

# INTRODUCTION TO DIFFERENTIAL GEOMETRY

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## 1. FUNDAMENTAL DEFINITIONS AND RESULTS

### 1.1. Set Theory.

**Definition 1.1.1.** Let  $\{A_i\}_{i \in I}$  be a collection of sets. The **disjoint union** of  $\{A_i\}_{i \in I}$ , denoted  $\coprod_{i \in I} A_i$ , is defined by

$$\coprod_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$$

**Note 1.1.2.** In these notes, we will identify  $\{i\} \times A_i$  and  $A_i$ .

**Definition 1.1.3.** Let  $\{A_i\}_{i \in I}$  be a collection of sets and  $\sigma : I \rightarrow \coprod_{i \in I} A_i$ . Then  $\sigma$  is said to be a **section** of  $\coprod_{i \in I} A_i$  if for each  $i \in I$ ,  $\sigma(i) \in A_i$ .

## 2. CALCULUS

## 2.1. Differentiation.

**Definition 2.1.1.** Let  $n \geq 1$ . For  $i = 1, \dots, n$ , define  $x_i : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $x_i(a_1, \dots, a_n) = a_i$ . The functions  $(x_i)_{i=1}^n$  are called the **standard coordinate functions on  $\mathbb{R}^n$** .

**Definition 2.1.2.** Let  $U \subset \mathbb{R}^n$  be open,  $f : U \rightarrow \mathbb{R}$  and  $a \in U$ . Then  $f$  is said to be **differentiable with respect to  $x_i$  at  $a$**  if

$$\lim_{h \rightarrow 0} \frac{f(a + he_i) - f(a)}{h}$$

exists. If  $f$  is differentiable with respect to  $x_i$  at  $a$ , we define the **partial derivative of  $f$  with respect to  $x_i$  at  $a$** , denoted

$$\frac{\partial f}{\partial x_i}(a) \text{ or } \left. \frac{\partial}{\partial x_i} \right|_a f$$

to be the limit above.

**Definition 2.1.3.** Let  $U \subset \mathbb{R}^n$  be open and  $f : U \rightarrow \mathbb{R}$ . Then  $f$  is said to be **differentiable with respect to  $x_i$**  if for each  $a \in U$ ,  $f$  is differentiable with respect to  $x_i$  at  $a$ .

**Exercise 2.1.4.** Let  $U \subset \mathbb{R}^n$  be open,  $f : U \rightarrow \mathbb{R}$  and  $a \in U$ . Suppose that  $\frac{\partial^2 f}{\partial x_i \partial x_j}$  and  $\frac{\partial^2 f}{\partial x_j \partial x_i}$  exist and are continuous at  $a$ . Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a)$$

*Proof.* □

**Definition 2.1.5.** Let  $U \subset \mathbb{R}^n$  be open and  $f : U \rightarrow \mathbb{R}$ . Then  $f$  is said to be **smooth** if for each  $i_1, \dots, i_k \in \{1, \dots, n\}$ ,  $\frac{\partial^k f}{\partial i_1 \dots i_k}$  exists and is continuous on  $U$ .

**Definition 2.1.6.** Let  $U \subset \mathbb{R}^n$ ,  $f : U \rightarrow \mathbb{R}$ . Then  $f$  is said to be **smooth** if there exists  $U' \subset \mathbb{R}^n$  and  $f' : U' \rightarrow \mathbb{R}$  such that  $U \subset U'$ ,  $U'$  is open,  $f'|_U = f$  and  $f'$  is smooth. The set of smooth functions on  $U$  is denoted  $C^\infty(U)$ .

**Definition 2.1.7.** Let  $U \subset \mathbb{R}^n$  and  $p \in U$ . Then  $U$  is said to be **star-shaped** if for each  $q \in U$ ,  $\{p + t(q - p) : 0 \leq t \leq 1\} \subset U$ .

**Theorem 2.1.1. (Taylor's Theorem)** Let  $U \subset \mathbb{R}^n$  be open,  $p \in U$  and  $f \in C^\infty(U)$ . Suppose that  $U$  is star-shaped with respect to  $p$ . Then there exist  $g_1, \dots, g_n \in C^\infty(U)$  such that for each  $x \in U$ ,

$$f(x) = f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x)$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \frac{\partial f}{\partial x_i}(p)$$

*Proof.* Let  $x \in U$ . Since  $U$  is star-shaped with respect to  $p$ ,  $\{p + t(x - p) : 0 \leq t \leq 1\} \subset U$ . By the chain rule,

$$\frac{d}{dt} \left[ f(p + t(x - p)) \right] = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p + t(x - p))(x_i - p_i)$$

Integrating both sides with respect to  $t$  from 0 to 1, we obtain

$$f(x) - f(p) = \sum_{i=1}^n (x_i - p_i) \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p)) dt$$

For  $i \in \{1, \dots, n\}$ , define  $g_i \in C^\infty(U)$  by

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p)) dt$$

Then for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \frac{\partial f}{\partial x_i}(p)$$

□

## 2.2. Smooth Maps.

**Definition 2.2.1.** Let  $U \subset \mathbb{R}^n$  and  $F : U \rightarrow \mathbb{R}^m$ . Let  $x_1, \dots, x_n$  be the standard coordinate functions on  $\mathbb{R}^n$  and  $y_1, \dots, y_m$  be the standard coordinate functions on  $\mathbb{R}^m$ . For  $i \in \{1, \dots, m\}$ , we define the  **$i$ th component of  $F$** , denoted  $F_i : U \rightarrow \mathbb{R}$ , by

$$F_i = y_i \circ F$$

Thus  $F = (F_1, \dots, F_m)$

**Definition 2.2.2.** Let  $U \subset \mathbb{R}^n$  and  $F : U \rightarrow \mathbb{R}^m$ . Then  $F$  is said to be **smooth** if for each  $i \in \{1, \dots, m\}$ , the  $i$ th component of  $F$ ,  $F_i : U \rightarrow \mathbb{R}$ , is smooth.

**Definition 2.2.3.** Let  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  and  $F : U \rightarrow V$ . Then  $F$  is said to be a **diffeomorphism** if  $F$  is a homeomorphism and  $F, F^{-1}$  are smooth.

**Definition 2.2.4.** Let  $U \subset \mathbb{R}^n$  be open,  $p \in U$  and  $F : U \rightarrow \mathbb{R}^m$ . We define the **Jacobian of  $F$  at  $p$** , denoted  $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$ , by

$$\left( \frac{\partial F}{\partial x}(p) \right)_{i,j} = \frac{\partial F_i}{\partial x_j}$$

### Exercise 2.2.5. Inverse Function Theorem:

Let  $U, V \subset \mathbb{R}^n$  be open and  $F : U \rightarrow V$ .

**Exercise 2.2.6.** Let  $U, V \subset \mathbb{R}^n$  and  $F : U \rightarrow V$ . Then  $F$  is a diffeomorphism iff for each  $p \in U$ , there exists a relatively open neighborhood  $N \subset U$  of  $p$  such that  $F|_N : N \rightarrow F(N)$  is a diffeomorphism

*Proof.* content...

□

### 2.3. Topology.

**Definition 2.3.1.** Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is said to be **continuous** if for each  $U \in \mathcal{T}$ ,  $f^{-1}(U) \in \mathcal{T}_X$ .

**Definition 2.3.2.** Let  $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$  be topological spaces and  $f : X \rightarrow Y$ . Then  $f$  is said to be a homeomorphism if  $f$  is a bijection and  $f, f^{-1}$  are continuous.

**Definition 2.3.3.** Let  $X, Y$  be topological spaces. Then  $X$  and  $Y$  are said to be **homeomorphic** if there exists  $f : X \rightarrow Y$  such that  $f$  is a homeomorphism. If  $X$  and  $Y$  are homeomorphic, we write  $X \cong Y$ .

**Theorem 2.3.1.** *Let  $m, n \in \mathbb{N}$ . If  $m \neq n$ , then  $\mathbb{R}^m \not\cong \mathbb{R}^n$*

## 3. MULTILINEAR ALGEBRA

3.1.  $(r, s)$ -Tensors.

**Definition 3.1.1.** Let  $V_1, \dots, V_k, W$  be vector spaces and  $\alpha : \prod_{i=1}^n V_i \rightarrow W$ . Then  $\alpha$  is said to be **multilinear** if for each  $i \in \{1, \dots, k\}$ ,  $v \in V$ ,  $c \in \mathbb{R}$  and  $v_1, \dots, v_k \in V$ ,

$$\alpha(v_1, \dots, v_i + cv, \dots, v_k) = \alpha(v_1, \dots, v_i, \dots, v_k) + c\alpha(v_1, \dots, v, \dots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{ \alpha : \prod_{i=1}^n V_i \rightarrow W : \alpha \text{ is multilinear} \right\}$$

**Note 3.1.2.** For the remainder of this section we let  $V$  denote an  $n$ -dimensional vector space with basis  $\{e_1, \dots, e_n\}$  with dual space  $V^*$  and dual basis  $\{\epsilon_1, \dots, \epsilon_n\}$  defined by  $\epsilon_i(e_j) = \delta_{i,j}$ . We identify  $V$  with  $V^{**}$  by the isomorphism  $V \rightarrow V^{**}$  defined by  $v \mapsto \hat{v}$  where  $\hat{v}(\alpha) = \alpha(v)$  for each  $\alpha \in V^*$ .

**Definition 3.1.3.** Let  $\alpha : (V^*)^r \times V^s \rightarrow \mathbb{R}$ . Then  $\alpha$  is said to be an  $(r, s)$ -tensor on  $V$  if  $\alpha \in L(\underbrace{V^*, \dots, V^*}_r, \underbrace{V, \dots, V}_s; \mathbb{R})$ . The set of all  $(r, s)$ -tensors on  $V$  is denoted  $T_s^r(V)$ .

