INTRODUCTION TO PROBABILITY

CARSON JAMES

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1. Basic Probability

2. Kolomogorov Extension Theorem

Show that for each x, $\pi_{st}(x) = \pi_s(x)\pi_t(x)$ and this implies that for each feasible (s, A), (t, B) so that $\pi_{st}-1(A \times B) = \pi_s(A) \cap \pi_t(B)$.

For $u \subset \mathbb{C}^{\mathbb{N}}$, set $\pi_u \mathbb{C}^{\mathbb{N}} = \{x \in \mathbb{C}^{\mathbb{N}} : \text{for each } j \in u^c, x_j = 0\}$. Consider lattice of subspaces of $\pi_u \mathbb{C}^{\mathbb{N}}$, ordered by inclusion indexed by finite subsets $u \subset \mathbb{N}$.

show that if $u \subset v$, then $\mathcal{B}(\pi_u \mathbb{C}^{\mathbb{N}}) = \iota_{u,v}^*(\mathcal{B}(\pi_v \mathbb{C}^{\mathbb{N}}))$ where $\iota_{u,v} : \pi_u \mathbb{C}^{\mathbb{N}} \to \pi_v \mathbb{C}^{\mathbb{N}}$ is inclusion. Equip $(\pi_u \mathbb{C}^{\mathbb{N}}, \mathcal{B}(\pi_u \mathbb{C}^{\mathbb{N}}))$ with measure μ_u such that π_u

with pullback sigma algebra $\iota_{u,v}^*(\mathcal{B}(\pi_v\mathbb{C}^{\mathbb{N}}))$ where $u \subset v$ and $\iota_{u,v} : \pi_u\mathbb{C}^{\mathbb{N}} \to \pi_v\mathbb{C}^{\mathbb{N}}$. Equip each object $(\pi_u\mathbb{C}^{\mathbb{N}}, \mathcal{B}(\pi_u\mathbb{C}^{\mathbb{N}}))$ with measures μ_u satisfying μ_u of in stoch has maximum

Definition 2.0.1. Let

3. Probability

3.1. Distributions.

Definition 3.1.1. Let Ω be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then \mathcal{P} is said to be a π -system on Ω if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Definition 3.1.2. Let Ω be a set and $\mathcal{L} \subset \mathcal{P}(\Omega)$. Then \mathcal{L} is said to be a λ -system on Ω if

- (1) $\mathcal{L} \neq \emptyset$
- (2) for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$
- (3) for each $(A_n)_{n\in\mathbb{N}}\subset\mathcal{L}$, if $(A_n)_{n\in\mathbb{N}}$ is disjoint, then $\bigcup_{n\in\mathbb{N}}A_n\in\mathcal{L}$

Exercise 3.1.3. Let Ω be a set and \mathcal{L} a λ -system on Ω . Then

(1) $\Omega, \varnothing \in \mathcal{L}$

Proof. Straightforward.

Definition 3.1.4. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. Put

$$\mathcal{S} = \{ \mathcal{L} \subset \mathcal{P}(\Omega) : \mathcal{L} \text{ is a } \lambda \text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L} \}$$

We define the λ -system on Ω generated by \mathcal{C} , $\lambda(\mathcal{C})$, to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L} \in \mathcal{S}} \mathcal{L}$$

Exercise 3.1.5. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. If \mathcal{C} is a λ -system and \mathcal{C} is a π -system, then \mathcal{C} is a σ -algebra.

Proof. Suppose that \mathcal{C} is a λ -system and \mathcal{C} is a π -system. Then we need only verify the third axiom in the definition of a σ -algebra. Let $(A_n)_{n\in\mathbb{N}}\subset\mathcal{C}$. Define $B_1=A_1$ and for $n\geq 2$, define $B_n=A_n\cap\left(\bigcup_{k=1}^{n-1}A_k\right)^c=A_n\cap\left(\bigcap_{k=1}^{n-1}A_k^c\right)\in\mathcal{C}$. Then $(B_n)_{n\in\mathbb{N}}$ is disjoint and therefore $\bigcup_{n\in\mathbb{N}}A_n=\bigcup_{n\in\mathbb{N}}B_n\in\mathcal{C}$.

Theorem 3.1.6. (Dynkin's Theorem)

Let Ω be a set.

- (1) Let \mathcal{P} be a π -system on Ω and \mathcal{L} a λ -system on Ω . If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
- (2) Let \mathcal{P} be a π -system on Ω . Then $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

Exercise 3.1.7. Let (Ω, \mathcal{F}) be a measurable space and μ, ν probability measures on (Ω, \mathcal{F}) . Put $\mathcal{L}_{\mu,\nu} = \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$. Then $\mathcal{L}_{\mu,\nu}$ is a λ -system on Ω .

Proof.

- $(1) \varnothing \in \mathcal{L}_{\mu,\nu}.$
- (2) Let $A \in \mathcal{L}_{\mu,\nu}$. Then $\mu(A) = \nu(A)$. Thus

$$\mu(A^c) = 1 - \mu(A)$$
$$= 1 - \nu(A)$$
$$= \nu(A^c)$$

So $A^c \in \mathcal{L}_{\mu,\nu}$.

(3) Let $(A_n)_{n\in\mathbb{N}}\subset\mathcal{L}_{\mu,\nu}$. So for each $n\in\mathbb{N}$, $\mu(A_n)=\nu(A_n)$. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$
$$= \sum_{n\in\mathbb{N}} \nu(A_n)$$
$$= \nu\left(\bigcup_{n\in\mathbb{N}} A_n\right)$$

Hence $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}_{\mu,\nu}$.

Exercise 3.1.8. Let (Ω, \mathcal{F}) be a measurable space, μ, ν probability measures on (Ω, \mathcal{F}) and $\mathcal{P} \subset \mathcal{F}$ a π -system on Ω . Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Proof. Using the previous exercise, we see that $\mathcal{P} \subset \mathcal{L}_{\mu,\nu}$. Dynkin's theorem implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu,\nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Definition 3.1.9. Let $F : \mathbb{R} \to \mathbb{R}$. Then F is said to be a **probability distribution** function if

- (1) F is right continuous
- (2) F is increasing
- (3) $F(-\infty) = 0$ and $F(\infty) = 1$

Definition 3.1.10. Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We define $F_P : \mathbb{R} \to \mathbb{R}$, by

$$F_P(x) = P((-\infty, x])$$

We call F_P the probability distribution function of P.

Exercise 3.1.11. Let (Ω, \mathcal{F}, P) be a probability measure. Then F_P is a probability distribution function.

Proof. (1) Let $x \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}} \subset [x, \infty)$. Suppose that $x_n \to x$. Then $(x, x_n] \to \emptyset$ because $\limsup_{n \to \infty} (x, x_n] = \emptyset$. Thus

$$F(x_n) - F(x) = P((x, x_n]) \to P(\emptyset) = 0$$

This implies that

$$F(x_n) \to F(x)$$

So F is right continuous.

