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Introduction to Quantum Physics

Carson James

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Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Set Theory

1.1 Operations and Relations

Definition 1.1.0.1.

- We define $[0] := \varnothing$ and for $k \in \mathbb{N}$, we define $[k] := \{1, \ldots, k\}$.
- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-tupels with entries in** S, denoted S^k, by

$$S^k := \{u : [k] \to S\}$$

- Let S be a set. We define the **set of all tuples with entries in** S, denoted S^*, by

$$S^* := \bigcup_{k \in \mathbb{N}_0} S^k$$

- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary operation on** S, denoted $\mathcal{F}^k(S)$, by $\mathcal{F}^k(S) := S^{(S^k)}$. We define the **set of finitary operations on** S, denoted $\mathcal{F}^*(S)$, by

$$\mathcal{F}^*(S) := \bigcup_{k \in \mathbb{N}_0} \mathcal{F}^k(S)$$

- Let S be a set. We define the **operation arity map**, denoted $\operatorname{arity} : \mathcal{F}^*(S) \to \mathbb{N}_0$, by

$$\operatorname{arity} f := k, \quad f \in \mathcal{F}^k(S)$$

- Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{F}$, denoted $\mathcal{F}_k$, by

$$\mathcal{F}_k := \mathcal{F} \cap \mathcal{F}^k(S)$$

- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary relations on** S, denoted $\mathcal{R}^k(S)$, by $\mathcal{R}^k(S) := \mathcal{P}(S^k)$. We define the **set of finitary relations on** S, denoted $\mathcal{R}^*(S)$, by

$$\mathcal{R}^*(S) := \bigcup_{k \in \mathbb{N}_0} \mathcal{R}^k(S)$$

- Let S be a set. We define the **arity map**, denoted $\operatorname{arity} : \mathcal{R}^*(S) \to \mathbb{N}_0$, by

$$\operatorname{arity} R := k, \quad f \in \mathcal{R}^k(S)$$

- Let S be a set, $\mathcal{R} \subset \mathcal{R}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{R}$, denoted $\mathcal{R}_k$, by

$$\mathcal{R}_k := \mathcal{R} \cap \mathcal{R}^k(S)$$

Definition 1.1.0.2. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $C \subset S$. Then C is said to be **$\mathcal{F}$-closed** if for each $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$ and $a_1, \dots, a_k \in C$, $f(a_1, \dots, a_k) \in C$.

Exercise 1.1.0.3. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $\mathcal{C} \subset \mathcal{P}(S)$. If for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed, then $\bigcap\limits_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed

Proof. Suppose that for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed. Let $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$, $a_1, \dots, a_k \in \bigcap\limits_{C \in \mathcal{C}} C$ and $C_0 \in \mathcal{C}$. Since $C_0 \in \mathcal{C}$, we have that

$$\begin{aligned} a_1, \dots, a_k &\in \bigcap_{C \in \mathcal{C}} C \\ &\subset C_0 \end{aligned}$$

Since C_0 is $\mathcal{F}$-closed, we have that $f(a_1, \dots, a_k) \in C_0$. Since $C_0 \in \mathcal{C}$ is arbitrary, we have that for each $C \in \mathcal{C}$, $f(a_1, \dots, a_k) \in C$. Hence $f(a_1, \dots, a_k) \in \bigcap\limits_{C \in \mathcal{C}} C$. Since $k \in \mathbb{N}_0$ and $a_1, \dots, a_k \in \bigcap\limits_{C \in \mathcal{C}} C$ are arbitrary, we have that $\bigcap\limits_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed. $\square$

Chapter 2

Quantization

Maybe change repo title to "into to quantum physics" instead of mechanics, that way we can cover field theory