When  $r = s = 0$ , we set  $T_s^r = \mathbb{R}$ .

**Exercise 3.1.4.** We have that  $T_s^r(V)$  is a vector space.

*Proof.* Clear. □

**Exercise 3.1.5.** Under the identification of  $V$  with  $V^{**}$  as noted above, we have that  $V = T_0^1(V)$ .

*Proof.* By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T_0^1(V) \end{aligned}$$

□

**Definition 3.1.6.** Let  $\alpha \in T_{s_1}^{r_1}(V)$  and  $\beta \in T_{s_2}^{r_2}(V)$ . We define the **tensor product of  $\alpha$  with  $\beta$** , denoted  $\alpha \otimes \beta \in T_{s_1+s_2}^{r_1+r_2}(V)$ , by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each  $v^* \in (V^*)^{r_1}$ ,  $w^* \in (V^*)^{r_2}$ ,  $v \in V^{s_1}$  and  $w \in V^{s_2}$ .

When  $r_1 = s_1 = r_2 = s_2 = 0$  (so that  $\alpha, \beta \in \mathbb{R}$ ), we set  $\alpha \otimes \beta = \alpha\beta$ .

**Definition 3.1.7.** We define the **tensor product**, denoted  $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$  by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

**Exercise 3.1.8.** The tensor product  $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$  is well defined.

*Proof.* Tedious but straightforward. □

**Exercise 3.1.9.** The tensor product  $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$  is associative.

*Proof.* Let  $\alpha \in T_{s_1}^{r_1}(V)$ ,  $\beta \in T_{s_2}^{r_2}(V)$  and  $\gamma \in T_{s_3}^{r_3}(V)$ . Then for each  $u^* \in (V^*)^{r_1}$ ,  $v^* \in (V^*)^{r_2}$ ,  $w^* \in (V^*)^{r_3}$ ,  $u \in V^{s_1}$ ,  $v \in V^{s_2}$ ,  $w \in V^{s_3}$ ,

$$\begin{aligned} (\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v) \gamma(w^*, w) \\ &= [\alpha(u^*, u) \beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(u^*, u) [\beta(v^*, v) \gamma(w^*, w)] \\ &= \alpha(u^*, u) (\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w) \end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

**Exercise 3.1.10.** The tensor product  $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$  is bilinear.

*Proof.*

(1) Linearity in the first argument:

Let  $\alpha, \beta \in T_{s_1}^{r_1}(V)$ ,  $\gamma \in T_{s_2}^{r_2}(V)$ ,  $\lambda \in \mathbb{R}$ ,  $v^* \in (V^*)^{r_1}$ ,  $w^* \in (V^*)^{r_2}$ ,  $v \in V^{s_1}$  and  $w \in V^{s_2}$ . To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned} [(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v) \gamma(w^*, w) \\ &= [\alpha(v^*, v) + \lambda\beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(v^*, v) \gamma(w^*, w) + \lambda\beta(v^*, v) \gamma(w^*, w) \\ &= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w) \end{aligned}$$

So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

(2) Linearity in the second argument:

Similar to (1).

□

**Definition 3.1.11.**

(1) Define  $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1, \dots, i_k \leq n\}$ . Each element  $I \in \mathcal{I}_k$  is called an **unordered multi-index of length  $k$** . Recall that  $\#\mathcal{I}_{\otimes k} = n^k$ .

(2) Define  $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$ . Each element  $I \in \mathcal{I}_k$  is called an **ordered multi-index of length  $k$** . Recall that  $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$ .

**Note 3.1.12.** For the remainder of this section we will write  $\mathcal{I}_k$  in place of  $\mathcal{I}_{\otimes k}$ .

**Definition 3.1.13.** Let  $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k\}$ .

(1) Define  $\epsilon_I \in (V^*)^k$  and  $e_I \in V^k$  by

$$\epsilon_I = (\epsilon_{i_1}, \dots, \epsilon_{i_k})$$

and

$$e_I = (e_{i_1}, \dots, e_{i_k})$$

(2) Define  $e_{\otimes I} \in T_0^k(V)$  and  $\epsilon_{\otimes I} \in T_k^0(V)$  by

$$e_{\otimes I} = e_{i_1} \otimes \cdots \otimes e_{i_k}$$

and

$$\epsilon_{\otimes I} = \epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_k}$$

**Exercise 3.1.14.** Let  $\alpha, \beta \in T_s^r(V)$ . If for each  $I \in \mathcal{I}_r, J \in \mathcal{I}_s$ ,  $\alpha(\epsilon_I, e_J) = \beta(\epsilon_I, e_J)$ , then  $\alpha = \beta$ .

*Proof.* Suppose that for each  $I \in \mathcal{I}_r, J \in \mathcal{I}_s$ ,  $\alpha(\epsilon_I, e_J) = \beta(\epsilon_I, e_J)$ . Let  $v_1^*, \dots, v_r^* \in V^*$  and  $v_1, \dots, v_s \in V$ . For each  $i \in \{1, \dots, r\}$  and  $j \in \{1, \dots, s\}$ , write

$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon_{k_i}$$

and

$$v_j = \sum_{l_j=1}^n b_{j,l_j} e_{l_j}$$

Then

$$\begin{aligned} \alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon_{k_1}, \dots, \epsilon_{k_r}, e_{l_1}, \dots, e_{l_s}) \\ &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon_{k_1}, \dots, \epsilon_{k_r}, e_{l_1}, \dots, e_{l_s}) \\ &= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s) \end{aligned}$$

So that  $\alpha = \beta$ . □

**Exercise 3.1.15.** Let  $I, K \in \mathcal{I}_r$  and  $J, L \in \mathcal{I}_s$ . Then  $e_{\otimes I} \otimes \epsilon_{\otimes J}(\epsilon_K, e_L) = \delta_{I,K} \delta_{J,L}$ .

*Proof.* Write  $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$  and  $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$ . Then

$$\begin{aligned} e_{\otimes I} \otimes \epsilon_{\otimes J}(\epsilon_K, e_L) &= e_{\otimes I}(\epsilon_K) \epsilon_{\otimes J}(e_L) \\ &= e_{i_1} \otimes \cdots \otimes e_{i_r}(\epsilon_{k_1}, \dots, \epsilon_{k_r}) \epsilon_{j_1} \otimes \cdots \otimes \epsilon_{j_s}(e_{l_1}, \dots, e_{l_s}) \\ &= \left[ \prod_{m=1}^r e_{i_m}(\epsilon_{k_m}) \right] \left[ \prod_{n=1}^s \epsilon_{j_n}(e_{l_n}) \right] \\ &= \left[ \prod_{m=1}^r \delta_{i_m, k_m} \right] \left[ \prod_{n=1}^s \delta_{j_n, l_n} \right] \\ &= \delta_{I,K} \delta_{J,L} \end{aligned}$$

□

**Exercise 3.1.16.** The set  $\{e_{\otimes I} \otimes \epsilon_{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$  is a basis for  $T_s^r(V)$  and  $\dim T_s^r(V) = n^{r+s}$ .

*Proof.* Let  $(a_{I,J})_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ . Let  $\alpha = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a_{I,J} e_{\otimes I} \otimes \epsilon_{\otimes J}$ . Suppose that  $\alpha = 0$ . Then for each  $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$ ,  $\alpha(\epsilon_I, e_J) = a_{I,J} = 0$ . Thus  $\{e_{\otimes I} \otimes \epsilon_{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$  is linearly independent. Let  $\beta \in T_s^r(V)$ . For  $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$ , put  $b_{I,J} = \beta(\epsilon_J, e_I)$ . Define  $\mu =$

$\sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b_{I,J} e_{\otimes I} \otimes \epsilon_{\otimes J} \in T_s^r(V)$ . Then for each  $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$ ,  $\mu(\epsilon_I, e_J) = b_{I,J} = \beta(\epsilon_I, e_J)$ . Hence  $\mu = \beta$  and therefore  $\beta \in \text{span}\{e_{\otimes I} \otimes \epsilon_{\otimes J}\}$ .  $\square$



### 3.2. $k$ -Tensors.

**Definition 3.2.1.** Let  $\alpha : V^k \rightarrow \mathbb{R}$ . Then  $\alpha$  is said to be a  **$k$ -tensor on  $V$**  if  $\alpha \in T_k^0(V)$ . We will write  $T_k(V)$  in place of  $T_k^0(V)$ .

**Definition 3.2.2.** For  $\sigma \in S_k$  and  $\alpha \in T_k(V)$ , define the  $\sigma\alpha : V^k \rightarrow \mathbb{R}$  by

$$\sigma\alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

The map  $\alpha \mapsto \sigma\alpha$  is called the **permutation action** of  $S_k$  on  $T_k(V)$

**Exercise 3.2.3.** The permutation action of  $S_k$  on  $T_k(V)$  is a group action.

*Proof.*

- (1) Clearly for each  $\sigma \in S_k$  and  $\alpha \in T_k(V)$ ,  $\sigma\alpha \in T_k(V)$ .
- (2) Clearly for each  $\alpha \in T_k(V)$ ,  $e\alpha = \alpha$ .
- (3) Let  $\tau, \sigma \in S_k$  and  $\alpha \in T_k(V)$ . Then for each  $v_1, \dots, v_k \in V$ ,

$$\begin{aligned} (\tau\sigma)\alpha(v_1, \dots, v_k) &= \alpha(v_{\tau\sigma(1)}, \dots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \dots, v_k) \end{aligned}$$

□

**Exercise 3.2.4.** Let  $\sigma \in S_k$ . Then  $L_\sigma : T_k(V) \rightarrow T_k(V)$  given by  $L_\sigma(\alpha) = \sigma\alpha$  is a linear transformation.

