each $p, q \in M$, $\operatorname{rank}_p F = \operatorname{rank}_q F$. If F has constant rank, we define the **rank of** F, denoted $\operatorname{rank} F$, by $\operatorname{rank} F = \operatorname{rank}_p F$ for $p \in M$.

Exercise 4.1.0.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$ and $p \in M$. Suppose that $\operatorname{rank}_p F = k$. Then there exist $(U, \phi) \in \mathcal{A}_M$, $(V, \psi) \in \mathcal{A}_N$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi,\psi})_{i,j} = A_{i,j}$$

Proof. Define $q \in V$ by $q = F(p)$. Choose $(U', \phi') \in \mathcal{A}$ and $(V', \psi') \in \mathcal{B}$ such that $p \in U'$ and $q \in V'$. Set $Z = [DF(p)]_{\phi',\psi'}$. By assumption, $\operatorname{rank} Z = k$. An exercise in the subsection on linear algebra implies that there exist $\sigma \in S_m$, $\tau \in S_n$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Define $\phi : U \to \sigma\phi(U)$ and $\psi : V \to \tau\psi(V)$ by

$$\phi = \sigma\phi', \quad \psi = \tau\psi'$$

A previous exercise implies that

$$[DF(p)]_{\phi,\psi} = P_\tau Z P_\sigma^*$$

□

Exercise 4.1.0.3. Constant Rank Theorem:
Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$. Suppose that F has constant rank and $\operatorname{rank} F = k$. Then for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$ and

$$\psi \circ F \circ \phi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

Hint: Needs a hint

Proof. Let $p \in M$. The previous exercise implies that there exist $(U_0, \phi_0) \in \mathcal{A}$, $(V_0, \psi_0) \in \mathcal{B}$ and $L \in GL(k, \mathbb{R})$ such that $p \in U$, $F(p) \in V_0$ and for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi_0,\psi_0})_{i,j} = L_{i,j}$$

Define $\hat{M} \subset \mathbb{R}^m$, $\hat{N} \subset \mathbb{R}^n$ and $\hat{F} : \hat{M} \to \hat{N}$ by $\hat{M} := \phi_0(U_0)$, $\hat{N} := \psi_0(V_0)$ and $\hat{F} := \psi_0 \circ F \circ \phi_0^{-1}$. Set $\hat{p} := \phi_0(p)$. Let (x, y) be the standard coordinates on $\mathbb{R}^m$, with $\pi_x : \mathbb{R}^m \to \mathbb{R}^k$ and $\pi_y : \mathbb{R}^m \to \mathbb{R}^{m-k}$ the standard projection maps. Write $\hat{p} = (x_0, y_0)$. There exist $Q : \hat{M} \to \mathbb{R}^k$ and $R : \hat{M} \to \mathbb{R}^{n-k}$ such that $\hat{F} = (Q, R)$. By construction, $[D_x Q(x_0, y_0)] = L$. Define $G : \hat{M} \to \mathbb{R}^m$ by $G(x, y) := (Q(x, y), y)$. Then

$$\begin{aligned}
[DG(x_0, y_0)] &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_x Q(x_0, y_0)] \\ [D_x \pi_y(x_0, y_0)] & [D_y \pi_y(x_0, y_0)] \end{pmatrix} \\
&= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} L & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\det([DG(x_0, y_0)]) &= \det(L)\det(I) \\
&= \det(L) \\
&\neq 0
\end{aligned}$$

The inverse function theorem implies that there exist $\hat{U} \subset \hat{M}$ such that $\hat{U}$ is open, $\hat{p} \in \hat{U}$ and $G|_{\hat{U}} : \hat{U} \to G(\hat{U})$ is a diffeomorphism. Since

$$\{U_1 \times U_2 : U_1 \subset \mathbb{R}^k, U_2 \subset \mathbb{R}^{m-k} \text{ and } U_1,\, U_2 \text{ are open}\}$$

is a basis for the topology on $\mathbb{R}^m$, there exist $\hat{U}_1 \subset \mathbb{R}^k$ and $\hat{U}_2 \subset \mathbb{R}^{m-k}$ such that $\hat{U}_1$, $\hat{U}_2$ are open, $\hat{p} \in \hat{U}_1 \times \hat{U}_2$ and $\hat{U}_1 \times \hat{U}_2 \subset \hat{U}$. Set $\hat{U}_{12} := \hat{U}_1 \times \hat{U}_2$ and define $G_{12} : \hat{U}_{12} \to Q(\hat{U}_{12}) \times \hat{U}_2$ by $G_{12} := G|_{\hat{U}_{12}}$. Since $G|_{\hat{U}} : \hat{U} \to G(\hat{U})$ is a diffeomorphism, $\hat{U}_{12} \subset \hat{U}$ and

$$\begin{aligned}
G(\hat{U}_{12}) &= G(\hat{U}_1 \times \hat{U}_2) \\
&= Q(\hat{U}_{12}) \times \hat{U}_2
\end{aligned}$$

we have that $G_{12} : \hat{U}_{12} \to Q(\hat{U}_{12}) \times \hat{U}_2$ is a diffeomorphism. Since G_{12} is a homeomorphism and π_x is open, $Q(\hat{U}_{12})$ is open. Since $G_{12}^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_{12}$, there exist $A : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_1$ and $B : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_2$ such that A, B are smooth and $G_{12}^{-1} = (A, B)$. Define $\tilde{R} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \mathbb{R}^{n-k}$ by $\tilde{R}(x, y) := R(A(x, y), y)$. Then $\tilde{R}$ is smooth. Let $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$. Then

$$\begin{aligned}
(x, y) &= G_{12} \circ G_{12}^{-1}(x, y) \\
&= G(A(x, y), B(x, y)) \\
&= (Q(A(x, y), B(x, y)), B(x, y))
\end{aligned}$$

This implies that $B(x, y) = y$,

$$\begin{aligned}
x &= Q(A(x, y), B(x, y)) \\
&= Q(A(x, y), y)
\end{aligned}$$

and

$$\begin{aligned}
G_{12}^{-1}(x, y) &= (A(x, y), B(x, y)) \\
&= (A(x, y), y)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{F} \circ G_{12}^{-1}(x, y) &= \hat{F}(A(x, y), y) \\
&= (Q(A(x, y), y), R(A(x, y), y)) \\
&= (x, R(A(x, y), y)) \\
&= (x, \tilde{R}(x, y))
\end{aligned}$$

We note that

$$\begin{aligned}[D(\hat{F} \circ G_{12}^{-1})(x,y)] &= \begin{pmatrix} [D_x\pi_x(x,y)] & [D_y\pi_x(x,y)] \\ [D_x\tilde{R}(x,y)] & [D_y\tilde{R}(x,y)] \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ [D_x\tilde{R}(x,y)] & [D_y\tilde{R}(x,y)] \end{pmatrix}\end{aligned}$$

Since $G_{12}^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_{12}$ is a diffeomorphism, we have that $[DG^{-1}(x,y)] \in GL(m,\mathbb{R})$. Since $\hat{F}$ has constant rank and $\operatorname{rank} \hat{F} = k$, we have that

$$\begin{aligned}\operatorname{rank}[D(\hat{F} \circ G_{12}^{-1})(x,y)] &= \operatorname{rank}([D\hat{F}(G_{12}^{-1}(x,y))][DG_{12}^{-1}(x,y)]) \\ &= \operatorname{rank}[D\hat{F}(G_{12}^{-1}(x,y))] \\ &= k\end{aligned}$$

Since $\operatorname{rank}\begin{pmatrix} I \\ [D_x\tilde{R}(x,y)] \end{pmatrix} = k$, we have that $\operatorname{rank}\begin{pmatrix} 0 \\ [D_y\tilde{R}(x,y)] \end{pmatrix} = 0$. Thus $[D_y\tilde{R}(x,y)] = 0$. Since $(x,y) \in Q(\hat{U}_{12}) \times \hat{U}_2$ is arbitrary, for each $(x,y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\tilde{R}(x,y) = \tilde{R}(x,y_0)$$

Define $\tilde{S} : Q(\hat{U}_{12}) \to \mathbb{R}^{n-k}$ by $\tilde{S}(x) := \tilde{R}(x,y_0)$. Then $\tilde{S}$ is smooth and for each $(x,y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\hat{F} \circ G_{12}^{-1}(x,y) = (x, \tilde{S}(x))$$

Let (a,b) be the standard coordinates on $\mathbb{R}^n$, with $\pi_a : \mathbb{R}^n \to \mathbb{R}^k$ and $\pi_b : \mathbb{R}^n \to \mathbb{R}^{n-k}$ the standard projection maps. Write $\hat{F}(\hat{p}) = (a_0, b_0)$. Set

$$\begin{aligned}\hat{V}_{12} &:= \pi_a|_{\hat{N}}^{-1}(Q(\hat{U}_{12})) \\ &= \pi_a^{-1}(Q(\hat{U}_{12})) \cap \hat{N}\end{aligned}$$

Since $Q(\hat{U}_{12})$ is open, $\hat{N}$ is open and π_a is continuous, we have that $\hat{V}_{12}$ is open. Since

$$\begin{aligned}Q(\hat{U}_{12}) &= \pi_a|_{\hat{N}} \circ \hat{F} \circ G^{-1}(Q(\hat{U}_{12}) \times \hat{U}_2) \\ &= \pi_a|_{\hat{N}} \circ \hat{F}(\hat{U}_{12})\end{aligned}$$

we have that

$$\begin{aligned}\hat{F}(\hat{U}_{12}) &\subset \pi_a|_{\hat{N}}^{-1}(Q(\hat{U}_{12})) \\ &\subset \hat{V}_{12}\end{aligned}$$

In particular, $\hat{F}(\hat{p}) \in \hat{V}_{12}$. Define $H : Q(\hat{U}_{12}) \times \mathbb{R}^{n-k} \to Q(\hat{U}_{12}) \times \mathbb{R}^{n-k}$ by $H := (\pi_a, \pi_b - \tilde{S} \circ \pi_a)$, i.e. for each $(a,b) \in Q(\hat{U}_{12}) \times \mathbb{R}^{n-k}$, $H(a,b) = (a, b - \tilde{S}(a))$. Then H is a bijection and $H^{-1}(a,b) = (\pi_a, \pi_b + \tilde{S} \circ \pi_a)$. Thus H and H^{-1} are smooth and therefore H is a diffeomorphism. Define $H_{12} : \hat{V}_{12} \to H(\hat{V}_{12})$ by $H_{12} = H|_{\hat{V}_{12}}$. Then H_{12} is a diffeomorphism and for each $x, y \in Q(\hat{U}_{12} \times \hat{U}_2)$, $H_{12} \circ \hat{F} \circ G_{12}^{-1}(x,y) = (x,0)$. Define $(U,\phi) \in \mathcal{A}$ and $(V,\psi) \in \mathcal{B}$ by $U := \phi_0^{-1}(\hat{U}_{12})$, $V := \psi_0^{-1}(\hat{V}_{12})$, $\phi := G_{12} \circ \phi_0|_U$ and $\psi := H_{12} \circ \psi_0|_V$. Then for each $(x,y) \in \phi(U)$,

$$\begin{aligned}\psi \circ F \circ \phi^{-1}(x,y) &= H_{12} \circ \psi_0|_V \circ F \circ \phi_0|_U^{-1} \circ G_{12}^{-1}(x,y) \\ &= H_{12} \circ \hat{F} \circ G_{12}^{-1}(x,y) \\ &= (x,0)\end{aligned}$$

□

Definition 4.1.0.4. Let $(M,\mathcal{A})$ and $(N,\mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. Then F is said to be

- an **immersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is injective
- a **submersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is surjective

Exercise 4.1.0.5. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map.

Definition 4.1.0.6. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$ smooth. Then F is said to be an **embedding** if

1. F is an immersion
2. $F : M \to F(M)$.

Note 4.1.0.7. Here the topology on $F(M)$ is the subspace topology.

4.2 Submanifolds

Exercise 4.2.0.1. Let $(M, \mathcal{A})$ be a smooth manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $\tilde{U} \subset S$ and $\tilde{\phi} : \tilde{U} \to \phi(\tilde{U})$ by $\tilde{U} = U \cap S$ and $\tilde{\phi} = \phi|_{U \cap S}$. Set $\mathcal{B} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \in \mathcal{A}\}$. Then $\mathcal{B}$ is a smooth structure on S.

Proof. $\square$

Definition 4.2.0.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. Suppose that $M \subset N$. Then $(M, \mathcal{A})$ is said to be

1. an **immersed submanifold** of $(N, \mathcal{B})$ if $\mathrm{id} : M \to N$ is a smooth immersion
2. an **embedded submanifold** of $(N, \mathcal{B})$ if $\mathrm{id} : M \to N$ is a smooth embedding

Note 4.2.0.3. Essentially, embedded submanifolds are immersed submanifolds with the subspace topology.

