1

Introduction to Algebra

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[1]

Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Part I

Sets and Order

Chapter 1

Set Theory

1.1 Operations and Relations

Definition 1.1.0.1.

- We define $[0] := \varnothing$ and for $k \in \mathbb{N}$, we define $[k] := \{1, \ldots, k\}$.
- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-tupels with entries in** S, denoted S^k, by
$$S^k := \{u : [k] \to S\}$$
- Let S be a set. We define the **set of all tuples with entries in** S, denoted S^*, by
$$S^* := \bigcup_{k \in \mathbb{N}_0} S^k$$
- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary operations on** S, denoted $\mathcal{F}^k(S)$, by $\mathcal{F}^k(S) := S^{(S^k)}$. We define the **set of finitary operations on** S, denoted $\mathcal{F}^*(S)$, by
$$\mathcal{F}^*(S) := \bigcup_{k \in \mathbb{N}_0} \mathcal{F}^k(S)$$
- Let S be a set. We define the **operation arity map**, denoted $\operatorname{arity} : \mathcal{F}^*(S) \to \mathbb{N}_0$, by
$$\operatorname{arity} f := k, \quad f \in \mathcal{F}^k(S)$$
- Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{F}$, denoted $\mathcal{F}_k$, by
$$\mathcal{F}_k := \mathcal{F} \cap \mathcal{F}^k(S)$$
- Let S be a set and $k \in \mathbb{N}_0$. We define the **set of k-ary relations on** S, denoted $\mathcal{R}^k(S)$, by $\mathcal{R}^k(S) := \mathcal{P}(S^k)$. We define the **set of finitary relations on** S, denoted $\mathcal{R}^*(S)$, by
$$\mathcal{R}^*(S) := \bigcup_{k \in \mathbb{N}_0} \mathcal{R}^k(S)$$
- Let S be a set. We define the **arity map**, denoted $\operatorname{arity} : \mathcal{R}^*(S) \to \mathbb{N}_0$, by
$$\operatorname{arity} R := k, \quad f \in \mathcal{R}^k(S)$$
- Let S be a set, $\mathcal{R} \subset \mathcal{R}^*(S)$ and $k \in \mathbb{N}_0$. We define the **k-ary members of** $\mathcal{R}$, denoted $\mathcal{R}_k$, by
$$\mathcal{R}_k := \mathcal{R} \cap \mathcal{R}^k(S)$$

Definition 1.1.0.2. Let S be a set, $k \geq 2$ and $f \in \mathcal{F}^k(S)$. Then f is said to be

- **associative** if for each $x_1, \dots, x_k, x_{k+1}, \dots, x_{k+(k-1)} \in S$,

$$\begin{aligned} f(f(x_1, \dots, x_k)x_{k+1}, \dots, x_{k+(k-1)}) &= f(x_1, f(x_2, \dots, x_{k+1}), x_{k+2}, \dots, x_{k+(k-1)}) \\ &\vdots \\ &= f(x_1, \dots, x_{k-1}, f(x_k, \dots, x_{k+(k-1)})) \end{aligned}$$

- **symmetric** if for each $x_1, \dots, x_k \in S$, $\sigma \in S_k$, $f(x_1, \dots, x_k) = f(x_{\sigma(1)}, \dots, x_{\sigma(k)})$.
- **idempotent** if for each $x \in S$, $f(x, \dots, x) = x$

Definition 1.1.0.3. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $C \subset S$. Then C is said to be **$\mathcal{F}$-closed** if for each $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$ and $a \in C^k$, $f(a) \in C$.

Exercise 1.1.0.4. Let S be a set, $\mathcal{F} \subset \mathcal{F}^*(S)$ and $\mathcal{C} \subset \mathcal{P}(S)$. If for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed, then $\bigcap_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed need special case where $k = 0$? maybe trivially true?

Proof. Suppose that for each $C \in \mathcal{C}$, C is $\mathcal{F}$-closed. Let $k \in \mathbb{N}_0$, $f \in \mathcal{F}_k$, $a_1, \dots, a_k \in \bigcap_{C \in \mathcal{C}} C$ and $C_0 \in \mathcal{C}$. Since $C_0 \in \mathcal{C}$, we have that

$$\begin{aligned} a_1, \dots, a_k &\in \bigcap_{C \in \mathcal{C}} C \\ &\subset C_0 \end{aligned}$$

Since C_0 is $\mathcal{F}$-closed, we have that $f(a_1, \dots, a_k) \in C_0$. Since $C_0 \in \mathcal{C}$ is arbitrary, we have that for each $C \in \mathcal{C}$, $f(a_1, \dots, a_k) \in C$. Hence $f(a_1, \dots, a_k) \in \bigcap_{C \in \mathcal{C}} C$. Since $k \in \mathbb{N}_0$ and $a_1, \dots, a_k \in \bigcap_{C \in \mathcal{C}} C$ are arbitrary, we have that $\bigcap_{C \in \mathcal{C}} C$ is $\mathcal{F}$-closed. $\square$

Chapter 2

Ordered Sets

2.1 To Do

at this point, there should be a simple structure capturing prosets and $\downarrow S$, measurable spaces and $\sigma(\mathcal{A})$, topological spaces and $\tau(\mathcal{E})$, groups/rings/$\cdots$/algebras and $\langle S\rangle/(E)/\cdots/(F)$. It seems like we need a category $\mathcal{C}$ with some structure, another category $\mathcal{C}_F$ which is like $\mathcal{C}$, but less structured and some forgetful-like functor $F : \mathcal{C} \to \mathcal{C}_F$. eg, $F : \mathbf{Pro} \to \mathbf{Set}$ or $F : \mathbf{Top} \to \mathbf{Set}$ and we need an associated poset $\mathcal{P}$ containing the structure forgotten by F, e.g. the lower sets of X or the set of topologies on X and we need a "generating" or "contained" object $\mathcal{E}$ in $\mathcal{C}_F$ in the sense that there is at least one object A in $\mathcal{C}$ and monomirphism $\iota_A : \mathcal{E} \to F(A)$ in $\mathcal{C}_F$. We then need a minimal element in the structure poset $\mathcal{P}$ in some universal sense relating to these monomorphisms. Ask people who know category theory

2.2 Prosets

2.2.1 Introduction

Definition 2.2.1.1. Preordered Set:
Let X be a set and $\leq \subset X \times X$ a binary relation on X. Then

- $\leq$ is said to be a **preorder on** X if
  1. for each $a \in X$, $a \leq a$
  2. for each $a, b, c \in X$, $a \leq b$ and $b \leq c$ implies that $a \leq c$
- $(X, \leq)$ is said to be a **preordered set** or **proset** if $\leq$ is a preorder on X.

Definition 2.2.1.2. Let $(X, \leq)$ be a proset. We define the **dual order of** $\leq$ **on** X, denoted $\leq^{\mathrm{op}}$, by $a \leq^{\mathrm{op}} b$ iff $b \leq a$.

Exercise 2.2.1.3. Let $(X, \leq)$ be a proset. Then $\leq^{\mathrm{op}}$ is a preorder on X.

Proof.

1. Let $a \in X$. Since $a \leq a$, we have that $a \leq^{\mathrm{op}} a$.
2. Let $a, b, c \in X$. Suppose that $a \leq^{\mathrm{op}} b$ and $b \leq^{\mathrm{op}} c$. Then $b \leq a$ and $c \leq b$. Hence $c \leq a$. Thus $a \leq^{\mathrm{op}} c$.

Therefore $\leq^{\mathrm{op}}$ is a preorder on X. $\square$

2.2.2 Products

Definition 2.2.2.1. Let $(A, \leq_A)$ and $(B, \leq_B)$ be prosets. We define the

- **product preorder of** $\leq_A$ **and** $\leq_B$ **on** $A \times B$, denoted $\leq_A \otimes \leq_B$ by $(a_1, b_1) \leq_A \otimes \leq_B (a_2, b_2)$ iff $a_1 \leq_A a_2$ and $b_1 \leq_B b_2$.

- **product proset of** $(A, \leq_A)$ **and** $(B, \leq_B)$, denoted $(A, \leq_A) \otimes (B, \leq_B)$ by $(A, \leq_A) \otimes (B, \leq_B) := (A \times B, \leq_A \otimes \leq_B)$

Exercise 2.2.2.2. probably need to change notation since $\otimes$ might be reserved for something else. Let $(A, \leq_A)$ and $(B, \leq_B)$ be prosets. Then

1. $\leq_A \otimes \leq_B$ is a preorder on $A \times B$,
2. $(A, \leq_A) \otimes (B, \leq_B)$ is a proset.

Proof.

1. (a) Let $(a, b) \in A \times B$. Then $a \leq_A a$ and $b \leq_B b$. Therefore $(a, b) \leq_A \otimes \leq_B (a, b)$.

   (b) Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A \times B$. Suppose that $(a_1, b_1) \leq_A \otimes \leq_B (a_2, b_2)$ and $(a_2, b_2) \leq_A \otimes \leq_B (a_3, b_3)$. Then $a_1 \leq_A a_2$, $a_2 \leq_A a_3$, $b_1 \leq_B b_2$ and $b_2 \leq_B b_3$. Therefore $a_1 \leq_A a_3$ and $b_1 \leq_B b_3$. Hence $(a_1, b_1) \leq_A \otimes \leq_B (a_3, b_3)$.

   Hence $\leq_A \otimes \leq_B$ is a preorder on $A \times B$.
2. Since $\leq_A \otimes \leq_B$ is a preorder on $A \times B$, $(X, \leq)$ is a proset.

$\square$

Definition 2.2.2.3. Let $(X, \leq)$ be a proset. We define $\sim_\leq \subset X \times X$ by $a \sim_\leq b$ iff $a \leq b$ and $b \leq a$.

Exercise 2.2.2.4. Let $(X, \leq)$ be a proset. Then $\sim_\leq$ is an equivalence relation on X.

Proof. Let $x, y, z \in X$.

1. Since $x \leq x$, $x \sim_\leq x$.
2. Suppose that $x \sim_\leq y$. Then $x \leq y$ and $y \leq x$. Thus $y \sim_\leq x$.
3. Suppose that $x \sim_\leq y$ and $y \sim_\leq z$. Then $x \leq y$, $y \leq x$, $y \leq z$ and $z \leq y$. Therefore $x \leq z$ and $z \leq x$. Hence $x \sim_\leq z$.

$\square$

2.2.3 Upper and Lower Sets

Definition 2.2.3.1. Let $(X, \leq)$ be a proset and $A \subset X$. Then

- A is said to be a $\leq$**-upper set** if for each $a \in A$ and $x \in X$, $a \leq x$ implies that $x \in A$.
- A is said to be an $\leq$**-lower set** if for each $a \in A$ and $x \in X$, $x \leq a$ implies that $x \in A$.

Note 2.2.3.2. When the context is clear, we say A is a

- "upper set" instead of "$\leq$-upper set"
- "lower set" instead of "$\leq$-lower set"

Exercise 2.2.3.3. Let $(X, \leq)$ be a proset. Then

1. X is a $\leq$-upper set
2. X is a $\leq$-lower set

Proof.

1. Let $a, x \in X$. Suppose that $a \leq x$. By assumption, $x \in X$. Since $a, x \in X$ with $a \leq x$ are arbitrary, we have that for each $a, x \in X$, $a \leq x$ implies that $x \in A$. Hence X is a $\leq$-upper set.
2. Similar to (1).

$\square$

Exercise 2.2.3.4. Let $(X, \leq)$ be a proset and $A \subset X$. Then

1. A is a $\leq$-upper set iff A is a $\leq^{\text{op}}$-lower set

2. A is a $\leq$-lower set iff A is a $\leq^{\text{op}}$-upper set

Proof.

1.
   - ($\implies$):
     Suppose that A is a $\leq$-upper set. Let $a \in A$ and $x \in X$. Suppose that $x \leq^{\text{op}} a$. Then $a \leq x$. Since A is a $\leq$-upper set, $x \in A$. Since $a \in A$ and $x \in X$ with $x \leq^{\text{op}} a$ is arbitrary, we have that for each $a \in A$ and $x \in X$, $x \leq^{\text{op}} a$ implies that $x \in A$. Hence A is a $\leq^{\text{op}}$-lower set.
   - ($\impliedby$):
     Suppose that A is a $\leq^{\text{op}}$-lower set. Let $a \in A$ and $x \in X$. Suppose that $a \leq x$. Then $x \leq^{\text{op}} a$. Since A is a $\leq^{\text{op}}$-lower set, $x \in A$. Since $a \in A$ and $x \in X$ with $a \leq x$ is arbitrary, we have that for each $a \in A$ and $x \in X$, $a \leq x$ implies that $x \in A$. Hence A is a $\leq$-upper set.

2. Similar to (1).

$\square$

Exercise 2.2.3.5. Let $(X, \leq)$ be a proset and $A \subset X$. Then

1. A is an upper set iff A^c is a lower set

2. A is a lower set iff A^c is an upper set

Proof.

1.
   - ($\implies$):
     Suppose that A is an upper set. Let $b \in A^c$ and $x \in X$. Suppose that $x \leq b$. For the sake of contradiction, suppose that $x \in A$. Since $x \leq b$ and A is an upper set, $b \in A$. This is a contradiction since $b \in A^c$. Hence $x \in A^c$. Since $b \in A^c$ and $x \in X$ with $x \leq b$ are arbitrary, we have that for each $b \in A^c$ and $x \in X$, $x \leq b$ implies that $x \in A^c$. Thus A^c is a lower set.
   - ($\impliedby$):
     Suppose that A^c is a $\leq$-lower set. Exercise 2.2.3.4 implies that A^c is a $\leq^{\text{op}}$-upper set. The previous part implies that A is a $\leq^{\text{op}}$-lower set. Another application of Exercise 2.2.3.4 implies that A is a $\leq$-upper set.

2. Similar to (1).

$\square$

Exercise 2.2.3.6. Let $(X, \leq)$ be a proset, $(E_\alpha)_{\alpha \in A} \subset \mathcal{P}(X)$.

1. If for each $\alpha \in A$, E_α is an upper set, then
   (a) $\bigcup\limits_{\alpha \in A} E_\alpha$ is an upper set
   (b) $\bigcap\limits_{\alpha \in A} E_\alpha$ is an upper set

2. If for each $\alpha \in A$, E_α is a lower set, then
   (a) $\bigcup\limits_{\alpha \in A} E_\alpha$ is a lower set
   (b) $\bigcap\limits_{\alpha \in A} E_\alpha$ is an lower set

Proof.

1. Suppose that for each $\alpha \in A$, E_α is an upper set. Set $E := \bigcup\limits_{\alpha \in A} E_\alpha$.

(a) Let $e \in E$ and $x \in X$. Suppose that $e \leq x$. Since $e \in E$, there exists $\alpha \in A$ such that $e \in E_\alpha$. Since E_α is an upper set and $e \leq x$, we have that

$$\begin{aligned} x &\in E_\alpha \\ &\subset \bigcup_{\alpha \in A} E_\alpha \\ &= E. \end{aligned}$$

Since $e \in E$ and $x \in X$ with $e \leq x$ are arbitrary, we have that for each $e \in E$ and $x \in X$, $e \leq x$ implies that $x \in E$. Hence E is an upper set.

(b) Let $\alpha \in A$. Since E_α is a $\leq$-upper set, Exercise 2.2.3.5 implies that E_α^c is a $\leq$-lower set. Exercise 2.2.3.4 then implies that E_α^c is a $\leq^{\mathrm{op}}$-upper set. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, E_α^c is a $\leq^{\mathrm{op}}$-upper set. The previous part implies that $\bigcup_{\alpha \in A} E_\alpha^c$ is a $\leq^{\mathrm{op}}$-upper set. Since $\bigcap_{\alpha \in A} E_\alpha = \left(\bigcup_{\alpha \in A} E_\alpha^c \right)^c$, Exercise 2.2.3.5 implies that $\bigcap_{\alpha \in A} E_\alpha$ is a $\leq^{\mathrm{op}}$-lower set. Exercise 2.2.3.4 then implies that $\bigcap_{\alpha \in A} E_\alpha$ is a $\leq$-upper set.

2. Similar to (1).

□

Definition 2.2.3.7. Let $(X, \leq)$ be a proset and $S \subset X$. We define the

- **$\leq$-upper sets containing** S, denoted $\mathcal{U}(S, \leq) \subset \mathcal{P}(X)$, by $\mathcal{U}(S, \leq) := \{U \subset X : U \text{ is a } \leq\text{-upper set and } S \subset U\}$
- **$\leq$-lower sets containing** S, denoted $\mathcal{L}(S, \leq) \subset \mathcal{P}(X)$, by $\mathcal{L}(S, \leq) := \{L \subset X : L \text{ is a } \leq\text{-lower set and } S \subset L\}$.
- **$\leq$-upper set generated by** S, denoted $\uparrow(S, \leq)$, by $\uparrow(S, \leq) := \bigcap_{U \in \mathcal{U}(S, \leq)} U$
- **$\leq$-lower set generated by** S, denoted $\downarrow(S, \leq)$, by $\downarrow(S, \leq) := \bigcap_{L \in \mathcal{L}(S, \leq)} L$

Note 2.2.3.8.

- When the context is clear, we write $\mathcal{U}(S)$, $\mathcal{L}(S)$, $\uparrow S$ and $\downarrow S$ in place of $\mathcal{U}(S, \leq)$, $\mathcal{L}(S, \leq)$, $\uparrow(S, \leq)$ and $\downarrow(S, \leq)$ respectively.
- If $S = \{s\}$, we write $\uparrow s$ and $\downarrow s$ in place of $\uparrow S$ and $\downarrow S$ respectively.
- Exercise 2.2.3.3 implies that $X \in \mathcal{U}(S)$ and $X \in \mathcal{L}(S)$.
- Exercise 2.2.3.6 implies that $\uparrow S \in \mathcal{U}(S)$ and $\downarrow S \in \mathcal{L}(S)$.

Exercise 2.2.3.9. Let $(X, \leq)$ be a proset and $S \subset X$. Then

1. $\mathcal{U}(S, \leq^{\mathrm{op}}) = \mathcal{L}(S, \leq)$,
2. $\mathcal{L}(S, \leq^{\mathrm{op}}) = \mathcal{U}(S, \leq)$,
3. $\uparrow(S, \leq^{\mathrm{op}}) = \downarrow(S, \leq)$,
4. $\downarrow(S, \leq^{\mathrm{op}}) = \uparrow(S, \leq)$.

Proof.

1.
   - Let $L \in \mathcal{U}(S, \leq^{\mathrm{op}})$. Then $S \subset L$ and L is a $\leq^{\mathrm{op}}$-upper set. Exercise 2.2.3.4 then implies that L is a $\leq$-lower set. Hence $L \in \mathcal{L}(S, \leq)$. Since $L \in \mathcal{U}(S, \leq^{\mathrm{op}})$ is arbitrary, we have that for each $L \in \mathcal{U}(S, \leq^{\mathrm{op}})$, $L \in \mathcal{L}(S, \leq)$. Thus $\mathcal{U}(S, \leq^{\mathrm{op}}) \subset \mathcal{L}(S, \leq)$.
   - Let $L \in \mathcal{L}(S, \leq)$. Then $S \subset L$ and L is a $\leq$-lower set. Another application of Exercise 2.2.3.4 implies that L is a $\leq^{\mathrm{op}}$-upper set. Hence $L \in \mathcal{U}(S, \leq^{\mathrm{op}})$. Since $L \in \mathcal{L}(S, \leq)$ is arbitrary, we have that for each $L \in \mathcal{L}(S, \leq)$, $L \in \mathcal{U}(S, \leq^{\mathrm{op}})$. Thus $\mathcal{L}(S, \leq) \subset \mathcal{U}(S, \leq^{\mathrm{op}})$.

   Since $\mathcal{U}(S, \leq^{\mathrm{op}}) \subset \mathcal{L}(S, \leq)$ and $\mathcal{L}(S, \leq) \subset \mathcal{U}(S, \leq^{\mathrm{op}})$, we have that $\mathcal{U}(S, \leq^{\mathrm{op}}) = \mathcal{L}(S, \leq)$.

2. By (1), we have that

$$\begin{aligned}\mathcal{L}(S, \leq^{\mathrm{op}}) &= \mathcal{U}(S, (\leq^{\mathrm{op}})^{\mathrm{op}}) \\ &= \mathcal{U}(S, \leq).\end{aligned}$$

3. Part (1) implies that

$$\begin{aligned}\uparrow(S, \leq^{\mathrm{op}}) &= \bigcap_{U \in \mathcal{U}(S, \leq^{\mathrm{op}})} U \\ &= \bigcap_{U \in \mathcal{L}(S, \leq)} U \\ &= \downarrow(S, \leq^{\mathrm{op}}).\end{aligned}$$

4. Similar to (3).

$\square$

Exercise 2.2.3.10. Let $(X, \leq)$ be a proset and $S \subset X$. Then

1. $S \in \mathcal{U}(S, \leq)$ iff $\uparrow S = S$
2. $S \in \mathcal{L}(S, \leq)$ iff $\downarrow S = S$

Proof.

