INTRODUCTION TO CATEGORY THEORY

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Contents

Preface

1. Basic Concepts

1.1. von Neumann–Bernays–Gödel Set Theory.

Definition 1.1.1. Let x be a class. Then x is said to be a set iff there exists a class A such that $x \in A$.

Note 1.1.2. We can define cartesion products, relations, and functions for classes just like for sets.

Axiom 1.1.3. Axiom of Replacement:
Let A, B be classes and $f : A \to B$. If A is a set, then $f(A)$ is a set.

Axiom 1.1.4. Schema of Specification:
Let ϕ a propositional function on sets. Then there exists a class A such that for each set x, $x \in A$ iff $\phi(x)$.

Exercise 1.1.5. There exists a class A such that for each class x, $x \in A$ iff x is a set.

Proof. Define ϕ by

$$\phi(x) : x = x$$

Axiom 1.1.4 implies that there exists a class A such that for each set x, $x \in A$ iff $x = x$. Let x be a class. If $x \in A$, then by definition, x is a set.
Conversely, if x is a set, then by construction, $x \in A$. □

Exercise 1.1.6. There exists a class A such that for each class G and $* : G \times G \to G$, $(G, *) \in A$ iff $(G, *)$ is a group.

Proof. Define ϕ_1, ϕ_2 and ϕ_3 by

- $\phi_1(G, *) : * : G \times G \to G$ is associative
- $\phi_2(G, *)$: there exists $e \in G$ such that for each $g \in G$, $e * g = g * e = g$
- $\phi_3(G, *)$: for each $g \in G$ there exists $h \in G$ such that $g * h = h * g = e$

Define ϕ by

$$\phi(G, *) : \phi_1(G, *) \text{ and } \phi_2(G, *) \text{ and } \phi_3(G, *)$$

Then there exists a class A such that for each set G and $* : G \times G \to G$, $(G, *) \in A$ iff $\phi(G, *)$ $(G, *)$ "is a group". Therefore, for each group $(G, *)$, $(G, *) \in A$. **FINISH!!!** □

1.2. Categories.

1.2.1. *Introduction.*

Definition 1.2.1. Let $\mathcal{C}_0$, $\mathcal{C}_1$ be classes and $\mathrm{dom}, \mathrm{cod} : \mathcal{C}_1 \to \mathcal{C}_0$ class functions. Set $\mathcal{C} = (C_0, C_1, \mathrm{dom}, \mathrm{cod})$. Then $\mathcal{C}$ is said to be a **category** if

(1) (composition): for each $f, g \in C_1$, if $\mathrm{cod}(f) = \mathrm{dom}(g)$, then there exists $g \circ f \in C_1$ such that $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$

(2) (associativity): for each $f, g, h \in C_1$, if $\mathrm{cod}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(g) = \mathrm{dom}(h)$, then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

(3) (identity): for each $X \in \mathcal{C}_0$, there exists $\mathrm{id}_X \in C_1$ such that $\mathrm{dom}(\mathrm{id}_X) = \mathrm{cod}(\mathrm{id}_X) = X$ and for each $f, g \in C_1$, if $\mathrm{dom}(f) = X$ and $\mathrm{cod}(g) = X$, then

$$f \circ \mathrm{id}_X = f \text{ and } \mathrm{id}_X \circ g = g$$

We define the

- **objects of** $\mathcal{C}$, denoted $\mathrm{Obj}(\mathcal{C})$, by $\mathrm{Obj}(\mathcal{C}) = C_0$
- **morphisms of** $\mathcal{C}$, denoted $\mathrm{Hom}_{\mathcal{C}}$, by $\mathrm{Hom}_{\mathcal{C}} = C_1$

For $X, Y \in \mathrm{Obj}(\mathcal{C})$, we define the **morphisms from** X **to** Y, denoted $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, by $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{f \in \mathrm{Hom}(\mathcal{C}) : \mathrm{dom}(f) = X \text{ and } \mathrm{cod}(f) = Y\}$.

Note 1.2.2. We typically define a category $\mathcal{C}$ by specifying

- $\mathrm{Obj}(\mathcal{C})$
- for $X, Y \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(X, Y)$
- for $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, the composite morphism $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$.

and then show

- well-definedness of composition
- associativity of composition
- existence of identities

Definition 1.2.3. We define the **empty category**, denoted $\mathbf{0}$, by

- $\mathrm{Obj}(\mathbf{0}) = \varnothing$
- $\mathrm{Hom}_{\mathbf{0}} = \varnothing$

Exercise 1.2.4. We have that $\mathbf{0}$ is a category

Proof. Vacuously true. ☐

Definition 1.2.5. We define the **trivial category**, denoted $\mathbf{1}$, by

- $\mathrm{Obj}(\mathbf{1}) = \{*\}$
- $\mathrm{Hom}_{\mathbf{1}} = \{\mathrm{id}_*\}$

Exercise 1.2.6. We have that $\mathbf{1}$ is a category.

Proof. Clear. ☐

Definition 1.2.7. We define **Set** by

- $\mathrm{Obj}(\mathbf{Set}) = \{A : A \text{ is a set}\}$
- for each $A, B \in \mathrm{Obj}(\mathbf{Set})$, $\mathrm{Hom}_{\mathbf{Set}}(A, B) = \{f : f : A \to B\}$
- for $A, B, C \in \mathbf{Set}$, $f \in \mathrm{Hom}_{\mathbf{Set}}(A, B)$ and $g \in \mathrm{Hom}_{\mathbf{Set}}(B, C)$, $g \circ_{\mathbf{Set}} f = g \circ f$.

Exercise 1.2.8. We have that **Set** is a category.

Proof.

- **well-definedness of composition:**
- **associativity of composition:**
- **existence of identities:**

FINISH!!! □

Definition 1.2.9. Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is said to be

- **small** if $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}$ are sets
- **locally small** if for each $A, B \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set

Exercise 1.2.10. Let $\mathcal{C}$ be a category. If $\mathcal{C}$ is small, then $\mathcal{C}$ is a set.

Proof. Suppose that $\mathcal{C}$ is small. Then $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}$ are sets. Then $\mathcal{P}(\mathrm{Obj}(\mathcal{C}))$, $\mathcal{P}(\mathrm{Hom}_{\mathcal{C}})$ and $\mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}}$ are sets. Hence $\mathrm{Obj}(\mathcal{C}) \times \mathrm{Hom}_{\mathcal{C}} \times \mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}} \times \mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}}$ is a set. By definition, $\mathcal{C} = (\mathrm{Obj}(\mathcal{C}), \mathrm{Hom}_{\mathcal{C}}, \mathrm{dom}, \mathrm{cod}) \in \mathrm{Obj}(\mathcal{C}) \times \mathrm{Hom}_{\mathcal{C}} \times \mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}} \times \mathrm{Obj}(\mathcal{C})^{\mathrm{Hom}_{\mathcal{C}}}$. By definition, $\mathcal{C}$ is a set. □

Exercise 1.2.11. There exists a class A such that $\mathcal{C} \in A$ iff $\mathcal{C}$ is a small category.

Proof. Exercise 1.2.10 implies that for each category $\mathcal{C}$, $\mathcal{C}$ is small implies that $\mathcal{C}$ is a set. Define ϕ by

$$\phi(\mathcal{C}) : \mathcal{C} \text{ is a small category}$$

Then Axiom 1.1.4 implies that there exists a class A such that $\mathcal{C} \in A$ iff $\mathcal{C}$ is a small category. □

1.2.2. *Opposite Category.*

Definition 1.2.12. Let $\mathcal{C}$ be a category, we define the dual of $\mathcal{C}$ or the **opposite of** $\mathcal{C}$, denoted $\mathcal{C}^{\mathrm{op}}$, by

- $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$
- for $X, Y \in \mathrm{Obj}(\mathcal{C}^{\mathrm{op}})$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$
- for $X, Y, Z \in \mathrm{Obj}(\mathcal{C}^{\mathrm{op}})$ and $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z)$, $g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} g$

Exercise 1.2.13. Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathrm{op}}$ is a category.

Proof.

- for $W, X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ and $h \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z)$. Then

$$\begin{aligned}
(h \circ_{\mathcal{C}^{\mathrm{op}}} g) \circ_{\mathcal{C}^{\mathrm{op}}} f &= f \circ_{\mathcal{C}} (h \circ_{\mathcal{C}^{\mathrm{op}}} g) \\
&= f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h) \\
&= (f \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} h \\
&= h \circ_{\mathcal{C}^{\mathrm{op}}} (f \circ_{\mathcal{C}} g) \\
&= h \circ_{\mathcal{C}^{\mathrm{op}}} (g \circ_{\mathcal{C}^{\mathrm{op}}} f)
\end{aligned}$$

  So composition is associative.

- Let $X \in \mathrm{Obj}(\mathcal{C})$ and $f, g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}$. Suppose that $\mathrm{dom}(f) = X$ and $\mathrm{cod}(g) = X$ Then

$$\begin{aligned} f \circ_{\mathcal{C}^{\mathrm{op}}} \mathrm{id}_X &= \mathrm{id}_X \circ_{\mathcal{C}} f \\ &= f \end{aligned}$$

and

$$\begin{aligned} \mathrm{id}_X \circ_{\mathcal{C}^{\mathrm{op}}} g &= g \circ_{\mathcal{C}} \mathrm{id}_X \\ &= g \end{aligned}$$

So $(\mathrm{id}_X)_{\mathcal{C}^{\mathrm{op}}} = (\mathrm{id}_X)_{\mathcal{C}}$.

□

1.2.3. *Slice Category.*

Definition 1.2.14. Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. We define the **slice category of $\mathcal{C}$ over X**, denoted $\mathcal{C}/X$, by

- $\mathrm{Obj}(\mathcal{C}/X) = \{f \in \mathrm{Hom}_{\mathcal{C}} : \mathrm{cod}(f) = X\}$
- for $f, g \in \mathrm{Obj}(\mathcal{C}/X)$,

$$\mathrm{Hom}_{\mathcal{C}/X}(f, g) = \{\alpha \in \mathrm{Hom}_{\mathcal{C}} : \mathrm{dom}(\alpha) = \mathrm{dom}(f),\ \mathrm{cod}(\alpha) = \mathrm{dom}(g) \text{ and } f = g \circ \alpha\}$$

i.e. for $f \in \mathrm{Hom}_{\mathcal{C}}(A, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f, g)$ iff the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

- for $f, g, h \in \mathrm{Obj}(\mathcal{C}/X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}/X}(g, h)$,

$$\beta \circ_{\mathcal{C}/X} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

Exercise 1.2.15. Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. Then $\mathcal{C}/X$ is a category.

