INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Contents

1. Fundamental Definitions and Results

1.1. Set Theory.

Definition 1.1.1. Let $\{A_i\}_{i\in I}$ be a collection of sets. The **disjoint union of** $\{A_i\}_{i\in I}$, denoted $\coprod_{i\in I} A_i$, is defined by

$$\coprod_{i\in I} A_i = \bigcup_{i\in I} \{i\} \times A_i$$

Note 1.1.1. In these notes, we will identify $\{i\} \times A_i$ and A_i.

Definition 1.1.2. Let Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod_{i\in I} A_i$. Then σ is said to be a **section of** $\coprod_{i\in I} A_i$ if for each $i \in I$, $\sigma(i) \in A_i$.

1.2. **Differentiation.**

Definition 1.2.1. Let $n \geq 1$. For $i = 1, \cdots, n$, define $x_i : \mathbb{R}^n \to \mathbb{R}$ by $x_i(a_1, \cdots, a_n) = a_i$. The functions $(x_i)_{i=1}^n$ are called the **standard coordinate functions on** $\mathbb{R}^n$.

Definition 1.2.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to** x_i **at** a if

$$\lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}$$

exists. If f is differentiable with respect to x_i at a, we define the **partial derivative of** f **with respect to** x_i **at** a, denoted

$$\frac{\partial f}{\partial x_i}(a) \text{ or } \left.\frac{\partial}{\partial x_i}\right|_a f$$

to be the limit above.

Definition 1.2.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **differentiable with respect to** x_i if for each $a \in U$, f is differentiable with respect to x_i at a.

Exercise 1.2.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x_i x_j}$ and $\frac{\partial^2 f}{\partial x_j x_i}$ exist and are continuous at a. Then

$$\frac{\partial^2 f}{\partial x_i x_j}(a) = \frac{\partial^2 f}{\partial x_j x_i}(a)$$

Proof. $\square$

Definition 1.2.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $\frac{\partial^k f}{\partial i_1 \cdots i_k}$ exists and is continuous on U.

Definition 1.2.6. Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \to \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Definition 1.2.7. Let $U \subset \mathbb{R}^n$ and $p \in U$. Then U is said to be **star-shaped** if for each $q \in U$, $\{p + t(q - p) : 0 \leq t \leq 1\} \subset U$.

Theorem 1.2.1. ***(Taylor's Theorem)*** *Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $f \in C^\infty(U)$. Suppose that U is star-shaped with respect to p. Then there exist $g_1, \cdots, g_n \in C^\infty(U)$ such that for each $x \in U$,*

$$f(x) = f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x)$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial f}{\partial x_i}(p)$$

Proof. Let $x \in U$. Since U is star-shaped with respect to p, $\{p + t(x - p) : 0 \leq t \leq 1\} \subset U$. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[f(p + t(x - p))\Big] = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p + t(x - p))(x_i - p_i)$$

Integrating both sides with respect to t from 0 to 1, we obtain

$$f(x) - f(p) = \sum_{i=1}^{n}(x_i - p_i)\int_0^1 \frac{\partial f}{\partial x_i}(p + t(x-p))dt$$

For $i \in \{1, \cdots, n\}$, define $g_i \in C^\infty(U)$ by

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x-p))dt$$

Then for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial f}{\partial x_i}(p))$$

□

1.3. **Smooth Maps.**

Definition 1.3.1. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Let $x_1, \cdots, x_n$ be the standard coordinate functions on $\mathbb{R}^n$ and $y_1, \cdots, y_m$ be the standard coordinate functions on $\mathbb{R}^m$. For $i \in \{1, \cdots, m\}$, we define the **ith component of F**, denoted $F_i : U \to \mathbb{R}$, by

$$F_i = y_i \circ F$$

Thus $F = (F_1, \cdots, F_m)$

Definition 1.3.2. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \cdots, m\}$, the ith component of F, $F_i : U \to \mathbb{R}$, is smooth.

Definition 1.3.3. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. Then F is said to be a **diffeomorphism** if F is a homeomorphism and F, F^{-1} are smooth.

Definition 1.3.4. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \to \mathbb{R}^m$. We define the **Jacobian of F at p**, denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p)\right)_{i,j} = \frac{\partial F_i}{\partial x_j}$$

Exercise 1.3.5. Inverse Function Theorem:
Let $U, V \subset \mathbb{R}^n$ be open and $F : U \to V$.

Exercise 1.3.6. Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \to F(N)$ is a diffeomorphism

Proof. content... □

1.4. **Topology.**

Definition 1.4.1. Let $(X, \mathbb{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 1.4.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Definition 1.4.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \to Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 1.4.1. *Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \ncong \mathbb{R}^n$*

2. Multilinear Algebra

2.1. Introduction.

Definition 2.1.1. Let $V_1, \dots, V_k, W$ be vector spaces and $\alpha : \prod_{i=1}^n V_i \to W$. Then α is said to be **multilinear** if for each $i \in \{1, \cdots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$,

$$\alpha(v_1, \cdots, v_i + cv, \cdots, v_k) = \alpha(v_1, \cdots, v_i, \cdots, v_k) + c\alpha(v_1, \cdots, v, \cdots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \{\alpha : \prod_{i=1}^n V_i \to W : \alpha \text{ is multilinear}\}$$

Definition 2.1.2. Let Let $V_1, \cdots, V_k, W_1, \cdots, W_l$ be vector spaces, $\alpha \in L(V_1, \dots, V_k; \mathbb{R})$ and $\beta \in L(W_1, \dots, W_l; \mathbb{R})$. We define the **tensor product** of α and β, denoted $\alpha \otimes \beta \in L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$, by

$$\alpha \otimes \beta(v_1, \cdots, v_k, w_1, \cdots, w_l) = \alpha(v_1, \cdots, v_k)\beta(w_1, \cdots, w_l)$$

Thus

$$\otimes : L(V_1, \dots, V_k; \mathbb{R}) \times L(W_1, \dots, W_l; \mathbb{R}) \to L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$$

Exercise 2.1.3. The tensor product

$$\otimes : L(V_1, \dots, V_k; \mathbb{R}) \times L(W_1, \dots, W_l; \mathbb{R}) \to L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$$

is associative.

Proof. Clear. □

Exercise 2.1.4. The tensor product

$$\otimes : L(V_1, \dots, V_k; \mathbb{R}) \times L(W_1, \dots, W_l; \mathbb{R}) \to L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$$

is bilinear.

Proof. Clear. □

Definition 2.1.5.

2.2. k-Tensors.

