

Introduction to Differential Geometry

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Contents

Notation	vii
Preface	1
1 Review of Fundamentals	3
1.1 Set Theory	3
1.2 Linear Algebra	4
1.3 Calculus	7
1.3.1 Differentiation	7
1.3.2 Integration	8
1.4 Topology	9
2 Multilinear Algebra	11
2.1 (r, s) -Tensors	11
2.2 Covariant k -Tensors	14
2.2.1 Symmetric and Alternating Covariant k -Tensors	14
2.2.2 Exterior Product	17
2.2.3 Interior Product	21
2.3 $(0, 2)$ -Tensors	22
2.3.1 Scalar Product Spaces	23
2.3.2 Symplectic Vector Spaces	25
3 Smooth Manifolds	29
3.1 Topological Manifolds	29
3.2 Smooth Manifolds	37
3.3 Smooth Maps	40
3.4 Partitions of Unity	44
3.5 The Tangent Space	45
3.6 The Cotangent Space	50
4 Submersions and Immersions	53
4.1 Maps of Constant Rank	53
4.2 Submanifolds	56
5 Bundles and Sections	59
5.1 Introduction	59
5.1.1 Fiber Bundles	59
5.1.2 Vector Bundles	60
5.2 Bundle Morphisms	62
5.3 Subbundles	63
5.4 Vertical and Horizontal Subbundles	64
5.5 The Tangent Bundle	66
5.6 The cotangent Bundle	67

5.7	The (r, s) -Tensor Bundle	67
5.8	Vector Fields	68
5.9	1-Forms	69
5.10	(r, s) -Tensor Fields	70
5.11	Differential Forms	72
6	Connections	77
6.1	Koszul Connections	77
7	Riemannian Geometry	79
8	Symplectic Geometry	85
8.1	Symplectic Manifolds	86
9	Extra	87
9.1	Integration of Differential Forms	89
A	Summation	91
B	Asymptotic Notation	93

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on (X, \mathcal{A})
v	velocity

Preface

cc-by-nc-sa

Chapter 1

Review of Fundamentals

1.1 Set Theory

Definition 1.1.0.1. Let $\{A_i\}_{i \in I}$ be a collection of sets. The **disjoint union of** $\{A_i\}_{i \in I}$, denoted $\coprod_{i \in I} A_i$, is defined by

$$\coprod_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$$

We define the **natural projection map**, denoted $\pi : \coprod_{i \in I} A_i \rightarrow I$, by $\pi(i, a) = i$.

Definition 1.1.0.2. Let E and M be sets, $\pi : E \rightarrow B$ a surjection and $\sigma : B \rightarrow E$. Then σ is said to be a section of (E, M, π) if $\pi \circ \sigma = \text{id}_M$.

Note 1.1.0.3. Let $\{A_i\}_{i \in I}$ be a collection of sets and $\sigma : I \rightarrow \coprod_{i \in I} A_i$. We will typically be interested in sections σ of $\left(\coprod_{i \in I} A_i, I, \pi \right)$.

Exercise 1.1.0.4. Let $\{A_i\}_{i \in I}$ be a collection of sets and $\sigma : I \rightarrow \coprod_{i \in I} A_i$. Then σ is a section of $\coprod_{i \in I} A_i$ iff for each $i \in I$, $\sigma(i) \in A_i$

Proof. Clear. □

1.2 Linear Algebra

Note 1.2.0.1. We denote the standard basis on \mathbb{R}^n by (e_1, \dots, e_n) .

Definition 1.2.0.2. Let $A \in \mathbb{R}^{n \times n}$. Then A is said to be **invertible** if $\det(A) \neq 0$. We denote the set of $n \times n$ invertible matrices by $GL(n, \mathbb{R})$.

$$O(n)$$

Exercise 1.2.0.3. Let $A, B \in \mathbb{R}^{n \times n}$. Then $AB = I$ iff $BA = I$.

Proof.

- (\implies):
Suppose that $AB = I$. Then

$$\begin{aligned} \ker B &\subset \ker AB \\ &= \ker I \\ &= \{0\} \end{aligned}$$

so that $\ker B = \{0\}$. Hence $\text{Im } B = \mathbb{R}^n$ and B is surjective. Then

$$\begin{aligned} IB &= BI \\ &= B(AB) \\ &= (BA)B \end{aligned}$$

Since B is surjective, $I = BA$.

- (\impliedby):
Immediate by the previous part.

□

Definition 1.2.0.4. Let $A \in \mathbb{R}^{n \times p}$. Then A is said to be an **orthogonal matrix** if $A^*A = I$. We denote the set of $n \times p$ orthogonal matrices by $O(n, p)$. We write $O(n)$ in place of $O(n, n)$.

$$O(n)$$

Exercise 1.2.0.5. Define $\phi : S_n \rightarrow GL(n, \mathbb{R})$ by

$$\phi(\sigma) = \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix}$$

Then

1. for each $A \in \mathbb{R}^{n \times p}$,

$$(\phi(\sigma)A)_{i,j} = A_{\sigma(i),j}$$

i.e. left multiplying A by $\phi(\sigma)$ the the same as permuting the rows of A by σ

2. ϕ is a group homomorphism

Proof. 1. Let $A \in \mathbb{R}^{n \times p}$. Then

$$\begin{aligned} (\phi(\sigma)A)_{i,j} &= \langle e_{\sigma(i)}^*, Ae_j \rangle \\ &= A_{\sigma(i),j} \end{aligned}$$

2. Let $\sigma, \tau \in S_n$. Part (1) implies that

$$\begin{aligned}\phi(\sigma\tau) &= \begin{pmatrix} e_{\sigma\tau(1)}^* \\ \vdots \\ e_{\sigma\tau(n)}^* \end{pmatrix} \\ &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} \begin{pmatrix} e_{\tau(1)}^* \\ \vdots \\ e_{\tau(n)}^* \end{pmatrix} \\ &= \phi(\sigma)\phi(\tau)\end{aligned}$$

Since $\sigma, \tau \in S_n$ are arbitrary, ϕ is a group homomorphism. □

Definition 1.2.0.6. Define $\phi : S_n \rightarrow GL(n, \mathbb{R})$ as in the previous exercise. Let $P \in GL(n, \mathbb{R})$. Then P is said to be a **permutation matrix** if there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. We denote the set of $n \times n$ permutation matrices by $\text{Perm}(n)$.

Exercise 1.2.0.7. We have that

1. $\text{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$
2. $\text{Perm}(n)$ is a subgroup of $O(n)$

Proof.

1. By definition, $\text{Perm}(n) = \text{Im } \phi$. Since $\phi : S_n \rightarrow GL(n, \mathbb{R})$ is a group homomorphism, $\text{Im } \phi$ is a subgroup of $GL(n, \mathbb{R})$. Hence $\text{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$.
2. Let $P \in \text{Perm}(n)$. Then there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. Then

$$\begin{aligned}PP^* &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix}^* \\ &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} (e_{\sigma(1)} \quad \cdots \quad e_{\sigma(n)}) \\ &= (\langle e_{\sigma(i)}, e_{\sigma(j)} \rangle)_{i,j} \\ &= I\end{aligned}$$

A previous exercise implies that $P^*P = I$. Hence $P \in O(n)$. Since $P \in \text{Perm}(n)$ is arbitrary, $\text{Perm}(n) \subset O(n)$. Part (1) implies that $\text{Perm}(n)$ is a group. Hence $\text{Perm}(n)$ is a subgroup of $O(n)$ □

Note 1.2.0.8. We will write P_σ in place of $\phi(\sigma)$.

Exercise 1.2.0.9. Let $Z \in \mathbb{R}^{p \times n}$. If $\text{rank } Z = k$, then there exist $\sigma \in S_n$, $\tau \in S_p$ and $A \in GL(k, \mathbb{R})$, such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Proof. Suppose that $\text{rank } Z = k$. Then there exist $i_1, \dots, i_k \in \{1, \dots, p\}$ such that $i_1 < \dots < i_k$ and $\{e_{i_1}^* Z, \dots, e_{i_k}^* Z\}$ is linearly independent. Set

$$Z' = \begin{pmatrix} e_{i_1}^* Z \\ \vdots \\ e_{i_k}^* Z \end{pmatrix}$$

Then $\text{rank } Z' = k$. Hence there exist $j_1, \dots, j_k \in \{1, \dots, n\}$ such that $j_1 < \dots < j_k$, and $\{Z'e_{i_1}, \dots, Z'e_{i_k}\}$ is linearly independent. Set

$$A = (Z'e_{i_1} \quad \dots \quad Z'e_{i_k})$$

Then $A \in \mathbb{R}^{k \times k}$ and $\text{rank } A = k$. Thus $A \in GL(k, \mathbb{R})$. Choose $\sigma \in S_n$ and $\tau \in S_p$ such that $\sigma(1) = j_1, \dots, \sigma(k) = j_k$ and $\tau(1) = i_1, \dots, \tau(k) = i_k$. Let $a, b \in \{1, \dots, k\}$. By construction,

$$\begin{aligned} (P_\tau Z P_\sigma^*)_{a,b} &= Z_{\tau(a), \sigma(b)} \\ &= Z_{i_a, j_b} \\ &= A_{a,b} \end{aligned}$$

□

Definition 1.2.0.10. Let $A \in \mathbb{R}^{n \times p}$. Then A is said to be a **diagonal matrix** if for each $i \in [n]$ and $j \in [p]$, $i \neq j$ implies that $A_{i,j} = 0$. We denote the set of $n \times p$ diagonal matrices by $D(n, p, \mathbb{R})$. We write $D(n, \mathbb{R})$ in place of $D(n, n, \mathbb{R})$.

Definition 1.2.0.11. For $(n, k), (m, l)$ $\text{diag}_{p, (n \times p)} : \mathbb{R}^p \rightarrow \mathbb{R}^{n \times p}$ and $\text{diag}_{n, (n \times p)} : \mathbb{R}^p \rightarrow \mathbb{R}^{n \times p}$ by $\text{diag}(v)$
FINISH!!!

Definition 1.2.0.12. Let $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \sigma(A)$. Suppose that A is symmetric. We define the **geometric multiplicity** of λ , denoted $\mu(\lambda)$, by

$$\mu(\lambda) = \dim \ker([\phi_\alpha] - \lambda I)$$

Definition 1.2.0.13. Let V be an n -dimensional vector space, $U \subset V$ a k -dimensional subspace and $(e_j)_{j=1}^n \subset V$ a basis. Then $(e_j)_{j=1}^n$ is said to be **adapted to** U if $(e_j)_{j=1}^k$ is a basis for U .

1.3 Calculus

1.3.1 Differentiation

Definition 1.3.1.1. Let $n \geq 1$. For $i = 1, \dots, n$, define $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$ by $x^i(a^1, \dots, a^n) = a^i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on \mathbb{R}^n** .

Definition 1.3.1.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to x^i at a** if

$$\lim_{h \rightarrow 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If f is differentiable with respect to x^i at a , we define the **partial derivative of f with respect to x^i at a** , denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \frac{\partial}{\partial x^i} f$$

to be the limit above.

Definition 1.3.1.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$. Then f is said to be **differentiable with respect to x^i** if for each $a \in U$, f is differentiable with respect to x^i at a .

Exercise 1.3.1.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i \partial x^j}$ and $\frac{\partial^2 f}{\partial x^j \partial x^i}$ exist and are continuous at a . Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(a) = \frac{\partial^2 f}{\partial x^j \partial x^i}(a)$$

Proof. □

Definition 1.3.1.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \dots, i_k \in \{1, \dots, n\}$, $\frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_k}}$ exists and is continuous on U .

Definition 1.3.1.6. Let $U \subset \mathbb{R}^n$, $f : U \rightarrow \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \rightarrow \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Theorem 1.3.1.7. Taylor's Theorem:

Let $U \subset \mathbb{R}^n$ be open and convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that for each $x \in U$,

$$f(x) = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each $|\alpha| = T+1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. See analysis notes □

Definition 1.3.1.8. Let $U \subset \mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^m$. Let x^1, \dots, x^n be the standard coordinate functions on \mathbb{R}^n and y_1, \dots, y_m be the standard coordinate functions on \mathbb{R}^m . For $i \in \{1, \dots, m\}$, we define the **i th component of F** , denoted $F^i : U \rightarrow \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \dots, F_m)$

Definition 1.3.1.9. Let $U \subset \mathbb{R}^n$ be open and $F : U \rightarrow \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \dots, m\}$, the i th component of F , $F^i : U \rightarrow \mathbb{R}$, is smooth.

Definition 1.3.1.10. Let $U \subset \mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^m$. Then F is said to be **smooth** if for each $x \in U$, there exists $U_x \in \mathcal{N}_x$ and $\tilde{F} : U_x \rightarrow \mathbb{R}^m$ such that U_x is open, \tilde{F} is smooth and $\tilde{F}|_{U \cap U_x} = F|_{U \cap U_x}$.

Definition 1.3.1.11. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \rightarrow V$. Then F is said to be a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 1.3.1.12. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \rightarrow V$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F is a bijection and F and F^{-1} are smooth. Thus, F and F^{-1} are continuous and F is a homeomorphism. \square

Definition 1.3.1.13. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \rightarrow \mathbb{R}^m$. We define the **Jacobian of F at p** , denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p) \right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Exercise 1.3.1.14. Inverse Function Theorem:

Let $U, V \subset \mathbb{R}^n$ be open and $F : U \rightarrow V$.

Exercise 1.3.1.15. Let $U, V \subset \mathbb{R}^n$ and $F : U \rightarrow V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \rightarrow F(N)$ is a diffeomorphism

Proof. content... \square

Exercise 1.3.1.16. Let $\sigma \in S_n$. Define $\phi_\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\phi(x^1, \dots, x^n) = \phi(x^{\sigma(1)}, \dots, x^{\sigma(n)})$. Then $D\phi = P_\sigma$

Definition 1.3.1.17. Let $\sigma \in S_n$ and $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. We define $\sigma x \in \mathbb{R}^n$ by

$$\sigma x = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of S_n on \mathbb{R}^n to be the map $S_n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $(\sigma, x) \mapsto \sigma x$

Definition 1.3.1.18. Let $\sigma \in S_n$, U a set, $V \subset \mathbb{R}^n$ and $\phi : U \rightarrow \mathbb{R}^n$ with $\phi = (x^1, \dots, x^n)$. We define $\sigma\phi : U \rightarrow \mathbb{R}^n$ by

$$\sigma\phi = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of S_n on $(\mathbb{R}^n)^U$ to be the map $S_n \times (\mathbb{R}^n)^U \rightarrow (\mathbb{R}^n)^U$ given by $(\sigma, \phi) \mapsto \sigma\phi$.

Exercise 1.3.1.19. Let $\sigma \in S_m$. Then for each $p \in \mathbb{R}^n$, $D(\sigma \text{id}_{\mathbb{R}^n})(p) = P_\sigma$.

Proof. Note that since $\text{id}_{\mathbb{R}^n} = (\pi_1, \dots, \pi_n)$, we have that $\sigma \text{id}_{\mathbb{R}^n} = (\pi_{\sigma(1)}, \dots, \pi_{\sigma(n)})$. Let $p \in \mathbb{R}^n$. Then

$$\begin{aligned} D(\sigma \text{id}_{\mathbb{R}^n})(p) &= \left(\frac{\partial \pi_i \circ \sigma \text{id}_{\mathbb{R}^n}}{\partial x^j}(p) \right)_{i,j} \\ &= \left(\frac{\partial \pi_{\sigma(i)}}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma \left(\frac{\partial \pi_i}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma \left(\frac{\partial \pi_i \circ \text{id}_{\mathbb{R}^n}}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma D \text{id}_{\mathbb{R}^n}(p) \\ &= P_\sigma I \\ &= P_\sigma \end{aligned}$$

\square

1.3.2 Integration

1.4 Topology

Definition 1.4.0.1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \rightarrow Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 1.4.0.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \rightarrow Y$. Then f is said to be a **homeomorphism** if f is a bijection and f, f^{-1} are continuous.

Definition 1.4.0.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \rightarrow Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 1.4.0.4. Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \not\cong \mathbb{R}^n$

Chapter 2

Multilinear Algebra

2.1 (r, s) -Tensors

Definition 2.1.0.1. Let V_1, \dots, V_k, W be vector spaces and $\alpha : \prod_{i=1}^n V_i \rightarrow W$. Then α is said to be **multilinear** if for each $i \in \{1, \dots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \dots, v_k \in V$,

$$\alpha(v_1, \dots, v_i + cv, \dots, v_k) = \alpha(v_1, \dots, v_i, \dots, v_k) + c\alpha(v_1, \dots, v, \dots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{ \alpha : \prod_{i=1}^n V_i \rightarrow W : \alpha \text{ is multilinear} \right\}$$

Note 2.1.0.2. For the remainder of this section we let V denote an n -dimensional vector space with basis $\{e^1, \dots, e^n\}$ with dual space V^* and dual basis $\{\epsilon_1, \dots, \epsilon_n\}$ defined by $\epsilon^i(e^j) = \delta_{i,j}$. We identify V with V^{**} by the isomorphism $V \rightarrow V^{**}$ defined by $v \mapsto \hat{v}$ where $\hat{v}(\alpha) = \alpha(v)$ for each $\alpha \in V^*$.

Definition 2.1.0.3. Let $\alpha : (V^*)^r \times V^s \rightarrow \mathbb{R}$. Then α is said to be an (r, s) -tensor on V if $\alpha \in L(\underbrace{V^*, \dots, V^*}_r, \underbrace{V, \dots, V}_s; \mathbb{R})$. The set of all (r, s) -tensors on V is denoted $T_s^r(V)$.

When $r = s = 0$, we set $T_s^r = \mathbb{R}$.

Exercise 2.1.0.4. We have that $T_s^r(V)$ is a vector space.

Proof. Clear. □

Exercise 2.1.0.5. Under the identification of V with V^{**} as noted above, we have that $V = T_0^1(V)$.

Proof. By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T_0^1(V) \end{aligned}$$

□

Definition 2.1.0.6. Let $\alpha \in T_{s_1}^{r_1}(V)$ and $\beta \in T_{s_2}^{r_2}(V)$. We define the **tensor product of α with β** , denoted $\alpha \otimes \beta \in T_{s_1+s_2}^{r_1+r_2}(V)$, by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$.

When $r_1 = s_1 = r_2 = s_2 = 0$ (so that $\alpha, \beta \in \mathbb{R}$), we set $\alpha \otimes \beta = \alpha\beta$.

Definition 2.1.0.7. We define the **tensor product**, denoted $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

Exercise 2.1.0.8. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is well defined.

Proof. Tedious but straightforward. □

Exercise 2.1.0.9. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is associative.

Proof. Let $\alpha \in T_{s_1}^{r_1}(V)$, $\beta \in T_{s_2}^{r_2}(V)$ and $\gamma \in T_{s_3}^{r_3}(V)$. Then for each $u^* \in (V^*)^{r_1}$, $v^* \in (V^*)^{r_2}$, $w^* \in (V^*)^{r_3}$, $u \in V^{s_1}$, $v \in V^{s_2}$, $w \in V^{s_3}$,

$$\begin{aligned} (\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v) \gamma(w^*, w) \\ &= [\alpha(u^*, u) \beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(u^*, u) [\beta(v^*, v) \gamma(w^*, w)] \\ &= \alpha(u^*, u) (\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w) \end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

Exercise 2.1.0.10. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is bilinear.

Proof.

1. Linearity in the first argument:

Let $\alpha, \beta \in T_{s_1}^{r_1}(V)$, $\gamma \in T_{s_2}^{r_2}(V)$, $\lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned} [(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v) \gamma(w^*, w) \\ &= [\alpha(v^*, v) + \lambda\beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(v^*, v) \gamma(w^*, w) + \lambda\beta(v^*, v) \gamma(w^*, w) \\ &= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w) \end{aligned}$$

So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

2. Linearity in the second argument:

Similar to (1).

□

Definition 2.1.0.11.

1. Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1, \dots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length k** . Recall that $\#\mathcal{I}_{\otimes k} = n^k$.
2. Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length k** . Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

Note 2.1.0.12. For the remainder of this section we will write \mathcal{I}_k in place of $\mathcal{I}_{\otimes k}$.

Definition 2.1.0.13. Let $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k\}$.

1. Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by

$$\epsilon^I = (\epsilon^{i_1}, \dots, \epsilon^{i_k})$$

and

$$e^I = (e^{i_1}, \dots, e^{i_k})$$

2. Define $e^{\otimes I} \in T_0^k(V)$ and $\epsilon^{\otimes I} \in T_k^0(V)$ by

$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$

and

$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

Exercise 2.1.0.14. Let $\alpha, \beta \in T_s^r(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v_1^*, \dots, v_r^* \in V^*$ and $v_1, \dots, v_s \in V$. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, write

$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$

and

$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$

Then

$$\begin{aligned} \alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s) \end{aligned}$$

So that $\alpha = \beta$. □

Exercise 2.1.0.15. Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K} \delta_{J,L}$.

