INTRODUCTION TO MEASURE AND INTEGRATION

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Contents

Preface

This aim of this book is to help students develope a basic grasp of the theory of integration. A typical student's first exposure to integration is in the context of the Darboux integral. This integral is applicable to a relatively small class of complex-valued functions of one or more real variables. Although this is integral is of critical importance, it is not sufficient for many purposes. Extending the Darboux integral to the Lebesgue integral, we can define a notion of integration for certain complex-valued functions on more general spaces, like for example topological spaces. To do so, we must first develop some measure theory, which is useful in its own right. Further extending the Lebesgue integral to the Bochner integral, we can define a notion of integration for certain vector valued functions.

The target audience of this book is composed of those who wish to deepen their understanding of integration beyond the Darboux integral. In practice, a basic understanding of the integral would benefit anyone who works with integration and limits in more exotic spaces. For example, students of statistics, physics and disciplines which utilize numerical solutions to differential equations would benefit greatly from a deeper understanding of the integral.

1. The Darboux Integral

1.1. Definition and Properties.

Definition 1.1.1. Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Define
$$B([a, b]) = \{f : [a, b] \to \mathbb{R} : f \text{ is bounded}\}$$

Definition 1.1.2. Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Let $x_0, \cdots, x_n \in [a, b]$. Suppose that $a = x_0 < x_1 < \cdots < x_n = b$. Put $\mathcal{P} = \{x_0, \cdots, x_n\}$. Then $\mathcal{P}$ is said to be a **partion** of $[a, b]$.

Definition 1.1.3. Let $f \in B([a, b])$ and $\mathcal{P} = \{x_0, \cdots, x_n\}$ a partion of $[a, b]$. Suppose that f is bounded. For $i = 1, \cdots, n$, put
$$M_i^f = \sup_{x \in [x_{i-1}, x_i]} f(x)$$
and
$$m_i^f = \inf_{x \in [x_{i-1}, x_i]} f(x)$$
We define the **upper Darboux sum** of f with respect to $\mathcal{P}$, denoted $U_\mathcal{P} f$, to be
$$U_\mathcal{P} f = \sum_{i=1}^n M_i^f (x_i - x_{i-1})$$
and we define the **lower Darboux sum** of f with respect to $\mathcal{P}$, denoted $L_\mathcal{P} f$, to be
$$L_\mathcal{P} f = \sum_{i=1}^n m_i^f (x_i - x_{i-1})$$

Exercise 1.1.4. Let $f \in B([a, b])$ and $\mathcal{P}$ a partition of $[a, b]$. Then
$$\left[\inf_{x \in [a,b]} f(x)\right](b - a) \leq L_\mathcal{P} f \leq U_\mathcal{P} f \leq \left[\sup_{x \in [a,b]} f(x)\right](b - a)$$

Proof. Clear. $\square$

Exercise 1.1.5. Let $f \in B([a, b])$ and $\mathcal{P}, \mathcal{P}'$ partitions of $[a, b]$. If $\mathcal{P} \subset \mathcal{P}'$, then
  (1) $U_{\mathcal{P}'} f \leq U_\mathcal{P} f$
  (2) $L_\mathcal{P} f \leq L_{\mathcal{P}'} f$

Proof.
  (1) Assume that $\mathcal{P} = \{x_0, \cdots, x_n\}$ and $\mathcal{P}' = \mathcal{P} \cup \{x'\}$. Then there exists $j \in \{1, \cdots, n\}$ such that $x_{j-1} < x' < x_j$. Define
  $$M_1' = \sup_{x \in [x_{j-1}, x']} f(x), \quad M_2' = \sup_{x \in [x', x_j]} f(x)$$
  Since $[x_{j-1}, x'], [x', x_j] \subset [x_{j-1}, x_j]$, we have that $M_1', M_2' \leq M_j^f$. Then
  $$U_{P'} f = \sum_{i=1}^{j-1} M_i^f (x_i - x_{i-1}) + M_1'(x' - x_{j-1}) + M_2'(x_j - x') + \sum_{i=j+1}^n M_i^f (x_i - x_{i-1})$$
  $$\leq \sum_{i=1}^n M_i^f (x_i - x_{i-1})$$
  $$= U_P f$$

By induction, this is true for general partitions $P \subset \mathcal{P}'$.
(2) Similar to (1).

□

Exercise 1.1.6. Let $f, g \in B([a, b])$ and $\mathcal{P} = \{x_0, \cdots, x_n\}$ a partition of $[a, b]$. Then

(1) $U_{\mathcal{P}}(f + g) \leq U_{\mathcal{P}} f + U_{\mathcal{P}} g$
(2) $L_{\mathcal{P}}(f + g) \geq L_{\mathcal{P}} f + L_{\mathcal{P}} g$

Proof.

(1) For each $i \in \{1, \cdots, n\}$, $M_i^{f+g} \leq M_i^f + M_i^g$. So

$$
\begin{aligned}
U_{\mathcal{P}}(f + g) &= \sum_{i=1}^{n} M_i^{f+g}(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} (M_i^f + M_i^g)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} M_i^f(x_i - x_{i-1}) + \sum_{i=1}^{n} M_i^g(x_i - x_{i-1}) \\
&= U_{\mathcal{P}} f + U_{\mathcal{P}} g
\end{aligned}
$$

(2) Similar to (1)

□

Exercise 1.1.7. Let $f \in B([a, b])$ and $\mathcal{P} = \{x_0, \cdots, x_n\}$ a partition of $[a, b]$. Then

(1) $U_{\mathcal{P}}(-f) = -L_P f$
(2) $L_{\mathcal{P}}(-f) = -U_{\mathcal{P}} f$

Proof.

(1) Since for $i \in \{1, \cdots, n\}$, $M_i^{-f} = -m_i^f$ we see that

$$
\begin{aligned}
U_{\mathcal{P}}(-f) &= \sum_{i=1}^{n} M_i^{-f}(x_i - x_{i-1}) \\
&= -\sum_{i=1}^{n} m_i^f(x_i - x_{i-1}) \\
&= -L_{\mathcal{P}} f
\end{aligned}
$$

(2) Similar to (1).

□

Exercise 1.1.8. Let $f \in B([a, b])$, $c > 0$ and $\mathcal{P} = \{x_0, \cdots, x_n\}$ a partition of $[a, b]$. Then

(1) $U_{\mathcal{P}}(cf) = cU_{\mathcal{P}} f$
(2) $L_{\mathcal{P}}(cf) = cL_{\mathcal{P}} f$

Proof.

(1) Since for $i \in \{1, \cdots, n\}$, $M_i^{cf} = cM_i^f$, we see that

$$U_{\mathcal{P}}(cf) = \sum_{i=1}^{n} M_i^{cf}(x_i - x_{i-1})$$

$$= c \sum_{i=1}^{n} M_i^f(x_i - x_{i-1})$$

$$= cU_{\mathcal{P}}f$$

(2) Similar to (1)

$\square$

Definition 1.1.9. Let $f \in B([a,b])$. We define the **upper Darboux integral** of f, denoted Uf, to be

$$Uf = \inf\{U_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a,b]\}$$

and we define the **lower Darboux integral** of f, denoted Lf, to be

$$Lf = \sup\{L_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a,b]\}$$

Exercise 1.1.10. Let $f \in B([a,b])$. Then

$$\left[\inf_{x \in [a,b]} f(x)\right](b-a) \leq Lf \leq Uf \leq \left[\sup_{x \in [a,b]} f(x)\right](b-a)$$

Proof. Clearly

$$\left[\inf_{x \in [a,b]} f(x)\right](b-a) \leq Lf \quad \text{and} \quad Uf \leq \left[\sup_{x \in [a,b]} f(x)\right](b-a)$$

Let $\epsilon > 0$. Then there exist partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,b]$ such that $U_{\mathcal{P}_1}f < Uf + \epsilon/2$ and $L_{\mathcal{P}_2}f > Lf - \epsilon/2$. Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then

$$Uf \geq U_{\mathcal{P}_1} - \epsilon/2$$

$$> U_{\mathcal{P}}f - \epsilon/2$$

$$\geq L_{\mathcal{P}}f - \epsilon/2$$

$$\geq L_{\mathcal{P}_2}f - \epsilon/2$$

$$> Lf - \epsilon$$

Since $\epsilon > 0$ is arbitrary, we have that $Uf \geq Lf$. $\square$

Exercise 1.1.11. Let $f, g \in B([a,b])$. Then
    (1) $U(f+g) \leq Uf + Ug$
    (2) $L(f+g) \geq Lf + Lg$

Proof.
    (1) Let $\epsilon > 0$. Then there exists a partitions $\mathcal{P}_1$ of $[a,b]$ such that $U_{\mathcal{P}_1}f < Uf + \epsilon/2$ and $U_{\mathcal{P}_2}g < Uf + \epsilon/2$. Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then

$$U_{\mathcal{P}}(f+g) \leq U_{\mathcal{P}}f + U_{\mathcal{P}}g$$

$$\leq U_{\mathcal{P}_1}f + U_{\mathcal{P}_2}g$$

$$< Uf + \epsilon/2 + Ug + \epsilon/2$$

$$= Uf + Ug + \epsilon$$

Since $\epsilon > 0$ is arbitrary, $U_{\mathcal{P}}(f + g) \leq Uf + Ug$.
(2) Similar to (1).

$\square$

Exercise 1.1.12. Let $f \in B([a, b])$. Then
(1) $U(-f) = -Lf$
(2) $L(-f) = -Uf$

Proof.

(1) Using a previous exercise, we have that

$$
\begin{aligned}
U(-f) &= \inf\{U_{\mathcal{P}}(-f) : \mathcal{P} \text{ is a partition of } [a, b]\} \\
&= \inf\{-L_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\} \\
&= -\sup\{L_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\} \\
&= -Lf
\end{aligned}
$$

(2) Similar to (1)

$\square$

Exercise 1.1.13. Let $f \in B([a, b])$ and $c \geq 0$. Then
(1) $U(cf) = cUf$
(2) $L(cf) = cLf$

Proof.

(1) Using a previous exercise, we have that

$$
\begin{aligned}
U(cf) &= \inf\{U_{\mathcal{P}}(cf) : \mathcal{P} \text{ is a partition of } [a, b]\} \\
&= \inf\{cU_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\} \\
&= c\inf\{U_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a, b]\} \\
&= cUf
\end{aligned}
$$

(2) Similar to (1)

$\square$

Definition 1.1.14. Let $f \in B([a, b])$. Then f is said to be **Darboux integrable** if $Uf = Lf$. If f is Darboux integrable, we define the **Darboux integral** of f, denoted by

$$
\int f \quad \text{or} \quad \int f(x)dx
$$

to be

$$
\int f = Uf = Lf
$$

The set of bounded, Darboux integrable functions is denoted by $D([a, b])$.

Exercise 1.1.15. Let $f \in B([a, b])$. Then $f \in D([a, b])$ iff for each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$.

Proof. Suppose that $f \in D([a,b])$. Let $\epsilon > 0$. Then there exist partions $\mathcal{P}_1$, $\mathcal{P}_2$ of $[a,b]$ such that $U_{\mathcal{P}_1} f < Uf + \epsilon/2$ and $L_{\mathcal{P}_2} f > Lf - \epsilon/2$. Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $U_{\mathcal{P}} f \leq U_{\mathcal{P}_1} f$ and $L_P f \geq L_{\mathcal{P}_2} f$. So

$$U_{\mathcal{P}} f - L_{\mathcal{P}} f < Uf - Lf + \epsilon$$
$$= \epsilon$$

Conversely, suppose that for each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}} f - L_{\mathcal{P}} f < \epsilon$. For the sake of contradiction, suppose that $Uf - Lf > 0$. Choose $\epsilon = Uf - Lf$. Then there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}} f - L_{\mathcal{P}} f < \epsilon$. Since $Uf \leq U_{\mathcal{P}} f$ and $Lf \geq L_{\mathcal{P}} f$, we have that

$$\epsilon > U_{\mathcal{P}} f - L_{\mathcal{P}} f$$
$$\geq Uf - Lf$$
$$= \epsilon$$

which is a contradiction. Hence $Uf = Lf$ and $f \in D([a,b])$. $\qquad\square$

Exercise 1.1.16. Let $f, g \in D([a,b])$. Then $f + g \in D([a,b])$ and

$$\int (f+g) = \int f + \int g$$

Proof. Clearly $f + g \in B([a,b])$. Using some previous results, we have that

$$\int f + \int g = Lf + Lg$$
$$\leq L(f+g)$$
$$\leq U(f+g)$$
$$\leq Uf + Ug$$
$$= \int f + \int g$$

So $U(f+g) = L(f+g) = \int f + \int g$. Therefore $f + g \in D([a,b])$ and

$$\int (f+g) = \int f + \int g$$

. $\qquad\square$

Exercise 1.1.17. Let $f \in D([a,b])$ and $c \in \mathbb{R}$. Then $cf \in D([a,b])$ and

$$\int (cf) = c \int f$$

Proof. Clearly $cf \in B([a,b])$. If $c \geq 0$, then

$$L(cf) = cLf$$
$$= c \int f$$
$$= cUf$$
$$= U(cf)$$

So

$$L(cf) = U(cf) = c\int f$$

If $c < 0$, then

$$
\begin{aligned}
L(cf) &= L(-|c|f) \\
&= -U(|c|f) \\
&= -|c|Uf \\
&= c\int f \\
&= -|c|Lf \\
&= -L(|c|f) \\
&= U(-|c|f) \\
&= U(cf)
\end{aligned}
$$

So

$$L(cf) = U(cf) = c\int f$$

Therefore $cf \in D([a, b])$ and

$$\int (cf) = c\int f$$

$\square$

Corollary 1.1.18. We have that $D([a, b])$ is a vector space and the map $I : D([a, b]) \to \mathbb{R}$ given by $If = \int f$ is linear.

Proof. Clear. $\square$

Exercise 1.1.19. Let $f : [a, b] \to \mathbb{R}$. If f is continuous, then $f \in D([a, b])$.

Proof. Suppose that f is continuous. Then f is uniformly continuous. Let $\epsilon > 0$. Uniform continuity implies that there exists $\delta > 0$ such that for each $x, y \in [a, b]$, $|x - y| < \delta$ implies that $|f(x) - f(Y)| < \epsilon/(b - a)$. Choose $n \in \mathbb{N}$ such that $(b - a)/n < \delta$. For $i \in \{0, \cdots, n\}$, define $x_i = a + i(b-a)/n$. Put $\mathcal{P} = \{x_0, \cdots, x_n\}$. Continuity implies that for each $i \in \{1, \cdots, n\}$, there exists $x_i^M, x_i^m \in [x_{i-1}, x_i]$ such that $f(x_i^M) = M_i^f$ and $f(x_i^m) = m_i^f$.

Then

$$
\begin{aligned}
U_{\mathcal{P}}f - L_{\mathcal{P}}f &= \sum_{i=1}^{n} M_i^f (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i^f (x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (M_i^f - m_i^f)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} [f(x_i^M) - f(x_i^m)](x_i - x_{i-1}) \\
&< \sum_{i=1}^{n} \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\
&= \epsilon
\end{aligned}
$$

So for each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$. Hence $f \in D([a,b])$. $\qquad\square$

Exercise 1.1.20. Let $f : [a,b] \to \mathbb{R}$. If f is monotonic, then $f \in D([a,b])$.

Proof. Suppose that f is increasing. Let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $(b-a)[f(b) - f(a)]/n < \epsilon$. For $i \in \{0, \cdots, n\}$, define $x_i = a + i(b-a)/n$. Put $\mathcal{P} = \{x_0, \cdots, x_n\}$. Then

$$
\begin{aligned}
U_{\mathcal{P}}f - L_{\mathcal{P}}f &= \sum_{i=1}^{n} M_i^f (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i^f (x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (M_i^f - m_i^f)(x_i - x_{i-1}) \\
&= \frac{b-a}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\
&= \frac{b-a}{n} [f(b) - f(a)] \\
&< \epsilon
\end{aligned}
$$

So for each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$. Hence $f \in D([a,b])$. The case is similar if f is decreasing. $\qquad\square$

Exercise 1.1.21. Define $\chi_{\mathbb{Q}} : [0,1] \to \mathbb{R}$ by

$$
\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}
$$

Then $\chi_{\mathbb{Q}} \notin D([a,b])$.

Proof. Let $\mathcal{P} = \{x_0, \cdots, x_n\}$ be a partition of $[0,1]$. Then for each $i \in \{1, \cdots, n\}$, $M_i^{\chi_{\mathbb{Q}}} = 1$ and $m_i^{\chi_{\mathbb{Q}}} = 0$. So $U_{\mathcal{P}}\chi_{\mathbb{Q}} = 1$ and $L_{\mathcal{P}}\chi_{\mathbb{Q}} = 0$. Since $\mathcal{P}$ is arbitrary, we have that $U\chi_{\mathbb{Q}} = 1$ and $L\chi_{\mathbb{Q}} = 0$ $\qquad\square$

2. Measures

2.1. Limits of Sets.

Definition 2.1.1. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. We define

$$\inf \mathcal{A} = \bigcap_{A \in \mathcal{A}} A, \quad \sup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

Definition 2.1.2. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. We define

$$\liminf_{n \to \infty} A_n = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} A_k \right), \quad \limsup_{n \to \infty} A_n = \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} A_k \right)$$

Note 2.1.3.
  (1) $\liminf\limits_{n \to \infty} A_n$ is the set of elements that are in all A_n except for finitely many.
  (2) $\limsup\limits_{n \to \infty} A_n$ is the set of elements that are in infinitely many A_n.

Exercise 2.1.4. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then

  (1) $\liminf\limits_{n \to \infty} A_n = \left\{ x \in X : \liminf\limits_{n \to \infty} \chi_{A_n}(x) = 1 \right\}$

  (2) $\limsup\limits_{n \to \infty} A_n = \left\{ x \in X : \limsup\limits_{n \to \infty} \chi_{A_n}(x) = 1 \right\}$

Proof.
  (1) Let $x \in \liminf\limits_{n \to \infty} A_n$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x \in A_k$. So for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\chi_{A_k}(x) = 1$. Then $\inf\limits_{k \geq n^*} \chi_{A_k}(x) = 1$ and thus

$$1 = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \chi_{A_k}(x) \right) = \liminf_{n \to \infty} \chi_{A_n}(x)$$

  Conversely, if $1 = \liminf\limits_{n \to \infty} \chi_{A_n}(x)$, then choosing $\epsilon = \frac{1}{2}$, there exists $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n$ implies that $\chi_{A_k}(x) > 1 - \epsilon$. Hence for each $k \in \mathbb{N}$, $k \geq n$ implies that $\chi_{A_k}(x) = 1$. So for each for each $k \in \mathbb{N}$, $k \geq n$ implies that $x \in A_k$. So $x \in \liminf\limits_{n \to \infty} A_n$.
  (2) Similar to (1).

$\square$

Exercise 2.1.5. Let $A_k = [0, \frac{k}{k+1})$. Then
  (1) $\inf\limits_{k \geq n} A_k = [0, \frac{n}{n+1})$

  (2) $\sup\limits_{k \geq n} A_k = [0, 1)$

  (3) $\liminf\limits_{n \to \infty} A_n = [0, 1)$

  (4) $\liminf\limits_{n \to \infty} A_n = [0, 1)$

Proof. Straightforward.   □

Exercise 2.1.6. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then

$$\liminf_{n \to \infty} A_n \subset \limsup_{n \to \infty} A_n$$

Proof. Let $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq n*$, then $x \in A_k$. Let $n \in \mathbb{N}$. Choose $k = \max\{n^*, n\} \geq n^*$. Then $x \in A_k$. Hence for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x \in A_k$. So $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Thus $\liminf_{n \to \infty} A_n \subset \limsup_{n \to \infty} A_n$.   □

Definition 2.1.7. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. If

$$\liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n$$

then we define

$$\lim_{n \to \infty} A_n = \liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n$$

Exercise 2.1.8. Let X be a set and $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Suppose that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$ and $B_{n+1} \subset B_n$. Then

(1) $\lim_{n \to \infty} A_n = \sup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$

(2) $\lim_{n \to \infty} B_n = \inf_{n \in \mathbb{N}} B_n = \bigcap_{n=1}^{\infty} B_n$

Proof.

(1) Let $n \in \mathbb{N}$. Then

$$\inf_{k \geq n} A_k = \bigcap_{k=n}^{\infty} A_k$$
$$= A_n$$

Thus

$$\liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \inf_{k \geq n} A_k$$
$$= \bigcup_{n=1}^{\infty} A_n$$

In addition,

$$\sup_{n \geq k} A_k = \bigcup_{k=n}^{\infty} A_k$$
$$= \bigcup_{k=1}^{\infty} A_k$$

Therefore

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \inf_{k\geq n} A_k$$
$$= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_k$$
$$= \bigcup_{n=1}^{\infty} A_n$$

So

$$\lim_{n\to\infty} A_n = \sup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$$

(2) Similar

$\square$

Exercise 2.1.9. Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets and $(A_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(A_n)_{n\in\mathbb{N}}$. Then
  (1) $\limsup\limits_{k\to\infty} A_{n_k} \subset \limsup\limits_{n\to\infty}(A_n)$
  (2) $\liminf\limits_{n\to\infty} A_n \subset \liminf\limits_{k\to\infty}(A_{n_k})$

Proof.
  (1) The elements that are in A_{n_k} for infinitely many k are in A_n for infinitely many n.
  (2) Similar.

$\square$

Exercise 2.1.10. Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets, $(A_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(A_n)_{n\in\mathbb{N}}$ and $A \subset X$. If $A_{n_k} \to A$, then
$$\liminf_{n\to\infty} A_n \subset A \subset \limsup_{n\to\infty} A_n$$

Proof. The previous exercises tells us that

$$\liminf_{n\to\infty} A_n \subset \liminf_{k\to\infty} A_{n_k}$$
$$= A$$
$$= \limsup_{k\to\infty} A_{n_k}$$
$$\subset \limsup_{n\to\infty} A_n$$

$\square$

Exercise 2.1.11. Let X be a set and $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Suppose that for each $n \in \mathbb{N}$, $A_n \subset B_n$. Then
  (1) $\limsup\limits_{n\to\infty} A_n \subset \limsup\limits_{n\to\infty} B_n$
  (2) $\liminf\limits_{n\to\infty} A_n \subset \liminf\limits_{n\to\infty} B_n$

Proof.

(1) Let $x \in \limsup\limits_{n\to\infty} A_n$. Then for infinitely many $n \in \mathbb{N}$, $x \in A_n \subset B_n$. So for infinitely many $n \in \mathbb{N}$, $x \in B_n$. Hence $x \in \limsup\limits_{n\to\infty} B_n$. Therefore $\limsup\limits_{n\to\infty} A_n \subset \limsup\limits_{n\to\infty} B_n$.

(2) Similar.

$\square$

Exercise 2.1.12. Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then

(1) $\limsup\limits_{n\to\infty} A_n = \left(\liminf\limits_{n\to\infty} A_n^c \right)^c$

(2) $\liminf\limits_{n\to\infty} A_n = \left(\limsup\limits_{n\to\infty} A_n^c \right)^c$

Proof.

(1)

$$\left(\liminf\limits_{n\to\infty} A_n^c \right)^c = \left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c \right)^c$$
$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$
$$= \limsup\limits_{n\to\infty} A_n$$

(2) Similar.

$\square$

Exercise 2.1.13. For $n \in \mathbb{N}$, define

$$A_n = \left\{ \frac{m}{n} : m \in \mathbb{N} \right\}$$

Then

(1) $\liminf\limits_{n\to\infty} A_n = \mathbb{N}$

(2) $\limsup\limits_{n\to\infty} A_n = \mathbb{Q} \cap (0, \infty)$

Proof.

(1) For each $x \in \mathbb{N}$ and $n \in \mathbb{N}$, $x = \frac{nx}{n} \in A_n$ Hence $\mathbb{N} \subset \liminf\limits_{n\to\infty} A_n$. Conversely, let $x \in \liminf\limits_{n\to\infty} A_n$. Then there exists $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq n$, then $x \in A_k$. In particular, $x \in A_n$. Hence there exists $m_n \in \mathbb{N}$ such that $x = \frac{m_n}{n}$. Choose $s, t \in \mathbb{N}$ such that $x = \frac{s}{t}$ and $\gcd(s, t) = 1$. Choose a prime $p > n$. By assumption, $x \in A_p$. Then there exist $m_p \in \mathbb{N}$ such that $x = \frac{m_p}{p}$. Hence $\frac{s}{t} = \frac{m_p}{p}$ and $tm_p = sp$. Since $t|sp$ and $\gcd(s, t) = 1$, we see that $t|p$. If $t > 1$, then p is not prime, which is a contradiction. So $t = 1$. Hence $x \in \mathbb{N}$. Thus $\liminf\limits_{n\to\infty} A_n \subset \mathbb{N}$.

(2) Let $x \in \mathbb{Q} \cap (0, \infty)$. Then there exist $s, t \in \mathbb{N}$ such that $x = \frac{s}{t}$. Define the subsequence $(A_{n_k})_{k \in \mathbb{N}}$ by $A_{n_k} = A_{tk}$. Then for each $k \in \mathbb{N}$, $x = \frac{sk}{tk} \in A_{tk} = A_{n_k}$. Thus

$$x \in \inf_{k \in \mathbb{N}} A_{n_k}$$
$$\subset \liminf_{n \to \infty} A_{n_k}$$
$$\subset \limsup_{n \to \infty} A_{n_k}$$
$$\subset \limsup_{n \to \infty} A_n$$

Conversely, clearly $\limsup_{n \to \infty} A_n \subset \mathbb{Q} \cap (0, \infty)$

$\square$

Exercise 2.1.14. Let X be a set and $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Then

$$\limsup_{n \to \infty} A_n \cup B_n = \limsup_{n \to \infty} A_n \cup \limsup_{n \to \infty} B_n$$

Proof. Let $x \in \limsup_{n \to \infty} A_n \cup B_n$. Suppose that $x \notin \limsup_{n \to \infty} A_n$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ if $k \geq n^*$, then $x \notin A_k$. Let $n \in \mathbb{N}$. Then there exists k such that $k \geq \max\{n, n^*\}$ and $x \in A_k \cup B_k$. Since $k \geq n^*$, $x \notin A_k$ Thus $x \in B_k$. So for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x \in B_k$. Therefore $x \in \limsup_{n \to \infty} B_n$ and

$$\limsup_{n \to \infty} A_n \cup B_n \subset \limsup_{n \to \infty} A_n \cup \limsup_{n \to \infty} B_n$$

Conversely, a previous exercise tells us that $\limsup_{n \to \infty} A_n \subset \limsup_{n \to \infty} A_n \cup B_n$ and $\limsup_{n \to \infty} B_n \subset \limsup_{n \to \infty} A_n \cup B_n$. Thus

$$\limsup_{n \to \infty} A_n \cup \limsup_{n \to \infty} B_n \subset \limsup_{n \to \infty} A_n \cup B_n$$

$\square$

Exercise 2.1.15. Let X be a set and $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Then

$$\liminf_{n \to \infty} A_n \cap B_n = \liminf_{n \to \infty} A_n \cap \liminf_{n \to \infty} B_n$$

Proof. A previous exercise tells us that

$$\liminf_{n \to \infty} A_n \cap B_n = \left(\limsup_{n \to \infty} A_n^c \cup B_n^c \right)^c$$
$$= \left(\limsup_{n \to \infty} A_n^c \cup \limsup_{n \to \infty} B_n^c \right)^c$$
$$= \left(\limsup_{n \to \infty} A_n^c \right)^c \cap \left(\limsup_{n \to \infty} B_n^c \right)^c$$
$$= \liminf_{n \to \infty} A_n \cap \liminf_{n \to \infty} B_n$$

$\square$

2.2. **Classes of sets.**

Definition 2.2.1. Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$. Then X is said to be an **elementary family on X** if

(1) $\varnothing \in \mathcal{E}$
(2) for each $A, B \in \mathcal{E}$, $A \cap B \in \mathcal{E}$
(3) for each $A \in \mathcal{E}$, there exist $(A_j)_{j=1}^n \subset \mathcal{E}$ such that $(A_j)_{j=1}^n$ is disjoint and $A^c = \bigcup_{j=1}^n A_j$

Exercise 2.2.2. Define

$$\mathcal{E} = \{(a, b] : a, b \in \overline{\mathbb{R}}\}$$

where we take $(a, \infty] = (a, \infty)$. Then $\mathcal{E}$ is an elementary family on $\mathbb{R}$

Proof.

(1) $\varnothing = (0, 0] \in \mathcal{E}$
(2) Let $a_1, a_2, b_1, b_2 \in \overline{\mathbb{R}}$. Then

$$(a_1, b_1] \cap (a_2, b_2] = \begin{cases} \varnothing & b_1 \leq a_2 \\ (a_2, b_1] & b_1 > a_2 \end{cases}$$

So $(a_1, b_1] \cap (a_2, b_2] \in \mathcal{E}$.
(3) Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Then $(a, b]^c = (-\infty, a] \cup (b, \infty) \in \mathcal{E}$.

$\square$

Definition 2.2.3. Let X be a set and $\mathcal{A}_0 \subset \mathcal{P}(X)$. Then $\mathcal{A}_0$ is said to be an **algebra** on X if

(1) $\mathcal{A}_0 \neq \varnothing$
(2) for each $A \in \mathcal{A}_0$, $A^c \in \mathcal{A}_0$
(3) for each $A, B \in \mathcal{A}_0$, $A \cup B \in \mathcal{A}_0$

Exercise 2.2.4. Let X be a set and $\mathcal{E}$ an elementary family on X. Define

$$\mathcal{A}_0^{\mathcal{E}} = \left\{ \bigcup_{j=1}^n A_j : (A_j)_{j=1}^n \text{ is disjoint and } (A_j)_{j=1}^n \subset \mathcal{E} \right\}$$

Then $\mathcal{A}_0^{\mathcal{E}}$ is an algebra on X.

Proof.

(1) By definition, $\varnothing \in \mathcal{E} \subset \mathcal{A}_0^{\mathcal{E}}$. So $\mathcal{A}_0^{\mathcal{E}} \neq \varnothing$.
(2) Let $A \in \mathcal{A}_0^{\mathcal{E}}$, there exists $(A_j)_{j=1}^n \subset \mathcal{E}$ such that $(A_j)_{j=1}^n$ is disjoint and $A = \bigcup_{j=1}^n A_j$. By definition of $\mathcal{E}$, for each $j \in \{1, \dots, n\}$, there exist $(B_{j,k})_{k=1}^{n_j} \subset \mathcal{E}$ such that $(B_{j,k})_{k=1}^{n_j}$

is disjoint and $A_j^c = \bigcup\limits_{k=1}^{n_j} B_{j,k}$. Then

$$A^c = \bigcap_{j=1}^{n} A_j^c$$

$$= \bigcap_{j=1}^{n} \left(\bigcup_{k=1}^{n_j} B_{j,k} \right)$$

$$= \bigcup$$

(3) Let $A, B \in \mathcal{A}_0^{\mathcal{E}}$. Then there exist $(A_j)_{j=1}^{n}, (B_j)_{j=1}^{m} \subset \mathcal{E}$ such that $A = \bigcup\limits_{j=1}^{n} A_j$ and $B = \bigcup\limits_{j=1}^{m} B_j$. Then

$$A \cup B = \left(\bigcup_{j=1}^{n} A_j \right) \cup \left(\bigcup_{j=1}^{m} B_j \right)$$

FINISH!!! □

Definition 2.2.5. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. Then $\mathcal{A}$ is said to be a σ**-algebra** on X if

(1) $\mathcal{A} \neq \varnothing$
(2) for each $A \in \mathcal{A}$, $A^c \in \mathcal{A}$
(3) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $\bigcup\limits_{n \in \mathbb{N}} A_n \in \mathcal{A}$

Exercise 2.2.6. Let X be a set and $\mathcal{A}$ a σ-algebra on X. Then

(1) $X, \varnothing \in \mathcal{A}$
(2) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $\bigcap\limits_{n \in \mathbb{N}} \in \mathcal{A}$
(3) For each $A, B \in \mathcal{A}$, $A \setminus B \in \mathcal{A}$

Proof.

(1) Since $\mathcal{A} \neq \varnothing$, there exists $A \in \mathcal{A}$. Then $A^c \in \mathcal{A}$. Hence $X = A \cup A^c \in \mathcal{A}$ and $\varnothing = X^c \in \mathcal{A}$.
(2) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Then $(A_n^c)_{n \in \mathbb{N}} \subset MA$. So $\bigcup\limits_{n \in \mathbb{N}} A_n^c \in \mathcal{A}$. Therefore

$$\bigcap_{n \in \mathbb{N}} A_n = \left(\bigcup_{n \in \mathbb{N}} A_n^c \right)^c \in \mathcal{A}$$

(3) Let $A, B \in \mathcal{A}$. Then $A \setminus B = A \cap B^c \in \mathcal{A}$.

□

Exercise 2.2.7. Let X be a set and $(\mathcal{A}_i)_{i \in I}$ a collection of σ-algebras (resp. algebra) on X. Then $\bigcap\limits_{i \in I} \mathcal{A}_i$ is a σ-algebra (resp. algebra) on X.

Proof.

(1) For each $i \in I$, $X \in \mathcal{A}_i$. Thus $X \in \bigcap_{i \in I} \mathcal{A}_i$ and $\bigcap_{i \in I} \mathcal{A}_i \neq \varnothing$.

(2) Let $A \in \bigcap_{i \in I} \mathcal{A}_i$. Then for each $i \in I$, $A \in \mathcal{A}_i$. Hence for each $i \in I$, $A^c \in \mathcal{A}_i$. Thus $A^c \in \bigcap_{i \in I} \mathcal{A}_i$.

(3) Let $(A_n)_{n \in \mathbb{N}} \subset \bigcap_{i \in I} \mathcal{A}_i$. Then for each $i \in I$, $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}_i$. Thus for each $i \in I$, $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_i$. So $\bigcup_{n \in \mathbb{N}} A_n \in \bigcap_{i \in I} \mathcal{A}_i$.

$\square$

Definition 2.2.8. Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$. Put

$$\mathcal{S} = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the **σ-algebra generated by** $\mathcal{C}$ on X, $\sigma(\mathcal{C})$, by

$$\sigma(\mathcal{C}) = \bigcap_{\mathcal{A} \in \mathcal{S}} \mathcal{A}$$

Note 2.2.9. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $\mathcal{A}$ a σ-alg on X. By definition, if $\mathcal{C} \subset \mathcal{A}$, then $\sigma(\mathcal{C}) \subset \mathcal{A}$.