Note 2.2.1. For the remainder of this section we let V denote an n-dimensional vector space with basis $\{e_1, \cdots, e_n\}$ with dual space V^* and dual basis $\{\epsilon_1, \cdots, \epsilon_n\}$ defined by $\epsilon_i(e_j) = \delta_{i,j}$.

Definition 2.2.1. Let $\alpha : V^k \to \mathbb{R}$. Then α is said to be a **k-tensor on V** if $\alpha \in L(V, \dots, V; \mathbb{R})$. The set of all k-tensors on V is denoted by $T_k(V)$. Define $L_0(V) = \mathbb{R}$.

Exercise 2.2.2. We have that $T_k(V)$ is a vector space.

Proof. Clear. □

Definition 2.2.3. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \to \mathbb{R}$ by

$$\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

The map $\alpha \mapsto \sigma\alpha$ is called the **permutation action** of S_k on $T_k(V)$

Exercise 2.2.4. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

(1) Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V), \sigma\alpha \in T_k(V)$.
(2) Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
(3) Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \cdots, v_k \in V$,

$$\begin{aligned}(\tau\sigma)\alpha(v_1, \cdots, v_k) &= \alpha(v_{\tau\sigma(1)}, \cdots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \cdots, v_k)\end{aligned}$$

□

Exercise 2.2.5. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \to T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$. Then

$$\begin{aligned}\sigma(c\alpha + \beta)(v_1, \cdots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \cdots, v_k) + \sigma\beta(v_1, \cdots, v_k)\end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

Definition 2.2.6. Let $\alpha \in T_k(V)$. Then α is said to be **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$. and α is said to be **alternating** if for each $\sigma \in S_k$, $\sigma\alpha = \mathrm{sgn}(\sigma)\alpha$. The set of symmetric k-tensors on V is denoted $\Xi_k(V)$ and the set of alternating k-tensors on V is denoted $\Lambda_k(V)$.

Definition 2.2.7. Define the **symmetric operator** $S : T_k(V) \to \Xi_k(V)$ by

$$S(\alpha) = \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \to \Lambda_k(V)$ by

$$A(\alpha) = \frac{1}{k!}\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\sigma\alpha$$

Exercise 2.2.8.
(1) For $\alpha \in T_k(V)$, $S(\alpha)$ is symmetric.
(2) For $\alpha \in T_k(V)$, $A(\alpha)$ is alternating.

Proof.
(1) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}\sigma S(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \tau\alpha\right] \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \tau\alpha \\ &= S(\alpha)\end{aligned}$$

(2) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}\sigma A(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right] \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\sigma\tau\alpha \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\ &= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\ &= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \\ &= \operatorname{sgn}(\sigma)A(\alpha)\end{aligned}$$

□

Exercise 2.2.9.

(1) For $\alpha \in \Xi_k(V)$, $S(\alpha) = \alpha$.
(2) For $\alpha \in \Lambda_k(V)$, $A(\alpha) = \alpha$.

Proof.

(1) Let $\alpha \in \Xi_k(V)$. Then

$$\begin{aligned}S(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha \\ &= \frac{1}{k!}\sum_{\sigma \in S_k} \alpha \\ &= \alpha\end{aligned}$$

(2) Let $\alpha \in \Lambda_k(V)$. Then

$$\begin{aligned}A(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha \\ &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2\alpha \\ &= \alpha\end{aligned}$$

□

Exercise 2.2.10. The symmetric operator $S : T_k(V) \to \Xi_k(V)$ and the alternating operator $A : T_k(V) \to \Lambda_k(V)$ are linear.

Proof. Clear. □

Definition 2.2.11. Let $\alpha \in \Lambda_k(V)$ and $\beta \in \Lambda_l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda_{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to \Lambda_{k+l}(V)$.

Exercise 2.2.12. The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to T_{k+l}(V)$ is bilinear.

Proof. Clear. $\square$

Exercise 2.2.13. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

(1) $A(A(\alpha) \otimes \beta) = A(\alpha \otimes \beta)$
(2) $A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \to S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

(1) Then

$$\begin{aligned}
A(A(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \Big[A(\alpha) \otimes \beta\Big] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right) \otimes \beta\right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau)(\tau\alpha) \otimes \beta\right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau(\alpha \otimes \beta)\right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau(\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \operatorname{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \operatorname{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= A(\alpha \otimes \beta)
\end{aligned}$$

(2) Similar to (1).

$\square$

Exercise 2.2.14. The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to \Lambda_{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$ and $\gamma \in \Lambda_m(V)$. Then

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!}A(\alpha \otimes \beta)\right] \wedge \gamma \\
&= \frac{(k+l+m)!}{(k+l)!m!}A\left(\left[\frac{(k+l)!}{k!l!}A(\alpha \otimes \beta)\right] \otimes \gamma\right) \\
&= \frac{(k+l+m)!}{(k+l)!m!}\frac{(k+l)!}{k!l!}A(A(\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{m!}\frac{1}{k!l!}A((\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{k!(l+m)!}\frac{(l+m)!}{l!m!}A(\alpha \otimes (\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}\frac{(l+m)!}{l!m!}A(\alpha \otimes A(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}A(\alpha \otimes \frac{(l+m)!}{l!m!}A(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}A(\alpha \otimes (\beta \wedge \gamma)) \\
&= \alpha \wedge (\beta \wedge \gamma))
\end{aligned}$$

□

Exercise 2.2.15. Let $\alpha_i \in \Lambda_{k_i}(V)$ for $i = 1, \cdots, m$. Then

$$\bigwedge_{i=1}^{m} \alpha_i = \frac{(\sum_{i=1}^{m} k_i)!}{\prod_{i=1}^{m} k_i!} A\left(\bigotimes_{i=1}^{m} \alpha_i\right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} A(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i\right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A\left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A\left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} A\left(\bigotimes_{i=1}^{m_0-1} \alpha_i\right)\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(A\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(\bigotimes_{i=1}^{m_0+1} \alpha_i\right)
\end{aligned}$$

□

Exercise 2.2.16. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl. (Hint: inversion number)

Proof.

