INTRODUCTION TO CATEGORY THEORY

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Contents

Preface

1. Categories, Functors and Natural Transformations

1.1. von Neumann–Bernays–Gödel Set Theory.

Definition 1.1.1. Let x be a class. Then x is said to be a set iff there exists a class A such that $x \in A$.

Note 1.1.2. We can define cartesion products, relations, and functions for classes just like for sets.

Axiom 1.1.3. Axiom of Replacement:
Let A, B be classes and $f : A \to B$. If A is a set, then $f(A)$ is a set.

Axiom 1.1.4. Schema of Specification:
Let ϕ a propositional function on sets. Then there exists a class A such that for each set x, $x \in A$ iff $\phi(x)$.

Exercise 1.1.5. There exists a class A such that for each class x, $x \in A$ iff x is a set.

Proof. Define ϕ by

$$\phi(x) : x = x$$

Axiom 1.1.4 implies that there exists a class A such that for each set x, $x \in A$ iff $x = x$. Let x be a class. If $x \in A$, then by definition, x is a set.
Conversely, if x is a set, then by construction, $x \in A$. □

Exercise 1.1.6. There exists a class A such that for each class G and $* : G \times G \to G$, $(G, *) \in A$ iff $(G, *)$ is a group.

Proof. Define ϕ_1, ϕ_2 and ϕ_3 by

- $\phi_1(G, *) : * : G \times G \to G$ is associative
- $\phi_2(G, *)$: there exists $e \in G$ such that for each $g \in G$, $e * g = g * e = g$
- $\phi_3(G, *)$: for each $g \in G$ there exists $h \in G$ such that $g * h = h * g = e$

Define ϕ by

$$\phi(G, *) : \phi_1(G, *) \text{ and } \phi_2(G, *) \text{ and } \phi_3(G, *)$$

Then there exists a class A such that for each set G and $* : G \times G \to G$, $(G, *) \in A$ iff $\phi(G, *)$ $(G, *)$ "is a group". Therefore, for each group $(G, *)$, $(G, *) \in A$. **FINISH!!!** □

1.2. Categories.

Definition 1.2.1. Let $\mathcal{C}_0$, $\mathcal{C}_1$ be classes and $\text{dom}, \text{cod} : \mathcal{C}_1 \to \mathcal{C}_0$ class functions. Set $\mathcal{C} = (C_0, C_1, \text{dom}, \text{cod})$. Then $\mathcal{C}$ is said to be a **category** if

(1) (composition): for each $f, g \in C_1$, if $\text{cod}(f) = \text{dom}(g)$, then there exists $g \circ f \in C_1$ such that $\text{dom}(g \circ f) = \text{dom}(f)$ and $\text{cod}(g \circ f) = \text{cod}(g)$

(2) (associativity): for each $f, g, h \in C_1$, if $\text{cod}(f) = \text{dom}(g)$ and $\text{cod}(g) = \text{dom}(h)$, then
$$(h \circ g) \circ f = h \circ (g \circ f)$$

(3) (identity): for each $X \in \mathcal{C}_0$, there exists $1_X \in C_1$ such that $\text{dom}(1_X) = \text{cod}(1_X) = X$ and for each $f, g \in C_1$, if $\text{dom}(f) = X$ and $\text{cod}(g) = X$, then
$$f \circ 1_X = f \text{ and } 1_X \circ g = g$$

We define the

- **objects of** $\mathcal{C}$, denoted $\text{Obj}(\mathcal{C})$, by $\text{Obj}(\mathcal{C}) = C_0$
- **morphisms of** $\mathcal{C}$, denoted $\text{Hom}_{\mathcal{C}}$, by $\text{Hom}_{\mathcal{C}} = C_1$

For $X, Y \in \text{Obj}(\mathcal{C})$, we define the **morphisms from** X **to** Y, denoted $\text{Hom}_{\mathcal{C}}(X, Y)$, by $\text{Hom}_{\mathcal{C}}(X, Y) = \{f \in \text{Hom}(\mathcal{C}) : \text{dom}(f) = X \text{ and } \text{cod}(f) = Y\}$.