*Proof.* Let  $\alpha, \beta \in T_k(V)$ ,  $c \in \mathbb{R}$  and  $v_1, \dots, v_k \in V$ . Then

$$\begin{aligned} \sigma(c\alpha + \beta)(v_1, \dots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \dots, v_k) + \sigma\beta(v_1, \dots, v_k) \end{aligned}$$

So  $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$ . □

**Definition 3.2.5.** Let  $\alpha \in T_k(V)$ . Then  $\alpha$  is said to be **symmetric** if for each  $\sigma \in S_k$ ,  $\sigma\alpha = \alpha$ . and  $\alpha$  is said to be **alternating** if for each  $\sigma \in S_k$ ,  $\sigma\alpha = \text{sgn}(\sigma)\alpha$ . The set of symmetric  $k$ -tensors on  $V$  is denoted  $\Xi_k(V)$  and the set of alternating  $k$ -tensors on  $V$  is denoted  $\Lambda_k(V)$ .

**Definition 3.2.6.** Define the **symmetric operator**  $S : T_k(V) \rightarrow \Xi_k(V)$  by

$$S(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator**  $A : T_k(V) \rightarrow \Lambda_k(V)$  by

$$A(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma\alpha$$

**Exercise 3.2.7.**

- (1) For  $\alpha \in T_k(V)$ ,  $S(\alpha)$  is symmetric.
- (2) For  $\alpha \in T_k(V)$ ,  $A(\alpha)$  is alternating.

*Proof.*

(1) Let  $\alpha \in T_k(V)$  and  $\sigma \in S_k$ . Then

$$\begin{aligned}
 \sigma S(\alpha) &= \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \tau \alpha \right] \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma \tau \alpha \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \tau \alpha \\
 &= S(\alpha)
 \end{aligned}$$

(2) Let  $\alpha \in T_k(V)$  and  $\sigma \in S_k$ . Then

$$\begin{aligned}
 \sigma A(\alpha) &= \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \right] \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \sigma \tau \alpha \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\sigma) \text{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \text{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \text{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \\
 &= \text{sgn}(\sigma) A(\alpha)
 \end{aligned}$$

□

**Exercise 3.2.8.**

- (1) For  $\alpha \in \Xi_k(V)$ ,  $S(\alpha) = \alpha$ .
- (2) For  $\alpha \in \Lambda_k(V)$ ,  $A(\alpha) = \alpha$ .

*Proof.*

(1) Let  $\alpha \in \Xi_k(V)$ . Then

$$\begin{aligned}
 S(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} \alpha \\
 &= \alpha
 \end{aligned}$$

(2) Let  $\alpha \in \Lambda_k(V)$ . Then

$$\begin{aligned} A(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma \alpha \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)^2 \alpha \\ &= \alpha \end{aligned}$$

□

**Exercise 3.2.9.** The symmetric operator  $S : T_k(V) \rightarrow \Xi_k(V)$  and the alternating operator  $A : T_k(V) \rightarrow \Lambda_k(V)$  are linear.

*Proof.* Clear.

□

**Definition 3.2.10.** Let  $\alpha \in \Lambda_k(V)$  and  $\beta \in \Lambda_l(V)$ . The **exterior product** of  $\alpha$  and  $\beta$  is defined to be the map  $\alpha \wedge \beta \in \Lambda_{k+l}(V)$  given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)$$

Thus  $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$ .

**Exercise 3.2.11.** The exterior product  $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$  is bilinear.

*Proof.* Clear.

□

**Exercise 3.2.12.** Let  $\alpha \in T_k(V)$  and  $\beta \in T_l(V)$ . Then

- (1)  $A(A(\alpha) \otimes \beta) = A(\alpha \otimes \beta)$
- (2)  $A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$

*Proof.* First note that if we fix  $\mu \in S_{k+l}$ , then for each  $\tau \in S_k$ , choosing  $\sigma = \mu\tau^{-1}$  yields  $\sigma\tau = \mu$ . For each  $\mu \in S_{k+l}$ , the map  $\phi_\mu : S_k \rightarrow S_{k+l}$  given by  $\phi_\mu(\tau) = \mu\tau^{-1}$  is injective. Thus for each  $\mu \in S_{k+l}$ , we have that  $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

(1) Then

$$\begin{aligned}
A(A(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma \left[ A(\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma \left[ \left( \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) (\tau \alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau (\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau) \sigma\tau (\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \operatorname{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \operatorname{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= A(\alpha \otimes \beta)
\end{aligned}$$

(2) Similar to (1).

□

**Exercise 3.2.13.** The exterior product  $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$  is associative.

*Proof.* Let  $\alpha \in \Lambda_k(V)$ ,  $\beta \in \Lambda_l(V)$  and  $\gamma \in \Lambda_m(V)$ . Then

$$\begin{aligned}
 (\alpha \wedge \beta) \wedge \gamma &= \left[ \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \right] \wedge \gamma \\
 &= \frac{(k+l+m)!}{(k+l)!m!} A \left( \left[ \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \right] \otimes \gamma \right) \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} A(A(\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} A((\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes (\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes A(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes \frac{(l+m)!}{l!m!} A(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes (\beta \wedge \gamma)) \\
 &= \alpha \wedge (\beta \wedge \gamma)
 \end{aligned}$$

□

**Exercise 3.2.14.** Let  $\alpha_i \in \Lambda_{k_i}(V)$  for  $i = 1, \dots, m$ . Then

$$\bigwedge_{i=1}^m \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} A \left( \bigotimes_{i=1}^m \alpha_i \right)$$

*Proof.* To see that the statment is true in the case  $m = 3$ , the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} A(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each  $3 \leq m \leq m_0$ . Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left( \bigwedge_{i=1}^{m_0-1} \alpha_i \right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A \left( \left[ \bigwedge_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A \left( \left[ \frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} A \left( \bigotimes_{i=1}^{m_0-1} \alpha_i \right) \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left( A \left[ \bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left( \left[ \bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left( \bigotimes_{i=1}^{m_0+1} \alpha_i \right)
\end{aligned}$$

□

**Exercise 3.2.15.** Define  $\tau \in S_{k+l}$  by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of  $\tau$  is  $kl$ . (Hint: inversion number)

*Proof.*

$$\begin{aligned}
N(\tau) &= \sum_{i=1}^l k \\
&= kl
\end{aligned}$$

Since  $\text{sgn}(\tau) = (-1)^{N(\tau)}$  we know that  $\text{sgn}(\tau) = (-1)^{kl}$ .

□

**Exercise 3.2.16.** Let  $\alpha \in \Lambda_k(V)$ ,  $\beta \in \Lambda_l(V)$ . Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

*Proof.* Define  $\tau \in S_{k+l}$  as in the previous exercise. Note that For  $\sigma \in S_{k+l}$  and  $v_1, \dots, v_{k+l} \in V$ , we have that

$$\begin{aligned}
\sigma\tau(\beta \otimes \alpha)(v_1, \dots, v_l, v_{l+1}, \dots, v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}) \alpha(v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\
&= \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\
&= \alpha \otimes \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}, v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\
&= \sigma(\alpha \otimes \beta)(v_1, \dots, v_k, v_{1+k}, \dots, v_{l+k})
\end{aligned}$$

Thus  $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$ . Then

$$\begin{aligned}
 \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} A(\beta \otimes \alpha) \\
 &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\beta \otimes \alpha) \\
 &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau) \sigma\tau(\beta \otimes \alpha) \\
 &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha \otimes \beta) \\
 &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \\
 &= \text{sgn}(\tau) \alpha \wedge \beta \\
 &= (-1)^{kl} \alpha \wedge \beta
 \end{aligned}$$

□

**Exercise 3.2.17.** Let  $\alpha \in \Lambda_k(V)$ . If  $k$  is odd, then  $\alpha \wedge \alpha = 0$ .

*Proof.* Suppose that  $k$  is odd. The previous exercise tells us that

$$\begin{aligned}
 \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\
 &= -\alpha \wedge \alpha
 \end{aligned}$$

Thus  $\alpha \wedge \alpha = 0$ .

□

**Exercise 3.2.18. (Fundamental Example)** Let  $\alpha_1, \dots, \alpha_m \in \Lambda_1(V)$  and  $v_1, \dots, v_m \in V$ . Then

$$\left( \bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) = \det(\alpha_i(v_j))$$

*Proof.* The previous exercises tell us that

$$\begin{aligned}
 \left( \bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) &= m! A \left( \bigotimes_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) \\
 &= m! \left[ \frac{1}{m!} \sum_{\sigma \in S_m} \text{sgn}(\sigma) \sigma \left( \bigotimes_{i=1}^m \alpha_i \right) \right] (v_1, \dots, v_m) \\
 &= \sum_{\sigma \in S_m} \text{sgn}(\sigma) \left( \bigotimes_{i=1}^m \alpha_i \right) (v_{\sigma(1)}, \dots, v_{\sigma(m)}) \\
 &= \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
 &= \det(\alpha_i(v_j))
 \end{aligned}$$

□

**Note 3.2.19.** Recall that  $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$  and that  $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$ . For the remainder of this section, we will write  $\mathcal{I}_k$  in place of  $\mathcal{I}_{\wedge k}$ .

**Definition 3.2.20.** Let  $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_k$ .

Define  $\epsilon_{\wedge I} \in \Lambda_k(V)$  by

$$\epsilon_{\wedge I} = \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_k}$$

**Exercise 3.2.21.** Let  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_k) \in \mathcal{I}_k$ . Then  $\epsilon_{\wedge I}(e_J) = \delta_{I,J}$ .

*Proof.* Put  $A = \begin{pmatrix} \epsilon_{i_1}(e_{j_1}) & \dots & \epsilon_{i_1}(e_{j_k}) \\ \vdots & & \vdots \\ \epsilon_{i_k}(e_{j_1}) & \dots & \epsilon_{i_k}(e_{j_k}) \end{pmatrix}$ . A previous exercise tells us that  $\epsilon_{\wedge I}(e_J) = \det A$ .

