

INTRODUCTION TO PROBABILITY

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1. INTRODUCTION

1.1. Purpose.

2. PROBABILITY FRAMEWORK

3. PROBABILITY

3.1. Distributions.

Definition 3.1.1. Let Ω be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then \mathcal{P} is said to be a π -**system** on Ω if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Definition 3.1.2. Let Ω be a set and $\mathcal{L} \subset \mathcal{P}(\Omega)$. Then \mathcal{L} is said to be a λ -**system** on Ω if

- (1) $\mathcal{L} \neq \emptyset$
- (2) for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$
- (3) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}$, if $(A_n)_{n \in \mathbb{N}}$ is disjoint, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}$

Exercise 3.1.3. Let Ω be a set and \mathcal{L} a λ -system on Ω . Then

- (1) $\Omega, \emptyset \in \mathcal{L}$

Proof. Straightforward. □

Definition 3.1.4. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. Put

$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(\Omega) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the λ -system on Ω generated by \mathcal{C} , $\lambda(\mathcal{C})$, to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L} \in \mathcal{S}} \mathcal{L}$$

Exercise 3.1.5. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. If \mathcal{C} is a λ -system and \mathcal{C} is a π -system, then \mathcal{C} is a σ -algebra.

Proof. Suppose that \mathcal{C} is a λ -system and \mathcal{C} is a π -system. Then we need only verify the third axiom in the definition of a σ -algebra. Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$. Define $B_1 = A_1$ and for $n \geq 2$, define $B_n = A_n \cap \left(\bigcup_{k=1}^{n-1} A_k \right)^c = A_n \cap \left(\bigcap_{k=1}^{n-1} A_k^c \right) \in \mathcal{C}$. Then $(B_n)_{n \in \mathbb{N}}$ is disjoint and therefore $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{C}$. \square

Theorem 3.1.1. (*Dynkin's Theorem*)

Let Ω be a set.

- (1) Let \mathcal{P} be a π -system on Ω and \mathcal{L} a λ -system on Ω . If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
- (2) Let \mathcal{P} be a π -system on Ω . Then $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

Exercise 3.1.6. Let (Ω, \mathcal{F}) be a measurable space and μ, ν probability measures on (Ω, \mathcal{F}) . Put $\mathcal{L}_{\mu, \nu} = \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$. Then $\mathcal{L}_{\mu, \nu}$ is a λ -system on Ω .

Proof.

- (1) $\emptyset \in \mathcal{L}_{\mu, \nu}$.
- (2) Let $A \in \mathcal{L}_{\mu, \nu}$. Then $\mu(A) = \nu(A)$. Thus

$$\begin{aligned} \mu(A^c) &= 1 - \mu(A) \\ &= 1 - \nu(A) \\ &= \nu(A^c) \end{aligned}$$

So $A^c \in \mathcal{L}_{\mu, \nu}$.

- (3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}_{\mu, \nu}$. So for each $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$. Suppose that $(A_n)_{n \in \mathbb{N}}$ is disjoint. Then

$$\begin{aligned} \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \sum_{n \in \mathbb{N}} \mu(A_n) \\ &= \sum_{n \in \mathbb{N}} \nu(A_n) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \end{aligned}$$

Hence $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}_{\mu, \nu}$. \square

Exercise 3.1.7. Let (Ω, \mathcal{F}) be a measurable space, μ, ν probability measures on (Ω, \mathcal{F}) and $\mathcal{P} \subset \mathcal{A}$ a π -system on Ω . Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Proof. Using the previous exercise, we see that $\mathcal{P} \subset \mathcal{L}_{\mu,\nu}$. Dynkin's theorem implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu,\nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$. \square

Definition 3.1.8. Let $F : \mathbb{R} \rightarrow \mathbb{R}$. Then F is said to be a **probability distribution function** if

- (1) F is right continuous
- (2) F is increasing
- (3) $F(-\infty) = 0$ and $F(\infty) = 1$

Definition 3.1.9. Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We define $F_P : \mathbb{R} \rightarrow \mathbb{R}$, by

$$F_P(x) = P((-\infty, x])$$

We call F_P the **probability distribution function** of P .

Exercise 3.1.10. Let (Ω, \mathcal{F}, P) be a probability measure. Then F_P is a probability distribution function.

Proof. (1) Let $x \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}} \subset [x, \infty)$. Suppose that $x_n \rightarrow x$. Then $(x, x_n] \rightarrow \emptyset$ because $\limsup_{n \rightarrow \infty} (x, x_n] = \emptyset$. Thus

$$F(x_n) - F(x) = P((x, x_n]) \rightarrow P(\emptyset) = 0$$

This implies that

$$F(x_n) \rightarrow F(x)$$

. So F is right continuous.