Note 4.2.0.4. For the remainder of this section, we assume that $k \leq n$.

Definition 4.2.0.5. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Then S is said to be a k**-slice** of U if $S = \{u \in U : u^{k+1}, \dots, u^n = 0\}$.

Exercise 4.2.0.6. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Suppose that S is a k-slice of U. Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then $\pi|_S \to \pi(S)$ is a diffeomorphism.

Proof. Clear. □

Definition 4.2.0.7. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ and $S \subset U$. Then S is said to be a k**-slice** of U if $\phi(S)$ is a k-slice of $\phi(U)$.

Definition 4.2.0.8. Let $(M, \mathcal{A})$ be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$. Then (U, ϕ) is said to be a k**-slice chart for** S if $U \cap S$ is a k-slice of U.

Exercise 4.2.0.9. Let $(M, \mathcal{A})$ be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. If (U, ϕ) is a k-slice chart for S, then $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$.

Proof. Clear. □

Definition 4.2.0.10. Let $(M, \mathcal{A})$ be a smooth manifold and $S \subset M$. Then S is said to satisfy the **local** k**-slice condition** if for each $p \in S$, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k-slice chart of S.

Exercise 4.2.0.11. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold and $S \subset M$ a subspace. If S satisfies the local k-slice condition, then there exists a smooth structure $\tilde{\mathcal{A}}$ on S such that $(S, \tilde{\mathcal{A}})$ is an embedded submanifold of M.

Proof. Suppose that S satisfies the local k-slice condition. Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ as above Let $(U, \phi) \in \mathcal{A}$. Suppose that (U, ϕ) is a k-slice chart for S. Define $\tilde{U} = U \cap S$ and $\tilde{\phi} : \tilde{U} \to \pi \circ \phi(\tilde{U})$ by

$$\tilde{\phi} = \pi \circ \phi|_{\tilde{U}}$$

By definition, $\phi(\tilde{U})$ is a k-slice of $\phi(U)$. A previous exercise implies that $\pi|_{\phi(\tilde{U})} \to \pi \circ \phi(\tilde{U})$ is a diffeomorphism and hence a homeomorphism. Thus $\tilde{\phi}$ is a homeomorphism.
Define

$$\tilde{\mathcal{B}} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \text{ is a } k\text{-slice for } S\}$$

Let $p \in S$. By assumption, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k-slice chart of S. Then $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{B}}$ and $\mathcal{A}$ is an atlas on S. By construction of $\tilde{\mathcal{B}}$, S is locally half Euclidean of dimension k. Since M is second countable Hausdorff, so is S in the subspace topology. Thus $(S, \tilde{\mathcal{B}})$ is a k-dimensional manifold.
Let $(\tilde{U}, \tilde{\phi})$, $(\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$. Then

$$\tilde{\phi} \circ \tilde{\psi}^{-1}|_{\tilde{U} \cap \tilde{V}} = \pi|_{\phi(\tilde{U} \cap \tilde{V})} \circ \phi|_{\tilde{U} \cap \tilde{V}} \circ \psi|^{-1}_{\tilde{U} \cap \tilde{V}} \circ \pi|^{-1}_{\psi(\tilde{U} \cap \tilde{V})}$$

which is a diffeomorphism. So $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ smoothly compatible. Hence $\tilde{\mathcal{B}}$ is smooth. An exercise in section 4.1 implies that there exists a unique smooth structure $\tilde{\mathcal{A}}$ on S such that $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$. So $(S, \tilde{\mathcal{A}})$ is a smooth k-dimensional manifold.
Clearly $\mathrm{id} : S \to S$ is a homeomorphism. Let $(V, \psi) \in \mathcal{A}$ and $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$.
Finish!! □

Definition 4.2.0.12.

Exercise 4.2.0.13.

Chapter 5

Vector Fields

5.1 The Tangent Bundle

Definition 5.1.0.1. Let $(M, \mathcal{A}_M)$ be an n-dimensional smooth manifold. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p \in M} T_p M$$

and we define the **tangent bundle projection**, denoted $\pi : TM \to M$, by

$$\pi(p, v) = p$$

Let $(U, \phi) \in \mathcal{A}_M$ with $\phi = (x^1, \ldots, x^n)$. We define $\Phi_\phi : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\Phi_\phi\left(p, \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p\right) = (\phi(p), \xi^1, \ldots, \xi^n)$$

We define $\mathcal{T}_{TM} = \tau_{TM}(\iota_p : p \in M)$.

Exercise 5.1.0.2. $\psi : \bigcup_{p \in U} T_p M \to \mathbb{R}^n$ is given by

$$\psi\left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p\right) = (\xi^1, \ldots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}[x^k \circ \pi] &= \frac{\partial}{\partial v^i}\bigg|_{\Phi_\phi(p,\xi)}[x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \frac{\partial}{\partial v^i}\bigg|_{(\phi(p),\psi(\xi))}[x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \frac{\partial}{\partial v^i}\bigg|_{\phi(p)}[x^k \circ \phi^{-1}] \\
&= 0
\end{aligned}$$

This implies that for each $i \in \{1, \ldots, n\}$, we have that

$$\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi))\frac{\partial x^k \circ \pi}{\partial \tilde{x}^i}(p,\xi) \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)\delta_{i,k} \\
&= \frac{\partial f}{\partial x^i}(p)
\end{aligned}$$

and

$$\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi))\frac{\partial x^k \circ \pi}{\partial \tilde{y}^i}(p,\xi) \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)0 \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \ker D\pi(p,\xi) \\
&= \coprod_{(p,\xi)\in\pi^{-1}(U)} \operatorname{span}\left\{\frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \ldots, n\}\right\}
\end{aligned}$$

Chapter 6

Lie Theory

6.1 Lie Groups

Definition 6.1.0.1. Let G be a smooth manifold and group. Then G is said to be a **Lie group** if

- multiplication $G \times G \to G$ given by $(g, h) \mapsto gh$ is smooth
- inversion $G \to G$ given by $g \mapsto g^{-1}$ is smooth

Definition 6.1.0.2. Let $\mathfrak{g}$ be a vector space and $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $[\cdot, \cdot]$ is said to be a **Lie bracket** on $\mathfrak{g}$ if

1. $[\cdot, \cdot]$ is bilinear
2. $[\cdot, \cdot]$ is antisymmetric
3. $[\cdot, \cdot]$ satisfies the Jacobi identity:
   for each $x, w, y \in \mathcal{F}g$,
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

In this case, $(\mathfrak{g}, [\cdot, \cdot])$ is said to be a **Lie algebra**.

Definition 6.1.0.3. Let $X \in$

Chapter 7

Bundles and Sections

7.1 Fiber Bundles

7.1.1 Fibered Manifolds

Definition 7.1.1.1. Let $E, M \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$. Then (E, M, π) is said to be a **smooth fibered manifold** if π is a surjective submersion.

Note 7.1.1.2. We write $\mathrm{proj}_1^n : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ to denote the projection onto M.

Definition 7.1.1.3. Let (E, M, π) be a smooth fibered manifold and $(V, \psi) \in \mathcal{A}_E$. Set $n := \dim M$ and $k := \dim E - n$. Then (V, ψ) is said to be a **π-fibered chart on** E if there exists $(U, \phi) \in \mathcal{A}_M$ such that

1. $U = \pi(V)$
2. $\phi \circ \pi|_V = \mathrm{proj}_1^n \circ \psi$

i.e. if $\psi = (y^1, \dots, y^{n+k})$ and $\phi = (x^1, \dots, x^n)$, then $\psi = (x^1 \circ \pi, \dots, x^n \circ \pi, y^{n+1}, \dots, y^{n+k})$.

Exercise 7.1.1.4. Let (E, M, π) be a smooth fibered manifold. Then for each $a \in E$, there exists $(V, \psi) \in \mathcal{A}_E$ such that $a \in V$ and (V, ψ) is a π-fibered chart on E.
Hint: Constant rank theorem

Proof. Set $n := \dim M$, $k := \dim E - n$. Let $a \in E$. Set $p := \pi(a)$. Since $\pi : E \to M$ is a submersion, π has constant rank and $\mathrm{rank}\, \pi = n$. The constant rank theorem implies that there exist $(V_0, \psi_0) \in \mathcal{A}_E$, $(U_0, \phi_0) \in \mathcal{A}_M$ such that $a \in V_0$, $p \in U$ and $\phi_0 \circ \pi \circ \psi_0^{-1} = \mathrm{proj}_1^n |_{\psi_0(V \cap \pi^{-1}(U))}$. Hence $\phi_0 \circ \pi = \mathrm{proj}_1^n \circ \psi_0$. Define $V := V_0 \cap \pi^{-1}(U_0)$, $U = U_0 \cap \pi(V_0)$, $\psi = \psi_0|_V$ and $\phi = \phi_0|_U$. Then

1. $$\begin{aligned} \pi(V) &= \pi(\pi^{-1}(U_0) \cap V_0) \\ &= U_0 \cap \pi(V_0) \\ &= U \end{aligned}$$

2. $$\begin{aligned} \phi \circ \pi|_V &= \phi_0|_U \circ \pi|_V \\ &= \mathrm{proj}_1^n \circ \psi_0|_V \\ &= \mathrm{proj}_1^n \circ \psi \end{aligned}$$

So that (V, ψ) is a π-fibered chart on E. $\square$

7.1.2 Local Trivializations

Note 7.1.2.1. Let M, F be sets, we write $\text{proj}_1 : M \times F \to M$ to denote the projection onto M.

Definition 7.1.2.2. Let $E, M, F \in \text{Obj}(\mathbf{Set})$, $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **local trivialization with respect to π of E over U with fiber F** if

1. Φ is a bijection
2. $\text{proj}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\text{proj}_1"] \\
 & U
\end{tikzcd}$$

Exercise 7.1.2.3. Let $E, M, F \in \text{Obj}(\mathbf{Set})$ and $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$ a local trivialization with respect to π of E over U with fiber F. Then for each $A \subset U$,

$$\Phi(\pi^{-1}(A)) = A \times F$$

Hint: consider $\Phi^{-1}(A \times F)$

Proof. Let $A \subset U$. Since $\text{proj}_1^{-1}(A) = A \times F$, we have that

$$\begin{aligned}
\Phi^{-1}(A \times F) &= \Phi^{-1}(\text{proj}_1^{-1}(A)) \\
&= (\text{proj}_1 \circ \Phi)^{-1}(A) \\
&= (\pi|_{\pi^{-1}(U)})^{-1}(A) \\
&= \pi^{-1}(A) \cap \pi^{-1}(U) \\
&\pi^{-1}(A \cap U) \\
&= \pi^{-1}(A)
\end{aligned}$$

Since Φ is a bijection, we have that

$$\begin{aligned}
\Phi(\pi^{-1}(A)) &= \Phi \circ \Phi^{-1}(A \times F) \\
&= A \times F
\end{aligned}$$

□

7.1.3 $\mathbf{Man}^0$ Fiber Bundles

Definition 7.1.3.1. Let $E, M, F \in \text{Obj}(\mathbf{Man}^0)$ and $\pi \in \text{Hom}_{\mathbf{Man}^0}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **continuous local trivialization with respect to π of E over U with fiber F** if

1. U is open
2. (U, Φ) is a local trivialization with respect to π of E over U with fiber F
3. Φ is a homeomorphism

Definition 7.1.3.2. Let $E, M, F \in \text{Obj}(\mathbf{Man}^0)$ and $\pi \in \text{Hom}_{\mathbf{Man}^0}(E, M)$ a surjection. Then (E, M, π, F) is said to be a **$\mathbf{Man}^0$ fiber bundle with total space E, base space M, fiber F and projection π** if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that (U, Φ) is a continuous local trivialization with respect to π of E over U with fiber F. For $p \in M$, we define the **fiber over** p, denoted E_p, by $E_p = \pi^{-1}(\{p\})$.

Exercise 7.1.3.3. Man^0 Fiber Bundle Chart Lemma:
Let $E \in \text{Obj}(\mathbf{Set})$, $M, F \in \text{Obj}(\mathbf{Man}^0)$, $\pi : E \to M$ a surjection, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. Set $n = \dim M$ and $k = \dim F$. Suppose that

- for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}_M$
- $M \subset \bigcup\limits_{\alpha \in \Gamma} U_\alpha$
- for each $\alpha \in \Gamma$, (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
- for each $\alpha, \beta \in \Gamma$, $\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is continuous.