$$\begin{aligned}
N(\tau) &= \sum_{i=1}^{l} k \\
&= kl
\end{aligned}$$

Since $\operatorname{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\operatorname{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.2.17. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \cdots, v_{k+l} \in V$, we have that

$$\begin{aligned}
\sigma\tau(\beta \otimes \alpha)(v_1, \cdots, v_l, v_{l+1}, \cdots v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}) \alpha(v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \\
&= \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\
&= \alpha \otimes \beta(v_{\sigma(1)}, \cdots v_{\sigma(k)}, v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\
&= \sigma(\alpha \otimes \beta)(v_1, \cdots, v_k, v_{1+k}, \cdots v_{l+k})
\end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned}
\beta \wedge \alpha &= \frac{(k+l)!}{k!l!} A(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma\tau)\sigma\tau(\beta \otimes \alpha) \\
&= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha \otimes \beta) \\
&= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \\
&= \operatorname{sgn}(\tau) \alpha \wedge \beta \\
&= (-1)^{kl} \alpha \wedge \beta
\end{aligned}$$

□

Exercise 2.2.18. Let $\alpha \in \Lambda_k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned}
\alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\
&= -\alpha \wedge \alpha
\end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.2.19. (Fundamental Example) Let $\alpha_1, \cdots, \alpha_m \in \Lambda_1(V)$ and $v_1, \cdots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}
\left(\bigwedge_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) &= m! A\left(\bigotimes_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) \\
&= m!\left[\frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\sigma\left(\bigotimes_{i=1}^m \alpha_i\right)\right](v_1, \cdots, v_m) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\left(\bigotimes_{i=1}^m \alpha_i\right)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
&= \det(\alpha_i(v_j))
\end{aligned}$$

□

Definition 2.2.20. Define $\mathcal{I}_k = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called a **multi-index**. Recall that $\#\mathcal{I}_k = \binom{n}{k}$.

Definition 2.2.21. Let $I = \{(i_1, i_2, \cdots, i_k) \in I_k$.
Define $e_I \in V^k$ by

$$e_I = (e_{i_1}, \cdots, e_{i_k})$$

Define $\epsilon_I \in \Lambda_k(V)$ by

$$\epsilon_I = \epsilon_{i_1} \wedge \cdots, \wedge \epsilon_{i_k}$$

Exercise 2.2.22. Let $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k) \in \mathcal{I}_k$. Then $\epsilon_I(e_J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon_{i_1}(e_{j_1}) & \cdots & \epsilon_{i_1}(e_{j_k}) \\ & \vdots & \\ \epsilon_{i_k}(e_{j_1}) & \cdots & \epsilon_{i_k}(e_{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon_I(e_J) = \det A$. If $I = J$, then $A = I_{k \times k}$ and therefore $\epsilon_I(e_J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the l_0th row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the l_0th column of A are 0. $\square$

Exercise 2.2.23. Let $\alpha, \beta \in \Lambda_k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e_I) = \beta(e_I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e_I) = \beta(e_I)$. Let $v_1, \cdots, v_k \in V$. For $i = 1, \cdots, k$, write $v_i = \sum_{j_i=1}^n a_{i,j_i} e_{j_i}$. Then

$$\begin{aligned}
\alpha(v_1, \cdots, v_k) &= \sum_{j_1, \cdots, j_k = 1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e_{j_1}, \cdots, e_{j_k}) \\
&= \sum_{j_1 \neq \cdots \neq j_k} \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e_{j_1}, \cdots, e_{j_k}) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^k a_{i,\sigma(j_i)} \right) \right] \alpha(e_J) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^k a_{i,\sigma(j_i)} \right) \right] \beta(e_J) \\
&= \sum_{j_1, \cdots, j_k = 1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \beta(e_{j_1}, \cdots, e_{j_k}) \\
&= \beta(v_1, \cdots, v_k)
\end{aligned}$$

$\square$

Exercise 2.2.24. The set $\{\epsilon_I : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(V)$ and $\dim \Lambda_k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon_I$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e_J) = a_J = 0$. Thus $\{e_I : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda_k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e_I)$. define $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon_I$. Then for each $J \in \mathcal{I}_k$, $\mu(e_J) = b_J = \beta(e_J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{\epsilon_I : I \in \mathcal{I}_k\}$. $\square$

2.3. (r, s)-Tensors.

Definition 2.3.1. Let $\alpha : (V^*)^r \times V^s \to \mathbb{R}$. Then α is said to be an (r, s)-tensor on V if $\alpha \in L(V^*, \dots, V^*, V, \dots, V; \mathbb{R})$. The set of all (r, s)-tensors on V is denoted $T^r_s(V)$.

3. Manifolds

3.1. Smooth Manifolds.

Definition 3.1.1. Define the **upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}_n$, by

$$\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\partial\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$$
$$(\mathbb{H}^n)^\circ = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

Definition 3.1.2. Let M be a topological space and $n \geq 1$.

(1) Let $U \subset M$, $V \subset \mathbb{H}^n$ open and $\phi : U \to V$. Then (U, ϕ) is said to be a **coordinate chart** on M if ϕ is a homeomorphism.
(2) Let $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$ be a collection of coordinate charts on M. Then $\mathcal{A}$ is said to be an **atlas** on M if $\bigcup_{a \in A} U_a = M$.
(3) The space M is said to be **locally half Euclidean of dimension** n if there exists an atlas $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$ on M such that for each $a \in A$, $\phi_a(U_a) \subset \mathbb{H}^n$.
(4) The space M is said to be an n**-dimensional manifold** if M is Hausdorff, second countable and locally half Euclidean of dimension n.

Note 3.1.1. For the remainder of this section, we assume M is an n-dimensional manifold.

Definition 3.1.3.

(1) Define the **boundary** of M, denoted ∂M, by
$$\partial M = \{p \in M : \text{ there exists a chart } (U, \phi) \text{ on } M \text{ such that } p \in U \text{ and } \phi(p) \in \partial\mathbb{H}^n\}$$
(2) Define the **interior** of M, denoted M°, by
$$M^\circ = M \setminus \partial M$$

Exercise 3.1.4. Let $p \in M$. Then $p \in \partial M$ iff for each chart (U, ϕ) on M, $p \in U$ implies that $\phi(p) \in \partial\mathbb{H}^n$. (Hint: simply connected)

Proof. Supposet that $p \in \partial M$. Then there exists a coordinate chart (V, ψ) on M such that $\psi(p) \in \partial\mathbb{H}^n$. Let (U, ϕ) be a coordinate chart on M. Suppose that $p \in U$. Note that $\phi \circ \psi^{-1} : \psi(V \cap U) \to \phi(V \cap U)$ is a homeomorphism. Choose open n-balls B_ϕ, $B_\psi \subset \mathbb{H}^n$ such that $B_\phi \subset \phi(V \cap U)$, $B_\psi \subset \psi(V \cap U)$, $\phi(p) \in B_\phi$ and $\psi(p) \in B_\psi$. For the sake of contradiction, suppose that $\phi(p) \notin \partial\mathbb{H}^n$. Put $U' = B_\phi \setminus \{\phi(p)\}$ and $V' = B_\psi \setminus \{\psi(p)\}$. Define $\lambda : V' \to U'$ by $\lambda = \phi \circ \psi|_{B_\psi}$. Then λ is a homeomorphism. Note that V' is simply connected and U' is not. This is a contradiction. □

Exercise 3.1.5. If $\partial M \neq \varnothing$, then

(1) ∂M is an $n-1$-dimensional manifold
(2) $\partial(\partial M) = \varnothing$.