- discuss how composite systems for particles are described by cartesian products. Then discuss how a particle at (x_0, p_0) can be described by a "field" (reall a prob dist) $\delta_{(x_0,p_0)}$. Then generally systems can be treated with prob dists and composite systems are described by tensor products, and that this generalizes the particle picture since $\delta_{(x_1,p_1)} \otimes \delta_{(x_2,p_2)} = \delta_{(x_1,p_1,x_2,p_2)}$.
- discuss Weil quantization, how as $\hbar \to 0$, we recover the posson bracket and commutative structure, discuss wigner transform of position and momentum functions giving position and momentum operators
- discuss rigged hilbert spaces to give meaning to "position basis", but treat as useful tool to get results like nonstandard analysis
- derive schrodinger equation from heisenberg picture
- free particle, harmonic oscillator, ladder operators, maybe hydrogen, do this in n-dimensions
- cover path integral, no complex measure exists, after rotation, prob measure on paths exists, then rotate back
- introduce field theory as first a theory of multiple particles and then as a continuum limit
- in the N particle case, the calculus of variations still works, remember, a quantum obervable operator $A \in HS(L^2(\mathbb{R}^N))$ corresponds to a classical observable function $f_A \in L^2(\mathbb{R}^{2N})$. We can write the lagrangian $L = \sum_j \mathcal{L}(x_1, \ldots, x_N, p_1, \ldots, p_N)$, which becomes $\int \mathcal{L}(\phi, \pi)$ in the continuum limit. Here $x_j = f_{X_j}$ and $p_j = f_{P_j}$ are the observeables corresponding to the position and momentum operators. We can then use calculus to find the $(x_j)_{j\in[N]}$, $(p_j)_{j\in[N]}$, i.e. ϕ and π which are stationary for $\mathcal{L}$. The question is then what quantization (wigner transform) means for these stationary ϕ and π. We can minimize the action $S[\phi] = \int \mathcal{L}(\phi, \pi)$ in $L^2(\mathbb{R}^{2N})$, but what does this correspond to in $HS(L^2(\mathbb{R}^N))$? does quantization preserve integration? i.e. $Q(\int \mathcal{L}((x_j), (p_j))) = \int Q(\mathcal{L}((x_j), (p_j))) = \mathcal{L}((X_j), (P_j))$?
- Since the isomorphism is between $HS(L^2\mathbb{R}^N)$ and $L^2(\mathbb{R}^{2N})$ with the star product, are we unable to guess what the true lagrangian is? For example, what if our system has classical lagrangian is xp. Since $x * p = p * x + O(\hbar)$, we wouldn't know if the true lagrangian was $x * p$, $p * x$, $(x * p + p * x)/2$ or something else. Can we distinguish this by experiment?
- Does the star product mess up the calculus of variations and do we still get the same euler lagrange equations with the star product? For lagrangians like the Klein-Gordon lagrangian with

2.1 Introduction

2.2

2.3 TODO

- Can we prepare various atoms in a certain state such that they bond in a way that yields interesting behavior? In other words, can we prepare atoms or subsystems in states, not necessarily all the same, such that we get interesting band structure or structure like cooper pairs, etc

Chapter 3

Quantum Fields

- discuss Schrodinger field as a continuum limit of a bunch of harmonic oscillators using creation annihilation operators and a lagrangian $L = \sum_{n \in \mathbb{Z}/N\mathbb{Z}} a_n^* a_n + a_{n+1}^* a_n + a_n^* a_{n+1}$ and explain how $a_{n+1}^* a_n + a_n^* a_{n+1}$ represents particles being destroyed at a location and created at an adjacent location. Show that this continuum limit is free particle. Then discuss more general lagrangians which allow for interactions like two-body interactions $V(n,m) a_n^* a_m^* a_m a_n$ and explain how we get interaction energy $V(n,m)$ if we have particles at sites n and m and so on with 3,4, ... particles see hitoshi murayama lectures

- Klein Gordon field as continuum limit of harmonic oscillators. Explore the case when the spring constants are not all constant, for instance maybe near $n = 0$, the spring constants get stronger/weaker. If we do the quantum field simulation done by ZAP physics, can we get represent the continuum limit using curvature? The idea here is that if the spring constant is stronger, the particle may pass by location $n = 0$ faster or slower and if we can get the particle to slow down enough as it approaches $n = 0$, can we get something like a schwarzchild radius to emerge?