- (2) Clearly F_P is increasing.
- (3) Continuity from below tells us that

$$F(-\infty) = \lim_{n \rightarrow -\infty} F(n) = \lim_{n \rightarrow -\infty} P((-\infty, n]) = 0$$

and continuity from above tell us that

$$F(\infty) = \lim_{n \rightarrow \infty} F(n) = \lim_{n \rightarrow \infty} P((-\infty, n]) = 1$$

\square

Exercise 3.1.11. Let μ, ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $F_\mu = F_\nu$ iff $\mu = \nu$.

Proof. Clearly if $\mu = \nu$, then $F_\mu = F_\nu$. Conversely, suppose that $F_\mu = F_\nu$. Then for each $x \in \mathbb{R}$,

$$\begin{aligned} \mu((-\infty, x]) &= F_\mu(x) \\ &= F_\nu(x) \\ &= \nu((-\infty, x]) \end{aligned}$$

Put $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then \mathcal{C} is a π -system and for each $A \in \mathcal{C}$, $\mu(A) = \nu(A)$. Hence for each $A \in \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, $\mu(A) = \nu(A)$. So $\mu = \nu$. \square

Definition 3.1.12. Let (Ω, \mathcal{F}, P) be a probability space and $X : \Omega \rightarrow \mathbb{R}^n$. Then X is said to be a **random vector** on (Ω, \mathcal{F}) if X is \mathcal{F} - $\mathcal{B}(\mathbb{R}^n)$ measurable. If $n = 1$, then X is said to be a **random variable**. We define

$$L_n^0(\Omega, \mathcal{F}, P) = \{X : \Omega \rightarrow \mathbb{R}^n : X \text{ is a random vector}\}$$

and

$$L_n^p(\Omega, \mathcal{F}, P) = \left\{ X \in L_n^0 : \int \|X\|^p dP < \infty \right\}$$

Definition 3.1.13. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . We define the **probability distribution** of X , $P_X : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$, to be the measure

$$P_X = X_*P$$

That is, for each $A \in \mathcal{B}(\mathbb{R})$,

$$P_X(A) = P(X^{-1}(A))$$

We define the **probability distribution function** of X , $F_X : \mathbb{R} \rightarrow [0, 1]$, to be

$$F_X = F_{P_X}$$

Definition 3.1.14. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . If $P_X \ll m$, we define the **probability density** of X , $f_X : \mathbb{R} \rightarrow \mathbb{R}$, by

$$f_X = \frac{dP_X}{dm}$$

Exercise 3.1.15. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on (Ω, \mathcal{F}) . Then for each $x \in \mathbb{R}$,

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} X_n > x\right) \leq \liminf_{n \rightarrow \infty} P(X_n > x)$$

Proof. Let $\omega \in \left\{ \liminf_{n \rightarrow \infty} X_n > x \right\}$. Then $x < \liminf_{n \rightarrow \infty} X_n(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} X_k(\omega) \right)$. So there exists $n^* \in \mathbb{N}$ such that $x < \inf_{k \geq n^*} X_k(\omega)$. Then for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x < X_k(\omega)$. So there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Hence $\inf_{k \geq n^*} \mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Thus $\liminf_{n \rightarrow \infty} \mathbf{1}_{\{X_k > x\}}(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \mathbf{1}_{\{X_k > x\}}(\omega) \right) = 1$. Therefore $\omega \in \liminf_{n \rightarrow \infty} \{X_k > x\}$ and we have shown that

$$\left\{ \liminf_{n \rightarrow \infty} X_n > x \right\} \subset \liminf_{n \rightarrow \infty} \{X_k > x\}$$

Then

$$\begin{aligned} P\left(\liminf_{n \rightarrow \infty} X_n > x\right) &\leq P\left(\liminf_{n \rightarrow \infty} \{X_k > x\}\right) \\ &\leq \liminf_{n \rightarrow \infty} P(\{X_k > x\}) \end{aligned}$$

□

Definition 3.1.16. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+(\Omega) \cup L^1$. Define the **expectation of X** , $E[X]$, to be

$$\mathbb{E}[X] = \int X dP$$

3.2. Independence.

Definition 3.2.1. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C} \subset \mathcal{F}$. Then \mathcal{C} is said to be **independent** if for each $(A_i)_{i=1}^n \subset \mathcal{C}$,

$$P\left(\bigcap_{k=1}^n A_k\right) = \prod_{k=1}^n P(A_k)$$

Definition 3.2.2. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Then $\mathcal{C}_1, \dots, \mathcal{C}_n$ are said to be **independent** if for each $A_1 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$, A_1, \dots, A_n are independent.