Proof. (1) Since subspaces of Hausdorff, second countable spaces are Hausdorff and second countable, we need only show that ∂M is locally half euclidean of dimension $n-1$. Let $p \in \partial M$. Then there exists a coordinate chart (U, ϕ) on M such that $p \in U$ and $\phi(p) \in \partial\mathbb{H}^n$.
Put $U' = U \cap \partial M$. Note that U' is open in ∂M and $\phi(U) \cap \partial\mathbb{H}^n$ is open in $\partial\mathbb{H}^n$.

Define $\phi' : U' \to \phi(U) \cap \partial\mathbb{H}^n$ by $\phi' = \phi|_{U'}$. Then ϕ' is a homeomorphism.
Since $\partial\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n-1}$ which is homeomorphic to $(\mathbb{H}^{n-1})^\circ$ there exists $\psi : \partial\mathbb{H}^n \to (\mathbb{H}^{n-1})^\circ$ such that ψ is a homeomorphism.
Define $V' = \psi(\phi(U) \cap \partial\mathbb{H}^n)$ and $\psi' : \phi(U) \cap \partial\mathbb{H}^n \to V'$ by and $\psi' = \psi|_{\phi(U)\cap\partial\mathbb{H}^n}$. Then V' is open in $(\mathbb{H}^{n-1})^\circ$ and ψ' is a homeomrophism.
Define $\lambda : U' \to V'$ by $\lambda = \psi' \circ \phi'$. Then λ is a homeomorhism and (U', λ) is a cooridnate chart on ∂M. So ∂M is locally Euclidean of dimension $n-1$.

(2) Let $p \in \partial M$. Define $(U \cap \partial M, \lambda \circ \psi)$ as in (1). Since $\lambda \circ \psi(p) \in (\mathbb{H}^{n-1})^\circ$, we have that $p \in M^\circ$. Thus $\partial M = (\partial M)^\circ$ and $\partial(\partial M) = \varnothing$.

□

Definition 3.1.6.

(1) Let $(U, \phi), (V, \psi)$ be coordinate charts on M. Then (U, ϕ) and (V, ψ) are said to be **smoothly compatible** if
$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \text{ is a diffeomorphism}$$
(2) Let $\mathcal{A} = ((U_a, \phi_a))_{a\in A}$ be an atlas on M. Then $\mathcal{A}$ is said to be **smooth** if for each $a, b \in A$, (U_a, ϕ_a) and (U_b, ϕ_b) are smoothly compatible.
(3) Let $\mathcal{A}$ be a smooth atlas on M. Then $\mathcal{A}$ is said to be **maximal** if for each smooth atlas $\mathcal{B}$ on M, $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on** M.
(4) Let $\mathcal{A}$ be a smooth structure on M. Then $(M, \mathcal{A})$ is said to be a **smooth n-dimensional manifold**.

Exercise 3.1.7. Let $\mathcal{B}$ be a smooth atlas on M. Then there exists a unique smooth structure $\mathcal{A}$ on M such that $\mathcal{B} \subset \mathcal{A}$.

Proof. Define $\mathcal{A}$ to be the set of all coordinate charts (U, ϕ) on M such that for each coordinate chart $(V, \psi) \in \mathcal{B}$, (U, ϕ) and (V, ψ) are smoothly compatible.
Clearly $\mathcal{B} \subset \mathcal{A}$.
Let $(U, \phi), (V, \psi) \in \mathcal{A}$ and $p \in U \cap V$. Then there exists $(W, \chi) \in \mathcal{B}$ such that $p \in W$. By assumption, $\phi \circ \chi^{-1} : \chi(U \cap W) \to \phi(U \cap W)$ and $\chi \circ \psi^{-1} : \psi(W \cap V) \to \chi(W \cap V)$ are diffeomorphisms. Then $(\phi\circ\chi^{-1})\circ(\chi\circ\psi^{-1}) = \phi\circ\psi^{-1} : \psi(U\cap W\cap V) \to \phi(U\cap W\cap V)$ is a diffeomorphism. Since for each $q \in \psi(U \cap V)$, there exits an open neighborhood $N \subset \psi(U \cap V)$ of q on which $\phi\circ\psi^{-1}$ are diffeomorphic, we have that $\phi\circ\psi^{-1}$ is a diffeomorphism on $\psi(U\cap V)$ and therefore (U, ϕ) and (V, ψ) are smoothly compatible. Hence $\mathcal{A}$ is a smooth atlas.
To see that $\mathcal{A}$ is maximal, let $\mathcal{B}'$ be a smooth atlas on M. Suppose that $\mathcal{A} \subset \mathcal{B}'$ and let $(U, \phi) \in \mathcal{B}'$. By definition, for each chart $(V, \psi) \in \mathcal{B}'$, (U, ϕ) and (V, ψ) are smoothly compatible. Since $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$, we have that $(U, \phi) \in \mathcal{A}$. So $\mathcal{A} = \mathcal{B}'$ and $\mathcal{A}$ is a maximal smooth atlas on M. □

Exercise 3.1.8. Let $\mathcal{A}$ be a smooth atlas on M. Define $\lambda : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ by $\lambda(x_1, \cdots, x_{n-1}, 0) = (x_1, \cdots, x_{n-1})$. Put $\mathcal{A}|_{\partial M} = \{(U \cap \partial M, \lambda \circ \phi_{U\cap\partial M}) : (U, \phi) \in \mathcal{A}\}$. Then

(1) $\mathcal{A}|_{\partial M}$ is a smooth atlas on ∂M.
(2) if $\mathcal{A}$ is maximal, then $\mathcal{A}|_{\partial M}$ is maximal.

Proof. □

Note 3.1.2. For the rest of this section, we assume that $(M, \mathcal{A})$ is a smooth n-dimensional manifold and we denote the standard coordinate functions on $\mathbb{R}^n$ by $u_1, \cdots, u_n$. For a coordinate chart $(U, \phi) \in \mathcal{A}$ and $i \in \{1, \cdots, n\}$, we will typically denote the ith coordinate of ϕ by x_i, that is, $x_i = u_i(\phi)$.