Then there exist a unique topology, $\mathcal{T}_E$, on E such that

1. $(E, \mathcal{T}_E)$ is a $n+k$-dimensional topological manifold
2. for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism
3. $\pi : E \to M$ is continuous
4. (E, M, π, F) is an $\mathbf{Man}^0$ fiber bundle

Proof.

1. For $\alpha \in \Gamma$, we define $X^n_\alpha(M, \mathcal{T}_M) \subset X^n(M, \mathcal{T}_M)$ by
$$X^n_\alpha(M, \mathcal{T}_M) = \{(V^M, \psi^M) \in X^n(M, \mathcal{T}_M) : V^M \subset U_\alpha\}$$
Choose index sets $(\Pi^M_\alpha)_{\alpha \in \Gamma}$ and Π^F such that for each $\alpha \in \Gamma$, $X^n_\alpha(M, \mathcal{T}_M) = (V^M_{\alpha,\mu}, \psi^M_{\alpha,\mu})_{\mu \in \Pi^M_\alpha}$ and $X^k(F, \mathcal{T}_F) = (V^F_\nu, \psi^F_\nu)_{\nu \in \Pi^F}$. Set $\Pi^M = \coprod_{\alpha \in \Gamma} \Pi^M_\alpha$ and $\Pi^E = \Pi^M \times \Pi^F$. For $(\alpha, \mu, \nu) \in \Pi^E$, we define $V^E_{\alpha,\mu,\nu} \subset E$ and $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ by
   - $V^E_{\alpha,\mu,\nu} = \Phi^{-1}_\alpha(V^M_{\alpha,\mu} \times V^F_\nu)$
   - $\psi^E_{\alpha,\mu,\nu} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu) \circ \Phi_\alpha|_{V^E_{\alpha,\mu,\nu}}$

   We have the following:

   - For each $(\alpha, \mu, \nu) \in \Pi^E$, $\psi^E_{\alpha,\mu,\nu}(V^E_{\alpha,\mu,\nu}) = \psi^M_\mu(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ and thus $\psi^E_{\alpha,\mu,\nu}(V^E_{\alpha,\mu,\nu}) \in \mathcal{T}_{\mathbb{H}^{n+k}}$
   - For each $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$,
   $$\begin{aligned}\psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}) &= (\psi^M_{\alpha_1,\mu_1} \times \psi^F_{\nu_1}) \circ \Phi_{\alpha_1}|_{V^E_{\alpha_1,\mu_1,\nu_1}}(\Phi^{-1}_{\alpha_1}([V^M_{\alpha_1,\mu_1} \times V^F_{\nu_1}] \cap [V^M_{\alpha_2,\mu_2} \times V^F_{\nu_2}])) \\ &= (\psi^M_{\alpha_1,\mu_1} \times \psi^F_{\nu_1})([V^M_{\alpha_1,\mu_1} \times V^F_{\nu_1}] \cap [V^M_{\alpha_2,\mu_2} \times V^F_{\nu_2}]) \\ &= (\psi^M_{\alpha_1,\mu_1} \times \psi^F_{\nu_1})([V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}] \times [V^F_{\nu_1} \cap V^F_{q_2}]) \\ &= \psi^M_{\alpha_1,\mu_1}(V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}) \times \psi^F_{\nu_1}(V^F_{\nu_1} \cap V^F_{\nu_2}) \\ &\in \mathcal{T}_{\mathbb{H}^{n+k}}\end{aligned}$$
   - For each $(\alpha, \mu, \nu) \in \Pi^E$, $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a bijection
   - Let $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$. For notational convenience, set $\psi^E_1 = \psi^E_{\alpha_1,\mu_1,\nu_1}$, $\psi^E_2 = \psi_{\alpha_2,\mu_2,\nu_2}$, $V^E = V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}$, $V^M = V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}$ and $V^F = V^F_{\nu_1} \cap V^F_{\nu_2}$. Then $\psi_2|_{V^E} \circ (\psi_1|_{V^E})^{-1} : \psi_1(V^E) \to \psi_2(V^E)$ is given by
   $$\begin{aligned}\psi^E_2|_{V^E} \circ (\psi^E_1|_{V^E})^{-1} &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F}) \circ \Phi_{\alpha_1}|_{V^E}]^{-1} \\ &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\Phi_{\alpha_1}|_{V^E})^{-1} \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}] \\ &= (\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ [\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}] \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}\end{aligned}$$
   Since $\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}$ is continuous, we have that $\psi^E_{\alpha_2,\mu_2,\nu_2}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}} \circ (\psi^E_{\alpha_1,\mu_1,\nu_1}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}})^{-1} : \psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}) \to \psi^E_{\alpha_2,\mu_2,\nu_2}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2})$ is continuous.

- A previous exercise in the section on topological manifolds implies that $(V^M_{\alpha,\mu})_{(\alpha,\mu)\in\Pi^M}$ is an open cover of M and $(V^F_\nu)_{\nu\in\Pi^F}$ is an open cover of F. Since M,F are second-countable M,F are Lindelöf and there exists $S^M\subset\Pi^M$, $S^F\subset\Pi^F$ such that S^M,S^F are countable, $(V^M_{\alpha,\mu})_{(\alpha,\mu)\in S^M}$ is an open cover of M and $(V^F_\nu)_{\nu\in\Pi^F}$ is an open cover of F. Then $S^M\times S^F$ is countable and $(V^M_{\alpha,\mu}\times V^F_\nu)_{(\alpha,\mu,\nu)\in S^M\times S^F}$ is an open cover of $M\times F$.
  Let $a\in E$. Set $p=\pi(a)$. Choose $(\alpha,\mu)\in S^M$ such that $p\in V^M_{\alpha,\mu}$. Since $V^M_{\alpha,\mu}\subset U_\alpha$, $a\in\pi^{-1}(U_\alpha)$ which implies that
  $$\begin{aligned} p&=\pi(a)\\ &=\mathrm{proj}_1\circ\Phi_\alpha(a)\end{aligned}$$
  Set $q=\mathrm{proj}_2\circ\Phi_\alpha(a)$. Choose $\nu\in S^F$ such that $q\in V^F_\nu$. Then
  $$\begin{aligned}\Phi_\alpha(a)&=(\mathrm{proj}_1\circ\Phi_\alpha(a),\mathrm{proj}_2\circ\Phi_\alpha(a))\\ &=(p,q)\\ &\in V^M_{\alpha,\mu}\times V^F_\nu\end{aligned}$$
  Thus
  $$\begin{aligned} a&\in\Phi_\alpha^{-1}(V^M_{\alpha,\mu}\times V^F_\nu)\\ &=V^E_{\alpha,\mu,\nu}\end{aligned}$$
  Since $a\in E$ is arbitrary, we have that for each $a\in E$, there exists $(\alpha,\mu,\nu)\in S^M\times S^F\subset\Pi^E$ such that $a\in V^E_{\alpha,\mu,\nu}$. Thus
  $$E\subset\bigcup_{(\alpha,\mu,\nu)\in S^M\times S^F}V_{\alpha,\mu,\nu}$$
- Let $a_1,a_2\in E$.
  For now, suppose that $\pi(a_1)\neq\pi(a_2)$. Set $p_1=\pi(a_1)$ and $p_2=\pi(a_2)$. Since M is Hausdorff, there exist $(\alpha_1,\mu_1),(\alpha_2,\mu_2)\in\Pi^M$ such that $p_1\in V^M_{\alpha_1,\mu_1}$, $p_2\in V^M_{\alpha_2,\mu_2}$ and $V^M_{\alpha_1,\mu_1}\cap V^M_{\alpha_2,\mu_2}=\varnothing$. Set $q_1=\mathrm{proj}_2\circ\Phi_{\alpha_1}(a_1)$ and $q_2=\mathrm{proj}_2\circ\Phi_{\alpha_2}(a_2)$. Choose $\nu_1,\nu_2\in\Pi^F$ such that $q_1\in V^F_{\nu_1}$ and $q_2\in V^F_{\nu_2}$. Then similarly to the previous part, $a_1\in V^E_{\alpha_1,\mu_1,\nu_1}$ and $a_2\in V^E_{\alpha_2,\mu_2,\nu_2}$ and therefore
  $$\begin{aligned} V^E_{\alpha_1,\mu_1,\nu_1}\cap V^E_{\alpha_2,\mu_2,\nu_2}&=\Phi_{\alpha_1}^{-1}(V^M_{\alpha_1,\mu_1}\times V^F_{\nu_1})\cap\Phi_{\alpha_2}^{-1}(V^M_{\alpha_2,\mu_2}\times V^F_{\nu_2})\\ &\subset\pi^{-1}(V^M_{\alpha_1,\mu_1})\cap\pi^{-1}(V^M_{\alpha_2,\mu_2})\\ &=\pi^{-1}(V^M_{\alpha_1,\mu_1}\cap V^M_{\alpha_2,\mu_2})\\ &=\pi^{-1}(\varnothing)\\ &=\varnothing\end{aligned}$$
  Now suppose that $\pi(a_1)=\pi(a_2)$. Set $p=\pi(a_1)$. Then there exists $(\alpha,\mu)\in\Pi^M$ such that $p\in V^M_{\alpha,\mu}\subset U_\alpha$.
  For now, suppose that $\mathrm{proj}_2\circ\Phi_\alpha(a_1)\neq\mathrm{proj}_2\circ\Phi_\alpha(a_2)$. Set $q_1=\mathrm{proj}_2\circ\Phi_\alpha(a_1)$ and $q_2=\mathrm{proj}_2\circ\Phi_\alpha(a_2)$. Since F is Hausdorff, there exist $\nu_1,\nu_2\in\Pi^F$ such that $q_1\in V^F_{\nu_1}$ and $q_2\in V^F_{\nu_2}$ and $V^F_{\nu_1}\cap V^F_{\nu_2}=\varnothing$. Then $a_1\in V^E_{\alpha,\mu,\nu_1}$, $a_2\in V^E_{\alpha,\mu,\nu_2}$ and
  $$\begin{aligned} V^E_{\alpha,\mu,\nu_1}\cap V^E_{\alpha,\mu,\nu_2}&=\Phi_\alpha^{-1}(V^M_{\alpha,\mu}\times V^F_{\nu_1})\cap\Phi_\alpha^{-1}(V^M_{\alpha,\mu}\times V^F_{\nu_2})\\ &=\Phi_\alpha^{-1}([V^M_{\alpha,\mu}\times V^F_{\nu_1}]\cap[V^M_{\alpha,\mu}\times V^F_{\nu_2}])\\ &=\Phi_\alpha^{-1}([V^M_{\alpha,\mu}\cap V^M_{\alpha,\mu}]\times[V^F_{\nu_1}\cap V^F_{\nu_2}])\\ &=\Phi_\alpha^{-1}(V^M_{\alpha,\mu}\times[V^F_{\nu_1}\cap V^F_{\nu_2}])\\ &=\Phi_\alpha^{-1}(V^M_{\alpha,\mu}\times\varnothing)\\ &=\Phi_\alpha^{-1}(\varnothing)\\ &=\varnothing\end{aligned}$$

Now, suppose that $\mathrm{proj}_2 \circ \Phi_\alpha(a_1) = \mathrm{proj}_2 \circ \Phi_\alpha(a_2)$. Set $q = \mathrm{proj}_2 \circ \Phi_\alpha(a_1)$. Choose $\nu \in \Pi^F$ such that $q \in V_\nu^F$. Since

$$\begin{aligned}\Phi_\alpha(a_1) &= (\mathrm{proj}_1 \circ \Phi_\alpha(a_1), \mathrm{proj}_2 \circ \Phi_\alpha(a_1)) \\ &= (p, q) \\ &= (\mathrm{proj}_1 \circ \Phi_\alpha(a_2), \mathrm{proj}_2 \circ \Phi_\alpha(a_2)) \\ &= \Phi_\alpha(a_2)\end{aligned}$$

we have that $a_1 = a_2$ and $a_1, a_2 \in V^E_{\alpha,\mu,\nu}$. Therefore, for each $a_1, a_2 \in E$, there exists $(\alpha, \mu, \nu) \in \Pi^E$ such that $p, q \in V^E_{\alpha,\mu,\nu}$ or there exist $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$ such that $a_1 \in V^E_{\alpha_1,\mu_1,\nu_1}$, $a_2 \in V^E_{\alpha_2,\mu_2,\nu_2}$ and $V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2} = \varnothing$.

The topological manifold chart lemma implies that there exists a unique topology $\mathcal{T}_E$ on E such that $(E, \mathcal{T}_E)$ is an $n+k$-dimensional topological manifold and $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E} \subset X^{n+k}(E, \mathcal{T}_E)$.