Note 2.2.10. Let X be a set, $\mathcal{T}$ an ordered set and $(\mathcal{A}_t)_{t \in \mathcal{T}}$ a collection of σ-algebras on X. Suppose that for each $s, t \in \mathcal{T}$, if $s \leq t$, then $\mathcal{A}_s \subset \mathcal{A}_t$. If there exists $t \in \mathcal{T}$ such that $\mathcal{A}_t = \bigcup_{t \in \mathcal{T}} \mathcal{A}_t$, then $\bigcup_{t \in \mathcal{T}} \mathcal{A}_t$ is a σ-algebra on X. So if $\mathcal{T}$ is finite or if $(\mathcal{A}_t)_{t \in \mathcal{T}}$ termintates, the union is σ-algebra.

Definition 2.2.11. Let $(X, \mathcal{T})$ be a topological space. We define the **Borel σ-algebra** on X, $\mathcal{B}(X)$, to be

$$\mathcal{B}(X) = \sigma(\mathcal{T})$$

The sets of $\mathcal{B}(X)$ are called **Borel sets**.

Exercise 2.2.12. The Borel σ-algebra on $\mathbb{R}$ with the standard topology is given by

$$\mathcal{B}(\mathbb{R}) = \begin{cases} \sigma(\{(a, b] : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{[a, b] : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{[a, b) : a, b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{(a, b) : a, b \in \mathbb{R} \text{ and } a < b\}) \end{cases}$$

Proof. Define

(1) $\mathcal{C}_{lo} = \{(a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$

(2) $\mathcal{C}_c = \{[a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$

(3) $\mathcal{C}_{ro} = \{[a, b) : a, b \in \mathbb{R} \text{ and } a < b\}$

(4) $\mathcal{C}_o = \{(a, b) : a, b \in \mathbb{R} \text{ and } a < b\}$

Recall that for each open set $A \subset \mathbb{R}$, there exist $(a_i)_{n \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}} \subset \mathbb{R}$ such that for each $i \in \mathbb{N}$, $a_i < b_i$, for each $i, j \in \mathbb{N}$, if $i \neq j$, then $(a_i, b_i) \cap (a_j, b_j) = \varnothing$ and $A = \bigcup_{i \in \mathbb{N}} (a_i, b_i)$. This implies that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o)$.

Now, let $a, b \in \mathbb{R}$. Suppose that $a < b$. Then

(1) $[a, b] = \bigcap_{n \in \mathbb{N}} (a - \frac{1}{n}, b]$, so $\sigma(\mathcal{C}_c) \subset \sigma(\mathcal{C}_{lo})$

(2) $[a, b) = \bigcup_{n \in \mathbb{N}} [a, b - \frac{1}{n}]$, so $\sigma(\mathcal{C}_{ro}) \subset \sigma(\mathcal{C}_c)$

(3) $(a, b) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, b)$, so $\sigma(\mathcal{C}_o) \subset \sigma(\mathcal{C}_{ro})$

(4) $(a, b] = \bigcap_{n \in \mathbb{N}} (a, b + \frac{1}{n})$, so $\sigma(\mathcal{C}_{lo}) \subset \sigma(\mathcal{C}_o)$

Hence $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o) = \sigma(\mathcal{C}_{ro}) = \sigma(\mathcal{C}_c) = \sigma(\mathcal{C}_{lo}) = \sigma(\mathcal{C}_o)$. $\qquad \square$

Exercise 2.2.13. Let X be a set. Define $\mathcal{A} = \{A \in \mathcal{A} : A \text{ is countable or } A^c \text{is countable}\}$. Then $\mathcal{A}$ is a σ-algebra on X.

Proof.

(1) Since $X^c = \varnothing$ is countable, $X \in \mathcal{A}$.
(2) Let $A \in \mathcal{A}$. Suppose that A^c is uncountable. Then by assumption, $A = (A^c)^c$ is countable. Hence $A^c \in \mathcal{A}$.
(3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Then for each $n \in \mathbb{N}$, A_n is countable or A_n^c is countable. Suppose that $\bigcup_{n \in \mathbb{N}} A_n$ is uncountable. Then there exists $N \in \mathbb{N}$ such that A_N is uncountable. Hence A_N^c is countable. Thus

$$\left(\bigcup_{n \in \mathbb{N}} A_n \right)^c = \bigcap_{n \in \mathbb{N}} A_n^c$$
$$\subset A_N^c$$

So $\left(\bigcup_{n \in \mathbb{N}} A_n \right)^c$ is countable and $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

$\qquad \square$

Definition 2.2.14. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $E \subset X$. We define $\mathcal{C} \cap E \subset \mathcal{P}(X)$ by

$$\mathcal{C} \cap E = \{S \cap E : S \in \mathcal{C}\}$$

Exercise 2.2.15. Let X be a set, $\mathcal{A}$ a σ-algebra on X and $E \subset X$. Then $\mathcal{A} \cap E$ is a σ-algebra on E.

Proof.

(1) Clearly $\varnothing, E \in \mathcal{A} \cap E$.

(2) Let $B \in \mathcal{A} \cap E$. Then there exists $A \in \mathcal{A}$ such that $B = A \cap E$. Since $A^c \in \mathcal{A}$, we have that

$$
\begin{aligned}
E \setminus B &= E \cap (A \cap E)^c \\
&= E \cap (A^c \cup E^c) \\
&= (E \cap A^c) \cup (E \cap E^c) \\
&= A^c \cap E \\
&\in \mathcal{A} \cap E
\end{aligned}
$$

(3) Let $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A} \cap E$. Then for each $n \in \mathbb{N}$, there exists $A_n \in \mathcal{A}$ such that $B_n = A_n \cap A$. So $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$. Hence

$$
\begin{aligned}
\bigcup_{n \in \mathbb{N}} (B_n) &= \bigcup_{n \in \mathbb{N}} (A_n \cap E) \\
&= \left(\bigcup_{n \in \mathbb{N}} A_n \right) \cap E \\
&\in \mathcal{A} \cap E
\end{aligned}
$$

$\square$

Exercise 2.2.16. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. Let $\sigma_A(\mathcal{C} \cap A)$ be the σ-algebra on A generated by $\mathcal{C} \cap A$. Define

$$
\mathcal{G} = \{S \subset X : S \cap A \in \sigma_A(\mathcal{C} \cap A)\}
$$

Then $\mathcal{G}$ is a σ-algebra on X.

Proof.
(1) Clearly $\varnothing, X \in \mathcal{G}$.
(2) Let $S \in \mathcal{G}$. Then $S \cap A \in \sigma_A(\mathcal{C} \cap A)$. Hence

$$
\begin{aligned}
S^c \cap A &= A \setminus (S \cap A) \\
&\in \sigma_A(\mathcal{C} \cap A)
\end{aligned}
$$

So $S^c \in \mathcal{G}$.
(3) Let $(S_n)_{n \in \mathbb{N}} \subset \mathcal{G}$. Then for each $n \in \mathbb{N}$, $S_n \cap A \in \sigma_A(\mathcal{C} \cap A)$. Thus

$$
\left(\bigcup_{n \in \mathbb{N}} S_n \right) \cap A = \bigcup_{n \in \mathbb{N}} (S_n \cap A) \in \sigma_A(\mathcal{C} \cap A)
$$

Thus $\bigcup_{n \in \mathbb{N}} S_n \in \mathcal{G}$.

$\square$

Exercise 2.2.17. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. Then

$$
\sigma(\mathcal{C}) \cap A = \sigma_A(\mathcal{C} \cap A)
$$

Proof. Clearly $\mathcal{C} \cap A \subset \sigma(\mathcal{C}) \cap A$. A previous exercise tells us that $\sigma(\mathcal{C}) \cap A$ is a σ-algebra on A. Thus $\sigma_A(\mathcal{C} \cap A) \subset \sigma(\mathcal{C}) \cap A$.

Conversely, from the previous exercise, we have that $\mathcal{G} = \{S \subset X : S \cap A \in \sigma_A(\mathcal{C} \cap A)\}$

is a σ-algebra on X. Clearly $\mathcal{C} \subset \mathcal{G}$. Then $\sigma(\mathcal{C}) \subset \mathcal{G}$. The definition of $\mathcal{G}$ implies that $\sigma(\mathcal{C}) \cap A \subset \sigma_A(\mathcal{C} \cap A)$. Hence $\sigma(\mathcal{C}) \cap A = \sigma_A(\mathcal{C} \cap A)$. $\square$

Definition 2.2.18. Let X be a set and $\mathcal{A}$ be a σ-algebra on X. Then $(X, \mathcal{A})$ is called a **measurable space**.

2.3. **Measures.**

Definition 2.3.1. Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [0, \infty]$. Then μ is said to be a **measure** on $(X, \mathcal{A})$ if
  (1) there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$
  (2) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. If $(A_n)_{n \in \mathbb{N}}$ is disjoint, then
$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n \right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

Definition 2.3.2. Let $(X, \mathcal{A})$ be a measurable space and μ a measure on $(A, \mathcal{A})$. Then $(A, \mathcal{A}, \mu)$ is called a **measure space**.

Exercise 2.3.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, A and index set and $(E_\alpha)_{\alpha \in A} \subset \mathcal{A}$. Suppose that $\mu(X) < \infty$ and $(E_\alpha)_{\alpha \in A}$ is disjoint. Then $\{\alpha \in A : \mu(E_\alpha) > 0\}$ is countable. **Hint:** set $A_n = \{\alpha \in A : \mu(E_\alpha) \geq 1/n\}$

Proof. For $n \in \mathbb{N}$, set $A_n = \{\alpha \in A : \mu(E_\alpha) \geq 1/n\}$ and define $A_> = \{\alpha \in A : \mu(E_\alpha) > 0\}$. Then
$$A_> = \bigcup_{n \in \mathbb{N}} A_n$$
For the sake of contradiction, suppose that $A_>$ is uncountable. Then there exists $N \in \mathbb{N}$ such that A_N is uncountable. So there exists a sequence $(\alpha_j)_{j \in \mathbb{N}} \subset A_N$. Then
$$\infty > \mu(X)$$
$$\geq \mu\left(\bigcup_{j \in \mathbb{N}} E_{\alpha_j} \right)$$
$$= \sum_{j \in \mathbb{N}} \mu(E_{\alpha_j})$$
$$\geq \sum_{j \in \mathbb{N}} \frac{1}{N}$$
$$= \infty$$
which is a contradiction. So $A_>$ is countable. $\qquad\square$

Exercise 2.3.4. Let $(X, \mathcal{A}, mu)$ be a measure space. Then
  (1) (monotonicity): for each $A, B \in \mathcal{A}$, if $A \subset B$, then $\mu(A) \leq \mu(B)$.
  (2) (subadditivity): for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$,
$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$$
  (3) (continuity from below): for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, if for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$, then
$$\mu\left(\sup_{n \in \mathbb{N}} A_n \right) = \sup_{n \in \mathbb{N}} \mu(A_n)$$
  (4) (continuity from above): for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, if for each $n \in \mathbb{N}$, $A_{n+1} \subset A_n$ and $\mu(A_1) < \infty$, then
$$\mu\left(\inf_{n \in \mathbb{N}} A_n \right) = \inf_{n \in \mathbb{N}} \mu(A_n)$$

Proof.

(1) Let $A, B \in \mathcal{A}$. Suppose that $A \subset B$. Then

$$\mu(B) = \mu\left((B \cap A) \cup (B \cap A^c) \right)$$
$$= \mu(B \cap A) + \mu(B \cap A^c)$$
$$= \mu(A) + \mu(B \cap A^c)$$
$$\geq \mu(A)$$

(2) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Define $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k \right)$. Then $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n$, $(B_n)_{n \in \mathbb{N}}$ disjoint and for each $n \in \mathbb{N}$, $B_n \subset A_n$. Thus

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n \right) = \mu\left(\bigcup_{n \in \mathbb{N}} B_n \right)$$
$$= \sum_{n \in \mathbb{N}} \mu(B_n)$$
$$\leq \sum_{n \in \mathbb{N}} \mu(A_n)$$

(3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$. Then for each $n \in \mathbb{N}$, $\mu(A_n) \leq \mu(A_{n+1})$ and $\lim_{n \to \infty} \mu(A_n) = \sup_{n \in \mathbb{N}} \mu(A_n)$. Recall that $\sup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} A_n$. Define $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus A_{n-1}$. Then $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $(B_n)_{n \in \mathbb{N}}$ is disjoint, $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$ and for each $n \in \mathbb{N}$, $\bigcup_{n=1}^{k} B_n = A_k$. Then

$$\mu\left(\sup_{n \in \mathbb{N}} A_n \right) = \mu\left(\bigcup_{n \in \mathbb{N}} A_n \right)$$
$$= \mu\left(\bigcup_{n \in \mathbb{N}} B_n \right)$$
$$= \sum_{n \in \mathbb{N}} \mu(B_n)$$
$$= \lim_{k \to \infty} \sum_{n=1}^{k} \mu(B_n)$$
$$= \lim_{k \to \infty} \mu\left(\bigcup_{n=1}^{k} B_n \right)$$
$$= \lim_{k \to \infty} \mu(A_k)$$
$$= \sup_{n \in \mathbb{N}} \mu(A_n)$$

(4) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that for each $n \in \mathbb{N}$, $A_{n+1} \subset A_n$ and $\mu(A_1) < \infty$. Then for each $n \in \mathbb{N}$ $\mu(A_{n+1}) \leq \mu(A_n) \leq \mu(A_1) < \infty$ and the arithmetic that follows is

well defined. Recall that $\inf_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} A_n$. For each $n \in \mathbb{N}$, define $B_n = A_1 \cap A_n$. Then for each $n \in \mathbb{N}$, $B_n \subset B_{n+1}$ and

$$
\begin{aligned}
\sup_{n\in\mathbb{N}} B_n &= \bigcup_{n\in\mathbb{N}} B_n \\
&= A_1 \setminus \bigcap_{n\in\mathbb{N}} A_n \\
&= A_1 \setminus \inf_{n\in\mathbb{N}} A_n
\end{aligned}
$$

So (3) implies that

$$
\begin{aligned}
\sup_{n\in\mathbb{N}} \mu(B_n) &= \mu\left(\sup_{n\in\mathbb{N}} B_n \right) \\
&= \mu\left(A_1 \setminus \inf_{n\in\mathbb{N}} A_n \right) \\
&= \mu(A_1) - \mu\left(\inf_{n\in\mathbb{N}} A_n \right)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sup_{n\in\mathbb{N}} \mu(B_n) &= \sup_{n\in\mathbb{N}} \mu(A_1 \setminus A_n) \\
&= \sup_{n\in\mathbb{N}} \left[\mu(A_1) - \mu(A_n) \right] \\
&= \mu(A_1) - \inf_{n\in\mathbb{N}} \mu(A_n)
\end{aligned}
$$

Therefore

$$
\mu\left(\inf_{n\in\mathbb{N}} A_n \right) = \inf_{n\in\mathbb{N}} \mu(A_n)
$$

$\square$

Exercise 2.3.5. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$. Then

(1) $\mu\left(\liminf_{n\to\infty} A_n \right) \leq \liminf_{n\to\infty} \mu(A_n)$

(2) If $\mu\left(\sup_{n\in\mathbb{N}} A_n \right) < \infty$, then $\limsup_{n\to\infty} \mu(A_n) \leq \mu\left(\limsup_{n\to\infty} A_n \right)$

Proof.

(1) Since $\left(\inf_{k \geq n} A_k \right)_{n \in \mathbb{N}}$ is an increasing sequence and for each $n \in \mathbb{N}$ $\inf_{k \geq n} A_k \subset A_n$, we have that

$$\mu\left(\liminf_{n \to \infty} A_n \right) = \mu\left[\sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} A_k \right) \right]$$

$$= \sup_{n \in \mathbb{N}} \mu\left(\inf_{k \geq n} A_k \right)$$

$$= \liminf_{n \to \infty} \mu\left(\inf_{k \geq n} A_k \right)$$

$$\leq \liminf_{n \to \infty} \mu(A_n)$$

(2) Since $\mu\left(\sup_{\geq 1} A_k \right) < \infty$, $\left(\sup_{k \geq n} \right)_{n \in \mathbb{N}}$ is a decreasing and for each $n \in \mathbb{N}$, $A_n \subset \sup_{k \geq n} A_n$, we have that

$$\mu\left(\limsup_{n \to \infty} A_n \right) = \mu\left[\inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} A_k \right) \right]$$

$$= \inf_{n \in \mathbb{N}} \mu\left(\sup_{k \geq n} A_k \right)$$

$$= \limsup_{n \to \infty} \mu\left(\sup_{k \geq n} A_k \right)$$

$$\geq \limsup_{n \to \infty} \mu(A_n)$$

$\qquad\qquad\square$

Exercise 2.3.6. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$. Suppose that $\mu\left(\sup_{n \in \mathbb{N}} A_n \right) < \infty$. Then $A_n \to A$ implies that $\mu(A_n) \to \mu(A)$.

Proof. Suppose that $A_n \to A$. Then the previous exercise tells us that

$$\mu(A) = \mu\left(\liminf_{n \to \infty} A_n \right)$$

$$\leq \liminf_{n \to \infty} \mu(A_n)$$

$$\leq \limsup_{n \to \infty} \mu(A_n)$$

$$\leq \mu(\limsup_{n \to \infty} A_n)$$

$$= \mu(A)$$

Thus $\mu(A) = \limsup_{n \to \infty} \mu(A_n) = \liminf_{n \to \infty} \mu(A_n)$ and $\mu(A_n) \to \mu(A)$ $\qquad\square$

2.4. **Outer Measures.**

Definition 2.4.1. Let X be a set and $\mu* : \mathcal{P}(X) \to [0, \infty]$. Then μ^* is said to be an **outer measure on X** if

(1) $\mu^*(\varnothing) = 0$
(2) for each $A, B \subset X$, if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$.
(3) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$,

$$\mu^*\Big(\bigcup_{n \in \mathbb{N}} A_n \Big) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n)$$

Theorem 2.4.2. Construction of Outer Measures:
Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$. Suppose that $\varnothing, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \rho(E_n) : (E_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } A \subset \bigcup_{n \in \mathbb{N}} E_n \right\}$$

Then μ^* is an outer measure on X.

Note 2.4.3. In particular, for each $A \in \mathcal{E}$, $\mu^*(A) = \rho(A)$.

Definition 2.4.4. Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$. Suppose that $\varnothing, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. Let μ^* be the outer measure on X defined as in the last theorem. Then μ^* is called the **outer measure on X induced by ρ**.

Definition 2.4.5. Let X be a set, μ^* an outer measure on X and $A \subset X$. Then A is said to be μ^*-**outer measurable** if for each $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Theorem 2.4.6. Let X be a set and μ^* an outer measure on X. Define $\mathcal{A} = \{A \subset X : A \text{ is } \mu^*\text{-measurable}\}$. Then $\mathcal{A}$ is a σ-algebra on X and $\mu^*|_{\mathcal{A}}$ is a complete measure on $(X, \mathcal{A})$.

Definition 2.4.7. Let X be a set, $\mathcal{A}_0$ be an algebra on X and $\mu_0 : \mathcal{A}_0 \to [0, \infty]$. Then μ_0 is said to be a **premeasure on** $(X, \mathcal{A}_0)$ if

(1) there exists $A \in \mathcal{A}_0$ such that $\mu_0(A) < \infty$
(2) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}_0$, if $(A_n)_{n \in \mathbb{N}}$ is disjoint and $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_0$, then

$$\mu_0\Big(\bigcup_{n \in \mathbb{N}} A_n \Big) = \sum_{n \in \mathbb{N}} \mu_0(A_n)$$

Note 2.4.8. The same reasoning applied to measures shows that $\mu_0(\varnothing) = 0$.

Theorem 2.4.9. Let X be a set, $\mathcal{A}_0$ an algebra on X, μ_0 a premeasure on $(X, \mathcal{A}_0)$ and μ^* the outer measure on X induced by μ_0. Put $\mathcal{A} = \sigma(\mathcal{A}_0)$. If μ_0 is σ-finite, then there exists a unique measure μ on $(X, \mathcal{A})$ such that $\mu|_{\mathcal{A}_0} = \mu^*|_{\mathcal{A}_0} = \mu_0$.

2.5. **Product Measures.**

Definition 2.5.1. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measurable spaces. Put $\mathcal{E} = \{A \times B : A \in \mathcal{A}$ and $B \in \mathcal{B}\}$. Then $\mathcal{E}$ is an elementary family and thus $\mathcal{M}_0 = \{\bigcup_{i=1}^{n} M_i : (M_i)_{i=1}^{n} \subset \mathcal{E}$ are disjoint$\}$ is an algebra on $X \times Y$. We define $\pi_0 : \mathcal{M}_0 \to [0, \infty]$ by

$$\pi_0\left(\bigcup_{i=1}^{n} A_i \times B_i\right) = \sum_{i=1}^{n} \mu(A_i)\nu(B_i)$$

Then π_0 is a premeasure on $(X \times Y, M_0)$. Since $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{M}_0)$, we define the **product measure**, $\mu \times \nu$ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$, to be the unique extension of π_0 to $\mathcal{A} \otimes \mathcal{B}$. The existence of which is guaranteed by a theorem in the previous section. In particular,

$$\mu \times \nu(E) = \inf\left\{\sum_{n \in \mathbb{N}} \pi_0(E_i) : (E_i)_{i \in \mathbb{N}} \subset \mathcal{M}_0 \text{ and } E \subset \bigcup_{i \in \mathbb{N}} E_i\right\}$$

$$= \inf\left\{\sum_{n \in \mathbb{N}} \mu(A_i)\nu(B_i) : (A_i \times B_i)_{i \in \mathbb{N}} \subset \mathcal{E} \text{ and } E \subset \bigcup_{i \in \mathbb{N}} A_i \times B_i\right\}$$

3. The Lebesgue Integral

3.1. Measurable Functions.

Definition 3.1.1. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then f is said to be **$\mathcal{A}$-$\mathcal{B}$ measurable** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. When $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say that f is **$\mathcal{A}$-measurbale**. If $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}, \mathcal{L})$, then we say that f is **Borel measurable** or **Lebsgue measurbale** respectively.

Definition 3.1.2. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. We define the

(1) **push-forward of $\mathcal{A}$**, denoted $f_*\mathcal{A}$, by
$$f_*\mathcal{A} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$$

(2) **pull-back of $\mathcal{B}$**, denoted $f^*\mathcal{B}$, by
$$f^*\mathcal{B} = \{f^{-1}(B) : B \in \mathcal{B}\}$$

Note 3.1.3. It is also common to write $\sigma(f)$ or $f^{-1}(\mathcal{B})$ in place of $f^*\mathcal{B}$.

Exercise 3.1.4. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then

(1) $f_*\mathcal{A}$ is a σ-algebra on Y
(2) $f^*\mathcal{B}$ is a σ-algebra on X

Proof.

(1) • Since $f^{-1}(Y) = X \in \mathcal{A}$, $Y \in f_*\mathcal{A}$ and $f_*\mathcal{A} \neq \varnothing$.
   • Let $B \in f_*\mathcal{A}$. Then $f^{-1}(B) \in \mathcal{A}$. Hence
$$f^{-1}(B^c) = (f^{-1}(B))^c \in \mathcal{A}$$
   Thus $B^c \in f_*\mathcal{A}$.
   • Now, let $(B_n)_{n \in \mathbb{N}} \subset f_*\mathcal{A}$. Then for each $n \in \mathbb{N}$, $f^{-1}(B_n) \in \mathcal{A}$. Thus
$$f^{-1}\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \bigcup_{n \in \mathbb{N}} f^{-1}(B_n) \in \mathcal{A}$$
   Hence $\bigcup_{n \in \mathbb{N}} B_n \in f_*\mathcal{A}$.
(2) Similar to (1).

$\square$

Exercise 3.1.5. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Suppose that there exists $\mathcal{E} \subset Y$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Let $f : X \to Y$. Then f is $\mathcal{A}$-$\mathcal{B}$ measurable iff for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$.

Proof. By definition, if f is $\mathcal{A}$-$\mathcal{B}$ measurable, then for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. Conversely, suppose that for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. The previous exercise tells us that $f_*\mathcal{A}$ is a σ-algebra on Y. Since $\mathcal{E} \subset f_*\mathcal{A}$, we have that $\mathcal{B} = \sigma(\mathcal{E}) \subset f_*\mathcal{A}$. So f is $\mathcal{A}$-$\mathcal{B}$ measurable. $\square$

Exercise 3.1.6. Let X, Y be sets, $f : X \to Y$ and $\mathcal{E} \subset \mathcal{P}(Y)$. Then $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$.

Proof. Clealy $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$. Since $f^{-1}(\sigma(\mathcal{E}))$ is a σ-algebra, we have that $\sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. Since $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$, the previous exercise tells us that f is $f^{-1}(\sigma(\mathcal{E}))$-$\sigma(\mathcal{E})$ measurable. Then $f^{-1}(\sigma(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. So $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$. $\square$

Exercise 3.1.7. Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X \to Y$. If f is continuous, then f is $\mathcal{B}(X)$-$\mathcal{B}(Y)$ measurable.

Proof. Recall that $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$ and continuity tells us that for each $U \in \mathcal{T}_2$, $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$. $\qquad\square$

Definition 3.1.8. Let X be a set and $f : X \to \mathbb{C}$. Then f is said to be **simple** if $f(X)$ is finite.

Definition 3.1.9. Let $(X, \mathcal{A})$ be a measurable space. We define $S^+(X, \mathcal{A}) = \{f : X \to [0, \infty) : f \text{ is simple, measurable}\}$ and $S(X, \mathcal{A}) = \{f : X \to \mathbb{C} : f \text{ is simple, measurable}\}$

Theorem 3.1.10. Let $(X, \mathcal{A})$ be a measurable space. Then
  (1) If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.
  (2) If $f : X \to \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S$ such that for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.

Exercise 3.1.11. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. If f is $\mathcal{A}$-$\mathcal{B}$ measurable iff f is $\mathcal{A}$-$\mathcal{B} \cap f(X)$ measurable.

Proof. Suppose that f is $\mathcal{A}$-$\mathcal{B}$ measurable. Let $E \in \mathcal{B} \cap f(X)$. Then there exists $B \in \mathcal{B}$ such that $E = B \cap f(X)$. Then

$$
\begin{aligned}
f^{-1}(E) &= f^{-1}(B \cap f(X)) \\
&= f^{-1}(B) \cap f^{-1}(f(X)) \\
&= f^{-1}(B) \cap X \\
&= f^{-1}(B) \\
&\in \mathcal{A}
\end{aligned}
$$

Conversely, suppose that f is $\mathcal{A}$-$\mathcal{B} \cap f(X)$ measurable. Let $B \in \mathcal{B}$. Then as before,

$$
\begin{aligned}
f^{-1}(B) &= f^{-1}(B \cap f(X)) \\
&\in \mathcal{A}
\end{aligned}
$$

$\qquad\square$

Exercise 3.1.12. Doob-Dynkin Lemma:
Let $(X_1, \mathcal{A}_1), (X_2, \mathcal{A}_2)$ and $(X_3, \mathcal{A}_3)$ be measurable spaces and $f : X_1 \to X_2$ and $g : X_1 \to X_3$. Suppose that f is surjective and $\mathcal{A}_1$-$\mathcal{A}_2$ measurable and g is $\mathcal{A}_1$-$\mathcal{A}_3$ measurable and for each $t \in X_3$, $\{t\} \in \mathcal{A}_3$. Then g is $f^*\mathcal{A}_2$-$\mathcal{A}_3$ measurable iff there exists $\phi : X_2 \to X_3$ such that ϕ is $\mathcal{A}_2$-$\mathcal{A}_3$ measurable and $g = \phi \circ f$.
Hint: For each $t \in X_3$, set $A_t = g^{-1}(\{t\}) \in f^*\mathcal{A}_2$ and choose $B_t \in \mathcal{A}_2$ such that $A_t = f^{-1}(B_t)$. Set $\phi(y) = t$ for $y \in B_t \cap f(X_1)$ and $t \in g(X_1)$.

Proof. Suppose that there exists $\phi : X_2 \to X_3$ such that ϕ is $\mathcal{A}_2$ - $\mathcal{A}_3$ measurable and $g = \phi \circ f$. Since f is $f^*\mathcal{A}_2$ - $\mathcal{A}_2$ measurable, we have that $g = \phi \circ f$ is $f^*\mathcal{A}_2$-$\mathcal{A}_3$ measurable. Conversely, suppose that g is $f^*\mathcal{A}_2$-$\mathcal{A}_3$ measurable. For each $t \in X_3$, set $A_t = g^{-1}(\{t\}) \in f^*\mathcal{A}_2$ and choose $B_t \in \mathcal{A}_2$ such that $A_t = f^{-1}(B_t)$.
Note that

- for $t_1, t_2 \in g(X_1)$,

$$f^{-1}(B_{t_1} \cap B_{t_2}) = A_{t_1} \cap A_{t_2}$$
$$= g^{-1}(\{t_1\} \cap \{t_2\})$$
$$= \varnothing$$

So

$$B_{t_1} \cap B_{t_2} \subset f(X^1)^c$$
$$= \varnothing$$

- we have that

$$f^{-1}\left(\bigcup_{t \in g(X_1)} B_t \right) = \bigcup_{t \in g(X_1)} A_t$$
$$= \bigcup_{t \in g(X_1)} g^{-1}(\{t\})$$
$$= g^{-1}(g(X_1))$$
$$= X_1$$

So $X_2 = f(X_1) \subset \bigcup_{t \in g(X_1)} B_t$.

Therefore $(B_t)_{t \in g(X_1)}$ is a partition of X_2 and $(A_t)_{t \in g(X_1)}$ is a partion of X_1. Define $\phi : X_2 \to X_3$ by $\phi(y) = t$ for $y \in B_t$ and $t \in g(X_1)$. So $\phi(X_2) = g(X_1)$. Since for each $t \in g(X_1)$ and $x \in A_t$, $f(x) \in B_t$ and $g(x) = t$, we have that $\phi \circ f(x) = t = g(x)$. So $\phi \circ f = g$.

Let $C \in \mathcal{A}_3$. Choose $B \in \mathcal{A}_2$ such that $g^{-1}(C) = f^{-1}(B)$.

Let $y \in \phi^{-1}(C) \subset X_2$. Set $t = \phi(y) \in C$ and choose $x \in X_1$ such that $y = f(x)$. Since

$$g(x) = \phi \circ f(x)$$
$$= \phi(y)$$
$$= t$$
$$\in C$$

$x \in g^{-1}(C) = f^{-1}(B)$. Therefore, $y = f(x) \in B$. So $\phi^{-1}(C) \subset B$.

Let $y \in B$. Choose $x \in X_1$ such that $f(x) = y$. Then $x \in f^{-1}(B) = g^{-1}(C)$. So

$$\phi(y) = \phi \circ f(x)$$
$$= g(x)$$
$$\in C$$

and $y \in \phi^{-1}(C)$. So $B \subset \phi^{-1}(C)$. Hence $\phi^{-1}(C) = B \in \mathcal{A}_2$ and ϕ is $\mathcal{A}_2$ - $\mathcal{A}_3$ measurable. $\square$

Exercise 3.1.13. Let $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2)$ and $(X_3, \mathcal{A}_3)$ be measurable spaces and $f : X_1 \to X_2$ and $g : X_1 \to X_3$. Suppose that f is $\mathcal{A}_1$-$\mathcal{A}_2$ measurable and g is $\mathcal{A}_1$-$\mathcal{A}_3$ measurable and for each $t \in X_3$, $\{t\} \in \mathcal{A}_3$. Then g is $f^*\mathcal{A}_2$-$\mathcal{A}_3$ measurable iff there exists $\phi : f(X_1) \to X_3$ such that ϕ is $\mathcal{A}_2 \cap f(X_1)$ - $\mathcal{A}_3$ measurable and $g = \phi \circ f$.

Proof. A previous exercise implies that $f : X_1 \to f(X_1)$ is $\mathcal{A}_1$ - $\mathcal{A}_2 \cap f(X_1)$ measurable. Now apply the previous exercise. $\square$

3.2. **Integration of Nonnegative Functions.**

Definition 3.2.1. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define
$$L^+(X, \mathcal{A}, \mu) = \{f : X \to [0, \infty] : f \text{ is measurable}\}$$
We will typically just write L^+.

Theorem 3.2.2. Monotone Convergence Theorem: Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then
$$\sup_{n \in \mathbb{N}} \int f_n = \int \sup_{n \in \mathbb{N}} f_n$$
.

Exercise 3.2.3. Let μ_1, μ_2 be measures on $(X, \mathcal{A})$, $\lambda \geq 0$ and $f \in L^+$. Then
$$\int f d(\mu_1 + \lambda \mu_2) = \int f d\mu_1 + \lambda \int f d\mu_2$$
.

Proof. Suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset [0, \infty)$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\int f d(\mu_1 + \lambda \mu_2) = \sum_{i=1}^n a_i (\mu_1 + \lambda \mu_2)(E_i)$$
$$= \sum_{i=1}^n a_i (\mu_1(E_i) + \lambda \mu_2(E_i))$$
$$= \sum_{i=1}^n a_i \mu_1(E_i) + \lambda \sum_{i=1}^n a_i \mu_2(E_i)$$
$$= \int f d\mu_1 + \lambda \int f d\mu_2$$

Now for a general f, choose $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that $\phi_n \to f$ pointwise and for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$. Then monotone convergence tells us that

$$\int f d(\mu_1 + \lambda \mu_2) = \lim_{n \to \infty} \int \phi_n d(\mu_1 + \lambda \mu_2)$$
$$= \lim_{n \to \infty} \int \phi_n d\mu_1 + \lim_{n \to \infty} \lambda \int \phi_n d\mu_2$$
$$= \int f d\mu_1 + \lambda \int f d\mu_2$$

$\square$

Exercise 3.2.4. Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Suppose that $\mu_1 \leq \mu_2$. Then for each $f \in L^+$,
$$\int f d\mu_1 \leq \int f d\mu_2$$

Proof. First suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset [0,\infty)$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum\limits_{i=1}^n a_i \chi_{E_i}$. Then

$$
\begin{aligned}
\int f d\mu_1 &= \sum_{i=1}^n a_i \mu_1(E_i) \\
&\leq \sum_{i=1}^n a_i \mu_2(E_i) \\
&= \int f d\mu_2
\end{aligned}
$$

for general f,

$$
\begin{aligned}
\int f d\mu_1 &= \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_1 \\
&\leq \sup_{\substack{s \in S^+ \\ s \leq f}} \int s d\mu_2 \\
&= \int f d\mu_2
\end{aligned}
$$

$\square$

Theorem 3.2.5. Fatou's Lemma Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Then

$$
\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n.
$$

Theorem 3.2.6. Let $(f_n)_{n \in \mathbb{N}} \subset L^+$. Then

$$
\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n.
$$

Exercise 3.2.7. Let $f \in L^+$ and suppose that $\int f < \infty$. Put $N = \{x \in X : f(x) = \infty\}$ and $S = \{x \in X : f(x) > 0\}$. Then $\mu(N) = 0$ and S is σ-finite.