3.2. **Smooth Maps.**

Definition 3.2.1. Let $f : M \to \mathbb{R}$. Then f is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M)$.

Exercise 3.2.2. We have that $C^\infty(M)$ is a vector space.

Proof. Clear. □

Definition 3.2.3. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$. Then F is said to be **smooth** if for each $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(F(U) \cap V)$$

is smooth and F is said to be a **diffeomorphism** if F is a homeomorphism and F, F^{-1} are smooth.

Exercise 3.2.4. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ a diffeomorphism. Then for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

Proof. Let $(V, \psi) \in \mathcal{B}$.

(1) Since ϕ and F^{-1} are homeomorphisms, $\phi \circ F^{-1} : F(U) \cap V \to \phi(U \cap F^{-1}(V))$ is a homeomorphism

(2) Since F is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \to \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \psi(V \cap F(U))$$

are smooth.

Therefore $(F(U), \phi \circ F^{-1})$ and (V, ψ) are smoothly compatible. Since $\mathcal{B}$ is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. □

Exercise 3.2.5. Let $(M, \mathcal{A})$ be smooth m-dimensional manifold, $(N, \mathcal{B})$ a smooth n-dimensional manifold and $F : M \to N$. If F is a diffeomorphism, then $m = n$.

Proof. Suppose that F is a diffeomorphism. Let $(U, \phi) \in \mathcal{A}$. The previous exercise implies that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Then

$$\begin{aligned} \phi(U) &= \phi \circ F^{-1}(F(U)) \\ &=\subset \mathbb{H}^n \end{aligned}$$

By definition, $\phi(U) \subset H^m$. So $m = n$. □

3.3. **The Tangent Space.**

Definition 3.3.1. Let $p \in M$. Define the relation $\sim_p$ on $C^\infty(M)$ by $f \sim_p g$ iff there exists $U \in \mathcal{N}_p$ such that U is open and $f|_U = g|_U$. Clearly $\sim_p$ is an equivalence relation on $C^\infty(M)$. We denote $C^\infty(M)/\sim_p$ by $C^\infty_p(M)$. For $f \in C^\infty(M)$, we define the **germ of** f **at** p to be the equivalence class of f under $\sim_p$.

Exercise 3.3.2. Let $p \in$We have that $C^\infty_p(M)$ is a vector space.

Proof. Clear. □

Definition 3.3.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$, $p \in U$ and $f \in C^\infty_p(M)$. For $i \in \{1, \cdots, n\}$, define the partial derivative of f with respect to x_i at p, denoted

$$\frac{\partial f}{\partial x_i}(p), \; \frac{\partial}{\partial x_i}\bigg|_p f, \; \partial_{x_i} f(p) \text{ or } \; \partial_{x_i}\bigg|_p f$$

by

$$\frac{\partial}{\partial x_i}\bigg|_p f = \frac{\partial}{\partial u_i}\bigg|_{\phi(p)} f \circ \phi^{-1}$$

Exercise 3.3.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$, we have that

$$\frac{\partial x_i}{\partial x_j}(p) = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \cdots, n\}$. Then

$$\begin{aligned}
\frac{\partial}{\partial x_j}\bigg|_p x_i &= \frac{\partial}{\partial u_j}\bigg|_{\phi(p)} x_i \circ \phi^{-1} \\
&= \frac{\partial}{\partial u_j}\bigg|_{\phi(p)} u_i \circ \phi \circ \phi^{-1} \\
&= \frac{\partial}{\partial u_j}\bigg|_{\phi(p)} u_i \\
&= \delta_{i,j}
\end{aligned}$$

□

Exercise 3.3.5. (Change of Coordinates): Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$ and $\psi = (y_1, \cdots, y_n)$, $p \in U \cap V$ and $f \in C^\infty_p(M)$. Then for each $i \in \{1, \cdots, n\}$, we have

$$\frac{\partial f}{\partial y_i}(p) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p) \frac{\partial x_j}{\partial y_i}(p)$$

Proof. Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \cdots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned}
\frac{\partial}{\partial y_i}\bigg|_p f &= \frac{\partial}{\partial u_i}\bigg|_{\psi(p)} f \circ \psi^{-1} \\
&= \frac{\partial}{\partial u_i}\bigg|_{\psi(p)} f \circ \phi^{-1} \circ h \\
&= \sum_{j=1}^{n} \bigg(\frac{\partial}{\partial u_j}\bigg|_{h\circ\psi(p)} f \circ \phi^{-1} \bigg) \bigg(\frac{\partial}{\partial u_i}\bigg|_{\psi(p)} h_j \bigg) \\
&= \sum_{j=1}^{n} \bigg(\frac{\partial}{\partial u_j}\bigg|_{\phi(p)} f \circ \phi^{-1} \bigg) \bigg(\frac{\partial}{\partial u_i}\bigg|_{\psi(p)} x_j \circ \psi^{-1} \bigg) \\
&= \sum_{j=1}^{n} \bigg(\frac{\partial}{\partial x_j}\bigg|_{p} f \bigg) \bigg(\frac{\partial}{\partial y_i}\bigg|_{p} x_j \bigg)
\end{aligned}$$

□

Exercise 3.3.6. Taylor's Theorem:
Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$, $p \in U$ and $f \in C_p^\infty(M)$. Then there exist $g_1, \cdots g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^{n} (x_i - x_i(p)) g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x_i}\bigg|_p f$$

Proof. Since we are interested in the germ of f at p, we may assume that $\phi(U)$ is star-shaped with respect to $\phi(p)$. Let $q \in U$. From Taylor's theorem in section 1, we know that there exist $\tilde{g}_1, \cdots, \tilde{g}_n \in C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = f \circ \phi^{-1}(\phi(p)) + \sum_{i=1}^{n} [u_i \circ \phi(q) - u_i \circ \phi(p)] \tilde{g}_i(\phi(q))$$

and for each $i \in \{1, \cdots, n\}$,

$$\tilde{g}_i(\phi(p)) = \frac{\partial}{\partial u_i}\bigg|_{\phi(p)} f \circ \phi^{-1}$$

For each $i \in \{1, \cdots, n\}$, define $g_i = \tilde{g}_i \circ \phi$. Then for each $q \in U$,

$$f(q) = f(p) + \sum_{i=1}^{n} [x_i(q) - x_i(p)] g_i(q)$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x_i}\bigg|_p f$$

□

Definition 3.3.7. Let $p \in M$ and $v : C_p^\infty(M) \to \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C_p^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation at** p if for each $f, g \in C_p^\infty(M)$ and $a \in \mathbb{R}$,

(1) v is linear
(2) v is Leibnizian

We define the **tangent space of** M **at** p, denoted T_pM, by

$$T_pM = \{v : C_p^\infty(M) \to \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 3.3.8. Let $f \in C_p^\infty(M)$ and $v \in T_pM$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. $\square$

Exercise 3.3.9. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$ and $p \in U$. Then

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \cdots, \frac{\partial}{\partial x_n}\bigg|_p \right\}$$

is a basis for T_pM and $\dim T_pM = n$.

