REAL ANALYSIS NOTES

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Contents

1. Algebra and Analysis of Sets

1.1. Limits.

Definition 1.1.1. *Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. We define*

$$\inf \mathcal{A} = \bigcap_{A \in \mathcal{A}} A, \quad \sup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

Definition 1.1.2. *Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. We define*

$$\liminf_{n \to \infty} A_n = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} A_k \right), \quad \limsup_{n \to \infty} A_n = \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} A_k \right)$$

Note 1.1.3.

(1) $\liminf_{n \to \infty} A_n$ *is the set of elements that are in all A_n except for finitely many.*
(2) $\limsup_{n \to \infty} A_n$ *is the set of elements that are in infinitely many A_n.*

Exercise 1.1.4. *Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then*

(1) $\liminf_{n \to \infty} A_n = \left\{ x \in X : \liminf_{n \to \infty} \chi_{A_n}(x) = 1 \right\}$
(2) $\limsup_{n \to \infty} A_n = \left\{ x \in X : \limsup_{n \to \infty} \chi_{A_n}(x) = 1 \right\}$

Proof.

(1) Let $x \in \liminf_{n \to \infty} A_n$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x \in A_k$. So for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\chi_{A_k}(x) = 1$. Then $\inf_{k \geq n^*} \chi_{A_k}(x) = 1$ and thus

$$1 = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \chi_{A_k}(x) \right) = \liminf_{n \to \infty} \chi_{A_n}(x)$$

Conversely, if $1 = \liminf_{n \to \infty} \chi_{A_n}(x)$, then choosing $\epsilon = \frac{1}{2}$, there exists $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n$ implies that $\chi_{A_k}(x) > 1 - \epsilon$. Hence for each $k \in \mathbb{N}$, $k \geq n$ implies that $\chi_{A_k}(x) = 1$. So for each for each $k \in \mathbb{N}$, $k \geq n$ implies that $x \in A_k$. So $x \in \liminf_{n \to \infty} A_n$.

(2) Similar to (1).

□

Exercise 1.1.5. *Let $A_k = [0, \frac{k}{k+1})$. Then*

(1) $\inf_{k \geq n} A_k = [0, \frac{n}{n+1})$
(2) $\sup_{k \geq n} A_k = [0, 1)$

(3) $\liminf_{n\to\infty} A_n = [0,1)$

(4) $\liminf_{n\to\infty} A_n = [0,1)$

Proof. Straightforward. $\square$

Exercise 1.1.6. *Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then*

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$$

Proof. Let $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq n*$, then $x \in A_k$. Let $n \in \mathbb{N}$. Choose $k = \max\{n^*, n\} \geq n^*$. Then $x \in A_k$. Hence for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x \in A_k$. So $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Thus $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$. $\square$

Definition 1.1.7. *Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. If*

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

then we define

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

Exercise 1.1.8. *Let X be a set and $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Suppose that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$ and $B_{n+1} \subset B_n$. Then*

(1) $\lim_{n\to\infty} A_n = \sup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$

(2) $\lim_{n\to\infty} B_n = \inf_{n\in\mathbb{N}} B_n = \bigcap_{n=1}^{\infty} B_n$

Proof.

(1) Let $n \in \mathbb{N}$. Then

$$\begin{aligned}\inf_{k\geq n} A_k &= \bigcap_{k=n}^{\infty} A_k \\ &= A_n\end{aligned}$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} A_n &= \bigcup_{n=1}^{\infty} \inf_{k\geq n} A_k \\ &= \bigcup_{n=1}^{\infty} A_n\end{aligned}$$

In addition,

$$\begin{aligned}\sup_{n\geq k} A_k &= \bigcup_{k=n}^{\infty} A_k \\ &= \bigcup_{k=1}^{\infty} A_k\end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{n\to\infty} A_n &= \bigcap_{n=1}^{\infty} \inf_{k\geq n} A_k \\ &= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_k \\ &= \bigcup_{n=1}^{\infty} A_n\end{aligned}$$

So

$$\lim_{n\to\infty} A_n = \sup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$$

(2) Similar

□

Exercise 1.1.9. *Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets and $(A_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(A_n)_{n\in\mathbb{N}}$. Then*

(1) $\limsup_{k\to\infty} A_{n_k} \subset \limsup_{n\to\infty}(A_n)$
(2) $\liminf_{n\to\infty} A_n \subset \limsup_{k\to\infty}(A_{n_k})$

Proof.

(1) The elements that are in A_{n_k} for infinitely many k are in A_n for infinitely many n.
(2) Similar.

□

Exercise 1.1.10. *Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets, $(A_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(A_n)_{n\in\mathbb{N}}$ and $A \subset X$. If $A_{n_k} \to A$, then*

$$\liminf_{n\to\infty} A_n \subset A \subset \limsup_{n\to\infty} A_n$$

Proof. The previous exercises tells us that

$$\begin{aligned}\liminf_{n\to\infty} A_n &\subset \liminf_{k\to\infty} A_{n_k} \\ &= A \\ &= \limsup_{k\to\infty} A_{n_k} \\ &\subset \limsup_{n\to\infty} A_n\end{aligned}$$

□

Exercise 1.1.11. *Let X be a set and $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Suppose that for each $n \in \mathbb{N}$, $A_n \subset B_n$. Then*

(1) $\limsup\limits_{n\to\infty} A_n \subset \limsup\limits_{n\to\infty} B_n$
(2) $\liminf\limits_{n\to\infty} A_n \subset \liminf\limits_{n\to\infty} B_n$

Proof.

(1) Let $x \in \limsup\limits_{n\to\infty} A_n$. Then for infinitely many $n \in \mathbb{N}$, $x \in A_n \subset B_n$. So for infinitely many $n \in \mathbb{N}$, $x \in B_n$. Hence $x \in \limsup\limits_{n\to\infty} B_n$. Therefore $\limsup\limits_{n\to\infty} A_n \subset \limsup\limits_{n\to\infty} B_n$.
(2) Similar.

□

Exercise 1.1.12. *Let*

Exercise 1.1.13. *Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then*

(1) $\limsup\limits_{n\to\infty} A_n = \left(\liminf\limits_{n\to\infty} A_n^c\right)^c$
(2) $\liminf\limits_{n\to\infty} A_n = \left(\limsup\limits_{n\to\infty} A_n^c\right)^c$

Proof.

(1)

$$\begin{aligned}\left(\liminf_{n\to\infty} A_n^c\right)^c &= \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^c\right)^c \\ &= \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \\ &= \limsup_{n\to\infty} A_n\end{aligned}$$

(2) Similar.

□

Exercise 1.1.14. *For $n \in \mathbb{N}$, define*

$$A_n = \left\{\frac{m}{n} : m \in \|\right\}$$

Then

(1) $\liminf\limits_{n\to\infty} A_n = \mathbb{N}$
(2) $\limsup\limits_{n\to\infty} A_n = \mathbb{Q} \cap (0, \infty)$

Proof.

(1) For each $x \in \mathbb{N}$ and $n \in \mathbb{N}$, $x = \frac{nx}{n} \in A_n$ Hence $\| \subset \liminf\limits_{n\to\infty} A_n$. Conversely, let $x \in \liminf\limits_{n\to\infty} A_n$. Then there exists $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq n$, then $x \in A_k$. In particular, $x \in A_n$. Hence there exists $m_n \in \mathbb{N}$ such that $x = \frac{m_n}{n}$. Choose $s, t \in \mathbb{N}$ such that $x = \frac{s}{t}$ and $\gcd(s,t) = 1$. Suppose that $t \neq 1$. Then choose a prime

$p > n$. By assumption, $x \in A_p$. Then there exist $m_p \in \mathbb{N}$ such that $x = \frac{m_p}{p}$. Hence $\frac{s}{t} = \frac{m_p}{p}$ and $tm_p = sp$. Since $t|sp$ and $\gcd(s,t) = 1$, we see that $t|p$. If $t \geq 1$, then p is not prime, a contradiction. So $t = 1$. Hence $x \in \mathbb{N}$. Thus $\liminf\limits_{n\to\infty} A_n \subset \mathbb{N}$.

(2) Let $x \in \mathbb{Q} \cap (0,\infty)$. Then there exist $s,t \in \mathbb{N}$ such that $x = \frac{s}{t}$. Define the subsequence $(A_{n_k})_{k\in\mathbb{N}}$ by $A_{n_k} = A_{tk}$. Then for each $k \in \mathbb{N}$, $x = \frac{sk}{tk} \in A_{tk} = A_{n_k}$. Thus $x \in \limsup\limits_{n\to\infty} A_n$. Conversely, clearly $\limsup\limits_{n\to\infty} A_n \subset \mathbb{Q} \cap (0,\infty)$

$\square$

Exercise 1.1.15. *Let X be a set and $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Then*

$$\limsup_{n\to\infty} A_n \cup B_n = \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n$$

Proof. Let $x \in \limsup\limits_{n\to\infty} A_n \cup B_n$. Suppose that $x \notin \limsup\limits_{n\to\infty} A_n$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ if $k \geq n^*$, then $x \notin A_k$. Let $n \in \mathbb{N}$. Then there exists k such that $k \geq \max\{n, n^*\}$ and $x \in A_k \cup B_k$. Since $k \geq n^*$, $x \notin A_k$ Thus $x \in B_k$. So for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x \in B_k$. Therefore $x \in \limsup\limits_{n\to\infty} B_n$ and

$$\limsup_{n\to\infty} A_n \cup B_n \subset \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n$$

Conversely, a previous exercise tells us that $\limsup\limits_{n\to\infty} A_n \subset \limsup\limits_{n\to\infty} A_n \cup B_n$ and $\limsup\limits_{n\to\infty} B_n \subset \limsup\limits_{n\to\infty} A_n \cup B_n$. Thus

$$\limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n \subset \limsup_{n\to\infty} A_n \cup B_n$$

$\square$

Exercise 1.1.16. *Let X be a set and $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Then*

$$\liminf_{n\to\infty} A_n \cap B_n = \liminf_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n$$

Proof. A previous exercise tells us that

$$\begin{aligned}
\liminf_{n\to\infty} A_n \cap B_n &= \left(\limsup_{n\to\infty} A_n^c \cup B_n^c\right)^c \\
&= \left(\limsup_{n\to\infty} A_n^c \cup \limsup_{n\to\infty} B_n^c\right)^c \\
&= \left(\limsup_{n\to\infty} A_n^c\right)^c \cap \left(\limsup_{n\to\infty} B_n^c\right)^c \\
&= \liminf_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n
\end{aligned}$$

$\square$

1.2. Classes of sets.

Definition 1.2.1. *Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. Then $\mathcal{A}$ is said to be an* ***algebra*** *on X if*

1. $\mathcal{A} \neq \varnothing$
2. *for each $A \in \mathcal{A}$, $A^c \in \mathcal{A}$*
3. *for each $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$*

Definition 1.2.2. *Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. Then $\mathcal{A}$ is said to be a **σ-algebra** on X if*

(1) $\mathcal{A} \neq \varnothing$
(2) *for each* $A \in \mathcal{A}$, $A^c \in \mathcal{A}$
(3) *for each* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}$

Exercise 1.2.3. *Let X be a set and $\mathcal{A}$ a σ-algebra on X. Then*

(1) $X, \varnothing \in \mathcal{A}$
(2) *for each* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, $\bigcap\limits_{n\in\mathbb{N}} \in \mathcal{A}$
(3) *For each* $A, B \in \mathcal{A}$, $A \setminus B \in \mathcal{A}$

Proof.

(1) Since $\mathcal{A} \neq \varnothing$, there exists $A \in \mathcal{A}$. Then $A^c \in \mathcal{A}$. Hence $X = A \cup A^c \in \mathcal{A}$ and $\varnothing = X^c \in \mathcal{A}$.
(2) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Then $(A_n^c)_{n\in\mathbb{N}} \subset MA$. So $\bigcup\limits_{n\in\mathbb{N}} A_n^c \in \mathcal{A}$. Therefore
$$\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^c\right)^c \in \mathcal{A}$$
(3) Let $A, B \in \mathcal{A}$. Then $A \setminus B = A \cap B^c \in \mathcal{A}$.

□

Exercise 1.2.4. *Let X be a set and $(\mathcal{A}_i)_{i\in I}$ a collection of σ-algebras (resp. algebra) on X. Then $\bigcap\limits_{i\in I} \mathcal{A}_i$ is a σ-algebra (resp. algebra) on X.*

Proof.

(1) For each $i \in I$, $X \in \mathcal{A}_i$. Thus $X \in \bigcap\limits_{i\in I} \mathcal{A}_i$ and $\bigcap\limits_{i\in I} \mathcal{A}_i \neq \varnothing$.
(2) Let $A \in \bigcap\limits_{i\in I} \mathcal{A}_i$. Then for each $i \in I$, $A \in \mathcal{A}_i$. Hence for each $i \in I$, $A^c \in \mathcal{A}_i$. Thus $A^c \in \bigcap\limits_{i\in I} \mathcal{A}_i$.
(3) Let $(A_n)_{n\in\mathbb{N}} \subset \bigcap\limits_{i\in I} \mathcal{A}_i$. Then for each $i \in I$, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_i$. Thus for each $i \in I$, $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}_i$. So $\bigcup\limits_{n\in\mathbb{N}} A_n \in \bigcap\limits_{i\in I} \mathcal{A}_i$.

□

Definition 1.2.5. *Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$. Put*
$$\mathcal{S} = \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ and } \mathcal{C} \subset \mathcal{L}\}$$
*We define the **σ-algebra generated by** $\mathcal{C}$ on X, $\sigma(\mathcal{C})$, by*
$$\sigma(\mathcal{C}) = \bigcap_{\mathcal{A}\in\mathcal{S}} \mathcal{A}$$

Note 1.2.6. *Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $\mathcal{A}$ a σ-alg on X. By definition, if $\mathcal{C} \subset \mathcal{A}$, then $\sigma(\mathcal{C}) \subset \mathcal{A}$.*

Note 1.2.7. *Let X be a set, $\mathcal{T}$ an ordered set and $(\mathcal{A}_t)_{t\in\mathcal{T}}$ a collection of σ-algebras on X. Suppose that for each $s,t \in \mathcal{T}$, if $s \leq t$, then $\mathcal{A}_s \subset \mathcal{A}_t$. If there exists $t \in \mathcal{T}$ such that $\mathcal{A}_t = \bigcup\limits_{t\in\mathcal{T}} \mathcal{A}_t$, then $\bigcup\limits_{t\in\mathcal{T}} \mathcal{A}_t$ is a σ-algebra on X. So if $\mathcal{T}$ is finite or if $(\mathcal{A}_t)_{t\in\mathcal{T}}$ termintates, the union is σ-algebra.*

Definition 1.2.8. *Let $(X,\mathcal{T})$ be a topological space. We define the **Borel σ-algebra** on X, $\mathcal{B}(X)$, to be*

$$\mathcal{B}(X) = \sigma(\mathcal{T})$$

*The sets of $\mathcal{B}(X)$ are called **Borel sets**.*

Exercise 1.2.9. *The Borel σ-algebra on $\mathbb{R}$ with the standard topology is given by*

$$\mathcal{B}(\mathbb{R}) = \begin{cases} \sigma(\{(a,b] : a,b \in \mathbb{R} \text{ and } a<b\}) \\ \sigma(\{[a,b] : a,b \in \mathbb{R} \text{ and } a<b\}) \\ \sigma(\{[a,b) : a,b \in \mathbb{R} \text{ and } a<b\}) \\ \sigma(\{(a,b) : a,b \in \mathbb{R} \text{ and } a<b\}) \end{cases}$$

Proof. Define

(1) $\mathcal{C}_{lo} = \{(a,b] : a,b \in \mathbb{R} \text{ and } a<b\}$

(2) $\mathcal{C}_{c} = \{[a,b] : a,b \in \mathbb{R} \text{ and } a<b\}$

(3) $\mathcal{C}_{ro} = \{[a,b) : a,b \in \mathbb{R} \text{ and } a<b\}$

(4) $\mathcal{C}_{o} = \{(a,b) : a,b \in \mathbb{R} \text{ and } a<b\}$

Recall that for each open set $A \subset \mathbb{R}$, there exist $(a_i)_{n\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ such that for each $i \in \mathbb{N}$, $a_i < b_i$, for each $i,j \in \mathbb{N}$, if $i \neq j$, then $(a_i,b_i)\cap(a_j,b_j) = \varnothing$ and $A = \bigcup\limits_{i\in\mathbb{N}}(a_i,b_i)$. This implies that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o)$.

Now, let $a,b \in \mathbb{R}$. Suppose that $a<b$. Then

(1) $[a,b] = \bigcap\limits_{n\in\mathbb{N}}(a-\frac{1}{n},b]$, so $\sigma(\mathcal{C}_c) \subset \sigma(\mathcal{C}_{lo})$

(2) $[a,b) = \bigcup\limits_{n\in\mathbb{N}}[a,b-\frac{1}{n}]$, so $\sigma(\mathcal{C}_{ro}) \subset \sigma(\mathcal{C}_c)$

(3) $(a,b) = \bigcup\limits_{n\in\mathbb{N}}[a+\frac{1}{n},b)$, so $\sigma(\mathcal{C}_o) \subset \sigma(\mathcal{C}_{ro})$

(4) $(a,b] = \bigcap\limits_{n\in\mathbb{N}}(a,b+\frac{1}{n})$, so $\sigma(\mathcal{C}_{lo}) \subset \sigma(\mathcal{C}_o)$

Hence $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o) = \sigma(\mathcal{C}_{ro}) = \sigma(\mathcal{C}_c) = \sigma(\mathcal{C}_{lo}) = \sigma(\mathcal{C}_o)$. □

Exercise 1.2.10. *Let X be a set. Define $\mathcal{A} = \{A \in \mathcal{A} : A \text{ is countable or } A^c \text{is countable}\}$. Then $\mathcal{A}$ is a σ-algebra on X.*

Proof.

(1) Since $X^c = \varnothing$ is countable, $X \in \mathcal{A}$.
(2) Let $A \in \mathcal{A}$. Suppose that A^c is uncountable. Then by assumption, $A = (A^c)^c$ is countable. Hence $A^c \in \mathcal{A}$.

(3) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Then for each $n \in \mathbb{N}$, A_n is countable or A_n^c is countable. Suppose that $\bigcup\limits_{n\in\mathbb{N}} A_n$ is uncountable. Then there exists $N \in \mathbb{N}$ such that A_N is uncountable. Hence A_N^c is countable. Thus

$$\begin{aligned}\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c &= \bigcap_{n\in\mathbb{N}} A_n^c \\ &\subset A_N^c\end{aligned}$$

So $\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right)^c$ is countable and $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

□

Definition 1.2.11. *Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. We define*

$$\mathcal{C} \cap A := \{S \cap A : S \in \mathcal{C}\}$$

Exercise 1.2.12. *Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. Then $\sigma(\mathcal{C}) \cap A$ is a σ-algebra on A.*

Proof.

(1) Clearly $\varnothing, A \in \sigma(\mathcal{C}) \cap A$.
(2) Let $B \in \sigma(\mathcal{C}) \cap A$. Then there exists $S \in \sigma(\mathcal{C})$ such that $B = S \cap A$. Then $S^c \in \sigma(\mathcal{C})$. Thus

$$A \setminus B = S^c \cap A \in \sigma(\mathcal{C}) \cap A$$

(3) Let $(B_n)_{n\in\mathbb{N}} \subset \sigma(\mathcal{C}) \cap A$. Then for each $n \in \mathbb{N}$, there exists $S_n \in \sigma(\mathcal{C})$ such that $B_n = S_n \cap A$. So $\bigcup\limits_{n\in\mathbb{N}} S_n \in \sigma(\mathcal{C})$. Hence

$$\begin{aligned}\bigcup_{n\in\mathbb{N}} (B_n) &= \bigcup_{n\in\mathbb{N}} (S_n \cap A) \\ &= \left(\bigcup_{n\in\mathbb{N}} S_n\right) \cap A \\ &\in \sigma(\mathcal{C}) \cap A\end{aligned}$$

□

Exercise 1.2.13. *Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. Let $\sigma_A(\mathcal{C} \cap A)$ be the σ-algebra on A generated by $\mathcal{C} \cap A$. Define*

$$\mathcal{G} = \{S \subset X : S \cap A \in \sigma_A(\mathcal{C} \cap A)\}$$

Then $\mathcal{G}$ is a σ-algebra on X.

Proof. (1) Clearly $\varnothing, X \in \mathcal{G}$.
(2) Let $S \in \mathcal{G}$. Then $S \cap A \in \sigma_A(\mathcal{C} \cap A)$. Hence $A \setminus (S \cap A) = S^c \cap A \in \sigma_A(\mathcal{C} \cap A)$. So $S^c \in \mathcal{G}$.
(3) Let $(S_n)_{n\in\mathbb{N}} \subset \mathcal{G}$. Then for each $n \in \mathbb{N}$, $S_n \cap A \in \sigma_A(\mathcal{C} \cap A)$. Thus

$$\left(\bigcup_{n\in\mathbb{N}} S_n\right) \cap A = \bigcup_{n\in\mathbb{N}} (S_n \cap A) \in \sigma_A(\mathcal{C} \cap A)$$

Thus $\bigcup_{n\in\mathbb{N}} S_n \in \mathcal{G}$.

$\square$

Exercise 1.2.14. *Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $A \subset X$. Then*

$$\sigma(\mathcal{C}) \cap A = \sigma_A(\mathcal{C} \cap A)$$

Proof. Clearly $\mathcal{C} \cap A \subset \sigma(\mathcal{C}) \cap A$. A previous exercise tells us that $\sigma(\mathcal{C}) \cap A$ is a σ-algebra on A. Thus $\sigma_A(\mathcal{C} \cap A) \subset \sigma(\mathcal{C}) \cap A$.

Conversely, from the previous exercise, we have that $\mathcal{G} = \{S \subset X : S \cap A \in \sigma_A(\mathcal{C} \cap A)\}$ is a σ-algebra on X. Clearly $\mathcal{C} \subset \mathcal{G}$. Then $\sigma(\mathcal{C}) \subset \mathcal{G}$. The definition of $\mathcal{G}$ implies that $\sigma(\mathcal{C}) \cap A \subset \sigma_A(\mathcal{C} \cap A)$. Hence $\sigma(\mathcal{C}) \cap A = \sigma_A(\mathcal{C} \cap A)$. $\square$

Definition 1.2.15. *Let X be a set and $\mathcal{A}$ be a σ-algebra on X. Then $(X, \mathcal{A})$ is called a* ***measurable space****.*

2. Measures

2.1. Measures.

Definition 2.1.1. *Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [0, \infty]$. Then μ is said to be a* ***measure*** *on $(X, \mathcal{A})$ if*

(1) *there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$*
(2) *for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. If $(A_n)_{n\in\mathbb{N}}$ is disjoint, then*

$$\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

Definition 2.1.2. *Let $(X, \mathcal{A})$ be a measurable space and μ a measure on $(A, \mathcal{A})$. Then $(A, \mathcal{A}, \mu)$ is called a* ***measure space****.*

Exercise 2.1.3. *Let $(X, \mathcal{A}, mu)$ be a measure space. Then*

(1) *(monotonicity): for each $A, B \in \mathcal{A}$, if $A \subset B$, then $\mu(A) \leq \mu(B)$.*
(2) *(subadditivity): for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$,*

$$\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$$

(3) *(continuity from below): for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$, then*

$$\mu\Big(\sup_{n\in\mathbb{N}} A_n\Big) = \sup_{n\in\mathbb{N}} \mu(A_n)$$

(4) *(continuity from above): for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if for each $n \in \mathbb{N}$, $A_{n+1} \subset A_n$ and $\mu(A_1) < \infty$, then*

$$\mu\Big(\inf_{n\in\mathbb{N}} A_n\Big) = \inf_{n\in\mathbb{N}} \mu(A_n)$$

Proof.

(1) Let $A, B \in \mathcal{A}$. Suppose that $A \subset B$. Then

$$\begin{aligned}\mu(B) &= \mu\Big((B \cap A) \cup (B \cap A^c)\Big) \\ &= \mu(B \cap A) + \mu(B \cap A^c) \\ &= \mu(A) + \mu(B \cap A^c) \\ &\geq \mu(A)\end{aligned}$$

(2) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Define $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus \left(\bigcup\limits_{k=1}^{n-1} A_k\right)$. Then $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $\bigcup\limits_{n \in \mathbb{N}} B_n = \bigcup\limits_{n \in \mathbb{N}} A_n$, $(B_n)_{n \in \mathbb{N}}$ disjoint and for each $n \in \mathbb{N}$, $B_n \subset A_n$. Thus

$$\begin{aligned}\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) \\ &= \sum_{n \in \mathbb{N}} \mu(B_n) \\ &\leq \sum_{n \in \mathbb{N}} \mu(A_n)\end{aligned}$$

(3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$. Then for each $n \in \mathbb{N}$, $\mu(A_n) \leq \mu(A_{n+1})$ and $\lim\limits_{n \to \infty} \mu(A_n) = \sup\limits_{n \in \mathbb{N}} \mu(A_n)$. Recall that $\sup\limits_{n \in \mathbb{N}} A_n = \bigcup\limits_{n \in \mathbb{N}} A_n$. Define $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus A_{n-1}$. Then $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, $(B_n)_{n \in \mathbb{N}}$ is disjoint, $\bigcup\limits_{n \in \mathbb{N}} A_n = \bigcup\limits_{n \in \mathbb{N}} B_n$ and for each $n \in \mathbb{N}$, $\bigcup\limits_{n=1}^{k} B_n = A_k$. Then

$$\begin{aligned}\mu\left(\sup_{n \in \mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \\ &= \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) \\ &= \sum_{n \in \mathbb{N}} \mu(B_n) \\ &= \lim_{k \to \infty} \sum_{n=1}^{k} \mu(B_n) \\ &= \lim_{k \to \infty} \mu\left(\bigcup_{n=1}^{k} B_n\right) \\ &= \lim_{k \to \infty} \mu(A_k) \\ &= \sup_{n \in \mathbb{N}} \mu(A_n)\end{aligned}$$

(4) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Suppose that for each $n \in \mathbb{N}$, $A_{n+1} \subset A_n$ and $\mu(A_1) < \infty$. Then for each $n \in \mathbb{N}$ $\mu(A_{n+1}) \leq \mu(A_n) \leq \mu(A_1) < \infty$ and the arithmetic that follows is well defined. Recall that $\inf\limits_{n \in \mathbb{N}} A_n = \bigcap\limits_{n \in \mathbb{N}} A_n$. For each $n \in \mathbb{N}$, define $B_n = A_1 \cap A_n$.

Then for each $n \in \mathbb{N}$, $B_n \subset B_{n+1}$ and

$$\begin{aligned}\sup_{n\in\mathbb{N}} B_n &= \bigcup_{n\in\mathbb{N}} B_n \\ &= A_1 \setminus \bigcap_{n\in\mathbb{N}} A_n \\ &= A_1 \setminus \inf_{n\in\mathbb{N}} A_n\end{aligned}$$

So (3) implies that

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mu(B_n) &= \mu\bigg(\sup_{n\in\mathbb{N}} B_n\bigg) \\ &= \mu\bigg(A_1 \setminus \inf_{n\in\mathbb{N}} A_n\bigg) \\ &= \mu(A_1) - \mu\bigg(\inf_{n\in\mathbb{N}} A_n\bigg)\end{aligned}$$

On the other hand,

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mu(B_n) &= \sup_{n\in\mathbb{N}} \mu(A_1 \setminus A_n) \\ &= \sup_{n\in\mathbb{N}} \bigg[\mu(A_1) - \mu(A_n)\bigg] \\ &= \mu(A_1) - \inf_{n\in\mathbb{N}} \mu(A_n)\end{aligned}$$

Therefore

$$\mu\bigg(\inf_{n\in\mathbb{N}} A_n\bigg) = \inf_{n\in\mathbb{N}} \mu(A_n)$$

$\square$

Exercise 2.1.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$. Then*

(1) $\mu\bigg(\liminf_{n\to\infty} A_n\bigg) \leq \liminf_{n\to\infty} \mu(A_n)$
(2) *If* $\mu\bigg(\sup_{n\in\mathbb{N}} A_n\bigg) < \infty$, *then* $\limsup_{n\to\infty} \mu(A_n) \leq \mu\bigg(\liminf_{n\to\infty} A_n\bigg)$

Proof.

(1) Since $\left(\inf_{k\geq n} A_k\right)_{n\in\mathbb{N}}$ is an increasing sequence and for each $n \in \mathbb{N}$ $\inf\limits_{k\geq n} A_k \subset A_n$, we have that

$$\begin{aligned}\mu\left(\liminf_{n\to\infty} A_n\right) &= \mu\left[\sup_{n\in\mathbb{N}}\left(\inf_{k\geq n} A_k\right)\right] \\ &= \sup_{n\in\mathbb{N}} \mu\left(\inf_{k\geq n} A_k\right) \\ &= \liminf_{n\to\infty} \mu\left(\inf_{k\geq n} A_k\right) \\ &\leq \liminf_{n\to\infty} \mu(A_n)\end{aligned}$$

(2) Since $\mu\left(\sup\limits_{\geq 1} A_k\right) < \infty$, $\left(\sup\limits_{k\geq n}\right)_{n\in\mathbb{N}}$ is a decreasing and for each $n \in \mathbb{N}$, $A_n \subset \sup_{k\geq n} A_n$, we have that

$$\begin{aligned}\mu\left(\limsup_{n\to\infty} A_n\right) &= \mu\left[\inf_{n\in\mathbb{N}}\left(\sup_{k\geq n} A_k\right)\right] \\ &= \inf_{n\in\mathbb{N}} \mu\left(\sup_{k\geq n} A_k\right) \\ &= \limsup_{n\to\infty} \mu\left(\sup_{k\geq n} A_k\right) \\ &\geq \limsup_{n\to\infty} \mu(A_n)\end{aligned}$$

□

Exercise 2.1.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$. Suppose that $\mu\left(\sup\limits_{n\in\mathbb{N}} A_n\right) < \infty$. Then $A_n \to A$ implies that $\mu(A_n) \to \mu(A)$.*

Proof. Suppose that $A_n \to A$. Then the previous exercise tells us that

$$\begin{aligned}\mu(A) &= \mu\left(\liminf_{n\to\infty} A_n\right) \\ &\leq \liminf_{n\to\infty} \mu(A_n) \\ &\leq \limsup_{n\to\infty} \mu(A_n) \\ &\leq \mu(\limsup_{n\to\infty} A_n) \\ &= \mu(A)\end{aligned}$$

Thus $\mu(A) = \limsup\limits_{n\to\infty} \mu(A_n) = \liminf\limits_{n\to\infty} \mu(A_n)$ and $\mu(A_n) \to \mu(A)$ □

2.2. **Outer Measures.**

Definition 2.2.1. *Let X be a set and $\mu* : \mathcal{P}(X) \to [0, \infty]$. Then μ^* is said to be an* ***outer measure on X*** *if*

(1) $\mu^*(\varnothing) = 0$
(2) *for each* $A, B \subset X$*, if* $A \subset B$*, then* $\mu^*(A) \leq \mu^*(B)$*.*
(3) *for each* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$*,*

$$\mu^*\big(\bigcup_{n\in\mathbb{N}} A_n\big) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$$

Theorem 2.2.2. ***Construction of Outer Measures:***
Let X *be a set and* $\mathcal{E} \subset \mathcal{P}(X)$ *and* $\rho : \mathcal{E} \to [0, \infty]$*. Suppose that* $\varnothing, X \in \mathcal{E}$ *and* $\rho(\varnothing) = 0$*. Define* $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *by*

$$\mu^*(A) = \inf\left\{ \sum_{n\in\mathbb{N}} \rho(E_n) : (E_n)_{n\in\mathbb{N}} \subset \mathcal{E} \text{ and } A \subset \bigcup_{n\in\mathbb{N}} E_n \right\}$$

Then μ^* *is an outer measure on* X*.*

Note 2.2.3. *In particular, for each* $A \in \mathcal{E}$*,* $\mu^*(A) = \rho(A)$*.*

Definition 2.2.4. *Let* X *be a set and* $\mathcal{E} \subset \mathcal{P}(X)$ *and* $\rho : \mathcal{E} \to [0, \infty]$*. Suppose that* $\varnothing, X \in \mathcal{E}$ *and* $\rho(\varnothing) = 0$*. Let* μ^* *be the outer measure on* X *defined as in the last theorem. Then* μ^* *is called the* ***outer measure on*** X ***induced by*** ρ*.*

Definition 2.2.5. *Let* X *be a set,* μ^* *an outer measure on* X *and* $A \subset X$*. Then* A *is said to be* μ^**-**outer measurable** if for each* $E \subset X$*,*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Theorem 2.2.6. *Let* X *be a set and* μ^* *an outer measure on* X*. Define* $\mathcal{A} = \{A \subset X : A \text{ is } \mu^*\text{-measurable}\}$*. Then* $\mathcal{A}$ *is a* σ*-algebra on* X *and* $\mu^*|_{\mathcal{A}}$ *is a complete measure on* $(X, \mathcal{A})$*.*

Definition 2.2.7. *Let* X *be a set,* $\mathcal{A}_0$ *be an algebra on* X *and* $\mu_0 : \mathcal{A}_0 \to [0, \infty]$*. Then* μ_0 *is said to be a* ***premeasure on*** $(X, \mathcal{A}_0)$ *if*

(1) *there exists* $A \in \mathcal{A}_0$ *such that* $\mu_0(A) < \infty$
(2) *for each* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$*, if* $(A_n)_{n\in\mathbb{N}}$ *is disjoint and* $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}_0$*, then*

$$\mu_0(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu_0(A_n)$$

Note 2.2.8. *The same reasoning applied to measures shows that* $\mu_0(\varnothing) = 0$*.*

Theorem 2.2.9. *Let* X *be a set,* $\mathcal{A}_0$ *an algebra on* X*,* μ_0 *a premeasure on* $(X, \mathcal{A}_0)$ *and* μ^* *the outer measure on* X *induced by* μ_0*. Put* $\mathcal{A} = \sigma(\mathcal{A}_0)$*. If* μ_0 *is* σ*-finite, then there exists a unique measure* μ *on* $(X, \mathcal{A})$ *such that* $\mu|_{\mathcal{A}_0} = \mu^*|_{\mathcal{A}_0} = \mu_0$*.*

2.3. **Product Measures.**

Definition 2.3.1. *Let* $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ *be* σ*-finite measurable spaces. Put* $\mathcal{E} = \{A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$*. Then* $\mathcal{E}$ *is an elementary family and thus* $\mathcal{M}_0 = \{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n \subset \mathcal{E} \text{ are disjoint}\}$ *is an algebra on* $X \times Y$*. We define* $\pi_0 : \mathcal{M}_0 \to [0, \infty]$ *by*

$$\pi_0\left(\bigcup_{i=1}^n A_i \times B_i\right) = \sum_{i=1}^n \mu(A_i)\nu(B_i)$$

*Then π_0 is a premeasure on $(X \times Y, M_0)$. Since $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{M}_0)$, we define the **product measure**, $\mu \times \nu$ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$, to be the unique extension of π_0 to $\mathcal{A} \otimes \mathcal{B}$. The existence of which is guaranteed by a theorem in the previous section. In particular,*

$$\begin{aligned}\mu \times \nu(E) &= \inf\left\{\sum_{n\in\mathbb{N}} \pi_0(E_i) : (E_i)_{i\in\mathbb{N}} \subset \mathcal{M}_0 \text{ and } E \subset \bigcup_{i\in\mathbb{N}} E_i\right\} \\ &= \inf\left\{\sum_{n\in\mathbb{N}} \mu(A_i)\nu(B_i) : (A_i \times B_i)_{i\in\mathbb{N}} \subset \mathcal{E} \text{ and } E \subset \bigcup_{i\in\mathbb{N}} A_i \times B_i\right\}\end{aligned}$$

3. Integration

3.1. Measurable Functions.

Definition 3.1.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then f is said to be $\mathcal{A}$-$\mathcal{B}$ **measurable** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. When $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say that f is $\mathcal{A}$-**measurbale**. If $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}, \mathcal{L})$, then we say that f is **Borel measurable** or **Lebsgue measurbale** respectively.*

Exercise 3.1.2. *Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then*
1. *$\{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y*
2. *$\{f^{-1}(B) : B \in \mathcal{B}\}$ is a σ-algebra on X*

Proof.
1. Define $\mathcal{L} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$. Clearly $Y \in \mathcal{L}$. Let $B \in \mathcal{L}$. Then $f^{-1}(B) \in \mathcal{A}$. Hence
$$f^{-1}(B^c) = (f^{-1}(B))^c \in \mathcal{A}$$
Thus $B^c \in \mathcal{L}$. Now, let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{L}$. Then for each $n \in \mathbb{N}$, $f^{-1}(B_n) \in \mathcal{A}$. Thus
$$f^{-1}\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \bigcup_{n\in\mathbb{N}} f^{-1}(B_n) \in \mathcal{A}$$
Hence $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{L}$.
2. Similar to (1).