Proof. Suppose that $\mu(N) > 0$. Define $f_n = n\chi_N \in L^+$. Then for each $n \in \mathbb{N}$, $f_n \leq f_{n+1} \leq f$ on N. So

$$
\begin{aligned}
\int f &\geq \int_N f \\
&= \lim_{n \to \infty} \int_N f_n \\
&= \lim_{n \to \infty} n\mu(N) \\
&= \infty, \text{ a contradiction.}
\end{aligned}
$$

Hence N is a null set. Now, put $S_n = \{x \in X : f(x) > 1/n\}$. Then $S = \bigcup_{n \in \mathbb{N}} S_n$. Suppose that there exists some $n \in \mathbb{N}$ such that $\mu(S_n) = \infty$. Then

$$\int f \geq \int_{S_n} f$$
$$\geq \frac{1}{n}\mu(S_n)$$
$$= \infty, \text{ a contradiction.}$$

So for each $n \in \mathbb{N}$, $\mu(S_n) < \infty$ and S is σ-finite.

$\square$

Exercise 3.2.8. Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.

Proof. $f = 0$ a.e. implies that for each $E \in \mathcal{A}$, $\int_E f = 0$ is clear. Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f = 0$. For $n \in \mathbb{N}$ put $N_n = \{x \in X : f(x) > 1/n\}$ and define $N = \{x \in X : f(x) > 0\}$. So $N = \bigcup_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$0 = \int_{N_n} f$$
$$\geq \frac{1}{n}\mu(N_n)$$
$$\geq 0.$$

Hence for each $n \in \mathbb{N}$, $\mu(N_n) = 0$. Thus $\mu(N) = 0$ and $f = 0$ a.e. as required.

$\square$

Exercise 3.2.9. Let $(f_n)_{n \in \mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that $f_n \xrightarrow{\text{p.w.}} f$, $\lim_{n \to \infty} \int f_n = \int f$ and $\int f < \infty$. Then for each $E \in \mathcal{A}$, $\lim_{n \to \infty} \int_E f_n = \int_E f$. This result may fail to be true if $\int f = \infty$

Proof. Let $E \in \mathcal{A}$. By Fatou's lemma, $\int_E f \leq \liminf_{n \to \infty} \int_E f_n$. Note that since $\int f < \infty$, we have that $\int_{E^c} f \leq \int f < \infty$. Thus we may write

$$\int_E f = \int f - \int_{E^c} f$$
$$\geq \int f - \liminf_{n \to \infty} \int_{E^c} f_n$$
$$= \int f - \liminf_{n \to \infty} \left(\int f_n - \int_E f_n \right)$$
$$= \int f - \int f + \limsup_{n \to \infty} \int_E f_n$$
$$= \limsup_{n \to \infty} \int_E f_n.$$

Hence

$$\limsup_{n \to \infty} \int_E f_n \leq \int_E f \leq \liminf_{n \to \infty} \int_E f_n$$

and therefore

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

If we drop the assumption that $\int f < \infty$, then the result would fail to be true for the functions $f = \infty\chi_{(0,1)}$ and $f_n = \infty\chi_{(0,1)} + n\chi_{(1,1+1/n)}$. Here $f_n \xrightarrow{\text{p.w.}} f$, $\lim\limits_{n\to\infty} \int f_n = \int f = \infty$ and $\lim\limits_{n\to\infty} \int_{(1,\infty)} f_n = 1$ while $\int_{(1,\infty)} f = 0$.

$\square$

Exercise 3.2.10. Let $f \in L^+$. Define $\lambda : \mathcal{A} \to [0,\infty]$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{A}$ Then λ is a measure on $(X, \mathcal{A})$ and for each $g \in L^+$, $\int g d\lambda = \int g f d\mu$.

Proof. Clearly $\lambda(\varnothing) = 0$. Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ and suppose that for each $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j = \varnothing$. For now, suppose that f is simple. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0,\infty)$ such that $f = \sum\limits_{i=1}^{n} a_i \chi_{E_i}$. Then

$$\begin{aligned}
\lambda\left(\bigcup_{j\in\mathbb{N}} A_j\right) &= \int_{\bigcup_{j\in\mathbb{N}} A_j} f \\
&= \sum_{i=1}^{n} a_i \mu\left(E_i \cap \left(\bigcup_{j\in\mathbb{N}} A_j\right)\right) \\
&= \sum_{i=1}^{n} a_i \mu\left(\bigcup_{j\in\mathbb{N}} E_i \cap A_j\right) \\
&= \sum_{i=1}^{n} a_i \sum_{j\in\mathbb{N}} \mu(E_i \cap A_j) \\
&= \sum_{j\in\mathbb{N}} \sum_{i=1}^{n} a_i \mu(E_i \cap A_j) \\
&= \sum_{j\in\mathbb{N}} \int_{A_j} f \\
&= \sum_{j\in\mathbb{N}} \lambda(A_j)
\end{aligned}$$

Hence λ is a measure on $(X, \mathcal{A})$. Now, for a general f, there exist $(\phi_n)_{n\in\mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \xrightarrow{\text{p.w.}} f$. Put $A = \bigcup_{j\in\mathbb{N}} A_j$ and define the measures λ_n by $\lambda_n(E) = \int_E \phi_n$. Note that we may define a monotonically increasing sequence of functions $g_n : \| \to [0,\infty]$ by $g_n(j) = \int_{A_j} \phi_n$. Using monotone convergence three times and a nice application of the counting measure on $\mathbb{N}$, we may write

$$\lambda(A) = \int_A f$$

$$= \lim_{n\to\infty} \int_A \phi_n$$

$$= \lim_{n\to\infty} \sum_{j\in\mathbb{N}} \int_{A_j} \phi_n$$

$$= \sum_{j\in\mathbb{N}} \lim_{n\to\infty} \int_{A_j} \phi_n \quad \text{(by the above)}$$

$$= \sum_{j\in\mathbb{N}} \int_{A_j} f$$

$$= \sum_{j\in\mathbb{N}} \lambda(A_j).$$

Hence λ is a measure on $(X, \mathcal{A})$. Let $g \in L^+$. First assume that g is simple. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0, \infty)$ such that $g = \sum_{i=1}^{n} a_i \chi_{E_i}$. In this case, we have that

$$\int g d\lambda = \sum_{i=1}^{n} a_i \lambda(E_i)$$

$$= \sum_{i=1}^{n} a_i \int_{E_i} f d\mu$$

$$= \int \left(\sum_{i=1}^{n} a_i \chi_{E_i} \right) f d\mu$$

$$= \int g f d\mu.$$

Now for a general $g \in L^+$, there exist $(\psi_n)_{n\in\mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ψ_n is simple, $\psi_n \le \psi_{n+1} \le f$ and $\psi_n \xrightarrow{\text{p.w.}} g$. Monotone convergence then gives us

$$\int g d\lambda = \lim_{n\to\infty} \int \psi_n d\lambda$$

$$= \lim_{n\to\infty} \int \psi_n f d\mu$$

$$= \int g f d\mu \text{ as required.}$$

$\square$

Exercise 3.2.11. Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \ge f_{n+1}$, $f_n \xrightarrow{\text{p.w.}} f$ and $\int f_1 < \infty$. Then $\lim_{n\to\infty} \int f_n = \int f$.

Proof. First we note that since $\int f_1 < \infty$, $f_1 < \infty$ a.e., for each $n \in \mathbb{N}$, $f_1 - f_n$ and $\int f_1 - \int f_n$ are well defined and $\int f_n \leq \int f_1 < \infty$. Also, for $n \in \mathbb{N}$, $f_1 - f_n \in L^+$. So we may write

$$\int (f_1 - f_n) = \int (f_1 - f_n) + \int f_n - \int f_n$$

$$= \int [(f_1 - f_n) + f_n] - \int f_n$$

$$= \int f_1 - \int f_n$$

Put $g_n = f + (f_1 - f_n)$. Then $g_n \in L^+$, for each $n \in \mathbb{N}$, $g_n \leq g_{n+1}$ and $g_n \xrightarrow{\text{p.w.}} f_1$. Monotone convergence tells us that

$$\int f_1 = \lim_{n \to \infty} \int g_n$$

$$= \lim_{n \to \infty} \left[\int f + (f_1 - f_n) \right]$$

$$= \lim_{n \to \infty} \left[\int f + \int (f_1 - f_n) \right]$$

$$= \lim_{n \to \infty} \left[\int f + \int f_1 - \int f_n \right]$$

Since $\lim_{n \to \infty} \int f$ and $\lim_{n \to \infty} \int f_1$ exist, $\lim_{n \to \infty} \int f_n = \int f$ as required.

$\square$

3.3. **Integration of Complex Valued Functions.**

Definition 3.3.1. Let $f : X \to \mathbb{C}$ be measurable. Then f is said to be **integrable** if

$$\int |f| d\mu < \infty$$

Definition 3.3.2. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^1(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} : f$ is measurable and $\int |f| < \infty\}$

Lemma 3.3.3. Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff f^+ and f^- are integrable.

Proof. $f^+, f^- \leq |f| = f^+ + f^-$ $\qquad\qquad\square$

Definition 3.3.4. Let $f : X \to \mathbb{R}$ be measurable. Then f is said to be **extended integrable** if

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

Lemma 3.3.5. Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff $Re(f)$ and $Im(f)$ are integrable.

Proof. $|Re(f)|, |Im(f)| \leq |f| \leq |Re(f)| + |Im(f)|$ $\qquad\qquad\square$

Exercise 3.3.6. Dominated Convergence Theorem:

Let $(f_n)_{n \in \mathbb{N}} \subset L^0$, $f \in L^0$ and $g \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$ and there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f \in L^1$ and

$$\int_X |f_n - f| d\mu \to 0$$

Hint: Fatou's lemma

Proof. Continuity imples that $|f| \leq g$ a.e. Since

$$|f_n - f| \leq |f_n| + |f|$$
$$\leq 2g$$

Fatou's lemma implies that

$$\int 2g d\mu = \int \liminf_{n \to \infty} (2g - |f_n - f|) d\mu$$
$$\leq \liminf_{n \to \infty} \int 2g - |f_n - f| d\mu$$
$$= \int 2g d\mu - \limsup_{n \to \infty} \int |f_n - f| d\mu$$

Hence

$$\limsup_{n \to \infty} \int |f_n - f| d\mu \leq 0$$

and thus

$$\int |f_n - f| d\mu \to 0$$

$\qquad\qquad\square$

Exercise 3.3.7. Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Then

(1) $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$
(2) for each $f \in L^1(\mu_1 + \mu_2)$, we have that

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

Proof.   (1) The firt part is clear since similar exercise from the section on nonnegative funtions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f| d\mu_1 + \int |f| d\mu_2$$

(2) Suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset \mathbb{C}$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\int f d(\mu_1 + \mu_2) = \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i)$$
$$= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i))$$
$$= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i)$$
$$= \int f d\mu_1 + \int f d\mu_2$$

Now for general f, choose $(\phi_n)_{n \in \mathbb{N}} \subset S$ such that $\phi_n \to f$ pointwise and for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$. Then dominated convergence tells us that

$$\int f d(\mu_1 + \mu_2) = \lim_{n \to \infty} \int \phi_n d(\mu_1 + \mu_2)$$
$$= \lim_{n \to \infty} \int \phi_n d\mu_1 + \lim_{n \to \infty} \int \phi_n d\mu_2$$
$$= \int f d\mu_1 + \int f d\mu_2$$

$\square$

Theorem 3.3.8. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$. Suppose that

$$\sum_{n \in \mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n \in \mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n \in \mathbb{N}} f_n = \sum_{n \in \mathbb{N}} \int f_n$$

Theorem 3.3.9. Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$.

Exercise 3.3.10. Generalized Fatou's Lemma: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \geq -g$. Then $\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n$. What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?

Proof. First note that for each $n \in \mathbb{N}$, $\int f_n$ is well defined since $f_n^- \leq g \in L^1$. Since $g + f_n \geq 0$, we may use Fatou's lemma to write

$$\int g + \int \liminf_{n \to \infty} f_n = \int \liminf_{n \to \infty} (g + f_n)$$

$$\leq \liminf_{n \to \infty} \int (g + f_n)$$

$$= \int g + \liminf_{n \to \infty} \int f_n$$

Since $\int g < \infty$, $\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n$ as required. The analogue is as follows: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \leq g$. Then $\limsup_{n \to \infty} \int f_n \leq \int \limsup_{n \to \infty} f_n$. To show this, just use the result from above with the sequence $(g_n)_{n \in \mathbb{N}}$ given by $g_n = -f_n$. $\square$

Exercise 3.3.11. Let $(f_n)_{n \in \mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \to \mathbb{C}$. Suppose that $f_n \xrightarrow{\text{uni}} f$. Then

  (1) if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim_{n \to \infty} \int f_n = \int f$
  (2) if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on $\mathbb{R}$ with Lebesgue measure).

Proof. Choose $N \in \mathbb{N}$ such that for $n \geq N$ and $x \in X$, $|f(x) - f_n(x)| < 1$. Then $||f| - |f_N|| < 1$ and so $|f| < |f_N| + 1$. Thus $\int |f| \leq \int |f_N| + \mu(X) < \infty$ and $f \in L^1$. Similarly for $n \geq N$, $|f_n| < |f| + 1$. Dominated convergence then gives us that $\lim_{n \to \infty} \int f_n = \int f$ as required. To see the necessity that $\mu(X) < \infty$, consider $f \equiv 0$ and $f_n = (1/n)\chi_{(0,n)}$. Then $f_n \xrightarrow{\text{uni}} f$, but $1 = \lim_{n \to \infty} \int f_n \neq \int f = 0$. $\square$

Exercise 3.3.12. Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$, $g_n \xrightarrow{\text{a.e.}} g$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. If

$$\int g_n d\mu \to \int g d\mu$$

then

$$\int f_n d\mu \to \int f d\mu$$

.

Proof. We simply use Fatou's lemma. Put $h_n = (g + g_n) - |f_n - f|$. Since for each $n \in \mathbb{N}$, $|f_n| \leq g_n$, we know that $|f| \leq g$. So $h_n \geq 0$ and $h_n \xrightarrow{\text{p.w.}} 2g$. Thus

$$2 \int g = \int \liminf_{n \to \infty} h_n$$
$$\leq \liminf_{n \to \infty} \left[\left(\int g + \int g_n \right) - \int |f_n - f| \right]$$
$$= 2 \int g + \liminf_{n \to \infty} \left(- \int |f_n - f| \right)$$
$$= 2 \int g - \limsup_{n \to \infty} \int |f_n - f|$$

Hence $\limsup\limits_{n \to \infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $\int f_n \to \int f$ as required. $\qquad \square$

Exercise 3.3.13. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.

Proof. Suppose that $\int |f_n - f| \to 0$. Since

$$\left| \int |f_n| - \int |f| \right| = \left| \int (|f_n| - |f|) \right|$$
$$\leq \int ||f_n| - |f||$$
$$\leq \int |f_n - f|,$$

we see that $\int |f_n| \to \int |f|$. Conversely, suppose that $\int |f_n| \to \int |f|$. Put $h_n = |f_n - f|$, $g_n = |f_n| + |f|$, $h \equiv 0$ and $g = 2f$. Then $h_n \xrightarrow{\text{a.e.}} h$, $g_n \xrightarrow{\text{a.e.}} g$ and for each $n \in \mathbb{N}$, $h_n \leq g_n$. Our assumption implies that $\int g_n \to \int g$. Thus the last exercise tells us that $\int h_n \to \int h$ as required.

$\qquad \square$

Exercise 3.3.14. Let $(r_n)_{n \in \mathbb{N}}$ be an enummeration of the rationals. Define $f : \mathbb{R} \to [0, \infty)$ by

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}$$

and define $g : X \to [0, \infty]$ by

$$g(x) = \sum_{n \in \mathbb{N}} 2^{-n} f(x - r_n).$$

Then

(1) $g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.
(2) $g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of $\mathbb{R}$
(3) Taking $g \in L^1$, g is unbounded on each subinterval of $\mathbb{R}$ and discontinuous everywhere and remains so after redefinition on a null set

Proof. For convenience, define $f_n : \mathbb{R} \to [0, \infty)$ by $f_n(x) = f(x - r_n)$ for $x \in \mathbb{R}$. To show (1) we note that for each $n \in \mathbb{N}$, $f_n \in L^1$ and

$$\int |2^{-n} f_n| = 2^{-n} \int_0^1 x^{-1/2} dx$$
$$= 2^{n-1}$$

Hence

$$\sum_{n \in \mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set, $g \in L^1$. In particular $\int |g| < \infty$ and so $|g|$ (and hence g) are finite almost everywhere. For (2), since $g < \infty$ a.e., so too is g^2. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Choose $N \in \mathbb{N}$ such that $r_N \in (a, b)$. Since all the terms in the sum are nonnegative, $g^2 \geq \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2$ and so

$$\int_{(a,b)} g^2 \geq \int_{(a,b)} \sum_{n \in \mathbb{N}} 2^{-2n} f_n^2$$
$$= \sum_{n \in \mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2$$
$$\geq 2^{-2N} \int_{(a,b)} f_N^2$$
$$\geq 2^{-2N} \int_{r_N}^{b \wedge (r_N + 1)} \frac{1}{x - r_N} dx$$
$$= \infty$$

So g^2 is not integrable on any subinterval of $\mathbb{R}$. For (3), note that redefining g on a null set does not change the result of (2). Suppose that there is a finite subinterval $I \subset \mathbb{R}$ such that g is bounded on I. Hence there exists $M > 0$ such that for each $x \in I$, $g(x)^2 \leq M$. Then

$$\int_I g^2 \leq M^2 m(I)$$
$$< \infty$$

which is a contradiction. So g is not bounded on any subinterval of $\mathbb{R}$. Now, suppose that there exists $x_0 \in \mathbb{R}$ such that g is continuous at x_0. Choose $\delta > 0$ such that for each $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then $|g(x) - g(x_0)| < 1$. The reverse triangle inequality tells us that for each $x \in (x_0 - \delta, x_0 + \delta)$, $|g(x)| < 1 + |g(x_0)|$. Hence g is bounded on $(x_0 - \delta, x_0 + \delta)$ which is a contradiction. So g is discontinuous everywhere. $\square$

Exercise 3.3.15. Let $f \in L^1$.

    (1) If f is bounded, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\int_E |f| < \epsilon$.

    (2) The same conclusion holds for general $f \in L^1$.

Proof. (1) Since f is bounded, there exists $M > 0$ such that $|f| \leq M$. Let $\epsilon > 0$. Choose $\delta = \epsilon/2M$. Let $E \in \mathcal{A}$. Suppose that $\mu(A) < \delta$. Then

$$\int_E |f| \leq M\mu(E)$$
$$= M\frac{\epsilon}{2M}$$
$$= \frac{\epsilon}{2}$$
$$< \epsilon$$

(2) Suppose that f is unbounded. Let $\epsilon > 0$. Then there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon/2$. Since ϕ is bounded, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\int_E |\phi| < \epsilon/2$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then

$$\int_E |f| \leq \int_E |f - \phi| + \int_E |\phi|$$
$$< \epsilon/2 + \epsilon/2$$
$$= \epsilon$$

$\square$

Exercise 3.3.16. Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_{(-\infty, x]} f \, dm.$$

Then F is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Since $f \in L^1$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| \, dm < \epsilon.$$

Let $x \in \mathbb{R}$. Suppose that $|x - x_0| < \delta$. Then

$$|F(x) - F(x_0)| = \left| \int_{(x \wedge x_0, x \vee x_0]} f \, dm \right|$$
$$\leq \int_{(x \wedge x_0, x \vee x_0]} |f| \, dm$$
$$< \epsilon$$

So F is continuous.

$\square$

Exercise 3.3.17. Let $x \in X$ and denote by δ_x the point mass measure at $x \in X$ on measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$. Then

$$\int f \, d\delta_x = f(x)$$

Proof. First assume that f is simple. Then there exist $(a_j)_{j=1}^n \subset \mathbb{C}$ and $(E_j)_{j=1}^n \subset \mathcal{P}(X)$ such that $(E_j)_{j=1}^n$ is disjoint and $f = \sum_{i=1}^n a_i \chi_{E_i}$. Choose $j^* \in \{1, \ldots, n\}$ such that $x \in E_{j^*}$. Thus

$$\int f d\delta_x = \int \sum_{j=1}^n c_j \chi_{E_j} d\delta_x$$
$$= \sum_{j=1}^n c_j \delta_x(E_j)$$
$$= c_j \delta_x(E_{j^*})$$
$$= c_j$$
$$= f(x)$$

Now for $f \in L^+$, choose a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1}$ and $\phi_n \xrightarrow{\text{p.w}} f$. Then monotone convergence implies that

$$\int f d\delta_x = \int \lim_{n \to \infty} \phi_n \delta_x$$
$$= \lim_{n \to \infty} \int \phi_n \delta_x$$
$$= \lim_{n \to \infty} \phi_n(x)$$
$$= f(x)$$

Now just extend to complex valued functions.

$\square$

Exercise 3.3.18. Denote by $\#$ the counting measure on the measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$ and suppose that $f \in L^1$. Then

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if f is integrable, then $\{x \in X : f(x) \neq 0\}$ is countable.

Proof. Please refer to the definition of the sum in the appendix. First suppose that $f(X) \subset [0, \infty)$. For $n \in \mathbb{N}$, put $X_n = \{x \in X : f(x) > 1/n\}$ and define $X_+ = \{x \in X : f(x) > 0\}$, $X_0 = \{x \in X : f(x) = 0\}$ Then $X_+ = \bigcup_{n \in \mathbb{N}} X_n$. Since $f \in L^1$, we have that for each $n \in \mathbb{N}$,

$$\infty > \int f d\#$$
$$\geq \int_{X_n} f d\#$$
$$\geq \frac{1}{n} \#(X_n).$$

Thus for each $n \in \mathbb{N}$, X_n is finite and X_+ is countable. Thus there exists $\{x_n\}_{n \in \mathbb{N}} \subset X$ such that $X_+ = \{x_n\}_{n \in \mathbb{N}}$. For $n \in \mathbb{N}$, define $E_n = \{x_1, x_2, \cdots, x_n\}$ and

$$f_n = f\chi_{E_n}$$

$$= \sum_{i=1}^{n} f(x_i)\chi_{\{x_i\}}$$

Then $f_n \xrightarrow{\text{p.w.}} f\chi_{X_+} = f$ and for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. So

$$\int f = \sup_{n \in \mathbb{N}} \int f_n$$

$$= \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} f(x_i)$$

$$= \sum_{x \in X_+} f(x)$$

$$= \sum_{x \in X} f(x).$$

For $f : X \to \mathbb{C}$, our L^1 assumption and the result above tell us that

$$\sum_{x \in X} |f(x)| < \infty.$$

Thus writing $f = g + ih$, we see that the same is true for f^+, f^-, g^+, g^-. Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f\, d\# = \sum_{x \in X} f(x).$$

$\square$

Exercise 3.3.19. Let $f, g : X \to \mathbb{R}$. Suppose that $f, g \in L^1$. Then $f \leq g$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$.

Proof. Suppose $f \leq g$ a.e. Put $N = \{x \in X : f(x) > g(x)\} \subset N$. Then $\mu(N) = 0$ and $g - f \geq 0$ on N^c. So for each $E \in \mathcal{A}$,

$$\int_E g - \int_E f = \int_E (g - f)$$

$$= \int_{E \cap N^c} (g - f)$$

$$\geq 0$$

Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$. Put $N_n = \{x \in X : f(x) - g(x) > 1/n\}$ and $N = \{x \in X : f(x) > g(x)\}$. Then $N = \bigcup_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption

tells us that

$$0 \geq \int_{N_n} f - g$$
$$\geq \frac{1}{n}\mu(N_n)$$
$$\geq 0.$$

So that $\mu(N_n) = 0$. Thus for each $n \in \mathbb{N}$, $\mu(N_n) = 0$ which implies $\mu(N) = 0$. Therefore $f \leq g$ a.e. as required. $\qquad \square$

Exercise 3.3.20. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \times \mathbb{R} \to \mathbb{C}$. Suppose that for each $t \in \mathbb{R}$, $f(\cdot, t) \in L^1(\mu)$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \int_X f(x, t)d\mu(x)$$

(1) Suppose that there exists $g \in L^1(\mu)$ such that for each $(x, t) \in X \times \mathbb{R}$, $|f(x, t)| \leq g(x)$. Let $t_0 \in \mathbb{R}$. If for each $x \in X$, $f(x, \cdot)$ is continuous at t_0, then F is continuous at t_0.
(2) Suppose that $\partial f / \partial t$ exits and there exists $g \in L^1(\mu)$ such that for each $(x, t) \in X \times \mathbb{R}$, $|\partial f / \partial t(x, t)| \leq g(x)$. Then F is differentiable and for each $t \in \mathbb{R}$,

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t)d\mu(x)$$

Proof.

(1) Suppose that for each $x \in X$, $f(x, \cdot)$ is continuous at t_0. Let $(t_n) \subset \mathbb{R}$. Suppose that $t_n \to t_0$. Then $f(\cdot, t_n) \xrightarrow{\text{p.w.}} f(\cdot, t_0)$. Since for each $n \in \mathbb{N}$, $|f(x, t_n)| \leq g(x)$, the dominated convergence theorem implies that $F(t_n) \to F(t_0)$.
(2) Let $t_0 \in \mathbb{R}$. Choose $(t_n)_{n \in \mathbb{N}}$ such that $t_n \to t_0$ and for each $n \in \mathbb{N}$, $t_n < t_0$. For $n \in \mathbb{N}$, define $q_n : X \to \mathbb{R}$ by

$$q_n(x) = \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}$$

So $q_n(\cdot) \xrightarrow{\text{p.w.}} \partial f / \partial t(\cdot, t_0)$. The mean value theorem implies that for each $x \in X$ and $n \in \mathbb{N}$, there exists $c_{n,x} \in (t_n, t_0)$ such that $q_n(x) = \partial f / \partial t(x, c_{n,x})$. Then for each $n \in \mathbb{N}$, $|q_n| \leq g$. The dominated convergence theorem then implies that $\partial f / \partial t(\cdot, t_0) \in L^1(\mu)$ and

$$\int \frac{\partial f}{\partial t}(x, t_0)d\mu(x) = \lim_{n \to \infty} \int_X q_n d\mu$$
$$= \lim_{n \to \infty} \frac{F(t_n) - F(t_0)}{t_n - t_0}$$
$$= F'(t_0^-)$$

So that F is differentiable at t_0 from the left. Similarly, F is differentiable at t_0 from the right.

$\qquad \square$

Definition 3.3.21. Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is said to be **uniformly integrable** if for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$. (i.e. $\lim_{k \to \infty} \sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$).

Exercise 3.3.22. Suppose that μ is finite. Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is uniformly integrable iff
  (1) there exists $M > 0$ such that $\sup_{f \in \mathcal{F}} \int |f| \leq M$
  (2) for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\sup_{f \in \mathcal{F}} \int_E |f| < \epsilon$.

Proof. ($\implies$): (1) Suppose that $\mathcal{F}$ is uniformly integrable. Then there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < 1$. Choose $M = \mu(X)K + 1$. Then for each $f \in \mathcal{F}$,

$$\int |f| = \int_{\{|f| > K\}} |f| + \int_{\{|f| \leq K|\}} |f|$$
$$\leq 1 + K\mu(X)$$
$$= M$$

  (2) Let $\epsilon > 0$. Then choose $K \in \mathbb{N}$ such that $\sup_{f \in \mathcal{F}} \int_{\{|f| > K\}} |f| < \epsilon/2$ and choose $\delta = \epsilon/2K$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then for $f \in \mathcal{F}$,

$$\int_E |f| = \int_{E \cap \{|f| > K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f|$$
$$\leq \epsilon/2 + K\delta$$
$$= \epsilon$$

  ($\impliedby$): Choose $M > 0$ as in (1). Suppose that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $f \in \mathcal{F}$ such that $\mu(\{|f| > K\}) \geq \epsilon$. Choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $f_K \in \mathcal{F}$ such that $\mu(\{|f_K| > K\}) \geq \epsilon$. Then

$$\int |f_K| \geq \int_{\{|f_K| > K\}} |f|$$
$$\geq K\mu(\{|f_K| > K\})$$
$$> \frac{M}{\epsilon} \cdot \epsilon$$
$$= M,$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $f \in \mathcal{F}$, $\mu(\{|f| > K\}) < \epsilon$. Since $\mu(\{|f| > k\})$ is a decreasing sequence in k, we have that $\lim_{k \to \infty} \sup_{f \in \mathcal{F}} \mu(\{|f| > k\}) = 0$. Now, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Choose $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$, $\mu(\{|f| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$,

$$\int_{\{|f| > k\}} |f| < \epsilon.$$

Thus

$$\limsup_{\substack{k \to \infty \\ f \in \mathcal{F}}} \int_{\{|f| > k\}} |f| = 0$$

as required.

$\square$

Exercise 3.3.23. Let $(X, \mathcal{A}, \mu)$ be a measure space and $\mathcal{B} \subset \mathcal{A}$ a sub σ-algebra. Define $\mu_{\mathcal{B}} = \mu|_{\mathcal{B}}$. Let $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$. Then $f \in L^1(X, \mathcal{A}, \mu)$ and for each $B \in \mathcal{B}$,

$$\int_B f d\mu_{\mathcal{B}} = \int_B f d\mu$$

Proof. Clearly f is $\mathcal{A}$-measurable. If f is simple, then there exist $(b_i)_{i=1}^n \subset [0, \infty)$ and $(B_i)_{i=1}^n \subset \mathcal{B}$ such that

$$f = \sum_{i=1}^n b_i \chi_{B_i}$$

such that for each $i \in \{1, \cdots, n\}$,

$$\infty > \mu_{\mathcal{B}}(B_i)$$
$$= \mu(B_i)$$

So $f \in L^1(X, \mathcal{A}, \mu)$ and

$$\int_B f d\mu_{\mathcal{B}} = \int_B \sum_{i=1}^n b_i \chi_{B_i} d\mu_{\mathcal{B}}$$
$$= \sum_{i=1}^n b_i \mu_{\mathcal{B}}(B_i \cap B)$$
$$= \sum_{i=1}^n b_i \mu(B_i \cap B)$$
$$= \int_B \sum_{i=1}^n b_i \chi_{B_i} d\mu$$
$$= \int_B f d\mu$$

If $f \geq 0$, then there exist $(\phi_n)_{n \in \mathbb{N}} \subset S^+(X, \mathcal{B})$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \xrightarrow{\text{p.w.}} f$. Then monotone convergence implies that for each $B \in \mathcal{B}$,

$$\int_B f d\mu_{\mathcal{B}} = \lim_{n \to \infty} \int_B \phi_n d\mu_{\mathcal{B}}$$
$$= \lim_{n \to \infty} \int_B \phi_n d\mu$$
$$= \int_B f d\mu$$

The statement also holds for general $f \in L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$, by writing $f = g + ih$ and applying the above to g^+, g^-, h^+ and h^-. $\square$

Definition 3.3.24. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $\|\cdot\|_* : L^1(\mu) \to [0, \infty)$ by

$$\|f\|_* = \sup_{A \in \mathcal{A}} \left| \int_A f \, d\mu \right|$$

Exercise 3.3.25. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $\|\cdot\|_*$ is a norm on $L^1(\mu)$ and there exists $C > 0$ such that $C\|\cdot\|_1 \leq \|\cdot\|_* \leq \|\cdot\|_1$.

3.4. **Integration on Product Spaces.**

Definition 3.4.1. Let X, Y, and Z be sets, $E \subset X \times Y$ and $f : X \times Y \to Z$. For each $x \in X$, define $E_x = \{y \in Y : (x, y) \in E\}$ and $f_x : Y \to Z$ by $f_x(y) = f(x, y)$. For each $y \in Y$, define $E^y = \{x \in X : (x, y) \in E\}$ and $f^y : X \to Z$ by $f^y(x) = f(x, y)$.

Note 3.4.2. It is often helpful to observe that $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.

Lemma 3.4.3. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces, $Z = [0, \infty]$ or $\mathbb{C}$ and $f : X \times Y \to Z$.
  (1) For each $E \in \mathcal{A} \otimes \mathcal{B}$, $x \in X$, $y \in Y$, we have that $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$
  (2) If f is $\mathcal{A} \otimes \mathcal{B}$-measurable, then for each $x \in X$, $y \in Y$, we have that f_x is $\mathcal{B}$-measurable and f^y is $\mathcal{A}$-measurable.