Proof. Clearly $\frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p \in T_pM$. Let $a_1, \cdots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\bigg|_p = 0$$

Then

$$\begin{aligned} 0 &= vx_j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\bigg|_p x_j \\ &= a_j \end{aligned}$$

Hence $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p \right\}$ is independent.
Now, let $v \in T_pM$ and $f \in \mathbb{C}_p^\infty(M)$. By Taylor's theorem, there exist $g_1, \cdots g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x_i - x_i(p))g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x_i}\bigg|_p f$$

Then

$$\begin{aligned}
v(f) &= \sum_{i=1}^{n} v(x_i - x_i(p))g_i(p) + \sum_{i=1}^{n}(x_i(p) - x_i(p))v(g_i) \\
&= \sum_{i=1}^{n} v(x_i)g_i(p) \\
&= \sum_{i=1}^{n} v(x_i)\, \frac{\partial}{\partial x_i}\bigg|_p f \\
&= \left[\sum_{i=1}^{n} v(x_i)\, \frac{\partial}{\partial x_i}\bigg|_p\right] f
\end{aligned}$$

So

$$v = \sum_{i=1}^{n} v(x_i)\, \frac{\partial}{\partial x_i}\bigg|_p$$

and

$$v \in \operatorname{span}\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \cdots, \frac{\partial}{\partial x_n}\bigg|_p \right\}$$

□

Definition 3.3.10. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. We define the **differential of** F **at** p, denoted $dF_p : T_pM \to T_{F(p)}N$, by

$$\Big[dF_p(v)\Big](f) = v(f \circ F)$$

for $v \in T_pM$ and $f \in C^\infty_{F(p)}(N)$.

Exercise 3.3.11. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. Then dF_p is well defined.

Proof. Let $v \in T_pM$, $f, g \in C^\infty_{F(p)}(N)$ and $c \in \mathbb{R}$. Then

(1)

$$\begin{aligned}
dF_p(v)(f + cg) &= v((f + cg) \circ F) \\
&= v(f \circ F + cg \circ F) \\
&= v(f \circ F) + cv(g \circ F) \\
&= dF_p(v)(f) + cdF_p(v)(g)
\end{aligned}$$

So $dF_p(v)$ is linear.

(2)

$$\begin{aligned}
dF_p(v)(fg) &= v(fg \circ F) \\
&= v((f \circ F) * (g \circ F)) \\
&= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\
&= dF_p(v)(f) * g(F(p)) + f(F(p)) * dF_p(v)(g)
\end{aligned}$$

So $dF_p(v)$ is Leibnizian and hence $dF_p(v) \in T_{F(p)}N$ $\square$

Exercise 3.3.12. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. If F is a diffeomorphism, then dF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x_1, \cdots, x_n)$ and $\phi \circ F^{-1} = (y_1, \cdots, y_n)$. Let $f \in C^\infty_{F(p)}(N)$ Then

$$\begin{aligned}
\frac{\partial}{\partial y_i}\bigg|_{F(p)} f &= \frac{\partial}{\partial u_i}\bigg|_{\phi\circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\
&= \frac{\partial}{\partial u_i}\bigg|_{\phi(p)} f \circ F \circ \phi^{-1} \\
&= \frac{\partial}{\partial x_i}\bigg|_{p} f \circ F
\end{aligned}$$

Therefore

$$\begin{aligned}
\left[dF_p\left(\frac{\partial}{\partial x_i}\bigg|_p\right)\right](f) &= \frac{\partial}{\partial x_i}\bigg|_{p} f \circ F \\
&= \frac{\partial}{\partial y_i}\bigg|_{F(p)} f
\end{aligned}$$

Hence

$$dF_p\left(\frac{\partial}{\partial x_i}\bigg|_p\right) = \frac{\partial}{\partial y_i}\bigg|_{F(p)}$$

Since $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p \right\}$ is a basis for T_pM and $\left\{ \frac{\partial}{\partial y_1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y_n}\Big|_{F(p)} \right\}$ is a basis for $T_{F(p)}N$, dF_p is an isomorphism. $\square$

Definition 3.3.13. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ a diffeomorphism. Define the **push forward of** F, denoted

$$F_* : M \to \coprod_{p\in M} \operatorname{Iso}(T_pM, T_{F(p)}N)$$

by

$$p \mapsto dF_p$$

Definition 3.3.14. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p\in M} T_pM$$

and we define the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p\in M} T^*_pM$$

3.4. **Submanifolds.**

4. Fields and Forms

4.1. Vector Fields.

Definition 4.1.1. Let $X : M \to TM$. Then X is said to be a **vector field on** M if for each $p \in M$, $X_p \in T_pM$.
For $f \in \mathbb{C}^\infty(M)$, we define $Xf : M \to \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in \mathbb{C}^\infty(M)$, Xf is smooth.
We denote the set of smooth vector fields on M by $\Gamma(M)$.