Proof.

- $f, g, h \in \mathrm{Obj}(\mathcal{C}/X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}/X}(g, h)$. Then $f = g \circ_{\mathcal{C}} \alpha$ and $g = h \circ_{\mathcal{C}} \beta$, i.e. the following diagrams commute:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & \mathrm{dom}(g) \arrow[dl, "g"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{dom}(g) \arrow[rr, "\beta"] \arrow[dr, "g"'] & & \mathrm{dom}(h) \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

Therefore, we have that

$$\begin{aligned} f &= g \circ_{\mathcal{C}} \alpha \\ &= (h \circ_{\mathcal{C}} \beta) \circ_{\mathcal{C}} \alpha \\ &= h \circ_{\mathcal{C}} (\beta \circ_{\mathcal{C}} \alpha) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\beta \circ_{\mathcal{C}} \alpha"] \arrow[dr, "f"'] & & \mathrm{dom}(h) \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

which implies that

$$\begin{aligned} \beta \circ_{\mathcal{C}/X} \alpha &= \beta \circ_{\mathcal{C}} \alpha \\ &\in \mathrm{Hom}_{\mathcal{C}/X}(f, h) \end{aligned}$$

and composition is well defined.

- Associativity of $\circ_{\mathcal{C}/X}$ follows from associativity of $\circ_{\mathcal{C}}$.
- Let $f \in \mathrm{Obj}(\mathcal{C}/X)$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}/X}$. Since $f \circ \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} = f$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{dom}_{\mathcal{C}}(f) \arrow[rr, "\mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)}"] \arrow[dr, "f"'] & & \mathrm{dom}_{\mathcal{C}}(f) \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

we have that $\mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} \in \mathrm{Hom}_{\mathcal{C}/X}(f, f)$. Suppose that $\mathrm{dom}_{\mathcal{C}/X}(\alpha) = f$ and $\mathrm{cod}_{\mathcal{C}/X}(\beta) = f$. Then

$$\begin{aligned} \alpha \circ_{\mathcal{C}/X} \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} &= \alpha \circ_{C} \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} \\ &= \alpha \end{aligned}$$

and

$$\begin{aligned} \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} \circ_{\mathcal{C}/X} \beta &= \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)} \circ_{\mathcal{C}} \beta \\ &= \beta \end{aligned}$$

So $\mathrm{id}_f = \mathrm{id}_{\mathrm{dom}_{\mathcal{C}}(f)}$.

□

1.2.4. *Product Category.*

Definition 1.2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We define the **product category of $\mathcal{C}$ and $\mathcal{D}$**, denoted $\mathcal{C} \times \mathcal{D}$ by

- $\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) = \{(A, B) : A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D})\}$
- for each $(A, A'), (B, B') \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B')) = \{(f, g) : f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \text{ and } g \in \mathrm{Hom}_{\mathcal{C}}(A', B')\}$
- for each $(A, A'), (B, B'), (C, C') \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $(f, f') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$ and $(g, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'))$,

$$(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') = (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f')$$

Exercise 1.2.17. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $\mathcal{C} \times \mathcal{D}$ is a category.

Proof.

- **well-definedness of composition:**
Let $(A, A'), (B, B'), (C, C') \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $(f, f') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$ and $(g, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'))$. Then $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $f' \in \mathrm{Hom}_{\mathcal{D}}(A', B')$, and $g' \in \mathrm{Hom}_{\mathcal{D}}(B', C')$. Hence $g \circ_{\mathcal{C}} f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$ and $g' \circ_{\mathcal{D}} f' \in \mathrm{Hom}_{\mathcal{D}}(A', C')$. Thus
$$\begin{aligned}(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') &= (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f') \\ &\in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (C, C'))\end{aligned}$$
Thus, composition is well defined.

- **associativity of composition:**
Let $(A, A'), (B, B'), (C, C'), (D, D') \in \mathrm{Obj}(\mathcal{C}\times\mathcal{D})$, $(f, f') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$, $(g, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'))$ and $(h, h') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C, C'), (D, D'))$. Then
$$\begin{aligned}[(h, h') \circ_{\mathcal{C}\times\mathcal{D}} (g, g')] \circ_{\mathcal{C}\times\mathcal{D}} (f, f') &= (h \circ_{\mathcal{C}} g, h' \circ_{\mathcal{D}} g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') \\ &= ((h \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} f, (h' \circ_{\mathcal{D}} g') \circ_{\mathcal{D}} f') \\ &= (h \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} f), h' \circ_{\mathcal{D}} (g' \circ_{\mathcal{D}} f')) \\ &= (h, h') \circ_{\mathcal{C}\times\mathcal{D}} (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f') \\ &= (h, h') \circ_{\mathcal{C}\times\mathcal{D}} [(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f')]\end{aligned}$$
Thus composition is associative.

- **existence of identities:**
Let $(A, B) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, $(f, f'), (g, g') \in \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}$. Suppose that $\mathrm{dom}_{\mathcal{C}\times\mathcal{D}}(f, f') = (A, B)$ and $\mathrm{cod}_{\mathcal{C}\times\mathcal{D}}(g, g') = (A, B)$. Then $\mathrm{dom}_{\mathcal{C}}(f) = A$, $\mathrm{dom}_{\mathcal{D}}(f') = B$, $\mathrm{cod}_{\mathcal{C}}(g) = A$ and $\mathrm{cod}_{\mathcal{D}}(g') = B$. Hence
$$\begin{aligned}(f, f') \circ_{\mathcal{C}\times\mathcal{D}} (\mathrm{id}_A, \mathrm{id}_B) &= (f \circ_{\mathcal{C}} \mathrm{id}_A, f' \circ_{\mathcal{D}} \mathrm{id}_B) \\ &= (f, f)\end{aligned}$$
and
$$\begin{aligned}(\mathrm{id}_A, \mathrm{id}_B) \circ_{\mathcal{C}\times\mathcal{D}} (g, g') &= (\mathrm{id}_A \circ_{\mathcal{C}} g, \mathrm{id}_B \circ g') \\ &= (g, g')\end{aligned}$$
Therefore $(\mathrm{id}_{(A,B)})_{\mathcal{C}\times\mathcal{D}} = (\mathrm{id}_A, \mathrm{id}_B)$.

$\square$

1.3. **Functors.**

1.3.1. *Introduction.*

Definition 1.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, $F_1 : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}$ class functions. Set $F = (F_0, F_1)$. Then F is said to be a functor from $\mathcal{C}$ to $\mathcal{D}$, denoted $F : \mathcal{C} \to \mathcal{D}$, if

(1) for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F_1(f) \in \mathrm{Hom}_{\mathcal{D}}(F_0(A), F_0(B))$
(2) for each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$
(3) for each $A \in \mathrm{Obj}(\mathcal{C})$, $F_1(\mathrm{id}_A) = \mathrm{id}_{F_0(A)}$

Note 1.3.2. For $A \in \mathrm{Obj}(C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}$, we typically write $F(A)$ and $F(f)$ instead of $F_0(A)$ and $F_1(f)$ respectively.

Definition 1.3.3. Let $\mathcal{C}$ be a category. We define the **empty functor** from $\mathbf{0}$ to $\mathcal{C}$, denoted $E_{\mathcal{C}} : \mathbf{0} \to \mathcal{C}$ by $(E_{\mathcal{C}})_0 = (E_{\mathcal{C}})_1 = \varnothing$.

Exercise 1.3.4. Let $\mathcal{C}$ be a category. Then $E_{\mathcal{C}} : \mathbf{0} \to \mathcal{C}$ is a functor.

Proof. Since $\mathrm{Obj}(\mathbf{0}) = \varnothing$ and $\mathrm{Hom}_{\mathbf{0}} = \varnothing$, this is vacuously true. □

Definition 1.3.5. Let $\mathcal{C}, \mathcal{D}$ be categories and $X \in \mathcal{D}$. We define the **constant functor** from $\mathcal{C}$ onto X, denoted $\Delta_X^{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ by

- $\Delta_X^{\mathcal{C}}(A) = X$
- $\Delta_X^{\mathcal{C}}(f) = \mathrm{id}_X$

Exercise 1.3.6. Let $\mathcal{C}, \mathcal{D}$ be categories and $X \in \mathrm{Obj}(\mathcal{D})$. Then $\Delta_X^{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ is a functor.

Proof.

(1) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned} \Delta_X^{\mathcal{C}}(f) &= \mathrm{id}_X \\ &\in \mathrm{Hom}_{\mathcal{D}}(X, X) \\ &= \mathrm{Hom}_{\mathcal{D}}(\Delta_X^{\mathcal{C}}(A), \Delta_X^{\mathcal{C}}(B)) \end{aligned}$$

(2) Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

$$\begin{aligned} \Delta_X^{\mathcal{C}}(g \circ f) &= \mathrm{id}_X \\ &= \mathrm{id}_X \circ \mathrm{id}_X \\ &= \Delta_X^{\mathcal{C}}(g) \circ \Delta_X^{\mathcal{C}}(f) \end{aligned}$$

(3) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned} \Delta_X^{\mathcal{C}}(\mathrm{id}_A) &= \mathrm{id}_X \\ &= \mathrm{id}_{\Delta_X^{\mathcal{C}}(A)} \end{aligned}$$

So $\Delta_X^{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ is a functor. □

1.3.2. *Category of Small Categories.*

Definition 1.3.7. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ functors. We define the **composition of** G **with** F, denoted $G \circ F : \mathcal{C} \to \mathcal{E}$, by

- $G \circ F(A) = G(F(A))$
- $G \circ F(f) = G(F(f))$

Exercise 1.3.8. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ functors. Then $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor.

Proof.

(1) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$, we have that $G(F(f)) \in \mathrm{Hom}_{\mathcal{E}}(G(F(A)), G(F(B)))$. Then

$$\begin{aligned} G \circ F(f) &= G(F(f)) \\ &\in \mathrm{Hom}_{\mathcal{E}}(G(F(A)), G(F(B))) \\ &= \mathrm{Hom}_{\mathcal{E}}(G \circ F(A), G \circ F(B)) \end{aligned}$$

(2) Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

$$\begin{aligned} G \circ F(g \circ f) &= G(F(g \circ f)) \\ &= G(F(g) \circ F(f)) \\ &= G(F(g)) \circ G(F(f)) \\ &= G \circ F(g) \circ G \circ F(f) \end{aligned}$$

(3) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned} G \circ F(\mathrm{id}_A) &= G(F(\mathrm{id}_A)) \\ &= G(\mathrm{id}_{F(A)}) \\ &= \mathrm{id}_{G(F(A))} \\ &= \mathrm{id}_{G \circ F(A)} \end{aligned}$$

So $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor. □

Exercise 1.3.9. Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, $H : \mathcal{E} \to \mathcal{F}$ functors. Then $(H \circ G) \circ F = H \circ (G \circ F)$.