Note 3.2.1. We will explicitly say that for each $i = 1, \dots, n$, \mathcal{C}_i is independent when talking about the independence of the elements of \mathcal{C}_i to avoid ambiguity.

Definition 3.2.3. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are said to be **independent** if for each $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent.

Exercise 3.2.4. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff $\sigma(X_1), \dots, \sigma(X_n)$ are independent.

Proof. Suppose that X_1, \dots, X_n are independent. Let $A_1 \in \sigma(X_1), \dots, A_n \in \sigma(X_n)$. Then for each $i = 1, \dots, n$, there exists $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X_i^{-1}(B_i)$. Then A_1, \dots, A_n are independent. Hence $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Conversely, suppose that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$. Then for each $i = 1, \dots, n$, $X_i^{-1}B_i \in \sigma(X_i)$. Then $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent. Hence X_1, \dots, X_n are independent. \square

Exercise 3.2.5. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{F}$ a collection of σ -algebras on Ω . Suppose that for each $i = 1, \dots, n$, X_i is \mathcal{F}_i -measurable. If $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent, then X_1, \dots, X_n are independent.

Proof. For each $i = 1, \dots, n$, $\sigma(X_i) \subset \mathcal{F}_i$. So $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Hence X_1, \dots, X_n are independent. \square

Exercise 3.2.6. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Suppose that for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent, then $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent.

Proof. Let $A_2 \in \mathcal{C}_2$. Define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap A_2) = P(A)P(A_2)\}$. Then

- (1) $\Omega \in \mathcal{L}$
- (2) If $A \in \mathcal{L}$, then

$$\begin{aligned} P(A^c \cap A_2) &= P(A_2) - P(A_2 \cap A) \\ &= P(A_2) - P(A_2)P(A) \\ &= (1 - P(A))P(A_2) \\ &= P(A^c)P(A_2) \end{aligned}$$

So $A^c \in \mathcal{L}$

(3) If $(B_n)_{n \in \mathbb{N}} \subset \mathcal{L}$ is disjoint, then

$$\begin{aligned}
 P\left(\left[\bigcup_{n \in \mathbb{N}} B_n\right] \cap A_2\right) &= P\left(\bigcup_{n \in \mathbb{N}} B_n \cap A_2\right) \\
 &= \sum_{n \in \mathbb{N}} P(B_n \cap A_2) \\
 &= \sum_{n \in \mathbb{N}} P(B_n)P(A_2) \\
 &= \left[\sum_{n \in \mathbb{N}} P(B_n)\right]P(A_2) \\
 &= P\left(\bigcup_{n \in \mathbb{N}} A_n\right)P(A_2)
 \end{aligned}$$

So $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{L}$.

Thus \mathcal{L} is a λ -system. Since $\mathcal{C}_1 \subset \mathcal{L}$ is a π -system, Dynkin's theorem tells us that $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Since $A_2 \in \mathcal{C}_2$ is arbitrary $\sigma(\mathcal{C}_1)$ and \mathcal{C}_2 are independent. The same reasoning implies that $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent. Let $A_2 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$. We may do the same process with

$$\mathcal{L} = \left\{ A \in \mathcal{F} : P\left(A \cap \left(\bigcap_{i=2}^n A_i\right)\right) = P(A) \prod_{i=2}^n P(A_i) \right\}$$

and conclude that $\sigma(\mathcal{C}_1), \mathcal{C}_2, \dots, \mathcal{C}_n$ are independent. Which, using the same reasoning would imply that $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent. \square

Exercise 3.2.7. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff for each $x_1, \dots, x_n \in \mathbb{R}$,

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$$

Proof. Suppose that X_1, \dots, X_n are independent. Then $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $x_1, \dots, x_n \in \mathbb{R}$. Then for each $i = 1, \dots, n$, $\{X_i \leq x_i\} \in \sigma(X_i)$. Hence

$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Conversely, suppose that for each

$x_1, \dots, x_n \in \mathbb{R}$, $P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Define $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$.

Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$. For each $i = 1, \dots, n$, define $\mathcal{C}_i = X_i^{-1}\mathcal{C}$. Then for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and

$$\begin{aligned}
 \sigma(\mathcal{C}_i) &= \sigma(X_i^{-1}(\mathcal{C})) \\
 &= X_i^{-1}(\sigma(\mathcal{C})) \\
 &= X_i^{-1}(\mathcal{B}(\mathbb{R})) \\
 &= \sigma(X_i)
 \end{aligned}$$

By assumption, $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent. The previous exersices tells us that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Then X_1, \dots, X_n are independent. \square

Exercise 3.2.8. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Define $X = (X_1, \dots, X_n)$. If X_1, \dots, X_n are independent, then

$$P_X = \prod_{i=1}^n P_{X_i}$$

.