Proof. Write $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$ and $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$. Then

$$\begin{aligned} e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K) \epsilon^{\otimes J}(e^L) \\ &= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r}) \epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\ &= \left[\prod_{m=1}^r e^{i_m}(\epsilon^{k_m}) \right] \left[\prod_{n=1}^s \epsilon^{j_n}(e^{l_n}) \right] \\ &= \left[\prod_{m=1}^r \delta_{i_m, k_m} \right] \left[\prod_{n=1}^s \delta_{j_n, l_n} \right] \\ &= \delta_{I,K} \delta_{J,L} \end{aligned}$$

□

Exercise 2.1.0.16. The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T_s^r(V)$ and $\dim T_s^r(V) = n^{r+s}$.

Proof. Let $(a_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a_J^I e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a_J^I = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T_s^r(V)$. For $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b_J^I = \beta(\epsilon^I, e^J)$. Define $\mu = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b_J^I e^{\otimes I} \otimes \epsilon^{\otimes J} \in T_s^r(V)$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b_J^I = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. □

2.2 Covariant k -Tensors

2.2.1 Symmetric and Alternating Covariant k -Tensors

Definition 2.2.1.1. Let $\alpha : V^k \rightarrow \mathbb{R}$. Then α is said to be a **covariant k -tensor on V** if $\alpha \in T_k^0(V)$. We denote the set of covariant k -tensors by $T_k(V)$.

Definition 2.2.1.2. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \rightarrow \mathbb{R}$ by

$$\sigma\alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

We define the **permutation action** of S_k on $T_k(V)$ to be the map $S_k \times T_k(V) \rightarrow T_k(V)$ given by $(\sigma, \alpha) \mapsto \sigma\alpha$

Exercise 2.2.1.3. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

1. Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V)$, $\sigma\alpha \in T_k(V)$.
2. Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
3. Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \dots, v_k \in V$,

$$\begin{aligned} (\tau\sigma)\alpha(v_1, \dots, v_k) &= \alpha(v_{\tau\sigma(1)}, \dots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \dots, v_k) \end{aligned}$$

□

Exercise 2.2.1.4. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \rightarrow T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \dots, v_k \in V$. Then

$$\begin{aligned} \sigma(c\alpha + \beta)(v_1, \dots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \dots, v_k) + \sigma\beta(v_1, \dots, v_k) \end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$.

□

Definition 2.2.1.5. Let $\alpha \in T_k(V)$. Then α is said to be

- **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$
- **antisymmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \text{sgn}(\sigma)\alpha$
- **alternating** if for each $v_1, \dots, v_k \in V$, if there exists $i, j \in \{1, \dots, k\}$ such that $v_i = v_j$, then $\alpha(v_1, \dots, v_k) = 0$.

We denote the set of symmetric k -tensors on V by $\Sigma^k(V)$. We denote the set of alternating k -tensors on V by $\Lambda^k(V)$.

Exercise 2.2.1.6. Let $\alpha \in T_k(V)$. Then α is antisymmetric iff α is alternating.

Proof. Suppose that α is antisymmetric. Let $v_1, \dots, v_k \in V$. Suppose that there exists $i, j \in \{1, \dots, k\}$ such that $v_i = v_j$. Define $\sigma \in S_k$ by $\sigma = (i, j)$. Then

$$\begin{aligned} \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) &= \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ &= \sigma(\alpha)(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\ &= \text{sgn}(\sigma)\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\ &= -\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \end{aligned}$$

Therefore $2\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = 0$ which implies that $\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = 0$. Hence α is alternating.

Conversely, suppose that α is alternating. Let $i, j \in \{1, \dots, k\}$ and $v_1, \dots, v_k \in V$. Then

$$\begin{aligned} 0 &= \alpha(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) \\ &= \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \end{aligned}$$

Since $i, j \in \{1, \dots, k\}$ and $v_1, \dots, v_k \in V$ are arbitrary, we have that for each $\tau \in S_k$, τ is a transposition implies that

$$\begin{aligned} \tau\alpha &= -\alpha \\ &= \text{sgn}(\tau)\alpha \end{aligned}$$

Let $n \in \mathbb{N}$. Suppose that for each $\tau_1, \dots, \tau_{n-1} \in S_k$ if for each $j \in \{1, \dots, n-1\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_{n-1})\alpha = \text{sgn}(\tau_1 \cdots \tau_{n-1})\alpha$. Let $\tau_1, \dots, \tau_n \in S_k$. Suppose that for each $j \in \{1, \dots, n\}$, τ_j is a transposition. Then

$$\begin{aligned} (\tau_1 \cdots \tau_n)\alpha &= (\tau_1 \cdots \tau_{n-1})(\tau_n\alpha) \\ &= (\tau_1 \cdots \tau_{n-1})(\text{sgn}(\tau_n)\alpha) \\ &= (\text{sgn}(\tau_n)(\tau_1 \cdots \tau_{n-1})\alpha) \\ &= (\text{sgn}(\tau_n) \text{sgn}(\tau_1 \cdots \tau_{n-1})\alpha) \\ &= \text{sgn}(\tau_1 \cdots \tau_n)\alpha \end{aligned}$$

By induction, for each $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in S_k$, if for each $j \in \{1, \dots, n\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_n)\alpha = \text{sgn}(\tau_1 \cdots \tau_n)\alpha$. Now let $\sigma \in S_k$. Then there exist $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in S_k$ such that $\sigma = \tau_1 \cdots \tau_n$ and for each $j \in \{1, \dots, n\}$, τ_j is a transposition. Hence

$$\begin{aligned} \sigma\alpha &= (\tau_1 \cdots \tau_n)\alpha \\ &= \text{sgn}(\tau_1 \cdots \tau_n)\alpha \\ &= \text{sgn}(\sigma)\alpha \end{aligned}$$

Therefore α is antisymmetric. □

Definition 2.2.1.7. Define the **symmetric operator** $S : T_k(V) \rightarrow \Sigma^k(V)$ by

$$\text{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \rightarrow \Lambda^k(V)$ by

$$\text{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma\alpha$$

Exercise 2.2.1.8.

1. For $\alpha \in T_k(V)$, $\text{Sym}(\alpha)$ is symmetric.
2. For $\alpha \in T_k(V)$, $\text{Alt}(\alpha)$ is alternating.

Proof.

1. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned} \sigma \text{Sym}(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \\ &= \text{Sym}(\alpha) \end{aligned}$$

2. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
 \sigma \operatorname{Alt}(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \right] \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \sigma \tau \alpha \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \operatorname{Alt}(\alpha)
 \end{aligned}$$

□

Exercise 2.2.1.9.

1. For $\alpha \in \Sigma^k(V)$, $\operatorname{Sym}(\alpha) = \alpha$.
2. For $\alpha \in \Lambda^k(V)$, $\operatorname{Alt}(\alpha) = \alpha$.

Proof.

1. Let $\alpha \in \Sigma^k(V)$. Then

$$\begin{aligned}
 \operatorname{Sym}(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} \alpha \\
 &= \alpha
 \end{aligned}$$

2. Let $\alpha \in \Lambda^k(V)$. Then

$$\begin{aligned}
 \operatorname{Alt}(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma \alpha \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2 \alpha \\
 &= \alpha
 \end{aligned}$$

□

Exercise 2.2.1.10. The symmetric operator $S : T_k(V) \rightarrow \Sigma^k(V)$ and the alternating operator $A : T_k(V) \rightarrow \Lambda^k(V)$ are linear.

Proof. Clear.

□

Exercise 2.2.1.11. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

1. $\operatorname{Alt}(\operatorname{Alt}(\alpha) \otimes \beta) = \operatorname{Alt}(\alpha \otimes \beta)$
2. $\operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta)) = \operatorname{Alt}(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \rightarrow S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

1. Then

$$\begin{aligned}
\text{Alt}(\text{Alt}(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\text{Alt}(\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) (\tau \alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau (\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \text{sgn}(\sigma\tau) \sigma\tau (\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= \text{Alt}(\alpha \otimes \beta)
\end{aligned}$$

2. Similar to (1).

□

2.2.2 Exterior Product

Definition 2.2.2.1. Let $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$.

Exercise 2.2.2.2. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$ is bilinear.

Proof. Clear.

□

Exercise 2.2.2.3. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$ and $\gamma \in \Lambda^m(V)$. Then

$$\begin{aligned}
 (\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \right] \wedge \gamma \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \text{Alt} \left(\left[\frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \right] \otimes \gamma \right) \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} \text{Alt}(\text{Alt}(\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} \text{Alt}((\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \text{Alt}(\alpha \otimes (\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \text{Alt}(\alpha \otimes \text{Alt}(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \text{Alt}(\alpha \otimes \frac{(l+m)!}{l!m!} \text{Alt}(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \text{Alt}(\alpha \otimes (\beta \wedge \gamma)) \\
 &= \alpha \wedge (\beta \wedge \gamma)
 \end{aligned}$$

□

Exercise 2.2.2.4. Let $\alpha_i \in \Lambda^{k_i}(V)$ for $i = 1, \dots, m$. Then

$$\bigwedge_{i=1}^m \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} \text{Alt} \left(\bigotimes_{i=1}^m \alpha_i \right)$$

Proof. To see that the statement is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} \text{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
 \bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i \right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
 &= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \text{Alt} \left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \text{Alt} \left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} \text{Alt} \left(\bigotimes_{i=1}^{m_0-1} \alpha_i \right) \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left(\text{Alt} \left[\bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left(\bigotimes_{i=1}^{m_0+1} \alpha_i \right)
 \end{aligned}$$

□

Exercise 2.2.2.5. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl . (Hint: inversion number)

Proof.

$$\begin{aligned} N(\tau) &= \sum_{i=1}^l k \\ &= kl \end{aligned}$$

Since $\text{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\text{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.2.2.6. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \dots, v_{k+l} \in V$, we have that

$$\begin{aligned} \sigma\tau(\beta \otimes \alpha)(v_1, \dots, v_l, v_{l+1}, \dots, v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)})\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\ &= \alpha \otimes \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}, v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\ &= \sigma(\alpha \otimes \beta)(v_1, \dots, v_k, v_{1+k}, \dots, v_{l+k}) \end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned} \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} \text{Alt}(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau) \sigma\tau(\beta \otimes \alpha) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha \otimes \beta) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \\ &= \text{sgn}(\tau) \alpha \wedge \beta \\ &= (-1)^{kl} \alpha \wedge \beta \end{aligned}$$

□

Exercise 2.2.2.7. Let $\alpha \in \Lambda^k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned} \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\ &= -\alpha \wedge \alpha \end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.2.2.8. Fundamental Example:

Let $\alpha_1, \dots, \alpha_m \in \Lambda^1(V)$ and $v_1, \dots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned} \left(\bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) &= m! \operatorname{Alt} \left(\bigotimes_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) \\ &= m! \left[\frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \sigma \left(\bigotimes_{i=1}^m \alpha_i \right) \right] (v_1, \dots, v_m) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \left(\bigotimes_{i=1}^m \alpha_i \right) (v_{\sigma(1)}, \dots, v_{\sigma(m)}) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\ &= \det(\alpha_i(v_j)) \end{aligned}$$

□

Note 2.2.2.9. Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write \mathcal{I}_k in place of $\mathcal{I}_{\wedge k}$.

Definition 2.2.2.10. Let $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k\}$. Define $\epsilon^{\wedge I} \in \Lambda^k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \dots \wedge \epsilon^{i_k}$$

Exercise 2.2.2.11. Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \dots & \epsilon^{i_1}(e^{j_k}) \\ \vdots & & \vdots \\ \epsilon^{i_k}(e^{j_1}) & \dots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$. If $I = J$, then

$A = I_{k \times k}$ and therefore $\epsilon^{\wedge I}(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the l_0 -th row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the l_0 -th column of A are 0. □

Exercise 2.2.2.12. Let $\alpha, \beta \in \Lambda^k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \dots, v_k \in V$. For $i = 1, \dots, k$, write $v_i =$

$\sum_{j_i=1}^n a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned}
 \alpha(v_1, \dots, v_k) &= \sum_{j_1, \dots, j_k=1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\
 &= \sum_{j_1 \neq \dots \neq j_k}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\
 &= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \text{sgn}(\sigma) \left(\prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \alpha(e^J) \\
 &= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \text{sgn}(\sigma) \left(\prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \beta(e^J) \\
 &= \sum_{j_1, \dots, j_k=1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \beta(e^{j_1}, \dots, e^{j_k}) \\
 &= \beta(v_1, \dots, v_k)
 \end{aligned}$$

□

Exercise 2.2.2.13. The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(V)$ and $\dim \Lambda^k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$.

Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda^k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda^k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$.

□

2.2.3 Interior Product

Definition 2.2.3.1. Let V be a finite dimensional vector space and $v \in V$. We define **interior multiplication by v** , denoted $\iota_v : T_k \rightarrow T_{k-1}$, by

$$\iota_v \alpha(w_1, \dots, w_{k-1}) = \alpha(v, w_1, \dots, w_{k-1})$$

Exercise 2.2.3.2. Let V be a finite dimensional vector space and $v \in V$. Then $\iota_v|_{\Lambda^k(V)} : \Lambda^k(V) \rightarrow \Lambda^{k-1}(V)$.

Proof. Let $\alpha \in \Lambda^k(V)$. Define $\beta \in \Lambda^k(V)$ by $\beta(w_1, \dots, w_k) = \alpha(w_k, w_1, \dots, w_{k-1})$. Let $\sigma \in S_{k-1}$. Define $\tau \in S_k$ by $\tau(j) = \begin{cases} 1 & j = k \\ \sigma(j) & j \neq k \end{cases}$. Let $w_1, \dots, w_{k-1} \in V$. Set $w_k = v$. Then

$$\begin{aligned}
 \sigma(\iota_v \alpha)(w_1, \dots, w_{k-1}) &= \iota_v \alpha(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}) \\
 &= \alpha(v, w_{\sigma(1)}, \dots, w_{\sigma(k-1)}) \\
 &= \beta(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}, v) \\
 &= \beta(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}, w_k) \\
 &= \beta(w_{\tau(1)}, \dots, w_{\tau(k-1)}, w_{\tau(k)}) \\
 &= \text{sgn}(\tau) \beta(w_1, \dots, w_{k-1}, w_k) \\
 &= \text{sgn}(\sigma) \beta(w_1, \dots, w_{k-1}, v) \\
 &= \text{sgn}(\sigma) \alpha(v, w_1, \dots, w_{k-1}) \\
 &= \text{sgn}(\sigma) (\iota_v \alpha)(w_1, \dots, w_{k-1})
 \end{aligned}$$

Since $w_1, \dots, w_{k-1} \in V$ are arbitrary, $\sigma(\iota_v \alpha) = \text{sgn}(\sigma) \iota_v \alpha$. Hence $\iota_v \alpha \in \Lambda^{k-1}(V)$.

□

2.3 $(0, 2)$ -Tensors

Definition 2.3.0.1. Let V be a finite dimensional vector space, $v \in V$ and $\alpha \in T_2^0(V)$. Then α is said to be **degenerate** if there exists $v \in V$ such that for each $w \in V$, $\alpha(v, w) = 0$ and $v \neq 0$.

Definition 2.3.0.2. Let V be a finite dimensional vector space, $\alpha \in T_2^0(V)$. We define $\phi_\alpha : V \rightarrow V^*$ by

$$\phi_\alpha(v) = \iota_v \alpha$$

Exercise 2.3.0.3. Let V be a finite dimensional vector space, $\alpha \in T_2^0(V)$. Then $\phi_\alpha \in L(V; V^*)$.

Proof. Let $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$. Then for each $w \in V$,

$$\begin{aligned} \phi_\alpha(v_1 + \lambda v_2)(w) &= (\iota_{v_1 + \lambda v_2} \alpha)(w) \\ &= \alpha(v_1 + \lambda v_2, w) \\ &= \alpha(v_1, w) + \lambda \alpha(v_2, w) \\ &= (\iota_{v_1} \alpha)(w) + \lambda (\iota_{v_2} \alpha)(w) \\ &= \phi_\alpha(v_1)(w) + \lambda \phi_\alpha(v_2)(w) \\ &= [\phi_\alpha(v_1) + \lambda \phi_\alpha(v_2)](w) \end{aligned}$$

Therefore, $\phi_\alpha(v_1 + \lambda v_2) = \phi_\alpha(v_1) + \lambda \phi_\alpha(v_2)$. Thus $\phi_\alpha \in L(V; V^*)$. \square

Exercise 2.3.0.4. Let V be a finite dimensional vector space and $\alpha \in T_2^0(V)$. Then α is nondegenerate iff ϕ_α is an isomorphism.

Proof.

- (\implies :)

Suppose that α is nondegenerate. Let $v \in \ker \phi_\alpha$. Then for each $w \in V$,

$$\begin{aligned} \alpha(v, w) &= (\iota_v \alpha)(w) \\ &= \phi_\alpha(v)(w) \\ &= 0 \end{aligned}$$

Since α is nondegenerate, $v = 0$. Since $v \in \ker \phi_\alpha$ is arbitrary, $\ker \phi_\alpha = \{0\}$. Hence ϕ_α is injective. Since $\dim V = \dim V^*$, ϕ_α is surjective. Hence ϕ_α is an isomorphism.

- (\impliedby :)

Suppose that ϕ_α is an isomorphism. Let $v \in V$. Suppose that for each $w \in V$, $\alpha(v, w) = 0$. Then for each $w \in V$,

$$\begin{aligned} \phi_\alpha(v)(w) &= (\iota_v \alpha)(w) \\ &= \alpha(v, w) \\ &= 0 \end{aligned}$$

Thus $\phi_\alpha(v) = 0$ which implies that $v \in \ker \phi_\alpha$. Since ϕ_α is an isomorphism, $v = 0$. Hence α is nondegenerate. \square

Exercise 2.3.0.5. Let V be a finite dimensional vector space and $\alpha \in T_2^0(V)$. Then

1. $[\phi_\alpha]_{i,j} = \alpha(e_j, e_i)$
2. for each $v, w \in V$,

$$\alpha(v, w) = [w]^* [\phi_\alpha] [v]$$

Proof. 1. Set $A = [\phi_\alpha]$. Let $i, j \in \{1, \dots, n\}$. By definition,

$$\phi_\alpha(e_j) = \sum_{k=1}^n A_{k,j} \epsilon^k$$

Then

$$\begin{aligned} \phi_\alpha(e_j)(e_i) &= \sum_{k=1}^n A_{k,j} \epsilon^k(e_i) \\ &= \sum_{k=1}^n A_{k,j} \delta_{k,i} \\ &= A_{i,j} \end{aligned}$$

2. Let $v, w \in V$. Then there exist $(v^i)_{i=1}^n, (w^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{i=1}^n v^i e_i$ and $w = \sum_{j=1}^n w^j e_j$. Part (1) implies that

$$\begin{aligned} \alpha(v, w) &= \sum_{i=1}^n \sum_{j=1}^n v^i w^j \alpha(e_i, e_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n v^i w^j [\phi_\alpha]_{j,i} \\ &= \sum_{i=1}^n \sum_{j=1}^n [v]_i [w]_j [\phi_\alpha]_{j,i} \\ &= [w]^* [\phi_\alpha] [v] \end{aligned}$$

□

2.3.1 Scalar Product Spaces

Definition 2.3.1.1. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be

- **positive semidefinite** if for each $v \in V$, $\alpha(v, v) \geq 0$
- **positive definite** if for each $v \in V$, $v \neq 0$ implies that $\alpha(v, v) > 0$
- **negative semidefinite** if $-\alpha$ is positive semidefinite
- **negative definite** if $-\alpha$ is positive definite

Exercise 2.3.1.2. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then

1. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda > 0$
2. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda \geq 0$

Proof.

1. Suppose that α is positive definite. Write $\sigma(\phi_\alpha) = \{\lambda_1, \dots, \lambda_n\}$. Define $\Lambda \in \mathbb{R}^{n \times n}$ by $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. Since α is symmetric, $[\phi_\alpha]$ is symmetric. There exists $U \in O(n)$ such that $[\phi_\alpha] = U \Lambda U^*$.

□

Definition 2.3.1.3. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be a **scalar product** if α is nondegenerate. In this case, (V, α) is said to be a **scalar product space**.

Definition 2.3.1.4. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$ a scalar product on V . We define the **index** of α , denoted $\text{ind } \alpha$ by

$$\text{ind } \alpha = \max\{\dim W : W \text{ is a subspace of } V \text{ and } \alpha|_{W \times W} \text{ is negative definite}\}$$

Definition 2.3.1.5. Let (V, α) be a scalar product space.

- Let $v_1, v_2 \in V$. Then v_1 and v_2 are said to be **orthogonal** if $\alpha(v_1, v_2) = 0$.
- Let $U \subset V$ be a subspace. We define the **orthogonal subspace of U** , denoted by U^\perp , by

$$U^\perp = \{v \in V : \text{for each } u \in U, \alpha(u, v) = 0\}$$

Exercise 2.3.1.6. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then U^\perp is a subspace of V .