Proof. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

- $$\begin{aligned} (H \circ G) \circ F(A) &= H \circ G(F(A)) \\ &= H(G(F(A))) \\ &= H(G \circ F(A)) \\ &= H \circ (G \circ F)(A) \end{aligned}$$

- $$\begin{aligned}(H \circ G) \circ F(f) &= H \circ G(F(f)) \\ &= H(G(F(f))) \\ &= H(G \circ F(f)) \\ &= H \circ (G \circ F)(f)\end{aligned}$$

Hence $(H \circ G) \circ F = H \circ (G \circ F)$. □

Definition 1.3.10. Let $\mathcal{C}$ be a category. We define the **identity functor from $\mathcal{C}$ to $\mathcal{C}$**, denoted $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, by

- $\mathrm{id}_{\mathcal{C}}(A) = A$, $(A \in \mathrm{Obj}(\mathcal{C}))$
- $\mathrm{id}_{\mathcal{C}}(f) = f$, $(f \in \mathrm{Hom}_{\mathcal{C}})$

Exercise 1.3.11. Let $\mathcal{C}$ be a category. Then $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a functor.

Proof.

(1) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(f) &= f \\ &\in \mathrm{Hom}_{\mathcal{C}}(A, B) \\ &= \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))\end{aligned}$$

(2) Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g \circ f) &= g \circ f \\ &= \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f)\end{aligned}$$

(3) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) &= \mathrm{id}_A \\ &= \mathrm{id}_{\mathrm{id}_{\mathcal{C}}(A)}\end{aligned}$$

□

Exercise 1.3.12. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. Then

(1) $\mathrm{id}_{\mathcal{D}} \circ F = F$
(2) $F \circ \mathrm{id}_{\mathcal{C}} = F$

Proof.

(1) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned}\mathrm{id}_{\mathcal{D}} \circ F(A) &= \mathrm{id}_{\mathcal{D}}(F(A)) \\ &= F(A)\end{aligned}$$

and

$$\begin{aligned}\mathrm{id}_{\mathcal{D}} \circ F(f) &= \mathrm{id}_{\mathcal{D}}(F(f)) \\ &= F(f)\end{aligned}$$

Since $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ are arbitrary, $\mathrm{id}_{\mathcal{D}} \circ F = F$.

(2) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned} F \circ \mathrm{id}_{\mathcal{C}}(A) &= F(\mathrm{id}_{\mathcal{C}}(A)) \\ &= F(A) \end{aligned}$$

and

$$\begin{aligned} F \circ \mathrm{id}_{\mathcal{C}}(f) &= F(\mathrm{id}_{\mathcal{C}}(f)) \\ &= F(f) \end{aligned}$$

Since $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ are arbitrary, $F \circ \mathrm{id}_{\mathcal{C}} = F$.

□

Exercise 1.3.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. If $\mathcal{C}$ is small, then F is a set.

Proof. Suppose that $\mathcal{C}$ is small. Then $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}$ are sets. By definition, there exist $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $F_1 : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}$ such that $F = (F_0, F_1)$. Axiom 1.1.3 implies that $F_0(\mathrm{Obj}(\mathcal{C}))$ and $F_1(\mathrm{Hom}_{\mathcal{C}})$ are sets. Therefore, $\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C))$ and $\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}})$ are sets. Hence $\mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C)))$ and $\mathcal{P}(\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}}))$ are sets. Since $F_0 \subset \mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C))$ and $F_1 \subset \mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}})$, we have that $F_0 \in \mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C)))$ and $F_1 \in \mathcal{P}(\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}}))$. Hence F_0 and F_1 are sets. Thus $F = (F_0, F_1)$ is a set. □

Exercise 1.3.14. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Then there exists a class A such that for each class F, $F \in A$ iff $F : \mathcal{C} \to \mathcal{D}$.

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Define ϕ by

$$\phi(F) : F : \mathcal{C} \to \mathcal{D}$$

Then there exists a class A such that for each set F, $F \in A$ iff $\phi(F)$. Let F be a class. Suppose that $F \in A$. By Definition 1.1.1, F is a set. Since F is a set and $F \in A$, we have that $\phi(F)$. Hence $F : \mathcal{C} \to \mathcal{D}$.
Conversely, suppose that $F : \mathcal{C} \to \mathcal{D}$. Exercise 1.3.13 implies that F is a set. Since F is a set and $\phi(F)$ is true, we have that $F \in A$. □

Definition 1.3.15. We define **Cat** by

- $\mathrm{Obj}(\mathbf{Cat}) = \{\mathcal{C} : \mathcal{C} \text{ is a small category}\}$.
- for $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathbf{Cat})$,
$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) = \{F : F : \mathcal{C} \to \mathcal{D}\}$$
- for $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathbf{Cat})$, $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ and $G \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{D}, \mathcal{E})$,
$$G \circ_{\mathbf{Cat}} F = G \circ F$$

Exercise 1.3.16. We have that **Cat** is

(1) a category
(2) locally small

Proof.

(1) Exercise 1.3.8 implies that composition is well defined. Exercise 1.3.9 implies that composition is associative. Exercise 1.3.11 and Exercise 1.3.12 imply the existence of identities.

(2) Let $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathbf{Cat})$ and $F\in\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{D})$. Definition 1.2.9 implies that $\mathrm{Obj}(\mathcal{C})$, $\mathrm{Obj}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{C}}$ and $\mathrm{Hom}_{\mathcal{D}}$ are sets. Then $\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}$ and $\mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ are sets. Hence $\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}\times\mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ is a set. Let $F\in\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{D})$. Then there exist $F_0\in\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}$ and $F_1\in\mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ such that $F=(F_0,F_1)$. Therefore $F\in\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}\times\mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$. Since $F\in\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{D})$ is arbitrary,

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{D})\subset\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}\times\mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$$

which implies that $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{D})$ is a set. Therefore, **Cat** is locally small.

$\square$

1.3.3. *Comma Categories.*

Definition 1.3.17. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be a categories and $S:\mathcal{A}\to\mathcal{C}$, $T:\mathcal{B}\to\mathcal{C}$ functors. We define the **comma category of** S **to** T, denoted $(S\downarrow T)$, by

- $\mathrm{Obj}(S\downarrow T)=\{(A,B,h):A\in\mathrm{Obj}(\mathcal{A}),B\in\mathrm{Obj}(\mathcal{B}),\text{ and }h\in\mathrm{Hom}_{\mathcal{C}}(S(A),T(B))\}$
- For $(A_1,B_1,h_1),(A_2,B_2,h_2)\in\mathrm{Obj}(S\downarrow T)$,

  $\mathrm{Hom}_{(S\downarrow T)}((A_1,B_1,h_1),(A_2,B_2,h_2))=$
  $\{(\alpha,\beta):\alpha\in\mathrm{Hom}_{\mathcal{A}}(A_1,A_2),\ \beta\in\mathrm{Hom}_{\mathcal{B}}(B_1,B_2)\text{ and }T(\beta)\circ_{\mathcal{C}}h_1=h_2\circ_{\mathcal{C}}S(\alpha)\}$

  i.e. for $(A_1,B_1,h_1),(A_2,B_2,h_2)\in\mathrm{Obj}(S\downarrow T)$, $\alpha\in\mathrm{Hom}_{\mathcal{A}}(A_1,A_2)$ and $\beta\in\mathrm{Hom}_{\mathcal{B}}(B_1,B_2)$, $(\alpha,\beta)\in\mathrm{Hom}_{(S\downarrow T)}((A_1,B_1,h_1),(A_2,B_2,h_2))$ iff the following diagram commutes:

$$\begin{tikzcd} S(A_1) \arrow[r, "S(\alpha)"] \arrow[d, "h_1"'] & S(A_2) \arrow[d, "h_2"] \\ T(B_1) \arrow[r, "T(\beta)"'] & T(B_2) \end{tikzcd}$$

- For
  - $(A_1,B_1,h_1),(A_2,B_2,h_2),(A_3,B_3,h_3)\in\mathrm{Obj}(S\downarrow T)$
  - $(\alpha_{12},\beta_{12})\in\mathrm{Hom}_{(S\downarrow T)}((A_1,B_1,h_1),(A_2,B_2,h_2))$
  - $(\alpha_{23},\beta_{23})\in\mathrm{Hom}_{(S\downarrow T)}((A_2,B_2,h_2),(A_3,B_3,h_3))$

  we define

$$(\alpha_{23},\beta_{23})\circ_{(S\downarrow T)}(\alpha_{12},\beta_{12})=(\alpha_{23}\circ_{\mathcal{A}}\alpha_{12},\beta_{23}\circ_{\mathcal{B}}\beta_{12})$$

Exercise 1.3.18. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be a categories and $S:\mathcal{A}\to\mathcal{C}$, $T:\mathcal{B}\to\mathcal{C}$ functors. Then $(S\downarrow T)$ is a category.

Proof.