If  $I = J$ , then  $A = I_{k \times k}$  and therefore  $\epsilon_I(e_I) = 1$ . Suppose that  $I \neq J$ . Put  $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$ . If  $i_{l_0} < j_{l_0}$ , then all entries on the  $l_0$ th row of  $A$  are 0. If  $i_{l_0} > j_{l_0}$ , then all entries on the  $l_0$ th column of  $A$  are 0.  $\square$

**Exercise 3.2.22.** Let  $\alpha, \beta \in \Lambda_k(V)$ . If for each  $I \in \mathcal{I}_k$ ,  $\alpha(e_I) = \beta(e_I)$ , then  $\alpha = \beta$ .

*Proof.* Suppose that for each  $I \in \mathcal{I}_k$ ,  $\alpha(e_I) = \beta(e_I)$ . Let  $v_1, \dots, v_k \in V$ . For  $i = 1, \dots, k$ , write  $v_i = \sum_{j_i=1}^n a_{i,j_i} e_{j_i}$ . Then

$$\begin{aligned} \alpha(v_1, \dots, v_k) &= \sum_{j_1, \dots, j_k=1}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \alpha(e_{j_1}, \dots, e_{j_k}) \\ &= \sum_{j_1 \neq \dots \neq j_k}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \alpha(e_{j_1}, \dots, e_{j_k}) \\ &= \sum_{J \in \mathcal{I}_k} \left[ \sum_{\sigma \in S_J} \text{sgn}(\sigma) \left( \prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \alpha(e_J) \\ &= \sum_{J \in \mathcal{I}_k} \left[ \sum_{\sigma \in S_J} \text{sgn}(\sigma) \left( \prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \beta(e_J) \\ &= \sum_{j_1, \dots, j_k=1}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \beta(e_{j_1}, \dots, e_{j_k}) \\ &= \beta(v_1, \dots, v_k) \end{aligned}$$

$\square$

**Exercise 3.2.23.** The set  $\{\epsilon_{\wedge I} : I \in \mathcal{I}_k\}$  is a basis for  $\Lambda_k(V)$  and  $\dim \Lambda_k(V) = \binom{n}{k}$ .

*Proof.* Let  $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$ . Let  $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon_{\wedge I}$ . Suppose that  $\alpha = 0$ . Then for each  $J \in \mathcal{I}_k$ ,  $\alpha(e_J) = a_J = 0$ . Thus  $\{\epsilon_{\wedge I} : I \in \mathcal{I}_k\}$  is linearly independent. Let  $\beta \in \Lambda_k(V)$ . For  $I \in \mathcal{I}_k$ , put  $b_I = \beta(e_I)$ . Define  $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon_{\wedge I} \in \Lambda_k(V)$ . Then for each  $J \in \mathcal{I}_k$ ,  $\mu(e_J) = b_J = \beta(e_J)$ . Hence  $\mu = \beta$  and therefore  $\beta \in \text{span}\{\epsilon_{\wedge I} : I \in \mathcal{I}_k\}$ .  $\square$



## 4. MANIFOLDS

## 4.1. Smooth Manifolds.

**Definition 4.1.1.** Define the **upper half space** of  $\mathbb{R}^n$ , denoted  $\mathbb{H}_n$ , by

$$\mathbb{H}_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\begin{aligned}\partial\mathbb{H}_n &= \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n = 0\} \\ (\mathbb{H}^n)^\circ &= \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}\end{aligned}$$

**Definition 4.1.2.** Let  $M$  be a topological space and  $n \geq 1$ .

- (1) Let  $U \subset M$ ,  $V \subset \mathbb{H}^n$  open and  $\phi : U \rightarrow V$ . Then  $(U, \phi)$  is said to be a **coordinate chart** on  $M$  if  $\phi$  is a homeomorphism.
- (2) Let  $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$  be a collection of coordinate charts on  $M$ . Then  $\mathcal{A}$  is said to be an **atlas** on  $M$  if  $\bigcup_{a \in A} U_a = M$ .
- (3) The space  $M$  is said to be **locally half Euclidean of dimension  $n$**  if there exists an atlas  $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$  on  $M$  such that for each  $a \in A$ ,  $\phi_a(U_a) \subset \mathbb{H}^n$ .
- (4) The space  $M$  is said to be an  **$n$ -dimensional manifold** if  $M$  is Hausdorff, second countable and locally half Euclidean of dimension  $n$ .

**Note 4.1.3.** For the remainder of this section, we assume  $M$  is an  $n$ -dimensional manifold.

**Definition 4.1.4.**

- (1) Define the **boundary** of  $M$ , denoted  $\partial M$ , by
 
$$\partial M = \{p \in M : \text{there exists a chart } (U, \phi) \text{ on } M \text{ such that } p \in U \text{ and } \phi(p) \in \partial\mathbb{H}^n\}$$
- (2) Define the **interior** of  $M$ , denoted  $M^\circ$ , by

$$M^\circ = M \setminus \partial M$$

**Exercise 4.1.5.** Let  $p \in M$ . Then  $p \in \partial M$  iff for each chart  $(U, \phi)$  on  $M$ ,  $p \in U$  implies that  $\phi(p) \in \partial\mathbb{H}^n$ . (Hint: simply connected)

*Proof.* Supposet that  $p \in \partial M$ . Then there exists a coordinate chart  $(V, \psi)$  on  $M$  such that  $\psi(p) \in \partial\mathbb{H}^n$ . Let  $(U, \phi)$  be a coordinate chart on  $M$ . Suppose that  $p \in U$ . Note that  $\phi \circ \psi^{-1} : \psi(V \cap U) \rightarrow \phi(V \cap U)$  is a homeomorphism. Choose open  $n$ -balls  $B_\phi, B_\psi \subset \mathbb{H}^n$  such that  $B_\phi \subset \phi(V \cap U)$ ,  $B_\psi \subset \psi(V \cap U)$ ,  $\phi(p) \in B_\phi$  and  $\psi(p) \in B_\psi$ . For the sake of contradiction, suppose that  $\phi(p) \notin \partial\mathbb{H}^n$ . Put  $U' = B_\phi \setminus \{\phi(p)\}$  and  $V' = B_\psi \setminus \{\psi(p)\}$ . Define  $\lambda : V' \rightarrow U'$  by  $\lambda = \phi \circ \psi|_{V'}$ . Then  $\lambda$  is a homeomorphism. Note that  $V'$  is simply connected and  $U'$  is not. This is a contradiction.  $\square$

**Exercise 4.1.6.** If  $\partial M \neq \emptyset$ , then

- (1)  $\partial M$  is an  $n - 1$ -dimensional manifold
- (2)  $\partial(\partial M) = \emptyset$ .

*Proof.* (1) Since subspaces of Hausdorff, second countable spaces are Hausdorff and second countable, we need only show that  $\partial M$  is locally half euclidean of dimension  $n - 1$ . Let  $p \in \partial M$ . Then there exists a coordinate chart  $(U, \phi)$  on  $M$  such that  $p \in U$  and  $\phi(p) \in \partial\mathbb{H}^n$ .

Put  $U' = U \cap \partial M$ . Note that  $U'$  is open in  $\partial M$  and  $\phi(U) \cap \partial\mathbb{H}^n$  is open in  $\partial\mathbb{H}^n$ .

Define  $\phi' : U' \rightarrow \phi(U) \cap \partial\mathbb{H}^n$  by  $\phi' = \phi|_{U'}$ . Then  $\phi'$  is a homeomorphism.

Since  $\partial\mathbb{H}^n$  is homeomorphic to  $\mathbb{R}^{n-1}$  which is homeomorphic to  $(\mathbb{H}^{n-1})^\circ$  there exists  $\psi : \partial\mathbb{H}^n \rightarrow (\mathbb{H}^{n-1})^\circ$  such that  $\psi$  is a homeomorphism.

Define  $V' = \psi(\phi(U) \cap \partial\mathbb{H}^n)$  and  $\psi' : \phi(U) \cap \partial\mathbb{H}^n \rightarrow V'$  by  $\psi' = \psi|_{\phi(U) \cap \partial\mathbb{H}^n}$ . Then  $V'$  is open in  $(\mathbb{H}^{n-1})^\circ$  and  $\psi'$  is a homeomorphism.

Define  $\lambda : U' \rightarrow V'$  by  $\lambda = \psi' \circ \phi'$ . Then  $\lambda$  is a homeomorphism and  $(U', \lambda)$  is a coordinate chart on  $\partial M$ . So  $\partial M$  is locally Euclidean of dimension  $n - 1$ .

- (2) Let  $p \in \partial M$ . Define  $(U \cap \partial M, \lambda \circ \psi)$  as in (1). Since  $\lambda \circ \psi(p) \in (\mathbb{H}^{n-1})^\circ$ , we have that  $p \in M^\circ$ . Thus  $\partial M = (\partial M)^\circ$  and  $\partial(\partial M) = \emptyset$ .

□

**Definition 4.1.7.**

- (1) Let  $(U, \phi), (V, \psi)$  be coordinate charts on  $M$ . Then  $(U, \phi)$  and  $(V, \psi)$  are said to be **smoothly compatible** if

$$\phi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \phi(U \cap V) \text{ is a diffeomorphism}$$

- (2) Let  $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$  be an atlas on  $M$ . Then  $\mathcal{A}$  is said to be **smooth** if for each  $a, b \in A$ ,  $(U_a, \phi_a)$  and  $(U_b, \phi_b)$  are smoothly compatible.
- (3) Let  $\mathcal{A}$  be a smooth atlas on  $M$ . Then  $\mathcal{A}$  is said to be **maximal** if for each smooth atlas  $\mathcal{B}$  on  $M$ ,  $\mathcal{A} \subset \mathcal{B}$  implies that  $\mathcal{A} = \mathcal{B}$ . A maximal smooth atlas on  $M$  is called a **smooth structure on  $M$** .
- (4) Let  $\mathcal{A}$  be a smooth structure on  $M$ . Then  $(M, \mathcal{A})$  is said to be a **smooth  $n$ -dimensional manifold**.