Proof. We note that since $U^\perp = \bigcap_{u \in U} \ker \phi_\alpha(u)$, U^\perp is a subspace of V . □

Exercise 2.3.1.7. Let (V, α) be an n -dimensional scalar product space, $U \subset V$ a k -dimensional subspace and $(e_j)_{j=1}^n \subset V$ a basis for V . Suppose that $(e_j)_{j=1}^k$ is a basis for U . Then for each $v \in V$, $v \in U^\perp$ iff for each $j \in [k]$, $\alpha(v, e_j) = 0$.

Proof. Let $v \in V$.

- (\implies): Suppose that $v \in U^\perp$. Since $(e_j)_{j=1}^k \subset U$, we have that for each $j \in [k]$, $\alpha(v, e_j) = 0$.
- (\impliedby): Suppose that for each $j \in [k]$, $\alpha(v, e_j) = 0$. Let $u \in U$. Then there exist $(a^j)_{j=1}^k \subset \mathbb{R}$ such that $u = \sum_{j=1}^k a^j u_j$. This implies that

$$\begin{aligned} \alpha(v, u) &= \sum_{j=1}^k a^j \alpha(v, u_j) \\ &= 0 \end{aligned}$$

Since $u \in U$ is arbitrary, we have that $v \in U^\perp$. □

Exercise 2.3.1.8. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then

1. $\dim V = \dim U + \dim U^\perp$
2. $(U^\perp)^\perp = U$

Proof. 1. Set $n = \dim V$ and $k = \dim U$. Choose a basis $(e_j)_{j=1}^n$ such that $(e_j)_{j=1}^k$ is a basis for U .

2. □

Exercise 2.3.1.9. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Set $\sigma([\phi_\alpha])^- = \{\lambda \in \sigma([\phi_\alpha]) : \lambda < 0\}$. Then

$$\text{ind } \alpha = \sum_{\lambda \in \sigma([\phi_\alpha])^-} \mu(\lambda)$$

Proof. Since α is symmetric, there exist $U \in O(n)$ and $\Lambda \in D(n, \mathbb{R})$ such that $[\phi_\alpha] = U\Lambda U^*$. Define $(u_j)_{j=1}^n \subset V$ by $u_j = \sum_{i=1}^n U_{i,j} e_i$. Define $J^- = \{j \in [n] : \Lambda_{j,j} < 0\}$, $n^- = \#J^-$ and $V^- = \text{span}\{u_j : j \in J^-\}$. Let $v \in V^-$. Then there exist $(a^j)_{j \in J^-}$ such that $v = \sum_{j \in J^-} a^j u_j$. We note that

$$\begin{aligned} U^*[\phi_\alpha]U &= U^*(U\Lambda U^*)U \\ &= (U^*U)\Lambda(U^*U) \\ &= I\Lambda I \\ &= \Lambda \end{aligned}$$

A previous exercise implies that

$$\begin{aligned} \alpha(v, v) &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k \alpha(u_j, u_k) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k [u_j]^* [\phi_\alpha] [u_k] \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k ([e_j]^* U^*) [\phi_\alpha] (U[e_k]) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (U^*[\phi_\alpha]U)_{j,k} \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (\Lambda)_{j,k} \\ &= \sum_{j \in J^-} |a^j|^2 \Lambda_{j,j} \\ &< 0 \end{aligned}$$

Since $v \in V^-$ is arbitrary, $\alpha|_{V^- \times V^-}$ is negative definite. Thus

$$\begin{aligned} \text{ind } \alpha &\geq \dim V^- \\ &= n^- \end{aligned}$$

Set $J^+ = (J^-)^c$. Let $W \subset V$ be a subspace. Suppose that $\alpha|_{W \times W}$ is negative definite. For the sake of contradiction, suppose that there exists $j_0 \in J^+$ such that $u_{j_0} \in W$. Then

$$\begin{aligned} \alpha(u_{j_0}, u_{j_0}) &= [u_{j_0}]^* [\phi_\alpha] [u_{j_0}] \\ &= [u_{j_0}]^* U\Lambda U^* [u_{j_0}] \\ &= \Lambda_{j_0, j_0} \\ &\geq 0 \end{aligned}$$

which is a contradiction since $\alpha|_{W \times W}$ is negative definite. Thus for each $j \in J^+$, $u_j \notin W$. □

2.3.2 Symplectic Vector Spaces

Definition 2.3.2.1. Let V be a finite dimensional vector space and $\omega \in \Lambda^2(V)$. Then ω is said to be a **symplectic form** if ω is nondegenerate. In this case (V, ω) is said to be a **symplectic space**.

Exercise 2.3.2.2. Let V be a $2n$ -dimensional vector space with basis $(a_j, b_j)_{j=1}^n$ and corresponding dual basis $(\alpha^j, \beta^j)_{j=1}^n$. Define $\omega \in \Lambda^2(V)$ by

$$\omega = \sum_{j=1}^n \alpha^j \wedge \beta^j$$

Then

1. for each $j, k \in \{1, \dots, n\}$,

(a) $\omega(a_j, a_k) = 0$

(b) $\omega(b_j, b_k) = 0$

(c) $\omega(a_j, b_k) = \delta_{j,k}$

2. (V, ω) is a symplectic space

Proof.

1. Let $j, k \in \{1, \dots, n\}$.

(a)

$$\begin{aligned} \omega(a_j, a_k) &= \sum_{l=1}^n \alpha^l \wedge \beta^l(a_j, a_k) \\ &= \sum_{l=1}^n [\alpha^l(a_j)\beta^l(a_k) - \alpha^l(a_k)\beta^l(a_j)] \\ &= 0 \end{aligned}$$

(b) Similar to (a)

(c)

$$\begin{aligned} \omega(a_j, b_k) &= \sum_{l=1}^n \alpha^l \wedge \beta^l(a_j, b_k) \\ &= \sum_{l=1}^n [\alpha^l(a_j)\beta^l(b_k) - \alpha^l(b_k)\beta^l(a_j)] \\ &= \sum_{l=1}^n \alpha^l(a_j)\beta^l(b_k) \\ &= \sum_{l=1}^n \delta_{j,l}\delta_{l,k} \\ &= \delta_{j,k} \end{aligned}$$

2. Let $v \in V$. Then there exist $(q^j, p^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{j=1}^n q^j a_j + p^j b_j$. Suppose that for each $w \in V$, $\omega(v, w) = 0$. Let $k \in \{1, \dots, n\}$. Then

$$\begin{aligned} 0 &= \omega(v, a_k) \\ &= \sum_{j=1}^n q^j \omega(a_j, a_k) + p^j \omega(b_j, a_k) \\ &= \sum_{j=1}^n p^j \delta_{j,k} \\ &= p^k \end{aligned}$$

Similarly,

$$\begin{aligned}
 0 &= \omega(v, b_k) \\
 &= \sum_{j=1}^n q^j \omega(a_j, b_k) + p^j \omega(b_j, b_k) \\
 &= \sum_{j=1}^n q^j \delta_{j,k} \\
 &= q^k
 \end{aligned}$$

Since $k \in \{1, \dots, n\}$ is arbitrary, $v = 0$. Hence ω is nondegenerate. Therefore (V, ω) is symplectic. \square

Exercise 2.3.2.3. Let (V, ω) be a symplectic space. Then $\dim V$ is even.

Proof. Set $n = \dim V$. Let $(e_j)_{j=1}^n$ be a basis for V . Define $[\omega] \in \mathbb{R}^{n \times n}$ by $[\omega]_{i,j} = \omega(e_i, e_j)$. Since $\omega \in \Lambda^2(V)$, $[\omega]^* = -[\omega]$. Therefore

$$\begin{aligned}
 \det[\omega] &= \det[\omega]^* \\
 &= \det(-[\omega]) \\
 &= (-1)^n \det[\omega]
 \end{aligned}$$

For the sake of contradiction, suppose that n is odd. Then $\det[\omega] = -\det[\omega]$ which implies that $\det[\omega] = 0$. Since ω is nondegenerate, $[\omega] \in GL(n, \mathbb{R})$. This is a contradiction. Hence n is even. \square

Definition 2.3.2.4. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. We define the **symplectic complement of V** , denoted S^\perp , by

$$S^\perp = \{v \in V : \text{for each } w \in S, \omega(v, w) = 0\}$$

Exercise 2.3.2.5. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then S^\perp is a subspace.

Proof. We note that

$$S^\perp = \bigcap_{v \in S} \ker \iota_v \omega$$

Hence S^\perp is a subspace. \square

Exercise 2.3.2.6. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then

$$\dim V = \dim S + \dim S^\perp$$

Proof. \square

Exercise 2.3.2.7. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then $(S^\perp)^\perp = S$.

Proof. Let $v \in (S^\perp)^\perp$. Then for each $w \in S^\perp$, $\omega(v, w) = 0$. \square

Chapter 3

Smooth Manifolds

3.1 Topological Manifolds

Exercise 3.1.0.1. We have that \mathbb{R} is homeomorphic to $(0, \infty)$

Proof. Define $f : \mathbb{R} \rightarrow (0, \infty)$ by $f(x) = e^x$. Then f is a homeomorphism. □

Definition 3.1.0.2. Let $n \in \mathbb{N}$. We define the **upper half space** of \mathbb{R}^n , denoted \mathbb{H}^n , by

$$\mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and we define

$$\partial\mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$$

$$\text{Int } \mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

We endow \mathbb{H}^n , $\partial\mathbb{H}^n$ and $\text{Int } \mathbb{H}^n$ with the subspace topology inherited from \mathbb{R}^n .

We define the projection map $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ by

$$\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$$

Definition 3.1.0.3. We define $\mathbb{R}^0 = \{0\}$ and $\mathbb{H}^0 = \emptyset$ endowed with the discrete topology.

Exercise 3.1.0.4. Let $n \in \mathbb{N}$.

1. $\partial\mathbb{H}^n$ is homeomorphic to \mathbb{R}^{n-1}
2. $\text{Int } \mathbb{H}^n$ is homeomorphic to \mathbb{R}^n

Proof.

1. Let $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ be the projection map given by

$$\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$$

Then π is a homeomorphism.

2. Define $f : \mathbb{R}^n \rightarrow \text{Int } \mathbb{H}^n$ by $f(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, e^{x_n})$. Then f is a homeomorphism. □

Definition 3.1.0.5. Let M be a topological space and $n \in \mathbb{N}_0$. Let $U \subset M$ and $V \subset \mathbb{R}^n$ and $\phi : U \rightarrow V$. Then (U, ϕ) is said to be a **n -coordinate chart on M** if

- U is open in M
- V is open in \mathbb{R}^n or V is open in \mathbb{H}^n

- ϕ is a homeomorphism

We denote the set of all n -coordinate charts on M by $X^n(M)$.

Definition 3.1.0.6. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be **locally Euclidean of dimension n** if for each $p \in M$, there exists $(U, \phi) \in X^n(M)$ such that $p \in U$.

Definition 3.1.0.7. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be an **n -dimensional topological manifold** if

1. M is Hausdorff
2. M is second-countable
3. M is locally Euclidean of dimension n

Theorem 3.1.0.8. Topological Invariance of Dimension:

Let M be an n -dimensional topological manifold and N a p -dimensional topological manifold. If M and N are homeomorphic, then $n = p$.

Note 3.1.0.9. In light of the previous theorem, we write $X(M)$ in place of $X^n(M)$ and refer to n -coordinate charts as coordinate charts when the context is clear.

Definition 3.1.0.10. Let M be an n -dimensional topological manifold and $(U, \phi) \in X(M)$. Then (U, ϕ) is said to be an

- **interior chart** if $\phi(U)$ is open in \mathbb{R}^n
- **boundary chart** if $\phi(U)$ is open in \mathbb{H}^n and $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$

We denote the set of all interior charts on M and the set of all boundary charts on M by $X_{\text{Int}}(M)$ and $X_{\partial}(M)$ respectively.

Exercise 3.1.0.11. Let M be an n -dimensional topological manifold. Then

1. $X(M) = X_{\text{Int}}(M) \cup X_{\partial}(M)$
2. $X_{\text{Int}}(M) \cap X_{\partial}(M) = \emptyset$

Proof.

1. By definition, $X_{\text{Int}}(M) \cup X_{\partial}(M) \subset X(M)$. Let $(U, \phi) \in X(M)$. Since (U, ϕ) is a coordinate chart on M , $\phi(U)$ is open in \mathbb{R}^n or $\phi(U)$ is open in \mathbb{H}^n . If $\phi(U)$ is open in \mathbb{R}^n , then

$$\begin{aligned} (U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

Suppose that $\phi(U)$ is open in \mathbb{H}^n . If $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$, then $\phi(U)$ is open in \mathbb{R}^n and

$$\begin{aligned} (U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

Suppose that $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$. Then

$$\begin{aligned} (U, \phi) &\in X_{\partial}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

So $X(M) \subset X_{\text{Int}}(M) \cup X_{\partial}(M)$.

2. For the sake of contradiction, suppose that $X_{\text{Int}}(M) \cup X_{\partial}(M) \neq X(M)$. Then there exists $(U, \phi) \in X(M)$ such that $(U, \phi) \in X_{\text{Int}}(M)$ and $(U, \phi) \in X_{\partial}(M)$. Therefore $\phi(U)$ is open in \mathbb{R}^n , $\phi(U)$ is open in \mathbb{H}^n and $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$. Since $\phi(U)$ is open in \mathbb{R}^n and $\phi(U) \subset \mathbb{H}^n$, $\phi(U) \subset \text{Int } \mathbb{H}^n$ and therefore $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$ which is a contradiction.

□

Definition 3.1.0.12. Let M be an n -dimensional topological manifold. We define the

- **interior** of M , denoted $\text{Int } M$, by

$$\text{Int } M = \{p \in M : \text{there exists } (U, \phi) \in X_{\text{Int}}(M) \text{ such that } p \in U\}$$

- **boundary** of M , denoted ∂M , by

$$\partial M = \{p \in M : \text{there exists } (V, \psi) \in X_{\partial}(M) \text{ such that } p \in V \text{ and } \psi(p) \in \partial \mathbb{H}^n\}$$

Exercise 3.1.0.13. Let M be an n -dimensional topological manifold, $(U, \phi) \in X_{\partial}(M)$ and $p \in U$. If $\phi(p) \notin \partial \mathbb{H}^n$, then $p \in \text{Int } M$.

Proof. Suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \text{Int } \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in \mathbb{R}^n and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then U' is open in M and $\phi' : U' \rightarrow B'$ is a homeomorphism. Hence $(U', \phi') \in X_{\text{Int}}(M)$. Since $\phi(p) \in B'$, we have that $p \in U'$. By definition, $p \in \text{Int } M$. □

Exercise 3.1.0.14. Let M be an n -dimensional topological manifold. Then

1. $M = \text{Int } M \cup \partial M$
2. $\text{Int } M \cap \partial M = \emptyset$

Hint: simply connected

Proof.

1. By definition, $\text{Int } M \cup \partial M \subset M$. Let $p \in M$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. A previous exercise implies that $(U, \phi) \in X_{\text{Int}}(M) \cup X_{\partial}(M)$. If $(U, \phi) \in X_{\text{Int}}(M)$, then by definition,

$$\begin{aligned} p &\in \text{Int } M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Suppose that $(U, \phi) \in X_{\partial}(M)$. If $\phi(p) \in \partial \mathbb{H}^n$, then by definition,

$$\begin{aligned} p &\in \partial M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Suppose that $\phi(p) \notin \partial \mathbb{H}^n$. The previous exercise implies that $p \in \text{Int } M$. Therefore,

$$\begin{aligned} p &\in \text{Int } M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Hence $M \subset \text{Int } M \cup \partial M$.

2. For the sake of contradiction, suppose that $\text{Int } M \cap \partial M \neq \emptyset$. Then there exists $p \in M$ such that $p \in \text{Int } M \cap \partial M$. By definition, there exists $(U, \phi) \in X_{\text{Int}}(M)$, $(V, \psi) \in X_{\partial}(M)$ such that $p \in U \cap V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in \mathbb{H}^n , $\phi(U \cap V)$ is open in \mathbb{R}^n and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$ is a homeomorphism. Since $\psi(U \cap V)$ is open in \mathbb{H}^n , there exists an $B_{\psi} \subset \psi(U \cap V)$ such that B_{ψ} is open in \mathbb{H}^n , B_{ψ} is simply connected and $\psi(p) \in B_{\psi}$. Set $B_{\phi} = \phi \circ \psi^{-1}(B_{\psi})$. Since $\phi(U \cap V)$ is open in \mathbb{R}^n , B_{ϕ} is open in \mathbb{R}^n . Since B_{ψ} is simply connected and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$ is a homeomorphism, B_{ϕ} is simply connected. Set $B'_{\phi} = B_{\phi} \setminus \{\phi(p)\}$ and $B'_{\psi} = B_{\psi} \setminus \{\psi(p)\}$. Then $\phi \circ \psi^{-1} : B'_{\psi} \rightarrow B'_{\phi}$ is a homeomorphism. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_{ψ} is simply connected. Since B_{ϕ} is open in \mathbb{R}^n , B'_{ϕ} is not simply connected. This is a contradiction since B'_{ϕ} is homeomorphic to B'_{ψ} . So $\partial M \cap \text{Int } M = \emptyset$.

□

Exercise 3.1.0.15. Let M be an n -dimensional topological manifold, $(U, \phi) \in X(M)$ and $p \in U$. If $p \in \partial M$, then $(U, \phi) \in X_{\partial}(M)$.

Hint: simply connected

Proof. Suppose that $p \in \partial M$. Then there exists a $(V, \psi) \in X_{\partial}(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in \mathbb{H}^n , $\phi(U \cap V)$ is open in \mathbb{R}^n and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$ is a homeomorphism.

Since $\psi(U \cap V)$ is open in \mathbb{H}^n , there exists $B_{\psi} \subset \psi(U \cap V)$ such B_{ψ} is open in \mathbb{H}^n , B_{ψ} is simply connected and $\psi(p) \in B_{\psi}$. Set $B_{\phi} = \phi \circ \psi^{-1}(B_{\psi})$.

For the sake of contradiction, suppose that $(U, \phi) \in X_{\text{Int}}(M)$. Then $\phi(U)$ is open in \mathbb{R}^n . Hence $\phi(U \cap V)$ is open in \mathbb{R}^n and B_{ϕ} is open in \mathbb{R}^n . Since $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$ is a homeomorphism, B_{ϕ} is simply connected. Set $B'_{\phi} = B_{\phi} \setminus \{\phi(p)\}$ and $B'_{\psi} = B_{\psi} \setminus \{\psi(p)\}$. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_{ψ} is simply connected. Since B_{ϕ} is open in \mathbb{R}^n , B'_{ϕ} is not simply connected. This is a contradiction since B'_{ϕ} is homeomorphic to B'_{ψ} . So $(U, \phi) \notin X_{\text{Int}}(M)$. Since $(X_{\text{Int}}(M))^c = X_{\partial}(M)$, we have that $(U, \phi) \in X_{\partial}(M)$. □

Exercise 3.1.0.16. Let M be an n -dimensional topological manifold, $(U, \phi) \in X_{\partial}(M)$ and $p \in U$. Then

1. $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$
2. $p \in \text{Int } M$ iff $\phi(p) \in \text{Int } \mathbb{H}^n$

Proof.

1. Suppose that $p \in \partial M$. For the sake of contradiction, suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \text{Int } \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in \mathbb{R}^n and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then $p \in U'$ and $(U', \phi') \in X_{\text{Int}}(M)$. Since $p \in U'$, the previous exercise implies that $(U', \phi') \in X_{\partial}(M)$. This is a contradiction since $X_{\text{Int}}(M) \cap X_{\partial}(M) = \emptyset$. So $\phi(p) \in \partial \mathbb{H}^n$. Conversely, suppose that $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$.

2. A previous exercise implies that $\text{Int } M = (\partial M)^c$. Part (1) implies that

$$\begin{aligned} p &\in (\partial M)^c \\ &= \text{Int } M \end{aligned}$$

if and only if

$$\begin{aligned} \phi(p) &\in (\partial \mathbb{H}^n)^c \\ &= \text{Int } \mathbb{H}^n \end{aligned}$$

□

Exercise 3.1.0.17. Let M be an n -dimensional topological manifold and $p \in M$. Then $p \in \partial M$ iff for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_{\partial}(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.

Proof. Suppose that $p \in \partial M$. Let $(U, \phi) \in X(M)$. Suppose that $p \in U$. The previous two exercises imply that $(U, \phi) \in X_{\partial}(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.