Note 1.2.2. We typically define a category $\mathcal{C}$ by specifying

- $\text{Obj}(\mathcal{C})$
- for $X, Y \in \text{Obj}(\mathcal{C})$, the class $\text{Hom}_{\mathcal{C}}(X, Y)$
- for $X, Y, Z \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$, the composite morphism $g \circ f \in \text{Hom}_{\mathcal{C}}(X, Z)$.

and then show

- well-definedness of composition
- associativity of composition
- existence of identities

Definition 1.2.3. Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is said to be

- **small** if $\text{Obj}(\mathcal{C})$ and $\text{Hom}_{\mathcal{C}}$ are sets
- **locally small** if for each $A, B \in \text{Obj}(\mathcal{C})$, $\text{Hom}_{\mathcal{C}}(A, B)$ is a set

Exercise 1.2.4. Let $\mathcal{C}$ be a category. If $\mathcal{C}$ is small, then $\mathcal{C}$ is a set.

Proof. Suppose that $\mathcal{C}$ is small. Then $\text{Obj}(\mathcal{C})$ and $\text{Hom}_{\mathcal{C}}$ are sets. Then $\mathcal{P}(\text{Obj}(\mathcal{C}))$, $\mathcal{P}(\text{Hom}_{\mathcal{C}})$ and $\text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}}$ are sets. Hence $\text{Obj}(\mathcal{C}) \times \text{Hom}_{\mathcal{C}} \times \text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}} \times \text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}}$ is a set. By definition, $\mathcal{C} = (\text{Obj}(\mathcal{C}), \text{Hom}_{\mathcal{C}}, \text{dom}, \text{cod}) \in \text{Obj}(\mathcal{C}) \times \text{Hom}_{\mathcal{C}} \times \text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}} \times \text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}}$. By definition, $\mathcal{C}$ is a set. $\square$

Exercise 1.2.5. There exists a class A such that $\mathcal{C} \in A$ iff $\mathcal{C}$ is a small category.

Proof. Exercise 1.2.4 implies that for each category $\mathcal{C}$, $\mathcal{C}$ is small implies that $\mathcal{C}$ is a set. Define ϕ by
$$\phi(\mathcal{C}) : \mathcal{C} \text{ is a small category}$$
Then Axiom 1.1.4 implies that there exists a class A such that $\mathcal{C} \in A$ iff $\mathcal{C}$ is a small category. $\square$

Definition 1.2.6. Let $\mathcal{C}$ be a category, we define the dual of $\mathcal{C}$ or the **opposite of** $\mathcal{C}$, denoted $\mathcal{C}^{\text{op}}$, by

- $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$
- for $X, Y \in \mathrm{Obj}(\mathcal{C}^{\mathrm{op}})$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$
- for $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y), g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z)$, $g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} g$

Exercise 1.2.7. Let $\mathcal{C}$ be a category. Then $\mathcal{C}^{\mathrm{op}}$ is a category.

Proof.

- for $W, X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ and $h \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z)$. Then

$$\begin{aligned}(h \circ_{\mathcal{C}^{\mathrm{op}}} g) \circ_{\mathcal{C}^{\mathrm{op}}} f &= f \circ_{\mathcal{C}} (h \circ_{\mathcal{C}^{\mathrm{op}}} g) \\ &= f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h) \\ &= (f \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} h \\ &= h \circ_{\mathcal{C}^{\mathrm{op}}} (f \circ_{\mathcal{C}} g) \\ &= h \circ_{\mathcal{C}^{\mathrm{op}}} (g \circ_{\mathcal{C}^{\mathrm{op}}} f)\end{aligned}$$

  So composition is associative.
- Let $X \in \mathrm{Obj}(\mathcal{C})$ and $f, g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}$. Suppose that $\mathrm{dom}(f) = X$ and $\mathrm{cod}(g) = X$ Then

$$\begin{aligned}f \circ_{\mathcal{C}^{\mathrm{op}}} 1_X &= 1_X \circ_{\mathcal{C}} f \\ &= f\end{aligned}$$

  and

$$\begin{aligned}1_X \circ_{\mathcal{C}^{\mathrm{op}}} g &= g \circ_{\mathcal{C}} 1_X \\ &= g\end{aligned}$$

  So $(1_X)_{\mathcal{C}^{\mathrm{op}}} = (1_X)_{\mathcal{C}}$.