- **well-definedness of composition:**
  Let
  - $(A_1,B_1,h_1),(A_2,B_2,h_2),(A_3,B_3,h_3)\in\mathrm{Obj}(S\downarrow T)$
  - $(\alpha_{12},\beta_{12})\in\mathrm{Hom}_{(S\downarrow T)}((A_1,B_1,h_1),(A_2,B_2,h_2))$
  - $(\alpha_{23},\beta_{23})\in\mathrm{Hom}_{(S\downarrow T)}((A_2,B_2,h_2),(A_3,B_3,h_3))$

  By definition, $\alpha_{12}\in\mathrm{Hom}_{\mathcal{A}}(A_1,A_2)$, $\alpha_{23}\in\mathrm{Hom}_{\mathcal{A}}(A_2,A_3)$, $\beta_{12}\in\mathrm{Hom}_{\mathcal{B}}(B_1,B_2)$, $\beta_{23}\in\mathrm{Hom}_{\mathcal{B}}(B_2,B_3)$, $T(\beta_{12})\circ_{\mathcal{C}}h_1=h_2\circ S(\alpha_{12})$ and $T(\beta_{23})\circ_{\mathcal{C}}h_2=h_3\circ_{\mathcal{C}}S(\alpha_{23})$, i.e. the following diagram commutes:

$$\begin{tikzcd} S(A_1) \arrow[r, "S(\alpha_{12})"] \arrow[d, "h_1"'] & S(A_2) \arrow[r, "S(\alpha_{23})"] \arrow[d, "h_2"] & S(A_3) \arrow[d, "h_3"] \\ T(B_1) \arrow[r, "T(\beta_{12})"'] & T(B_2) \arrow[r, "T(\beta_{23})"'] & T(B_3) \end{tikzcd}$$

Then $\alpha_{23} \circ_{\mathcal{A}} \alpha_{12} \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_3)$, $\beta_{23} \circ_{\mathcal{B}} \beta_{12} \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_3)$ and

$$\begin{aligned} T(\beta_{23} \circ_{\mathcal{B}} \beta_{12}) \circ_{\mathcal{C}} h_1 &= (T(\beta_{23}) \circ_{\mathcal{C}} T(\beta_{12})) \circ_{\mathcal{C}} h_1 \\ &= T(\beta_{23}) \circ_{\mathcal{C}} (T(\beta_{12}) \circ_{\mathcal{C}} h_1) \\ &= T(\beta_{23}) \circ_{\mathcal{C}} (h_2 \circ_{\mathcal{C}} S(\alpha_{12})) \\ &= (T(\beta_{23}) \circ_{\mathcal{C}} h_2) \circ_{\mathcal{C}} S(\alpha_{12}) \\ &= (h_3 \circ_{\mathcal{C}} S(\alpha_{23})) \circ_{\mathcal{C}} S(\alpha_{12}) \\ &= h_3 \circ_{\mathcal{C}} (S(\alpha_{23}) \circ_{\mathcal{C}} S(\alpha_{12})) \\ &= h_3 \circ_{\mathcal{C}} S(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} S(A_1) \arrow[r, "S(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12})"] \arrow[d, "h_1"'] & S(A_3) \arrow[d, "h_3"] \\ T(B_1) \arrow[r, "T(\beta_{23} \circ_{\mathcal{B}} \beta_{12})"'] & T(B_3) \end{tikzcd}$$

Hence $(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}, \beta_{23} \circ_{\mathcal{B}} \beta_{12}) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_3, B_3, h_3))$ and composition is well defined.

- **associativity of composition:**
  Let
  - $(A_1, B_1, h_1), (A_2, B_2, h_2), (A_3, B_3, h_3), (A_4, B_4, h_4) \in \mathrm{Obj}(S \downarrow T)$
  - $(\alpha_{12}, \beta_{12}) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$
  - $(\alpha_{23}, \beta_{23}) \in \mathrm{Hom}_{(S\downarrow T)}((A_2, B_2, h_2), (A_3, B_3, h_3))$
  - $(\alpha_{34}, \beta_{34}) \in \mathrm{Hom}_{(S\downarrow T)}((A_3, B_3, h_3), (A_4, B_4, h_4))$

  Then

$$\begin{aligned} [(\alpha_{34}, \beta_{34}) \circ_{(S\downarrow T)} (\alpha_{23}, \beta_{23})] \circ_{(S\downarrow T)} (\alpha_{12}, \beta_{12}) &= (\alpha_{34} \circ_{\mathcal{A}} \alpha_{23}, \beta_{34} \circ_{\mathcal{B}} \beta_{23}) \circ_{(S\downarrow T)} (\alpha_{12}, \beta_{12}) \\ &= ([\alpha_{34} \circ_{\mathcal{A}} \alpha_{23}] \circ_{\mathcal{A}} \alpha_{12}, [\beta_{34} \circ_{\mathcal{B}} \beta_{23}] \circ_{\mathcal{B}} \beta_{12}) \\ &= (\alpha_{34} \circ_{\mathcal{A}} [\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}], \beta_{34} \circ_{\mathcal{B}} [\beta_{23} \circ_{\mathcal{B}} \beta_{12}]) \\ &= (\alpha_{34}, \beta_{34}) \circ_{(S\downarrow T)} (\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}, \beta_{23} \circ_{\mathcal{B}} \beta_{12}) \\ &= (\alpha_{34}, \beta_{34}) \circ_{(S\downarrow T)} [(\alpha_{23}, \beta_{23}) \circ_{(S\downarrow T)} (\alpha_{12}, \beta_{12})] \end{aligned}$$

  So composition is associative.

- **existence of identities:**
  Let
  - $(A_1, B_1, h_1), (A_2, B_2, h_2), \in \mathrm{Obj}(S \downarrow T)$
  - $(\alpha, \beta) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$

By definition,
- $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$, $\beta \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_2)$
- $h_1 \in \mathrm{Hom}_{\mathcal{C}}(S(A_1), T(B_1))$, $h_2 \in \mathrm{Hom}_{\mathcal{C}}(S(A_2), T(B_2))$
- $T(\beta) \circ h_1 = h_2 \circ S(\alpha)$

Since $\mathrm{id}_{A_1} \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_1)$, $\mathrm{id}_{B_1} \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_1)$, and

$$\begin{aligned} T(\mathrm{id}_{B_1}) \circ_{\mathcal{C}} h_1 &= \mathrm{id}_{T(B_1)} \circ_{\mathcal{C}} h_1 \\ &= h_1 \\ &= h_1 \circ_{\mathcal{C}} \mathrm{id}_{S(A_1)} \\ &= h_1 \circ_{\mathcal{C}} S(\mathrm{id}_{A_1}) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} S(A_1) \arrow[r, "S(\mathrm{id}_{A_1})"] \arrow[d, "h_1"'] & S(A_1) \arrow[d, "h_1"] \\ T(B_1) \arrow[r, "T(\mathrm{id}_{B_1})"'] & T(B_1) \end{tikzcd}$$

we have that $(\mathrm{id}_{A_1}, \mathrm{id}_{B_1}) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_1, B_1, h_1))$. Similarly $(\mathrm{id}_{A_2}, \mathrm{id}_{B_2}) \in \mathrm{Hom}_{(S\downarrow T)}((A_2, B_2, h_2), (A_2, B_2, h_2))$. Therefore

$$\begin{aligned} (\alpha, \beta) \circ_{(S\downarrow T)} (\mathrm{id}_{A_1}, \mathrm{id}_{B_1}) &= (\alpha \circ_{\mathcal{A}} \mathrm{id}_{A_1}, \beta \circ_{\mathcal{B}} \mathrm{id}_{B_1}) \\ &= (\alpha, \beta) \end{aligned}$$

and

$$\begin{aligned} (\mathrm{id}_{A_2}, \mathrm{id}_{B_2}) \circ_{(S\downarrow T)} (\alpha, \beta) &= (\mathrm{id}_{A_2} \circ_{\mathcal{A}} \alpha, \mathrm{id}_{B_2} \circ_{\mathcal{B}} \beta) \\ &= (\alpha, \beta) \end{aligned}$$

Since $(A_1, B_1, h_1), (A_2, B_2, h_2), \in \mathrm{Obj}(S \downarrow T)$ and $(\alpha, \beta) \in \mathrm{Hom}_{(S\downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$ are arbitrary, we have that for each $(A, B, h) \in \mathrm{Obj}(S \downarrow T)$, $\mathrm{id}_{(A,B,h)} = (\mathrm{id}_A, \mathrm{id}_B)$.

$\square$

Definition 1.3.19. Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. We define the **comma category from X to F**, denoted $(X \downarrow F)$, by $(X \downarrow F) = (\Delta^{\mathbf{1}}_X \downarrow F)$.
We may make the following identification:

- $\mathrm{Obj}(X \downarrow F) = \{(A, f) : A \in \mathrm{Obj}(\mathcal{C}) \text{ and } f \in \mathrm{Hom}_{\mathcal{D}}(X, F(A))\}$
- For $(A_1, f_1), (A_2, f_2) \in \mathrm{Obj}(X \downarrow F)$,

$$\mathrm{Hom}_{(X\downarrow F)}((A_1, f_1), (A_2, f_2)) = \{\alpha \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2) \text{ and } F(\alpha) \circ f_1 = f_2\}$$

i.e. for $(A_1, f_1), (A_2, f_2) \in \mathrm{Obj}(X \downarrow F)$ and $\alpha \in \mathrm{Hom}_{A_1,A_2}$, $\alpha \in \mathrm{Hom}_{(X\downarrow F)}((A_1, f_1), (A_2, f_2))$ iff the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ F(A_1) \arrow[rr, "F(\alpha)"'] & & F(A_2) \end{tikzcd}$$

- For
  - $(A_1, f_1), (A_2, f_2), (A_3, f_3) \in \mathrm{Obj}(X \downarrow F)$
  - $\alpha \in \mathrm{Hom}_{(X\downarrow F)}((A_1, f_1), (A_2, f_2))$
  - $\beta \in \mathrm{Hom}_{(X\downarrow F)}((A_2, f_2), (A_3, f_3))$

  we define
  $$\beta \circ_{(X\downarrow F)} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

Definition 1.3.20. Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. We define the **comma category from** F **to** X, denoted $(F \downarrow X)$, by $(F \downarrow X) = (F \downarrow \Delta^{\mathbf{1}}_X)$.
We may make the following identification:

- $\mathrm{Obj}(F \downarrow X) = \{(A, f) : A \in \mathrm{Obj}(\mathcal{C}) \text{ and } f \in \mathrm{Hom}_{\mathcal{D}}(F(A), X)\}$
- For $(A_1, f_1), (A_2, f_2) \in \mathrm{Obj}(F \downarrow X)$,
  $$\mathrm{Hom}_{(X\downarrow F)}((A_1, f_1), (A_2, f_2)) = \{\alpha \in \mathrm{Hom}_{\mathcal{C}}(A_1, A_2) \text{ and } f_2 \circ F(\alpha) = f_1\}$$
  i.e. for $(A_1, f_1), (A_2, f_2) \in \mathrm{Obj}(F \downarrow X)$ and $\alpha \in \mathrm{Hom}_{A_1,A_2}$, $\alpha \in \mathrm{Hom}_{(F\downarrow X)}((A_1, f_1), (A_2, f_2))$ iff the following diagram commutes:
  $$\begin{tikzcd} F(A_1) \arrow[rr, "F(\alpha)"] \arrow[dr, "f_1"'] & & F(A_2) \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$
- For
  - $(A_1, f_1), (A_2, f_2), (A_3, f_3) \in \mathrm{Obj}(F \downarrow X)$
  - $\alpha \in \mathrm{Hom}_{(F\downarrow X)}((A_1, f_1), (A_2, f_2))$
  - $\beta \in \mathrm{Hom}_{(F\downarrow X)}((A_2, f_2), (A_3, f_3))$

  we define
  $$\beta \circ_{(F\downarrow X)} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

1.4. **Natural Transformations.**

1.4.1. *Introduction.*

Definition 1.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Hom}_{\mathcal{D}}$. Then α is said to be a **natural transformation from** F **to** G, denoted $\alpha : F \Rightarrow G$, if

(1) for each $A \in \mathrm{Obj}(\mathcal{C})$, $\alpha_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$
(2) for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$, i.e. the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\alpha_B"] & G(B)
\end{tikzcd}$$

1.4.2. *Category of Functors.*

Definition 1.4.2. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ natural transformations. We define the **composition of** β **with** α, denoted $\beta \circ \alpha : F \Rightarrow H$, by

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$$

Exercise 1.4.3. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ natural transformations. Then $\beta \circ \alpha : F \Rightarrow H$ is a natural transformation.