- (2) Clearly F_P is increasing.
- (3) Continuity from below tells us that

$$F(-\infty) = \lim_{n \to -\infty} F(n) = \lim_{n \to -\infty} P((-\infty, n]) = 0$$

and continuity from above tell us that

$$F(\infty) = \lim_{n \to \infty} F(n) = \lim_{n \to \infty} P((-\infty, n]) = 1$$

Exercise 3.1.12. Let μ, ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $F_{\mu} = F_{\nu}$ iff $\mu = \nu$.

Proof. Clearly if $\mu = \nu$, then $F_{\mu} = F_{\nu}$. Conversely, suppose that $F_{\mu} = F_{\nu}$. Then for each $x \in \mathbb{R}$,

$$\mu((-\infty, x]) = F_{\mu}(x)$$

$$= F_{\nu}(x)$$

$$= \nu((-\infty, x])$$

Put $C = \{(-\infty, x] : x \in \mathbb{R}\}$. Then C is a π -system and for each $A \in C$, $\mu(A) = \nu(A)$. Hence for each $A \in \sigma(C) = \mathcal{B}(\mathbb{R})$, $\mu(A) = \nu(A)$. So $\mu = \nu$.

Definition 3.1.13. Let (Ω, \mathcal{F}, P) be a probability space and $X : \Omega \to \mathbb{R}^n$. Then X is said to be a **random vector** on (Ω, \mathcal{F}) if X is \mathcal{F} - $\mathcal{B}(\mathbb{R}^n)$ measurable. If n = 1, then X is said to be a **random variable**. We define

$$L_n^0(\Omega, \mathcal{F}, P) = \{X : \Omega \to \mathbb{R}^n : X \text{ is a random vector}\}$$

and

$$L_n^p(\Omega, \mathcal{F}, P) = \left\{ X \in L_n^0 : \int ||X||^p dP < \infty \right\}$$

Definition 3.1.14. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . We define the **probability distribution** of $X, P_X : \mathcal{B}(R) \to [0, 1]$, to be the measure

$$P_X = X_*P$$

That is, for each $A \in \mathcal{B}(\mathbb{R})$,

$$P_X(A) = P(X^{-1}(F))$$

We define the **probability distribution function** of $X, F_X : \mathbb{R} \to [0, 1]$, to be

$$F_X = F_{P_X}$$

Definition 3.1.15. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . If $P_X \ll m$, we define the **probability density** of X, $f_X : \mathbb{R} \to \mathbb{R}$, by

$$f_X = \frac{dP_X}{dm}$$

Exercise 3.1.16. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on (Ω, \mathcal{F}) . Then for each $x \in \mathbb{R}$,

$$\mathbb{P}\bigg(\liminf_{n\to\infty} X_n > x\bigg) \le \liminf_{n\to\infty} P(X_n > x)$$

Proof. Let $\omega \in \left\{ \liminf_{n \to \infty} X_n > x \right\}$. Then $x < \liminf_{n \to \infty} X_n(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \ge n} X_k(\omega) \right)$. So there exists $n^* \in \mathbb{N}$ such that $x < \inf_{k \ge n^*} X_k(\omega)$. Then for each $k \in \mathbb{N}$, $k \ge n^*$ implies that $x < X_k(\omega)$. So there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \ge n^*$ implies that $\mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Hence $\inf_{k \ge n^*} \mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Thus $\liminf_{n \to \infty} \mathbf{1}_{\{X_k > x\}}(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \ge n} \mathbf{1}_{\{X_k > x\}}(\omega) \right) = 1$. Therefore

 $\omega \in \liminf_{n \to \infty} \{X_k > x\}$ and we have shown that

$$\left\{ \liminf_{n \to \infty} X_n > x \right\} \subset \liminf_{n \to \infty} \{X_k > x\}$$

Then

$$P\left(\liminf_{n\to\infty} X_n > x\right) \le P\left(\liminf_{n\to\infty} \{X_k > x\}\right)$$

$$\le \liminf_{n\to\infty} P(\{X_k > x\})$$

Definition 3.1.17. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+(\Omega) \cup L^1$. Define the **expectation of X**, E(X), to be

$$E(X) = \int X \, dP$$

.

3.2. Independence.

Definition 3.2.1. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C} \subset \mathcal{F}$. Then \mathcal{C} is said to be **independent** if for each $(A_i)_{i=1}^n \subset \mathcal{C}$,

$$P\bigg(\bigcap_{k=1}^{n} A_k\bigg) = \prod_{k=1}^{n} P(A_k)$$

Definition 3.2.2. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Then $\mathcal{C}_1, \dots, \mathcal{C}_n$ are said to be **independent** if for each $A_1 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n, A_1, \dots, A_n$ are independent.

Note 3.2.3. We will explicitly say that for each $i = 1, \dots, n$, C_i is independent when talking about the independence of the elements of C_i to avoid ambiguity.

Definition 3.2.4. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_2 random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are said to be **independent** if for each $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent.

Exercise 3.2.5. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff $\sigma(X_1), \dots, \sigma(X_n)$ are independent.

Proof. Suppose that X_1, \dots, X_n are independent. Let $A_1, \in \sigma(X_1), \dots, A_n \in \sigma(A_n)$. Then for each $i = 1, \dots, n$, there exists $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X_i^{-1}(B_i)$. Then A_1, \dots, A_n are independent. Hence $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Conversely, suppose that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$. Then for each $i = 1, \dots, n, X_i^{-1}B_i \in \sigma(X_i)$. Then $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent. Hence X_1, \dots, X_n are independent. \square

Exercise 3.2.6. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{F}$ a collection of σ -algebras on Ω . Suppose that for each $i = 1, \dots, n, X_i$ is \mathcal{F}_i -measurable. If $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent, then X_1, \dots, X_n are independent.

Proof. For each $i=1,\cdots,n,\ \sigma(X_i)\subset\mathcal{F}_i$. So $\sigma(X_1),\cdots,\sigma(X_n)$ are independent. Hence X_1,\cdots,X_n are independent. \square

Exercise 3.2.7. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Suppose that for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent, then $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent.

Proof. Let $A_2 \in \mathcal{C}_2$. Define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap A_2) = P(A)P(A_2)\}$. Then

- (1) $\Omega \in \mathcal{L}$
- (2) If $A \in \mathcal{L}$, then

$$P(A^{c} \cap A_{2}) = P(A_{2}) - P(A_{2} \cap A)$$

$$= P(A_{2}) - P(A_{2})P(A)$$

$$= (1 - P(A))P(A_{2})$$

$$= P(A^{c})P(A_{2})$$

So $A^c \in \mathcal{L}$

(3) If $(B_n)_{n\in\mathbb{N}}\subset\mathcal{L}$ is disjoint, then

$$P\left(\left[\bigcup_{n\in\mathbb{N}}B_n\right]\cap A_2\right) = P\left(\bigcup_{n\in\mathbb{N}}B_n\cap A_2\right)$$

$$= \sum_{n\in\mathbb{N}}P(B_n\cap A_2)$$

$$= \sum_{n\in\mathbb{N}}P(B_n)P(A_2)$$

$$= \left[\sum_{n\in\mathbb{N}}P(B_n)\right]P(A_2)$$

$$= P\left(\bigcup_{n\in\mathbb{N}}A_n\right)P(A_2)$$

So
$$\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{L}$$
.

