

# INTRODUCTION TO LATENT SPACE NETWORK STATISTICS

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## CONTENTS

1. General Model	1
1.1. Introduction	1
2. Random Inner Product Graphs	2
2.1. Introduction	2
3. Random Kernel Graphs	3
3.1. Introduction	3

## 1. GENERAL MODEL

### 1.1. Introduction.

**Definition 1.1.1.** Let  $(M, d)$  be a metric space,  $(G, \tau)$  a topological group, and  $\cdot : G \times M \rightarrow M$  a group action. Suppose that for each  $g \in G$ , the map  $x \mapsto g \cdot x$  is an isometry. We define  $\bar{d} : M/G \rightarrow [0, \infty)$  by

$$\begin{aligned}\bar{d}(o_x, o_y) &= \inf_{\substack{a \in o_x \\ b \in o_y}} d(a, b) \\ &= \inf_{g \in G} d(g \cdot x, y)\end{aligned}$$

**Exercise 1.1.2.** If for each  $x \in M$ ,  $o_x$  is closed, then  $\bar{d}$  is a metric.

*Proof.* Suppose that for each  $x \in M$ ,  $o_x$  is closed. We need only show that for each  $x, y \in M$ ,  $\bar{d}(o_x, o_y) = 0$  implies that  $o_x = o_y$ . Suppose that  $\bar{d}(o_x, o_y) = 0$ . Then  $\inf_{g \in G} d(g \cdot x, y) = 0$ . Hence there exists  $(\tau_n)_{n \in \mathbb{N}} \subset G$  such that  $\tau_n \cdot x \rightarrow y$ . Since  $(\tau_n \cdot x)_{n \in \mathbb{N}} \subset o_x$  and  $o_x$  is closed,  $y \in o_x$ . Thus  $o_x = o_y$ .  $\square$

**Example 1.1.3.** Consider the metric space  $(\mathbb{C}, |\cdot|)$ , topological group  $(S^1, |\cdot|)$  and the (right) action  $x \cdot u = xu$ . Then the orbits are concentric circles, which are closed.

## 2. RANDOM INNER PRODUCT GRAPHS

## 2.1. Introduction.

**Example 2.1.1.** Consider the metric space  $(\mathbb{C}^{n \times d}, \|\cdot\|_F)$ , topological group  $(U_d, \|\cdot\|_F)$  and the (right) action  $X \cdot U = XU$

## 3. RANDOM KERNEL GRAPHS

## 3.1. Introduction.

**Definition 3.1.1.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Define  $\|\cdot\|_* : L^1(X, \mathcal{A}, \mu) \rightarrow [0, \infty)$  by

$$\|f\|_* = \sup_{A \in \mathcal{A}} \left| \int_A f d\mu \right|$$

**Exercise 3.1.2.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. Then  $\|\cdot\|_*$  is a norm on  $L^1(X, \mathcal{A}, \mu)$ .

*Proof.* Clear. □

**Definition 3.1.3.** Let  $(X, \tau)$  be a topological space. Define

$$\text{Aut}(X) = \{\sigma : X \rightarrow X : \sigma \text{ is a homeomorphism}\}$$

**Definition 3.1.4.** Let  $(X, d)$  be a compact metric space. Define

$$\text{Aut}(X, d) = \{\sigma : X \rightarrow X : \sigma \text{ is an isometric isomorphism}\}$$

We metrize  $\text{Aut}(X, d)$  with the supremum metric  $d_u$ . It is known that this topology is equivalent to the compact-open topology.

**Exercise 3.1.5.** Let  $(X, d)$  be a compact metric space. Then  $(\text{Aut}(X), d_u)$  is a topological group and  $(\text{Aut}(X, d), d_u)$  is a compact subgroup of  $(\text{Aut}(X), d_u)$ .

*Proof.* Please see section on topological groups: [Analysis Notes](#) □

**Definition 3.1.6.** Let  $(X, d)$  be a compact metric space and  $\mu : \mathcal{B}(X) \rightarrow \mathbb{R}$  a Borel measure. Define

$$\text{Aut}(X, \mathcal{B}(X), \mu) = \{\sigma \in \text{Aut}(X) : \sigma_*\mu = \mu\}$$

So that  $(\text{Aut}(X, \mathcal{B}(X), \mu), d_u)$  is a subspace of  $(\text{Aut}(X), d_u)$ .

**Exercise 3.1.7.** Let  $(X, d)$  be a compact metric space and  $\mu : \mathcal{B}(X) \rightarrow \mathbb{R}$  an outer-regular Borel measure. Then  $\text{Aut}(X, \mu)$  is a closed subgroup of  $\text{Aut}(X)$ .

*Proof.* Please see section on topological groups: [Analysis Notes](#) □

**Example 3.1.8.** Let  $X$  be a compact metric space. Define the (right) group action  $\cdot : (L^1(X, \mathcal{B}(X), \mu), \|\cdot\|_*) \times \text{Aut}(X, \mathcal{B}(X), \mu) \rightarrow (L^1(X, \mathcal{B}(X), \mu), \|\cdot\|_*)$  by  $f \cdot \sigma = f \circ \sigma$ . Then for each  $\sigma \in \text{Aut}(X, \mathcal{B}(X), \mu)$ , the map  $f \mapsto f \cdot \sigma$  is an isometry.

*Proof.* Clear. □

**Definition 3.1.9.** Let  $(X, d)$  be a compact metric space and  $\mu : \mathcal{B}(X) \rightarrow \mathbb{R}$  an outer-regular Borel measure. Define  $\text{Aut}(X, d, \mu) = \text{Aut}(X, d) \cap \text{Aut}(X, \mu)$ .

**Exercise 3.1.10.** Let  $X$  be a compact metric space.

*Proof.* IDK, would like to show. I dont think  $\text{Aut}(X, \mathcal{B}(X), \mu)$  is compact. So still thinking about how to show this. □