Definition 4.1.2. Let $f \in C^\infty(M)$ and $X, Y \in \Gamma(M)$. We define

- $fX \in \Gamma(M)$ by
$$(fX)_p = f(p)X_p$$
- $X + Y \in \Gamma(M)$ by
$$(X+Y)_p = X_p + Y_p$$

Exercise 4.1.3. The set $\Gamma(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Exercise 4.1.4. Let $X \in \Gamma(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$. Then

$$X|_U = \sum_{i=1}^n (Xx_i)\frac{\partial}{\partial x_i}$$

Proof. Let $p \in M$. Then $X_p \in T_pM$ and $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p \right\}$ is a basis of T_pM. So there exist $f_1(p), \cdots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum\limits_{i=1}^n f_i(p)\, \frac{\partial}{\partial x_i}\Big|_p$. Let $j \in \{1, \cdots, n\}$. Then,

$$\begin{aligned} X_p(x_j) &= \sum_{i=1}^n f_i(p)\frac{\partial x_j}{\partial x_i}(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx_j = f_j$ and $X|_U = \sum\limits_{i=1}^n (Xx_i)\frac{\partial}{\partial x_i}$. □

Exercise 4.1.5. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$. Then for each $i \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial x_i} \in \Gamma(U)$$

Proof. Let $i \in \{1, \cdots, n\}$ and $f \in C^\infty(M)$. Define $g : M \to \mathbb{R}$ by $g = \frac{\partial}{\partial x_i} f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x_i}\bigg|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u_i}\bigg|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u_i}\big(f \circ \phi^{-1}\big) \circ (\phi \circ \psi^{-1}) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence g is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x_i}$ is smooth. $\square$

4.2. **Differential Forms.**

Definition 4.2.1. We define

$$\Lambda_k(TM) = \coprod_{p \in M} \Lambda_k(T_pM)$$

Definition 4.2.2. Let $\omega : M \to \Lambda_k(TM)$. Then ω is said to be a k**-form on** M if for each $p \in M$, $\omega_p \in \Lambda_k(T_pM)$.
For each $X_1, \cdots, X_k \in \Gamma(M)$, we define $\omega(X_1, \cdots, X_k) : M \to \mathbb{R}$ by

$$\omega(X_1, \cdots, X_k)_p = \omega_p(X_{1p}, \cdots, X_{kp})$$

and ω is said to be **smooth** if for each $X_1, \cdots, X_k \in \Gamma(M)$, $\omega(X_1, \cdots, X_k)$ is smooth.
The set of smooth k-forms on M is denoted $\Omega_k(M)$.

Note 4.2.1. Observe that $\Omega_0(M) = C^\infty(M)$.

Definition 4.2.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Omega_k(M)$. We define

- $f\alpha \in \Omega_k(M)$ by
$$(f\alpha)_p = f(p)\alpha_p$$
- $\alpha + \beta \in \Omega_k(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 4.2.4. The set $\Omega_k(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Definition 4.2.5. Define the **exterior product**

$$\wedge : \Omega_k(M) \times \Omega_l(M) \to \Omega_{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Define the **permutation action of** S_k **on** $\Omega_k(M)$ by

$$(\sigma\omega)_p = \sigma\omega_p$$

Note 4.2.2. For $f \in \Omega_0(M)$ and $\alpha \in \Omega_k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 4.2.6. The exterior product $\wedge : \Omega_k(M) \times \Omega_l(M) \to \Omega_{k+l}(M)$ is $C^\infty(M)$-bilinear.

Proof. Clear. □

Note 4.2.3. All of the results from multilinear algebra apply here.

Definition 4.2.7. We define the **exterior derivative** $d : \Omega_k(M) \to \Omega_{k+1}(M)$ inductively by

(1) $d(d\alpha) = 0$ for $\alpha \in \Omega_p(M)$
(2) $df(X) = Xf$ for $f \in \Omega_0(M)$
(3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega_p(M)$ and $\beta \in \Omega_q(M)$
(4) extending linearly

Exercise 4.2.8. Let (U, ϕ) be a chart on M with $\phi = (x_1, \cdots, x_n)$. Then on U, for each $i, j \in \{1, \cdots, n\}$,

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{(dx_1)_p, \cdots, (dx_n)_p\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p \right\}$ and $T_p^*M = \text{span}\{(dx_1)_p, \cdots, (dx_n)_p\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned}\left[dx_i\left(\frac{\partial}{\partial x_j}\right)\right]_p &= \left(\frac{\partial}{\partial x_j} x_i\right)_p \\ &= \frac{\partial}{\partial x_j}\bigg|_p x_i \\ &= \delta_{i,j}\end{aligned}$$

□

Exercise 4.2.9. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x_1, \cdots, x_n)$. Then on U, $df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$

Proof. Let $p \in U$. Since $\{dx_1, \cdots, dx_n\}$ is a basis for $\Lambda(T_pM)$, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $(df)_p = \sum\limits_{i=1}^n a_i(p)(dx_i)_p$. Therefore, we have that

$$\begin{aligned}(df)_p\left(\frac{\partial}{\partial x_j}\bigg|_p\right) &= \sum_{i=1}^n a_i(p)(dx_i)_p\left(\frac{\partial}{\partial x_j}\bigg|_p\right) \\ &= a_j(p)\end{aligned}$$

By definition, we have that

$$\begin{aligned}(df)_p\left(\frac{\partial}{\partial x_j}\bigg|_p\right) &= \frac{\partial}{\partial x_j}\bigg|_p f \\ &= \frac{\partial f}{\partial x_j}(p)\end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x_j}(p)$ and

$$(df)_p = \sum_{i=1}^n \frac{\partial f}{\partial x_j}(p)(dx_i)_p$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

□

Exercise 4.2.10. Let $f \in \Omega_0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \dots, x_n)$. Then for each $i \in \{1, \dots, n\}$,

$$\frac{\partial}{\partial x_i}\bigg|_p f = 0$$

This implies that

$$\begin{aligned}(df)_p &= \sum_{i=1}^n \frac{\partial f}{\partial x_j}(p)(dx_i)_p \\ &= 0\end{aligned}$$

□

Exercise 4.2.11.

Definition 4.2.12. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x_1, \cdots, x_n)$ and $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$. We define

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega_k(M)$$

and we define

$$\frac{\partial}{\partial x_I} = \left(\frac{\partial}{\partial x_{i_1}}, \cdots, \frac{\partial}{\partial x_{i_k}}\right)$$

Exercise 4.2.13. Let $\omega \in \Omega_k(M)$ and (U, ϕ) be a chart on M with $\phi = (x_1, \cdots, x_n)$. Then there exists $(f_I)_{I \in \mathcal{I}_k} \subset C^\infty(U)$ such that for each $p \in U$,

$$\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p)(dx_I)_p$$

Proof. Let $p \in U$. For each $I \in \mathcal{I}_k$, put

$$f_I(p) = \omega_p\left(\frac{\partial}{\partial x_I}\bigg|_p\right) \in \mathbb{R}$$

Since $\{(dx_I)_p : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(T_pM)$, we have that $\omega_p = \sum\limits_{I \in \mathcal{I}_k} f_I(p)(dx_I)_p$. Since ω is smooth, we have that for each $J \in \mathcal{I}_k$,

$$\begin{aligned}\omega\left(\frac{\partial}{\partial x_J}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx_I\left(\frac{\partial}{\partial x_J}\right) \\ &= f_J\end{aligned}$$

is smooth. □

Exercise 4.2.14. Let $\omega \in \Omega_k(M)$ and (U, ϕ) be a chart on M with $\phi = (x_1, \cdots, x_n)$. If $\omega = \sum\limits_{I \in \mathcal{I}_k} f_I dx_I$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I$$

.