□

Exercise 3.1.3. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Suppose that there exists $\mathcal{E} \subset Y$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Let $f : X \to Y$. Then f is $\mathcal{A}$-$\mathcal{B}$ measurable iff for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$.*

Proof. By definition, if f is $\mathcal{A}$-$\mathcal{B}$ measurable, then for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. Conversely, suppose that for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. The previous lemma tells us that $\mathcal{L} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra on Y. Since $\mathcal{E} \subset \mathcal{L}$, we have that $\mathcal{B} = \sigma(\mathcal{E}) \subset \mathcal{L}$. So f is $\mathcal{A}$-$\mathcal{B}$ measurable. □

Exercise 3.1.4. *Let X, Y be sets, $f : X \to Y$ and $\mathcal{E} \subset \mathcal{P}(Y)$. Then $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$.*

Proof. Clealy $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$. Since $f^{-1}(\sigma(\mathcal{E}))$ is a σ-algebra, we have that $\sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. Since $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$, the previous exercise tells us that f is $f^{-1}(\sigma(\mathcal{E}))$-$\sigma(\mathcal{E})$ measurable. Then $f^{-1}(\sigma(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. So $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$. □

Exercise 3.1.5. *Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X \to Y$. If f is continuous, then f is $\mathcal{B}(X)$-$\mathcal{B}(Y)$ measurable.*

Proof. Recall that $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$ and continuity tells us that for each $U \in \mathcal{T}_2$, $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$. $\square$

Definition 3.1.6. *Let X be a set and $f : X \to \mathbb{C}$. Then f is said to be* ***simple*** *if $f(X)$ is finite.*

Definition 3.1.7. *Let $(X, \mathcal{A})$ be a measurable space. We define $S^+(X, \mathcal{A}) = \{f : X \to [0, \infty) : f$ is simple, measurable$\}$ and $S(X, \mathcal{A}) = \{f : X \to \mathbb{C} : f$ is simple, measurable$\}$*

Theorem 3.1.8. *Let $(X, \mathcal{A})$ be a measurable space. Then*
1. *If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.*
2. *If $f : X \to \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset S$ such that for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.*

3.2. **Integration of Nonnegative Functions.**

Definition 3.2.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define*

$$L^+(X, \mathcal{A}, \mu) = \{f : X \to [0, \infty] : f \text{ is measurable}\}$$

We will typically just write L^+.

Theorem 3.2.2. ***Monotone Convergence Theorem:*** *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then*

$$\sup_{n\in\mathbb{N}} \int f_n = \int \sup_{n\in\mathbb{N}} f_n$$

.

Exercise 3.2.3. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$ and $f \in L^+$. Then*

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

.

Proof. Suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset [0, \infty)$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum\limits_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}
\int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\
&= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\
&= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\
&= \int f d\mu_1 + \int f d\mu_2
\end{aligned}$$

Now for a general f, choose $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that $\phi_n \to f$ pointwise and for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$. Then monotone convergence tells us that

$$\begin{aligned}\int f d(\mu_1 + \mu_2) &= \lim_{n\to\infty} \int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n\to\infty} \int \phi_n d\mu_1 + \lim_{n\to\infty} \int \phi_n d\mu_2 \\ &= \int f d\mu_1 + \int f d\mu_2\end{aligned}$$

□

Exercise 3.2.4. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Suppose that $\mu_1 \leq \mu_2$. Then for each $f \in L^+$,*

$$\int f d\mu_1 \leq \int f d\mu_2$$

Proof. First suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset [0, \infty)$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum\limits_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}\int f d\mu_1 &= \sum_{i=1}^n a_i \mu_1(E_i) \\ &\leq \sum_{i=1}^n a_i \mu_2(E_i) \\ &= \int f d\mu_2\end{aligned}$$

for general f,

$$\begin{aligned}\int f d\mu_1 &= \sup_{\substack{s\in S^+ \\ s\leq f}} \int s d\mu_1 \\ &\leq \sup_{\substack{s\in S^+ \\ s\leq f}} \int s d\mu_2 \\ &= \int f d\mu_2\end{aligned}$$

□

Theorem 3.2.5. *Fatou's Lemma Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then*

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Theorem 3.2.6. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then*

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n.$$

Exercise 3.2.7. *Let $f \in L^+$ and suppose that $\int f < \infty$. Put $N = \{x \in X : f(x) = \infty\}$ and $S = \{x \in X : f(x) > 0\}$. Then $\mu(N) = 0$ and S is σ-finite.*

Proof. Suppose that $\mu(N) > 0$. Define $f_n = n\chi_N \in L^+$. Then for each $n \in \mathbb{N}$, $f_n \leq f_{n+1} \leq f$ on N. So

$$\begin{aligned}\int f &\geq \int_N f \\ &= \lim_{n\to\infty} \int_N f_n \\ &= \lim_{n\to\infty} n\mu(N) \\ &= \infty, \text{ a contradiction.}\end{aligned}$$

Hence N is a null set. Now, put $S_n = \{x \in X : f(x) > 1/n\}$. Then $S = \bigcup_{n\in\mathbb{N}} S_n$. Suppose that there exists some $n \in \mathbb{N}$ such that $\mu(S_n) = \infty$. Then

$$\begin{aligned}\int f &\geq \int_{S_n} f \\ &\geq \frac{1}{n}\mu(S_n) \\ &= \infty, \text{ a contradiction.}\end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(S_n) < \infty$ and S is σ-finite.

□

Exercise 3.2.8. *Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.*

Proof. $f = 0$ a.e. implies that for each $E \in \mathcal{A}$, $\int_E f = 0$ is clear. Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f = 0$. For $n \in \mathbb{N}$ put $N_n = \{x \in X : f(x) > 1/n\}$ and define $N = \{x \in X : f(x) > 0\}$. So $N = \bigcup_{n\in\mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned}0 &= \int_{N_n} f \\ &\geq \frac{1}{n}\mu(N_n) \\ &\geq 0.\end{aligned}$$

Hence for each $n \in \mathbb{N}$, $\mu(N_n) = 0$. Thus $\mu(N) = 0$ and $f = 0$ a.e. as required.

□

Exercise 3.2.9. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that $f_n \xrightarrow{p.w.} f$, $\lim_{n\to\infty} \int f_n = \int f$ and $\int f < \infty$. Then for each $E \in \mathcal{A}$, $\lim_{n\to\infty} \int_E f_n = \int_E f$. This result may fail to be true if $\int f = \infty$*

Proof. Let $E \in \mathcal{A}$. By Fatou's lemma, $\int_E f \leq \liminf\limits_{n\to\infty} \int_E f_n$. Note that since $\int f < \infty$, we have that $\int_{E^c} f \leq \int f < \infty$. Thus we may write

$$\begin{aligned}
\int_E f &= \int f - \int_{E^c} f \\
&\geq \int f - \liminf_{n\to\infty} \int_{E^c} f_n \\
&= \int f - \liminf_{n\to\infty} \left(\int f_n - \int_E f_n \right) \\
&= \int f - \int f + \limsup_{n\to\infty} \int_E f_n \\
&= \limsup_{n\to\infty} \int_E f_n.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \int_E f_n \leq \int_E f \leq \liminf_{n\to\infty} \int_E f_n$$

and therefore

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

If we drop the assumption that $\int f < \infty$, then the result would fail to be true for the functions $f = \infty\chi_{(0,1)}$ and $f_n = \infty\chi_{(0,1)} + n\chi_{(1,1+1/n)}$. Here $f_n \xrightarrow{\text{p.w.}} f$, $\lim\limits_{n\to\infty} \int f_n = \int f = \infty$ and $\lim\limits_{n\to\infty} \int_{(1,\infty)} f_n = 1$ while $\int_{(1,\infty)} f = 0$.

□

Exercise 3.2.10. *Let $f \in L^+$. Define $\lambda : \mathcal{A} \to [0,\infty]$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{A}$ Then λ is a measure on $(X, \mathcal{A})$ and for each $g \in L^+$, $\int g d\lambda = \int g f d\mu$.*

Proof. Clearly $\lambda(\varnothing) = 0$. Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ and suppose that for each $i, j \in \mathbb{N}$, if $i \neq j$, then $A_i \cap A_j = \varnothing$. For now, suppose that f is simple. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and

$a_1, a_2, \cdots, a_n \in [0, \infty)$ such that $f = \sum_{i=1}^{n} a_i \chi_{E_i}$. Then

$$\begin{aligned}
\lambda\bigg(\bigcup_{j\in\mathbb{N}} A_j\bigg) &= \int_{\bigcup_{j\in\mathbb{N}} A_j} f \\
&= \sum_{i=1}^{n} a_i \mu\bigg(E_i \cap \bigg(\bigcup_{j\in\mathbb{N}} A_j\bigg)\bigg) \\
&= \sum_{i=1}^{n} a_i \mu\bigg(\bigcup_{j\in\mathbb{N}} E_i \cap A_j\bigg) \\
&= \sum_{i=1}^{n} a_i \sum_{j\in\mathbb{N}} \mu(E_i \cap A_j) \\
&= \sum_{j\in\mathbb{N}} \sum_{i=1}^{n} a_i \mu(E_i \cap A_j) \\
&= \sum_{j\in\mathbb{N}} \int_{A_j} f \\
&= \sum_{j\in\mathbb{N}} \lambda(A_j)
\end{aligned}$$

Hence λ is a measure on $(X, \mathcal{A})$. Now, for a general f, there exist $(\phi_n)_{n\in\mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \xrightarrow{\text{p.w.}} f$. Put $A = \bigcup_{j\in\mathbb{N}} A_j$ and define the measures λ_n by $\lambda_n(E) = \int_E \phi_n$. Note that we may define a monotonically increasing sequence of functions $g_n : \| \to [0, \infty]$ by $g_n(j) = \int_{A_j} \phi_n$. Using monotone convergence three times and a nice application of the counting measure on $\mathbb{N}$, we may write

$$\begin{aligned}
\lambda(A) &= \int_A f \\
&= \lim_{n\to\infty} \int_A \phi_n \\
&= \lim_{n\to\infty} \sum_{j\in\mathbb{N}} \int_{A_j} \phi_n \\
&= \sum_{j\in\mathbb{N}} \lim_{n\to\infty} \int_{A_j} \phi_n \quad \text{(by the above)} \\
&= \sum_{j\in\mathbb{N}} \int_{A_j} f \\
&= \sum_{j\in\mathbb{N}} \lambda(A_j).
\end{aligned}$$

Hence λ is a measure on $(X, \mathcal{A})$. Let $g \in L^+$. First assume that g is simple. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0, \infty)$ such that $g = \sum_{i=1}^{n} a_i \chi_{E_i}$. In this case, we have that

$$\begin{aligned}\int g d\lambda &= \sum_{i=1}^{n} a_i \lambda(E_i)\\ &= \sum_{i=1}^{n} a_i \int_{E_i} f d\mu\\ &= \int \left(\sum_{i=1}^{n} a_i \chi_{E_i}\right) f d\mu\\ &= \int g f d\mu.\end{aligned}$$

Now for a general $g \in L^+$, there exist $(\psi_n)_{n\in\mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ψ_n is simple, $\psi_n \leq \psi_{n+1} \leq f$ and $\psi_n \xrightarrow{\text{p.w.}} g$. Monotone convergence then gives us

$$\begin{aligned}\int g d\lambda &= \lim_{n\to\infty} \int \psi_n d\lambda\\ &= \lim_{n\to\infty} \int \psi_n f d\mu\\ &= \int g f d\mu \text{ as required.}\end{aligned}$$

□

Exercise 3.2.11. *Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$, $f_n \xrightarrow{p.w.} f$ and $\int f_1 < \infty$. Then $\lim\limits_{n\to\infty} \int f_n = \int f$.*

Proof. First we note that since $\int f_1 < \infty$, $f_1 < \infty$ a.e., for each $n \in \mathbb{N}$, $f_1 - f_n$ and $\int f_1 - \int f_n$ are well defined and $\int f_n \leq \int f_1 < \infty$. Also, for $n \in \mathbb{N}$, $f_1 - f_n \in L^+$. So we may write

$$\begin{aligned}\int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n\\ &= \int [(f_1 - f_n) + f_n] - \int f_n\\ &= \int f_1 - \int f_n\end{aligned}$$

Put $g_n = f + (f_1 - f_n)$. Then $g_n \in L^+$, for each $n \in \mathbb{N}$, $g_n \leq g_{n+1}$ and $g_n \xrightarrow{\text{p.w.}} f_1$. Monotone convergence tells us that

$$\begin{aligned}\int f_1 &= \lim_{n\to\infty} \int g_n\\ &= \lim_{n\to\infty} \left[\int f + (f_1 - f_n)\right]\\ &= \lim_{n\to\infty} \left[\int f + \int (f_1 - f_n)\right]\\ &= \lim_{n\to\infty} \left[\int f + \int f_1 - \int f_n\right]\end{aligned}$$

Since $\lim_{n\to\infty} \int f$ and $\lim_{n\to\infty} \int f_1$ exist, $\lim_{n\to\infty} \int f_n = \int f$ as required.

□

3.3. **Integration of Complex Valued Functions.**

Definition 3.3.1. *Let $f : X \to \mathbb{C}$ be measurable. Then f is said to be* ***integrable*** *if*

$$\int |f| d\mu < \infty$$

Definition 3.3.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $L^1(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} : f \text{ is measurable and } \int |f| < \infty\}$*

Lemma 3.3.3. *Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff f^+ and f^- are integrable.*

Proof. $f^+, f^- \leq |f| = f^+ + f^-$ □

Definition 3.3.4. *Let $f : X \to \mathbb{R}$ be measurable. Then f is said to be* ***extended integrable*** *if*

$$\int f^+ d\mu < \infty \text{ or } \int f^- d\mu < \infty$$

Lemma 3.3.5. *Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff $Re(f)$ and $Im(f)$ are integrable.*

Proof. $|Re(f)|, |Im(f)| \leq |f| \leq |Re(f)| + |Im(f)|$ □

Theorem 3.3.6. *Dominated Convergence Let $(f_n)_{n\in\mathbb{N}} \subset L^1$, f measurable and $g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. Then $f \in L^1$ and $\int f_n \to \int f$.*

Exercise 3.3.7. *Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Then*

(1) $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$
(2) *for each $f \in L^1(\mu_1 + \mu_2)$, we have that*

$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2$$

Proof. (1) The firt part is clear since similar exercise from the section on nonnegative funtions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f| d\mu_1 + \int |f| d\mu_2$$

(2) Suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset \mathbb{C}$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum\limits_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}
\int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i (\mu_1 + \mu_2)(E_i) \\
&= \sum_{i=1}^n a_i (\mu_1(E_i) + \mu_2(E_i)) \\
&= \sum_{i=1}^n a_i \mu_1(E_i) + a_i \mu_2(E_i) \\
&= \int f d\mu_1 + \int f d\mu_2
\end{aligned}$$

Now for general f, choose $(\phi_n)_{n\in\mathbb{N}} \subset S$ such that $\phi_n \to f$ pointwise and for each $n \in \mathbb{N}$, $|\phi_n| \le |\phi_{n+1}| \le |f|$. Then dominated convergence tells us that

$$\begin{aligned}
\int f d(\mu_1 + \mu_2) &= \lim_{n\to\infty} \int \phi_n d(\mu_1 + \mu_2) \\
&= \lim_{n\to\infty} \int \phi_n d\mu_1 + \lim_{n\to\infty} \int \phi_n d\mu_2 \\
&= \int f d\mu_1 + \int f d\mu_2
\end{aligned}$$

□

Theorem 3.3.8. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$. Suppose that*

$$\sum_{n\in\mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n\in\mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n$$

Theorem 3.3.9. *Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$.*

Exercise 3.3.10. *Generalized Fatou's Lemma: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \ge 0$ and for each $n \in \mathbb{N}$, $f_n \ge -g$. Then $\int \liminf\limits_{n\to\infty} f_n \le \liminf\limits_{n\to\infty} \int f_n$. What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?*

Proof. First note that for each $n \in \mathbb{N}$, $\int f_n$ is well defined since $f_n^- \leq g \in L^1$. Since $g + f_n \geq 0$, we may use Fatou's lemma to write

$$\begin{aligned}\int g + \int \liminf_{n\to\infty} f_n &= \int \liminf_{n\to\infty}(g + f_n) \\ &\leq \liminf_{n\to\infty} \int (g + f_n) \\ &= \int g + \liminf_{n\to\infty} \int f_n\end{aligned}$$

Since $\int g < \infty$, $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$ as required. The analogue is as follows: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \leq g$. Then $\limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n$. To show this, just use the result from above with the sequence $(g_n)_{n\in\mathbb{N}}$ given by $g_n = -f_n$. □

Exercise 3.3.11. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \to \mathbb{C}$. Suppose that $f_n \xrightarrow{uni} f$. Then*

(1) *if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim_{n\to\infty} \int f_n = \int f$*
(2) *if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on $\mathbb{R}$ with Lebesgue measure).*

Proof. Choose $N \in \mathbb{N}$ such that for $n \geq N$ and $x \in X$, $|f(x) - f_n(x)| < 1$. Then $||f| - |f_N|| < 1$ and so $|f| < |f_N| + 1$. Thus $\int |f| \leq \int |f_N| + \mu(X) < \infty$ and $f \in L^1$. Similarly for $n \geq N$, $|f_n| < |f| + 1$. Dominated convergence then gives us that $\lim_{n\to\infty} \int f_n = \int f$ as required. To see the necessity that $\mu(X) < \infty$, consider $f \equiv 0$ and $f_n = (1/n)\chi_{(0,n)}$. Then $f_n \xrightarrow{\text{uni}} f$, but $1 = \lim_{n\to\infty} \int f_n \neq \int f = 0$. □

Exercise 3.3.12. *Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$, $g_n \xrightarrow{a.e.} g$, $|f_n| \leq g_n$ and $\int g_n \to \int g$. Then $\int f_n \to \int f$.*

Proof. We simply use Fatou's lemma. Put $h_n = (g + g_n) - |f_n - f|$. Since for each $n \in \mathbb{N}$, $|f_n| \leq g_n$, we know that $|f| \leq g$. So $h_n \geq 0$ and $h_n \xrightarrow{\text{p.w.}} 2g$. Thus

$$\begin{aligned}2\int g &= \int \liminf_{n\to\infty} h_n \\ &\leq \liminf_{n\to\infty} \left[\left(\int g + \int g_n\right) - \int |f_n - f|\right] \\ &= 2\int g + \liminf_{n\to\infty}\left(-\int |f_n - f|\right) \\ &= 2\int g - \limsup_{n\to\infty} \int |f_n - f|\end{aligned}$$

Hence $\limsup_{n\to\infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $\int f_n \to \int f$ as required. □

Exercise 3.3.13. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{a.e.} f$. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.*

Proof. Suppose that $\int |f_n - f| \to 0$. Since

$$\begin{aligned}\left| \int |f_n| - \int |f| \right| &= \left| \int (|f_n| - |f|) \right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|,\end{aligned}$$

we see that $\int |f_n| \to \int |f|$. Conversely, suppose that $\int |f_n| \to \int |f|$. Put $h_n = |f_n - f|$, $g_n = |f_n| + |f|$, $h \equiv 0$ and $g = 2f$. Then $h_n \xrightarrow{\text{a.e.}} h$, $g_n \xrightarrow{\text{a.e.}} g$ and for each $n \in \mathbb{N}$, $h_n \leq g_n$. Our assumption implies that $\int g_n \to \int g$. Thus the last exercise tells us that $\int h_n \to \int h$ as required.

□

Exercise 3.3.14. *Let $(r_n)_{n\in\mathbb{N}}$ be an enummeration of the rationals. Define $f : \mathbb{R} \to [0, \infty)$ by*

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

and define $g : X \to [0, \infty]$ by

$$g(x) = \sum_{n\in\mathbb{N}} 2^{-n} f(x - r_n).$$

Then

1. *$g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.*
2. *$g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of $\mathbb{R}$*
3. *Taking $g \in L^1$, g is unbounded on each subinterval of $\mathbb{R}$ and discontinuous everywhere and remains so after redefinition on a null set*

Proof. For convenience, define $f_n : \mathbb{R} \to [0, \infty)$ by $f_n(x) = f(x - r_n)$ for $x \in \mathbb{R}$. To show (1) we note that for each $n \in \mathbb{N}$, $f_n \in L^1$ and

$$\begin{aligned}\int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1}\end{aligned}$$

Hence

$$\sum_{n\in\mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set, $g \in L^1$. In particular $\int |g| < \infty$ and so $|g|$ (and hence g) are finite almost everywhere. For (2), since $g < \infty$ a.e., so too is g^2. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Choose $N \in \mathbb{N}$ such that $r_N \in (a, b)$. Since all the terms in the sum are nonnegative, $g^2 \geq \sum_{n\in\mathbb{N}} 2^{-2n} f_n^2$ and so

$$\begin{aligned}\int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n\in\mathbb{N}} 2^{-2n} f_n^2 \\ &= \sum_{n\in\mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\ &\geq 2^{-2N} \int_{(a,b)} f_N^2 \\ &\geq 2^{-2N} \int_{r_N}^{b\wedge(r_N+1)} \frac{1}{x-r_N} dx \\ &= \infty\end{aligned}$$

So g^2 is not integrable on any subinterval of $\mathbb{R}$. For (3), note that redefining g on a null set does not change the result of (2). Suppose that there is a finite subinterval $I \subset \mathbb{R}$ such that g is bounded on I. Hence there exists $M > 0$ such that for each $x \in I$, $g(x)^2 \leq M$. Then

$$\begin{aligned}\int_I g^2 &\leq M^2 m(I) \\ &< \infty\end{aligned}$$

which is a contradiction. So g is not bounded on any subinterval of $\mathbb{R}$. Now, suppose that there exists $x_0 \in \mathbb{R}$ such that g is continuous at x_0. Choose $\delta > 0$ such that for each $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then $|g(x) - g(x_0)| < 1$. The reverse triangle inequality tells us that for each $x \in (x_0 - \delta, x_0 + \delta)$, $|g(x)| < 1 + |g(x_0)|$. Hence g is bounded on $(x_0 - \delta, x_0 + \delta)$ which is a contradiction. So g is discontinuous everywhere. □

Exercise 3.3.15. *Let $f \in L^1$.*

(1) *If f is bounded, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\int_E |f| < \epsilon$.*
(2) *The same conclusion holds for f unbounded.*

Proof. (1) Since f is bounded, there exists $M > 0$ such that $|f| \leq M$. Let $\epsilon > 0$. Choose $\delta = \epsilon/2M$. Let $E \in \mathcal{A}$. Suppose that $\mu(A) < \delta$. Then

$$\begin{aligned}\int_E |f| &\leq M\mu(E) \\ &= M\frac{\epsilon}{2M} \\ &= \frac{\epsilon}{2} \\ &< \epsilon\end{aligned}$$

(2) Suppose that f is unbounded. Let $\epsilon > 0$. Then there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon/2$. Since ϕ is bounded, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$,

if $\mu(E) < \delta$, then $\int_E |\phi| < \epsilon/2$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then

$$\begin{aligned}\int_E |f| &\leq \int_E |f-\phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon\end{aligned}$$

□

Exercise 3.3.16. *Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Define $F : \mathbb{R} \to \mathbb{R}$ by*

$$F(x) = \int_{(-\infty,x]} f dm.$$

Then F is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Since $f \in L^1$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then

$$\int_{(x \wedge x_0, x \vee x_0]} |f| dm < \epsilon.$$

Let $x \in \mathbb{R}$. Suppose that $|x - x_0| < \delta$. Then

$$\begin{aligned}|F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f| dm \\ &< \epsilon\end{aligned}$$

So F is continuous.

□

Exercise 3.3.17. *Denote by δ_x the point mass measure at $x \in X$ on measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$. Then*

$$\int f d\delta_x = f(x)$$

Proof. First assume that f is simple. Then there exist $a_1, a_2, \cdots, a_n \in \mathbb{C}$ and $E_1, E_2, \cdots, E_n \in \mathcal{P}(X)$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$ Thus $\int f d\delta_x = f(x)$. Now assume that f, which is measurable by choice of σ-algebra, satisfies $f(X) \subset [0, \infty)$. Choose a sequence $(\phi_n)_{n \in \mathbb{N}} \subset L^+$ such that for each $n \in \mathbb{N}$, ϕ_n is simple, $\phi_n \leq \phi_{n+1}$ and $\phi_n \xrightarrow{\text{p.w}} f$. From before, we see that for each $n \in \mathbb{N}$, $\int \phi_n d\delta_x = \phi_n(x)$. Monotone convergence tells us that $\int f d\delta_x = f(x)$. Now just extend to complex valued functions.

□

Exercise 3.3.18. *Denote by # the counting measure on the measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$ and suppose that $f \in L^1$. Then*

$$\int f d\# = \sum_{x \in X} f(x).$$

In particular, if f is integrable, then $\{x \in X : f(x) \neq 0\}$ is countable.

Proof. Please refer to the definition of the sum in the appendix. First suppose that $f(X) \subset [0, \infty)$. For $n \in \mathbb{N}$, put $X_n = \{x \in X : f(x) > 1/n\}$ and define $X^* = \{x \in X : f(x) > 0\}$, $X_0 = \{x \in X : f(x) = 0\}$ Then $X^* = \bigcup_{n\in\mathbb{N}} X_n$. Since $f \in L^1$, we have that for each $n \in \mathbb{N}$,

$$\begin{aligned}\infty &> \int f d\# \\ &\geq \int_{X_n} f d\# \\ &\geq \frac{1}{n}\#(X_n).\end{aligned}$$

Thus for each $n \in \mathbb{N}$, X_n is finite and X^* is countable. Thus there exists $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $X^* = \{x_n\}_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$, define $E_n = \{x_1, x_2, \cdots, x_n\}$ and

$$\begin{aligned}f_n &= f\chi_{E_n} \\ &= \sum_{i=1}^n f(x_i)\chi_{\{x_i\}}\end{aligned}$$

Then $f_n \xrightarrow{\text{p.w.}} f\chi_{X^*} = f$ and for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. So

$$\begin{aligned}\int f &= \sup_{n\in\mathbb{N}} \int f_n \\ &= \sup_{n\in\mathbb{N}} \sum_{i=1}^n f(x_i) \\ &= \sum_{x\in X^*} f(x) \\ &= \sum_{x\in X} f(x).\end{aligned}$$

For $f : X \to \mathbb{C}$, our L^1 assumption and the result above tell us that

$$\sum_{x\in X} |f(x)| < \infty.$$

Thus writing $f = g + ih$, we see that the same is true for f^+, f^-, g^+, g^-. Simply using the definitions of the sum and the integral, as well as the result from above, we have that

$$\int f d\# = \sum_{x\in X} f(x).$$

$\square$

Exercise 3.3.19. *Let $f, g : X \to \mathbb{R}$. Suppose that $f, g \in L^1$. Then $f \leq g$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$.*

Proof. Suppose $f \leq g$ a.e. Put $N = \{x \in X : f(x) > g(x)\} \subset N$. Then $\mu(N) = 0$ and $g - f \geq 0$ on N^c. So for each $E \in \mathcal{A}$,

$$\begin{aligned}\int_E g - \int_E f &= \int_E (g - f) \\ &= \int_{E \cap N^c} (g - f) \\ &\geq 0\end{aligned}$$

Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f \leq \int_E g$. Put $N_n = \{x \in X : f(x) - g(x) > 1/n\}$ and $N = \{x \in X : f(x) > g(x)\}$. Then $N = \bigcup\limits_{n \in \mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned}0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n}\mu(N_n) \\ &\geq 0.\end{aligned}$$

So that $\mu(N_n) = 0$. Thus for each $n \in \mathbb{N}$, $\mu(N_n) = 0$ which implies $\mu(N) = 0$. Therefore $f \leq g$ a.e. as required. $\square$

Definition 3.3.20. *Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is said to be **uniformly integrable** if for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup\limits_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < \epsilon$. (i.e. $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| = 0$).*

Exercise 3.3.21. *Suppose that μ is finite. Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is uniformly integrable iff*

(1) *there exists $M > 0$ such that $\sup\limits_{f \in \mathcal{F}} \int |f| \leq M$*
(2) *for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\sup\limits_{f \in \mathcal{F}} \int_E |f| < \epsilon$.*

Proof. ($\Rightarrow$): (1) Suppose that $\mathcal{F}$ is uniformly integrable. Then there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup\limits_{f \in \mathcal{F}} \int_{\{|f| > k\}} |f| < 1$. Choose $M = \mu(X)K + 1$. Then for each $f \in \mathcal{F}$,

$$\begin{aligned}\int |f| &= \int_{\{|f| > K\}} |f| + \int_{\{|f| \leq K|\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M\end{aligned}$$

(2) Let $\epsilon > 0$. Then choose $K \in \mathbb{N}$ such that $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>K\}} |f| < \epsilon/2$ and choose $\delta = \epsilon/2K$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then for $f \in \mathcal{F}$,

$$\begin{aligned}\int_E |f| &= \int_{E\cap\{|f|>K\}} |f| + \int_{E\cap\{|f|\leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon\end{aligned}$$

($\Leftarrow$): Choose $M > 0$ as in (1). Suppose that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $f \in \mathcal{F}$ such that $\mu(\{|f| > K\}) \geq \epsilon$. Choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $f_K \in \mathcal{F}$ such that $\mu(\{|f_K| > K\}) \geq \epsilon$. Then

$$\begin{aligned}\int |f_K| &\geq \int_{\{|f_K|>K\}} |f| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M,\end{aligned}$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $f \in \mathcal{F}$, $\mu(\{|f| > K\}) < \epsilon$. Since $\mu(\{|f| > k\})$ is a decreasing sequence in k, we have that $\lim\limits_{k\to\infty} \sup\limits_{f\in\mathcal{F}} \mu(\{|f| > k\}) = 0$. Now, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Choose $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$, $\mu(\{|f| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$,

$$\int_{\{|f|>k\}} |f| < \epsilon.$$

Thus $\lim\limits_{k\to\infty} \sup\limits_{f\in\mathcal{F}} \int_{\{|f|>k\}} |f| = 0$ as required.

$\square$

3.4. **Integration on Product Spaces.**

Definition 3.4.1. *Let X, Y, and Z be sets, $E \subset X \times Y$ and $f : X \times Y \to Z$. For each $x \in X$, define $E_x = \{y \in Y : (x, y) \in E\}$ and $f_x : Y \to Z$ by $f_x(y) = f(x, y)$. For each $y \in Y$, define $E^y = \{x \in X : (x, y) \in E\}$ and $f^y : X \to Z$ by $f^y(x) = f(x, y)$.*

Note 3.4.2. *It is often helpful to observe that $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.*

Lemma 3.4.3. *Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces, $Z = [0, \infty]$ or $\mathbb{C}$ and $f : X \times Y \to Z$.*

(1) *For each $E \in \mathcal{A} \otimes \mathcal{B}$, $x \in X$, $y \in Y$, we have that $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$*
(2) *If f is $\mathcal{A}\otimes\mathcal{B}$-measurable, then for each $x \in X$, $y \in Y$, we have that f_x is $\mathcal{B}$-measurable and f^y is $\mathcal{A}$-measurable.*

Theorem 3.4.4. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then for each $E \in \mathcal{A} \otimes \mathcal{B}$, the maps $\phi : X \to [0, \infty]$ and $\psi : Y \to [0, \infty]$ defined by $\phi(x) = \nu(E_x)$ and $\psi(y) = \mu(E^y)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, respectively and*

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

Theorem 3.4.5. *Fubini, Tonelli: Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.*

(1) *(Tonelli) For each $f \in L^+(X \times Y)$, the functions $g : X \to [0, \infty]$, $h : Y \to [0, \infty]$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable respectively and*
$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$
(2) *(Fubini) For each $f \in L^1(X \times Y)$, $f_x \in L^1(\nu)$ for μ-a.e. $x \in X$ and $f^y \in L^1(\mu)$ for ν-a.e. $y \in Y$, respectively and the functions (after redefinition of f on a null set) $g : X \to \mathbb{C}$, $h : Y \to \mathbb{C}$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x) d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Furthermore*
$$\int_{X \times Y} f d\mu \times \nu = \int_X g d\mu = \int_Y h d\nu$$

Note 3.4.6. *We usually just write $\int \int f d\mu d\nu$ and $\int \int f d\nu d\mu$ instead of $\int h d\nu$ and $\int g d\mu$ respectively. We have a similar result for complete product measure spaces. See*

Exercise 3.4.7. *Take $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1]), \mathcal{B} = \mathcal{P}([0, 1])$ and μ, ν to be Lebesgue measure and counting measure respectively. Define $D = \{(x, y) \in [0, 1]^2 : x = y\}$ Show that*
$$\int \chi_D d\mu \times \nu, \int \int \chi_D d\mu d\nu \text{ and } \int \int \chi_D d\nu d\mu$$
are all different. (Hint: for the first integral, use the definition of $\mu \times \nu$)

Proof. Let $x, y \in [0, 1]$. Then $(\chi_D)_x = \chi_{D_x} = \chi_x$ and $(\chi_D)^y = \chi_{D^y} = \chi_y$. Thus
$$\begin{aligned}\int \int \chi_D d\mu d\nu &= \int \mu(\{y\}) d\nu \\ &= \int 0 d\nu \\ &= 0\end{aligned}$$
and
$$\begin{aligned}\int \int \chi_D d\mu d\nu &= \int \nu(\{x\}) d\mu \\ &= \int 1 d\mu \\ &= 1\end{aligned}$$
Now, Observe that $\int \chi_D d\mu \times \nu = \mu \times \nu(D)$. Recall from the section on product measures that $\mu \times \nu(D) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n)\nu(B_n) : (A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ and $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n\}$. Let $(A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$. Suppose that $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n$. Then for each $x \in [0, 1]$, $(x, x) \in \bigcup_{n \in \mathbb{N}} A_n \times B_n$. So for each $x \in [0, 1]$, there exists $n \in \mathbb{N}$, such that $x \in A_n \cap B_n$. Thus $[0, 1] \subset \bigcup_{n \in \mathbb{N}} A_n \cap B_n$. Since $1 = \mu([0, 1]) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap B_n)$, we know that there exists $n \in \mathbb{N}$ such that $0 < \mu(A_n \cap B_n)$. Thus $\mu(A_n) > 0$ and $\mu(B_n) > 0$. Since $\mu(B_n) > 0$, B_n must be infinite and therefore $\nu(B_n) = \infty$. So $\sum_{n \in \mathbb{N}} \mu(A_n)\nu(B_n) = \infty$.