Conversely, suppose that for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_{\partial}(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. By assumption, $(U, \phi) \in X_{\partial}(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$. □

Exercise 3.1.0.18. Let M be an n -dimensional topological manifold. Let $(U, \phi) \in X_{\partial}(M)$. Then

1. $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$
2. $\phi(U \cap \text{Int } M) = \phi(U) \cap \text{Int } \mathbb{H}^n$

Proof.

1. Since $(U, \phi) \in X_{\partial}(M)$, a previous exercise implies that for each $p \in U$, $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$. Let $q \in \phi(U \cap \partial M)$. Then there exists $p \in U \cap \partial M$ such that $\phi(p) = q$. Since $p \in \partial M$, $\phi(p) \in \partial \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \partial \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \partial M)$ is arbitrary, $\phi(U \cap \partial M) \subset \phi(U) \cap \partial \mathbb{H}^n$.

Let $q \in \phi(U) \cap \partial \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \partial \mathbb{H}^n$, we have that $p \in \partial M$. Hence $p \in U \cap \partial M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \partial M)$. Thus $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$.

2. Since $(U, \phi) \in X_{\partial}(M)$, a previous exercise implies that for each $p \in U$, $p \in \text{Int } M$ iff $\phi(p) \in \text{Int } \mathbb{H}^n$. Let $q \in \phi(U \cap \text{Int } M)$. Then there exists $p \in U \cap \text{Int } M$ such that $\phi(p) = q$. Since $p \in \text{Int } M$, $\phi(p) \in \text{Int } \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \text{Int } \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \text{Int } M)$ is arbitrary, $\phi(U \cap \text{Int } M) \subset \phi(U) \cap \text{Int } \mathbb{H}^n$.

Let $q \in \phi(U) \cap \text{Int } \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \text{Int } \mathbb{H}^n$, we have that $p \in \text{Int } M$. Hence $p \in U \cap \text{Int } M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \text{Int } M) \end{aligned}$$

Since $q \in \phi(U) \cap \text{Int } \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \text{Int } \mathbb{H}^n \subset \phi(U \cap \text{Int } M)$. Thus $\phi(U \cap \text{Int } M) = \phi(U) \cap \text{Int } \mathbb{H}^n$. □

Exercise 3.1.0.19. Let M be an n -dimensional topological manifold. Then

1. $\text{Int } M$ is open
2. ∂M is closed

Proof.

1. Let $p \in \text{Int } M$. Then there exists $(U, \phi) \in X_{\text{Int}}(M)$ such that $p \in U$. By definition of coordinate charts, U is open. By definition of $\text{Int } M$, for each $q \in U$, $q \in \text{Int } M$. Hence $U \subset \text{Int } M$. Since $p \in \text{Int } M$ is arbitrary, we have that for each $p \in \text{Int } M$, there exists $U \subset \text{Int } M$ such that U is open. Hence $\text{Int } M$ is open.
2. Since $\partial M = (\text{Int } M)^c$, and $\text{Int } M$ is open, we have that ∂M is closed. □

Definition 3.1.0.20. Let M be an n -dimensional topological manifold and $\pi : \partial \mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ the projection map. For $(U, \phi) \in X_{\partial}(M)$, we define $\bar{U} \subset \partial M$ and $\bar{\phi} : \bar{U} \rightarrow \pi(\phi(\bar{U}))$ by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi \circ \phi|_{\bar{U}}$ respectively.

Exercise 3.1.0.21. Let M be an n -dimensional topological manifold, and $\lambda : \partial \mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ a homeomorphism. Then $\{(\bar{U}, \bar{\phi}) : (U, \phi) \in X_{\partial}(M)\} \subset X_{\text{Int}}^{n-1}(\partial M)$.

Proof. Let $(U, \phi) \in X_{\partial}(M)$.

1. Since U is open in M , $\bar{U} = U \cap \partial M$ is open in ∂M .
2. Since $(U, \phi) \in X_{\partial}(M)$, $\phi(U)$ is open in \mathbb{H}^n . A previous exercise implies that $\phi(\bar{U}) = \phi(U) \cap \partial \mathbb{H}^n$ which is open in $\partial \mathbb{H}^n$. Since $\pi : \partial \mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ is a homeomorphism, we have that $\pi(\phi(\bar{U}))$ is open in \mathbb{R}^{n-1} .
3. Since $\phi|_{\bar{U}} : \bar{U} \rightarrow \phi(U) \cap \partial \mathbb{H}^n$ and $\pi|_{\phi(\bar{U})} : \phi(\bar{U}) \rightarrow \pi(\phi(\bar{U}))$ are homeomorphisms, we have that $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$ is a homeomorphism.

Hence $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$. □

Exercise 3.1.0.22. Let M be an n -dimensional topological manifold. Then

1. ∂M is an $(n-1)$ -dimensional topological manifold
2. $\partial(\partial M) = \emptyset$

Proof.

1. (a) Since M is Hausdorff, ∂M is Hausdorff.
- (b) Since M is second-countable, ∂M is second countable.
- (c) Let $p \in \partial M$. Then there exists $(U, \phi) \in X_{\partial}(M)$ such that $\phi(p) \in \partial \mathbb{H}^n$. Then $p \in \bar{U}$ and the previous exercise implies that $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$. Thus ∂M is locally Euclidean of dimension $n-1$.

Hence ∂M is an $(n-1)$ -dimensional topological manifold.

2. Let $p \in \partial M$. Part (1) implies that there exists $(U, \phi) \in X_{\text{Int}}^{n-1}(\partial M)$ such that $p \in U$. Thus $p \in \text{Int } \partial M$. Since $p \in \partial M$ is arbitrary, $\text{Int } \partial M = \partial M$. Hence

$$\begin{aligned} \partial(\partial M) &= (\text{Int}(\partial M))^c \\ &= (\partial M)^c \\ &= \emptyset \end{aligned}$$

□

Exercise 3.1.0.23. Let M be an n -dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M , then $(U', \phi|_{U'}) \in X^n(M)$.

Proof. Suppose that U' is open in M . Set $\phi' = \phi|_{U'}$.

- By assumption U' is open in M .
- Since U' is open in M , we have that $U' = U' \cap U$ is open in U . Since ϕ is a homeomorphism and U' is open in U , we have that $\phi(U')$ is open in $\phi(U)$. By assumption $\phi(U)$ is open in \mathbb{R}^n or $\phi(U)$ is open in \mathbb{H}^n . Therefore $\phi'(U')$ is open in \mathbb{R}^n or $\phi'(U')$ is open in \mathbb{H}^n .
- Since $\phi : U \rightarrow V$ is a homeomorphism, $\phi' : U' \rightarrow \phi'(U')$ is a homeomorphism.

So $(U', \phi') \in X^n(M)$. □

Note 3.1.0.24. Since U is open in M , U' being open in U is equivalent to U' being open in M , so we could have also assumed that U' is open in U .

Exercise 3.1.0.25. Let M be an n -dimensional topological manifold and $U \subset M$. If U is open, then

$$X^n(U) = \{(V, \psi) \in X^n(M) : V \subset U\}$$

Proof. Suppose that U is open and set $A = \{(V, \psi) \in X^n(M) : V \subset U\}$. Let $(V, \psi) \in X^n(U)$. By definition of $X^n(U)$, V is open in U . Thus, there exists $W \subset M$ such that W is open in M and $V = U \cap W$. Since U is open in M , we have that $V = U \cap W$ is open in M . Hence $(V, \psi) \in X^n(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X^n(U)$ is arbitrary, $X^n(U) \subset A$.

Conversely, suppose that $(V, \psi) \in A$. Then $(V, \psi) \in X^n(M)$ and $V \subset U$. By definition of $X^n(M)$, V is open in M . Since $V \subset U$, we have that $V = V \cap U$ is open in U . Hence $(V, \psi) \in X^n(U)$. Since $(V, \psi) \in X^n(U)$ is arbitrary, $A \subset X^n(U)$. Hence $X^n(A) = A$. \square

Exercise 3.1.0.26. Let M be an n -dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M , then $(U', \phi|_{U'}) \in X^n(U)$.

Proof. Suppose that U' is open in M . A previous exercise implies that $(U', \phi') \in X^n(M)$. The previous exercise implies that $(U', \phi') \in X^n(U)$. \square

Exercise 3.1.0.27. Topological Open Submanifolds:

Let M be an n -dimensional topological manifold and $U \subset M$ open. Then U is an n -dimensional topological manifold.

Proof.

1. Since M is Hausdorff, U is Hausdorff.
2. M is second-countable, U is second countable.
3. Let $p \in U$. Since then there exists $(V, \psi) \in X^n(M)$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{U \cap V}$. The previous exercise implies that $(V', \psi') \in X^n(U)$. Therefore U is locally Euclidean of dimension n .

Hence U is an n -dimensional topological manifold. \square

Exercise 3.1.0.28. Let M be an n -dimensional topological manifold and $U \subset M$. If U is open, then

1. $X_{\text{Int}}(U) = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$
2. $X_{\partial}(U) = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$

Proof. Suppose that U is open in M .

1. Set $A = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\text{Int}}(U)$. By definition of $X_{\text{Int}}(U)$, V is open in U and $\phi(V)$ is open in \mathbb{R}^n . Since U is open in M , V is open in M . Hence $(V, \psi) \in X_{\text{Int}}(M)$. Since U is open in M , V is open in M . Hence $(V, \psi) \in X_{\text{Int}}(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X_{\text{Int}}(U)$ is arbitrary, $X_{\text{Int}}(U) \subset A$.
Conversely, let $(V, \psi) \in A$. Then $(V, \psi) \in X_{\text{Int}}(M)$ and $V \subset U$. By definition of $X_{\text{Int}}(M)$, V is open in M and $\phi(V)$ is open in \mathbb{R}^n . Thus $V = V \cap U$ is open in U . So $(V, \psi) \in X_{\text{Int}}(U)$. Since $(V, \psi) \in A$ is arbitrary, $A \subset X_{\text{Int}}(U)$. Thus $X_{\text{Int}}(U) = A$.
2. Set $B = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\partial}(U)$. By definition of $X_{\partial}(U)$, V is open in U , $\phi(V)$ is open in \mathbb{H}^n and $\partial\mathbb{H}^n \cap \phi(V) \neq \emptyset$. Since U is open in M , V is open in M . Hence $(V, \psi) \in X_{\partial}(M)$, which implies that $(V, \psi) \in B$. Since $(V, \psi) \in X_{\partial}(U)$ is arbitrary, $X_{\partial}(U) \subset B$.
Conversely, let $(V, \psi) \in B$. Then $(V, \psi) \in X_{\partial}(M)$ and $V \subset U$. By definition of $X_{\partial}(M)$, V is open in M , $\phi(V)$ is open in \mathbb{H}^n and $\partial\mathbb{H}^n \cap \phi(V) \neq \emptyset$. Thus $V = V \cap U$ is open in U . So $(V, \psi) \in X_{\partial}(U)$. Since $(V, \psi) \in B$ is arbitrary, $B \subset X_{\partial}(U)$. Thus $X_{\partial}(U) = B$.

\square

Exercise 3.1.0.29. Let M be an n -dimensional topological manifold and $U \subset M$. If U is open, then $\partial U = \partial M \cap U$.

Proof. Suppose that U is open. Let $p \in \partial U$. Then there exists $(V, \psi) \in X_\partial(U)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Since U is open, the previous exercise implies that $(V, \psi) \in X_\partial(M)$. Thus $p \in \partial M$. Since $p \in \partial U$ is arbitrary, $\partial U \subset \partial M$. Since $\partial U \subset U$, we have that $\partial U \subset \partial M \cap U$.

Conversely, let $p \in \partial M \cap U$. Since $p \in \partial M$, there exists $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Set $V' = V \cap U$ and $\psi' = \psi|_{V'}$. Then $p \in V'$ since V and U are open in M , V' is open in M . A previous exercise implies that $(V', \psi') \in X(M)$. Since $p \in \partial M$, a previous exercise implies that $(V', \psi') \in X_\partial(M)$. The previous exercise implies that $(V', \psi') \in X_\partial(U)$. Since $\psi'(p) \in \partial \mathbb{H}^n$, $p \in \partial U$. Since $p \in \partial M \cap U$ is arbitrary, $\partial M \cap U \subset \partial U$. Hence $\partial U = \partial M \cap U$.

label exercises and reference them!!! □

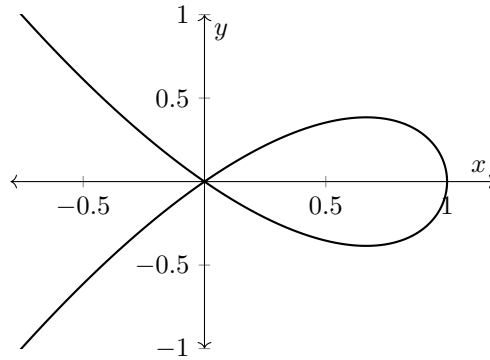
Exercise 3.1.0.30. Graph of Continuous Function:

Let $f \in C(\mathbb{R})$. Set $M = \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$ (i.e. the graph of f). Then M is a 1-dimensional manifold.

Proof. Set $U = \mathbb{R}$ and define $\phi : U \rightarrow M$ by $\phi(x) = (x, f(x))$. Then $\phi^{-1} = \pi_1$. Since f is continuous, ϕ is continuous. Since π_1 is continuous, ϕ is a homeomorphism. □

Exercise 3.1.0.31. Nodal Cubic:

Let $M = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2 - x^3\}$. We equip M with the subspace topology.



Then M is not a 1-dimensional topological manifold.

Hint: connected components

Proof. Suppose that M is a 1-dimensional manifold. Set $p = (0, 0)$. Then there exists $(U, \phi) \in X(M)$ such that $p \in U$. Since $\phi(U)$ is open (in \mathbb{R} or \mathbb{H}), there exists a $B \subset \phi(U)$ such that B is open (in \mathbb{R} or \mathbb{H}), B is connected and $\phi(p) \in B$. Set $V = \phi^{-1}(B)$, $V' = V \setminus \{p\}$ and $B' = B \setminus \{\phi(p)\}$. Then $\phi : V \rightarrow B$ and $\phi' : V' \rightarrow B'$ are homeomorphisms. Since B is open (in \mathbb{R} or \mathbb{H}) and connected, B' has at most two connected components. Then V' This is a contradiction since V' has four connected components and B' and V' are homeomorphic. □

3.2 Smooth Manifolds

Definition 3.2.0.1. Let M be an n -dimensional topological manifold and $(U, \phi), (V, \psi) \in X(M)$. Then (U, ϕ) and (V, ψ) are said to be **smoothly compatible** if

$$\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V) \text{ is a diffeomorphism}$$

Definition 3.2.0.2. Let M be an n -dimensional topological manifold.

- Let $\mathcal{A} \subset X(M)$. Then \mathcal{A} is said to be an **atlas on M** if $\bigcup_{(U, \phi) \in \mathcal{A}} U = M$.
- Let \mathcal{A} be an atlas on M . Then \mathcal{A} is said to be **smooth** if for each $(U, \phi), (V, \psi) \in \mathcal{A}$, (U, ϕ) and (V, ψ) are smoothly compatible.
- Let \mathcal{A} be a smooth atlas on M . Then \mathcal{A} is said to be **maximal** if for each smooth atlas \mathcal{B} on M , $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on M** .
- Let \mathcal{A} be an atlas on M . Then (M, \mathcal{A}) is said to be an **n -dimensional smooth manifold** if \mathcal{A} is a smooth structure on M .

Exercise 3.2.0.3. Let M be an n -dimensional topological manifold and \mathcal{B} a smooth atlas on M . Then there exists a unique smooth structure \mathcal{A} on M such that $\mathcal{B} \subset \mathcal{A}$.

Proof. Define

$$\mathcal{A} = \{(U, \phi) \in X(M) : \text{for each } (V, \psi) \in \mathcal{B}, (U, \phi) \text{ and } (V, \psi) \text{ are smoothly compatible}\}$$

Clearly $\mathcal{B} \subset \mathcal{A}$. Let (U, ϕ) and $(V, \psi) \in \mathcal{A}$. Define $F : \phi(U \cap V) \rightarrow \psi(U \cap V)$ by

$$F = \psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$$

Let $q \in \phi(U \cap V)$. Set $p = \phi^{-1}(q)$. Since $p \in U \cap V \subset M$, there exists $(W, \chi) \in \mathcal{B}$ such that $p \in W$. By definition of \mathcal{A} , $\psi|_{W \cap V} \circ (\chi|_{W \cap V})^{-1} : \chi(W \cap V) \rightarrow \psi(W \cap V)$ and $\chi|_{U \cap W} \circ (\phi|_{U \cap W})^{-1} : \phi(U \cap W) \rightarrow \chi(U \cap W)$ are diffeomorphisms. Set $N = U \cap W \cap V$. Then $q \in \phi(N) \subset \phi(U \cap V)$ and

$$\begin{aligned} F|_{\phi(N)} &= \psi|_N \circ (\phi|_N)^{-1} \\ &= [\psi|_N \circ (\chi|_N)^{-1}] \circ [\chi|_N \circ (\phi|_N)^{-1}] \end{aligned}$$

is a diffeomorphism. Thus, for each $q \in \phi(U \cap V)$, there exists $N' \subset \phi(U \cap V)$ such that $F|_{N'}$ is a diffeomorphism. Hence F is a diffeomorphism and $(U, \phi), (V, \psi)$ are smoothly compatible. Therefore \mathcal{A} is a smooth atlas.

To see that \mathcal{A} is maximal, let \mathcal{B}' be a smooth atlas on M . Suppose that $\mathcal{A} \subset \mathcal{B}'$ and let $(U, \phi) \in \mathcal{B}'$. By definition, for each chart $(V, \psi) \in \mathcal{B}'$, (U, ϕ) and (V, ψ) are smoothly compatible. Since $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$, we have that $(U, \phi) \in \mathcal{A}$. So $\mathcal{A} = \mathcal{B}'$ and \mathcal{A} is a maximal smooth atlas on M . \square

Exercise 3.2.0.4. Let (M, \mathcal{A}) be an n -dimensional smooth manifold, $(U, \phi) \in \mathcal{A}$ and $U' \subset U$. If U' is open, then $(U', \phi|_{U'}) \in \mathcal{A}$.

Proof. Set $\phi' = \phi|_{U'}$. A previous exercise implies that $(U', \phi') \in X(U)$. Define $\mathcal{B} = \mathcal{A} \cup \{(U', \phi')\}$. Let $(V, \psi) \in \mathcal{B}$. If $(V, \psi) = (U', \phi')$, then

$$\phi' \circ \psi^{-1} = \text{id}_{U'}$$

which is a diffeomorphism. Thus $(U', \phi'), (V, \psi)$ are smoothly compatible. Suppose that $(V, \psi) \in \mathcal{A}$. Since \mathcal{A} is smooth, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ is a diffeomorphism. Therefore $\psi|_{U' \cap V} \circ (\phi'|_{U' \cap V})^{-1} : \phi'(U' \cap V) \rightarrow \psi(U' \cap V)$ is a diffeomorphism and $(U', \phi'), (V, \psi)$ are smoothly compatible. Since $(V, \psi) \in \mathcal{A}$ is arbitrary, \mathcal{B} is smooth. Since \mathcal{A} is maximal and $\mathcal{A} \subset \mathcal{B}$, we have that $\mathcal{A} = \mathcal{B}$ and $(U', \phi') \in \mathcal{A}$. \square

Exercise 3.2.0.5. Let (M, \mathcal{A}) be a n -dimensional smooth manifold and $U \subset M$ open. Set $\mathcal{B} = \{(V, \psi) \in \mathcal{A} : V \subset U\}$. Then \mathcal{B} is a smooth atlas on U .

Proof.

- Some previous exercises imply that U is an n -dimensional topological manifold and $X(U) = \{(V, \psi) \in X(M) : V \subset U\}$. Since

$$\begin{aligned} \mathcal{B} &\subset \mathcal{A} \\ &\subset X(M) \end{aligned}$$

we have that $\mathcal{B} \subset X(U)$. Let $p \in U$. Then there exists $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{V'}$. The previous exercise implies that $(V', \psi') \in \mathcal{A}$. By definition, $(V', \psi') \in \mathcal{B}$. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $(V', \psi') \in \mathcal{B}$ such that $p \in V'$. Hence \mathcal{B} is an atlas on U .

- Let $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$. Then $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{A}$. Since \mathcal{A} is smooth, (V_1, ψ_1) and (V_2, ψ_2) are smoothly compatible. Since $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ are arbitrary, \mathcal{B} is smooth. □

Definition 3.2.0.6. Smooth Open Submanifold:

Let (M, \mathcal{A}) be an n -dimensional smooth manifold and $U \subset M$ open. A previous exercise implies that U is an n -dimensional topological manifold. We define $\mathcal{A}|_U \subset X(U)$ to be the unique smooth structure on U such that $\{(V, \psi) \in \mathcal{A} : V \subset U\} \subset \mathcal{A}|_U$. Then $(U, \mathcal{A}|_U)$ is said to be a **smooth open submanifold of (M, \mathcal{A})** .