Proof. Let $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned} P_X(A_1 \times \dots \times A_n) &= P(X \in A_1 \times \dots \times A_n) \\ &= P(X_1 \in A_1, \dots, X_n \in A_n) \\ &= P(X_1 \in A_1) \cdots P(X_n \in A_n) \\ &= P_{X_1}(A_1) \cdots P_{X_n}(A_n) \\ &= \prod_{i=1}^n P_{X_i}(A_1 \times \dots \times A_n) \end{aligned}$$

Put

$$\mathcal{P} = \{A_1 \times \dots \times A_n : A_1 \in \mathcal{B}(R), \dots, A_n \in \mathcal{B}(R)\}$$

Then \mathcal{P} is a π -system and

$$\sigma(\mathcal{P}) = \mathcal{B}(R) \otimes \dots \otimes \mathcal{B}(R) = \mathcal{B}(\mathbb{R}^n)$$

A previous exercise then tells us that $P_X = \prod_{i=1}^n P_{X_i}$ □

Exercise 3.2.9. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R} \in L^0$. Suppose that $f_1 \circ X_1, \dots, f_n \circ X_n \in L^+(\Omega)$ or $f_1 \circ X_1, \dots, f_n \circ X_n \in L^1(\Omega)$. If X_1, \dots, X_n are independent, then

$$E[f_1(X_1) \cdots f_n(X_n)] = \prod_{i=1}^n E[f_i(X_i)]$$

Proof. Define the random vector $X : \Omega \rightarrow \mathbb{R}^n$ by $X = (X_1, \dots, X_n)$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ by $g(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$. Suppose that for each $i = 1, \dots, n$, $f_i \in L^+(\mathbb{R})$. Then

$g \in L^+(\mathbb{R}^n)$ and by change of variables,

$$\begin{aligned}
E[f_1(X_1) \cdots f_n(X_n)] &= E[g(X)] \\
&= \int_{\Omega} g \circ X dP \\
&= \int_{\mathbb{R}^n} g(x) dP_X(x) \\
&= \int_{\mathbb{R}^n} g(x) d \prod_{i=1}^n P_{X_i}(x) \\
&= \prod_{i=1}^n \int_{\mathbb{R}} f_i(x) dP_{X_i}(x) \\
&= \prod_{i=1}^n \int_{\Omega} f_i \circ X dP \\
&= \prod_{i=1}^n E[f_i(X_i)]
\end{aligned}$$

If for each $i = 1, \dots, n$, $f_i \in L^1(\mathbb{R}, P_{X_i})$, then following the above reasoning with $|g|$ tells us that $g \in L^1(\mathbb{R}^n, P_X)$ and we use change of variables and Fubini's theorem to get the same result. \square

3.3. L^p Spaces for Probability.

Note 3.3.1. Recall that for a probability space (Ω, \mathcal{F}, P) and $1 \leq p \leq q \leq \infty$ we have $L^q \subset L^p$ and for each $X \in L^q$, $\|X\|_p \leq \|X\|_q$. Also recall that for $X, Y \in L^2$, we have that $\|XY\|_1 \leq \|X\|_2 \|Y\|_2$.

Definition 3.3.1. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Define the **variance of X** , $Var(X)$, to be

$$Var(X) = \mathbb{E}[(X - E[X])^2]$$

.

Definition 3.3.2. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the

Definition 3.3.3. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the **covariance of X and Y** , $Cov(X, Y)$, to be

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Exercise 3.3.4. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Then the covariance is well defined and $Cov(X, Y)^2 \leq Var(X)Var(Y)$

Proof. By Holder's inequality,

$$\begin{aligned}
 |Cov(X, Y)| &= \left| \int (X - E[X])(Y - E[Y])dP \right| \\
 &\leq \int |(X - E[X])(Y - E[Y])|dP \\
 &= \|(X - E[X])(Y - E[Y])\|_1 \\
 &\leq \|X - E[X]\|_2 \|Y - E[Y]\|_2 \\
 &= \left(\int |X - E[X]|^2 dP \right)^{\frac{1}{2}} \left(\int |Y - E[Y]|^2 dP \right)^{\frac{1}{2}} \\
 &= Var(X)^{\frac{1}{2}} Var(Y)^{\frac{1}{2}}
 \end{aligned}$$

So $Cov(X, Y)^2 \leq Var(X)Var(Y)$. □

Exercise 3.3.5. Let (Ω, \mathcal{F}, P) be a measure space and $X, Y \in L^2$. Then

- (1) $Cov(X, Y) = E[XY] - E[X]E[Y]$
- (2) If X, Y are independent, then $Cov(X, Y) = 0$
- (3) $Var(X) = E[X^2] - E[X]^2$
- (4) for each $a, b \in \mathbb{R}$, $Var(aX + b) = a^2 Var(X)$.
- (5) $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Proof.