Thus \mathcal{L} is a λ -system. Since $\mathcal{C}_1 \subset \mathcal{L}$ is a π -system, Dynkin's theorem tells us that $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Since $A_2 \in \mathcal{C}_2$ is arbitrary $\sigma(\mathcal{C}_1)$ and \mathcal{C}_2 are independent. The same reasoning implies that $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent. Let $A_2 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$ We may do the same process with

$$\mathcal{L} = \left\{ A \in \mathcal{F} : P\left(A \cap \left(\bigcap_{i=2}^{n} A_i\right)\right) = P(A) \prod_{i=2}^{n} P(A_i) \right\}$$

and conclude that $\sigma(\mathcal{C}_1), \mathcal{C}_2, \cdots, \mathcal{C}_n$ are independent. Which, using the same reasoning would imply that $\sigma(\mathcal{C}_1), \cdots, \sigma(\mathcal{C}_n)$ are independent.

Exercise 3.2.8. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff for each $x_1, \dots, x_n \in \mathbb{R}$,

$$P(X_1 \le x_1, \dots, X_n \le x_n) = \prod_{i=1}^n P(X_i \le x_i)$$

Proof. Suppose that X_1, \dots, X_n are independent. Then $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $x_1, \dots, x_n \in \mathbb{R}$. Then for each $i = 1, \dots, n$, $\{X_i \leq x_i\} \in \sigma(X_i)$. Hence

 $P(X_1 \le x_1, \dots, X_n \le x_n) = \prod_{i=1}^n P(X_i \le x_i)$. Conversely, suppose that for each

$$x_1, \dots, x_n \in \mathbb{R}, P(X_1 \le x_1, \dots, X_n \le x_n) = \prod_{i=1}^n P(X_i \le x_i). \text{ Define } \mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}.$$

Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$. For each $i = 1, \dots, n$, define $\mathcal{C}_i = X_i^{-1}\mathcal{C}$. Then for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and

$$\sigma(C_i) = \sigma(X^{-1}(C))$$

$$= X_i^{-1}(\sigma(C))$$

$$= X_i^{-1}(\mathcal{B}(\mathbb{R}))$$

$$= \sigma(X_i)$$

By assumption, C_1, \dots, C_n are independent. The previous exercise tells us that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Then X_1, \dots, X_n are independent.

Exercise 3.2.9. Let Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Define $X = (X_1, \dots, X_n)$. If X_1, \dots, X_n are independent, then

$$P_X = \prod_{i=1}^n P_{X_i}$$

.

Proof. Let $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. Then

$$P_X(A_1 \times \dots \times A_n) = P(X \in A_1 \times \dots \times \in A_n)$$

$$= P(X_1 \in A_1, \dots, X_n \in A_n)$$

$$= P(X_1 \in A_1) \dots P(X_n \in A_n)$$

$$= P_{X_1}(A_1) \dots P_{X_n}(A_n)$$

$$= \prod_{i=1}^n P_{X_i}(A_1 \times \dots \times A_n)$$

Put

$$\mathcal{P} = \{A_1 \times \cdots \times A_n : A_1 \in \mathcal{B}(R), \cdots, A_n \in \mathcal{B}(R)\}$$

Then \mathcal{P} is a π -system and

$$\sigma(\mathcal{P}) = \mathcal{B}(R) \otimes \cdots \otimes \mathcal{B}(R) = \mathcal{B}(\mathbb{R}^n)$$

A previous exercise then tells us that $P_X = \prod_{i=1}^n P_{X_i}$

Exercise 3.2.10. Let Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R} \in L^0$. Suppose that $f_1 \circ X_1, \dots, f_n \circ X_n \in L^+(\Omega)$ or $f_1 \circ X_1, \dots, f_n \circ X_n \in L^1(\Omega)$. If X_1, \dots, X_n are independent, then

$$E(f_1(X_1)\cdots f_n(X_n)) = \prod_{i=1}^n E(f_i(X_i))$$

Proof. Define the random vector $X: \Omega \to \mathbb{R}^n$ by $X = (X_1, \dots, X_n)$ and $g: \mathbb{R}^n \to \mathbb{R}$ by $g(x_1, \dots, x_n) = f_1(x_1) \dots f_n(x_n)$. Suppose that for each $i = 1, \dots, n, f_i \in L^+(\mathbb{R})$. Then $g \in L^+(\mathbb{R}^n)$ and by change of variables,

$$E(f_1(X_1)\cdots f_n(X_n)) = E(g(X))$$

$$= \int_{\Omega} g \circ X \, dP$$

$$= \int_{\mathbb{R}^n} g(x) \, dP_X(x)$$

$$= \int_{R^n} g(x) d \prod_{i=1}^n P_{X_i}(x)$$

$$= \prod_{i=1}^n \int_{\mathbb{R}} f_i(x) dP_{X_i}(x)$$

$$= \prod_{i=1}^n \int_{\Omega} f_i \circ X dP$$

$$= \prod_{i=1}^n E(f_i(X_i))$$

If for each $i = 1, \dots, n$, $f_i \in L^1(\mathbb{R}, P_{X_i})$, then following the above reasoning with |g| tells us that $g \in L^1(\mathbb{R}^n, P_X)$ and we use change of variables and Fubini's theorem to get the same result.

3.3. L^p Spaces for Probability.

Note 3.3.1. Recall that for a probability space (Ω, \mathcal{F}, P) and $1 \leq p \leq q \leq \infty$ we have $L^q \subset L^p$ and for each $X \in L^q$, $||X||_p \leq ||X||_q$. Also recall that for $X, Y \in L^2$, we have that $||XY||_1 \leq ||X||_2 ||X||_2$.

Definition 3.3.2. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Define the **variance** of X, Var(X), to be

$$Var(X) = E(\left[\left(X - E(X)\right)^{2}\right)$$

.

Definition 3.3.3. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the

Definition 3.3.4. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the **covariance of** X **and** Y, Cov(X, Y), to be

$$Cov(X,Y) = E([X - E(X)][Y - E(Y)])$$

Exercise 3.3.5. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Then the covariance is well defined and $Cov(X, Y)^2 \leq Var(X)Var(Y)$

Proof. By Holder's inequality,

$$|Cov(X,Y)| = \left| \int (X - E(X))(Y - E(Y)) dP \right|$$

$$\leq \int |(X - E(X))(Y - E(Y))| dP$$

$$= ||(X - E(X))(Y - E(Y))||_1$$

$$\leq ||X - E(X)||_2 ||(Y - E(Y))||_2$$

$$= \left(\int |X - E(X)|^2 dP \right)^{\frac{1}{2}} \left(|Y - E(Y)|^2 \right)^{\frac{1}{2}}$$

$$= Var(X)^{\frac{1}{2}} Var(Y)^{\frac{1}{2}}$$

So $Cov(X,Y)^2 \leq Var(X)Var(Y)$.