1.
   - ($\implies$):
     Suppose that $S \in \mathcal{U}(S, \leq)$. Then

     $$\begin{aligned}\uparrow S &= \bigcap_{U \in \mathcal{U}(S, \leq)} U \\ &\subset S \\ &\subset \uparrow S.\end{aligned}$$

     Hence $\uparrow S = S$.
   - ($\impliedby$):
     Suppose that $\uparrow S = S$. Then

     $$\begin{aligned}S &= \uparrow S \\ &\in \mathcal{U}(S, \leq).\end{aligned}$$

2. Similar to (1).

$\square$

Exercise 2.2.3.11. Let $(X, \leq)$ be a proset and $a \in X$. Then

1. $\uparrow a = \{x \in X : a \leq x\}$
2. $\downarrow a = \{x \in X : x \leq a\}$

Proof.

1. Define $U \subset X$ by $U := \{x \in X : a \leq x\}$.

- Let $u \in U$ and $x \in X$. Suppose that $u \leq x$. Then

$$\begin{aligned} a &\leq u \\ &\leq x. \end{aligned}$$

Hence $x \in U$. Since $u \in U$ and $x \in X$ with $u \leq x$ are arbitrary, we have that for each $u \in U$ and $x \in X$, $u \leq x$ implies that $x \in U$. Hence U is an upper set. Since $a \leq a$, $a \in U$. By definition, $U \in \mathcal{U}(a)$. Therefore

$$\begin{aligned} \uparrow a &= \bigcap_{U' \in \mathcal{U}(a)} U' \\ &\subset U. \end{aligned}$$

- Let $x \in U$ and $U' \in \mathcal{U}(a)$. Since $x \in U$, $a \leq x$. Since $U' \in \mathcal{U}(a)$, U' is an upper set and $a \in U'$. Thus $x \in U'$. Since $U' \in \mathcal{U}(a)$ is arbitrary, we have that for each $U' \in \mathcal{U}(a)$, $x \in U'$. Thus

$$\begin{aligned} x &\in \bigcap_{U' \in \mathcal{U}(a)} U' \\ &= \uparrow a. \end{aligned}$$

Since $x \in U$ is arbitrary, we have that for each $x \in U$, $x \in \uparrow a$. Hence $U \subset \uparrow a$.

Since $\uparrow a \subset U$ and $U \subset \uparrow a$, we have that $\uparrow a = U$.

2. Exercise 2.2.3.4 and (1) imply that

$$\begin{aligned} \downarrow(a, \leq) &= \uparrow(a, \leq^{\text{op}}) \\ &= \{x \in X : a \leq^{\text{op}} x\} \\ &= \{x \in X : x \leq a\}. \end{aligned}$$

□

Exercise 2.2.3.12. Let $(X, \leq)$ be a proset and $S_1, S_2 \subset X$. If $S_1 \subset S_2$, then

1. $\uparrow S_1 \subset \uparrow S_2$
2. $\downarrow S_1 \subset \downarrow S_2$

Proof. Suppose that $S_1 \subset S_2$.

1. Since $\uparrow S_2 \in \mathcal{U}(S_2)$, $S_2 \subset \uparrow S_2$ and $\uparrow S_2$ is an upper set. Since

$$\begin{aligned} S_1 &\subset S_2 \\ &\subset \uparrow S_2, \end{aligned}$$

we have that $\uparrow S_2 \in \mathcal{U}(S_1)$. Therefore

$$\begin{aligned} \uparrow S_1 &= \bigcap_{U \in \mathcal{U}(S_1)} U \\ &\subset \uparrow S_2. \end{aligned}$$

2. Part (1) and Exercise 2.2.3.9 implies that

$$\begin{aligned} \downarrow(S_1, \leq) &= \uparrow(S_1, \leq^{\text{op}}) \\ &\subset \uparrow(S_2, \leq^{\text{op}}) \\ &= \downarrow(S_2, \leq). \end{aligned}$$

$\square$

Exercise 2.2.3.13. Let $(X, \leq)$ be a proset and $a, b \in X$. Then the following are equivalent:

1. $a \leq b$,
2. $\uparrow b \subset \uparrow a$,
3. $\downarrow a \subset \downarrow b$.

Proof.

1. (1) $\implies$ (2):
   Suppose that $a \leq b$. Since $\uparrow a \in \mathcal{U}(a)$, we have that $a \in \uparrow a$ and $\uparrow a$ is an upper set. Since $a \leq b$, we have that $b \in \uparrow a$. Hence $\uparrow a \in \mathcal{U}(b)$. Therefore

$$\begin{aligned}\uparrow b &= \bigcap_{U \in \mathcal{U}(b)} U \\ &\subset \uparrow a.\end{aligned}$$

2. (2) $\implies$ (3):
   Suppose that $\uparrow(b, \leq) \subset \uparrow(a, \leq)$. Exercise 2.2.3.11 then implies that

$$\begin{aligned} b &\in \uparrow(b, \leq) \\ &\subset \uparrow(a, \leq) \\ &= \{x \in X : a \leq x\}. \end{aligned}$$

   Hence $a \leq b$. Thus $b \leq^{\text{op}} a$. Exercise 2.2.3.9 and part (1) $\implies$ (2) then imply that

$$\begin{aligned} \downarrow(a, \leq) &= \uparrow(a, \leq^{\text{op}}) \\ &\subset \uparrow(b, \leq^{\text{op}}) \\ &= \downarrow(b, \leq). \end{aligned}$$

3. (3) $\implies$ (1):
   Suppose that $\downarrow a \subset \downarrow b$. Exercise 2.2.3.11 then implies that

$$\begin{aligned} a &\in \downarrow a \\ &\subset \downarrow b \\ &= \{x \in X : x \leq b\}. \end{aligned}$$

   Hence $a \leq b$.

$\square$

Exercise 2.2.3.14. Let $(X, \leq)$ be a proset and $S \subset X$. Then

1. $\uparrow S = \bigcup_{s \in S} \uparrow s$
2. $\downarrow S = \bigcup_{s \in S} \downarrow s$

Proof.

1. Define $U \subset X$ by $U := \bigcup_{s \in S} \uparrow s$.

- Since for each $s \in S$, $\uparrow s$ is an upper set, Exercise 2.2.3.6 implies that U is an upper set. Let $s \in S$. Then

$$\begin{aligned} s &\in \uparrow s \\ &\subset \bigcup_{s \in S} \uparrow s \\ &= U. \end{aligned}$$

Since $s \in S$ is arbitrary, we have that for each $s \in S$, $s \in U$. Hence $S \subset U$. Therefore $U \in \mathcal{U}(S)$ and

$$\begin{aligned} \uparrow S &= \bigcap_{U' \in \mathcal{U}(S)} U' \\ &\subset U. \end{aligned}$$

- Let $x \in U$ and $U' \in \mathcal{U}(S)$. Since $x \in U$, there exists $s \in S$ such that $x \in \uparrow s$. Exercise 2.2.3.11 then implies that $s \leq x$. Since $U' \in \mathcal{U}(S)$, U' is an upper set and $S \subset U'$. Then

$$\begin{aligned} s &\in S \\ &\subset U'. \end{aligned}$$

Since U' is an upper set and $s \leq x$, $x \in U'$. Since $U' \in \mathcal{U}(S)$ is arbitrary, we have that for each $U' \in \mathcal{U}(S)$, $x \in U'$. Thus

$$\begin{aligned} x &\in \bigcap_{U' \in \mathcal{U}(S)} U' \\ &= \uparrow S. \end{aligned}$$

Since $x \in U$ is arbitrary, we have that for each $x \in U$, $x \in \uparrow S$. Hence $U \subset \uparrow S$.

Since $\uparrow S \subset U$ and $U \subset \uparrow S$, we have that $\uparrow S = U$.

2. Similar to (1).

□

Exercise 2.2.3.15. Let $(X, \leq)$ be a proset and $(E_\alpha)_{\alpha \in A} \subset \mathcal{P}(X)$. Then

1. $\uparrow \bigcup_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} \uparrow E_\alpha$
2. $\downarrow \bigcup_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} \downarrow E_\alpha$

Proof.

1. - Let $x \in \uparrow \bigcup_{\alpha \in A} E_\alpha$. Exercise 2.2.3.14 implies that there exists $y \in \bigcup_{\alpha \in A} E_\alpha$ such that $x \in \uparrow y$. Then there exists $\alpha_0 \in A$ such that $y \in E_{\alpha_0}$. Since $\{y\} \subset E_{\alpha_0}$, Exercise 2.2.3.12 implies that $\uparrow y \subset \uparrow E_{\alpha_0}$. Therefore

$$\begin{aligned} x &\in \uparrow y \\ &\subset \uparrow E_{\alpha_0} \\ &\subset \bigcup_{\alpha \in A} \uparrow E_\alpha. \end{aligned}$$

Since $x \in \uparrow \bigcup_{\alpha \in A} E_\alpha$ is arbitrary, we have that for each $x \in \uparrow \bigcup_{\alpha \in A} E_\alpha$, $x \in \bigcup_{\alpha \in A} \uparrow E_\alpha$. Hence $\uparrow \bigcup_{\alpha \in A} E_\alpha \subset \bigcup_{\alpha \in A} \uparrow E_\alpha$.

- Let $x \in \bigcup_{\alpha \in A} \uparrow E_\alpha$. Then there exists $\alpha_0 \in A$ such that $x \in \uparrow E_{\alpha_0}$. Exercise 2.2.3.14 implies that there exists $y \in E_{\alpha_0}$ such that $x \in \uparrow y$. Since

$$\begin{aligned} \{y\} &\subset E_{\alpha_0} \\ &\subset \bigcup_{\alpha \in A} E_\alpha, \end{aligned}$$

Exercise 2.2.3.12 implies that

$$\begin{aligned} x &\in \uparrow y \\ &\subset \uparrow \bigcup_{\alpha \in A} E_\alpha. \end{aligned}$$

Since $x \in \bigcup_{\alpha \in A} \uparrow E_\alpha$ is arbitrary, we have that for each $x \in \bigcup_{\alpha \in A} \uparrow E_\alpha$, $x \in \uparrow \bigcup_{\alpha \in A} E_\alpha$. Hence $\bigcup_{\alpha \in A} \uparrow E_\alpha \subset \uparrow \bigcup_{\alpha \in A} E_\alpha$.

Since $\uparrow \bigcup_{\alpha \in A} E_\alpha \subset \bigcup_{\alpha \in A} \uparrow E_\alpha$ and $\bigcup_{\alpha \in A} \uparrow E_\alpha \subset \uparrow \bigcup_{\alpha \in A} E_\alpha$, we have that $\uparrow \bigcup_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} \uparrow E_\alpha$.

2. Similar to (1).

□

Definition 2.2.3.16. Let $(X, \leq)$ be a proset and $a \in X$. Then a is said to be

- **$\leq$-maximal** if for each $x \in X$, $a \leq x$ implies that $x \sim_\leq a$.
- **$\leq$-minimal** if for each $x \in X$, $x \leq a$ implies that $x \sim_\leq a$.

Note 2.2.3.17. When the context is clear, we write "maximal" and "minimal" instead of "$\leq$-maximal" and "$\leq$-minimal" respectively.

Exercise 2.2.3.18. Let $(X, \leq)$ be a proset and $a \in X$. Then

1. a is maximal iff $\uparrow a = \pi_{X/\sim_\leq}(a)$
2. a is minimal iff $\downarrow a = \pi_{X/\sim_\leq}(a)$

Proof.

1. - ($\implies$):
     Suppose that a is maximal.
     - Let $x \in \uparrow a$. Exercise 2.2.3.11 implies $a \leq x$. Since a is maximal, $x \sim_\leq a$. Thus $x \in \pi_{X/\sim_\leq}(a)$. Since $x \in \uparrow a$ is arbitrary, we have that for each $x \in \uparrow a$, $x \in \pi_{X/\sim_\leq}(a)$. Hence $\uparrow a \subset \pi_{X/\sim_\leq}(a)$.
     - Let $x \in \pi_{X/\sim_\leq}(a)$. Then $a \sim_\leq x$. Hence $a \leq x$ and $x \leq a$. Since $a \leq x$, we have that $x \in \uparrow a$. Since $x \in \pi_{X/\sim_\leq}(a)$ is arbitrary, we have that for each $x \in \pi_{X/\sim_\leq}(a)$, $x \in \uparrow a$. Hence $\pi_{X/\sim_\leq}(a) \subset \uparrow a$.

     Since $\uparrow a \subset \pi_{X/\sim_\leq}(a)$ and $\pi_{X/\sim_\leq}(a) \subset \uparrow a$, we have that $\uparrow a = \pi_{X/\sim_\leq}(a)$.
   - ($\impliedby$):
     Suppose that $\uparrow a = \pi_{X/\sim_\leq}(a)$. Let $x \in X$. Suppose that $a \leq x$. Then

$$\begin{aligned} x &\in \uparrow a \\ &= \pi_{X/\sim_\leq}(a). \end{aligned}$$

     Thus $x \sim_\leq a$. Since $x \in X$ with $a \leq x$ is arbitrary, we have that for each $x \in X$, $a \leq x$ implies that $x \sim_\leq a$. Hence a is maximal.
2. Similar to (1).

□

Definition 2.2.3.19. Let $(X, \leq)$ be a proset and $A \subset X$.

- Let $x \in X$. Then x is said to be a
  - **$\leq$-upper bound of** A if for each $a \in A$, $a \leq x$
  - **$\leq$-lower bound of** A if for each $a \in A$, $x \leq a$
- We define
  - $\mathrm{ub}(A, \leq) := \{x \in X : x \text{ is a } \leq\text{-upper bound of } A\}$
  - $\mathrm{lb}(A, \leq) := \{x \in X : x \text{ is a } \leq\text{-lower bound of } A\}$
- Then A is said to be
  - **$\leq$-bounded above** if $\mathrm{ub}(A, \leq) \neq \varnothing$
  - **$\leq$-bounded below** if $\mathrm{lb}(A, \leq) \neq \varnothing$

Note 2.2.3.20. When the context is clear, we write

- "upper bound" and "lower bound" instead of "$\leq$-upper bound" and "$\leq$-lower bound" respectively
- $\mathrm{ub}\, A$ and $\mathrm{lb}\, A$ in place of $\mathrm{ub}(A, \leq)$ and $\mathrm{lb}(A, \leq)$ respectively
- "bounded above" and "bounded below" instead of "$\leq$-bounded above" and "$\leq$-bounded below" respectively

Exercise 2.2.3.21. Let $(X, \leq)$ be a proset and $A \subset X$. Then

1. $\mathrm{ub}\, A = \bigcap\limits_{x \in A} \uparrow x$
2. $\mathrm{lb}\, A = \bigcap\limits_{x \in A} \downarrow x$

Proof.

1. - Let $a \in \mathrm{ub}\, A$ and $x \in A$. Then $x \leq a$. Hence $a \in \uparrow x$. Since $x \in A$ is arbitrary, we have that for each $x \in A$, $a \in \uparrow x$. Thus $a \in \bigcap\limits_{x \in A} \uparrow x$. Since $a \in \mathrm{ub}\, A$ is arbitrary, we have that for each $a \in \mathrm{ub}\, A$, $a \in \bigcap\limits_{x \in A} \uparrow x$. Hence $\mathrm{ub}\, A \subset \bigcap\limits_{x \in A} \uparrow x$.
   - Let $a \in \bigcap\limits_{x \in A} \uparrow x$ and $x_0 \in A$. Then $a \in \uparrow x_0$. Hence $x_0 \leq a$. Since $x_0 \in A$ is arbitrary, we have that for each $x_0 \in A$, $x_0 \leq a$. Hence $a \in \mathrm{ub}\, A$. Since $a \in \bigcap\limits_{x \in A} \uparrow x$ is arbitrary, we have that for each $a \in \bigcap\limits_{x \in A} \uparrow x$, $a \in \mathrm{ub}\, A$. Thus $\bigcap\limits_{x \in A} \uparrow x \subset \mathrm{ub}\, A$.

   Since $\mathrm{ub}\, A \subset \bigcap\limits_{x \in A} \uparrow x$ and $\bigcap\limits_{x \in A} \uparrow x \subset \mathrm{ub}\, A$, we have that $\mathrm{ub}\, A = \bigcap\limits_{x \in A} \uparrow x$.
2. Similar to (1).

□

Definition 2.2.3.22. Let $(X, \leq)$ be a proset and $A \subset X$.

- Let $x \in X$. Then x is said to be a
  - **supremum of** A or **least upper bound of** A, if
    1. $x \in \mathrm{ub}\, A$
    2. for each $y \in \mathrm{ub}\, A$, $x \leq y$
  - **infimum of** A or **greatest lower bound of** A, denoted $x \in \inf A$, if
    1. $x \in \mathrm{lb}\, A$

2. for each $y \in \operatorname{lb} A$, $y \leq x$.

- – We define $\sup A \subset X$ by $\sup A := \{x \in X : x \text{ is a supremum of } A\}$
  – We define $\inf A \subset X$ by $\inf A := \{x \in X : x \text{ is a infimum of } A\}$

Exercise 2.2.3.23. Let $(X, \leq)$ be a poset and $A \subset X$. Then

1. for each $x, y \in \sup A$, $x \sim_{\leq} y$.
2. for each $x, y \in \inf A$, $x \sim_{\leq} y$.

Proof.

1. Let $x, y \in \sup A$. Then $x, y \in \operatorname{ub} A$. Since $x \in \sup A$ and $y \in \operatorname{ub} A$, $x \leq y$. Since $y \in \sup A$ and $x \in \operatorname{ub} A$, $y \leq x$. Hence $x \sim_{\leq} y$.
2. Similar to (1).

□

2.2.4 Order Preserving Maps

Definition 2.2.4.1. Let $(X, \leq_X)$ $(Y, \leq_Y)$ be prosets and $f : X \to Y$. Then f is said to be $(\leq_X, \leq_Y)$**-order preserving** or $(\leq_X, \leq_Y)$**-monotone** if for each $a, b \in X$, $a \leq_X b$ implies that $f(a) \leq_Y f(b)$.

Note 2.2.4.2. When the context is clear we say that f is "monotone" instead of "$(\leq_X, \leq_Y)$-monotone".

Exercise 2.2.4.3. Let $(X, \leq_X)$ $(Y, \leq_Y)$ be prosets and $f : X \to Y$. Then f is $(\leq_X, \leq_Y)$-monotone iff f is $(\leq_X^{\text{op}}, \leq_Y^{\text{op}})$-monotone.

Proof.

- $(\implies)$:
  Suppose that f is $(\leq_X, \leq_Y)$-monotone. Let $a, b \in X$. Suppose that $a \leq_X^{\text{op}} b$. Then $b \leq_X a$. Hence $f(b) \leq_Y f(a)$. Thus $f(a) \leq_Y^{\text{op}} f(b)$. Since $a, b \in X$ with $a \leq_X^{\text{op}} b$ are arbitrary, we have that for each $a, b \in X$, $a \leq_X^{\text{op}} b$ implies that $f(a) \leq_Y^{\text{op}} f(b)$. Therefore f is $(\leq_X^{\text{op}}, \leq_Y^{\text{op}})$-monotone.
- $(\impliedby)$:
  Suppose that f is $(\leq_X^{\text{op}}, \leq_Y^{\text{op}})$-monotone. Since $(\leq_X^{\text{op}})^{\text{op}} = \leq_X$, the previous part implies that f is $(\leq_X, \leq_Y)$-monotone.

□

Exercise 2.2.4.4. Let $(X, \leq_X)$ $(Y, \leq_Y)$ be prosets and $f : X \to Y$. Then for each $U \subset Y$, U is a $\leq_Y$-upper set implies that $f^{-1}(U)$ is a $\leq_X$-upper set iff for each $L \subset Y$, L is a $\leq_Y$-lower set implies that $f^{-1}(L)$ is a $\leq_X$-lower set.

Proof.

- $(\implies)$:
  Suppose that for each $U \subset Y$, U is a $\leq_Y$-upper set implies that $f^{-1}(U)$ is a $\leq_X$-upper set. Let $L \subset Y$. Suppose that L is a $\leq_Y$-lower set. Exercise 2.2.3.5 then implies that L^c is a $\leq_Y$-upper set. By assumption, $f^{-1}(L^c)$ is a $\leq_X$-upper set. Another application of Exercise 2.2.3.5 implies that $f^{-1}(L^c)^c$ is a $\leq_X$-lower set. Since

  $$\begin{aligned} f^{-1}(L) &= [f^{-1}(L)^c]^c \\ &= f^{-1}(L^c)^c, \end{aligned}$$

  we have that $f^{-1}(L)$ is a $\leq_X$-lower set. Since $L \subset Y$ such that L is a $\leq_Y$-lower set is arbitrary, we have that for each $L \subset Y$, L is a $\leq_Y$-lower set implies that $f^{-1}(L)$ is a $\leq_X$-lower set.
- $(\impliedby)$:
  Similar to $(\implies)$.