Theorem 3.4.4. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then for each $E \in \mathcal{A} \otimes \mathcal{B}$, the maps $\phi : X \to [0, \infty]$ and $\psi : Y \to [0, \infty]$ defined by $\phi(x) = \nu(E_x)$ and $\psi(y) = \mu(E^y)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, respectively and

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

Theorem 3.4.5. Fubini, Tonelli: Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.
  (1) (Tonelli) For each $f \in L^+(X \times Y)$, the functions $g : X \to [0, \infty]$, $h : Y \to [0, \infty]$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable respectively and

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

  (2) (Fubini) For each $f \in L^1(X \times Y)$, $f_x \in L^1(\nu)$ for μ-a.e. $x \in X$ and $f^y \in L^1(\mu)$ for ν-a.e. $y \in Y$, respectively and the functions (after redefinition of f on a null set) $g : X \to \mathbb{C}$, $h : Y \to \mathbb{C}$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Furthermore

$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

Note 3.4.6. We usually just write $\int \int f d\mu d\nu$ and $\int \int f d\nu d\mu$ instead of $\int h d\nu$ and $\int g d\mu$ respectively. We have a similar result for complete product measure spaces. See

Exercise 3.4.7. Take $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1]), \mathcal{B} = \mathcal{P}([0, 1])$ and μ, ν to be Lebesgue measure and counting measure respectively. Define $D = \{(x, y) \in [0, 1]^2 : x = y\}$ Show that

$$\int \chi_D d\mu \times \nu, \int \int \chi_D d\mu d\nu \text{ and } \int \int \chi_D d\nu d\mu$$

are all different. (Hint: for the first integral, use the definition of $\mu \times \nu$)

Proof. Let $x, y \in [0, 1]$. Then $(\chi_D)_x = \chi_{D_x} = \chi_x$ and $(\chi_D)^y = \chi_{D^y} = \chi_y$. Thus

$$\int \int \chi_D d\mu d\nu = \int \mu(\{y\}) d\nu$$
$$= \int 0 d\nu$$
$$= 0$$

and

$$\int \int \chi_D d\mu d\nu = \int \nu(\{x\}) d\mu$$
$$= \int 1 d\mu$$
$$= 1$$

Now, Observe that $\int \chi_D d\mu \times \nu = \mu \times \nu(D)$. Recall from the section on product measures that $\mu \times \nu(D) = \inf\{\sum_{n\in\mathbb{N}} \mu(A_n)\nu(B_n) : (A_n \times B_n)_{n\in\mathbb{N}} \subset \mathcal{E}$ and $D \subset \bigcup_{n\in\mathbb{N}} A_n \times B_n\}$. Let $(A_n \times B_n)_{n\in\mathbb{N}} \subset \mathcal{E}$. Suppose that $D \subset \bigcup_{n\in\mathbb{N}} A_n \times B_n$. Then for each $x \in [0,1]$, $(x,x) \in \bigcup_{n\in\mathbb{N}} A_n \times B_n$. So for each $x \in [0,1]$, there exists $n \in \mathbb{N}$, such that $x \in A_n \cap B_n$. Thus $[0,1] \subset \bigcup_{n\in\mathbb{N}} A_n \cap B_n$. Since $1 = \mu([0,1]) \leq \sum_{n\in\mathbb{N}} \mu(A_n \cap B_n)$, we know that there exists $n \in \mathbb{N}$ such that $0 < \mu(A_n \cap B_n)$. Thus $\mu(A_n) > 0$ and $\mu(B_n) > 0$. Since $\mu(B_n) > 0$, B_n must be infinite and therefore $\nu(B_n) = \infty$. So $\sum_{n\in\mathbb{N}} \mu(A_n)\nu(B_n) = \infty$.

$\square$

Exercise 3.4.8. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $f : X \to [0,\infty) \in L^+$. Show that $G = \{(x,y) \in X \times [0,\infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0,\infty))$ and $\mu \times m(G) = \int_X f d\mu$. The same is true if we replace "$\geq$" with "$>$". (Hint: to show that G is measurable, split up $(x,y) \mapsto f(x) - y$ into the composition of measurable functions.

Proof. Define $\phi : X \times [0,\infty) \to [0,\infty)^2$ and $\psi : [0,\infty)^2 \to [0,\infty)$ by $\phi(x,y) = (f(x),y)$ and $\psi(z,y) = z - y$. Then $G = \{(x,y) \in X \times [0,\infty) : \psi \circ \phi(x,y) \geq 0\}$. Let $A, B \in \mathcal{B}([0,\infty))$. Then $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0,\infty))$. Since $\mathcal{B}([0,\infty)^2) = \mathcal{B}([0,\infty)) \otimes \mathcal{B}([0,\infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0,\infty))\})$, we have that ϕ is $\mathcal{A} \otimes \mathcal{B}([0,\infty))$-$\mathcal{B}([0,\infty)^2)$ measurable. Since ψ is continuous, we have that ψ is $\mathcal{B}([0,\infty)^2)$-$\mathcal{B}([0,\infty))$ measurable. This implies that $\psi \circ \phi$ is $\mathcal{A} \otimes \mathcal{B}([0,\infty))$-$\mathcal{B}([0,\infty))$ measurable. Thus $G = \psi \circ \phi^{-1}([0,\infty)) \in \mathcal{A} \otimes \mathcal{B}([0,\infty))$. Now for $x \in X$, $G_x = \{y \in [0,\infty) : f(x) \geq y\} = [0, f(x)]$. Thus

$$\mu \times m(G) = \int \chi_G d\mu \times m$$
$$= \int_X \int_{[0,\infty)} \chi_{G_x} dm d\mu(x)$$
$$= \int_X f(x) d\mu(x)$$

The same reasoning holds if we replace "$\geq$" with "$>$".

$\square$

Exercise 3.4.9. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$. Define $h : X \times Y \to \mathbb{C}$ by $h(x,y) = f(x)g(y)$.

(1) If f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable, then h is $\mathcal{A} \otimes \mathcal{B}$-measurable.
(2) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and

$$\int_{X \times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu$$

Proof. (1) First suppose that f, g are simple. Then there exist $(A_i)_{i=1}^n \subset \mathcal{A}$, $(B_j)_{j=1}^m \subset \mathcal{B}$ and $(a_i)_{i=1}^n, (b_i)_{j=1}^m \subset \mathbb{C}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$ and $g = \sum_{j=1}^m b_j \chi_{B_j}$. Then

$h = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \chi_{A_i \times B_j}$. So h is $\mathcal{A} \otimes \mathcal{B}$-measurable. For general f, g, there exist $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$ and $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$ such that $f_n \to f$ pointwise, $g_n \to g$ pointwise and for each $n \in \mathbb{N}$, $|f_n| \leq |f_{n+1}| \leq |f|$ and $|g_n| \leq |g_{n+1}| \leq |g|$. For $n \in \mathbb{N}$, define $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$ by $h_n = f_n g_n$. Then $h_n \to h$ pointwise and for each $n \in \mathbb{N}$, $|h_n| \leq |h_{n+1}| \leq |h|$. Thus h is $\mathcal{A} \otimes \mathcal{B}$-measurable.

(2) First suppose f and g are simple as before. Then

$$
\begin{aligned}
\int_{X \times Y} |h| d\mu \times \nu &\leq \sum_{i=1}^{n} \sum_{j=1}^{m} |a_i b_j| \mu(A_i) \nu(B_j) \\
&= \Big(\sum_{i=1}^{n} |a_i| \mu(A_i) \Big) \Big(\sum_{j=1}^{m} |b_j| \nu(B_j) \Big) \\
&= \int_X |f| d\mu \int_Y |g| d\nu \\
&< \infty
\end{aligned}
$$

So $h \in L^1(\mu \times \nu)$. Furthermore,

$$
\begin{aligned}
\int_{X \times Y} h \, d\mu \times \nu &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \mu(A_i) \nu(B_j) \\
&= \Big(\sum_{i=1}^{n} a_i \mu(A_i) \Big) \Big(\sum_{j=1}^{m} b_j \nu(B_j) \Big) \\
&= \int_X f \, d\mu \int_Y g \, d\nu
\end{aligned}
$$

For general $f \in L^1(\mu)$, $g \in L^1(\nu)$, take $(h_n)_{n \in \mathbb{N}}$ as before. Monotone convergence and the result above say that

$$
\begin{aligned}
\int_{X \times Y} |h| d\mu \times d\nu &= \lim_{n \to \infty} \int_{X \times Y} |h_n| d\mu \times \nu \\
&= \lim_{n \to \infty} \Big(\int_X |f_n| d\mu \int_Y |g_n| d\nu \Big) \\
&= \int_X |f| d\mu \int_Y |g| d\nu \\
&< \infty
\end{aligned}
$$

So $h \in L^1(\mu \times \nu)$. Dominated convergence and the result above then tell us that

$$
\begin{aligned}
\int_{X \times Y} h \, d\mu \times d\nu &= \lim_{n \to \infty} \int_{X \times Y} h_n \, d\mu \times d\nu \\
&= \lim_{n \to \infty} \Big(\int_X f_n \, d\mu \int_Y g_n \, d\nu \Big) \\
&= \int_X f \, d\mu \int_Y g \, d\nu
\end{aligned}
$$

□

Note 3.4.10. In the above exercise part (2), we can replace L^1 with L^+ and get the same result by the same method.

Exercise 3.4.11. Let $f : \mathbb{R} \to [0, \infty) \in L^+$. Show that

$$\int_{\mathbb{R}} f \, dm = \int_{[0,\infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) dm(t)$$

Proof. Note that

$$\int_{[0,\infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) = \int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}} dm \right] dm(t)$$

Comparing this with Tonelli's theorem, we can put $\chi_{\{x \in \mathbb{R} : f(x) \geq t\}} = (\chi_E)^t = \chi_{E^t}$. Then $E = \{(x, t) \in \mathbb{R} \times [0, \infty) : f(x) \geq t\}$ and $E_x = \{t \in [0, \infty) : f(x) \geq t\} = [0, f(x)]$. Tonelli's theorem tells us that

$$\int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x \in \mathbb{R} : f(x) \geq t\}}(x) dm(x) \right] dm(t) = \int_{\mathbb{R}} \left[\int_{[0,\infty)} \chi_{[0,f(x)]}(t) dm(t) \right] dm(x)$$

$$= \int_{\mathbb{R}} f(x) dm(x)$$

□

3.5. **Modes of Convergence.**

Definition 3.5.1. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Then $(f_n)_{n \in \mathbb{N}}$ is said to **converge to f in measure**, denoted $f_n \xrightarrow{\mu} f$, if for each $\epsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \quad \text{as } n \to \infty$$

Definition 3.5.2. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$. Then $(f_n)_{n \in \mathbb{N}}$ is said to be **Cauchy in measure** if for each $\epsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \quad \text{as } n, m \to \infty$$

i.e. for each $\epsilon, \delta > 0$, there exists $N \in \mathbb{N}$ such that for each $n, m \in \mathbb{N}$, $n, m \geq N$ implies that $\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) < \delta$.

Exercise 3.5.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then $(f_n)_{n \in \mathbb{N}}$ is Cauchy in measure.

Proof. Suppose that $f_n \xrightarrow{\mu} f$. For $\epsilon > 0$ and $n, m \in \mathbb{N}$, set

$$A_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$$

and

$$B_{n,m,\epsilon} = \{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}$$

Let $\epsilon > 0$, $n, m \in \mathbb{N}$ and $x \in A^c_{n,\frac{\epsilon}{2}} \cap A^c_{m,\frac{\epsilon}{2}}$. Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon$$

and $x \in B^c_{n,m,\epsilon}$. Therefore $A^c_{n,\frac{\epsilon}{2}} \cap A^c_{m,\frac{\epsilon}{2}} \subset B^c_{n,m,\epsilon}$. This implies that $B_{n,m,\epsilon} \subset A_{n,\frac{\epsilon}{2}} \cup A_{m,\frac{\epsilon}{2}}$. Let $\delta > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $\mu(A_{n,\frac{\epsilon}{2}}) < \delta/2$. Then for each $n, m \in \mathbb{N}$, $n, m \geq N$ implies that

$$\mu(B_{n,m,\epsilon}) \leq \mu(A_{n,\frac{\epsilon}{2}}) + \mu(A_{m,\frac{\epsilon}{2}})$$
$$< \frac{\delta}{2} + \frac{\delta}{2}$$
$$= \delta$$

So for each $\epsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \quad \text{as } n, m \to \infty$$

and $(f_n)_{n \in \mathbb{N}}$ is Cauchy in measure. $\qquad \square$

Note 3.5.4. It is useful to observe that

$$\bigcup_{\epsilon > 0} \limsup_{n \to \infty} \{x \in X : |f_n(x) - f(x)| \geq \epsilon\} = \{x \in X : f_n(x) \not\to f(x)\}$$

and

$$\bigcap_{\epsilon > 0} \liminf_{n \to \infty} \{x \in X : |f_n(x) - f(x)| < \epsilon\} = \{x \in X : f_n(x) \to f(x)\}$$

Exercise 3.5.5. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$. Then $f_n \xrightarrow{\mu} f$.

Proof. For $n \in \mathbb{N}$ and $\epsilon > 0$, set

$$E_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$$

As noted above,

$$\{x \in X : f_n(x) \nrightarrow f(x)\} = \bigcup_{\epsilon > 0} \limsup_{n \to \infty} E_{n,\epsilon}$$

$$= \bigcup_{\epsilon > 0} \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} E_{n,\epsilon}$$

$\square$

Exercise 3.5.6. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$. Suppose that $(f_n)_{n \in \mathbb{N}}$ is Cauchy in measure.

(1) Then there exists a subsequence $(f_{n_j})_{j \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$ such that for each $j \in \mathbb{N}$,

$$\mu(\{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}) < 2^{-j}$$

(2) For $j, k \in \mathbb{N}$ set

$$E_j = \{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}$$

and

$$F_k = \bigcup_{j \geq k} E_j$$

Then for each $i, j, k \in \mathbb{N}$, $\mu(F_k) \leq 2^{1-k}$ $i \geq j \geq k$ implies that for each $x \in F_k^c$,

$$|f_{n_i}(x) - f_{n_j}(x)| \leq 2^{1-k}$$

So for each $k \in \mathbb{N}$, $(f_{n_j})_{j \in \mathbb{N}}$ is pointwise Cauchy on F_k^c.
Hint: get a telescoping sum via the triangle inequality
(3) Set

$$F = \bigcap_{k \in \mathbb{N}} F_k$$

Then $\mu(F) = 0$ and there exists $f \in L^0$ such that $f_{n_j} \xrightarrow{\text{a.e.}} f$.
(4) Finally, $f_{n_j} \xrightarrow{\mu} f$, $f_n \xrightarrow{\mu} f$ and for each $g \in L^0$, if $f_n \xrightarrow{\mu} g$, then $f = g$ a.e
Hint: consider showing $\{x \in X : |f_{n_k}(x) - f(x)| \geq \epsilon\} \subset F_k$

Proof.

(1) By definition, for each $j \in \mathbb{N}$, there exists $N_j \in \mathbb{N}$ such that for each $n, m \in \mathbb{N}$, $n, m \geq N_j$ implies that

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq 2^{-j}\}) < 2^{-j}$$

Thus we may obtain a subsequence (f_{n_j}) such that for each $j \in \mathbb{N}$,

$$\mu(\{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}) < 2^{-j}$$

(2) Let $k \in \mathbb{N}$. Part (1) implies that

$$\mu(F_k) \leq \sum_{j \geq k} 2^{-j}$$
$$= 2^{1-k} \sum_{j \geq 1} 2^{-j}$$
$$= 2^{1-k}$$

Let $i, j \in \mathbb{N}$. Suppose that $i \geq j \geq k$. Let $x \in F_k^c$. Then

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{l=j}^{i-1} |f_{n_{l+1}}(x) - f_{n_l}(x)|$$
$$< \sum_{l=j}^{i-1} 2^{-l}$$
$$< \sum_{l \geq j} 2^{-l}$$
$$= 2^{1-j}$$
$$\leq 2^{1-k}$$

Let $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $2^{1-k} < \epsilon$. Then for each $i, j \in \mathbb{N}$, $i, j \geq k$ implies that

$$|f_{n_i}(x) - f_{n_j}(x)| < 2^{1-k}$$
$$< \epsilon$$

So $(f_{n_j})_{j \in \mathbb{N}}$ is pointwise Cauchy on F_k^c

(3) Since $\mu(F_1) \leq 1$, and $F = \inf_{k \in \mathbb{N}} F_k$, we have that

$$\mu(F) = \inf_{k \in \mathbb{N}} \mu(F_k)$$
$$\leq \inf_{k \in \mathbb{N}} 2^{1-k}$$
$$= 0$$

Since for each $k \in \mathbb{N}$, $(f_{n_j})_{j \in \mathbb{N}}$ is pointwise Cauchy on F_k^c, $(f_{n_j})_{j \in \mathbb{N}}$ is pointwise Cauchy on F^c. Then $(f_{n_j} \chi_{F^c})_{j \in \mathbb{N}}$ is pointwise Cauchy.
Define $f : X \to \mathbb{C}$ pointwise by

$$f = \lim_{j \to \infty} f_{n_j} \chi_{F^c}$$

Then $f \in L^0$ since $(f_{n_j} \chi_{F^c})_{j \in \mathbb{N}} \subset L^0$ and $f_{n_j} \chi_{F^c} \xrightarrow{\text{p.w.}} f$. Since $\mu(F) = 0$ and $\{x \in X : f_{n_j}(x) \not\to f(x)\} \subset F$, we have that $f_{n_j} \xrightarrow{\text{a.e.}} f$.

(4) For $n, m \in \mathbb{N}$ and $\epsilon > 0$, set

$$A_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$$

and

$$B_{m,n,\epsilon} = \{x \in X : |f_m(x) - f_n(x)| \geq \epsilon\}$$

Let $\epsilon, \delta > 0$. Choose $k \in \mathbb{N}$ such that $2^{3-k} < \epsilon$ and $\mu(F_k) < \delta$. Let $x \in F_k^c$. Since $f_{n_j}(x) \to f(x)$, there exists $J \in \mathbb{N}$ such that $J \geq k$ and for each $j \in \mathbb{N}$, $j \geq J$ implies that $|f_{n_j}(x) - f(x)| < 2^{1-k}$. Let $l \in \mathbb{N}$. Suppose that $l \geq k$. Then part (2) implies that

$$|f_{n_l}(x) - f(x)| \leq |f_{n_l}(x) - f_{n_k}(x)| + |f_{n_k}(x) - f_{n_J}(x)| + |f_{n_J}(x) + f(x)|$$
$$\leq 2^{1-k} + 2^{1-k} + 2^{1-k}$$
$$\leq 2^{3-k}$$
$$< \epsilon$$

So $x \in A_{n_l,\epsilon}^c$. Hence $A_{n_l,\epsilon} \subset F_k$ and $\mu(A_{n_l},\epsilon) < \delta$. So $f_{n_j} \xrightarrow{\mu} f$.

Similarly to proof of Exercise 3.5.3, we have that for each $\epsilon > 0$ and $j \in \mathbb{N}$,

$$\mu(A_{n,\epsilon}) \leq \mu(B_{n,n_j,\frac{\epsilon}{2}}) + \mu(A_{n_j,\frac{\epsilon}{2}})$$

So $f_n \xrightarrow{\mu} f$. **FINISH!!!** $\square$

Exercise 3.5.7. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$. If $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, then there exists a $f \in L^0$ and a subsequence $(f_{n_j})_{j\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_n \xrightarrow{\mu} f$ and $f_{n_j} \xrightarrow{a.e.} f$. In addition, for each $g \in L^0$, if $f_n \xrightarrow{\mu} g$, then $f = g$ a.e.

Proof. Previous exercise. $\square$

Exercise 3.5.8. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{a.e.} f$.

Proof. Since $f_n \xrightarrow{\mu} f$, $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure. The previous exercise implies that there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{a.e.} f$. $\square$

Definition 3.5.9. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Then $(f_n)_{n\in\mathbb{N}}$ is said to **converge to** f **almost uniformly**, denoted $f_n \xrightarrow{a.u.} f$, if for each $\epsilon > 0$, there exists $N \in \mathcal{A}$ such that $\mu(N) < \epsilon$ and $f_n \xrightarrow{u} f$ on N^c.

Exercise 3.5.10. Egoroff's Theorem: Suppose that $\mu(X) < \infty$. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.e.} f$. Then $f_n \xrightarrow{a.u.} f$.

Proof. Let $\epsilon > 0$. For each $n, k \in \mathbb{N}$, define $E_{n,k} = \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{k}\}$ and $F_{n,k} = \bigcup_{m\geq n} E_{m,k}$. Then $F_{n,k}$ is decreasing in n and

$$\bigcap_{n\in\mathbb{N}} F_{n,k} \subset \{x : f_n(x) \not\to f(x)\}$$

Thus $\mu(\bigcap_{n\in\mathbb{N}} F_{n,k}) = 0$. Since $\mu(X) < \infty$, $\inf_{n\in\mathbb{N}} \mu(F_{n,k}) = 0$. Hence we may choose a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\mu(F_{n_k,k}) \leq \frac{\epsilon}{2^k}$. Put $N = \bigcup_{k\in\mathbb{N}} F_{n_k,k}$. Then

$$\mu(N) \leq \sum_{k\in\mathbb{N}} \mu(F_{n_k,k})$$
$$\leq \sum_{k\in\mathbb{N}} \frac{\epsilon}{2^k}$$
$$= \epsilon$$

Let $\delta > 0$. Choose $K \in \mathbb{N}$ such that $\frac{1}{K} < \delta$. Then for each $m \geq n_K$ and $x \in N^c = \bigcap_{k \in \mathbb{N}} \bigcap_{m \geq n_k} E_{m,k}^c$, $|f_m(x) - f(x)| < \frac{1}{K} < \delta$. So $f_n \xrightarrow{u} f$ on N^c. $\square$

Exercise 3.5.11. Let $(f_n)_{n \in \mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.

Proof. Let $\epsilon > 0$. for $n \in \mathbb{N}$, define $E_{\epsilon,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$. Then for $n \in \mathbb{N}$,

$$\int |f - f_n| \geq \int_{E_{\epsilon,n}} |f - f_n|$$
$$\geq \epsilon \mu(E_{\epsilon,n}).$$

So for each $n \in \mathbb{N}$, $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$. Since $\int |f - f_n| \to 0$, we have that $\mu(E_{\epsilon,n}) \to 0$. Since $\epsilon > 0$ is arbitrary, $f_n \xrightarrow{\mu} f$ as required. $\square$

Exercise 3.5.12. Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose $\mu(X) < \infty$. Define $d : L^0 \times L^0 \to [0, \infty)$ by

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} d\mu$$

Then d is a metric on L^0 if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each $f, g \in L^0$, $d(f, g) \leq \mu(X)$.

Proof. Let $f, g \in L^0$. Clearly $d(f, g) = d(g, f)$. If $f = g$ a.e. then clearly $d(f, g) = 0$. Conversely, if $d(f, g) = 0$, then $\frac{|f-g|}{1+|f-g|} = 0$ a.e and so $|f - g| = 0$ a.e. which implies $f = g$ a.e. It is not hard to show that $\phi : [0, \infty) \to [0, \infty)$ given by $\phi(x) = \frac{x}{1+x}$ satisfies $\phi(x + y) \leq \phi(x) + \phi(y)$. Thus satisfies the triangle inequality. Now, let $(f_n)_{n \in \mathbb{N}} \subset L^0$. Suppose that $f_n \xrightarrow{\mu} f$. Then there exists $\epsilon > 0, \delta > 0$ and a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that for each $k \in \mathbb{N}$, $\mu(E_{\epsilon,n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$. It is not hard to show that ϕ from earlier is increasing. Thus for each $k \in \mathbb{N}$,

$$d(f_{n_k}, f) = \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|}$$
$$\geq \int_{E_{\epsilon,n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|}$$
$$\geq \int_{E_{\epsilon,n_k}} \frac{\epsilon}{1 + \epsilon}$$
$$\geq \frac{\epsilon \delta}{1 + \epsilon}$$

So $f_{n_k} \xrightarrow{d} f$. Hence $f_{n_k} \xrightarrow{d} f$ implies that $f_{n_k} \xrightarrow{\mu} f$. Conversely, suppose that $f_{n_k} \xrightarrow{\mu} f$. Let $\epsilon > 0$. Then $\delta = \frac{\epsilon}{1+\mu(X)} > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then $\mu(E_{\delta,n}) < \frac{\delta}{1+\delta}$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Since ϕ is increasing and $\phi \leq 1$, we have

that

$$d(f_n, f) = \int \frac{|f_n - f|}{1 + |f_n - f|}$$

$$= \int_{E_{\delta,n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E_{\delta,n}^c} \frac{|f_n - f|}{1 + |f_n - f|}$$

$$\leq \mu(E_{\delta,n}) + \mu(X)\frac{\delta}{1 + \delta}$$

$$< \frac{\delta}{1 + \delta}(1 + \mu(X))$$

$$\leq \delta(1 + \mu(X))$$

$$= \epsilon$$

$\square$

Exercise 3.5.13. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \xrightarrow{\mu} f$. Then $f \geq 0$ a.e. and

$$\int f d\mu \leq \liminf_{n \to \infty} \int f_n d\mu$$

Proof. Since $f_n \xrightarrow{\mu} f$, there is a subsequence converging to f a.e. So clearly $f \geq 0$ a.e. Now, choose a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $\int f_{n_k} \to \liminf_{n \to \infty} \int f_n$. Since $f_n \xrightarrow{\mu} f$ so does $(f_{n_k})_{k \in \mathbb{N}}$. Therefore there exists a subsequence $(f_{n_{k_j}})_{k \in \mathbb{N}}$ of $(f_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Thus $f \geq 0$ a.e. and Fatou's lemma tells us that

$$\int f \leq \liminf_{j \in \mathbb{N}} \int f_{n_{k_j}}$$

$$= \liminf_{n \to \infty} \int f_n.$$

$\square$

Exercise 3.5.14. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \xrightarrow{\mu} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.

Proof. Clearly $(f_n)_{n \in \mathbb{N}} \subset L^1$. Since $f_n \xrightarrow{\mu} f$, there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}} \subset (f_n)_{n \in \mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$. This implies that $|f| \leq g$ a.e. and so $f \in L^1$. For $n \in \mathbb{N}$, put $h_n = 2g - |f_n - f|$. Then for each $n \in \mathbb{N}$, $h_n \geq 0$ and $h_n \xrightarrow{\mu} 2g$. By the previous exercise

$$\int 2g \leq \liminf_{n \to \infty} \int (2g - |f_n - f|)$$

$$= \int 2g - \limsup_{n \to \infty} \int |f_n - f|.$$

So $\limsup_{n \to \infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $f_n \xrightarrow{L^1} f$ as required. $\square$

Exercise 3.5.15. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$.

(1) If ϕ is continuous, and $f_n \xrightarrow{\text{a.e.}} f$ then $\phi \circ f_n \xrightarrow{\text{a.e.}} \phi \circ f$.

(2) If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly or in measure, respectively.

(3) Find a counter example to (2) if we drop the word "uniform".

Proof.   (1) Clear

(2) Suppose that ϕ is unifomly continuous.

(uniform conv.) Suppose that $f_n \xrightarrow{u} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Now choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq n$ then for each $x \in X$, $|f_n(x) - f(x)| < \delta$. Let $n \in \mathbb{N}$, suppose $n \geq N$, Let $x \in X$. Then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Thus $\phi \circ f_n \xrightarrow{u} \phi \circ f$.

(almost uni.) Suppose that $f_n \xrightarrow{a.u.} f$. Let $\epsilon > 0$. Choose $N \in \mathcal{A}$ such $\mu(N) < \epsilon$ and $f_n \xrightarrow{u} f$ on N^c. Then from above, we know that $\phi \circ f_n \xrightarrow{u} \phi \circ f$ on N^c. Thus $\phi \circ f_n \xrightarrow{a.u.} \phi \circ f$.

(measure) Suppose that $f_n \xrightarrow{\mu} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Observe that for $x \in X$, if $|f_n(x) - f(x)| < \delta$, then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Hence $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$. By definition of convergence in measure, $\mu(F_{n,\delta}) \to 0$. Thus $\mu(E_{n,\epsilon}) \to 0$. Hence $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.

(3)

$\square$

Exercise 3.5.16. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.u} f$. Then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{a.e.} f$.

Proof. (measure) Let $\epsilon > 0$, $\delta > 0$. Choose $M \in \mathcal{A}$ such that $\mu(M) < \delta$ and $f_n \xrightarrow{u} f$ on M^c. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then for each $x \in M^c$, $|f_n(x) - f(x)| < \epsilon$. Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then $E_{\epsilon,n} \subset M$ and $\mu(E_{\epsilon,n}) < \delta$. Thus $\mu(E_{\epsilon,n}) \to 0$ and $f_n \xrightarrow{\mu} f$.

(a.e.) For each $n \in \mathbb{N}$, Choose $N_n \in \mathcal{A}$ such that $\mu(N_n) < 1/n$ and $f_n \xrightarrow{u} f$ on N_n^c. Observe that for $x \in X$, if $x \in \bigcup_{n \in \mathbb{N}} N_n^c$, then $f_n(x) \to f(x)$. Thus $N = \{x \in X : f_n(x) \not\to f(x)\} \subset \bigcap_{n \in \mathbb{N}} N_n$. Therefor $\mu(N) = 0$ and $f_n \xrightarrow{a.e.} f$. $\square$

Exercise 3.5.17. Let $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then

(1) $f_n + g_n \xrightarrow{\mu} f + g$

(2) if $\mu(X) < \infty$, then $f_n g_n \xrightarrow{\mu} fg$

Proof.   (1) Let $\epsilon > 0$. For convenience, put $F_{n,\epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$, $G_{n,\epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$, and $(F + G)_{n,\epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g_n(x))| \geq \epsilon\}$ Observe that for $x \in X$, $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$. Thus $(F + G)_{n,\epsilon} \subset F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$. Since $\mu(F_{n,\epsilon/2} \cup G_{n,\epsilon/2}) \leq \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}) \to 0$, we have that $\mu((F + G)_{n,\epsilon}) \to 0$. Hence $f_n + g_n \xrightarrow{\mu} f + g$.

(2) Suppose that $\mu(X) < \infty$. Let $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(f_n g_n)_{n \in \mathbb{N}}$. Choose a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{a.e} f$ and $g_{n_{k_j}} \xrightarrow{a.e} g$. Then $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{a.e.} fg$. Egoroff's theorem tells us that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{a.u.} fg$, which implies that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu}$

fg. Thus for each subsequence $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ of $(f_n g_n)_{n \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ of $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$. Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that $f_n g_n \xrightarrow{\mu} fg$.

$\square$

Exercise 3.5.18. Let $(f_n)_{n \in \mathbb{N}}, \subset L^0$ and $f \in L^0$. Suppose that $\mu(X) < \infty$. Then $f_n \xrightarrow{\mu} f_n$ iff for each subsequence $(f_{n_k})_{k \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$.

Proof. Suppose that $f_n \xrightarrow{\mu} f$. Let $(f_{n_k})_{k \in \mathbb{N}}$ be a subsequence. Then $f_{n_k} \xrightarrow{\mu} f$. By a previous theorem, there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Conversely, suppose that for each subsequence $(f_{n_k})_{k \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Let $\epsilon > 0$. For $n \in \mathbb{N}$, define $E_n = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$ and define $E = \{x \in X : f_n(x) \not\to f(x)\}$. Let $(f_{n_k})_{k \in \mathbb{N}}$ be a subsequence. Choose a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Since $\left\{ x \in X : \limsup_{j \to \infty} \chi_{E_{n_{k_j}}}(x) = 1 \right\} = \limsup_{j \to \infty} E_{n_{k_j}} \subset E$ and $\mu(E) = 0$, we have that $\limsup_{j \to \infty} \chi_{E_{n_{k_j}}} = 0$ a.e. and $\chi_{E_{n_{k_j}}} \xrightarrow{\text{a.e}} 0$. Since $\mu(X) < \infty$, the dominated convergence theorem implies that

$$\mu(E_{n_{k_j}}) = \int \chi_{E_{n_{k_j}}} d\mu \to 0$$

So for each subsequence $(\mu(E_{n_k}))_{k \in \mathbb{N}}$, there exists a subsequence $(\mu(E_{n_{k_j}}))_{j \in \mathbb{N}}$ such that $\mu(E_{n_{k_j}}) \to 0$. Thus $\mu(E_n) \to 0$ and $f_n \xrightarrow{\mu} f$. $\square$

Exercise 3.5.19. Let $(f_n)_{n \in \mathbb{N}}, \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$. Suppose that $\mu(X) < \infty$. If ϕ is continuous and $f_n \xrightarrow{\mu} f$, then $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.

Proof. Suppose that ϕ is continuous and $f_n \xrightarrow{\mu} f$. Let $(\phi \circ f_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(\phi \circ f_n)_{n \in \mathbb{N}}$. Then $(f_{n_k})_{k \in \mathbb{N}}$ is a subsequence of $(f_n)_{n \in \mathbb{N}}$. Since $f_n \xrightarrow{\mu} f$, the previous exercise tells us that there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. A previous exercise implies that $\phi \circ f_{n_{k_j}} \xrightarrow{\text{a.e.}} \phi \circ f$. The previous exercise implies that $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$. $\square$

Exercise 3.5.20. Let $(f_n)_{n \in \mathbb{N}} L^0$ and $f \in L^0$. Suppose that for each $\epsilon > 0$,
$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$
Then $f_n \xrightarrow{\text{a.e.}} f$.

Proof. Let $\epsilon > 0$. By assumption we know that

$$\int \left[\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} \right] d\mu = \sum_{n \in \mathbb{N}} \int \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} d\mu$$
$$= \sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\})$$
$$< \infty$$

Thus we also know that $\sum_{n\in\mathbb{N}} \chi_{\{x\in X:|f_n(x)-f(x)|>\epsilon\}} < \infty$ a.e. Equivalently, we could say that for a.e. $x \in X$, $|\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\}| < \infty$. For $k \in \mathbb{N}$, define $N_k = \{x \in X : \sum_{n\in\mathbb{N}} \chi_{\{x\in X:|f_n(x)-f(x)|>1/k\}} = \infty\}$. Then for each $k \in \mathbb{N}$, $\mu(N_k) = 0$. Define $N = \bigcup_{k\in\mathbb{N}} N_k$. Then $\mu(N) = 0$. Let $x \in N^c$ and $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \epsilon$. Then $\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\} \subset \{n \in \mathbb{N} : f_n(x) - f(x) > 1/k\}$ which is finite because $x \in N_k^c$. Put $M = \max\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\}$. Then for $m \geq M$, $|f_m(x) - f(x) \leq \epsilon|$. Thus $f_n(x) \to f(x)$. Hence $f_n \xrightarrow{\text{a.e.}} f$. $\square$

4. Signed and Complex Measures

4.1. Signed Measures.

Definition 4.1.1. Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$. Then ν is said to be a **signed measure** if
  (1) for each $E \in \mathcal{A}$, $\nu(E) < \infty$ or for each $E \in \mathcal{A}$, $\nu(E) > -\infty$.
  (2) $\nu(\varnothing) = 0$
  (3) for each $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ if $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is disjoint, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$ and if $|\sum_{n \in \mathbb{N}} \nu(E_n)| < \infty$, then $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely.