Proof. First we note that

$$\begin{aligned} d(f_I dx_I) &= df_I \wedge dx_I + (-1)^0 f d(dx_I) \\ &= df_I \wedge dx_I \\ &= \left(\sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \right) \wedge dx_I \\ &= \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I \end{aligned}$$

Then we extend linearly. □

Definition 4.2.15. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ be a diffeomorphism. Define the **pullback of** F, denoted $F^* : \Omega_k(N) \to \Omega_k(M)$ by

$$(F^*\omega)_p(v_1, \cdots, v_k) = \omega_{F(p)}(dF_p(v_1), \cdots, dF_p(v_k))$$

for $\omega \in \Omega_k(N)$, $p \in M$ and $v_1, \cdots, v_k \in T_pM$

4.3. **Tensor Fields.**

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Definition 4.3.1. When working in $\mathbb{R}^n$, we introduce the formal objects $dx_1, dx_2, \cdots, dx_n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.

Definition 4.3.2. Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx_I : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

(1) the exterior product is bilinear
(2) for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$
(3) for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$
(4) for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k-forms on $\mathbb{R}^n$. Let ω be a k-form on $\mathbb{R}^n$. If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx_I$

Note 4.3.1. The terms $dx_1, dx_2, \cdots, dx_n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du_1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx_1, dx_2, \cdots, dx_n\}$ when referencing the coordinates on $\mathbb{R}^n$.

Exercise 4.3.3. Let $B_{n\times n} = (b_{i,j}) \in [C^\infty(M)]^{n\times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx_j \right) = (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned} \bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx_j \right) &= \left(\sum_{j=1}^n b_{1,j} dx_j \right) \wedge \left(\sum_{j=1}^n b_{2,j} dx_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n b_{n,j} dx_j \right) \\ &= \sum_{j_1, \cdots, j_n = 1}^n \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\ &= \sum_{j_1 \neq \cdots \neq j_n} \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\ &= \left[\sum_{\sigma \in S_n} \text{sgn}(\sigma) \left(\prod_{i=1}^n b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\ &= (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \end{aligned}$$

$\square$

Definition 4.3.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted df, on $\mathbb{R}^n$ by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Let $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx_I$ be a k-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx_I$$

Exercise 4.3.5. On $\mathbb{R}^3$, put

(1) $\omega_0 = f_0$,
(2) $\omega_1 = f_1 dx_1 + f2 dx_2 + f_2 dx_3$,
(3) $\omega_2 = f_1 dx_2 \wedge dx_3 - f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2$

Show that

(1) $d\omega_0 = \frac{\partial f_0}{\partial x_1} dx_1 + \frac{\partial f_0}{\partial x_2} dx_2 + \frac{\partial f_0}{\partial x_3} dx_3$
(2) $d\omega_1 = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \right) dx_1 \wedge dx_3 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 \wedge dx_2$
(3) $d\omega_2 = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3$

Proof. Straightforward. □

Exercise 4.3.6. Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx_I \wedge dx_{I_*} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.

Definition 4.3.7. We define a linear map $* : \Phi_k(\mathbb{R}^n) \to \Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge $*$-operator** by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 4.3.8. Let $\phi : \mathbb{R}^k \to \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We define $\phi^* : \Phi_k(\mathbb{R}^n) \to \Phi_k(\mathbb{R}^k)$ via the following properties:

(1) for each 0-form f on $\mathbb{R}^n$, $\phi^* f = f \circ \phi$
(2) for $i = 1, \cdots, n$, $\phi^* dx_i = d\phi_i$
(3) for an s-form ω, and a t-form χ on $\mathbb{R}^n$, $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
(4) for l-forms ω, χ on $\mathbb{R}^n$, $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 4.3.9. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold of $\mathbb{R}^n$, $\phi : U \to V$ a smooth parametrization of M, $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$ an k-form on $\mathbb{R}^n$. Then

$$\phi^*\omega = \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du_1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof. Using the definitions, we see that

$$\begin{aligned}\phi^*\omega &= \phi^* \sum_{I\in\mathcal{I}_{k,n}} f_I dx_I \\ &= \sum_{I\in\mathcal{I}_{k,n}} (\phi^* f_I)\phi^* dx_I \\ &= \sum_{I\in\mathcal{I}_{k,n}} (f_I\circ\phi) d\phi_I\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}d\phi_I &= d\phi_{i_1}\wedge d\phi_{i_2}\wedge\cdots\wedge d\phi_{i_n} \\ &= (\sum_{j=1}^n \frac{\partial\phi_{i_1}}{\partial u_j} du_j)\wedge(\sum_{j=1}^n \frac{\partial\phi_{i_2}}{\partial u_j} du_j)\wedge\cdots\wedge(\sum_{j=1}^n \frac{\partial\phi_{i_k}}{\partial u_j} du_j) \\ &= (\det v\phi_I) du_1\wedge du_2\wedge\cdots\wedge du_k\end{aligned}$$

Therefore

$$\begin{aligned}\phi^*\omega &= \sum_{I\in\mathcal{I}_{k,n}} (f_I\circ\phi) d\phi_I \\ &= \sum_{I\in\mathcal{I}_{k,n}} (f_I\circ\phi)(\det v\phi_I) du_1\wedge du_2\wedge\cdots\wedge du_k \\ &= \left(\sum_{I\in\mathcal{I}_{k,n}} (f_I\circ\phi)(\det v\phi_I)\right) du_1\wedge du_2\wedge\cdots\wedge du_k\end{aligned}$$

□

4.4. **Integration of Differential Forms.**

Definition 4.4.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx_1\wedge dx_2\wedge\cdots\wedge dx_k$ a k-form on $\mathbb{R}^k$. Define

$$\int_U \omega = \int_U f dx$$

Definition 4.4.2. Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$, ω a k-form on $\mathbb{R}^n$ and $\phi : U \to V$ a local smooth, orientation-preserving parametrization of M. Define

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 4.4.3.

Theorem 4.4.1. ***(Stokes Theorem)*** *Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$ and ω a $k-1$-form on $\mathbb{R}^n$. Then*

$$\int_{\partial M} \omega = \int_M d\omega$$