□

Exercise 3.4.8. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $f : X \to [0, \infty) \in L^+$. Show that $G = \{(x, y) \in X \times [0, \infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$ and $\mu \times m(G) = \int_X f d\mu$. The same is true if we replace "$\geq$" with "$>$". (Hint: to show that G is measurable, split up $(x, y) \mapsto f(x) - y$) into the composition of measurable functions.*

Proof. Define $\phi : X \times [0, \infty) \to [0, \infty)^2$ and $\psi : [0, \infty)^2 \to [0, \infty)$ by $\phi(x, y) = (f(x), y)$ and $\psi(z, y) = z - y$. Then $G = \{(x, y) \in X \times [0, \infty) : \psi \circ \phi(x, y) \geq 0\}$. Let $A, B \in \mathcal{B}([0, \infty))$. Then $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0, \infty))$. Since $\mathcal{B}([0, \infty)^2) = \mathcal{B}([0, \infty)) \otimes \mathcal{B}([0, \infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0, \infty))\})$, we have that ϕ is $\mathcal{A} \otimes \mathcal{B}([0, \infty))$-$\mathcal{B}([0, \infty)^2)$ measurable. Since ψ is continuous, we have that ψ is $\mathcal{B}([0, \infty)^2)$-$\mathcal{B}([0, \infty))$ measurable. This implies that $\psi \circ \phi$ is $\mathcal{A} \otimes \mathcal{B}([0, \infty))$-$\mathcal{B}([0, \infty))$ measurable. Thus $G = \psi \circ \phi^{-1}([0, \infty)) \in \mathcal{A} \otimes \mathcal{B}([0, \infty))$. Now for $x \in X$, $G_x = \{y \in [0, \infty) : f(x) \geq y\} = [0, f(x)]$. Thus

$$\begin{aligned}
\mu \times m(G) &= \int \chi_G d\mu \times m \\
&= \int_X \int_{[0,\infty)} \chi_{G_x} dm d\mu(x) \\
&= \int_X f(x) d\mu(x)
\end{aligned}$$

The same reasoning holds if we replace "$\geq$" with "$>$". $\square$

Exercise 3.4.9. *Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$. Define $h : X \times Y \to \mathbb{C}$ by $h(x, y) = f(x)g(y)$.*

1. *If f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable, then h is $\mathcal{A} \otimes \mathcal{B}$-measurable.*
2. *If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and*
$$\int_{X \times Y} h d\mu \times \nu = \int_X f d\mu \int_Y g d\nu$$

Proof.
1. First suppose that f, g are simple. Then there exist $(A_i)_{i=1}^n \subset \mathcal{A}$, $(B_j)_{j=1}^m \subset \mathcal{B}$ and $(a_i)_{i=1}^n, (b_i)_{j=1}^m \subset \mathbb{C}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$ and $g = \sum_{j=1}^m b_j \chi_{B_j}$. Then $h = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{A_i \times B_j}$. So h is $\mathcal{A} \otimes \mathcal{B}$-measurable. For general f, g, there exist $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$ and $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$ such that $f_n \to f$ pointwise, $g_n \to g$ pointwise and for each $n \in \mathbb{N}$, $|f_n| \leq |f_{n+1}| \leq |f|$ and $|g_n| \leq |g_{n+1}| \leq |g|$. For $n \in \mathbb{N}$, define $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$ by $h_n = f_n g_n$. Then $h_n \to h$ pointwise and for each $n \in \mathbb{N}$, $|h_n| \leq |h_{n+1}| \leq |h|$. Thus h is $\mathcal{A} \otimes \mathcal{B}$-measurable.
2. First suppose f and g are simple as before. Then
$$\begin{aligned}
\int_{X \times Y} |h| d\mu \times \nu &\leq \sum_{i=1}^n \sum_{j=1}^m |a_i b_j| \mu(A_i) \nu(B_j) \\
&= \big(\sum_{i=1}^n |a_i| \mu(A_i)\big)\big(\sum_{j=1}^m |b_j| \nu(B_j)\big) \\
&= \int_X |f| d\mu \int_Y |g| d\nu \\
&< \infty
\end{aligned}$$

So $h \in L^1(\mu \times \nu)$. Furthermore,

$$\begin{aligned}\int_{X\times Y} h d\mu \times \nu &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \mu(A_i)\nu(B_j)\\ &= \big(\sum_{i=1}^{n} a_i\mu(A_i)\big)\big(\sum_{j=1}^{m} b_j\nu(B_j)\big)\\ &= \int_X f d\mu \int_Y g d\nu\end{aligned}$$

For general $f \in L^1(\mu), g \in L^1(\nu)$, take $(h_n)_{n\in\mathbb{N}}$ as before. Monotone convergence and the result above say that

$$\begin{aligned}\int_{X\times Y} |h| d\mu \times d\nu &= \lim_{n\to\infty}\int_{X\times Y} |h_n| d\mu \times \nu\\ &= \lim_{n\to\infty}\left(\int_X |f_n| d\mu \int_Y |g_n| d\nu\right)\\ &= \int_X |f| d\mu \int_Y |g| d\nu\\ &< \infty\end{aligned}$$

So $h \in L^1(\mu \times \nu)$. Dominated convergence and the result above then tell us that

$$\begin{aligned}\int_{X\times Y} h d\mu \times d\nu &= \lim_{n\to\infty}\int_{X\times Y} h_n d\mu \times d\nu\\ &= \lim_{n\to\infty}\left(\int_X f_n d\mu \int_Y g_n d\nu\right)\\ &= \int_X f d\mu \int_Y g d\nu\end{aligned}$$

□

Note 3.4.10. *In the above exercise part (2), we can replace L^1 with L^+ and get the same result by the same method.*

Exercise 3.4.11. *Let $f : \mathbb{R} \to [0,\infty) \in L^+$. Show that*

$$\int_{\mathbb{R}} f dm = \int_{[0,\infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) dm(t)$$

Proof. Note that

$$\int_{[0,\infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) = \int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x\in\mathbb{R}: f(x)\geq t\}} dm\right] dm(t)$$

Comparing this with Tonelli's theorem, we can put $\chi_{\{x\in\mathbb{R}:f(x)\geq t\}} = (\chi_E)^t = \chi_{E^t}$. Then $E = \{(x,t) \in \mathbb{R} \times [0,\infty) : f(x) \geq t\}$ and $E_x = \{t \in [0,\infty) : f(x) \geq t\} = [0, f(x)]$. Tonelli's theorem tells us that

$$\int_{[0,\infty)}\left[\int_{\mathbb{R}}\chi_{\{x\in\mathbb{R}:f(x)\geq t\}}(x)dm(x)\right]dm(t)=\int_{\mathbb{R}}\left[\int_{[0,\infty)}\chi_{[0,f(x)]}(t)dm(t)\right]dm(x)$$
$$=\int_{\mathbb{R}}f(x)dm(x)$$

□

3.5. **Convergence.**

Definition 3.5.1. *Let $(X,\mathcal{A})$ be a measurable space. For convencience we will define $L^0=\{f:X\to\mathbb{C}:f$ is measurable$\}$.*

Definition 3.5.2. *Let $(f_n)_{n\in\mathbb{N}}\subset L^0$ and $f\in L^0$. Then f_n converges to f* ***in measure****, denoted $f_n\xrightarrow{\mu}f$, if for each $\epsilon>0$, $\mu(\{x\in X:|f_n(x)-f(x)|\geq\epsilon\})\to 0$.*

Note 3.5.3. *It is useful to observe that*

$$\bigcup_{\epsilon>0}\limsup_{n\to\infty}\{x\in X:|f_n(x)-f(x)|\geq\epsilon\}=\{x\in X:f_n(x)\not\to f(x)\}$$

and

$$\bigcap_{\epsilon>0}\liminf_{n\to\infty}\{x\in X:|f_n(x)-f(x)|<\epsilon\}=\{x\in X:f_n(x)\to f(x)\}$$

Definition 3.5.4. *Let $(f_n)_{n\in\mathbb{N}}\subset L^0$ and $f\in L^0$. Then f_n converges to f* ***almost uniformly*** *if for each $\epsilon>0$, there exists $N\in\mathcal{A}$ such that $\mu(N)<\epsilon$ and $f_n\xrightarrow{uni}f$ on N^c. This is written $f_n\xrightarrow{a.u.}f$.*

Theorem 3.5.5. *Let $(f_n)_{n\in\mathbb{N}}\subset L^0$ and $f\in L^0$. If $f_n\xrightarrow{\mu}f$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_k}\xrightarrow{a.e.}f$.*

Exercise 3.5.6. ***Egoroff's Theorem:*** *Suppose that $\mu(X)<\infty$. Let $(f_n)_{n\in\mathbb{N}}\subset L^0$ and $f\in L^0$. Suppose that $f_n\xrightarrow{a.e.}$. Then $f_n\xrightarrow{a.u.}f$.*

Proof. Let $\epsilon>0$. For each $n,k\in\mathbb{N}$, define $E_{n,k}=\{x\in X:|f_n(x)-f(x)|\geq\frac{1}{k}\}$ and $F_{n,k}=\bigcup\limits_{m\geq n}E_{m,k}$. Then $F_{n,k}$ is decreasing in n and $\bigcap\limits_{n\in\mathbb{N}}F_{n,k}\subset\{x:f_n(x)\not\to f(x)\}$. Thus $\mu(\bigcap\limits_{n\in\mathbb{N}}F_{n,k})=0$. Since $\mu(X)<\infty$, $\inf\limits_{n\in\mathbb{N}}\mu(F_{n,k})=0$. Hence we may choose a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ such that $\mu(F_{n_k,k})\leq\frac{\epsilon}{2^k}$. Put $N=\bigcup\limits_{k\in\mathbb{N}}F_{n_k,k}$. Then

$$\begin{aligned}\mu(N)&\leq\sum_{k\in\mathbb{N}}\mu(F_{n_k,k})\\&\leq\sum_{k\in\mathbb{N}}\frac{\epsilon}{2^k}\\&=\epsilon\end{aligned}$$

Let $\delta>0$. Choose $K\in\mathbb{N}$ such that $\frac{1}{K}<\delta$. Then for each $m\geq n_K$ and $x\in N^c=\bigcap\limits_{k\in\mathbb{N}}\bigcap\limits_{m\geq n_k}E^c_{m,k}$, $|f_m(x)-f(x)|<\frac{1}{K}<\delta$. So $f_n\xrightarrow{\text{uni}}f$ on N^c. □

Exercise 3.5.7. *Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.*

Proof. Let $\epsilon > 0$. for $n \in \mathbb{N}$, define $E_{e,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$. Then for $n \in \mathbb{N}$,

$$\begin{aligned} \int |f - f_n| &\geq \int_{E_{\epsilon,n}} |f - f_n| \\ &\geq \epsilon\mu(E_{\epsilon,n}). \end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$. Since $\int |f - f_n| \to 0$, we have that $\mu(E_{\epsilon,n}) \to 0$. Since $\epsilon > 0$ is arbitrary, $f_n \xrightarrow{\mu} f$ as required. □

Exercise 3.5.8. *Suppose $\mu(X) < \infty$. Define $d : L^0 \times L^0 \to [0, \infty)$ by*

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \quad f, g \in L^0$$

Then d is a metric on L^0 if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each $f, g \in L^0$, $d(f, g) \leq \mu(X)$.

Proof. Let $f, g \in L^0$. Clearly $d(f, g) = d(g, f)$. If $f = g$ a.e. then clearly $d(f, g) = 0$. Conversely, if $d(f, g) = 0$, then $\frac{|f-g|}{1+|f-g|} = 0$ a.e and so $|f - g| = 0$ a.e. which implies $f = g$ a.e. It is not hard to show that $\phi : [0, \infty) \to [0, \infty)$ given by $\phi(x) = \frac{x}{1+x}$ satisfies $\phi(x + y) \leq \phi(x) + \phi(y)$. Thus satisfies the triangle inequality. Now, let $(f_n)_{n\in\mathbb{N}} \subset L^0$. Suppose that $f_n \overset{\mu}{\not\to} f$. Then there exists $\epsilon > 0, \delta > 0$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$, $\mu(E_{\epsilon,n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$. It is not hard to show that ϕ from earlier is increasing. Thus for each $k \in \mathbb{N}$,

$$\begin{aligned} d(f_{n_k}, f) &= \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon,n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\ &\geq \int_{E_{\epsilon,n_k}} \frac{\epsilon}{1 + \epsilon} \\ &\geq \frac{\epsilon\delta}{1 + \epsilon} \end{aligned}$$

So $f_{n_k} \overset{d}{\not\to} f$. Hence $f_{n_k} \xrightarrow{d} f$ implies that $f_{n_k} \xrightarrow{\mu} f$. Conversely, suppose that $f_{n_k} \xrightarrow{\mu} f$. Let $\epsilon > 0$. Then $\delta = \frac{\epsilon}{1+\mu(X)} > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then $\mu(E_{\delta,n}) < \frac{\delta}{1+\delta}$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Since ϕ is increasing and $\phi \leq 1$, we have that

$$\begin{aligned}
d(f_n, f) &= \int \frac{|f_n - f|}{1 + |f_n - f|} \\
&= \int_{E_{\delta,n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E^c_{\delta,n}} \frac{|f_n - f|}{1 + |f_n - f|} \\
&\leq \mu(E_{\delta,n}) + \mu(X)\frac{\delta}{1+\delta} \\
&< \frac{\delta}{1+\delta}(1 + \mu(X)) \\
&\leq \delta(1 + \mu(X)) \\
&= \epsilon
\end{aligned}$$

□

Exercise 3.5.9. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \xrightarrow{\mu} f$. Then $f \geq 0$ a.e. and $\int f \leq \liminf\limits_{n\to\infty} \int f_n$.*

Proof. Since $f_n \xrightarrow{\mu} f$, there is a subsequence converging to f a.e. So clearly $f \geq 0$ a.e. Now, choose a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $\int f_{n_k} \to \liminf\limits_{n\to\infty} \int f_n$. Since $f_n \xrightarrow{\mu} f$ so does $(f_{n_k})_{k\in N}$. Therefore there exists a subsequence $(f_{n_{k_j}})_{k\in\mathbb{N}}$ of $(f_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Thus $f \geq 0$ a.e. and Fatou's lemma tells us that

$$\begin{aligned}
\int f &\leq \liminf_{j\in\mathbb{N}} \int f_{n_{k_j}} \\
&= \liminf_{n\to\infty} \int f_n.
\end{aligned}$$

□

Exercise 3.5.10. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \xrightarrow{\mu} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.*

Proof. Clearly $(f_n)_{n\in\mathbb{N}} \subset L^1$. Since $f_n \xrightarrow{\mu} f$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$. This implies that $|f| \leq g$ a.e. and so $f \in L^1$. For $n \in \mathbb{N}$, put $h_n = 2g - |f_n - f|$. Then for each $n \in \mathbb{N}$, $h_n \geq 0$ and $h_n \xrightarrow{\mu} 2g$. By the previous exercise

$$\begin{aligned}
\int 2g &\leq \liminf_{n\to\infty} \int (2g - |f_n - f|) \\
&= \int 2g - \limsup_{n\to\infty} \int |f_n - f|.
\end{aligned}$$

So $\limsup\limits_{n\to\infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $f_n \xrightarrow{L^1} f$ as required. □

Exercise 3.5.11. *Let $(f_n)_{n\in\mathbb{N}} \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$.*

1. *If ϕ is continuous, and $f_n \xrightarrow{a.e.} f$ then $\phi \circ f_n \xrightarrow{a.e.} \phi \circ f$.*
2. *If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly or in measure, respectively.*

(3) *Find a counter example to (2) if we drop the word "uniform".*

Proof. (1) Clear

(2) Suppose that ϕ is unifomly continuous.

(uniform conv.) Suppose that $f_n \xrightarrow{\text{uni}} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Now choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq n$ then for each $x \in X$, $|f_n(x) - f(x)| < \delta$. Let $n \in \mathbb{N}$, suppose $n \geq N$, Let $x \in X$. Then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Thus $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$.

(almost uni.) Suppose that $f_n \xrightarrow{\text{a.u.}} f$. Let $\epsilon > 0$. Choose $N \in \mathcal{A}$ such $\mu(N) < \epsilon$ and $f_n \xrightarrow{\text{uni}} f$ on N^c. Then from above, we know that $\phi \circ f_n \xrightarrow{\text{uni}} \phi \circ f$ on N^c. Thus $\phi \circ f_n \xrightarrow{\text{a.u.}} \phi \circ f$.

(measure) Suppose that $f_n \xrightarrow{\mu} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Observe that for $x \in X$, if $|f_n(x) - f(x)| < \delta$, then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Hence $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$. By definition of convergence in measure, $\mu(F_{n,\delta}) \to 0$. Thus $\mu(E_{n,\epsilon}) \to 0$. Hence $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.

(3)

$\square$

Exercise 3.5.12. *Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{a.u} f$. Then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{a.e.} f$.*

Proof. (measure) Let $\epsilon > 0$, $\delta > 0$. Choose $M \in \mathcal{A}$ such that $\mu(M) < \delta$ and $f_n \xrightarrow{\text{uni}} f$ on M^c. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then for each $x \in M^c$, $|f_n(x) - f(x)| < \epsilon$. Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then $E_{\epsilon,n} \subset M$ and $\mu(E_{\epsilon,n}) < \delta$. Thus $\mu(E_{\epsilon,n}) \to 0$ and $f_n \xrightarrow{\mu} f$.

(a.e.) For each $n \in \mathbb{N}$, Choose $N_n \in \mathcal{A}$ such that $\mu(N_n) < 1/n$ and $f_n \xrightarrow{\text{uni}} f$ on N_n^c. Observe that for $x \in X$, if $x \in \bigcup_{n \in \mathbb{N}} N_n^c$, then $f_n(x) \to f(x)$. Thus $N = \{x \in X : f_n(x) \not\to f(x)\} \subset \bigcap_{n \in \mathbb{N}} N_n$. Therefor $\mu(N) = 0$ and $f_n \xrightarrow{\text{a.e.}} f$. $\square$

Exercise 3.5.13. *Let $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then*

(1) *$f_n + g_n \xrightarrow{\mu} f + g$*
(2) *if $\mu(X) < \infty$, then $f_n g_n \xrightarrow{\mu} fg$*

Proof. (1) Let $\epsilon > 0$. For convenience, put $F_{n,\epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$, $G_{n,\epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$, and $(F + G)_{n,\epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g_n(x))| \geq \epsilon\}$ Observe that for $x \in X$, $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$. Thus $(F + G)_{n,\epsilon} \subset F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$. Since $\mu(F_{n,\epsilon/2} \cup G_{n,\epsilon/2}) \leq \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}) \to 0$, we have that $\mu((F+G)_{n,\epsilon}) \to 0$. Hence $f_n + g_n \xrightarrow{\mu} f + g$.

(2) Suppose that $\mu(X) < \infty$. Let $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(f_n g_n)_{n \in \mathbb{N}}$. Choose a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e}} f$ and $g_{n_{k_j}} \xrightarrow{\text{a.e}} g$. Then $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.e.}} fg$. Egoroff's theorem tells us that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.u.}} fg$, which implies that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$. Thus for each subsequence $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ of $(f_n g_n)_{n \in \mathbb{N}}$, there exists a subsequence

$(f_{n_{k_j}} g_{n_{k_j}})_{j\in\mathbb{N}}$ of $(f_{n_k} g_{n_k})_{k\in\mathbb{N}}$ such that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$. Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that $f_n g_n \xrightarrow{\mu} fg$.

□

Exercise 3.5.14. *Let $(f_n)_{n\in\mathbb{N}}, \subset L^0$ and $f \in L^0$. Suppose that $\mu(X) < \infty$. Then $f_n \xrightarrow{\mu} f_n$ iff for each subsequence $(f_{n_k})_{k\in\mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}})_{j\in\mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{a.e.} f$.*

Proof. Suppose that $f_n \xrightarrow{\mu} f$. Let $(f_{n_k})_{k\in\mathbb{N}}$ be a subsequence. Then $f_{n_k} \xrightarrow{\mu} f$. By a previous theorem, there exists a subsequence $(f_{n_{k_j}})_{j\in\mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Conversely, suppose that for each subsequence $(f_{n_k})_{k\in\mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}})_{j\in\mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Let $\epsilon > 0$. For $n \in \mathbb{N}$, define $E_n = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$ and define $E = \{x \in X : f_n(x) \not\to f(x)\}$. Let $(f_{n_k})_{k\in\mathbb{N}}$ be a subsequence. Choose a subsequence $(f_{n_{k_j}})_{j\in\mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Since $\left\{x \in X : \limsup\limits_{j\to\infty} \chi_{E_{n_{k_j}}}(x) = 1\right\} = \limsup\limits_{j\to\infty} E_{n_{k_j}} \subset E$ and $\mu(E) = 0$, we have that $\limsup\limits_{j\to\infty} \chi_{E_{n_{k_j}}} = 0$ a.e. and $\chi_{E_{n_{k_j}}} \xrightarrow{\text{a.e}} 0$. Since $\mu(X) < \infty$, the dominated convergence theorem implies that

$$\mu(E_{n_{k_j}}) = \int \chi_{E_{n_{k_j}}} d\mu \to 0$$

So for each subsequence $(\mu(E_{n_k}))_{k\in\mathbb{N}}$, there exists a subsequence $(\mu(E_{n_{k_j}}))_{j\in\mathbb{N}}$ such that $\mu(E_{n_{k_j}}) \to 0$. Thus $\mu(E_n) \to 0$ and $f_n \xrightarrow{\mu} f$. □

Exercise 3.5.15. *Let $(f_n)_{n\in\mathbb{N}}, \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$. Suppose that $\mu(X) < \infty$. If ϕ is continuous and $f_n \xrightarrow{\mu} f$, then $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.*

Proof. Suppose that ϕ is continuous and $f_n \xrightarrow{\mu} f$. Let $(\phi \circ f_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(\phi \circ f_n)_{n\in\mathbb{N}}$. Then $(f_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$. Since $f_n \xrightarrow{\mu} f$, the previous exercise tells us that there exists a subsequence $(f_{n_{k_j}})_{j\in\mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. A previous exercise implies that $\phi \circ f_{n_{k_j}} \xrightarrow{\text{a.e.}} \phi \circ f$. The previous exercise implies that $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$. □

Exercise 3.5.16. *Let $(f_n)_{n\in\mathbb{N}} L^0$ and $f \in L^0$. Suppose that for each $\epsilon > 0$,*

$$\sum_{n\in\mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then $f_n \xrightarrow{a.e.} f$.

Proof. Let $\epsilon > 0$. By assumption we know that

$$\begin{aligned}\int \left[\sum_{n\in\mathbb{N}} \chi_{\{x\in X:|f_n(x)-f(x)|>\epsilon\}}\right] d\mu &= \sum_{n\in\mathbb{N}} \int \chi_{\{x\in X:|f_n(x)-f(x)|>\epsilon\}} d\mu \\ &= \sum_{n\in\mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \\ &< \infty\end{aligned}$$

Thus we also know that $\sum_{n\in\mathbb{N}} \chi_{\{x\in X:|f_n(x)-f(x)|>\epsilon\}} < \infty$ a.e. Equivalently, we could say that for a.e. $x \in X$, $|\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\}| < \infty$. For $k \in \mathbb{N}$, define $N_k = \{x \in X :$

$\sum_{n\in\mathbb{N}} \chi_{\{x\in X:|f_n(x)-f(x)|>1/k\}} = \infty\}$. Then for each $k \in \mathbb{N}$, $\mu(N_k) = 0$. Define $N = \bigcup_{k\in\mathbb{N}} N_k$. Then $\mu(N) = 0$. Let $x \in N^c$ and $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \epsilon$. Then $\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\} \subset \{n \in \mathbb{N} : f_n(x) - f(x) > 1/k\}$ which is finite because $x \in N_k^c$. Put $M = \max\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\}$. Then for $m \geq M$, $|f_m(x) - f(x) \leq \epsilon|$. Thus $f_n(x) \to f(x)$. Hence $f_n \xrightarrow{\text{a.e.}} f$. □

4. Differentiation

4.1. Signed Measures.

Definition 4.1.1. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$. Then ν is said to be a **signed measure** if*

(1) *for each $E \in \mathcal{A}$, $\nu(E) < \infty$ or for each $E \in \mathcal{A}$, $\nu(E) > -\infty$.*
(2) *$\nu(\varnothing) = 0$*
(3) *for each $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ if $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is disjoint, then $\nu(\bigcup\limits_{n\in\mathbb{N}} E_n) = \sum\limits_{n\in\mathbb{N}} \nu(E_n)$ and if $|\sum\limits_{n\in\mathbb{N}} \nu(E_n)| < \infty$, then $\sum\limits_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely.*

Exercise 4.1.2. *Let $\nu : \mathcal{A} \to [0, \infty]$ be a signed measure and $(E_n)_{n\in\mathbb{N}}, (F_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. If $(E_n)_{n\in\mathbb{N}}$ is increasing, then $\nu(\bigcup\limits_{n\in\mathbb{N}} E_n) = \lim\limits_{n\to\infty} \nu(E_n)$. If $(F_n)_{n\in\mathbb{N}}$ is decreasing and $|\nu(E_1)| < \infty$, then $\nu(\bigcap\limits_{n\in\mathbb{N}} F_n) = \lim\limits_{n\to\infty} \nu(F_n)$.*

Proof. Put $E_1' = E_1$, $F_1' = F_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $E_n' = E_n \setminus E_{n-1}$ and $F_n' = F_1 \setminus F_n$. Then $(E_n')_{n\in\mathbb{N}} \subset \mathcal{A}$ is disjoint. Thus

$$\begin{aligned}
\nu(\bigcup_{n\in\mathbb{N}} E_n) &= \nu(\bigcup_{n\in\mathbb{N}} E_n') \\
&= \sum_{n\in\mathbb{N}} \nu(E_n') \\
&= \lim_{n\to\infty} \sum_{n=1}^{n} \nu(E_n') \\
&= \lim_{n\to\infty} \nu(E_n)
\end{aligned}$$

Since $(F_n')_{n\in\mathbb{N}}$ is increasing, we now know that

$$\begin{aligned}
\nu(F_1) - \nu(\bigcap_{n\in\mathbb{N}} F_n) &= \nu(F_1 \setminus \bigcap_{n\in\mathbb{N}} F_n) \\
&= \nu(\bigcup_{n\in\mathbb{N}} F_n') \\
&= \lim_{n\to\infty} \nu(F_n') \\
&= \lim_{n\to\infty} \nu(F_1 \setminus F_n) \\
&= \nu(F_1) - \lim_{n\to\infty} \nu(F_n)
\end{aligned}$$

Since $|\nu(F_1)| < \infty$, we see that $\nu(\bigcap\limits_{n\in\mathbb{N}} F_n) = \lim\limits_{n\to\infty} \nu(F_n)$. □

Definition 4.1.3. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$ a signed measure and $E \in \mathcal{A}$. Then E is said to be ν-**positive**, ν-**negative** and ν-**null** if for each $F \in \mathcal{A}$, $F \subset E$ implies that $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ respectively.*

Exercise 4.1.4. *Let $E \subset \mathcal{A}$. If E is positive, negative or null, then for each $F \in \mathcal{A}$, if $F \subset E$, then F is positive, negative or null respectively.*

Proof. Clear □

Exercise 4.1.5. *Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ be positive, negative or null. Then $\bigcup\limits_{n\in\mathbb{N}} E_n$ is positive, negative or null respectively.*

Proof. Suppose that $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is positive. Let $F \in \mathcal{A}$. Suppose that $F \subset \bigcup\limits_{n\in\mathbb{N}} E_n$. Put $P_1 = E_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $P_n = E_n \setminus (\bigcup\limits_{j=1}^{n-1} E_j)$. So $\bigcup\limits_{n\in\mathbb{N}} P_n = \bigcup\limits_{n\in\mathbb{N}} E_n$ and $(P_n)_{n\in\mathbb{N}}$ is disjoint. Thus

$$\begin{aligned}
\nu(F) &= \nu(F \cap \bigcup_{n\in\mathbb{N}} P_n) \\
&= \nu(\bigcup_{n\in\mathbb{N}} (F \cap P_n)) \\
&= \sum_{n\in\mathbb{N}} \nu(F \cap P_n) \\
&\geq 0
\end{aligned}$$

The process is the same if $(E_n)_{n\in\mathbb{N}}$ is negative and null. □

Theorem 4.1.6. *Hahn Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist $P, N \in \mathcal{A}$ such that P is positive, N is negative, $X = N \cup P$ and $N \cap P = \varnothing$. Furthermore, these two sets are unique in the following sense: For any $P', N' \in \mathcal{A}$, if N, P satisfy the properties above, $P' \Delta P = N' \Delta N$ is null.*

Definition 4.1.7. *Let ν be a signed measure on $(X, \mathcal{A})$ and $P, N \in \mathcal{A}$. Then P and N are said to form a **Hahn decomposition** of X with respect to ν if P, N satisfy the results in the above theorem.*

Definition 4.1.8. *Let μ, ν be signed measures on $(X, \mathcal{A})$. Then μ and ν are said to be **mutually singular** if there exist $E, F \in \mathcal{A}$ such that $X = E \cup F$, $E \cap F = \varnothing$ and E is μ-null and F is ν-null. We will denote this by $\mu \perp \nu$.*

Theorem 4.1.9. *Jordan Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{A})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. Choose a Hahn decomposition P, N of X with respect to ν. Define ν^+, ν^- by $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$. □

Definition 4.1.10. *Let ν be a signed measure on $(X, \mathcal{A})$. Then ν^+ and ν^- from the last theorem are called the **positive** and **negative variations** of ν respectively. We define the **total variation** measure $|\nu|$ on $(X, \mathcal{A})$ by $|\nu| = \nu^+ + \nu^-$.*

Definition 4.1.11. *Let ν be a signed measure on $(X, \mathcal{A})$. Then ν is said to be σ-finite if $|\nu|$ is σ-finite.*

Exercise 4.1.12. *Let ν be a signed measure and λ, μ positive measures on $(X, \mathcal{A})$. Suppose that $\nu = \lambda - \mu$. Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.*

Proof. Choose a Hahn decomposition P, N of X with respect to ν. Let $E \in \mathcal{A}$. Then

$$\begin{aligned}\lambda(E \cap P) - \mu(E \cap P) &= \nu(E \cap P) \\ &= \nu^+(E \cap P)\end{aligned}$$

So $\lambda(E \cap P) \geq \nu^+(E \cap P)$ and therefore

$$\begin{aligned}\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) + \lambda(E \cap N) \\ &\geq \nu^+(E \cap P) \\ &= \nu^+(E)\end{aligned}$$

Similarly $\mu(E \cap N) \geq \nu^-(E \cap N)$ and $\mu(E) \geq \nu^-(E)$. □

Exercise 4.1.13. *Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Hint: use the last exercise)*

Proof. Since

$$\begin{aligned}\nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\ &= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)\end{aligned}$$

the previous exercise tells us that $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1 + \nu_2)^+$ and $\mu = \nu_1^- + \nu_2^- \geq (\nu_1 + \nu_2)^-$. Therefore

$$\begin{aligned}|\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ &\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\ &= |\nu_1| + |\nu_2|\end{aligned}$$

□

Note 4.1.14. *Recall that a previous exercise from the section on complex valued functions tells us that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.*

Definition 4.1.15. *Let ν be a signed measure on $(X, \mathcal{A})$. Then we define $L^1(\nu) = L^1(|\nu|)$. For $f \in L^1(\nu)$, we define*

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

Exercise 4.1.16. *Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$*

Proof. The previous exercise tells us that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. Two previous exercises from the section on nonnegative functions tells us that

$$\begin{aligned}\int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\ &= \int |f| d|\nu_1| + \int |f| d|\nu_2|\end{aligned}$$

□

Exercise 4.1.17. *Let ν, μ be signed measures on $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then*

1. *E is ν-null iff $|\nu|(E) = 0$*
2. *$\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.*

Proof. (1) Suppose that E is ν-null. Choose a Hahn decomposition P, N of X with respect to ν. Then $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = \nu(E \cap N) = 0$. Therefore $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Conversely, suppose that $|\nu|(E) = 0$. Then $\nu^+(E) = \nu^-(E) = 0$. Let $F \in \mathcal{A}$. Suppose that $F \subset E$. Then $\nu^+(F) = 0$ and $\nu^-(F) = 0$. Therefore $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. So E is ν-null.

(2) Suppose that $\nu \perp \mu$. Then there exist $E, F \in \mathcal{A}$ such that $E \cup F = X$, $E \cap F = \varnothing$, E is μ-null and F is ν-null. By (1), F is $|\nu|$-null and thus $|\nu| \perp \mu$. If $|\nu| \perp \mu$, choose $E, F \in \mathcal{A}$ as before. Since F is $|\nu|$-null, we know that $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$. This implies that F is ν^+-null and F is ν^--null. So $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Finally assume that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. **FINISH!!!!**

□

Exercise 4.1.18. *Let ν be a signed measure on $(X, \mathcal{A})$. Then*

1. *for $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$*
2. *if ν is finite, then for each $E \in \mathcal{A}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : f$ is measurable and $|f| \leq 1\}$*

Proof. (1) Let $f \in L^1(\nu)$. Then

$$\begin{aligned}
\left|\int f d\nu\right| &= \left|\int f d\nu^+ - \int f d\nu^-\right| \\
&\leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \\
&\leq \int |f| d\nu^+ + \int |f| d\nu^- \\
&= \int |f| d(\nu^+ + \nu^-) \\
&= \int |f| d|\nu|
\end{aligned}$$

(2) Let $E \in \mathcal{A}$. Let $f : X \to \mathbb{R}$ be measurable and suppose that $|f| \leq 1$. Since ν is finite, so is $|\nu|$ and thus $f \in L^1(\nu)$. Then (1) tells us that

$$\begin{aligned}
|\int_E f d\nu| &\leq \int_E |f| d|\nu| \\
&\leq |\nu|(E)
\end{aligned}$$

Now, choose a Hahn decomposition P, N of X with respect to ν. Define $f = \chi_P - \chi_N$. Then $|f| \leq 1$, f is measurable and

$$\begin{aligned}\left|\int_E f d\nu\right| &= \left|\int_E f d\nu^+ - \int_E f d\nu^-\right| \\ &= |\nu^+(E \cap P) + \nu^-(E \cap N)| \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E).\end{aligned}$$

□

Exercise 4.1.19. *Let μ be a positive measure on $(X, \mathcal{A})$ and $f \in L^0(X, \mathcal{A})$ extended μ-integrable. Define ν on $(X, \mathcal{A})$ by $\nu(E) = \int_E f d\mu$. Then*

(1) *ν is a signed measure*
(2) *for each $E \in \mathcal{A}$, $|\nu|(E) = \int_E |f| d\mu$.*

Proof. (1) Clearly $\nu(\varnothing) = 0$ and ν is finte by assumption. Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_n)_{n\in\mathbb{N}}$ is disjoint. Then

$$\begin{aligned}\nu(\bigcup_{n\in\mathbb{N}} E_n) &= \int_{\bigcup_{n\in\mathbb{N}} E_n} f d\mu \\ &= \int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu \\ &= \sum_{n\in\mathbb{N}} \int_{E_n} f^+ d\mu - \sum_{n\in\mathbb{N}} \int_{E_n} f^- d\mu \\ &= \sum_{n\in\mathbb{N}} \left[\int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu\right] \\ &= \sum_{n\in\mathbb{N}} \int_{E_n} f d\mu \\ &= \sum_{n\in\mathbb{N}} \nu(E_n)\end{aligned}$$

If $|\nu(\bigcup_{n\in\mathbb{N}} E_n)| < \infty$, then $\int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu < \infty$ and $\int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu < \infty$ because

$$\begin{aligned}|\nu(\bigcup_{n\in\mathbb{N}} E_n)| &= \left|\int_{\bigcup_{n\in\mathbb{N}} E_n} f d\mu\right| \\ &= \left|\int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu - \int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu\right|\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\sum_{n\in\mathbb{N}}|\nu(E_n)| &= \sum_{n\in\mathbb{N}}\bigg|\int_{E_n} f d\mu\bigg| \\ &= \sum_{n\in\mathbb{N}}\bigg|\int_{E_n} f^+ d\mu - \int_{E_n} f^- d\mu\bigg| \\ &\leq \sum_{n\in\mathbb{N}}\int_{E_n} f^+ d\mu + \sum_{n\in\mathbb{N}}\int_{E_n} f^- d\mu \\ &= \int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu + \int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu \\ &< \infty\end{aligned}$$

So the sum $\sum_{n\in\mathbb{N}}\nu(E_n)$ converges absolutely and ν is a signed measure.