Exercise 3.2.0.7. Let $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ be the projection map given by $\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$. Then π is a diffeomorphism.

Proof. Define projection map $\pi' : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ by $\pi'(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1})$. Then \mathbb{R}^n is an open neighborhood of $\partial\mathbb{H}^n$, $\pi'|_{\partial\mathbb{H}^n} = \pi$ and π' is smooth. Then by definition, π is smooth. Clearly, π^{-1} is smooth. So π is a diffeomorphism. □

Definition 3.2.0.8. Let (M, \mathcal{A}) be a n -dimensional smooth manifold and $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ the projection map. Recall that for $(U, \phi) \in X_{\partial}^n(M)$, the $(n-1)$ -coordinate chart $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$ is defined by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$.

We define

$$\bar{\mathcal{A}} = \{(\bar{U}, \bar{\phi}) : (U, \phi) \in \mathcal{A} \cap X_{\partial}^n(M)\}$$

Exercise 3.2.0.9. Let (M, \mathcal{A}) be a n -dimensional smooth manifold. Then $\bar{\mathcal{A}}$ is a smooth atlas on ∂M .

Proof.

- A previous exercise implies that ∂M is an $(n-1)$ -dimensional topological manifold. Let $p \in \partial M$. Then there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Since $\mathcal{A} \subset X^n(M)$ and $p \in \partial M$, we have that $p \in \bar{U}$ and a previous exercise implies that $(U, \phi) \in X_{\partial}^n(M)$. By definition of $\bar{\mathcal{A}}$, $(\bar{U}, \bar{\phi}) \in \bar{\mathcal{A}}$. Since $p \in \partial M$ is arbitrary, $\bar{\mathcal{A}}$ is an atlas on ∂M .
- Let $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \bar{\mathcal{A}}$. Since (U, ϕ) and (V, ψ) are smoothly compatible, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$ is a diffeomorphism. Thus $\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}$ is a diffeomorphism. Since $\pi|_{\phi(U \cap V)}$ and $\pi|_{\psi(U \cap V)}$ are diffeomorphisms, $\pi|_{\phi(\bar{U} \cap \bar{V})}$ and $\pi|_{\psi(\bar{U} \cap \bar{V})}$ are diffeomorphisms. Then

$$\begin{aligned} \bar{\psi}|_{\bar{U} \cap \bar{V}} \circ (\bar{\phi}|_{\bar{U} \cap \bar{V}})^{-1} &= \left[\pi|_{\psi(\bar{U} \cap \bar{V})} \circ \psi|_{\bar{U} \cap \bar{V}} \right] \circ \left[(\phi|_{\bar{U} \cap \bar{V}})^{-1} \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \right] \\ &= \pi|_{\psi(\bar{U} \cap \bar{V})} \circ \left[\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1} \right] \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \end{aligned}$$

is a diffeomorphism. Therefore $(\bar{U}, \bar{\phi})$ and $(\bar{V}, \bar{\psi})$ are smoothly compatible. Since $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \bar{\mathcal{A}}$ are arbitrary, $\bar{\mathcal{A}}$ is smooth. □

Definition 3.2.0.10. Let (M, \mathcal{A}) be a n -dimensional smooth manifold. We define $\mathcal{A}|_{\partial M}$ to be the unique smooth structure on ∂M such that $\overline{\mathcal{A}} \subset \mathcal{A}|_{\partial M}$. We define the **smooth boundary submanifold of M** to be $(\partial M, \mathcal{A}|_{\partial M})$.

Note 3.2.0.11. For the rest of this section, we assume that (M, \mathcal{A}) is a n -dimensional smooth manifold and we denote the standard coordinate functions on \mathbb{R}^n by u^1, \dots, u^n . For a coordinate chart $(U, \phi) \in \mathcal{A}$ and $i \in \{1, \dots, n\}$, we will typically denote the i th coordinate of ϕ by x^i , that is, $x^i = u^i(\phi)$.

3.3 Smooth Maps

Definition 3.3.0.1. Let (M, \mathcal{A}) be a smooth manifold and $f : M \rightarrow \mathbb{R}$. Then f is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M)$.

Exercise 3.3.0.2. Let (M, \mathcal{A}) be a smooth manifold. Then $C^\infty(M)$ is a vector space.

Proof. Let $f, g \in C^\infty(M)$, $\lambda \in \mathbb{R}$ and $(U, \phi) \in \mathcal{A}$. By assumption, $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are smooth. Hence

$$(f + \lambda g) \circ \phi^{-1} = f \circ \phi^{-1} + \lambda g \circ \phi^{-1}$$

is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, $f + \lambda g \in C^\infty(M)$. Since $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$ are arbitrary, $C^\infty(M)$ is a vector space. \square

Exercise 3.3.0.3. Let (M, \mathcal{A}) be a smooth manifold, \mathcal{B} an atlas on M and $f : M \rightarrow \mathbb{R}$. Suppose that $\mathcal{B} \subset \mathcal{A}$. Then f is smooth iff for each $(U, \phi) \in \mathcal{B}$, $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is smooth.

Proof.

- (\implies) :
Suppose that f is smooth. Let $(U, \phi) \in \mathcal{B}$. Since $\mathcal{B} \subset \mathcal{A}$, $(U, \phi) \in \mathcal{A}$. Since f is smooth, $f \circ \phi^{-1}$ is smooth. Since $(U, \phi) \in \mathcal{B}$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{B}$, $f \circ \phi^{-1}$ is smooth.
- (\impliedby) :
Suppose that for each $(V, \psi) \in \mathcal{B}$, $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$ is smooth. Let $(U, \phi) \in \mathcal{A}$ and $q \in \phi(U)$. Set $p = \phi^{-1}(q)$. Since \mathcal{B} is an atlas, there exists $(V, \psi) \in \mathcal{B}$ such that $p \in V$. Since $\mathcal{B} \subset \mathcal{A}$, $(V, \psi) \in \mathcal{A}$. Set $W = U \cap V$ and $\tilde{\phi} = \phi|_W$ and $\tilde{\psi} = \psi|_W$. We note that $\phi(W) \in \mathcal{N}_q$ and $\phi(W)$ is open. An exercise in the section on smooth manifolds implies that $(W, \tilde{\phi}), (W, \tilde{\psi}) \in \mathcal{A}$. Therefore $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(W) \rightarrow \psi(W)$ is smooth. By assumption, $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$ is smooth. This implies that $(f \circ \psi^{-1})|_{\psi(W)} : \psi(W) \rightarrow \mathbb{R}$ is smooth. Hence

$$\begin{aligned} (f \circ \phi^{-1})|_{\phi(W)} &= f \circ \tilde{\phi}^{-1} \\ &= f \circ (\tilde{\psi}^{-1} \circ \tilde{\psi}) \circ \tilde{\phi}^{-1} \\ &= (f \circ \tilde{\psi}^{-1}) \circ (\tilde{\psi} \circ \tilde{\phi}^{-1}) \end{aligned}$$

is smooth. Since $q \in \phi(U)$ is arbitrary, for each $q \in \phi(U)$, there exists $A \in \mathcal{N}_q$ such that A is open and $(f \circ \phi^{-1})|_A : A \rightarrow \mathbb{R}$ is smooth. This implies that $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, f is smooth. \square

Exercise 3.3.0.4. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $p \in U$ and $f \in C^\infty(M)$. Then $f|_U \in C^\infty(U)$.

Proof. Let \square

Definition 3.3.0.5. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \dots, n\}$. We define the **partial derivative of f with respect to x^i** , denoted

$$\partial f / \partial x^i : U \rightarrow \mathbb{R} \quad \text{or} \quad \partial_i f : U \rightarrow \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left(\frac{\partial}{\partial u^i}[f \circ \phi^{-1}] \right) \circ \phi$$

Exercise 3.3.0.6. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \dots, n\}$. Then $\partial/\partial x^i : C^\infty(U) \rightarrow C^\infty(U)$ is linear.

Proof. **FINISH!!!** □

Exercise 3.3.0.7. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i, j \in \{1, \dots, n\}$. Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} f \right) \\ &= \frac{\partial}{\partial x^i} \left(\left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \circ \phi \right) \\ &= \left(\frac{\partial}{\partial u^i} \left[\left(\left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \circ \phi \right) \circ \phi^{-1} \right] \right) \circ \phi \\ &= \left(\frac{\partial}{\partial u^i} \left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \right) \circ \phi \\ &= \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi \end{aligned}$$

□

Exercise 3.3.0.8. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $i, j \in \{1, \dots, n\}$. Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i}$$

Proof. Let $f \in C^\infty(U)$. Since $f \circ \phi^{-1}$ is smooth,

$$\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] = \frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i} [f \circ \phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi \\ &= \left(\frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i} [f \circ \phi^{-1}] \right) \circ \phi \\ &= \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f \end{aligned}$$

□

Exercise 3.3.0.9. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $f \in C^\infty(U)$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha f = (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi$$

Proof. The claim is clearly true when $|\alpha| = 0$ or by definition if $|\alpha| = 1$. Let $n \in \mathbb{N}$ and suppose the claim is true for each $|\alpha| \in \{1, \dots, n-1\}$. Then there exists $i \in \{1, \dots, n\}$ such that $\alpha_i \geq 1$. Hence

$$\begin{aligned} \partial^\alpha f &= \partial^{e^i} (\partial^{\alpha-e^i} f) \\ &= \partial^{e^i} (\partial^{\alpha-e^i} [f \circ \phi^{-1}] \circ \phi) \\ &= (\partial^{e^i} [(\partial^{\alpha-e^i} [f \circ \phi^{-1}] \circ \phi) \circ \phi^{-1}]) \circ \phi \\ &= (\partial^{e^i} [\partial^{\alpha-e^i} [f \circ \phi^{-1}]]) \circ \phi \\ &= (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi \end{aligned}$$

□

Exercise 3.3.0.10. Taylor's Theorem:

Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\phi(U)$ convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that

$$f = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(x_0) \right] + \sum_{|\alpha|=T+1} (x^i - x^i(p))^\alpha g_\alpha$$

and for each $|\alpha| = T+1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. Since $\phi(U)$ is open and convex and $f \circ \phi^{-1} \in C^\infty(\phi(U))$, Taylors thorem in section 2.1 implies that there exist $(\tilde{g}_\alpha)_{|\alpha|=T+1} \subset C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q))$$

and for each $|\alpha| = T+1$,

$$\begin{aligned} \tilde{g}_\alpha(\phi(p)) &= \frac{1}{(T+1)!} \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \\ &= \frac{1}{(T+1)!} \partial^\alpha f(p) \end{aligned}$$

For $|\alpha| = T+1$, set $g_\alpha = \tilde{g}_\alpha \circ \phi$. Then

$$\begin{aligned} f(q) &= f \circ \phi^{-1}(\phi(q)) \\ &= \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\ &= \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q) \end{aligned}$$

□

Definition 3.3.0.11. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$. Then F is said to be

- **smooth** if for each $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \psi(F(U) \cap V)$$

is smooth

- a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 3.3.0.12. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifold and $F : M \rightarrow N$. If F is smooth, then F is continuous.

Proof. Suppose that F is smooth. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $F(p) \in V$. Put $\tilde{U} = U \cap F^{-1}(V)$ and $\tilde{V} = F(U) \cap V$. Define $\tilde{\phi} : \tilde{U} \rightarrow \phi(\tilde{U})$ and $\tilde{\psi} : \tilde{V} \rightarrow \psi(\tilde{V})$ by

$$\tilde{\phi} = \phi|_{\tilde{U}}, \quad \tilde{\psi} = \psi|_{\tilde{V}}$$

Then $\tilde{\phi}$ and $\tilde{\psi}$ are homeomorphisms, $p \in \tilde{U}$ and $F(\tilde{U}) \subset \tilde{V}$. Define $\tilde{F} : \phi(\tilde{U}) \rightarrow \psi(\tilde{V})$ by

$$\tilde{F} = \tilde{\psi} \circ F \circ \tilde{\phi}^{-1}$$

By definition, \tilde{F} is smooth and therefore continuous. Since ϕ and ψ are homeomorphisms and $F|_{\tilde{U}} = \tilde{\psi}^{-1} \circ \tilde{F} \circ \tilde{\phi}$, we have that $F|_{\tilde{U}}$ is continuous. In particular, F is continuous at p and since $p \in M$ is arbitrary, F is continuous. \square

Exercise 3.3.0.13. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifold and $F : M \rightarrow N$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F and F^{-1} are smooth. The previous exercise implies that F and F^{-1} are continuous. Hence F is a homeomorphism. \square

Exercise 3.3.0.14. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$ a diffeomorphism. Then for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

Proof. Let $(V, \psi) \in \mathcal{B}$.

1. Since ϕ and F^{-1} are homeomorphisms, $\phi \circ F^{-1} : F(U) \cap V \rightarrow \phi(U \cap F^{-1}(V))$ is a homeomorphism
2. Since F is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \rightarrow \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \psi(V \cap F(U))$$

are smooth.

Therefore $(F(U), \phi \circ F^{-1})$ and (V, ψ) are smoothly compatible. Since \mathcal{B} is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. \square

Definition 3.3.0.15. Let (N, \mathcal{B}) be a smooth n -dimensional manifold, $F : M \rightarrow N$ smooth and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. For $i \in \{1, \dots, n\}$, We define the **i -th component of F with respect to (V, ψ)** , denoted $F^i : V \rightarrow \mathbb{R}$, by

$$F^i = y^i \circ F$$

3.4 Partitions of Unity

Definition 3.4.0.1. Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then ρ is said to be a **bump function at p supported in U** if

1. $\rho \geq 0$
2. there exists $V \in \mathcal{N}_p$ such that V is open and $\rho|_V = 1$
3. $\text{supp } \rho \subset U$

Exercise 3.4.0.2. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

Proof.

□

3.5 The Tangent Space

Definition 3.5.0.1. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. For $i \in \{1, \dots, n\}$, define the partial derivative with respect to x^i at p , denoted

$$\left. \frac{\partial}{\partial x^i} \right|_p : C^\infty(M) \rightarrow \mathbb{R}, \text{ or } \partial_i|_p : C^\infty(M) \rightarrow \mathbb{R}$$

by

$$\left. \frac{\partial}{\partial x^i} \right|_p f = \frac{\partial f}{\partial x^i}(p)$$

Exercise 3.5.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \dots, n\}$, we have that

$$\left. \frac{\partial}{\partial x^i} \right|_p x^j = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \dots, n\}$. Then

$$\begin{aligned} \left. \frac{\partial}{\partial x^i} \right|_p x^j &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} x^j \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} u^j \circ \phi \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} u^j \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 3.5.0.3. Change of Coordinates:

Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$, $p \in U \cap V$ and $f \in C_p^\infty(M)$. Then for each $i \in \{1, \dots, n\}$,

$$\left. \frac{\partial}{\partial y^i} \right|_p = \sum_{j=1}^n \left. \frac{\partial}{\partial x^j} \right|_p y^j(p) \left. \frac{\partial}{\partial x^i} \right|_p$$

Proof. Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \dots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned} \left. \frac{\partial}{\partial y^i} \right|_p f &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \psi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \phi^{-1} \circ h \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial u^j} \right|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left(\left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} h_j \right) \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} f \circ \phi^{-1} \right) \left(\left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} x^j \circ \psi^{-1} \right) \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial x^j} \right|_p f \right) \left(\left. \frac{\partial}{\partial y^i} \right|_p x^j \right) \end{aligned}$$

□

Definition 3.5.0.4. Let $p \in M$ and $v : C_p^\infty(M) \rightarrow \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C_p^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation at p** if for each $f, g \in C_p^\infty(M)$ and $a \in \mathbb{R}$,

1. v is linear
2. v is Leibnizian

We define the **tangent space of M at p** , denoted T_pM , by

$$T_pM = \{v : C_p^\infty(M) \rightarrow \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 3.5.0.5. Let $f \in C_p^\infty(M)$ and $v \in T_pM$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. \square

Exercise 3.5.0.6. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

is a basis for T_pM and $\dim T_pM = n$.

Proof. Clearly $\frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \in T_pM$. Let $a_1, \dots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p x^j \\ &= a_j \end{aligned}$$

Hence $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is independent.

Now, let $v \in T_pM$ and $f \in C_p^\infty(M)$. By Taylor's theorem, there exist $g_1, \dots, g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \dots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x^i} \Big|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p f \\ &= \left[\sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p \right] f \end{aligned}$$

So

$$v = \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p$$

and

$$v \in \text{span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

□

Definition 3.5.0.7. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. We define the **differential of F at p** , denoted $dF_p : T_p M \rightarrow T_{F(p)} N$, by

$$\left[dF_p(v) \right] (f) = v(f \circ F)$$

for $v \in T_p M$ and $f \in C_{F(p)}^\infty(N)$.

Exercise 3.5.0.8. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. Then for each $v \in T_p M$, $dF_p(v)$ is a derivation.

Proof. Let $v \in T_p M$, $f, g \in C_{F(p)}^\infty(N)$ and $c \in \mathbb{R}$. Then

1.

$$\begin{aligned} dF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= dF_p(v)(f) + cdF_p(v)(g) \end{aligned}$$

So $dF_p(v)$ is linear.

2.

$$\begin{aligned} dF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= dF_p(v)(f) * g(F(p)) + f(F(p)) * dF_p(v)(g) \end{aligned}$$

So $dF_p(v)$ is Leibnizian and hence $dF_p(v) \in T_{F(p)} N$

□

Exercise 3.5.0.9. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. If F is a diffeomorphism, then dF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \dots, x^n)$ and $\phi \circ F^{-1} = (y^1, \dots, y^n)$. Let $f \in C_{F(p)}^\infty(N)$. Then

$$\begin{aligned} \frac{\partial}{\partial y^i} \Big|_{F(p)} f &= \frac{\partial}{\partial u^i} \Big|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \frac{\partial}{\partial x^i} \Big|_p f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[dF_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) \right] (f) &= \frac{\partial}{\partial x^i} \Big|_p f \circ F \\ &= \frac{\partial}{\partial y^i} \Big|_{F(p)} f \end{aligned}$$

Hence

$$dF_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) = \frac{\partial}{\partial y^i} \Big|_{F(p)}$$

Since $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is a basis for $T_p M$ and $\left\{ \frac{\partial}{\partial y^1} \Big|_{F(p)}, \dots, \frac{\partial}{\partial y^n} \Big|_{F(p)} \right\}$ is a basis for $T_{F(p)} N$, dF_p is an isomorphism. \square

Exercise 3.5.0.10. Let (M, \mathcal{A}) be a smooth m -dimensional manifold, (N, \mathcal{B}) a n -dimensional smooth manifold, $F : M \rightarrow N$ smooth, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^m)$ and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. Suppose that $p \in U$ and $F(p) \in V$. Define the ordered bases $B_\phi = \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^m} \Big|_p \right\}$ and $B_\psi = \left\{ \frac{\partial}{\partial y^1} \Big|_{F(p)}, \dots, \frac{\partial}{\partial y^n} \Big|_{F(p)} \right\}$. Then the matrix representation of dF_p with respect to the bases B_ϕ and B_ψ is

$$dF_p^{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Proof. Let $(dF_p)_{B_\phi, B_\psi} = (a_{i,j})_{i,j} \in \mathbb{R}^{n \times m}$. Then for each $j \in \{1, \dots, m\}$,

$$dF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i} \Big|_{F(p)}$$

This implies that

$$\begin{aligned} dF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) (y^k) &= \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i} \Big|_{F(p)} (y^k) \\ &= \sum_{i=1}^n a_{i,j} \delta_{i,k} \\ &= a_{k,j} \end{aligned}$$

By definition,

$$\begin{aligned} dF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) (y^k) &= \frac{\partial}{\partial x^j} \Big|_p y^k \circ F \\ &= \frac{\partial}{\partial x^j} \Big|_p F^k \\ &= \frac{\partial F^k}{\partial x^j}(p) \end{aligned}$$

\square

Note 3.5.0.11. Since $\text{rank } dF_p$ is independent of basis, it is independent of coordinate charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$.

Definition 3.5.0.12. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ a diffeomorphism. Define the **push forward of F** , denoted

$$F_* : M \rightarrow \coprod_{p \in M} \text{Iso}(T_p M, T_{F(p)} N)$$

by

$$p \mapsto dF_p$$

3.6 The Cotangent Space

Definition 3.6.0.1. Let $p \in M$. We define the **cotangent space of M at p** , denoted T_p^*M , by

$$T_p^*M = (T_pM)^*$$

Definition 3.6.0.2. Let $f \in C^\infty(M)$. We define the **differential of f at p** , denoted $df_p : T_pM \rightarrow \mathbb{R}$, by

$$df_p(v) = vf$$

Exercise 3.6.0.3. Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

Proof. Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1f + \lambda v_2f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that df_p is linear and hence $df_p \in T_p^*M$. □

Exercise 3.6.0.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \dots, n\}$,

$$dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \delta_{i,j}$$

In particular, $\{dx_p^1, \dots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ and $T_p^*M = \text{span}\{dx_p^1, \dots, dx_p^n\}$.