**Exercise 4.1.8.** Let  $\mathcal{B}$  be a smooth atlas on  $M$ . Then there exists a unique smooth structure  $\mathcal{A}$  on  $M$  such that  $\mathcal{B} \subset \mathcal{A}$ .

*Proof.* Define  $\mathcal{A}$  to be the set of all coordinate charts  $(U, \phi)$  on  $M$  such that for each coordinate chart  $(V, \psi) \in \mathcal{B}$ ,  $(U, \phi)$  and  $(V, \psi)$  are smoothly compatible.

Clearly  $\mathcal{B} \subset \mathcal{A}$ .

Let  $(U, \phi), (V, \psi) \in \mathcal{A}$  and  $p \in U \cap V$ . Then there exists  $(W, \chi) \in \mathcal{B}$  such that  $p \in W$ . By assumption,  $\phi \circ \chi^{-1} : \chi(U \cap W) \rightarrow \phi(U \cap W)$  and  $\chi \circ \psi^{-1} : \psi(W \cap V) \rightarrow \chi(W \cap V)$  are diffeomorphisms. Then  $(\phi \circ \chi^{-1}) \circ (\chi \circ \psi^{-1}) = \phi \circ \psi^{-1} : \psi(U \cap W \cap V) \rightarrow \phi(U \cap W \cap V)$  is a diffeomorphism. Since for each  $q \in \psi(U \cap V)$ , there exists an open neighborhood  $N \subset \psi(U \cap V)$  of  $q$  on which  $\phi \circ \psi^{-1}$  are diffeomorphic, we have that  $\phi \circ \psi^{-1}$  is a diffeomorphism on  $\psi(U \cap V)$  and therefore  $(U, \phi)$  and  $(V, \psi)$  are smoothly compatible. Hence  $\mathcal{A}$  is a smooth atlas.

To see that  $\mathcal{A}$  is maximal, let  $\mathcal{B}'$  be a smooth atlas on  $M$ . Suppose that  $\mathcal{A} \subset \mathcal{B}'$  and let  $(U, \phi) \in \mathcal{B}'$ . By definition, for each chart  $(V, \psi) \in \mathcal{B}'$ ,  $(U, \phi)$  and  $(V, \psi)$  are smoothly compatible. Since  $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$ , we have that  $(U, \phi) \in \mathcal{A}$ . So  $\mathcal{A} = \mathcal{B}'$  and  $\mathcal{A}$  is a maximal smooth atlas on  $M$ . □

**Exercise 4.1.9.** Let  $\mathcal{A}$  be a smooth atlas on  $M$ . Define  $\lambda : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$  by  $\lambda(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$ . Put  $\mathcal{A}|_{\partial M} = \{(U \cap \partial M, \lambda \circ \phi_{U \cap \partial M}) : (U, \phi) \in \mathcal{A}\}$ . Then

- (1)  $\mathcal{A}|_{\partial M}$  is a smooth atlas on  $\partial M$ .
- (2) if  $\mathcal{A}$  is maximal, then  $\mathcal{A}|_{\partial M}$  is maximal.

*Proof.*

□

**Note 4.1.10.** For the rest of this section, we assume that  $(M, \mathcal{A})$  is a smooth  $n$ -dimensional manifold and we denote the standard coordinate functions on  $\mathbb{R}^n$  by  $u_1, \dots, u_n$ . For a

coordinate chart  $(U, \phi) \in \mathcal{A}$  and  $i \in \{1, \dots, n\}$ , we will typically denote the  $i$ th coordinate of  $\phi$  by  $x_i$ , that is,  $x_i = u_i(\phi)$ .

## 4.2. Smooth Maps.

**Definition 4.2.1.** Let  $f : M \rightarrow \mathbb{R}$ . Then  $f$  is said to be smooth if for each coordinate chart  $(U, \phi) \in \mathcal{A}$ ,  $f \circ \phi^{-1}$  is smooth. The set of all smooth functions on  $M$  is denoted  $C^\infty(M)$ .

**Exercise 4.2.2.** We have that  $C^\infty(M)$  is a vector space.

*Proof.* Clear. □

**Definition 4.2.3.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F : M \rightarrow N$ . Then  $F$  is said to be **smooth** if for each  $(U, \phi) \in \mathcal{A}$  and  $(V, \psi) \in \mathcal{B}$ ,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \psi(F(U) \cap V)$$

is smooth and  $F$  is said to be a **diffeomorphism** if  $F$  is a homeomorphism and  $F, F^{-1}$  are smooth.

**Exercise 4.2.4.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F : M \rightarrow N$  a diffeomorphism. Then for each  $(U, \phi) \in \mathcal{A}$ ,  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ .

*Proof.* Let  $(V, \psi) \in \mathcal{B}$ .

- (1) Since  $\phi$  and  $F^{-1}$  are homeomorphisms,  $\phi \circ F^{-1} : F(U) \cap V \rightarrow \phi(U \cap F^{-1}(V))$  is a homeomorphism
- (2) Since  $F$  is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \rightarrow \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \psi(V \cap F(U))$$

are smooth.

Therefore  $(F(U), \phi \circ F^{-1})$  and  $(V, \psi)$  are smoothly compatible. Since  $\mathcal{B}$  is maximal,  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ . □

**Exercise 4.2.5.** Let  $(M, \mathcal{A})$  be smooth  $m$ -dimensional manifold,  $(N, \mathcal{B})$  a smooth  $n$ -dimensional manifold and  $F : M \rightarrow N$ . If  $F$  is a diffeomorphism, then  $m = n$ .

*Proof.* Suppose that  $F$  is a diffeomorphism. Let  $(U, \phi) \in \mathcal{A}$ . The previous exercise implies that  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ . Then

$$\begin{aligned} \phi(U) &= \phi \circ F^{-1}(F(U)) \\ &= \subset \mathbb{H}^n \end{aligned}$$

By definition,  $\phi(U) \subset \mathbb{H}^m$ . So  $m = n$ . □

### 4.3. Partitions of Unity.

**Definition 4.3.1.** Let  $p \in M$ ,  $U \in \mathcal{N}_a$  open and  $\rho \in C_c^\infty(M)$ . Then  $\rho$  is said to be a **bump function at  $p$  supported in  $U$**  if

- (1)  $\rho \geq 0$
- (2) there exists  $V \in \mathcal{N}_p$  such that  $V$  is open and  $\rho|_V = 1$
- (3)  $\text{supp } \rho \subset U$

**Exercise 4.3.2.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then  $f \in C_c^\infty(\mathbb{R})$ .

*Proof.*

□

#### 4.4. The Tangent Space.

**Definition 4.4.1.** Let  $p \in M$ . Define the relation  $\sim_p$  on  $C^\infty(M)$  by  $f \sim_p g$  iff there exists  $U \in \mathcal{N}_p$  such that  $U$  is open and  $f|_U = g|_U$ . Clearly  $\sim_p$  is an equivalence relation on  $C^\infty(M)$ . We denote  $C^\infty(M)/\sim_p$  by  $C_p^\infty(M)$ . For  $f \in C^\infty(M)$ , we define the **germ of  $f$  at  $p$**  to be the equivalence class of  $f$  under  $\sim_p$ .

**Exercise 4.4.2.** Let  $p \in M$ . We have that  $C_p^\infty(M)$  is a vector space.

*Proof.* Clear. □

**Definition 4.4.3.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$ ,  $p \in U$  and  $f \in C_p^\infty(M)$ . For  $i \in \{1, \dots, n\}$ , define the partial derivative of  $f$  with respect to  $x_i$  at  $p$ , denoted

$$\frac{\partial f}{\partial x_i}(p), \quad \left. \frac{\partial}{\partial x_i} \right|_p f, \quad \partial_{x_i} f(p) \quad \text{or} \quad \left. \partial_{x_i} \right|_p f$$

by

$$\left. \frac{\partial}{\partial x_i} \right|_p f = \left. \frac{\partial}{\partial u_i} \right|_{\phi(p)} f \circ \phi^{-1}$$

**Exercise 4.4.4.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$  and  $p \in U$ . Then for each  $i, j \in \{1, \dots, n\}$ , we have that

$$\frac{\partial x_i}{\partial x_j}(p) = \delta_{i,j}$$

*Proof.* Let  $i, j \in \{1, \dots, n\}$ . Then

$$\begin{aligned} \left. \frac{\partial}{\partial x_j} \right|_p x_i &= \left. \frac{\partial}{\partial u_j} \right|_{\phi(p)} x_i \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u_j} \right|_{\phi(p)} u_i \circ \phi \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u_j} \right|_{\phi(p)} u_i \\ &= \delta_{i,j} \end{aligned}$$

□

**Exercise 4.4.5. (Change of Coordinates):** Let  $(U, \phi), (V, \psi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$  and  $\psi = (y_1, \dots, y_n)$ ,  $p \in U \cap V$  and  $f \in C_p^\infty(M)$ . Then for each  $i \in \{1, \dots, n\}$ , we have

$$\frac{\partial f}{\partial y_i}(p) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p) \frac{\partial x_j}{\partial y_i}(p)$$