(2) Put $P = \{x \in X : f(x) \geq 0\}$ and $N = \{x \in X : f(x) < 0\}$. Then P, N form a Hahn decomposition of X with respect to ν. Thus for $E \in \mathcal{A}$,

$$\nu^+(E) = \int_{E\cap P} f d\mu = \int_E f^+ d\mu$$

and

$$\nu^-(E) = \int_{E\cap N} f d\mu = \int_E f^- d\mu$$

. So for $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f| d\mu$$

□

4.2. **The Lebesgue-Radon-Nikodym Theorem.**

Definition 4.2.1. *Let $(X, \mathcal{A})$ be a measureable space, ν be a signed measure on $(X, \mathcal{A})$ and μ a measure on $(X, \mathcal{A})$. Then ν is said to be* ***absolutely continuous*** *with respect to μ, denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.*

Note 4.2.2. *If there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that for each $E \in \mathcal{A}$, $\nu(E) = \int_E f d\mu$, then we write $d\nu = f d\mu$.*

Theorem 4.2.3. *Let $(X, \mathcal{A})$ be a measureable space, ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that $d\rho = f d\mu$ and f is unique μ-a.e.*

Definition 4.2.4. *The decomposition $\nu = \lambda + \rho$ is referred to as the* ***Lebesgue decomposition of ν with respect to*** *μ. In the case $\nu \ll \mu$, we have $\lambda = 0$ and $\rho = \nu$ and we define the* ***Radon-Nikodym derivative of ν with respect to*** *μ, denoted by $d\nu/d\mu$, to be $d\nu/d\mu = f$ where $d\nu = f d\mu$.*

Theorem 4.2.5. *Let ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then*

(1) *for each* $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ *and*

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$

(2) $\nu \ll \lambda$ *and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Exercise 4.2.6. *Let* $(\nu_n)_{n\in\mathbb{N}}$ *be a sequence of measures and* μ *a measure.*

(1) *If for each* $n \in \mathbb{N}$, $\nu_n \ll \mu$, *then* $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.
(2) *If for each* $n \in \mathbb{N}$, $\nu_n \perp \mu$, *then* $\sum_{n\in\mathbb{N}} \nu_n \perp \mu$.

Proof. (1) Let $E \in \mathcal{A}$. Suppose that $\mu(E) = 0$. Then for each $n \in \mathbb{N}$, $\nu_i(E) = 0$ and thus $\sum_{n\in\mathbb{N}} \nu_n(E) = 0$. Hence $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.

(2) For each $n \in \mathbb{N}$, there exist $N_i, M_i \in \mathcal{A}$ such that $N_i \cap M_i = \varnothing$, $N_i \cup M_i = X$ and $\nu_i(M_i) = \mu(N_i) = 0$. Put $N = \bigcup_{n\in\mathbb{N}} N_i$ and $M = N^c$. Note that for each $n \in \mathbb{N}$, $M \subset N_i^c = M_i$. So $\mu(N) \leq \sum_{n\in\mathbb{N}} \mu(N_i) = 0$ and $(\sum_{n\in\mathbb{N}} \nu_i)(M) \leq \sum_{n\in\mathbb{N}} \nu_i(M_i) = 0$. Thus $\sum_{n\in\mathbb{N}} \nu_i \perp \mu$.

□

Exercise 4.2.7. *Choose* $X = [0,1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$. *Let* m *be Lebesgue measure and* μ *the counting measure.*

Then

(1) $m \ll \mu$ *but for each* $f \in L^+$, $dm \neq f d\mu$
(2) *There is no Lebesgue decomposition of* μ *with respect to* m.

Proof. (1) Let $E \in \mathcal{A}$. If $\mu(E) = 0$, then $E = \varnothing$ and $m(E) = 0$. So $m \ll \mu$. Suppose for the sake of contradiction that there exists $f \in L^+$ such that $dm = f d\mu$. Then

$$\begin{aligned} 1 &= m(X) \\ &= \sum_{x\in X} f(x) \end{aligned}$$

Put $Z = \{x \in X : f(x) \neq 0\}$. Then Z is countable. So

$$\begin{aligned} 1 &= m(X \setminus Z) \\ &= \sum_{x\in X\setminus Z} f(x) \\ &= 0 \end{aligned}$$

This is a contradiction, so no such f exists.

(2) Suppose for the sake of contradiction that there is a Lebesgue decomposition for μ with respect to m given by $\mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho \ll m$. We may assume λ and ρ are positive. Then for each $x \in X$, $m(\{x\}) = 0$ which implies that $\rho(\{x\}) = 0$. Let $E \subset X$, if E is countable, then $\lambda(E) = \mu(E)$. If E is uncountable, choose $F \subset E$ such that F is countable. Then

$$\begin{aligned} \lambda(E) &\geq \lambda(F) \\ &= \mu(F) \\ &= \infty \end{aligned}$$

So $\lambda = \mu$. This is a contradiction since $\mu \not\ll m$.

$\square$

Exercise 4.2.8. *Let $(X, \mathcal{F}, \mu)$ be a measure space and $\mathcal{E}$ a sub σ-alg of $\mathcal{F}$ and $f \in L^1(\mu)$. Define $\nu : \mathcal{E} \to [0, \infty]$ by $\nu(E) = \int_E f d\mu$. Then ν is σ-finite. Let $\overline{\mu}$ be the restriction of μ to $\mathcal{E}$. So $\nu \ll \overline{\mu}$. Define the* ***expectation of*** *f* ***given*** *$\mathcal{E}$ to be $E[f|\mathcal{E}] = d\nu/d\overline{\mu} \in L^1(X, \mathcal{F}, \overline{\mu})$. Then for each $E \in \mathcal{E}$,*

$$\int_E E[f|\mathcal{E}]d\mu = \int_E f d\mu$$

Proof. Let $E \in \mathcal{E}$. By definition,

$$\begin{aligned}
\int_E E[f|\mathcal{E}]d\mu &= \int_E d\nu/d\overline{\mu}d\mu \\
&= \int_E d\nu/d\overline{\mu}d\overline{\mu} \qquad \text{(since } E \in \mathcal{E}\text{)} \\
&= \nu(E) \\
&= \int_E f d\mu
\end{aligned}$$

$\square$

4.3. Complex Measures.

Definition 4.3.1. *Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to \mathbb{C}$. Then ν is said to be a* ***complex measure*** *if*

1. *$\nu(\varnothing) = 0$*
2. *for each sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if $(E_n)_{n\in\mathbb{N}}$ is disjoint, then $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n\in\mathbb{N}} \nu(E_n)$ and $\sum_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely.*

Note 4.3.2. *We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.*

Definition 4.3.3. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. We define $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$. For $f \in L^1(\nu)$, we define*

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

Theorem 4.3.4. *Let $(X, \mathcal{A})$ be a measurable space, ν a complex measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exists a complex measure λ on $(X, \mathcal{A})$ and $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$ and such that for each complex measure λ' on $(X, \mathcal{A})$, $f' \in L^1(\mu)$, if $\nu = d\lambda' + f' d\mu$, then $\lambda = \lambda'$ and $f = f'$ μ-a.e.*

Theorem 4.3.5. *Let ν be a complex measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then*

1. *for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and*
$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$$
2. *$\nu \ll \lambda$ and*
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Definition 4.3.6. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. Define $\mu = |\nu_1| + |\nu_2|$. Then $\nu \ll \mu$ and thus There exists $f \in L^1(\mu)$ such that $d\nu = f d\mu$. Define $|\nu| : \mathcal{A} \to [0, \infty)$ by $|\nu|(E) = \int_E |f| d\mu$ for each $E \in \mathcal{A}$. We call $|\nu|$ the* ***total variation of*** *ν.*

Exercise 4.3.7. *Let ν be a complex measure on $(X, \mathcal{A})$ and μ a σ-finite measures on $(X, \mathcal{A})$. If $\nu \ll \mu$, then $\{x \in X : d\nu/d\mu(x) = 0\}$ is ν-null.*

Proof. Define $f = d\nu/d\mu$ and $E = \{x : f(x) = 0\}$. Let $A \in \mathcal{A}$ and suppose that $A \subset E$. Then

$$\begin{aligned} \nu(A) &= \int_A f d\mu \\ &= 0 \end{aligned}$$

□

Exercise 4.3.8. *Let $(X, \mathcal{A})$ be a measurable space and $\nu = \nu_1 + i\nu_2$ a complex measure on $(X, \mathcal{A})$. Then $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu 2|$.*

Proof. Let μ and f be as in the definition of $|\nu|$. Since for each $E \in \mathcal{A}$, we have

$$\begin{aligned} \nu(E) &= \int_E f d\mu \\ &= \int_E f_1 d\mu + i \int_E f_2 d\mu \end{aligned}$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E)$$

we know that $\nu_1 = f_1 d\mu$ and $\nu_2 = f_2 d\mu$.

A previous exercise tells us that $d|\nu_1| = |f_1| d\mu$ and $d|\nu_2| = |f_2| d\mu$. Since $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$, we have that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \end{aligned}$$

□

Exercise 4.3.9. *Let $(X, \mathcal{A})$ be a measurable space, ν a complex measure on $(X, \mathcal{A})$ and $c \in \mathbb{C}$. Then $|c\nu| = |c||\nu|$.*

Proof. Define μ and f as before so that $d\nu = f d\mu$. Then $d(c\nu) = cf d\mu$. Hence

$$\begin{aligned} d|c\nu| &= |cf| d\mu \\ &= |c||f| d\mu \\ &= |c| d|\nu| \end{aligned}$$

So $|c\nu| = |c||\nu|$. □

Exercise 4.3.10. *Let $(X, \mathcal{A})$ be a measurable space and ν a complex measure on $(X, \mathcal{A})$. Then*

(1) *for each $E \in \mathcal{A}$, $|\nu(E)| \leq |\nu|(E)$.*
(2) *$\nu \ll |\nu|$ and $\big|d\nu/d|\nu|\big| = 1$ $|\nu|$-a.e.*

(3) *$L^1(\nu) = L^1(|\nu|)$ and for each $g \in L^1(\nu)$, $|\int g d\nu| \leq \int |g| d|\nu|$*

Proof. Let μ, $f \in L^1(\mu)$ be as in the definition of $|\nu|$.

(1) Let $E \in \mathcal{A}$. Then

$$\begin{aligned}|\nu(E)| &= \left| \int_E f d\mu \right| \\ &\leq \int_E |f| d\mu \\ &= |\nu|(E)\end{aligned}$$

(2) Let $E \in \mathcal{A}$ and suppose that $|\nu|(E) = 0$. The previous part implies $|\nu(E)| = 0$ and $\nu \ll |\nu|$. Put $g = d\nu/d|\nu|$. Then

$$\begin{aligned}f &= \frac{d\nu}{d\mu} \\ &= g|f| \;\; \mu\text{-a.e.}\end{aligned}$$

Hence $|f| = |g||f|$ μ-a.e. Since $|\nu| \ll \mu$, $|f| = |g||f|$ $|\nu|$-a.e.
A previous exercise tells us that $|f| \neq 0$ $|\nu|$-a.e. Thus $|g| = 1$ $|\nu|$-a.e.

(3) Write $\nu = \nu_1 + i\nu_2$ and $f = f_1 + if_2$. First we observe that

$$\begin{aligned}L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\ &= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\ &= L^1(|\nu_1| + |\nu_2|) \\ &= L^1(\mu)\end{aligned}$$

The previous exercise tells us that

$$\begin{aligned}|\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \\ &= \mu\end{aligned}$$

Let $g \in L^1(\mu)$. Then

$$\begin{aligned}\int |g| d|\nu| &\leq \int |g| d\mu \\ &< \infty\end{aligned}$$

So $g \in L^1(|\nu|)$.
Conversely, let $g \in L^1(|\nu|)$. Then

$$\begin{aligned}\int |g| d|\nu_1|, \int |g| d|\nu_2| &\leq \int |g| d|\nu| \\ &< \infty\end{aligned}$$

So

$$\begin{aligned}\int |g| d\mu &= \int |g| d|\nu_1| + \int |g| d|\nu_2| \\ &< \infty\end{aligned}$$

and $g \in L^1(\mu)$. Hence $L^1(\nu) = L^1(|\nu|)$.
Now, let $g \in L^1(\nu) = L^1(|\nu|)$, then

$$\begin{aligned}\left|\int g d\nu\right| &= \left|\int g f d\mu\right| \\ &\leq \int |g||f| d\mu \\ &= \int |g| d|\nu|\end{aligned}$$

□

4.4. Differentiation.

Definition 4.4.1. *Let $f : \mathbb{R}^n \to \mathbb{C}$. Then f is said to be* ***locally integrable*** *(with respect to Lebesgue measure) if f is measurable and for each $K \subset \mathbb{R}$, K is compact implies $\int_K |f| dm < \infty$. We define $L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f$ is locally integrable$\}$*

Definition 4.4.2. *For $f \in L^1_{loc}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$, we define the* ***average of f over*** *$B(x,r)$, denoted by $Af(x,r)$, to be*

$$Af(x,r) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f dm$$

Exercise 4.4.3. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Define*

$$H^* f(x) = \sup\{\frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B\} \quad (x \in \mathbb{R}^n)$$

Then $Hf \leq H^* f \leq 2^n Hf$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\left\{\frac{1}{m(B(x,r))} \int_{B(x,r)} |f| dm : r > 0\right\} \subset \left\{\frac{1}{m(B)} \int_B |f| dm : B \text{ is a ball and } x \in B\right\}$$

So $Hf(x) \leq H^* f(x)$. Let B be a ball. Then there exists $y \in \mathbb{R}^n$, $R > 0$ such that $B = B(y,R)$ Suppose that $x \in B$. Then $B \subset B(x, 2R)$. Since $m(B(x,2R)) = 2^n m(B(y,R))$, we have that

$$\begin{aligned}\frac{1}{m(B)} \int_B |f| dm &\leq \frac{1}{m(B)} \int_{m(B(x,2R))} |f| dm \\ &= \frac{2^n}{m(B(x,2R))} \int_{m(B(x,2R))} |f| dm\end{aligned}$$

Thus $H^* f(x) \leq 2^n Hf(x)$. □

Lemma 4.4.4. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then $Af : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is continuous.*

Definition 4.4.5. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define its* ***Hardy Littlewood maximal function***, *denoted by Hf to be*

$$Hf(x) = \sup_{r>0} A|f|(x,r) \quad x \in \mathbb{R}^n$$

Theorem 4.4.6. *There exists $C > 0$ such that for each $f \in L^1(m)$ and $\alpha > 0$,*

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \leq \frac{C}{a} \int |f| dm$$

Exercise 4.4.7. *Let $f \in L^1(\mathbb{R}^n)$. Suppose that $||f||_1 > 0$. Then there exist $C, R > 0$ such that for each $x \in \mathbb{R}^n$, if $|x| > R$, then $Hf(x) \geq C|x|^{-n}$. Hence there exists $C' > 0$ such that for each $\alpha > 0$, $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$ when α is small.*

Proof. Since $||f||_1 > 0$, there exists $R > 0$ such that $\int_{B(0,R)} |f| dm > 0$. Recall that there exists $K > 0$ such that for each $x \in R^n$ and $r > 0$, $m(B(x, r)) = Kr^n$ Choose

$$C = \frac{\int_{B(0,R)} |f| dm}{K 2^n}$$

. Let $x \in \mathbb{R}^n$. Suppose that $|x| > R$. Then $B(0, R) \subset B(x, 2|x|)$. Thus

$$\begin{aligned} Hf(x) &\geq \frac{1}{m(B(x, 2|x|))} \int_{B(x,2|x|)} |f| dm \\ &= \frac{1}{K 2^n |x|^n} \int_{B(x,2|x|)} |f| dm \\ &\geq \frac{1}{K 2^n |x|^n} \int_{B(0,R)} |f| dm \\ &= \frac{C}{|x^n|} \end{aligned}$$

Let $a < \frac{C}{2R^n}$. Then $R^n < \frac{C}{2\alpha}$. Choose $C' = \frac{KC}{2}$. Let $A = \{x \in \mathbb{R}^n : R < |x| < (\frac{C}{\alpha})^{\frac{1}{n}}\}$. For $x \in A$,

$$\begin{aligned} Hf(x) &\geq \frac{C}{|x|^n} \\ &> \alpha \end{aligned}$$

Thus $A \subset m(\{x \in R^n : Hf(x) > \alpha\})$ and therefore

$$\begin{aligned} m(\{x \in R^n : Hf(x) > \alpha\}) &\geq m(A) \\ &= m(B(0, (C/\alpha)^{1/n})) - m(B(0, R)) \\ &= K\left[\frac{C}{\alpha} - R^n\right] \\ &> K\left[\frac{C}{\alpha} - \frac{C}{2\alpha}\right] \\ &= \frac{KC}{2\alpha} \\ &= \frac{C'}{\alpha} \end{aligned}$$

□

Theorem 4.4.8. *Let $f \in L^1_{loc}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,*

$$\lim_{r \to 0} Af(x, r) = f(x)$$

. Equivalently, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} [f(y) - f(x)] dm(y) \right] = 0$$

Note 4.4.9. *We can a stronger result of the same flavor.*

Definition 4.4.10. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define the **Lebesgue set of** f, denoted by L_f, to be*

$$\begin{aligned} L_f &= \{x \in \mathbb{R}^n : \lim_{r \to 0} A|f - f(x)|(x,r) = 0\} \\ &= \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) \right] = 0 \right\} \end{aligned}$$

Exercise 4.4.11. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. If f is continuous at x, then $x \in L_f$.*

Proof. Suppose that f is continuous at x. Let $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that for each $y \in \mathbb{R}^n$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Let $r > 0$. Suppose that $r < \delta$. Then for each $y \in \mathbb{R}^n$, $y \in B(x,r)$ implies that $|f(x) - f(y)| < \epsilon$ and thus

$$\begin{aligned} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(B(x,r))} \epsilon m(B(x,r)) \\ &= \epsilon \end{aligned}$$

Hence

$$\lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) \right] = 0$$

and $x \in L_f$. □

Theorem 4.4.12. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then $m((L_f)^c) = 0$*

Definition 4.4.13. *Let $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then $(E_r)_{r>0}$ is said to **shrink nicely to** x if*

1. *for each $r > 0$, $E_r \subset B(x,r)$*
2. *there exists $\alpha > 0$ such that for each $r > 0$, $m(E_r) > \alpha m(B(x,r))$*

Theorem 4.4.14. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then for each $x \in L_f$,*

$$\lim_{r \to 0} \left[\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \right] = 0$$

and

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f dm = f(x)$$

Definition 4.4.15. *Let $\mu : \mathcal{B}(\mathbb{R}^n) \to [0, \infty]$ be a Borel measure. Then μ is said to be **regular** if*

1. *for each $K \subset \mathbb{R}^n$, if K is compact, then $\mu(K) < \infty$*
2. *for each $E \in \mathcal{B}(\mathbb{R}^n)$, $\mu(E) = \inf\{\mu(U) : U$ is open and $E \subset U\}$*

Let ν be a signed or complex Borel measure on $\mathbb{R}^n$. Then ν is said to be regular if $|\nu|$ is regular.

Theorem 4.4.16. *Let ν be a regular signed or complex measure on $\mathbb{R}^n$. Let $d\nu = d\lambda + f dm$ be the Lebesgue decomposition of ν with respect to m. Then for m-a.e. $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(R^n)$, if $(E_r)_{r>0}$ shrinks nicely to x, then*

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

4.5. Functions of Bounded Variation.

Definition 4.5.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Define $F_+ : \mathbb{R} \to \mathbb{R}$ by*

$$F_+(x) = \lim_{t\to x^+} F(t) = \inf\{F(t) : t > x\}$$

Note 4.5.2. *Observe that $F \leq F_+$ and F_+ is increasing.*

Exercise 4.5.3. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then for each $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in (x, x+\delta)$, $0 \leq F_+(y) - F(y) \leq \epsilon$.*

Proof. For the sake of contradiction, suppose not. Then there exists $x \in R$ and $\epsilon > 0$ such that for each $\delta > 0$, there exist $y \in (x, x+\delta)$ such that $F_+(y) - F(y) > \epsilon$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that for each $n \in \mathbb{N}$, $y_n \in (x, x+\frac{1}{n})$, $y_n > y_{n+1}$ and $F_+(y_n) - F(y_n) > \epsilon$. Choose $N \in \mathbb{N}$ such that $(N-1)\epsilon > F(y_1) - F(x)$. Then

$$\begin{aligned}
F(y_1) - F(x) &= \sum_{i=1}^{N-1} \Big[F(y_i) - F_+(y_{i+1}) + F_+(y_{i+1}) - F(y_{i+1})\Big] + F(y_N) - F(x) \\
&= \sum_{i=1}^{N-1} \Big[F(y_i) - F_+(y_{i+1})\Big] + \sum_{i=1}^{N-1} \Big[F_+(y_{i+1}) - F(y_{i+1})\Big] + F(y_N) - F(x) \\
&\geq (N-1)\epsilon \\
&> F(y_1) - F(x)
\end{aligned}$$

This is a contradiction, so the claim holds. □

Exercise 4.5.4. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then F_+ is right continuous.*

Proof. Let $x \in \mathbb{R}$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that for each $y \in (x, x+\delta_1)$ $0 \leq F(y) - F_+(x) < \epsilon/2$. There exists $\delta_2 > 0$ such that for each $y \in (x, x+\delta_2)$, $0 \leq F_+(y) - F(y) < \epsilon/2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Let $y \in (x, x+\delta)$.

$$\begin{aligned}
|F_+(x) - F_+(y)| &\leq |F_+(x) - F(y)| + |F(y) - F_+(y)| \\
&= (F(y) - F_+(x)) + (F_+(y) - F(y)) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So $\lim_{t\to x^+} F_+(t) = F_+(x)$ and F_+ is right continuous. □

Theorem 4.5.5. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then*

1. *$\{x \in \mathbb{R} : F$ is not continuous at $x\}$ is countable*
2. *F and F_+ are differentiable a.e. and $F' = F'_+$ a.e.*

Definition 4.5.6. *Let $F : \mathbb{R} \to \mathbb{C}$. Define $T_F : \mathbb{R} \to \mathbb{R}$ by*

$$T_F(x) = \sup\left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \quad (x \in \mathbb{R})$$

*T_F is called the **total variation function of** F.*

Exercise 4.5.7. *Let $F : \mathbb{R} \to \mathbb{C}$. Then T_F is increasing.*

Proof. Let $x, y \in \mathbb{R}$. Suppose that $x < y_2$.
Define $A_x = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\}$ and
$A_y = \left\{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = y \right\}$. Let $z \in A_x$. Then there exists $(x_i)_{i=0}^n \subset \mathbb{R}$ such that $(x_i)_{i=0}^n$ is increasing,
$x_n = x$ and $z = \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$. Then

$$\begin{aligned} z &\leq z + |F(y) - F(x)| \\ &= \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \\ &\in A_y \end{aligned}$$

So $z \leq \sup A_y = T_F(y)$ and thus $F_T(x) = \sup A_x \leq T_F(y)$ □

Lemma 4.5.8. *Let $F : \mathbb{R} \to \mathbb{R}$. Then $T_F + F$ and $T_F - F$ are increasing.*

Exercise 4.5.9. *For each $F : \mathbb{R} \to \mathbb{C}$, $T_{|F|} \leq T_F$.*

Proof. Let $F : \mathbb{R} \to \mathbb{C}$, $x \in R$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then by the reverse triangle inequality,

$$\sum_{i=1}^{n} \big| |F(x_i)| - |F(x_{i-1})| \big| \leq \sum_{i=1}^{n} \big| F(x_i) - |F(x_{i-1}) \big|$$

Thus

$$\begin{aligned} T_{|F|}(x) &= \sup\left\{ \sum_{i=1}^{n} \big| |F(x_i)| - |F(x_{i-1})| \big| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\ &\leq \sup\left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\ &= T_F(x) \end{aligned}$$

Hence $T_{|F|} \leq T_F$ □

Definition 4.5.10. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to have **bounded variation** if $\lim\limits_{x\to\infty} T_F(x) < \infty$. The **total variation of** F, denoted by $TV(F)$, is defined to be $TV(F) = \lim\limits_{x\to\infty} T_F(x)$. We define $BV = \{F : \mathbb{R} \to \mathbb{C} : TV(F) < \infty\}$.*

Definition 4.5.11. *Let $a, b \in \mathbb{R}$ and $F : [a, b] \to \mathbb{C}$. Define $G_F : \mathbb{R} \to \mathbb{C}$ by $G_F = F(a)\chi_{(-\infty,a)} + F\chi_{[a,b]} + F(b)\chi_{(b,\infty)}$. Then F is said to have **bounded variation on** $[a, b]$ if $G_F \in BV$. The **total variation of** F **on** $[a, b]$, denoted by $TV(F, [a, b])$, is defined to be $TV(F, [a, b]) = TV(G_F)$ We define $BV([a, b]) = \{F : [a, b] \to \mathbb{C} : TV(F, [a, b]) < \infty\}$.*

Note 4.5.12. *Equivalently,* $TV(F,[a,b]) = \sup\left\{\sum_{i=1}^n |F(x_i)-F(x_{i-1})| : (x_i)_{i=0}^n \subset [a,b]\right.$ *is increasing,* $x_0 = a$, *and* $\left. x_n = b\right\}$ *and* $F \in BV([a,b])$ *iff* $TV(F,[a,b]) < \infty$. *In general,*

Exercise 4.5.13. *Let* $F \in BV$. *Then* F *is bounded.*

Proof. If F is unbounded, then the supremum in the previous definition is clearly infinite. □

Exercise 4.5.14. *Let* $F : \mathbb{R} \to \mathbb{R}$. *If* F *is bounded and increasing, then* $F \in BV$.

Proof. Suppose that F is bounded and increasing. Then $-\infty < \inf_{x\in\mathbb{R}} F(x) \leq \sup_{x\in\mathbb{R}} F(x) < \infty$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then

$$\begin{aligned}\sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n F(x_i) - F(x_{i-1}) \\ &= F(x) - F(x_0)\end{aligned}$$

Thus

$$T_F(x) = F(x) - \inf_{x\in\mathbb{R}} F(x)$$

. This implies that

$$\begin{aligned}TV(F) &= \sup_{x\in\mathbb{R}} F(x) - \inf_{x\in\mathbb{R}} F(x) \\ &< \infty\end{aligned}$$

Hence $F \in BV$. □

Exercise 4.5.15. *Let* $F : \mathbb{R} \to \mathbb{C}$. *If* F *is differentiable and* F' *is bounded on* $[a,b]$, *then,* $F \in BV([a,b])$.

Proof. Suppose that F is differentiable and F' is bounded on $[a,b]$. Then there exists $M > 0$ such that for each $x \in [a,b]$, $|F(x)| \leq M$. Let $(x_i)_{i=1}^n \subset [a,b]$. Suppose that $(x_i)_{i=1}^n$ is strictly increasing, $x_0 = a$ and $x_n = b$. By the mean value theorem, for each $i = 1, 2, \cdots, n$, there exists $c_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$. Then

$$\begin{aligned}\sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n |F'(c_i)(x_i - x_{i-1})| \\ &\leq \sum_{i=1}^n M(x_i - x_{i-1}) \\ &= M(b-a)\end{aligned}$$

Hence $TV(F,[a,b]) \leq M(b-a)$. □

Exercise 4.5.16. *Define* $F, G : \mathbb{R} \to \mathbb{R}$ *by*

$$F(x) = \begin{cases} x^2 sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then F *and* G *are differentiable,* $F \in BV([-1,1])$ *and* $G \notin BV([-1,1])$.

Proof. On $\mathbb{R} \setminus \{0\}$,

$$\begin{aligned} F'(x) &= 2xsin(x^{-1}) - sin(x^{-1}) \\ &= sin(x^{-1})(2x-1) \end{aligned}$$

We see that F is also differentiable at $x = 0$ since

$$\begin{aligned} F'(0) &= \lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 sin(x^{-1})}{x} \\ &= \lim_{x \to 0} xsin(x^{-1}) \\ &= 0 \end{aligned}$$

Therefore for each $x \in [-1, 1]$, $|F'(x)| \leq 3$. Which by a previous exercise implies that $F \in BV([-1, 1])$.

On $\mathbb{R} \setminus \{0\}$,

$$\begin{aligned} G'(x) &= 2xsin(x^{-2}) - \frac{2sin(x^{-2})}{x} \\ &= sin(x^{-2})(2x - \frac{2}{x}) \end{aligned}$$

We see that G is also differentiable at $x = 0$ since

$$\begin{aligned} G'(0) &= \lim_{x \to 0} \frac{G(x) - G(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 sin(x^{-2})}{x} \\ &= \lim_{x \to 0} xsin(x^{-2}) \\ &= 0 \end{aligned}$$

For $n \in \mathbb{N}$, define $(x_i)_{i=0}^n \subset [-1, 1]$ by

$$x_i = \frac{-1}{\sqrt{\frac{\pi}{2} + i\pi}}$$

Then for each $n \in \mathbb{N}$, $(x_i)_{i=1}^n$ is strictly increasing and for each $i = 1, 2, \cdots, n$ we have that

$$\begin{aligned} |G(x_i) - G(x_{i-1})| &= \frac{1}{\frac{\pi}{2} + i\pi} + \frac{1}{\frac{\pi}{2} + (i-1)\pi} \\ &= \frac{2}{\pi} \left[\frac{(2i-1) + (2i+1)}{(2i+1)(2i-1)} \right] \\ &= \frac{2}{\pi} \left[\frac{4i}{4i^2 - 1} \right] \\ &> \frac{2}{i\pi} \end{aligned}$$

Hence for each $n \in \mathbb{N}$,

$$\begin{aligned} TV(G,[-1,1]) &\geq \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})| \\ &> \frac{2}{\pi} \sum_{i=1}^{n} \frac{1}{i} \end{aligned}$$

Therefore $G \notin BV([-1,1])$. □

Exercise 4.5.17. *The following is stated for BV, but is also true for $BV([a,b])$.*

(1) *For each $F, G \in BV$, $T_{F+G} \leq T_F + T_G$ and therefore BV is a vector space.*
(2) *For each $F : \mathbb{R} \to \mathbb{C}$, $F \in BV$ iff $Re(f) \in BV$ and $Im(F) \in BV$.*
(3) *For each $F : \mathbb{R} \to \mathbb{R}$, $F \in BV$ iff there exist functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$*
(4) *For each $F \in BV$ and $x \in \mathbb{R}$, $\lim_{t \to x^+} F(t)$ and $\lim_{t \to x^-} F(t)$ exist.*
(5) *For each $F \in BV$, $\{x \in R : F$ is not continuous at $x\}$ is countable.*
(6) *For each $F \in BV$, F and F_+ are differentiable a.e. and $F' = (F_+)'$ a.e.*
(7) *For each $F \in BV, c \in \mathbb{R}$, $F - c \in BV$*

Proof. (1) Let $F, G \in BV$, $x \in \mathbb{R}$ and $\epsilon > 0$. Since $T_{F+G}(x) < \infty$, $T_{F+G}(x) - \epsilon < T_{F+G}(x)$. Thus there exists $(x_i)_{i=0}^n \subset \mathbb{R}$ such that $(x_i)_{i=0}^n$ is increasing, $x_n = x$ and $T_{F+G}(x) < \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1}))| + \epsilon$. Thuerefore

$$\begin{aligned} T_{F+G}(x) &< \sum_{i=1}^{n} |(F+G)(x_i) - (F+G)(x_{i-1}))| + \epsilon \\ &\leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})| + \epsilon \\ &\leq T_F(x) + T_G(x) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $T_{F+G}(x) \leq T_F(x) + T_G(x)$. Therefore $TV(F+G) \leq TV(F) + TV(G) < \infty$. Thus $F + G \in BV$. It is straight forward to verify the other requirements needed to show that BV is a vector space.

(2) Let $F : \mathbb{R} \to \mathbb{C}$. Write $F = F_1 + iF_2$ with $F_1, F_2 : \mathbb{R} \to \mathbb{R}$. Suppose that $F \in BV$. Note that for each $x_1, x_2 \in \mathbb{R}$ and $j = 1, 2$, $|F_j(x_1) - F_j(x_2)| \leq |F(x_1) - F(x_2)|$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then for $j = 1, 2$

$$\sum_{i=1}^{n} |F_j(x_i) - F_j(x_{i-1})| \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|$$

. Thus for $j = 1, 2$ we have that $T_{F_j}(x) \leq T_F(x)$ which implies that $Re(f), Im(F) \in BV$. Conversely, Suppose that $Re(f), Im(F) \in BV$. Then $F = Re(f) + iIm(f) \in BV$ by (1).

(3) Suppose that $F \in BV$. Choose $F_1 = \frac{1}{2}(T_F - F)$ and $F_2 = \frac{1}{2}(T_F + F)$. Then F_1, F_2 are bounded, increasing and $F = F_1 + F_2$. Conversely, if there exist $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$, then $F_1, F_2 \in BV$. By (1) $F \in BV$.

(4) This is clear by previous results and (3)

(5) This is clear by previous results and (3)
(6) This is clear by previous results and (3)
(7) Clearly constant functions have zero total variation. The rest is implied by (1).