2. Let $\alpha \in \Gamma$. By assumption $U_\alpha \in \mathcal{T}_M$. Let $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$. Then $(\alpha, \mu, \nu) \in \Pi^E$. Since

   - $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a homeomorphism
   - $\psi^M_{\alpha,\mu} \times \psi^F_\nu : V^M_{\alpha,\mu} \times V^F_\nu \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a homeomorphism
   - $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is given by $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu)^{-1} \circ \psi^E_{\alpha,\mu,\nu}$,

   we have that $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a homeomorphism. Since $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$ are arbitrary we have that for each $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$, $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a homeomorphism. Since $(V^M_{\alpha,\mu})_{\mu\in\Pi^M_\alpha}$ is an open cover of U_α and $(V^M_{\alpha,\mu} \times V^F_\nu)_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F}$ is an open cover of $U_\alpha \times F$, we have that

$$\begin{aligned}\pi^{-1}(U_\alpha) &= \pi^{-1}\bigg(\bigcup_{\mu\in\Pi^M_\alpha} V^M_{\alpha,\mu}\bigg) \\ &= \bigcup_{\mu\in\Pi^M_\alpha} \pi^{-1}(V^M_{\alpha,\mu}) \\ &= \bigcup_{\mu\in\Pi^M_\alpha} \Phi_\alpha^{-1}(V^M_{\alpha,\mu} \times F) \\ &= \bigcup_{\mu\in\Pi^M_\alpha} \Phi_\alpha^{-1}\bigg(V^M_{\alpha,\mu} \times \bigg[\bigcup_{\nu\in\Pi^F} V^F_\nu\bigg]\bigg) \\ &= \bigcup_{\mu\in\Pi^M_\alpha} \Phi_\alpha^{-1}\bigg(\bigcup_{\nu\in\Pi^F} [V^M_{\alpha,\mu} \times V^F_\nu]\bigg) \\ &= \bigcup_{\mu\in\Pi^M_\alpha} \bigg[\bigcup_{\nu\in\Pi^F} \Phi_\alpha^{-1}(V^M_{\alpha,\mu} \times V^F_\nu)\bigg] \\ &= \bigcup_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F} V^E_{\alpha,\mu,\nu}\end{aligned}$$

   Hence $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$, $(V^E_{\alpha,\mu,\nu})_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F}$ is an open cover of $\pi^{-1}(U_\alpha)$ and Φ_α is a local homeomorphism. Since Φ_α is a bijection, Φ_α is a homeomorphism. Since $\alpha \in \Gamma$ is arbitrary, we have that for each $\alpha \in \Gamma$, $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism.

3. Let $(\alpha, \mu, \nu) \in \Pi^E$. Since

   - $V^E_{\alpha,\mu,\nu} \subset \pi^{-1}(U_\alpha)$
   - $\mathrm{proj}_1 : M \times F \to M$ is continuous

- $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is continuous
- $\pi|_{V^E_{\alpha,\mu,\nu}} = \mathrm{proj}_1 \circ \Phi|_{V^E_{\alpha,\mu,\nu}}$

we have that $\pi|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu}$ is continuous. Since $(\alpha,\mu,\nu) \in \Pi^E$ is arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is an open cover of E, we have that $\pi : E \to M$ is continuous.

4. Let $p \in M$. By assumption, there exists $\alpha \in \Gamma$ such that $p \in U_\alpha$, $U_\alpha \in \mathcal{T}_M$. Since $E, M, F \in \mathrm{Obj}(\mathbf{Man}^0)$, $\pi \in \mathrm{Hom}_{\mathbf{Man}^0}(E,M)$ is a surjection, and

   - U_α is open
   - (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism

   we have that (U_α, Φ_α) is a continuous local trivialization with respect to π of E over U_α with fiber F. Since $p \in M$ is arbitrary, (E, M, π, F) is a $\mathbf{Man}^0$ fiber bundle.

□

7.1.4 $\mathbf{Man}^\infty$ Fiber Bundles

Definition 7.1.4.1. Let $E, M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E,M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **smooth local trivialization of E over U with fiber F** if

1. U is open
2. (U, Φ) is a local trivialization of E over U with fiber F
3. Φ is a diffeomorphism

Definition 7.1.4.2. Let $E, M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E,M)$ a surjection. Then (E, M, π, F) is said to be a **$\mathbf{Man}^\infty$ fiber bundle with total space E, base space M, fiber F and projection π** if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that U is open and (U, Φ) is a smooth local trivialization of E over U with fiber F. For $p \in M$, we define the **fiber over** p, denoted E_p, by $E_p = \pi^{-1}(\{p\})$.

Exercise 7.1.4.3. Let (E, M, π, F) be a $\mathbf{Man}^\infty$ fiber bundle with total space E, base space M, fiber F and projection π. Then (E, M, π) is a smooth fibered manifold.

Proof. Let $a \in E$. Set $p = \pi(a)$. Then there exists $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that U is open and (U, Φ) is a smooth local trivialization of E over U with fiber F. Then Φ is a diffeomorphim and

$$\begin{aligned}\mathrm{rank}_a\, \pi &= \mathrm{rank}\, D\pi(a) \\ &= \mathrm{rank}\, D\, \mathrm{proj}_1(\Phi(a)) \\ &= \dim M\end{aligned}$$

Since $a \in E$ is arbitrary, π has constant rank. Thus π is a submersion. Hence (E, M, π) is a smooth fibered manifold. □

Exercise 7.1.4.4. $\mathbf{Man}^\infty$ Fiber Bundle Chart Lemma:
Let $E \in \mathrm{Obj}(\mathbf{Set})$, $M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $\pi : E \to M$ a surjection, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. Set $n = \dim M$ and $k = \dim F$. Suppose that

- for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}_M$
- $M \subset \bigcup\limits_{\alpha\in\Gamma} U_\alpha$
- for each $\alpha \in \Gamma$, (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F

- for each $\alpha, \beta \in \Gamma$, $\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is smooth.

Then there exist a unique topology $\mathcal{T}_E$ on E and smooth structure $\mathcal{A}_E \subset X^{n+k}(M, \mathcal{T}_E)$ on E such that

1. $(E, \mathcal{A}_E)$ is an $n+k$-dimensional smooth manifold
2. for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism
3. $\pi : E \to M$ is smooth
4. (E, M, π, F) is an $\mathbf{Man}^\infty$ fiber bundle

Proof. The $\mathbf{Man}^0$ fiber bundle chart lemma implies that there exists a unique topology $\mathcal{T}_E$ on E such that

- $(E, \mathcal{T}_E)$ is a $n+k$-dimensional topological manifold
- for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism
- $\pi : E \to M$ is continuous
- (E, M, π, F) is an $\mathbf{Man}^0$ fiber bundle

1. Define $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E} \subset X^{n+k}(E, \mathcal{T}_E)$ as in the proof of the $\mathbf{Man}^0$ fiber bundle chart lemma. Let $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$. For notational convenience, set $\psi^E_1 = \psi^E_{\alpha_1,\mu_1,\nu_1}$, $\psi^E_2 = \psi_{\alpha_2,\mu_2,\nu_2}$, $V^E = V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}$, $V^M = V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}$ and $V^F = V^F_{\nu_1} \cap V^F_{\nu_2}$. Then $\psi_2|_{V^E} \circ (\psi_1|_{V^E})^{-1} : \psi_1(V^E) \to \psi_2(V^E)$ is given by
$$\begin{aligned}\psi^E_2|_{V^E} \circ (\psi^E_1|_{V^E})^{-1} &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F}) \circ \Phi_{\alpha_1}|_{V^E}]^{-1} \\ &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\Phi_{\alpha_1}|_{V^E})^{-1} \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}] \\ &= (\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ [\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}] \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}\end{aligned}$$
Since $\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}$ is smooth, we have that $\psi^E_{\alpha_2,\mu_2,\nu_2}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}} \circ (\psi^E_{\alpha_1,\mu_1,\nu_1}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}})^{-1} : \psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}) \to \psi^E_{\alpha_2,\mu_2,\nu_2}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2})$ is smooth. Since $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$ are arbitrary, we have that $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is a smooth atlas on E. An exercise in the section on smooth manifolds implies that there exists a unique smooth structure $\mathcal{A}_E$ on E such that $(E, \mathcal{A}_E)$ is an $n+k$-dimensional smooth manifold.

2. Let $\alpha \in \Gamma$. By assumption $U_\alpha \in \mathcal{T}_M$. Let $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$. Then $(\alpha, \mu, \nu) \in \Pi^E$. Since
   - $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a diffeomorphism
   - $\psi^M_{\alpha,\mu} \times \psi^F_\nu : V^M_{\alpha,\mu} \times V^F_\nu \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a diffeomorphism
   - $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is given by $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu)^{-1} \circ \psi^E_{\alpha,\mu,\nu}$,

   we have that $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a diffeomorphism. Since $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$ are arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F}$ is an open cover of $\pi^{-1}(U_\alpha)$, we have that $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a local diffeomorphism. Since Φ_α is a bijection, Φ_α is a diffeomorphism. Since $\alpha \in \Gamma$ is arbitrary, we have that for each $\alpha \in \Gamma$, $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism.

3. Let $(\alpha, \mu, \nu) \in \Pi^E$. Since
   - $V^E_{\alpha,\mu,\nu} \subset \pi^{-1}(U_\alpha)$
   - $\mathrm{proj}_1 : M \times F \to M$ is smooth
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is smooth
   - $\pi|_{V^E_{\alpha,\mu,\nu}} = \mathrm{proj}_1 \circ \Phi|_{V^E_{\alpha,\mu,\nu}}$

   we have that $\pi|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu}$ is smooth. Since $(\alpha, \mu, \nu) \in \Pi^E$ is arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is an open cover of E, we have that $\pi : E \to M$ is smooth.

4. Let $p \in M$. By assumption, there exists $\alpha \in \Gamma$ such that $p \in U_\alpha$, $U_\alpha \in \mathcal{T}_M$. Since $E, M, F \in \text{Obj}(\mathbf{Man}^\infty)$, $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ is a surjection, and

   - U_α is open
   - (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism

   we have that (U_α, Φ_α) is a smooth local trivialization with respect to π of E over U_α with fiber F. Since $p \in M$ is arbitrary, (E, M, π, F) is a $\mathbf{Man}^\infty$ fiber bundle.

□

Definition 7.1.4.5. Let (E_1, M_1, π_1, F_1) and (E_2, M_2, π_2, F_2) be $\mathbf{Man}^\infty$ fiber bundles, $\Phi \in \text{Hom}_{\mathbf{Man}^\infty}(E_1, E_2)$ and $\phi \in \text{Hom}_{\mathbf{Man}^\infty}(M_1, M_2)$. Then (Φ, ϕ) is said to be a **smooth bundle morphism** from (E_1, M_1, π_1, F_1) to (E_2, M_2, π_2, F_2) if $\pi_2 \circ \Phi = \phi \circ \pi_1$, i.e. the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\phi"'] & M_2 \end{tikzcd}$$

Definition 7.1.4.6. We define the category of $\mathbf{Man}^\infty$ fiber bundles, denoted $\mathbf{Bun}^\infty$, by

- $\text{Obj}(\mathbf{Bun}^\infty) = \{(E, M, \pi, F) : (E, M, \pi, F) \text{ is a } \mathbf{Man}^\infty \text{ fiber bundle}\}$
- For $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \text{Obj}(\mathbf{Bun}^\infty)$,

$$\begin{aligned} &\text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) = \\ &\{(\Phi, \phi) : (\Phi, \phi) \text{ is a smooth bundle morphism from } (E_1, M_1, \pi_1, F_1) \text{ to } (E_2, M_2, \pi_2, F_2)\} \end{aligned}$$

- For
  - $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3) \in \text{Obj}(\mathbf{Bun}^\infty)$
  - $(\Phi_{12}, \phi_{12}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$
  - $(\Phi_{23}, \phi_{23}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3))$

  we define $(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_3, M_3, \pi_3))$ by

$$(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) = (\Phi_{23} \circ \Phi_{12}, \phi_{23} \circ \phi_{12})$$

Exercise 7.1.4.7. We have that $\mathbf{Bun}^\infty$ is a full subcategory of $(\text{id}_{\mathbf{Man}^\infty} \downarrow \text{id}_{\mathbf{Man}^\infty})$.

