INTRODUCTION TO COMMUTATIVE ALGEBRA

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Contents

Preface

This aim of this book is to help students develope a basic grasp of the theory of integration. A typical student's first exposure to integration is in the context of the Darboux integral. This integral is applicable to a relatively small class of complex-valued functions of one or more real variables. Although this is integral is of critical importance, it is not sufficient for many purposes. Extending the Darboux integral to the Lebesgue integral, we can define a notion of integration for certain complex-valued functions on more general spaces, like for example topological spaces. To do so, we must first develop some measure theory, which is useful in its own right. Further extending the Lebesgue integral to the Bochner integral, we can define a notion of integration for certain vector valued functions.

The target audience of this book is composed of those who wish to deepen their understanding of integration beyond the Darboux integral. In practice, a basic understanding of the integral would benefit anyone who works with integration and limits in more exotic spaces. For example, students of statistics, physics and disciplines which utilize numerical solutions to differential equations would benefit greatly from a deeper understanding of the integral.

1. Rings

Definition 1.0.1. Let R be a set and $+, * : R \times R \to R$ (we write $a + b$ and ab in place of $+(a, b)$ and $*(a, b)$ respectively). Then R is said to be a **ring** if for each $a, b, c \in R$,

(1) R is an abelian group with respect to $+$. The identity element with respect to $+$ is denoted by 0.
(2) R is a monoid with respect to $*$. The identity element of R with respect to $*$ is denoted 1.
(3) R is commutative with respect to $*$.
(4) $*$ distributes over $+$.

Definition 1.0.2. Let R be a ring and $I \subset R$. Then I is said to be an **ideal** of R if for each $a \in R$ and $x, y \in I$,

(1) $x + y \in I$
(2) $ax \in I$

Definition 1.0.3. Let R be a ring and $A, B \subset R$. We define the **product** of A and B, denoted AB, to be

$$AB = \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in A, b_i \in B, n \in \mathbb{N} \right\}$$

Exercise 1.0.4. Let R be a ring and $I \subset R$. Then I is an ideal of R iff $RI \subset I$.

Proof. Suppose that $RI \subset I$. Let $a \in R$ and $x, y \in I$. Then by assumption $x+y = 1x+1y \in I$ and $ax \in I$. So I is an ideal of R
Conversely, suppose that I is an ideal of R. Let $a_1, \cdots, a_n \in R$ and $x_1, \cdots, x_n \in I$. Then by assumption, for each $i = 1, \cdots, n$, $a_i x_i \in I$ and therefore $\sum_{i=1}^{n} a_i b_i \in I$. Hence $RI \subset I$. □

2. Modules

2.1. Modules.

Definition 2.1.1. Let R be a ring, M a set, $+ : M \times M \to M$ and $* : R \times M \to M$ (we write rx in place of $*(r, x)$). Then M is said to be an R-**module** if

(1) M is an abelian group with respect to $+$. The identity element of M with respect to $+$ is denoted by 0.
(2) for each $r \in R$, $*(r, \cdot)$ is a group endomorphism of M
(3) for each $x \in M$, $*(\cdot, x)$ is a group homomorphism from R to M
(4) $*$ is a monoid action of R on M

Note 2.1.1. For the remainder of this section, we assume that R is a ring.

Exercise 2.1.2. Let M be an R-module. Then for each $r \in R$ and $x \in M$,

(1) $r0 = 0$
(2) $0x = 0$
(3) $(-1)x = -x$

Proof. Let $r \in R$ and $x \in M$. Then

(1)
$$\begin{aligned} r0 &= r(0+0) \\ &= r0 + r0 \end{aligned}$$
which implies that $r0 = 0$.

(2)
$$\begin{aligned} 0x &= (0+0)x \\ &= 0x + 0x \end{aligned}$$
which implies that $0x = 0$.

(3)
$$\begin{aligned} (-1)x + x &= (-1)x + 1x \\ &= (-1+1)x \\ &= 0x \\ &= 0 \end{aligned}$$
which implies that $(-1)x = -x$.

$\square$

Definition 2.1.3. Let M an R-module and $N \subset M$. Then N is said to be a **submodule** of M if for each $r \in R$ and $x, y \in N$, we have that $rx \in N$ and $x + y \in N$.

Definition 2.1.4. Let M be an R-module. We define $\mathcal{S}(M) = \{N \subset M : N \text{ is a submodule of } M\}$.

Exercise 2.1.5. Let M be an R-module and $N \in \mathcal{S}(M)$. Then N is a subgroup of M.

Proof. Let $x, y \in M$. Then $x - y = 1x + (-1)y \in N$. So N is a subgroup of M. $\square$

Definition 2.1.6. Let M be an R-module and $N \in \mathcal{S}(M)$. We define

(1) $M/N = \{x + N : x \in M\}$
(2) $+ : M/N \times M/N \to M/N$ by
$$(x + N) + (y + N) = (x + y) + N$$
(3) $* : R \times M/N \to M/N$ by
$$r(x + N) = (rx) + N$$

Under these operations (see next exercise), M/N is an R-module known as the **quotient module** of M by N.