□

Exercise 2.2.4.5. Let $(X, \leq_X)$ $(Y, \leq_Y)$ be prosets and $f : X \to Y$. If f is $(\leq_X, \leq_Y)$-monotone, then for each $U \subset Y$, U is a $\leq_Y$-upper set implies that $f^{-1}(U)$ is a $\leq_X$-upper set.

Proof. Suppose that f is $(\leq_X, \leq_Y)$-monotone. Let $U \subset Y$. Suppose that U is a $\leq_Y$-upper set. Let $a \in f^{-1}(U)$ and $x \in X$. Suppose that $a \leq x$. Since $a \in f^{-1}(U)$, $f(a) \in U$. Since f is monotone, $f(a) \leq f(x)$. Since U is a $\leq_Y$-upper set and $f(a) \in U$, we have that $f(x) \in U$. Hence $x \in f^{-1}(U)$. Since $a \in f^{-1}(U)$ and $x \in X$ with $a \leq x$ are arbitrary, we have that for each $a \in f^{-1}(U)$ and $x \in X$, $a \leq x$ implies that $x \in f^{-1}(U)$. Thus $f^{-1}(U)$ is a $\leq_X$-upper set. Since $U \subset Y$ such that U is a $\leq_Y$-upper set is arbitrary, we have that for each $U \subset Y$, U is a $\leq_Y$-upper set implies that $f^{-1}(U)$ is a $\leq_X$-upper set. □

Exercise 2.2.4.6. Let $(X, \leq_X)$ $(Y, \leq_Y)$ be prosets, $f : X \to Y$ and $A \subset X$. If f is $(\leq_X, \leq_Y)$-monotone, then $f(\mathrm{ub}(A, \leq_X)) \subset \mathrm{ub}(f(A), \leq_Y)$.

Proof. Suppose that f is $(\leq_X, \leq_Y)$-monotone. Let $x \in \mathrm{ub}(A, \leq_X)$ and $y_0 \in f(A)$. Then there exists $x_0 \in A$ such that $f(x_0) = y$. Since $x \in \mathrm{ub}(A, \leq_X)$, $x_0 \leq x$. Since f is $(\leq_X, \leq_Y)$-monotone,

$$\begin{aligned} y_0 &= f(x_0) \\ &\leq f(x). \end{aligned}$$

Since $y_0 \in f(A)$ is arbitrary, we have that for each $y_0 \in f(A)$, $y_0 \leq f(x)$. Therefore $f(x) \in \mathrm{ub}(F(A), \leq_Y)$. Since $x \in \mathrm{ub}(A, \leq_X)$ is arbitrary, ew have that for each $x \in \mathrm{ub}(A, \leq_X)$, $f(x) \in \mathrm{ub}(F(A), \leq_Y)$. Hence $f(\mathrm{ub}(A, \leq_X)) \subset \mathrm{ub}(f(A), \leq_Y)$. □

2.3 Posets

2.3.1 Introduction

Definition 2.3.1.1. Poset:
Let X be a set and $\leq \subset X \times X$ a binary relation on X. Then

- $\leq$ is said to be a **partial order on** X if
  1. $\leq$ is a preorder on X
  2. for each $a, b \in X$, $a \leq b$ and $b \leq a$ implies that $a = b$,
- $(X, \leq)$ is $(X, \leq)$ is said to be a **partially ordered set** or **poset** if $\leq$ is a partial ordering on X.

Exercise 2.3.1.2. Let $(X, \leq)$ be a poset and $a, b \in X2$ Then $a \sim_{\leq} b$ iff $a = b$.

Proof.

- $(\implies)$:
  Suppose that $a \sim_{\leq} b$. Then $a \leq b$ and $b \leq a$. Since $\leq$ is a partial order, $a = b$.
- $(\impliedby)$:
  Since $\sim_{\leq}$ is reflexive, $a = b$ implies that $a \sim_{\leq} b$.

□

Definition 2.3.1.3. Let $(X, \leq_X)$ be a proset. Set $Y := X/\sim_{\leq_X}$. We define $\leq_Y \subset Y \times Y$ by $y_1 \leq_Y y_2$ iff there exists $x_1 \in y_1$ and $x_2 \in y_2$ such that $x_1 \leq_X x_2$.

Exercise 2.3.1.4. Let $(X, \leq_X)$ be a proset. Set $Y := X/\sim_{\leq_X}$. Then $(Y, \leq_Y)$ is a poset.

Proof.

1. (a) Let $y \in Y$. Then there exists $x \in y$. Since $x \leq_X x$, we have that $y \leq_Y y$.
   (b) Let $y_1, y_2, y_3 \in Y$. Suppose that $y_1 \leq_Y y_2$ and $y_2 \leq_Y y_3$. Then there exist $x_1 \in y_1$, $x_2 \in y_2$ and $x_3 \in y_3$ such that $x_1 \leq_X x_2$ and $x_2 \leq_X x_3$. Therefore $x_1 \leq x_3$. Hence $y_1 \leq_Y y_3$.

   Thus $\leq_Y$ is a preorder on Y.
2. Let $y_1, y_2 \in Y$. Suppose that $y_1 \leq_Y y_2$ and $y_2 \leq_Y y_1$. Then there exist $a_1, b_1 \in y_1$, $a_2, b_2 \in y_2$ such that $a_1 \leq_X a_2$ and $b_2 \leq_X b_1$. Since $a_1, b_1 \in y_1$ and $a_2, b_2 \in y_2$, $a_1 \sim_{\leq_X} b_1$ and $a_2 \sim_{\leq_X} b_2$. Thus

$$\begin{aligned} a_1 &\leq_X a_2 \\ &\leq_X b_2. \end{aligned}$$

   and

$$\begin{aligned} b_2 &\leq_X b_1 \\ &\leq_X a_1. \end{aligned}$$

   Hence $a_1 \sim_{\leq_X} b_2$ and

$$\begin{aligned} y_1 &= \pi(a_1) \\ &= \pi(b_2) \\ &= y_2. \end{aligned}$$

Therefore $\leq_Y$ is a partial order on Y. □

Exercise 2.3.1.5. Let $(X, \leq)$ be a poset and $A \subset X$. Then

1. for each $x, y \in \sup A$, $x = y$.
2. for each $x, y \in \inf A$, $x = y$.

Proof.

1. Let $x, y \in \sup A$. Exercise 2.2.3.23 implies that $x \sim_{\leq} y$. Exercise 2.3.1.2 then implies that $x = y$.
2. Similar to (1).

□

Definition 2.3.1.6. Let $(X, \leq)$ be a poset and $A \subset X$.

- We say that $\sup A$ **(resp.** $\inf A$**) exists** if $\sup A \neq \varnothing$ (resp. $\inf A \neq \varnothing$).
- Let $x \in X$. We write $x = \sup A$ (resp. $x = \inf A$) if $x \in \sup A$ (resp. $x \in \inf A$)

Exercise 2.3.1.7. Associativity of Supremum:
Let $(X, \leq)$ be a poset and $(E_\alpha)_{\alpha \in A} \subset \mathcal{P}(X)$.

1. Suppose that
   - for each $\alpha \in A$, $\sup E_\alpha$ exists
   - $\sup\limits_{\alpha \in A} \left[\sup E_\alpha\right]$ exists.

   Then
$$\sup \bigcup_{\alpha \in A} E_\alpha = \sup_{\alpha \in A} \left[\sup E_\alpha\right]$$
2. Suppose that
   - for each $\alpha \in A$, $\inf E_\alpha$ exists
   - $\inf\limits_{\alpha \in A} \left[\inf E_\alpha\right]$ exists.

   Then
$$\inf \bigcup_{\alpha \in A} E_\alpha = \inf_{\alpha \in A} \left[\inf E_\alpha\right]$$

Proof.

1. Define $s_1, s_2 \in X$ by $s_1 := \bigcup\limits_{\alpha \in A} E_\alpha$ and $s_2 := \sup\limits_{\alpha \in A} \left[\sup E_\alpha\right]$. We note by definition, s_2 is an upper bound for $\{\sup E_\alpha : \alpha \in A\}$. Then for each $\alpha \in A$, $s_2 \geq \sup E_\alpha$.
   - Let $x \in \bigcup\limits_{\alpha \in A} E_\alpha$. Then there exists $\alpha_0 \in A$ such that $x \in E_{\alpha_0}$. Since $\sup E_{\alpha_0}$ is an upper bound of E_{α_0}, we have that
$$\begin{aligned} s_2 &\geq \sup E_{\alpha_0} \\ &\geq x. \end{aligned}$$
     Since $x \in \bigcup\limits_{\alpha \in A} E_\alpha$ is arbitrary, we have that s_2 is an upper bound for $\bigcup\limits_{\alpha \in A} E_\alpha$. Therefore
$$\begin{aligned} s_1 &= \sup \bigcup_{\alpha \in A} E_\alpha \\ &\leq s_2. \end{aligned}$$

- Let $\alpha_0 \in A$ and $x \in E_{\alpha_0}$. Then

$$\begin{aligned} x &\in E_{\alpha_0} \\ &\subset \bigcup_{\alpha \in A} E_\alpha. \end{aligned}$$

Since s_1 is an upper bound of $\bigcup_{\alpha \in A} E_\alpha$, we have that $s_1 \geq x$. Since $x \in E_{\alpha_0}$ is arbitrary, we have that for each $x \in E_{\alpha_0}$, $s_1 \geq x$. Hence s_1 is an upper bound of E_{α_0}. Therefore $\sup E_{\alpha_0} \leq s_1$. Since $\alpha_0 \in A$ is arbitrary, we have that for each $\alpha \in A$, $s_1 \geq \sup E_\alpha$. Therfore s_1 is an upper bound of $\{\sup E_\alpha : \alpha \in A\}$. Hence

$$\begin{aligned} s_2 &= \sup_{\alpha \in A} \left[\sup E_\alpha \right] \\ &\leq s_1. \end{aligned}$$

Since $s_1 \leq s_2$ and $s_2 \leq s_1$, we have that $s_1 = s_2$.

2. Similar to (1). Maybe fill out

□

Exercise 2.3.1.8. Let $(X, \leq)$ be a poset and $a, b \in X$. Then $a \leq b$ iff $b = \sup\{a, b\}$.

Proof.

- ($\Longrightarrow$):
Suppose that $a \leq b$. Since $b \leq b$, we have that for each $c \in \{a, b\}$, $c \leq b$. Hence $b \in \text{lb}\{a, b\}$. Let $c \in \text{ub}\{a, b\}$. Then $b \leq c$. Since $c \in \text{ub}\{a, b\}$ is arbitrary, we have that for each $c \in \text{ub}\{a, b\}$, $b \leq c$. Hence $b = \sup\{a, b\}$.

- ($\Longleftarrow$):
Suppose that $b = \sup\{a, b\}$. Then $b \in \text{ub}\{a, b\}$. Therefore $a \leq b$.

□

Exercise 2.3.1.9. Let $(X, \leq)$ be a poset. Then

1. for each $a, b \in \text{ub}\, X$, $a = b$,
2. for each $a, b \in \text{lb}\, X$, $a = b$.

Proof.

1. Let $a, b \in \text{ub}\, X$. Since $a \in X$ and $b \in \text{ub}\, X$, $a \leq b$. Similarly, $b \leq a$. Hence $a = b$.
2. Similar to (1).

□

2.4 Directed Sets

Definition 2.4.0.1. Directed Set:
Let A be a set and $\leq \subset A \times A$ a binary relation on A. Then

- $\leq$ is said to be a **direction on** A if
  1. $\leq$ is a preorder on A
  2. for each $\alpha, \beta \in A$, $\{\alpha, \beta\}$ is bounded above.
- $(A, \leq)$ is said to be a **directed set** if
  1. $A \neq \varnothing$
  2. $\leq$ is a direction on A

Exercise 2.4.0.2. Let $(A, \leq_A)$ and $(B, \leq_B)$ be directed sets. Then

1. $\leq_A \otimes \leq_B$ is a direction on $A \times B$,
2. $(A, \leq_A) \otimes (B, \leq_B)$ is a directed set.

Proof.

1. (a) Exercise **??** implies that $\leq_A \otimes \leq_B$ is a preorder of $A \times B$.
   (b) Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then there exist $a \in A$ and $b \in B$ such that $a_1, a_2 \leq_A a$ and $b_1, b_2 \leq_B b$. Hence $(a_1, b_1), (a_2, b_2) \leq_A \otimes \leq_B (a, b)$.

   Hence $\leq_A \otimes \leq_B$ is a direction on $A \times B$.
2. (a) Since $A \neq \varnothing$ and $B \neq \varnothing$, we have that $A \times B \neq \varnothing$.
   (b) From above, $\leq_A \otimes \leq_B$ is a direction on $A \times B$.

   Hence $(A \times B, \leq_A \otimes \leq_B)$ is a directed set.

□

Part II

Algebraic Structures

Chapter 3

Lattices

3.1 Introduction

3.1.1 Skew Semilattices

cite Leech, skew lattices in rings

Definition 3.1.1.1. Let X be a set and $\diamond : X \times X \to X$ a binary operator on X. Then

- $\diamond$ is said to be a **skew semilattice operator on** X if $\diamond$ is associative and idempotent.
- $(X, \diamond)$ is said to be a **skew semilattice** if $\diamond$ is a skew semilattice operator on X.

Definition 3.1.1.2. Let $(X, \diamond)$ be an skew semilattice. We define the

- **join preorder on** X **induced by** $\diamond$, denoted $\leq_\diamond^\vee \subset X \times X$, by $a \leq_\diamond^\vee b$ iff $(b \diamond a) \diamond b = b$
- **meet preorder on** X **induced by** $\diamond$, denoted $\leq_\diamond^\wedge \subset X \times X$, by $a \leq_\diamond^\wedge b$ iff $(a \diamond b) \diamond a = a$.

Exercise 3.1.1.3. Let $(X, \diamond)$ be an skew semilattice. Then

1. $\leq_\diamond^\vee$ is a preorder on X.
   Hint: If $cbc = c$ and $bab = b$, then
   (a) $c = (cb)(abc)(abc)$ and $c = cabc$
   (b) $cac = (ca)(ca)(bc)$
2. $\leq_\diamond^\wedge$ is a preorder on X.

Proof.

1. Let $a, b, c \in X$.
   (a) Since $\diamond$ is idempotent, we have that
   $$\begin{aligned}(a \diamond a) \diamond a &= a \diamond a \\ &= a\end{aligned}$$
   and therefore $a \leq_\diamond^\vee a$.

(b) Suppose that $a \leq_\diamond^\vee b$ and $b \leq_\diamond^\vee c$. Then $(b \diamond a) \diamond b = b$ and $(c \diamond b) \diamond c = c$. Since $\diamond$ is associative and idempotent,

$$\begin{aligned} c &= cbc \\ &= cbabc \\ &= (cb)(abc) \\ &= (cb)(abc)(abc) \\ &= c(bab)cabc \\ &= cbcabc \\ &= (cbc)abc \\ &= cabc. \end{aligned}$$

Therefore

$$\begin{aligned} cac &= (ca)(c) \\ &= (ca)(cabc) \\ &= (ca)(ca)(bc) \\ &= (ca)(bc) \\ &= cabc \\ &= c. \end{aligned}$$

Hence $a \leq_\diamond^\vee c$.

So $\leq_\diamond^\vee$ is a preorder on X.

2. Similar to (1).

$\square$

Exercise 3.1.1.4. Let $(X, \diamond)$ be an skew semilattice. Then

1. $(\leq_\diamond^\vee)^{\text{op}} = \leq_\diamond^\wedge$
2. $(\leq_\diamond^\wedge)^{\text{op}} = \leq_\diamond^\vee$

Proof.

1. Let $a, b \in X$. Then

$$\begin{aligned} a(\leq_\diamond^\vee)^{\text{op}} b &\iff b \leq_\diamond^\vee a \\ &\iff a \diamond b \diamond a = a \\ &\iff a \leq_\diamond^\wedge b. \end{aligned}$$

2. Similar to (1).

$\square$

Exercise 3.1.1.5. Let $(X, \diamond)$ be an skew semilattice. Then for each $a, b \in X$,

1. $a \diamond b \in \text{ub}(\{a, b\}, \leq_\diamond^\vee)$
2. $a \diamond b \in \text{lb}(\{a, b\}, \leq_\diamond^\wedge)$

Proof. Let $a, b \in X$.

1. We note that

$$\begin{aligned}(a \diamond b) \diamond [a \diamond (a \diamond b)] &= (a \diamond b) \diamond [(a \diamond a) \diamond b] \\ &= (a \diamond b) \diamond (a \diamond b) \\ &= a \diamond b.\end{aligned}$$

Similarly, $[(a \diamond b) \diamond b] \diamond (a \diamond b) = a \diamond b$. Therefore $a \leq_{\diamond}^{\vee} a \diamond b$ and $b \leq_{\diamond}^{\vee} a \diamond b$. Hence $a \diamond b \in \mathrm{ub}(\{a, b\}, \leq_{\diamond}^{\vee})$.

2. FINISH!!!

□

Definition 3.1.1.6. Let X be a set and $\vee, \wedge$ skew semilattice operators on X. Then $\vee$ and $\wedge$ are said to **satisfy the skew lattice absorption identities** if for each $a, b \in X$,

1. $a \wedge (a \vee b) = a$
2. $a \vee (a \wedge b) = a$
3. $(b \vee a) \wedge a = a$
4. $(b \wedge a) \vee a = a$

Exercise 3.1.1.7. Let X be a set and $\vee, \wedge$ skew semilattice operators on X. Then $\vee$ and $\wedge$ satisfy the skew lattice absorption identities iff for each $a, b \in X$,

1. $a \vee b = b$ iff $a \wedge b = a$
2. $a \vee b = a$ iff $a \wedge b = b$

Proof.

- ($\implies$):
  Suppose that $\vee$ and $\wedge$ satisfy the skew lattice absorption identities. Let $a, b \in X$.

  1. – ($\implies$):
       If $a \vee b = b$, then
       $$\begin{aligned}a \wedge b &= a \wedge (a \vee b) \\ &= a.\end{aligned}$$
     – ($\impliedby$):
       If $a \wedge b = a$, then
       $$\begin{aligned}a \vee b &= (a \wedge b) \vee b \\ &= b.\end{aligned}$$

     Thus $a \vee b = b$ iff $a \wedge b = a$.
  2. Similar to (1)

- ($\impliedby$):
  Suppose that for each $a, b \in X$,

  1. $a \vee b = b$ iff $a \wedge b = a$,
  2. $a \vee b = a$ iff $a \wedge b = b$.

  Let $a, b \in X$.

1. Set $c := a \vee b$. By assumption,

$$\begin{aligned} a \wedge (a \vee b) = a &\iff a \wedge c = a \\ &\iff a \vee c = c \\ &\iff a \vee (a \vee b) = c \\ &\iff (a \vee a) \vee b = c \\ &\iff a \vee b = c \\ &\iff c = c. \end{aligned}$$

   Since $c = c$, we have that $a \wedge (a \vee b) = a$.
2. Similar to (1)
3. Similar to (1)
4. Similar to (1)

□

Exercise 3.1.1.8. Let X be a set and $\vee, \wedge$ skew semilattice operators on X. If $\vee$ and $\wedge$ satisfy the skew lattice absorption identities, then for each $a, b \in X$, $b \vee a \vee b = b$ iff $a \wedge b \wedge a = a$.
Hint: Exercise 3.1.1.7 implies that $b \vee (a \vee b) = b$ iff $b \wedge (a \vee b) = a \vee b$. Use, skew lattice absorption identities and associativity.

Proof. Suppose that $\vee$ and $\wedge$ satisfy the skew lattice absorption identities. Let $a, b \in X$.

- $(\implies)$:
  Suppose that $b \vee a \vee b = b$. Then Exercise 3.1.1.7 implies that $b \wedge (a \vee b) = (a \vee b)$. Therefore

$$\begin{aligned} a \wedge b &= (a \wedge b) \vee [(a \wedge b) \wedge (a \vee b)] \quad \text{(absorption)} \\ &= (a \wedge b) \vee [a \wedge (b \wedge [a \vee b])] \quad \text{(associativity)} \\ &= (a \wedge b) \vee [a \wedge (a \vee b)] \quad \text{(from earlier)} \\ &= (a \wedge b) \vee a \quad \text{(absorption)}. \end{aligned}$$

  Another application of Exercise 3.1.1.7 implies that $(a \wedge b) \wedge a = a$.
- $(\impliedby)$:
  Similar to $(\implies)$.

□

3.1.2 Skew lattices

Definition 3.1.2.1. Let X be a set and $\vee, \wedge$ skew semilattice operators on X. Then $(X, \vee, \wedge)$ is said to be a **skew lattice** if $\vee$ and $\wedge$ satisfy the skew lattice absorption identities.