Proof. Set $\mathcal{C} = (\text{id}_{\mathbf{Man}^\infty} \downarrow \text{id}_{\mathbf{Man}^\infty})$. We note that

- $\text{Obj}(\mathbf{Bun}^\infty) \subset \text{Obj}(\mathcal{C})$
- for each $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \text{Obj}(\mathbf{Bun}^\infty)$,

$$\text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) = \text{Hom}_{\mathcal{C}}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$$

So $\mathbf{Bun}^\infty$ is a full subcategory of $\mathcal{C}$. □

Exercise 7.1.4.8. Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$ and (U, Φ) a local trivialization of E over U and (V, Ψ) a local trvialization of E over V. Then

1. $\text{proj}_{U \cap V} \circ \Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1} = \text{proj}_1$

2. there exists $\sigma \in \mathrm{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that for each $p \in U \cap V$, $\sigma(p, \cdot) : F \to F$ is a diffeomorphism.

Proof.

1. By definition, the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times F \arrow[dr, "\mathrm{proj}_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\Phi"'] \arrow[r, "\Psi"] \arrow[d, "\pi"] & (U \cap V) \times F \arrow[dl, "\mathrm{proj}_1"] \\
& N &
\end{tikzcd}$$

   $\mathrm{proj}_1 \circ \Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1} = \mathrm{proj}_1$

2. there exists $\sigma \in \mathrm{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that for each $p \in U \cap V$ and $x \in F$,

$$\Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1}(p, x) = (p, \sigma(p, x))$$

   and $\sigma(p, \cdot) : F \to F$ is a diffeomorphism.

$\square$

Definition 7.1.4.9. Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$ and $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ a collection of smooth local trivializations of E. Then $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ is said to be a **fiber bundle atlas** if for each $p \in M$, there exists $\alpha \in A$ such that $p \in U_\alpha$. For $\alpha, \beta \in A$, we define ϕ

7.2 G-Bundles

Definition 7.2.0.1. Let G be a Lie group and $(E, M, \pi, F) \in \mathrm{Obj}(\mathbf{Bun}^{\infty})$. Then

7.3 Vector Bundles

Note 7.3.0.1. Let M be a set and $p \in M$. We endow $\{p\} \times \mathbb{R}^n$ with the natural vector space structure such that $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

Definition 7.3.0.2. Let $E, M \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then (E, M, π) is said to be a **rank k smooth vector bundle** if

1. $(E, M, \pi, \mathbb{R}^k) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$
2. for each $p \in M$, E_p is a k-dimensional real vector space
3. for each smooth local trivialization (U, Φ) of E over U with fiber $\mathbb{R}^k$ and $p \in U$,
$$\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$$
is a vector space isomorphism

In this case we define the **rank of** (E, M, π), denoted $\mathrm{rank}(E, M, \pi)$, by $\mathrm{rank}(E, M, \pi) = k$.

Definition 7.3.0.3. We define the category of smooth vector bundles, denoted $\mathbf{VecBun}^\infty$, by

- $\mathrm{Obj}(\mathbf{VecBun}^\infty) = \{(E, M, \pi) : (E, M, \pi) \text{ is a smooth vector bundle}\}$
- For $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ with $\mathrm{rank}(E_1, M_1, \pi_1) = k_1$ and $\mathrm{rank}(E_2, M_2, \pi_2) = k_2$,
$$\mathrm{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

Exercise 7.3.0.4. We have that $\mathbf{VecBun}^\infty$ is a full subcategory of $\mathbf{Bun}^\infty$.

Proof. We note that

- $\mathrm{Obj}(\mathbf{VecBun}^\infty) \subset \mathrm{Obj}(\mathbf{Bun}^\infty)$
- for each $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$ with $\mathrm{rank}(E_1, M_1, \pi_1) = k_1$ and $\mathrm{rank}(E_2, M_2, \pi_2) = k_2$,
$$\mathrm{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

So $\mathbf{Bun}^\infty$ is a full subcategory of $\mathcal{C}$. □

Exercise 7.3.0.5. Let $M \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Set $n = \dim M$, $E = M \times \mathbb{R}^k$ and define $\pi : E \to M$ by $\pi(p, x) = p$. Then (E, M, π) is a rank k smooth vector bundle.

Proof.

1. For each $p \in M$, $\pi_1^{-1}(\{p\}) = \{p\} \times \mathbb{R}^k$ is an n-dimensional real vector space.
2. Let $p \in M$. Set $U = M$. Then $\pi^{-1}(U) = E$. Define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ by $\Phi = \mathrm{id}_E$. Then (U, Φ) is a smooth local trivialization of E over U.
3. Let $p \in M$. Then $\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \to \{p\} \times \mathbb{R}^k$ is clearly an isomorphism.

□

Exercise 7.3.0.6. Smooth Vector Bundle Chart Lemma:
Let $M \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Denote the topology on M by $\mathcal{T}_M$. Suppose that for each $p \in M$, there exists $E_p \in \mathrm{Obj}(\mathbf{Vect}_\mathbb{R})$ such that $\dim E_p = k$. We define $E \in \mathrm{Obj}(\mathbf{Set})$ and $\pi \in \mathrm{Hom}_{\mathbf{Set}}(E, M)$ by
$$E = \coprod_{p \in M} E_p$$
and $\pi(p, v) = p$. Let Γ be an index set and $(U_\alpha)_{\alpha \in \Gamma} \subset \mathcal{T}_M$. Suppose that

1. $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
2. for each $\alpha \in \Gamma$, there exists $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ is a bijection
   - $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a vector space isomorphism
3. for each $\alpha, \beta \in \Gamma$, there exists $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that
   - $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ is smooth
   - $\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is given by

7.4 Bundle Morphisms

Definition 7.4.0.1. Let (E, M, π_E) and (F, N, π_F) be $\mathbf{Man}^\infty$ fiber bundles and $\Phi : E \to F$ and $\phi : M \to N$. Then (Φ, ϕ) is said to be a $\mathbf{Man}^\infty$ **fiber bundle morphism** from (E, M, π_E) to (F, N, π_F) if Φ is smooth, ϕ is smooth and $\pi_F \circ \Phi = \phi \circ \pi_E$, i.e. the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

and we write $(\Phi, \phi) : (E, M, \pi_E) \to (F, N, \pi_F)$.

Exercise 7.4.0.2. Let (E, M, π_E) and (F, N, π_F) be $\mathbf{Man}^\infty$ fiber bundles and $(\Phi, \phi) : (E, M, \pi_E) \to (F, N, \pi_F)$. Suppose that (Φ, ϕ) is smooth. Then for each $p \in M$,

$$\Phi^{-1}(F_{\phi(p)}) = E_p$$

Proof. Let $p \in M$. Set $q = \phi(p)$. Then

$$\begin{aligned} \Phi^{-1}(F_q) &= \Phi^{-1}(\pi_F^{-1}(\{q\})) \\ &= (\pi_F \circ \Phi)^{-1}(\{q\}) \\ &= (\phi \circ \pi_E)^{-1}(\{q\}) \\ &= \pi_E^{-1}(\phi^{-1}(\{\phi(p)\})) \end{aligned}$$

FINISH!!!, multiple fibers get mapped to same fiber $\square$

7.5 Subbundles

7.6 Vertical and Horizontal Subbundles

Definition 7.6.0.1. Let $(E, M, \pi_M) \in \text{Obj}(\mathbf{Bun}^\infty)$. We define the **vertical bundle associated to** (E, M, π_M) , denoted $(VE, M, \pi_V) \in \mathbf{Bun}^\infty$, by

$$VE = \coprod_{q \in E} \ker D\pi(q)$$

relocate this to after tangent bundle is introduced

Exercise 7.6.0.2. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $(\pi^{-1}(U), \Phi_\phi) \in \mathcal{A}_{TM}$ the induced chart on TM with $\Phi_\phi = (\tilde{x}^1, \dots, \tilde{x}^n, \tilde{y}^1, \dots, \tilde{y}^n)$. Then

$$V(TM)|_{\pi^{-1}(U)} = \coprod_{(p,\xi) \in \pi^{-1}(U)} \text{span}\left\{ \frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \dots, n\} \right\}$$

Split into smaller exercises

Proof. Let $f \in C^\infty(M)$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$ the standard coordinates on $\mathbb{R}^n \times \mathbb{R}^n$. We note that by definition, $\Phi_\phi(p, \xi) = (\phi(p), \psi(\xi))$ where $\psi : \bigcup_{p \in U} T_pM \to \mathbb{R}^n$ is given by

$$\psi\left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{(\phi(p),\psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial v^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i}\bigg|_{(\phi(p),\psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= 0 \end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned} D\pi(p,\xi)\left(\left.\frac{\partial}{\partial \tilde{x}^i}\right|_{(p,\xi)}\right)(f) &= \left.\frac{\partial}{\partial \tilde{x}^i}\right|_{(p,\xi)} f \circ \pi \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i}(p,\xi) \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)\delta_{i,k} \\ &= \frac{\partial f}{\partial x^i}(p) \end{aligned}$$

and

$$\begin{aligned} D\pi(p,\xi)\left(\left.\frac{\partial}{\partial \tilde{y}^i}\right|_{(p,\xi)}\right)(f) &= \left.\frac{\partial}{\partial \tilde{y}^i}\right|_{(p,\xi)} f \circ \pi \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i}(p,\xi) \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)0 \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \ker D\pi(p,\xi) \\ &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \operatorname{span}\left\{ \left.\frac{\partial}{\partial \tilde{y}^j}\right|_{(p,\xi)} : j \in \{1, \dots, n\} \right\} \end{aligned}$$

$\square$

7.7 The Tangent Bundle

Definition 7.7.0.1. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p \in M} T_p M$$

We denote the natrual projection map by $\pi : TM \to M$.

Definition 7.7.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Define $\tilde{U} \subset TM$ and $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}^n$ by

- $\tilde{U} = \pi^{-1}(U)$
- $$\begin{aligned} \tilde{\phi}\bigg(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\bigg|_p\bigg) &= (\phi(p), v) \\ &= (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \end{aligned}$$
,

Exercise 7.7.0.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}$ is a bijection.

7.8 The cotangent Bundle

Definition 7.8.0.1. We define the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p \in M} T_p^* M$$

7.9 The (r, s)-Tensor Bundle

Definition 7.9.0.1.

1. the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p \in M} T_p^* M$$

2. the (r, s)**-tensor bundle of** M, denoted $T_s^r M$, by

$$T_s^r M = \coprod_{p \in M} T_s^r (T_p M)$$

3. the k**-alternating tensor bundle of** M, denoted $\Lambda^k(M)$, by

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k (T_p M)$$

7.10 Vector Fields

Definition 7.10.0.1. Let $X : M \to TM$. Then X is said to be a **vector field on** M if for each $p \in M$, $X_p \in T_pM$.
For $f \in \mathbb{C}^\infty(M)$, we define $Xf : M \to \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in \mathbb{C}^\infty(M)$, Xf is smooth.
We denote the set of smooth vector fields on M by $\Gamma^1(M)$.

Definition 7.10.0.2. Let $f \in C^\infty(M)$ and $X, Y \in \Gamma^1(M)$. We define

- $fX \in \Gamma^1(M)$ by
$$(fX)_p = f(p)X_p$$
- $X + Y \in \Gamma^1(M)$ by
$$(X+Y)_p = X_p + Y_p$$

Exercise 7.10.0.3. The set $\Gamma^1(M)$ is a $C^\infty(M)$-module.

Proof. Clear. $\square$

Exercise 7.10.0.4. Let $X \in \Gamma^1(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then

$$X|_U = \sum_{i=1}^n (Xx^i)\frac{\partial}{\partial x^i}$$

Proof. Let $p \in M$. Then $X_p \in T_pM$ and $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis of T_pM. So there exist $f_1(p), \cdots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum\limits_{i=1}^n f^i(p)\frac{\partial}{\partial x^i}\Big|_p$. Let $j \in \{1, \cdots, n\}$. Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^n f^i(p)\frac{\partial}{\partial x^j}x^i(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx^j = f_j$ and $X|_U = \sum\limits_{i=1}^n (Xx^i)\frac{\partial}{\partial x^i}$. $\square$

Exercise 7.10.0.5. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then for each $i \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

Proof. Let $i \in \{1, \cdots, n\}$ and $f \in C^\infty(M)$. Define $g : M \to \mathbb{R}$ by $g = \frac{\partial}{\partial x^i}f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i}\Big|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i}\Big|_{\phi\circ\psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence g is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x^i}$ is smooth. $\square$

7.11 1-Forms

Definition 7.11.0.1. Let $\omega : M \to T^*M$. Then ω is said to be a **1-form on** M if for each $p \in M$, $\omega_p \in T_p^*M$.
For each $X \in \Gamma^1(M)$, we define $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)$, $\omega(X)$ is smooth.
The set of smooth 1-forms on M is denoted $\Gamma_1(M)$.