- we can think of the classical fields as expected values of coherent states. Does this mean that the coherent states of an operator contain all information about operator?

3.1 Introduction

Appendix A

Summation

Definition A.0.0.1. Let $f : X \to [0, \infty)$, Then we define

$$\sum_{x\in X} f(x) := \sup_{\substack{F\subset X \\ F \text{ finite}}} \sum_{x\in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x\in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x\in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x\in X} f(x)$ where $f : X \to \mathbb{C}$.

Note A.0.0.2. Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x\in X} |f(x)| < \infty$, then $\sum_{x\in X} f(\alpha(x)) = \sum_{x\in X} f(x)$.

Appendix B

Asymptotic Notation

Definition B.0.0.1. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = o(g) \qquad \text{as } x \to x_0$$

if for each $\epsilon > 0$, there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U$,

$$\|f(x)\| \leq \epsilon \|g(x)\|$$

Exercise B.0.0.2. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. If there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U \setminus \{x_0\}$, $g(x) > 0$, then

$$f = o(g) \text{ as } x \to x_0 \quad \text{iff} \quad \lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = 0$$

Exercise B.0.0.3. Let X and Y a be normed vector spaces, $A \subset X$ open and $f : A \to Y$. Suppose that $0 \in A$. If $f(h) = o(\|h\|)$ as $h \to 0$, then for each $h \in X$, $f(th) = o(|t|)$ as $t \to 0$.

Proof. Suppose that $f(h) = o(\|h\|)$ as $h \to 0$. Let $h \in X$ and $\epsilon > 0$. Choose $\delta' > 0$ such that for each $h' \in B(0, \delta')$, $h' \in A$ and

$$\|f(h')\| \leq \frac{\epsilon}{\|h\| + 1} \|h'\|$$

Choose $\delta > 0$ such that for each $t \in B(0, \delta)$, $th \in B(0, \delta')$. Let $t \in B(0, \delta)$. Then

$$\begin{aligned} \|f(th)\| &\leq \frac{\epsilon}{\|h\| + 1} |t| \|h\| \\ &< \epsilon |t| \end{aligned}$$

So $f(th) = o(|t|)$ as $t \to 0$. $\square$

Definition B.0.0.4. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = O(g) \qquad \text{as } x \to x_0$$

if there exists $U \in \mathcal{N}(x_0)$ and $M \geq 0$ such that for each $x \in U$,

$$\|f(x)\| \leq M \|g(x)\|$$

Appendix C

Categories

move to notation?

Definition C.0.0.1. We define the category of topological measure spaces, denoted $\mathbf{TopMsr}_+$, by

- $\mathrm{Obj}(\mathbf{TopMsr}_+) := \{(X, \mu) : X \in \mathrm{Obj}(\mathbf{Top}) \text{ and } \mu \in M(X)\}$
- $\mathrm{Hom}_{\mathbf{TopMsr}_+}((X, \mu), (Y, \nu)) := \mathrm{Hom}_{\mathbf{Top}}(X, Y) \cap \mathrm{Hom}_{\mathbf{Msr}_+}((X, \mathcal{B}(X), \mu), (Y, \mathcal{B}(Y), \nu))$

Appendix D

Vector Spaces

it might be better to cover some category theory and write everything in terms of $\text{Hom}_{\mathbf{Vect}_{\mathbb{K}}}$ and $\text{Obj}(\mathbf{Vect}_{\mathbb{K}})$

D.1 Introduction

Definition D.1.0.1. Let X be a set, $\mathbb{K}$ a field, $+ : X \times X \to X$ and $\cdot : \mathbb{K} \times X \to X$. Then $(X, +, \cdot)$ is said to be a $\mathbb{K}$**-vector space** if

1. $(X, +)$ is an abelian group
2.