Proof.

(1) Let $A \in \mathrm{Obj}(\mathcal{C})$. Since $\alpha_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$ and $\beta_A \in \mathrm{Hom}_{\mathcal{D}}(G(A), H(A))$, we have that

$$\begin{aligned}
(\beta \circ \alpha)_A &= \beta_A \circ \alpha_A \\
&\in \mathrm{Hom}_{\mathcal{D}}(F(A), H(A))
\end{aligned}$$

(2) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ and $H(f) \circ \beta_A = \beta_B \circ G(f)$. Therefore

$$\begin{aligned}
H(f) \circ (\beta \circ \alpha)_A &= H(f) \circ (\beta_A \circ \alpha_A) \\
&= (H(f) \circ \beta_A) \circ \alpha_A \\
&= (\beta_B \circ G(f)) \circ \alpha_A \\
&= \beta_B \circ (G(f) \circ \alpha_A) \\
&= \beta_B \circ (\alpha_B \circ F(f)) \\
&= (\beta_B \circ \alpha_B) \circ F(f) \\
&= (\beta \circ \alpha)_B \circ F(f)
\end{aligned}$$

So $\beta \circ \alpha : F \Rightarrow H$ is a natural transformation. $\square$

Exercise 1.4.4. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H, I : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ and $\gamma : H \Rightarrow I$ natural transformations. Then

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

Proof. Let $A \in \mathrm{Obj}(\mathcal{C})$. By definition,

$$\begin{aligned}[(\gamma \circ \beta) \circ \alpha]_A &= (\gamma \circ \beta)_A \circ \alpha_A \\ &= (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &= \gamma_A \circ (\beta \circ \alpha)_A \\ &= [\gamma \circ (\beta \circ \alpha)]_A\end{aligned}$$

Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary,

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

□

Definition 1.4.5. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. We define the **identity natural transformation from F to F**, denoted $\mathrm{id}_F : F \Rightarrow F$, by

$$(\mathrm{id}_F)_A = \mathrm{id}_{F(A)}$$

Exercise 1.4.6. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. Then $\mathrm{id}_F : F \Rightarrow F$ is a natural transformation from F to F.

Proof.

(1) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\mathrm{id}_F)_A &= \mathrm{id}_{F(A)} \\ &\in \mathrm{Hom}_{\mathcal{D}}(F(A), F(A))\end{aligned}$$

(2) Let $A, B \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned}F(f) \circ (\mathrm{id}_F)_A &= F(f) \circ \mathrm{id}_{F(A)} \\ &= F(f) \\ &= \mathrm{id}_{F(B)} \circ F(f) \\ &= (\mathrm{id}_F)_B \circ F(f)\end{aligned}$$

□

Exercise 1.4.7. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Then

(1) $\mathrm{id}_G \circ \alpha = \alpha$
(2) $\alpha \circ \mathrm{id}_F = \alpha$

Proof.

(1) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\mathrm{id}_G \circ \alpha)_A &= (\mathrm{id}_G)_A \circ \alpha_A \\ &= \mathrm{id}_{G(A)} \circ \alpha_A \\ &= \alpha_A\end{aligned}$$

Since $A \in \mathrm{Obj}(C)$ is arbitrary, $\mathrm{id}_G \circ \alpha = \alpha$

(2) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ (\mathrm{id}_F)_A \\ &= \alpha_A \circ \mathrm{id}_{F(A)} \\ &= \alpha_A\end{aligned}$$

Since $A \in \mathrm{Obj}(C)$ is arbitrary, $\alpha \circ \mathrm{id}_F = \alpha$.

□

Exercise 1.4.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. If $\mathcal{C}$ is small, then α is a set.

Proof. Suppose that $\mathcal{C}$ is small. Then $\mathrm{Obj}(\mathcal{C})$ is a set. Since $\alpha : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Hom}_{\mathcal{D}}$, Axiom 1.1.3 implies that $\alpha(\mathrm{Obj}(\mathcal{C}))$ is a set. Then $\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C}))$ is a set. Therefore $\mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C})))$ is a set. Since $\alpha \subset \mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C}))$, we have that $\alpha \in \mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C})))$ which implies that α is a set. □

Exercise 1.4.9. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$. If $\mathcal{C}$ is small, then there exists a class A such that for each class α, $\alpha \in A$ iff $\alpha : F \Rightarrow G$.

Proof. Suppose that $\mathcal{C}$ is small. Define ϕ by

$$\phi(\alpha) : \alpha : F \Rightarrow G$$

Axiom 1.1.4 implies that there exists a class A such that for each set α, $\alpha \in A$ iff $\phi(\alpha)$. Let α be a class. Suppose that $\alpha \in A$. By Definition 1.1.1, α is a set. Since α is a set and $\alpha \in A$, we have that $\phi(\alpha)$. Hence $\alpha : F \Rightarrow G$.
Conversely, suppose that $\alpha : F \Rightarrow G$. Since $\mathcal{C}$ is small, Exercise 1.4.8 implies that α is a set. Since $\phi(\alpha)$, we have that $\alpha \in A$. □

Definition 1.4.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. We define the **functor category from $\mathcal{C}$ to $\mathcal{D}$**, denoted $\mathcal{D}^{\mathcal{C}}$, by

- $\mathrm{Obj}(\mathcal{D}^{\mathcal{C}}) = \{F : F : \mathcal{C} \to \mathcal{D}\}$
- For $F, G \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G) = \{\alpha : \alpha : F \Rightarrow G\}$
- For $F, G, H \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$ and $\beta \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(G, H)$, $\beta \circ_{\mathcal{D}^{\mathcal{C}}} \alpha = \beta \circ \alpha$

Exercise 1.4.11. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Then $\mathcal{D}^{\mathcal{C}}$ is a category.

Proof. Exercise 1.4.3 implies that composition is well-defined. Exercise 1.4.4 implies that composition is associative. Exercise 1.4.6 and Exercise 1.4.7 imply the existence of identities.

□

1.4.3. *Diagonal Functor.*

Definition 1.4.12. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $X, Y \in \mathrm{Obj}(\mathcal{D})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$. We define the **constant natural transformation on $\mathcal{C}$ at f**, denoted $\delta^{\mathcal{C}}_f : \Delta^{\mathcal{C}}_X \Rightarrow \Delta^{\mathcal{C}}_Y$, by

$$(\delta^{\mathcal{C}}_f)_A = f$$

Exercise 1.4.13. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $X, Y \in \mathrm{Obj}(\mathcal{D})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$. Then $\delta^{\mathcal{C}}_f : \Delta^{\mathcal{C}}_X \Rightarrow \Delta^{\mathcal{C}}_Y$ is a natural transformation.

Proof.

(1) By definition, for each $A \in \mathrm{Obj}(\mathcal{C})$ $(\delta^{\mathcal{C}}_f)_A \in \mathrm{Hom}_{\mathcal{D}}(\Delta^{\mathcal{C}}_X(A), \Delta^{\mathcal{C}}_Y(A))$.

(2) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned} \Delta^{\mathcal{C}}_Y(g) \circ (\delta^{\mathcal{C}}_f)_A &= \mathrm{id}_Y \circ f \\ &= f \\ &= f \circ \mathrm{id}_X \\ &= (\delta^{\mathcal{C}}_f)_B \circ \Delta^{\mathcal{C}}_X(g) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} \Delta^{\mathcal{C}}_X(A) & \xrightarrow{(\delta^{\mathcal{C}}_f)_A} & \Delta^{\mathcal{C}}_Y(A) \\ {\scriptstyle \Delta^{\mathcal{C}}_X(g)}\downarrow & & \downarrow{\scriptstyle \Delta^{\mathcal{C}}_Y(g)} \\ \Delta^{\mathcal{C}}_X(B) & \xrightarrow{(\delta^{\mathcal{C}}_f)_B} & \Delta^{\mathcal{C}}_Y(B) \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

So $\delta^{\mathcal{C}}_f : \Delta^{\mathcal{C}}_X \Rightarrow \Delta^{\mathcal{C}}_Y$ is a natural transformation. □

Exercise 1.4.14. Let $\mathcal{C}, \mathcal{D}$ be categories, $X, Y, Z \in \mathrm{Obj}(\mathcal{D})$, $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$. Then $\delta^{\mathcal{C}}_{g \circ f} = \delta^{\mathcal{C}}_g \circ \delta^{\mathcal{C}}_f$.

Proof. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned} (\delta^{\mathcal{C}}_{g \circ f})_A &= g \circ f \\ &= (\delta^{\mathcal{C}}_g)_A \circ (\delta^{\mathcal{C}}_f)_A \\ &= (\delta^{\mathcal{C}}_g \circ \delta^{\mathcal{C}}_f)_A \end{aligned}$$

Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\delta^{\mathcal{C}}_{g \circ f} = \delta^{\mathcal{C}}_g \circ \delta^{\mathcal{C}}_f$. □

Exercise 1.4.15. Let $\mathcal{C}, \mathcal{D}$ be categories and $X \in \mathrm{Obj}(\mathcal{D})$. Then $\delta^{\mathcal{C}}_{\mathrm{id}_X} = \mathrm{id}_{\Delta^{\mathcal{C}}_X}$.

Proof. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned} (\delta^{\mathcal{C}}_{\mathrm{id}_X})_A &= \mathrm{id}_X \\ &= \mathrm{id}_{\Delta^{\mathcal{C}}_X(A)} \\ &= (\mathrm{id}_{\Delta^{\mathcal{C}}_X})_A \end{aligned}$$

Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\delta^{\mathcal{C}}_{\mathrm{id}_X} = \mathrm{id}_{\Delta^{\mathcal{C}}_X}$ □

Definition 1.4.16. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. We define the **$\mathcal{C}$-ary diagonal functor** on $\mathcal{D}$, denoted by $\Delta^{\mathcal{C}} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$, by

- $\Delta^{\mathcal{C}}(X) = \Delta^{\mathcal{C}}_X$
- $\Delta^{\mathcal{C}}(f) = \delta^{\mathcal{C}}_f$

Exercise 1.4.17. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Then $\Delta^{\mathcal{C}} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$ is a functor.