Exercise 4.1.2. Let $\nu : \mathcal{A} \to [0, \infty]$ be a signed measure and $(E_n)_{n \in \mathbb{N}}$, $(F_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. If $(E_n)_{n \in \mathbb{N}}$ is increasing, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \lim_{n \to \infty} \nu(E_n)$. If $(F_n)_{n \in \mathbb{N}}$ is decreasing and $|\nu(E_1)| < \infty$, then $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \to \infty} \nu(F_n)$.

Proof. Put $E_1' = E_1$, $F_1' = F_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $E_n' = E_n \setminus E_{n-1}$ and $F_n' = F_1 \setminus F_n$. Then $(E_n')_{n \in \mathbb{N}} \subset \mathcal{A}$ is disjoint. Thus

$$\nu(\bigcup_{n \in \mathbb{N}} E_n) = \nu(\bigcup_{n \in \mathbb{N}} E_n')$$
$$= \sum_{n \in \mathbb{N}} \nu(E_n')$$
$$= \lim_{n \to \infty} \sum_{n=1}^{n} \nu(E_n')$$
$$= \lim_{n \to \infty} \nu(E_n)$$

Since $(F_n')_{n \in \mathbb{N}}$ is increasing, we now know that

$$\nu(F_1) - \nu(\bigcap_{n \in \mathbb{N}} F_n) = \nu(F_1 \setminus \bigcap_{n \in \mathbb{N}} F_n)$$
$$= \nu(\bigcup_{n \in \mathbb{N}} F_n')$$
$$= \lim_{n \to \infty} \nu(F_n')$$
$$= \lim_{n \to \infty} \nu(F_1 \setminus F_n)$$
$$= \nu(F_1) - \lim_{n \to \infty} \nu(F_n)$$

Since $|\nu(F_1)| < \infty$, we see that $\nu(\bigcap_{n \in \mathbb{N}} F_n) = \lim_{n \to \infty} \nu(F_n)$. $\square$

Definition 4.1.3. Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$ a signed measure and $E \in \mathcal{A}$. Then E is said to be ν-**positive**, ν-**negative** and ν-**null** if for each $F \in \mathcal{A}$, $F \subset E$ implies that $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ respectively.

Exercise 4.1.4. Let $E \subset \mathcal{A}$. If E is positive, negative or null, then for each $F \in \mathcal{A}$, if $F \subset E$, then F is positive, negative or null respectively.

Proof. Clear $\square$

Exercise 4.1.5. Let $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ be positive, negative or null. Then $\bigcup_{n \in \mathbb{N}} E_n$ is positive, negative or null respectively.

Proof. Suppose that $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ is positive. Let $F \in \mathcal{A}$. Suppose that $F \subset \bigcup_{n \in \mathbb{N}} E_n$. Put $P_1 = E_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $P_n = E_n \setminus (\bigcup_{j=1}^{n-1} E_j)$. So $\bigcup_{n \in \mathbb{N}} P_n = \bigcup_{n \in \mathbb{N}} E_n$ and $(P_n)_{n \in \mathbb{N}}$ is disjoint. Thus

$$\nu(F) = \nu(F \cap \bigcup_{n \in \mathbb{N}} P_n)$$
$$= \nu(\bigcup_{n \in \mathbb{N}} (F \cap P_n))$$
$$= \sum_{n \in \mathbb{N}} \nu(F \cap P_n)$$
$$\geq 0$$

The process is the same if $(E_n)_{n \in \mathbb{N}}$ is negative and null. $\square$

Theorem 4.1.6. Hahn Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist $P, N \in \mathcal{A}$ such that P is positive, N is negative, $X = N \cup P$ and $N \cap P = \varnothing$. Furthermore, these two sets are unique in the following sense: For any $P', N' \in \mathcal{A}$, if N, P satisfy the properties above, $P' \Delta P = N' \Delta N$ is ν-null.

Definition 4.1.7. Let ν be a signed measure on $(X, \mathcal{A})$ and $P, N \in \mathcal{A}$. Then P and N are said to form a **Hahn decomposition** of X with respect to ν if P, N satisfy the results in the above theorem.

Definition 4.1.8. Let μ, ν be signed measures on $(X, \mathcal{A})$. Then μ and ν are said to be **mutually singular** if there exist $E, F \in \mathcal{A}$ such that $X = E \cup F$, $E \cap F = \varnothing$ and E is μ-null and F is ν-null. We will denote this by $\mu \perp \nu$.

Theorem 4.1.9. Jordan Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{A})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof. Choose a Hahn decomposition P, N of X with respect to ν. Define ν^+, ν^- by $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$. $\square$

Definition 4.1.10. Let ν be a signed measure on $(X, \mathcal{A})$. Then ν^+ and ν^- from the last theorem are called the **positive** and **negative variations** of ν respectively. We define the **total variation of** ν, denoted $|\nu| : \mathcal{A} \to [0, \infty]$ by

$$|\nu| = \nu^+ + \nu^-$$

Definition 4.1.11. Let ν be a signed measure on $(X, \mathcal{A})$. Then ν is said to be σ-finite if $|\nu|$ is σ-finite.

Exercise 4.1.12. Let ν be a signed measure and λ, μ positive measures on $(X, \mathcal{A})$. Suppose that $\nu = \lambda - \mu$. Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. Choose a Hahn decomposition P, N of X with respect to ν. Let $E \in \mathcal{A}$. Then

$$\lambda(E \cap P) - \mu(E \cap P) = \nu(E \cap P)$$
$$= \nu^+(E \cap P)$$

So $\lambda(E \cap P) \geq \nu^+(E \cap P)$ and therefore

$$\begin{aligned}
\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N) \\
&\geq \nu^+(E \cap P) + \lambda(E \cap N) \\
&\geq \nu^+(E \cap P) \\
&= \nu^+(E)
\end{aligned}$$

Similarly $\mu(E \cap N) \geq \nu^-(E \cap N)$ and $\mu(E) \geq \nu^-(E)$. □

Exercise 4.1.13. Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Hint: use the last exercise)

Proof. Since

$$\begin{aligned}
\nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\
&= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)
\end{aligned}$$

the previous exercise tells us that $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$ and $\mu = \nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$.
Therefore

$$\begin{aligned}
|\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\
&\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\
&= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\
&= |\nu_1| + |\nu_2|
\end{aligned}$$

□

Note 4.1.14. Recall that a previous exercise from the section on complex valued functions tells us that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.

Definition 4.1.15. Let ν be a signed measure on $(X, \mathcal{A})$. Then we define $L^1(\nu) = L^1(|\nu|)$. For $f \in L^1(\nu)$, we define

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

Exercise 4.1.16. Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$

Proof. The previous exercise tells us that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. Two previous exercises from the section on nonnegative functions tells us that

$$\begin{aligned}
\int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\
&= \int |f| d|\nu_1| + \int |f| d|\nu_2|
\end{aligned}$$

□

Exercise 4.1.17. Let ν, μ be signed measures on $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then

(1) E is ν-null iff $|\nu|(E) = 0$
(2) $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof. (1) Suppose that E is ν-null. Choose a Hahn decomposition P, N of X with respect to ν. Then $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = \nu(E \cap N) = 0$. Therefore $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Conversely, suppose that $|\nu|(E) = 0$. Then $\nu^+(E) = \nu^-(E) = 0$. Let $F \in \mathcal{A}$. Suppose that $F \subset E$. Then $\nu^+(F) = 0$ and $\nu^-(F) = 0$. Therefore $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. So E is ν-null.

(2) Suppose that $\nu \perp \mu$. Then there exist $E, F \in \mathcal{A}$ such that $E \cup F = X$, $E \cap F = \varnothing$, E is μ-null and F is ν-null. By (1), F is $|\nu|$-null and thus $|\nu| \perp \mu$. If $|\nu| \perp \mu$, choose $E, F \in \mathcal{A}$ as before. Since F is $|\nu|$-null, we know that $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$. This implies that F is ν^+-null and F is ν^--null. So $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Finally assume that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. **FINISH!!!!**

$\square$

Exercise 4.1.18. Let ν be a signed measure on $(X, \mathcal{A})$. Then
(1) for $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$
(2) if ν is finite, then for each $E \in \mathcal{A}$,

$$|\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| : f \text{ is measurable and } |f| \leq 1 \right\}$$

Proof. (1) Let $f \in L^1(\nu)$. Then

$$\left| \int f d\nu \right| = \left| \int f d\nu^+ - \int f d\nu^- \right|$$

$$\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right|$$

$$\leq \int |f| d\nu^+ + \int |f| d\nu^-$$

$$= \int |f| d(\nu^+ + \nu^-)$$

$$= \int |f| d|\nu|$$

(2) Let $E \in \mathcal{A}$. Let $f : X \to \mathbb{R}$ be measurable and suppose that $|f| \leq 1$. Since ν is finite, so is $|\nu|$ and thus $f \in L^1(\nu)$. Then (1) tells us that

$$\left| \int_E f d\nu \right| \leq \int_E |f| d|\nu|$$

$$\leq |\nu|(E)$$

Now, choose a Hahn decomposition P, N of X with respect to ν. Define $f = \chi_P - \chi_N$. Then $|f| \leq 1$, f is measurable and

$$\left| \int_E f d\nu \right| = \left| \int_E f d\nu^+ - \int_E f d\nu^- \right|$$

$$= |\nu^+(E \cap P) + \nu^-(E \cap N)|$$

$$= \nu^+(E) + \nu^-(E)$$

$$= |\nu|(E).$$

$\square$

Exercise 4.1.19. Let μ be a positive measure on $(X, \mathcal{A})$ and $f \in L^0(X, \mathcal{A})$ extended μ-integrable. Define ν on $(X, \mathcal{A})$ by

$$\nu(E) = \int_E f d\mu$$

Then

(1) ν is a signed measure
(2) for each $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E |f| d\mu$$

Proof.   (1) Clearly $\nu(\varnothing) = 0$ and ν is finte by assumption. Let $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_n)_{n \in \mathbb{N}}$ is disjoint. Then

$$
\begin{aligned}
\nu(\bigcup_{n \in \mathbb{N}} E_n) &= \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \\
&= \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \\
&= \sum_{n \in \mathbb{N}} \int_{E_n} f^+ d\mu - \sum_{n \in \mathbb{N}} \int_{E_n} f^- d\mu \\
&= \sum_{n \in \mathbb{N}} \left[\int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu \right] \\
&= \sum_{n \in \mathbb{N}} \int_{E_n} f d\mu \\
&= \sum_{n \in \mathbb{N}} \nu(E_n)
\end{aligned}
$$

If $|\nu(\bigcup_{n \in \mathbb{N}} E_n)| < \infty$, then $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu < \infty$ and $\int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu < \infty$ because

$$
\begin{aligned}
|\nu(\bigcup_{n \in \mathbb{N}} E_n)| &= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f d\mu \right| \\
&= \left| \int_{\bigcup_{n \in \mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n \in \mathbb{N}} E_n} f^- d\mu \right|
\end{aligned}
$$

Therefore, we have that

$$\sum_{n\in\mathbb{N}}|\nu(E_n)| = \sum_{n\in\mathbb{N}}\left|\int_{E_n}fd\mu\right|$$

$$= \sum_{n\in\mathbb{N}}\left|\int_{E_n}f^+d\mu - \int_{E_n}f^-d\mu\right|$$

$$\leq \sum_{n\in\mathbb{N}}\int_{E_n}f^+d\mu + \sum_{n\in\mathbb{N}}\int_{E_n}f^-d\mu$$

$$= \int_{\bigcup_{n\in\mathbb{N}}E_n}f^+d\mu + \int_{\bigcup_{n\in\mathbb{N}}E_n}f^-d\mu$$

$$< \infty$$

So the sum $\sum_{n\in\mathbb{N}}\nu(E_n)$ converges absolutely and ν is a signed measure.

(2) Put $P = \{x \in X : f(x) \geq 0\}$ and $N = \{x \in X : f(x) < 0\}$. Then P, N form a Hahn decomposition of X with respect to ν. Thus for $E \in \mathcal{A}$,

$$\nu^+(E) = \int_{E\cap P}fd\mu = \int_E f^+d\mu$$

and

$$\nu^-(E) = \int_{E\cap N}fd\mu = \int_E f^-d\mu$$

So for $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E f^+d\mu + \int_E f^-d\mu = \int_E |f|d\mu$$

$\square$

Definition 4.1.20. Let $(X, \mathcal{A})$ be a measureable space, ν be a signed measure on $(X, \mathcal{A})$ and μ a measure on $(X, \mathcal{A})$. Then ν is said to be **absolutely continuous** with respect to μ, denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.

Note 4.1.21. If there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that for each $E \in \mathcal{A}$, $\nu(E) = \int_E fd\mu$, then we write $d\nu = fd\mu$.

Theorem 4.1.22. Let $(X, \mathcal{A})$ be a measureable space, ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that $d\rho = fd\mu$ and f is unique μ-a.e.

Definition 4.1.23. The decomposition $\nu = \lambda + \rho$ is referred to as the **Lebesgue decomposition of ν with respect to μ**. In the case $\nu \ll \mu$, we have $\lambda = 0$ and $\rho = \nu$ and we define the **Radon-Nikodym derivative of ν with respect to** μ, denoted by $d\nu/d\mu$, to be $d\nu/d\mu = f$ where $d\nu = fd\mu$.

Theorem 4.1.24. Let ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then

(1) for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int gd\nu = \int g\frac{d\nu}{d\mu}d\mu$$

(2) $\nu \ll \lambda$ and
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Exercise 4.1.25. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures and μ a measure.
  (1) If for each $n \in \mathbb{N}$, $\nu_n \ll \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.
  (2) If for each $n \in \mathbb{N}$, $\nu_n \perp \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \perp \mu$.

Proof.   (1) Let $E \in \mathcal{A}$. Suppose that $\mu(E) = 0$. Then for each $n \in \mathbb{N}$, $\nu_i(E) = 0$ and
      thus $\sum_{n\in\mathbb{N}} \nu_n(E) = 0$. Hence $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.
  (2) For each $n \in \mathbb{N}$, there exist $N_i, M_i \in \mathcal{A}$ such that $N_i \cap M_i = \varnothing$, $N_i \cup M_i = X$ and
      $\nu_i(M_i) = \mu(N_i) = 0$. Put $N = \bigcup_{n\in\mathbb{N}} N_i$ and $M = N^c$. Note that for each $n \in \mathbb{N}$,
      $M \subset N_i^c = M_i$. So $\mu(N) \le \sum_{n\in\mathbb{N}} \mu(N_i) = 0$ and $(\sum_{n\in\mathbb{N}} \nu_i)(M) \le \sum_{n\in\mathbb{N}} \nu_i(M_i) = 0$.
      Thus $\sum_{n\in\mathbb{N}} \nu_i \perp \mu$.
                                                                                      $\square$

Exercise 4.1.26. Choose $X = [0,1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$. Let m be Lebesgue measure and μ the
counting measure.
    Then
    (1) $m \ll \mu$ but for each $f \in L^+$, $dm \ne f d\mu$
    (2) There is no Lebesgue decomposition of μ with respect to m.

Proof.   (1) Let $E \in \mathcal{A}$. If $\mu(E) = 0$, then $E = \varnothing$ and $m(E) = 0$. So $m \ll \mu$. Suppose for
      the sake of contradiction that there exists $f \in L^+$ such that $dm = f d\mu$. Then
$$1 = m(X)$$
$$= \sum_{x\in X} f(x)$$
      Put $Z = \{x \in X : f(x) \ne 0\}$. Then Z is countable. So
$$1 = m(X \setminus Z)$$
$$= \sum_{x\in X\setminus Z} f(x)$$
$$= 0$$
      This is a contradiction, so no such f exists.
    (2) Suppose for the sake of contradiction that there is a Lebesgue decomposition for μ
      with respect to m given by $\mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho \ll m$. We may assume λ
      and ρ are positive. Then for each $x \in X$, $m(\{x\}) = 0$ which implies that $\rho(\{x\}) = 0$.
      Let $E \subset X$, if E is countable, then $\lambda(E) = \mu(E)$. If E is uncountable, choose $F \subset E$
      such that F is countable. Then
$$\lambda(E) \ge \lambda(F)$$
$$= \mu(F)$$
$$= \infty$$
      So $\lambda = \mu$. This is a contradiction since $\mu \not\perp m$.
                                                                                      $\square$

4.2. **Complex Measures.**

Definition 4.2.1. Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to \mathbb{C}$. Then ν is said to be a **complex measure** if

(1) $\nu(\varnothing) = 0$
(2) for each sequence $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, if $(E_n)_{n \in \mathbb{N}}$ is disjoint, then $\nu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \nu(E_n)$ and $\sum_{n \in \mathbb{N}} \nu(E_n)$ converges absolutely.

Definition 4.2.2. Let $(X, \mathcal{A})$ be a measurable space. We define

$$\mathcal{M}(X, \mathcal{A}) = \{\mu : \mathcal{A} \to \mathbb{C} : \mu \text{ is a complex measure}\}$$

When X is a topological space, we write $\mathcal{M}(X)$ in place of $\mathcal{M}(X, \mathcal{B}(X))$.

Note 4.2.3. We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.

Definition 4.2.4. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in \mathcal{M}(X, \mathcal{A})$ with $\nu = \nu_1 + i\nu_2$. We define $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$. For $f \in L^1(\nu)$, we define

$$\int f \, d\nu = \int f \, d\nu_1 + i \int f \, d\nu_2$$

Theorem 4.2.5. Lebesgue-Radon-Nikodym Theorem:
Let $(X, \mathcal{A})$ be a measurable space, $\nu \in \mathcal{M}(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exists unique $\lambda, \rho \in \mathcal{M}(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists $f \in L^1(\mu)$ such that $d\rho = f \, d\mu$ and f is unique μ-a.e.

Exercise 4.2.6. Let $\nu \in \mathcal{M}(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then

(1) for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g \, d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2) $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Definition 4.2.7. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in \mathcal{M}(X, \mathcal{A})$ with $\nu = \nu_1 + i\nu_2$. Define $\mu = |\nu_1| + |\nu_2|$. Then $\nu \ll \mu$ and thus there exists $f \in L^1(\mu)$ such that $d\nu = f \, d\mu$. We define the **total variation of** ν, denoted $|\nu| : \mathcal{A} \to [0, \infty)$, by

$$|\nu|(E) = \int_E |f| \, d\mu$$

Exercise 4.2.8. Let $(X, \mathcal{A})$ be a measurable space, $\nu \in \mathcal{M}(X, \mathcal{A})$ and λ a σ-finite measure on $(X, \mathcal{A})$. Suppose that $\nu \ll \lambda$. Set $g = d\nu/d\lambda$. Then for each $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E |g| \, d\lambda$$

Proof. Write $\nu = \nu_1 + i\nu_2$. Then $\nu_1, \nu_2 \ll \lambda$. Set $f_1 = d\nu_1/d\lambda$ and $f_2 = d\nu_2/d\lambda$. Then Exercise 4.1.19 implies that $d|\nu_1| = |f_1|d\lambda$ and $d|\nu_2| = |f_2|d\lambda$. Set $\mu = |\nu_1| + |\nu_2|$ and $f = d\nu/d\mu$ as in Definition 4.2.7. Then by construction,

$$d\mu = d|\nu_1| + d|\nu_2|$$
$$= |f_1|d\lambda + |f_2|d\lambda$$
$$= (|f_1| + |f_2|)d\lambda$$

So that $\mu \ll \lambda$ with $d\mu/d\lambda = |f_1| + |f_2|$. Then Exercise 4.2.6 implies that $\nu \ll \lambda$ with

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$
$$= f(|f_1| + |f_2|)$$
$$= g$$

and for each $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E |f| d\mu$$
$$= \int_E |f|(|f_1| + |f_2|)\, d\lambda$$
$$= \int_E |g|\, d\lambda$$

$\square$

Exercise 4.2.9. Let $\nu \in \mathcal{M}(X, \mathcal{A})$ and μ a σ-finite measures on $(X, \mathcal{A})$. If $\nu \ll \mu$, then $\{x \in X : d\nu/d\mu(x) = 0\}$ is ν-null.

Proof. Define $f = d\nu/d\mu$ and $E = \{x : f(x) = 0\}$. Let $A \in \mathcal{A}$ and suppose that $A \subset E$. Then

$$\nu(A) = \int_A f d\mu$$
$$= 0$$

$\square$

Exercise 4.2.10. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in \mathcal{M}(X, \mathcal{A})$ with $\nu = \nu_1 + i\nu_2$. Then $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu_2|$.

Proof. Let μ and $f = f_1 + if_2$ be as in the definition of $|\nu|$. Since for each $E \in \mathcal{A}$, we have

$$\nu(E) = \int_E f d\mu$$
$$= \int_E f_1 d\mu + i \int_E f_2 d\mu$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E)$$

we know that $\nu_1 = f_1 d\mu$ and $\nu_2 = f_2 d\mu$.

A previous exercise tells us that $d|\nu_1| = |f_1|d\mu$ and $d|\nu_2| = |f_2|d\mu$. Since $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$, we have that

$$|\nu_1|, |\nu_2| \leq |\nu|$$
$$\leq |\nu_1| + |\nu_2|$$

$\square$

Exercise 4.2.11. Let $(X, \mathcal{A})$ be a measurable space, $\nu \in \mathcal{M}(X, \mathcal{A})$ and $c \in \mathbb{C}$. Then $|c\nu| = |c||\nu|$.

Proof. Define μ and f as before so that $d\nu = f d\mu$. Then $d(c\nu) = cf d\mu$. Hence

$$d|c\nu| = |cf|d\mu$$
$$= |c||f|d\mu$$
$$= |c|d|\nu|$$

So $|c\nu| = |c||\nu|$. $\square$

Exercise 4.2.12. Define $\| \cdot \| : \mathcal{M}(X, \mathcal{A}) \to [0, \infty)$ by

$$\|\mu\| = |\mu|(X)$$

Then $\| \cdot \|$ is a norm on $\mathcal{M}(X, \mathcal{A})$.

Proof. Let $\mu, \nu \in \mathcal{M}(X, \mathcal{A})$ and $\alpha \in \mathbb{C}$. The previous exercises tell us that $|\mu + \nu| \leq |\mu| + |\nu|$ and $|\alpha\mu| = |\alpha||\mu|$. So clearly $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ and $\|c\mu\| = |c|\|\mu\|$. If $\|\mu\| = 0$, then X is μ-null and μ is the zero measure. $\square$

Exercise 4.2.13. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in \mathcal{M}(X, \mathcal{A})$. Then
   (1) for each $E \in \mathcal{A}$, $|\nu(E)| \leq |\nu|(E)$.
   (2) $\nu \ll |\nu|$ and $\big|d\nu/d|\nu|\big| = 1$ $|\nu|$-a.e.
   (3) $L^1(\nu) = L^1(|\nu|)$ and for each $g \in L^1(\nu)$,

$$\left| \int g d\nu \right| \leq \int |g| d|\nu|$$

Proof. Let $\mu, f \in L^1(\mu)$ be as in the definition of $|\nu|$.
   (1) Let $E \in \mathcal{A}$. Then

$$|\nu(E)| = \left| \int_E f d\mu \right|$$
$$\leq \int_E |f| d\mu$$
$$= |\nu|(E)$$

   (2) Let $E \in \mathcal{A}$ and suppose that $|\nu|(E) = 0$. The previous part implies $|\nu(E)| = 0$ and $\nu \ll |\nu|$. Put $g = d\nu/d|\nu|$. Then

$$f = \frac{d\nu}{d\mu}$$
$$= g|f| \quad \mu\text{-a.e.}$$

Hence $|f| = |g||f|$ μ-a.e. Since $|\nu| \ll \mu$, $|f| = |g||f|$ $|\nu|$-a.e.
A previous exercise tells us that $|f| \neq 0$ $|\nu|$-a.e. Thus $|g| = 1$ $|\nu|$-a.e.

(3) Write $\nu = \nu_1 + i\nu_2$ and $f = f_1 + if_2$. First we observe that

$$
\begin{aligned}
L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\
&= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\
&= L^1(|\nu_1| + |\nu_2|) \\
&= L^1(\mu)
\end{aligned}
$$

The previous exercise tells us that

$$
\begin{aligned}
|\nu_1|, |\nu_2| &\leq |\nu| \\
&\leq |\nu_1| + |\nu_2| \\
&= \mu
\end{aligned}
$$

Let $g \in L^1(\mu)$. Then

$$
\int |g|d|\nu| \leq \int |g|d\mu \\
< \infty
$$

So $g \in L^1(|\nu|)$. Conversely, let $g \in L^1(|\nu|)$. Then

$$
\int |g|d|\nu_1|, \int |g|d|\nu_2| \leq \int |g|d|\nu| \\
< \infty
$$

So

$$
\int |g|d\mu = \int |g|d|\nu_1| + \int |g|d|\nu_2| \\
< \infty
$$

and $g \in L^1(\mu)$. Hence $L^1(\nu) = L^1(|\nu|)$. Now, let $g \in L^1(\nu) = L^1(|\nu|)$, then

$$
\begin{aligned}
\left| \int g d\nu \right| &= \left| \int gf d\mu \right| \\
&\leq \int |g||f|d\mu \\
&= \int |g|d|\nu|
\end{aligned}
$$

$\square$

Exercise 4.2.14. Let $(X, \mathcal{A})$ be a measurable space, $\mu_1, \mu_2 \in \mathcal{M}(X, \mathcal{A})$ and $\lambda \in \mathbb{C}$. Then for each $f \in L^1(\mu_1 + \lambda\mu_2)$,

$$
\int f d(\mu_1 + \lambda\mu_2) = \int f d\mu_1 + \lambda \int f d\mu_2
$$

Proof. Clear by an exercise in section 3.2. $\square$

4.3. **Weak Convergence of Measures.**

Definition 4.3.1. Let X be a topological space. For $f \in C_b(X)$, define $\lambda_f : \mathcal{M}(X) \to \mathbb{C}$ by

$$\lambda_f(\mu) = \int f \, d\mu$$

.

Exercise 4.3.2. Let X be a topological space. For each $f \in C_b(X)$, $\lambda_f \in \mathcal{M}(X)^*$.

Proof. Let $f \in C_b(X)$ and $\mu \in \mathcal{M}(X)$. Then

$$|\lambda_f(\mu)| = \left| \int f \, d\mu \right|$$
$$\leq \int |f| \, d|\mu|$$
$$\leq \|f\|_u \|\mu\|$$

Exercise 4.2.14 implies that λ_f is linear. So $\lambda_f \in \mathcal{M}(X)^*$. $\qquad\square$

Definition 4.3.3. Let X be a topological space. We define the **weak topology on $\mathcal{M}(X)$** to be the weak topology generated by $\{\lambda_f \in \mathcal{M}(X)^* : f \in C_b(X)\}$.

Definition 4.3.4. Let X be a topological space and $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Then $(\mu_n)_{n\in\mathbb{N}}$ is said to **converge weakly** to μ, denoted $\mu_n \xrightarrow{w} \mu$, if $(\mu_n)_{n\in\mathbb{N}}$ converges to μ in the weak topology, i.e. for each $f \in C_b(X)$,

$$\int f \, d\mu_n \to \int f \, d\mu$$

Exercise 4.3.5. Portmanteau Theorem: Let X be a topological space and $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Suppose that for each $n \in \mathbb{N}$, $\mu_n(X) = \mu(X)$. Then the following are equivalent:

(1) $\mu_n \xrightarrow{w} \mu$
(2) for each $A \in \mathcal{B}(X)$, A is open implies that $\mu(A) \leq \liminf_{n\to\infty} \mu_n(A)$
(3) for each $A \in \mathcal{B}(X)$, A is closed implies that $\mu(A) \geq \limsup_{n\to\infty} \mu_n(A)$
(4) for each $A \in \mathcal{B}(X)$, $\mu(\partial A) = 0$ implies that $\mu_n(A) \to \mu(A)$

Proof.

- (2) $\iff$ (3):
  Suppose (2). Let $A \in \mathcal{B}(X)$. Suppose that A is closed. Then A^c is open. By

assumption, $\mu(A^c) \leq \liminf\limits_{n\to\infty} \mu_n(A^c)$. Hence

$$
\begin{aligned}
\mu(A) &= \mu(X) - \mu(A^c) \\
&\geq \mu(X) - \liminf_{n\to\infty} \mu_n(A^c) \\
&= \mu(X) + \limsup_{n\to\infty} [-\mu_n(A^c)] \\
&= \limsup_{n\to\infty} \left[\mu(X) - \mu_n(A^c) \right] \\
&= \limsup_{n\to\infty} \left[\mu_n(X) - \mu_n(A^c) \right] \\
&= \limsup_{n\to\infty} \mu_n(A)
\end{aligned}
$$

So (3) holds. Similarly, (3) implies (2).

- (2) $\iff$ (4):
  Suppose (2). From above, (3) holds. Let $A \in \mathcal{B}(X)$. Then

$$
\begin{aligned}
\mu(A^\circ) &\leq \liminf_{n\to\infty} \mu_n(A^\circ) \\
&\leq \liminf_{n\to\infty} \mu_n(A) \\
&\leq \limsup_{n\to\infty} \mu_n(A) \\
&\leq \limsup_{n\to\infty} \mu_n(\overline{A}) \\
&\leq \mu(\overline{A})
\end{aligned}
$$

Suppose that $\mu(\partial A) = 0$. Then

$$
\begin{aligned}
\mu(A^\circ) &\leq \mu(A) \\
&\leq \mu(\overline{A}) \\
&= \mu(A^\circ) + \mu(\partial A) \\
&= \mu(A^\circ)
\end{aligned}
$$

which implies that $\mu_n(A) \to \mu(A)$. Conversely, suppose (4).

-

$\square$

5. Applications to Differentiation

5.1. Differentiation on $\mathbb{R}^n$.

Definition 5.1.1. Let $B \subset \mathbb{R}^n$. Then B is said to be a **ball** if there exists $x \in \mathbb{R}^n$ and $r > 0$ such that $B = B(x, r)$.

Definition 5.1.2. Let $f \in L^0(\mathbb{R}^n)$. Then f is said to be **locally integrable** (with respect to Lebesgue measure) if f is measurable and for each $K \subset \mathbb{R}$, K is compact implies $\int_K |f| dm < \infty$. We define $L^1_{\text{loc}}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is locally integrable}\}$

Definition 5.1.3. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$, we define the **average of f over** $B(x, r)$, denoted by $Af(x, r)$, to be

$$Af(x, r) = \frac{1}{m(B(x, r))} \int_{B(x,r)} f dm$$

Exercise 5.1.4. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Define

$$H^* f(x) = \sup\{\frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B\} \quad (x \in \mathbb{R}^n)$$

Then $Hf \leq H^* f \leq 2^n Hf$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\left\{ \frac{1}{m(B(x,r))} \int_{B(x,r)} |f| dm : r > 0 \right\} \subset \left\{ \frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B \right\}$$

So $Hf(x) \leq H^* f(x)$. Let B be a ball. Then there exists $y \in \mathbb{R}^n$, $R > 0$ such that $B = B(y, R)$ Suppose that $x \in B$. Then $B \subset B(x, 2R)$. Since $m(B(x, 2R)) = 2^n m(B(y, R))$, we have that

$$\frac{1}{m(B)} \int_B |f| dm \leq \frac{1}{m(B)} \int_{m(B(x,2R))} |f| dm$$
$$= \frac{2^n}{m(B(x, 2R))} \int_{m(B(x,2R))} |f| dm$$

Thus $H^* f(x) \leq 2^n Hf(x)$. $\square$

Lemma 5.1.5. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $Af : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ is continuous.

Definition 5.1.6. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We define its **Hardy Littlewood maximal function**, denoted by Hf to be

$$Hf(x) = \sup_{r > 0} A|f|(x, r) \quad x \in \mathbb{R}^n$$

Theorem 5.1.7. There exists $C > 0$ such that for each $f \in L^1(m)$ and $\alpha > 0$,

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{a} \int |f| dm$$

Exercise 5.1.8. Let $f \in \mathrm{L}^1(\mathbb{R}^n)$. Suppose that $||f||_1 > 0$. Then there exist $C, R > 0$ such that for each $x \in \mathbb{R}^n$, if $|x| > R$, then $Hf(x) \geq C|x|^{-n}$. Hence there exists $C' > 0$ such that for each $\alpha > 0$, $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$ when α is small.

Proof. Since $||f||_1 > 0$, there exists $R > 0$ such that $\int_{B(0,R)} |f| dm > 0$. Recall that there exists $K > 0$ such that for each $x \in R^n$ and $r > 0$, $m(B(x,r)) = Kr^n$. Choose

$$C = \frac{1}{K2^n} \int_{B(0,R)} |f| \, dm$$

. Let $x \in \mathbb{R}^n$. Suppose that $|x| > R$. Then $B(0,R) \subset B(x, 2|x|)$. Thus

$$Hf(x) \geq \frac{1}{m(B(x, 2|x|))} \int_{B(x,2|x|)} |f| dm$$

$$= \frac{1}{K2^n |x|^n} \int_{B(x,2|x|)} |f| dm$$

$$\geq \frac{1}{K2^n |x|^n} \int_{B(0,R)} |f| dm$$

$$= \frac{C}{|x^n|}$$

Let $a < \frac{C}{2R^n}$. Then $R^n < \frac{C}{2a}$. Choose $C' = \frac{KC}{2}$. Let $A = \{x \in \mathbb{R}^n : R < |x| < (\frac{C}{\alpha})^{\frac{1}{n}}\}$. For $x \in A$,

$$Hf(x) \geq \frac{C}{|x|^n}$$

$$> \alpha$$

Thus $A \subset m(\{x \in R^n : Hf(x) > \alpha\})$ and therefore

$$m(\{x \in R^n : Hf(x) > \alpha\}) \geq m(A)$$

$$= m(B(0, (C/\alpha)^{1/n})) - m(B(0, R))$$

$$= K\left[\frac{C}{\alpha} - R^n\right]$$

$$> K\left[\frac{C}{\alpha} - \frac{C}{2\alpha}\right]$$

$$= \frac{KC}{2\alpha}$$

$$= \frac{C'}{\alpha}$$

$\square$

Theorem 5.1.9. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} Af(x, r) = f(x)$$

Equivalently, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} [f(y) - f(x)] dm(y)\right] = 0$$

Note 5.1.10. We can a stronger result of the same flavor.