Exercise 3.1.2.2. Let $(X, \vee, \wedge)$ be an skew lattice. Then

1. $\leq^{\vee}_{\vee} = \leq^{\wedge}_{\wedge}$.
2. $\leq^{\wedge}_{\vee} = \leq^{\vee}_{\wedge} = (\leq^{\vee}_{\vee})^{\mathrm{op}}$.

Proof.

1. Let $a, b \in X$. Since $(X, \vee, \wedge)$ is an skew lattice, $\vee$ and $\wedge$ satisfy the skew lattice absorption identities. Exercise 3.1.1.8 then implies that

$$\begin{aligned} a \leq^{\vee}_{\vee} b &\iff b \vee a \vee b = b \\ &\iff a \wedge b \wedge a = a \\ &\iff a \leq^{\wedge}_{\wedge} b. \end{aligned}$$

   Since $a, b \in X$ are arbitrary, we have that for each $a, b \in X$, $a \leq^{\vee}_{\vee} b$ iff $a \leq^{\wedge}_{\wedge} b$. Hence $\leq^{\vee}_{\vee} = \leq^{\wedge}_{\wedge}$.

2. Exercise 3.1.1.4 and part (1) imply that

$$\begin{aligned}\leq_{\vee}^{\wedge} &= (\leq_{\vee}^{\vee})^{\mathrm{op}} \\ &= (\leq_{\wedge}^{\wedge})^{\mathrm{op}} \\ &= \leq_{\wedge}^{\vee} .\end{aligned}$$

$\square$

Note 3.1.2.3. Let $(X, \vee, \wedge)$ be an skew lattice. When the context is clear, we write $\leq$ in place of $\leq_{\vee}^{\vee}$.

3.1.3 Semilattices

Definition 3.1.3.1. Let X be a set and $\diamond : X \times X \to X$. Then

- $\diamond$ is said to be a **semilattice operator on** X if $\diamond$ associative, idempotent and commutative.
- $(X, \diamond)$ is said to be an **semilattice** if $\diamond$ is a semilattice operator on X.

Exercise 3.1.3.2. Let X be a set and $\diamond : X \times X \to X$. Then $\diamond$ is a semilattice operator on X iff

1. $\diamond$ is a skew semilattice operator on X,
2. $\diamond$ is commutative.

Proof. Clear by definition. $\square$

Exercise 3.1.3.3. Let $(X, \diamond)$ be a semilattice. Then

1. for each $a, b \in X$,
   (a) $a \leq_\diamond^\vee b$ iff $a \diamond b = b$
   (b) $a \leq_\diamond^\wedge b$ iff $a \diamond b = a$
2. (a) $\leq_\diamond^\vee$ is a partial order on X
   (b) $\leq_\diamond^\wedge$ is a partial order on X

Proof.

1. Let $a, b \in X$.
   (a) Since $\diamond$ is commutative, associative and idempotent,
$$\begin{aligned} a \leq_\diamond^\vee b &\iff (b \diamond a) \diamond b = b \\ &\iff (a \diamond b) \diamond b = b \\ &\iff a \diamond (b \diamond b) = b \\ &\iff a \diamond b = b. \end{aligned}$$
   (b) Similar to $1(a)$.
2. (a) Exercise 3.1.1.3 implies that $\leq_\diamond^\vee$ is a preorder on X. Let $a, b \in X$. Suppose that $a \leq_\diamond^\vee b$ and $b \leq_\diamond^\vee a$. Part (1) then implies that $a \diamond b = b$ and $b \diamond a = a$. Since $\diamond$ is commutative,
$$\begin{aligned} a &= b \diamond a \\ &= a \diamond b \\ &= b. \end{aligned}$$
   Since $a, b \in X$ are arbitrary, we have that for each $a, b \in X$, if $a \leq_\diamond^\vee b$ and $b \leq_\diamond^\vee a$, then $a = b$. Thus $\leq_\diamond^\vee$ is a a partial order on X.
   (b) Similar to $2(a)$.

$\square$

Exercise 3.1.3.4. Let $(X, \diamond)$ be a semilattice. Then for each $a, b \in X$,

1. $a \diamond b = \sup(\{a, b\}, \leq_\diamond^\vee)$
2. $a \diamond b = \inf(\{a, b\}, \leq_\diamond^\wedge)$

Proof. Let $a, b \in X$.

1. Exercise 3.1.1.5 implies that $a \diamond b \in \mathrm{ub}(\{a,b\}, \leq_\diamond^\vee)$. Let $c \in \mathrm{ub}(\{a,b\}, \leq_\diamond^\vee)$. Since $a \leq_\diamond^\vee c$ and $b \leq_\diamond^\vee c$, Exercise 3.1.3.3 implies that $a \diamond c = c$ and $b \diamond c = c$. Therefore

$$\begin{aligned}(a \diamond b) \diamond c &= a \diamond (b \diamond c) \\ &= a \diamond c \\ &= c.\end{aligned}$$

   Another application of Exercise 3.1.3.3 implies that $a \diamond b \leq_\diamond^\vee c$. Since $c \in \mathrm{ub}(\{a,b\}, \leq_\diamond^\vee)$ is arbitrary, we have that for each $c \in \mathrm{ub}(\{a,b\}, \leq_\diamond^\vee)$, $a \diamond b \leq_\diamond^\vee c$. Therefore $a \diamond b = \sup(\{a,b\}, \leq_\diamond^\vee)$.

2. FINISH!!!

$\square$

Definition 3.1.3.5. Let $(X, \leq)$ be a poset. Then $(X, \leq)$ is said to be a

- **join-semilattice** if for each $a, b \in X$, $\sup\{a,b\}$ exists.
- **meet-semilattice** if for each $a, b \in X$, $\inf\{a,b\}$ exists.

Exercise 3.1.3.6. Let $(X, \diamond)$ be a semilattice. Then

1. $(X, \leq_\diamond^\vee)$ is a join-semilattice
2. $(X, \leq_\diamond^\wedge)$ is a meet-semilattice

Proof. Exercise 3.1.3.4 implies that

1. For each $a, b \in X$, $\sup(\{a,b\}, \leq_\diamond^\vee) = a \diamond b$. Thus $(X, \leq_\diamond^\vee)$ is a join-semilattice.
2. for each $a, b \in X$, $\inf(\{a,b\}, \leq_\diamond^\wedge) = a \diamond b$. Thus $(X, \leq_\diamond^\wedge)$ is a meet-semilattice.

$\square$

Definition 3.1.3.7. Let $(X, \leq)$ be a poset.

- If $(X, \leq)$ is a join-semilattice, we define the **join operator on** X **induced by** $\leq$, denoted $\vee_\leq : X \times X \to X$, by $a \vee_\leq b := \sup\{a,b\}$.
- If $(X, \leq)$ is a meet-semilattice, we define the **meet operator on** X **induced by** $\leq$, denoted $\wedge_\leq : X \times X \to X$, by $a \wedge_\leq b := \inf\{a,b\}$.

Exercise 3.1.3.8. Let $(X, \leq)$ be a poset.

1. If $(X, \leq)$ an join-semilattice, then $\vee_\leq$ is a semilattice operator on X.
2. If $(X, \leq)$ an meet-semilattice, then $\wedge_\leq$ is a semilattice operator on X.

Proof.

1. Suppose that $(X, \leq)$ an join-semilattice.

   (a) • Let $a, b, c \in X$. Exercise 2.3.1.7 implies that

$$\begin{aligned}(a \vee_\leq b) \vee_\leq c &= \sup\{\sup\{a,b\}, c\} \\ &= \sup\{\sup\{a,b\}, \sup\{c\}\} \\ &= \sup\{a,b,c\} \\ &= \sup\{\sup\{a\}, \sup\{b,c\}\} \\ &= \sup\{a, \sup\{b,c\}\} \\ &= a \vee_\leq (b \vee_\leq c).\end{aligned}$$

   Hence $\vee_\leq$ is associative.

- Let $a \in X$. Then

$$\begin{aligned} a \vee_{\leq} a &= \sup\{a, a\} \\ &= \sup\{a\} \\ &= a. \end{aligned}$$

Hence $\vee_{\leq}$ is idempotent.

- Let $a, b \in X$. Then

$$\begin{aligned} a \vee_{\leq} b &= \sup\{a, b\} \\ &= \sup\{b, a\} \\ &= b \vee_{\leq} a. \end{aligned}$$

Hence $\vee_{\leq}$ is commutative.

Since $\vee_{\leq}$ is associative, idempotent and commutative, $\vee_{\leq}$ is a semilattice operator on X.

2. Similar to (1).

□

Exercise 3.1.3.9. Let $(X, \leq)$ be a poset.

1. If $(X, \leq)$ is a join-semilattice, then
   (a) $\leq^{\vee}_{\vee_{\leq}} = \leq$,
   (b) $\leq^{\wedge}_{\vee_{\leq}} = \leq^{\mathrm{op}}$.
2. If $(X, \leq)$ is a meet-semilattice, then
   (a) $\leq^{\wedge}_{\wedge_{\leq}} = \leq$,
   (b) $\leq^{\vee}_{\wedge_{\leq}} = \leq^{\mathrm{op}}$.

Proof.

1. Suppose that $(X, \leq)$ is a join-semilattice.
   (a) Let $a, b \in X$. Then

$$\begin{aligned} a \leq^{\vee}_{\vee_{\leq}} b &\iff a \vee_{\leq} b = b \quad \text{(Exercise 3.1.3.3)} \\ &\iff \sup(\{a, b\}, \leq) = b \quad \text{(Definition 3.1.3.7)} \\ &\iff a \leq b \quad \text{(Exercise 2.3.1.8).} \end{aligned}$$

   Since $a, b \in X$ are arbitrary, we have that $\leq^{\vee}_{\vee_{\leq}} = \leq$.

   (b) Exercise 3.1.1.4 implies that

$$\begin{aligned} \leq^{\wedge}_{\vee_{\leq}} &= (\leq^{\vee}_{\vee_{\leq}})^{\mathrm{op}} \\ &= \leq^{\mathrm{op}}. \end{aligned}$$

2. Similar to (1).

□

Exercise 3.1.3.10. Let $(X, \diamond)$ be a semilattice. Then

1. $\leq^{\vee}_{\diamond}$ is the unique partial order $\leq$ on X such that $(X, \leq)$ is a join-semilattice and for each $a, b \in X$, $a \diamond b = \sup(\{a, b\}, \leq)$.
2. $\leq^{\wedge}_{\diamond}$ is the unique partial order $\leq$ on X such that $(X, \leq)$ is an meet-semilattice and for each $a, b \in X$, $a \diamond b = \inf(\{a, b\}, \leq)$.

Proof.

1. Let $\leq$ be partial order on X. Suppose that $(X, \leq)$ is a join-semilattice and for each $a, b \in X$, $a \diamond b = \sup(\{a, b\}, \leq)$. Let $a, b \in X$. Then

$$\begin{aligned} a \leq b &\iff b = \sup(\{a, b\}, \leq) \quad \text{(Exercise 2.3.1.8)} \\ &\iff a \diamond b = b \quad \text{(by assumption)} \\ &\iff a \leq_{\diamond}^{\vee} b \quad \text{(Exercise 3.1.3.3).} \end{aligned}$$

   Since $a, b \in X$ are arbitrary, we have that for each $a, b \in X$, $a \leq b$ iff $a \leq_{\diamond}^{\vee} b$. Hence $\leq = \leq_{\diamond}^{\vee}$.

2. Similar to (1).

□

Exercise 3.1.3.11. Let $(X, \leq)$ be a poset.

1. If $(X, \leq)$ is a join-semilattice, then $\vee_{\leq}$ is the unique semilattice operator $\diamond$ on X such that for each $a, b \in X$, $a \diamond b = \sup(\{a, b\}, \leq)$.
2. If $(X, \leq)$ is a meet-semilattice, then $\wedge_{\leq}$ is the unique semilattice operator $\diamond$ on X such that for each $a, b \in X$, $a \diamond b = \inf(\{a, b\}, \leq)$.

Proof.

1. Let $\diamond : X \times X \to X$ be a semilattice operator on X. Suppose that $a, b \in X$, $a \diamond b = \sup(\{a, b\}, \leq)$. Then for each $a, b \in X$,

$$\begin{aligned} a \diamond b &= \sup(\{a, b\}, \leq) \quad \text{(by assumption)} \\ &= a \vee_{\leq} b \quad \text{(Definition 3.1.3.7).} \end{aligned}$$

   Hence $\diamond = \vee_{\leq}$.

2. Similar to (1).

□

3.1.4 Lattices

Definition 3.1.4.1. Let X be a set and $\vee, \wedge$ semilattice operators on X. Then $\vee$ and $\wedge$ are said to **satisfy the lattice absorption identities** if for each $a, b \in X$,

1. $a \wedge (a \vee b) = a$
2. $a \vee (a \wedge b) = a$

Exercise 3.1.4.2. Let X be a set and $\vee, \wedge$ semilattice operators on X. Then $\vee$ and $\wedge$ satisfy the lattice absorption identities iff $\vee$ and $\wedge$ satisfy the skew lattice absorption identities.

Proof.

- $(\implies)$:
  Clear by commutativity.
- $(\impliedby)$:
  Immediate.

□

Definition 3.1.4.3. Let X be a set and $\vee, \wedge$ semilattice operators on X. Then $(X, \vee, \wedge)$ is said to be an **lattice** if $\vee$ and $\wedge$ satisfy the skew lattice absorption identities.

Definition 3.1.4.4. Let $(X, \leq)$ be a poset. Then $(X, \leq)$ is said to be an **ordered lattice** if $(X, \leq)$ is a join-semilattice and $(X, \leq)$ is a meet-semilattice.

Exercise 3.1.4.5. Let $(X, \vee, \wedge)$ be an lattice. Then $(X, \leq_\vee^\vee)$ is an ordered lattice.

Proof. Exercise 3.1.3.6 implies that $(X, \leq_\vee^\vee)$ is a join-semilattice and $(X, \leq_\wedge^\wedge)$ is a meet-semilattice. Exercise 3.1.2.2 implies that $\leq_\vee^\vee = \leq_\wedge^\wedge$. Therefore $(X, \leq_\vee^\vee)$ is a join-semilattice and $(X, \leq_\vee^\vee)$ is a meet-semilattice. Hence $(X, \leq_\vee^\vee)$ is an ordered lattice. □

Exercise 3.1.4.6. Let $(X, \leq)$ be an ordered lattice. Then $(X, \vee_\leq, \wedge_\leq)$ is an lattice.

Proof. Let $a, b \in X$.

1.
   - Since $\sup\{a, b\} \in \operatorname{ub}\{a, b\}$ is an upper bound of $\{a, b\}$, we have that $a \leq \sup\{a, b\}$. Since $a \leq a$, we have that $a \in \operatorname{lb}\{a, \sup\{a, b\}\}$. Hence $a \leq \inf\{a, \sup\{a, b\}\}$.
   - Since $\inf\{a, \sup\{a, b\}\} \in \operatorname{lb}\{a, \sup\{a, b\}\}$, we have that $\inf\{a, \sup\{a, b\}\} \leq a$.

   Since $a \leq \inf\{a, \sup\{a, b\}\}$ and $\inf\{a, \sup\{a, b\}\} \leq a$, we have that

$$\begin{aligned} a \wedge_\leq (a \vee_\leq b) &= \inf\{a, \sup\{a, b\}\} \\ &= a. \end{aligned}$$

2. Similarly, $a \vee_\leq (a \wedge_\leq b) = a$.

Since $a, b \in X$ are arbitrary, we have that for each $a, b \in X$,

1. $a \wedge_\leq (a \vee_\leq b) = a$
2. $a \vee_\leq (a \wedge_\leq b) = a$

Hence $\vee_\leq$ and $\wedge_\leq$ satisfy the lattice absorption identities. Therefore $(X, \vee_\leq, \wedge_\leq)$ is an lattice. □

Exercise 3.1.4.7. Let $(X, \leq)$ be an ordered lattice. Then

1. $\leq_{\vee_\leq}^\vee = \leq_{\wedge_\leq}^\wedge = \leq$
2. $\leq_{\vee_\leq}^\wedge = \leq_{\wedge_\leq}^\vee = \leq^{\mathrm{op}}$

Proof. Since $(X, \leq)$ is an ordered lattice, $(X, \leq)$ is a join-semilattice $(X, \leq)$ is a meet-semilattice. Exercise 3.1.3.9 then implies that

1.

$$\begin{aligned} \leq_{\vee_\leq}^\vee &= \leq \\ &= \leq_{\wedge_\leq}^\wedge \end{aligned}$$

2.

$$\begin{aligned} \leq_{\vee_\leq}^\wedge &= \leq^{\mathrm{op}} \\ &= \leq_{\wedge_\leq}^\vee \end{aligned}$$

□

Exercise 3.1.4.8. Let $(X, \vee, \wedge)$ be an lattice. Then $\leq_\vee^\vee$ is the unique partial order $\leq$ on X such that $(X, \leq)$ is an ordered lattice and for each $a, b \in X$, $a \vee b = \sup(\{a, b\}, \leq)$.

Proof. Let $\leq$ be a partial order on X. Suppose that $(X, \leq)$ is an ordered lattice and for each $a, b \in X$, $a \vee b = \sup(\{a, b\}, \leq)$. Since $(X, \vee)$ is a semilattice, $(X, \leq_\vee^\vee)$ and $(X, \leq)$ are join-semilattices and for each $a, b \in X$, $a \vee b = \sup(\{a, b\}, \leq_\vee^\vee)$ and $a \vee b = \sup(\{a, b\}, \leq)$, Exercise 3.1.3.10 implies that $\leq = \leq_\vee^\vee$. □

Exercise 3.1.4.9. Let $(X, \leq)$ be an ordered lattice. Then $\vee_{\leq}, \wedge_{\leq}$ are the unique semilattice operators $\vee, \wedge$ on X such that $(X, \vee, \wedge)$ is an lattice and for each $a, b \in X$, $a \vee b = \sup(\{a, b\}, \leq)$

Proof. Let $\vee, \wedge$ be semilattice operators on X. Suppose that $(X, \vee, \wedge)$ is a lattice and for each $a, b \in X$, $a \vee b = \sup(\{a, b\}, \leq)$. Then for each $a, b \in X$,

$$\begin{aligned} a \vee_{\leq} b &= \sup(\{a, b\}, \leq) \quad \text{(Definition 3.1.3.7)} \\ &= a \vee b \quad \text{(by assumption)}. \end{aligned}$$

Thus $\vee = \vee_{\leq}$. Since $(X, \vee, \wedge)$ is a lattice and for each $a, b \in X$, $a \vee b = \sup(\{a, b\}, \leq)$, Exercise 3.1.4.8 implies that $\leq_{\vee}^{\vee} = \leq$. Exercise 3.1.2.2 then implies that

$$\begin{aligned} \leq_{\wedge}^{\wedge} &= \leq_{\vee}^{\vee} \\ &= \leq . \end{aligned}$$

Therefore for each $a, b \in X$,

$$\begin{aligned} a \wedge b &= \inf(\{a, b\}, \leq_{\wedge}^{\wedge}) \quad \text{(Exercise 3.1.3.4)} \\ &= \inf(\{a, b\}, \leq) \\ &= a \wedge_{\leq} b \quad \text{(Definition 3.1.3.7)}. \end{aligned}$$

Hence $\wedge = \wedge_{\leq}$. $\square$

3.2 Basic Structures

3.2.1 Bounded Lattices

Definition 3.2.1.1. Let L be a lattice.

- Let $a \in L$. Then
  - a is said to be a **one** of L if for each $x \in L$, $x \wedge a = x$
  - a is said to be a **zero** of L if for each $x \in L$, $x \vee a = x$.
- Then
  - L is said to **have a zero** if there exists $0 \in L$ such that 0 is a zero of L
  - L is said to **have a one** if there exists $1 \in L$ such that 1 is a one of L
  - L is said to be **bounded** if L has a zero and L has a one.

Exercise 3.2.1.2. Let L be a lattice and $a, b \in L$.

1. If a, b are zeros of L, then $a = b$.
2. If a, b are ones of L, then $a = b$.

Proof.

1. Suppose that a, b are zeros of L. Then

$$\begin{aligned} a &= a \wedge b \\ &= b \wedge a \\ &= b. \end{aligned}$$

2. Similar to (1).

□

Note 3.2.1.3.

- If L has a one, we denote the unique one of L by 1.
- If L has a zero, we denote the unique zero of L by 0.

Exercise 3.2.1.4. Let L be a lattice. Then

1. there exists $a \in L$ such that a is a one of L iff $\operatorname{ub} L = \{a\}$.
2. there exists $a \in L$ such that a is a zero of L iff $\operatorname{lb} L = \{a\}$.

Proof.

1.
   - $(\Longrightarrow)$:
     Suppose that there exists $a \in L$ such that x is a one of L. Let $x \in L$. Since $x \wedge a = x$, we have that $x \leq a$. Since $x \in L$ is arbitrary, we have that for each $x \in L$, $x \leq a$. Hence $a \in \operatorname{ub} L$. Exercise 2.3.1.9 then implies that $\operatorname{ub} L = \{a\}$.
   - $(\Longleftarrow)$:
     Suppose that $\operatorname{ub} L = \{a\}$. Let $x \in L$. Then Since $a \in \operatorname{ub} L$, $x \leq a$. Therefore

$$\begin{aligned} x \wedge a &= \inf\{x, a\} \\ &= x. \end{aligned}$$

     Since $x \in L$ is arbitrary, we have that for each $x \in L$, $x \wedge a = x$. Hence a is a one of L.