Definition D.1.0.2. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$. Then $(E, +_E, \cdot_E)$ is said to be a subspace of X if

1. $+_E = +_X|_{E \times E}$
2. $\cdot_E = \cdot_X|_{\mathbb{K} \times E}$

Exercise D.1.0.3. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$.

Exercise D.1.0.4. Let $(X, +, \cdot)$ be a vector space and $E \subset X$. Then E is a subspace of X

Definition D.1.0.5. Let X be a vector space and $(E_j)_{j \in J}$ a collection of subspaces of X. Then $\bigcap\limits_{j \in J} E_j$ is a subspace of X.

Proof. Set $E := \bigcap\limits_{j \in J} E_j$. Let $x, y \in E$ and $\lambda \in \mathbb{K}$. Then for each $j \in J$, $x, y \in E_j$. Since for each $j \in J$, E_j is a subspace of X, we have that for each $j \in J$, $x + \lambda y \in E_j$. Thus $x + \lambda y \in E$. Since $x, y \in E$ and $\lambda \in \mathbb{K}$ are arbitrary, (cite exercise here) we have that E is a subspace of X. □

Definition D.1.0.6. Let X, Y be vector spaces and $T : X \to Y$. Then T is said to be **linear** if for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

1. $T(x_1 + x_2) = T(x_1) + T(x_2)$,
2. $T(\lambda x_1) = \lambda T(x_1)$.

We define $L(X; Y) := \{T : X \to Y : T \text{ is linear}\}$.

Exercise D.1.0.7. Let X, Y be vector spaces and $T : X \to Y$. Then T is linear iff for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

$$T(x_1 + \lambda x_2) = T(x_1) + \lambda T(x_2)$$

Proof. Clear. (add details) □

Definition D.1.0.8. define addition/scalar multiplication of linear maps

Exercise D.1.0.9. Let X, Y be vector spaces. Then $L(X;Y)$ is a $\mathbb{K}$-vector space.

Proof. Clear □

Definition D.1.0.10. Let X be a vector space over $\mathbb{K}$ and $T : X \to \mathbb{K}$. Then T is said to be a **linear functional on** X if T is linear. We define the **dual space of** X, denoted X^*, by $X^* := \{T : X \to \mathbb{K} : T \text{ is linear}\}$.

Exercise D.1.0.11. Let X be a vector space. Then X^* is a vector space.

Proof. Clear. □

D.2 Bases

Definition D.2.0.1. Let X be a vector space and $(e_\alpha)_{\alpha \in A} \subset X$. Then $(e_\alpha)_{\alpha \in A}$ is said to be

- **linearly independent** if for each $(\alpha_j)_{j=1}^n \subset A$, $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $\sum\limits_{j=1}^n \lambda_j e_{\alpha_j} = 0$ implies that for each $j \in [n]$, $\lambda_j = 0$.
- a **Hamel basis for** X if $(e_\alpha)_{\alpha \in A}$ is linearly independent and $\text{span}(e_\alpha)_{\alpha \in A} = X$.

Exercise D.2.0.2. every vector space has a Hamel basis

Proof. □

Exercise D.2.0.3.

Exercise D.2.0.4. Let X be a $\mathbb{K}$-vector space and $x \in X$. Then $x = 0$ iff for each $\phi \in X^*$, $\phi(x) = 0$.

Proof.

- $(\implies)$:
  Suppose that $x = 0$. Linearity implies that for each $\phi \in X^*$ $\phi(x) = 0$.
- $(\impliedby)$:
  Conversely, suppose that $x \neq 0$. Define $\epsilon_x : \text{span}(x) \to \mathbb{K}$ by $\epsilon_x(\lambda x) := \lambda$. Let $u, v \in \text{span}(x)$. Then there exists $\lambda_u, \lambda_v \in \mathbb{K}$ such that $u = \lambda_u x$ and $v = \lambda_v x$. Suppose that $u = v$. Then

$$\begin{aligned}(\lambda_u - \lambda_v)x &= \lambda_u x - \lambda_v x \\ &= u - v \\ &= 0\end{aligned}$$

  Since $x \neq 0$, we have that $\lambda_u - \lambda_v = 0$ and therefore $\lambda_u = \lambda_v$. Hence

$$\begin{aligned}\lambda_u &= \epsilon_x(u) \\ &= \epsilon_x(v) \\ &= \lambda_v.\end{aligned}$$

  Thus ϵ_x is well defined.