Proof.

(1) Exercise 1.4.13 implies that for each $X, Y \in \mathrm{Obj}(\mathcal{D})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$, $\Delta^{\mathcal{C}}(f) \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta^{\mathcal{C}}(X), \Delta^{\mathcal{C}}(Y))$

(2) Exercise 1.4.14 implies that for each $X, Y, Z \in \mathrm{Obj}(\mathcal{D})$, $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$, $\Delta^{\mathcal{C}}(g \circ f) = \Delta^{\mathcal{C}}(g) \circ \Delta^{\mathcal{C}}(f)$
(3) Exercise 1.4.15 implies that for each $X \in \mathrm{Obj}(\mathcal{D})$, $\Delta^{\mathcal{C}}(\mathrm{id}_X) = \mathrm{id}_{\Delta^{\mathcal{C}}(X)}$

So $\Delta^{\mathcal{C}} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$ is a functor. $\square$

1.5. **Algebra of Morphisms.**

1.5.1. *Isomorphisms.*

Exercise 1.5.1. Uniqueness of Identities:
Let $\mathcal{C}$ be a category. Then for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists a unique $e_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ such that for each $B \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$, $f \circ e_A = f$ and $e_A \circ g = g$.

Proof. Let $A \in \mathrm{Obj}(\mathcal{C})$. Let $e_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$. Suppose that for each $B \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$, $f \circ e_A = f$ and $e_A \circ g = g$. Then

$$\begin{aligned} e_A &= e_A \circ \mathrm{id}_A \\ &= \mathrm{id}_A \end{aligned}$$

□

Definition 1.5.2. Let $\mathcal{C}$ be a category, $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then f is said to be an **isomorphism** if there exists $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Exercise 1.5.3. Uniqueness of Inverses:
Let $\mathcal{C}$ be a category, $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. If f is an isomorphism, then there exists a unique $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Proof. Suppose that f is an isomorphism. Let $g, h \in \mathrm{Hom}_{\mathcal{C}}(B, A)$. Suppose that $g \circ f = \mathrm{id}_A$, $f \circ g = \mathrm{id}_B$ and $h \circ f = \mathrm{id}_A$, $f \circ h = \mathrm{id}_B$. Then

$$\begin{aligned} g &= g \circ \mathrm{id}_B \\ &= g \circ (f \circ h) \\ &= (g \circ f) \circ h \\ &= \mathrm{id}_A \circ h \\ &= h \end{aligned}$$

□

Definition 1.5.4. Let $\mathcal{C}$ be a category, $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Suppose that f is an isomorphism. We define the **inverse of** f, denoted f^{-1}, to be the unique $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Exercise 1.5.5. Let $\mathcal{C}$ be a category and $A \in \mathrm{Obj}(\mathcal{C})$. Then id_A is an isomorphism and $(\mathrm{id}_A)^{-1} = \mathrm{id}_A$.

Proof. Since $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$, we have that id_A is an isomorphism and $(\mathrm{id}_A)^{-1} = \mathrm{id}_A$. □

Exercise 1.5.6. Let $\mathcal{C}$ be a category and $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. If f is an isomorphism, then f^{-1} is an isomorphism and $(f^{-1})^{-1} = f$.

Proof. Suppose that f is an isomorphism. By definition, $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. Hence f^{-1} is an isomorphism and $(f^{-1})^{-1} = f$. □

Exercise 1.5.7. Let $\mathcal{C}$ be a category, $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. If f and g are isomorphisms, then $g \circ f$ is an isomorphism and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. Suppose that f and g are isomorphisms. Then

$$\begin{aligned}(f^{-1} \circ g^{-1}) \circ (g \circ f) &= ((f^{-1} \circ g^{-1}) \circ g) \circ f \\ &= (f^{-1} \circ (g^{-1} \circ g)) \circ f \\ &= (f^{-1} \circ \mathrm{id}_B) \circ f \\ &= f^{-1} \circ f \\ &= \mathrm{id}_A\end{aligned}$$

and

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= ((g \circ f) \circ f^{-1}) \circ g^{-1} \\ &= (g \circ (f \circ f^{-1})) \circ g^{-1} \\ &= (g \circ \mathrm{id}_B) \circ g^{-1} \\ &= g \circ g^{-1} \\ &= \mathrm{id}_C\end{aligned}$$

So $g \circ f$ is an isomorphism and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. ☐

Definition 1.5.8. Let $\mathcal{C}$ be a category and $A, B \in \mathrm{Obj}(\mathcal{C})$. Then A is said to be **isomorphic** to B if there exists $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that f is an isomorphism.

Exercise 1.5.9. Let $\mathcal{C}$ be a category. We define the relation $\cong$ on $\mathrm{Obj}(\mathcal{C})$ by $A \cong B$ iff A is isomorphic to B. Then $\cong$ is an equivalence relation on $\mathrm{Obj}(\mathcal{C})$.

Proof.

(1) **reflexivity:**
Let $A \in \mathrm{Obj}(\mathcal{C})$. Exercise 1.5.5 implies that id_A is an isomorphism. So $A \cong A$. Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, we have that for each $A \in \mathrm{Obj}(\mathcal{C})$, $A \cong A$ and thus $\cong$ is reflexive.

(2) **symmetry:**
Let $A, B \in \mathrm{Obj}(\mathcal{C})$. Suppose that $A \cong B$. Then there exists $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that f is an isomorphism. Exercise 1.5.6 implies that f^{-1} is an isomorphism. Since $f^{-1} \in \mathrm{Hom}_{\mathcal{C}}(B, A)$, $B \cong A$. Since $A, B \in \mathrm{Obj}(\mathcal{C})$ are arbitrary, we have that for each $A, B \in \mathrm{Obj}(\mathcal{C})$, $A \cong B$ implies that $B \cong A$ and thus $\cong$ is reflexive.

(3) **transitivity:** Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$. Suppose that $A \cong B$ and $B \cong C$. Then there exist $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ such that that f and g are isomorphisms. Exercise 1.5.7 implies that $g \circ f$ is an isomorphism. Since $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A, C)$, $A \cong C$. Since $A, B, C \in \mathrm{Obj}(\mathcal{C})$ are arbitrary, we have that for each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $A \cong B$ and $B \cong C$ implies that $A \cong C$ and thus $\cong$ is transitive.

Since $\cong$ is reflexive, symmetric and transitive, $\cong$ is an equivalence relation on $\mathrm{Obj}(\mathcal{C})$. ☐

Definition 1.5.10. Let $\mathcal{C}$ be a category, $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f : A \to B$. Then

- f is said to be a **monomorphism** if for each $C \in \mathrm{Obj}(C)$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(C, A)$, $f \circ g = f \circ h$ implies that $g = h$, i.e. we have the following implication of commutative diagrams:

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "h"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \implies \begin{tikzcd} C \arrow[rr, bend left, "g"] \arrow[rr, bend right, "h"'] & & A \end{tikzcd}$$

- f is said to be an **epimorphism** if for each $C \in \mathrm{Obj}(C)$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $g \circ f = h \circ f$ implies that $g = h$, i.e. we have the following implication of commutative diagrams:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ B \arrow[r, "h"'] & C \end{tikzcd} \implies \begin{tikzcd} B \arrow[rr, bend left, "g"] \arrow[rr, bend right, "h"'] & & C \end{tikzcd}$$

Exercise 1.5.11. Let $A, B \in \mathrm{Obj}(\mathbf{Set})$ and $f \in \mathrm{Hom}_{\mathbf{Set}}(A, B)$. Then

(1) f is a monomorphism iff f is injective
(2) f is an epimorphism iff f is surjective

Hint: consider $C = \{0\}$ and $C = \{0, 1\}$.

Proof.

(1) Suppose that f is injective. Let $C \in \mathrm{Obj}(\mathbf{Set})$ and $g, h \in \mathrm{Hom}_{\mathbf{Set}}(C, A)$. Suppose that $f \circ g = f \circ h$. Let $x \in C$. Then $f(g(x)) = f(h(x))$. Injectivity of f implies that $g(x) = h(x)$. Since $x \in C$ is arbitrary, $g = h$. Hence f is a monomorphism.
Conversely, suppose that f is a monomorphism. Let $a, b \in A$. Suppose that $f(a) = f(b)$. Set $C = \{0\}$ and define $g, h : C \to A$ by $g(0) = a$ and $h(0) = b$. Then

$$\begin{aligned} f \circ g(0) &= f(g(0)) \\ &= f(a) \\ &= f(b) \\ &= f(h(0)) \\ &= f \circ h(0) \end{aligned}$$

Therefore $f \circ g = f \circ h$. Since f is a monomorphism, we have that $g = h$. Hence

$$\begin{aligned} a &= g(0) \\ &= h(0) \\ &= b \end{aligned}$$

(2) Suppose that f is surjective. Let $C \in \mathrm{Obj}(\mathcal{C})$ and $g, h \in \mathrm{Hom}_{\mathbf{Set}}(B, C)$. Suppose that $g \circ f = h \circ f$. Let $y \in B$. Surjective of f implies that there exists $x \in A$ such

that $y = f(x)$. Then

$$\begin{aligned} g(y) &= g(f(x)) \\ &= g \circ f(x) \\ &= h \circ f(x) \\ &= h(f(x)) \\ &= h(y) \end{aligned}$$

Since $y \in B$ is arbitrary, $g = h$. Hence f is an epimorphism.
Conversely, suppose that f is an epimorphism. Set $C = \{0, 1\}$ and define $g, h : B \to C$ by $g = \chi_{f(A)}$ and $h = \chi_B$. Then $g \circ f = h \circ f$. Since f is an epimorphism, $g = h$ and $f(A) = B$. Hence f is surjective.

□

Exercise 1.5.12. Let $\mathcal{C}$ be a category, $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$. If f is an isomorphism, then f is a monomorphism and f is an epimorphism.

Proof. Suppose that f is an isomorphism.