2. Similar to (1).

□

3.2.2 Complete Lattices

3.2.3 Irreducibility and Primality

Definition 3.2.3.1. Let $(L, \leq)$ be a poset.

- Suppose that $(L, \leq)$ is a join-semilattice.
  - Let $x \in L$. Then x is said to be
    - **join-irreducible** if
      1. x is not a zero of L,
      2. for each $a, b \in L$, $x = a \vee b$ implies that $x = a$ or $x = b$
    - **join-prime**
      1. x is not a zero of L,
      2. if for each $a, b \in L$, $x \leq a \vee b$ implies that $x \leq a$ or $x \leq b$
  - We define
    - $JI(L) := \{x \in L : x \text{ is join-irreducible.}\}$
    - $JP(L) := \{x \in L : x \text{ is join-prime.}\}$
- Suppose that $(L, \leq)$ is a meet-semilattice.
  - Let $x \in L$. Then x is said to be
    - **meet-irreducible**
      1. x is not a one of L,
      2. if for each $a, b \in L$, $x = a \wedge b$ implies that $x = a$ or $x = b$
    - **meet-prime** if
      1. x is not a one of L,
      2. for each $a, b \in L$, $a \wedge b \leq x$ implies that $a \leq x$ or $b \leq x$
  - We define
    - $MI(L) := \{x \in L : x \text{ is meet-irreducible.}\}$
    - $MP(L) := \{x \in L : x \text{ is meet-prime.}\}$

Exercise 3.2.3.2. Let $(L, \leq)$ be a poset.

1. Suppose that $(L, \leq)$ is a join-semilattice. Then for each $x \in L$, if x is join-prime, then x is join-irreducible.
2. Suppose that $(L, \leq)$ is a meet-semilattice. Then for each $x \in L$, if x is meet-prime, then x is meet-irreducible.

Proof.

1. Let $x \in L$. Suppose that x is join-prime.
   (a) Since x is join-prime, x is not at zero of L.
   (b) Let $a, b \in L$. Suppose that $x = a \vee b$. Then $x \leq a \vee b$. Since x is join-prime, $x \leq a$ or $x \leq b$.
   - Suppose that $x \leq a$. Then
     $$\begin{aligned} a \vee b &= x \\ &\leq a \\ &\leq a \vee b \end{aligned}$$
     and therefore $x = a$. Thus $x \leq a$ implies that $x = a$.
   - Similarly, $x \leq b$ implies that $x = b$.

   Since $x \leq a$ or $x \leq b$, we have that $x = a$ or $x = b$. Since $a, b \in L$ with $x = a \vee b$ are arbitrary, we have that for each $a, b \in L$, $x = a \vee b$ implies that $x = a$ or $x = b$.

Thus x is join-irreducible. Since $x \in L$ such that x is join-prime is arbitrary, we have that for each $x \in L$, if x is join-prime, then x is join-irreducible.

2. Similar to (1).

□

3.2.4 Lattice Homomorphisms

Definition 3.2.4.1. Let $(X, \leq_X), (Y, \leq_Y)$ be posets and $f : X \to Y$.

1. Suppose that $(X, \leq_X), (Y, \leq_Y)$ are join-semilattices. Then f is said to be **finite** $(\leq_X, \leq_Y)$**-join preserving** if for each $a, b \in X$, $f(a \vee_{\leq_X} b) = f(a) \vee_{\leq_Y} f(b)$
2. Suppose that $(X, \leq_X), (Y, \leq_Y)$ are meet-semilattices. Then f is said to be **finite** $(\leq_X, \leq_Y)$**-meet preserving** if for each $a, b \in X$, $f(a \wedge_{\leq_X} b) = f(a) \wedge_{\leq_Y} f(b)$

Exercise 3.2.4.2. Let $(X, \leq_X), (Y, \leq_Y)$ be posets.

1. Suppose that $(X, \leq_X)$ and $(Y, \leq_Y)$ are join semilattices. Then f is $(\leq_X, \leq_Y)$-monotone iff for each $a, b \in X$, $f(a) \vee_{\leq_Y} f(b) \leq_Y f(a \vee_{\leq_X} b)$.
2. Suppose that $(X, \leq_X)$ and $(Y, \leq_Y)$ are meet semilattices. Then f is $(\leq_X, \leq_Y)$-monotone iff for each $a, b \in X$, $f(a) \vee_{\leq_Y} f(b) \leq_Y f(a \vee_{\leq_X} b)$.

Proof.

1. - $(\implies)$:
     Suppose that f is $(\leq_X, \leq_Y)$-monotone. Let $a, b \in X$. Exercise 2.2.4.6 implies that $f(a \vee_{\leq_X} b) \in \mathrm{ub}(\{f(a), f(b)\}, \leq_Y)$. Therefore
     $$\begin{aligned} f(a) \vee_{\leq_Y} f(b) &= \sup(\{f(a), f(b)\}, \leq_Y) \\ &\leq f(a \vee_{\leq_X} b). \end{aligned}$$
     Since $a, b \in X$ are arbitrary, we have that for each $a, b \in X$, $f(a) \vee_{\leq_Y} f(b) \leq_Y f(a \vee_{\leq_X} b)$.
   - $(\impliedby)$:
     Suppose that for each $a, b \in X$, $f(a) \vee_{\leq_Y} f(b) \leq_Y f(a \vee_{\leq_X} b)$. Let $a, b \in X$. Suppose that $a \leq_X b$. Exercise 2.3.1.8 then implies that $a \vee_{\leq_X} b = b$. By assumption,
     $$\begin{aligned} f(a) \vee_{\leq_Y} f(b) &\leq_Y f(a \vee_{\leq_X} b) \\ &= f(b) \\ &\leq f(a) \vee_{\leq_Y} f(b). \end{aligned}$$
     Therefore $f(a) \vee_{\leq_Y} f(b) = f(b)$. Another application of Exercise 2.3.1.8 implies that $f(a) \leq_Y f(b)$. Since $a, b \in X$ with $a \leq b$ are arbitrary, we have that for each $a, b \in X$, $a \leq_X b$ implies that $f(a) \leq_Y f(b)$. Hence f is $(\leq_X, \leq_Y)$-monotone.
2. use duality FINISH!!!

□

Exercise 3.2.4.3. Let $(X, \leq_X), (Y, \leq_Y)$ be posets.

1. Suppose that $(X, \leq_X)$ and $(Y, \leq_Y)$ are join semilattices. If f preserves finite joins, then f is $(\leq_X, \leq_Y)$-monotone.
2. Suppose that $(X, \leq_X)$ and $(Y, \leq_Y)$ are meet semilattices. If f preserves finite meets, then f is $(\leq_X, \leq_Y)$-monotone.

Proof. □

Definition 3.2.4.4. move Let $(X, \leq_X), (Y, \leq_Y)$ be complete lattices and $f : X \to Y$. Then f is said to

1. **preserve arbitrary joins** if for each $A \subset X$, $f(\sup(A, \leq_X)) = \sup(f(A), \leq_Y)$
2. **preserve arbitrary meets** if for each $A \subset X$, $f(\inf(A, \leq_X)) = \inf(f(A), \leq_Y)$

3.3 Lattice Ideals and Filters

3.3.1 Introduction

Definition 3.3.1.1. Let L be a lattice.

- Let $J \subset L$. Then
  - J is said to be an **ideal of** L if
    1. $J \neq \varnothing$,
    2. for each $a, b \in J$, $a \vee b \in J$,
    3. for each $x \in L$ and $x \in J$, $x \wedge a \in J$,
  - J is said to be an **filter of** L if
    1. $J \neq \varnothing$,
    2. for each $a, b \in J$, $a \wedge b \in J$,
    3. for each $x \in L$ and $a \in J$, $x \vee a \in J$.
- We define
  - $\mathcal{I}(L) := \{J \subset L : J \text{ is an ideal of } L\}$
  - $\mathcal{F}(L) := \{J \subset L : J \text{ is a filter of } L\}$

Exercise 3.3.1.2. Let L be a lattice. If $L \neq \varnothing$, then

1. $L \in \mathcal{I}(L)$,
2. $L \in \mathcal{F}(L)$.

Proof. Clear. (maybe fill out later) □

Exercise 3.3.1.3. Let L be a lattice and $J \subset L$. Set $\leq_J := \leq|_{J^2}$. Then

1. $J \in \mathcal{I}(L)$ iff J is a $\leq$-lower set and $(J, \leq_J)$ is upward directed. rework after making some exercises about subprosets, subdirected sets and subposets and showing facts about sup inf of subprosets and upper/lower sets in sub-prosets
2. $J \in \mathcal{F}(L)$ iff J is an $\leq$-upper set and $(J, \leq_J)$ is downward directed.

Proof.

1. - ($\Longrightarrow$):
     Suppose that $J \in \mathcal{I}(L)$.
     - Let $x \in L$ and $a \in J$. Suppose that $x \leq a$. Then $x = \inf(\{a, b\}, \leq)$ and therefore
       $$\begin{aligned} x &= \inf(\{x, a\}, \leq) \\ &= x \wedge_{\leq} a \\ &\in J. \end{aligned}$$
       Since $a \in J$ and $x \in L$ with $x \leq a$ are arbitrary, we have that J is a down set.
     - Since $J \in \mathcal{I}(L)$, we have that $J \neq \varnothing$. Let $a, b \in J$. Since $J \in \mathcal{I}(L)$,
       $$\begin{aligned} &\sup(\{a, b\}, \leq) \\ &\qquad = a \vee_{\leq} b \\ &\qquad \in J. \end{aligned}$$
       Hence (ref ex here)
       $$\begin{aligned} \sup(\{a, b\}, \leq) &= \sup(\{a, b\}, \leq_J) \\ &\in \operatorname{ub}(\{a, b\}, \leq_J) \end{aligned}$$
       and $\operatorname{ub}(\{a, b\}, \leq_J) \neq \varnothing$. Since $a, b \in J$ are arbitrary, we have that for each $a, b \in J$, $\operatorname{ub}(\{a, b\}, \leq_J) \neq \varnothing$. Since $J \neq \varnothing$ and for each $a, b \in J$, $\operatorname{ub}(J, \leq_J) \neq \varnothing$, we have that $(J, \leq_J)$ is upward directed.

- ($\Longleftarrow$):
  Suppose that J is a $\leq$-lower set and $(J, \leq_J)$ is upward directed.
  (a) Since $(J, \leq_J)$ is upward directed, $J \neq \varnothing$.
  (b) Let $a, b \in J$. Since $(J, \leq_J)$ is upward directed, $\mathrm{ub}(\{a,b\}, \leq_J) \neq \varnothing$. Thus there exists (ref ex here)

$$c \in \mathrm{ub}(\{a,b\}, \leq_J) \tag{3.1}$$
$$\subset \mathrm{ub}(\{a,b\}, \leq). \tag{3.2}$$

  Therefore $\sup(\{a,b\}, \leq) \leq c$. Since J is a $\leq$-lower set, $c \in J$ and $\sup(\{a,b\}, \leq) \leq c$, we have that

$$\begin{aligned} a \vee b &= \sup(\{a,b\}, \leq) \\ &\in J. \end{aligned}$$

  Since $a, b \in J$ are arbitrary, we have that for each $a, b \in J$, $a \vee b \in J$. rework after making some exercises about subprosets, subdirected sets and subposets and showing facts about sup inf of subprosets and upper/lower sets in sub-prosets
  (c) Let $x \in L$ and $a \in J$. Then $x \wedge a \leq a$. Since J is a $\leq$-lower set, $x \wedge a \in J$. Since $x \in L$ and $a \in J$ are arbitrary, we have that for each $x \in L$ and $a \in J$, $x \wedge a \in J$.

  Therefore $J \in \mathcal{I}(L)$.

2.

□

Exercise 3.3.1.4. Let L be a lattice and $J \subset L$.

1. If L has a zero and $J \in \mathcal{I}(L)$, then $0 \in J$.
2. If L has a one and $J \in \mathcal{F}(L)$, then $1 \in J$.

Proof.

1. Suppose that L has a zero and $J \in \mathcal{I}(L)$. Since J is an ideal of L, $J \neq \varnothing$ and a previous ex implies J is a lower set. Hence there exists $a \in J$. a prev ex implies that $0 \in \mathrm{lb}\, L$. Thus $0 \leq a$. Since J is a lower set, $0 \in J$.
2. Similar to (1).

□

Exercise 3.3.1.5. Let L be a lattice.

1. Let $(J_\alpha)_{\alpha \in A} \subset \mathcal{I}(L)$.
   (a) If $\bigcap_{\alpha \in A} J_\alpha \neq \varnothing$, then $\bigcap_{\alpha \in A} J_\alpha \in \mathcal{I}(L)$.
   (b) If $A = \{1, 2\}$, then $\bigcap_{\alpha \in A} J_\alpha \in \mathcal{I}(L)$.
   (c) If L has a zero, then $\bigcap_{\alpha \in A} J_\alpha \in \mathcal{I}(L)$.
2. Let $(J_\alpha)_{\alpha \in A} \subset \mathcal{F}(L)$.
   (a) If $\bigcap_{\alpha \in A} J_\alpha \neq \varnothing$, then $\bigcap_{\alpha \in A} J_\alpha \in \mathcal{F}(L)$.
   (b) If $A = \{1, 2\}$, then $\bigcap_{\alpha \in A} J_\alpha \in \mathcal{F}(L)$.
   (c) If L has a one, then $\bigcap_{\alpha \in A} J_\alpha \in \mathcal{F}(L)$.

Proof. Set $J := \bigcap_{\alpha \in A} J_\alpha$.

1. (a) Suppose that $J \neq \varnothing$.
   i. By assumption $J \neq \varnothing$.
   ii. Let $a, b \in J$ and $\alpha \in A$. Then $a, b \in J_\alpha$. Since $J_\alpha \in \mathcal{I}(L)$, $a \vee b \in J_\alpha$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $a \vee b \in J_\alpha$. Hence $a \vee b \in J$.
   iii. Let $x \in L$, $a \in J$ and $\alpha \in A$. Then $a \in J_\alpha$. Since $J_\alpha \in \mathcal{I}(L)$, $x \in L$ and $a \in J_\alpha$, we have that $x \wedge a \in J_\alpha$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $x \wedge a \in J_\alpha$. Hence $x \wedge a \in J$. Since $x \in L$ and $a \in J$ are arbitrary, we have that for each $x \in L$ and $a \in J$, $x \wedge a \in J$.

   Thus $J \in \mathcal{I}(L)$.

   (b) Suppose that $A = \{1, 2\}$.
   i. Since $J_1, J_2 \in \mathcal{I}(L)$, $J_1 \neq \varnothing$ and $J_2 \neq \varnothing$. Thus there exist $x_1 \in J_1$ and $x_2 \in J_2$. Since $J_1 \in \mathcal{I}(L)$, $x_2 \in L$ and $x_1 \in J_1$, we have that $x_1 \wedge x_2 \in J_1$. Similarly, $x_1 \wedge x_2 \in J_2$. Hence $x_1 \wedge x_2 \in J_1 \cap J_2$ and $J_1 \cap J_2 \neq \varnothing$. Part $1(a)$ implies that $J_1 \cap J_2 \in \mathcal{I}(L)$.

   (c) Suppose that L has a zero. Let $\alpha \in A$. Since $J_\alpha \in \mathcal{I}(L)$, a previous exercise implies that $0 \in J_\alpha$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $0 \in J_\alpha$. Hence $0 \in J$. Thus $J \neq \varnothing$. The

2. Similar to (1).

□

Exercise 3.3.1.6. Let L be a lattice. Then

1. for each $J_1, J_2 \in \mathcal{I}(L)$, $\{x \in L : \text{there exist } a \in J_1 \text{ and } b \in J_2 \text{ such that } x \leq a \vee b\} \in \mathcal{I}(L)$.
2. for each $J_1, J_2 \in \mathcal{F}(L)$, $\{x \in L : \text{there exist } a \in J_1 \text{ and } b \in J_2 \text{ such that } a \vee b \leq x\} \in \mathcal{F}(L)$ (maybe FIX!!!).

Proof.

1. Let $J_1, J_2 \in \mathcal{I}(L)$. Set $J := \{x \in L : \text{there exist } a \in J_1 \text{ and } b \in J_2 \text{ such that } x \leq a \vee b\}$.

   (a) Since $J_1, J_2 \in \mathcal{I}(L)$, we have that $J_1 \neq \varnothing$ and $J_2 \neq \varnothing$. Hence there exist $x_1 \in J_1$ and $x_2 \in J_2$. Since $x_1 \leq x_1 \vee x_2$, we have that $x_1 \in J$. Thus $J \neq \varnothing$.

   (b) Let $a, b \in J$. Then there exist $x_a, x_b \in J_1$ and $y_a, y_b \in J_2$ such that $a \leq x_a \vee y_a$ and $b \leq x_b \vee y_b$. Define $z_1, z_2 \in L$ by $z_1 := x_a \vee x_b$ and $z_2 := y_a \vee y_b$. Since $J_1, J_2 \in \mathcal{I}(L)$, we have that $z_1 \in J_1$ and $z_2 \in J_2$. We note that

$$\begin{aligned} a &\leq x_a \vee y_a \\ &\leq (x_a \vee y_a) \vee (x_b \vee y_b) \\ &= (x_a \vee x_b) \vee (y_a \vee y_b) \\ &= z_1 \vee z_2. \end{aligned}$$

   and similarly $b \leq z_1 \vee z_2$. Therefore $z_1 \vee z_2 \in \operatorname{ub}(\{a, b\}, \leq)$. Hence

$$\begin{aligned} a \vee b &= \sup(\{a, b\}, \leq) \\ &\leq z_1 \vee z_2. \end{aligned}$$

   Thus $a \vee b \in J$. Since $a, b \in J$ are arbitrary, we have that for each $a, b \in J$, $a \vee b \in J$.

   (c) Let $x \in L$ and $y \in J$. Then there exists $a \in J_1$ and $b \in J_2$ such that $y \leq a \vee b$. Then

$$\begin{aligned} x \wedge y &\leq y \\ &\leq a \vee b. \end{aligned}$$

   Therefore $x \wedge y \in J$. Since $x \in L$ and $y \in J$ are arbitrary, we have that for each $x \in L$ and $y \in J$, $x \wedge y \in J$.

   Therefore $J \in \mathcal{I}(L)$.

2. Similar to (1). FINISH!!!

□

Definition 3.3.1.7. Let L be a lattice. Define $\leq_{\mathcal{I}(L)} := \subset|_{\mathcal{I}(L)}$ (maybe standardize notation).

Exercise 3.3.1.8. Let L be a lattice.

1. Let $J_1, J_2 \in \mathcal{I}(L)$.
   (a) $\sup(\{J_1, J_2\}, \leq_{\mathcal{I}(L)}) = \{x \in L : \text{there exist } a \in J_1 \text{ and } b \in J_2 \text{ such that } x \leq a \vee b\}$.
   (b) $\inf(\{J_1, J_2\}, \leq_{\mathcal{I}(L)}) = J_1 \cap J_2$
2. Set $\leq_{\mathcal{F}(L)} := \subset|_{\mathcal{F}(L)}$ (maybe standardize notation). Let $J_1, J_2 \in \mathcal{F}(L)$. FIX!!!
   (a) $\sup(\{J_1, J_2\}, \leq_{\mathcal{I}(L)}) = \{x \in L : \text{there exist } a \in J_1 \text{ and } b \in J_2 \text{ such that } x \leq a \vee b\}$.
   (b) $\inf(\{J_1, J_2\}, \leq_{\mathcal{I}(L)}) = J_1 \cap J_2$

Proof.

1. (a) Set $J := \{x \in L : \text{there exist } a \in J_1 \text{ and } b \in J_2 \text{ such that } x \leq a \vee b\}$. (ref previous ex) implies that $J \in \mathcal{I}(L)$.
      i. • Let $a \in J_1$. Since $J_2 \in \mathcal{I}(L)$, $J_2 \neq \varnothing$. Hence there exists $b \in J_2$. Then $a \leq a \vee b$. Since $a \in J_1$, $b \in J_2$ and $a \leq a \vee b$, we have that $a \in J$. Since $a \in J_1$ is arbitrary, we have that for each $a \in J_1$, $a \in J$. Thus $J_1 \subset J$.
         • Similarly, $J_2 \subset J$.
         Since $J_1, J_2 \subset J$, we have that $J \in \text{ub}(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$.
      ii. Let $K \in \text{ub}(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$ and $x \in J$. Then $J_1, J_2 \subset K$ and there exist $a \in J_1$ and $b \in J_2$ such that $x \leq a \vee b$. Since $a \in J_1$, $b \in J_2$ and $J_1, J_2 \subset K$, we have that $a, b \in K$. Since $K \in \mathcal{I}(L)$, $a \vee b \in K$. Since $K \in \mathcal{I}(L)$ (ref a prev ex) implies that K is a lower set. Since K is a lower set, $a \vee b \in K$ and $x \leq a \vee b$, we have that $x \in K$. Since $x \in J$ is arbitrary, we have that for each $x \in J$, $x \in K$. Hence $J \subset K$. Since $K \in \text{ub}(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$ is arbitrary, we have that for each $K \in \text{ub}(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$, $J \leq_{\mathcal{I}(L)} K$.