Proof. Let $i, j \in \{1, \dots, n\}$. Then by definition,

$$\begin{aligned} \left[dx_p^i \left(\frac{\partial}{\partial x^i} \Big|_p \right) \right]_p &= \frac{\partial}{\partial x^i} \Big|_p x^i \\ &= \delta_{i,i} \end{aligned}$$

□

Exercise 3.6.0.5. Let $f \in C^\infty(M)$, (U, ϕ) a chart on M with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

Proof. Since $\{dx_p^1, \dots, dx_p^n\}$ is a basis for T_p^*M , for each there exist $a_1(p), \dots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left(\frac{\partial}{\partial x^i} \Big|_p \right) \\ &= a_i(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) &= \frac{\partial f}{\partial x^i} \Big|_p \\ &= \frac{\partial f}{\partial x^i}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

□

Chapter 4

Submersions and Immersions

4.1 Maps of Constant Rank

Definition 4.1.0.1. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map. We define the **rank map of F** , denoted $\text{rank } F : M \rightarrow \mathbb{N}_0$ by

$$\text{rank}_p F = \dim \text{Im } DF(p)$$

and F is said to have **constant rank** if for each $p, q \in M$, $\text{rank}_p F = \text{rank}_q F$. If F has constant rank, we define the **rank of F** , denoted $\text{rank } F$, by $\text{rank } F = \text{rank}_p F$ for $p \in M$.

Exercise 4.1.0.2. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$ and $p \in M$. Suppose that $\text{rank}_p F = k$. Then there exist $(U, \phi) \in \mathcal{A}_M$, $(V, \psi) \in \mathcal{A}_N$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi, \psi})_{i,j} = A_{i,j}$$

Proof. Define $q \in V$ by $q = F(p)$. Choose $(U', \phi') \in \mathcal{A}$ and $(V', \psi') \in \mathcal{B}$ such that $p \in U'$ and $q \in V'$. Set $Z = [DF(p)]_{\phi', \psi'}$. By assumption, $\text{rank } Z = k$. An exercise in the subsection on linear algebra implies that there exist $\sigma \in S_m$, $\tau \in S_n$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Define $\phi : U \rightarrow \sigma\phi(U)$ and $\psi : V \rightarrow \tau\psi(V)$ by

$$\phi = \sigma\phi', \quad \psi = \tau\psi'$$

A previous exercise implies that

$$[DF(p)]_{\phi, \psi} = P_\tau Z P_\sigma^*$$

□

Exercise 4.1.0.3. Constant Rank Theorem:

Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$. Suppose that F has constant rank and $\text{rank } F = k$. Then for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$ and

$$\psi \circ F \circ \phi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

Proof. Let $p \in M$. The previous exercise implies that there exist $(U_0, \phi_0) \in \mathcal{A}$, $(V_0, \psi_0) \in \mathcal{B}$ and $L \in GL(k, \mathbb{R})$ such that $p \in U$, $F(p) \in V_0$ and for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi_0, \psi_0})_{i,j} = L_{i,j}$$

Define $\hat{M} \subset \mathbb{R}^m$, $\hat{N} \subset \mathbb{R}^n$ and $\hat{F} : \hat{M} \rightarrow \hat{N}$ by $\hat{M} = \phi_0(U_0)$, $\hat{N} = \psi_0(V_0)$ and $\hat{F} = \psi_0 \circ F \circ \phi_0^{-1}$. Set $\hat{p} = \phi_0(p)$. Let (x, y) be the standard coordinates on \mathbb{R}^m , with $\pi_x : \mathbb{R}^m \rightarrow \mathbb{R}^k$ and $\pi_y : \mathbb{R}^m \rightarrow \mathbb{R}^{m-k}$ the standard projection maps. Write $\hat{p} = (x_0, y_0)$. There exist $Q : \hat{M} \rightarrow \mathbb{R}^k$ and $R : \hat{M} \rightarrow \mathbb{R}^{n-k}$ such that $\hat{F} = (Q, R)$. By construction, $[D_x Q(x_0, y_0)] = A$. Define $G : \hat{M} \rightarrow \mathbb{R}^m$ by $G(x, y) = (Q(x, y), y)$. Then

$$\begin{aligned} [DG(x_0, y_0)] &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_x Q(x_0, y_0)] \\ [D_x \pi_y(x_0, y_0)] & [D_y \pi_y(x_0, y_0)] \end{pmatrix} \\ &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} L & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \det([DG(x_0, y_0)]) &= \det(L) \det(I) \\ &= \det(L) \\ &\neq 0 \end{aligned}$$

The inverse function theorem implies that there exist $\hat{U} \subset \hat{M}$ such that \hat{U} is open, $\hat{p} \in \hat{U}$ and $G|_{\hat{U}} : \hat{U} \rightarrow G(\hat{U})$ is a diffeomorphism. Since

$$\{U_1 \times U_2 : U_1 \subset \mathbb{R}^k, U_2 \subset \mathbb{R}^{m-k} \text{ and } U_1, U_2 \text{ are open}\}$$

is a basis for the topology on \mathbb{R}^m , there exist $\hat{U}_1 \subset \mathbb{R}^k$ and $\hat{U}_2 \subset \mathbb{R}^{m-k}$ such that \hat{U}_1, \hat{U}_2 are open, $\hat{p} \in \hat{U}_1 \times \hat{U}_2$ and $\hat{U}_1 \times \hat{U}_2 \subset \hat{U}$. Set $\hat{U}_{12} = \hat{U}_1 \times \hat{U}_2$. Since $G(\hat{U}_1 \times \hat{U}_2) = Q(\hat{U}_{12}) \times \hat{U}_2$, we have that $G|_{\hat{U}_{12}} : \hat{U}_{12} \rightarrow Q(\hat{U}_{12}) \times \hat{U}_2$ is a diffeomorphism. Since π_x is open, $Q(\hat{U}_{12})$ is open. There exist $A : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_1$ and $B : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_2$ such that $G^{-1} = (A, B)$. Define $\tilde{R} : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \mathbb{R}^{n-k}$ by $\tilde{R}(x, y) = R(A(x, y), y)$. Let $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$. Then

$$\begin{aligned} (x, y) &= G \circ G^{-1}(x, y) \\ &= G(A(x, y), B(x, y)) \\ &= (Q(A(x, y), B(x, y)), B(x, y)) \end{aligned}$$

This implies that $B(x, y) = y$,

$$\begin{aligned} x &= Q(A(x, y), B(x, y)) \\ &= Q(A(x, y), y) \end{aligned}$$

Hand

$$\begin{aligned} G^{-1}(x, y) &= (A(x, y), B(x, y)) \\ &= (A(x, y), y) \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{F} \circ G^{-1}(x, y) &= \hat{F}(A(x, y), y) \\ &= (Q(A(x, y), y), R(A(x, y), y)) \\ &= (x, R(A(x, y), y)) \\ &= (x, \tilde{R}(x, y)) \end{aligned}$$

We note that

$$\begin{aligned} [D(\hat{F} \circ G^{-1})(x, y)] &= \begin{pmatrix} [D_x \pi_x(x, y)] & [D_y \pi_x(x, y)] \\ [D_x \tilde{R}(x, y)] & [D_y \tilde{R}(x, y)] \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ [D_x \tilde{R}(x, y)] & [D_y \tilde{R}(x, y)] \end{pmatrix} \end{aligned}$$

Since $G^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_{12}$ is a diffeomorphism, we have that $[DG^{-1}(x, y)] \in GL(m, \mathbb{R})$. Since \hat{F} has constant rank and $\text{rank } \hat{F} = k$, we have that

$$\begin{aligned} \text{rank}[D(\hat{F} \circ G^{-1})(x, y)] &= \text{rank}([D\hat{F}(G^{-1}(x, y))][DG^{-1}(x, y)]) \\ &= \text{rank}[D\hat{F}(G^{-1}(x, y))] \\ &= k \end{aligned}$$

Since $\text{rank} \begin{pmatrix} I \\ [D_x \tilde{R}(x, y)] \end{pmatrix} = k$, we have that $\text{rank} \begin{pmatrix} 0 \\ [D_y \tilde{R}(x, y)] \end{pmatrix} = 0$. Thus $[D_y \tilde{R}(x, y)] = 0$. Since $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$ is arbitrary, for each $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\tilde{R}(x, y) = \tilde{R}(x, y_0)$$

Define $\tilde{S} : Q(\hat{U}_{12}) \rightarrow \mathbb{R}^{n-k}$ by $\tilde{S}(x) = \tilde{R}(x, y_0)$. Then for each $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\hat{F} \circ G^{-1}(x, y) = (x, \tilde{S}(x))$$

Let (a, b) be the standard coordinates on \mathbb{R}^n , with $\pi_a : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $\pi_b : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ the standard projection maps. Write $\hat{F}(\hat{p}) = (a_0, b_0)$. Set

$$\begin{aligned} \hat{V} &= [(\pi_a)|_{\hat{N}}]^{-1}(Q(\hat{U}_{12})) \\ &= \pi_a^{-1}(Q(\hat{U}_{12})) \cap \hat{N} \end{aligned}$$

Since $Q(\hat{U}_{12})$ is open, \hat{N} is open and π_a is continuous, we have that \hat{V} is open. Since

$$\begin{aligned} Q(\hat{U}_{12}) &= (\pi_a)|_{\hat{N}} \circ \hat{F} \circ G^{-1}(Q(\hat{U}_{12}) \times \hat{U}_2) \\ &= (\pi_a)|_{\hat{N}} \circ \hat{F}(\hat{U}_{12}) \end{aligned}$$

we have that $\hat{F}(\hat{U}_{12}) \subset \hat{V}$. In particular, $\hat{F}(\hat{p}) \in \hat{V}$. Define $H : \hat{V} \rightarrow \mathbb{R}^n$ by $H(a, b) = (a, b - \tilde{S}(a))$. Then $H \circ \hat{F} \circ G^{-1}(x, y) = (x, 0)$. Define $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{N}$ by $U = \phi_0^{-1}(\hat{U}_{12})$, $V = \psi_0^{-1}(\hat{V})$, $\phi = G \circ \phi_0$ and $\psi = H \circ \psi_0$. Then for each $(x, y) \in \phi(U)$,

$$\begin{aligned} \psi \circ F \circ \phi^{-1}(x, y) &= H \circ \psi_0 \circ F \circ \phi_0^{-1} \circ G^{-1}(x, y) \\ &= H \circ \hat{F} \circ G^{-1}(x, y) \\ &= (x, 0) \end{aligned}$$

□

Definition 4.1.0.4. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map. Then F is said to be

- an **immersion** if for each $p \in M$, $DF(p) : T_p M \rightarrow T_{F(p)} N$ is injective
- a **submersion** if for each $p \in M$, $DF(p) : T_p M \rightarrow T_{F(p)} N$ is surjective

Exercise 4.1.0.5. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map.

Definition 4.1.0.6. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds and $F : M \rightarrow N$ smooth. Then F is said to be an **embedding** if

1. F is an immersion
2. $F : M \rightarrow F(M)$.

Note 4.1.0.7. Here the topology on $F(M)$ is the subspace topology.

4.2 Submanifolds

Exercise 4.2.0.1. Let (M, \mathcal{A}) be a smooth manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $\tilde{U} \subset S$ and $\tilde{\phi} : \tilde{U} \rightarrow \phi(\tilde{U})$ by $\tilde{U} = U \cap S$ and $\tilde{\phi} = \phi|_{U \cap S}$. Set $\mathcal{B} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \in \mathcal{A}\}$. Then \mathcal{B} is a smooth structure on S .

Proof.

□

Definition 4.2.0.2. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds. Suppose that $M \subset N$. Then (M, \mathcal{A}) is said to be

1. an **immersed submanifold** of (N, \mathcal{B}) if $\text{id} : M \rightarrow N$ is a smooth immersion
2. an **embedded submanifold** of (N, \mathcal{B}) if $\text{id} : M \rightarrow N$ is a smooth embedding

Note 4.2.0.3. Essentially, embedded submanifolds are immersed submanifolds with the subspace topology.

Note 4.2.0.4. For the remainder of this section, we assume that $k \leq n$.

Definition 4.2.0.5. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Then S is said to be a **k -slice** of U if $S = \{u \in U : u^{k+1}, \dots, u^n = 0\}$.

Exercise 4.2.0.6. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Suppose that S is a k -slice of U . Define $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then $\pi|_S \rightarrow \pi(S)$ is a diffeomorphism.

Proof. Clear. □

Definition 4.2.0.7. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ and $S \subset U$. Then S is said to be a **k -slice** of U if $\phi(S)$ is a k -slice of $\phi(U)$.

Definition 4.2.0.8. Let (M, \mathcal{A}) be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$. Then (U, ϕ) is said to be a **k -slice chart for S** if $U \cap S$ is a k -slice of U .

Exercise 4.2.0.9. Let (M, \mathcal{A}) be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. If (U, ϕ) is a k -slice chart for S , then $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$.

Proof. Clear. □

Definition 4.2.0.10. Let (M, \mathcal{A}) be a smooth manifold and $S \subset M$. Then S is said to satisfy the **local k -slice condition** if for each $p \in S$, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k -slice chart of S .

Exercise 4.2.0.11. Let (M, \mathcal{A}) be a n -dimensional smooth manifold and $S \subset M$ a subspace. If S satisfies the local k -slice condition, then there exists a smooth structure $\tilde{\mathcal{A}}$ on S such that $(S, \tilde{\mathcal{A}})$ is an embedded submanifold of M .

Proof. Suppose that S satisfies the local k -slice condition. Define $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ as above. Let $(U, \phi) \in \mathcal{A}$. Suppose that (U, ϕ) is a k -slice chart for S . Define $\tilde{U} = U \cap S$ and $\tilde{\phi} : \tilde{U} \rightarrow \pi \circ \phi(\tilde{U})$ by

$$\tilde{\phi} = \pi \circ \phi|_{\tilde{U}}$$

By definition, $\phi(\tilde{U})$ is a k -slice of $\phi(U)$. A previous exercise implies that $\pi|_{\phi(\tilde{U})} \rightarrow \pi \circ \phi(\tilde{U})$ is a diffeomorphism and hence a homeomorphism. Thus $\tilde{\phi}$ is a homeomorphism.

Define

$$\tilde{\mathcal{B}} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \text{ is a } k\text{-slice for } S\}$$

Let $p \in S$. By assumption, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k -slice chart of S . Then $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{B}}$ and $\tilde{\mathcal{A}}$ is an atlas on S . By construction of $\tilde{\mathcal{B}}$, S is locally half Euclidean of dimension k . Since M is second countable Hausdorff, so is S in the subspace topology. Thus $(S, \tilde{\mathcal{B}})$ is a k -dimensional manifold. Let $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$. Then

$$\tilde{\phi} \circ \tilde{\psi}^{-1}|_{\tilde{U} \cap \tilde{V}} = \pi|_{\phi(\tilde{U} \cap \tilde{V})} \circ \phi|_{\tilde{U} \cap \tilde{V}} \circ \psi|_{\tilde{U} \cap \tilde{V}}^{-1} \circ \pi|_{\psi(\tilde{U} \cap \tilde{V})}^{-1}$$

which is a diffeomorphism. So $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ smoothly compatible. Hence $\tilde{\mathcal{B}}$ is smooth. An exercise in section 4.1 implies that there exists a unique smooth structure $\tilde{\mathcal{A}}$ on S such that $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$. So $(S, \tilde{\mathcal{A}})$ is a smooth k -dimensional manifold.

Clearly $\text{id} : S \rightarrow S$ is a homeomorphism. Let $(V, \psi) \in \mathcal{A}$ and $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$.

Finish!! □

Definition 4.2.0.12.

Exercise 4.2.0.13.

Chapter 5

Bundles and Sections

5.1 Introduction

5.1.1 Fiber Bundles

Note 5.1.1.1. Let U, F be sets, we write $\text{proj}_U : U \times F \rightarrow U$ to denote the projection onto U .

Definition 5.1.1.2. Let E, M and F be smooth manifolds and $\pi : E \rightarrow M$ a smooth surjection, $U \subset M$ open and $\Phi : \pi^{-1}(U) \rightarrow U \times F$. Then (U, Φ) is said to be a **smooth local trivialization of E over U with fiber F** if

1. Φ is a diffeomorphism
2. $\text{proj}_U \circ \Phi = \pi|_{\pi^{-1}(U)}$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ & \searrow \pi & \downarrow \text{proj}_U \\ & & U \end{array}$$

Exercise 5.1.1.3. Let E, M and F be sets and $\pi : E \rightarrow M$ and $\Phi : \pi^{-1}(U) \rightarrow U \times F$ a bijection. If $\text{proj}_U \circ \Phi = \pi|_{\pi^{-1}(U)}$, then for each $A \subset U$,

$$\Phi(\pi^{-1}(A)) = A \times F$$

Hint: consider $\Phi^{-1}(A \times F)$

Proof. Let $A \subset U$. Since $\text{proj}_U^{-1}(A) = A \times F$, we have that

$$\begin{aligned} \Phi^{-1}(A \times F) &= \Phi^{-1}(\text{proj}_U^{-1}(A)) \\ &= (\text{proj}_U \circ \Phi)^{-1}(A) \\ &= (\pi|_{\pi^{-1}(U)})^{-1}(A) \\ &= \pi^{-1}(A) \cap \pi^{-1}(U) \\ &= \pi^{-1}(A \cap U) \\ &= \pi^{-1}(A) \end{aligned}$$

Since Φ is a bijection, we have that

$$\begin{aligned} \Phi(\pi^{-1}(A)) &= \Phi \circ \Phi^{-1}(A \times F) \\ &= A \times F \end{aligned}$$

□

Definition 5.1.1.4. Let E , M and F be smooth manifolds and $\pi : E \rightarrow M$ a smooth surjection. Then (E, M, π, F) is said to be a **smooth fiber bundle with total space E , base space M , fiber F and projection π** if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \rightarrow U \times F$ such that U is open and (U, Φ) is a smooth local trivialization of E over U with fiber F . For $p \in M$, we define the **fiber over p** , denoted E_p , by $E_p = \pi^{-1}(\{p\})$.

Note 5.1.1.5. When the context is clear, we will suppress the fiber manifold F .

Definition 5.1.1.6. Let (E_1, M_1, π_1) and (E_2, M_2, π_2) be smooth fiber bundles, $\Phi \in \text{Hom}_{\mathbf{Man}^\infty}(E_1, E_2)$ and $\phi \in \text{Hom}_{\mathbf{Man}^\infty}(M_1, M_2)$. Then (Φ, ϕ) is said to be a **smooth bundle morphism** from (E_1, M_1, π_1) to (E_2, M_2, π_2) if $\pi_2 \circ \Phi = \phi \circ \pi_1$, i.e. the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{\phi} & M_2 \end{array}$$

Definition 5.1.1.7. We define the category of smooth fiber bundles, denoted \mathbf{Bun}^∞ , by

- $\text{Obj}(\mathbf{Bun}^\infty) = \{(E, M, \pi) : (E, M, \pi) \text{ is a smooth fiber bundle}\}$
- For $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{Bun}^\infty)$,

$$\begin{aligned} \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \\ \{(\Phi, \phi) : (\Phi, \phi) \text{ is a smooth bundle morphism from } (E, M, \pi_E) \text{ to } (F, N, \pi_F)\} \end{aligned}$$

- For
 - $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2), (E_3, M_3, \pi_3) \in \text{Obj}(\mathbf{Bun}^\infty)$
 - $(\Phi_{12}, \phi_{12}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2))$
 - $(\Phi_{23}, \phi_{23}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_2, M_2, \pi_2), (E_3, M_3, \pi_3))$

we define $(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1), (E_3, M_3, \pi_3))$ by

$$(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) = (\Phi_{23} \circ \Phi_{12}, \phi_{23} \circ \phi_{12})$$

Exercise 5.1.1.8. We have that \mathbf{Bun}^∞ is a full subcategory of $(\text{id}_{\mathbf{Man}^\infty} \downarrow \text{id}_{\mathbf{Man}^\infty})$.

Proof. Set $\mathcal{C} = (\text{id}_{\mathbf{Man}^\infty} \downarrow \text{id}_{\mathbf{Man}^\infty})$. We note that

- $\text{Obj}(\mathbf{Bun}^\infty) \subset \text{Obj}(\mathcal{C})$
- for each $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{Bun}^\infty)$,

$$\text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \text{Hom}_{\mathcal{C}}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2))$$

So \mathbf{Bun}^∞ is a full subcategory of \mathcal{C} . □

Exercise 5.1.1.9.

5.1.2 Vector Bundles

Note 5.1.2.1. Let M be a set and $p \in M$. We endow $\{p\} \times \mathbb{R}^n$ with the natural vector space structure such that $\{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$.