- (monomorphism)
  Let $C \in \text{Obj}(\mathcal{C})$ and $g, h \in \text{Hom}_{\mathcal{C}}(C, A)$. Suppose that $f \circ g = f \circ h$. Then

$$\begin{aligned} g &= \text{id}_A \circ g \\ &= (f^{-1} \circ f) \circ g \\ &= f^{-1} \circ (f \circ g) \\ &= f^{-1} \circ (f \circ h) \\ &= (f^{-1} \circ f) \circ h \\ &= \text{id}_A \circ h \\ &= h \end{aligned}$$

  So f is a monomorphism.
- (epimorphism)
  Let $C \in \text{Obj}(\mathcal{C})$ and $g, h \in \text{Hom}_{\mathcal{C}}(B, C)$. Suppose that $g \circ f = h \circ f$. Then

$$\begin{aligned} g &= g \circ \text{id}_B \\ &= g \circ (f \circ f^{-1}) \\ &= (g \circ f) \circ f^{-1} \\ &= (h \circ f) \circ f^{-1} \\ &= h \circ (f \circ f^{-1}) \\ &= h \circ \text{id}_B \\ &= h \end{aligned}$$

  So f is an epimorphism.

□

Definition 1.5.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Then α is said to be a **natural isomorphism** if for each $A \in \text{Obj}(\mathcal{C})$, α_A is an isomorphism.

Definition 1.5.14. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Suppose that α is a natural isomorphism. We define $\alpha^{-1} : G \Rightarrow F$ by $(\alpha^{-1})_A = \alpha_A^{-1}$.

Exercise 1.5.15. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Suppose that α is a natural isomorphism. Then $\alpha^{-1} : G \Rightarrow F$ is a natural transformation.

Proof.

(1) Let $A \in \mathrm{Obj}(\mathcal{C})$. Since $\alpha_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A))$, we have that

$$\begin{aligned}(\alpha^{-1})_A &= \alpha_A^{-1} \\ &\in \mathrm{Hom}_{\mathcal{D}}(G(A), F(A))\end{aligned}$$

(2) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since $G(f) \circ \alpha_A = \alpha_B \circ F(f)$, i.e. the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$

we have that

$$\begin{aligned}F(f) \circ (\alpha^{-1})_A &= F(f) \circ \alpha_A^{-1} \\ &= \mathrm{id}_{F(B)} \circ (F(f) \circ \alpha_A^{-1}) \\ &= (\alpha_B^{-1} \circ \alpha_B) \circ (F(f) \circ \alpha_A^{-1}) \\ &= \alpha_B^{-1} \circ (\alpha_B \circ (F(f) \circ \alpha_A^{-1})) \\ &= \alpha_B^{-1} \circ ((\alpha_B \circ F(f)) \circ \alpha_A^{-1}) \\ &= \alpha_B^{-1} \circ ((G(f) \circ \alpha_A) \circ \alpha_A^{-1}) \\ &= \alpha_B^{-1} \circ (G(f) \circ (\alpha_A \circ \alpha_A^{-1})) \\ &= \alpha_B^{-1} \circ (G(f) \circ \mathrm{id}_{G(A)}) \\ &= \alpha_B^{-1} \circ G(f) \\ &= (\alpha^{-1})_B \circ G(f)\end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} G(A) \arrow[r, "(\alpha^{-1})_A"] \arrow[d, "G(f)"'] & F(A) \arrow[d, "F(f)"] \\ G(B) \arrow[r, "(\alpha^{-1})_B"] & F(B) \end{tikzcd}$$

So $\alpha^{-1} : G \Rightarrow F$.

□

Exercise 1.5.16. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Suppose that α is a natural isomorphism. Then $\alpha^{-1} \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \alpha^{-1} = \mathrm{id}_G$.

Proof. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\alpha^{-1} \circ \alpha)_A &= (\alpha^{-1})_A \circ \alpha_A \\ &= \alpha_A^{-1} \circ \alpha_A \\ &= \mathrm{id}_{F(A)} \\ &= (\mathrm{id}_F)_A\end{aligned}$$

and

$$\begin{aligned}(\alpha \circ \alpha^{-1})_A &= \alpha_A \circ (\alpha^{-1})_A \\ &= \alpha_A \circ \alpha_A^{-1} \\ &= \mathrm{id}_{G(A)} \\ &= (\mathrm{id}_G)_A\end{aligned}$$

Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\alpha^{-1} \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \alpha^{-1} = \mathrm{id}_G$. □

Exercise 1.5.17. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Let $F, G \in \mathcal{D}^{\mathcal{C}}$ and $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$. Then α is an isomorphism iff α is a natural isomorphism.

Proof. Suppose that α is an isomorphism. Then there exists $\beta \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(G, F)$ such that $\beta \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \beta - \mathrm{id}_G$. Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}\beta_A \circ \alpha_A &= (\beta \circ \alpha)_A \\ &= (\mathrm{id}_F)_A \\ &= \mathrm{id}_{F(A)}\end{aligned}$$

and

$$\begin{aligned}\alpha_A \circ \beta_A &= (\alpha \circ \beta)_A \\ &= (\mathrm{id}_G)_A \\ &= \mathrm{id}_{G(A)}\end{aligned}$$

Hence α_A is an isomorphism. Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, α is a natural isomorphism.
Conversely, suppose that α is a natural isomorphism. Exercise 1.5.15 and Exercise 1.5.16 imply that α is an isomorphism. □

1.5.2. *Initial and Final Objects.*

Definition 1.5.18. Let $\mathcal{C}$ be a category and $0 \in \mathrm{Obj}(\mathcal{C})$. Then 0 is said to be **initial** if for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $f \in \mathrm{Hom}_{\mathcal{C}}(0, A)$ such that $\mathrm{Hom}_{\mathcal{C}}(0, A) = \{f\}$.

Definition 1.5.19. Let $\mathcal{C}$ be a category and $1 \in \mathrm{Obj}(\mathcal{C})$. Then 1 is said to be **final** if for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $f \in \mathrm{Hom}_{\mathcal{C}}(A, 1)$ such that $\mathrm{Hom}_{\mathcal{C}}(A, 1) = \{f\}$.

Exercise 1.5.20. Let $\mathcal{C}$ be a category and $0 \in \mathrm{Obj}(\mathcal{C})$. If 0 is initial, then $\mathrm{Hom}_{\mathcal{C}}(0, 0) = \{\mathrm{id}_0\}$.

Proof. Suppose that 0 is initial. Then there exists a $f \in \mathrm{Hom}_{\mathcal{C}}(0, 0)$ such that $\mathrm{Hom}_{\mathcal{C}}(0, 0) = \{f\}$. Since $\mathrm{id}_0 \in \mathrm{Hom}_{\mathcal{C}}(0, 0)$, $f = \mathrm{id}_0$ and therefore $\mathrm{Hom}_{\mathcal{C}}(0, 0) = \{\mathrm{id}_0\}$. □

Exercise 1.5.21. Let $\mathcal{C}$ be a category and $1 \in \mathrm{Obj}(\mathcal{C})$. If 1 is final, then $\mathrm{Hom}_{\mathcal{C}}(1, 1) = \{\mathrm{id}_1\}$.

Proof. Similar to Exercise 1.5.20 □

Exercise 1.5.22. Let $\mathcal{C}$ be a category and $0, 0' \in \mathrm{Obj}(\mathcal{C})$. If 0 and $0'$ are initial, then 0 and $0'$ are isomorphic.

Proof. Suppose that 0 and $0'$ are initial. By definition, there exist $f \in \mathrm{Hom}_{\mathcal{C}}(0, 0')$ and $f' \in \mathrm{Hom}_{\mathcal{C}}(0', 0)$ such that $\mathrm{Hom}_{\mathcal{C}}(0, 0') = \{f\}$ and $\mathrm{Hom}_{\mathcal{C}}(0', 0) = \{f'\}$, i.e. we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[loop left, "f' \circ f"] \arrow[rr, bend right, "f"'] & & 0' \arrow[ll, bend right, "f'"'] \arrow[loop right, "f \circ f'"] \end{tikzcd}$$

Exercise 1.5.20 implies that $f' \circ f = \mathrm{id}_0$ and $f \circ f' = \mathrm{id}_{0'}$. Hence f is an isomorphism. Since $f \in \mathrm{Hom}_{\mathcal{C}}(0, 0')$, we have that $0 \cong 0'$. □

Exercise 1.5.23. Let $\mathcal{C}$ be a category and $1, 1' \in \mathrm{Obj}(\mathcal{C})$. If 1 and $1'$ are final, then 1 and $1'$ are isomorphic.

Proof. Similar to Exercise 1.5.22 □

Exercise 1.5.24. We have that $\varnothing$ is initial in **Set**.

Proof. Let $A \in \mathrm{Obj}(\mathbf{Set})$. Define $f \in \mathrm{Hom}_{\mathbf{Set}}(\varnothing, A)$ by $f = \varnothing$. Let $g \in \mathrm{Hom}_{\mathbf{Set}}(\varnothing, A)$. Then $g = f$. Since $g \in \mathrm{Hom}_{\mathbf{Set}}(\varnothing, A)$ is arbitrary, $\mathrm{Hom}_{\mathbf{Set}}(\varnothing, A) = \{f\}$. Hence $\varnothing$ is initial. □

Exercise 1.5.25. We have that $\{\varnothing\}$ is terminal in **Set**.

Proof. Let $A \in \mathrm{Obj}(\mathbf{Set})$. Define $f \in \mathrm{Hom}_{\mathbf{Set}}(A, \{\varnothing\})$ by $f(x) = \varnothing$. Let $g \in \mathrm{Hom}_{\mathbf{Set}}(A, \{\varnothing\})$. Then $g = f$. Since $g \in \mathrm{Hom}_{\mathbf{Set}}(A, \{\varnothing\})$ is arbitrary, $\mathrm{Hom}_{\mathbf{Set}}(A, \{\varnothing\}) = \{f\}$. Hence $\{\varnothing\}$ is final. □

Exercise 1.5.26. We have that $\mathbf{0}$ is initial in **Cat**.

Proof. Let $\mathcal{C} \in \mathrm{Obj}(\mathbf{Cat})$. It is clear that $\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{0}, \mathcal{C}) = \{E_{\mathcal{C}}\}$. Hence $\mathbf{0}$ is initial in **Cat**. □

Exercise 1.5.27. We have that $\mathbf{1}$ is final in **Cat**.

Proof. Let $\mathcal{C} \in \mathrm{Obj}(\mathbf{Cat})$. It is clear that $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathbf{1}) = \{\Delta^{\mathcal{C}}_*\}$. Hence $\mathbf{1}$ is final in **Cat**. □

Definition 1.5.28. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $0 \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. Suppose that 0 is initial in $\mathcal{D}$. Then for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $f_A \in \mathrm{Hom}_{\mathcal{D}}(0, F(A))$ such that $\mathrm{Hom}_{\mathcal{D}}(0, F(A)) = \{f_A\}$. We define the **initial natural transformation induced by** 0 from $\Delta^{\mathcal{C}}_0$ to F, denoted $\zeta_0 : \Delta^{\mathcal{C}}_0 \Rightarrow F$, by $(\eta_0)_A = f_A$.