$\square$

Definition 1.2.8. Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. We define the **slice category of $\mathcal{C}$ over X**, denoted $\mathcal{C}/X$, by

- $\mathrm{Obj}(\mathcal{C}/X) = \{f \in \mathrm{Hom}_{\mathcal{C}} : \mathrm{cod}(f) = X\}$
- for $f, g \in \mathrm{Obj}(\mathcal{C}/X)$,

$$\mathrm{Hom}_{\mathcal{C}/X}(f, g) = \{\alpha \in \mathrm{Hom}_{\mathcal{C}} : \mathrm{dom}(\alpha) = \mathrm{dom}(f),\ \mathrm{cod}(\alpha) = \mathrm{dom}(g) \text{ and } f = g \circ \alpha\}$$

  i.e. for $f \in \mathrm{Hom}_{\mathcal{C}}(A, X)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f, g)$ iff the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

- for $f, g, h \in \mathrm{Obj}(\mathcal{C}/X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}/X}(g, h)$,

$$\beta \circ_{\mathcal{C}/X} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

Exercise 1.2.9. Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. Then $\mathcal{C}/X$ is a category.

Proof.

- $f, g, h \in \mathrm{Obj}(\mathcal{C}/X)$, $\alpha \in \mathrm{Hom}_{\mathcal{C}/X}(f,g)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}/X}(g,h)$. Then $f = g \circ_{\mathcal{C}} \alpha$ and $g = h \circ_{\mathcal{C}} \beta$, i.e. the following diagrams commute:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & \mathrm{dom}(g) \arrow[dl, "g"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{dom}(g) \arrow[rr, "\beta"] \arrow[dr, "g"'] & & \mathrm{dom}(h) \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

  Therefore, we have that

$$\begin{aligned} f &= g \circ_{\mathcal{C}} \alpha \\ &= (h \circ_{\mathcal{C}} \beta) \circ_{\mathcal{C}} \alpha \\ &= h \circ_{\mathcal{C}} (\beta \circ_{\mathcal{C}} \alpha) \end{aligned}$$

  i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "\beta \circ_{\mathcal{C}} \alpha"] \arrow[dr, "f"'] & & \mathrm{dom}(h) \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

  which implies that

$$\begin{aligned} \beta \circ_{\mathcal{C}/X} \alpha &= \beta \circ_{\mathcal{C}} \alpha \\ &\in \mathrm{Hom}_{\mathcal{C}/X}(f,h) \end{aligned}$$

  and composition is well defined.
- Associativity of $\circ_{\mathcal{C}/X}$ follows from associativity of $\circ_{\mathcal{C}}$.
- Let $f \in \mathrm{Obj}(\mathcal{C}/X)$ and $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}/X}$. Since $f \circ 1_{\mathrm{dom}_{\mathcal{C}}(f)} = f$, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathrm{dom}(f) \arrow[rr, "1_{\mathrm{dom}(f)}"] \arrow[dr, "f"'] & & \mathrm{dom}(f) \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

  we have that $1_{\mathrm{dom}(f)} \in \mathrm{Hom}_{\mathcal{C}/X}(f,f)$. Suppose that $\mathrm{dom}_{\mathcal{C}/X}(\alpha) = f$ and $\mathrm{cod}_{\mathcal{C}/X}(\beta) = f$. Then

$$\begin{aligned} \alpha \circ_{\mathcal{C}/X} 1_{\mathrm{dom}(f)} &= \alpha \circ_{C} 1_{\mathrm{dom}(f)} \\ &= \alpha \end{aligned}$$

  and

$$\begin{aligned} 1_{\mathrm{dom}(f)} \circ_{\mathcal{C}/X} \beta &= 1_{\mathrm{dom}(f)} \circ_{\mathcal{C}} \beta \\ &= \beta \end{aligned}$$

  So $(1_f)_{\mathcal{C}/X} = (1_{\mathrm{dom}(f)})_{\mathcal{C}}$.

