INTRODUCTION TO FOURIER ANALYSIS

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Contents

1. The Fourier Transform on $\mathbb{R}^n$ 2
1.1. Schwartz Space 2
1.2. The Convolution 3
1.3. The Fourier Transform on $L^1(\mathbb{R}^n)$ 3

1. The Fourier Transform on $\mathbb{R}^n$

1.1. **Schwartz Space.**

Definition 1.1.1. Let $\alpha \in \mathbb{N}_0^n$ and $x, y \in \mathbb{R}^n$. We define

(1) $\langle x, y\rangle = \sum_j x_j y_j$
(2) $|x| = \langle x, x\rangle^{1/2}$
(3) $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
(4) $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

Definition 1.1.2. Let $f \in C^\infty(\mathbb{R}^n)$,$\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$. We define

$$\|f\|_{\alpha,N} = \sup_{x \in \mathbb{R}^n} (1 + |x|^N)|\partial^\alpha f(x)|$$

We define Schwartz space, denoted $\mathcal{S}$, by

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \text{ for each } \alpha \in \mathbb{N}_0^n,\ N \in \mathbb{N}_0,\ \|f\|_{\alpha,N} < \infty\}$$

Exercise 1.1.3. For each $f \in \mathcal{S}$ and $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{S}$, $\alpha \in \mathbb{N}_0^n$. Then there exists $C \geq 0$ such that for each $x \in \mathbb{R}^n$,

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^{-1}$$

Define $g : \mathbb{R}^n \to [0, \infty)$ defined by $g(x) = (1 + |x|^2)^{-1}$. Then $g \in L^1(\mathbb{R}^n)$ which implies that $\partial^\alpha f \in L^1(\mathbb{R}^n)$. □

Definition 1.1.4.

1.2. **The Convolution.**

Definition 1.2.1. Let $f, g \in L^1(\mathbb{R}^n)$. We define the **convolution of** f **with** g, denoted $f * g : \mathbb{R}^n \to \mathbb{C}$, by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

Exercise 1.2.2. Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1(\mathbb{R}^n)$.

Proof. By Tonelli's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f * g| dm &\leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |f(x-y)g(y)| dm(y) \right] dm(x) \\
&= \int_{\mathbb{R}^n} |g(y)| \left[\int_{\mathbb{R}^n} |f(x-y)| dm(y) \right] dm(x) \\
&= \|f\|_1 \int_{\mathbb{R}^n} |g(y)| dm(x) \\
&= \|f\|_1 \|g\|_1 \\
&< \infty
\end{aligned}$$

□

1.3. The Fourier Transform on $L^1(\mathbb{R}^n)$.

Definition 1.3.1. Let $f \in L^1(\mathbb{R}^n)$. We define the **Fourier transform of** f, denoted $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i}$$