- (1) We have that

$$\begin{aligned}
 Cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
 &= E[XY - E[Y]X - E[X]Y + E[X]E[Y]] \\
 &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
 &= E[XY] - E[X]E[Y]
 \end{aligned}$$

- (2) Suppose that X, Y are independent. Then $E[XY] = E[X]E[Y]$. Hence

$$\begin{aligned}
 Cov(X, Y) &= E[XY] - E[X]E[Y] \\
 &= E[X]E[Y] - E[X]E[Y] \\
 &= 0
 \end{aligned}$$

- (3) Part (1) implies that

$$\begin{aligned}
 Var(X) &= Cov(X, X) \\
 &= E[X^2] - E[X]^2
 \end{aligned}$$

- (4) Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
 Var(aX + b) &= E[(aX + b)^2] - E[aX + b]^2 \\
 &= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\
 &= a^2E[X^2] + 2abE[X] + b^2 - (a^2E[X]^2 + 2abE[X] + b^2) \\
 &= a^2(E[X^2] - E[X]^2) \\
 &= a^2Var(X)
 \end{aligned}$$

(5) We have that

$$\begin{aligned}
 \text{Var}(X + Y) &= E[(X + Y)^2] - E[X + Y]^2 \\
 &= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
 &= E[X^2] + 2E[XY] + E[Y^2] - (E[X]^2 + 2E[X]E[Y] + E[Y]^2) \\
 &= (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2) + 2(E[XY] - E[X]E[Y]) \\
 &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)
 \end{aligned}$$

□

Definition 3.3.6. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. The **correlation of X and Y**, $\text{Cor}(X, Y)$, is defined to be

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Exercise 3.3.7.

Exercise 3.3.8. Jensen's Inequality Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^1$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$. If ϕ is convex, then

$$\phi(E[X]) \leq E[\phi(X)]$$

Proof. Put $x_0 = E[X]$. Since ϕ is convex, there exist $a, b \in \mathbb{R}$ such that $\phi(x_0) = ax_0 + b$ and for each $x \in \mathbb{R}$, $\phi(x) \geq ax + b$. Then

$$\begin{aligned}
 E[\phi(X)] &= \int \phi(X) dP \\
 &\geq \int [aX + b] dP \\
 &= a \int X dP + b \\
 &= aE[X] + b \\
 &= ax_0 + b \\
 &= \phi(x_0) \\
 &= \phi(E[X])
 \end{aligned}$$

□

Exercise 3.3.9. Markov's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+$. Then for each $a \in (0, \infty)$,

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Proof. Let $a \in (0, \infty)$. Then $a\mathbf{1}_{\{X \geq a\}} \leq X\mathbf{1}_{\{X \geq a\}}$. Thus

$$\begin{aligned} aP(X \geq a) &= \int a\mathbf{1}_{\{X \geq a\}} dP \\ &= \int X\mathbf{1}_{\{X \geq a\}} dP \\ &\leq \int X dP \\ &= E[X] \end{aligned}$$

Therefore

$$P(X \geq a) \leq \frac{E[X]}{a}$$

□

Exercise 3.3.10. Chebychev's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a \in (0, \infty)$,

$$P(|X - E[X]| \geq a) \leq \frac{\text{Var}(X)}{a^2}$$

Proof. Let $a \in (0, \infty)$. Then

$$\begin{aligned} P(|X - E[X]| \geq a) &= P((X - E[X])^2 \geq a^2) \\ &\leq \frac{E[(X - E[X])^2]}{a^2} \\ &= \frac{\text{Var}(X)}{a^2} \end{aligned}$$

□

Exercise 3.3.11. Chernoff's Bound: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a, t \in (0, \infty)$,

$$P(X \geq a) \leq e^{-ta} E[e^{tX}]$$

Proof. Let $a, t \in (0, \infty)$. Then

$$\begin{aligned} P(X \geq a) &= P(tX \geq ta) \\ &= P(e^{tX} \geq e^{ta}) \\ &\leq e^{-ta} E[e^{tX}] \end{aligned}$$

□

Exercise 3.3.12. Weak Law of Large Numbers: Let (Ω, \mathcal{F}, P) be a probability space $(X_i)_{i \in \mathbb{N}} \subset L^2$. Suppose that $(X_i)_{i \in \mathbb{N}}$ are iid. Then