$\square$

1.3. **Functors.**

Definition 1.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, $F_1 : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}$ class functions. Set $F = (F_0, F_1)$. Then F is said to be a functor from $\mathcal{C}$ to $\mathcal{D}$, denoted $F : \mathcal{C} \to \mathcal{D}$, if

(1) for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F_1(f) \in \mathrm{Hom}_{\mathcal{D}}(F_0(A), F_0(B))$
(2) for each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$
(3) for each $A \in \mathrm{Obj}(\mathcal{C})$, $F_1(\mathrm{id}_A) = \mathrm{id}_{F_0(A)}$

Note 1.3.2. For $A \in \mathrm{Obj}(C)$ and $f \in \mathrm{Hom}_{\mathcal{C}}$, we typically write $F(A)$ and $F(f)$ instead of $F_0(A)$ and $F_1(f)$ respectively.

Definition 1.3.3. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ functors. We define the **composition of G with F**, denoted $G \circ F : \mathcal{C} \to \mathcal{E}$, by

- $G \circ F(A) = G(F(A))$
- $G \circ F(f) = G(F(f))$

Exercise 1.3.4. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ functors. Then $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor.

Proof.

(1) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Since $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$, we have that $G(F(f)) \in \mathrm{Hom}_{\mathcal{E}}(G(F(A)), G(F(B)))$. Then

$$\begin{aligned} G \circ F(f) &= G(F(f)) \\ &\in \mathrm{Hom}_{\mathcal{E}}(G(F(A)), G(F(B))) \\ &= \mathrm{Hom}_{\mathcal{E}}(G \circ F(A), G \circ F(B)) \end{aligned}$$

(2) Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

$$\begin{aligned} G \circ F(g \circ f) &= G(F(g \circ f)) \\ &= G(F(g) \circ F(f)) \\ &= G(F(g)) \circ G(F(f)) \\ &= G \circ F(g) \circ G \circ F(f) \end{aligned}$$

(3) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned} G \circ F(\mathrm{id}_A) &= G(F(\mathrm{id}_A)) \\ &= G(\mathrm{id}_{F(A)}) \\ &= \mathrm{id}_{G(F(A))} \\ &= \mathrm{id}_{G \circ F(A)} \end{aligned}$$

So $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor. □

Exercise 1.3.5. Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, $H : \mathcal{E} \to \mathcal{F}$ functors. Then $(H \circ G) \circ F = H \circ (G \circ F)$.

Proof. Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

- $$\begin{aligned}(H \circ G) \circ F(A) &= H \circ G(F(A)) \\ &= H(G(F(A))) \\ &= H(G \circ F(A)) \\ &= H \circ (G \circ F)(A)\end{aligned}$$

- $$\begin{aligned}(H \circ G) \circ F(f) &= H \circ G(F(f)) \\ &= H(G(F(f))) \\ &= H(G \circ F(f)) \\ &= H \circ (G \circ F)(f)\end{aligned}$$

Hence $(H \circ G) \circ F = H \circ (G \circ F)$. □

Definition 1.3.6. Let $\mathcal{C}$ be a category. We define the **identity functor from $\mathcal{C}$ to $\mathcal{C}$**, denoted $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, by

- $\mathrm{id}_{\mathcal{C}}(A) = A$, $(A \in \mathrm{Obj}(\mathcal{C}))$
- $\mathrm{id}_{\mathcal{C}}(f) = f$, $(f \in \mathrm{Hom}_{\mathcal{C}})$

Exercise 1.3.7. Let $\mathcal{C}$ be a category. Then $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a functor.

Proof.

(1) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(f) &= f \\ &\in \mathrm{Hom}_{\mathcal{C}}(A, B) \\ &= \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))\end{aligned}$$

(2) Let $A, B, C \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$. Then

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g \circ f) &= g \circ f \\ &= \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f)\end{aligned}$$

(3) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) &= \mathrm{id}_A \\ &= \mathrm{id}_{\mathrm{id}_{\mathcal{C}}(A)}\end{aligned}$$

□

Exercise 1.3.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. Then

(1) $\mathrm{id}_{\mathcal{D}} \circ F = F$
(2) $F \circ \mathrm{id}_{\mathcal{C}} = F$

Proof.