□

D.3 Multilinear Maps

Definition D.3.0.1. Let $X_1, \cdots, X_n, Y$ be vector spaces and $T : \prod_{j=1}^{n} X_j \to \mathbb{K}$. Then T is said to be **multilinear** if for each $j_0 \in [n]$ and $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$, $T(x_1, \ldots, x_{j_0-1}, \cdot, x_{j_0+1})$ is linear.

$$L^n(X_1, \ldots, X_n; Y) = \left\{ T : \prod_{j=1}^{n} X_j \to Y : T \text{ is multilinear} \right\}$$

If $X_1 = \cdots = X_n = X$, we write $L^n(X; Y)$ in place of $L^n(X, \ldots, X; Y)$.

Definition D.3.0.2. define addition and scalar mult of multilinear maps

Exercise D.3.0.3. Let $X_1, \cdots, X_n, Y$ be vector spaces. Then $L^n(X_1, \ldots, X_n; Y)$ is a $\mathbb{K}$-vector space.

Proof. content... $\square$

Exercise D.3.0.4. Let $X_1, \cdots, X_n, Y, Z$ be $\mathbb{K}$-vector spaces, $\alpha \in L^n(X_1, \ldots, X_n; Y)$ and $\phi \in L^1(Y; Z)$. Then $\phi \circ \alpha \in L^n(X_1, \ldots, X_n; Z)$.

Proof. Let $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$. Define $f : X_{j_0} \to Y$ by

$$f(a) := \alpha(x_1, \ldots, x_{j_0-1}, a, x_{j_0+1}, \ldots, x_n)$$

Since $\alpha \in L^n(X_1, \ldots, X_n; Y)$, f is linear. Since ϕ is linear, and $\phi \circ f$ is linear. Since $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$ are arbitrary, we have that $\phi \circ \alpha \in L^n(X_1, \ldots, X_n; Y)$. $\square$

D.4 Tensor Products

Definition D.4.0.1. Let X, Y and T be vector spaces over $\mathbb{K}$ and $\alpha \in L^2(X, Y; T)$. Then (T, α) is said to be a **tensor product of** X **and** Y if for each vector space Z and $\beta \in L^2(X, Y; Z)$, there exists a unique $\phi \in L^1(T; Z)$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & T \arrow[d, dashed, "\phi"] \\ & Z \end{tikzcd}$$

Exercise D.4.0.2. Let X, Y, S, T be vector spaces, $\alpha \in L^2(X, Y; S)$ and $\beta \in L^2(X, Y; T)$. Suppose that (S, α) and (T, β) are tensor products of X and Y. Then S and T are isomorphic.

Proof. Since (T, β) is a tensor product of X and Y, $\beta \in L^2(X, Y; T)$ there exists a unique $f \in L^1(T; T)$ such that $f\ circ\beta = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dd, dashed, "f"] \\ X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T \end{tikzcd}$$

Since $\mathrm{id}_T \in L^1(T; T)$ and $\mathrm{id}_T \circ \beta = \beta$, we have that $f = \mathrm{id}_T$. Since (S, α) is a tensor product of X and Y, there exists a unique $\phi : S \to T$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] \\ & T \end{tikzcd}$$

Similarly, since (T, β) is a tensor product of X and Y, there exists a unique $\psi : T \to S$ such that $\psi \circ \beta = \alpha$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & T \arrow[d, dashed, "\psi"] \\ & S \end{tikzcd}$$