      Since
      i. $J \in \text{ub}(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$,
      ii. for each $K \in \text{ub}(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$, $J \leq_{\mathcal{I}(L)} K$,

      we have that $J = \sup(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$.

   (b) (ref previous ex) implies that $J_1 \cap J_2 \in \mathcal{I}(L)$. Since $J_1 \cap J_2 \subset J_1$ and $J_1 \cap J_2 \subset J_2$, we have that $J_1 \cap J_2 \in \text{lb}(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$. Since $(\mathcal{I}(L), \leq_{\mathcal{I}(L)})$ is a subposet of $(\mathcal{P}(L), \subset)$, (define subproset/subposet and make exercise for the following fact) implies that

$$\begin{aligned}\inf(\{J_1, J_2\}, \leq_{\mathcal{I}(L)}) &\leq \inf(\{J_1, J_2\}, \subset) \\ &= J_1 \cap J_2 \\ &\leq \inf(\{J_1, J_2\}, \leq_{\mathcal{I}(L)}).\end{aligned}$$

      Hence $J_1 \cap J_2 = \inf(\{J_1, J_2\}, \leq_{\mathcal{I}(L)})$.

2. Similar to (1). FINISH!!!

□

Definition 3.3.1.9. Let L be a lattice. Define $\vee_{\mathcal{I}(L)}, \wedge_{\mathcal{I}(L)} : \mathcal{I}(L) \times \mathcal{I}(L) \to \mathcal{I}(L)$ and $\vee_{\mathcal{F}(L)}, \wedge_{\mathcal{F}(L)} : \mathcal{F}(L) \times \mathcal{F}(L) \to \mathcal{F}(L)$ by

1. • $J_1 \vee_{\mathcal{I}(L)} J_2 := \{x \in L : \text{there exist } a \in J_1 \text{ and } b \in J_2 \text{ such that } x \leq a \vee b\}$
   • $J_1 \wedge_{\mathcal{I}(L)} J_2 := J_1 \cap J_2$.
2. FIX!!!
   • $J_1 \vee_{\mathcal{F}(L)} J_2 := \{x \in L : \text{there exist } a \in J_1 \text{ and } b \in J_2 \text{ such that } x \leq a \vee b\}$

- $J_1 \wedge_{\mathcal{F}(L)} J_2 := J_1 \cap J_2$.

Exercise 3.3.1.10. Let L be a lattice. Then

1. (a) $(\mathcal{I}(L), \vee_{\mathcal{I}(L)}, \wedge_{\mathcal{I}(L)})$ is a lattice
   (b) L has a zero implies that $(\mathcal{I}(L), \vee_{\mathcal{I}(L)}, \wedge_{\mathcal{I}(L)})$ is a complete lattice.
2. (a) $(\mathcal{F}(L), \vee_{\mathcal{F}(L)}, \wedge_{\mathcal{F}(L)})$ is a lattice
   (b) L has a zero implies that $(\mathcal{F}(L), \vee_{\mathcal{F}(L)}, \wedge_{\mathcal{F}(L)})$ is a complete lattice.

Proof.

1. (a) the previous exercise implies that $(\mathcal{I}(L), \leq_{\mathcal{I}(L)})$ is an ordered lattice. Exercise 3.1.4.6 then implies that $(\mathcal{I}(L), \vee_{\mathcal{I}(L)}, \wedge_{\mathcal{I}(L)})$ is a lattice.
   (b)
2. (a) FINISH!!!
   (b)

$\square$

3.3.2 Properties of Ideals and Filters

Definition 3.3.2.1. Let L be a lattice.

- Let $J \in \mathcal{I}(L)$. Then J is said to be
  - **proper** if $J \neq L$.
  - **maximal** if J is maximal in $(\mathcal{I}(L) \setminus \{L\}, \subset)$.
  - **prime** if
    1. J is proper
    2. for each $a, b \in L$, $a \wedge b \in J$ implies that $a \in J$ or $b \in J$.
  - **principal** if there exists $a \in L$ such that $J = \downarrow a$.
- Let $J \in \mathcal{F}(L)$. Then J is said to be
  - **proper** if $J \neq L$.
  - **maximal** if J is maximal in $(\mathcal{F}(L) \setminus \{L\}, \subset)$.
  - **prime** if
    1. J is proper
    2. for each $a, b \in L$, $a \vee b \in J$ implies that $a \in J$ or $b \in J$.
  - **principal** if there exists $a \in L$ such that $J = \uparrow a$.

Exercise 3.3.2.2. Let L be a lattice and $J \subset L$.

1. Suppose $J \in \mathcal{I}(L)$.
   (a) If J is proper, then for each $x \in J$, x is not a one of L.
   (b) If J is prinicpal, then J is proper iff for each $x \in J$, x is not a one of L.
2. Suppose $J \in \mathcal{F}(L)$.
   (a) If J is proper, then for each $x \in J$, x is not a zero of L.
   (b) If J is prinicpal, then J is proper iff for each $x \in J$, x is not a zero of L.

Proof.

1. (a) Suppose that J is proper. Let $x \in J$. For the sake of contradiction, suppose that x is a one of L. (previous ex) then implies that $(L, \leq) = \{x\}$. Let $y \in L$. Then $y \leq x$. Since J is a lower set, $x \in J$ and $y \leq x$, we have that $y \in J$. Since $y \in L$ is arbitrary, we have that for each $y \in L$, $y \in J$. Thus

$$\begin{aligned} L &\subset J \\ &\subset L. \end{aligned}$$

Thus $J = L$ and J is not proper. Therefore x is not a one of L. Since $x \in J$ is arbitrary, we have that for each $x \in J$, x is not a one of L.

(b) Suppose that J is principal.

- $(\implies)$:
If J is proper, then $1(a)$ implies that for each $x \in J$, x is not a one of L.
- $(\impliedby)$:
Suppose that J is not proper. Then $J = L$. Since J is principal, there exists $a \in J$ such that $J = \downarrow a$. Let $x \in L$. Since

$$\begin{aligned} x &\in L \\ &= J \\ &= \downarrow a, \end{aligned}$$

we have that $x \leq a$. Hence

$$\begin{aligned} x \wedge a &= \inf\{x, a\} \\ &= x. \end{aligned}$$

Since $x \in L$ is arbitrary, we have that for each $x \in L$, $x \wedge a = x$. Hence a is a one of L. Therefore J is not proper implies that there exists $a \in J$ such that a is a one of L. By contrapositive, we have that if for each $x \in J$, x is not a one of L, then J is proper.

2. FINISH!!

$\square$

Exercise 3.3.2.3. Let L be a lattice and $J \subset L$.

1. Suppose that $J \in \mathcal{I}(L)$ and J is principal. Then J is prime iff there exists $x \in J$ such that x is meet-prime and $J = \downarrow x$.
2. Suppose that $J \in \mathcal{F}(L)$ and J is principal. Then J is prime iff there exists $x \in J$ such that x is join-prime and $J = \uparrow x$.

Proof.

1. - $(\implies)$:
Suppose that J is prime. Since J is principal, there exists $x \in L$ such that $J = \downarrow x$. Since J is prime, J is proper and for each $a, b \in L$, $a \wedge b \in J$ implies that $a \in J$ or $b \in J$.
     - Since J is proper, previous exercise implies that x is not a one of L.
     - Let $a, b \in L$. Suppose that $a \wedge b \leq x$. Since J is a lower set, $x \in J$ and $a \wedge b \leq x$, we have that $a \wedge b \in J$. Therefore $a \in J$ or $b \in J$. Since $J = \downarrow x$, we have that $a \leq x$ or $b \leq x$. Since $a, b \in L$ with $a \wedge b \leq x$ are arbitrary, we have that for each $a, b \in L$, $a \wedge b \leq x$ implies that $a \leq x$ or $b \leq x$.

   Thus x is meet-prime.
   - $(\impliedby)$:
Suppose that there exists $x \in J$ such that x is meet-prime and $J = \downarrow x$. Then J is principal.
     - For the sake of contradiction, suppose that J is not proper. Then

$$\begin{aligned} \downarrow x &= J \\ &= L. \end{aligned}$$

Thus $\operatorname{ub} L = \{x\}$. A previous ex implies that x is a one of L. This is a contracition since x is meet-prime. Thus J is proper.

  - Let $a, b \in L$. Suppose that $a \wedge b \in J$. Since $J = \downarrow x$, we have that $a \wedge b \leq x$. Since x is meet-prime, $a \leq x$ or $b \leq x$. Hence $a \in J$ or $b \in J$. Since $a, b \in L$ with $a \wedge b \in J$ are arbitrary, we have that for each $a, b \in L$, $a \wedge b \in J$ implies that $a \in J$ or $b \in J$.

  Therefore J is prime.

2. Similar to (1) FIX/CHECK/FINISH!!!

□

3.3.3 Completely Prime Ideals and Filters

Definition 3.3.3.1. Let L be a complete lattice.

- Let $J \in \mathcal{I}(L)$. Then J is said to be
  - **completely prime** if
    1. J is proper
    2. for each $(a_\alpha)_{\alpha \in A} \subset L$, $\bigwedge\limits_{\alpha \in A} a_\alpha \in J$ implies that there exists $\alpha \in A$ such that $a_\alpha \in J$.
- Let $J \in \mathcal{F}(L)$. Then J is said to be
  - **completely prime** if
    1. J is proper
    2. for each $(a_\alpha)_{\alpha \in A} \subset L$, $\bigvee\limits_{\alpha \in A} a_\alpha \in J$ implies that there exists $\alpha \in A$ such that $a_\alpha \in J$.

3.4 Complete Lattices

Definition 3.4.0.1. Let $(X, \leq)$ be a poset. Then $(X, \leq)$ is said to satisfy the

- **least upper bound (LUB) property** if for each $A \subset X$, if $A \neq \varnothing$ and A is bounded above, then there exists $x \in X$ such that $x = \sup A$
- **greatest lower bound (GLB) property** if for each $A \subset X$, if $A \neq \varnothing$ and A is bounded below, then there exists $x \in X$ such that $x = \inf A$.

Exercise 3.4.0.2. LUB iff GLB

Proof. FINISH!!!! □

Definition 3.4.0.3. Suplattice:
Let $(L, \leq)$ be a poset. Then $(L, \leq)$ is said to be a **suplattice** if for each $A \subset L$, there exists $x \in L$ such that $x = \sup A$.

3.5 Modular and Distributive Lattices

3.5.1 Introduction

Definition 3.5.1.1. Let L be a lattice. Then L is said to be a **distributive lattice** if for each $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Exercise 3.5.1.2. Let L be a lattice. Then for each $a, b, c \in L$,

1. $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$.
2. $a \wedge (b \vee c) \leq (a \vee b) \wedge (a \vee c)$

Proof. Let $a, b, c \in L$.

1. - Since $a \wedge b \leq a$ and $a \wedge c \leq a$, we have that $a \in \mathrm{ub}\{a \wedge b, a \wedge c\}$. Therefore
$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= \sup\{a \wedge b, a \wedge c\} \\ &\leq a.\end{aligned}$$
   - Since
$$\begin{aligned}a \wedge b &\leq b \\ &\leq b \vee c\end{aligned}$$
and
$$\begin{aligned}a \wedge c &\leq c \\ &\leq b \vee c,\end{aligned}$$
we have that $b \vee c \in \mathrm{ub}\{a \wedge b, a \wedge c\}$. Therefore
$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &= \sup\{a \wedge b, a \wedge c\} \\ &\leq b \vee c.\end{aligned}$$

   Then $(a \wedge b) \vee (a \wedge c) \in \mathrm{lb}\{a, b \vee c\}$. Hence
$$\begin{aligned}(a \wedge b) \vee (a \wedge c) &\leq \inf\{a, b \vee c\} \\ &= a \wedge (b \vee c).\end{aligned}$$

2. FINISH!!! (use duality)

□

Exercise 3.5.1.3. Let L be a lattice. Then for each $a, b, c \in L$,

1. $c \leq a$ implies that $(a \wedge b) \vee c \leq a \wedge (b \vee c)$
2. $a \leq c$ implies that $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Proof. Let $a, b, c \in L$.

1. Suppose that $c \leq a$. Then $a \wedge c = c$. The previous exercise then implies that

$$\begin{aligned}(a \wedge b) \vee c &= (a \wedge b) \vee (a \wedge c) \\ &\leq a \wedge (b \vee c).\end{aligned}$$

2. FINISH!!! use duality

□

Exercise 3.5.1.4. Let L be a lattice. Then for each $a, b, c \in L$,

$$(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

Proof. Let $a, b, c \in L$.

- We first note that

$$\begin{aligned}a \wedge b \leq a \leq a \vee b, \\ b \wedge c \leq b \leq a \vee b, \\ c \wedge a \leq a \leq a \vee b.\end{aligned}$$

  Therefore $a \vee b \in \operatorname{ub}\{a \wedge b, b \wedge c, c \wedge a\}$ and

$$\begin{aligned}(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) &= \sup\{a \wedge b, b \wedge c, c \wedge a\} \\ &\leq a \vee b.\end{aligned}$$

- Similarly,

$$\begin{aligned}a \wedge b \leq b \leq b \vee c, \\ b \wedge c \leq c \leq b \vee c, \\ c \wedge a \leq c \leq b \vee c.\end{aligned}$$

  Therefore $b \vee c \in \operatorname{ub}\{a \wedge b, b \wedge c, c \wedge a\}$ and

$$\begin{aligned}(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) &= \sup\{a \wedge b, b \wedge c, c \wedge a\} \\ &\leq b \vee c.\end{aligned}$$

- Finally, we have that

$$\begin{aligned}a \wedge b \leq a \leq c \vee a, \\ b \wedge c \leq c \leq c \vee a, \\ c \wedge a \leq c \leq c \vee a.\end{aligned}$$

  Therefore $c \vee a \in \operatorname{ub}\{a \wedge b, b \wedge c, c \wedge a\}$ and

$$\begin{aligned}(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) &= \sup\{a \wedge b, b \wedge c, c \wedge a\} \\ &\leq c \vee a.\end{aligned}$$

Hence $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \in \text{lb}\{a \vee b, b \vee c, c \vee a\}$ and

$$\begin{aligned}(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) &\leq \inf\{a \vee b, b \vee c, c \vee a\} \\ &= (a \vee b) \wedge (b \vee c) \wedge (c \vee a).\end{aligned}$$

□

Exercise 3.5.1.5. Let L be a lattice. Then the following are equivalent:

1. For each $a, b, c \in L$, $c \leq a$ implies that $a \wedge (b \vee c) = (a \wedge b) \vee c$.
2. For each $a, b, c \in L$, $c \leq a$ implies that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
3. For each $p, q, r \in L$, $p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)$.

Proof.

1. (1) $\implies$ (2):
   Suppose that for each $a, b, c \in L$, $c \leq a$ implies that $a \wedge (b \vee c) = (a \wedge b) \vee c$. Let $a, b, c \in L$. Suppose that $c \leq a$. Then $a \wedge c = c$. By assumption

   $$\begin{aligned}a \wedge (b \vee c) &= (a \wedge b) \vee c \\ &= (a \wedge b) \vee (a \wedge c)\end{aligned}$$

   Since $a, b, c \in L$ with $c \leq a$ are arbitrary, we have that for each $a, b, c \in L$, $c \leq a$ implies that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

2. (2) $\implies$ (3):
   Suppose that for each $a, b, c \in L$, $c \leq a$ implies that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Let $p, q, r \in L$. Define $a, b, c \in L$ by $a := p$, $b := q$ and $c := p \wedge r$. Then

   $$\begin{aligned}a \wedge c &= p \wedge (p \wedge r) \\ &= (p \wedge p) \wedge r \\ &= p \wedge r.\end{aligned}$$

   By assumption,

   $$\begin{aligned}p \wedge (q \vee (p \wedge r)) &= a \wedge (b \vee c) \\ &= (a \wedge b) \vee (a \wedge c) \\ &= (p \wedge q) \vee (p \wedge r).\end{aligned}$$

   Since $p, q, r \in L$ are arbitrary, we have that for each $p, q, r \in L$, $p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)$.

3. (3) $\implies$ (1):
   Suppose that for each $p, q, r \in L$, $p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)$. Let $a, b, c \in L$. Suppose that $c \leq a$. Define $p, q, r \in L$ by $p := a$, $q := b$ and $r := c$. Since $c \leq a$, we have that

   $$\begin{aligned}p \wedge r &= a \wedge c \\ &= c.\end{aligned}$$

   By assumption,

   $$\begin{aligned}a \wedge (b \vee c) &= p \wedge (q \vee (p \wedge r)) \\ &= (p \wedge q) \vee (p \wedge r) \\ &= (a \wedge b) \vee c.\end{aligned}$$

   Since $a, b, c \in L$ with $c \leq a$ are arbitrary, we have that for each $a, b, c \in L$, $c \leq a$ implies that $a \wedge (b \vee c) = (a \wedge b) \vee c$.

□

Definition 3.5.1.6. Let L be a lattice. Then L is said to be **modular** if for each $a, b, c \in L$,

$$c \leq a \text{ implies that } a \wedge (b \vee c) = (a \wedge b) \vee c.$$

Exercise 3.5.1.7. Let L be a lattice. Then the following are equivalent:

1. For each $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
2. For each $p, q, r \in L$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.

Proof.

1. (1) $\implies$ (2):
   Suppose that for each $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Let $p, q, r \in L$. Define $a, b, c \in L$ by $a := p \vee q$, $b := p$ and $c := r$. By absorption,
   $$\begin{aligned} a \wedge b &= (p \vee q) \wedge p \\ &= p \wedge (p \vee q) \\ &= p. \end{aligned}$$
   By assumption,
   $$\begin{aligned} a \wedge c &= (p \vee q) \wedge r \\ &= r \wedge (p \vee q) \\ &= (r \wedge p) \vee (r \wedge q). \end{aligned}$$
   Then by assumption and absorption,
   $$\begin{aligned} (p \vee q) \wedge (p \vee r) &= a \wedge (b \vee c) \\ &= (a \wedge b) \vee (a \wedge c) \\ &= p \vee [(r \wedge p) \vee (r \wedge q)] \\ &= [p \vee (r \wedge p)] \vee (r \wedge q) \\ &= [p \vee (p \wedge r)] \vee (q \wedge r) \\ &= p \vee (q \wedge r). \end{aligned}$$
   Since $p, q, r \in L$ are arbitrary, we have that for each $p, q, r \in L$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.
2. (2) $\implies$ (1):
   Suppose that for each $p, q, r \in L$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$. Let $a, b, c \in L$. Define $p, q, r \in L$ by $p := a \wedge b$, $q := a$ and $r := c$. By absorption,
   $$\begin{aligned} p \vee q &= (a \wedge b) \vee a \\ &= a \vee (a \wedge b) \\ &= a. \end{aligned}$$
   By assumption,
   $$\begin{aligned} p \vee r &= (a \wedge b) \vee c \\ &= c \vee (a \wedge b) \\ &= (c \vee a) \wedge (c \vee b). \end{aligned}$$
   Then by assumption and absorption,
   $$\begin{aligned} (a \wedge b) \vee (a \wedge c) &= p \vee (q \wedge r) \\ &= (p \vee q) \wedge (p \vee r) \\ &= a \wedge [(c \vee a) \wedge (c \vee b)] \\ &= [a \wedge (c \vee a)] \wedge (c \vee b) \\ &= [a \wedge (a \vee c)] \wedge (b \vee c) \\ &= a \wedge (b \vee c). \end{aligned}$$
   Since $a, b, c \in L$ are arbitrary, we have that for each $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

□

Definition 3.5.1.8. Let L be a lattice. Then L is said to be **distributive** if for each $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Exercise 3.5.1.9. Let L be a lattice. If L is distributive, then L is modular.

Proof. Suppose that L is distributive. FINISH!!! □

Exercise 3.5.1.10. Let L be a lattice. Then for each $a, b, c \in L$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ iff $a \wedge (b \vee c) = (a \wedge b) \vee (a \vee c)$.

Proof. Let $a, b, c \in L$.

- ($\Longrightarrow$):
  Suppose that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.
- ($\Longleftarrow$):

□

3.6 Galois Connections

Definition 3.6.0.1. Let

Chapter 4

Frames and Locales

4.1 Introduction

4.1.1 Frames

Definition 4.1.1.1. Let L be a complete lattice. Then L is said to be a **frame** if for each $a \in L$ and $(b_\alpha)_{\alpha \in A} \subset L$, $a \wedge \left(\bigvee_{\alpha \in A} b_\alpha \right) = \bigvee_{\alpha \in A} a \wedge b_\alpha$.