(1) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then
$$\begin{aligned} \mathrm{id}_{\mathcal{D}} \circ F(A) &= \mathrm{id}_{\mathcal{D}}(F(A)) \\ &= F(A) \end{aligned}$$
and
$$\begin{aligned} \mathrm{id}_{\mathcal{D}} \circ F(f) &= \mathrm{id}_{\mathcal{D}}(F(f)) \\ &= F(f) \end{aligned}$$
Since $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ are arbitrary, $\mathrm{id}_{\mathcal{D}} \circ F = F$.

(2) Let $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then
$$\begin{aligned} F \circ \mathrm{id}_{\mathcal{C}}(A) &= F(\mathrm{id}_{\mathcal{C}}(A)) \\ &= F(A) \end{aligned}$$
and
$$\begin{aligned} F \circ \mathrm{id}_{\mathcal{C}}(f) &= F(\mathrm{id}_{\mathcal{C}}(f)) \\ &= F(f) \end{aligned}$$
Since $A, B \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ are arbitrary, $F \circ \mathrm{id}_{\mathcal{C}} = F$.

□

Exercise 1.3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. If $\mathcal{C}$ is small, then F is a set.

Proof. Suppose that $\mathcal{C}$ is small. Then $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}}$ are sets. By definition, there exist $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $F_1 : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}$ such that $F = (F_0, F_1)$. Axiom 1.1.3 implies that $F_0(\mathrm{Obj}(\mathcal{C}))$ and $F_1(\mathrm{Hom}_{\mathcal{C}})$ are sets. Therefore, $\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C))$ and $\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}})$ are sets. Hence $\mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C)))$ and $\mathcal{P}(\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}}))$ are sets. Since $F_0 \subset \mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C))$ and $F_1 \subset \mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}})$, we have that $F_0 \in \mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times F_0(\mathrm{Obj}(C)))$ and $F_1 \in \mathcal{P}(\mathrm{Hom}_{\mathcal{C}} \times F_1(\mathrm{Hom}_{\mathcal{C}}))$. Hence F_0 and F_1 are sets. Thus $F = (F_0, F_1)$ is a set. □

Exercise 1.3.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Then there exists a class A such that for each class F, $F \in A$ iff $F : \mathcal{C} \to \mathcal{D}$.

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Define ϕ by
$$\phi(F) : F : \mathcal{C} \to \mathcal{D}$$
Then there exists a class A such that for each set F, $F \in A$ iff $\phi(F)$. Let F be a class. Suppose that $F \in A$. By Definition 1.1.1, F is a set. Since F is a set and $F \in A$, we have that $\phi(F)$. Hence $F : \mathcal{C} \to \mathcal{D}$.
Conversely, suppose that $F : \mathcal{C} \to \mathcal{D}$. Exercise 1.3.9 implies that F is a set. Since F is a set and $\phi(F)$ is true, we have that $F \in A$. □

Definition 1.3.11. We define $\mathbf{Cat}$ by

- $\mathrm{Obj}(\mathbf{Cat}) = \{\mathcal{C} : \mathcal{C} \text{ is a small category}\}$.
- for $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathbf{Cat})$,
$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) = \{F : F : \mathcal{C} \to \mathcal{D}\}$$
- for $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathbf{Cat})$, $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ and $G \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{D}, \mathcal{E})$,
$$G \circ_{\mathbf{Cat}} F = G \circ F$$

Exercise 1.3.12. We have that **Cat** is

(1) a category
(2) locally small

Proof.

(1) The previous exercises imply the associativity of composition and the existence of identities.
(2) Let $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathbf{Cat})$ and $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$. Definition 1.2.3 implies that $\mathrm{Obj}(\mathcal{C})$, $\mathrm{Obj}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{C}}$ and $\mathrm{Hom}_{\mathcal{D}}$ are sets. Then $\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}$ and $\mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ are sets. Hence $\mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})} \times \mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ is a set. Let $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$. Then there exist $F_0 \in \mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})}$ and $F_1 \in \mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$ such that $F = (F_0, F_1)$. Therefore $F \in \mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})} \times \mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$. Since $F \in \mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ is arbitrary,
$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Obj}(\mathcal{D})^{\mathrm{Obj}(\mathcal{C})} \times \mathrm{Hom}_{\mathcal{D}}^{\mathrm{Hom}_{\mathcal{C}}}$$
which implies that $\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ is a set. Therefore, **Cat** is locally small.