Exercise 3.3.6. Let (Ω, \mathcal{F}, P) be a measure space and $X, Y \in L^2$. Then

- (1) Cov(X,Y) = E(XY) E(X)E(Y)
- (2) If X, Y are independent, then Cov(X, Y) = 0
- (3) $Var(X) = E(X^2) E(X)^2$
- (4) for each $a, b \in \mathbb{R}$, $Var(aX + b) = a^2Var(X)$.
- (5) Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)

Proof.

(1) We have that

$$Cov(X,Y) = E \Big[(X - E(X))(Y - E(Y)) \Big]$$

$$= E(XY - E(Y)X - E(X)Y + E(X)E(Y))$$

$$= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y)$$

$$= E(XY) - E(X)E(Y)$$

(2) Suppose that X, Y are independent. Then E(XY) = E(X)E(Y). Hence

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= E(X)E(Y) - E(X)E(Y)$$
$$= 0$$

(3) Part (1) implies that

$$Var(X) = Cov(X, X)$$
$$= E(X^{2}) - E(X)^{2}$$

(4) Let $a, b \in \mathbb{R}$. Then

$$Var(aX + b) = E[(aX + b)^{2}] - E(aX + b)^{2}$$

$$= E[a^{2}X^{2} + 2abX + b^{2}] - (aE(X) + b)^{2}$$

$$= a^{2}E(X^{2}) + 2abE(X) + b^{2} - (a^{2}E(X)^{2} + 2abE(X) + b^{2})$$

$$= a^{2}(E(X^{2}) - E(X)^{2})$$

$$= a^{2}Var(X)$$

(5) We have that

$$\begin{split} Var(X+Y) &= E[(X+Y)^2] - E[X+Y]^2 \\ &= E[X^2 + 2XY + Y^2] - (E(X) + E[Y])^2 \\ &= E(X^2) + 2E[XY] + E[Y^2] - (E(X)^2 + 2E(X)E[Y] + E[Y]^2) \\ &= (E(X^2) - E(X)^2) + (E[Y^2] - E[Y]^2) + 2(E[XY] - E(X)E[Y]) \\ &= Var(X) + Var(Y) + 2Cov(X, Y) \end{split}$$

Definition 3.3.7. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. The **correlation** of **X** and **Y**, Cor(X, Y), is defined to be

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Exercise 3.3.8.

Exercise 3.3.9. Jensen's Inequality:

Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^1$ and $\phi : \mathbb{R} \to \mathbb{R}$. If ϕ is convex, then

$$\phi(E(X)) \le E[\phi(X)]$$

Proof. Put $x_0 = E(X)$. Since ϕ is convex, there exist $a, b \in \mathbb{R}$ such that $\phi(x_0) = ax_0 + b$ and for each $x \in \mathbb{R}$, $\phi(x) \ge ax + b$. Then

$$E[\phi(X)] = \int \phi(X) dP$$

$$\geq \int [aX + b] dP$$

$$= a \int X dP + b$$

$$= aE(X) + b$$

$$= ax_0 + b$$

$$= \phi(x_0)$$

$$= \phi(E(X))$$

Exercise 3.3.10. Markov's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+$. Then for each $a \in (0, \infty)$,

$$P(X \ge a) \le \frac{E(X)}{a}$$

Proof. Let $a \in (0, \infty)$. Then $a\mathbf{1}_{\{X \geq a\}} \leq X\mathbf{1}_{\{X \geq a\}}$. Thus

$$aP(X \ge a) = \int a\mathbf{1}_{\{X \ge a\}} dP$$
$$= \int X\mathbf{1}_{\{X \ge a\}} dP$$
$$\le \int X dP$$
$$= E(X)$$

Therefore

$$P(X \ge a) \le \frac{E(X)}{a}$$

.

Exercise 3.3.11. Chebychev's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a \in (0, \infty)$,

$$P(|X - E(X)| \ge a) \le \frac{Var(X)}{a^2}$$

Proof. Let $a \in (0, \infty)$. Then

$$P(|X - E(X)| \ge a) = P((X - E(X))^2 \ge a^2)$$

$$\le \frac{E[(X - E(X))^2]}{a^2}$$

$$= \frac{Var(X)}{a^2}$$

Exercise 3.3.12. Chernoff's Bound: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a, t \in (0, \infty)$,

$$P(X \ge a) \le e^{-ta} E[e^{tX}]$$

Proof. Let $a, t \in (0, \infty)$. Then

$$P(X \ge a) = P(tX \ge ta)$$
$$= P(e^{tX} \ge e^{ta})$$
$$\le e^{-ta}E[e^{tX}]$$

Exercise 3.3.13. Weak Law of Large Numbers: Let (Ω, \mathcal{F}, P) be a probability space $(X_i)_{i \in \mathbb{N}} \subset L^2$. Suppose that $(X_i)_{i \in \mathbb{N}}$ are iid. Then

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} E[X_1]$$

Proof. Put $\mu = E[X_1]$ and $\sigma^2 = Var(X_1)$. Then

$$E[\frac{1}{n}\sum_{i=1}^{n} X_{i}] = \frac{1}{n}\sum_{i=1}^{n} E[X_{i}]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu$$

$$= \mu$$

and

$$Var(\frac{1}{n}\sum_{i=1}^{n}X_i) = \frac{1}{n^2}Var(\sum_{i=1}^{n}X_i)$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i)$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2$$
$$= \frac{\sigma^2}{n}$$

Let $\epsilon > 0$. Then

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_{i} - E[X_{1}]\right| \geq \epsilon\right) = P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_{i} - \mu\right| \geq \epsilon\right)$$

$$= P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_{i} - E\left(\left[\frac{1}{n}\sum_{i=1}^{n}X_{i}\right]\right| \geq \epsilon\right)\right)$$

$$\leq \frac{Var\left(\frac{1}{n}\sum_{i=1}^{n}X_{i}\right)}{\epsilon^{2}}$$

$$= \frac{\sigma^{2}/n}{\epsilon^{2}}$$

$$= \frac{\sigma^{2}}{n\epsilon^{2}} \to 0$$

So

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} E[X_1]$$

3.4. Borel Cantelli Lemma.

Exercise 3.4.1. Borel Cantelli Lemma:

Let (Ω, \mathcal{F}, P) be a probability space and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{F}$.

- (1) If $\sum_{n\in\mathbb{N}} P(A_n) < \infty$, then $P(\limsup_{n\to\infty} A_n) = 0$. (2) If $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum_{n\in\mathbb{N}} P(A_n) = \infty$, then $P(\limsup_{n\to\infty} A_n) = 1$

Proof.