Definition 5.1.11. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We define the **Lebesgue set of** f, denoted by L_f, to be

$$L_f = \{x \in \mathbb{R}^n : \lim_{r \to 0} A|f - f(x)|(x, r) = 0\}$$

$$= \left\{x \in \mathbb{R}^n : \lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y)\right] = 0\right\}$$

Exercise 5.1.12. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. If f is continuous at x, then $x \in L_f$.

Proof. Suppose that f is continuous at x. Let $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that for each $y \in \mathbb{R}^n$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Let $r > 0$. Suppose that $r < \delta$. Then for each $y \in \mathbb{R}^n$, $y \in B(x, r)$ implies that $|f(x) - f(y)| < \epsilon$ and thus

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) \leq \frac{1}{m(B(x,r))} \epsilon m(B(x,r))$$

$$= \epsilon$$

Hence

$$\lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y)\right] = 0$$

and $x \in L_f$. $\qquad \square$

Theorem 5.1.13. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then $m((L_f)^c) = 0$

Definition 5.1.14. Let $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then $(E_r)_{r>0}$ is said to **shrink nicely to** x if

(1) for each $r > 0$, $E_r \subset B(x, r)$
(2) there exists $\alpha > 0$ such that for each $r > 0$, $m(E_r) > \alpha m(B(x, r))$

Theorem 5.1.15. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then for each $x \in L_f$,

$$\lim_{r \to 0} \left[\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y)\right] = 0$$

and

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

Definition 5.1.16. Let $\mu : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$ be a Borel measure. Then μ is said to be **regular** if

(1) for each $K \subset \mathbb{R}^n$, if K is compact, then $\mu(K) < \infty$
(2) for each $E \in \mathcal{B}(\mathbb{R}^n)$, $\mu(E) = \inf\{\mu(U) : U$ is open and $E \subset U\}$

Let ν be a signed or complex Borel measure on $\mathbb{R}^n$. Then ν is said to be regular if $|\nu|$ is regular.

Theorem 5.1.17. Let ν be a regular signed or complex measure on $\mathbb{R}^n$. Let $d\nu = d\lambda + f dm$ be the Lebesgue decomposition of ν with respect to m. Then for m-a.e. $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(R^n)$, if $(E_r)_{r>0}$ shrinks nicely to x, then

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

5.2. **Functions of Bounded Variation.**

Definition 5.2.1. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Define $F_+ : \mathbb{R} \to \mathbb{R}$ by
$$F_+(x) = \lim_{t \to x^+} F(t) = \inf\{F(t) : t > x\}$$

Note 5.2.2. Observe that $F \leq F_+$ and F_+ is increasing.

Exercise 5.2.3. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then for each $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in (x, x + \delta)$, $0 \leq F_+(y) - F(y) \leq \epsilon$.

Proof. For the sake of contradiction, suppose not. Then there exists $x \in R$ and $\epsilon > 0$ such that for each $\delta > 0$, there exist $y \in (x, x + \delta)$ such that $F_+(y) - F(y) > \epsilon$. Then there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that for each $n \in \mathbb{N}$, $y_n \in (x, x + \frac{1}{n})$, $y_n > y_{n+1}$ and $F_+(y_n) - F(y_n) > \epsilon$. Choose $N \in \mathbb{N}$ such that $(N-1)\epsilon > F(y_1) - F(x)$. Then

$$\begin{aligned}
F(y_1) - F(x) &= \sum_{i=1}^{N-1}\left[F(y_i) - F_+(y_{i+1}) + F_+(y_{i+1}) - F(y_{i+1})\right] + F(y_N) - F(x) \\
&= \sum_{i=1}^{N-1}\left[F(y_i) - F_+(y_{i+1})\right] + \sum_{i=1}^{N-1}\left[F_+(y_{i+1}) - F(y_{i+1})\right] + F(y_N) - F(x) \\
&\geq (N-1)\epsilon \\
&> F(y_1) - F(x)
\end{aligned}$$

This is a contradiction, so the claim holds. $\square$

Exercise 5.2.4. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then F_+ is right continuous.

Proof. Let $x \in \mathbb{R}$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that for each $y \in (x, x + \delta_1)$ $0 \leq F(y) - F_+(x) < \epsilon/2$. There exists $\delta_2 > 0$ such that for each $y \in (x, x + \delta_2)$, $0 \leq F_+(y) - F(y) < \epsilon/2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Let $y \in (x, x + \delta)$.

$$\begin{aligned}
|F_+(x) - F_+(y)| &\leq |F_+(x) - F(y)| + |F(y) - F_+(y)| \\
&= (F(y) - F_+(x)) + (F_+(y) - F(y)) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So $\lim_{t \to x^+} F_+(t) = F_+(x)$ and F_+ is right continuous. $\square$

Theorem 5.2.5. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then
  (1) $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$ is countable
  (2) F and F_+ are differentiable a.e. and $F' = F'_+$ a.e.

Definition 5.2.6. Let $F : \mathbb{R} \to \mathbb{C}$. Define $T_F : \mathbb{R} \to \mathbb{R}$ by

$$T_F(x) = \sup\left\{\sum_{i=1}^{n}|F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^{n} \subset \mathbb{R} \text{ is increasing and } x_n = x\right\} \quad (x \in \mathbb{R})$$

T_F is called the **total variation function of** F.

Exercise 5.2.7. Let $F : \mathbb{R} \to \mathbb{C}$. Then T_F is increasing.

Proof. Let $x, y \in \mathbb{R}$. Suppose that $x < y_2$.
Define $A_x = \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^{n} \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$ and
$A_y = \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^{n} \subset \mathbb{R} \text{ is increasing and } x_n = y \right\}$. Let $z \in A_x$. Then
there exists $(x_i)_{i=0}^{n} \subset \mathbb{R}$ such that $(x_i)_{i=0}^{n}$ is increasing,
$x_n = x$ and $z = \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|$. Then

$$z \leq z + |F(y) - F(x)|$$
$$= \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)|$$
$$\in A_y$$

So $z \leq \sup A_y = T_F(y)$ and thus $F_T(x) = \sup A_x \leq T_F(y)$ $\qquad\square$

Lemma 5.2.8. Let $F : \mathbb{R} \to \mathbb{R}$. Then $T_F + F$ and $T_F - F$ are increasing.

Exercise 5.2.9. For each $F : \mathbb{R} \to \mathbb{C}$, $T_{|F|} \leq T_F$.

Proof. Let $F : \mathbb{R} \to \mathbb{C}$, $x \in R$ and $(x_i)_{i=0}^{n} \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^{n}$ is increasing and
$x_n = x$. Then by the reverse triangle inequality,

$$\sum_{i=1}^{n} \big||F(x_i)| - |F(x_{i-1})|\big| \leq \sum_{i=1}^{n} \big|F(x_i) - |F(x_{i-1})|\big|$$

Thus

$$T_{|F|}(x) = \sup \left\{ \sum_{i=1}^{n} \big||F(x_i)| - |F(x_{i-1})|\big| : (x_i)_{i=0}^{n} \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$$
$$\leq \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^{n} \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$$
$$= T_F(x)$$

Hence $T_{|F|} \leq T_F$ $\qquad\square$

Definition 5.2.10. Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to have **bounded variation** if
$\lim_{x \to \infty} T_F(x) < \infty$. The **total variation of** F, denoted by $TV(F)$, is defined to be $TV(F) = \lim_{x \to \infty} T_F(x)$. We define $BV = \{F : \mathbb{R} \to \mathbb{C} : TV(F) < \infty\}$.

Definition 5.2.11. Let $a, b \in \mathbb{R}$ and $F : [a, b] \to \mathbb{C}$. Define $G_F : \mathbb{R} \to \mathbb{C}$ by $G_F = F(a)\chi_{(-\infty,a)} + F\chi_{[a,b]} + F(b)\chi_{(b,\infty)}$. Then F is said to have **bounded variation on** $[a, b]$ if $G_F \in BV$. The **total variation of** F **on** $[a, b]$, denoted by $TV(F, [a, b])$, is defined to be $TV(F, [a, b]) = TV(G_F)$ We define $BV([a, b]) = \{F : [a, b] \to \mathbb{C} : TV(F, [a, b]) < \infty\}$.

Note 5.2.12. Equivalently, $TV(F, [a, b]) = \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^{n} \subset [a, b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$ and $F \in BV([a, b])$ iff $TV(F, [a, b]) < \infty$. In general,

Exercise 5.2.13. Let $F \in BV$. Then F is bounded.

Proof. If F is unbounded, then the supremum in the previous definition is clearly infinite. $\qquad\square$

Exercise 5.2.14. Let $F : \mathbb{R} \to \mathbb{R}$. If F is bounded and increasing, then $F \in BV$.

Proof. Suppose that F is bounded and increasing. Then $-\infty < \inf_{x \in \mathbb{R}} F(x) \leq \sup_{x \in \mathbb{R}} F(x) < \infty$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then

$$\sum_{i=1}^n |F(x_i) - F(x_{i-1})| = \sum_{i=1}^n F(x_i) - F(x_{i-1})$$
$$= F(x) - F(x_0)$$

Thus

$$T_F(x) = F(x) - \inf_{x \in \mathbb{R}} F(x)$$

. This implies that

$$TV(F) = \sup_{x \in \mathbb{R}} F(x) - \inf_{x \in \mathbb{R}} F(x)$$
$$< \infty$$

Hence $F \in BV$. $\square$

Exercise 5.2.15. Let $F : \mathbb{R} \to \mathbb{C}$. If F is differentiable and F' is bounded on $[a, b]$, then, $F \in BV([a, b])$.

Proof. Suppose that F is differentiable and F' is bounded on $[a, b]$. Then there exists $M > 0$ such that for each $x \in [a, b]$, $|F(x)| \leq M$. Let $(x_i)_{i=1}^n \subset [a, b]$. Suppose that $(x_i)_{i=1}^n$ is strictly increasing, $x_0 = a$ and $x_n = b$. By the mean value theorem, for each $i = 1, 2, \cdots, n$, there exists $c_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$. Then

$$\sum_{i=1}^n |F(x_i) - F(x_{i-1})| = \sum_{i=1}^n |F'(c_i)(x_i - x_{i-1})|$$
$$\leq \sum_{i=1}^n M(x_i - x_{i-1})$$
$$= M(b - a)$$

Hence $TV(F, [a, b]) \leq M(b - a)$. $\square$

Exercise 5.2.16. Define $F, G : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} x^2 sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then F and G are differentiable, $F \in BV([-1, 1])$ and $G \notin BV([-1, 1])$.

Proof. On $\mathbb{R} \setminus \{0\}$,

$$F'(x) = 2x sin(x^{-1}) - sin(x^{-1})$$
$$= sin(x^{-1})(2x - 1)$$

We see that F is also differentiable at $x = 0$ since

$$F'(0) = \lim_{x \to 0} \frac{F(x) - F(0)}{x - 0}$$
$$= \lim_{x \to 0} \frac{x^2 sin(x^{-1})}{x}$$
$$= \lim_{x \to 0} x sin(x^{-1})$$
$$= 0$$

Therefore for each $x \in [-1, 1]$, $|F'(x)| \leq 3$. Which by a previous exercise implies that $F \in BV([-1, 1])$.

On $\mathbb{R} \setminus \{0\}$,

$$G'(x) = 2x sin(x^{-2}) - \frac{2 sin(x^{-2})}{x}$$
$$= sin(x^{-2})(2x - \frac{2}{x})$$

We see that G is also differentiable at $x = 0$ since

$$G'(0) = \lim_{x \to 0} \frac{G(x) - G(0)}{x - 0}$$
$$= \lim_{x \to 0} \frac{x^2 sin(x^{-2})}{x}$$
$$= \lim_{x \to 0} x sin(x^{-2})$$
$$= 0$$

For $n \in \mathbb{N}$, define $(x_i)_{i=0}^n \subset [-1, 1]$ by

$$x_i = \frac{-1}{\sqrt{\frac{\pi}{2} + i\pi}}$$

Then for each $n \in \mathbb{N}$, $(x_i)_{i=1}^n$ is strictly increasing and for each $i = 1, 2, \cdots, n$ we have that

$$|G(x_i) - G(x_{i-1})| = \frac{1}{\frac{\pi}{2} + i\pi} + \frac{1}{\frac{\pi}{2} + (i-1)\pi}$$
$$= \frac{2}{\pi}\left[\frac{(2i-1) + (2i+1)}{(2i+1)(2i-1)}\right]$$
$$= \frac{2}{\pi}\left[\frac{4i}{4i^2 - 1}\right]$$
$$> \frac{2}{i\pi}$$

Hence for each $n \in \mathbb{N}$,

$$TV(G, [-1, 1]) \geq \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})|$$

$$> \frac{2}{\pi} \sum_{i=1}^{n} \frac{1}{i}$$

Therefore $G \notin BV([-1, 1])$.   □

Exercise 5.2.17. The following is stated for BV, but is also true for $BV([a, b])$.
  (1) For each $F, G \in BV$, $T_{F+G} \leq T_F + T_G$ and therefore BV is a vector space.
  (2) For each $F : \mathbb{R} \to \mathbb{C}$, $F \in BV$ iff $Re(f) \in BV$ and $Im(F) \in BV$.
  (3) For each $F : \mathbb{R} \to \mathbb{R}$, $F \in BV$ iff there exist functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$
  (4) For each $F \in BV$ and $x \in \mathbb{R}$, $\lim_{t \to x^+} F(t)$ and $\lim_{t \to x^-} F(t)$ exist.
  (5) For each $F \in BV$, $\{x \in R : F$ is not continuous at $x\}$ is countable.
  (6) For each $F \in BV$, F and F_+ are differentiable a.e. and $F' = (F_+)'$ a.e.
  (7) For each $F \in BV, c \in \mathbb{R}$, $F - c \in BV$

Proof.   (1) Let $F, G \in BV$, $x \in \mathbb{R}$ and $\epsilon > 0$. Since $T_{F+G}(x) < \infty$, $T_{F+G}(x) - \epsilon < T_{F+G}(x)$. Thus there exists $(x_i)_{i=0}^{n} \subset \mathbb{R}$ such that $(x_i)_{i=0}^{n}$ is increasing, $x_n = x$ and $T_{F+G}(x) < \sum_{i=1}^{n} |(F + G)(x_i) - (F + G)(x_{i-1}))| + \epsilon$. Thuerefore

$$T_{F+G}(x) < \sum_{i=1}^{n} |(F + G)(x_i) - (F + G)(x_{i-1}))| + \epsilon$$

$$\leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})| + \epsilon$$

$$\leq T_F(x) + T_G(x) + \epsilon$$

Since $\epsilon > 0$ is arbitrary, $T_{F+G}(x) \leq T_F(x) + T_G(x)$. Therefore $TV(F + G) \leq TV(F) + TV(G) < \infty$. Thus $F + G \in BV$. It is straight forward to verify the other requirements needed to show that BV is a vector space.

(2) Let $F : \mathbb{R} \to \mathbb{C}$. Write $F = F_1 + iF_2$ with $F_1, F_2 : \mathbb{R} \to \mathbb{R}$. Suppose that $F \in BV$. Note that for each $x_1, x_2 \in \mathbb{R}$ and $j = 1, 2$, $|F_j(x_1) - F_j(x_2)| \leq |F(x_1) - F(x_2)|$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^{n} \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^{n}$ is increasing and $x_n = x$. Then for $j = 1, 2$

$$\sum_{i=1}^{n} |F_j(x_i) - F_j(x_{i-1})| \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|$$

. Thus for $j = 1, 2$ we have that $T_{F_j}(x) \leq T_F(x)$ which implies that $Re(f), Im(F) \in BV$. Conversely, Suppose that $Re(f), Im(F) \in BV$. Then $F = Re(f) + iIm(f) \in BV$ by (1).

(3) Suppose that $F \in BV$. Choose $F_1 = \frac{1}{2}(T_F - F)$ and $F_2 = \frac{1}{2}(T_F + F)$. Then F_1, F_2 are bounded, increasing and $F = F_1 + F_2$. Conversely, if there exist $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$, then $F_1, F_2 \in BV$. By (1) $F \in BV$.

(4) This is clear by previous results and (3)

(5) This is clear by previous results and (3)

(6) This is clear by previous results and (3)

(7) Clearly constant functions have zero total variation. The rest is implied by (1).

$\square$

Lemma 5.2.18. Let $F \in BV$. Then $\lim_{x \to -\infty} T_F(x) = 0$ and if F is right continuous, then T_F is right continuous.

Definition 5.2.19. Define $NBV = \{F \in BV : F$ is right continuous and $\lim_{x \to -\infty} F(x) = 0\}$.

Theorem 5.2.20. Let $M(\mathbb{R})$ be the set of complex Borel measures on $\mathbb{R}$. For $F \in NBV$, define $\mu_F \in M(\mathbb{R})$ by $\mu_F((-\infty, x]) = F(x)$. Then $F \mapsto \mu_F$ defines a bijection $NBV \to M(\mathbb{R})$. In addition, $|\mu_F| = \mu_{T_F}$

Theorem 5.2.21. Let $F \in NBV$. Then $F' \in L^1(m)$, $\mu_F \perp m$ iff $F' = 0$ a.e. and $\mu_F \ll m$ iff for each $x \in \mathbb{R}$, $\int_{(-\infty, x]} F' dm = F(x)$

Definition 5.2.22. Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to be **absolutely continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.

Definition 5.2.23. Let $F : [a, b] \to \mathbb{C}$. Then F is said to be **absolutely continuous on** $[a, b]$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.

Proposition 5.2.24. Let $F : [a, b] \to \mathbb{C}$. If F is absolutely continuous on $[a, b]$, then $F \in BV[a, b]$.

Exercise 5.2.25. Let $F : \mathbb{R} \to \mathbb{C}$. Suppose that there exists $f \in L^1(m)$ such that $F(x) = \int_{(-\infty, x)} f dm$. Then $F \in NBV$.

Proof. Let $x \in \mathbb{R}$ and $(x_i)_{i=1}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=1}^n$ is increasing and $x_n = x$. Then

$$\sum_{i=1}^n |F(x_i) - F(x_{i-1})| = \sum_{i=1}^n \left| \int_{(x_{i-1}, x_i]} f dm \right|$$

$$\leq \sum_{i=1}^n \int_{(x_{i-1}, x_i]} |f| dm$$

$$= \int_{(x_0, x]} |f| dm$$

$$< \int |f| dm$$

Hence $T_F(x) \leq \int |f| dm$. Since $x \in \mathbb{R}$ is arbitrary, $TV(F) \leq \int |f| dm$. Therefore $F \in BV$. By the continuity from above and below for measures and the fact that $m(x) = 0$ for each $x \in \mathbb{R}$, F is continuous. By continuity from above for measures, $\lim_{x \to -\infty} F(x) = 0$. So $F \in NBV$.

$\square$

Lemma 5.2.26. Let $F \in NBV$. Then F is absolutely continuous iff $\mu_F \ll m$.

Exercise 5.2.27. The Fundamental Theorem of Calculus:
Let $F : [a, b] \to \mathbb{C}$. The following are equivalent:

(1) F is absolutely continuous on $[a,b]$.
(2) there exists $f \in L^1([a,b],m)$ such that for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} f dm$
(3) F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b],m)$ and for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} F' dm$

Proof. (1) $\implies$ (3)
Suppose that F is absolutely continuous on $[a,b]$. Then $F \in BV[a,b]$. Extend F to $\mathbb{R}$ by setting $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$. Then $G = F - F(a) \in NBV$ and is absolutely continuus. The previous lemma implies that there exists $f \in L^1(m)$ such that $\mu_G = f dm$. A previous theorem implies that for a.e. $x \in [a,b]$

$$F'(x) = \lim_{r \to x} \frac{\mu_G((x, x+r])}{m((x, x+r])}$$
$$= f(x)$$

So F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b],m)$ and by construction, for each $x \in [a,b]$, we have that

$$F(x) - F(a) = \mu_G((a,x])$$
$$= \int_{(a,x]} f dm$$
$$= \int_{(a,x]} F' dm$$

(3) $\implies$ (2)
Trivial.
(2) $\implies$ (1)
Suppose that there exists $f \in L^1([a,b],m)$ such that for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} f dm$. Extend F as before and obtain G as before. Note that a previous exercise implies that $G \in NBV$. Since $\mu_G \ll m$, the previous lemma implies that G is absolutely continuous. $\square$

Exercise 5.2.28. Let $F : \mathbb{R} \to \mathbb{C}$. If F is absolutely continuous. Then F is differentiable a.e.

Proof. Let $n \in \mathbb{N}$. Since F is absolutely continuous on $\mathbb{R}$, F is absolutely continuous on $[-n,n]$. The FTC implies that F is differentiable a.e. on $[-n,n]$. Since $n \in \mathbb{N}$ is arbitrary, F is differentiable a.e on $\mathbb{R}$. $\square$

Exercise 5.2.29. Let $F : \mathbb{R} \to \mathbb{C}$. Then F is Lipschitz continuous iff F is absolutely continuous and F' is bounded a.e.

Proof. Suppose that F is Lipschitz continuous. Then there exists $M > 0$ such that for each $x, y \in \mathbb{R}$, $|F(x) - F(y)| \leq M|x - y|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{M}$. Let $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, Suppose that $\sum_{i=1}^n b_i - a_i < \delta$. Then

$$\sum_{i=1}^n |F(b_i) - F(a_i)| \leq \sum_{i=1}^n M(b_i - a_i)$$
$$< M\delta$$
$$= \epsilon$$

Hence F is absolutely continuous. For each $x, y \in \mathbb{R}$, if $x \neq y$, then $\left| \frac{F(x) - F(y)}{x - y} \right| \leq M$. Hence for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Conversely, suppose that F is absolutely continuous and F' is bounded a.e. Then there exits $M > 0$ such that for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Let $x, y \in \mathbb{R}$. Suppose $x < y$. Then the FTC implies that

$$\begin{aligned}
|F(y) - F(x)| &= \left| \int_{(x,y]} F' dm \right| \\
&\leq \int_{(x,y]} |F'| dm \\
&= M|y - x|
\end{aligned}$$

and F is Lipschitz continuous. $\square$

Exercise 5.2.30. Construct an increasing function $F : \mathbb{R} \to \mathbb{R}$ whose discontinuities is $\mathbb{Q}$.

Proof. Let $(q_n)_{n \in \mathbb{N}}$ be an ennumeration of $\mathbb{Q}$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F = \sum_{n \in \mathbb{N}} 2^{-n} \chi_{[q_n, \infty)}$$

Equivalently, if we define $S_x = \{n \in \mathbb{N} : q_n \leq x\}$, then we may write

$$F(x) = \sum_{n \in S_x} 2^{-n}$$

Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Then $S_x \subsetneq S_y$. So $F(x) < F(y)$ and therefore F is strictly increasing.

For each $x, y \in R$ with $x < y$, define $S_{x,y} = \{n \in \mathbb{N} : x < q_n \leq y\}$. Note that $\lim_{y \to x^+} \min(S_{x,y}) = \infty$ and if $y \in \mathbb{R} \setminus \mathbb{Q}$, then $\lim_{x \to y^-} \min(S_{x,y}) = \infty$.

Now, let $x \in \mathbb{R}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$. Choose $\delta > 0$ such that $\min(S_{x,x+\delta}) \geq N$. Let $y \in [x, \infty)$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned}
|F(x) - F(y)| &= \sum_{n \in S_y} 2^{-n} - \sum_{n \in S_x} 2^{-n} \\
&= \sum_{n \in S_{x,y}} 2^{-n} \\
&\leq \sum_{n=N}^{\infty} 2^{-n} \\
&< \epsilon
\end{aligned}$$

Hence F is right continuous. Now let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ as before and $\delta > 0$ such that $\min(S_{x-\delta,x}) \geq N$. Let $y \in (-\infty, x]$. Suppose that $|x - y| < \delta$. Then

$$|F(x) - F(y)| = \sum_{n \in S_x} 2^{-n} - \sum_{n \in S_y} 2^{-n}$$

$$= \sum_{n \in S_{y,x}} 2^{-n}$$

$$\leq \sum_{n=N}^{\infty} 2^{-n}$$

$$< \epsilon$$

Hence F is left continuous on $\mathbb{R} \setminus \mathbb{Q}$.

Now, let $x \in \mathbb{Q}$. Then there exists $j \in \mathbb{N}$ such that $q_j = x$. Choose $\epsilon = 2^{-j}$. Let $\delta > 0$. Choose $y = x - \frac{\delta}{2}$. Then $|x - y| < \delta$ and

$$|F(x) - F(y)| = \sum_{n \in S_{y,x}} 2^{-n}$$

$$\geq 2^{-j}$$

$$= \epsilon$$

Hence F is discontinuous from the left at x. Since $x \in \mathbb{Q}$ is arbitrary, F is discontinuous from the left on $\mathbb{Q}$. $\qquad\square$

Exercise 5.2.31. Let $(F_n)_{n \in \mathbb{N}} \in NBV$ be a sequence of nonnegative, increasing functions. If for each $x \in \mathbb{R}$, $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$, then for a.e. $x \in \mathbb{R}$, F is differentiable at x and $F'(x) = \sum_{n \in \mathbb{N}} F_n'(x)$.

Proof. Define $\mu = \sum_{n \in \mathbb{N}} \mu_{F_n}$. Note that

$$\mu((-\infty, x]) = \sum_{n \in \mathbb{N}} \mu_{F_n}((-\infty, x])$$

$$= \sum_{n \in \mathbb{N}} F_n(x)$$

$$= F(x)$$

Hence $F \in NBV$ and $\mu = \mu_F$. For each $n \in \mathbb{N}$, there exist $\lambda_n \in M(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ such that $d\mu_{F_n} = d\lambda_n + f_n dm$ and $\lambda \perp m$. Since for each $n \in \mathbb{N}$, λ_n, f_n are nonnegative, we have that $d\mu_F = \sum_{n \in \mathbb{N}} d\lambda_n + (\sum_{n \in \mathbb{N}} f_n) dm$. By a previous theorem, for a.e. $x \in \mathbb{R}$,

$$F'(x) = \lim_{r \to 0} \frac{\mu_F((x, x+r])}{m((x, x+r])}$$

$$= \sum_{n \in \mathbb{N}} f_n(x)$$

$$= \sum_{n \in \mathbb{N}} \lim_{r \to 0} \frac{\mu_{F_n}((x, x+r])}{m((x, x+r])}$$

$$= \sum_{n \in \mathbb{N}} F_n'(x)$$

□

Exercise 5.2.32. Let $F : [0,1] \to [0,1]$ be the Cantor function. Extend F to $\mathbb{R}$ by setting $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $([a_n, b_n])_{n \in \mathbb{N}}$ be an ennumeration of the closed subintervals of $[0,1]$ with rational endpoints. For $n \in \mathbb{N}$, define $F_n : \mathbb{R} \to [0,1]$ by $F_n(x) = F(\frac{x-a_n}{b_n-a_n})$. Define $G : \mathbb{R} \to \mathbb{R}$ by $G = \sum_{n \in \mathbb{N}} 2^{-n} F_n$. Then G is continuous, strictly increasing on $[0,1]$ and $G' = 0$ a.e.

Proof. Since F is continuous on $\mathbb{R}$, we have that for each $n \in \mathbb{N}$, F_n is continuous on $\mathbb{R}$. We observe that for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $|2^{-n} F_n(x)| \leq 2^{-n}$. Thus the Weierstrass M-test implies that G converges uniformly on $\mathbb{R}$ and is therefore continuous. Since F is increasing, for each $n \in \mathbb{N}$, F_n is increasing. Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Choose $j \in \mathbb{N}$ such that $x < a_j < y < b_j$. Then

$$
\begin{aligned}
G(x) &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(x) \\
&= \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(x) + 0 \\
&< \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(y) + 2^{-j} F_n(y) \\
&= \sum_{n \in \mathbb{N}} 2^{-n} F_n(y) \\
&= G(y)
\end{aligned}
$$

So G is strictly increasing.

Now we observe that for each $n \in \mathbb{N}$, $F_n \in NBV$. The previous exercise implies that

$$ G' = \sum 2^{-n} F_n' = 0 \text{ a.e.} $$

□

5.3. **Weak Differentiation.**

6. L^p Spaces

6.1. Introduction.

Definition 6.1.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in [1, \infty]$. Define $\| \cdot \|_p : L^0(X, \mathcal{A}, \mu) \to [0, \infty]$ by

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{\frac{1}{p}} \qquad (p < \infty)$$

and

$$\|f\|_\infty = \inf \left\{ \lambda > 0 : \mu(\{x \in X : \lambda < |f(x)|\}) = 0 \right\}$$

We define

$$L^p(X, \mathcal{A}, \mu) = \{f \in L^0(X, \mathcal{A}, \mu) : \|f\|_p < \infty\}$$

Theorem 6.1.2. Hölder's Inequality: Let $(X, \mathcal{A}, \mu)$ be a measure space, $p, q \in [1, \infty)$ and $f, g \in L^0$. Suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Exercise 6.1.3. Minkowski Inequality: Let $(X, \mathcal{A}, \mu)$ be a measure space, $p \in [1, \infty)$ and $f, g \in L^p$. Then $f + g \in L^p$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. Define $\phi : \mathbb{R} \to [0, \infty)$ by $\phi(x) = |x|^p$. Then ϕ is convex because it is the composition of an increasing convex function with a convex function. By Jensen's inequality, we have that

$$\phi\left(\frac{1}{2}[f + g]\right) \leq \frac{1}{2}[\phi(f) + \phi(g)]$$

This implies that

$$\frac{1}{2^p}|f + g|^p \leq \frac{1}{2}\left(|f|^p + |g|^p\right)$$

Hence

$$\int |f + g|^p d\mu \leq 2^{p-1} \int |f|^p + |g|^p d\mu$$

$$= 2^{p-1}\left(\int |f|^p d\mu + \int |g|^p d\mu\right)$$

$$= 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right)$$

$$< \infty$$

So $f + g \in L^p$. Now, it is not hard to see that $|f + g|^p \leq \left(|f| + |g|\right)|f + g|^{p-1}$. Let q be the conjugate of p, so that $\frac{1}{p} + \frac{1}{q} = 1$. Then $q(p - 1) = p$. We use Hölder's inequality to show

that

$$\|f + g\|_p^p = \int |f + g|^p d\mu$$

$$\leq \int |f||f + g|^{p-1} d\mu + \int |g||f + g|^{p-1} d\mu$$

$$\leq \|f\|_p \left(\int |f + g|^{(p-1)q} d\mu \right)^{\frac{1}{q}} + \|g\|_p \left(\int |f + g|^{(p-1)q} d\mu \right)^{\frac{1}{q}}$$

$$= \|f\|_p \left(\int |f + g|^p d\mu \right)^{\frac{1}{q}} + \|g\|_p \left(\int |f + g|^p d\mu \right)^{\frac{1}{q}}$$

$$= (\|f\|_p + \|g\|_p) \left(\int |f + g|^p d\mu \right)^{\frac{1}{q}}$$

$$= (\|f\|_p + \|g\|_p) \|f + g\|_p^{p/q}$$

Since $\|f + g\|_p < \infty$, we see that

$$\|f\|_p + \|g\|_p \geq \|f + g\|_p^{p-p/q}$$

$$= \|f + g\|_p^{p(1-1/q)}$$

$$= \|f + g\|_p^{p/p}$$

$$= \|f + g\|_p$$

$$\square$$

Exercise 6.1.4. Let $(X, \mathcal{A}, \mu)$ be a measure space, $p, q \in (0, \infty]$. Suppose that $\mu(X) < \infty$ and $p < q$. Then $L^q \subset L^p$. In particular, if $\mu(X) = 1$, then for each $f \in L^q$, $\|f\|_p \leq \|f\|_q$.

Proof. Suppose that $q = \infty$. Let $f \in L^q$. Then

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{\frac{1}{p}}$$

$$\leq \left(\int \|f\|_\infty^p d\mu \right)^{\frac{1}{p}}$$

$$= \|f\|_\infty \mu(X)^{\frac{1}{p}}$$

If $q < \infty$, then $\frac{q}{p} > 1$ and the conjugate of $\frac{q}{p}$ is $\frac{1}{1-p/q}$. By Hölder's inequality, we have that

$$\|f\|_p^p = \|f^p\|_1$$

$$\leq \|f^p\|_{\frac{q}{p}} \|1\|_{\frac{1}{1-p/q}}$$

$$= \left(\int |f|^{\frac{pq}{p}} d\mu \right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}}$$

$$= \left(\int |f|^q d\mu \right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}}$$

$$= \|f\|_q^p \mu(X)^{1-\frac{p}{q}}$$

Hence

$$\|f\|_p \le \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}}$$
$$< \infty$$

$\square$

Exercise 6.1.5. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $K \in L^0(X \times Y)$. Suppose that there exists $C > 0$ such that for μ-a.e $x \in X$,

$$\int_Y |K(x,y)|d\nu(y) < C$$

and for ν-a.e $y \in Y$,

$$\int_X |K(x,y)|d\mu(x) < C$$

Let $f \in L^p(\nu)$.

(1) Then for μ-a.e. $x \in X$,

$$\int_Y K(x,y)f(y)d\nu(y)$$

exists.

Hint: Note that $|K(x,y)f(y)| = (|K(x,y)|^{1/q})(|K(x,y)|^{1/p}|f(y)|)$

(2) Define $Tf \in L^0(X)$ by

$$Tf(x) = \int_Y K(x,y)f(y)d\nu(y)$$

Then $Tf \in L^p(\mu)$ and $\|Tf\|_p \le C\|f\|_p$.

Proof. Let $p, q \in (0, \infty)$ be conjugate.