$\square$

1.4. **Natural Transformations.**

Definition 1.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : \text{Obj}(\mathcal{C}) \to \text{Hom}_{\mathcal{D}}$. Then α is said to be a **natural transformation from** F **to** G, denoted $\alpha : F \Rightarrow G$, if

(1) for each $A \in \text{Obj}(\mathcal{C})$, $\alpha_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$
(2) for each $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$, i.e. the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$

Definition 1.4.2. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ natural transformations. We define the **composition of** β **with** α, denoted $\beta \circ \alpha : F \Rightarrow H$, by

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$$

Exercise 1.4.3. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ natural transformations. Then $\beta \circ \alpha : F \Rightarrow H$ is a natural transformation.

Proof.

(1) Let $A \in \text{Obj}(\mathcal{C})$. Since $\alpha_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$ and $\beta_A \in \text{Hom}_{\mathcal{D}}(G(A), H(A))$, we have that

$$\begin{aligned} (\beta \circ \alpha)_A &= \beta_A \circ \alpha_A \\ &\in \text{Hom}_{\mathcal{D}}(F(A), H(A)) \end{aligned}$$

(2) Let $A, B \in \text{Obj}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$. Since $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ and $H(f) \circ \beta_A = \beta_B \circ G(f)$. Therefore

$$\begin{aligned} H(f) \circ (\beta \circ \alpha)_A &= H(f) \circ (\beta_A \circ \alpha_A) \\ &= (H(f) \circ \beta_A) \circ \alpha_A \\ &= (\beta_B \circ G(f)) \circ \alpha_A \\ &= \beta_B \circ (G(f) \circ \alpha_A) \\ &= \beta_B \circ (\alpha_B \circ F(f)) \\ &= (\beta_B \circ \alpha_B) \circ F(f) \\ &= (\beta \circ \alpha)_B \circ F(f) \end{aligned}$$

So $\beta \circ \alpha : F \Rightarrow H$ is a natural transformation. $\square$

Exercise 1.4.4. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G, H, I : \mathcal{C} \to \mathcal{D}$ functors and $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ and $\gamma : H \Rightarrow I$ natural transformations. Then

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

Proof. Let $A \in \mathrm{Obj}(\mathcal{C})$. By definition,

$$\begin{aligned}[(\gamma \circ \beta) \circ \alpha]_A &= (\gamma \circ \beta)_A \circ \alpha_A \\ &= (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &= \gamma_A \circ (\beta \circ \alpha)_A \\ &= [\gamma \circ (\beta \circ \alpha)]_A\end{aligned}$$

Since $A \in \mathrm{Obj}(\mathcal{C})$ is arbitrary,

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

□

Definition 1.4.5. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. We define the **identity natural transformation from F to F**, denoted $\mathrm{id}_F : F \Rightarrow F$, by

$$(\mathrm{id}_F)_A = \mathrm{id}_{F(A)}$$

Exercise 1.4.6. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$. Then $\mathrm{id}_F : F \Rightarrow F$ is a natural transformation from F to F.

Proof.

(1) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\mathrm{id}_F)_A &= \mathrm{id}_{F(A)} \\ &\in \mathrm{Hom}_{\mathcal{D}}(F(A), F(A)) \\ &= \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))\end{aligned}$$

(2) Let $A, B \in \mathrm{Obj}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then

$$\begin{aligned}F(f) \circ (\mathrm{id}_F)_A &= F(f) \circ \mathrm{id}_{F(A)} \\ &= F(f) \\ &= \mathrm{id}_{F(B)} \circ F(f) \\ &= (\mathrm{id}_F)_B \circ F(f)\end{aligned}$$

□

Exercise 1.4.7. Let $\mathcal{C}$, $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. Then

(1) $\mathrm{id}_G \circ \alpha = \alpha$
(2) $\alpha \circ \mathrm{id}_F = \alpha$

Proof.