(1) Suppose that $\sum_{n\in\mathbb{N}} P(A_n) < \infty$. Recall that

$$\limsup_{n \to \infty} A_n = \left\{ \omega \in \Omega : \sum_{n \in \mathbb{N}} 1_{A_n}(\omega) = \infty \right\}$$

Then

$$\infty > \sum_{n \in \mathbb{N}} P(A_n)$$

$$= \sum_{n \in \mathbb{N}} \int 1_{A_n} dP$$

$$= \int \sum_{n \in \mathbb{N}} 1_{A_n} dP$$

Thus $\sum_{n\in\mathbb{N}} 1_{A_n} < \infty$ a.e. and $P(\limsup_{n\to\infty} A_n) = 0$. (2) Suppose that $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum_{n\in\mathbb{N}} P(A_n) = \infty$.

Exercise 3.4.2. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}} \subset L^0$ and $X \in L^0$.

- (1) If for each $\epsilon > 0$, $\sum_{n \in \mathbb{N}} P(|X_n X| \ge \epsilon) < \infty$, then $X_n \to X$ a.e.
- (2) If $(X_n)_{n\in\mathbb{N}}$ are independent and there exists $\epsilon > 0$ such that $\sum_{n\in\mathbb{N}} P(|X_n X| \ge \epsilon) = \infty$, then $X_n \not\to X$ a.e.

Proof.

(1) For $\epsilon > 0$ and $n \in \mathbb{N}$, set $A_n(\epsilon) = \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \ge \epsilon\}$. Suppose that for each $\epsilon > 0$, $\sum_{n \in \mathbb{N}} P(|X_n - X| \ge \epsilon) < \infty$. The Borel-Cantelli lemma implies that for each $m \in \mathbb{N}$,

$$P(\limsup_{n\to\infty} A_n(1/m)) = 0$$

Let $\omega \in \Omega$. Then $X_n(\omega) \not\to X(\omega)$ iff

$$\omega \in \bigcup_{m \in \mathbb{N}} \limsup_{n \to \infty} A_n(1/m)$$

So

$$P(X_n \not\to X) = P\bigg(\bigcup_{m \in \mathbb{N}} \limsup_{n \to \infty} A_n(1/m)\bigg)$$

$$\leq \sum_{m \in \mathbb{N}} P(\limsup_{n \to \infty} A_n(1/m))$$

$$= 0$$

Hence $X_n \to X$ a.e.

(2)

4. Probability on locally compact Groups

Note 4.0.1. In this section, familiarity with Haar measure will be assumed. This section is intended as a continuation of section 7 of [3].

4.1. Action on Probability Measures.

Note 4.1.1. We recall some notation from section 7.1 of [3].

- $l_g \in \text{Homeo}(G), \ l_g(x) = gc$ $L_g \in \text{Sym}(L_0(G)), \ L_g f = f \circ l_g^{-1}$ We continue from section 7

Note 4.1.2. The next exercise generalizes the notion of a scale-family.

Exercise 4.1.3. Let (Ω, \mathcal{F}, P) be a probability space, G a locally compact group, μ a left Haar measure on $G, X \in L_G^0$ and $g \in G$. If $P_X \ll \mu$, then $f_{gX} = L_g f_X$.

Proof. Suppose that $P_X \ll \mu$. Let $A \in \mathcal{B}(G)$. Then

$$P_{gX}(A) = P(gX \in A)$$

$$= P(X \in g^{-1}A)$$

$$= P_X(g^{-1}A)$$

$$= P_X(l_g^{-1}(A))$$

$$= l_{g_*}P_X(A)$$

$$= g \cdot P_X(A)$$

The previous exercise tells us that $f_{gX} = L_g f_X$.

5. Weak Convergence of Measures

6. Concentration Inequalities

6.1. Introduction.

Exercise 6.1.1. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$. Then for each $s, t \in \mathbb{R}$,

$$P(X + Y \ge s + t) \le P(X \ge s) + P(Y \ge t)$$

Proof. For $Z \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ and $t \in \mathbb{R}$, define $A_Z^t \in \mathcal{F}$ by

$$A_Z^t = \{ \omega \in \Omega : Z(\omega) \ge t \}$$

Let $s,t \in \mathbb{R}$. Since $(A_X^s)^c \cap (A_Y^t)^c \subset (A_{X+Y}^{s+t})^c$, we have that $A_{X+Y}^{s+t} \subset A_X^s \cup A_Y^t$. Then

$$\begin{split} P(X+Y \geq s+t) &= P(A_{X+Y}^{s+t}) \\ &\leq P(A_X^s) + P(A_Y^t) \\ &= P(X \geq s) + P(Y \geq t) \end{split}$$

Exercise 6.1.2. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^+(\Omega, \mathcal{F}, P)$. Then for each $s, t \geq 0$,

$$P(XY \ge st) \le P(X \ge s) + P(Y \ge t)$$

Proof. For $Z \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ and $t \in \mathbb{R}$, define define $A_Z^t \in \mathcal{F}$ by

$$A_Z^t = \{\omega \in \Omega : Z(\omega) \ge t\}$$

Let $s,t \in \mathbb{R}$. Since $(A_X^s)^c \cap (A_Y^t)^c \subset (A_{XY}^{st})^c$, we have that $A_{XY}^{st} \subset A_X^s \cup A_Y^t$. Then

$$\begin{split} P(XY \geq st) &= P(A_{XY}^{st}) \\ &\leq P(A_X^s) + P(A_Y^t) \\ &= P(X \geq s) + P(Y \geq t) \end{split}$$

6.2. Sub α -Exponential Random Variables.

Definition 6.2.1. Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then X is said to be **sub** α -exponential if there exist M, K > 0 such that for each $t \geq 0$,

$$P(|X| \ge t) \le Me^{-Kt^{\alpha}}$$

Exercise 6.2.2. Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then the following are equivalent:

- (1) X is sub α -exponential
- (2) there exists K > 0 such that for each $p \ge 1$, $||X||_p \le Kp^{1/\alpha}$
- (3)
- (4)

Proof.

