INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Contents

1. Review of Fundamentals

1.1. Set Theory.

Definition 1.1.1. Let $\{A_i\}_{i\in I}$ be a collection of sets. The **disjoint union of** $\{A_i\}_{i\in I}$, denoted $\coprod\limits_{i\in I} A_i$, is defined by

$$\coprod_{i\in I} A_i = \bigcup_{i\in I} \{i\} \times A_i$$

We define the **natural projection map**, denoted $\pi : \coprod_{i\in I} A_i \to I$, by $\pi(i, a) = i$.

Definition 1.1.2. Let E and M be sets, $\pi : E \to B$ a surjection and $\sigma : B \to E$. Then σ is said to be a section of (E, M, π) if $\pi \circ \sigma = \mathrm{id}_M$.

Note 1.1.3. Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod_{i\in I} A_i$. We will typically be interested in sections σ of $\left(\coprod_{i\in I} A_i, I, \pi\right)$.

Exercise 1.1.4. Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod_{i\in I} A_i$. Then σ is a section of $\coprod_{i\in I} A_i$ iff for each $i \in I$, $\sigma(i) \in A_i$

Proof. Clear. □

1.2. **Smooth Maps.**

Definition 1.2.1. Let $n \geq 1$. For $i = 1, \cdots, n$, define $x^i : \mathbb{R}^n \to \mathbb{R}$ by $x^i(a^1, \cdots, a^n) = a^i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on** $\mathbb{R}^n$.

Definition 1.2.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to x^i at a** if

$$\lim_{h\to 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If f is differentiable with respect to x^i at a, we define the **partial derivative of f with respect to x^i at a**, denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \left.\frac{\partial}{\partial x^i}\right|_a f$$

to be the limit above.

Definition 1.2.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **differentiable with respect to x^i** if for each $a \in U$, f is differentiable with respect to x^i at a.

Exercise 1.2.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i x^j}$ and $\frac{\partial^2 f}{\partial x^j x^i}$ exist and are continuous at a. Then

$$\frac{\partial^2 f}{\partial x^i x^j}(a) = \frac{\partial^2 f}{\partial x^j x^i}(a)$$

Proof. □

Definition 1.2.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $\frac{\partial^k f}{\partial i_1 \cdots i_k}$ exists and is continuous on U.

Definition 1.2.6. Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \to \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Theorem 1.2.7. Taylor's Theorem:
Let $U \subset \mathbb{R}^n$ be open and convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that for each $x \in U$,

$$f(x) = \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each $|\alpha| = T+1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. See analysis notes □

Definition 1.2.8. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Let $x^1, \cdots, x^n$ be the standard coordinate functions on $\mathbb{R}^n$ and $y_1, \cdots, y_m$ be the standard coordinate functions on $\mathbb{R}^m$. For $i \in \{1, \cdots, m\}$, we define the **ith component of** F, denoted $F^i : U \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \cdots, F_m)$

Definition 1.2.9. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \cdots, m\}$, the ith component of F, $F^i : U \to \mathbb{R}$, is smooth.

Definition 1.2.10. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. Then F is said to be a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 1.2.11. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F is a bijection and F and F^{-1} are smooth. Thus, F and F^{-1} are continuous and F is a homeomorphism. □

Definition 1.2.12. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \to \mathbb{R}^m$. We define the **Jacobian of F at p**, denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p) \right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Exercise 1.2.13. Inverse Function Theorem:
Let $U, V \subset \mathbb{R}^n$ be open and $F : U \to V$.

Exercise 1.2.14. Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \to F(N)$ is a diffeomorphism

Proof. content... □

1.3. **Topology.**

Definition 1.3.1. Let $(X, \mathbb{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 1.3.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Definition 1.3.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \to Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 1.3.4. Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \ncong \mathbb{R}^n$

2. Multilinear Algebra

2.1. (r, s)-Tensors.

Definition 2.1.1. Let $V_1, \dots, V_k, W$ be vector spaces and $\alpha : \prod_{i=1}^n V_i \to W$. Then α is said to be **multilinear** if for each $i \in \{1, \cdots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$,

$$\alpha(v_1, \cdots, v_i + cv, \cdots, v_k) = \alpha(v_1, \cdots, v_i, \cdots, v_k) + c\alpha(v_1, \cdots, v, \cdots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{\alpha : \prod_{i=1}^n V_i \to W : \alpha \text{ is multilinear}\right\}$$

Note 2.1.2. For the remainder of this section we let V denote an n-dimensional vector space with basis $\{e^1, \cdots, e^n\}$ with dual space V^* and dual basis $\{\epsilon_1, \cdots, \epsilon_n\}$ defined by $\epsilon^i(e^j) = \delta_{i,j}$. We identify V with V^{**} by the isomorphism $V \to V^{**}$ defined by $v \mapsto \hat{v}$ where $\hat{v}(\alpha) = \alpha(v)$ for each $\alpha \in V^*$.

Definition 2.1.3. Let $\alpha : (V^*)^r \times V^s \to \mathbb{R}$. Then α is said to be an (r, s)-tensor on V if $\alpha \in L(\underbrace{V^*, \dots, V^*}_{r}, \underbrace{V, \dots, V}_{s}; \mathbb{R})$. The set of all (r, s)-tensors on V is denoted $T^r_s(V)$.
When $r = s = 0$, we set $T^r_s = \mathbb{R}$.

Exercise 2.1.4. We have that $T^r_s(V)$ is a vector space.

Proof. Clear. □

Exercise 2.1.5. Under the identification of V with V^{**} as noted above, we have that $V = T^1_0(V)$.

Proof. By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T^1_0(V) \end{aligned}$$

□

Definition 2.1.6. Let $\alpha \in T^{r_1}_{s_1}(V)$ and $\beta \in T^{r_2}_{s_2}(V)$. We define the **tensor product of** α **with** β, denoted $\alpha \otimes \beta \in T^{r_1+r_2}_{s_1+s_2}(V)$, by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$.
When $r_1 = s_1 = r_2 = s_2 = 0$ (so that $\alpha, \beta \in \mathbb{R}$), we set $\alpha \otimes \beta = \alpha\beta$.

Definition 2.1.7. We define the **tensor product**, denoted $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

Exercise 2.1.8. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is well defined.

Proof. Tedious but straightforward. □

Exercise 2.1.9. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is associative.

Proof. Let $\alpha \in T^{r_1}_{s_1}(V)$, $\beta \in T^{r_2}_{s_2}(V)$ and $\gamma \in T^{r_3}_{s_3}(V)$. Then for each $u^* \in (V^*)^{r_1}, v^* \in (V^*)^{r_2}, w^* \in (V^*)^{r_3}, u \in V^{s_1}, v \in V^{s_2}, w \in V^{s_3}$,

$$\begin{aligned}(\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v)\gamma(w^*, w) \\ &= [\alpha(u^*, u)\beta(v^*, v)]\gamma(w^*, w) \\ &= \alpha(u^*, u)[\beta(v^*, v)\gamma(w^*, w)] \\ &= \alpha(u^*, u)(\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w)\end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

Exercise 2.1.10. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is bilinear.

Proof.

(1) Linearity in the first argument:
Let $\alpha, \beta \in T^{r_1}_{s_1}(V)$, $\gamma \in T^{r_2}_{s_2}(V)$, $\lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $vinV^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned}[(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v)\gamma(w^*, w) \\ &= [\alpha(v^*, v) + \lambda\beta(v^*, v)]\gamma(w^*, w) \\ &= \alpha(v^*, v)\gamma(w^*, w) + \lambda\beta(v^*, v)\gamma(w^*, w) \\ &= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w)\end{aligned}$$

So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

(2) Linearity in the second argument:
Similar to (1).

□

Definition 2.1.11.

(1) Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1, \cdots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length** k. Recall that $\#\mathcal{I}_{\otimes k} = n^k$.

(2) Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length** k. Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

Note 2.1.12. For the remainder of this section we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\otimes k}$.

Definition 2.1.13. Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.

(1) Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by

$$\epsilon^I = (\epsilon^{i_1}, \cdots, \epsilon^{i_k})$$

and

$$e^I = (e^{i_1}, \cdots, e^{i_k})$$

(2) Define $e^{\otimes I} \in T^k_0(V)$ and $\epsilon^{\otimes I} \in T^0_k(V)$ by
$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$
and
$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

Exercise 2.1.14. Let $\alpha, \beta \in T^r_s(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v_1^*, \dots, v_r^* \in V^*$ and $v_1, \dots, v_s \in V$. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, write
$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$
and
$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$
Then
$$\begin{aligned}
\alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1,\dots,k_r=1}^n \sum_{l_1,\dots,l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \sum_{k_1,\dots,k_r=1}^n \sum_{l_1,\dots,l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s)
\end{aligned}$$
So that $\alpha = \beta$. □

Exercise 2.1.15. Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K}\delta_{J,L}$.

Proof. Write $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$ and $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$. Then
$$\begin{aligned}
e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K)\epsilon^{\otimes J}(e^L) \\
&= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r})\epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\
&= \left[\prod_{m=1}^r e^{i_m}(\epsilon^{k_m})\right]\left[\prod_{n=1}^s \epsilon^{j_n}(e^{l_n})\right] \\
&= \left[\prod_{m=1}^r \delta_{i_m,k_m}\right]\left[\prod_{n=1}^s \delta_{j_n,l_n}\right] \\
&= \delta_{I,K}\delta_{J,L}
\end{aligned}$$
□

Exercise 2.1.16. The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T^r_s(V)$ and $\dim T^r_s(V) = n^{r+s}$.

Proof. Let $(a^I_J)_{I\in\mathcal{I}_r, J\in\mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum_{(I,J)\in\mathcal{I}_r\times\mathcal{I}_s} a^I_J e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a^I_J = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T^r_s(V)$. For $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b^I_J = \beta(\epsilon^J, e^I)$. Define

$\mu = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b^I_J e^{\otimes I} \otimes \epsilon^{\otimes J} \in T^r_s(V)$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b^I_J = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. $\square$

2.2. k-Tensors.

Definition 2.2.1. Let $\alpha : V^k \to \mathbb{R}$. Then α is said to be a **k-tensor on V** if $\alpha \in T^0_k(V)$. We will write $T_k(V)$ in place of $T^0_k(V)$.

Definition 2.2.2. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \to \mathbb{R}$ by

$$\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

The map $\alpha \mapsto \sigma\alpha$ is called the **permutation action** of S_k on $T_k(V)$

Exercise 2.2.3. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

(1) Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V), \sigma\alpha \in T_k(V)$.
(2) Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
(3) Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \cdots, v_k \in V$,

$$\begin{aligned}(\tau\sigma)\alpha(v_1, \cdots, v_k) &= \alpha(v_{\tau\sigma(1)}, \cdots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \cdots, v_k)\end{aligned}$$

□

Exercise 2.2.4. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \to T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$. Then

$$\begin{aligned}\sigma(c\alpha + \beta)(v_1, \cdots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \cdots, v_k) + \sigma\beta(v_1, \cdots, v_k)\end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

Definition 2.2.5. Let $\alpha \in T_k(V)$. Then α is said to be **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$. and α is said to be **alternating** if for each $\sigma \in S_k$, $\sigma\alpha = \operatorname{sgn}(\sigma)\alpha$. The set of symmetric k-tensors on V is denoted $\Xi_k(V)$ and the set of alternating k-tensors on V is denoted $\Lambda_k(V)$.