*Proof.* Put  $h = \phi \circ \psi^{-1}$  and write  $h = (h_1, \dots, h_n)$ . Then  $\phi = h \circ \psi$  and  $\psi^{-1} = \phi^{-1} \circ h$ . By definition and the chain rule, we have that

$$\begin{aligned} \left. \frac{\partial}{\partial y_i} \right|_p f &= \left. \frac{\partial}{\partial u_i} \right|_{\psi(p)} f \circ \psi^{-1} \\ &= \left. \frac{\partial}{\partial u_i} \right|_{\psi(p)} f \circ \phi^{-1} \circ h \\ &= \sum_{j=1}^n \left( \left. \frac{\partial}{\partial u_j} \right|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left( \left. \frac{\partial}{\partial u_i} \right|_{\psi(p)} h_j \right) \\ &= \sum_{j=1}^n \left( \left. \frac{\partial}{\partial u_j} \right|_{\phi(p)} f \circ \phi^{-1} \right) \left( \left. \frac{\partial}{\partial u_i} \right|_{\psi(p)} x_j \circ \psi^{-1} \right) \\ &= \sum_{j=1}^n \left( \left. \frac{\partial}{\partial x_j} \right|_p f \right) \left( \left. \frac{\partial}{\partial y_i} \right|_p x_j \right) \end{aligned}$$

□

**Exercise 4.4.6. Taylor's Theorem:**

Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$ ,  $p \in U$  and  $f \in C_p^\infty(M)$ . Then there exist  $g_1, \dots, g_n \in C_p^\infty(M)$  such that

$$f = f(p) + \sum_{i=1}^n (x_i - x_i(p)) g_i$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \left. \frac{\partial}{\partial x_i} \right|_p f$$

*Proof.* Since we are interested in the germ of  $f$  at  $p$ , we may assume that  $\phi(U)$  is star-shaped with respect to  $\phi(p)$ . Let  $q \in U$ . From Taylor's theorem in section 1, we know that there exist  $\tilde{g}_1, \dots, \tilde{g}_n \in C^\infty(\phi(U))$  such that for each  $q \in U$ ,

$$f \circ \phi^{-1}(\phi(q)) = f \circ \phi^{-1}(\phi(p)) + \sum_{i=1}^n [u_i \circ \phi(q) - u_i \circ \phi(p)] \tilde{g}_i(\phi(q))$$

and for each  $i \in \{1, \dots, n\}$ ,

$$\tilde{g}_i(\phi(p)) = \left. \frac{\partial}{\partial u_i} \right|_{\phi(p)} f \circ \phi^{-1}$$

For each  $i \in \{1, \dots, n\}$ , define  $g_i = \tilde{g}_i \circ \phi$ . Then for each  $q \in U$ ,

$$f(q) = f(p) + \sum_{i=1}^n [x_i(q) - x_i(p)] g_i(q)$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \left. \frac{\partial}{\partial x_i} \right|_p f$$

□

**Definition 4.4.7.** Let  $p \in M$  and  $v : C_p^\infty(M) \rightarrow \mathbb{R}$ . Then  $v$  is said to be **Leibnizian** if for each  $f, g \in C_p^\infty(M)$ ,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and  $v$  is said to be a **derivation at  $p$**  if for each  $f, g \in C_p^\infty(M)$  and  $a \in \mathbb{R}$ ,

- (1)  $v$  is linear
- (2)  $v$  is Leibnizian

We define the **tangent space of  $M$  at  $p$** , denoted  $T_pM$ , by

$$T_pM = \{v : C_p^\infty(M) \rightarrow \mathbb{R} : v \text{ is a derivation at } p\}$$

**Exercise 4.4.8.** Let  $f \in C_p^\infty(M)$  and  $v \in T_pM$ . If  $f$  is constant, then  $vf = 0$ .

*Proof.* Suppose that  $f = 1$ . Then  $f^2 = f$  and  $v(f^2) = 2v(f)$ . So  $v(f) = 2v(f)$  which implies that  $v(f) = 0$ . If  $f \neq 1$ , then there exists  $c \in \mathbb{R}$  such that  $f = c$ . Since  $v$  is linear,  $v(f) = cv(1) = 0$ .  $\square$

**Exercise 4.4.9.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$  and  $p \in U$ . Then

$$\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\}$$

is a basis for  $T_pM$  and  $\dim T_pM = n$ .

*Proof.* Clearly  $\frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \in T_pM$ . Let  $a_1, \dots, a_n \in \mathbb{R}$ . Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \Big|_p = 0$$

Then

$$\begin{aligned} 0 &= vx_j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \Big|_p x_j \\ &= a_j \end{aligned}$$

Hence  $\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\}$  is independent.

Now, let  $v \in T_pM$  and  $f \in C_p^\infty(M)$ . By Taylor's theorem, there exist  $g_1, \dots, g_n \in C_p^\infty(M)$  such that

$$f = f(p) + \sum_{i=1}^n (x_i - x_i(p))g_i$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \frac{\partial}{\partial x_i} \Big|_p f$$



Then

$$\begin{aligned}
v(f) &= \sum_{i=1}^n v(x_i - x_i(p))g_i(p) + \sum_{i=1}^n (x_i(p) - x_i(p))v(g_i) \\
&= \sum_{i=1}^n v(x_i)g_i(p) \\
&= \sum_{i=1}^n v(x_i) \left. \frac{\partial}{\partial x_i} \right|_p f \\
&= \left[ \sum_{i=1}^n v(x_i) \left. \frac{\partial}{\partial x_i} \right|_p \right] f
\end{aligned}$$

So

$$v = \sum_{i=1}^n v(x_i) \left. \frac{\partial}{\partial x_i} \right|_p$$

and

$$v \in \text{span} \left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$$

□

**Definition 4.4.10.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F : M \rightarrow N$  smooth and  $p \in M$ . We define the **differential of  $F$  at  $p$** , denoted  $dF_p : T_p M \rightarrow T_{F(p)} N$ , by

$$\left[ dF_p(v) \right] (f) = v(f \circ F)$$

for  $v \in T_p M$  and  $f \in C_{F(p)}^\infty(N)$ .

**Exercise 4.4.11.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F : M \rightarrow N$  smooth and  $p \in M$ . Then  $dF_p$  is well defined.

*Proof.* Let  $v \in T_p M$ ,  $f, g \in C_{F(p)}^\infty(N)$  and  $c \in \mathbb{R}$ . Then

(1)

$$\begin{aligned}
dF_p(v)(f + cg) &= v((f + cg) \circ F) \\
&= v(f \circ F + cg \circ F) \\
&= v(f \circ F) + cv(g \circ F) \\
&= dF_p(v)(f) + cdF_p(v)(g)
\end{aligned}$$

So  $dF_p(v)$  is linear.

(2)

$$\begin{aligned}
dF_p(v)(fg) &= v(fg \circ F) \\
&= v((f \circ F) * (g \circ F)) \\
&= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\
&= dF_p(v)(f) * g(F(p)) + f(F(p)) * dF_p(v)(g)
\end{aligned}$$

So  $dF_p(v)$  is Leibnizian and hence  $dF_p(v) \in T_{F(p)}N$   $\square$

**Exercise 4.4.12.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F : M \rightarrow N$  smooth and  $p \in M$ . If  $F$  is a diffeomorphism, then  $dF_p$  is an isomorphism.

*Proof.* Suppose that  $F$  is a diffeomorphism. Since  $F$  is a homeomorphism,  $\dim N = n$ . Choose  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . A previous exercise tells us that  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ . Write  $\phi = (x_1, \dots, x_n)$  and  $\phi \circ F^{-1} = (y_1, \dots, y_n)$ . Let  $f \in C_{F(p)}^\infty(N)$ . Then

$$\begin{aligned} \left. \frac{\partial}{\partial y_i} \right|_{F(p)} f &= \left. \frac{\partial}{\partial u_i} \right|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \left. \frac{\partial}{\partial u_i} \right|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial x_i} \right|_p f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[ dF_p \left( \left. \frac{\partial}{\partial x_i} \right|_p \right) \right] (f) &= \left. \frac{\partial}{\partial x_i} \right|_p f \circ F \\ &= \left. \frac{\partial}{\partial y_i} \right|_{F(p)} f \end{aligned}$$

Hence

$$dF_p \left( \left. \frac{\partial}{\partial x_i} \right|_p \right) = \left. \frac{\partial}{\partial y_i} \right|_{F(p)}$$

Since  $\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$  is a basis for  $T_p M$  and  $\left\{ \left. \frac{\partial}{\partial y_1} \right|_{F(p)}, \dots, \left. \frac{\partial}{\partial y_n} \right|_{F(p)} \right\}$  is a basis for  $T_{F(p)}N$ ,  $dF_p$  is an isomorphism.  $\square$

**Definition 4.4.13.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F : M \rightarrow N$  a diffeomorphism. Define the **push forward of  $F$** , denoted

$$F_* : M \rightarrow \coprod_{p \in M} \text{Iso}(T_p M, T_{F(p)}N)$$

by

$$p \mapsto dF_p$$

**Definition 4.4.14.** We define the **tangent bundle of  $M$** , denoted  $TM$ , by

$$TM = \coprod_{p \in M} T_p M$$

and we define the **cotangent bundle of  $M$** , denoted  $T^*M$ , by

$$T^*M = \coprod_{p \in M} T_p^* M$$

#### 4.5. Submanifolds.

## 5. TENSOR FIELDS

## 5.1. Vector Fields.

**Definition 5.1.1.** Let  $X : M \rightarrow TM$ . Then  $X$  is said to be a **vector field on  $M$**  if for each  $p \in M$ ,  $X_p \in T_p M$ .

For  $f \in C^\infty(M)$ , we define  $Xf : M \rightarrow \mathbb{R}$  by

$$(Xf)_p = X_p(f)$$

and  $X$  is said to be **smooth** if for each  $f \in C^\infty(M)$ ,  $Xf$  is smooth. We denote the set of smooth vector fields on  $M$  by  $\Gamma(M)$ .