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

Proof. Put $\mu = E[X_1]$ and $\sigma^2 = \text{Var}(X_1)$. Then

$$\begin{aligned} E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] &= \frac{1}{n} \sum_{i=1}^n E[X_i] \\ &= \frac{1}{n} \sum_{i=1}^n \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) &= \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) \\ &= \frac{1}{n^2} \sum_{i=1}^n \sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Let $\epsilon > 0$. Then

$$\begin{aligned} P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - E[X_1]\right| \geq \epsilon\right) &= P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) \\ &= P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n X_i\right]\right| \geq \epsilon\right) \\ &\leq \frac{\text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right)}{\epsilon^2} \\ &= \frac{\sigma^2/n}{\epsilon^2} \\ &= \frac{\sigma^2}{n\epsilon^2} \rightarrow 0 \end{aligned}$$

So

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

□

3.4. Borel Cantelli Lemma.

Definition 3.4.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{F}$. We will define

$$P(A_n \text{ i.o.}) := P(\limsup_{n \rightarrow \infty} A_n)$$

and

$$P(A_n \text{ ev.}) := P(\liminf_{n \rightarrow \infty} A_n)$$

to be the **probability that A_n happens infinitely often** and the **probability that A_n happens eventually** respectively.

Exercise 3.4.2. Borel Cantelli Lemma: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{F}$.

- (1) If $\sum_{n \in \mathbb{N}} P(A_n) < \infty$, then $P(A_n \text{ i.o.}) = 0$.
- (2) If $(A_n)_{n \in \mathbb{N}}$ are independent and $\sum_{n \in \mathbb{N}} P(A_n) = \infty$, then $P(A_n \text{ i.o.}) = 1$

Proof.

- (1) Suppose that $\sum_{n \in \mathbb{N}} P(A_n) < \infty$. Recall that

$$\limsup_{n \rightarrow \infty} A_n = \left\{ \omega \in \Omega : \sum_{n \in \mathbb{N}} \mathbf{1}_{A_n}(\omega) = \infty \right\}$$

Then

$$\begin{aligned} \infty &> \sum_{n \in \mathbb{N}} P(A_n) \\ &= \sum_{n \in \mathbb{N}} \int \mathbf{1}_{A_n} dP \\ &= \int \sum_{n \in \mathbb{N}} \mathbf{1}_{A_n} dP \end{aligned}$$

Thus $\sum_{n \in \mathbb{N}} \mathbf{1}_{A_n} < \infty$ a.e. and $P(A_n \text{ i.o.}) = 0$.

- (2) Suppose that $(A_n)_{n \in \mathbb{N}}$ are independent and $\sum_{n \in \mathbb{N}} P(A_n) = \infty$.

□

Exercise 3.4.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_n)_{n \in \mathbb{N}} \subset L^0$ and $X \in L^0$.

- (1) If there exists $\epsilon > 0$ such that $\sum_{n \in \mathbb{N}} P(|X_n - X| > \epsilon) < \infty$, then $X_n \rightarrow X$ a.s.
- (2) If $(X_n)_{n \in \mathbb{N}}$ are independent and there exists $\epsilon > 0$ such that $\sum_{n \in \mathbb{N}} P(|X_n - X| > \epsilon) = \infty$, then $X_n \not\rightarrow X$ a.s.

Proof. (1)

□

4. CONDITIONAL EXPECTATION AND PROBABILITY

4.1. Conditional Expectation.

Exercise 4.1.1. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -alg of \mathcal{F} and $X \in L^1(\Omega, \mathcal{F}, P)$. Define $P_{\mathcal{G}} = P|_{\mathcal{G}}$ and $Q : \mathcal{G} \rightarrow [0, \infty)$ by $Q(G) = \int_G X dP$. Then Q is finite. and $Q \ll P_{\mathcal{G}}$.

Proof. Since $X \in L^1$, for each $G \in \mathcal{G}$,

$$\begin{aligned} |Q(G)| &= \left| \int_G X dP \right| \\ &\leq \int_G |X| dP \\ &< \infty \end{aligned}$$

So Q is finite. Let $G \in \mathcal{G}$. Suppose that $P_{\mathcal{G}}(G) = 0$. By definition then, $P(G) = 0$. So $Q(G) = 0$ and $Q \ll P_{\mathcal{G}}$. \square

Definition 4.1.2. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -alg of \mathcal{F} and $X, Y \in L^1(\Omega, \mathcal{F}, P)$. Then Y is said to be a **conditional expectation of X given \mathcal{G}** if

- (1) Y is \mathcal{G} -measurable
- (2) for each $G \in \mathcal{G}$,

$$\int_G Y dP = \int_G X dP$$

To denote this, we write $Y = E[X|\mathcal{G}]$

Exercise 4.1.3. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -alg of \mathcal{F} and $X \in L^1(\Omega, \mathcal{F}, P)$. Define Q and $P_{\mathcal{G}}$ as in the previous exercise. Define $Y = dQ/dP_{\mathcal{G}}$. Then Y is a conditional expectation of X given \mathcal{G} .