 \bullet (1) \Longrightarrow (2):

Choose $C_{\alpha} > 0$ such that for each $x \geq \alpha^{-1}$, $\Gamma(x) \leq C_{\alpha}x^{x}$. Since X is sub α -exponential, there exist $M, K_{0} > 0$ such that for each $t \geq 0$, $P(|X| \geq t) \leq Me^{-Kt^{\alpha}}$. Set $K = \max(M\alpha^{-1}C_{\alpha}, 1)2K_{0}^{-1/\alpha}\alpha^{-1/\alpha}$. Let $p \geq 1$. Then $p\alpha^{-1} \geq \alpha^{-1}$

$$\begin{split} \|X\|_p^p &= E(|X|^p) \\ &= \int_0^\infty P(|X|^p \ge t) dt \\ &= \int_0^\infty P(|X| \ge t^{1/p}) dt \\ &\le \int_0^\infty M e^{-K_0 t^{\alpha/p}} dt \\ &= M p \alpha^{-1} \int_0^\infty u^{p/\alpha - 1} e^{-K_0 u} du \\ &= M p \alpha^{-1} \Gamma(p/\alpha) K_0^{-p/\alpha} \\ &\le M p \alpha^{-1} C_\alpha (p \alpha^{-1})^{p/\alpha} K_0^{-p/\alpha} \end{split}$$

Therefore

$$||X||_{p} \leq (M\alpha^{-1}C_{\alpha})^{1/p} p^{1/p} K_{0}^{-1/\alpha} \alpha^{-1/\alpha} p^{1/\alpha}$$

$$\leq \max(M\alpha^{-1}C_{\alpha}, 1) 2K_{0}^{-1/\alpha} \alpha^{-1/\alpha} p^{1/\alpha}$$

$$= Kp^{1/\alpha}$$

 \bullet (2) \Longrightarrow (3):

Definition 6.2.3. Let $\psi:[0,\infty)\to[0,\infty)$. Then ψ is said to be an **Orlicz function** if

- (1) ψ is convex
- (2) ψ is increasing
- (3) $\psi(x) \to \infty \text{ as } x \to \infty$
- (4) $\psi(0) = 0$

Definition 6.2.4. Let (Ω, \mathcal{F}, P) be a probability space and $\psi : [0, \infty) \to [0, \infty)$ and Orlicz function. We define the **Orlicz** ψ -norm, denoted $\|\cdot\|_{\psi} : L^0(\Omega, \mathcal{F}, P) \to [0, \infty]$, by

$$||X||_{\psi} = \inf\{t > 0 : E[\psi(|X|/t)] \le 1\}$$

We define the **Orlicz** ψ -space, denoted $L^{\psi}(\Omega, \mathcal{F}, P)$, by

$$L^{\psi}(\Omega, \mathcal{F}, P) = \{ X \in L^{1}(\Omega, \mathcal{F}, P) : ||X||_{\psi} < \infty \}$$

Exercise 6.2.5. Let (Ω, \mathcal{F}, P) be a probability space and $\psi : [0, \infty) \to [0, \infty)$ an Orlicz function. Then L^{ψ} is a vector space and $\|\cdot\|_{\psi} : L^{\psi} \to [0, \infty)$ is a norm. **Hint:** note that

• for s, t > 0,

$$\frac{|X|}{s+t} + \frac{|Y|}{s+t} = \frac{s}{s+t} \frac{|X|}{s} + \frac{t}{s+t} \frac{|Y|}{t}$$

• ψ is star-shaped, i.e. for each $x, t \ge 0$ and $f(tx) \le tf(x)$.

Proof. For $X \in L^0(\Omega, \mathcal{F}, P)$, define $A_X \in \mathcal{F}$ by

$$A_X = \{t > 0 : E[\psi(|X|/t)] \le 1\}$$

Let $X, Y \in L^{\psi}(\Omega, \mathcal{F}, P)$ and $\lambda \in \mathbb{C}$. Since $||X||_{\psi} < \infty$ and $||Y||_{\psi} < \infty$, we have that $A_X \neq \emptyset$ and $A_Y \neq \emptyset$,

(1) **subadditivity:** Let $\epsilon > 0$. Then there exists $s \in A_X$ and $t \in A_Y$ such that $s < \inf A_X + \epsilon/2$ and $t < \inf A_Y + \epsilon/2$. Since is convex and increasing, we have that

$$\psi\left(\frac{|X+Y|}{s+t}\right) \le \psi\left(\frac{|X|}{s+t} + \frac{|Y|}{s+t}\right)$$

$$= \psi\left(\frac{s}{s+t} \frac{|X|}{s} + \frac{t}{s+t} \frac{|Y|}{t}\right)$$

$$\le \frac{s}{s+t} \psi\left(\frac{|X|}{s}\right) + \frac{t}{s+t} \psi\left(\frac{|Y|}{t}\right)$$

Therefore,

$$E\left[\psi\left(\frac{|X+Y|}{s+t}\right)\right] \le \frac{s}{s+t} E\left[\psi\left(\frac{|X|}{s}\right)\right] + \frac{t}{s+t} E\left[\psi\left(\frac{|Y|}{t}\right)\right]$$
$$\le \frac{s}{s+t} + \frac{t}{s+t}$$
$$\le 1$$

Hence $s + t \in A_{X+Y}$. Thus $A_{X+Y} \neq \emptyset$. Since $s < \inf A_X + \epsilon/2$ and $t < \inf A_Y + \epsilon/2$, we have that

$$||X + Y||_{\psi} = \inf A_{X+Y}$$

$$\leq s + t$$

$$< \inf A_X + \inf A_Y + \epsilon$$

$$= ||X||_{\psi} + ||Y||_{\psi} + \epsilon$$

Since $\epsilon > 0$ is arbitrary, $||X + Y||_{\psi} \le ||X||_{\psi} + ||Y||_{\psi}$. So $X + Y \in L^{\psi}(\Omega, \mathcal{F}, P)$

(2) absolute homogeneity: Suppose that $\lambda = 0$. Then for each t > 0,

$$E[\psi(|\lambda X|/t)] = E[\psi(0)]$$

$$= E[0]$$

$$= 0$$

$$\leq 1$$

Thus

$$\|\lambda X\|_{\psi} = 0$$
$$= |\lambda| \|X\|_{\psi}$$

Suppose that $\lambda \neq 0$. Since for each t > 0,

$$\psi\left(\frac{|X|}{t}\right) = \psi\left(\frac{|\lambda X|}{|\lambda|t}\right)$$

we have that for each t > 0, $t \in A_X$ iff $|\lambda|t \in A_{\lambda X}$. Therefore $A_{\lambda X} = |\lambda|A_X$ and

$$\|\lambda X\|_{\psi} = \inf A_{\lambda X}$$
$$= |\lambda| \inf A_{X}$$
$$= |\lambda| \|X\|_{\psi}$$

So $|\lambda X| \in L^{\psi}(\Omega, \mathcal{F}, P)$.

(3) **positive definiteness:** Note that since ψ is increasing, for each $t_0 > 0$, $t_0 \in A_X$ implies that $[t_0, \infty) \subset A_X$. Suppose that $||X||_{\psi} = 0$. Then $(0, \infty) \subset A_X$. Jensen's inequality implies that for each t > 0,

$$\psi\left(\frac{E|X|}{t}\right) \le E\left[\psi\left(\frac{|X|}{t}\right)\right] < 1$$

For the sake of contradiction, suppose that E|X|>0. Since $\psi(x)\to\infty$ as $x\to\infty$, Choose x>0 such that $\psi(x)>1$. Set t=E|X|/x. Then t>0 and

$$1 < \psi(x)$$

$$= \psi\left(\frac{E|X|}{t}\right)$$

$$\leq 1$$

which is a contradiction. Hence E|X| = 0. Thus X = 0 a.s.