**Definition 5.1.2.** Let  $f \in C^\infty(M)$  and  $X, Y \in \Gamma(M)$ . We define

- $fX \in \Gamma(M)$  by

$$(fX)_p = f(p)X_p$$

- $X + Y \in \Gamma(M)$  by

$$(X + Y)_p = X_p + Y_p$$

**Exercise 5.1.3.** The set  $\Gamma(M)$  is a  $C^\infty(M)$ -module.

*Proof.* Clear. □

**Exercise 5.1.4.** Let  $X \in \Gamma(M)$  and  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$ . Then

$$X|_U = \sum_{i=1}^n (Xx_i) \frac{\partial}{\partial x_i}$$

*Proof.* Let  $p \in M$ . Then  $X_p \in T_p M$  and  $\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\}$  is a basis of  $T_p M$ . So there exist  $f_1(p), \dots, f_n(p) \in \mathbb{R}$  such that  $X_p = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x_i} \Big|_p$ . Let  $j \in \{1, \dots, n\}$ . Then,

$$\begin{aligned} X_p(x_j) &= \sum_{i=1}^n f_i(p) \frac{\partial x_j}{\partial x_i}(p) \\ &= f_j(p) \end{aligned}$$

Hence  $Xx_j = f_j$  and  $X|_U = \sum_{i=1}^n (Xx_i) \frac{\partial}{\partial x_i}$ . □

**Exercise 5.1.5.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$ . Then for each  $i \in \{1, \dots, n\}$ ,

$$\frac{\partial}{\partial x_i} \in \Gamma(U)$$

*Proof.* Let  $i \in \{1, \dots, n\}$  and  $f \in C^\infty(M)$ . Define  $g : M \rightarrow \mathbb{R}$  by  $g = \frac{\partial}{\partial x_i} f$ . Let  $(V, \psi) \in \mathcal{A}$ . Then for each  $x \in \psi(U \cap V)$ ,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x_i} \Big|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u_i} \Big|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u_i} (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) \end{aligned}$$

Since  $f \circ \phi^{-1}$  and  $\phi \circ \psi^{-1}$  are smooth,  $g \circ \psi^{-1}$  is smooth and hence  $g$  is smooth. Since  $f \in C^\infty(M)$  was arbitrary, by definition,  $\frac{\partial}{\partial x_i}$  is smooth.  $\square$

## 5.2. Differential Forms.

**Definition 5.2.1.** We define

$$\Lambda_k(TM) = \coprod_{p \in M} \Lambda_k(T_p M)$$

**Definition 5.2.2.** Let  $\omega : M \rightarrow \Lambda_k(TM)$ . Then  $\omega$  is said to be a  **$k$ -form on  $M$**  if for each  $p \in M$ ,  $\omega_p \in \Lambda_k(T_p M)$ .

For each  $X_1, \dots, X_k \in \Gamma(M)$ , we define  $\omega(X_1, \dots, X_k) : M \rightarrow \mathbb{R}$  by

$$\omega(X_1, \dots, X_k)_p = \omega_p(X_1(p), \dots, X_k(p))$$

and  $\omega$  is said to be **smooth** if for each  $X_1, \dots, X_k \in \Gamma(M)$ ,  $\omega(X_1, \dots, X_k)$  is smooth. The set of smooth  $k$ -forms on  $M$  is denoted  $\Omega_k(M)$ .

**Note 5.2.3.** Observe that  $\Omega_0(M) = C^\infty(M)$ .

**Definition 5.2.4.** Let  $f \in C^\infty(M)$  and  $\alpha, \beta \in \Omega_k(M)$ . We define

- $f\alpha \in \Omega_k(M)$  by

$$(f\alpha)_p = f(p)\alpha_p$$

- $\alpha + \beta \in \Omega_k(M)$  by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

**Exercise 5.2.5.** The set  $\Omega_k(M)$  is a  $C^\infty(M)$ -module.

*Proof.* Clear. □

**Definition 5.2.6.** Define the **exterior product**

$$\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

**Note 5.2.7.** For  $f \in \Omega_0(M)$  and  $\alpha \in \Omega_k(M)$ , we have that  $f \wedge \alpha = f\alpha$ .

**Exercise 5.2.8.** The exterior product  $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$  is well defined.

*Proof.* Let  $\alpha \in \Omega_k(M)$ ,  $\beta \in \Omega_l(M)$ ,  $(X_i)_{i=1}^k \subset \Gamma(M)$ ,  $(Y_j)_{j=1}^l \subset \Gamma(M)$  and  $p \in M$ . Then

$$\begin{aligned} \alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{(k+l)!}{k!l!} A(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) (\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta_p(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \end{aligned}$$

□

**Exercise 5.2.9.** The exterior product  $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$  is  $C^\infty(M)$ -bilinear.

*Proof.*

(1)  $C^\infty(M)$ -linearity in the first argument:

Let  $\alpha \in \Omega_k(M)$ ,  $\beta, \gamma \in \Omega_l(M)$ ,  $f \in C^\infty(M)$  and  $p \in M$ . Bilinearity of  $\wedge : \Lambda_k(T_p M) \times \Lambda_l(T_p M) \rightarrow \Lambda_{k+l}(T_p M)$  implies that

$$\begin{aligned} [(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p \end{aligned}$$

So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

and  $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$  is  $C^\infty(M)$ -linear in the first argument.

(2)  $C^\infty(M)$ -linearity in the second argument:

Similar to (1).

□

**Note 5.2.10.** All of the results from multilinear algebra apply here.

**Definition 5.2.11.** We define the **exterior derivative**  $d : \Omega_k(M) \rightarrow \Omega_{k+1}(M)$  inductively by

- (1)  $d(d\alpha) = 0$  for  $\alpha \in \Omega_p(M)$
- (2)  $df(X) = Xf$  for  $f \in \Omega_0(M)$
- (3)  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$  for  $\alpha \in \Omega_p(M)$  and  $\beta \in \Omega_q(M)$
- (4) extending linearly

**Exercise 5.2.12.** Let  $(U, \phi)$  be a chart on  $M$  with  $\phi = (x_1, \dots, x_n)$ . Then on  $U$ , for each  $i, j \in \{1, \dots, n\}$ ,

$$dx_i \left( \frac{\partial}{\partial x_j} \right) = \delta_{i,j}$$

In particular, for each  $p \in U$ ,  $\{(dx_1)_p, \dots, (dx_n)_p\}$  is the dual basis to  $\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\}$  and  $T_p^* M = \text{span}\{(dx_1)_p, \dots, (dx_n)_p\}$ .

*Proof.* Let  $p \in U$  and  $i, j \in \{1, \dots, n\}$ . Then by definition,

$$\begin{aligned} \left[ dx_i \left( \frac{\partial}{\partial x_j} \right) \right]_p &= \left( \frac{\partial}{\partial x_j} x_i \right)_p \\ &= \frac{\partial}{\partial x_j} \Big|_p x_i \\ &= \delta_{i,j} \end{aligned}$$

□

**Exercise 5.2.13.** Let  $f \in C^\infty(M)$  and  $(U, \phi)$  be a chart on  $M$  with  $\phi = (x_1, \dots, x_n)$ . Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

*Proof.* Let  $p \in U$ . Since  $\{dx_1, \dots, dx_n\}$  is a basis for  $\Lambda(T_p M)$ , for each there exist  $a_1(p), \dots, a_n(p) \in \mathbb{R}$  such that  $(df)_p = \sum_{i=1}^n a_i(p)(dx_i)_p$ . Therefore, we have that

$$\begin{aligned} (df)_p \left( \frac{\partial}{\partial x_j} \Big|_p \right) &= \sum_{i=1}^n a_i(p)(dx_i)_p \left( \frac{\partial}{\partial x_j} \Big|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} (df)_p \left( \frac{\partial}{\partial x_j} \Big|_p \right) &= \frac{\partial}{\partial x_j} \Big|_p f \\ &= \frac{\partial f}{\partial x_j}(p) \end{aligned}$$

So  $a_j(p) = \frac{\partial f}{\partial x_j}(p)$  and

$$(df)_p = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(dx_i)_p$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

□

**Exercise 5.2.14.** Let  $f \in \Omega_0(M)$ . If  $f$  is constant, then  $df = 0$ .

*Proof.* Suppose that  $f$  is constant. Let  $p \in M$ . Choose  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . Write  $\phi = (x_1, \dots, x_n)$ . Then for each  $i \in \{1, \dots, n\}$ ,

$$\frac{\partial}{\partial x_i} \Big|_p f = 0$$

This implies that

$$\begin{aligned} (df)_p &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(dx_i)_p \\ &= 0 \end{aligned}$$

□

**Exercise 5.2.15.**

**Definition 5.2.16.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$  and  $I = (i_1, \dots, i_k) \in \mathcal{I}_k$ . We define

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Omega_k(M)$$



and we define

$$\frac{\partial}{\partial x_I} = \left( \frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}} \right)$$

**Note 5.2.17.** We have that

(1)

$$dx_I \left( \frac{\partial}{\partial x_J} \right) = \delta_{I,J}$$

(2) Since  $\frac{\partial}{\partial x_I} \in \Gamma(U)^k$ , by definition, for each  $\omega \in \Omega_k(U)$ ,

$$\omega \left( \frac{\partial}{\partial x_I} \right) \in C^\infty(U)$$

**Exercise 5.2.18.** Let  $\omega \in \Omega_k(M)$  and  $(U, \phi)$  be a chart on  $M$  with  $\phi = (x_1, \dots, x_n)$ . Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega \left( \frac{\partial}{\partial x_I} \right) dx_I$$

*Proof.* Let  $p \in U$ . Since  $\{(dx_I)_p : I \in \mathcal{I}_k\}$  is a basis for  $\Lambda_k(T_p M)$ , there exists  $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$  such that  $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p)(dx_I)_p$ . So for each  $J \in \mathcal{I}_k$ ,

$$\begin{aligned} \omega \left( \frac{\partial}{\partial x_J} \right) &= \sum_{I \in \mathcal{I}_k} f_I dx_I \left( \frac{\partial}{\partial x_J} \right) \\ &= f_J \end{aligned}$$

□

**Exercise 5.2.19.** Let  $\omega \in \Omega_k(M)$  and  $(U, \phi)$  be a chart on  $M$  with  $\phi = (x_1, \dots, x_n)$ . If  $\omega = \sum_{I \in \mathcal{I}_k} f_I dx_I$ , then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I$$

*Proof.* First we note that

$$\begin{aligned} d(f_I dx_I) &= df_I \wedge dx_I + (-1)^0 f_I d(dx_I) \\ &= df_I \wedge dx_I \\ &= \left( \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \right) \wedge dx_I \\ &= \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I \end{aligned}$$

Then we extend linearly.

