

INTRODUCTION TO CATEGORY THEORY

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PREFACE

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1. CATEGORIES AND FUNCTORS

1.1. Categories.

Definition 1.1.1. Let $\mathcal{C}_0, \mathcal{C}_1$ be classes and $\text{dom}, \text{cod} : \mathcal{C}_1 \rightarrow \mathcal{C}_0$. Set $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \text{dom}, \text{cod})$. Then \mathcal{C} is said to be a **category** if

- (composition): for each $f, g \in \mathcal{C}_1$, if $\text{cod}(f) = \text{dom}(g)$, there exists $g \circ f \in \mathcal{C}_1$ such that $\text{dom}(g \circ f) = \text{dom}(f)$ and $\text{cod}(g \circ f) = \text{cod}(g)$
- (associativity): for each $f, g, h \in \mathcal{C}_1$, if $\text{cod}(f) = \text{dom}(g)$ and $\text{cod}(g) = \text{dom}(h)$, then
$$(h \circ g) \circ f = h \circ (g \circ f)$$
- (identity): for each $X \in \mathcal{C}_0$, there exists $1_X \in \mathcal{C}_1$ such that $\text{dom}(1_X) = \text{cod}(1_X) = X$ and for each $f, g \in \mathcal{C}_1$, if $\text{dom}(f) = X$ and $\text{cod}(g) = X$, then

$$f \circ 1_X = f \text{ and } 1_X \circ g = g$$

We define the

- **objects of \mathcal{C}** , denoted $\text{Obj}(\mathcal{C})$, by $\text{Obj}(\mathcal{C}) = \mathcal{C}_0$
- **morphisms of \mathcal{C}** , denoted $\text{Hom}_{\mathcal{C}}$, by $\text{Hom}_{\mathcal{C}} = \mathcal{C}_1$

For $X, Y \in \text{Obj}(\mathcal{C})$, we define the **morphisms from X to Y** , denoted $\text{Hom}_{\mathcal{C}}(X, Y)$, by $\text{Hom}_{\mathcal{C}}(X, Y) = \{f \in \text{Hom}(\mathcal{C}) : \text{dom}(f) = X \text{ and } \text{cod}(f) = Y\}$.

Note 1.1.2. We typically define a category \mathcal{C} by specifying

- $\text{Obj}(\mathcal{C})$
- for $X, Y \in \text{Obj}(\mathcal{C})$, the class $\text{Hom}_{\mathcal{C}}(X, Y)$
- for $X, Y, Z \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$, the composite morphism $g \circ f \in \text{Hom}_{\mathcal{C}}(X, Z)$.

and then show

- associativity of composition
- existence of identities

Definition 1.1.3. Let \mathcal{C} be a category, we define the dual of \mathcal{C} or the **opposite of \mathcal{C}** , denoted \mathcal{C}^{op} , by

- $\text{Obj}(\mathcal{C}^{\text{op}}) = \text{Obj}(\mathcal{C})$
- for $X, Y \in \text{Obj}(\mathcal{C}^{\text{op}})$, $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$
- for $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$, $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$, $g \circ_{\mathcal{C}^{\text{op}}} f = f \circ_{\mathcal{C}} g$

Exercise 1.1.4. Let \mathcal{C} be a category. Then \mathcal{C}^{op} is a category.

Proof.

- for $W, X, Y, Z \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(W, X)$ and $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$ and $h \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$. Then

$$\begin{aligned} (h \circ_{\mathcal{C}^{\text{op}}} g) \circ_{\mathcal{C}^{\text{op}}} f &= f \circ_{\mathcal{C}} (h \circ_{\mathcal{C}^{\text{op}}} g) \\ &= f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h) \\ &= (f \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} h \\ &= h \circ_{\mathcal{C}^{\text{op}}} (f \circ_{\mathcal{C}} g) \\ &= h \circ_{\mathcal{C}^{\text{op}}} (g \circ_{\mathcal{C}^{\text{op}}} f) \end{aligned}$$

So composition is associative.