(1) Define $h \in L^0(X \times Y)$ by $h(x,y) = K(x,y)f(y)$. By assumption, there exists $N \in \mathcal{A}$ such that $\mu(N) = 0$ and

$$\left\{ x \in X : \int_Y |K(x,y)|d\nu(y) < C \right\} \subset N^c$$

Let $x \in N^c$. Then Holder's inequality implies that

$$\int_Y |h(x,y)|d\nu(y) = \int_Y (|K(x,y)|^{1/q})(|K(x,y)|^{1/p}|f(y)|)d\nu(y)$$
$$\le \left(\int_Y |K(x,y)|d\nu(y) \right)^{1/q} \left(\int_Y |K(x,y)||f(y)|^p d\nu(y) \right)^{1/p}$$
$$\le C^{1/q} \left(\int_Y |K(x,y)||f(y)|^p d\nu(y) \right)^{1/p}$$

Tonelli's theorem implies that the map

$$x \mapsto \int_Y |h(x,y)|d\nu(y)$$

is measurable and that

$$\int_X \left[\int_Y |h(x,y)| d\nu(y) \right]^p d\mu(x) \leq C^{p/q} \int_X \left[\int_Y |K(x,y)||f(y)|^p d\nu(y) \right] d\mu(x)$$

$$= C^{p/q} \int_Y \left[\int_X |K(x,y)||f(y)|^p d\mu(x) \right] d\nu(y)$$

$$= C^{p/q} \int_Y \left[\int_X |K(x,y)| d\mu(x) \right] |f(y)|^p d\nu(y)$$

$$\leq C^{1+p/q} \int_Y |f(y)|^p d\nu(y)$$

$$= C^{1+p/q} \|f\|_p^p$$

So for μ-a.e. $x \in X$,

$$\int_Y |h(x,y)| d\nu(y) < \infty$$

which implies that for μ-a.e. $x \in X$, $h(x, \cdot) \in L^1(\nu)$. Therefore, for μ-a.e. $x \in X$,

$$\int_Y h(x,y) d\nu(y)$$

exists. The case is similar when $p \in \{1, \infty\}$.

(2) Let $x \in X$. Then

$$|Tf(x)| \leq \int_Y |K(x,y)f(y)| d\nu(y)$$

which implies that

$$|Tf(x)|^p \leq \left(\int_Y |K(x,y)f(y)| d\nu(y) \right)^p$$

By part (1),

$$\int_X |Tf|^p d\mu \leq C^{1+p/q} \|f\|_p^p$$

So $Tf \in L^p(\mu)$ and $\|Tf\|_p \leq C\|f\|_p$. The case is similar when $p \in \{1, \infty\}$.

$\square$

7. Radon Measures

7.1. Introduction.

Definition 7.1.1. Let X be a topological space, $\mu : \mathcal{B}(X) \to [0, \infty]$ a Borel measure and $E \in \mathcal{B}(X)$. Then μ is said to be

(1) **inner regular on E** if
$$\mu(E) = \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}$$

(2) **outer regular on E** if
$$\mu(E) = \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$$

(3) **regular on E** if μ is inner regular on E and μ is outer regular on E

Definition 7.1.2. Let X be a topological space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Borel measure. Then μ is said to be

(1) **inner regular** if for each $E \in \mathcal{A}$, μ is inner regular on E
(2) **outer regular** if for each $E \in \mathcal{A}$, μ is outer regular on E
(3) **regular** if μ is inner regular and μ is outer regular

Definition 7.1.3. Let X be a topological space, $\mu : \mathcal{B}(X) \to [0, \infty]$ and $E \in \mathcal{B}(X)$. Then μ is said to be a **Radon measure** if for each $E \in \mathcal{B}(X)$,

(1) E is compact implies that $\mu(E) < \infty$
(2) μ is outer regular on E
(3) E is open implies that μ is inner regular on E

Exercise 7.1.4. Let X be a topological space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure. Set
$$\mathcal{N}_\mu = \{U \subset X : U \text{ is open and } \mu(U) = 0\}$$
and
$$N_\mu = \bigcup_{U \in \mathcal{N}_\mu} U$$
Then N_μ is open, N_μ^c is closed and $\mu(N_\mu) = 0$.
Hint: use inner regularity and compactness

Proof. Since N_μ is the union of open sets, it is open and N_μ^c is closed since N_μ is open. Let $K \subset N_\mu$. Suppose that K is compact. Since $\mathcal{N}_\mu$ is an open cover for K, there exist $U_1, \ldots, U_n \in \mathcal{N}_\mu$ such that
$$K \subset \bigcup_{j=1}^{n} U_j$$
This implies that
$$\mu(K) \leq \mu\left(\bigcup_{j=1}^{n} U_j\right)$$
$$\leq \sum_{j=1}^{n} \mu(U_j)$$
$$= 0$$

Inner regularity implies that

$$\mu(N_\mu) = \sup\{\mu(K) : K \subset N_\mu \text{ and } K \text{ is compact}\}$$
$$= 0$$

$\square$

Definition 7.1.5. Let X be a topological space and $\mu : \mathcal{B}(X) \to [0,\infty]$ a Radon measure. Define $\mathcal{N}_\mu$ and N_μ as in the previous exercise. We define the **support of** μ, denoted $\text{supp}(\mu)$, by

$$\text{supp}(\mu) = N_\mu^c$$

Exercise 7.1.6. Let X be a topological space, $\mu : \mathcal{B}(X) \to [0,\infty]$ a Radon measure and $E \in \mathcal{B}(X)$. If $\mu(E) < \infty$, then for each $\epsilon > 0$,
  (1) there exists $U \in \mathcal{B}(X)$ such that U is open, $E \subset U$ and $\mu(U \setminus E) < \epsilon$
  (2) there exists $C \in \mathcal{B}(X)$ such that C is compact, $C \subset U$ and $\mu(U) - \epsilon < \mu(C)$
  (3) there exists $V \in \mathcal{B}(X)$ such that U and V are open, $U \setminus E \subset V$ and $\mu(V) < \epsilon$

Proof. Suppose that $\mu(E) < \infty$. Let $\epsilon > 0$.
  (1) Outer regularity om E implies that there exists $U \in \mathcal{B}(X)$ such that U is open, $E \subset U$ and $\mu(U \setminus E) < \epsilon$.
  (2) Inner regularity on U implies that there exists $C \in \mathcal{B}(X)$ such that C is compact, $C \subset U$ and $\mu(U) - \epsilon < \mu(C)$.
  (3) Outer regularity on $U \setminus E$ implies that there exists $V \in \mathcal{B}(X)$ such that U and V are open, $U \setminus E \subset V$ and $\mu(V) < \epsilon$.

$\square$

Exercise 7.1.7. Let X be a topological space, $\mu : \mathcal{B}(X) \to [0,\infty]$ a Radon measure and $E \in \mathcal{A}$. Suppose that $\mu(E) < \infty$. Let $\epsilon > 0$. Define U, C and V as in the previous exercise. Set $K = C \setminus V$. Then K is compact, $K \subset E$ and $\mu(K) > \mu(E) - 2\epsilon$

Proof. Since C is closed and V is open, $C \setminus V = C \cap V^c$ is closed. Since C is compact and $C \setminus V \subset C$, we have that $K = C \setminus V$ is compact. Set algebra implies that

$$K = C \cap V^c$$
$$\subset U \cap V^c$$
$$\subset U \cap (U^c \cup E)$$
$$= (U \cap U^c) \cup (U \cap E)$$
$$= U \cap E$$
$$\subset E$$

The previous exercise implies that

$$\mu(K) = \mu(C \cap V^c)$$
$$= \mu(C) - \mu(C \cap V)$$
$$> \mu(U) - \epsilon - \mu(V)$$
$$> \mu(E) - 2\epsilon$$

$\square$

Exercise 7.1.8. Let X be a topological space, $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure and $E \in \mathcal{B}(X)$. If E is σ-finite, then μ is inner regular on E.
Hint: use the previous exercise

Proof. Suppose that E is σ-finite.
If $\mu(E) < \infty$, the previous exercise implies that for each $\epsilon > 0$, there exists $K \in \mathcal{B}(X)$ such that K is compact, $E \subset K$ and $\mu(K) > \mu(E) - \epsilon$. Hence μ is inner regular on E.
If $\mu(E) = \infty$, then σ-finiteness implies that there exists $(E_j)_{j \in \mathbb{N}} \subset \mathcal{B}(X)$ such that $E = \bigcup_{j \in N} E_j$, for each $j \in \mathbb{N}$, $\mu(E_j) < \infty$ and $\mu(E_j) \to \infty$. Let $N \in \mathbb{N}$. Choose $J \in \mathbb{N}$ such that $\mu(E_J) > N$. The above argument implies that there exists $K \in \mathcal{B}(X)$ such that K is compact, $K \subset E_J \subset E$ and $\mu(K) > N$. So

$$\mu(E) = \infty$$
$$= \sup_{\substack{K \subset E \\ K \text{ is compact}}} \mu(K)$$

and μ is inner regular on E. $\qquad\square$

Exercise 7.1.9. Let X be a topological space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure. If μ is σ-finite, then μ is regular.

Proof. Clear by previous exercise. $\qquad\square$

Exercise 7.1.10. Let X be a topological space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure. If X is σ-compact, then μ is σ-finite. The previous exercise implies that μ is regular.

Proof. If X is σ-compact, then μ is σ-finite. Hence μ is regular. $\qquad\square$

Exercise 7.1.11. Let X be a topological space and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure. Then for each $p \in [1, \infty]$, $C_c(X) \subset L^p(\mu)$.

Proof. Let $p \in [1, \infty]$ and $f \in C_c(X)$. Then $|f|^p \in C_c(X)$ and

$$\|f\|_p = \int |f|^p d\mu$$
$$\leq \|f\|_\infty^p \mu(\text{supp}(f))$$
$$< \infty$$

$\qquad\square$

7.2. Representation Theorems.

Definition 7.2.1. Let X be a topological space and $I : C_c(X) \to \mathbb{C}$ a linear functional. Then I is said to be **positive** if for each $f \in C_c(X, \mathbb{R})$, $f \geq 0$ implies that $I(f) \geq 0$.

Exercise 7.2.2. Let X be a topological space, $I : C_c(X) \to \mathbb{C}$ a positive linear functional and $f, g \in C_c(X, \mathbb{R})$. If $f \geq g$, then $I(f) \geq I(g)$.

Proof. Suppose that $f \geq g$. Then $f - g \geq 0$. So

$$I(f) - I(g) = I(f - g)$$
$$\geq 0$$

$\square$

Exercise 7.2.3. Let X be a LCH space, $I : C_c(X) \to \mathbb{C}$ a positive linear functional. Then for each $K \subset X$, K is compact implies that there exists $C_K \geq 0$ such that for each $f \in C_c(X)$, if $\operatorname{supp}(f) \subset K$, then $I(f) \leq C_K \|f\|_\infty$.
Hint: Urysohn's lemma

Proof. Let $K \subset X$. Suppose that K is compact. Then Urysohn's lemma implies that there exists $\phi \in C_c(X)$ such that $0 \leq \phi \leq 1$ and $\phi|_K = 1$. Then $I(\phi) \geq 0$. Choose $C_K = I(\phi)$. Let $f \in C_c(X)$. Suppose that $\operatorname{supp}(f) \subset K$. Then

$$f, -f \leq |f|$$
$$\leq \|f\|_\infty \phi$$

The previous exercise implies that $I(f), -I(f) \leq \|f\|_\infty I(\phi)$. So

$$|I(f)| \leq \|f\|_\infty I(\phi)$$
$$\leq C_K \|f\|_\infty$$

$\square$

Note 7.2.4. Let X be a LCH space, $U \subset X$ open and $f \in C_c(X)$. We write $f \prec U$ to mean $0 \leq f \leq 1$ and $\operatorname{supp}(f) \subset U$.

Exercise 7.2.5. Let X be a LCH space, $I : C_c(C) \to \mathbb{C}$ a positive linear functional and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure. Suppose that for each $f \in C_c(X)$,

$$I(f) = \int f d\mu$$

Then

(1) for each $U \subset X$, U is open implies that

$$\mu(U) = \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\}$$

(2) μ is the unique Radon measure such that for each $f \in C_c(X)$,

$$I(f) = \int f d\mu$$

Proof.

(1) Let $U \subset X$. Suppose that U is open. For $f \in C_c(X)$, if $f \prec U$, then

$$I(f) = \int f d\mu$$
$$\leq \mu(U)$$

Let $K \subset U$. Suppose that K is compact. Then Urysohn's lemma implies that there exists $f \in C_c(X)$ such that $f \prec U$ and $f|_K = 1$. Then

$$\mu(K) \leq \int f d\mu$$
$$= I(f)$$

Inner regularity implies that

$$\mu(U) = \sup\{\mu(K) : K \subset X \text{ and } K \text{ is compact}$$
$$\leq \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\}$$
$$\leq \mu(U)$$

(2) Let $\nu : \mathcal{B}(X) \to [0, \infty]$ be a Radon measure. Suppose that for each $f \in C_c(X)$,

$$I(f) = \int f d\nu$$

Part (1) implies that for each $U \subset X$, if U is open, then

$$\nu(U) = \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\}$$
$$= \mu(U)$$

Outer regularity implies that for each $E \in \mathcal{B}(X)$,

$$\nu(E) = \inf\{\nu(U) : E \subset U \text{ and } U \text{ is open}\}$$
$$= \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$$
$$= \mu(E)$$

So $\nu = \mu$ and μ is unique.

$\square$

Theorem 7.2.6. Representation Theorem 1:
Let X be a LCH space and $I : C_c(C) \to \mathbb{C}$ a positive linear functional. Then there exists a unique Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$ such that for each $f \in C_c(X)$,

$$I(f) = \int f d\mu$$

In addition,
(1) for each $U \subset X$, U is open implies that
$$\mu(U) = \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\}$$

(2) for each $K \subset X$, K is compact implies that
$$\mu(U) = \inf\{I(f) : f \in C_c(X) \text{ and } f \geq \chi_K\}$$

Note 7.2.7. Let X be a topological space. Recall from section (4.3) that we define
$$\mathcal{M}(X) = \{\mu : \mathcal{B}(X) \to \mathbb{C} : \mu \text{ is a complex measure}\}$$
and that $\mu \mapsto |\mu|(X)$ is a norm on $\mathcal{M}(X)$.

Definition 7.2.8. Let X be a topological space. For $\mu \in \mathcal{M}(X)$, define $I_\mu : C_0(X) \to \mathbb{C}$ by
$$I_\mu(f) = \int f d\mu$$
.

Exercise 7.2.9. Let X be a topological space. For each $\mu \in \mathcal{M}(X)$, $I_\mu \in C_0(X)^*$.

Proof. Let $\mu \in \mathcal{M}(X)$ and $f \in C_0(X)$. An exercise in section (4.3) implies that
$$|I_\mu(f)| = \left| \int f d\mu \right|$$
$$\leq \int |f| d|\mu|$$
$$\leq \|\mu\|\|f\|_\infty$$
So I_μ is bounded and $I_\mu \in C_0(X)^*$. $\qquad\qquad\square$

Theorem 7.2.10. Let $I \in C_0(X, \mathbb{R})^*$, then there exist positive linear functionals $I^+, I^- \in C_0(X, \mathbb{R})^*$ such that $I = I^+ - I^-$

Exercise 7.2.11. Let X be a LCH space. Then the map $\phi : \mathcal{M}(X) \to C_0(X)^*$ given by $\phi(\mu) = I_\mu$ is a linear surjection.

Proof. An exercise in section (4.3) implies that ϕ is linear. Let $I \in C_0(X)^*$. Then there exists positive linear functionals $I^\pm$, $J^\pm \in C_0(X)^*$ such that $I = I^+ - I^- + i(J^+ - J^-)$. The first representation theorem implies that there exist Radon measures $\mu^\pm, \nu^\pm$ such that $I^\pm = I_{\mu^\pm}$ and $J^\pm = I_{\mu^\pm}$. Set $\mu = \mu^+ - \mu^- + i(\nu^+ - \nu^-)$. Then $I = \phi(\mu)$ $\qquad\square$

Theorem 7.2.12. Representation Theorem 2:
Let X be a LCH space. Then the map $\phi : \mathcal{M}(X) \to C_0(X)^*$ given by $\phi(\mu) = I_\mu$ is an isometric linear isomorphism.

Definition 7.2.13. Let X be a LCH space, $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Then μ_n is said to **converge to μ in weak-***, denoted $\mu_n \xrightarrow{w^*} \mu$, if $I_{\mu_n} \xrightarrow{w^*} I_\mu$, i.e. for each $f \in C_0(X)$,
$$\int f d\mu_n \to \int f d\mu$$

Exercise 7.2.14.

8. Haar Measure

8.1. Topological Groups.

Definition 8.1.1. Let G be a group and $\mathcal{T}$ a topology on G. Then $(G, \mathcal{T})$ is said to be a **topological group** if the maps

(1) $G \times G \to G$ given by $(x, y) \mapsto xy$
(2) $G \to G$ given by $x \mapsto x^{-1}$

are continuous.

Note 8.1.2. For the remainder of this chapter, measurablility is in reference to $(G, \mathcal{B}(\mathcal{T}))$. That is, the measurable sets are the Borel sets.

Definition 8.1.3. Let G be a topological group. We define

$$\text{Homeo}(G) = \{\phi : G \to G : \phi \text{ is a homeomorphism}\}$$

Note 8.1.4. Let G be a topological group. Then $\text{Homeo}(G)$ is a group.

Definition 8.1.5. Let G be a group. Define $\iota : G \to G$ by $\iota(x) = x^{-1}$.

Exercise 8.1.6. Let G be a topological group. Then $\iota \in \text{Homeo}(G)$.

Proof. By assumption ι is continuous. We know from basic group theory that ι is a bijection with $\iota^{-1} = \iota$. $\qquad\square$

Definition 8.1.7. Let G be a group and $S \subset G$, then S is said to be **symmetric** if $\iota(S) = S$, (i.e. $S^{-1} = S$).

Definition 8.1.8. Let G be a topological group and $\phi : G \to G$. Then ϕ is said to be an **automorphism** of G if ϕ is a homomorphism and a homeomorphism. We define

$$\text{Aut}(G) = \{\phi : G \to G : \phi \text{ is an automorphism}\}$$

Exercise 8.1.9. Let G be a topological group. Then $\iota \in \text{Aut}(G)$ iff G is abelian.

Proof. Basic group theory tells us that ι is a homomorphism iff G is abelian. $\qquad\square$

Definition 8.1.10. Let G be a group and $g \in G$. Define $l_g : G \to G$ and $r_g : G \to G$ by $l_g(x) = gx$ and $r_g(x) = xg^{-1}$.

Exercise 8.1.11. Let G be a topological group and $g \in G$. Then $l_g, r_g \in \text{Homeo}(G)$.

Proof. By assumption l_g and r_g are continuous. We know from basic group theory that l_g and r_g are bijections with $l_g^{-1} = l_{g^{-1}}$ and $r_g^{-1} = r_{g^{-1}}$ so l_g and r_g. are homeomorphisms. $\quad\square$

Exercise 8.1.12. Let G be a toplogical group. Define $\phi, \psi : G \to \text{Homeo}(G)$ by $\phi(g) = l_g$ and $\psi(g) = r_g$. Then ϕ, ψ are homomorphisms.

Proof. Let $g_1, g_2 \in G$. Then

$$l_{g_1} \circ l_{g_2}(x) = l_{g_1}(g_2 x) = g_1 g_2 x = l_{g_1 g_2}(x)$$

and

$$r_{g_1} \circ r_{g_2}(x) = r_{g_1}(xg_2^{-1}) = xg_2^{-1}g_1^{-1} = x(g_1 g_2)^{-1} = r_{g_1 g_2}(x)$$

$\square$

Exercise 8.1.13. Let G be a topological group. Then for each $U \subset G$ and $g \in G$, if U is open, then gU, Ug and U^{-1} are open.

Proof. Let $U \subset G$ and $g \in G$. Suppose that U is open. Since l_g, r_g and ι are homeomorphisms, $l_g(U) = gU$, $r_g(U) = Ug$ and $\iota(U) = U^{-1}$ are open. $\qquad\square$

Definition 8.1.14. Let G be a topological group, $y \in G$ and $f \in L^0$. Define $L_y, R_y : L^0(G) \to L^0(G)$ by $L_y f = f \circ l_y^{-1}$ and $R_y f = f \circ r_y^{-1}$, that is, $L_y f(x) = f(y^{-1}x)$ and $R_y f(x) = f(xy)$.

Exercise 8.1.15. Let G be a topological group and $y \in G$. Then $L_y, R_y \in \mathrm{Sym}(L^0(G))$.

Proof. It is straight forward to show that $L_y^{-1} = L_{y^{-1}}$ and $R_y^{-1} = R_{y^{-1}}$. $\qquad\square$

Exercise 8.1.16. Let G be a topological group. Define $\phi, \psi : G \to \mathrm{Sym}(L^0(G))$ by $\phi(y) = L_y$ and $\psi(y) = R_y$. Then ϕ and ψ are homomorphisms.

Proof. Let $y, z \in G$ and $f \in L^0(G)$. Then

$$
\begin{aligned}
L_y \circ L_z(f) &= L_y(L_z(f)) \\
&= L_y(f \circ l_z^{-1}) \\
&= (f \circ l_z^{-1}) \circ l_y^{-1} \\
&= f \circ (l_z^{-1} \circ l_y^{-1}) \\
&= f \circ (l_y \circ l_z)^{-1} \\
&= f \circ l_{yz}^{-1} \\
&= L_{yz}(f)
\end{aligned}
$$

The case is similar for R_y and R_z. $\qquad\square$

Exercise 8.1.17. Let G be a topological group, $U \in \mathcal{B}(G)$ and $y \in G$. Then $L_y \chi_U = \chi_{yU}$ and $R_y \chi_U = \chi_{Uy^{-1}}$.

Proof. Let $x \in G$. Then

$$
\begin{aligned}
L_y \chi_U(x) = 1 &\iff y^{-1}x \in U \\
&\iff x \in yU \\
&\iff \chi_{yU}(x) = 1
\end{aligned}
$$

The case is similar for R_y $\qquad\square$

Exercise 8.1.18. Let G be a topological group, $y \in G$ and $f \in L^0(G)$. Then $\mathrm{supp}(L_y f) = y\,\mathrm{supp}(f)$ and $\mathrm{supp}(R_y f) = \mathrm{supp}(f)y^{-1}$

Proof. Put $A = \{x \in G : L_y f(x) \neq 0\}$ and $B = \{x \in G : f(x) \neq 0\}$. Then

$$
\begin{aligned}
x \in A &\iff L_y f(x) \neq 0 \\
&\iff f(y^{-1}x) \neq 0 \\
&\iff y^{-1}x \in B \\
&\iff x \in yB
\end{aligned}
$$

Thus $A = yB$ which implies that $\overline{A} = y\overline{B}$. Therefore $\mathrm{supp}(L_y f) = y\,\mathrm{supp}(f)$. $\qquad\square$

Exercise 8.1.19. Let G be a topological group and $y \in G$. Then L_y, R_y are linear and if we restrict to the bounded measurable functions, then $L_y, R_y \in L(B(G))$ and $\|L_y\|, \|R_y\| = 1$.

Proof. Let $f, g \in L^0(G)$ and $\lambda \in \mathbb{C}$. Then

$$L_y(\lambda f + g)(x) = (\lambda f + g)(y^{-1}x)$$
$$= \lambda f(y^{-1}x) + g(y^{-1}x)$$
$$= \lambda L_y f(x) + L_y g(x)$$

So L_y is linear. Next, we restrict to $B(G) \cap L^0$. We note that

$$\{|f(y^{-1}x)| : x \in y\operatorname{supp}(f)\} = \{|f(x)| : x \in \operatorname{supp}(f)\}$$

This implies that

$$\|L_y f\|_u = \sup_{x \in \operatorname{supp}(L_y f)} |L_y f(x)|$$
$$= \sup_{x \in y\operatorname{supp}(f)} |f(y^{-1}x)|$$
$$= \sup_{x \in \operatorname{supp}(f)} |f(x)|$$
$$= \|f\|_u$$

So L_y is bounded. Hence $L_y \in L(L^0)$. The case is similar for R_y. $\qquad\square$

Definition 8.1.20. Let G be a topological group. We say that G is a **locally compact group** if G is locally compact and Hausdorff.

8.2. **Haar Measure.**

Definition 8.2.1. Let G be a topological group and μ a Radon measure on G. Then μ is said to be a **left Haar measure on** G if

(1) μ is nonzero
(2) for each $U \in \mathcal{B}(G)$ and $g \in G$, $\mu(gU) = \mu(U)$.

Similarly, μ is said to be a **right Haar measure on** G if

(1) μ is nonzero
(2) for each $U \in \mathcal{B}(G)$ and $g \in G$, $\mu(Ug) = \mu(U)$.

Exercise 8.2.2. Let G be a topological group, μ a Radon measure on G. Then μ is a left Haar measure on G iff $\iota_* \mu$ is a right Haar measure on G.

Proof. Suppose that μ is a left Haar measure on G. Let $U \in \mathcal{B}(G)$ and $g \in G$. Then

$$\begin{aligned}
\iota_* \mu(Ug) &= \mu(\iota^{-1}(Ug)) \\
&= \mu(g^{-1}U^{-1}) \\
&= \mu(U^{-1}) \\
&= \mu(\iota^{-1}(U)) \\
&= \iota_* \mu(U)
\end{aligned}$$

So $\iota_* \mu$ is a right Haar measure on G. The converse is similar. $\square$

Exercise 8.2.3. Let G be a topological group, and μ a left Haar measure on G. Then for each $g \in G$, $r_{g_*} \mu$ is a left Haar measure on G.

Proof. Let $g \in G$ and $U \in \mathcal{B}(G)$. Observe that $r_{g_*} \mu(U) = \mu(Ug)$. So for each $h \in G$,

$$\begin{aligned}
r_{g_*} \mu(hU) &= \mu(hUg) \\
&= \mu(Ug) \\
&= r_{g_*} \mu(U)
\end{aligned}$$

$\square$

Exercise 8.2.4. Let G be a topological group, μ a left Haar measure on G and ν a right Haar measure on G. Then for each $f \in L^1 \cup L^+$ and $y \in G$,

(1)
$$\int L_y f \, d\mu = \int f \, d\mu$$

(2)
$$\int R_y f \, d\nu = \int f \, d\nu$$

Proof.

(1) Let $y \in G$ and $E \in \mathcal{B}(G)$. Put $f = \chi_E$. Then

$$\int L_y f d\mu = \int L_y \chi_E d\mu$$
$$= \int \chi_{yE} d\mu$$
$$= \mu(yE)$$
$$= \mu(E)$$
$$= \int \chi_E d\mu$$
$$= \int f d\mu$$

By linearity of L_y, for $f \in S^+$ we have that,

$$\int L_y f d\mu = \int f d\mu$$

For $f \in L^+$, choose $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$ $\phi_n \le \phi_{n+1} \le f$ and $\phi_n \to f$. Then for each $n \in \mathbb{N}$ $L_y \phi_n \le L_y \phi_{n+1} \le L_y f$ and $L_y \phi \to L_y f$. So MCT implies that

$$\int L_y f d\mu = \lim_{n \to \infty} \int L_y \phi_n d\mu$$
$$= \lim_{n \to \infty} \int \phi_n d\mu$$
$$= \int f d\mu$$

Let $f \in L^1$. If f is real valued, write $f = f^+ - f^-$. Then $L_y f = L_y f^+ - L_y f^-$ and

$$\int L_y f d\mu = \int L_y f^+ d\mu - \int L_y f^- d\mu$$
$$= \int f^+ d\mu - \int f^- d\mu$$
$$= \int f d\mu$$

If f is complex valued, write $f = g + ih$ with $g, h \in L^1$ real valued. Then

$$\int L_y f d\mu = \int L_y g d\mu + i \int L_y h d\mu$$
$$= \int g d\mu + i \int h d\mu$$
$$= \int f d\mu$$

(2) Similar

$\square$

Exercise 8.2.5. Let G be a topological group and μ a left Haar measure on G. Then for each $U \subset G$, if U is open and $U \neq \varnothing$, then $\mu(U) > 0$

Proof. Let $U \subset G$. Suppose that U is open and $U \neq \varnothing$. Suppose that $\mu(U) = 0$. Since μ is nonzero, inner regularity implies that there exists $K \subset G$ such that K is compact and $\mu(K) > 0$. Then $\{xU : x \in K\}$ is an open cover of K. Then there exist $x_1, \cdots, x_n \in K$ such that $K \subset \bigcap_{k=1}^{n} x_k U$. Then

$$(3) \qquad\qquad\qquad \mu(K) \leq \sum_{k=1}^{n} \mu(x_k U)$$

$$(4) \qquad\qquad\qquad\qquad = \sum_{k=1}^{n} \mu(U)$$

$$(5) \qquad\qquad\qquad\qquad = 0$$

This is a contradiction. So $\mu(U) > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Exercise 8.2.6. Let G be a locally compact group and μ a left Haar measure on G. Then there exists $S \in \mathcal{B}(G)$ such that S is symmetric, $e \in S$ and $\mu(E) > 0$

Proof. Since G is locally compact, there exists a compact neighborhood K of e. Then $\mu(K) > 0$. Put $S = KK^{-1} \in \mathcal{B}(G)$. Then S is symmetric. Since $e \in K$, $K \subset S$ and $0 < \mu(K) \leq \mu(S)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Exercise 8.2.7. Let G be a locally compact group and μ a left Haar measure on G. Then
    (1) $\mu(\{e\}) > 0$ iff there exists $\lambda > 0$ such that $\mu = \lambda\#$.
    (2) μ is finite iff G is compact

Proof.
    (1) If there exists $\lambda > 0$ such that $\mu = \lambda\#$, then $\mu(\{e\}) > 0$ Conversely, suppose that $\mu(\{e\}) > 0$. Define $\lambda = \mu(\{e\}) > 0$. Let $B \in \mathcal{B}(G)$. If B is finite, then

$$\mu(B) = \sum_{x \in B} \mu(\{x\})$$
$$= \sum_{x \in B} \mu(x\{e\})$$
$$= \sum_{x \in B} \mu(\{e\})$$
$$= \sum_{x \in B} \lambda$$
$$= \lambda\#(\{e\})$$

        If B is infinite, then we may choose a countable subset and the same reasoning as above tells us that

$$\mu(B) = \infty = \lambda\#(B)$$

    (2) If G is compact, then μ is finite since μ is Radon. Conversely, suppose that μ is finite. Then **FINISH**

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Theorem 8.2.8. Let G be a locally compact group. Then there exists a left Haar measure on G.

Theorem 8.2.9. Let G be a locally compact group and μ_1, μ_2 left Haar measures on G. Then there exists $\lambda > 0$ such that $\mu_1 = \lambda\mu_2$.

Definition 8.2.10. Let G be a locally compact group and μ a left Haar measure on G. A previous exercise tells us that for each $g \in G$, $r_{g_*}\mu$ is a left Haar measure on G. The previous result tells us that for each $g \in G$ there exists $\lambda_g > 0$ such that $r_{g_*}\mu = \lambda_g\mu$. Define $\Delta : G \to (0, \infty)$ by $\Delta(g) = \lambda_g$. We call Δ the **modular function of** G.

Exercise 8.2.11. Let G be a locally compact group and μ a left Haar measure on G. Then
  (1) Δ is a homomorphism
  (2) for each $f \in L^1 \cup L^+$,

$$\int R_y f d\mu = \Delta(y^{-1}) \int f d\mu$$

Proof.
  (1) Recall that for each $g \in G$, $\Delta(g)\mu(U) = r_{g_*}\mu(U) = \mu(Ug)$. Let $g, h \in G$ and $U \in \mathcal{B}(G)$. Then $\Delta(gh)\mu(U) = \mu(Ugh) = \Delta(h)\mu(Ug) = \Delta(g)\Delta(h)\mu(U)$. So $\Delta(gh) = \Delta(g)\Delta(h)$.
  (2) Let $y \in G$ and $U \in \mathcal{B}(G)$. Put $f = \chi_U$ Then

$$\int R_y f d\mu = \int R_y \chi_U d\mu$$
$$= \int \chi_{Uy^{-1}} d\mu$$
$$= \mu(Uy^{-1})$$
$$= \Delta(y^{-1})\mu(U)$$
$$= \Delta(y^{-1}) \int \chi_U d\mu$$
$$= \Delta(y^{-1}) \int f d\mu$$

By linearity of R_y, for $f \in S^+$,

$$\int R_y f d\mu = \Delta(y^{-1}) \int f d\mu$$

For $f \in L^+$, choose $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$ $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$. Then for each $n \in \mathbb{N}$ $R_y\phi_n \leq R_y\phi_{n+1} \leq R_yf$ and $R_y\phi \to R_yf$. So MCT implies that

$$\int R_y f d\mu = \lim_{n \to \infty} \int R_y \phi_n d\mu$$
$$= \lim_{n \to \infty} \Delta(y^{-1}) \int \phi_n d\mu$$
$$= \Delta(y^{-1}) \int f d\mu$$

Let $f \in L^1$. If f is real valued, write $f = f^+ - f^-$. Then $R_y f = R_y f^+ - R_y f^-$ and

$$\int R_y f d\mu = \int R_y f^+ d\mu - \int R_y f^- d\mu$$

$$= \Delta(y^{-1}) \int f^+ d\mu - \Delta(y^{-1}) \int f^- d\mu$$

$$= \Delta(y^{-1}) \int f d\mu$$

If f is complex valued, write $f = g + ih$ with $g, h \in L^1$ real valued. Then

$$\int R_y f d\mu = \int R_y g d\mu + i \int R_y h d\mu$$

$$= \Delta(y^{-1}) \int g d\mu + i\Delta(y^{-1}) \int h d\mu$$

$$= \Delta(y^{-1}) \int f d\mu$$

$\square$

Definition 8.2.12. Let G be a locally compact group. Then G is said to be **unimodular** if $\ker \Delta = G$.

Exercise 8.2.13. Let G be a locally compact group. Then the following are quivalent:
   (1) G is unimodular
   (2) there exists a left Haar measure μ on G such that μ is a right Haar measure on G.
   (3) for each nonzero Radon measure μ on G, μ is a left Haar measure on G iff μ is a right Haar measure on G.

Proof.
   • (1) $\implies$ (2):
     Since G is a locally compact group, there exists a left Haar measure μ on G. Let $g \in G$ and $U \in \mathcal{B}(G)$. Then

     $$\mu(Ug) = \Delta(g)\mu(U) = \mu(U)$$

     Since G is unimodular, $\Delta(g) = 1$. Then μ is a right Haar measure on G.
   • (2) $\implies$ (3):
     By assumption, there exists a left Haar measure μ' on G such that μ' is a right Haar measure on G. Let μ be a nonzero Radon measure on G. If μ is a left Haar measure on G, then there exists $\lambda > 0$ such that $\mu = \lambda\mu'$ and therefore μ is a right Haar measure. The same reasoning implies that if μ is a right Haar measure on G, then μ is a left Haar measure on G.
   • (3) $\implies$ (1):
     Since G is locally compact, there exists a left *Haar* measure μ on G. By assumption, μ is a right Haar measure on G. By inner regularity there exists $K \in \mathcal{B}(G)$ such that $\mu(K) > 0$. Let $g \in G$. Then

     $$\Delta(g)\mu(K) = \mu(Kg) = \mu(K)$$

   So $\Delta(g) = 1$.