□

**Definition 5.2.20.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F : M \rightarrow N$  be a diffeomorphism. Define the **pullback of  $F$** , denoted  $F^* : \Omega_k(N) \rightarrow \Omega_k(M)$  by

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))$$

for  $\omega \in \Omega_k(N)$ ,  $p \in M$  and  $v_1, \dots, v_k \in T_pM$

### 5.3. $(r, s)$ -Tensor Fields.

**Definition 5.3.1.** We define

$$T_k(TM) = \coprod_{p \in M} \Lambda_k(T_p M)$$

**Definition 5.3.2.** Define the **tensor product**

$$\otimes : T_{l_1}^{k_1}(M) \times T_{l_2}^{k_2}(M) \rightarrow \Omega_{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

## 6. EXTRA

**Definition 6.0.1.** When working in  $\mathbb{R}^n$ , we introduce the formal objects  $dx_1, dx_2, \dots, dx_n$ . Let  $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_{k,n}$  and  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ . Write  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ . We formally define  $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$  and  $\phi_I = (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$ .

**Definition 6.0.2.** Let  $k \in \{0, 1, \dots, n\}$ . We define a  $C^\infty(\mathbb{R}^n)$ -module of dimension  $\binom{n}{k}$ , denoted  $\Phi^k(\mathbb{R}^n)$  to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx_I : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each  $\omega \in \Phi_k(\mathbb{R}^n)$  and  $\chi \in \Gamma^l(\mathbb{R}^n)$ , we may form their **exterior product**, denoted by  $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$ . Thus the exterior product is a map  $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \rightarrow \Gamma^{k+l}(\mathbb{R}^n)$ . The exterior product is characterized by the following properties:

- (1) the exterior product is bilinear
- (2) for each  $\omega \in \Phi_k(\mathbb{R}^n)$  and  $\chi \in \Gamma^l(\mathbb{R}^n)$ ,  $\omega \wedge \chi = -\chi \wedge \omega$
- (3) for each  $\omega \in \Phi_k(\mathbb{R}^n)$ ,  $\omega \wedge \omega = 0$
- (4) for each  $f \in C^\infty(\mathbb{R}^n)$  and  $\omega \in \Phi_k(\mathbb{R}^n)$ ,  $f \wedge \omega = f\omega$

We call  $\Phi_k(\mathbb{R}^n)$  the differential  $k$ -forms on  $\mathbb{R}^n$ . Let  $\omega$  be a  $k$ -form on  $\mathbb{R}^n$ . If  $k \geq 1$ , then for each  $I \in \mathcal{I}_{k,n}$ , there exists  $f_I \in C^\infty(\mathbb{R}^n)$  such that  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$

**Note 6.0.3.** The terms  $dx_1, dx_2, \dots, dx_n$  are a sort of place holder for the coordinates of a point  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ . When we work with functions  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ , we will have different coordinates and to avoid confusion, we will write  $\{du_1, du_2, \dots, du_k\}$  when referencing the coordinates on  $\mathbb{R}^k$  and  $\{dx_1, dx_2, \dots, dx_n\}$  when referencing the coordinates on  $\mathbb{R}^n$ .

**Exercise 6.0.4.** Let  $B_{n \times n} = (b_{i,j}) \in [C^\infty(M)]^{n \times n}$  be an  $n \times n$  matrix. Then

$$\bigwedge_{i=1}^n \left( \sum_{j=1}^n b_{i,j} dx_j \right) = (\det B) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

*Proof.* Bilinearity of the exterior product implies that

$$\begin{aligned} \bigwedge_{i=1}^n \left( \sum_{j=1}^n b_{i,j} dx_j \right) &= \left( \sum_{j=1}^n b_{1,j} dx_j \right) \wedge \left( \sum_{j=1}^n b_{2,j} dx_j \right) \wedge \dots \wedge \left( \sum_{j=1}^n b_{n,j} dx_j \right) \\ &= \sum_{j_1, \dots, j_n=1}^n \left( \prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_n} \\ &= \sum_{j_1 \neq \dots \neq j_n} \left( \prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_n} \\ &= \left[ \sum_{\sigma \in S_n} \text{sgn}(\sigma) \left( \prod_{i=1}^n b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \\ &= (\det B) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \end{aligned}$$

□

**Definition 6.0.5.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a 0-form on  $\mathbb{R}^n$ . We define a 1-form, denoted  $df$ , on  $\mathbb{R}^n$  by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

Let  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$  be a  $k$ -form on  $\mathbb{R}^n$ . We can define a differential  $k+1$ -form, denoted  $d\omega$ , on  $\mathbb{R}^n$  by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx_I$$

**Exercise 6.0.6.** On  $\mathbb{R}^3$ , put

- (1)  $\omega_0 = f_0$ ,
- (2)  $\omega_1 = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ ,
- (3)  $\omega_2 = f_1 dx_2 \wedge dx_3 - f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2$

Show that

- (1)  $d\omega_0 = \frac{\partial f_0}{\partial x_1} dx_1 + \frac{\partial f_0}{\partial x_2} dx_2 + \frac{\partial f_0}{\partial x_3} dx_3$
- (2)  $d\omega_1 = \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left( \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \right) dx_1 \wedge dx_3 + \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 \wedge dx_2$
- (3)  $d\omega_2 = \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3$

*Proof.* Straightforward. □

**Exercise 6.0.7.** Let  $I \in \mathcal{I}_{k,n}$ . Then there is a unique  $I_* \in \mathcal{I}_{n-k,n}$  such that  $dx_I \wedge dx_{I_*} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ .

**Definition 6.0.8.** We define a linear map  $*$  :  $\Phi_k(\mathbb{R}^n) \rightarrow \Gamma^{n-k}(\mathbb{R}^n)$  called the **Hodge \***-operator by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

**Definition 6.0.9.** Let  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$  be smooth. Write  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ . We define  $\phi^* : \Phi_k(\mathbb{R}^n) \rightarrow \Phi_k(\mathbb{R}^k)$  via the following properties:

- (1) for each 0-form  $f$  on  $\mathbb{R}^n$ ,  $\phi^* f = f \circ \phi$
- (2) for  $i = 1, \dots, n$ ,  $\phi^* dx_i = d\phi_i$
- (3) for an  $s$ -form  $\omega$ , and a  $t$ -form  $\chi$  on  $\mathbb{R}^n$ ,  $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
- (4) for  $l$ -forms  $\omega, \chi$  on  $\mathbb{R}^n$ ,  $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

**Exercise 6.0.10.** Let  $M \subset \mathbb{R}^n$  be a  $k$ -dimensional smooth submanifold of  $\mathbb{R}^n$ ,  $\phi : U \rightarrow V$  a smooth parametrization of  $M$ ,  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$  an  $k$ -form on  $\mathbb{R}^n$ . Then

$$\phi^*\omega = \left( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du_1 \wedge du_2 \wedge \cdots \wedge du_k$$

*Proof.* Using the definitions, we see that

$$\begin{aligned}\phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I\end{aligned}$$

A previous exercise tells us that for each  $I \in \mathcal{I}_{k,n}$ ,

$$\begin{aligned}d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\ &= \left( \sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u_j} du_j \right) \wedge \left( \sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u_j} du_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^n \frac{\partial \phi_{i_n}}{\partial u_j} du_j \right) \\ &= (\det v\phi_I) du_1 \wedge du_2 \wedge \cdots \wedge du_k\end{aligned}$$

Therefore

$$\begin{aligned}\phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det v\phi_I) du_1 \wedge du_2 \wedge \cdots \wedge du_k \\ &= \left( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det v\phi_I) \right) du_1 \wedge du_2 \wedge \cdots \wedge du_k\end{aligned}$$

□

### 6.1. Integration of Differential Forms.

**Definition 6.1.1.** Let  $U \subset \mathbb{R}^k$  be open and  $\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$  a  $k$ -form on  $\mathbb{R}^k$ . Define

$$\int_U \omega = \int_U f dx$$

**Definition 6.1.2.** Let  $M \subset \mathbb{R}^n$  be a  $k$ -dimensional oriented smooth submanifold of  $\mathbb{R}^n$ ,  $\omega$  a  $k$ -form on  $\mathbb{R}^n$  and  $\phi : U \rightarrow V$  a local smooth, orientation-preserving parametrization of  $M$ . Define

$$\int_V \omega = \int_U \phi^* \omega$$

### Exercise 6.1.3.

**Theorem 6.1.1. (Stokes Theorem)** Let  $M \subset \mathbb{R}^n$  be a  $k$ -dimensional oriented smooth submanifold of  $\mathbb{R}^n$  and  $\omega$  a  $k-1$ -form on  $\mathbb{R}^n$ . Then

$$\int_{\partial M} \omega = \int_M d\omega$$