Exercise 6.2.6. Let (Ω, \mathcal{F}, P) be a probability space and $\psi : [0, \infty) \to [0, \infty)$ an Orlicz function. Then $L^{\psi}(\Omega, \mathcal{F}, P)$ is a Banach space.

Proof.

7. CONDITIONAL EXPECTATION AND PROBABILITY

7.1. Conditional Expectation.

Exercise 7.1.1. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -algebra of \mathcal{F} and $X \in L^1(\Omega, \mathcal{F}, P)$. Define $P_{\mathcal{G}} = P|_{\mathcal{G}}$ and $Q : \mathcal{G} \to [0, \infty)$ by $Q(G) = \int_G X dP$. Then $Q \ll P_{\mathcal{G}}$.

Proof. Let $G \in \mathcal{G}$. Suppose that $P_{\mathcal{G}}(G) = 0$. By definition, P(G) = 0. So Q(G) = 0 and $Q \ll P_{\mathcal{G}}$.

Definition 7.1.2. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -algebra of \mathcal{F} and $X, Y \in L^1(\Omega, \mathcal{F}, P)$. Then Y is said to be a **conditional expectation of** X **given** \mathcal{G} if

- (1) Y is \mathcal{G} -measurable
- (2) for each $G \in \mathcal{G}$,

$$\int_{G} Y \, dP = \int_{G} X \, dP$$

Since (2) implies that conditional expectations of X given \mathcal{G} are equal $P_{\mathcal{G}}$ -a.e., we write $Y = E(X|\mathcal{G})$.

Note 7.1.3. Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. We typically write E(X|Y) instead of $E(X|Y^*\mathcal{S})$.

Exercise 7.1.4. Existence of Conditional Expectation:

Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -algebra of \mathcal{F} and $X \in L^1(\Omega, \mathcal{F}, P)$. Define Q and $P_{\mathcal{G}}$ as in the previous exercise. Define $Y = dQ/dP_{\mathcal{G}}$. Then Y is a conditional expectation of X given \mathcal{G} .

Proof. The Radon-Nikodym theorem implies that Y is \mathcal{G} -measurable. Since Q is finite, so is |Q|. Since $d|Q| = |Y| dP_{\mathcal{G}}$, we have that $Y \in L^1(\Omega, \mathcal{G}, P_{\mathcal{G}})$. An exercise in section 3.3 of [3], implies that for each $G \in \mathcal{G}$

$$\int_{G} Y dP = \int_{G} Y dP_{\mathcal{G}}$$
$$= Q(G)$$
$$= \int_{G} X dP$$

Definition 7.1.5. Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. Let $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$. Then ϕ is said to be a **conditional expectation function of** X **given** Y if for each $B \in \mathcal{S} \cap Y(\Omega)$,

$$\int_{Y^{-1}(B)} X \, dP = \int_B \phi \, dP_Y$$

To denote this, we write $\phi(y) = E[X|Y = y]$.

Exercise 7.1.6. Existence of Conditional Expectation Function:

Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. Suppose that for each $y \in S$, $\{y\} \in \mathcal{S}$. Then there exists $\phi \in L^0(Y(S), \mathcal{S} \cap Y(\Omega))$ such that ϕ is a conditional expectation function of X given Y.

Hint: Doob-Dynkin lemma

Proof. Since $E[X|Y] \in L^0(\Omega, Y^*S)$, the Doob-Dynkin lemma implies that there exists $\phi \in L^0(Y(\Omega), S \cap Y(\Omega))$ such that $\phi \circ Y = E(X|Y)$. Let $B \in S \cap Y(\Omega)$. Then

$$\int_{B} \phi \, dP_Y = \int_{Y^{-1}(B)} \phi \circ Y \, dP$$

$$= \int_{Y^{-1}(B)} E(X|Y) \, dP$$

$$= \int_{Y^{-1}(B)} X \, dP$$

7.2. Conditional Probability.

Definition 7.2.1. Let (A, \mathcal{A}) be a measurable space, (B, \mathcal{B}, P_Y) a probability space and $Q: B \times \mathcal{A} \to [0, 1]$. Then Q is said to be a **stochastic transition kernel from** (B, \mathcal{B}, P) **to** (A, \mathcal{A}) if

- (1) for each $E \in \mathcal{A}$, $Q(\cdot, E)$ is \mathcal{B} -measurable
- (2) for P-a.e. $b \in B$, $Q(b,\cdot)$ is a probability measure on (A, A)

Definition 7.2.2. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0(\Omega, \mathcal{F}, P)$ and $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$. Then Q is said to be a **conditional probability distribution of** X **given** Y if

- (1) Q is a stochastic transition kernel from $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_Y)$ to $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$
- (2) for each $A, B \in \mathcal{F}$,

$$\int_{B} Q(y, A)dP_{Y}(y) = P(X \in A, Y \in B)$$

Note 7.2.3. It is helpful to connect this notion of conditional probability with the elementary one by writing $Q(y, A) = P(X \in A|Y = y)$. If $P_Y \ll \mu$, then property (2) in the definition becomes

$$P(X \in A, Y \in B) = \int_{B} Q(y, A) dP_{Y}(y)$$
$$= \int_{B} P(X \in A | Y = y) f_{Y}(y) d\mu(y)$$

as in a first course on probability.

Exercise 7.2.4. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$. Suppose that for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable, for P_Y -a.e. $y \in \mathbb{R}^n$, $P_{X|Y}(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) and $Q(Y, A) = P(X \in A|Y)$ a.e. Then Q is a conditional probability of X given Y.

Proof. By assumption, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable and for P_Y -a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) . Let $A, B \in \mathcal{F}$. Then

$$\int_{B} Q(y, A)dP_{Y}(y) = \int_{Y^{-1}(B)} Q(Y(\omega), A)dP(\omega)$$

$$= \int_{Y^{-1}(B)} P(X \in A|Y)dP$$

$$= \int_{Y^{-1}(B)} E[1_{X^{-1}(A)}|Y]dP$$

$$= \int_{Y^{-1}(B)} 1_{X^{-1}(A)}dP$$

$$= \int 1_{X^{-1}(A)} 1_{Y^{-1}(B)}dP$$

$$= \int 1_{X^{-1}(A)\cap Y^{-1}(B)}dP$$

$$= P(X \in A, Y \in B)$$

So Q is a conditional probability distribution of X given Y.

Definition 7.2.5. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and μ a σ -finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Then $P_{X,Y} \ll \mu^2$. Let $f_X = dP_X/d\mu$, $f_Y = dP_Y/d\mu$ and $f_{X,Y} = dP_{X,Y}/d\mu^2$. Define $f_{X|Y} : \mathbb{R}^n \times \mathbb{R}^n$ by

$$f_{X|Y}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)}, & y \in \text{supp } Y \\ 0, & y \notin \text{supp } Y \end{cases}$$

Then $f_{X|Y}$ is called the **conditional probability density of** X **given** Y.