□

Lemma 4.5.18. *Let $F \in BV$. Then $\lim_{x\to-\infty} T_F(x) = 0$ and if F is right continuous, then T_F is right continuous.*

Definition 4.5.19. *Define $NBV = \{F \in BV : F$ is right continuous and $\lim_{x\to-\infty} F(x) = 0\}$.*

Theorem 4.5.20. *Let $M(\mathbb{R})$ be the set of complex Borel measures on $\mathbb{R}$. For $F \in NBV$, define $\mu_F \in M(\mathbb{R})$ by $\mu_F((-\infty, x]) = F(x)$. Then $F \mapsto \mu_F$ defines a bijection $NBV \to M(\mathbb{R})$. In addition, $|\mu_F| = \mu_{T_F}$*

Theorem 4.5.21. *Let $F \in NBV$. Then $F' \in L^1(m)$, $\mu_F \perp m$ iff $F' = 0$ a.e. and $\mu_F \ll m$ iff for each $x \in \mathbb{R}$, $\int_{(-\infty,x]} F'dm = F(x)$*

Definition 4.5.22. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to be* ***absolutely continuous*** *if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Definition 4.5.23. *Let $F : [a, b] \to \mathbb{C}$. Then F is said to be* ***absolutely continuous on*** *$[a, b]$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.*

Proposition 4.5.24. *Let $F : [a, b] \to \mathbb{C}$. If F is absolutely continuous on $[a, b]$, then $F \in BV[a, b]$.*

Exercise 4.5.25. *Let $F : \mathbb{R} \to \mathbb{C}$. Suppose that there exists $f \in L^1(m)$ such that $F(x) = \int_{(-\infty,x} fdm$. Then $F \in NBV$.*

Proof. Let $x \in \mathbb{R}$ and $(x_i)_{i=1}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=1}^n$ is increasing and $x_n = x$. Then

$$\begin{aligned}
\sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \left| \int_{(x_{i-1},x_i]} fdm \right| \\
&\leq \sum_{i=1}^n \int_{(x_{i-1},x_i]} |f|dm \\
&= \int_{(x_0,x]} |f|dm \\
&< \int |f|dm
\end{aligned}$$

Hence $T_F(x) \leq \int |f|dm$. Since $x \in \mathbb{R}$ is arbitrary, $TV(F) \leq \int |f|dm$. Therefore $F \in BV$. By the continuity from above and below for measures and the fact that $m(x) = 0$ for each $x \in \mathbb{R}$, F is continuous. By continuity from above for measures, $\lim\limits_{x\to-\infty} F(x) = 0$. So $F \in NBV$. □

Lemma 4.5.26. *Let $F \in NBV$. Then F is absolutely continuous iff $\mu_F \ll m$.*

Exercise 4.5.27. *Fundamental Theorem of Calculus: Let $F : [a, b] \to \mathbb{C}$. The following are equivalent:*

(1) *F is absolutely continuous on $[a,b]$.*
(2) *there exists $f \in L^1([a,b],m)$ such that for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} f dm$*
(3) *F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b],m)$ and for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} F' dm$*

Proof. (1) $\implies$ (3)
Suppose that F is absolutely continuous on $[a,b]$. Then $F \in BV[a,b]$. Extend F to $\mathbb{R}$ by setting $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$. Then $G = F - F(a) \in NBV$ and is absolutely continuus. The previous lemma implies that there exists $f \in L^1(m)$ such that $\mu_G = f dm$. A previous theorem implies that for a.e. $x \in [a,b]$

$$\begin{aligned} F'(x) &= \lim_{r \to x} \frac{\mu_G((x, x+r])}{m((x, x+r])} \\ &= f(x) \end{aligned}$$

So F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b],m)$ and by construction, for each $x \in [a,b]$, we have that

$$\begin{aligned} F(x) - F(a) &= \mu_G((a,x]) \\ &= \int_{(a,x]} f dm \\ &= \int_{(a,x]} F' dm \end{aligned}$$

(3) $\implies$ (2)
Trivial.
(2) $\implies$ (1)
Suppose that there exists $f \in L^1([a,b],m)$ such that for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} f dm$. Extend F as before and obtain G as before. Note that a previous exercise implies that $G \in NBV$. Since $\mu_G \ll m$, the previous lemma implies that G is absolutely continuous. □

Exercise 4.5.28. *Let $F : \mathbb{R} \to \mathbb{C}$. If F is absolutely continuous. Then F is differentiable a.e.*

Proof. Let $n \in \mathbb{N}$. Since F is absolutely continuous on $\mathbb{R}$, F is absolutely continuous on $[-n,n]$. The FTC implies that F is differentiable a.e. on $[-n,n]$. Since $n \in \mathbb{N}$ is arbitrary, F is differentiable a.e on $\mathbb{R}$. □

Exercise 4.5.29. *Let $F : \mathbb{R} \to \mathbb{C}$. Then F is Lipschitz continuous iff F is absolutely continuous and F' is bounded a.e.*

Proof. Suppose that F is Lipschitz continuous. Then there exists $M > 0$ such that for each $x, y \in \mathbb{R}, |F(x) - F(y)| \leq M|x - y|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{M}$. Let $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, Suppose that $\sum_{i=1}^n b_i - a_i < \delta$. Then

$$\begin{aligned} \sum_{i=1}^n |F(b_i) - F(a_i)| &\leq \sum_{i=1}^n M(b_i - a_i) \\ &< M\delta \\ &= \epsilon \end{aligned}$$

Hence F is absolutely continuous. For each $x, y \in \mathbb{R}$, if $x \neq y$, then $\left|\frac{F(x)-F(y)}{x-y}\right| \leq M$. Hence for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Conversely, suppose that F is absolutely continuous and F' is bounded a.e. Then there exits $M > 0$ such that for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Let $x, y \in \mathbb{R}$. Suppose $x < y$. Then the FTC implies that

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_{(x,y]} F' dm \right| \\ &\leq \int_{(x,y]} |F'| dm \\ &= M|y - x| \end{aligned}$$

and F is Lipschitz continuous. □

Exercise 4.5.30. *Construct an increasing function $F : \mathbb{R} \to \mathbb{R}$ whose discontinuities is $\mathbb{Q}$.*

Proof. Let $(q_n)_{n\in\mathbb{N}}$ be an ennumeration of $\mathbb{Q}$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F = \sum_{n\in\mathbb{N}} 2^{-n} \chi_{[q_n,\infty)}$$

. Equivalently, if we define $S_x = \{n \in \mathbb{N} : q_n \leq x\}$, then we may write

$$F(x) = \sum_{n\in S_x} 2^{-n}$$

Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Then $S_x \subsetneq S_y$. So $F(x) < F(y)$ and therefore F is strictly increasing.
For each $x, y \in R$ with $x < y$, define $S_{x,y} = \{n \in \mathbb{N} : x < q_n \leq y\}$. Note that $\lim\limits_{y\to x^+} \min(S_{x,y}) = \infty$ and if $y \in \mathbb{R} \setminus \mathbb{Q}$, then $\lim\limits_{x\to y^-} \min(S_{x,y}) = \infty$.
Now, let $x \in \mathbb{R}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum\limits_{n=N}^{\infty} 2^{-n} < \epsilon$. Choose $\delta > 0$ such that $\min(S_{x,x+\delta}) \geq N$. Let $y \in [x, \infty)$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n\in S_y} 2^{-n} - \sum_{n\in S_x} 2^{-n} \\ &= \sum_{n\in S_{x,y}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence F is right continuous. Now let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ as before and $\delta > 0$ such that $\min(S_{x-\delta,x}) \geq N$. Let $y \in (-\infty, x]$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned}|F(x) - F(y)| &= \sum_{n \in S_x} 2^{-n} - \sum_{n \in S_y} 2^{-n} \\ &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon\end{aligned}$$

Hence F is left continuous on $\mathbb{R} \setminus \mathbb{Q}$.
Now, let $x \in \mathbb{Q}$. Then there exists $j \in \mathbb{N}$ such that $q_j = x$. Choose $\epsilon = 2^{-j}$. Let $\delta > 0$. Choose $y = x - \frac{\delta}{2}$. Then $|x - y| < \delta$ and

$$\begin{aligned}|F(x) - F(y)| &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\geq 2^{-j} \\ &= \epsilon\end{aligned}$$

Hence F is discontinuous from the left at x. Since $x \in \mathbb{Q}$ is arbitrary, F is discontinuous from the left on $\mathbb{Q}$. □

Exercise 4.5.31. *Let $(F_n)_{n \in \mathbb{N}} \in NBV$ be a sequence of nonnegative, increasing functions. If for each $x \in \mathbb{R}$, $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$, then for a.e. $x \in \mathbb{R}$, F is differentiable at x and $F'(x) = \sum_{n \in \mathbb{N}} F_n'(x)$.*

Proof. Define $\mu = \sum_{n \in \mathbb{N}} \mu_{F_n}$. Note that

$$\begin{aligned}\mu((-\infty, x]) &= \sum_{n \in \mathbb{N}} \mu_{F_n}((-\infty, x]) \\ &= \sum_{n \in \mathbb{N}} F_n(x) \\ &= F(x)\end{aligned}$$

Hence $F \in NBV$ and $\mu = \mu_F$. For each $n \in \mathbb{N}$, there exist $\lambda_n \in M(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ such that $d\mu_{F_n} = d\lambda_n + f_n dm$ and $\lambda \perp m$. Since for each $n \in \mathbb{N}$, λ_n, f_n are nonnegative, we have that $d\mu_F = \sum_{n \in \mathbb{N}} d\lambda_n + (\sum_{n \in \mathbb{N}} f_n) dm$. By a previous theorem, for a.e. $x \in \mathbb{R}$,

$$\begin{aligned}F'(x) &= \lim_{r \to 0} \frac{\mu_F((x, x + r])}{m((x, x + r])} \\ &= \sum_{n \in \mathbb{N}} f_n(x) \\ &= \sum_{n \in \mathbb{N}} \lim_{r \to 0} \frac{\mu_{F_n}((x, x + r])}{m((x, x + r])} \\ &= \sum_{n \in \mathbb{N}} F_n'(x)\end{aligned}$$

□

Exercise 4.5.32. *Let $F : [0,1] \to [0,1]$ be the Cantor function. Extend F to $\mathbb{R}$ by setting $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $([a_n, b_n])_{n\in\mathbb{N}}$ be an ennumeration of the closed subintervals of $[0,1]$ with rational endpoints. For $n \in \mathbb{N}$, define $F_n : \mathbb{R} \to [0,1]$ by $F_n(x) = F(\frac{x-a_n}{b_n-a_n})$. Define $G : \mathbb{R} \to \mathbb{R}$ by $G = \sum_{n\in\mathbb{N}} 2^{-n}F_n$. Then G is continuous, strictly increasing on $[0,1]$ and $G' = 0$ a.e.*

Proof. Since F is continuous on $\mathbb{R}$, we have that for each $n \in \mathbb{N}$, F_n is continuous on $\mathbb{R}$. We observe that for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $|2^{-n}F_n(x)| \leq 2^{-n}$. Thus the Weierstrass M-test implies that G converges uniformly on $\mathbb{R}$ and is therefore continuous. Since F is increasing, for each $n \in \mathbb{N}$, F_n is increasing. Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Choose $j \in \mathbb{N}$ such that $x < a_j < y < b_j$. Then

$$\begin{aligned} G(x) &= \sum_{n\in\mathbb{N}} 2^{-n}F_n(x) \\ &= \sum_{\substack{n\in\mathbb{N}\\ n\neq j}} 2^{-n}F_n(x) + 0 \\ &< \sum_{\substack{n\in\mathbb{N}\\ n\neq j}} 2^{-n}F_n(y) + 2^{-j}F_n(y) \\ &= \sum_{n\in\mathbb{N}} 2^{-n}F_n(y) \\ &= G(y) \end{aligned}$$

So G is strictly increasing.
Now we observe that for each $n \in \mathbb{N}$, $F_n \in NBV$. The previous exercise implies that

$$G' = \sum 2^{-n}F_n' = 0 \text{ a.e.}$$

□

5. Topology

Definition 5.0.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be topological spaces and $f : X \to Y$. Then*

(1) *f is said to be* ***continuous*** *if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$.*
(2) *f is said to be open if for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$.*
(3) *f is said to be* ***closed*** *if for each $A \subset X$, if $A^c \in \mathcal{A}$, then $f(A)^c \in \mathcal{B}$.*

Exercise 5.0.2. *Let X, Y be topological spaces and $\phi : X \to Y$ a homeomorphism. Then for each $A \subset X$,*

(1) $\overline{\phi(A)} = \phi(\overline{A})$
(2) $\phi(A)^\circ = \phi(A^\circ)$

Proof.

(1) Let $A \subset X$. Since $A \subset \overline{A}$, we have that $\phi(A) \subset \phi(\overline{A})$. Since $\overline{A}$ is closed, $\phi(\overline{A})$ is closed and thus $\overline{\phi(A)} \subset \phi(\overline{A})$. Conversely, let $x \in \phi(\overline{A})$. Then $\phi^{-1}(x) \in \overline{A}$. Then there exists a net $\langle y_\alpha \rangle \subset A$ such that $y_\alpha \to \phi^{-1}(x)$. Then $\langle \phi(y_\alpha) \rangle \subset \phi(A)$ and $\phi(y_\alpha) \to x$. Thus $x \in \overline{\phi(A)}$ and $\phi(\overline{A}) \subset \overline{\phi(A)}$.

(2) Similar

□

6. L^p Spaces

Definition 6.0.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in (0, \infty]$. Define $\|\cdot\|_p : L^0(X, \mathcal{A}, \mu) \to [0, \infty]$ by*

$$\|f\|_p = \bigg(\int |f|^p d\mu \bigg)^{\frac{1}{p}} \qquad (p < \infty)$$

and

$$\|f\|_\infty = \inf \bigg\{ \lambda > 0 : \mu\big(\{x \in X : \lambda < |f(x)|\} \big) = 0 \bigg\}$$

We define

$$L^p(X, \mathcal{A}, \mu) = \{f \in L^0(X, \mathcal{A}, \mu) : \|f\|_p < \infty\}$$

Theorem 6.0.2. *Hölder's Inequality:* *Let $(X, \mathcal{A}, \mu)$ be a measure space, $p, q \in [1, \infty)$ and $f, g \in L^0$. Suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Exercise 6.0.3. *Minkowski Inequality:* *Let $(X, \mathcal{A}, \mu)$ be a measure space, $p \in [1, \infty)$ and $f, g \in L^p$. Then $f + g \in L^p$ and*

$$\|f + g\| \leq \|f\|_p + \|g\|_p$$

Proof. Define $\phi : \mathbb{R} \to [0, \infty)$ by $\phi(x) = |x|^p$. Then ϕ is convex because it is the composition of an increasing convex function with a convex function. By Jensen's inequality, we have that

$$\phi\bigg(\frac{1}{2}[f + g]\bigg) \leq \frac{1}{2}[\phi(f) + \phi(g)]$$

This implies that

$$\frac{1}{2^p}|f + g|^p \leq \frac{1}{2}\bigg(|f|^p + |g|^p\bigg)$$

Hence

$$\begin{aligned}
\int |f + g|^p d\mu &\leq 2^{p-1} \int |f|^p + |g|^p d\mu \\
&= 2^{p-1}\bigg(\int |f|^p d\mu + \int |g|^p d\mu \bigg) \\
&= 2^{p-1}\bigg(\|f\|_p^p + \|g\|_p^p \bigg) \\
&< \infty
\end{aligned}$$

So $f + g \in L^p$. Now, it is not hard to see that $|f + g|^p \leq \big(|f| + |g|\big)|f + g|^{p-1}$. Let q be the conjugate of p, so that $\frac{1}{p} + \frac{1}{q} = 1$. Then $q(p - 1) = p$. We use Hölder's inequality to show that

$$\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p d\mu \\
&\leq \int |f||f+g|^{p-1} d\mu + \int |g||f+g|^{p-1} d\mu \\
&\leq \|f\|_p \bigg(\int |f+g|^{(p-1)q} d\mu \bigg)^{\frac{1}{q}} + \|g\|_p \bigg(\int |f+g|^{(p-1)q} d\mu \bigg)^{\frac{1}{q}} \\
&= \|f\|_p \bigg(\int |f+g|^p d\mu \bigg)^{\frac{1}{q}} + \|g\|_p \bigg(\int |f+g|^p d\mu \bigg)^{\frac{1}{q}} \\
&= (\|f\|_p + \|g\|_p) \bigg(\int |f+g|^p d\mu \bigg)^{\frac{1}{q}} \\
&= (\|f\|_p + \|g\|_p) \|f+g\|_p^{p/q}
\end{aligned}$$

Since $\|f+g\|_p < \infty$, we see that

$$\begin{aligned}
\|f\|_p + \|g\|_p &\geq \|f+g\|_p^{p-p/q} \\
&= \|f+g\|_p^{p(1-1/q)} \\
&= \|f+g\|_p^{p/p} \\
&= \|f+g\|_p
\end{aligned}$$

□

Exercise 6.0.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $p, q \in (0, \infty]$. Suppose that $\mu(X) < \infty$ and $p < q$. Then $L^q \subset L^p$. In particular, if $\mu(X) = 1$, then for each $f \in L^q$, $\|f\|_p \leq \|f\|_q$.*

Proof. Suppose that $q = \infty$. Let $f \in L^q$. Then

$$\begin{aligned}
\|f\|_p &= \bigg(\int |f|^p d\mu \bigg)^{\frac{1}{p}} \\
&\leq \bigg(\int \|f\|_\infty^p d\mu \bigg)^{\frac{1}{p}} \\
&= \|f\|_\infty \mu(X)^{\frac{1}{p}}
\end{aligned}$$

If $q < \infty$, then $\frac{q}{p} > 1$ and the conjugate of $\frac{q}{p}$ is $\frac{1}{1-p/q}$. By Hölder's inequality, we have that

$$\begin{aligned}
\|f\|_p^p &= \|f^p\|_1 \\
&\leq \|f^p\|_{\frac{q}{p}} \|1\|_{\frac{1}{1-p/q}} \\
&= \bigg(\int |f|^{\frac{pq}{p}} d\mu \bigg)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}} \\
&= \bigg(\int |f|^q d\mu \bigg)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}} \\
&= \|f\|_q^p \mu(X)^{1-\frac{p}{q}}
\end{aligned}$$

Hence

$$\begin{aligned}\|f\|_p &\leq \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}} \\ &< \infty\end{aligned}$$

□

7. Functional Analysis

7.1. Normed Vector Spaces.

Note 7.1.1. *In the following, we will consider vector spaces over $\mathbb{C}$. There are analogous results for real vector spaces as well, just replace every $\mathbb{C}$ with $\mathbb{R}$.*

Definition 7.1.2. *Let X be a normed vector space. Then X is said to be a* ***Banach space*** *if X is complete.*

Definition 7.1.3. *Let X be a normed vector space and $(x_i)_{i=1}^n \subset X$. The series $\sum_{i=1}^\infty x_i$ is said to* ***converge*** *if the sequence $s_n := \sum_{i=1}^n x_i$ converges. The series $\sum_{i=1}^\infty x_i$ is said to* ***converge absolutely*** *if $\sum_{i\in\mathbb{N}} \|x_i\| < \infty$.*

Theorem 7.1.4. *Let X be a normed vector space. Then X is complete iff for each $(i_\subset)_{\subset\in\mathbb{N}}X$, $\sum_{i=1}^\infty x_i$ converges absolutely implies that $\sum_{i=1}^\infty x_i$ converges.*

Proof. Suppose that X is complete. Let $(i_\subset)_{\subset\in\mathbb{N}}X$. Suppose that $\sum_{i=1}^\infty x_i$ converges absolutely. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq N$ and $m < n$, then $\sum_{m+1}^n \|x_i\| < \epsilon$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then

$$\begin{aligned}\|s_n - s_m\| &= \left\| \sum_{i=1}^n x_i - \sum_{i=1}^m x_i \right\| \\ &= \left\| \sum_{i=m+1}^n x_i \right\| \\ &\leq \sum_{i=m+1}^n \|x_i\| \\ &< \epsilon\end{aligned}$$

Thus $(s_n)_{n\in N}$ is cauchy. Since X is complete, $\sum_{i=1}^\infty x_i$ converges.
Conversely, Suppose that for each $(i_\subset)_{\subset\in\mathbb{N}}X$, $\sum_{i=1}^\infty x_i$ converges absolutely implies that $\sum_{i=1}^\infty x_i$ converges. Let $(i_\subset)_{\subset\in\mathbb{N}}X$ be cauchy. Proceed inductively to create a strictly increasing sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that for each $m, n \in \mathbb{N}$, if $m, n \geq n_i$, then $\|x_m - x_n\| < 2^{-i}$. Define $(y_i)_{i\in\mathbb{N}} \subset X$ by

$$y_i = \begin{cases} x_{n_1} & i = 1 \\ x_{n_i} - x_{n_{i-1}} & i \geq 2 \end{cases}$$

Then $\sum_{i=1}^k y_i = x_{n_k}$ and

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|y_i\| &= \|x_{n_1}\| + \sum_{i\in\mathbb{N}} \|x_{n_i} - x_{n_{i-1}}\| \\ &\leq \|x_{n_1}\| + \sum_{i\in\mathbb{N}} 2^{-i} \\ &= \|x_{n_1}\| + 1\end{aligned}$$

Hence $(x_{n_k})_{k\in\mathbb{N}} = (\sum_{i=1}^k y_i)_{i\in\mathbb{N}}$ converges. Since $(x_i)_{i\in\mathbb{N}}$ is cauchy and has a convergent subsequence, it converges. So X is complete. □

Definition 7.1.5. *Let X, Y be a normed vector spaces. A linear map $T : X \to Y$ is said to be **bounded** if there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$.*

Exercise 7.1.6. *Let X, Y be a normed vector spaces and $T : X \to Y$ a linear map. Then T is bounded iff there exists $r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$*

Proof. Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Thus $T(B(0, 1)) \subset B(0, C + 1)$. Conversely. Suppose that there exists $r, s > 0$ such that $T(B(0, r)) \subset B(0, s)$. Define $C = \frac{2s}{r}$. Let $x \in X$. Put $\alpha = \frac{r}{2\|x\|}$ Then $\alpha x \in B(0, r)$. So $T(\alpha x) = \alpha T(x) \in B(0, s)$. Hence

$$\begin{aligned}\|T(\alpha x)\| &= \|\alpha T(x)\| \\ &= |\alpha|\|T(x)\| \\ &= \frac{r}{2\|x\|}\|T(x)\| \\ &< s.\end{aligned}$$

Thus

$$\|Tx\| < \frac{2s}{r}\|x\| = C\|x\|$$

So T is bounded. □

Theorem 7.1.7. *Let X, Y be normed vector spaces and $T : X \to Y$ a linear map. Then the following are equivalent:*

(1) *T is continuous*
(2) *T is continuous at $x = 0$*
(3) *T is bounded*

Proof. (1) $\implies$ (2):
Trivial

(2) $\implies$ (3):
Suppose that T is continuous at $x = 0$. Then there exists $\delta > 0$ such that for each $x \in X$, if $\|x\| < \delta$, then $\|Tx\| < 1$. Choose $C = \frac{2}{\delta}$. If $x = 0$, then $\|Tx\| \leq C\|x\|$. Suppose that $\|x\| \neq 0$. Define $y = \frac{\delta}{2\|x\|}x$. Then $\|y\| < \delta$. So

$$\|Ty\| = \frac{\delta}{2\|x\|}\|Tx\| < 1$$

Thus

$$\begin{aligned}\|Tx\| &< \frac{2}{\delta}\|x\| \\ &= C\|x\|\end{aligned}$$

Hence T is bounded.

(3) $\implies$ (1)
Suppose that T is bounded. Then there exists $C \geq 0$ such that for each $x \in X$, $\|Tx\| \leq C\|x\|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{C+1}$. Let $x, y \in X$ Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned}\|Tx - Ty\| &= \|T(x-y)\| \\ &\leq C\|x-y\| \\ &< (C+1)\delta \\ &= \epsilon\end{aligned}$$

So T is continuous. □

Definition 7.1.8. *Let X, Y be normed vector spaces. Define $L(X,Y) = \{T : X \to Y : T \text{ is bounded}\}$. Define $\|\cdot\| : L(X,Y) \to [0,\infty)$ by*

$$\|T\| = \inf\{C \geq 0 : \textit{for each } x \in X, \ \|Tx\| \leq C\|x\|\}$$

We call $\|\cdot\|$ the ***operator norm on*** *$L(X,Y)$*

Exercise 7.1.9. *Let X, Y be normed vector spaces. If $X \neq \{0\}$, then the operater norm on $L(X,Y)$ is given by:*
1. $\|T\| = \sup\limits_{\|x\|=1} \|Tx\|$
2. $\|T\| = \sup\limits_{x \neq 0} \|x\|^{-1}\|Tx\|$
3. $\|T\| = \inf\{C \geq 0 : \textit{for each } x \in X, \ \|Tx\| \leq C\|x\|\}$

Proof. Since $X \neq \{0\}$, the supremums in (1) and (2) are well defined. Let $T \in L(X,Y)$. By linearity of T, the sets over which the supremums are taken in (1) and (2) are the same. So (1) and (2) are equal.

Now, put $M = \sup\limits_{\|x\|=1} \|Tx\|$, $m = \inf\{C \geq 0 : \text{ for each } x \in X, \|Tx\| \leq C\|x\|\}$ and let $x \in X$. If $\|x\| = 0$, then $\|Tx\| \leq M\|x\|$. Suppose that $\|x\| \neq 0$. Then

$$\begin{aligned}\|Tx\| &= \left(\big\|T(x/\|x\|)\big\|\right)\|x\| \\ &\leq M||x||\end{aligned}$$

Hence $M \in \{C \geq 0 : \text{ for each } x \in X, \|Tx\| \leq C\|x\|\}$. Therefore $m \leq M$

Let $C \in \{C \geq 0 : \text{ for each } x \in X, \|Tx\| \leq C\|x\|\}$. Suppose that $\|x\| = 1$. Then $\|Tx\| \leq C\|x\| = C$. So $M \leq C$. Therefore $M \leq m$. So $M = m$ and the supremum in (1) is the same as the infimum in (3). □

Note 7.1.10. *From here on, unless stated otherwise, we assume $X \neq 0$.*

Exercise 7.1.11. *Let X, Y be normed vector spaces and $T \in L(X, Y)$. Then for each $x \in X$, $\|Tx\| \leq \|T\|\|x\|$*

Proof. This is just part of the previous exercise. Let $x \in X$. If $x = 0$, then $\|Tx\| \leq \|T\|\|x\|$. Suppose that $x \neq 0$. Then $\|Tx\| = T(x/\|x\|)\|x\| \leq \|T\|\|x\|$ □

Exercise 7.1.12. *Let X, Y be normed vector spaces. Then the operator norm is a norm on $L(X, Y)$.*

Proof. Let $S, T \in L(X, Y)$ and $\alpha \in \mathbb{C}$. For each $x \in X$, we have that

$$\begin{aligned}\|(S+T)x\| &= \|Sx + Tx\| \\ &\leq \|Sx\| + \|Tx\| \\ &\leq \|S\|\|x\| + \|T\|\|x\| \\ &= \big(\|S\| + \|T\|\big)\|x\|\end{aligned}$$

So $\|S + T\| \leq \|S\| + \|T\|$.

Using the definition of $\|T\|$, we see that

$$\begin{aligned}\|\alpha T\| &= \sup_{\|x\|=1} \|(\alpha T)x\| \\ &= \sup_{\|x\|=1} |\alpha|\|Tx\| \\ &= |\alpha| \sup_{\|x\|=1} \|Tx\| \\ &= |\alpha|\|T\|\end{aligned}$$

So $\|\alpha S\| = |\alpha|\|S\|$.

Suppose that $\|T\| = 0$. Let $x \in X$. Then $\|Tx\| \leq \|T\|\|x\| = 0$. So $Tx = 0$. Since $x \in X$ is arbitrary, we have that $T = 0$. □

Exercise 7.1.13. *Let X be a normed vector space. Then addition and scalar multiplication are continuous on $X \times X$ and $\|\cdot\| : X \to [0, \infty)$ is continuous.*

Proof. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Let $(x_1, y_1), (x_2, y_2) \in X \times X$. Suppose that $\|(x_1, y_1) - (x_2, y_2)\| = \max\{\|x_1 - x_2\|, \|y_1 - y_2\|\} < \delta$. Then

$$\begin{aligned}\|(x_1 + y_1) - (x_2 + y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\ &\leq \|x_1 - x_2\| + \|y_1 - y_2\| \\ &< 2\delta \\ &= \epsilon\end{aligned}$$

Hence addition is uniformly continuous.

Let $(\lambda_1, x_1) \in \mathbb{C} \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(|\lambda_1|+\|x_1\|+1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(\lambda_2, x_2) \in \mathbb{C} \times X$. Suppose that $\|(\lambda_1, x_1) - (\lambda_2, x_2)\| = \max\{|\lambda_1 - \lambda_2|, \|x_1 - x_2\|\} < \delta$. Then

$$\begin{aligned}
\|\lambda_1 x_1 - \lambda_2 x_2\| &= \|\lambda_1 x_1 - \lambda_1 x_2 + \lambda_1 x_2 - \lambda_2 x_2\| \\
&= \|\lambda_1(x_1 - x_2) + (\lambda_1 - \lambda_2)x_2\| \\
&\leq |\lambda_1|\|x_1 - x_2\| + |\lambda_1 - \lambda_2|\|x_2\| \\
&\leq |\lambda_1|\|x_1 - x_2\| + |\lambda_1 - \lambda_2|(\|x_1 - x_2\| + \|x_1\|) \\
&< |\lambda_1|\delta + \delta(\delta + \|x_1\|) \\
&= (|\lambda_1| + \|x_1\|)\delta + \delta^2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Since $(\lambda_1, x_1) \in \mathbb{C} \times X$ is arbitrary, scalar multiplication is continuous.

Let $\epsilon > 0$. Choose $\delta = \epsilon$. Let $x, y \in X$. Suppose that $\|x - y\| < \delta$. Then

$$\begin{aligned}
\big|\|x\| - \|y\|\big| &\leq \|x - y\| \\
&< \delta \\
&= \epsilon
\end{aligned}$$

So $\|\cdot\| : X \to [0, \infty)$ is uniformly continuous. □

Exercise 7.1.14. *Let X, Y be normed vector spaces. If Y is complete, then so is $L(X, Y)$.*

Proof. Suppose that Y is complete. Let $(T_n)_{n\in\mathbb{N}} \subset L(X, Y)$. Suppose that $(T_n)_{n\in\mathbb{N}}$ is Cauchy. Since for each $m, n \in \mathbb{N}$, $\big|\|T_m\| - \|T_n\|\big| \leq \|T_m - T_n\|$, we have that $(\|T_n\|)_{n\in\mathbb{N}} \subset [0, \infty)$ is Cauchy. Hence $\lim\limits_{n\to\infty} \|T_n\|$ exists.

Let $x \in X$ and $m, n \in \mathbb{N}$. Then

$$\begin{aligned}
\|T_m x - T_n x\| &= \|(T_m - T_n)x\| \\
&\leq \|T_m - T_n\|\|x\|
\end{aligned}$$

So $(T_n x)_{n\in\mathbb{N}} \subset Y$ is Cauchy and hence converges. Define $T : X \to Y$ by $Tx = \lim\limits_{n\to\infty} T_n x$.

Since addition and scalar multiplication are continuous, T is linear. Let $x \in X$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in N$, if $n \geq N$, then $\|Tx - T_n x\| < \epsilon$. Then for each $n \in \mathbb{N}$, if $n \geq N$ we have that

$$\begin{aligned}
\|Tx\| &\leq \|Tx - T_n x\| + \|T_n x\| \\
&< \epsilon + \|T_n x\| \\
&\leq \epsilon + \|T_n\|\|x\|
\end{aligned}$$

Thus $\|Tx\| \leq \epsilon + (\lim_{n\to\infty} \|T_n\|)\|x\|$. Since $\epsilon > 0$ is arbitrary, $\|Tx\| \leq (\lim_{n\to\infty} \|T_n\|)\|x\|$. Thus $T \in L(X,Y)$ and $\|T\| \leq \lim_{n\to\infty} \|T_n\|$.

Note that since addition, scalar multiplication and $\|\cdot\|$ are continuous, we have that for each $n \in \mathbb{N}$ and $x \in X$, $\|(T_n - T_m)x\|$ converges to $\|(T_n - T)x\|$ because

$$\begin{aligned}\lim_{m\to\infty} \|(T_n - T_m)x\| &= \lim_{m\to\infty} \|T_n x - T_m x\| \\ &= \|T_n x - \lim_{m\to\infty} T_m x\| \\ &= \|T_n x - Tx\| \\ &= \|(T_n - T)x\|\end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that for each $m, n \in \mathbb{N}$ if $n, m \geq N$, then $\|T_n - T_m\| < \epsilon$. Then for each $n \in \mathbb{N}$ if $n \geq N$, then for each $x \in X$,

$$\|(T_n - T_m)x\| \leq \|(T_n - T_m)\|\|x\| < \epsilon\|x\|$$

Combining this with the previous fact, we see that for each $n \in N$, if $n \geq N$, then for each $x \in X$,

$$\|(T_n - T)x\| \leq \epsilon\|x\|$$

In particular, for each $n \in \mathbb{N}$, if $n \geq N$, then

$$\|T_n - T\| = \sup_{\|x\|=1} \|(T_n - T)x\| \leq \epsilon$$

This implies that T_n converges to T in $L(X,Y)$. Since

$$\big|\|T_n\| - \|T\|\big| \leq \|T_n - T\|$$

It is clear that $\lim_{n\to\infty} \|T_n\| = \|T\|$ □

Definition 7.1.15. *Let X be a normed vector space and $M \subset X$ a closed subspace. Define $\|\cdot\| : X/M \to [0,\infty)$ by*

$$\|x + M\| := \inf_{y\in M} \|x + y\|$$

We call $\|\cdot\|$ the ***subspace norm on*** *X/M*

Exercise 7.1.16. *Let X be a normed vector space and $M \subsetneq X$ a proper, closed subspace of M. Then*

(1) *The previously defined subspace norm on X/M is well defined and is a norm.*
(2) *For each $\epsilon > 0$, there exists $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.*
(3) *The projection map $\pi : X \to X/M$ defined by $\pi(x) = x + M$ is continuous and $\|\pi\| = 1$.*
(4) *If X is complete, then X/M is complete.*

Proof. (1) Let $x, y \in X$ and $\alpha \in \mathbb{C}$. Suppose that $x + M = y + M$. Then there exists $m \in M$ such that $x = y + m$. Since M is a subspace, the map $T : M \to M$ given by $Tx = x + m$ is a bijection. So

$$\inf_{z\in M} \|y + m + z\| = \inf_{z\in M} \|y + z\|$$

which implies that

$$\begin{aligned}\|x+M\| &= \inf_{z\in M}\|x+z\| \\ &= \inf_{z\in M}\|y+m+z\| \\ &= \inf_{z\in M}\|y+z\| \\ &= \|y+M\|\end{aligned}$$

So $\|\cdot\| : X/M \to [0,\infty)$ is well defined.