Definition 7.11.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma^1(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta \in \Gamma^1(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 7.11.0.3. The set $\Gamma_1(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Exercise 7.11.0.4.

7.12 (r, s)-Tensor Fields

Definition 7.12.0.1. Let $\alpha : M \to T^r_s M$. Then α is said to be an (r, s)**-tensor field on** M if for each $p \in M$, $\alpha_p \in T^r_s(T_p M)$.
For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \to \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and α is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth.
The set of smooth (r, s)-tensor fields on M is denoted $T^r_s(M)$.

Definition 7.12.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T^r_s(M)$. We define

- $f\alpha : M \to T^r_s M$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta : M \to T^r_s M$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 7.12.0.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T^r_s(M)$. Then

1. $f\alpha \in T^r_s(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
2. $\alpha + \beta \in T^r_s(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Proof. Clear. □

Exercise 7.12.0.4. The set $T^r_s(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Definition 7.12.0.5. Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. We define the **tensor product of** α **with** β, denoted $\alpha \otimes \beta : M \to T^{r_1+r_2}_{s_1+s_2} M$, by
$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

Exercise 7.12.0.6. Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma^{r_1+r_2}_{s_1+s_2}(M)$

Proof. Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since α_1 and α_2 are smooth by assumption. □

Definition 7.12.0.7. We define the **tensor product**, denoted $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

Exercise 7.12.0.8. The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is associative.

Proof. Clear. □

Exercise 7.12.0.9. The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is $C^\infty(M)$-bilinear.

Proof. Clear. □

Definition 7.12.0.10. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ a smooth map and $\alpha \in \Gamma^0_k(N)$. We define the **pullback of** α **by** F, denoted $F^*\alpha \in \Gamma^0_k(M)$, by

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k))$$

for $p \in M$ and $v_1, \dots, v_k \in T_pM$

Exercise 7.12.0.11. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ and $(L, \mathcal{C})$ be smooth manifolds, $F : M \to N$ and $G : N \to L$ smooth maps, $\alpha \in \Gamma^0_k(N)$, $\beta \in \Gamma^0_l(N)$, $\gamma \in \Gamma^0_k(L)$ and $f \in C^\infty(N)$. Then

1. $F^*(f\alpha) = (f \circ F)F^*\alpha$
2. $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
3. $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
4. $(G \circ F)^*\gamma = F^*(G^*\gamma)$
5. $id^*_N\alpha = \alpha$

Proof.

1.
$$\begin{aligned}
[F^*(f\alpha)]_p(v_1, \dots, v_k) &= (f\alpha)_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\
&= f(F(p))\alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\
&= (f \circ F)(p)(F^*\alpha)_p(v_1, \dots, v_k)
\end{aligned}$$

   So that $F^*(f\alpha) = (f \circ F)F^*\alpha$

2.
$$F^*$$

□

Definition 7.12.0.12.

Exercise 7.12.0.13.

Proof. □

Exercise 7.12.0.14. Let $\alpha \in T^r_s(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then there exist $(f^I_J)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

Proof. Let $p \in M$. Then $\omega_p \in T^r_s(T_pM)$ and $\left\{\partial_{x^{\otimes I}}|_p \otimes dx^{\otimes J}_p\right\}$ is a basis of $T^r_s(T_pM)$. So there exist $(f^I_J(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p \otimes dx^{\otimes J}_p$$

Let $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned}
\alpha_p(dx^K_p, \partial_{x^L}|_p) &= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p \otimes dx^{\otimes J}_p(dx^K_p, \partial_{x^L}|_p) \\
&= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p(dx^K_p) dx^{\otimes J}_p(\partial_{x^L}|_p) \\
&= f^K_L(p)
\end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx^K, \partial_{x^L})_p$ is smooth, so that $f^K_L \in C^\infty(U)$.

□

Definition 7.12.0.15.

7.13 Differential Forms

Definition 7.13.0.1. We define

$$\Lambda^k(TM) = \coprod_{p \in M} \Lambda^k(T_pM)$$

Definition 7.13.0.2. Let $\omega : M \to \Lambda^k(TM)$. Then ω is said to be a **k-form on** M if for each $p \in M$, $\omega_p \in \Lambda^k(T_pM)$.
For each $X \in \Gamma^1(M)^k$, we define $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)^k$, $\omega(X)$ is smooth.
The set of smooth k-forms on M is denoted $\Omega^k(M)$.

Note 7.13.0.3. Observe that

1. $\Omega^k(M) \subset \Gamma^0_k(M)$
2. $\Omega^0(M) = C^\infty(M)$

Exercise 7.13.0.4. The set $\Omega^k(M)$ is a $C^\infty(M)$-submodule of $\Gamma^0_k(M)$.

Proof. Clear. □

Definition 7.13.0.5. Define the **exterior product**

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Note 7.13.0.6. For $f \in \Omega^0(M)$ and $\alpha \in \Omega^k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 7.13.0.7. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is well defined.

Proof. Let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{i=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then

$$\begin{aligned}
\alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)(\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \cdots, X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p))
\end{aligned}$$

□

Exercise 7.13.0.8. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is $C^\infty(M)$-bilinear.

Proof.

1. $C^\infty(M)$-linearity in the first argument:
   Let $\alpha \in \Omega^k(M)$, $\beta, \gamma \in \Omega^l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda^k(T_pM) \times \Lambda^l(T_pM) \to \Lambda^{k+l}(T_pM)$ implies that

$$\begin{aligned}
[(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\
&= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\
&= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\
&= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p
\end{aligned}$$

   So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

   and $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is $C^\infty(M)$-linear in the first argument.

2. $C^\infty(M)$-linearity in the second argument:
   Similar to (1).

□

Note 7.13.0.9. All of the results from multilinear algebra apply here.

Definition 7.13.0.10. We define the **exterior derivative** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ inductively by

1. $d(d\alpha) = 0$ for $\alpha \in \Omega^p(M)$
2. $df(X) = Xf$ for $f \in \Omega^0(M)$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$
4. extending linearly

Exercise 7.13.0.11. Let (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then on U, for each $i, j \in \{1, \cdots, n\}$,

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx^1_p, \cdots, dx^n_p\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ and $T^*_pM = \operatorname{span}\{dx^1_p, \cdots, dx^n_p\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned}
\left[dx^i\left(\frac{\partial}{\partial x^j}\right)\right]_p &= \left(\frac{\partial}{\partial x^j} x^i\right)_p \\
&= \frac{\partial}{\partial x^i}\Big|_p x^i \\
&= \delta_{i,j}
\end{aligned}$$

□

Exercise 7.13.0.12. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Proof. Let $p \in U$. Since $\{dx^1, \cdots, dx^n\}$ is a basis for $\Lambda(T_pM)$, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^{n} a^i(p)dx^i_p$. Therefore, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) &= \sum_{i=1}^{n} a^i(p)dx^i_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) &= \frac{\partial}{\partial x^i}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p)dx^i_p$$

Therefore

$$df|_U = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}dx^i$$

□

Exercise 7.13.0.13. Let $f \in \Omega^0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \ldots, x_n)$. Then for each $i \in \{1, \ldots, n\}$,

$$\frac{\partial}{\partial x^i}\bigg|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p)dx^i_p \\ &= 0 \end{aligned}$$

□

Exercise 7.13.0.14.

Definition 7.13.0.15. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$. We define

$$dx^i = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_k}}\right)$$

Note 7.13.0.16. We have that

1. $$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{I,J}$$

2. Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega^k(U)$,

$$\omega\left(\frac{\partial}{\partial x^i}\right) \in C^\infty(U)$$

Exercise 7.13.0.17. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega\left(\frac{\partial}{\partial x^i}\right) dx^i$$

Proof. Let $p \in U$. Since $\{dx^i_p : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(T_pM)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p) dx^i_p$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial x^j}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j}\right) \\
&= f_J
\end{aligned}$$

□

Exercise 7.13.0.18. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. If $\omega = \sum_{I \in \mathcal{I}_k} f_I dx^i$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

.

Proof. First we note that

$$\begin{aligned}
d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f d(dx^i) \\
&= df_I \wedge dx^i \\
&= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i\right) \wedge dx^i \\
&= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i
\end{aligned}$$

Then we extend linearly. □

Definition 7.13.0.19. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ be a diffeomorphism. Define the **pullback of** F, denoted $F^* : \Omega^k(N) \to \Omega^k(M)$ by

$$(F^*\omega)_p(v_1, \cdots, v_k) = \omega_{F(p)}(DF_p(v_1), \cdots, DF_p(v_k))$$

for $\omega \in \Omega^k(N)$, $p \in M$ and $v_1, \cdots, v_k \in T_pM$

Chapter 8

de Rham Cohomology

8.1 TO DO

1. de Rham cohomology
2. de Rham homology
3. in de Rham homology, measures on the manifold can be identified with the 0th Homology, group
4. think about how the other homology groups can be used in statistics

8.2 Introduction

Note 8.2.0.1. We recall that $d : \Omega^*(M) \to \Omega^*(M)$ satisfies the properties:

1. $d^2 = 0$
2.
3.

Definition 8.2.0.2. Let M be an n-dimensional smooth manifold. For $k \in \{1, \ldots, n\}$, we define the

- **k-th coboundary operator**, denoted $d^k : \Omega^k(M) \to \Omega^{k+1}(M)$, by $d^k = d|_{\Omega^k(M)}$
-
-

Chapter 9

Connections

9.1 Koszul Connections

Definition 9.1.0.1. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ and $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then ∇ is said to be a **Koszul connection on E in the first representation** if

1. for each $\sigma \in \Gamma(E)$, $\nabla(\cdot, \sigma)$ is $C^\infty(M)$-linear
2. for each $X \in \mathfrak{X}(M)$, $\nabla(X, \cdot)$ is $\mathbb{R}$-linear
3. for each $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\nabla(X, f\sigma) = f\,\nabla(X, \sigma) + X(f)\sigma$$

Definition 9.1.0.2. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ be a smooth vector bundle and $\nabla : \Gamma(E) \to T^*M \otimes \Gamma(E)$. Then ∇ is said to be a **Koszul connection on E in the second representation** if

1. ∇ is $\mathbb{R}$-linear
2. for each $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\nabla(f\sigma) = f\,\nabla\,\sigma + df \otimes \sigma$$

Note 9.1.0.3. When the context is clear, we will write $\nabla_X Y$ in place of $\nabla(X, Y)$ and we will refer to ∇ as a connection.

Exercise 9.1.0.4. Define $\phi : \Gamma(E)^{\mathfrak{X}(M)\times\Gamma(E)} \to [T^*M \otimes \Gamma(E)]^{\Gamma(E)}$ by

$$\phi(\nabla)(X) = \nabla_X\,\sigma$$

Then ∇ is a Koszul connection on E in the first representation iff $\phi(\nabla)$ Koszul connection on E in the second representation.

Proof. $\square$

Exercise 9.1.0.5. Let $(E, M, \pi) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$, ∇ a connection on E, $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$. If $X = 0$ or $Y = 0$, then $\nabla_X Y = 0$.

Proof.

- If $X = 0$, then

$$\begin{aligned}\nabla_X Y &= \nabla_{0X} Y \\ &= 0\,\nabla_X Y \\ &= 0\end{aligned}$$

- Similarly, if $Y = 0$, then $\nabla_X Y = 0$.

$\square$

Exercise 9.1.0.6. Let (E, M, π) be a smooth vector bundle, ∇ a connection on E, $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$ and $p \in M$. If $X \sim_p 0$ or $Y \sim_p 0$, then $[\nabla_X Y]_p = 0$.

Proof.

- Suppose that $X \sim_p 0$. Then there exists $U \subset M$ such that U is open and $X|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\operatorname{supp} \phi \subset U$ and $\phi \sim_p 1$. Then $\phi X = 0$. The previous exercise implies that $\nabla_{\phi X} Y = 0$. Therefore

$$\begin{aligned}
\nabla_X Y &= \nabla_{\phi X + (1-\phi)X} Y \\
&= \nabla_{\phi X} Y + \nabla_{(1-\phi)X} Y \\
&= 0 + (1-\phi)\nabla_X Y \\
&= (1-\phi)\nabla_X Y
\end{aligned}$$

Hence

$$\begin{aligned}
[\nabla_X Y]_p &= [(1-\phi)\nabla_X Y]_p \\
&= (1-\phi(p))[\nabla_X Y]_p \\
&= 0
\end{aligned}$$

- Suppose that $Y \sim_p 0$. Then there exists $U \subset M$ such that U is open and $Y|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\operatorname{supp} \phi \subset U$ and $\phi \sim_p = 1$. Then $\phi Y = 0$. The previous exercise implies that $\nabla_X \phi Y = 0$. Since $\phi \sim_p 1$, we have that $1 - \phi \sim_p 0$. Thus $X(1-\phi) \sim_p 0$ and

$$\begin{aligned}
\nabla_X Y &= \nabla_X[\phi Y + (1-\phi)Y] \\
&= \nabla_X[\phi Y] + \nabla_X[(1-\phi)Y] \\
&= \nabla_X[(1-\phi)Y] \\
&= (1-\phi)\nabla_X Y + [X(1-\phi)]\nabla_X Y
\end{aligned}$$

Hence

$$\begin{aligned}
[\nabla_X Y]_p &= (1-\phi(p))[\nabla_X Y]_p + [X(1-\phi)](p)[\nabla_X Y]_p \\
&= 0
\end{aligned}$$

$\square$

Exercise 9.1.0.7. Let (E, M, π) be a smooth vector bundle and ∇ a connection on E. Then for each $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$, $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$ implies that $[\nabla_{X_1} Y_1]_p = [\nabla_{X_2} Y_2]_p$.