Definition 2.2.6. Define the **symmetric operator** $S : T_k(V) \to \Xi_k(V)$ by

$$S(\alpha) = \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \to \Lambda_k(V)$ by

$$A(\alpha) = \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha$$

Exercise 2.2.7.

(1) For $\alpha \in T_k(V)$, $S(\alpha)$ is symmetric.
(2) For $\alpha \in T_k(V)$, $A(\alpha)$ is alternating.

Proof.

(1) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}\sigma S(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \tau\alpha\right] \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \tau\alpha \\ &= S(\alpha)\end{aligned}$$

(2) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}\sigma A(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right] \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\sigma\tau\alpha \\ &= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\ &= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\ &= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \\ &= \operatorname{sgn}(\sigma)A(\alpha)\end{aligned}$$

□

Exercise 2.2.8.

(1) For $\alpha \in \Xi_k(V)$, $S(\alpha) = \alpha$.
(2) For $\alpha \in \Lambda_k(V)$, $A(\alpha) = \alpha$.

Proof.

(1) Let $\alpha \in \Xi_k(V)$. Then

$$\begin{aligned}S(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha \\ &= \frac{1}{k!}\sum_{\sigma \in S_k} \alpha \\ &= \alpha\end{aligned}$$

(2) Let $\alpha \in \Lambda_k(V)$. Then

$$\begin{aligned} A(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma\alpha \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2 \alpha \\ &= \alpha \end{aligned}$$

□

Exercise 2.2.9. The symmetric operator $S : T_k(V) \to \Xi_k(V)$ and the alternating operator $A : T_k(V) \to \Lambda_k(V)$ are linear.

Proof. Clear. □

Definition 2.2.10. Let $\alpha \in \Lambda_k(V)$ and $\beta \in \Lambda_l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda_{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to \Lambda_{k+l}(V)$.

Exercise 2.2.11. The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to T_{k+l}(V)$ is bilinear.

Proof. Clear. □

Exercise 2.2.12. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

(1) $A(A(\alpha) \otimes \beta) = A(\alpha \otimes \beta)$
(2) $A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \to S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

(1) Then

$$\begin{aligned}
A(A(\alpha)\otimes\beta) &= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\sigma\left[A(\alpha)\otimes\beta\right] \\
&= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\sigma\left[\left(\frac{1}{k!}\sum_{\tau\in S_k}\operatorname{sgn}(\tau)\tau\alpha\right)\otimes\beta\right] \\
&= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\sigma\left[\frac{1}{k!}\sum_{\tau\in S_k}\operatorname{sgn}(\tau)(\tau\alpha)\otimes\beta\right] \\
&= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}\operatorname{sgn}(\sigma)\sigma\left[\frac{1}{k!}\sum_{\tau\in S_k}\operatorname{sgn}(\tau)\tau(\alpha\otimes\beta)\right] \\
&= \frac{1}{k!(k+l)!}\sum_{\sigma\in S_{k+l}}\sum_{\tau\in S_k}\operatorname{sgn}(\sigma\tau)\sigma\tau(\alpha\otimes\beta) \\
&= \frac{k!}{k!(k+l)!}\sum_{\mu\in S_{k+l}}\operatorname{sgn}(\mu)\mu(\alpha\otimes\beta) \\
&= \frac{1}{(k+l)!}\sum_{\mu\in S_{k+l}}\operatorname{sgn}(\mu)\mu(\alpha\otimes\beta) \\
&= A(\alpha\otimes\beta)
\end{aligned}$$

(2) Similar to (1).

□

Exercise 2.2.13. The exterior product $\wedge : \Lambda_k(V)\times\Lambda_l(V)\to\Lambda_{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$ and $\gamma \in \Lambda_m(V)$. Then

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!}A(\alpha \otimes \beta)\right] \wedge \gamma \\
&= \frac{(k+l+m)!}{(k+l)!m!}A\left(\left[\frac{(k+l)!}{k!l!}A(\alpha \otimes \beta)\right] \otimes \gamma\right) \\
&= \frac{(k+l+m)!}{(k+l)!m!}\frac{(k+l)!}{k!l!}A(A(\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{m!}\frac{1}{k!l!}A((\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{k!(l+m)!}\frac{(l+m)!}{l!m!}A(\alpha \otimes (\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}\frac{(l+m)!}{l!m!}A(\alpha \otimes A(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}A(\alpha \otimes \frac{(l+m)!}{l!m!}A(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!}A(\alpha \otimes (\beta \wedge \gamma)) \\
&= \alpha \wedge (\beta \wedge \gamma))
\end{aligned}$$

□

Exercise 2.2.14. Let $\alpha_i \in \Lambda_{k_i}(V)$ for $i = 1, \cdots, m$. Then

$$\bigwedge_{i=1}^{m} \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!}A\left(\bigotimes_{i=1}^{m} \alpha_i\right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1+k_2+k_3)!}{k_1!k_2!k_3!}A(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i\right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A\left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A\left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} A\left(\bigotimes_{i=1}^{m_0-1} \alpha_i\right)\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(A\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A\left(\bigotimes_{i=1}^{m_0+1} \alpha_i\right)
\end{aligned}$$

□

Exercise 2.2.15. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl. (Hint: inversion number)

Proof.

$$\begin{aligned}
N(\tau) &= \sum_{i=1}^{l} k \\
&= kl
\end{aligned}$$

Since $\mathrm{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\mathrm{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.2.16. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \cdots, v_{k+l} \in V$, we have that

$$\begin{aligned}
\sigma\tau(\beta \otimes \alpha)(v_1, \cdots, v_l, v_{l+1}, \cdots v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}) \alpha(v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \\
&= \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\
&= \alpha \otimes \beta(v_{\sigma(1)}, \cdots v_{\sigma(k)}, v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\
&= \sigma(\alpha \otimes \beta)(v_1, \cdots, v_k, v_{1+k}, \cdots v_{l+k})
\end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned}
\beta \wedge \alpha &= \frac{(k+l)!}{k!l!} A(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\beta \otimes \alpha) \\
&= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma\tau)\sigma\tau(\beta \otimes \alpha) \\
&= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha \otimes \beta) \\
&= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \\
&= \operatorname{sgn}(\tau) \alpha \wedge \beta \\
&= (-1)^{kl} \alpha \wedge \beta
\end{aligned}$$

□

Exercise 2.2.17. Let $\alpha \in \Lambda_k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned}
\alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\
&= -\alpha \wedge \alpha
\end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.2.18. Fundamental Example:
Let $\alpha_1, \cdots, \alpha_m \in \Lambda_1(V)$ and $v_1, \cdots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}
\left(\bigwedge_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) &= m! A\left(\bigotimes_{i=1}^m \alpha_i \right)(v_1, \cdots, v_m) \\
&= m! \left[\frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\sigma\left(\bigotimes_{i=1}^m \alpha_i \right) \right](v_1, \cdots, v_m) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \left(\bigotimes_{i=1}^m \alpha_i \right)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
&= \det(\alpha_i(v_j))
\end{aligned}$$

□

Note 2.2.19. Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\wedge k}$.

Definition 2.2.20. Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.
Define $\epsilon^{\wedge I} \in \Lambda_k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$$

Exercise 2.2.21. Let $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \cdots & \epsilon^{i_1}(e^{j_k}) \\ & \vdots & \\ \epsilon^{i_k}(e^{j_1}) & \cdots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$. If $I = J$, then $A = I_{k \times k}$ and therefore $\epsilon^I(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the $l_0 th$ row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the $l_0 th$ column of A are 0. $\square$

Exercise 2.2.22. Let $\alpha, \beta \in \Lambda_k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \cdots, v_k \in V$. For $i = 1, \cdots, k$, write $v_i = \sum_{j_i=1}^n a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned}
\alpha(v_1, \cdots, v_k) &= \sum_{j_1, \cdots, j_k = 1}^n \left(\prod_{i=1}^k a_{i,j_i}\right) \alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{j_1 \neq \cdots \neq j_k}^n \left(\prod_{i=1}^k a_{i,j_i}\right) \alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \mathrm{sgn}(\sigma) \left(\prod_{i=1}^k a_{i,\sigma(j_i)}\right)\right] \alpha(e^J) \\
&= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \mathrm{sgn}(\sigma) \left(\prod_{i=1}^k a_{i,\sigma(j_i)}\right)\right] \beta(e^J) \\
&= \sum_{j_1, \cdots, j_k = 1}^n \left(\prod_{i=1}^k a_{i,j_i}\right) \beta(e^{j_1}, \cdots, e^{j_k}) \\
&= \beta(v_1, \cdots, v_k)
\end{aligned}$$

$\square$

Exercise 2.2.23. The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(V)$ and $\dim \Lambda_k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$. Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda_k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda_k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \mathrm{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$. $\square$

3. Smooth Manifolds

3.1. Manifolds.

Exercise 3.1.1. We have that $\mathbb{R}$ is homeomorphic to $(0, \infty)$

Proof. Define $f : \mathbb{R} \to (0, \infty)$ by $f(x) = e^x$. Then f is a homeomorphism. $\square$

Definition 3.1.2. We define $\mathbb{R}^0 = \{0\}$.

Definition 3.1.3. Let $n \in \mathbb{N}$. We define the **upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}^n$, by

$$\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\partial\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$$
$$\operatorname{Int}\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

We endow $\mathbb{H}^n$, $\partial\mathbb{H}^n$ and $\operatorname{Int}\mathbb{H}^n$ with the subspace topology inherited from $\mathbb{R}^n$.

Exercise 3.1.4. Let $n \in \mathbb{N}$.

(1) $\partial\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n-1}$
(2) $\operatorname{Int}\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^n$

Proof.

(1) Define $f : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ by
$$f(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$$
Then f is a homeomorphism.
(2) Define $f : \mathbb{R}^n \to \operatorname{Int}\mathbb{H}^n$ by $f(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, e^{x_n})$. Then f is a homeomorphism.

$\square$

Definition 3.1.5. Let M be a topological space and $n \in \mathbb{N}$. Let $U \subset M$ and $V \subset \mathbb{R}^n$. be open and $\phi : U \to V$. Then (U, ϕ) is said to be a **coordinate chart on** M if

- U is open in M
- V is open in $\mathbb{R}^n$ or V is open in $\mathbb{H}^n$
- ϕ is a homeomorphism

We denote the set of all coordinate charts on M by $X(M)$.

Definition 3.1.6. Let M be a topological space, $n \in \mathbb{N}$ and $(U, \phi) \in X(M)$. Then (U, ϕ) is said to be an

- **interior chart** if $\phi(U)$ is open in $\mathbb{R}^n$
- **boundary chart** if $\phi(U)$ is open in $\mathbb{H}^n$ and $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$

We denote the set of all interior charts on M and the set of all boundary charts on M by $X_{\text{Int}}(M)$ and $X_\partial(M)$ respectively.

Exercise 3.1.7. Let M be a topological space. Then

(1) $X(M) = X_{\text{Int}}(M) \cup X_\partial(M)$
(2) $X_{\text{Int}}(M) \cap X_\partial(M) = \varnothing$

Proof.

(1) By definition, $X_{\text{Int}}(M) \cup X_\partial(M) \subset X(M)$. Let $(U, \phi) \in X(M)$. Since (U, ϕ) is a coordinate chart on M, $\phi(U)$ is open in $\mathbb{R}^n$ or $\phi(U)$ is open in $\mathbb{H}^n$. If $\phi(U)$ is open in $\mathbb{R}^n$, then
$$\begin{aligned}(U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M)\end{aligned}$$
Suppose that $\phi(U)$ is open in $\mathbb{H}^n$. If $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$, then $\phi(U)$ is open in $\mathbb{R}^n$ and
$$\begin{aligned}(U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M)\end{aligned}$$
Suppose that $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$. Then
$$\begin{aligned}(U, \phi) &\in X_\partial(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M)\end{aligned}$$
So $X(M) \subset X_{\text{Int}}(M) \cup X_\partial(M)$.

(2) For the sake of contradiction, suppose that $X_{\text{Int}}(M) \cup X_\partial(M) \neq \varnothing$. Then there exists $(U, \phi) \in X(M)$ such that $(U, \phi) \in X_{\text{Int}}(M)$ and $(U, \phi) \in X_\partial(M)$. Therefore $\phi(U)$ is open in $\mathbb{R}^n$, $\phi(U)$ is open in $\mathbb{H}^n$ and $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$. Since $\phi(U)$ is open in $\mathbb{R}^n$ and $\phi(U) \subset \mathbb{H}^n$, $\phi(U) \subset \operatorname{Int}\mathbb{H}^n$ and therefore $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$ which is a contradiction.

□

Definition 3.1.8. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be an **n-dimensional manifold** if

(1) M is Hausdorff
(2) M is second countable
(3) for each $p \in M$, there exists $(U, \phi) \in X(M)$ such that $p \in U$

Definition 3.1.9. Let M be an n-dimensional manifold. We define the

- **interior** of M, denoted $\operatorname{Int} M$, by
$$\operatorname{Int} M = \{p \in M : \text{there exists } (U, \phi) \in X_{\text{Int}}(M) \text{ such that } p \in U\}$$
- **boundary** of M, denoted ∂M, by
$$\partial M = \{p \in M : \text{there exists } (V, \psi) \in X_\partial(M) \text{ such that } p \in V \text{ and } \psi(p) \in \partial\mathbb{H}^n\}$$

Exercise 3.1.10. Let M be an n-dimensional manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. If $\phi(p) \notin \partial\mathbb{H}^n$, then $p \in \operatorname{Int} M$.

Proof. Suppose that $\phi(p) \notin \partial\mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int}\mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then U' is open in M and $\phi' : U; \to B'$ is a homeomorphism. Hence $(U', \phi') \in X_{\text{Int}}(M)$. Since $\phi(p) \in B'$, $p \in U'$. By definition, $p \in \operatorname{Int} M$. □

Exercise 3.1.11. Let M be an n-dimensional manifold. Then

(1) $M = \operatorname{Int} M \cup \partial M$
(2) $\operatorname{Int} M \cap \partial M = \varnothing$

Hint: simply connected

Proof.

(1) By definition, $\operatorname{Int} M \cup \partial M \subset M$. Let $p \in M$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. The previous exercise implies that $(U, \phi) \in X_{\operatorname{Int}}(M) \cup X_\partial(M)$. If $(U, \phi) \in X_{\operatorname{Int}}(M)$, then by definition,

$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Suppose that $(U, \phi) \in X_\partial(M)$. If $\phi(p) \in \partial \mathbb{H}^n$, then by definition,

$$\begin{aligned} p &\in \partial M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Suppose that $\phi(p) \notin \partial \mathbb{H}^n$. The previous exercise implies that $p \in \operatorname{Int} M$. Therefore,

$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Hence $M \subset \operatorname{Int} M \cup \partial M$.

(2) For the sake of contradiction, suppose that $\operatorname{Int} M \cap \partial M \neq \varnothing$. Then there exists $p \in M$ such that $p \in \operatorname{Int} M \cap \partial M$. By definition, there exists $(U, \phi) \in X_{\operatorname{Int}}(M)$, $(V, \psi) \in X_\partial(M)$ such that $p \in U \cap V$ and $\psi(p) \in \partial \mathbb{H}^n$. Then $\psi(U \cap V)$ is open in $\mathbb{H}^n$ and $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi \circ \psi^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism. Since $\psi(U \cap V)$ is open in $\mathbb{H}^n$, there exists an $B_\psi \subset \psi(U \cap V)$ such that B_ψ is open in $\mathbb{H}^n$, B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$. Since $\phi(U \cap V)$ is open in $\mathbb{R}^n$, B_ϕ is open in $\mathbb{R}^n$. Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Then $\phi \circ \psi^{-1} : B'_\psi \to B'_\phi$ is a homeomorphism. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_ψ is simply connected. Since B_ϕ is open in $\mathbb{R}^n$, B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ. So $\partial M \cap \operatorname{Int} M = \varnothing$.

□

Exercise 3.1.12. Let M be an n-dimensional manifold, $(U, \phi) \in X(M)$ and $p \in U$. If $p \in \partial M$, then $(U, \phi) \in X_\partial(M)$.
Hint: simply connected

Proof. Suppose that $p \in \partial M$. Then there exists a $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Then $\phi \circ \psi^{-1} : \psi(V \cap U) \to \phi(V \cap U)$ is a homeomorphism. Since $\psi(U \cap V)$ is open in $\mathbb{H}^n$, there exists $B_\psi \subset \psi(U \cap V)$ such B_ψ is open in $\mathbb{H}^n$, B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$.
For the sake of contradiction, suppose that $(U, \phi) \in X_{\operatorname{Int}}(M)$. Then $\phi(U)$ is open in $\mathbb{R}^n$. Hence $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and B_ϕ is open in $\mathbb{R}^n$. Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Then $\phi \circ \psi^{-1} : B'_\psi \to B'_\phi$ is a homeomorphism. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_ψ is simply connected. Since B_ϕ is open in $\mathbb{R}^n$, B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ. So $(U, \phi) \notin X_{\operatorname{Int}}(M)$. A previous exercise implies that $(U, \phi) \in X_\partial(M)$.

□

Exercise 3.1.13. Let M be an n-dimensional manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. Then

(1) $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$
(2) $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n$

Proof.

(1) Suppose that $p \in \partial M$. For the sake of contradiction, suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then $p \in U'$ and $(U', \phi') \in X_{\mathrm{Int}}(M)$. Since $p \in U'$, the previous exercise implies that $(U', \phi') \in X_\partial(M)$. This is a contradiction since $X_{\mathrm{Int}}(M) \cap X_\partial(M) = \varnothing$. So $\phi(p) \in \partial \mathbb{H}^n$.
Conversely, suppose that $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$.

(2) A previous exercise implies that $\operatorname{Int} M = (\partial M)^c$. Part (1) implies that

$$\begin{aligned} p &\in (\partial M)^c \\ &= \operatorname{Int} M \end{aligned}$$

if and only if

$$\begin{aligned} \phi(p) &\in (\partial \mathbb{H}^n)^c \\ &= \operatorname{Int} \mathbb{H}^n \end{aligned}$$

□

Exercise 3.1.14. Let M be an n-dimensional manifold and $p \in M$. Then $p \in \partial M$ iff for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.

Proof. Suppose that $p \in \partial M$. Let $(U, \phi) \in X(M)$. Suppose that $p \in U$. The previous two exercises imply that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.
Conversely, suppose that for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. By assumption, $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that $\phi(p) \in \partial \mathbb{H}^n$ iff $p \in \partial M$. Since $\phi(p) \in \partial \mathbb{H}^n$, we have that $p \in \partial M$. □

Exercise 3.1.15. Let M be an n-dimensional manifold. Let $(U, \phi) \in X_\partial(M)$. Then

(1) $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$
(2) $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int} \mathbb{H}^n$

Proof.

(1) Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$. Let $q \in \phi(U \cap \partial M)$. Then there exists $p \in U \cap \partial M$ such that $\phi(p) = q$. Since $p \in \partial M$, $\phi(p) \in \partial \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \partial \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \partial M)$ is arbitrary, $\phi(U \cap \partial M) \subset \phi(U) \cap \partial \mathbb{H}^n$.
Let $q \in \phi(U) \cap \partial \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \partial \mathbb{H}^n$, we have that $p \in \partial M$. Hence $p \in U \cap \partial M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \partial M)$. Thus $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$.

(2) Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Let $q \in \phi(U \cap \operatorname{Int} M)$. Then there exists $p \in U \cap \operatorname{Int} M$ such that $\phi(p) = q$. Since $p \in \operatorname{Int} M$, $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \operatorname{Int} \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \operatorname{Int} M)$ is arbitrary, $\phi(U \cap \operatorname{Int} M) \subset \phi(U) \cap \operatorname{Int} \mathbb{H}^n$.
Let $q \in \phi(U) \cap \operatorname{Int} \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \operatorname{Int} \mathbb{H}^n$, we have that $p \in \operatorname{Int} M$. Hence $p \in U \cap \operatorname{Int} M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \operatorname{Int} M)$. Thus $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int} \mathbb{H}^n$.

□

Exercise 3.1.16. Let M be an n-dimensional manifold. Then

(1) $\operatorname{Int} M$ is open
(2) ∂M is closed

Proof.

(1) Let $p \in \operatorname{Int} M$. Then there exists $(U, \phi) \in X_{\text{Int}}(M)$ such that $p \in U$. By definition, for each $q \in U$, $q \in \operatorname{Int} M$. Hence $U \subset \operatorname{Int} M$. Since U is open and $p \in \operatorname{Int} M$ is arbitrary, for each $p \in \operatorname{Int} M$, there exists $U \subset \operatorname{Int} M$ such that U is open. Hence $\operatorname{Int} M$ is open.
(2) Since $\partial M = (\operatorname{Int} M)^c$, and $\operatorname{Int} M$ is open, we have that ∂M is closed.

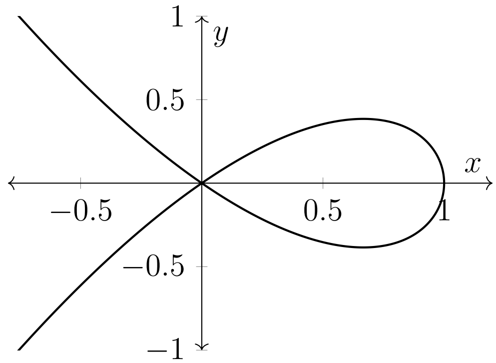
□

Theorem 3.1.17. Topological Invariance of Dimension Let M be an m-dimensional manifold, $(N, \mathcal{B})$ a n-dimensional manifold and $F : M \to N$. If F is a homeomorphism, then $m = n$.

Exercise 3.1.18. Graph of Continuous Function:
Let $f \in C(\mathbb{R})$. Set $M = \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$ (i.e. the graph of f). Then M is a 1-dimensional manifold.

Proof. Set $U = \mathbb{R}$ and define $\phi : U \to M$ by $\phi(x) = (x, f(x))$. Then $\phi^{-1} = \pi_1$. Since f is continuous, ϕ is continous. Since π_1 is continiuous, ϕ is a homeomorphism. □

Exercise 3.1.19. Nodal Cubic:
Let $M = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2 - x^3\}$. We equip M with the subspace topology.



Then M is not a 1-dimensional manifold.
Hint: connected components

Proof. Suppose that M is a 1-dimensional manifold. Set $p = (0, 0)$. Then there exists $(U, \phi) \in X(M)$ such that $p \in U$. Since $\phi(U)$ is open (in $\mathbb{R}$ or $\mathbb{H}$), there exists a $B \subset \phi(U)$ such that B is open (in $\mathbb{R}$ or $\mathbb{H}$), B is connected and $\phi(p) \in B$. Set $V = \phi^{-1}(B)$, $V' = V \setminus \{p\}$ and $B' = B \setminus \{\phi(p)\}$. Then $\phi : V \to B$ and $\phi' : V' \to B'$ are homeomorphisms. Since B is open (in $\mathbb{R}$ or $\mathbb{H}$) and connected, B' has at most two connected components. Then V' This is a contradiction since V' has four connected components and B' and V' are homeomorphic. $\square$

3.2. **Smooth Manifolds.**

Definition 3.2.1. Let M be an n-dimensional manifold and $(U,\phi),(V,\psi) \in X(M)$. Then (U,ϕ) and (V,ψ) are said to be **smoothly compatible** if

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \text{ is a diffeomorphism}$$

Definition 3.2.2. Let M be an n-dimensional manifold.

(1) Let $\mathcal{A} \subset X(M)$. Then $\mathcal{A}$ is said to be an **atlas on** M if $\bigcup\limits_{(U,\phi)\in\mathcal{A}} U = M$.
(2) Let $\mathcal{A}$ be an atlas on M. Then $\mathcal{A}$ is said to be **smooth** if for each $(U,\phi),(V,\psi) \in \mathcal{A}$, (U,ϕ) and (V,ψ) are smoothly compatible.
(3) Let $\mathcal{A}$ be a smooth atlas on M. Then $\mathcal{A}$ is said to be **maximal** if for each smooth atlas $\mathcal{B}$ on M, $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on** M.
(4) Let $\mathcal{A}$ be an atlas on M. Then $(M,\mathcal{A})$ is said to be an n**-dimensional smooth manifold** if $\mathcal{A}$ is a smooth structure on M.

Exercise 3.2.3. Let M be a topological space and $\mathcal{B}$ a smooth atlas on M. Then there exists a unique smooth structure $\mathcal{A}$ on M such that $\mathcal{B} \subset \mathcal{A}$.

Proof. Define $\mathcal{A}$ to be the set of all coordinate charts (U,ϕ) on M such that for each coordinate chart $(V,\psi) \in \mathcal{B}$, (U,ϕ) and (V,ψ) are smoothly compatible.
Clearly $\mathcal{B} \subset \mathcal{A}$.
Let $(U,\phi),(V,\psi) \in \mathcal{A}$ and $p \in U \cap V$. Then there exists $(W,\chi) \in \mathcal{B}$ such that $p \in W$. By assumption, $\phi \circ \chi^{-1} : \chi(U \cap W) \to \phi(U \cap W)$ and $\chi \circ \psi^{-1} : \psi(W \cap V) \to \chi(W \cap V)$ are diffeomorphisms. Then $(\phi\circ\chi^{-1})\circ(\chi\circ\psi^{-1}) = \phi\circ\psi^{-1} : \psi(U\cap W\cap V) \to \phi(U\cap W\cap V)$ is a diffeomorphism. Since for each $q \in \psi(U\cap V)$, there exits an open neighborhood $N \subset \psi(U\cap V)$ of q on which $\phi\circ\psi^{-1}$ are diffeomorphic, we have that $\phi\circ\psi^{-1}$ is a diffeomorphism on $\psi(U\cap V)$ and therefore (U,ϕ) and (V,ψ) are smoothly compatible. Hence $\mathcal{A}$ is a smooth atlas.
To see that $\mathcal{A}$ is maximal, let $\mathcal{B}'$ be a smooth atlas on M. Suppose that $\mathcal{A} \subset \mathcal{B}'$ and let $(U,\phi) \in \mathcal{B}'$. By definition, for each chart $(V,\psi) \in \mathcal{B}'$, (U,ϕ) and (V,ψ) are smoothly compatible. Since $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$, we have that $(U,\phi) \in \mathcal{A}$. So $\mathcal{A} = \mathcal{B}'$ and $\mathcal{A}$ is a maximal smooth atlas on M. $\square$

Exercise 3.2.4. Let $(M,\mathcal{A})$ be a n-dimensional smooth manifold, $(U,\phi) \in \mathcal{A}$ and $U' \subset U$. Define $\phi' : U' \to \phi(U')$ by $\phi' = \phi|_{U'}$. Then $(U',\phi') \in \mathcal{A}$.

Proof. Clearly $(U',\phi') \in X(M)$. Let $(V,\psi) \in \mathcal{A}$. Then $\phi \circ \psi : \psi(U \cap V) \to \phi(U \cap V)$ is a diffeomorphism. Since $U' \subset U$, $\phi \circ \psi : \psi(U' \cap V) \to \phi(U' \cap V)$ is a diffeomorphism. Since $\phi\circ\psi = \phi'\circ\psi$ on $U'\cap V$, $\phi'\circ\psi$ is a diffeomorphism. Therefore (U',ϕ') and (V,ψ) are smoothly compatible. Since $(V,\psi) \in \mathcal{A}$ is arbitrary, and $\mathcal{A}$ is maximal, $(U',\phi') \in \mathcal{A}$. $\square$

Definition 3.2.5. Let $(M,\mathcal{A})$ be a smooth n-dimensional manifold and $U \subset M$ open. We define $\mathcal{A}|_U = \{(U',\phi') \in \mathcal{A} : U' \subset U\}$.

Exercise 3.2.6. Smooth Open Submanifold:
Let $(M,\mathcal{A})$ be a smooth n-dimensional manifold and $U \subset M$ open. Then

(1) $\mathcal{A}|_U$ is a smooth structure on U
(2) $(U,\mathcal{A}|_U)$ is an smooth n-dimensional manifold
(3) $\partial U = \partial M \cap U$

Definition 3.2.7. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold. Define $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ by

$$\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$$

For $(U, \phi) \in \mathcal{A}$, set $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi \circ \phi|_{\bar{U}}$.
Define $\mathcal{A}|_{\partial M} = \{(\bar{U}, \bar{\phi}) : (U, \phi) \in \mathcal{A} \text{ and } U \cap \partial M \neq \varnothing\}$.

Exercise 3.2.8. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold. Then

(1) $\mathcal{A}|_{\partial M}$ is a smooth structure on ∂M
(2) $(\partial M, \mathcal{A}|_{\partial M})$ is a $(n-1)$-dimensional smooth manifold
(3) $\partial(\partial M) = \varnothing$

Proof.

(1) A previous exercise implies that π is a homeomorphism and it is clear that π is a diffeomorphism. Let $p \in \partial M$. Then there exists a boundary chart (U, ϕ) on M such that $p \in U$. Thus $p \in \bar{U}$ and the previous exercise implies that

$$\begin{aligned}\phi(\bar{U}) &= \phi(U \cap \partial M) \\ &= \phi(U) \cap \partial\mathbb{H}^n\end{aligned}$$

Since U is open in M and $\phi(U)$ is open in $\mathbb{H}^n$, we have that $\bar{U}$ is open in ∂M and $\phi(\bar{U})$ is open in $\partial\mathbb{H}^n$ which implies that $\pi(\phi(\bar{U}))$ is open in $\mathbb{R}^{n-1}$. Since $\phi : U \to \phi(U)$ is a homeomorphism, $\phi|_{\bar{U}} : \bar{U} \to \phi(\bar{U})$ is a homeomorphism. Hence $\bar{\phi} : \bar{U} \to \pi(\phi(\bar{U}))$ is a homeomorphism and $(\bar{U}, \bar{\phi})$ is an interior chart on ∂M. Therefore $\mathcal{A}|_{\partial M}$ is an atlas on ∂M. Let $(\bar{U}, \bar{\phi})$, $(\bar{V}, \bar{\psi}) \in \mathcal{A}|_{\partial M}$. Then

$$\bar{\phi} \circ \bar{\psi}^{-1} = \pi \circ \phi \circ \psi^{-1} \circ \pi^{-1}$$

which is a diffeomorphism. So $(\bar{U}, \bar{\phi})$ and $(\bar{V}, \bar{\psi})$ are smoothly compatible. Hence $\mathcal{A}$ is smooth.

(2) Subspaces of Hausdorff, second countable spaces are Hausdorff and second countable, so ∂M with the subspace topology is Hausdorff and second countable. Since $\mathcal{A}|_{\partial M}$ is an atlas on ∂M, $(\partial M, \mathcal{A}|_{\partial M})$ is and $(n-1)$-dimensional manifold.

(3) Since for each $(\bar{U}, \bar{\phi}) \in \mathcal{A}|_{\partial M}$, $\bar{\phi}(\bar{U})$ is open in $\mathbb{R}^n$, $(\bar{U}, \bar{\phi})$ is an interior chart. Hence $\partial(\partial M) = \varnothing$.

□

Note 3.2.9. For the rest of this section, we assume that $(M, \mathcal{A})$ is a n-dimensional smooth manifold and we denote the standard coordinate functions on $\mathbb{R}^n$ by $u^1, \cdots, u^n$. For a coordinate chart $(U, \phi) \in \mathcal{A}$ and $i \in \{1, \cdots, n\}$, we will typically denote the ith coordinate of ϕ by x^i, that is, $x^i = u^i(\phi)$.

Exercise 3.2.10. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $U' \subset S$ and $\phi' : \tilde{U} \to \phi(U')$ by $U' = U \cap S$ and $\phi' = \phi|_{U \cap S}$. Set $\mathcal{A}' = \{(U', \phi') : (U, \phi) \in \mathcal{A}\}$. Then $(S, \mathcal{A}')$ is a n-dimensional smooth manifold.

Proof. Since M is Hausdorff and second countable, so is S. Clearly $S = \bigcup\limits_{(U', \phi') \in \mathcal{A}'} U'$ and for each $(U', \phi') \in \mathcal{A}'$, ϕ' is a homeomorphism and $\phi'(U') \subset \mathbb{H}^n$. Let $(U', \phi'), (V', \psi') \in \mathcal{A}'$.

Then there exist (U, ϕ), $(V, \psi) \in \mathcal{A}$ such that $U' = U \cap S$, $V' = V \cap S$. Then $\psi' \circ \phi'^{-1}$: $\phi(U \cap V \cap S) \to \psi(U \cap V \cap S)$ is given by

$$\psi' \circ \phi'^{-1} : \phi(U' \cap V') \to \psi(U' \cap V') = \psi \circ \phi^{-1} : \phi(U \cap V \cap S) \to \psi(U \cap V \cap S)$$

which is smooth. So $\mathcal{A}'$ is a smooth atlas. Let $\mathcal{B}'$ be a smooth atlas on S. Suppose that $\mathcal{A}' \subset \mathcal{B}'$. □

Exercise 3.2.11. Let $(M, \mathcal{A})$ a smooth manifold an $U \subset M$ open. For $(V, \psi) \in \mathcal{A}$, define $\bar{V} \subset M$ and $\bar{\psi} : \bar{V} \to \psi(\bar{V})$ by $\bar{V} = V \cap U$ and $\bar{\psi} = \psi|_{V \cap U}$. Define

$$\mathcal{A} \cap U = \{(\bar{V}, \bar{\psi}) : (V, \psi) \in \mathcal{A}\}$$

Then

(1) $\mathcal{A}_U$ is an atlas on U
(2) $\mathcal{A}_U$ is a smooth structure on U
(3) $(U, \mathcal{A}_U)$ is an n-dimensional smooth manifold.

Definition 3.2.12. Let $(M, \mathcal{A})$ an n-dimensional smooth manifold an $U \subset M$ open. Define $(U, \mathcal{A}_U)$ as in the previous exercise.

3.3. **Smooth Maps.**

Definition 3.3.1. Let $f : M \to \mathbb{R}$. Then f is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M)$.

Exercise 3.3.2. We have that $C^\infty(M)$ is a vector space.

Proof. Clear. □

Definition 3.3.3. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$. We define

$$C^\infty(U) = \{f : U \to \mathbb{R} : f \circ \phi^{-1} \in C^\infty(\phi(U))\}$$

Note 3.3.4. Later we will give a subset $S \subset M$ the structure of a smooth manifold such that Definition 3.3.1 and Definition 3.3.3 coincide.

Exercise 3.3.5. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $p \in U$ and $f \in C^\infty(M)$. Then $f|_U \in C^\infty(U)$.

Proof. **FINISH!!!** □

Definition 3.3.6. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. We define the **partial derivative of f with respect to x^i**, denoted

$$\partial f / \partial x^i : U \to \mathbb{R} \text{ or } \partial_i f : U \to \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left(\frac{\partial}{\partial u^i}[f \circ \phi^{-1}]\right) \circ \phi$$

Exercise 3.3.7. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. Then $\partial / \partial x^i : C^\infty(U) \to C^\infty(U)$ is linear.

Proof. **FINISH!!!** □

Exercise 3.3.8. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f = \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}f &= \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j}f\right) \\
&= \frac{\partial}{\partial x^i}\left(\left[\frac{\partial}{\partial u^j}[f\circ\phi^{-1}]\right]\circ\phi\right) \\
&= \left(\frac{\partial}{\partial u^i}\left[\left(\left[\frac{\partial}{\partial u^j}[f\circ\phi^{-1}]\right]\circ\phi\right)\circ\phi^{-1}\right]\right)\circ\phi \\
&= \left(\frac{\partial}{\partial u^i}\left[\frac{\partial}{\partial u^j}[f\circ\phi^{-1}]\right]\right)\circ\phi \\
&= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f\circ\phi^{-1}]\right)\circ\phi
\end{aligned}$$

□

Exercise 3.3.9. Let $(M,\mathcal{A})$ be a smooth manifold, $(U,\phi)\in\mathcal{A}$ with $\phi=(x^1,\cdots,x^n)$ and $i,j\in\{1,\cdots,n\}$. Then

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}=\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}$$

Proof. Let $f\in C^\infty(U)$. Since $f\circ\phi^{-1}$ is smooth,

$$\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f\circ\phi^{-1}]=\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f\circ\phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned}
\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}f &= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f\circ\phi^{-1}]\right)\circ\phi \\
&= \left(\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f\circ\phi^{-1}]\right)\circ\phi \\
&= \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}f
\end{aligned}$$

□

Exercise 3.3.10. Let $(M,\mathcal{A})$ be a smooth manifold, $(U,\phi)\in\mathcal{A}$ with $\phi=(x^1,\cdots,x^n)$ and $f\in C^\infty(U)$. Then for each $\alpha\in\mathbb{N}_0^n$,

$$\partial^\alpha f=(\partial^\alpha[f\circ\phi^{-1}])\circ\phi$$

Proof. The claim is clearly true when $|\alpha|=0$ or by definition if $|\alpha|=1$. Let $n\in\mathbb{N}$ and suppose the claim is true for each $|\alpha|\in\{1,\ldots,n-1\}$. Then there exists $i\in\{1,\ldots,n\}$ such that $\alpha_i \geq 1$. Hence

$$\begin{aligned}\partial^\alpha f &= \partial^{e^i}(\partial^{\alpha - e^i} f) \\ &= \partial^{e^i}(\partial^{\alpha - e^i}[f \circ \phi^{-1}] \circ \phi) \\ &= (\partial^{e^i}[(\partial^{\alpha - e^i}[f \circ \phi^{-1}] \circ \phi) \circ \phi^{-1}]) \circ \phi \\ &= (\partial^{e^i}[\partial^{\alpha - e^i}[f \circ \phi^{-1}]]) \circ \phi \\ &= (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi\end{aligned}$$

□

Exercise 3.3.11. Taylor's Theorem:
Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\phi(U)$ convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that

$$f = \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(x_0) \right] + \sum_{|\alpha|=T+1} (x^i - x^i(p))^\alpha g_\alpha$$

and for each $|\alpha| = T+1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. Since $\phi(U)$ is open and convex and $f \circ \phi^{-1} \in C^\infty(\phi(U))$, Taylors thorem in section 2.1 implies that there exist $(\tilde{g}_\alpha)_{|\alpha|=T+1} \subset C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q))$$

and for each $|\alpha| = T+1$,

$$\begin{aligned}\tilde{g}_\alpha(\phi(p)) &= \frac{1}{(T+1)!} \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \\ &= \frac{1}{(T+1)!} \partial^\alpha f(p)\end{aligned}$$

For $|\alpha| = T+1$, set $g_\alpha = \tilde{g} \circ \phi$. Then

$$\begin{aligned}f(q) &= f \circ \phi^{-1}(\phi(q)) \\ &= \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\ &= \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q)\end{aligned}$$

□

Definition 3.3.12. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$. Then F is said to be

- **smooth** if for each $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$,
$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(F(U) \cap V)$$
is smooth
- a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 3.3.13. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifold and $F : M \to N$. If F is smooth, then F is continuous.

Proof. Suppose that F is smooth. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $F(p) \in V$. Put $\tilde{U} = U \cap F^{-1}(V)$ and $\tilde{V} = F(U) \cap V$.
Define $\tilde{\phi} : \tilde{U} \to \phi(\tilde{U})$ and $\tilde{\psi} : \tilde{V} \to \psi(\tilde{V})$ by
$$\tilde{\phi} = \phi|_{\tilde{U}}, \;\; \tilde{\phi} = \psi|_{\tilde{V}}$$
Then $\tilde{\phi}$ and $\tilde{\psi}$ are homeomorphisms, $p \in \tilde{U}$ and $F(\tilde{U}) \subset \tilde{V}$. Define $\tilde{F} : \phi(\tilde{U}) \to \psi(\tilde{V})$ by
$$\tilde{F} = \tilde{\psi} \circ F \circ \tilde{\phi}^{-1}$$
By definition, $\tilde{F}$ is smooth and therefore continuous. Since ϕ and ψ are homeomorphisms and $F|_{\tilde{U}} = \tilde{\psi}^{-1} \circ \tilde{F} \circ \tilde{\phi}$, we have that $F|_{\tilde{U}}$ is continuous. In particular, F is continuous at p and since $p \in M$ is arbitrary, F is continuous. □

Exercise 3.3.14. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifold and $F : M \to N$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F and F^{-1} are smooth. The previous exercise implies that F and F^{-1} are continuous. Hence F is a homeomorphism. □

Exercise 3.3.15. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ a diffeomorphism. Then for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

Proof. Let $(V, \psi) \in \mathcal{B}$.

(1) Since ϕ and F^{-1} are homeomorphisms, $\phi \circ F^{-1} : F(U) \cap V \to \phi(U \cap F^{-1}(V))$ is a homeomorphism
(2) Since F is a diffeomorphism,
$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \to \phi(U \cap F^{-1}(V))$$
and
$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \psi(V \cap F(U))$$
are smooth.

Therefore $(F(U), \phi \circ F^{-1})$ and (V, ψ) are smoothly compatible. Since $\mathcal{B}$ is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. □

Definition 3.3.16. Let $(N, \mathcal{B})$ be a smooth n-dimensional manifold, $F : M \to N$ smooth and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. For $i \in \{1, \dots, n\}$, We define the ***i*-th component of *F* with respect to** (V, ψ), denoted $F^i : V \to \mathbb{R}$, by
$$F^i = y^i \circ F$$

3.4. **Partitions of Unity.**

Definition 3.4.1. Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then ρ is said to be a **bump function at p supported in** U if

(1) $\rho \geq 0$
(2) there exists $V \in \mathcal{N}_p$ such that V is open and $\rho|_V = 1$
(3) $\operatorname{supp} \rho \subset U$

Exercise 3.4.2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

Proof. □

3.5. **The Tangent Space.**

Definition 3.5.1. Let $p \in M$. Define the relation $\sim_p$ on $C^\infty(M)$ by $f \sim_p g$ iff there exists $U \in \mathcal{N}_p$ such that U is open and $f|_U = g|_U$. Clearly $\sim_p$ is an equivalence relation on $C^\infty(M)$. We denote $C^\infty(M)/\sim_p$ by $C_p^\infty(M)$. For $f \in C^\infty(M)$, we define the **germ of f at** p to be the equivalence class of f under $\sim_p$.

Exercise 3.5.2. Let $p \in$We have that $C_p^\infty(M)$ is a vector space.

Proof. Clear. □

Definition 3.5.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. For $i \in \{1, \cdots, n\}$, define the partial derivative with respect to x^i at p, denoted

$$\left.\frac{\partial}{\partial x^i}\right|_p : C_p^\infty(M) \to \mathbb{R}, \text{ or } \partial_i|_p : C_p^\infty(M) \to \mathbb{R}$$

by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p)$$

Exercise 3.5.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$, we have that

$$\frac{\partial x^i}{\partial x^j}(p) = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \cdots, n\}$. Then

$$\begin{aligned}
\left.\frac{\partial}{\partial x^j}\right|_p x^i &= \left.\frac{\partial}{\partial u^j}\right|_{\phi(p)} x^i \circ \phi^{-1} \\
&= \left.\frac{\partial}{\partial u^j}\right|_{\phi(p)} u^i \circ \phi \circ \phi^{-1} \\
&= \left.\frac{\partial}{\partial u^j}\right|_{\phi(p)} u^i \\
&= \delta_{i,j}
\end{aligned}$$

□

Exercise 3.5.5. Change of Coordinates:
Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $\psi = (y^1, \cdots, y^n)$, $p \in U \cap V$ and $f \in C_p^\infty(M)$. Then for each $i \in \{1, \cdots, n\}$,

$$\left.\frac{\partial}{\partial y^i}\right|_p = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i}(p) \left.\frac{\partial}{\partial x^j}\right|_p$$

Proof. Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \cdots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned}
\frac{\partial}{\partial y^i}\bigg|_p f &= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \psi^{-1} \\
&= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \phi^{-1} \circ h \\
&= \sum_{j=1}^n \bigg(\frac{\partial}{\partial u^j}\bigg|_{h\circ\psi(p)} f \circ \phi^{-1} \bigg) \bigg(\frac{\partial}{\partial u^i}\bigg|_{\psi(p)} h_j \bigg) \\
&= \sum_{j=1}^n \bigg(\frac{\partial}{\partial u^j}\bigg|_{\phi(p)} f \circ \phi^{-1} \bigg) \bigg(\frac{\partial}{\partial u^i}\bigg|_{\psi(p)} x^j \circ \psi^{-1} \bigg) \\
&= \sum_{j=1}^n \bigg(\frac{\partial}{\partial x^j}\bigg|_p f \bigg) \bigg(\frac{\partial}{\partial y^i}\bigg|_p x^j \bigg)
\end{aligned}$$

□

Definition 3.5.6. Let $p \in M$ and $v : C_p^\infty(M) \to \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C_p^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation at** p if for each $f, g \in C_p^\infty(M)$ and $a \in \mathbb{R}$,

(1) v is linear
(2) v is Leibnizian

We define the **tangent space of** M **at** p, denoted T_pM, by

$$T_pM = \{v : C_p^\infty(M) \to \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 3.5.7. Let $f \in C_p^\infty(M)$ and $v \in T_pM$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. □

Exercise 3.5.8. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

is a basis for T_pM and $\dim T_pM = n$.

Proof. Clearly $\frac{\partial}{\partial x^1}\big|_p, \cdots, \frac{\partial}{\partial x^n}\big|_p \in T_pM$. Let $a_1, \cdots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \left.\frac{\partial}{\partial x^i}\right|_p x^j \\ &= a_j \end{aligned}$$

Hence $\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$ is independent.
Now, let $v \in T_pM$ and $f \in \mathbb{C}_p^\infty(M)$. By Taylor's theorem, there exist $g_1, \cdots g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \left.\frac{\partial}{\partial x^i}\right|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i) \left.\frac{\partial}{\partial x^i}\right|_p f \\ &= \left[\sum_{i=1}^n v(x^i) \left.\frac{\partial}{\partial x^i}\right|_p \right] f \end{aligned}$$

So

$$v = \sum_{i=1}^n v(x^i) \left.\frac{\partial}{\partial x^i}\right|_p$$

and

$$v \in \operatorname{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

□

Definition 3.5.9. Let $(N, \mathcal{B})$ be a smooth manifold, $F: M \to N$ smooth and $p \in M$. We define the **differential of** F **at** p, denoted $dF_p : T_pM \to T_{F(p)}N$, by

$$\Big[dF_p(v)\Big](f) = v(f \circ F)$$

for $v \in T_pM$ and $f \in C^\infty_{F(p)}(N)$.

Exercise 3.5.10. Let $(N, \mathcal{B})$ be a smooth manifold, $F: M \to N$ smooth and $p \in M$. Then for each $v \in T_pM$, $dF_p(v)$ is a derivation.

Proof. Let $v \in T_pM$, $f, g \in C^\infty_{F(p)}(N)$ and $c \in \mathbb{R}$. Then

(1)
$$\begin{aligned} dF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= dF_p(v)(f) + cdF_p(v)(g) \end{aligned}$$

So $dF_p(v)$ is linear.

(2)
$$\begin{aligned} dF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= dF_p(v)(f) * g(F(p)) + f(F(p)) * dF_p(v)(g) \end{aligned}$$

So $dF_p(v)$ is Leibnizian and hence $dF_p(v) \in T_{F(p)}N$ □

Exercise 3.5.11. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. If F is a diffeomorphism, then dF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \cdots, x^n)$ and $\phi \circ F^{-1} = (y^1, \cdots, y^n)$. Let $f \in C^\infty_{F(p)}(N)$ Then

$$\begin{aligned} \frac{\partial}{\partial y^i}\bigg|_{F(p)} f &= \frac{\partial}{\partial u^i}\bigg|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \frac{\partial}{\partial x^i}\bigg|_{p} f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right)\right](f) &= \frac{\partial}{\partial x^i}\bigg|_p f \circ F \\ &= \frac{\partial}{\partial y^i}\bigg|_{F(p)} f \end{aligned}$$

Hence

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial y^i}\bigg|_{F(p)}$$

Since $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis for T_pM and $\left\{ \frac{\partial}{\partial y^1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}\Big|_{F(p)} \right\}$ is a basis for $T_{F(p)}N$, dF_p is an isomorphism. □

Exercise 3.5.12. Let $(M, \mathcal{A})$ be a smooth m-dimensional manifold, $(N, \mathcal{B})$ a n-dimensional smooth manifold, $F : M \to N$ smooth, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^m)$, $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$ and $p \in U$. Define the ordered bases $B_\phi = \left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^m}\Big|_p \right\}$ and $B_\psi = \left\{ \frac{\partial}{\partial y^1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}\Big|_{F(p)} \right\}$. Then the matrix representation of dF_p with respect to the bases B_ϕ and B_ψ is

$$dF_p^{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Proof. Let $(dF_p)_{B_\phi, B_\psi} = (a_{i,j})_{i,j} \in \mathbb{R}^{n \times m}$. Then for each $j \in \{1, \dots, m\}$,

$$dF_p\left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i}\bigg|_{F(p)}$$

This implies that

$$\begin{aligned} dF_p\left(\frac{\partial}{\partial x^j}\bigg|_p \right)(y^k) &= \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i}\bigg|_{F(p)}(y^k) \\ &= \sum_{i=1}^n a_{i,j}\delta_{i,k} \\ &= a_{k,j} \end{aligned}$$

By definition,

$$\begin{aligned} dF_p\left(\frac{\partial}{\partial x^j}\bigg|_p \right)(y^k) &= \frac{\partial}{\partial x^j}\bigg|_p y^k \circ F \\ &= \frac{\partial}{\partial x^j}\bigg|_p F^k \\ &= \frac{\partial F^k}{\partial x^j}(p) \end{aligned}$$

□

Note 3.5.13. Since $\operatorname{rank} dF_p$ is independent of basis, it is independent of coordinate charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$.

Definition 3.5.14. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ a diffeomorphism. Define the **push forward of** F, denoted

$$F_* : M \to \coprod_{p \in M} \mathrm{Iso}(T_p M, T_{F(p)} N)$$

by

$$p \mapsto dF_p$$

3.6. **The Cotangent Space.**

Definition 3.6.1. Let $p \in M$. We define the **cotangent space of** M **at** p, denoted T_p^*M, by

$$T_p^*M = (T_pM)^*$$

Definition 3.6.2. Let $f \in C^\infty(M)$. We define the **differential of** f **at** p, denoted $df_p : T_pM \to \mathbb{R}$, by

$$df_p(v) = vf$$

Exercise 3.6.3. Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

Proof. Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1 f + \lambda v_2 f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that df_p is linear and hence $df_p \in T_p^*M$. □

Exercise 3.6.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$,

$$dx^i_p\left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta_{i,j}$$

In particular, $\{dx^1_p, \cdots, dx^n_p\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1}\big|_p, \cdots, \frac{\partial}{\partial x^n}\big|_p \right\}$ and $T_p^*M = \operatorname{span}\{dx^1_p, \cdots, dx^n_p\}$.

Proof. Let $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned} \left[dx^i_p\left(\frac{\partial}{\partial x^j}\bigg|_p \right) \right]_p &= \frac{\partial}{\partial x^j}\bigg|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 3.6.5. Let $f \in C^\infty(M)$, (U, ϕ) a chart on M with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) {dx^i}_p$$

Proof. Since $\{dx^1_p, \cdots, dx^n_p\}$ is a basis for T_p^*M, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum\limits_{i=1}^n a_i(p) dx^i_p$. Therefore, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^j}\bigg|_p \right) &= \sum_{i=1}^n a_i(p) dx^i_p\left(\frac{\partial}{\partial x^j}\bigg|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^j}\bigg|_p \right) &= \frac{\partial}{\partial x^j}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p) dx^i_p$$

$\square$

3.7. Maps of Full Rank.

Definition 3.7.1. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map and $p \in M$. We define the **rank of F at** p, denoted $\operatorname{rank}_p F$, by $\operatorname{rank}_p F = \operatorname{rank} dF_p$. We say that F has **constant rank** if for each $p, q \in M$, $\operatorname{rank}_p F = \operatorname{rank}_q F$. If F has constant rank, we define the **rank of** F, denoted $\operatorname{rank} F$, by $\operatorname{rank} F = \operatorname{rank}_p F$.

Definition 3.7.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. Then F is said to be

- an **immersion** if for each $p \in M$, $dF_p : T_pM \to T_{F(p)}N$ is injective
- a **submersion** if for each $p \in M$, $dF_p : T_pM \to T_{F(p)}N$ is surjective

Exercise 3.7.3. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map.

Definition 3.7.4. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$ smooth. Then F is said to be an **embedding** if

(1) F is an immersion
(2) $F : M \to F(M)$.

Note 3.7.5. Here the topology on $F(M)$ is the subspace topology.

3.8. **Submanifolds.**

Exercise 3.8.1. Let $(M, \mathcal{A})$ be a smooth manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $\tilde{U} \subset S$ and $\tilde{\phi} : \tilde{U} \to \phi(\tilde{U})$ by $\tilde{U} = U \cap S$ and $\tilde{\phi} = \phi|_{U \cap S}$. Set $\mathcal{B} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \in \mathcal{A}\}$. Then $\mathcal{B}$ is a smooth structure on S.

Proof. $\square$

Definition 3.8.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. Suppose that $M \subset N$. Then $(M, \mathcal{A})$ is said to be

(1) an **immersed submanifold** of $(N, \mathcal{B})$ if $\mathrm{id} : M \to N$ is a smooth immersion
(2) an **embedded submanifold** of $(N, \mathcal{B})$ if $\mathrm{id} : M \to N$ is a smooth embedding

Note 3.8.3. Essentially, embedded submanifolds are immersed submanifolds with the subspace topology.

Note 3.8.4. For the remainder of this section, we assume that $k \leq n$.

Definition 3.8.5. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Then S is said to be a **k-slice** of U if $S = \{u \in U : u^{k+1}, \dots, u^n = 0\}$.

Exercise 3.8.6. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Suppose that S is a k-slice of U. Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then $\pi|_S \to \pi(S)$ is a diffeomorphism.

Proof. Clear. □

Definition 3.8.7. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ and $S \subset U$. Then S is said to be a **k-slice** of U if $\phi(S)$ is a k-slice of $\phi(U)$.

Definition 3.8.8. Let $(M, \mathcal{A})$ be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$. Then (U, ϕ) is said to be a **k-slice chart for** S if $U \cap S$ is a k-slice of U.

Exercise 3.8.9. Let $(M, \mathcal{A})$ be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. If (U, ϕ) is a k-slice chart for S, then $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$.

Proof. Clear. □

Definition 3.8.10. Let $(M, \mathcal{A})$ be a smooth manifold and $S \subset M$. Then S is said to satisfy the **local k-slice condition** if for each $p \in S$, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k-slice chart of S.

Exercise 3.8.11. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold and $S \subset M$ a subspace. If S satisfies the local k-slice condition, then there exists a smooth structure $\tilde{\mathcal{A}}$ on S such that $(S, \tilde{\mathcal{A}})$ is an embedded submanifold of M.

Proof. Suppose that S satisfies the local k-slice condition. Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ as above Let $(U, \phi) \in \mathcal{A}$. Suppose that (U, ϕ) is a k-slice chart for S. Define $\tilde{U} = U \cap S$ and $\tilde{\phi} : \tilde{U} \to \pi \circ \phi(\tilde{U})$ by

$$\tilde{\phi} = \pi \circ \phi|_{\tilde{U}}$$

By definition, $\phi(\tilde{U})$ is a k-slice of $\phi(U)$. A previous exercise implies that $\pi|_{\phi(\tilde{U})} \to \pi \circ \phi(\tilde{U})$ is a diffeomorphism and hence a homeomorphism. Thus $\tilde{\phi}$ is a homeomorphism.
Define

$$\tilde{\mathcal{B}} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \text{ is a } k\text{-slice for } S\}$$

Let $p \in S$. By assumption, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k-slice chart of S. Then $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{B}}$ and $\mathcal{A}$ is an atlas on S. By construction of $\tilde{\mathcal{B}}$, S is locally half

Euclidean of dimension k. Since M is second countable Hausdorff, so is S in the subspace topology. Thus $(S, \tilde{\mathcal{B}})$ is a k-dimensional manifold. Let $(\tilde{U}, \tilde{\phi})$, $(\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$. Then

$$\tilde{\phi} \circ \tilde{\psi}^{-1}|_{\tilde{U} \cap \tilde{V}} = \pi|_{\phi(\tilde{U} \cap \tilde{V})} \circ \phi|_{\tilde{U} \cap \tilde{V}} \circ \psi|^{-1}_{\tilde{U} \cap \tilde{V}} \circ \pi|^{-1}_{\psi(\tilde{U} \cap \tilde{V})}$$

which is a diffeomorphism. So $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ smoothly compatible. Hence $\tilde{\mathcal{B}}$ is smooth. An exercise in section 4.1 implies that there exists a unique smooth structure $\tilde{\mathcal{A}}$ on S such that $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$. So $(S, \tilde{\mathcal{A}})$ is a smooth k-dimensional manifold.
Clearly $\mathrm{id} : S \to S$ is a homeomorphism. Let $(V, \psi) \in \mathcal{A}$ and $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$.

Finish!! $\square$

Definition 3.8.12.

Exercise 3.8.13.

4. Vector Bundles and Tensor Fields

4.1. The Vector Bundle.

Definition 4.1.1. Let E, M and F be smooth manifolds and $\pi : E \to M$ a smooth surjection, $U \subset M$ open and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **smooth local trivialization of E over U** if

(1) Φ is a diffeomorphism
(2) $\pi_U \circ \Phi = \pi|_{\pi^{-1}(U)}$ (where $\pi_U : U \times F \to U$ denotes projection onto U)

Exercise 4.1.2. Let E, M and F be topological spaces and $\pi : E \to M$ a continuous surjection and (U, Φ) a local trivialization of E over U. Then for each $A \subset U$,

$$\Phi(\pi^{-1}(A)) = A \times F$$

Hint: show that $\pi^{-1}(A) = (\pi_U \circ \Phi)^{-1}(A)$

Proof. Let $A \subset U$. Since $\pi^{-1}(A) \subset \pi^{-1}(U)$, property (2) implies that $\pi^{-1}(A) = (\pi_U \circ \Phi)^{-1}(A)$. Since Φ is a bijection,

$$\begin{aligned}\Phi(\pi^{-1}(A)) &= \Phi \circ (\pi_U \circ \Phi)^{-1}(A)] \\ &= \Phi \circ \Phi^{-1}(\pi_U^{-1}(A)) \\ &= \pi_U^{-1}(A) \\ &= A \times F\end{aligned}$$

□

Definition 4.1.3. Let E and M be topological spaces and $\pi : E \to M$ a continuous surjection. Then (E, M, π) is said to be a **smooth vector bundle of rank** n if

(1) for each $p \in M$, $\pi^{-1}(\{p\})$ is a n-dimensional real vector space.
(2) for each $p \in M$, there exist open $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that (U, Φ) is a smooth local trivialization of E over U.
(3) for each $p \in M$,
$$\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \to \{p\} \times \mathbb{R}^n$$
is an isomorphism.

Exercise 4.1.4. Let M be a n-dimensional smooth manifold. Set $E = M \times \mathbb{R}^n$ and define $\pi : E \to M$ by $\pi(p, x) = p$. Then (E, M, π) is a smooth vector bundle of rank n.

Proof.

(1) For each $p \in M$, $\pi_1^{-1}(\{p\}) = \{p\} \times \mathbb{R}^n$ which may be given the obvious vector space structure.
(2) Let $p \in M$. Set $U = M$. Then $\pi^{-1}(U) = E$. Define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by $\Phi = \mathrm{id}_E$. Then (U, Φ) is a smooth local trivialization of E over U.
(3) Let $p \in M$. Then $\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \to \{p\} \times \mathbb{R}^n$ is clearly an isomorphism.

□

Theorem 4.1.5. Let E and M be smooth manifolds and $\pi : E \to M$ a smooth surjection.

Definition 4.1.6. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p \in M} T_pM$$

We denote the natrual projection map by $\pi : TM \to M$.

Definition 4.1.7. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Define $\tilde{U} \subset TM$ and $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}^n$ by

- $\tilde{U} = \pi^{-1}(U)$
- $$\begin{aligned} \tilde{\phi}\bigg(\sum_{i=1}^n v^i \, \frac{\partial}{\partial x^i}\bigg|_p \bigg) &= (\phi(p), v) \\ &= (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \end{aligned}$$
  ,

Exercise 4.1.8. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}$ is a bijection.

4.2. The cotangent Bundle.

Definition 4.2.1. We define the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p\in M} T_p^*M$$

4.3. The (r, s)-Tensor Bundle.

Definition 4.3.1.

(1) the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p\in M} T_p^*M$$

(2) the (r, s)**-tensor bundle of** M, denoted T_s^rM, by

$$T_s^rM = \coprod_{p\in M} T_s^r(T_pM)$$

(3) the k**-alternating tensor bundle of** M, denoted $\Lambda_k(M)$, by

$$\Lambda_kM = \coprod_{p\in M} \Lambda_k(T_pM)$$

4.4. **Vector Fields.**

Definition 4.4.1. Let $X : M \to TM$. Then X is said to be a **vector field on** M if for each $p \in M$, $X_p \in T_pM$.
For $f \in \mathbb{C}^\infty(M)$, we define $Xf : M \to \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in \mathbb{C}^\infty(M)$, Xf is smooth.
We denote the set of smooth vector fields on M by $\Gamma^1(M)$.

Definition 4.4.2. Let $f \in C^\infty(M)$ and $X, Y \in \Gamma^1(M)$. We define

- $fX \in \Gamma^1(M)$ by

$$(fX)_p = f(p)X_p$$

- $X + Y \in \Gamma^1(M)$ by

$$(X + Y)_p = X_p + Y_p$$

Exercise 4.4.3. The set $\Gamma^1(M)$ is a $C^\infty(M)$-module.

Proof. Clear. $\square$

Exercise 4.4.4. Let $X \in \Gamma^1(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then

$$X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$$

Proof. Let $p \in M$. Then $X_p \in T_pM$ and $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis of T_pM. So there exist $f_1(p), \cdots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum_{i=1}^n f^i(p) \frac{\partial}{\partial x^i}\Big|_p$. Let $j \in \{1, \cdots, n\}$. Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^n f^i(p) \frac{\partial x^j}{\partial x^i}(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx^j = f_j$ and $X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$. $\square$

Exercise 4.4.5. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then for each $i \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

Proof. Let $i \in \{1, \cdots, n\}$ and $f \in C^\infty(M)$. Define $g : M \to \mathbb{R}$ by $g = \frac{\partial}{\partial x^i} f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i}\bigg|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence g is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x^i}$ is smooth. $\square$

4.5. 1-Forms.

Definition 4.5.1. Let $\omega : M \to T^*M$. Then ω is said to be a **1-form on** M if for each $p \in M$, $\omega_p \in T_p^*M$.
For each $X \in \Gamma^1(M)$, we define $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)$, $\omega(X)$ is smooth.
The set of smooth 1-forms on M is denoted $\Gamma_1(M)$.

Definition 4.5.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma^1(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta \in \Gamma^1(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 4.5.3. The set $\Gamma_1(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Exercise 4.5.4.

4.6. (r, s)-Tensor Fields.

Definition 4.6.1. Let $\alpha : M \to T^r_s M$. Then α is said to be a **(r, s)-tensor field on** M if for each $p \in M$, $\alpha_p \in T^r_s(T_p M)$.
For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \to \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and α is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth. The set of smooth (r, s)-tensor fields on M is denoted $\Gamma^r_s(M)$.

Definition 4.6.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma^r_s(M)$. We define

- $f\alpha : M \to T^r_s M$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta : M \to T^r_s M$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 4.6.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma^r_s(M)$. Then

(1) $f\alpha \in \Gamma^r_s(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
(2) $\alpha + \beta \in \Gamma^r_s(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Proof. Clear. □

Exercise 4.6.4. The set $\Gamma^r_s(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Definition 4.6.5. Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. We define the **tensor product of** α **with** β, denoted $\alpha \otimes \beta : M \to T^{r_1+r_2}_{s_1+s_2} M$, by

$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

Exercise 4.6.6. Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma^{r_1+r_2}_{s_1+s_2}(M)$

Proof. Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since α_1 and α_2 are smooth by assumption. □

Definition 4.6.7. We define the **tensor product**, denoted $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

Exercise 4.6.8. The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is associative.

Proof. Clear. □

Exercise 4.6.9. The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is $C^\infty(M)$-bilinear.

Proof. Clear. □

Definition 4.6.10. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ a smooth map and $\alpha \in \Gamma^0_k(N)$. We define the **pullback of** α **by** F, denoted $F^*\alpha \in \Gamma^0_k(M)$, by

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))$$

for $p \in M$ and $v_1, \dots, v_k \in T_pM$

Exercise 4.6.11. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ and $(L, \mathcal{C})$ be smooth manifolds, $F : M \to N$ and $G : N \to L$ smooth maps, $\alpha \in \Gamma^0_k(N)$, $\beta \in \Gamma^0_l(N)$, $\gamma \in \Gamma^0_k(L)$ and $f \in C^\infty(N)$. Then

(1) $F^*(f\alpha) = (f \circ F)F^*\alpha$
(2) $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
(3) $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
(4) $(G \circ F)^*\gamma = F^*(G^*\gamma)$
(5) $id_N^*\alpha = \alpha$

Proof.

(1)

$$\begin{aligned}[F^*(f\alpha)]_p(v_1, \dots, v_k) &= (f\alpha)_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)) \\ &= f(F(p))\alpha_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)) \\ &= (f \circ F)(p)(F^*\alpha)_p(v_1, \dots, v_k)\end{aligned}$$

So that $F^*(f\alpha) = (f \circ F)F^*\alpha$

(2)

$$F^*$$

□

Definition 4.6.12.

Exercise 4.6.13.

Proof. □

Exercise 4.6.14. Let $\alpha \in \Gamma^r_s(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then there exist $(f^I_J)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

Proof. Let $p \in M$. Then $\omega_p \in T^r_s(T_pM)$ and $\left\{\partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}\right\}$ is a basis of $T^r_s(T_pM)$. So there exist $(f^I_J(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned}\alpha_p(dx^K_p, \partial_{x^L}|_p) &= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}(dx^K_p, \partial_{x^L}|_p) \\ &= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p(dx^K_p) dx_p^{\otimes J}(\partial_{x^L}|_p) \\ &= f^K_L(p)\end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx^K, \partial_{x^L})_p$ is smooth, so that $f^K_L \in C^\infty(U)$.

$\square$

Definition 4.6.15.

4.7. **Differential Forms.**

Definition 4.7.1. We define

$$\Lambda_k(TM) = \coprod_{p \in M} \Lambda_k(T_pM)$$

Definition 4.7.2. Let $\omega : M \to \Lambda_k(TM)$. Then ω is said to be a k**-form on** M if for each $p \in M$, $\omega_p \in \Lambda_k(T_pM)$.
For each $X \in \Gamma^1(M)^k$, we define $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)^k$, $\omega(X)$ is smooth.
The set of smooth k-forms on M is denoted $\Omega_k(M)$.

Note 4.7.3. Observe that

1. $\Omega_k(M) \subset \Gamma^0_k(M)$
2. $\Omega_0(M) = C^\infty(M)$

Exercise 4.7.4. The set $\Omega_k(M)$ is a $C^\infty(M)$-submodule of $\Gamma^0_k(M)$.

Proof. Clear. □

Definition 4.7.5. Define the **exterior product**

$$\wedge : \Omega_k(M) \times \Omega_l(M) \to \Omega_{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Note 4.7.6. For $f \in \Omega_0(M)$ and $\alpha \in \Omega_k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 4.7.7. The exterior product $\wedge : \Omega_k(M) \times \Omega_l(M) \to \Omega_{k+l}(M)$ is well defined.

Proof. Let $\alpha \in \Omega_k(M)$, $\beta \in \Omega_l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{i=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then

$$\begin{aligned}
\alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{(k+l)!}{k!l!} A(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)(\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \cdots, X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p))
\end{aligned}$$

□

Exercise 4.7.8. The exterior product $\wedge : \Omega_k(M) \times \Omega_l(M) \to \Omega_{k+l}(M)$ is $C^\infty(M)$-bilinear.

Proof.

(1) $C^\infty(M)$-linearity in the first argument:
Let $\alpha \in \Omega_k(M)$, $\beta, \gamma \in \Omega_l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda_k(T_pM) \times \Lambda_l(T_pM) \to \Lambda_{k+l}(T_pM)$ implies that

$$\begin{aligned}[(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p\end{aligned}$$

So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

and $\wedge : \Omega_k(M) \times \Omega_l(M) \to \Omega_{k+l}(M)$ is $C^\infty(M)$-linear in the first argument.

(2) $C^\infty(M)$-linearity in the second argument:
Similar to (1).

□

Note 4.7.9. All of the results from multilinear algebra apply here.

Definition 4.7.10. We define the **exterior derivative** $d : \Omega_k(M) \to \Omega_{k+1}(M)$ inductively by

(1) $d(d\alpha) = 0$ for $\alpha \in \Omega_p(M)$
(2) $df(X) = Xf$ for $f \in \Omega_0(M)$
(3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega_p(M)$ and $\beta \in \Omega_q(M)$
(4) extending linearly

Exercise 4.7.11. Let (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then on U, for each $i, j \in \{1, \cdots, n\}$,

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx^1_p, \cdots, dx^n_p\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ and $T^*_pM = \operatorname{span}\{dx^1_p, \cdots, dx^n_p\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned}\left[dx^i\left(\frac{\partial}{\partial x^j}\right)\right]_p &= \left(\frac{\partial}{\partial x^j} x^i\right)_p \\ &= \frac{\partial}{\partial x^j}\bigg|_p x^i \\ &= \delta_{i,j}\end{aligned}$$

□

Exercise 4.7.12. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Proof. Let $p \in U$. Since $\{dx^1, \cdots, dx^n\}$ is a basis for $\Lambda(T_pM)$, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum\limits_{i=1}^n a^i(p) dx^i_p$. Therefore, we have that

$$\begin{aligned} df_p\bigg(\frac{\partial}{\partial x^j}\bigg|_p\bigg) &= \sum_{i=1}^n a^i(p) dx^i_p\bigg(\frac{\partial}{\partial x^j}\bigg|_p\bigg) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p\bigg(\frac{\partial}{\partial x^j}\bigg|_p\bigg) &= \frac{\partial}{\partial x^j}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx^i_p$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

□

Exercise 4.7.13. Let $f \in \Omega_0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \ldots, x_n)$. Then for each $i \in \{1, \ldots, n\}$,

$$\frac{\partial}{\partial x^i}\bigg|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx^i_p \\ &= 0 \end{aligned}$$

□

Exercise 4.7.14.

Definition 4.7.15. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$. We define

$$dx^i = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega_k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x_{i_1}}, \cdots, \frac{\partial}{\partial x_{i_k}}\right)$$

Note 4.7.16. We have that

(1)

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{I,J}$$

(2) Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega_k(U)$,

$$\omega\left(\frac{\partial}{\partial x^i}\right) \in C^\infty(U)$$

Exercise 4.7.17. Let $\omega \in \Omega_k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega\left(\frac{\partial}{\partial x^i}\right) dx^i$$

Proof. Let $p \in U$. Since $\{dx_p^i : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(T_pM)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum\limits_{I \in \mathcal{I}_k} f_I(p) dx_p^i$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial x^j}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j}\right) \\
&= f_J
\end{aligned}$$

□

Exercise 4.7.18. Let $\omega \in \Omega_k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. If $\omega = \sum\limits_{I \in \mathcal{I}_k} f_I dx^i$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

.

Proof. First we note that

$$\begin{aligned}
d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f d(dx^i) \\
&= df_I \wedge dx^i \\
&= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i\right) \wedge dx^i \\
&= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i
\end{aligned}$$

Then we extend linearly. □

Definition 4.7.19. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ be a diffeomorphism. Define the **pullback of** F, denoted $F^* : \Omega_k(N) \to \Omega_k(M)$ by

$$(F^*\omega)_p(v_1, \cdots, v_k) = \omega_{F(p)}(dF_p(v_1), \cdots, dF_p(v_k))$$

for $\omega \in \Omega_k(N)$, $p \in M$ and $v_1, \cdots, v_k \in T_pM$

5. Extra

Definition 5.0.1. When working in $\mathbb{R}^n$, we introduce the formal objects $dx^1, dx_2, \cdots, dx^n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx^i = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.

Definition 5.0.2. Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \operatorname{span}\{dx^i : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

(1) the exterior product is bilinear
(2) for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$
(3) for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$
(4) for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k-forms on $\mathbb{R}^n$. Let ω be a k-form on $\mathbb{R}^n$. If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx^i$

Note 5.0.3. The terms $dx^1, dx_2, \cdots, dx^n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx^1, dx_2, \cdots, dx^n\}$ when referencing the coordinates on $\mathbb{R}^n$.

Exercise 5.0.4. Let $B_{n\times n} = (b_{i,j}) \in [C^\infty(M)]^{n\times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned} \bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) &= \left(\sum_{j=1}^n b_{1,j} dx^j \right) \wedge \left(\sum_{j=1}^n b_{2,j} dx^j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n b_{n,j} dx^j \right) \\ &= \sum_{j_1, \cdots, j_n = 1}^n \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\ &= \sum_{j_1 \neq \cdots \neq j_n} \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\ &= \left[\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^n b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\ &= (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \end{aligned}$$

□

Definition 5.0.5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted df, on $\mathbb{R}^n$ by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

Let $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx^i$ be a k-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx^i$$

Exercise 5.0.6. On $\mathbb{R}^3$, put

(1) $\omega_0 = f_0$,
(2) $\omega_1 = f_1 dx^1 + f2dx_2 + f_2 dx_3$,
(3) $\omega_2 = f_1 dx_2 \wedge dx_3 - f_2 dx^1 \wedge dx_3 + f_3 dx^1 \wedge dx_2$

Show that

(1) $d\omega_0 = \frac{\partial f_0}{\partial x_1} dx^1 + \frac{\partial f_0}{\partial x_2} dx_2 + \frac{\partial f_0}{\partial x_3} dx_3$
(2) $d\omega_1 = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx^1 \wedge dx_3 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx^1 \wedge dx_2$
(3) $d\omega_2 = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx^1 \wedge dx_2 \wedge dx_3$

Proof. Straightforward. □

Exercise 5.0.7. Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx^i \wedge dx_{I_*} = dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$.

Definition 5.0.8. We define a linear map $* : \Phi_k(\mathbb{R}^n) \to \Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge $*$-operator** by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 5.0.9. Let $\phi : \mathbb{R}^k \to \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We define $\phi^* : \Phi_k(\mathbb{R}^n) \to \Phi_k(\mathbb{R}^k)$ via the following properties:

(1) for each 0-form f on $\mathbb{R}^n$, $\phi^* f = f \circ \phi$
(2) for $i = 1, \cdots, n$, $\phi^* dx^i = d\phi_i$
(3) for an s-form ω, and a t-form χ on $\mathbb{R}^n$, $\phi^*(\omega \wedge \chi) = (\phi^* \omega) \wedge (\phi^* \chi)$
(4) for l-forms ω, χ on $\mathbb{R}^n$, $\phi^*(\omega + \chi) = \phi^* \omega + \phi^* \chi$

Exercise 5.0.10. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold of $\mathbb{R}^n$, $\phi : U \to V$ a smooth parametrization of M, $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$ an k-form on $\mathbb{R}^n$. Then

$$\phi^* \omega = \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof. By definition,

$$\begin{aligned}\phi^*\omega &= \phi^* \sum_{I\in\mathcal{I}_{k,n}} f_I dx^i \\ &= \sum_{I\in\mathcal{I}_{k,n}} (\phi^* f_I)\phi^* dx^i \\ &= \sum_{I\in\mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\ &= (\sum_{j=1}^{n} \frac{\partial \phi_{i_1}}{\partial u^j} du^j) \wedge (\sum_{j=1}^{n} \frac{\partial \phi_{i_2}}{\partial u^j} du^j) \wedge \cdots \wedge (\sum_{j=1}^{n} \frac{\partial \phi_{i_k}}{\partial u^j} du^j) \\ &= (\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k\end{aligned}$$

Therefore

$$\begin{aligned}\phi^*\omega &= \sum_{I\in\mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I\in\mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k \\ &= \left(\sum_{I\in\mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \cdots \wedge du_k\end{aligned}$$

□

5.1. **Integration of Differential Forms.**

Definition 5.1.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \cdots \wedge dx_k$ a k-form on $\mathbb{R}^k$. Define

$$\int_U \omega = \int_U f dx$$

Definition 5.1.2. Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$, ω a k-form on $\mathbb{R}^n$ and $\phi : U \to V$ a local smooth, orientation-preserving parametrization of M. Define

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 5.1.3.

Theorem 5.1.4. Stokes Theorem:
Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$ and ω a $k-1$-form on $\mathbb{R}^n$. Then

$$\int_{\partial M} \omega = \int_M d\omega$$