We observe that for each $z, w \in M$,

$$\|x+y+z\| \leq \|x+w\| + \|y+w+z\|$$

Taking infimums over M with respect to z in this inequality implies that for each $w \in M$,

$$\begin{aligned}\inf_{z\in M}\|x+y+z\| &\leq \inf_{z\in M}\Big(\|x+w\| + \|y+w+z\|\Big) \\ &= \|x+w\| + \inf_{z\in M}\|y+w+z\|\end{aligned}$$

Again we use the fact that for each $w \in M$,

$$\inf_{z\in M}\|y+w+z\| = \inf_{z\in M}\|y+z\|$$

This implies that for each $w \in M$,

$$\inf_{z\in M}\|x+y+z\| \leq \|x+w\| + \inf_{z\in M}\|y+z\|$$

Therefore, taking infimums over M with respect to w in this inequality yields

$$\begin{aligned}\|x+y+M\| &= \inf_{z\in M}\|x+y+z\| \\ &\leq \inf_{w\in M}\Big(\|x+w\| + \inf_{z\in M}\|y+z\|\Big) \\ &= \inf_{w\in M}\|x+w\| + \inf_{z\in M}\|y+z\| \\ &= \|x+M\| + \|y+M\|\end{aligned}$$

If $\alpha = 0$, then $\alpha x = 0$. Choosing $z = 0 \in M$ gives $\|\alpha x + M\| = 0 = |\alpha|\|x+M\|$. Suppose that $\alpha \neq 0$. Then the map $T : M \to M$ given by $Tx = \alpha^{-1}x$ is a bijection and thus $\inf_{z\in M}\|x+\alpha^{-1}z\| = \inf_{z\in M}\|x+z\|$. Hence we have that

$$\begin{aligned}\|\alpha x+M\| &= \inf_{z\in M}\|\alpha x+z\| \\ &= \inf_{z\in M}|\alpha|\|x+\alpha^{-1}z\| \\ &= |\alpha|\inf_{z\in M}\|x+\alpha^{-1}z\| \\ &= |\alpha|\inf_{z\in M}\|x+z\| \\ &= |\alpha|\|x+M\|\end{aligned}$$

Suppose that $\|x\| = 0$. Choose a sequence $(z_n)_{n\in N} \subset M$ such that

$$\begin{aligned}\lim_{n\to\infty} \|x - z_n\| &= \inf_{z\in M} \|x + z\| \\ &= 0\end{aligned}$$

Then $\lim_{n\to\infty} z_n = x$. Since M is closed, $x \in M$. Hence $x + M = 0 + M$.

(2) Since M is a proper subspace, there exists $v \in X$ such that $v \notin M$. Then $\|v+M\| \neq 0$. Let $\epsilon > 0$. Then $(1-\epsilon)^{-1}\|v + M\| > \|v + M\|$. So there exists $z \in M$ such that

$$0 < \|v + M\| \leq \|v + z\| < (1-\epsilon)^{-1}\|v + M\|$$

Choose $x = \|v + z\|^{-1}(v + z)$. Then $\|x\| = 1$ and

$$\begin{aligned}\|x + M\| &= \|v + z\|^{-1}\|v + z + M\| \\ &= \|v + z\|^{-1}\|v + M\| \\ &> 1 - \epsilon\end{aligned}$$

(3) Let $x \in X$. Taking $z = 0$, we we see that $\|\pi(x)\| = \|x + M\| \leq \|x + z\| = \|x\|$. So π is bounded and in particular,

$$\sup_{\|x\|=1} \|\pi(x)\| \leq 1$$

From (2) we see that

$$\sup_{\|x\|=1} \|\pi(x)\| \geq 1$$

Hence $\|\pi\| = 1$.

(4) Suppose that X is complete. Let $(x_i + M)_{i\in\mathbb{N}} \subset X/M$. Suppose that $\sum_{i\in\mathbb{N}} \|x_i + M\| < \infty$. Let $\epsilon > 0$. Then for each $i \in \mathbb{N}$, there exists $z_i \in M$ such that $\|x_i + z_i\| < \|x_i + M\| + \epsilon 2^{-i}$. Define the sequence $(a_i)_{i\in\mathbb{N}} \subset X$ by $a_i = x_i + z_i$. Then we have

$$\begin{aligned}\sum_{i\in\mathbb{N}} \|a_i\| &= \sum_{i\in N} \|x_i + z_i\| \\ &\leq \sum_{i\in N} \left(\|x_i + M\| + \epsilon 2^{-i}\right) \\ &= \sum_{i\in\mathbb{N}} \|x_i + M\| + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\sum_{i\in\mathbb{N}} \|a_i\| \leq \sum_{i\in\mathbb{N}} \|x_i + M\| < \infty$$

Since X is complete, $\sum_{i=1}^{\infty} a_i$ converges in X. Define $(s_n)_{n\in\mathbb{N}} \subset X$ and $s \in X$ by $s_n = \sum_{i=1}^{n} a_i$ and $s = \sum_{i=1}^{\infty} a_i$. Since $\lim_{n\to\infty} s_n = s$, and $\pi : X \to X/M$ is continuous, it follows that $\lim_{n\to\infty} \pi(s_n) = \pi(s)$. Since

$$\begin{aligned}\pi(s_n) &= \sum_{i=1}^{n} a_i + M \\ &= \sum_{i=1}^{n} x_i + M\end{aligned}$$

We have that $\sum_{i=1}^{\infty} x_i + M$ converges which implies that X/M is complete.

□

Exercise 7.1.17. *Let X, Y be normed vector spaces and $T \in L(X,Y)$. Then*

(1) $\ker T$ *is closed*
(2) *there exists a unique map $S : X/\ker T \to T(X)$ such that $T = S \circ \pi$. Furthermore S is a bounded linear bijection and $\|S\| = \|T\|$.*

Proof. (1) Since T is continuous and $\ker T = T^{-1}(\{0\})$, we have that $\ker T$ is closed.

(2) Suppose that there exists $S_1, S_2 \in L(X/\ker T, T(X))$ such that $T = S_1 \circ \pi$ and $T = S_2 \circ \pi$. Let $x \in X$. Then

$$S_1(x + \ker T) = S_1(\pi(x)) = T(x) = S_2(\pi(x)) = S_2(x + \ker T)$$

So $S_1 = S_2$. Therefore such a map is unique.
Define $S : X/\ker T \to T(X)$ by $S(x + \ker T) = T(x)$. Then S is clearly a linear bijection that satisfies $T = S \circ \pi$. Let $x \in X$ and $z \in \ker T$. Then

$$\begin{aligned}\|S(x + \ker T)\| &= \|T(x)\| \\ &= \|T(x+z)\| \\ &\leq \|T\|\|x+z\|\end{aligned}$$

Thus

$$\|S(x + \ker T)\| \leq \|T\| \inf_{z \in \ker T} \|x+z\| = \|T\|\|x + \ker T\|$$

So S is bounded and $\|S\| \leq \|T\|$. This implies that

$$\|T\| = \|S \circ \pi\| \leq \|S\|\|\pi\| = \|S\|$$

Thus $\|S\| = \|T\|$.

□

Exercise 7.1.18. *Let X, Y be normed vector spaces. Define $\phi : L(X,Y) \times X \to Y$ by $\phi(T,x) = Tx$. Then ϕ is continuous.*

Proof. Let $(T_1, x_1) \in L(X,Y) \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|x_1\| + \|T_1\| + 1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(t_2, x_2) \in L(X,Y) \times X$. Suppose that

$$\|(T_1, x_1) - (T_2, x_2)\| = \max\{\|T_1 - T_2\|, \|x_1 - x_2\|\} < \delta$$

. Then

$$\begin{aligned}
\|\phi(T_1, x_1) - \phi(T_2 - x_2)\| &= \|T_1 x_- T_2 x_2\| \\
&= \|T_1 x_1 - T_2 x_1 + T_2 x_1 - T_2 x_2\| \\
&\leq \|(T_1 - T_2)x_1\| + \|T_2(x_1 - x_2)\| \\
&\leq \|T_1 - T_2\|\|x_1\| + \|T_2\|\|x_1 - x_2\| \\
&\leq \|T_1 - T_2\|\|x_1\| + \big(\|T_1 - T_2\| + \|T_1\|\big)\|x_1 - x_2\| \\
&< \delta\|x_1\| + (\delta + \|T_1\|)\delta \\
&= \delta(\|T_1\| + \|x_1\|) + \delta^2 \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So ϕ is continuous. □

Exercise 7.1.19. *Let X be a normed vector space and $M \subset X$ a subspace. Then $\overline{M}$ is a subspace.*

Proof. Let $x, y \in \overline{M}$ and $\alpha \in \mathbb{C}$. Then there exist sequences $(x_n)_{n\in\mathbb{N}} \subset M$ and $(y_n)_{n\in\mathbb{N}} \subset M$ such that $x_n \to x$ and $y_n \to y$. Since M is a subspace, $(x_n+y_n)_{n\in\mathbb{N}} \subset M$ and $(\alpha x_n)_{n\in\mathbb{N}} \subset M$. Since addition and scalar multiplication are continuous, we have that $x_n + y_n \to x + y$ and $\alpha x_n \to \alpha x$. Thus $x + y \in \overline{M}$ and $\alpha x \in \overline{M}$ and hence $\overline{M}$ is a subspace. □

Exercise 7.1.20. *Let X, Y, Z be normed vector spaces, $T \in L(X, Y)$ and $S \in L(Y, Z)$. Define $ST : X \to Z$ by $STx = S(Tx)$. Then $ST \in L(X, Z)$ and $\|ST\| \leq \|S\|\|T\|$.*

Proof. Clearly ST is linear. Let $x \in X$. Then

$$\begin{aligned}
\|STx\| &= \|S(Tx)\| \\
&\leq \|S\|\|Tx\| \\
&\leq \|S\|\|T\|\|x\|
\end{aligned}$$

So $\|ST\| \leq \|S\|\|T\|$. □

Definition 7.1.21. *Let X be a Banach space and an associative algebra. Then X is said to be a Banach algebra if for each $S, T \in X$, $\|ST\| \leq \|S\|\|T\|$. If there exists $I \in X$ such that $I \neq 0$ and for each $T \in X$, $IT = TI = T$, then X is said to be **unital** with identity I. An element $T \in X$ is said to be **invertible** if there exists $S \in X$ such that $TS = ST = I$.*

Exercise 7.1.22. *Let X be a unital Banach algebra. Then $\|I\| \leq 1$.*

Proof. Since $I \neq 0$, $\|I\| \neq 0$. By definition,

$$\|I\| = \|II\| \leq \|I\|\|I\|$$

Hence $1 \leq \|I\|$. □

Note 7.1.23. *If X is a Banach space, then a previous exercise implies that $L(X, X)$ equipped with composition is a unital Banach algebra where I is the identity operator. It is easy to see that $\|I\| = 1$.*

Note 7.1.24. *Let X be a Banach algebra. Then the set of invertible elements in X is a group.*

Exercise 7.1.25. *Let X be a Banach algebra. Then mulitplication is continuous.*

Proof. Let $(S_1, T_1) \in X \times X$ and $\epsilon > 0$. Choose $\delta = \min\{\frac{\epsilon}{2(\|S_1\|+\|T_1\|+1)}, \frac{\sqrt{\epsilon}}{\sqrt{2}}\}$. Let $(S_2, T_2) \in X \times X$. Suppose that

$$\|(S_1, T_1) = (S_2, T_2)\| = \max\{\|S_2 - S_2\|, \|T_1 - T_2\|\} < \delta$$

. Then

$$\begin{aligned}\|S_1T_1 - S_2T_2\| &= \|S_1T_1 - S_2T_1 + S_2T_1 - S_2T_2\| \\ &\leq \|S_1 - S_2\|\|T_1\| + \|S_2\|\|T_1 - T_2\| \\ &\leq \|S_1 - S_2\|\|T_1\| + \big(\|S_1 - S_2\| + \|S_1\|\big)\|T_1 - T_2\| \\ &\leq \delta\|T_1\| + (\delta + \|S_1\|)\delta \\ &= \delta(\|S_1\| + \|T_1\|) + \delta^2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

□

Definition 7.1.26. *Let X, Y be a normed vector spaces and $T \in L(X, Y)$. Then T is said to be* ***invertible*** *or an* ***isomorphism*** *if T is a bijection and $T^{-1} \in L(Y, X)$.*

Definition 7.1.27. *Let X be a Banach space. Define $GL(X) := \{T \in L(X, X) : T$ is invertible$\}$.*

Exercise 7.1.28. *Let X be a Banach space. Then*

(1) *For each $T \in L(X, X)$, if $\|I - T\| < 1$, then T is invertible and*

$$T^{-1} = \sum_{n=0}^{\infty}(I - T)^n$$

(2) *For each $S, T \in L(X, X)$, if S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$, then T is invertible.*

(3) *$GL(X)$ is open.*

Proof. (1) Let $T \in L(X, X)$. Suppose that $\|I - T\| < 1$. Then

$$\sum_{n=0}^{\infty} \|(I - T)^n\| \leq \sum_{n=0}^{\infty} \|I - T\|^n < \infty$$

. Since X is a complete, so is $L(X, X)$ and thus $\sum\limits_{n=0}^{\infty}(I - T)^n$ converges in $L(X, X)$.

Define $(S_k)_{k=0}^{\infty} \subset L(X, X)$ and $S \in L(X, X)$ by $S_k = \sum\limits_{n=0}^{k}(I - T)^n$ and $S = \sum\limits_{n=0}^{\infty}(I - T)^n$. Then for each $k \in \mathbb{N}$,

$$\begin{aligned}S_kT &= S_k - S_k(I - T) \\ &= (I - T)^0 - (I - T)^{k+1} \\ &= I - (I - T)^{k+1}\end{aligned}$$

and $\|S_kT - I\| \leq \|I - T\|^{k+1}$. Since multiplication on Banach algebras is continuous, we have that
$$ST = (\lim_{k\to\infty} S_k)T = \lim_{k\to\infty} S_kT = I$$
Similarly $TS = I$. Thus T is invertible and $T^{-1} = S \in L(X, X)$.

(2) Let $S, T \in L(X, X)$. Suppose that S is invertible and $\|S - T\| < \|S^{-1}\|^{-1}$. Then
$$\begin{aligned}\|I - S^{-1}T\| &= \|S^{-1}(S - T)\| \\ &\leq \|S^{-1}\|\|S - T\| \\ &< 1\end{aligned}$$
So $S^{-1}T$ is invertible. Thus $T = S(S^{-1}T)$ is invertible.

(3) Let $T \in GL(X)$. Choose $\delta = \|T^{-1}\|^{-1}$. By (2), $B(T, \delta) \subset GL(X)$.

□

Exercise 7.1.29. *Let $M(X, \mathcal{A})$ denote the set of complex measures on the measurable space $(X, \mathcal{A})$. Define $\|\cdot\| : M(X, \mathcal{A}) \to [0, \infty)$ by $\|\mu\| = |\mu|(X)$. Then $\|\cdot\|$ is a norm on $M(X, \mathcal{A})$.*

Proof. Let $\mu, \nu \in M(X, \mathcal{A})$ and $\alpha \in \mathbb{C}$. Exercises in a previous section tell us that $|\mu + \nu| \leq |\mu| + |\nu|$ and $|\alpha\mu| = |\alpha||\mu|$. So clearly $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ and $\|c\mu\| = |c|\|\mu\|$. If $\|\mu\| = 0$, then X is $\mu - null$ and μ is the zero measure. □

7.2. **Linear Functionals.**

Definition 7.2.1. *Let X be a normed vector space and $T : X \to \mathbb{C}$. Then T is said to be a **linear functional on** X if T is linear and T is said to be a **bounded linear functional on** X if $T \in L(X, \mathbb{C})$. We define the **dual space of** X, denoted X^*, by $X^* = L(X, \mathbb{C})$.*

Definition 7.2.2. *Let X be a normed vector space and $p : X \to \mathbb{R}$. Then p is said to be a **sublinear functional** if for each $x, y \in X$, $\lambda \geq 0$,*

(1) $p(x + y) \leq p(x) + p(y)$
(2) $p(\lambda x) = \lambda p(x)$

Note 7.2.3. *Let X be a vector space and $\|\cdot\| : X \to [0, \infty)$ be a seminorm, then $\|\cdot\|$ is a sublinear functional.*

Theorem 7.2.4. *Hahn-Banach Theorem: Let X be a vector space, $p : X \to \mathbb{R}$ a sublinear functional, $M \subset X$ a subspace and $f : M \to C$ a linear functional. If for each $x \in M$, $|f(x)| \leq p(x)$, then there exists a linear functional $F : X \to \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x)$ and $F|_M = f$.*

Exercise 7.2.5. *Let X be a normed vector space, $M \subset X$ a subspace and $f \in M^*$. Then there exists $F \in X^*$ such that $\|F\| = \|f\|$ and $F|_M = f$.*

Proof. If $f = 0$, Choose $F = 0$. Suppose $f \neq 0$. Then $\|f\| \neq 0$ and there exists $x_0 \in M$ such that $x_0 \neq 0$. Thus $\|f\| = \sup\{|f(x)| : x \in M \text{ and } \|x\| = 1\}$. Define $p : X \to [0, \infty)$ by $p(x) = \|f\|\|x\|$. Then p is a sublinear functional on X and for each $x \in M$, $|f(x)| \leq p(x)$. So there exists a linear functional $F : X \to \mathbb{C}$ such that for each $x \in X$, $|F(x)| \leq p(x) = \|f\|\|x\|$ and $F|_M = f$. Thus $F \in X^*$ with $\|F\| \leq \|f\|$. Also

$$\|F\| = \sup_{\substack{x \in X \\ \|x\|=1}} |F(x)| \geq \sup_{\substack{x \in M \\ \|x\|=1}} |F(x)| = \sup_{\substack{x \in M \\ \|x\|=1}} |f(x)| = \|f\|$$

So $\|F\| = \|f\|$. □

Exercise 7.2.6. *Let X be a normed vector space, $M \subsetneq X$ a proper closed subspace and $x \in X \setminus M$. Then there exists $F \in X^*$ such that $F|_M = 0$, $\|F\| = 1$ and $F(x) = \|x+M\| \neq 0$. (Hint: Consider $f : M + \mathbb{C}x \to \mathbb{C}$ defined by $f(m + \lambda x) = \lambda\|x + M\|$.)*

Proof. Define $f : M + \mathbb{C}x \to \mathbb{C}$ as above. Clearly f is linear and $f|M = 0$. Let $m \in M$ and $\lambda \in \mathbb{C}$. If $\lambda = 0$, then $|f(m + \lambda x)| = 0 \leq \|m + \lambda x\|$. Suppose that $\lambda \neq 0$. Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \inf_{z \in M} \|z + \lambda x\| \\ &\leq \|m + \lambda x\| \end{aligned}$$

So $f \in (M + \mathbb{C}x)^*$ and $\|f\| \leq 1$. Let $\epsilon > 0$. A previous exercise tells us that there exist $m \in M, \lambda \in \mathbb{C}$ such that $\|m + \lambda x\| = 1$ and $\|m + \lambda x + M\| > 1 - \epsilon$. Then

$$\begin{aligned} |f(m + \lambda x)| &= |\lambda|\|x + M\| \\ &= \|\lambda x + M\| \\ &= \|m + \lambda x + M\| \\ &> 1 - \epsilon \end{aligned}$$

So

$$\|f\| = \sup_{\substack{z \in M + \mathbb{C}x \\ \|z\|=1}} |f(z)| \geq 1$$

Hence $\|f\| = 1$. The same exercise also tells us that $f(x) = \|x+M\| \neq 0$. Using the previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{M+\mathbb{C}x} = f$. □

Exercise 7.2.7. *Let X be a normed vector space and $x \in X$. If $x \neq 0$, then there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$.*

Proof. Define $f : \mathbb{C}x \to \mathbb{C}$ by $f(\lambda x) = \lambda\|x\|$. Then f is linear and $f(x) = \|x\|$. Clearly

$$\sup_{\substack{z \in \mathbb{C}x \\ \|z\|=1}} |f(z)| = 1$$

So $f \in (\mathbb{C}x)^*$ and $\|f\| = 1$. By a previous exercise, there exists $F \in X^*$ such that $\|F\| = \|f\| = 1$ and $F|_{\mathbb{C}x} = f$. □

Exercise 7.2.8. *Let X be a normed vector space. Then X^* separates the points of X.*

Proof. Let $x, y \in X$. Suppose that $x \neq y$. Then $x - y \neq 0$. The previous exercies implies that there exists $F \in X^*$ such that $\|F\| = 1$ and

$$F(x) - F(y) = F(x - y) = \|x - y\| \neq 0$$

Thus $F(x) \neq F(y)$ and X^* separates the points of X. □

Definition 7.2.9. *Let X, Y be metric spaces and $T : X \to Y$. Then T is said to be an* ***isometry*** *if for each $x_1, x_2 \in X$, $d(Tx_1, Tx_2) = d(x_1, x_2)$.*

Exercise 7.2.10. *Let X, Y be metric spaces and $T : X \to Y$ and isometry. Then T is injective.*

Proof. Let $x_1, x_2 \in X$. Suppose that $Tx_1 = Tx_2$. Then $0 = d(Tx_1, Tx_2) = d(x_1, x_2)$. So $x_1 = x_2$. Hence T is injective. $\square$

Note 7.2.11. *Let X, Y be metric spaces and $T : X \to Y$ an isometry. Then T is clearly continuous. If T is surjective, then T^{-1} is an isometry and therefore continuous. Hence T is a homeomorphism.*

Exercise 7.2.12. *Let X be a normed vector space and $x \in X$. Define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then $\hat{x} \in X^{**}$ and $\|\hat{x}\| = \|x\|$.*

Proof. Let $f, g \in X^*$ and $\lambda \in \mathbb{C}$. Then

$$\hat{x}(f + \lambda g) = (f + \lambda g)(x) = f(x) + \lambda g(x) = \hat{x}(f) + \lambda \hat{x}(g)$$

So $\hat{x}$ is linear. For each $f \in X^*$,

$$|\hat{x}(f)| = |f(x)| \leq \|x\|\|f\|$$

Hence $\hat{x} \in X^{**}$ with $\|\hat{x}\| \leq \|x\|$. If $x = 0$, then $\hat{x} = 0$ and $\|\hat{x}\| = \|x\|$. Suppose that $x \neq 0$. Then a previous exercise implies that there exists $F \in X^*$ such that $\|F\| = 1$ and $F(x) = \|x\|$. Then we have that

$$\sup_{\substack{f \in X^* \\ \|f\|=1}} |\hat{x}(f)| = \sup_{\substack{f \in X^* \\ \|f\|=1}} |f(x)| \geq |F(x)| = \|x\|$$

Hence $\|\hat{x}\| = \|x\|$. $\square$

Exercise 7.2.13. *Let X be a normed vector space. Define $\phi : X \to X^{**}$ by $\phi(x) = \hat{x}$. Then ϕ is a linear isometry.*

Proof. Let $x, y \in X$ and $\lambda \in \mathbb{C}$. Then for each $f \in X^*$, we have that

$$\begin{aligned}
\phi(x + \lambda y)(f) &= \widehat{x + \lambda y}(f) \\
&= f(x + \lambda y) \\
&= f(x) + \lambda f(y) \\
&= \hat{x}(f) + \lambda \hat{y}(f) \\
&= \phi(x)(f) + \lambda \phi(y)(f)
\end{aligned}$$

So $\phi(x + \lambda y) = \phi(x) + \lambda \phi(y)$ and ϕ is linear. The previous exercise tells us that

$$\begin{aligned}
\|\phi(x) - \phi(y)\| &= \|\phi(x - y)\| \\
&= \|\widehat{x - y}\| = \|x - y\|
\end{aligned}$$

So ϕ is an isometry. $\square$

Definition 7.2.14. *Let X be a normed vector space and define $\phi : X \to X^{**}$ as above. We define $\widehat{X} = \phi(X) \subset X^{**}$. Since $\widehat{X}$ and X are isomorphic, we may identify X as a subset of X^{**}.*

Definition 7.2.15. *Let X be a normed vector space and define $\phi : X \to X^{**}$ as above. Then X is said to be reflexive if ϕ is surjective. In this case ϕ is then an isomorphism*

Exercise 7.2.16. *Let X be a normed vector space and $f : X \to \mathbb{C}$ a linear functional on X. Then f is bounded iff $\ker f$ is closed.*

Proof. Suppose that f is continuous. Since $\{0\}$ is closed, we have that $\ker f = f^{-1}(\{0\})$ is closed. Conversely, suppose that $\ker f$ is closed. If $\ker f = X$, then $f = 0$ and f is continuous. Suppose that $\ker f \neq X$. Then $\ker f$ is a proper, closed subspace of X. A previous exercise tells us that there exists $x \in X$ such that $\|x\| = 1$ and $\|x + \ker f\| > \frac{1}{2}$. Let $y \in X$. Suppose that $\|y\| < \frac{1}{2}$. Then for each $z \in \ker f$,

$$\begin{aligned}\|z - (x + y)\| &= \|(z - x) - y\| \\ &\geq \|z - x\| - \|y\| \\ &> \frac{1}{2} - \frac{1}{2} \\ &= 0\end{aligned}$$

So $x + y \notin \ker f$. Therefore $f(B(x, \frac{1}{2})) \cap \{0\} = \varnothing$. If $f(B(x, \frac{1}{2}))$ is unbounded, then $f(B(x, \frac{1}{2})) = \mathbb{C}$ by linearity. This is a contradiction since $0 \notin f(B(x, \frac{1}{2}))$. So There exists $s > 0$ such that $f(B(x, \frac{1}{2})) \subset B(0, s)$ and thus f is bounded. □

Exercise 7.2.17. *Let X be a normed vector space.*

(1) *Let $M \subsetneq X$ be a proper closed subspace of X and $x \in X \setminus M$. Then $M + \mathbb{C}x$ is closed.*
(2) *Let $M \subset X$ be a finite dimensional subspace of X. Then M is closed.*

Proof. (1) Let $y \in X$ and $(y_n)_{n\in\mathbb{N}} \subset M + \mathbb{C}x$. Suppose that $y_n \to y$. If $y \in M$, then $y \in M + \mathbb{C}x$. Suppose that $y \notin M$. For each $n \in \mathbb{N}$, there exists $m_n \in M$ and $\lambda_n \in \mathbb{C}$ such that $y_n = m_n + \lambda_n x$. A previous exercise tells us that there exists $F \in X^*$ such that $\|F\| = 1$, $F|_M = 0$ and $F(x) = \|x + M\| \neq 0$. Since F is continuous, $F(y_n) \to F(y)$. Since for each $n \in \mathbb{N}$,

$$F(y_n) = F(m_n + \lambda_n x) = F(m_n) + \lambda_n(F_x) = \lambda_n F(x)$$

we have that $\lambda_n F(x) \to F(y)$. Since $F(x) \neq 0$, this implies that $\lambda_n \to F(x)^{-1}F(y)$. It follows that $\lambda_n x \to F(x)^{-1}F(y)x$. Since for each $n \in \mathbb{N}$, $m_n = y_n - \lambda_n x$, we know that $m_n \to y - F(x)^{-1}F(y)x$. Since $(m_n)_{n\in\mathbb{N}} \subset M$ and M is closed, we have that $y - F(x)^{-1}F(y)x \in M$ and therefore $y \in M + \mathbb{C}x$. Hence $M + \mathbb{C}x$ is closed.

(2) If $M = X$, then M is closed. Suppose that $M \neq X$. Let $(x_i)_{i=1}^n$ be a basis for M. Define $N_0 = \{0\}$ and for each $i = 1, 2, \cdots, n$, define $N_i = N_{i-1} + \mathbb{C}x_i$. Since N_0 is a proper closed subpace of X and $x_1 \in X \setminus N_0$, (1) implies that N_1 is closed. Proceed inductively to obtain that $M = N_n$ is closed.

□

Exercise 7.2.18. *Let X be an infinite-dimensional normed vector space.*

(1) *There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that for each $m, n \in \mathbb{N}$, $\|x_n\| = 1$ and if $m \neq n$, then $\|x_m - x_n\| > \frac{1}{2}$.*
(2) *X is not locally compact.*

Proof. (1) Define $N_0 = \{0\}$. Then N_0 is a closed proper subspace of X. Choose $x_1 \in X$ such that $\|x_1\| = 1$. Using the results of previous exercises, we proceed inductively. For each $n \geq 2$ we define $N_{n-1} = \text{span}(x_1, x_2, \cdots, x_{n-1})$. Then N_{n-1} is a closed proper subspace of X. Thus we may choose $x_n \in X$ such that $\|x_n\| = 1$ and $\|x_n + N_{n-1}\| > \frac{1}{2}$. Let $m, n \in \mathbb{N}$. Suppose that $m < n$. Then $x_m \in N_{n-1}$. Thus $\|x_n - x_m\| \geq \|x_n + N_{n-1}\| > \frac{1}{2}$

(2) Suppose that X is locally compact. Then $\overline{B(0,1)}$ is compact and therefore sequentially compact. Using $(x_n)_{n\in\mathbb{N}} \subset \overline{B(0,1)}$ defined in (1), we see that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, $x \in \overline{B(0,1)}$ such that $x_{n_k} \to x$. Then $(x_{n_k})_{k\in\mathbb{N}}$ is Cauchy. So there exists $N \in N$ such that for each $j, k \in \mathbb{N}$, if $j, k \geq N$, then $\|x_{n_j} - x_{n_k}\| < \frac{1}{2}$. Then $\|x_{n_N} - x_{n_{N+1}}\| < \frac{1}{2}$. This is a contradiction since by construction, $\|x_{n_N} - x_{n_{N+1}}\| > \frac{1}{2}$. Thus X is not locally compact.

□

Exercise 7.2.19. *Let X, Y be normed vector spaces and $T \in L(X, Y)$.*

(1) *Define the **adjoint of** T, denoted $T^* : Y^* \to X^*$ by $T^*(f) = f \circ T$. Then $T^* \in L(Y^*, X^*)$.*
(2) *Applying the result from (1) twice, we have that $T^{**} \in L(X^{**}, Y^{**})$. We have that for each $x \in X$, $T^{**}(\hat{x}) = \widehat{T(x)}$.*
(3) *T^* is injective iff $T(X)$ is dense in Y.*
(4) *If $T^*(Y^*)$ is dense in X^*, then T is injective. The converse is true if X is reflexive.*

Proof. (1) Let $f \in Y^*$. Then $\|T^*(f)\| = \|f \circ T\| \leq \|T\|\|f\|$. So $T^* \in L(Y^*, X^*)$ with $\|T^*\| \leq \|T\|$.

(2) Let $x \in X$. Let $f \in Y^*$. Then

$$\begin{aligned} T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= \widehat{T(x)}(f) \end{aligned}$$

Hence $T^{**}(\hat{x}) = \widehat{T(x)}$.

(3) Suppose that $T(X)$ is not dense in Y. Then $\overline{T(X)} \neq Y$. So $T(X)$ is a proper closed subspace of Y and there exists $y \in Y$ such that $y \notin \overline{T(X)}$. By a previous exercise, there exists $f \in Y^*$ such that $f(y) = \|y + \overline{T(X)}\| \neq 0$, $\|f\| = 1$ and $f|_{\overline{T(X)}} = 0$. Let $x \in X$. Then $T^*(f)(x) = f \circ T(x) = 0$. Hence $T^*(f) = 0 = T^*(0)$. Since $f \neq 0$, T^* is not injective.

Now suppose that $T(X)$ is dense in Y. Let $f, g \in Y^*$. Define $h \in Y^*$ by $h = f - g$ Suppose that $T*(f) = T^*(g)$ Then $T^*(h) = 0$. So for each $x \in X$, $h(T(x)) = 0$. Let $y \in Y$ and $\epsilon > 0$. By continuity, there exists $\delta > 0$ such that for each $y' \in Y$, if $\|y - y'\| < \delta$, then $\|h(y) - h(y')\| < \epsilon$. Since $T(X)$ is dense in Y, there exists $x \in X$ such that $\|y - T(x)\| < \delta$. Thus

$$\begin{aligned}\|h(y)\| &\leq \|h(y) - h(T(x))\| + \|h(T(x))\| \\ &= \|h(y) - h(T(x))\| \\ &< \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|h(y)\| = 0$. This implies that $h(y) = 0$ and therefore $f(y) = g(y)$. Since $y \in Y$ is arbitrary, $f = g$ and T^* is injective.

(4) For the sake of contradiction, suppose that $T^*(Y^*)$ is dense in X^* and T is not injective. Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $T(x_1) = T(x_2)$. Define $x = x_1 - x_2$. Then $x \neq 0$ and $T(x) = 0$. A previous exercise implies that there exists $F \in X^*$ such that $F(x) = \|x\| \neq 0$ and $\|F\| = 1$. Let $\epsilon > 0$. Choose $g \in Y^*$ such that $\|F - T^*(g)\| < \epsilon$. Then

$$\begin{aligned}\|x\| &= |F(x)| \\ &\leq |F(x) - T^*(g)(x)| + |T^*(g)(x)| \\ &< \epsilon\|x\| + |g(T(x))| \\ &= \epsilon\|x\|\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\|x\| = 0$ which is a contradiction. Hence if $T^*(Y^*)$ is dense in X^*, then T is injective.

Now, suppose that X is reflexive and T is injective. Let $\phi_1, \phi_2 \in X^{**}$. Suppose that $T^{**}(\phi_1) = T^{**}(\phi_2)$. Then $T^{**}(\phi_1 - \phi_2) = 0$. Since X is reflexive, there exist $x_1, x_2 \in X$ such that $\phi_1 = \hat{x_1}$ and $\phi_2 = \hat{x_2}$. Define $x = x_1 - x_2$. Then $T^{**}(\hat{x}) = 0$. So for each $f \in Y^*$,

$$\begin{aligned}T^{**}(\hat{x})(f) &= \hat{x} \circ T^*(f) \\ &= \hat{x}(T^*(f)) \\ &= \hat{x}(f \circ T) \\ &= f \circ T(x) \\ &= f(T(x)) \\ &= 0\end{aligned}$$

Suppose that $T(x) \neq 0$. Then a previous exercise implies that there exists $g \in Y^*$ such that $g(T(x)) = \|T(x)\| \neq 0$ and $\|g\| = 1$. This is a contradiction since $g(T(x)) = 0$. So $T(x) = 0$. Since T is injective, this implies that $x = 0$. Hence $\hat{x} = 0$ and thus $\phi_1 = \phi_2$. Thus T^{**} is injective. By (3), we have that $T^*(Y^*)$ is dense in X^*.

□

Exercise 7.2.20. *Let X be a normed vector space. Then X is reflexive iff X^* is reflexive.*

Proof. Suppose that X is reflexive. Let $\alpha \in X^{***}$. Define $f : X \to \mathbb{C}$ by $f(x) = \alpha(\hat{x})$. Clearly f is linear and a previous exercise tells us that for each $x \in X$,

$$\begin{aligned} |f(x)| &\leq \|\alpha\|\|\hat{x}\| \\ &= \|\alpha\|\|x\| \end{aligned}$$

So $f \in X^*$. Let $\phi \in X^{**}$. Since X is reflexive, there exists $x \in X$ such that $\phi = \hat{x}$. Then

$$\begin{aligned} \alpha(\phi) &= \alpha(\hat{x}) \\ &= f(x) \\ &= \hat{x}(f) \\ &= \hat{f}(\hat{x}) \\ &= \hat{f}(\phi) \end{aligned}$$

Hence $\alpha = \hat{f}$. Thus the map $X^* \to X^{***}$ given by $f \mapsto \hat{f}$ is surjective and so X^* is reflexive.