Exercise 7.2.6. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and μ a σ -finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Define $Q : \mathbb{R}^n \times \mathcal{F} \to [0, 1]$ by

$$Q(y,A) = \int_{A} f_{X|Y}(x,y) d\mu(x)$$

Then Q is a conditional probability distribution of X given Y.

Proof. By the Fubini-Tonelli Theorem, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable and for P_Y -a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) . Let $A, B \in \mathcal{F}$. Then

$$\begin{split} \int_{B} Q(y,A) dP_{Y}(y) &= \int_{B} \left[\int_{A} f_{X|Y}(x,y) d\mu(x) \right] dP_{Y}(y) \\ &= \int_{B \cap \text{supp } Y} \left[\int_{A \cap \text{supp } Y} \frac{f_{X,Y}(x,y)}{f_{Y}(y)} d\mu(x) f_{Y}(y) \right] d\mu(y) \\ &= \int_{B \cap \text{supp } Y} \left[\int_{A} f_{X,Y}(x,y) d\mu(x) \right] d\mu(y) \\ &= P(X \in A, Y \in B \cap \text{supp } Y) \\ &= P(X \in A, Y \in B) \end{split}$$

Theorem 7.2.7. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^1(\Omega, \mathcal{F}, P)$. Suppose that $\operatorname{Im} X \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a conditional probability distribution of Y given X.

8. Markov Chains

Definition 8.0.1. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}_0} \in L_n^0$. Then $(X_n)_{n \in \mathbb{N}_0}$ is said to be a **homogeneous Markov chain** if for each $A \in \mathcal{F}$ and $n \in \mathbb{N}$, $P(X_n \in A|X_1, \dots, X_{n-1}) = P(X_1 \in A|X_0)$ a.e.

9. Probabilities Induced by Isometric Group Actions

9.1. Applications to Bayesian Statistics.

Exercise 9.1.1. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, ϕ : $G \times \Theta \to \Theta$ an isometric group action. Suppose that d is a metric on Θ/G . Let

- H_p^{Θ} be the Hausdorff measure on Θ , $\mu_{\mathcal{X}}$ a measure on \mathcal{X} ,
- p a denisty on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on \mathcal{X} .
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_i \sim p(x|\theta_0)$

Suppose that μ_{Θ} is G-invariant, p is G-invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G-invariant and continuous on Θ . For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1,\ldots,X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}}: \mathcal{B}(\Theta) \to [0,1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)}) dH_p^{\Theta}(\theta)$$

Then there exists a density $\bar{p}(\cdot|X^{(n)})$ on Θ/G such that

$$d\bar{P}_{\Theta|X^{(n)}}(\theta) = \bar{p}(\theta|X^{(n)}) d\bar{H}^{\Theta}(\theta)$$

Proof. Clear from previous work.

Exercise 9.1.2. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, ϕ : $G \times \Theta \to \Theta$ an isometric group action. Suppose that \bar{d} is a metric on Θ/G . Let

- H_p^{Θ} be the Hausdorff measure on Θ , $\mu_{\mathcal{X}}$ a measure on \mathcal{X} , p a denisty on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on \mathcal{X} .
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_i \sim p(x|\theta_0)$

Suppose p is G-invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G-invariant and continuous on Θ . For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1,\ldots,X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}}: \mathcal{B}(\Theta) \to [0,1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)}) dH_p^{\Theta}(\theta)$$

Suppose that $(P_{\Theta|X^{(n)}})_{n\in\mathbb{N}}$ concentrates on $\bar{\theta}_0\subset\Theta$ a.s. or in probability. Then $(\bar{P}_{\Theta|X^{(n)}})_{n\in\mathbb{N}}$ concentrates a.s. or in probability on $\{\bar{\theta_0}\}\subset\Theta/G$ (i.e. is consistent a.s. or in probability)

Proof. Let $V \in \mathcal{N}_{\bar{\theta}_0}$. Then $\pi^{-1}(V) \in \mathcal{N}_{\bar{\theta}_0}$. By definition,

$$\bar{P}_{\Theta|X^{(n)}}(V^c) = P_{\Theta|X^{(n)}}(\pi^{-1}(V^c))$$

$$= P_{\Theta|X^{(n)}}(\pi^{-1}(V)^c)$$

$$\xrightarrow{\text{a.s.}/P} 0$$

Note 9.1.3. Some examples of G-invariant priors would be the uniform distribution, or $N_n(0,\sigma^2 I)$ on \mathbb{R}^n when acted on by O(n). An example of a G-invariant likelihood would be $f(A|Z) \sim \text{Ber}(ZZ^T)$ as in a latent position random graph model where $Z \in \mathbb{R}^{n \times d}$ is the parameter is invariant under right multiplication by $U \in O_d$.

10. Projective Systems

11. STOCHASTIC INTEGRATION

Exercise 11.0.1. Let (Ω, \mathcal{F}, P) be a probability space, X a set \mathcal{A}_0 an algebra, $\mu_0 : \mathcal{A}_0 \to \mathbb{C}$ and $B : \mathcal{A}_0 \to L^2(\Omega, \mathcal{F}, P)$. Suppose that

- (1) $B(\varnothing) = 0$
- (2) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $B(E \cup F) = B(E) + B(F)$
- (3) $E(B(E)B(F)^*) = \mu_0(E \cap F)$

Then

- (1) for each $E \in \mathcal{A}_0$, $\mu_0(E) = E(|B(E)|^2)$.
- (2) for each $E \in \mathcal{A}_0$, $0 \le \mu_0(E) < \infty$
- (3) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$

Proof.

- (1) Clear
- (2) Clear
- (3) Let $E, F \in \mathcal{A}_0$. Suppose that $E \cap F = \emptyset$. Then

$$E(B(E)B(F)^*) = \mu_0(E \cap F)$$

$$= \mu_0(\varnothing)$$

$$= E(|B(\varnothing)|^2)$$

$$= E(0)$$

$$= 0$$

This implies that

$$\mu_0(E \cup F) = E(|B(E \cup F)|^2)$$

$$= E(|B(E) + B(F)|^2)$$

$$= E(|B(E)|^2) + E(|B(F)|^2) + 2\text{Re}E(B(E)B(F)^*)$$

$$= \mu_0(E) + \mu_0(F) + 0$$

$$= \mu_0(E) + \mu_0(F)$$

Definition 11.0.2. Let (Ω, \mathcal{F}, P) be a probability space, X a set \mathcal{A}_0 an algebra, $\mu_0 : \mathcal{A}_0 \to [0, \infty)$ a premeasure and $B : \mathcal{A}_0 \to L^2(\Omega, \mathcal{F}, P)$. Suppose that

- (1) $B(\emptyset) = 0$
- (2) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $B(E \cup F) = B(E) + B(F)$
- (3) $E(B(E)B(F)^*) = \mu_0(E \cap F)$

Then B is said to be a stochastic premeasure with sturcture μ_0

12. TODO

Incorporate vector measures to have conditional expectations of Banach space valued functions given a σ -algebra

References

- Introduction to Analysis
 Introduction to Group Theory
 Introduction to Measure and Integration