Definition 5.1.2.2. Let E and M be smooth manifolds and $\pi : E \rightarrow M$ a smooth surjection. Then (E, M, π) is said to be a **smooth vector bundle of rank n** if

1. for each $p \in M$, $\pi^{-1}(\{p\})$ is a n -dimensional real vector space
2. for each $p \in M$, there exist open $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ such that (U, Φ) is a smooth local trivialization of E over U
3. for each $p \in M$,

$$\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \rightarrow \{p\} \times \mathbb{R}^n$$

is a vector space isomorphism

Exercise 5.1.2.3. Let M be a n -dimensional smooth manifold. Set $E = M \times \mathbb{R}^n$ and define $\pi : E \rightarrow M$ by $\pi(p, x) = p$. Then (E, M, π) is a smooth vector bundle of rank n .

Proof.

1. For each $p \in M$, $\pi^{-1}(\{p\}) = \{p\} \times \mathbb{R}^n$ is an n -dimensional real vector space.
2. Let $p \in M$. Set $U = M$. Then $\pi^{-1}(U) = E$. Define $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ by $\Phi = \text{id}_E$. Then (U, Φ) is a smooth local trivialization of E over U .
3. Let $p \in M$. Then $\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \rightarrow \{p\} \times \mathbb{R}^n$ is clearly an isomorphism.

□

Theorem 5.1.2.4. Let E and M be smooth manifolds and $\pi : E \rightarrow M$ a smooth surjection.

5.2 Bundle Morphisms

Definition 5.2.0.1. Let (E, M, π_E) and (F, N, π_F) be smooth fiber bundles and $\Phi : E \rightarrow F$ and $\phi : M \rightarrow N$. Then (Φ, ϕ) is said to be a **smooth fiber bundle morphism** from (E, M, π_E) to (F, N, π_F) if Φ is smooth, ϕ is smooth and $\pi_F \circ \Phi = \phi \circ \pi_E$, i.e. the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ \pi_E \downarrow & & \downarrow \pi_F \\ M & \xrightarrow{\phi} & N \end{array}$$

and we write $(\Phi, \phi) : (E, M, \pi_E) \rightarrow (F, N, \pi_F)$.

Exercise 5.2.0.2. Let (E, M, π_E) and (F, N, π_F) be smooth fiber bundles and $(\Phi, \phi) : (E, M, \pi_E) \rightarrow (F, N, \pi_F)$. Suppose that (Φ, ϕ) is smooth. Then for each $p \in M$,

$$\Phi^{-1}(F_{\phi(p)}) = E_p$$

Proof. Let $p \in M$. Set $q = \phi(p)$. Then

$$\begin{aligned} \Phi^{-1}(F_q) &= \Phi^{-1}(\pi_F^{-1}(\{q\})) \\ &= (\pi_F \circ \Phi)^{-1}(\{q\}) \\ &= (\phi \circ \pi_E)^{-1}(\{q\}) \\ &= \pi_E^{-1}(\phi^{-1}(\{\phi(p)\})) \end{aligned}$$

FINISH!!!, multiple fibers get mapped to same fiber

□

5.3 Subbundles

5.4 Vertical and Horizontal Subbundles

Definition 5.4.0.1. Let $(E, M, \pi_M) \in \text{Obj}(\mathbf{Bun}^\infty)$. We define the **vertical bundle associated to** (E, M, π_M) , denoted $(VE, M, \pi_V) \in \mathbf{Bun}^\infty$, by

$$VE = \coprod_{q \in E} \ker D\pi(q)$$

relocate this to after tangent bundle is introduced

Exercise 5.4.0.2. Let (M, \mathcal{A}) be an n -dimensional smooth manifold and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $(\pi^{-1}(U), \Phi_\phi) \in \mathcal{A}_{TM}$ the induced chart on TM with $\Phi_\phi = (\tilde{x}^1, \dots, \tilde{x}^n, \tilde{y}^1, \dots, \tilde{y}^n)$. Then

$$V(TM)|_{\pi^{-1}(U)} = \coprod_{(p, \xi) \in \pi^{-1}(U)} \text{span} \left\{ \frac{\partial}{\partial \tilde{y}^j} \Big|_{(p, \xi)} : j \in \{1, \dots, n\} \right\}$$

Split into smaller exercises

Proof. Let $f \in C^\infty(M)$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$ the standard coordinates on $\mathbb{R}^n \times \mathbb{R}^n$. We note that by definition, $\Phi_\phi(p, \xi) = (\phi(p), \psi(\xi))$ where $\psi : \bigcup_{p \in U} T_p M \rightarrow \mathbb{R}^n$ is given by

$$\psi \left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j} \Big|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i} \Big|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i} \Big|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} [x^k \circ \pi] &= \frac{\partial}{\partial v^i} \Big|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i} \Big|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i} \Big|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= 0 \end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
 D\pi(p, \xi) \left(\frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} \right) (f) &= \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} f \circ \pi \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i} (p, \xi) \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) \delta_{i,k} \\
 &= \frac{\partial f}{\partial x^i} (p)
 \end{aligned}$$

and

$$\begin{aligned}
 D\pi(p, \xi) \left(\frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} \right) (f) &= \frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} f \circ \pi \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i} (p, \xi) \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) 0 \\
 &= 0
 \end{aligned}$$

Hence

$$\begin{aligned}
 V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p, \xi) \in \pi^{-1}(U)} \ker D\pi(p, \xi) \\
 &= \coprod_{(p, \xi) \in \pi^{-1}(U)} \text{span} \left\{ \frac{\partial}{\partial \tilde{y}^j} \Big|_{(p, \xi)} : j \in \{1, \dots, n\} \right\}
 \end{aligned}$$

□

5.5 The Tangent Bundle

Definition 5.5.0.1. We define the **tangent bundle of M** , denoted TM , by

$$TM = \coprod_{p \in M} T_p M$$

We denote the natural projection map by $\pi : TM \rightarrow M$.

Definition 5.5.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Define $\tilde{U} \subset TM$ and $\tilde{\phi} : \tilde{U} \rightarrow \phi(U) \times \mathbb{R}^n$ by

- $\tilde{U} = \pi^{-1}(U)$
-

$$\begin{aligned} \tilde{\phi} \left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p \right) &= (\phi(p), v) \\ &= (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \end{aligned}$$

Exercise 5.5.0.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then $\tilde{\phi} : \tilde{U} \rightarrow \phi(U) \times \mathbb{R}^n$ is a bijection.

5.6 The cotangent Bundle

Definition 5.6.0.1. We define the **cotangent bundle of M** , denoted T^*M , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

5.7 The (r, s) -Tensor Bundle

Definition 5.7.0.1. 1. the **cotangent bundle of M** , denoted T^*M , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

2. the **(r, s) -tensor bundle of M** , denoted $T_s^r M$, by

$$T_s^r M = \coprod_{p \in M} T_s^r(T_p M)$$

3. the **k -alternating tensor bundle of M** , denoted $\Lambda^k(M)$, by

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k(T_p M)$$

5.8 Vector Fields

Definition 5.8.0.1. Let $X : M \rightarrow TM$. Then X is said to be a **vector field on M** if for each $p \in M$, $X_p \in T_p M$.

For $f \in C^\infty(M)$, we define $Xf : M \rightarrow \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in C^\infty(M)$, Xf is smooth.

We denote the set of smooth vector fields on M by $\Gamma^1(M)$.

Definition 5.8.0.2. Let $f \in C^\infty(M)$ and $X, Y \in \Gamma^1(M)$. We define

- $fX \in \Gamma^1(M)$ by

$$(fX)_p = f(p)X_p$$

- $X + Y \in \Gamma^1(M)$ by

$$(X + Y)_p = X_p + Y_p$$

Exercise 5.8.0.3. The set $\Gamma^1(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Exercise 5.8.0.4. Let $X \in \Gamma^1(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then

$$X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$$

Proof. Let $p \in M$. Then $X_p \in T_p M$ and $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is a basis of $T_p M$. So there exist $f_1(p), \dots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i} \Big|_p$. Let $j \in \{1, \dots, n\}$. Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i} x^j(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx^j = f_j$ and $X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$. □

Exercise 5.8.0.5. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then for each $i \in \{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

Proof. Let $i \in \{1, \dots, n\}$ and $f \in C^\infty(M)$. Define $g : M \rightarrow \mathbb{R}$ by $g = \frac{\partial}{\partial x^i} f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i} \Big|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i} [f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence g is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x^i}$ is smooth. □

5.9 1-Forms

Definition 5.9.0.1. Let $\omega : M \rightarrow T^*M$. Then ω is said to be a **1-form on M** if for each $p \in M$, $\omega_p \in T_p^*M$. For each $X \in \Gamma^1(M)$, we define $\omega(X) : M \rightarrow \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)$, $\omega(X)$ is smooth. The set of smooth 1-forms on M is denoted $\Gamma_1(M)$.

Definition 5.9.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma_1(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by

$$(f\omega)_p = f(p)\omega_p$$

- $\alpha + \beta \in \Gamma_1(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 5.9.0.3. The set $\Gamma_1(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Exercise 5.9.0.4.

5.10 (r, s) -Tensor Fields

Definition 5.10.0.1. Let $\alpha : M \rightarrow T_s^r M$. Then α is said to be an (r, s) -**tensor field on** M if for each $p \in M$, $\alpha_p \in T_p^r(T_p M)$.

For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \rightarrow \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and α is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth. The set of smooth (r, s) -tensor fields on M is denoted $T_s^r(M)$.

Definition 5.10.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T_s^r(M)$. We define

- $f\alpha : M \rightarrow T_s^r M$ by

$$(f\alpha)_p = f(p)\alpha_p$$

- $\alpha + \beta : M \rightarrow T_s^r M$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 5.10.0.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T_s^r(M)$. Then

1. $f\alpha \in T_s^r(M)$ by

$$(f\alpha)_p = f(p)\alpha_p$$

2. $\alpha + \beta \in T_s^r(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Proof. Clear. □

Exercise 5.10.0.4. The set $T_s^r(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Definition 5.10.0.5. Let $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$ and $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$. We define the **tensor product of** α **with** β , denoted $\alpha \otimes \beta : M \rightarrow T_{s_1+s_2}^{r_1+r_2} M$, by

$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

Exercise 5.10.0.6. Let $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$ and $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma_{s_1+s_2}^{r_1+r_2}(M)$

Proof. Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since α_1 and α_2 are smooth by assumption. □

Definition 5.10.0.7. We define the **tensor product**, denoted $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

Exercise 5.10.0.8. The tensor product $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ is associative.

Proof. Clear. □

Exercise 5.10.0.9. The tensor product $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ is $C^\infty(M)$ -bilinear.

Proof. Clear. □

Definition 5.10.0.10. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ a smooth map and $\alpha \in \Gamma_k^0(N)$. We define the **pullback of α by F** , denoted $F^*\alpha \in \Gamma_k^0(M)$, by

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))$$

for $p \in M$ and $v_1, \dots, v_k \in T_p M$

Exercise 5.10.0.11. Let (M, \mathcal{A}) , (N, \mathcal{B}) and (L, \mathcal{C}) be smooth manifolds, $F : M \rightarrow N$ and $G : N \rightarrow L$ smooth maps, $\alpha \in \Gamma_k^0(N)$, $\beta \in \Gamma_l^0(N)$, $\gamma \in \Gamma_k^0(L)$ and $f \in C^\infty(N)$. Then

1. $F^*(f\alpha) = (f \circ F)F^*\alpha$
2. $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
3. $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
4. $(G \circ F)^*\gamma = F^*(G^*\gamma)$
5. $id_N^*\alpha = \alpha$

Proof.

1.

$$\begin{aligned} [F^*(f\alpha)]_p(v_1, \dots, v_k) &= (f\alpha)_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)) \\ &= f(F(p))\alpha_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)) \\ &= (f \circ F)(p)(F^*\alpha)_p(v_1, \dots, v_k) \end{aligned}$$

So that $F^*(f\alpha) = (f \circ F)F^*\alpha$

2.

$$F^*$$

□

Definition 5.10.0.12.

Exercise 5.10.0.13.

Proof.

□

Exercise 5.10.0.14. Let $\alpha \in T_s^r(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then there exist $(f_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

Proof. Let $p \in M$. Then $\omega_p \in T_s^r(T_p M)$ and $\left\{ \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J} \right\}$ is a basis of $T_s^r(T_p M)$. So there exist $(f_J^I(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned} \alpha_p(dx_p^K, \partial_{x^L}|_p) &= \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}(dx_p^K, \partial_{x^L}|_p) \\ &= \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p(dx_p^K) dx_p^{\otimes J}(\partial_{x^L}|_p) \\ &= f_L^K(p) \end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx_p^K, \partial_{x^L}|_p)$ is smooth, so that $f_L^K \in C^\infty(U)$.

□

Definition 5.10.0.15.

5.11 Differential Forms

Definition 5.11.0.1. We define

$$\Lambda^k(TM) = \coprod_{p \in M} \Lambda^k(T_p M)$$

Definition 5.11.0.2. Let $\omega : M \rightarrow \Lambda^k(TM)$. Then ω is said to be a **k -form on M** if for each $p \in M$, $\omega_p \in \Lambda^k(T_p M)$.

For each $X \in \Gamma^1(M)^k$, we define $\omega(X) : M \rightarrow \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)^k$, $\omega(X)$ is smooth. The set of smooth k -forms on M is denoted $\Omega^k(M)$.

Note 5.11.0.3. Observe that

1. $\Omega^k(M) \subset \Gamma_k^0(M)$
2. $\Omega^0(M) = C^\infty(M)$

Exercise 5.11.0.4. The set $\Omega^k(M)$ is a $C^\infty(M)$ -submodule of $\Gamma_k^0(M)$.

Proof. Clear. □

Definition 5.11.0.5. Define the **exterior product**

$$\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Note 5.11.0.6. For $f \in \Omega^0(M)$ and $\alpha \in \Omega^k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 5.11.0.7. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$ is well defined.

Proof. Let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{j=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then

$$\begin{aligned} \alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{(k+l)!}{k!l!} \text{Alt}(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) (\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \end{aligned}$$

□

Exercise 5.11.0.8. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$ is $C^\infty(M)$ -bilinear.

Proof.

1. $C^\infty(M)$ -linearity in the first argument:

Let $\alpha \in \Omega^k(M)$, $\beta, \gamma \in \Omega^l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda^k(T_p M) \times \Lambda^l(T_p M) \rightarrow \Lambda^{k+l}(T_p M)$ implies that

$$\begin{aligned} [(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p \end{aligned}$$

So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

and $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$ is $C^\infty(M)$ -linear in the first argument.

2. $C^\infty(M)$ -linearity in the second argument:

Similar to (1).

□

Note 5.11.0.9. All of the results from multilinear algebra apply here.

Definition 5.11.0.10. We define the **exterior derivative** $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ inductively by

1. $d(d\alpha) = 0$ for $\alpha \in \Omega^p(M)$
2. $df(X) = Xf$ for $f \in \Omega^0(M)$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$
4. extending linearly

Exercise 5.11.0.11. Let (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then on U , for each $i, j \in \{1, \dots, n\}$,

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx_p^1, \dots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ and $T_p^* M = \text{span}\{dx_p^1, \dots, dx_p^n\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \dots, n\}$. Then by definition,

$$\begin{aligned} \left[dx^i \left(\frac{\partial}{\partial x^j} \right) \right]_p &= \left(\frac{\partial}{\partial x^j} x^i \right)_p \\ &= \frac{\partial}{\partial x^j} x^i \Big|_p \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 5.11.0.12. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Proof. Let $p \in U$. Since $\{dx^1, \dots, dx^n\}$ is a basis for $\Lambda(T_p M)$, for each there exist $a_1(p), \dots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a^i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) &= \sum_{i=1}^n a^i(p) dx_p^i \left(\frac{\partial}{\partial x^i} \Big|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) &= \frac{\partial}{\partial x^i} \Big|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx_p^i$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

□

Exercise 5.11.0.13. Let $f \in \Omega^0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \dots, x_n)$. Then for each $i \in \{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i} \Big|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx_p^i \\ &= 0 \end{aligned}$$

□

Exercise 5.11.0.14.

Definition 5.11.0.15. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $I = (i_1, \dots, i_k) \in \mathcal{I}_k$. We define

$$dx^i = dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Omega^k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right)$$

Note 5.11.0.16. We have that

1.

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{I,J}$$

2. Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega^k(U)$,

$$\omega\left(\frac{\partial}{\partial x^i}\right) \in C^\infty(U)$$

Exercise 5.11.0.17. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega\left(\frac{\partial}{\partial x^i}\right) dx^i$$

Proof. Let $p \in U$. Since $\{dx_p^i : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(T_p M)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p) dx_p^i$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned} \omega\left(\frac{\partial}{\partial x^j}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j}\right) \\ &= f_J \end{aligned}$$

□

Exercise 5.11.0.18. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. If $\omega = \sum_{I \in \mathcal{I}_k} f_I dx^i$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

.

Proof. First we note that

$$\begin{aligned} d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f_I d(dx^i) \\ &= df_I \wedge dx^i \\ &= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \right) \wedge dx^i \\ &= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i \end{aligned}$$

Then we extend linearly.

□

Definition 5.11.0.19. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$ be a diffeomorphism. Define the **pullback of F** , denoted $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$ by

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))$$

for $\omega \in \Omega^k(N)$, $p \in M$ and $v_1, \dots, v_k \in T_p M$

Chapter 6

Connections

6.1 Koszul Connections

Definition 6.1.0.1. Let (E, M, π) be a smooth vector bundle and $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$. Then ∇ is said to be a **Koszul connection** in E if

1. for each $Y \in \Gamma(E)$, $\nabla(\cdot, Y)$ is $C^\infty(M)$ -linear
2. for each $X \in \mathfrak{X}(M)$, $\nabla(X, \cdot)$ is \mathbb{R} -linear
3. for each $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\nabla(X, fY) = f \nabla(X, Y) + X(f)Y$$

Note 6.1.0.2. When the context is clear, we will write $\nabla_X Y$ in place of $\nabla(X, Y)$ and we will refer to ∇ as a connection.

Exercise 6.1.0.3. Let (E, M, π) be a smooth vector bundle, ∇ a connection in E , $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$. If $X = 0$ or $Y = 0$, then $\nabla_X Y = 0$.

Proof.

- If $X = 0$, then

$$\begin{aligned} \nabla_X Y &= \nabla_{0X} Y \\ &= 0 \nabla_X Y \\ &= 0 \end{aligned}$$

- Similarly, if $Y = 0$, then $\nabla_X Y = 0$.

□

Exercise 6.1.0.4. Let (E, M, π) be a smooth vector bundle, ∇ a connection in E , $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$ and $p \in M$. If $X \sim_p 0$ or $Y \sim_p 0$, then $[\nabla_X Y]_p = 0$.

Proof.

- Suppose that $X \sim_p 0$. Then there exists $U \subset M$ such that U is open and $X|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\text{supp } \phi \subset U$ and $\phi \sim_p 1$. Then $\phi X = 0$. The previous exercise implies that $\nabla_{\phi X} Y = 0$. Therefore

$$\begin{aligned} \nabla_X Y &= \nabla_{\phi X + (1-\phi)X} Y \\ &= \nabla_{\phi X} Y + \nabla_{(1-\phi)X} Y \\ &= 0 + (1-\phi) \nabla_X Y \\ &= (1-\phi) \nabla_X Y \end{aligned}$$

Hence

$$\begin{aligned} [\nabla_X Y]_p &= [(1 - \phi) \nabla_X Y]_p \\ &= (1 - \phi(p))[\nabla_X Y]_p \\ &= 0 \end{aligned}$$

- Suppose that $Y \sim_p 0$. Then there exists $U \subset M$ such that U is open and $Y|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\text{supp } \phi \subset U$ and $\phi \sim_p 1$. Then $\phi Y = 0$. The previous exercise implies that $\nabla_X \phi Y = 0$. Since $\phi \sim_p 1$, we have that $1 - \phi \sim_p 0$. Thus $X(1 - \phi) \sim_p 0$ and

$$\begin{aligned} \nabla_X Y &= \nabla_X [\phi Y + (1 - \phi)Y] \\ &= \nabla_X [\phi Y] + \nabla_X [(1 - \phi)Y] \\ &= \nabla_X [(1 - \phi)Y] \\ &= (1 - \phi) \nabla_X Y + [X(1 - \phi)] \nabla_X Y \end{aligned}$$

Hence

$$\begin{aligned} [\nabla_X Y]_p &= (1 - \phi(p))[\nabla_X Y]_p + [X(1 - \phi)](p)[\nabla_X Y]_p \\ &= 0 \end{aligned}$$

□

Exercise 6.1.0.5. Let (E, M, π) be a smooth vector bundle and ∇ a connection in E . Then for each $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$, $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$ implies that $[\nabla_{X_1} Y_1]_p = [\nabla_{X_2} Y_2]_p$.