(1) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\mathrm{id}_G \circ \alpha)_A &= (\mathrm{id}_G)_A \circ \alpha_A \\ &= \mathrm{id}_{G(A)} \circ \alpha_A \\ &= \alpha_A\end{aligned}$$

Since $A \in \mathrm{Obj}(C)$ is arbitrary, $\mathrm{id}_G \circ \alpha = \alpha$

(2) Let $A \in \mathrm{Obj}(\mathcal{C})$. Then

$$\begin{aligned}(\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ (\mathrm{id}_F)_A \\ &= \alpha_A \circ \mathrm{id}_{F(A)} \\ &= \alpha_A\end{aligned}$$

Since $A \in \mathrm{Obj}(C)$ is arbitrary, $\alpha \circ \mathrm{id}_F = \alpha$.

□

Exercise 1.4.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $F, G : \mathcal{C} \to \mathcal{D}$ and $\alpha : F \Rightarrow G$. If $\mathcal{C}$ is small, then α is a set.

Proof. Suppose that $\mathcal{C}$ is small. Then $\mathrm{Obj}(\mathcal{C})$ is a set. Since $\alpha : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Hom}_{\mathcal{D}}$, Axiom 1.1.3 implies that $\alpha(\mathrm{Obj}(\mathcal{C}))$ is a set. Then $\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C}))$ is a set. Therefore $\mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C})))$ is a set. Since $\alpha \subset \mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C}))$, we have that $\alpha \in \mathcal{P}(\mathrm{Obj}(\mathcal{C}) \times \alpha(\mathrm{Obj}(\mathcal{C})))$ which implies that α is a set. □

Exercise 1.4.9. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$. If $\mathcal{C}$ is small, then there exists a class A such that for each class α, $\alpha \in A$ iff $\alpha : F \Rightarrow G$.

Proof. Suppose that $\mathcal{C}$ is small. Define ϕ by

$$\phi(\alpha) : \alpha : F \Rightarrow G$$

Axiom 1.1.4 implies that there exists a class A such that for each set α, $\alpha \in A$ iff $\phi(\alpha)$. Let α be a class. Suppose that $\alpha \in A$. By Definition 1.1.1, α is a set. Since α is a set and $\alpha \in A$, we have that $\phi(\alpha)$. Hence $\alpha : F \Rightarrow G$.
Conversely, suppose that $\alpha : F \Rightarrow G$. Since $\mathcal{C}$ is small, Exercise 1.4.8 implies that α is a set. Since $\phi(\alpha)$, we have that $\alpha \in A$. □

Definition 1.4.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. We define the **functor category from $\mathcal{C}$ to $\mathcal{D}$**, denoted $\mathcal{D}^{\mathcal{C}}$, by

- $\mathrm{Obj}(\mathcal{D}^{\mathcal{C}}) = \{F : F : \mathcal{C} \to \mathcal{D}\}$
- For $F, G \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G) = \{\alpha : \alpha : F \Rightarrow G\}$
- For $F, G, H \in \mathrm{Obj}(\mathcal{D}^{\mathcal{C}})$, $\alpha \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$ and $\beta \in \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(G, H)$, $\beta \circ_{\mathcal{D}^{\mathcal{C}}} \alpha = \beta \circ \alpha$

Exercise 1.4.11. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose that $\mathcal{C}$ is small. Then $\mathcal{D}^{\mathcal{C}}$ is a category.

Proof. The previous exercises imply the associativity of composition and existence of identities. □

1.5. Product Categories.

Definition 1.5.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We define the **product category of $\mathcal{C}$ and $\mathcal{D}$**, denoted $\mathcal{C} \times \mathcal{D}$ by

- $\text{Obj}(\mathcal{C} \times \mathcal{D}) = \{(A, B) : A \in \text{Obj}(\mathcal{C}) \text{ and } B \in \text{Obj}(\mathcal{D})\}$
- for each $(A, A'), (B, B') \in \text{Obj}(\mathcal{C} \times \mathcal{D})$, $\text{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B')) = \{(f, g) : f \in \text{Hom}_{\mathcal{C}}(A, B) \text{ and } g \in \text{Hom}_{\mathcal{C}}(A', B')\}$
- for each $(A, A'), (B, B'), (C, C') \in \text{Obj}(\mathcal{C} \times \mathcal{D})$, $(f, f') \in \text{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$ and $(g, g') \in \text{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'))$,
$$(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') = (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f')$$

Exercise 1.5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $\mathcal{C} \times \mathcal{D}$ is a category.