- Let $X \in \text{Obj}(\mathcal{C})$ and $f, g \in \text{Hom}_{\mathcal{C}^{\text{op}}}$. Suppose that $\text{dom}(f) = X$ and $\text{cod}(g) = X$. Then

$$\begin{aligned} f \circ_{\mathcal{C}^{\text{op}}} 1_X &= 1_X \circ_{\mathcal{C}} f \\ &= f \end{aligned}$$

and

$$\begin{aligned} 1_X \circ_{\mathcal{C}^{\text{op}}} g &= g \circ_{\mathcal{C}} 1_X \\ &= g \end{aligned}$$

$$\text{So } (1_X)_{\mathcal{C}^{\text{op}}} = (1_X)_{\mathcal{C}}.$$

□

Definition 1.1.5. Let \mathcal{C} be a category and $X \in \text{Obj}(\mathcal{C})$. We define the **slice category of \mathcal{C} over X** , denoted \mathcal{C}/X , by

- $\text{Obj}(\mathcal{C}/X) = \{f \in \text{Hom}_{\mathcal{C}} : \text{cod}(f) = X\}$
- for $f, g \in \text{Obj}(\mathcal{C}/X)$,

$$\text{Hom}_{\mathcal{C}/X}(f, g) = \{\alpha \in \text{Hom}_{\mathcal{C}} : \text{dom}(\alpha) = \text{dom}(f), \text{cod}(\alpha) = \text{dom}(g) \text{ and } f = g \circ \alpha\}$$

i.e. for $f \in \text{Hom}_{\mathcal{C}}(A, X)$ and $g \in \text{Hom}_{\mathcal{C}}(B, X)$, $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$ iff the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

- for $f, g, h \in \text{Obj}(\mathcal{C}/X)$, $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \text{Hom}_{\mathcal{C}/X}(g, h)$, $\beta \circ_{\mathcal{C}/X} \alpha = \beta \circ_{\mathcal{C}} \alpha$

Exercise 1.1.6. Let \mathcal{C} be a category and $X \in \text{Obj}(\mathcal{C})$. Then \mathcal{C}/X is a category.

Proof.

- $f, g, h \in \text{Obj}(\mathcal{C}/X)$, $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \text{Hom}_{\mathcal{C}/X}(g, h)$. Then $f = g \circ_{\mathcal{C}} \alpha$ and $g = h \circ_{\mathcal{C}} \beta$, i.e. the following diagrams commute:

$$\begin{array}{ccc} \text{dom}(f) & \xrightarrow{\alpha} & \text{dom}(g) \\ & \searrow f & \swarrow g \\ & X & \end{array} \qquad \begin{array}{ccc} \text{dom}(g) & \xrightarrow{\beta} & \text{dom}(h) \\ & \searrow g & \swarrow h \\ & X & \end{array}$$

Therefore, we have that

$$\begin{aligned} f &= g \circ_{\mathcal{C}} \alpha \\ &= (h \circ_{\mathcal{C}} \beta) \circ_{\mathcal{C}} \alpha \\ &= h \circ_{\mathcal{C}} (\beta \circ_{\mathcal{C}} \alpha) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} \text{dom}(f) & \xrightarrow{\beta \circ_{\mathcal{C}} \alpha} & \text{dom}(h) \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

which implies that

$$\begin{aligned}\beta \circ_{\mathcal{C}/X} \alpha &= \beta \circ_{\mathcal{C}} \alpha \\ &\in \text{Hom}_{\mathcal{C}/X}(f, h)\end{aligned}$$

and composition is well defined. Then associativity of $\circ_{\mathcal{C}/X}$ follows from associativity of $\circ_{\mathcal{C}}$.

- Let $f \in \text{Obj}(\mathcal{C}/X)$ and $\alpha, \beta \in \text{Hom}_{\mathcal{C}/X}$. Since $f \circ 1_{\text{dom}_{\mathcal{C}}(f)} = f$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \text{dom}(f) & \xrightarrow{1_{\text{dom}(f)}} & \text{dom}(f) \\ & \searrow f \quad \swarrow f & \\ & X & \end{array}$$

we have that $1_{\text{dom}(f)} \in \text{Hom}_{\mathcal{C}/X}(f, f)$. Suppose that $\text{dom}_{\mathcal{C}/X}(\alpha) = f$ and $\text{cod}_{\mathcal{C}/X}(\beta) = f$. Then

$$\begin{aligned}\alpha \circ_{\mathcal{C}/X} 1_{\text{dom}(f)} &= \alpha \circ_{\mathcal{C}} 1_{\text{dom}(f)} \\ &= \alpha\end{aligned}$$

and

$$\begin{aligned}1_{\text{dom}(f)} \circ_{\mathcal{C}/X} \beta &= 1_{\text{dom}(f)} \circ_{\mathcal{C}} \beta \\ &= \beta\end{aligned}$$

So $(1_f)_{\mathcal{C}/X} = (1_{\text{dom}(f)})_{\mathcal{C}}$.

□

1.2. Functors.**Definition 1.2.1.**

1.3. Natural Transformations.