Conversely, suppose that X^* is reflexive. Since $\phi : X \to X^{**}$ given by $\phi(x) = \hat{x}$ is an isometry, $\widehat{X} \subset X^{**}$ is closed. For the sake of contradiction, suppose that $\widehat{X} \neq X^{**}$. Then there exists $\alpha \in X^{**}$ such that $\alpha \notin \widehat{X}$. Thus there exists $F \in X^{***}$ such that $\|F\| = 1$, $F(\alpha) = \|\alpha + \widehat{X}\| \neq 0$ and $F|_{\widehat{X}} = 0$. Since X^* is reflexive, there exists $f \in X^*$ such that $F = \hat{f}$. A previous exercise tells us that $\|f\| = \|\hat{f}\| = \|F\| = 1$. Since for each $x \in X$, $f(x) = \hat{x}(f) = \hat{f}(\hat{x}) = F(\hat{x}) = 0$, we have that $f = 0$. Thus $\|f\| = 0$, a contradiction. So $\widehat{X} = X^{**}$ and X is reflexive.

□

7.3. The Baire Category Theorem and Consequences.

Theorem 7.3.1. *Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is surjective, then T is open.*

Corollary 7.3.2. *Let X, Y be Banach spaces and $T \in L(X, Y)$. If T is a bijection, then $T^{-1} \in L(X, Y)$.*

Definition 7.3.3. *Let X, Y be sets and $f : X \to Y$. We define the* ***graph of f****, $\Gamma(f)$, by $\Gamma(f) = \{(x, y) \in X \times Y : f(x) = y\}$.*

Theorem 7.3.4. *Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If $\Gamma(T)$ is closed, then $T \in L(X, Y)$.*

Note 7.3.5. *We recall that $\Gamma(T)$ is closed iff for each $(x_n)_{n\in\mathbb{N}} \subset X$, $x \in X$ and $y \in Y$ if $x_n \to x$ and $T(x_n) \to y$, then $T(x) = y$.*

Theorem 7.3.6. *Let X, Y be Banach spaces and $S \subset L(X, Y)$. If for each $x \in X$,*

$$\sup_{T\in S} \|Tx\| < \infty$$

then

$$\sup_{T\in S} \|T\| < \infty$$

Exercise 7.3.7. *Let μ be counting measure on $(N, \mathcal{P}(\mathbb{N}))$. Define $h : \mathbb{N} \to \mathbb{N}$ and ν on $(N, \mathcal{P}(\mathbb{N}))$ by $h(n) = n$ and $d\nu = hd\mu$. Define $X = L^1(\nu)$ and $Y = L^1(\mu)$. Equip both X and Y with the L^1 norm with respect to μ.*

1. *We have that X is a proper subspace of Y and therefore X is not complete.*
2. *Define $T : X \to Y$ by $Tf(n) = nf(n)$. Then T is linear, $\Gamma(T)$ is closed, and T is unbounded.*
3. *Define $S : Y \to X$ by $Sg(n) = \frac{1}{n}g(n)$. Then $S \in L(Y, X)$, S is surjective and S is not open.*

Proof. (1) Note that for each $f : \| \to \mathbb{C}$,

$$\begin{aligned}\|f\|_{\mu,1} &= \sum_{n=1}^{\infty} |f(n)| \\ &\leq \sum_{n=1}^{\infty} n|f(n)| \\ &= \|f\|_{\nu,1}\end{aligned}$$

Hence X is a subspace of Y. Define $f : \| \to \mathbb{C}$ by $f(n) = \frac{1}{n^2}$. Then

$$\|f\|_{\mu,1} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

So $f \in Y$. However

$$\|f\|_{\nu,1} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

So $f \notin X$. Thus X is a proper subspace of Y. Let $g \in Y$ and $\epsilon > 0$. Since the simple functions are dense in $L^1(\mu)$, there exists $\phi \in L^1(\mu)$ such that ϕ is simple and $\|g - \phi\|_{\mu,1} < \epsilon$. Then there exist $(c_i)_{i=1}^k \subset \mathbb{C}$ and $(E_i)_{i=1}^k \subset \mathcal{P}(\mathbb{N})$ such that for each $i = 1, 2, \cdots, k$, E_i is finite and

$$\phi = \sum_{i=1}^{k} c_i \chi_{E_i}$$

Define $c = \max\{|c_i| : i = 1, 2, \cdots k\}$ and $m = \max \bigcup_{i=1}^k E_i$. Then

$$\begin{aligned}\|\phi\|_{\nu,1} &= \sum_{n=1}^{m} n|\phi(n)| \\ &\leq \sum_{n=1}^{m} mc \\ &= cm^2 \\ &< \infty\end{aligned}$$

Hence $\phi \in X$ and X is dense in Y. Since X is a dense, proper subspace, it is not closed. Since Y is complete and $X \subset Y$ is not closed, we have that X is not complete.

(2) Clearly T is linear. Let $(f_j)_{j\in\mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_j \xrightarrow{L^1(\mu)} f$ and $Tf_j \xrightarrow{L^1(\mu)} g$.

Note that for each $j \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$|f_j(n) - f(n)| \leq \sum_{n=1}^{\infty} |f_j(n) - f(n)| = \|f_j - f\|_{\mu,1}$$

and

$$|nf_j(n) - g(n)| \leq \sum_{n=1}^{\infty} |nf_j(n) - g(n)| = \|Tf_n - g\|_{\mu,1}$$

Thus for each $n \in \mathbb{N}$, $f_j(n) \xrightarrow{j} f(n)$ and $nf_j(n) \xrightarrow{j} g(n)$. This implies that for each $n \in \mathbb{N}$, $nf(n) = g(n)$. Thus $Tf = g$ which implies that $\Gamma(T)$ is closed. Suppose, for the sake of contradiction, that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\|_{\mu,1} \leq C\|f\|_{\mu,1}$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f : \| \to \mathbb{C}$ by $f = \chi_{\{n\}}$. As established above, $S^+ \subset L^1(\mu)$. Then $\|f\|_{\mu,1} = 1$ and

$$\begin{aligned}\|Tf\|_{\mu,1} &= n \\ &> C \\ &= C\|f\|_{\mu,1}\end{aligned}$$

which is a contradiction. So T is unbounded.

(3) Clearly S is linear. Let $g \in Y$. Then

$$\begin{aligned}\|Sg\|_{\mu,1} &= \sum_{n=1}^{\infty} \frac{1}{n}|g(n)| \\ &\leq \sum_{n=1}^{\infty} |g(n)| \\ &= \|g\|_{\mu,1}\end{aligned}$$

So S is bounded and $\|S\| \leq 1$. Thus $S \in L(Y, X)$. Let $f \in X$. Define $g : \| \to \mathbb{C}$ by $g(n) = nf(n)$. By defnition, $g \in Y$ and we have that

$$\begin{aligned}Sg(n) &= \frac{1}{n}g(n) \\ &= f(n)\end{aligned}$$

Hence $Sg = f$ and thus S is surjective. Let $g \in Y$. Suppsose that $Sg = 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n}|g(n)| = \|Sg\| = 0$$

Thus for each $n \in \mathbb{N}$, $g(n) = 0$. Hence $\ker g = \{0\}$ and g is injective. Note that $S^{-1} = T$. If g is open, then T is continuous which as shown above is a contradiction. So g is not open.

$\square$

Exercise 7.3.8. *Let $X = C^1([0,1])$ and $Y = C([0,1])$. Equip both X and Y with the uniform norm.*

(1) *Then X is not complete*

(2) *Define $T : X \to Y$ by $Tf = f'$. Then $\Gamma(T)$ is closed and T is not bounded.*

Proof. (1) Recall that for each $a, b \geq 0$ and $p \in \mathbb{N}$,

$$(a^{\frac{1}{p}} + b^{\frac{1}{p}})^p = \sum_{n=0}^{p} \binom{p}{n} a^{\frac{n}{p}} b^{\frac{p-n}{p}} \geq a + b$$

Thus $(a+b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$.

For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{C}$ by $f_n(x) = \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n^2}}$. Then $(f_n)_{n\in\mathbb{N}} \subset X$. Define $f : [0,1] \to \mathbb{C}$ by $f(x) = |x - \frac{1}{2}|$. Then $f \in Y \cap X^c$. Note that for each $n \in \mathbb{N}$, $f \leq f_n$. Our observation above implies that for each $x \in X$,

$$\begin{aligned} f_n(x) &= \left[(x - \frac{1}{2})^2 + \frac{1}{n^2}\right]^{\frac{1}{2}} \\ &\leq |x - \frac{1}{2}| + \frac{1}{n} \end{aligned}$$

Thus $0 \leq f_n - f \leq \frac{1}{n}$. This implies that $f_n \xrightarrow{\text{u}} f$. Since $f \notin X$, X is not complete.

(2) Let $(f_n)_{n\in\mathbb{N}} \subset X$, $f \in X$ and $g \in Y$. Suppose that $f_n \xrightarrow{\text{u}} f$ and $Tf_n \xrightarrow{\text{u}} g$. Let $x \in [0,1]$. Then $f_n(x) \to f(x)$ and $f_n(0) \to f(0)$ and $f_n' \xrightarrow{u} g$. Applying the DCT to this sequence of integrable functions that converges uniformly to an integrable function on a finite measure space (a previous exercise) we have that

$$\begin{aligned} f_n(x) - f_n(0) &= \int_{[0,x]} f_n' dm \\ &\to \int_{[0,x]} g dm \end{aligned}$$

Since $f_n(x) - f_n(0) \to f(x) - f(0)$, we know that

$$f(x) - f(0) = \int_{[0,x]} g dm$$

. Thus $Tf = g$ and $\Gamma(T)$ is closed.

Suppose for the sake of contradiction that T is bounded. Then there exists $C \geq 0$ such that for each $f \in X$, $\|Tf\| \leq C\|f\|$. Choose $n \in \mathbb{N}$ such that $n > C$. Define $f \in X$ by $f(x) = x^n$. Then $\|f\| = 1$ and

$$\begin{aligned} \|Tf\| &= \|f'\| \\ &= n \\ &> C \\ &= C\|f\| \end{aligned}$$

which is a contradiction. So T is not bounded.

$\square$

Exercise 7.3.9. *Let X, Y be Banach spaces and $T \in L(X,Y)$. Then $X/\ker T \cong T(X)$ iff $T(X)$ is closed.*

Proof. Since X is a banach space and T is continuous, we have that $\ker T$ is closed and $X/\ker T$ is a Banach space. Suppose that $X/\ker T \cong T(X)$. Then $T(X)$ is complete. Since Y is complete, this implies that $T(X)$ is closed.
Conversely Suppose that $T(X)$ is closed. Then $T(X)$ is complete. Define $S : X/\ker T \to T(X)$ by $S(x+\ker T) = T(x)$. A previous exercise tells us that the map $S : X/\ker T \to T(X)$ defined by $S(x + \ker T) = T(x)$ is a bounded linear bijection. Since $T(X)$ is complete and S is surjective, S^{-1} is bounded and thus S is an isomorphism. □

Exercise 7.3.10. *Let X be a separable Banach space. Define $B_X = \{x \in X : \|x\| < 1\}$. Let $(x_n)_{n\in\mathbb{N}} \subset B_X$ a dense subset of the unit ball and μ the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Define $T : L^1(\mu) \to X$ by*

$$Tf = \sum_{n=1}^{\infty} f(n)x_n$$

Then

(1) *T is well defined and $T \in L(L^1(\mu), X)$*
(2) *T is surjective*
(3) *There exists a closed subspace $K \subset L^1(\mu)$ such that $L^1(\mu)/K \cong X$*

Proof. (1) Let $f \in L^1(\mu)$. Since X is complete and

$$\begin{aligned}\sum_{n=1}^{\infty} \|f(n)x_n\| &= \sum_{n=1}^{\infty} |f(n)|\|x_n\| \\ &\le \sum_{n=1}^{\infty} |f(n)| \\ &< \infty\end{aligned}$$

we have that $\sum_{n=1}^{\infty} f(n)x_n$ converges and thus $Tf \in X$. Hence T is well defined.

Clearly T is linear. Let $f \in L^1(\mu)$. Then

$$\begin{aligned}\|Tf\| &= \|\sum_{n=1}^{\infty} f(n)x_n\| \\ &\le \sum_{n=1}^{\infty} \|f(n)x_n\| \\ &\le \sum_{n=1}^{\infty} |f(n)| \\ &= \|f\|_1\end{aligned}$$

So T is bounded with $\|T\| \le 1$.

(2) Let $x \in X$. Suppose that $\|x\| < 1$. Then $x \in B_X$. So there exists $n_1 \in \mathbb{N}$ such that $\|x - x_{n_1}\| < \frac{1}{2}$. Then $2(x - x_{n_1}) \in B_X$. Since for each $j \in \mathbb{N}$, $B_X \setminus (x_n)_{n=1}^{j}$ is dense in B_X, there exists $n_2 \in \mathbb{N}$ such that $x_{n_2} \notin (x_n)_{n=1}^{n_1}$ and $\|2(x - x_{n_1}) - x_{n_2}\| < \frac{1}{2}$ which implies that $\|x - (x_{n_1} - \frac{1}{2}x_{n_2})\| < \frac{1}{4}$.

Proceed inductively to obtain a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that for each $k \geq 2$, $x_{n_k} \notin (x_n)_{n=1}^{n_{k-1}}$ and $\|x - \sum_{j=1}^{k} 2^{1-j} x_{n_j}\| < \frac{1}{2^k}$. Then $x = \sum_{k=1}^{\infty} 2^{1-k} x_{n_k}$.

Define $f : \| \to \mathbb{C}$ by $f = \sum_{k=1}^{\infty} 2^{1-k} \chi_{\{n_k\}}$. Then $\|f\|_1 = \sum_{k=1}^{\infty} 2^{1-k} < \infty$, so $f \in L^1(\mu)$ and $Tf = \sum_{k=1}^{\infty} 2^{1-k} x_{n_k} = x$. Now, suppose that $\|x\| \geq 1$, then $\frac{1}{2\|x\|}x \in B_X$. The above argument shows that there exists $f \in L^1(\mu)$ such that $Tf = \frac{1}{2\|x\|}x$. Then $2\|x\|f \in L^1(\mu)$ and $T(2\|x\|f) = 2\|x\|Tf = x$.
So for each $x \in X$, there exists $f \in L^1(\mu)$ such that $Tf = x$ and thus T is surjective.

(3) Since X is a Banach space and T is surjective, the previous exercise implies that $L^1(\mu)/\ker T \cong X$.

□

Exercise 7.3.11. *Let X, Y be Banach spaces and $T : X \to Y$ a linear map. If for each $f \in Y^*$, $f \circ T \in X^*$, then $T \in L(X, Y)$.*

Proof. Suppose that for each $f \in Y^*$, $f \circ T \in X^*$. Let $x \in X$, □

7.4. **Hilbert Spaces.**

Definition 7.4.1. *Let H be a vector space and $\langle \cdot, \cdot \rangle : H \to \mathbb{C}$. Then $\langle \cdot, \cdot \rangle$ is said to be an* ***inner product*** *on H if for each $x, y, z \in H$ and $c \in \mathbb{C}$*

(1) $\langle x, y + cz \rangle = \langle x, y \rangle + c\langle x, z \rangle$
(2) $\langle x, y \rangle = \langle y, x \rangle^*$
(3) $\langle x, x \rangle \geq 0$
(4) *if $\langle x, x \rangle = 0$, then $x = 0$.*

8. Radon Measures

Theorem 8.0.1. *Let G be a locally compact group*

9. Haar Measure

9.1. Topological Groups.

Definition 9.1.1. *Let G be a group and $\mathcal{T}$ a topology on G. Then $(G, \mathcal{T})$ is said to be a **topological group** if the maps*

1. *$G \times G \to G$ given by $(x, y) \mapsto xy$*
2. *$G \to G$ given by $x \mapsto x^{-1}$*

are continuous.

Definition 9.1.2. *Let G be a group. Define $\iota : G \to G$ by $\iota(x) = x^{-1}$.*

Exercise 9.1.3. *Let G be a topological group. Then ι is a homeomorphism.*

Proof. By assumption ι is continuous. We know from basic group theory that ι is a bijection with $\iota^{-1} = \iota$. □

Definition 9.1.4. *Let G be a group and $S \subset G$, then S is said to be **symmetric** if $\iota(S) = S$, (i.e. $S^{-1} = S$).*

Definition 9.1.5. *Let G be a topological group and $\phi : G \to G$. Then ϕ is said to be an automorphism of G if ϕ is a homomorphism and a homeomorphism. We define $Aut(G) = \{\phi : G \to G : \phi \text{ is an automorphism}\}$*

Definition 9.1.6. *Let G be a group and $g \in G$. Define $l_g : G \to G$ and $r_g : G \to G$ by $l_g(x) = gx$ and $r_g(x) = xg^{-1}$.*

Exercise 9.1.7. *Let G be a topological group and $g \in G$. Then $l_g, r_g \in Aut(G)$.*

Proof. By assumption l_g and r_g are continuous. We know from basic group theory that l_g and r_g are bijections with $l_g^{-1} = l_{g^{-1}}$ and $r_g^{-1} = r_{g^{-1}}$ so l_g and r_g. are homeomorphisms. Let $g_1, g_2 \in G$. Then

$$l_{g_1} \circ l_{g_2}(x) = g_1 g_2 x = l_{g_1 g_2} x$$

and

$$r_{g_!} \circ r_{g_2} x = x g_2^{-1} g_1^{-1} = x(g_1 g_2)^{-1} = r_{g_1 g_2} x$$

So they are automorphisms. □

Exercise 9.1.8. *Let G be a topological group. Then for each $U \subset G$ and $g \in G$, if U is open, then gU, Ug and U^{-1} are open.*

Proof. Let $U \subset G$ and $g \in G$. Suppose that U is open. Since l_g, r_g and ι are homeomorphisms, $l_g(U) = gU$, $r_g(U) = Ug$ and $\iota(U) = U^{-1}$ are open. □

Definition 9.1.9. *Let G be a topological group and $y \in G$. Define $L_y, R_y : L^0 \to L^0$ by $L_y f(x) = f(y^{-1}x)$ and $R_y f(x) = f(xy)$, that is, $L_y f = f \circ l_y^{-1}$ and $R_y f = f \circ r_y^{-1}$.*

Exercise 9.1.10. *Let G be a topological group, $f \in L^0$ and $y, z \in G$. Then $L_y L_z = L_{yz}$ and $R_y R_z = R_{yz}$*

Proof. Let $x \in G$. Then

$$\begin{aligned}[L_y L_z]f(x) &= L_y[L_z f](x) \\ &= L_z f(y^{-1}x) \\ &= f(z^{-1}y^{-1}x) \\ &= f((yz)^{-1}x) \\ &= L_{yz}f(x)\end{aligned}$$

The case is similar for R_y and R_z. □

Exercise 9.1.11. *Let G be a topological group, $U \in \mathcal{B}(G)$ and $y \in G$. Then $L_y\chi_U = \chi_{yU}$ and $R_y\chi_U = \chi_{Uy^{-1}}$.*

Proof. Let $x \in G$. Then

$$\begin{aligned}L_y\chi_U(x) = 1 &\iff y^{-1}x \in U \\ &\iff x \in yU \\ &\iff \chi_{yU}(x) = 1\end{aligned}$$

The case is similar for R_y □

Exercise 9.1.12. *Let G be a topological group, $y \in G$ and $f \in L^0$. Then* $\operatorname{supp}(L_y f) = y\operatorname{supp}(f)$ *and* $\operatorname{supp}(R_y f) = \operatorname{supp}(f)y^{-1}$

Proof. Put $A = \{x \in G : L_y f(x) \neq 0\}$ and $B = \{x \in G : f(x) \neq 0\}$. Then

$$\begin{aligned}x \in A &\iff L_y f(x) \neq 0 \\ &\iff f(y^{-1}x) \neq 0 \\ &\iff y^{-1}x \in B \\ &\iff x \in yB\end{aligned}$$

Thus $A = yB$ which implies that $\overline{A} = y\overline{B}$. Therefore $\operatorname{supp}(L_y f) = y\operatorname{supp}(f)$. □

Exercise 9.1.13. *Let G be a topological group and $y \in G$. Then L_y, R_y are linear and if we restrict to the bounded measurable functions, then $L_y, R_y \in L(B(G))$ and $\|L_y\|, \|R_y\| = 1$.*

Proof. Let $f, g \in L^0(G)$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}L_y(\lambda f + g)(x) &= (\lambda f + g)(y^{-1}x) \\ &= \lambda f(y^{-1}x) + g(y^{-1}x) \\ &= \lambda L_y f(x) + L_y g(x)\end{aligned}$$

So L_y is linear. Next, we restrict to $B(G) \cap L^0$. We note that

$$\{|f(y^{-1}x)| : x \in y\operatorname{supp}(f)\} = \{|f(x)| : x \in \operatorname{supp}(f)\}$$

This implies that

$$\begin{aligned}\|L_y f\|_u &= \sup_{x\in\operatorname{supp}(L_y f)} |L_y f(x)| \\ &= \sup_{x\in y\operatorname{supp}(f)} |f(y^{-1}x)| \\ &= \sup_{x\in\operatorname{supp}(f)} |f(x)| \\ &= \|f\|_u\end{aligned}$$

So L_y is bounded. Hence $L_y \in L(L^0)$. The case is similar for R_y. $\square$

Definition 9.1.14. *Let G be a topological group. We say that G is a* ***locally compact group*** *if G is locally compact and Hausdorff.*

9.2. **Haar Measure.**

Definition 9.2.1. *Let G be a topological group and μ a Radon measure on G. Then μ is said to be a* ***left Haar measure on*** *G if*

(1) *μ is nonzero*
(2) *for each $U \in \mathcal{B}(G)$ and $g \in G$, $\mu(gU) = \mu(U)$.*

Similarly, μ is said to be a ***right Haar measure on*** *G if*

(1) *μ is nonzero*
(2) *for each $U \in \mathcal{B}(G)$ and $g \in G$, $\mu(Ug) = \mu(U)$.*

Exercise 9.2.2. *Let G be a topological group, μ a Radon measure on G. Then μ is a left Haar measure on G iff $\iota_*\mu$ is a right Haar measure on G.*

Proof. Suppose that μ is a left Haar measure on G. Let $U \in \mathcal{B}(G)$ and $g \in G$. Then

$$\begin{aligned}
\iota_*\mu(Ug) &= \mu(\iota^{-1}(Ug)) \\
&= \mu(g^{-1}U^{-1}) \\
&= \mu(U^{-1}) \\
&= \mu(\iota^{-1}(U)) \\
&= \iota_*\mu(U)
\end{aligned}$$

So $\iota_*\mu$ is a right Haar measure on G. The converse is similar. □

Exercise 9.2.3. *Let G be a topological group, and μ a left Haar measure on G. Then for each $g \in G$, ${r_g}_*\mu$ is a left Haar measure on G.*

Proof. Let $g \in G$ and $U \in \mathcal{B}(G)$. Observe that ${r_g}_*\mu(U) = \mu(Ug)$. So for each $h \in G$,

$$\begin{aligned}
{r_g}_*\mu(hU) &= \mu(hUg) \\
&= \mu(Ug) \\
&= {r_g}_*\mu(U)
\end{aligned}$$

□

Exercise 9.2.4. *Let G be a topological group, μ a left Haar measure on G and ν a right Haar measure on G. Then for each $f \in L^1 \cup L^+$ and $y \in G$,*

$$\int L_y f d\mu = \int f d\mu \tag{1}$$

$$\int R_y f d\nu = \int f d\nu \tag{2}$$

Proof.

(1) Let $y \in G$ and $E \in \mathcal{B}(G)$. Put $f = \chi_E$. Then

$$\begin{aligned}
\int L_y f d\mu &= \int L_y \chi_E d\mu \\
&= \int \chi_{yE} d\mu \\
&= \mu(yE) \\
&= \mu(E) \\
&= \int \chi_E d\mu \\
&= \int f d\mu
\end{aligned}$$

By linearity of L_y, for $f \in S^+$ we have that,

$$\int L_y f d\mu = \int f d\mu$$

For $f \in L^+$, choose $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$ $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$. Then for each $n \in \mathbb{N}$ $L_y\phi_n \leq L_y\phi_{n+1} \leq L_y f$ and $L_y\phi \to L_y f$. So MCT implies that

$$\begin{aligned}
\int L_y f d\mu &= \lim_{n\to\infty} \int L_y \phi_n d\mu \\
&= \lim_{n\to\infty} \int \phi_n d\mu \\
&= \int f d\mu
\end{aligned}$$

Let $f \in L^1$. If f is real valued, write $f = f^+ - f^-$. Then $L_y f = L_y f^+ - L_y f^-$ and

$$\begin{aligned}
\int L_y f d\mu &= \int L_y f^+ d\mu - \int L_y f^- d\mu \\
&= \int f^+ d\mu - \int f^- d\mu \\
&= \int f d\mu
\end{aligned}$$

If f is complex valued, write $f = g + ih$ with $g, h \in L^1$ real valued. Then

$$\begin{aligned}
\int L_y f d\mu &= \int L_y g d\mu + i \int L_y h d\mu \\
&= \int g d\mu + i \int h d\mu \\
&= \int f d\mu
\end{aligned}$$

(2) Similar

$\square$

Exercise 9.2.5. *Let G be a topological group and μ a left Haar measure on G. Then for each $U \subset G$, if U is open and $U \neq \varnothing$, then $\mu(U) > 0$*

Proof. Let $U \subset G$. Suppose that U is open and $U \neq \varnothing$. Suppose that $\mu(U) = 0$. Since μ is nonzero, inner regularity implies that there exists $K \subset G$ such that K is compact and $\mu(K) > 0$. Then $\{xU : x \in K\}$ is an open cover of K. Then there exist $x_1, \cdots, x_n \in K$ such that $K \subset \bigcap_{k=1}^{n} x_k U$. Then

$$\mu(K) \leq \sum_{k=1}^{n} \mu(x_k U) \tag{3}$$

$$= \sum_{k=1}^{n} \mu(U) \tag{4}$$

$$= 0 \tag{5}$$

This is a contradiction. So $\mu(U) > 0$. □

Exercise 9.2.6. *Let G be a locally compact group and μ a left Haar measure on G. Then there exists $S \in \mathcal{B}(G)$ such that S is symmetric, $e \in S$ and $\mu(E) > 0$*

Proof. Since G is locally compact, there exists a compact neighborhood K of e. Then $\mu(K) > 0$. Put $S = KK^{-1} \in \mathcal{B}(G)$. Then S is symmetric. Since $e \in K$, $K \subset S$ and $0 < \mu(K) \leq \mu(S)$. □

Exercise 9.2.7. *Let G be a locally compact group and μ a left Haar measure on G. Then*

1. $\mu(\{e\}) > 0$ *iff there exists $\lambda > 0$ such that $\mu = \lambda\#$.*
2. μ *is finite iff G is compact*

Proof.

1. If there exists $\lambda > 0$ such that $\mu = \lambda\#$, then $\mu(\{e\}) > 0$ Conversely, suppose that $\mu(\{e\}) > 0$. Define $\lambda = \mu(\{e\}) > 0$. Let $B \in \mathcal{B}(G)$. If B is finite, then
$$\begin{aligned}\mu(B) &= \sum_{x \in B} \mu(\{x\}) \\ &= \sum_{x \in B} \mu(x\{e\}) \\ &= \sum_{x \in B} \mu(\{e\}) \\ &= \sum_{x \in B} \lambda \\ &= \lambda\#(\{e\})\end{aligned}$$
If B is infinite, then we may choose a countable subset and the same reasoning as above tells us that
$$\mu(B) = \infty = \lambda\#(B)$$
2. If G is compact, then μ is finite since μ is Radon. Conversely, suppose that μ is finite. Then **FINISH**

□

Theorem 9.2.8. *Let G be a locally compact group. Then there exists a left Haar measure on G.*

Theorem 9.2.9. *Let G be a locally compact group and μ_1, μ_2 left Haar measures on G. Then there exists $\lambda > 0$ such that $\mu_1 = \lambda \mu_2$.*

Definition 9.2.10. *Let G be a locally compact group and μ a left Haar measure on G. A previous exercise tells us that for each $g \in G$, $r_{g*}\mu$ is a left Haar measure on G. The previous result tells us that for each $g \in G$ there exists $\lambda_g > 0$ such that $r_{g*}\mu = \lambda_g \mu$. Define $\Delta : G \to (0, \infty)$ by $\Delta(g) = \lambda_g$. We call Δ the* ***modular function of*** *G.*

Exercise 9.2.11. *Let G be a locally compact group and μ a left Haar measure on G. Then*

(1) *Δ is a homomorphism*
(2) *for each $f \in L^1 \cup L^+$,*

$$\int R_y f d\mu = \Delta(y^{-1}) \int f d\mu$$

Proof.

(1) Recall that for each $g \in G$, $\Delta(g)\mu(U) = r_{g*}\mu(U) = \mu(Ug)$. Let $g, h \in G$ and $U \in \mathcal{B}(G)$. Then $\Delta(gh)\mu(U) = \mu(Ugh) = \Delta(h)\mu(Ug) = \Delta(g)\Delta(h)\mu(U)$. So $\Delta(gh) = \Delta(g)\Delta(h)$.
(2) Let $y \in G$ and $U \in \mathcal{B}(G)$. Put $f = \chi_U$ Then

$$\begin{aligned}\int R_y f d\mu &= \int R_y \chi_U d\mu \\ &= \int \chi_{Uy^{-1}} d\mu \\ &= \mu(Uy^{-1}) \\ &= \Delta(y^{-1})\mu(U) \\ &= \Delta(y^{-1}) \int \chi_U d\mu \\ &= \Delta(y^{-1}) \int f d\mu\end{aligned}$$

By linearity of R_y, for $f \in S^+$,

$$\int R_y f d\mu = \Delta(y^{-1}) \int f d\mu$$

For $f \in L^+$, choose $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$ $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$. Then for each $n \in \mathbb{N}$ $R_y\phi_n \leq R_y\phi_{n+1} \leq R_y f$ and $R_y\phi \to R_y f$. So MCT implies that

$$\begin{aligned}\int R_y f d\mu &= \lim_{n\to\infty} \int R_y \phi_n d\mu \\ &= \lim_{n\to\infty} \Delta(y^{-1}) \int \phi_n d\mu \\ &= \Delta(y^{-1}) \int f d\mu\end{aligned}$$

Let $f \in L^1$. If f is real valued, write $f = f^+ - f^-$. Then $R_y f = R_y f^+ - R_y f^-$ and

$$\begin{aligned}\int R_y f d\mu &= \int R_y f^+ d\mu - \int R_y f^- d\mu \\ &= \Delta(y^{-1}) \int f^+ d\mu - \Delta(y^{-1}) \int f^- d\mu \\ &= \Delta(y^{-1}) \int f d\mu\end{aligned}$$

If f is complex valued, write $f = g + ih$ with $g, h \in L^1$ real valued. Then

$$\begin{aligned}\int R_y f d\mu &= \int R_y g d\mu + i \int R_y h d\mu \\ &= \Delta(y^{-1}) \int g d\mu + i\Delta(y^{-1}) \int h d\mu \\ &= \Delta(y^{-1}) \int f d\mu\end{aligned}$$

□

Definition 9.2.12. *Let G be a locally compact group. Then G is said to be* ***unimodular*** *if* $\ker \Delta = G$.

Exercise 9.2.13. *Let G be a locally compact group. Then the following are quivalent:*

1. *G is unimodular*
2. *there exists a left Haar measure μ on G such that μ is a right Haar measure on G.*
3. *for each nonzero Radon measure μ on G, μ is a left Haar measure on G iff μ is a right Haar measure on G.*

Proof.

(1) $\implies$ (2) Since G is a locally compact group, there exists a left Haar measure μ on G. Let $g \in G$ and $U \in \mathcal{B}(G)$. Then

$$\mu(Ug) = \Delta(g)\mu(U) = \mu(U)$$

Since G is unimodular, $\Delta(g) = 1$. Then μ is a right Haar measure on G.

(2) $\implies$ (3) By assumption, there exists a left Haar measure μ' on G such that μ' is a right Haar measure on G. Let μ be a nonzero Radon measure on G. If μ is a left Haar measure on G, then there exists $\lambda > 0$ such that $\mu = \lambda\mu'$ and therefore μ is a right Haar measure. The same reasoning implies that if μ is a right Haar measure on G, then μ is a left Haar measure on G.

(3) $\implies$ (1) Since G is locally compact, there exists a left $Haar$ measure μ on G. By assumption, μ is a right Haar measure on G. By inner regularity there exists $K \in \mathcal{B}(G)$ such that $\mu(K) > 0$. Let $g \in G$. Then

$$\Delta(g)\mu(K) = \mu(Kg) = \mu(K)$$

So $\Delta(g) = 1$.

□

Note 9.2.14. *If G is a locally compact abelian group, then G is unimodular.*

Exercise 9.2.15. *Let G be a locally compact group and μ a left Haar measure on G. If G is unimodular then $\iota_*\mu = \mu$.*

Proof. Suppose that G is unimodular. A previous exercise tells us that $\iota_*\mu$ is a right Haar measure on G. The unimodularity of G implies that $\iota_*\mu$ a left Haar measure on G. Then there exists $\lambda > 0$ such that $\iota_*\mu = \lambda\mu$. Since G is locally compact, there exists $S \in \mathcal{B}(G)$ such that S is symmetric and $\mu(S) > 0$. Then

$$\begin{aligned}\mu(S) &= \mu(S^{-1}) \\ &= \iota_*\mu(S) \\ &= \lambda\mu(S)\end{aligned}$$

So $\lambda = 1$ and $\iota_*\mu = \mu$.

it is also (Since G is locally compact, there exists $S \in \mathcal{B}(G)$ such that S is symmetric and $\mu(S) > 0$. Then

$$\mu(S) = \mu(S^{-1}) = \iota_*\mu(S)$$

Since $\iota_*\mu$ is a right Haar measure on G and G is unimodular, $\iota_*\mu(S)$ is also a left Haar measure on G. Then there exists $\lambda > 0$ such that $\mu(S) = \lambda\iota_*\mu(S)$. $\square$

9.3. **Generalization.**

Definition 9.3.1. *Let G be a locally compact group. For $\phi \in Aut(G)$, define $T_\phi : L^0 \to L^0$ by*

$$T_\phi f = f \circ \phi^{-1}$$

Exercise 9.3.2. *Let $\phi, \psi \in Aut(G)$. Then $T_{\phi \circ \psi} = T_\phi T_\psi$.*

Proof. Let $f \in L^0$. Then

$$\begin{aligned} T_{\phi\circ\psi} f &= f \circ (\phi \circ \psi)^{-1} \\ &= (f \circ \psi^{-1}) \circ \phi^{-1} \\ &= T_\phi(f \circ \psi^{-1}) \\ &= T_\phi T_\psi f \end{aligned}$$

□

Exercise 9.3.3. *Let G be a locally compact group and μ a left Haar measure on G. Then for each $\phi \in Aut(G)$, $\phi_*\mu$ is a left Haar measure on G.*

Proof. Let $\phi \in Aut(G)$, $g \in G$ and $E \in \mathcal{B}(G)$. Then

$$\begin{aligned} \phi_*\mu(gE) &= \mu(\phi^{-1}(gE)) \\ &= \mu(\phi^{-1}(g)\phi^{-1}(E)) \\ &= \mu(\phi^{-1}(E)) \\ &= \phi_*\mu(E) \end{aligned}$$

□

Definition 9.3.4. *Let G be a locally compact group and μ a left Haar measure on G. The previous exercise tells us that for each $\phi \in Aut(G)$, there exists $\lambda_\phi > 0$ such that $\phi_*\mu = \lambda_\phi \mu$. Define $\Delta : Aut(G) \to (0, \infty)$ by $\Delta(\phi) = \lambda_\phi$. Δ is called the **modular function of** G.*

Exercise 9.3.5. *Let G be a locally compact group and μ a left Haar measure on G. Then*

1. *Δ is a homomorphism*
2. *for each $f \in L^+ \cup L^1$,*

$$\int T_\phi f d\mu = \Delta(\phi)^{-1} \int f d\mu$$

Proof.