Proof. Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$. Suppose that $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$. Define $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$ by $X = X_2 - X_1$ and $Y = Y_2 - Y_1$. Then $X \sim_p 0$ and $Y \sim_p 0$. The previous exercise implies that $[\nabla_X Y_1]_p = 0$ and $[\nabla_{X_2} Y]_p = 0$. Therefore

$$\begin{aligned}
[\nabla_{X_1} Y_1]_p &= [\nabla_{X_1} Y_1]_p + [\nabla_X Y_1]_p \\
&= [\nabla_{X_1} Y_1 + \nabla_X Y_1]_p \\
&= [\nabla_{X_1+X} Y_1]_p \\
&= [\nabla_{X_2} Y_1]_p \\
&= [\nabla_{X_2} Y_1]_p + [\nabla_{X_2} Y]_p \\
&= [\nabla_{X_2} Y_1 + \nabla_{X_2} Y]_p \\
&= [\nabla_{X_2}(Y_1 + Y)]_p \\
&= [\nabla_{X_2} Y_2]_p
\end{aligned}$$

□

Exercise 9.1.0.8. Let (E, M, π) be a smooth vector bundle, ∇ a connection on E and $U \subset M$. If U is open, then there exists a unique connection $\nabla^U : \mathfrak{X}(U) \times \Gamma(E|_U) \to \Gamma(E|_U)$ such that for each $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$,

$$\nabla^U_{X|_U} Y|_U = (\nabla_X Y)|_U$$

Chapter 10

Semi-Riemannian Geometry

Definition 10.0.0.1. Let M be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then g is said to be nondegenerate if for each $p \in M$, g_p is nondegenerate.

Definition 10.0.0.2. Let M be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then g is said to be a **metric tensor field** on M if

1. g is nondegenerate
2. g has constant index

In this case (M, g) is said to be a **semi-Riemannian manifold**

Definition 10.0.0.3. Define Interval
FINISH!!!

Definition 10.0.0.4. Let $(E, M, \pi) \in \text{Obj}(\textbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\textbf{Man}^\infty}(I, M)$ and $\gamma \in \text{Hom}_{\textbf{Man}^\infty}(I, E)$. Then γ is said to be a **section of** E **over** α if $\pi \circ \gamma = \alpha$. We denote the set of sections of E over α by $\Gamma(E, \alpha)$.

Definition 10.0.0.5. Let $(E, M, \pi) \in \text{Obj}(\textbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\textbf{Man}^\infty}(I, M)$ and $\gamma \in \Gamma(E, \alpha)$. Then γ is said to be said to be **extendible** if there exists $U \in \mathcal{N}_{\alpha(I)}$ and $\tilde{\gamma} \in \Gamma(E|_U)$ such that U is open and $\tilde{\gamma} \circ \alpha = \gamma$.

Exercise 10.0.0.6. figure 8 not extendible FINISH!!!

Exercise 10.0.0.7. Let $(E, M, \pi) \in \text{Obj}(\textbf{VecBun}^\infty)$, ∇ a connection on E, $I \subset \mathbb{R}$ an interval and $\alpha \in \text{Hom}_{\textbf{Man}^\infty}(I, M)$. There exists a unique $D_\alpha : \Gamma(E, \alpha) \to \Gamma(E, \alpha)$ such that

1. for each $\lambda \in \mathbb{R}$ and $\gamma, \sigma \in \Gamma(E, \alpha)$,
$$D_\alpha(\gamma + \lambda\sigma) = D_\alpha\gamma + \lambda D_\alpha\sigma$$
2. for each $f \in C^\infty(I)$ and $\gamma \in \Gamma(E, \alpha)$,
$$D_\alpha(f\gamma) = f'\gamma + f D_\alpha\gamma$$
3. for each $\gamma \in \Gamma(E)$, if $\tilde{\gamma}$ extends γ, then
$$D_\alpha\gamma = \nabla_{\alpha'}\gamma$$

Proof. □

Chapter 11

Riemannian Geometry

Definition 11.0.0.1. Let M be a smooth manifold and $g \in T_2^0(M)$ a metric tensor on M. We define $\hat{g} \in T_0^2(M)$ by $\hat{g}(\omega, \eta) = g(\phi_g^{-1}(\omega), \phi_g^{-1}(\eta))$.

Exercise 11.0.0.2. content...

Exercise 11.0.0.3. Let (M, g) be a semi-Riemannian manifold and $(U, \phi) \in \mathcal{A}$. Then the induced metric $\langle\rangle_{T^*M \otimes TM}$ on $T^*M \otimes TM$ is given by

$$\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} = g^{i,j} g_{kl}$$

Proof. We have that

$$\begin{aligned}\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} &= \langle dx^i, dx^j \rangle_{T^*M} \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle_{TM} \\ &= g^{i,j} g_{k,l}\end{aligned}$$

$\square$

Exercise 11.0.0.4. Let (M, g) be an n-dimensional Riemannian manifold.

1. There exists $\lambda \in \Omega^n(M)$ such that for each orthonormal frame $e_1, \ldots, e_n$,

$$\lambda(e_1, \ldots, e_n) = 1$$

   Hint: Choose a frame $z_1, \ldots, z_n$ on M with corresponding dual frame $\zeta^1, \ldots, \zeta^n$. Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n$$

2. Let $N \in \mathfrak{X}(M)$ be the outward pointing normal to ∂M and $X \in \mathfrak{X}(M)$. Then

$$\int_M \mathrm{div} X \lambda = \int_{\partial M} g(X, N) \tilde{\lambda}$$

3. For each $u \in \mathbb{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, we have that

$$\mathrm{div}(uX) = u\mathrm{div}(X) + du(X)$$

   and therefore

$$\int_M du(X) \lambda = \int_{\partial M} ug(X, N) \tilde{\lambda} - \int_M u\mathrm{div}(X) \lambda$$

Proof.

1. Let $z_1, \dots, z_n$ be a frame on M and $\zeta^1, \dots, \zeta^n$ with corresponding dual frame $\zeta^1, \dots, \zeta^n$. Define

$$\lambda = \det[g(z_i, z_j)]^{1/2}\zeta^1 \wedge \cdots \wedge \zeta^n$$

Let $e_1, \dots, e_n$, be an orthonormal frame on M with corresponding dual coframe $\epsilon^1, \dots, \epsilon^n$. Let $i, j \in \{1, \dots, n\}$. Then there exist $(a_{k,i}) \subset \mathbb{R}$ such that $\zeta^i = \sum_{k=1}^{n} a_{k,i}\epsilon^k$. Then

$$\begin{aligned}
\hat{g}(\epsilon^j, \zeta^i) &= \sum_{k=1}^{n} a_{k,i}\hat{g}(\epsilon^j, \epsilon^k) \\
&= \sum_{k=1}^{n} a_{k,i} g(\phi_g^{-1}(\epsilon^j), \phi_g^{-1}(\epsilon^k)) \\
&= \sum_{k=1}^{n} a_{k,i} g(e_j, e_k) \\
&= \sum_{k=1}^{n} a_{k,i}\delta_{j,k} \\
&= a_{j,i}
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta_{i,j} &= \zeta^i(z_j) \\
&= \sum_{k=1}^{n} a_{k,i}\epsilon^k(z_j) \\
&= \sum_{k=1}^{n} a_{k,i} g(e_k, z_j) \\
&= \sum_{k=1}^{n} \hat{g}(\epsilon^k, \zeta^i) g(e_k, z_j)
\end{aligned}$$

Define $U, V \in \mathbb{R}^{n \times n}$ by $U_{i,k} = \hat{g}(\zeta^i, \epsilon^k)$ and $V_{k,j} = g(e_k, z_j)$. Then from above, we have that $UV = I$. Since $U, V \in \mathbb{R}^{n \times n}$, $VU = I$. Hence $U = V^{-1}$. Since

$$\begin{aligned}
\zeta^i(e_j) &= \sum_{k=1}^{n} a_{k,i}\epsilon^k(e_j) \\
&= \sum_{k=1}^{n} a_{k,i}\delta_{k,j} \\
&= a_{j,i} \\
&= \hat{g}(\epsilon^j, \zeta^i) \\
&= U_{i,j}
\end{aligned}$$

and

$$\begin{aligned}
g(z_i, z_j) &= \left(\sum_{k=1}^{n} g(e_k, z_i) e_k, \sum_{l=1}^{n} g(e_l, z_j) e_l \right) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} g(e_k, z_i) g(e_l, z_j) g(e_k, e_l) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} g(e_k, z_i) g(e_l, z_j) \delta_{k,l} \\
&= \sum_{k=1}^{n} g(e_k, z_i) g(e_k, z_j) \\
&= (V^* V)_{i,j}
\end{aligned}$$

we have that

$$\begin{aligned}
\lambda(e_1, \dots, e_n) &= \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n (e_1, \dots, e_n) \\
&= \det[g(z_i, z_j)]^{1/2} \det[\zeta^i(e_j)] \\
&= \det(V^* V)^{1/2} \det U \\
&= \det V (\det V)^{-1} \\
&= 1
\end{aligned}$$

2. Choose an orthonormal frame $e_1, \dots, e_{n-1} \in \mathfrak{X}(\partial M)$ with dual coframe $\epsilon^1, \dots, \epsilon^{n-1}$. Define $\nu \in \Omega^1(M)$ to be the dual covector to N. We note that $N, e_1, \dots, e_{n-1}$ is an orthonormal frame on $\mathfrak{X}(M)$. Let $X_1, \dots, X_{n-1} \in \mathfrak{X}(\partial M)$. Since for each $j \in \{1, \dots, n-1\}$, $X_j \in \mathfrak{X}(\partial M)$ and for each $p \in \partial M$, $N_p \in (T_p \partial M)^{\perp}$, we have that for each $j \in \{1, \dots, n-1\}$, $g(X_j, N) = 0$. This implies that

$$\begin{aligned}
\iota^* \iota_X \lambda(X_1, \dots, X_{n-1}) &= \lambda(X, X_1, \dots, X_{n-1}) \\
&= \nu \wedge \epsilon^1 \wedge \cdots \wedge \epsilon^{n-1}(X, X_1, \dots, X_{n-1}) \\
&= \det \begin{pmatrix} \nu(X) & \nu(X_1) & \cdots & \nu(X_{n-1}) \\ \epsilon^1(X) & \epsilon^1(X_1) & \cdots & \epsilon^1(X_{n-1}) \\ \vdots & & & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \cdots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= \det \begin{pmatrix} g(X, N) & g(X_1, N) & \cdots & g(X_{n-1}, N) \\ \epsilon^1(X) & \epsilon^1(X_1) & \cdots & \epsilon^1(X_{n-1}) \\ \vdots & & & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \cdots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= g(X, N) \det(\epsilon^i(X_j)) \\
&= g(X, N) \epsilon^1 \wedge \ldots \wedge \epsilon^{n-1}(X_1, \dots, X_{n-1}) \\
&= g(X, N) \tilde{\lambda}(X_1, \dots, X_n)
\end{aligned}$$

Therefore $\iota^* \iota_X \lambda = g(X, N) \tilde{\lambda}$ and

$$\begin{aligned}
\int_M \operatorname{div} X \lambda &= \int_M d(\iota_X \lambda) \\
&= \int_{\partial M} \iota^*(\iota_X \lambda) \\
&= \int_{\partial M} g(X, N) \tilde{\lambda}
\end{aligned}$$