Proof. Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$. Suppose that $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$. Define $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$ by $X = X_2 - X_1$ and $Y = Y_2 - Y_1$. Then $X \sim_p 0$ and $Y \sim_p 0$. The previous exercise implies that $[\nabla_X Y_1]_p = 0$ and $[\nabla_{X_2} Y]_p = 0$. Therefore

$$\begin{aligned} [\nabla_{X_1} Y_1]_p &= [\nabla_{X_1} Y_1]_p + [\nabla_X Y_1]_p \\ &= [\nabla_{X_1} Y_1 + \nabla_X Y_1]_p \\ &= [\nabla_{X_1 + X} Y_1]_p \\ &= [\nabla_{X_2} Y_1]_p \\ &= [\nabla_{X_2} Y_1]_p + [\nabla_{X_2} Y]_p \\ &= [\nabla_{X_2} Y_1 + \nabla_{X_2} Y]_p \\ &= [\nabla_{X_2} (Y_1 + Y)]_p \\ &= [\nabla_{X_2} Y_2]_p \end{aligned}$$

□

Exercise 6.1.0.6. Let (E, M, π) be a smooth vector bundle, ∇ a connection in E and $U \subset M$. If U is open, then there exists a unique connection $\nabla^U : \mathfrak{X}(U) \times \Gamma(E|_U) \rightarrow \Gamma(E|_U)$ such that for each $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$,

$$\nabla_{X|_U}^U Y|_U = (\nabla_X Y)|_U$$

Chapter 7

Riemannian Geometry

Definition 7.0.0.1. Let M be a manifold and $g \in T_2^0(M)$. Then g is said to be a **metric tensor** on M if

1. g is nondegenerate
2. g is symmetric
3. g has constant index

(M, g)

Definition 7.0.0.2. Let M be a smooth manifold and $g \in T_2^0(M)$ a metric tensor on M . We define $\hat{g} \in T_0^2(M)$ by $\hat{g}(\omega, \eta) = g(\phi_g^{-1}(\omega), \phi_g^{-1}(\eta))$.

Exercise 7.0.0.3. content...

Exercise 7.0.0.4. Let (M, g) be a semi-Riemannian manifold and $(U, \phi) \in \mathcal{A}$. Then the induced metric $\langle \rangle_{T^*M \otimes TM}$ on $T^*M \otimes TM$ is given by

$$\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} = g^{i,j} g_{kl}$$

Proof. We have that

$$\begin{aligned} \left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} &= \langle dx^i, dx^j \rangle_{T^*M} \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle_{TM} \\ &= g^{i,j} g_{k,l} \end{aligned}$$

□

Exercise 7.0.0.5. Let (M, g) be an n -dimensional Riemannian manifold.

1. There exists $\lambda \in \Omega^n(M)$ such that for each orthonormal frame e_1, \dots, e_n ,

$$\lambda(e_1, \dots, e_n) = 1$$

Hint: Choose a frame z_1, \dots, z_n on M with corresponding dual frame ζ^1, \dots, ζ^n . Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n$$

2. Let $N \in \mathfrak{X}(M)$ be the outward pointing normal to ∂M and $X \in \mathfrak{X}(M)$. Then

$$\int_M \operatorname{div} X \lambda = \int_{\partial M} g(X, N) \tilde{\lambda}$$

3. For each $u \in \mathbb{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, we have that

$$\operatorname{div}(uX) = u\operatorname{div}(X) + du(X)$$

and therefore

$$\int_M du(X)\lambda = \int_{\partial M} ug(X, N)\tilde{\lambda} - \int_M u\operatorname{div}(X)\lambda$$

Proof.

1. Let z_1, \dots, z_n be a frame on M and ζ^1, \dots, ζ^n with corresponding dual frame ζ^1, \dots, ζ^n . Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n$$

Let e_1, \dots, e_n , be an orthonormal frame on M with corresponding dual coframe $\epsilon^1, \dots, \epsilon^n$. Let $i, j \in \{1, \dots, n\}$. Then there exist $(a_{k,i}) \subset \mathbb{R}$ such that $\zeta^i = \sum_{k=1}^n a_{k,i} \epsilon^k$. Then

$$\begin{aligned} \hat{g}(\epsilon^j, \zeta^i) &= \sum_{k=1}^n a_{k,i} \hat{g}(\epsilon^j, \epsilon^k) \\ &= \sum_{k=1}^n a_{k,i} g(\phi_g^{-1}(\epsilon^j), \phi_g^{-1}(\epsilon^k)) \\ &= \sum_{k=1}^n a_{k,i} g(e_j, e_k) \\ &= \sum_{k=1}^n a_{k,i} \delta_{j,k} \\ &= a_{j,i} \end{aligned}$$

which implies that

$$\begin{aligned} \delta_{i,j} &= \zeta^i(z_j) \\ &= \sum_{k=1}^n a_{k,i} \epsilon^k(z_j) \\ &= \sum_{k=1}^n a_{k,i} g(e_k, z_j) \\ &= \sum_{k=1}^n \hat{g}(\epsilon^k, \zeta^i) g(e_k, z_j) \end{aligned}$$

Define $U, V \in \mathbb{R}^{n \times n}$ by $U_{i,k} = \hat{g}(\zeta^i, \epsilon^k)$ and $V_{k,j} = g(e_k, z_j)$. Then from above, we have that $UV = I$. Since $U, V \in \mathbb{R}^{n \times n}$, $VU = I$. Hence $U = V^{-1}$. Since

$$\begin{aligned} \zeta^i(e_j) &= \sum_{k=1}^n a_{k,i} \epsilon^k(e_j) \\ &= \sum_{k=1}^n a_{k,i} \delta_{k,j} \\ &= a_{j,i} \\ &= \hat{g}(\epsilon^j, \zeta^i) \\ &= U_{i,j} \end{aligned}$$

and

$$\begin{aligned}
g(z_i, z_j) &= \left(\sum_{k=1}^n g(e_k, z_i) e_k, \sum_{l=1}^n g(e_l, z_j) e_l \right) \\
&= \sum_{k=1}^n \sum_{l=1}^n g(e_k, z_i) g(e_l, z_j) g(e_k, e_l) \\
&= \sum_{k=1}^n \sum_{l=1}^n g(e_k, z_i) g(e_l, z_j) \delta_{k,l} \\
&= \sum_{k=1}^n g(e_k, z_i) g(e_k, z_j) \\
&= (V^* V)_{i,j}
\end{aligned}$$

we have that

$$\begin{aligned}
\lambda(e_1, \dots, e_n) &= \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n(e_1, \dots, e_n) \\
&= \det[g(z_i, z_j)]^{1/2} \det[\zeta^i(e_j)] \\
&= \det(V^* V)^{1/2} \det U \\
&= \det V (\det V)^{-1} \\
&= 1
\end{aligned}$$

2. Choose an orthonormal frame $e_1, \dots, e_{n-1} \in \mathfrak{X}(\partial M)$ with dual coframe $\epsilon^1, \dots, \epsilon^{n-1}$. Define $\nu \in \Omega^1(M)$ to be the dual covector to N . We note that N, e_1, \dots, e_{n-1} is an orthonormal frame on $\mathfrak{X}(M)$. Let $X_1, \dots, X_{n-1} \in \mathfrak{X}(\partial M)$. Since for each $j \in \{1, \dots, n-1\}$, $X_j \in \mathfrak{X}(\partial M)$ and for each $p \in \partial M$, $N_p \in (T_p \partial M)^\perp$, we have that for each $j \in \{1, \dots, n-1\}$, $g(X_j, N) = 0$. This implies that

$$\begin{aligned}
\iota^* \iota_X \lambda(X_1, \dots, X_{n-1}) &= \lambda(X, X_1, \dots, X_{n-1}) \\
&= \nu \wedge \epsilon^1 \wedge \dots \wedge \epsilon^{n-1}(X, X_1, \dots, X_{n-1}) \\
&= \det \begin{pmatrix} \nu(X) & \nu(X_1) & \dots & \nu(X_{n-1}) \\ \epsilon^1(X) & \epsilon^1(X_1) & \dots & \epsilon^1(X_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \dots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= \det \begin{pmatrix} g(X, N) & g(X_1, N) & \dots & g(X_{n-1}, N) \\ \epsilon^1(X) & \epsilon^1(X_1) & \dots & \epsilon^1(X_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \dots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= g(X, N) \det(\epsilon^i(X_j)) \\
&= g(X, N) \epsilon^1 \wedge \dots \wedge \epsilon^{n-1}(X_1, \dots, X_{n-1}) \\
&= g(X, N) \tilde{\lambda}(X_1, \dots, X_{n-1})
\end{aligned}$$

Therefore $\iota^* \iota_X \lambda = g(X, N) \tilde{\lambda}$ and

$$\begin{aligned}
\int_M \operatorname{div} X \lambda &= \int_M d(\iota_X \lambda) \\
&= \int_{\partial M} \iota^* (\iota_X \lambda) \\
&= \int_{\partial M} g(X, N) \tilde{\lambda}
\end{aligned}$$

3. We note that

$$\begin{aligned}
 0 &= \iota_X(du \wedge \lambda) \\
 &= \iota_X(du) \wedge \lambda - du \wedge (\iota_X \lambda) \\
 &= du(X)\lambda - du \wedge (\iota_X \lambda)
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \operatorname{div}(uX)\lambda &= d(\iota_{uX}\lambda) \\
 &= d(u\iota_X\lambda) \\
 &= du \wedge (\iota_X\lambda) + u d(\iota_X\lambda) \\
 &= du(X)\lambda + u \operatorname{div}(X)\lambda \\
 &= [du(X) + u \operatorname{div}(X)]\lambda
 \end{aligned}$$

This implies that $\operatorname{div}(uX) = du(X) + u \operatorname{div}(X)$. From before, we have that

$$\begin{aligned}
 \int_M du(X)\lambda &= \int_M \operatorname{div}(uX)\lambda - \int_M u \operatorname{div}(X)\lambda \\
 &= \int_{\partial M} g(uX, N) \tilde{\lambda} - \int_M u \operatorname{div}(X)\lambda \\
 &= \int_{\partial M} u g(X, N) \tilde{\lambda} - \int_M u \operatorname{div}(X)\lambda
 \end{aligned}$$

□

Exercise 7.0.0.6.

$$\operatorname{div}(X) = \sum_{j=1}^n (\nabla_{\partial_j} X)^j$$

Proof. We have that

$$\begin{aligned}
 \nabla_{\partial_i}(X) &= \sum_{j=1}^n \nabla_{\partial_i}(X^j \partial_j) \\
 &= \sum_{j=1}^n \left[X^j \nabla_{\partial_i} \partial_j + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n \left[X^j \left(\sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n \left[X^j \left(\sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n X^j \left(\sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \sum_{j=1}^n \partial_i(X^j) \partial_j \\
 &= \sum_{k=1}^n \left(\sum_{j=1}^n X^j \Gamma_{i,j}^k \right) \partial_k + \sum_{k=1}^n \partial_i(X^k) \partial_k \\
 &= \sum_{k=1}^n \left[\left(\sum_{j=1}^n X^j \Gamma_{i,j}^k \right) + \partial_i(X^k) \right] \partial_k
 \end{aligned}$$

so that $(\nabla_{\partial_i}(X))^i = \left(\sum_{j=1}^n X^j \Gamma_{i,j}^i \right) + \partial_i(X^i)$. We note that

$$\begin{aligned} \operatorname{div}(X) &= \sum_{i=1}^n \operatorname{div}(X^i \partial_i) \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + dx^i(\partial_i)] \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + 1] \end{aligned}$$

Since $\lambda = [\det g(\partial_i, \partial_j)]^{1/2} dx^1 \wedge \cdots \wedge dx^n = (\det g)^{1/2} dx$, we have that

$$\begin{aligned} d(\iota_{\partial_i} \lambda) &= d((\det g)^{1/2} \iota_{\partial_i} dx) \\ &= d[(\det g)^{1/2}] \iota_{\partial_i} dx + (\det g)^{1/2} d(\iota_{\partial_i} dx) \\ &= d[(\det g)^{1/2}] \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n + (\det g)^{1/2} \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n \end{aligned}$$

FINISH!!! □

Exercise 7.0.0.7. Let (M, g) be a Riemannian manifold.

1. For each $u, v \in C^\infty(M)$. Then

(a)

$$\int_M u \Delta v \lambda + \int_M g(\nabla u, \nabla v) \lambda = \int_{\partial M} u N(v) \tilde{\lambda}$$

(b)

$$\int_M [u \Delta v - v \Delta u] \lambda = \int_{\partial M} [u N(v) - v N(u)] \tilde{\lambda}$$

2. (a) If $\partial M \neq \emptyset$, then for each $u, v \in C^\infty(M)$, u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$ implies that $u = v$.

(b) If $\partial M = \emptyset$, then for each $u \in C^\infty(M)$, u is harmonic implies that u is constant.

Proof.

1. Let $u, v \in C^\infty(M)$. Then

(a)

$$\begin{aligned} \int_M u \Delta v \lambda &= \int_M u \operatorname{div}(\nabla v) \lambda \\ &= \int_{\partial M} u g(\nabla v, N) \tilde{\lambda} - \int_M du(\nabla v) \lambda \\ &= \int_{\partial M} u dv(N) \tilde{\lambda} - \int_M g(\nabla u, \nabla v) \lambda \\ &= \int_{\partial M} u N(v) \tilde{\lambda} - \int_M g(\nabla u, \nabla v) \lambda \end{aligned}$$

(b) From above, we have that

$$\begin{aligned}
 \int_M [u\Delta v - v\Delta u]\lambda &= \int_M u\Delta v\lambda - \int_M v\Delta u\lambda \\
 &= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda - \left(\int_{\partial M} vN(u)\tilde{\lambda} - \int_M g(\nabla v, \nabla u)\lambda \right) \\
 &= \int_{\partial M} uN(v)\tilde{\lambda} - \int_{\partial M} vN(u)\tilde{\lambda} \\
 &= \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}
 \end{aligned}$$

2. (a) Suppose that $\partial M \neq \emptyset$. Let $u, v \in C^\infty(M)$. Suppose that u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$. Then $u - v$ is harmonic and

$$\begin{aligned}
 \int_M \|\nabla(u - v)\|_g^2 \lambda &= \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
 &= 0 + \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
 &= \int_M (u - v)\Delta(u - v)\lambda + \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
 &= \int_{\partial M} (u - v)N(u - v)\tilde{\lambda} \\
 &= 0
 \end{aligned}$$

Thus $\nabla(u - v) = 0$ and $u - v$ is constant. Since $u|_{\partial M} = v|_{\partial M}$, we have that $u - v = 0$ and thus $u = v$.

- (b) Suppose that $\partial M = \emptyset$. Let $u \in C^\infty(M)$. Suppose that u is harmonic. Then

$$\begin{aligned}
 \int_M \|\nabla u\|_g^2 \lambda &= \int_M g(\nabla u, \nabla u)\lambda \\
 &= 0 + \int_M g(\nabla u, \nabla u)\lambda \\
 &= \int_M u\Delta u\lambda + \int_M g(\nabla u, \nabla u)\lambda \\
 &= \int_{\partial M} (u - v)g(\nabla(u - v), N)\tilde{\lambda} \\
 &= 0
 \end{aligned}$$

Therefore $\nabla u = 0$ and u is constant.

□

Chapter 8

Symplectic Geometry

8.1 Symplectic Manifolds

Definition 8.1.0.1. Let $M \in \mathbf{Man}^\infty$ and $\omega \in \Omega^2(M)$. Then ω is said to be **symplectic** if

1. ω is nondegenerate
2. ω is closed

Chapter 9

Extra

Definition 9.0.0.1. When working in \mathbb{R}^n , we introduce the formal objects dx^1, dx_2, \dots, dx^n . Let $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. We formally define $dx^i = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$.

Definition 9.0.0.2. Let $k \in \{0, 1, \dots, n\}$. We define a $C^\infty(\mathbb{R}^n)$ -module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx^i : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \rightarrow \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

1. the exterior product is bilinear
2. for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$
3. for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$
4. for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k -forms on \mathbb{R}^n . Let ω be a k -form on \mathbb{R}^n . If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$

Note 9.0.0.3. The terms dx^1, dx_2, \dots, dx^n are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \dots, du_k\}$ when referencing the coordinates on \mathbb{R}^k and $\{dx^1, dx_2, \dots, dx^n\}$ when referencing the coordinates on \mathbb{R}^n .

Exercise 9.0.0.4. Let $B_{n \times n} = (b_{i,j}) \in [C^\infty(M)]^{n \times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \dots \wedge dx^n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned}
 \bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) &= \left(\sum_{j=1}^n b_{1,j} dx^j \right) \wedge \left(\sum_{j=1}^n b_{2,j} dx^j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n b_{n,j} dx^j \right) \\
 &= \sum_{j_1, \dots, j_n=1}^n \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\
 &= \sum_{j_1 \neq \dots \neq j_n} \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\
 &= \left[\sum_{\sigma \in S_n} \text{sgn}(\sigma) \left(\prod_{i=1}^n b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\
 &= (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n
 \end{aligned}$$

□

Definition 9.0.0.5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a 0-form on \mathbb{R}^n . We define a 1-form, denoted df , on \mathbb{R}^n by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Let $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$ be a k -form on \mathbb{R}^n . We can define a differential $k+1$ -form, denoted $d\omega$, on \mathbb{R}^n by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx^I$$

Exercise 9.0.0.6. On \mathbb{R}^3 , put

1. $\omega_0 = f_0$,
2. $\omega_1 = f_1 dx^1 + f_2 dx^2 + f_3 dx^3$,
3. $\omega_2 = f_1 dx^2 \wedge dx^3 - f_2 dx^1 \wedge dx^3 + f_3 dx^1 \wedge dx^2$

Show that

1. $d\omega_0 = \frac{\partial f_0}{\partial x^1} dx^1 + \frac{\partial f_0}{\partial x^2} dx^2 + \frac{\partial f_0}{\partial x^3} dx^3$
2. $d\omega_1 = \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) dx^2 \wedge dx^3 + \left(\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3} \right) dx^1 \wedge dx^3 + \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) dx^1 \wedge dx^2$
3. $d\omega_2 = \left(\frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3$

Proof. Straightforward. □

Exercise 9.0.0.7. Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx^I \wedge dx_{I_*} = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$.

Definition 9.0.0.8. We define a linear map $*$: $\Phi_k(\mathbb{R}^n) \rightarrow \Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge *-operator** by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 9.0.0.9. Let $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. We define $\phi^* : \Phi_k(\mathbb{R}^n) \rightarrow \Phi_k(\mathbb{R}^k)$ via the following properties:

1. for each 0-form f on \mathbb{R}^n , $\phi^* f = f \circ \phi$

2. for $i = 1, \dots, n$, $\phi^* dx^i = d\phi_i$
3. for an s -form ω , and a t -form χ on \mathbb{R}^n , $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
4. for l -forms ω, χ on \mathbb{R}^n , $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 9.0.0.10. Let $M \subset \mathbb{R}^n$ be a k -dimensional smooth submanifold of \mathbb{R}^n , $\phi : U \rightarrow V$ a smooth parametrization of M , $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$ a k -form on \mathbb{R}^n . Then

$$\phi^*\omega = \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \dots \wedge du_k$$

Proof. By definition,

$$\begin{aligned} \phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx^I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned} d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \dots \wedge d\phi_{i_n} \\ &= \left(\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u^j} du^j \right) \wedge \left(\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u^j} du^j \right) \wedge \dots \wedge \left(\sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u^j} du^j \right) \\ &= (\det v\phi_I) du^1 \wedge du_2 \wedge \dots \wedge du_k \end{aligned}$$

Therefore

$$\begin{aligned} \phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) du^1 \wedge du_2 \wedge \dots \wedge du_k \\ &= \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \dots \wedge du_k \end{aligned}$$

□

9.1 Integration of Differential Forms

Definition 9.1.0.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \dots \wedge dx_k$ a k -form on \mathbb{R}^k . Define

$$\int_U \omega = \int_U f dx$$

Definition 9.1.0.2. Let $M \subset \mathbb{R}^n$ be a k -dimensional oriented smooth submanifold of \mathbb{R}^n , ω a k -form on \mathbb{R}^n and $\phi : U \rightarrow V$ a local smooth, orientation-preserving parametrization of M . Define

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 9.1.0.3.

Theorem 9.1.0.4. Stokes Theorem:

Let $M \subset \mathbb{R}^n$ be a k -dimensional oriented smooth submanifold of \mathbb{R}^n and ω a $k-1$ -form on \mathbb{R}^n . Then

$$\int_{\partial M} \omega = \int_M d\omega$$

Appendix A

Summation

Appendix B

Asymptotic Notation

Bibliography

- [1] [Introduction to Algebra](#)
- [2] [Introduction to Analysis](#)
- [3] [Introduction to Fourier Analysis](#)
- [4] [Introduction to Measure and Integration](#)