Therefore

$$\begin{aligned} (\phi \circ \psi) \circ \beta &= \phi \circ (\psi \circ \beta) \\ &= \phi \circ \alpha \\ &= \beta, \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, dashed, "\psi"] & & & T \arrow[dd, dashed, "\phi\circ\psi"] \\ X \times Y \arrow[ur, "\beta"] \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] & \Longrightarrow & X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T & & & T \end{tikzcd}$$

By uniqueness of $f \in L^1(T; T)$, we have that

$$\begin{aligned} \mathrm{id}_T &= f \\ &= \phi \circ \psi \end{aligned}$$

A similar argument implies that $\psi \circ \phi = \mathrm{id}_S$. Hence ϕ and ψ are isomorphisms with $\phi^{-1} = \psi$. Hence S and T are isomorphic. $\square$

Definition D.4.0.3. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. We define $x \otimes y : X^* \times Y^* \to \mathbb{K}$ by $x \otimes y(\phi, \psi) := \phi(x)\psi(y)$.

Exercise D.4.0.4. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. Then $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$.

Proof. Let $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} x \otimes y(\phi_1 + \lambda\phi_2, \psi) &= [\phi_1 + \lambda\phi_2(x)]\psi(y) \\ &= \phi_1(x)\psi(y) + \lambda\phi_2(x)\psi(y) \\ &= x \otimes y(\phi_1, \psi) + \lambda x \otimes y(\phi_2, \psi) \end{aligned}$$

Since $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$ are arbitrary, we have that for each $\psi \in Y^*$, $x \otimes y(\cdot, \psi)$ is linear. Similarly for each $\phi \in X^*$, $x \otimes y(\phi, \cdot)$ is linear. Hence $x \otimes y$ is bilinear and $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$. □

Definition D.4.0.5. Let X, Y be vector spaces. We define

- the **tensor product of** X **and** Y, denoted $X \otimes Y \subset L^2(X^*, Y^*; \mathbb{K})$, by

$$X \otimes Y := \operatorname{span}(x \otimes y : x \in X \text{ and } y \in Y),$$

- the **tensor map**, denoted $\otimes : X \times Y \to X \otimes Y$, by $\otimes(x, y) := x \otimes y$.

Exercise D.4.0.6. Let X, Y be vector spaces, $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$. The following are equivalent:

1. $\sum\limits_{j=1}^n x_j \otimes y_j = 0$
2. for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum\limits_{j=1}^n \phi(x_j)\psi(y_j) = 0$
3. for each $\phi \in X^*$, $\sum\limits_{j=1}^n \phi(x_j)y_j = 0$
4. for each $\psi \in Y^*$, $\sum\limits_{j=1}^n \psi(y_j)x_j = 0$

Proof.

1. (1) $\implies$ (2) :
   Suppose that $\sum\limits_{j=1}^n x_j \otimes y_j = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Then

$$\begin{aligned} \sum_{j=1}^n \phi(x_j)\psi(y_j) &= \phi\bigg(\sum_{j=1}^n \psi(y_j)x_j\bigg) \\ &= \end{aligned}$$

2.
3.

□

Exercise D.4.0.7. Let X, Y be vector spaces. Then $(X \otimes Y, \otimes)$ is a tensor product of X and Y.

Proof. Let Z be a vector space and $\alpha \in L^2(X, Y; Z)$. Define $\phi : X \otimes Y \to Z$ by $\phi\bigg(\sum\limits_{j=1}^n \lambda_j x_j \otimes y_j) := \sum\limits_{j=1}^n \lambda_j \alpha(x_j, y_j)$.

- **(well defined):**
  Let $u \in X \otimes Y$. Then there exist $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $(x_j)_{j=1}^n \subset X$, $(y_j)_{j=1}^n \subset Y$ such that $u = \sum\limits_{j=1}^n \lambda_j x_j \otimes y_j$. Suppose that $u = 0$. Let $\phi \in Z^*$. Then $\phi \circ \alpha \in L^2(X, Y; Z)$.

□

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