1. Let $\phi, \psi \in Aut(G)$. By inner regularity, there exists $E \in \mathcal{B}(G)$ such that $\mu(E) > 0$. Then

$$\begin{aligned} \Delta(\phi \circ \psi)\mu(E) &= (\phi \circ \psi)_*\mu(E) \\ &= \mu((\phi \circ \psi)^{-1}(E)) \\ &= \mu(\psi^{-1} \circ \phi^{-1}(E)) \\ &= \psi_*\mu(\phi^{-1}(E)) \\ &= \Delta(\psi)\mu(\phi^{-1}(E)) \\ &= \Delta(\psi)\phi_*\mu(E) \\ &= \Delta(\phi)\Delta(\psi)\mu(E) \end{aligned}$$

So $\Delta(\phi \circ \psi) = \Delta(\phi)\Delta(\psi)$.

(2) Let $\phi \in Aut(G)$ and $f \in L^+ \cup L^1$. From basic integration theory, we know that

$$\begin{aligned}\int T_\phi f d\mu &= \int f \circ \phi^{-1} d\mu \\ &= \int f d{\phi^{-1}}_* \mu \\ &= \Delta(\phi^{-1}) \int f d\mu \\ &= \Delta(\phi)^{-1} \int f d\mu\end{aligned}$$

$\square$

Note 9.3.6. *This generalizes the previous definition in which we used $\phi = r_g$. Choosing the subgroup $H = \{r_g : g \in G\}$ we have that G is unimodular if* $\ker \Delta|_H = H$.

Definition 9.3.7. *Let G be a topological group. For $g \in G$, define $c_g \in Aut(G)$ by $c_g(x) = gxg^{-1}$*

Exercise 9.3.8. *Let G be a locally compact group. Define the subgroup $H = \{c_G : g \in G\}$. Then G is unimodular iff* $\ker \Delta|_H = H$.

Proof. Choose a left Haar measure μ on G. Let $g \in G$ and $E \in \mathcal{B}(G)$. Then

$$\begin{aligned}\Delta(c_g)\mu(E) &= {c_g}_*\mu(E) \\ &= \mu(g^{-1}Eg) \\ &= \mu(Eg)\end{aligned}$$

If G is unimodular, then $\mu(Eg) = \mu(E)$ and $\Delta(c_g) = 1$. Conversely, if $\ker \Delta|_H = H$, then $\mu(E) = \mu(Eg)$ and G is unimodular. $\square$

9.4. **Fundamental Examples.**

Note 9.4.1. *The Haar measure on $(\mathbb{R}^n, +)$ is m.*

Exercise 9.4.2. *The Haar measure on $(\mathbb{R}^\times, \cdot)$ is $d\mu(x) = \frac{1}{|x|} dm(x)$*

Proof. Let $0 < a < b$ and $c > 0$. Then

$$\begin{aligned}
\mu(c(a,b)) &= \mu((ca,cb)) \\
&= \int_{(ca,cb)} \frac{1}{|x|} dm(x) \\
&= \int_{(ca,cb)} \frac{1}{x} dm(x) \\
&= \Big[\log|x|\Big]_{ca}^{cb} \\
&= \log(cb) - \log(ca) \\
&= \log b - \log a \\
&= \Big[\log|x|\Big]_{a}^{b} \\
&= \int_{(a,b)} \frac{1}{x} dm(x) \\
&= \mu((a,b))
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mu(-c(a,b)) &= \mu((-cb,-ca)) \\
&= \int_{(-cb,-ca)} \frac{1}{|x|} dm(x) \\
&= -\int_{(-cb,-ca)} \frac{1}{x} dm(x) \\
&= -\Big[\log|x|\Big]_{-cb}^{-ca} \\
&= \log(cb) - \log(ca) \\
&= \log b - \log a \\
&= \Big[\log|x|\Big]_{a}^{b} \\
&= \int_{(a,b)} \frac{1}{x} dm(x) \\
&= \mu((a,b))
\end{aligned}$$

□

Exercise 9.4.3. *Define $f : [0,1) \to \mathbb{T}$ by $f(x) = e^{i2\pi x}$. Let m be Lebesgue measure on $[0,1)$, then the Haar measure on $\mathbb{T}$ is f_*m.*

Proof. Note that f is a bijection and the topology on $\mathbb{T}$ is generated by sets of the form $f((a,b))$ where $a, b \in [0,1)$ and $a < b$. Let $a, b \in [0,1)$ and suppose that $a < b$. Put $A = f((a,b))$. Let $z \in \mathbb{T}$. Then there exists $\theta \in [0,1)$ such that $z = f(\theta)$. If $1 \notin zA$, then $f^{-1}(zA) = (\theta + a, \theta + b)$. If $1 \in zA$, then $f^{-1}(zA) = (\theta + a, 1) \cup [0, \theta + b - 1)$. Suppose that

$1 \notin zA$. Then

$$\begin{aligned} &= f_*m(zA) && = m(f^{-1}(zA)) \\ &= m((\theta + a, \theta + b)) \\ &= b - a \\ &= m((a, b)) \\ &= m(f^{-1}(A)) \\ &= f_*m(A) \end{aligned}$$

Similarly if $1 \in zA$, $f_*m(zA) = f_*m(A)$. □

Exercise 9.4.4. *Let p be a prime. Define $|\cdot|_p : \mathbb{Q} \to [0, \infty)]$ by*

$$\begin{cases} |\frac{a}{b}p^n|_p = p^{-n}, \ \textit{if} \ \gcd(a,p) = \gcd(b,p) = 1 \\ |0|_p = 0 \end{cases}$$

Then $|\cdot|_p$ is an absolute value on $\mathbb{Q}$. Define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to the metric induced by $|\cdot|_p$. Define $\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1\}$. It is well known that $\mathbb{Q}_p$ is a locally compact field and $\mathbb{Z}_p$ is compact. Define $P = \{0, 1, \cdots, p-1\}$. It is known that the topology is generated by

$$\{x + p^n\mathbb{Z}_p : \ \textit{for } n \in \mathbb{Z}, x \in \mathbb{Q}_p\}$$

Another useful fact is that

$$\mathbb{Q}_p = \{\sum_{j=-n}^{\infty} a_j p^j : a_j \in P, n \in \mathbb{N}_0\}$$

and

$$\mathbb{Z}_p = \{\sum_{j=0}^{\infty} a_j p^j : a_j \in P\}$$

Let μ be the Haar measure on $\mathbb{Q}_p$. Then μ is completely determined by the value $\mu(\mathbb{Z}_p)$

Proof. We observe that for $n \in \mathbb{Z}$, we may write $p^n\mathbb{Z}_p$ as the following disjoint union:

$$p^n\mathbb{Z}_p = \bigcup_{j \in P} jp^n + p^{n+1}\mathbb{Z}^p$$

Thus $\mu(p^n\mathbb{Z}^p) = p\mu(p^{n+1}\mathbb{Z}_p)$. If we set $\mu(\mathbb{Z}_p) = 1$, we obtain that $\mu(\mathbb{Z}_p) = p^n\mu(p^n\mathbb{Z}_p)$, which implies that

$$\mu(p^n\mathbb{Z}_p) = \frac{1}{p^n}\mu(\mathbb{Z}_p)$$

. □

Exercise 9.4.5. *Let ν be the Haar measure on $\mathbb{Q}_p$. Then the Haar measure on $\mathbb{Q}_p^\times$ is $d\mu = \frac{1}{|x|_p}d\nu$.*

Proof. Let $x, y \in P^\times$ and $\alpha = xp^{n-1} + p^n\mathbb{Z}_p$. Then

$$\alpha(yp^{k-1} + p^k\mathbb{Z}_p) = p^{(n-1)+(k-1)}(xy + p^{n+k}\mathbb{Z}_p)$$

□

10. Probability

10.1. Distributions.

Definition 10.1.1. *Let Ω be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then $\mathcal{P}$ is said to be a **π-system** on Ω if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.*

Definition 10.1.2. *Let Om be a set and $\mathcal{L} \subset \mathcal{P}(\Omega)$. Then $\mathcal{L}$ is said to be a **λ-system** on Ω if*
1. *$\mathcal{L} \neq \varnothing$*
2. *for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$*
3. *for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}$, if $(A_n)_{n\in\mathbb{N}}$ is disjoint, then $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{L}$*

Exercise 10.1.3. *Let Ω be a set and $\mathcal{L}$ a λ-system on Ω. Then*
1. $\Omega, \varnothing \in \mathcal{L}$

Proof. Straightforward. □

Definition 10.1.4. *Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. Put*
$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(\Omega) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$
*We define the **λ-system on Ω generated by** $\mathcal{C}$, $\lambda(\mathcal{C})$, to be*
$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L}\in\mathcal{S}} \mathcal{L}$$

Exercise 10.1.5. *Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. If $\mathcal{C}$ is a λ-system and $\mathcal{C}$ is a π-system, then $\mathcal{C}$ is a σ-algebra.*

Proof. Suppose that $\mathcal{C}$ is a λ-system and $\mathcal{C}$ is a π-system. Then we need only verify the third axiom in the definition of a σ-algebra. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{C}$. Define $B_1 = A_1$ and for $n \geq 2$, define $B_n = A_n \cap \left(\bigcup\limits_{k=1}^{n-1} A_k\right)^c = A_n \cap \left(\bigcap\limits_{k=1}^{n-1} A_k^c\right) \in \mathcal{C}$. Then $(B_n)_{n\in\mathbb{N}}$ is disjoint and therefore $\bigcup\limits_{n\in\mathbb{N}} A_n = \bigcup\limits_{n\in\mathbb{N}} B_n \in \mathcal{C}$. □

Theorem 10.1.6. *(Dynkin's Theorem)*
Let Ω be a set.
1. *Let $\mathcal{P}$ be a π-system on Ω and $\mathcal{L}$ a λ-system on Ω. If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.*
2. *Let $\mathcal{P}$ be a π-system on Ω. Then $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$*

Exercise 10.1.7. *Let $(\Omega, \mathcal{F})$ be a measurable space and μ, ν probability measures on $(\Omega, \mathcal{F})$. Put $\mathcal{L}_{\mu,\nu} = \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$. Then $\mathcal{L}_{\mu,\nu}$ is a λ-system on Ω.*

Proof.
1. $\varnothing \in \mathcal{L}_{\mu,\nu}$.
2. Let $A \in \mathcal{L}_{\mu,\nu}$. Then $\mu(A) = \nu(A)$. Thus
$$\begin{aligned}\mu(A^c) &= 1 - \mu(A)\\ &= 1 - \nu(A)\\ &= \nu(A^c)\end{aligned}$$
So $A^c \in \mathcal{L}_{\mu,\nu}$.

(3) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}_{\mu,\nu}$. So for each $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Then

$$\begin{aligned}\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) &= \sum_{n\in\mathbb{N}} \mu(A_n)\\ &= \sum_{n\in\mathbb{N}} \nu(A_n)\\ &= \nu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)\end{aligned}$$

Hence $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}_{\mu,\nu}$.

□

Exercise 10.1.8. *Let $(\Omega, \mathcal{F})$ be a measurable space, μ, ν probability measures on $(\Omega, \mathcal{F})$ and $\mathcal{P} \subset \mathcal{A}$ a π-system on Ω. Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.*

Proof. Using the previous exercise, we see that $\mathcal{P} \subset \mathcal{L}_{\mu,\nu}$. Dynkin's theorem implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu,\nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$. □

Definition 10.1.9. *Let $F : \mathbb{R} \to \mathbb{R}$. Then F is said to be a **probability distribution function** if*

1. *F is right continuous*
2. *F is increasing*
3. *$F(-\infty) = 0$ and $F(\infty) = 1$*

Definition 10.1.10. *Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We define $F_P : \mathbb{R} \to \mathbb{R}$, by*

$$F_P(x) = P((-\infty, x])$$

*We call F_P the **probability distribution function of** P.*

Exercise 10.1.11. *Let $(\Omega, \mathcal{F}, P)$ be a probability measure. Then F_P is a probability distribution function.*

Proof.

(1) Let $x \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}} \subset [x, \infty)$. Suppose that $x_n \to x$. Then $(x, x_n] \to \varnothing$ because $\limsup_{n\to\infty}(x, x_n] = \varnothing$. Thus

$$F(x_n) - F(x) = P((x, x_n]) \to P(\varnothing) = 0$$

This implies that

$$F(x_n) \to F(x)$$

. So F is right continuous.

(2) Clearly F_P is increasing.

(3) Continuity from below tells us that

$$F(-\infty) = \lim_{n\to-\infty} F(n) = \lim_{n\to-\infty} P((-\infty, n]) = 0$$

and continuity from above tell us that

$$F(\infty) = \lim_{n\to\infty} F(n) = \lim_{n\to\infty} P((-\infty, n]) = 1$$

□

Exercise 10.1.12. *Let μ, ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $F_\mu = F_\nu$ iff $\mu = \nu$.*

Proof. Clearly if $\mu = \nu$, then $F_\mu = F_\nu$. Conversely, suppose that $F_\mu = F_\nu$. Then for each $x \in \mathbb{R}$,

$$\begin{aligned} \mu((-\infty, x]) &= F_\mu(x) \\ &= F_\nu(x) \\ &= \nu((-\infty, x]) \end{aligned}$$

Put $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then $\mathcal{C}$ is a π-system and for each $A \in \mathcal{C}$, $\mu(A) = \nu(A)$. Hence for each $A \in \sigma(C) = \mathcal{B}(\mathbb{R})$, $\mu(A) = \nu(A)$. So $\mu = \nu$. □

Definition 10.1.13. *Let $(\Omega, \mathcal{F})$ be a measurable space and $X : \Omega \to \mathbb{R}$. Then X is said to be a **random variable** on $(\Omega, \mathcal{F})$ if X is $\mathcal{F}$-$\mathcal{B}(R)$ measurable.*

Definition 10.1.14. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable on $(\Omega, \mathcal{F})$. We define the **probability distribution** of X, $P_X : \mathcal{B}(R) \to [0, 1]$, to be the measure*

$$P_X = X_* P$$

so that for each $A \in \mathcal{B}(\mathbb{R})$,

$$P_X(A) = P(X^{-1}(F))$$

*We define the **probability distribution function** of X, $F_X : \mathbb{R} \to [0, 1]$, to be*

$$F_X = F_{P_X}$$

Definition 10.1.15. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and X a random variable on $(\Omega, \mathcal{F})$. If $P_X \ll m$, we define the **probability density** of X, $f_X : \mathbb{R} \to \mathbb{R}$, by*

$$f_X = \frac{dP_X}{dm}$$

Exercise 10.1.16. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{F})$. Then for each $x \in \mathbb{R}$,*

$$\mathbb{P}\left(\liminf_{n \to \infty} X_n > x\right) \leq \liminf_{n \to \infty} P(X_n > x)$$

Proof. Let $\omega \in \left\{\liminf_{n \to \infty} X_n > x\right\}$. Then $x < \liminf_{n \to \infty} X_n(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} X_k(\omega)\right)$. So there exists $n^* \in \mathbb{N}$ such that $x < \inf_{k \geq n^*} X_k(\omega)$. Then for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x < X_k(\omega)$. So there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Hence $\inf_{k \geq n^*} \mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Thus $\liminf_{n \to \infty} \mathbf{1}_{\{X_k > x\}}(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \mathbf{1}_{\{X_k > x\}}(\omega)\right) = 1$. Therefore $\omega \in \liminf_{n \to \infty} \{X_k > x\}$ and we have shown that

$$\left\{\liminf_{n \to \infty} X_n > x\right\} \subset \liminf_{n \to \infty} \{X_k > x\}$$

Then

$$P\Big(\liminf_{n\to\infty} X_n > x\Big) \leq P\Big(\liminf_{n\to\infty}\{X_k > x\}\Big)$$
$$\leq \liminf_{n\to\infty} P(\{X_k > x\})$$

□

Definition 10.1.17. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^+(\Omega) \cup L^1$. Define the* ***expectation of X****, $E[X]$, to be*

$$\mathbb{E}[X] = \int X dP$$

.

10.2. **Independence.**

Definition 10.2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C} \subset \mathcal{F}$. Then $\mathcal{C}$ is said to be* ***independent*** *if for each $(A_i)_{i=1}^n \subset \mathcal{C}$,*

$$P\bigg(\bigcap_{k=1}^n A_k\bigg) = \prod_{k=1}^n P(A_k)$$

Definition 10.2.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C}_1, \cdots, \mathcal{C}_n \subset \mathcal{F}$. Then $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are said to be* ***independent*** *if for each $A_1 \in \mathcal{C}_1, \cdots, A_n \in \mathcal{C}_n$, $A_1, \cdots, A_n$ are independent.*

Note 10.2.3. *We will explicitely say that for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is independent when talking about the independence of the elements of $\mathcal{C}_i$ to avoid ambiguity.*

Definition 10.2.4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_2$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are said to be* ***independent*** *if for each $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R})$, $X_1^{-1}B_1, \cdots, X_n^{-1}B_n$ are independent.*

Exercise 10.2.5. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are independent iff $\sigma(X_1), \cdots, \sigma(X_n)$ are independent.*

Proof. Suppose that $X_1, \cdots, X_n$ are independent. Let $A_1, \in \sigma(X_1), \cdots, A_n \in \sigma(A_n)$. Then for each $i = 1, \cdots, n$, there exists $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X_i^{-1}(B_i)$. Then $A_1, \cdots, A_n$ are independent. Hence $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Conversely, suppose that $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Let $B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R})$. Then for each $i = 1, \cdots, n$, $X_i^{-1}B_i \in \sigma(X_i)$. Then $X_1^{-1}B_1, \cdots, X_n^{-1}B_n$ are independent. Hence $X_1, \cdots, X_n$ are independent. □

Exercise 10.2.6. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$ and $\mathcal{F}_1, \cdots, \mathcal{F}_n \subset \mathcal{F}$ a collection of σ-algebras on Ω. Suppose that for each $i = 1, \cdots, n$, X_i is $\mathcal{F}_i$-measurable. If $\mathcal{F}_1, \cdots, \mathcal{F}_n$ are independent, then $X_1, \cdots, X_n$ are independent.*

Proof. For each $i = 1, \cdots, n$, $\sigma(X_i) \subset \mathcal{F}_i$. So $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Hence $X_1, \cdots, X_n$ are independent. □

Exercise 10.2.7. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{C}_1, \cdots, \mathcal{C}_n \subset \mathcal{F}$. Suppose that for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is a π-system and $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are independent, then $\sigma(\mathcal{C}_1), \cdots, \sigma(\mathcal{C}_n)$ are independent.*

Proof. Let $A_2 \in \mathcal{C}_2$. Define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap A_2) = P(A)P(A_2)\}$. Then

(1) $\Omega \in \mathcal{L}$
(2) If $A \in \mathcal{L}$, then

$$\begin{aligned} P(A^c \cap A_2) &= P(A_2) - P(A_2 \cap A) \\ &= P(A_2) - P(A_2)P(A) \\ &= (1 - P(A))P(A_2) \\ &= P(A^c)P(A_2) \end{aligned}$$

So $A^c \in \mathcal{L}$
(3) If $(B_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ is disjoint, then

$$\begin{aligned} P\left(\left[\bigcup_{n\in\mathbb{N}} B_n\right] \cap A_2\right) &= P\left(\bigcup_{n\in\mathbb{N}} B_n \cap A_2\right) \\ &= \sum_{n\in\mathbb{N}} P(B_n \cap A_2) \\ &= \sum_{n\in\mathbb{N}} P(B_n)P(A_2) \\ &= \left[\sum_{n\in\mathbb{N}} P(B_n)\right] P(A_2) \\ &= P\left(\bigcup_{n\in\mathbb{N}} A_n\right) P(A_2) \end{aligned}$$

So $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{L}$.

Thus $\mathcal{L}$ is a λ-system. Since $\mathcal{C}_1 \subset \mathcal{L}$ is a π-system, Dynkin's theorem tells us that $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Since $A_2 \in \mathcal{C}_2$ is arbitrary $\sigma(\mathcal{C}_1)$ and $\mathcal{C}_2$ are independent. The same reasoning implies that $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent. Let $A_2 \in \mathcal{C}_1, \cdots, A_n \in \mathcal{C}_n$ We may do the same process with

$$\mathcal{L} = \left\{A \in \mathcal{F} : P\left(A \cap \left(\bigcap_{i=2}^{n} A_i\right)\right) = P(A)\prod_{i=2}^{n} P(A_i)\right\}$$

and conclude that $\sigma(\mathcal{C}_1), \mathcal{C}_2, \cdots, \mathcal{C}_n$ are independent. Which, using the same reasoning would imply that $\sigma(\mathcal{C}_1), \cdots, \sigma(\mathcal{C}_n)$ are independent. □

Exercise 10.2.8. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Then $X_1, \cdots, X_n$ are independent iff for each $x_1, \cdots, x_n \in \mathbb{R}$,*

$$P(X_1 \le x_1, \cdots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i)$$

Proof. Suppose that $X_1, \cdots, X_n$ are independent. Then $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Let $x_1, \cdots, x_n \in \mathbb{R}$. Then for each $i = 1, \cdots, n$, $\{X_i \le x_i\} \in \sigma(X_i)$. Hence $P(X_1 \le x_1, \cdots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i)$. Conversely, suppose that for each $x_1, \cdots, x_n \in \mathbb{R}$, $P(X_1 \le x_1, \cdots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i)$. Define $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$.

Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$. For each $i = 1, \cdots, n$, define $\mathcal{C}_i = X_i^{-1}\mathcal{C}$. Then for each $i = 1, \cdots, n$, $\mathcal{C}_i$ is a π-system and

$$\begin{aligned}\sigma(\mathcal{C}_i) &= \sigma(X^{-1}(\mathcal{C})) \\ &= X_i^{-1}(\sigma(\mathcal{C})) \\ &= X_i^{-1}(\mathcal{B}(\mathbb{R})) \\ &= \sigma(X_i)\end{aligned}$$

By assumption, $\mathcal{C}_1, \cdots, \mathcal{C}_n$ are independent. The previous exercies tells us that $\sigma(X_1), \cdots, \sigma(X_n)$ are independent. Then $X_1, \cdots, X_n$ are independent. □

Exercise 10.2.9. *Let Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$. Define $X = (X_1, \cdots, X_n)$. If $X_1, \cdots, X_n$ are independent, then*

$$P_X = \prod_{i=1}^n P_{X_i}$$

.

Proof. Let $A_1, \cdots, A_n \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}P_X(A_1 \times \cdots \times A_n) &= P(X \in A_1 \times \cdots \times \in A_n) \\ &= P(X_1 \in A_1, \cdots, X_n \in A_n) \\ &= P(X_1 \in A_1) \cdots P(X_n \in A_n) \\ &= P_{X_1}(A_1) \cdots P_{X_n}(A_n) \\ &= \prod_{i=1}^n P_{X_i}(A_1 \times \cdots \times A_n)\end{aligned}$$

Put

$$\mathcal{P} = \{A_1 \times \cdots \times A_n : A_1 \in \mathcal{B}(R), \cdots, A_n \in \mathcal{B}(R)\}$$

Then $\mathcal{P}$ is a π-system and

$$\sigma(\mathcal{P}) = \mathcal{B}(R) \otimes \cdots \otimes \mathcal{B}(R) = \mathcal{B}(\mathbb{R}^n)$$

A previous exercise then tells us that $P_X = \prod_{i=1}^n P_{X_i}$ □

Exercise 10.2.10. *Let Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X_1, \cdots, X_n$ random variables on $(\Omega, \mathcal{F})$ and $f_1, \cdots, f_n : \mathbb{R} \to \mathbb{R} \in L^0$. Suppose that $f_1 \circ X_1, \cdots, f_n \circ X_n \in L^+(\Omega)$ or $f_1 \circ X_1, \cdots, f_n \circ X_n \in L^1(\Omega)$. If $X_1, \cdots, X_n$ are independent, then*

$$E[f_1(X_1) \cdots f_n(X_n)] = \prod_{i=1}^n E[f_i(X_i)]$$

Proof. Define the random vector $X : \Omega \to \mathbb{R}^n$ by $X = (X_1, \cdots, X_n)$ and $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x_1, \cdots, x_n) = f_1(x_1) \cdots f_n(x_n)$. Suppose that for each $i = 1, \cdots, n$, $f_i \in L^+(\mathbb{R})$. Then

$g \in L^+(\mathbb{R}^n)$ and by change of variables,

$$\begin{aligned} E[f_1(X_1)\cdots f_n(X_n)] &= E[g(X)] \\ &= \int_\Omega g \circ X dP \\ &= \int_{\mathbb{R}^n} g(x) dP_X(x) \\ &= \int_{R^n} g(x) d\prod_{i=1}^n P_{X_i}(x) \\ &= \prod_{i=1}^n \int_{\mathbb{R}} f_i(x) dP_{X_i}(x) \\ &= \prod_{i=1}^n \int_\Omega f_i \circ X dP \\ &= \prod_{i=1}^n E[f_i(X_i)] \end{aligned}$$

If for each $i = 1, \cdots, n$, $f_i \in L^1(\mathbb{R}, P_{X_i})$, then following the above reasoning with $|g|$ tells us that $g \in L^1(\mathbb{R}^n, P_X)$ and we use change of variables and Fubini's theorem to get the same result. $\square$

10.3. L^p Spaces for Probability.

Note 10.3.1. *Recall that for a probability space $(\Omega, \mathcal{F}, P)$ and $1 \leq p \leq q \leq \infty$ we have $L^q \subset L^p$ and for each $X \in L^q$, $\|X\|_p \leq \|X\|_q$. Also recall that for $X, Y \in L^2$, we have that $\|XY\|_1 \leq \|X\|_2 \|X\|_2$.*

Definition 10.3.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Define the* ***variance of X***, *$Var(X)$, to be*

$$Var(X) = \mathbb{E}\big[\big(X - E[X]\big)^2\big]$$

.

Definition 10.3.3. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Define the*

Definition 10.3.4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Define the* ***covariance of** X **and** Y*, *$Cov(X, Y)$, to be*

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Exercise 10.3.5. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. Then the covariance is well defined and $Cov(X, Y)^2 \leq Var(X)Var(Y)$*

Proof. By Holder's inequality,

$$\begin{aligned}
|Cov(X,Y)| &= \left| \int (X - E[X])(Y - E[Y]) dP \right| \\
&\leq \int |(X - E[X])(Y - E[Y])| dP \\
&= \|(X - E[X])(Y - E[Y])\|_1 \\
&\leq \|X - E[X]\|_2 \|(Y - E[Y]\|_2 \\
&= \left(\int |X - E[X]|^2 dP \right)^{\frac{1}{2}} \left(|Y - E[Y]|^2 \right)^{\frac{1}{2}} \\
&= Var(X)^{\frac{1}{2}} Var(Y)^{\frac{1}{2}}
\end{aligned}$$

So $Cov(X,Y)^2 \leq Var(X)Var(Y)$. □

Exercise 10.3.6. *Let $(\Omega, \mathcal{F}, P)$ be a measure space and $X, Y \in L^2$. Then*

1. $Cov(X,Y) = E[XY] - E[X]E[Y]$
2. *If X, Y are independent, then $Cov(X,Y) = 0$*
3. $Var(X) = E[X^2] - E[X]^2$
4. *for each $a, b \in \mathbb{R}$, $Var(aX + b) = a^2 Var(X)$.*
5. $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

Proof.

1. We have that
$$\begin{aligned}
Cov(X,Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY - E[Y]X - E[X]Y + E[X]E[Y]] \\
&= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$
2. Suppose that X, Y are independent. Then $E[XY] = E[X]E[Y]$. Hence
$$\begin{aligned}
Cov(X,Y) &= E[XY] - E[X]E[Y] \\
&= E[X]E[Y] - E[X]E[Y] \\
&= 0
\end{aligned}$$
3. Part (1) implies that
$$\begin{aligned}
Var(X) &= Cov(X,X) \\
&= E[X^2] - E[X]^2
\end{aligned}$$
4. Let $a, b \in \mathbb{R}$. Then
$$\begin{aligned}
Var(aX + b) &= E[(aX + b)^2] - E[aX + b]^2 \\
&= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - (a^2E[X]^2 + 2abE[X] + b^2) \\
&= a^2(E[X^2] - E[X]^2) \\
&= a^2 Var(X)
\end{aligned}$$

(5) We have that

$$\begin{aligned}
Var(X+Y) &= E[(X+Y)^2] - E[X+Y]^2 \\
&= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
&= E[X^2] + 2E[XY] + E[Y^2] - (E[X]^2 + 2E[X]E[Y] + E[Y]^2) \\
&= (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2) + 2(E[XY] - E[X]E[Y]) \\
&= Var(X) + Var(Y) + 2Cov(X,Y)
\end{aligned}$$

□

Definition 10.3.7. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X, Y \in L^2$. The* ***correlation of X and Y****, $Cor(X,Y)$, is defined to be*

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Exercise 10.3.8.

Exercise 10.3.9. *Jensen's Inequality Let $(\Omega, \mathcal{F}, P)$ be a probability space, $X \in L^1$ and $\phi : \mathbb{R} \to \mathbb{R}$. If ϕ is convex, then*

$$\phi(E[X]) \leq E[\phi(X)]$$

Proof. Put $x_0 = E[X]$. Since ϕ is convex, there exist $a, b \in \mathbb{R}$ such that $\phi(x_0) = ax_0 + b$ and for each $x \in \mathbb{R}$, $\phi(x) \geq ax + b$. Then

$$\begin{aligned}
E[\phi(X)] &= \int \phi(X)dP \\
&\geq \int [aX + b]dP \\
&= a\int XdP + b \\
&= aE[X] + b \\
&= ax_0 + b \\
&= \phi(x_0) \\
&= \phi(E[X])
\end{aligned}$$

□

Exercise 10.3.10. *Markov's Inequality: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^+$. Then for each $a \in (0, \infty)$,*

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Proof. Let $a \in (0, \infty)$. Then $a\mathbf{1}_{\{X \geq a\}} \leq X\mathbf{1}_{\{X \geq a\}}$. Thus

$$\begin{aligned} aP(X \geq a) &= \int a\mathbf{1}_{\{X \geq a\}} dP \\ &= \int X\mathbf{1}_{\{X \geq a\}} dP \\ &\leq \int X dP \\ &= E[X] \end{aligned}$$

Therefore

$$P(X \geq a) \leq \frac{E[X]}{a}$$

. $\square$

Exercise 10.3.11. *Chebychev's Inequality: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Then for each $a \in (0, \infty)$,*

$$P(|X - E[X]| \geq a) \leq \frac{Var(X)}{a^2}$$

Proof. Let $a \in (0, \infty)$. Then

$$\begin{aligned} P(|X - E[X]| \geq a) &= P((X - E[X])^2 \geq a^2) \\ &\leq \frac{E[(X - E[X])^2]}{a^2} \\ &= \frac{Var(X)}{a^2} \end{aligned}$$

$\square$

Exercise 10.3.12. *Chernoff's Bound: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^2$. Then for each $a, t \in (0, \infty)$,*

$$P(X \geq a) \leq e^{-ta} E[e^{tX}]$$

Proof. Let $a, t \in (0, \infty)$. Then

$$\begin{aligned} P(X \geq a) &= P(tX \geq ta) \\ &= P(e^{tX} \geq e^{ta}) \\ &\leq e^{-ta} E[e^{tX}] \end{aligned}$$

$\square$

Exercise 10.3.13. *Weak Law of Large Numbers: Let $(\Omega, \mathcal{F}, P)$ be a probability space $(X_i)_{i \in \mathbb{N}} \subset L^2$. Suppose that $(X_i)_{i \in \mathbb{N}}$ are iid. Then*

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} E[X_1]$$

Proof. Put $\mu = E[X_1]$ and $\sigma^2 = Var(X_1)$. Then

$$\begin{aligned} E[\frac{1}{n}\sum_{i=1}^n X_i] &= \frac{1}{n}\sum_{i=1}^n E[X_i] \\ &= \frac{1}{n}\sum_{i=1}^n \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} Var(\frac{1}{n}\sum_{i=1}^n X_i) &= \frac{1}{n^2}Var(\sum_{i=1}^n X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^n Var(X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^n \sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Let $\epsilon > 0$. Then

$$\begin{aligned} P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - E[X_1]\right| \geq \epsilon\right) &= P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) \\ &= P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i\right]\right| \geq \epsilon\right) \\ &\leq \frac{Var(\frac{1}{n}\sum_{i=1}^n X_i)}{\epsilon^2} \\ &= \frac{\sigma^2/n}{\epsilon^2} \\ &= \frac{\sigma^2}{n\epsilon^2} \to 0 \end{aligned}$$

So

$$\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

□

10.4. **Borel Cantelli Lemma.**

Definition 10.4.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$. We will define*

$$P(A_n \ i.o.) := P(\limsup_{n\to\infty} A_n)$$

and

$$P(A_n \ ev.) := P(\liminf_{n\to\infty} A_n)$$

to be the ***probability that*** A_n ***happens infinitely often*** *and the* ***probability that*** A_n ***happens eventually*** *respectively.*

Exercise 10.4.2. ***Borel Cantelli Lemma:*** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$.

(1) *If* $\sum_{n\in\mathbb{N}} P(A_n) < \infty$*, then* $P(A_n \text{ i.o.}) = 0$.
(2) *If* $(A_n)_{n\in\mathbb{N}}$ *are independent and* $\sum_{n\in\mathbb{N}} P(A_n) = \infty$*, then* $P(A_n \text{ i.o.}) = 1$

Proof.

(1) Suppose that $\sum_{n\in\mathbb{N}} P(A_n) < \infty$. Recall that

$$\limsup_{n\to\infty} A_n = \left\{\omega \in \Omega : \sum_{n\in\mathbb{N}} \mathbf{1}_{A_n}(\omega) = \infty\right\}$$

Then

$$\begin{aligned}\infty &> \sum_{n\in\mathbb{N}} P(A_n) \\ &= \sum_{n\in\mathbb{N}} \int \mathbf{1}_{A_n} dP \\ &= \int \sum_{n\in\mathbb{N}} \mathbf{1}_{A_n} dP\end{aligned}$$

Thus $\sum_{n\in\mathbb{N}} \mathbf{1}_{A_n} < \infty$ a.e. and $P(A_n \text{ i.o.}) = 0$.

(2) Suppose that $(A_n)_{n\in\mathbb{N}}$ are independent and $\sum_{n\in\mathbb{N}} P(A_n) = \infty$.

□

Exercise 10.4.3. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $(X_n)_{n\in\mathbb{N}} \subset L^0$ *and* $X \in L^0$.

(1) *If there exists* $\epsilon > 0$ *such that* $\sum_{n\in\mathbb{N}} P(|X_n - X| > \epsilon) < \infty$*, then* $X_n \to X$ *a.s.*
(2) *If* $(X_n)_{n\in\mathbb{N}}$ *are independent and there exists* $\epsilon > 0$ *such that* $\sum_{n\in\mathbb{N}} P(|X_n - X| > \epsilon) = \infty$*, then* $X_n \not\to X$ *a.s.*

Proof. (1)

□

11. Appendix

11.1. **Summation.**

Definition 11.1.1. *Let $f : X \to [0, \infty)$, Then we define*

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note 11.1.2. *Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.*