$\square$

Note 8.2.14. If G is a locally compact abelian group, then G is unimodular.

Exercise 8.2.15. Let G be a locally compact group and μ a left Haar measure on G. If G is unimodular then $\iota_*\mu = \mu$.

Proof. Suppose that G is unimodular. A previous exercise tells us that $\iota_*\mu$ is a right Haar measure on G. The unimodularity of G implies that $\iota_*\mu$ a left Haar measure on G. Then there exists $\lambda > 0$ such that $\iota_*\mu = \lambda\mu$. Since G is locally compact, there exists $S \in \mathcal{B}(G)$ such that S is symmetric and $\mu(S) > 0$. Then

$$\begin{aligned}
\mu(S) &= \mu(S^{-1}) \\
&= \iota_*\mu(S) \\
&= \lambda\mu(S)
\end{aligned}$$

So $\lambda = 1$ and $\iota_*\mu = \mu$.

it is also (Since G is locally compact, there exists $S \in \mathcal{B}(G)$ such that S is symmetric and $\mu(S) > 0$. Then

$$\mu(S) = \mu(S^{-1}) = \iota_*\mu(S)$$

Since $\iota_*\mu$ is a right Haar measure on G and G is unimodular, $\iota_*\mu(S)$ is also a left Haar measure on G. Then there exists $\lambda > 0$ such that $\mu(S) = \lambda\iota_*\mu(S)$. $\qquad \square$

8.3. **Fundamental Examples.**

Note 8.3.1. The Haar measure on $(\mathbb{R}^n, +)$ is m.

Exercise 8.3.2. The Haar measure on $(\mathbb{R}^\times, \cdot)$ is

$$d\mu(x) = \frac{1}{|x|} dm(x)$$

Proof. Let $0 < a < b$ and $c > 0$. Then

$$
\begin{aligned}
\mu(c(a,b)) &= \mu((ca, cb)) \\
&= \int_{(ca,cb)} \frac{1}{|x|} dm(x) \\
&= \int_{(ca,cb)} \frac{1}{x} dm(x) \\
&= \left[\log|x| \right]_{ca}^{cb} \\
&= \log(cb) - \log(ca) \\
&= \log b - \log a \\
&= \left[\log|x| \right]_a^b \\
&= \int_{(a,b)} \frac{1}{x} dm(x) \\
&= \mu((a,b))
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mu(-c(a,b)) &= \mu((-cb, -ca)) \\
&= \int_{(-cb,-ca)} \frac{1}{|x|} dm(x) \\
&= -\int_{(-cb,-ca)} \frac{1}{x} dm(x) \\
&= -\left[\log|x| \right]_{-cb}^{-ca} \\
&= \log(cb) - \log(ca) \\
&= \log b - \log a \\
&= \left[\log|x| \right]_a^b \\
&= \int_{(a,b)} \frac{1}{x} dm(x) \\
&= \mu((a,b))
\end{aligned}
$$

$\square$

Exercise 8.3.3. Define $f : [0, 1) \to \mathbb{T}$ by $f(x) = e^{i2\pi x}$. Let m be Lebesgue measure on $[0, 1)$, then the Haar measure on $\mathbb{T}$ is $f_* m$.

Proof. Note that f is a bijection and the topology on $\mathbb{T}$ is generated by sets of the form $f((a, b))$ where $a, b \in [0, 1)$ and $a < b$. Let $a, b \in [0, 1)$ and suppose that $a < b$. Put $A = f((a, b))$. Let $z \in \mathbb{T}$. Then there exists $\theta \in [0, 1)$ such that $z = f(\theta)$. If $1 \notin zA$, then $f^{-1}(zA) = (\theta + a, \theta + b)$. If $1 \in zA$, then $f^{-1}(zA) = (\theta + a, 1) \cup [0, \theta + b - 1)$. Suppose that $1 \notin zA$. Then

$$
\begin{aligned}
&= f_* m(zA) &&= m(f^{-1}(zA)) \\
&= m((\theta + a, \theta + b)) \\
&= b - a \\
&= m((a, b)) \\
&= m(f^{-1}(A)) \\
&= f_* m(A)
\end{aligned}
$$

Similarly if $1 \in zA$, $f_* m(zA) = f_* m(A)$.   □

Exercise 8.3.4. Let p be a prime. Define $|\cdot|_p : \mathbb{Q} \to [0, \infty)]$ by

$$
\begin{cases}
|\frac{a}{b} p^n|_p = p^{-n}, & \text{if } \gcd(a, p) = \gcd(b, p) = 1 \\
|0|_p = 0
\end{cases}
$$

Then $|\cdot|_p$ is an absolute value on $\mathbb{Q}$. Define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to the metric induced by $|\cdot|_p$. Define $\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1\}$. It is well known that $\mathbb{Q}_p$ is a locally compact field and $\mathbb{Z}_p$ is compact. Define $P = \{0, 1, \cdots, p - 1\}$. It is known that the topology is generated by

$$
\{x + p^n \mathbb{Z}_p : \text{ for } n \in \mathbb{Z}, x \in \mathbb{Q}_p\}
$$

Another useful fact is that

$$
\mathbb{Q}_p = \{\sum_{j=-n}^{\infty} a_j p^j : a_j \in P, n \in \mathbb{N}_0\}
$$

and

$$
\mathbb{Z}_p = \{\sum_{j=0}^{\infty} a_j p^j : a_j \in P\}
$$

Let μ be the Haar measure on $\mathbb{Q}_p$. Then μ is completely determined by the value $\mu(\mathbb{Z}_p)$

Proof. We observe that for $n \in \mathbb{Z}$, we may write $p^n \mathbb{Z}_p$ as the following disjoint union:

$$
p^n \mathbb{Z}_p = \bigcup_{j \in P} jp^n + p^{n+1} \mathbb{Z}^p
$$

Thus $\mu(p^n \mathbb{Z}^p) = p\mu(p^{n+1} \mathbb{Z}_p)$. If we set $\mu(\mathbb{Z}_p) = 1$, we obtain that $\mu(\mathbb{Z}_p) = p^n \mu(p^n \mathbb{Z}_p)$, which implies that

$$
\mu(p^n \mathbb{Z}_p) = \frac{1}{p^n} \mu(\mathbb{Z}_p)
$$

.   □

Exercise 8.3.5. Let ν be the Haar measure on $\mathbb{Q}_p$. Then the Haar measure on $\mathbb{Q}_p^\times$ is $d\mu = \frac{1}{|x|_p} d\nu$.

Proof. Let $x, y \in P^\times$ and $\alpha = xp^{n-1} + p^n\mathbb{Z}_p$. Then

$$\alpha(yp^{k-1} + p^k\mathbb{Z}_p) = p^{(n-1)+(k-1)}(xy + p^{n+k}\mathbb{Z}_p)$$

$\square$

8.4. **Action on Measures.**

Exercise 8.4.1. Let G be a locally compact group, μ a left Haar measure on G and $\nu \in \mathcal{M}(G)$. If $\nu \ll \mu$, then $l_{g_*}\nu \ll \mu$.

Proof. Suppose that $\nu \ll \mu$. Let $A \in \mathcal{B}(G)$. Then

$$
\begin{aligned}
\mu(A) = 0 &\implies \mu(g^{-1}A) = 0 \\
&\implies \nu(g^{-1}A) = 0 \\
&\implies \nu(l_{g^{-1}}(A)) = 0 \\
&\implies \nu(l_g^{-1}(A)) = 0 \\
&\implies l_{g_*}\nu(A) = 0
\end{aligned}
$$

So $l_{g_*}\nu \ll \mu$. $\square$

Definition 8.4.2. Let G be a locally compact group and μ a left Haar measure on G. Define $\mathcal{M}_\mu \subset \mathcal{M}(G)$ by

$$\mathcal{M}_\mu = \{\nu \in \mathcal{M}(G) : \nu \ll \mu\}$$

We define an action $\phi : G \times \mathcal{M}_\mu \to \mathcal{M}_\mu$ by

$$g \cdot \nu = l_{g_*}\nu$$

Exercise 8.4.3. Let G be a locally compact group, μ a σ-finite left Haar measure on G, $\nu \in \mathcal{M}_\mu$ and $g \in G$. Then

$$\frac{d(g \cdot \nu)}{d\mu} = L_g \frac{d\nu}{d\mu}$$

Proof. Set $f = d\nu/d\mu$. Let $A \in \mathcal{B}(X)$. Then

$$\int_A L_g f d\mu = \int_A f \circ l_g^{-1} \, d\mu$$

$$= \int_A f \circ l_g^{-1} \, d\mu$$

$$= \int_{l_g^{-1}(A)} f \, d(l_g^{-1}{}_* \mu)$$

$$= \int_{l_g^{-1}(A)} f \, d(l_{g^{-1}}{}_* \mu)$$

$$= \int_{l_g^{-1}(A)} f \, d\mu$$

$$= \nu(l_g^{-1}(A))$$

$$= l_{g_*} \nu(A)$$

$$= g \cdot \nu(A)$$

Since A is arbitrary, uniqueness implies that

$$\frac{d(g \cdot \nu)}{d\mu} = L_g \frac{d\nu}{d\mu}$$

$\square$

Exercise 8.4.4. Let G be a locally compact group, μ a σ-finite left Haar measure on G, $\nu \in \mathcal{M}_\mu$ and $g \in G$. Then $\|g \cdot \nu\| = \|\nu\|$.

Proof. Exercise 4.2.8 implies that

$$\|g \cdot \nu\| = \int \left| \frac{d(g \cdot \nu)}{d\mu} \right| d\mu$$

$$= \int \left| L_g \frac{d\nu}{d\mu} \right| d\mu$$

$$= \int L_g \left| \frac{d\nu}{d\mu} \right| d\mu$$

$$= \int \left| \frac{d\nu}{d\mu} \right| d\mu$$

$$= \|\nu\|$$

$\square$

9. The Bochner Integral

9.1. **Introduction.**

Definition 9.1.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. We define

$$L_Y^0(X, \mathcal{A}) = \left\{ f : X \to Y : f \text{ is } (\mathcal{A}, \mathcal{B}(Y))\text{-measurable} \right\}$$

Exercise 9.1.2. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. Then $L_Y^0(X, \mathcal{A})$ is a vector space.

Proof. Let $f, g \in L_Y^0(X, \mathcal{A})$ and $\lambda \in \mathbb{C}$. By definition, f and g are measurable. Since $f + \lambda g$ is a composition of measurable maps, $f + \lambda g$ is measurable. Therefore $f + \lambda g \in L_Y^0(X, \mathcal{A})$. Clearly constant maps are measurable and hence $0 \in L_Y^0(X, \mathcal{A})$. So $L_Y^0(X, \mathcal{A})$ is a vector space. $\square$

Definition 9.1.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space and $p \in [1, \infty]$. Define $\| \cdot \|_p : L_Y^0(X, \mathcal{A}, \mu) \to [0, \infty]$ by

$$\|f\|_p = \left(\int \|f\|^p d\mu \right)^{\frac{1}{p}} \qquad (p < \infty)$$

and

$$\|f\|_\infty = \inf \left\{ \lambda > 0 : \mu(\{x \in X : \lambda < \|f(x)\|\}) = 0 \right\}$$

We define

$$L_Y^p(X, \mathcal{A}, \mu) = \{ f \in L_Y^0(X, \mathcal{A}, \mu) : \|f\|_p < \infty \}$$

Exercise 9.1.4. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space and $p \in [1, \infty]$. Then $L_Y^p(X, \mathcal{A}, \mu)$ is a subspace of $L_Y^0(X, \mathcal{A}, \mu)$.

Proof. Let $f, g \in L_Y^p X, \mathcal{A}, \mu)$ and $\lambda \in \mathbb{C}$. Then $\|f\|_p, \|g\|_p < \infty$.

(1) Clearly $\|\lambda f\|_p = |\lambda| \|f\|_p < \infty$. So $\lambda f \in L_Y^p$.
(2) Let $\| \cdot \|_p' : L^0(X, \mathcal{A}, \mu) \to [0, \infty]$ denote the usual L^p norm. Since $\|f + g\| \leq \|f\| + \|g\|$, we have that

$$\begin{aligned}
\|f + g\|_p &= \| \|f + g\| \|_p' \\
&\leq \| \|f\| + \|g\| \|_p' \\
&\leq \| \|f\| \|_p' + \| \|g\| \|_p' \\
&= \|f\|_p + \|g\|_p \\
&< \infty
\end{aligned}$$

So $f + g \in L_Y^p$.

Hence L_Y^p is a subspace. $\square$

Exercise 9.1.5. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space and $p \in [1, \infty]$. Then

(1) $\| \cdot \|_p$ is a seminorm on $L_Y^p(X, \mathcal{A}, \mu)$
(2) if we identify functions that are equal μ-a.e., then $\| \cdot \|_p$ is a norm on $L_Y^p(X, \mathcal{A}, \mu)$

Proof. Let $f, g \in L_Y^p X, \mathcal{A}, \mu)$ and $\lambda \in \mathbb{C}$.

(1) The previous exercise implies that, $\|\lambda f\|_p = |\lambda|\|f\|_p$ and $\|f + g\|_p \le \|f\|_p + \|g\|_p$. So $\|\cdot\|_p$ is a seminorm on L_Y^p.

(2) If $f = 0$ μ-a.e., then $\|f\| = 0$ μ-a.e. Hence

$$\|f\|_p = \|\|f\|\|_p'$$
$$= 0$$

So if we identify functions that are equal μ-a.e., $\|\cdot\|_p$ becomes a norm on L_Y^p.

$\square$

Note 9.1.6. So for $(f_n)_{n\in\mathbb{N}} \subset L_Y^p$ and $f \in L_Y^p$,

$$f_n \xrightarrow{L_Y^p} f \text{ iff } \int \|f_n - f\|^p \to 0$$

Definition 9.1.7. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space and $\phi : X \to Y$. Then ϕ is said to be **simple** if ϕ is measurable, $\phi(X)$ is finite and for each $y \in \phi(X) \setminus \{0\}$, $\mu(\phi^{-1}(y)) < \infty$. If ϕ is simple then the **standard representation of** ϕ is defined to be the sum

$$\phi = \sum_{j=1}^n \chi_{E_j} y_j$$

where $(y_j)_{j=1}^n = \phi(X)$ and for each $j \in \{1, \cdots, n\}$, $E_j = \phi^{-1}(y_j)$. We define

$$S_Y(X, \mathcal{A}, \mu) = \{f \in L_Y^0(X, \mathcal{A}) : f \text{ is simple}\}$$

Note 9.1.8. If $\phi = \sum_{j=1}^n \chi_{E_j} y_j$ is in the standard representation, then $(E_j)_{j=1}^n$ are disjoint and $\bigcup_{j=1}^n E_j = X$.

Exercise 9.1.9. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. Then $S_Y \subset L_Y^1$.

Proof. Let $\phi \in S_Y$. Write $\phi = \sum_{j=1}^n \chi_{E_j} y_j$ in the standard representation. Then $\|\phi\| = \sum_{j=1}^n \|y_j\|\chi_{E_j}$. By definition, for each $j \in \{1, \cdots, n\}$, $y_j \ne 0$ implies that $\mu(E_j) < \infty$. Then

$$\int \|\phi\| d\mu = \sum_{j=1}^n \|y_j\|\mu(E_j)$$
$$< \infty$$

So $\phi \in L_Y^1$. $\square$

Exercise 9.1.10. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. Then $S_Y(X, \mathcal{A}, \mu)$ is a subspace of $L_Y^0(X, \mathcal{A})$

Proof. Clear. $\square$

Note 9.1.11. For the remainder of this section, we will use the shorthand notation L_Y^0, L_Y^p and S_Y unless the context underlying measure space $(X, \mathcal{A}, \mu)$ is unclear.

Definition 9.1.12. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. Let $\phi \in S_Y$. Write $\phi = \sum_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. With the convention that $\infty \cdot 0_Y = 0_Y$, we define

$$\int \phi d\mu = \sum_{j=1}^{n} \mu(E_j) y_j$$

For $A \in \mathcal{A}$, define

$$\int_A \phi d\mu = \int \chi_A \phi d\mu$$

Exercise 9.1.13. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $\phi \in S_Y$ and $A \in \mathcal{A}$. Write $\phi = \sum_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. Then

$$\int_A \phi d\mu = \sum_{j=1}^{n} \mu(A \cap E_j) y_j$$

Proof. Note that $\chi_A \phi = \sum_{j=1}^{n} \chi_{A \cap E_j} y_j$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Exercise 9.1.14. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $\phi, \psi \in S_Y$ and $\lambda \in \mathbb{C}$. Then

$$\int \phi + \lambda \psi d\mu = \int \phi d\mu + \lambda \int \psi d\mu$$

Proof. If $\lambda = 0$, then the result clearly holds. Suppose that $\lambda \neq 0$. Write $\phi = \sum_{j=1}^{n} \chi_{A_j} a_j$ and $\psi = \sum_{j=k}^{m} \chi_{B_k} b_k$ in the standard representation. Put

$$L = \{(j, k) \in \mathbb{N}^2 : j \leq n, k \leq m, \text{ and } A_j \cap B_k \neq \varnothing\}$$

Then the standard representation of $\phi + \lambda\psi$ is given by $\phi + \lambda\psi = \sum\limits_{(j,k)\in L} \chi_{A_j \cap B_k}(a_j + \lambda b_k)$. So

$$\int \phi + \lambda\psi d\mu = \int \sum_{(j,k)\in L} \chi_{A_j \cap B_k}(a_j + \lambda b_k)d\mu$$

$$= \sum_{(j,k)\in L} \mu(A_j \cap B_k)(a_j + \lambda b_k)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m} \mu(A_j \cap B_k)(a_j + \lambda b_k)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{m} \mu(A_j \cap B_k)a_j + \lambda\sum_{j=1}^{n}\sum_{k=1}^{m} \mu(A_j \cap B_k)b_k$$

$$= \sum_{j=1}^{n} \mu(A_j)a_j + \lambda\sum_{k=1}^{m} \mu(B_k)b_k$$

$$= \int \phi d\mu + \lambda\int \psi d\mu$$

$\square$

Exercise 9.1.15. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $\phi \in S_Y$. Then

$$\left\| \int \phi d\mu \right\| \leq \int \|\phi\| d\mu$$

Proof. Write $\phi = \sum\limits_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. Note that $\|\phi\| = \sum\limits_{j=1}^{n} \chi_{E_j}\|y_j\|$. Then

$$\left\| \int \phi d\mu \right\| = \left\| \int \sum_{j=1}^{n} \chi_{E_j} y_j d\mu \right\|$$

$$= \left\| \sum_{j=1}^{n} \mu(E_j) y_j \right\|$$

$$\leq \sum_{j=1}^{n} \mu(E_j)\|y_j\|$$

$$= \int \sum_{j=1}^{n} \|y_j\|\chi_{E_j} d\mu$$

$$= \int \|\phi\| d\mu$$

$\square$

Exercise 9.1.16. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $f \in L_Y^1$ and $(\phi_n)_{n\in\mathbb{N}} \subset S_Y$. If $\phi_n \xrightarrow{L_Y^1} f$, then

$$\lim_{n\to\infty} \int \phi_n d\mu$$

exists.

Proof. Suppose that $\phi \xrightarrow{L^1_Y} f$. Then by definition,

$$\int \|\phi_n - f\| d\mu \to 0$$

Let $m, n \in \mathbb{N}$. Then

$$\left\| \int \phi_m d\mu - \int \phi_n d\mu \right\| = \left\| \int \phi_m - \phi_n d\mu \right\|$$

$$\leq \int \|\phi_m - \phi_n\| d\mu$$

$$\leq \int \|\phi_m - f\| d\mu + \int \|f - \phi_n\| d\mu$$

Hence $(\int \phi_n d\mu)_{n \in \mathbb{N}} \subset Y$ is Cauchy and $\lim\limits_{n \to \infty} \int \phi_n d\mu$ exists. $\square$

Exercise 9.1.17. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $f \in L^1_Y$ and $(\phi_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}} \subset S_Y$. If $\phi_n \xrightarrow{L^1_Y} f$ and $\psi_n \xrightarrow{L^1_Y} f$, then

$$\lim_{n \to \infty} \int \phi_n d\mu = \lim_{n \to \infty} \int \psi_n d\mu$$

Proof. Suppose that $\phi_n \xrightarrow{L^1_Y} f$ and $\psi_n \xrightarrow{L^1_Y} f$. Let $\epsilon > 0$. By defintion, there exist $N_1 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N_1$ implies that $\int \|\phi_n - f\| d\mu < \frac{\epsilon}{6}$ and $\int \|\psi_n - f\| d\mu < \frac{\epsilon}{6}$. Similarly to the previous exercise we have that for each $n \in \mathbb{N}$, $n \geq N_1$ implies that

$$\left\| \int \phi_n d\mu - \int \psi_n d\mu \right\| = \left\| \int \phi_n - \psi_n d\mu \right\|$$

$$\leq \int \|\phi_n - \psi_n\| d\mu$$

$$\leq \int \|\phi_n - f\| d\mu + \int \|f - \psi_n\| d\mu$$

$$< \frac{\epsilon}{6} + < \frac{\epsilon}{6}$$

$$= \frac{\epsilon}{3}$$

Put $I_\phi = \lim\limits_{n \to \infty} \int \phi_n d\mu$ and $I_\psi = \lim\limits_{n \to \infty} \int \psi_n d\mu$. Then there exists $N_2 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N_2$, then

$$\left\| \int \phi_n d\mu - I_\phi \right\| < \frac{\epsilon}{3}$$

and

$$\left\| \int \psi_n d\mu - I_\psi \right\| < \frac{\epsilon}{3}$$

Choose $N = \max(N_1, N_2)$. Then for each $n \in \mathbb{N}$, $n \geq N$ implies that

$$\|I_\phi - I_\psi\| \leq \left\|I_\phi - \int \phi_n d\mu\right\| + \left\|\int \phi_n d\mu - \int \psi_n d\mu\right\| + \left\|\int \psi_n d\mu - I_\psi\right\|$$
$$=< \frac{\epsilon}{3} + < \frac{\epsilon}{3} + < \frac{\epsilon}{3}$$
$$= \epsilon$$

Since $\epsilon > 0$ is arbitrary, $I_\phi = I_\psi$. $\square$

Exercise 9.1.18. Let Y be a Banach space and $(y_n)_{n \in \mathbb{N}} \subset Y$ a countable dense subset. For $\epsilon > 0$ and $n \in \mathbb{N}$, define
$$B_n^\epsilon = \{y \in Y : \|y - y_n\| < \epsilon \|y_n\|\}$$
Then for each $\epsilon > 0$,
$$U \setminus \{0\} \subset \bigcup_{n \in \mathbb{N}} B_n^\epsilon$$

Proof. Let $\epsilon > 0$. For the sake of contradiction, suppose that $Y \setminus \{0\} \not\subset \bigcup_{n \in \mathbb{N}} B_n^\epsilon$. Then there exists $y \in Y$ such that $y \neq 0$ and for each $n \in \mathbb{N}$, $\|y - y_n\| \geq \epsilon \|y_n\|$. Since $(y_n)_{n \in \mathbb{N}}$ is dense in Y, there exists a subsequence $(y_{n_j})_{j \in \mathbb{N}} \subset (y_n)_{n \in \mathbb{N}}$ such that for each $j \in \mathbb{N}$, $\|y_{n_j} - y\| < 2^{-j}$. Then for each $j \in \mathbb{N}$,

$$\|y_{n_j}\| \leq \epsilon^{-1} \|y - y_{n_j}\|$$
$$< \epsilon^{-1} 2^{-j}$$

So that $y_{n_j} \to y$ and $y_{n_j} \to 0$. Since $y \neq 0$, this is a contradiction and thus

$$Y \setminus \{0\} \subset \bigcup_{n \in \mathbb{N}} B_n^\epsilon$$

$\square$

Exercise 9.1.19. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a separable Banach space and $f \in L_Y^1(X, \mathcal{A}, \mu)$. Then there exists $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{\text{a.e.}} f$ and $\phi_n \xrightarrow{L_Y^1} f$.
Hint: Choose a countable dense subset $(y_n)_{n \in \mathbb{N}} \subset f(X)$ and define
- $A_1^j = B_1^{1/j}$
- $A_n^j = B_n^{1/j} \setminus \left(\bigcup_{k=1}^{n-1} B_k^{1/j}\right)$
- $E_n^j = f^{-1}(A_n^j)$
- $\phi_j = \sum_{n \in \mathbb{N}} \chi_{E_n^j} y_n$

Proof. $\square$

Definition 9.1.20. Bochner Integral:
Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a separable Banach space and $f : X \to Y$. Then f is said to be **Bochner** integrable if $f \in L_Y^1$. If f is Bochner integrable, then there exists $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{\text{a.e.}} f$ and $\phi_n \xrightarrow{L_Y^1} f$ and the **Bochner integral of** f with respect to μ, denoted

$$\int f d\mu$$

is defined to be

$$\int f d\mu = \lim_{n \to \infty} \int \phi_n d\mu$$

Exercise 9.1.21. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a separable Banach space, $f, g \in L_Y^1$ and $\lambda \in \mathbb{C}$. Then

$$\int f + \lambda g d\mu = \int f d\mu + \lambda \int g d\mu$$

Proof. Choose $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{L_Y^1} f$ and $(\psi_n)_{n \in \mathbb{N}} \subset S_Y$ such that $\psi_n \xrightarrow{L_Y^1} g$. Since addition and scalar multiplication are continuous, $\phi_n + \lambda \psi_n \xrightarrow{L_Y^1} f + \lambda g$. By definition, we have that

$$\int \phi_n + \lambda \psi_n d\mu \to \int f + \lambda g d\mu$$

$$\int \phi_n d\mu \to \int f d\mu$$

and

$$\int \psi_n d\mu \to \int g d\mu$$

Hence

$$\int f + \lambda g d\mu = \lim_{n \to \infty} \int \phi_n + \lambda \psi_n d\mu$$

$$= \lim_{n \to \infty} \int \phi_n d\mu + \lambda \lim_{n \to \infty} \int \psi_n d\mu$$

$$= \int f d\mu + \lambda \int g d\mu$$

$\square$

Exercise 9.1.22. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a separable Banach space. Define $I : L_Y^1 \to Y$ by

$$If = \int f d\mu$$

Then $I \in L(L_Y^1, Y)$ and $\|I\| \leq 1$.

Proof. Let $f \in L_Y^1$. Choose $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{L_Y^1} f$. Then

$$\left| \int \|\phi_n\| d\mu - \int \|f\| d\mu \right| = \left| \int \|\phi_n\| - \|f\| d\mu \right|$$

$$\leq \int |\|\phi_n\| - \|f\|| d\mu$$

$$\leq \int \|\phi_n - f\| d\mu$$

$$\to 0$$

So

$$\int \|\phi_n\| d\mu \to \int \|f\| d\mu$$

By continuity of $\|\cdot\| : Y \to [0, \infty)$,

$$\|If\| = \left\| \int f d\mu \right\|$$

$$= \left\| \lim_{n\to\infty} \int \phi_n d\mu \right\|$$

$$= \lim_{n\to\infty} \left\| \int \phi_n d\mu \right\|$$

$$\leq \lim_{n\to\infty} \int \|\phi_n\| d\mu$$

$$= \int \|f\| d\mu$$

$$= \|f\|_1$$

$\square$

Exercise 9.1.23. Let Y be a separable Banach space and $f : [a, b] \to Y$ continuous. Then f is Banach-integrable.

Proof. Continuity implies that $f \in L_Y^\infty$ and

$$\int \|f\| dm \leq \|f\|_\infty (b - a)$$

$$< \infty$$

so that $f \in L_Y^1$ and f is Bochner integrable. $\square$

Exercise 9.1.24. Dominated Convergence Theorem:

Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a separable Banach space, $(f_n)_{n\in\mathbb{N}} \subset L_Y^1$ and $f \in L_Y^0$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$ and there exists $g \in L_Y^1$ such that for each $n \in \mathbb{N}$, $\|f_n\| \leq g$. Then $f \in L_Y^1$ and $f_n \xrightarrow{L^1} f$.

Proof. Since $f_n \xrightarrow{\text{a.e.}} f$, $\|f\| \leq g$ a.e. and $f \in L_Y^1$. Also,

$$\|f_n - f\| \leq \|f_n\| + \|f\|$$

$$\leq 2g \text{ a.e.}$$

Hence $2g - \|f_n - f\| \geq 0$ a.e. Fatou's lemma implies that

$$\int 2g \, d\mu = \int \liminf_{n\to\infty} (2g - \|f_n - f\|) \, d\mu$$

$$\leq \liminf_{n\to\infty} \left[\int 2g - \|f_n - f\| \, d\mu \right]$$

$$= \int 2g \, d\mu - \limsup_{n\to\infty} \int \|f_n - f\| \, d\mu$$

Hence

$$0 \leq \limsup_{n\to\infty} \int \|f_n - f\| \, d\mu \leq 0$$

and $f_n \xrightarrow{L_Y^1} f$. $\square$

Exercise 9.1.25. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y, Z separable Banach spaces and $f \in L^1_Y$ and $T \in L(Y, Z)$. Then $T \circ f \in L^1_Z$ and

$$\int T \circ f d\mu = T \left(\int f d\mu \right)$$

Note 9.1.26. The statement remains true if T is continuous and conjugate-linear.

Proof. Suppose that $f \in S_Y$. Write $f = \sum_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. Then $T \circ f = \sum_{j=1}^{n} \chi_{E_j} T(y_j)$ and

$$\int T \circ f d\mu = \sum_{j=1}^{n} \mu(E_j) T(y_j)$$
$$= T \left(\sum_{j=1}^{n} \mu(E_j) y_j \right)$$
$$= T \left(\int f d\mu \right)$$

For $f \in L^1_Y$, choose $(\phi_n)_{n \in \mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{\text{a.e.}} f$ and $\phi_n \xrightarrow{L^1_Y} f$. Then

$$\|T \circ \phi_n - T \circ f\| = \|T \circ (\phi_n - f)\|$$
$$\leq \|T\| \|\phi_n - f\|$$

So $T \circ \phi_n \xrightarrow{\text{a.e.}} T \circ f$ and $T \circ \phi_n \xrightarrow{L^1_Z} T \circ f$. Thus

$$\int T \circ f d\mu = \lim_{n \to \infty} \int T \circ \phi_n d\mu$$
$$= \lim_{n \to \infty} T \left(\int \phi_n d\mu \right)$$
$$= T \left(\lim_{n \to \infty} \int \phi_n d\mu \right)$$
$$= T \left(\int f d\mu \right)$$

$\square$

Note 9.1.27. Recall that for a function $f : X \times Y \to Z$, $x \in X$ and $y \in Y$, the functions $f_x : Y \to Z$ and $f^y : X \to Z$ are defined by $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$.

Exercise 9.1.28. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, Z a separable Banach space, $A \subset Y$ open and $f : X \times A \to Z$. Suppose that for each $y \in A$, $f^y \in L^1_Z(\mu)$. Define $F : Y \to Z$ by

$$F(y) = \int_X f^y \, d\mu$$

(1) Suppose that there exists $g \in L^1(\mu)$ such that for each $(x, y) \in X \times A$, $\|f(x, y)\| \leq g(x)$. Let $y_0 \in A$. If for each $x \in X$, f_x is continuous at y_0, then F is continuous at y_0.

(2) Suppose that for each $x \in X$, $f_x : A \to Z$ is Gateaux differentiable and there exists $g \in L^1(\mu)$ such that for each $(x,y) \in X \times A$, $h \in Y$, $|df_x(y)(h)| \le g(x)$. Then F is Gateaux differentiable and for each $y \in A$, $h \in Y$,

$$dF(y)(h) = \int_X df_x(y)(h)\, d\mu(x)$$

Proof.

(1) Suppose that for each $x \in X$, f_x is continuous at y_0. Let $(y_n) \subset A$. Suppose that $y_n \to y_0$. Then $f^{y_n} \xrightarrow{\text{p.w.}} f^{y_0}$. Since for each $n \in \mathbb{N}$, $|f^{y_n}| \le g$, the dominated convergence theorem implies that $F(y_n) \to F(y_0)$.

(2) Let $y_0 \in \mathbb{R}$. Choose $(y_n)_{n \in \mathbb{N}}$ such that $y_n \to y_0$ and for each $n \in \mathbb{N}$, $y_n \ne y_0$. For $n \in \mathbb{N}$, define $q_n : X \to \mathbb{R}$ by

$$q_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}$$

So $h_n(\cdot) \xrightarrow{\text{p.w.}} \partial f/\partial t(\cdot, t_0)$. The mean value theorem implies that for each $x \in X$ and $n \in \mathbb{N}$, there exists $c_{n,x} \in (t_n, t_0)$ such that $h_n(x) = \partial f/\partial t(x, c_{n,x})$. Then for each $n \in \mathbb{N}$, $|h_n| \le g$. The dominated convergence theorem then implies that $\partial f/\partial t(\cdot, t_0) \in L^1(\mu)$ and

$$\int \frac{\partial f}{\partial t}(x,t_0) d\mu(x) = \lim_{n \to \infty} \int_X h_n\, d\mu$$
$$= \lim_{n \to \infty} \frac{F(t_n) - F(t_0)}{t_n - t_0}$$
$$= F'(t_0^-)$$

So that F is differentiable at t_0 from the left. Similarly, F is differentiable at t_0 from the right.

FINISH!!! $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

9.2. **Applications to Hilbert Spaces.**

Exercise 9.2.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, H a separable Hilbert space, $f \in L^1_H$ and $a \in H$. Then

$$\int \langle f(x), a \rangle d\mu(x) = \left\langle \int f(x) d\mu(x), a \right\rangle$$

Proof. Define $T \in L^*(H, \mathbb{C})$ by $T(x) = \langle x, a \rangle$ and apply a previous exercise. □

10. Appendix

10.1. **Summation.**

Definition 10.1.1. Let $f : X \to [0, \infty)$, Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note 10.1.2. Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.

References

[1] bibliographic information
[2] bibliographic information ...