Definition 1.5.29. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $1 \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. Suppose that 1 is final in $\mathcal{D}$. Then for each $A \in \mathrm{Obj}(\mathcal{C})$, there exists $f_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), 1)$ such that $\mathrm{Hom}_{\mathcal{D}}(F(A), 1) = \{f_A\}$. We define the **final natural transformation induced by** 1 from F to $\Delta^{\mathcal{C}}_1$, denoted $\phi_1 : F \Rightarrow \Delta^{\mathcal{C}}_1$, by $(\phi_1)_A = f_A$.

Exercise 1.5.30. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $0 \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. Suppose that 0 is initial in $\mathcal{D}$. Then $\eta_0 : \Delta^{\mathcal{C}}_0 \Rightarrow F$ is a natural transformation.

Proof.

(1) By definition, for each $A \in \mathrm{Obj}(\mathcal{C})$, $(\eta_0)_A \in \mathrm{Hom}_{\mathcal{D}}(\Delta^{\mathcal{C}}_0(A), F(A))$

(2) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since

$$\begin{aligned} F(f) \circ (\eta_0)_A &\in \mathrm{Hom}_{\mathcal{D}}(0, F(B)) \\ &= \{(\eta_0)_B\} \end{aligned}$$

we have that

$$\begin{aligned} F(f) \circ (\eta_0)_A &= (\eta_0)_B \\ &= (\eta_0)_B \circ \mathrm{id}_0 \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} \Delta_0^{\mathcal{C}}(A) \arrow[r, "(\eta_0)_A"] \arrow[d, "\Delta_0^{\mathcal{C}}(f)"'] & F(A) \arrow[d, "F(f)"] \\ \Delta_0^{\mathcal{C}}(B) \arrow[r, "(\eta_0)_B"] & F(B) \end{tikzcd} = \begin{tikzcd} 0 \arrow[r, "(\eta_0)_A"] \arrow[d, "\mathrm{id}_0"'] & F(A) \arrow[d, "F(f)"] \\ 0 \arrow[r, "(\eta_0)_B"] & F(B) \end{tikzcd}$$

So $\eta_0 : \Delta_0^{\mathcal{C}} \Rightarrow F$ is a natural transformation. □

Exercise 1.5.31. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $1 \in \mathrm{Obj}(\mathcal{D})$ and $F : \mathcal{C} \to \mathcal{D}$. Suppose that 1 is final in $\mathcal{D}$. Then $\phi_1 : F \Rightarrow \Delta_0^{\mathcal{C}}$ is a natural transformation.

Proof. Similar to Exercise 1.5.30 □

Exercise 1.5.32. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $0 \in \mathrm{Obj}(\mathcal{D})$. Suppose that $\mathcal{C}$ is small. If 0 is initial in $\mathcal{D}$, then $\Delta_0^{\mathcal{C}}$ is initial in $\mathcal{D}^{\mathcal{C}}$.

Proof. Suppose that 0 is initial in $\mathcal{D}$. Let $F \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F)$ and $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned} \alpha_A &\in \mathrm{Hom}_{\mathcal{D}}(\Delta_0^{\mathcal{C}}(A), F(A)) \\ &= \mathrm{Hom}_{\mathcal{D}}(0, F(A)) \\ &= \{(\eta_0)_A\} \end{aligned}$$

Hence $\alpha_A = (\eta_0)_A$. Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary, $\alpha = \eta_0$. Since $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F)$ is arbitrary, $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F) = \{\eta_0\}$. Therefore $\Delta_0^{\mathcal{C}}$ is initial in $\mathcal{D}^{\mathcal{C}}$. □

Exercise 1.5.33. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $1 \in \mathrm{Obj}(\mathcal{D})$. Suppose that $\mathcal{C}$ is small. If 1 is final in $\mathcal{D}$, then $\Delta_1^{\mathcal{C}}$ is final in $\mathcal{D}^{\mathcal{C}}$.

Proof. Similar to Exercise 1.5.32. □

2. Universal Morphisms and Limits

2.0.1. *Universal Morphisms.*

Definition 2.0.1. Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$, $F : \mathcal{C} \to \mathcal{D}$, $A \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, F(A))$. Then (A, f) is said to be a **universal morphism** from X to F if for each $A' \in \mathrm{Obj}(\mathcal{C})$ $f' \in \mathrm{Hom}_{\mathcal{D}}(X, F(A'))$, there exists a unique $\alpha \in \mathrm{Hom}_{\mathcal{C}}(A, A')$ such that $f' = F(\alpha) \circ f$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & F(A) \arrow[d, dashed, "F(\alpha)"] & A \arrow[d, dashed, "\alpha"] \\ & F(A') & A' \end{tikzcd}$$

Definition 2.0.2. Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$, $F : \mathcal{C} \to \mathcal{D}$, $A \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(F(A), X)$. Then (A, f) is said to be a **universal morphism** from F to X if for each $A' \in \mathrm{Obj}(\mathcal{C})$ $f' \in \mathrm{Hom}_{\mathcal{D}}(F(A'), X)$, there exists a unique $\alpha \in \mathrm{Hom}_{\mathcal{C}}(A', A)$ such that $f' = f \circ F(\alpha)$, i.e. the following diagram commutes:

$$\begin{tikzcd} X & F(A) \arrow[l, "f"'] & A \\ & F(A') \arrow[ul, "f'"] \arrow[u, dashed, "F(\alpha)"'] & A' \arrow[u, dashed, "\alpha"'] \end{tikzcd}$$

Exercise 2.0.3. Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$, $F : \mathcal{C} \to \mathcal{D}$, $A \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, F(A))$. Then (A, f) is a universal morphism from X to F iff (A, f) is initial in $(X \downarrow F)$.

Proof. □

Exercise 2.0.4. Let $\mathcal{C}, \mathcal{D}$ be a categories, $X \in \mathrm{Obj}(\mathcal{D})$, $F : \mathcal{C} \to \mathcal{D}$ $A \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(F(A), X)$. Then (A, f) is a universal morphism from F to X iff (A, f) is terminal in $(F \downarrow X)$.

Proof. □

2.1. **Limits.**

Definition 2.1.1. Let $\mathcal{J}, \mathcal{C}$ be categories and $D : \mathcal{J} \to \mathcal{C}$. Then D is said to be a **diagram of type $\mathcal{J}$ in $\mathcal{C}$**.

Note 2.1.2. We are usually interested in the case that $\mathcal{J}$ is small. We will identify a diagram D with its image.

Example 2.1.3. Define $\mathcal{J}$ by

- $\text{Obj}(\mathcal{J}) = \{1, 2, 3\}$ and for $i, j \in \text{Obj}(\mathcal{J})$, $\text{Hom}_{\mathcal{J}}(i, j) = \{a_{i,j}\}$,
- for $i, j \in \text{Obj}(\mathcal{J})$, $\text{Hom}_{\mathcal{J}}(i, j) = \{a_{ij}\}$.

Let $\mathcal{C}$ be a category and $D : \mathcal{J} \to \mathcal{C}$. Without including the identity morphisms or compositions, we can visualize D as follows:

$$
\begin{tikzcd}
 & 1 \arrow[r, "b"] \arrow[dl, "a"'] & 2 \arrow[dl, "c"] & & & D_1 \arrow[r] \arrow[dl] & D2 \arrow[dl] \\
3 \arrow[r, "d"'] & 4 & & \arrow[r, "D"] & D_3 \arrow[r] & D_4 &
\end{tikzcd}
$$

Definition 2.1.4. Let $\mathcal{J}, \mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$. We define the **category of cones to** D, denoted $\mathbf{Cone}(D)$, by $\mathbf{Cone}(D) = (\Delta^{\mathcal{J}} \downarrow D)$.

Example 2.1.5. Let $\mathcal{J}$

$$
\begin{tikzcd}
 & X, \arrow[d, "\phi_1"] \arrow[drr, "\phi_2"] \arrow[ddl, "\phi_3"'] \arrow[dd, "\phi_4"] & & \\
 & D_1 \arrow[rr] \arrow[dl] & & D2 \arrow[dl] \\
D_3 \arrow[rr] & & D_4 &
\end{tikzcd}
$$

Definition 2.1.6. Let $\mathcal{J}, \mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$. We define the **category of cocones from** D, denoted $\mathbf{Cocone}(D)$, by $\mathbf{Cocone}(D) = (D \downarrow \Delta^{\mathcal{J}})$.

Definition 2.1.7. Let $\mathcal{J}, \mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$ and $(X, \phi) \in \mathbf{Cone}(D)$. Then (X, ϕ) is said to be a **limit of** D if (X, ϕ) is a universal morphism from $\Delta^{\mathcal{J}}$ to D.

Note 2.1.8. Let $\mathcal{J}$, $\mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$ and $(X, \phi) \in \mathbf{Cone}(D)$. Then

$$
\begin{aligned}
(X, \phi) \text{ is a limit of } D &\iff (X, \phi) \text{ is terminal in } \mathbf{Cone}(D) \\
&\iff \text{for each } (Y, \psi) \in \mathbf{Cone}(D), \text{ there exists a unique} \\
&\qquad f \in \text{Hom}_{\mathcal{C}}(Y, X) \text{ such that for each } j \in \mathcal{J},\ \psi_j = \phi_j \circ f
\end{aligned}
$$

Definition 2.1.9. Let $\mathcal{J}, \mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \mathrm{Obj}(\mathcal{C}^{\mathcal{J}})$ and $(X, \phi) \in \mathbf{Cocone}(D)$. Then (X, ϕ) is said to be a **colimit of** D if (X, ϕ) is a universal morphism from D to $\Delta^{\mathcal{J}}$.

Note 2.1.10. Let $\mathcal{J}$, $\mathcal{C}$ be categories. Suppose that $\mathcal{J}$ is small. Let $D \in \mathrm{Obj}(\mathcal{C}^{\mathcal{J}})$ and $(X, \phi) \in \mathbf{Cone}(D)$. Then

$$\begin{aligned}
(X, \phi) \text{ is a colimit of } D &\iff (X, \phi) \text{ is initial in } \mathbf{Cocone}(D) \\
&\iff \text{for each } (Y, \psi) \in \mathbf{Cocone}(D), \text{ there exists a unique} \\
&\qquad f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \text{ such that for each } j \in \mathcal{J},\ \psi_j = f \circ \phi_j
\end{aligned}$$

2.1.1. *Products and Coproducts.*

2.1.2. *Equalizers and Coequalizers.*