Exercise 2.1.7. Let M be an R-module and $N \in \mathcal{S}(M)$. Then

(1) the monoid action defined above is well defined
(2) the quotient M/N is an R-module

Proof.

(1) Let $r \in R$ and $x+N, y+N \in M/N$. Recall from group theory that $x+N = y+N$ iff $x-y \in N$. Suppose that $x+N = y+N$. Then $x-y \in N$ and there exists $n \in N$ such that $x-y = n$. Therefore

$$\begin{aligned} rx - ry &= r(x-y) \\ &= rn \\ &\in N \end{aligned}$$

So $rx + N = ry + N$.

(2) Properties (1) - (4) in the definition of a module are easily shown to be satisfied for M/N since they are true for M.

□

Definition 2.1.8. Let M and N be R-modules and $\phi : M \to N$. Then ϕ is said to be a **module homomorphism** if for each $r \in R$ and $x, y \in M$

(1) $\phi(rx) = r\phi(x)$
(2) $\phi(x+y) = \phi(x) + \phi(y)$

Exercise 2.1.9. Let M and N be R-modules and $\phi : M \to N$. Then ϕ is a iff for each $r \in R$ and $x, y \in M$, $\phi(x + ry) = \phi(x) + r\phi(y)$.

Proof. Clear. □

Exercise 2.1.10. Let M and N be R-modules and $\phi : M \to N$ a homomorphism. Then

(1) $\ker \phi$ is a submodule of M
(2) $\operatorname{Im} \phi$ is a submodule of N

Proof. Let $r \in R$, $x, y \in \ker \phi$ and $w, z \in \operatorname{Im} \phi$. Then

(1)

$$\begin{aligned} \phi(rx) &= r\phi(x) \\ &= r0 \\ &= 0 \end{aligned}$$

So $rx \in \ker \phi$. Group theory tells us that $\ker \phi$ is a subgroup of M, so $x + y \in \ker \phi$. Hence $\ker \phi$ is a submodule of M.

(2) Similar.

□

Definition 2.1.11. Let M be an R-module and $A \subset M$. We define the **submodule of** M **generated by** A, denoted $\operatorname{span}(A)$, to be

$$\operatorname{span}(A) = \bigcap_{N \in \mathcal{S}(M)} N$$

Exercise 2.1.12. Let M be an R-module and $A \subset M$. Then $\operatorname{span}(A) \in \mathcal{S}(M)$

Proof. Let $r \in R$ and $x, y \in \operatorname{span}(A)$. Basic group theory tells us that $\operatorname{span}(A)$ is a subgroup of M. So $x + y \in \operatorname{span}(A)$. For $N \in \mathcal{S}(M)$, by definition we have $x \in N$ and therefore $rx \in N$. So $rx \in \operatorname{span}(A)$. Hence $\operatorname{span}(A)$ is a submodule of M. □

Exercise 2.1.13. Let M be an R-module and $A \subset M$. If $A \neq \varnothing$, then

$$\text{span}(A) = \left\{ \sum_{i=1}^{n} r_i a_i : r_i \in R, a_i \in A, n \in \mathbb{N} \right\}$$

Proof. Clearly □

Definition 2.1.14. Let M

3. Appendix

3.1. Monoids.

Definition 3.1.1. Let G be a set and $* : G \times G \to G$ (we write ab in place of $*(a, b)$). Then

(1) $*$ is called a **binary operation** on G
(2) $*$ is said to be **associative** if for each $x, y, z \in G$, $(xy)z = x(yz)$
(3) $*$ is said to be **commutative** if for each $x, y \in G$, $xy = yx$

Definition 3.1.2. Let G be a set, $* : G \times G \to G$, $e, x, y \in G$. Then e is said to be an **identity element** if for each $x \in G$, $ex = xe = x$.

Definition 3.1.3. Let G be a set and $* : G \times G \to G$. Then G is said to be a **monoid** if

(1) $*$ is associative
(2) there exits $e \in G$ such that e is an identity element.

Exercise 3.1.4. Let G be a monoid. Then the identity element is unique.

Proof. Let $e, f \in G$. Suppose that e and f are identity elements. Then $e = ef = f$. □

Note 3.1.1. Unless otherwise specified, we will denote the identity element of a monoid by e.

Definition 3.1.5. Let G be a monoid, X a set and $* : G \times X \to X$ (we write gx in place of $*(g, x)$). Then $*$ is said to be a **monoid action** of G on X if for each $g, h \in G$ and $x \in X$,

(1) $(gh)x = g(hx)$
(2) $ex = x$