Definition 4.1.1.2. Let L, M be frames and $f : L \to M$. Then f is said to be a **frame homomorphism** if

1. for each $(x_\alpha)_{\alpha \in A} \subset L$, $f\left(\bigvee_{\alpha \in A} x_\alpha \right) = \sup_{\alpha \in A} f(x_\alpha)$.
2. for each $a, b \in L$, $f(a \wedge b) = f(a) \wedge f(b)$

maybe reword this with some vocab to make shorter, like "preserves arbitrary joins" and "preserves meets"

Definition 4.1.1.3. (check notation consistent with category theory notes) We define the **category of frames**, denoted **Frm**, by

- $\mathrm{Obj}(\mathbf{Frm}) := \{L : L \text{ is a frame}\}$
- $\mathrm{Hom}_{\mathbf{Frm}}(L, M) := \{f : L \to M : f \text{ is a frame homomorphism}\}$

Exercise 4.1.1.4. We have that **Frm** is a category

Proof. FINISH!!! □

4.1.2 Locales

Definition 4.1.2.1. We define the **category of locales**, denoted **Loc** by $\mathbf{Loc} := \mathbf{Frm}^{\mathrm{op}}$.

4.2 More Lattice Stuff to Come

- talk about join and meet irriducibility
- talk about join and meet primality
- talk about maximality.
- the goal is to get all the background for sober topological/measure spaces, locale theory for constructive topology and universal algebra

Chapter 5

Model Theory

5.1 Introduction

Chapter 6

Some Chapter

6.1 Closure Operators

Definition 6.1.0.1. Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Then C is said to be a **closure operator on** A if for each $X, Y \in \mathcal{P}(A)$,

1. $X \subset C(X)$,
2. $C^2(X) = C(X)$,
3. $X \subset Y$ implies that $C(X) \subset C(Y)$.

Exercise 6.1.0.2. Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Suppose that C is a closure operator on A. Then for each $(E_j)_{j \in J} \subset \mathcal{P}(A)$,

1. $C\bigg(\bigcap\limits_{j \in J} E_j\bigg) \subset \bigcap\limits_{k \in J} C(E_k)$,
2. $\bigcup\limits_{k \in J} C(E_k) \subset C\bigg(\bigcup\limits_{j \in J} E_j\bigg)$.

Proof. Let $(E_j)_{j \in J} \subset \mathcal{P}(A)$.

1. Let $k \in J$. Then $\bigcap\limits_{j \in J} E_j \subset E_k$. So $C\bigg(\bigcap\limits_{j \in J} E_j\bigg) \subset C(E_k)$. Since $k \in J$ is arbitrary, we have that
$$C\bigg(\bigcap_{j \in J} E_j\bigg) \subset \bigcap_{k \in J} C(E_k).$$
2. Let $k \in J$. Then $E_k \subset \bigcup_{j \in J} E_j$. Hence $C(E_k) \subset C\bigg(\bigcup\limits_{j \in J} E_j\bigg)$. Since $k \in J$ is arbitrary, we have that
$$\bigcup_{k \in J} C(E_k) \subset C\bigg(\bigcup_{j \in J} E_j\bigg)$$

$\square$

Definition 6.1.0.3. Let A be a set, $C : \mathcal{P}(A) \to \mathcal{P}(A)$ and $X \subset A$. Suppose that C is a closure operator on A. Then X is said to be C-closed if $C(X) = X$.

Definition 6.1.0.4. Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Suppose that C is a closure operator on A. We define the **lattice of C-closed subsets of A**, denoted $L_C(A) \subset \mathcal{P}(A)$, by

$$L_C(A) := \{X \subset A : X \text{ is } C\text{-closed}\}$$

.

Exercise 6.1.0.5. Let A be a set and $C : \mathcal{P}(A) \to \mathcal{P}(A)$. Suppose that C is a closure operator on A. Then

1. for each $(E_j)_{j\in J} \subset L_C(A)$, $\bigcap\limits_{j\in J} E_j \in L_C(A)$ and $\bigcup\limits_{j\in J} E_j \in L_C(A)$.
2. $(L_C(A), \subset)$ is a complete lattice define complete lattice

$$C\left(\bigcap_{j\in J} E_j\right) = \bigcap_{j\in J} E_j$$

and

$$C\left(\bigcup_{j\in J} E_j\right) = \bigcup_{j\in J} E_j.$$

Proof.

1. Let $(E_j)_{j\in J} \subset L_C(A)$.
   - A previous exercise Exercise B.0.0.3 implies that

$$\begin{aligned} C\left(\bigcap_{j\in J} E_j\right) &\subset \bigcap_{k\in J} C(E_k) \\ &= \bigcap_{k\in J} E_k \\ &\subset C\left(\bigcap_{k\in J} E_k\right). \end{aligned}$$

   Hence $C\left(\bigcap\limits_{j\in J} E_j\right) = \bigcap\limits_{k\in J} E_k$.
   - A previous exercise Exercise B.0.0.3 implies that

$$\begin{aligned} \bigcup_{k\in J} E_k &= \bigcup_{k\in J} C(E_k) \\ &\subset C\left(\bigcup_{j\in J} E_j\right) \\ &\subset \bigcap_{k\in J} C(E_k) \\ &= \bigcap_{k\in J} E_k \\ &\subset C\left(\bigcap_{k\in J} E_k\right). \end{aligned}$$

   Hence $C\left(\bigcap\limits_{j\in J} E_j\right) = \bigcap\limits_{k\in J} E_k$.
2.

FINISH!!!, don't need to show second part, □

Definition 6.1.0.6. then is said to be an **algebraic closure operator on A** if

Chapter 7

Universal Algebra

7.1 Introduction

Definition 7.1.0.1. Let $A, J \in \mathrm{Obj}(\mathbf{Set})$ be a set and $f \in \mathcal{F}^*(A)^J$. Then (A, f) is said to be an **algebra** if $A \neq \varnothing$ and $J \neq \varnothing$.

Definition 7.1.0.2. Let (A, f) be an algebra. Set $J := \operatorname{dom} f$.

- We define the **universe of** (A, f), denoted $\mathrm{Uni}(A, f)$, by $\mathrm{Uni}(A, f) := A$.
- We define the **operations of** (A, f), denoted $\mathrm{Oper}(A, f)$, by $\mathrm{Oper}(A, f) := f$.
- We define the **type of** (A, f), denoted $\mathrm{Type}(A, f) : J \to \mathbb{N}_0$, by $\mathrm{Type}(A, f)(j) := \operatorname{arity} f_j$.

Definition 7.1.0.3. Let $\mathcal{A}$, $\mathcal{B}$ be algebras. Then $\mathcal{A}$ and $\mathcal{B}$ are said to be **type-similar**, denoted $\mathcal{A} \sim_{\mathrm{Type}} \mathcal{B}$, if $\mathrm{Type}\,\mathcal{A} = \mathrm{Type}\,\mathcal{B}$.

Definition 7.1.0.4. Let $(A, f), (B, g)$ be algebras and $\alpha : A \to B$. Suppose that $(A, f) \sim_{\mathrm{Type}} (B, g)$. Set $J := \operatorname{dom} f$ and $\rho := \mathrm{Type}(A, f)$. Write $f = (f_j)_{j \in J}$ and $g = (g_j)_{j \in J}$. Then α is said to be an $((A, f), (B, g))$**-homomorphism** if for each $j \in J$ and $a \in A^{\rho(j)}$,

$$g_j(\alpha^{\rho(j)}(a)) = \alpha(f_j(a)).$$

Exercise 7.1.0.5. Let $(A, f), (B, g), (C, h)$ be algebras and $\alpha : A \to B$, $\beta : B \to C$. Suppose that $(A, f) \sim_{\mathrm{Type}} (B, g), (C, h)$. Set $J := \operatorname{dom} f$ and $\rho := \mathrm{Type}(A, f)$. Write $f = (f_j)_{j \in J}$, $g = (g_j)_{j \in J}$ and $h = (h_j)_{j \in J}$. If α is a $((A, f), (B, g))$-homomorphism and β is a $((B, g), (C, h))$-homomorphism, then $\beta \circ \alpha$ is a $((A, f), (C, h))$-homomorphism.

Proof. Suppose that α is a $((A, f), (B, g))$-homomorphism and β is a $((B, g), (C, h))$-homomorphism. Let $j \in J$ and $a \in A^{\rho(j)}$. Since α is a $((A, f), (B, g))$-homomorphism, $g_j(\alpha^{\rho(j)}(a)) = \alpha(f_j(a))$. Define $b \in B^{\rho(j)}$ (special case $\rho(j) = 0$?) by $b := \alpha^{\rho(j)}(a)$. Since β is a $((B, g), (C, h))$-homomorphism, $h_j(\beta^{\rho(j)}(b)) = \beta(g_j(b))$. Therefore

$$\begin{aligned}
h_j([\beta \circ \alpha]^{\rho(j)}(a)) &= h_j(\beta^{\rho(j)} \circ \alpha^{\rho(j)}(a)) \\
&= h_j(\beta^{\rho(j)}(\alpha^{\rho(j)}(a))) \\
&= h_j(\beta^{\rho(j)}(b)) \\
&= \beta(g_j(b)) \\
&= \beta(g_j(\alpha^{\rho(j)}(a))) \\
&= \beta(\alpha(f_j(a))) \\
&= \beta \circ \alpha(f_j(a)).
\end{aligned}$$

Since $j \in J$ and $a \in A^{\rho(j)}$ are arbitrary, we have that for each $j \in J$ and $a \in A^{\rho(j)}$, $h_j([\beta \circ \alpha]^{\rho(j)}(a)) = \beta \circ \alpha(f_j(a))$. Hence $\beta \circ \alpha$ is a $((A, f), (C, h))$-homomorphism. □

Definition 7.1.0.6. Define category of algebras $\mathbf{Alg}(\rho)$ of a given type ρ.

FINISH!!!!

7.2 Subalgebras

Definition 7.2.0.1. Let $(A, f), (B, g) \in \mathrm{Obj}(\mathbf{Alg})$. Suppose that (A, f) and (B, g) are type-similar. Set $J := \mathrm{dom}\, f$ and $\rho := \mathrm{Type}(A, f)$. Write $f = (f_j)_{j \in J}$ and $g = (g_j)_{j \in J}$.

- Then (B, g) is said to be a **subalgebra of** (A, f) if
  1. $B \subset A$
  2. for each $j \in J$, $f_j|_{B^{\rho(j)}} = g_j$.
- We define $\mathrm{SubAlg}(\mathcal{A}) := \{\mathcal{B} \in \mathrm{Obj}(\mathbf{Alg}) : \mathcal{A} \sim_{\mathrm{Type}} \mathcal{B}$ and $\mathcal{B}$ is a subalgebra of $\mathcal{A}\}$.

Definition 7.2.0.2. Let $(A, f) \in \mathrm{Obj}(\mathbf{Alg})$ and $B \in \mathrm{Obj}(\mathbf{Set})$.

- Then B is said to be a **subuniverse of** (A, f) if
  1. $B \subset A$,
  2. B is $\mathrm{Im}\, f$-closed.
- We define $\mathrm{SubUni}(\mathcal{A}) := \{B \in \mathrm{Obj}(\mathbf{Set}) : B$ is a subuniverse of $\mathcal{A}\}$.

Exercise 7.2.0.3. Let (A, f), (B, g) be algebras. Suppose that (A, f) and (B, g) are type-similar. If (B, g) is a subalgebra of (A, f), then B is a subuniverse of A.

Proof. Set $J := \mathrm{dom}\, f$ and $\rho := \mathrm{Type}(A, f)$. Suppose that (B, g) is a subalgebra of (A, f).

1. Since (B, g) is a subalgebra of (A, f), we have that $B \subset A$.
2. Let $j \in J$. Then for each $a \in B^{\rho}(j)$,

$$\begin{aligned} f_j(a_1, \dots, a_{\rho(j)}) &= f_j|_{B^{\rho(j)}}(a_1, \dots, a_{\rho(j)}) \\ &= g_j(a_1, \dots, a_{\rho(j)}) \\ &\in B. \end{aligned}$$

   Since $j \in J$ is arbitrary, we have that B is $\mathrm{Im}\, f$-closed. Thus B is a subuniverse of A.

□

Definition 7.2.0.4. Let $\mathcal{A}$ be an algebra and B a subuniverse of $\mathcal{A}$. Set $\mathcal{S}(B, \mathcal{A}) := \{S \subset A : S$ is a subuniverse of $\mathcal{A}$ and $B \subset S\}$. We define the **subuniverse of $\mathcal{A}$ generated by** B, denoted $\mathrm{Sg}(B, \mathcal{A})$, by

$$\mathrm{Sg}(B, \mathcal{A}) := \bigcap_{S \in \mathcal{S}(B, \mathcal{A})} S$$

show $\mathcal{S} \neq \varnothing$ and intersection of subiniverses is subuniverse

Exercise 7.2.0.5. Let (A, f) be an algebra and $B \subset A$. Then

1. $\mathrm{Sg}(B, f)$ is a subuniverse of A
2. $B \subset \mathrm{Sg}(B, f)$.

.

Proof.

1. Set $\mathcal{S} := \{S \subset A : S$ is an f-subuniverse of $A\}$. By construction, for each $S \in S$, S is f-closed. Since $\mathrm{Sg}(B, f) = \bigcap_{S \in \mathcal{S}} S$, Exercise B.0.0.3 A previous exercise in the set theory section implies that $\mathrm{Sg}(B, f)$ is f-closed. Hence $\mathrm{Sg}(B, f)$ is an f-subuniverse of A.

2. By construction, for each $S \in S$, $B \subset S$. Thus

$$\begin{aligned} B &\subset \bigcap_{S \in \mathcal{S}} S \\ &= \operatorname{Sg}(B, f). \end{aligned}$$

$\square$

Exercise 7.2.0.6. Let (A, f) be an algebra. Then $\operatorname{Sg}(\cdot, f)$ is an algebraic closure operator on A.

Proof. $\square$

Chapter 8

Groups

8.0.1 Direct Products

Definition 8.0.1.1. Let G, H be groups. Define a product $* : (G \times H) \times (G \times H) \to G \times H$ by

$$(x_1, y_1) * (x_2, y_2) = (x_1x_2, y_1y_2)$$

Then $(G \times H, *)$ is called the **direct product of** G **and** H.

Exercise 8.0.1.2. Let G, H be groups. Then the direct product $G \times H$ is a group.

Proof. Clear. □

Definition 8.0.1.3. Let G, H be groups. Define $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ by $\pi_G(x, y) = x$ and $\pi_H(x, y) = y$. Then π_G and π_H are respectively called the **projection maps onto** G **and** H.

Exercise 8.0.1.4. Let G, H be groups. Then

1. $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ are homomorphisms
2. $\ker \pi_G \cong H$ and $\ker \pi_H \cong G$

Proof.

1. Clear
2. Define $\iota_G : G \to \ker \pi_H$ by
$$\iota_G(x) = (x, e_H)$$
Then ι_G is an isomorphism. Similarly, we can define $\iota_H : H \to \ker \pi_G$ and show that it is an isomorphism.

□

Definition 8.0.1.5. Let G, H, K be groups, $\phi \in \mathrm{Hom}(G, K)$ and $\psi \in \mathrm{Hom}(H, K)$. We define $\phi \times \psi : G \times H \to K$ by $\phi \times \psi(x, y) = \phi(x)\psi(y)$

Exercise 8.0.1.6. Let G, H, K be groups, $\phi \in \mathrm{Hom}(G, K)$ and $\psi \in \mathrm{Hom}(H, K)$. If K is abelian, then $\phi \times \psi \in Hom(G \times H, K)$.

Proof. Let $x_1, x_2 \in G$ and $y_1, y_2 \in H$. Then

$$\begin{aligned} \phi \times \psi[(x_1, y_1)(x_2, y_2)] &= \phi \times \psi(x_1x_2, y_1y_2) \\ &= \phi(x_1x_2)\psi(y_1y_2) \\ &= \phi(x_1)\phi(x_2)\psi(y_1)\psi(y_2) \\ &= \phi(x_1)\psi(y_1)\phi(x_2)\psi(y_2) \\ &= [\phi \times \psi(x_1, y_1)][\phi \times \psi(x_2, y_2)] \end{aligned}$$

□

Exercise 8.0.1.7. Let G, H, K be groups and $\phi \in \text{Hom}(G \times H, K)$. Then there exist $\phi_G \in \text{Hom}(G, K)$, $\phi_H \in \text{Hom}(H, K)$ such that $\phi_G \times \phi_H = \phi$.

Proof. Suppose that K is abelian. Define $\iota_G \in \text{Hom}(G, \ker \pi_H)$ and $\iota_H \in \text{Hom}(H, \ker \pi_G)$ as in part (2) of Exercise 8.0.1.4 Define $\phi_G \in \text{Hom}(G, K)$ and $\phi_H \in \text{Hom}(H, K)$ by $\phi_G = \phi \circ \iota_G$ and $\phi_H = \phi \circ \iota_H$. Let $(x, y) \in G \times H$. Then

$$\begin{aligned}
\phi_G \times \phi_H(x, y) &= \phi_G(x)\phi_H(y) \\
&= \phi \circ \iota_G(x)\phi \circ \iota_H(y) \\
&= \phi(x, e_H)\phi(e_G, y) \\
&= \phi(x, y)
\end{aligned}$$

So $\phi = \phi_G \times \phi_H$ □

8.1 Rings

Definition 8.1.0.1. Let R be a set and $+, * : R \times R \to R$ (we write $a+b$ and ab in place of $+(a,b)$ and $*(a,b)$ respectively). Then R is said to be a **ring** if for each $a, b, c \in R$,

1. R is an abelian group with respect to $+$. The identity element with respect to $+$ is denoted by 0.
2. R is a monoid with respect to $*$. The identity element of R with respect to $*$ is denoted 1.
3. R is commutative with respect to $*$.
4. $*$ distributes over $+$.

Definition 8.1.0.2. Let R be a ring and $I \subset R$. Then I is said to be an **ideal** of R if for each $a \in R$ and $x, y \in I$,

1. $x + y \in I$
2. $ax \in I$

Definition 8.1.0.3. Let R be a ring and $A, B \subset R$. We define the **product** of A and B, denoted AB, to be

$$AB = \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in A, b_i \in B, n \in \mathbb{N} \right\}$$

Exercise 8.1.0.4. Let R be a ring and $I \subset R$. Then I is an ideal of R iff $RI \subset I$.

Proof. Suppose that $RI \subset I$. Let $a \in R$ and $x, y \in I$. Then by assumption $x + y = 1x + 1y \in I$ and $ax \in I$. So I is an ideal of R
Conversely, suppose that I is an ideal of R. Let $a_1, \cdots, a_n \in R$ and $x_1, \cdots, x_n \in I$. Then by assumption, for each $i = 1, \cdots, n$, $a_i x_i \in I$ and therefore $\sum\limits_{i=1}^{n} a_i b_i \in I$. Hence $RI \subset I$. $\square$

8.2 Modules

8.2.1 Introduction

Definition 8.2.1.1. Let R be a ring, M a set, $+ : M \times M \to M$ and $* : R \times M \to M$ (we write rx in place of $*(r, x)$). Then M is said to be an **R-module** if

1. M is an abelian group with respect to $+$. The identity element of M with respect to $+$ is denoted by 0.
2. for each $r \in R$, $*(r, \cdot)$ is a group endomorphism of M
3. for each $x \in M$, $*(\cdot, x)$ is a group homomorphism from R to M
4. $*$ is a monoid action of R on M

Note 8.2.1.2. For the remainder of this section, we assume that R is a commutative ring.

Exercise 8.2.1.3. Let M be an R-module. Then for each $r \in R$ and $x \in M$,

1. $r0 = 0$
2. $0x = 0$
3. $(-1)x = -x$

Proof. Let $r \in R$ and $x \in M$. Then

1. $$\begin{aligned} r0 &= r(0+0) \\ &= r0 + r0 \end{aligned}$$
   which implies that $r0 = 0$.
2. $$\begin{aligned} 0x &= (0+0)x \\ &= 0x + 0x \end{aligned}$$
   which implies that $0x = 0$.
3. $$\begin{aligned} (-1)x + x &= (-1)x + 1x \\ &= (-1+1)x \\ &= 0x \\ &= 0 \end{aligned}$$
   which implies that $(-1)x = -x$.

$\square$

Definition 8.2.1.4. Let M an R-module and $N \subset M$. Then N is said to be a **submodule** of M if for each $r \in R$ and $x, y \in N$, we have that $rx \in N$ and $x + y \in N$.

Definition 8.2.1.5. Let M be an R-module. We define $\mathcal{S}(M) = \{N \subset M : N \text{ is a submodule of } M\}$.

Exercise 8.2.1.6. Let M be an R-module and $N \in \mathcal{S}(M)$. Then N is a subgroup of M.