3. We note that

$$\begin{aligned} 0 &= \iota_X(du \wedge \lambda) \\ &= \iota_X(du) \wedge \lambda - du \wedge (\iota_X \lambda) \\ &= du(X)\lambda - du \wedge (\iota_X \lambda) \end{aligned}$$

which implies that

$$\begin{aligned} \mathrm{div}(uX)\lambda &= d(\iota_{uX}\lambda) \\ &= d(u\iota_X \lambda) \\ &= du \wedge (\iota_X \lambda) + ud(\iota_X \lambda) \\ &= du(X)\lambda + u\mathrm{div}(X)\lambda \\ &= [du(X) + u\mathrm{div}(X)]\lambda \end{aligned}$$

This implies that $\mathrm{div}(uX) = du(X) + u\mathrm{div}(X)$. From before, we have that

$$\begin{aligned} \int_M du(X)\lambda &= \int_M \mathrm{div}(uX)\lambda - \int_M u\mathrm{div}(X)\lambda \\ &= \int_{\partial M} g(uX, N)\tilde{\lambda} - \int_M u\mathrm{div}(X)\lambda \\ &= \int_{\partial M} ug(X, N)\tilde{\lambda} - \int_M u\mathrm{div}(X)\lambda \end{aligned}$$

□

Exercise 11.0.0.5.

$$\mathrm{div}(X) = \sum_{j=1}^{n} (\nabla_{\partial_j} X)^j$$

Proof. We have that

$$\begin{aligned} \nabla_{\partial_i}(X) &= \sum_{j=1}^{n} \nabla_{\partial_i}(X^j \partial_j) \\ &= \sum_{j=1}^{n} \left[X^j \nabla_{\partial_i} \partial_j + \partial_i(X^j)\partial_j \right] \\ &= \sum_{j=1}^{n} \left[X^j (\sum_{k=1}^{n} \Gamma_{i,j}^k \partial_k) + \partial_i(X^j)\partial_j \right] \\ &= \sum_{j=1}^{n} \left[X^j \left(\sum_{k=1}^{n} \Gamma_{i,j}^k \partial_k \right) + \partial_i(X^j)\partial_j \right] \\ &= \sum_{j=1}^{n} X^j \left(\sum_{k=1}^{n} \Gamma_{i,j}^k \partial_k \right) + \sum_{j=1}^{n} \partial_i(X^j)\partial_j \\ &= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} X^j \Gamma_{i,j}^k \right) \partial_k + \sum_{k=1}^{n} \partial_i(X^k)\partial_k \\ &= \sum_{k=1}^{n} \left[\left(\sum_{j=1}^{n} X^j \Gamma_{i,j}^k \right) + \partial_i(X^k) \right] \partial_k \end{aligned}$$

so that $(\nabla_{\partial_i}(X))^i = \left(\sum_{j=1}^n X^j \Gamma^i_{i,j}\right) + \partial_i(X^i)$. We note that

$$\begin{aligned}
\operatorname{div}(X) &= \sum_{i=1}^n \operatorname{div}(X^i \partial_i) \\
&= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + dx^i(\partial_i)] \\
&= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + 1]
\end{aligned}$$

Since $\lambda = [\det g(\partial_i, \partial_j)]^{1/2} dx^1 \wedge \cdots \wedge dx^n = (\det g)^{1/2} dx$, we have that

$$\begin{aligned}
d(\iota_{\partial_i}\lambda) &= d((\det g)^{1/2} \iota_{\partial_i} dx) \\
&= d[(\det g)^{1/2}]\iota_{\partial_i} dx + (\det g)^{1/2} d(\iota_{\partial_i} dx) \\
&= d[(\det(g)^{1/2}] \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots dx^n + (\det g)^{1/2} \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots dx^n)
\end{aligned}$$

FINISH!!! $\square$

Exercise 11.0.0.6. Let (M, g) be a Riemannian manifold.

1. For each $u, v \in C^\infty(M)$. Then

   (a)
   $$\int_M u \Delta v \lambda + \int_M g(\nabla u, \nabla v)\lambda = \int_{\partial M} u N(v) \tilde{\lambda}$$

   (b)
   $$\int_M [u\Delta v - v \Delta u]\lambda = \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}$$

2. (a) If $\partial M \neq \varnothing$, then for each $u, v \in C^{\infty(M)}$, u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$ implies that $u = v$.

   (b) If $\partial M = \varnothing$, then for each $u \in C^\infty(M)$, u is harmonic implies that u is constant.

Proof.

1. Let $u, v \in C^\infty(M)$. Then

   (a)
   $$\begin{aligned}
   \int_M u \Delta v \lambda &= \int_M u \operatorname{div}(\nabla v)\lambda \\
   &= \int_{\partial M} u g(\nabla v, N)\tilde{\lambda} - \int_M du(\nabla v)\lambda \\
   &= \int_{\partial M} u dv(N)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda \\
   &= \int_{\partial M} u N(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda
   \end{aligned}$$

(b) From above, we have that

$$\begin{aligned}
\int_M [u\Delta v - v\Delta u]\lambda &= \int_M u\Delta v\lambda - \int_M v\Delta u\lambda \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda - \left(\int_{\partial M} vN(u)\tilde{\lambda} - \int_M g(\nabla v, \nabla u)\lambda\right) \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_{\partial M} vN(u)\tilde{\lambda} \\
&= \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}
\end{aligned}$$

2. (a) Suppose that $\partial M \neq \varnothing$. Let $u, v \in C^{\infty(M)}$. Suppose that u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$. Then $u - v$ is harmonic and

$$\begin{aligned}
\int_M \| \nabla(u-v)\|_g^2\lambda &= \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= 0 + \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= \int_M (u-v)\Delta(u-v)\lambda + \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= \int_{\partial M} (u-v)N(u-v)\tilde{\lambda} \\
&= 0
\end{aligned}$$

Thus $\nabla(u-v) = 0$ and $u - v$ is constant. Since $u|_{\partial M} = v|_{\partial M}$, we have that $u - v = 0$ and thus $u = v$.

(b) Suppose that $\partial M = \varnothing$. Let $u \in C^\infty(M)$. Suppose that u is harmonic. Then

$$\begin{aligned}
\int_M \| \nabla u\|_g^2\lambda &= \int_M g(\nabla u, \nabla u)\lambda \\
&= 0 + \int_M g(\nabla u, \nabla u)\lambda \\
&= \int_M u\Delta u\lambda + \int_M g(\nabla u, \nabla u)\lambda \\
&= \int_{\partial M} (u-v)g(\nabla(u-v), N)\tilde{\lambda} \\
&= 0
\end{aligned}$$

Therefore $\nabla u - 0$ and u is constant.

$\square$

Chapter 12

Symplectic Geometry

12.1 Symplectic Manifolds

Definition 12.1.0.1. Let $M \in \mathrm{Obj}(\mathbf{Man}^{\infty})$ and $\omega \in \Omega^2(M)$. Then ω is said to be **symplectic** if

1. ω is nondegenerate
2. ω is closed

Chapter 13

Extra

Definition 13.0.0.1. When working in $\mathbb{R}^n$, we introduce the formal objects $dx^1, dx_2, \cdots, dx^n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx^i = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.

Definition 13.0.0.2. Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \operatorname{span}\{dx^i : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

1. the exterior product is bilinear

2. for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$

3. for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$

4. for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k-forms on $\mathbb{R}^n$. Let ω be a k-form on $\mathbb{R}^n$. If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx^i$

Note 13.0.0.3. The terms $dx^1, dx_2, \cdots, dx^n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx^1, dx_2, \cdots, dx^n\}$ when referencing the coordinates on $\mathbb{R}^n$.

Exercise 13.0.0.4. Let $B_{n\times n} = (b_{i,j}) \in [C^\infty(M)]^{n\times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^{n} \left(\sum_{j=1}^{n} b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned}\bigwedge_{i=1}^{n}\left(\sum_{j=1}^{n} b_{i,j}dx^j\right) &= \left(\sum_{j=1}^{n} b_{1,j}dx^j\right)\wedge\left(\sum_{j=1}^{n} b_{2,j}dx^j\right)\wedge\cdots\wedge\left(\sum_{j=1}^{n} b_{n,j}dx^j\right)\\ &= \sum_{j_1,\cdots,j_n=1}^{n}\left(\prod_{i=1}^{n} b_{i,j_i}\right)dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}\\ &= \sum_{j_1\neq\cdots\neq j_n}\left(\prod_{i=1}^{n} b_{i,j_i}\right)dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}\\ &= \left[\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\left(\prod_{i=1}^{n} b_{i,\sigma(i)}\right)\right]dx_1\wedge dx_2\wedge\cdots\wedge dx_n\\ &= (\det B)dx_1\wedge dx_2\wedge\cdots\wedge dx_n\end{aligned}$$

□

Definition 13.0.0.5. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted df, on $\mathbb{R}^n$ by

$$df=\sum_{i=1}^{n}\frac{\partial f}{\partial x^i}dx^i$$

Let $\omega=\sum_{I\in\mathcal{I}_{k,n}} f_I dx^i$ be a k-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega=\sum_{I\in\mathcal{I}_{k,n}} df_I\wedge dx^i$$

Exercise 13.0.0.6. On $\mathbb{R}^3$, put

1. $\omega_0=f_0$,
2. $\omega_1=f_1dx^1+f2dx_2+f_2dx_3$,
3. $\omega_2=f_1dx_2\wedge dx_3-f_2dx^1\wedge dx_3+f_3dx^1\wedge dx_2$

Show that

1. $d\omega_0=\frac{\partial f_0}{\partial x^1}dx^1+\frac{\partial f_0}{\partial x^2}dx_2+\frac{\partial f_0}{\partial x^3}dx_3$
2. $d\omega_1=\left(\frac{\partial f_3}{\partial x^2}-\frac{\partial f_2}{\partial x^3}\right)dx_2\wedge dx_3+\left(\frac{\partial f_3}{\partial x^1}-\frac{\partial f_1}{\partial x^3}\right)dx^1\wedge dx_3+\left(\frac{\partial f_2}{\partial x^1}-\frac{\partial f_1}{\partial x^2}\right)dx^1\wedge dx_2$
3. $d\omega_2=\left(\frac{\partial f_1}{\partial x^1}+\frac{\partial f_2}{\partial x^2}+\frac{\partial f_3}{\partial x^3}\right)dx^1\wedge dx_2\wedge dx_3$

Proof. Straightforward. □

Exercise 13.0.0.7. Let $I\in\mathcal{I}_{k,n}$. Then there is a unique $I_*\in\mathcal{I}_{n-k,n}$ such that $dx^i\wedge dx_{I_*}=dx^1\wedge dx_2\wedge\cdots\wedge dx^n$.

Definition 13.0.0.8. We define a linear map $*:\Phi_k(\mathbb{R}^n)\to\Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge $*$-operator** by

$$*\sum_{I\in\mathcal{I}_{k,n}} f_I dx^i=\sum_{I\in\mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 13.0.0.9. Let $\phi:\mathbb{R}^k\to\mathbb{R}^n$ be smooth. Write $\phi=(\phi_1,\phi_2,\cdots,\phi_n)$. We define $\phi^*:\Phi_k(\mathbb{R}^n)\to\Phi_k(\mathbb{R}^k)$ via the following properties:

1. for each 0-form f on $\mathbb{R}^n$, $\phi^* f = f \circ \phi$
2. for $i = 1, \cdots, n$, $\phi^* dx^i = d\phi_i$
3. for an s-form ω, and a t-form χ on $\mathbb{R}^n$, $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
4. for l-forms ω, χ on $\mathbb{R}^n$, $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 13.0.0.10. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold of $\mathbb{R}^n$, $\phi : U \to V$ a smooth parametrization of M, $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$ an k-form on $\mathbb{R}^n$. Then

$$\phi^*\omega = \bigg(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I)\bigg) du^1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof. By definition,

$$\begin{aligned}
\phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i \\
&= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I)\phi^* dx^i \\
&= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I
\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}
d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\
&= (\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u^j} du^j) \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u^j} du^j) \wedge \cdots \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u^j} du^j) \\
&= (\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\
&= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k \\
&= \bigg(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I)\bigg) du^1 \wedge du_2 \wedge \cdots \wedge du_k
\end{aligned}$$

□

13.1 Integration of Differential Forms

Definition 13.1.0.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \cdots \wedge dx_k$ a k-form on $\mathbb{R}^k$. Define

$$\int_U \omega = \int_U f dx$$

Definition 13.1.0.2. Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$, ω a k-form on $\mathbb{R}^n$ and $\phi : U \to V$ a local smooth, orientation-preserving parametrization of M. Define

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 13.1.0.3.

Theorem 13.1.0.4. Stokes Theorem:
Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$ and ω a $k-1$-form on $\mathbb{R}^n$. Then

$$\int_{\partial M} \omega = \int_M d\omega$$

Appendix A

Summation

Appendix B

Asymptotic Notation

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration