INTRODUCTION TO BAYESIAN STATISTICS

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Contents

1. Introduction

Definition 1.0.1. We define

$$\mathcal{D}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^d) : f \geq 0 \text{ and } \|f\|_1 = 1\}$$

2. Sampling

2.1. Inverse CDF Sampling.

2.2. Conditional Chain Sampling.

Definition 2.2.1. Let $A \subset \mathbb{R}$ be open, $a = (a_1, \dots, a_n) \in A$. Define

$$A_1 = \{x_1 \in \mathbb{R} : (x_1, a_2, \dots, a_n) \in A\}$$

Let $f_1 \in \mathcal{D}(A_1)$ and $a_1' \sim f_1$. For $j \in \{2, \cdots, n\}$, define $\tau_j : \mathbb{R} \to \mathbb{R}^n$, A_j, choose f_j and sample a_j' inductively by

$$\tau_j(x) = (a_1', \dots, a_{j-1}', x_j, a_{j+1}, \dots, a_n)$$
$$A_j = \tau_j^{-1}(A)$$
$$f_j \in \mathcal{D}(A_j)$$

and

$$a_j' \sim f_j$$

Note that τ_j is continuous which implies that $A_j = \tau_j^{-1}(A)$ is open.

Exercise 2.2.2. Let $A \subset \mathbb{R}$ be open and $a = (a_1, \dots, a_n) \in A$. Define A_j, f_j and a_j' as above and define $a' \in A$ and $f : A \to \mathbb{R}$ by

$$a' = (a_1', \dots, a_n')$$

and

$$f(x_1, \dots, x_n) = \prod_{j=1}^{n} f_j(x_j)$$

Then $f \in \mathcal{D}(A)$ and $a' \sim f$.

Proof. Fubini's theorem implies that

$$\begin{aligned}\int_A f dm^n &= \int f_1(x_1)\bigg(\int f_2(x_2)\bigg(\cdots \int f_n(x_n)dm(x_n)\dots\bigg)dm(x_2)\bigg)dm(x_1)\\ &= 1\end{aligned}$$

□

2.3. **Importance Sampling.**

2.4. **Rejection Sampling.**

Exercise 2.4.1. Let $f, g \in \mathcal{D}(\mathbb{R}^d)$ and $A \in \mathcal{B}(\mathbb{R}^d)$. Suppose that $m^d(A) > 0$. If $X \sim f$, then $X|X \in A \sim \|fI_A\|_1^{-1} f I_A$.

Proof. Let $C \in \mathcal{B}(\mathbb{R}^d)$. Then

$$\begin{aligned}P(X \in C|X \in A) &= P(X \in C \cap A)P(X \in A)^{-1}\\ &= \|fI_A\|_1^{-1}\int_C fI_A dm^d\end{aligned}$$

So $f_{X|X\in A} = \|fI_A\|_1^{-1} f I_A$. □

Exercise 2.4.2. Let $A, B \in \mathcal{B}(\mathbb{R}^d)$. Suppose that $A \subset B$ and $0 < m^d(A)$ and $m^d(B) < \infty$. If $X \sim \text{Uni}(B)$, then $X|X \in A \sim \text{Uni}(A)$.

Proof. Clear using the previous exercise with $f = I_B$. □

Exercise 2.4.3. (Fundamental Theorem of Simulation):
Let $f \in \mathcal{D}(\mathbb{R}^d)$ and $c > 0$. Define

$$G_c = \{(x, v) \in \mathbb{R}^{d+1} : 0 < v < cf(x)\}$$

(1) If $X \sim f$ and $U \sim \text{Uni}(0, 1)$ are independent, then $(X, cUf(X)) \sim \text{Uni}(G_c)$.
(2) If $(X, V) \sim \text{Uni}(G_c)$, then $X \sim f$.

Proof. First we note that $m^{d+1}(G_c) = c$.

(1) Suppose that $X \sim f$ and $U \sim \text{Uni}(0, 1)$ are independent and put $Y = cUf(X)$. Then $Y|X = x \sim cUf(x) \sim \text{Uni}(0, cf(x))$ and we have that for each $x \in \operatorname{supp} X$ and $y \in (0, cf(x))$,

$$\begin{aligned}f_{X,Y}(x, y) &= f_{Y|X}(y|x)f(x)\\ &= \frac{1}{cf(x)}f(x)\\ &= \frac{1}{c}\end{aligned}$$

So $(X, Y) \sim \text{Uni}(G_c)$

(2) Suppose that $(X, V) \sim \text{Uni}(G_c)$. Then $f_{X,V}(x, v) = \frac{1}{c} I_{G_c}(x, v)$. So

$$\begin{aligned} f_X(x) &= \int_{\mathbb{R}} \frac{1}{c} I_{G_c}(x, v) dm(v) \\ &= \int_0^{cf(x)} \frac{1}{c} dv \\ &= f(x) \end{aligned}$$

So $X \sim f$.

□

Exercise 2.4.4. Let $f, g \in \mathcal{D}(\mathbb{R}^d)$, $c_f, c_g > 0$ and $M > 0$. Put $\tilde{f} = c_f f$ and $\tilde{g} = c_g g$. Suppose that $\tilde{f} \leq M\tilde{g}$. If $Y \sim g$ and $U \sim \text{Uni}(0, 1)$ are independent, then $Y | U \leq \frac{\tilde{f}(Y)}{M\tilde{g}(Y)} \sim f$ and $P(U \leq \frac{\tilde{f}(Y)}{M\tilde{g}(Y)}) = \frac{c_f}{c_g M}$

Proof. Put

$$G_g = \{(y, v) \in \mathbb{R}^{d+1} : 0 < v < M\tilde{g}(y)\}$$

and

$$G_f = \{(y, v) \in \mathbb{R}^{d+1} : 0 < v < \tilde{f}(y)\}$$

Then $G_f \subset G_g$, $m^d(G_g) = c_g M$ and $m^d(G_f) = c_f$. By the first part of the fundamental theorem of simulation, we know that

$$(Y, MUc_g g(Y)) \sim \text{Uni}(G_g)$$

Since $\{(Y, MUc_g g(Y)) \in G_f\} = \{U \leq \frac{c_f f(Y)}{Mc_g g(Y)}\}$, a previous exercise tells us that

$$(Y, MUc_g g(Y)) | U \leq \frac{c_f f(Y)}{Mc_g g(Y)} \sim \text{Uni}(G_f)$$

Then the second part of the fundamental theorem of simulation tells us that

$$Y | U \leq \frac{c_f f(Y)}{Mc_g g(Y)} \sim f$$

Finally we have that

$$\begin{aligned} P\left(U \leq \frac{c_f f(Y)}{Mc_g g(Y)}\right) &= P[(Y, MUc_g g(Y)) \in G_f] \\ &= \frac{c_f}{c_g M} \end{aligned}$$

□

Definition 2.4.5. (Rejection Sampling Algorithm):
Let $f, g \in \mathcal{D}(\mathbb{R}^d)$, $c_f, c_g > 0$ and $M > 0$. Put $\tilde{f} = c_f f$ and $\tilde{g} = c_g g$. Suppose that $\tilde{f} \leq M\tilde{g}$. We define the **rejection sampling algorithm** as follows:

(1) sample $Y \sim g$ and $U \sim \text{Uni}(0, 1)$ independently
(2) if $U \leq \frac{\tilde{f}(Y)}{M\tilde{g}(Y)}$, accept Y, else return to (1).

If we sample $(X_n)_{n \in \mathbb{N}}$ independently using the rejection sampler, then the previous exercises imply that $(X_n)_{n \in \mathbb{N}} \overset{iid}{\sim} f$ and the acceptance rate is $\frac{c_f}{c_g M}$.

Note 2.4.1. Phrasing the rejection sampler in terms of $\tilde{f}$ and $\tilde{g}$ instead of f and g is usefule because we may not always be able to solve for the normalizing constants.