Proof. By definition of the Radon-Nikodym derivative, Y is \mathcal{G} -measurable and by the Radon-Nikodym theorem, $X \in L^1(\Omega, \mathcal{F}, P)$ implies that $Y \in L^1(\Omega, \mathcal{G}, P_{\mathcal{G}})$. An exersices in section 3.3 of [?], implies that for each $G \in \mathcal{G}$

$$\int_G Y dP = \int_G X dP$$

\square

Exercise 4.1.4. (Doob–Dynkin Lemma)

Let Ω be a nonempty set, (Ω', \mathcal{F}') a measurable space $X : \Omega \rightarrow \Omega'$ and $Z : \Omega \rightarrow \mathbb{R}^n$. Suppose that $\text{Im } X \in \mathcal{F}'$. Then Z is $\sigma(X)$ - $\mathcal{B}(\mathbb{R}^n)$ measurable iff there exists $\phi : \Omega' \rightarrow \mathbb{R}^n$ such that ϕ is \mathcal{F}' - $\mathcal{B}(\mathbb{R}^n)$ measurable and $Z = \phi \circ X$.

Proof. Suppose that there exists $\phi : \Omega' \rightarrow \mathbb{R}^n$ such that ϕ is \mathcal{F}' - $\mathcal{B}(\mathbb{R}^n)$ measurable and $Z = \phi \circ X$. Since X is $\sigma(X)$ - \mathcal{F}' measurable, $Z = \phi \circ X$ is $\sigma(X)$ - $\mathcal{B}(\mathbb{R}^n)$ measurable. Conversely, suppose that Z is $\sigma(X)$ - $\mathcal{B}(\mathbb{R}^n)$ measurable. For now, suppose that $n = 1$ and Z is simple. Then there exists a partition $(A_i)_{i=1}^k \subset \sigma(X)$ of Ω and $(a_i)_{i=1}^k \in \mathbb{R}$ such that

$$Z = \sum_{i=1}^k a_i 1_{A_i}$$

By definition of $\sigma(X)$, there exists a partition $(B_i)_{i=1}^k \subset \mathcal{F}'$ such that for each $i = 1, \dots, k$, $A_i = X^{-1}(B_i)$. Define

$$\phi = \sum_{i=1}^k a_i 1_{B_i}$$

Since $(B_i)_{i=1}^k$ partitions Ω' ,

$$\begin{aligned} \phi \circ X &= \sum_{i=1}^k a_i 1_{X^{-1}(B_i)} \\ &= \sum_{i=1}^k a_i 1_{A_i} \\ &= Z \end{aligned}$$

More generally, if Z is $\sigma(X)$ - $\mathcal{B}(\mathbb{R})$ measurable, there exists a sequence $(Z_j)_{j \in \mathbb{N}}$ of simple $\sigma(X)$ - $\mathcal{B}(\mathbb{R})$ measurable functions such that for each $j \in \mathbb{N}$ $0 \leq |Z_j| \leq |Z_{j+1}| \leq |Z|$ and $Z_j \xrightarrow{\text{p.w.}} Z$.

Therefore, as shown previously, there exists a sequence $(\phi_j)_{j \in \mathbb{N}}$ of \mathcal{F}' - $\mathcal{B}(\mathbb{R})$ -measurable simple functions such that for each $j \in \mathbb{N}$, $Z_j = \phi_j \circ X$. Let $b \in \text{Im } X$. Then there exists $a \in \Omega$ such that $X(a) = b$. So

$$\begin{aligned} \phi_j(b) &= \phi_j \circ X(a) \\ &= Z_j(a) \\ &\rightarrow Z(a) \end{aligned}$$

Thus we may define $\phi : \Omega' \rightarrow \mathbb{R}$ by

$$\phi = \lim_{j \rightarrow \infty} \phi_j 1_{\text{Im } X}$$

Then ϕ is measurable since $\text{Im } X \in \mathcal{F}'$ and $Z = \phi \circ X$. For $n \geq 1$, we may write $Z = (Z_1, \dots, Z_n)$ where for each $i = 1, \dots, n$, Z_i is $\sigma(X)$ - $\mathcal{B}(\mathbb{R})$ measurable and apply the result from above to obtain $\phi = (\phi_1, \dots, \phi_n)$ where for each $i = 1, \dots, n$, ϕ_i is \mathcal{F}' - $\mathcal{B}(\mathbb{R})$ measurable and $Z_i = \phi_i \circ X$. Then $Z = \phi \circ X$. \square