Proof.

- Let $(A, A'), (B, B'), (C, C') \in \text{Obj}(\mathcal{C} \times \mathcal{D})$, $(f, f') \in \text{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$ and $(g, g') \in \text{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'))$. Then $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $g \in \text{Hom}_{\mathcal{C}}(B, C)$, $f' \in \text{Hom}_{\mathcal{D}}(A', B')$, and $g' \in \text{Hom}_{\mathcal{D}}(B', C')$. Hence $g \circ_{\mathcal{C}} f \in \text{Hom}_{\mathcal{C}}(A, C)$ and $g' \circ_{\mathcal{D}} f' \in \text{Hom}_{\mathcal{D}}(A', C')$. Thus
$$\begin{aligned}(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') &= (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f') \\ &\in \text{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (C, C'))\end{aligned}$$
Thus, composition is well defined.

- Let $(A, A'), (B, B'), (C, C'), (D, D') \in \text{Obj}(\mathcal{C}\times\mathcal{D})$, $(f, f') \in \text{Hom}_{\mathcal{C}\times\mathcal{D}}((A, A'), (B, B'))$, $(g, g') \in \text{Hom}_{\mathcal{C}\times\mathcal{D}}((B, B'), (C, C'))$ and $(h, h') \in \text{Hom}_{\mathcal{C}\times\mathcal{D}}((C, C'), (D, D'))$. Then
$$\begin{aligned}[(h, h') \circ_{\mathcal{C}\times\mathcal{D}} (g, g')] \circ_{\mathcal{C}\times\mathcal{D}} (f, f') &= (h \circ_{\mathcal{C}} g, h' \circ_{\mathcal{D}} g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f') \\ &= ((h \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} f, (h' \circ_{\mathcal{D}} g') \circ_{\mathcal{D}} f') \\ &= (h \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} f), h' \circ_{\mathcal{D}} (g' \circ_{\mathcal{D}} f')) \\ &= (h, h') \circ_{\mathcal{C}\times\mathcal{D}} (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f') \\ &= (h, h') \circ_{\mathcal{C}\times\mathcal{D}} [(g, g') \circ_{\mathcal{C}\times\mathcal{D}} (f, f')]\end{aligned}$$
Thus composition is associative.

- Let $(A, B) \in \text{Obj}(\mathcal{C} \times \mathcal{D})$, $(f, f'), (g, g') \in \text{Hom}_{\mathcal{C}\times\mathcal{D}}$. Suppose that $\text{dom}_{\mathcal{C}\times\mathcal{D}}(f, f') = (A, B)$ and $\text{cod}_{\mathcal{C}\times\mathcal{D}}(g, g') = (A, B)$. Then $\text{dom}_{\mathcal{C}}(f) = A$, $\text{dom}_{\mathcal{D}}(f') = B$, $\text{cod}_{\mathcal{C}}(g) = A$ and $\text{cod}_{\mathcal{D}}(g') = B$. Hence
$$\begin{aligned}(f, f') \circ_{\mathcal{C}\times\mathcal{D}} (1_A, 1_B) &= (f \circ_{\mathcal{C}} 1_A, f' \circ_{\mathcal{D}} 1_B) \\ &= (f, f)\end{aligned}$$
and
$$\begin{aligned}(1_A, 1_B) \circ_{\mathcal{C}\times\mathcal{D}} (g, g') &= (1_A \circ_{\mathcal{C}} g, 1_B \circ g') \\ &= (g, g')\end{aligned}$$
Therefore $(1_{(A,B)})_{\mathcal{C}\times\mathcal{D}} = (1_A, 1_B)$.

$\square$