Proof. Let $x, y \in M$. Then $x - y = 1x + (-1)y \in N$. So N is a subgroup of M. $\square$

Definition 8.2.1.7. Let M be an R-module and $N \in \mathcal{S}(M)$. We define

1. $M/N = \{x + N : x \in M\}$
2. $+ : M/N \times M/N \to M/N$ by
$$(x + N) + (y + N) = (x + y) + N$$
3. $* : R \times M/N \to M/N$ by
$$r(x + N) = (rx) + N$$

Under these operations (see next exercise), M/N is an R-module known as the **quotient module** of M by N.

Exercise 8.2.1.8. Let M be an R-module and $N \in \mathcal{S}(M)$. Then

1. the monoid action defined above is well defined
2. the quotient M/N is an R-module

Proof.

1. Let $r \in R$ and $x + N, y + N \in M/N$. Recall from group theory that $x + N = y + N$ iff $x - y \in N$. Suppose that $x + N = y + N$. Then $x - y \in N$ and there exists $n \in N$ such that $x - y = n$. Therefore
$$\begin{aligned} rx - ry &= r(x - y) \\ &= rn \\ &\in N \end{aligned}$$
So $rx + N = ry + N$.
2. Properties (1) - (4) in the definition of a module are easily shown to be satisfied for M/N since they are true for M.

□

Definition 8.2.1.9. Let M and N be R-modules and $\phi : M \to N$. Then ϕ is said to be a **module homomorphism** if for each $r \in R$ and $x, y \in M$

1. $\phi(rx) = r\phi(x)$
2. $\phi(x + y) = \phi(x) + \phi(y)$

Exercise 8.2.1.10. Let M and N be R-modules and $\phi : M \to N$. Then ϕ is a iff for each $r \in R$ and $x, y \in M$, $\phi(x + ry) = \phi(x) + r\phi(y)$.

Proof. Clear. □

Exercise 8.2.1.11. Let M and N be R-modules and $\phi : M \to N$ a homomorphism. Then

1. $\ker \phi$ is a submodule of M
2. $\operatorname{Im} \phi$ is a submodule of N

Proof. Let $r \in R$, $x, y \in \ker \phi$ and $w, z \in \operatorname{Im} \phi$. Then

1.
$$\begin{aligned} \phi(rx) &= r\phi(x) \\ &= r0 \\ &= 0 \end{aligned}$$
So $rx \in \ker \phi$. Group theory tells us that $\ker \phi$ is a subgroup of M, so $x + y \in \ker \phi$. Hence $\ker \phi$ is a submodule of M.
2. Similar.

□

Definition 8.2.1.12. Let M be an R-module and $A \subset M$. We define the **submodule of** M **generated by** A, denoted $\operatorname{span}(A)$, to be

$$\operatorname{span}(A) = \bigcap_{N \in \mathcal{S}(M)} N$$

Exercise 8.2.1.13. Let M be an R-module and $A \subset M$. Then $\operatorname{span}(A) \in \mathcal{S}(M)$

Proof. Let $r \in R$ and $x, y \in \operatorname{span}(A)$. Basic group theory tells us that $\operatorname{span}(A)$ is a subgroup of M. So $x + y \in \operatorname{span}(A)$. For $N \in \mathcal{S}(M)$, by definition we have $x \in N$ and therefore $rx \in N$. So $rx \in \operatorname{span}(A)$. Hence $\operatorname{span}(A)$ is a submodule of M. □

Exercise 8.2.1.14. Let M be an R-module and $A \subset M$. If $A \neq \varnothing$, then

$$\operatorname{span}(A) = \left\{ \sum_{i=1}^{n} r_i a_i : r_i \in R, a_i \in A, n \in \mathbb{N} \right\}$$

Proof. Clearly □

Definition 8.2.1.15. Let M

8.3 Fields

8.4 Vector Spaces

8.5 Appendix

8.5.1 Monoids

Definition 8.5.1.1. Let G be a set and $* : G \times G \to G$ (we write ab in place of $*(a, b)$). Then

1. $*$ is called a **binary operation** on G
2. $*$ is said to be **associative** if for each $x, y, z \in G$, $(xy)z = x(yz)$
3. $*$ is said to be **commutative** if for each $x, y \in G$, $xy = yx$

Definition 8.5.1.2. Let G be a set, $* : G \times G \to G$, $e, x, y \in G$. Then e is said to be an **identity element** if for each $x \in G$, $ex = xe = x$.

Definition 8.5.1.3. Let G be a set and $* : G \times G \to G$. Then G is said to be a **monoid** if

1. $*$ is associative
2. there exits $e \in G$ such that e is an identity element.

Exercise 8.5.1.4. Let G be a monoid. Then the identity element is unique.

Proof. Let $e, f \in G$. Suppose that e and f are identity elements. Then $e = ef = f$. □

Note 8.5.1.5. Unless otherwise specified, we will denote the identity element of a monoid by e.

Definition 8.5.1.6. Let G be a monoid, X a set and $* : G \times X \to X$ (we write gx in place of $*(g, x)$). Then $*$ is said to be a **monoid action** of G on X if for each $g, h \in G$ and $x \in X$,

1. $(gh)x = g(hx)$
2. $ex = x$

Appendix A

Summation

Definition A.0.0.1. Let $f : X \to [0, \infty)$, Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note A.0.0.2. Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.

Appendix B

Asymptotic Notation

Definition B.0.0.1. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = o(g) \qquad \text{as } x \to x_0$$

if for each $\epsilon > 0$, there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U$,

$$\|f(x)\| \leq \epsilon \|g(x)\|$$

Exercise B.0.0.2. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. If there exists $U \in \mathcal{N}(x_0)$ such that for each $x \in U \setminus \{x_0\}$, $g(x) > 0$, then

$$f = o(g) \text{ as } x \to x_0 \quad \text{iff} \quad \lim_{x \to x_0} \frac{\|f(x)\|}{\|g(x)\|} = 0$$

Exercise B.0.0.3. Let X and Y a be normed vector spaces, $A \subset X$ open and $f : A \to Y$. Suppose that $0 \in A$. If $f(h) = o(\|h\|)$ as $h \to 0$, then for each $h \in X$, $f(th) = o(|t|)$ as $t \to 0$.

Proof. Suppose that $f(h) = o(\|h\|)$ as $h \to 0$. Let $h \in X$ and $\epsilon > 0$. Choose $\delta' > 0$ such that for each $h' \in B(0, \delta')$, $h' \in A$ and

$$\|f(h')\| \leq \frac{\epsilon}{\|h\| + 1} \|h'\|$$

Choose $\delta > 0$ such that for each $t \in B(0, \delta)$, $th \in B(0, \delta')$. Let $t \in B(0, \delta)$. Then

$$\begin{aligned} \|f(th)\| &\leq \frac{\epsilon}{\|h\| + 1} |t| \|h\| \\ &< \epsilon |t| \end{aligned}$$

So $f(th) = o(|t|)$ as $t \to 0$. $\square$

Definition B.0.0.4. Let X be a topological space, Y, Z be normed vector spaces, $f : X \to Y$, $g : X \to Z$ and $x_0 \in X \cup \{\infty\}$. Then we write

$$f = O(g) \qquad \text{as } x \to x_0$$

if there exists $U \in \mathcal{N}(x_0)$ and $M \geq 0$ such that for each $x \in U$,

$$\|f(x)\| \leq M \|g(x)\|$$

Appendix C

Categories

move to notation?

Definition C.0.0.1. We define the category of topological measure spaces, denoted $\mathbf{TopMsr}_+$, by

- $\mathrm{Obj}(\mathbf{TopMsr}_+) := \{(X, \mu) : X \in \mathrm{Obj}(\mathbf{Top}) \text{ and } \mu \in M(X)\}$
- $\mathrm{Hom}_{\mathbf{TopMsr}_+}((X, \mu), (Y, \nu)) := \mathrm{Hom}_{\mathbf{Top}}(X, Y) \cap \mathrm{Hom}_{\mathbf{Msr}_+}((X, \mathcal{B}(X), \mu), (Y, \mathcal{B}(Y), \nu))$

Appendix D

Vector Spaces

it might be better to cover some category theory and write everything in terms of $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}$ and $\mathrm{Obj}(\mathbf{Vect}_{\mathbb{K}})$

D.1 Introduction

Definition D.1.0.1. Let X be a set, $\mathbb{K}$ a field, $+ : X \times X \to X$ and $\cdot : \mathbb{K} \times X \to X$. Then $(X, +, \cdot)$ is said to be a $\mathbb{K}$**-vector space** if

1. $(X, +)$ is an abelian group
2.

Definition D.1.0.2. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$. Then $(E, +_E, \cdot_E)$ is said to be a subspace of X if

1. $+_E = +_X|_{E \times E}$
2. $\cdot_E = \cdot_X|_{\mathbb{K} \times E}$

Exercise D.1.0.3. Let $(X, +_X, \cdot_X)$ and $(E, +_E, \cdot_E)$ be vector spaces. Suppose that $E \subset X$.

Exercise D.1.0.4. Let $(X, +, \cdot)$ be a vector space and $E \subset X$. Then E is a subspace of X

Definition D.1.0.5. Let X be a vector space and $(E_j)_{j \in J}$ a collection of subspaces of X. Then $\bigcap\limits_{j \in J} E_j$ is a subspace of X.

Proof. Set $E := \bigcap\limits_{j \in J} E_j$. Let $x, y \in E$ and $\lambda \in \mathbb{K}$. Then for each $j \in J$, $x, y \in E_j$. Since for each $j \in J$, E_j is a subspace of X, we have that for each $j \in J$, $x + \lambda y \in E_j$. Thus $x + \lambda y \in E$. Since $x, y \in E$ and $\lambda \in \mathbb{K}$ are arbitrary, (cite exercise here) we have that E is a subspace of X. □

Definition D.1.0.6. Let X, Y be vector spaces and $T : X \to Y$. Then T is said to be **linear** if for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

1. $T(x_1 + x_2) = T(x_1) + T(x_2)$,
2. $T(\lambda x_1) = \lambda T(x_1)$.

We define $L(X; Y) := \{T : X \to Y : T \text{ is linear}\}$.

Exercise D.1.0.7. Let X, Y be vector spaces and $T : X \to Y$. Then T is linear iff for each $x_1, x_2 \in X$ and $\lambda \in \Lambda$,

$$T(x_1 + \lambda x_2) = T(x_1) + \lambda T(x_2)$$

Proof. Clear. (add details) □

Definition D.1.0.8. define addition/scalar multiplication of linear maps

Exercise D.1.0.9. Let X, Y be vector spaces. Then $L(X; Y)$ is a $\mathbb{K}$-vector space.

Proof. Clear □

Definition D.1.0.10. Let X be a vector space over $\mathbb{K}$ and $T : X \to \mathbb{K}$. Then T is said to be a **linear functional on** X if T is linear. We define the **dual space of** X, denoted X^*, by $X^* := \{T : X \to \mathbb{K} : T \text{ is linear}\}$.

Exercise D.1.0.11. Let X be a vector space. Then X^* is a vector space.

Proof. Clear. □

D.2 Bases

Definition D.2.0.1. Let X be a vector space and $(e_\alpha)_{\alpha \in A} \subset X$. Then $(e_\alpha)_{\alpha \in A}$ is said to be

- **linearly independent** if for each $(\alpha_j)_{j=1}^n \subset A$, $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $\sum\limits_{j=1}^n \lambda_j e_{\alpha_j} = 0$ implies that for each $j \in [n]$, $\lambda_j = 0$.
- a **Hamel basis for** X if $(e_\alpha)_{\alpha \in A}$ is linearly independent and $\operatorname{span}(e_\alpha)_{\alpha \in A} = X$.

Exercise D.2.0.2. every vector space has a Hamel basis

Proof. □

Exercise D.2.0.3.

Exercise D.2.0.4. Let X be a $\mathbb{K}$-vector space and $x \in X$. Then $x = 0$ iff for each $\phi \in X^*$, $\phi(x) = 0$.

Proof.

- $(\implies)$:
  Suppose that $x = 0$. Linearity implies that for each $\phi \in X^*$ $\phi(x) = 0$.
- $(\impliedby)$:
  Conversely, suppose that $x \neq 0$. Define $\epsilon_x : \operatorname{span}(x) \to \mathbb{K}$ by $\epsilon_x(\lambda x) := \lambda$. Let $u, v \in \operatorname{span}(x)$. Then there exists $\lambda_u, \lambda_v \in \mathbb{K}$ such that $u = \lambda_u x$ and $v = \lambda_v x$. Suppose that $u = v$. Then

  $$\begin{aligned}(\lambda_u - \lambda_v)x &= \lambda_u x - \lambda_v x \\ &= u - v \\ &= 0\end{aligned}$$

  Since $x \neq 0$, we have that $\lambda_u - \lambda_v = 0$ and therefore $\lambda_u = \lambda_v$. Hence

  $$\begin{aligned}\lambda_u &= \epsilon_x(u) \\ &= \epsilon_x(v) \\ &= \lambda_v.\end{aligned}$$

  Thus ϵ_x is well defined.

□

D.3 Multilinear Maps

Definition D.3.0.1. Let $X_1, \cdots, X_n, Y$ be vector spaces and $T : \prod_{j=1}^{n} X_j \to \mathbb{K}$. Then T is said to be **multilinear** if for each $j_0 \in [n]$ and $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$, $T(x_1, \dots, x_{j_0-1}, \cdot, x_{j_0+1})$ is linear.

$$L^n(X_1, \dots, X_n; Y) = \left\{T : \prod_{j=1}^{n} X_j \to Y : T \text{ is multilinear}\right\}$$

If $X_1 = \cdots = X_n = X$, we write $L^n(X;Y)$ in place of $L^n(X, \dots, X; Y)$.

Definition D.3.0.2. define addition and scalar mult of multilinear maps

Exercise D.3.0.3. Let $X_1, \cdots, X_n, Y$ be vector spaces. Then $L^n(X_1, \dots, X_n; Y)$ is a $\mathbb{K}$-vector space.

Proof. content... □

Exercise D.3.0.4. Let $X_1, \cdots, X_n, Y, Z$ be $\mathbb{K}$-vector spaces, $\alpha \in L^n(X_1, \dots, X_n; Y)$ and $\phi \in L^1(Y; Z)$. Then $\phi \circ \alpha \in L^n(X_1, \dots, X_n; Z)$.

Proof. Let $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$. Define $f : X_{j_0} \to Y$ by

$$f(a) := \alpha(x_1, \dots, x_{j_0-1}, a, x_{j_0+1}, \dots, x_n)$$

Since $\alpha \in L^n(X_1, \dots, X_n; Y)$, f is linear. Since ϕ is linear, and $\phi \circ f$ is linear. Since $(x_j)_{j=1}^n \in \prod_{j=1}^{n} X_j$ and $j_0 \in [n]$ are arbitrary, we have that $\phi \circ \alpha \in L^n(X_1, \dots, X_n; Y)$. □

D.4 Tensor Products

Definition D.4.0.1. Let X, Y and T be vector spaces over $\mathbb{K}$ and $\alpha \in L^2(X, Y; T)$. Then (T, α) is said to be a **tensor product of** X **and** Y if for each vector space Z and $\beta \in L^2(X, Y; Z)$, there exists a unique $\phi \in L^1(T; Z)$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & T \arrow[d, dashed, "\phi"] \\ & Z \end{tikzcd}$$

Exercise D.4.0.2. Let X, Y, S, T be vector spaces, $\alpha \in L^2(X, Y; S)$ and $\beta \in L^2(X, Y; T)$. Suppose that (S, α) and (T, β) are tensor products of X and Y. Then S and T are isomorphic.

Proof. Since (T, β) is a tensor product of X and Y, $\beta \in L^2(X, Y; T)$ there exists a unique $f \in L^1(T; T)$ such that $f\ circ\beta = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dd, dashed, "f"] \\ X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T \end{tikzcd}$$

Since $\mathrm{id}_T \in L^1(T; T)$ and $\mathrm{id}_T \circ \beta = \beta$, we have that $f = \mathrm{id}_T$. Since (S, α) is a tensor product of X and Y, there exists a unique $\phi : S \to T$ such that $\phi \circ \alpha = \beta$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] \\ & T \end{tikzcd}$$

Similarly, since (T, β) is a tensor product of X and Y, there exists a unique $\psi : T \to S$ such that $\psi \circ \beta = \alpha$, i.e. the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & T \arrow[d, dashed, "\psi"] \\ & S \end{tikzcd}$$

Therefore

$$\begin{aligned} (\phi \circ \psi) \circ \beta &= \phi \circ (\psi \circ \beta) \\ &= \phi \circ \alpha \\ &= \beta, \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, dashed, "\psi"] & & & T \arrow[dd, dashed, "\phi\circ\psi"] \\ X \times Y \arrow[ur, "\beta"] \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & S \arrow[d, dashed, "\phi"] & \Longrightarrow & X \times Y \arrow[ur, "\beta"] \arrow[dr, "\beta"'] & \\ & T & & & T \end{tikzcd}$$

By uniqueness of $f \in L^1(T; T)$, we have that

$$\begin{aligned} \mathrm{id}_T &= f \\ &= \phi \circ \psi \end{aligned}$$

A similar argument implies that $\psi \circ \phi = \mathrm{id}_S$. Hence ϕ and ψ are isomorphisms with $\phi^{-1} = \psi$. Hence S and T are isomorphic. $\square$

Definition D.4.0.3. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. We define $x \otimes y : X^* \times Y^* \to \mathbb{K}$ by $x \otimes y(\phi, \psi) := \phi(x)\psi(y)$.

Exercise D.4.0.4. Let X, Y be vector spaces, $x \in X$ and $y \in Y$. Then $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$.

Proof. Let $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} x \otimes y(\phi_1 + \lambda\phi_2, \psi) &= [\phi_1 + \lambda\phi_2(x)]\psi(y) \\ &= \phi_1(x)\psi(y) + \lambda\phi_2(x)\psi(y) \\ &= x \otimes y(\phi_1, \psi) + \lambda x \otimes y(\phi_2, \psi) \end{aligned}$$

Since $\phi_1, \phi_2 \in X^*$, $\psi \in Y^*$ and $\lambda \in \mathbb{K}$ are arbitrary, we have that for each $\psi \in Y^*$, $x \otimes y(\cdot, \psi)$ is linear. Similarly for each $\phi \in X^*$, $x \otimes y(\phi, \cdot)$ is linear. Hence $x \otimes y$ is bilinear and $x \otimes y \in L^2(X^*, Y^*; \mathbb{K})$. $\square$

Definition D.4.0.5. Let X, Y be vector spaces. We define

- the **tensor product of** X **and** Y, denoted $X \otimes Y \subset L^2(X^*, Y^*; \mathbb{K})$, by

$$X \otimes Y := \text{span}(x \otimes y : x \in X \text{ and } y \in Y),$$

- the **tensor map**, denoted $\otimes : X \times Y \to X \otimes Y$, by $\otimes(x, y) := x \otimes y$.

Exercise D.4.0.6. Let X, Y be vector spaces, $(x_j)_{j=1}^n \subset X$ and $(y_j)_{j=1}^n \subset Y$. The following are equivalent:

1. $\sum_{j=1}^n x_j \otimes y_j = 0$
2. for each $\phi \in X^*$ and $\psi \in Y^*$, $\sum_{j=1}^n \phi(x_j)\psi(y_j) = 0$
3. for each $\phi \in X^*$, $\sum_{j=1}^n \phi(x_j)y_j = 0$
4. for each $\psi \in Y^*$, $\sum_{j=1}^n \psi(y_j)x_j = 0$

Proof.

1. (1) $\implies$ (2) :
   Suppose that $\sum_{j=1}^n x_j \otimes y_j = 0$. Let $\phi \in X^*$ and $\psi \in Y^*$. Then

$$\begin{aligned} \sum_{j=1}^n \phi(x_j)\psi(y_j) &= \phi\bigg(\sum_{j=1}^n \psi(y_j)x_j\bigg) \\ &= \end{aligned}$$

2.
3.

$\square$

Exercise D.4.0.7. Let X, Y be vector spaces. Then $(X \otimes Y, \otimes)$ is a tensor product of X and Y.

Proof. Let Z be a vector space and $\alpha \in L^2(X, Y; Z)$. Define $\phi : X \otimes Y \to Z$ by $\phi\bigg(\sum_{j=1}^n \lambda_j x_j \otimes y_j) := \sum_{j=1}^n \lambda_j \alpha(x_j, y_j)$.

- **(well defined):**
  Let $u \in X \otimes Y$. Then there exist $(\lambda_j)_{j=1}^n \subset \mathbb{K}$, $(x_j)_{j=1}^n \subset X$, $(y_j)_{j=1}^n \subset Y$ such that $u = \sum_{j=1}^n \lambda_j x_j \otimes y_j$. Suppose that $u = 0$. Let $\phi \in Z^*$. Then $\phi \circ \alpha \in L^2(X, Y; Z)$.

$\square$

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