4.2. Conditional Probability.

Definition 4.2.1. Let (A, \mathcal{A}) be a measurable space, (B, \mathcal{B}, P_Y) a probability space and $Q : B \times \mathcal{A} \rightarrow [0, 1]$. Then Q is said to be a **stochastic transition kernel from (B, \mathcal{B}, P) to (A, \mathcal{A})** if

- (1) for each $E \in \mathcal{A}$, $Q(\cdot, E)$ is \mathcal{B} -measurable
- (2) for P -a.e. $b \in B$, $Q(b, \cdot)$ is a probability measure on (A, \mathcal{A})

Definition 4.2.2. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0(\Omega, \mathcal{F}, \mathbb{P})$ and $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$. Then Q is said to be a **conditional probability distribution of X given Y** if

- (1) Q is a stochastic transition kernel from $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_Y)$ to $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$
- (2) for each $A, B \in \mathcal{F}$,

$$\int_B Q(y, A) dP_Y(y) = P(X \in A, Y \in B)$$

Note 4.2.1. It is helpful to connect this notion of conditional probability with the elementary one by writing $Q(y, A) = P(X \in A | Y = y)$. If $P_Y \ll \mu$, then property (2) in the definition becomes

$$\begin{aligned} P(X \in A, Y \in B) &= \int_B Q(y, A) dP_Y(y) \\ &= \int_B P(X \in A | Y = y) f_Y(y) d\mu(y) \end{aligned}$$

as in a first course on probability.

Exercise 4.2.3. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$. Suppose that for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable, for P_Y -a.e. $y \in \mathbb{R}^n$, $P_{X|Y}(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) and $Q(Y, A) = P(X \in A | Y)$ a.e. Then Q is a conditional probability of X given Y .

Proof. By assumption, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable and for P_Y -a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) . Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned} \int_B Q(y, A) dP_Y(y) &= \int_{Y^{-1}(B)} Q(Y(\omega), A) dP(\omega) \\ &= \int_{Y^{-1}(B)} P(X \in A | Y) dP \\ &= \int_{Y^{-1}(B)} E[1_{X^{-1}(A)} | Y] dP \\ &= \int_{Y^{-1}(B)} 1_{X^{-1}(A)} dP \\ &= \int 1_{X^{-1}(A)} 1_{Y^{-1}(B)} dP \\ &= \int 1_{X^{-1}(A) \cap Y^{-1}(B)} dP \\ &= P(X \in A, Y \in B) \end{aligned}$$

So Q is a conditional probability distribution of X given Y . □

Definition 4.2.4. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and μ a σ -finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Then $P_{X,Y} \ll \mu^2$. Let $f_X = dP_X/d\mu$, $f_Y = dP_Y/d\mu$ and $f_{X,Y} = dP_{X,Y}/d\mu^2$. Define $f_{X|Y} : \mathbb{R}^n \times \mathbb{R}^n$ by

$$f_{X|Y}(x, y) = \begin{cases} \frac{f_{X,Y}(x, y)}{f_Y(y)}, & y \in \text{supp } Y \\ 0, & y \notin \text{supp } Y \end{cases}$$

Then $f_{X|Y}$ is called the **conditional probability density of X given Y** .

Exercise 4.2.5. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and μ a σ -finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Define $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$ by

$$Q(y, A) = \int_A f_{X|Y}(x, y) d\mu(x)$$

Then Q is a conditional probability distribution of X given Y .

Proof. By the Fubini-Tonelli Theorem, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable and for P_Y -a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) . Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned} \int_B Q(y, A) dP_Y(y) &= \int_B \left[\int_A f_{X|Y}(x, y) d\mu(x) \right] dP_Y(y) \\ &= \int_{B \cap \text{supp } Y} \left[\int_{A \cap \text{supp } Y} \frac{f_{X,Y}(x, y)}{f_Y(y)} d\mu(x) f_Y(y) \right] d\mu(y) \\ &= \int_{B \cap \text{supp } Y} \left[\int_A f_{X,Y}(x, y) d\mu(x) \right] d\mu(y) \\ &= P(X \in A, Y \in B \cap \text{supp } Y) \\ &= P(X \in A, Y \in B) \end{aligned}$$

□

Theorem 4.2.1. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^1(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\text{Im } X \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a conditional probability distribution of Y given X .

5. MARKOV CHAINS

Definition 5.0.1. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}_0} \in L_n^0$. Then $(X_n)_{n \in \mathbb{N}_0}$ is said to be a **homogeneous Markov chain** if for each $A \in \mathcal{F}$ and $n \in \mathbb{N}$, $P(X_n \in A | X_1, \dots, X_{n-1}) = P(X_1 \in A | X_0)$ a.e.