1

Introduction to Group Theory

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Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Prelimiaries

1.1 Category Theory

- **Hilb**:
  - $\text{Obj}(\mathbf{Hilb}) := \{H : H \text{ is a Hilbert space}\}$
  - $\text{Hom}_{\mathbf{Hilb}}(H_1, H_2) := \text{Hom}_{\mathbf{Ban}}(H_1, H_2)$
- **Mon**

1.1.1 The Unitary Group

Definition 1.1.1.1. Let $H_1, H_2 \in \text{Obj}(\mathbf{Hilb})$. We define the **unitary group from** H_1 **to** H_2, denoted $U(H_1, H_2)$, by

$$U(H_1, H_2) = \{T \in \text{Iso}_{\mathbf{Hilb}}(H_1, H_2) : T^* = T^{-1}\}$$

We write $U(H)$ in place of $U(H, H)$. We equip $U(H_1, H_2)$ with the strong operator topology.

Exercise 1.1.1.2. Let $H \in \text{Obj}(\mathbf{Hilb})$. Then $\mathcal{T}^s_{U(H)} = \mathcal{T}^w_{U(H)}$. strong weak operator topologies coincide

Exercise 1.1.1.3. Let $H \in \text{Obj}(\mathbf{Hilb})$. Then $U(H)$ is a topological group.

Proof. content... $\square$

Chapter 2

Representation Theory

2.1 Group Representations

2.1.1 Unitary representations

Definition 2.1.1.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $H \in \mathrm{Obj}(\mathbf{Hilb})$ and $\pi \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(H))$. Then (H, π) is said to be a **unitary representation of** G. We define the **dimension of** (H, π), denoted $\dim(H, \pi)$, by $\dim(H, \pi) := \dim V$.

Definition 2.1.1.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, (H_π, π), (H_ρ, ρ) unitary representations of G and $T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho)$. Then T is said to be (π, ρ)**-equivariant** if for each $g \in G$, $T \circ \pi(g) = \rho(g) \circ T$, i.e. the following diagram commutes:

$$\begin{tikzcd} H_\pi \arrow[r, "T"] \arrow[d, "\pi(g)"'] & H_\rho \arrow[d, "\rho(g)"] \\ H_\pi \arrow[r, "T"'] & H_\rho \end{tikzcd}$$

Definition 2.1.1.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. We define $\mathbf{URep}(G)$ by

- $\mathrm{Obj}(\mathbf{URep}(G)) = \{(H, \pi) : (H, \pi) \text{ is a unitary representation of } G\}$.
- for $(H_\pi, \pi), (H_\rho, \rho) \in \mathrm{Obj}(\mathbf{URep}(G))$,
$$\mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) = \{T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho) : T \text{ is } (\pi, \rho)\text{-equivariant}\}$$
- for $(H_\pi, \pi), (H_\rho, \rho), (H_\mu, \mu) \in \mathrm{Obj}(\mathbf{URep}(G))$, $T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$ and $S \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\rho, \rho), (H_\mu, \mu))$,
$$S \circ_{\mathbf{URep}(G)} T = S \circ T$$

Exercise 2.1.1.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. Then $\mathbf{URep}(G)$ is a category.

Proof. FINISH!!! □

Exercise 2.1.1.5. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H_\pi, \pi), (H_\rho, \rho) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then $\mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) \in \mathrm{Obj}(\mathbf{Vect}_{\mathbb{C}})$.

Proof. Let $S, T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$ and $\lambda \in \mathbb{C}$. Then for each $g \in G$,

$$\begin{aligned}(S + \lambda T) \circ \pi(g) &= S \circ \pi(g) + \lambda T \circ \pi(g) \\ &= \rho(g) \circ S + \rho(g) \circ (\lambda T) \\ &= \rho(g) \circ (S + \lambda T).\end{aligned}$$

Hence $S + \lambda T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$. Since $S, T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$ and $\lambda \in \mathbb{C}$ is arbitrary, we have that $\mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) \in \mathrm{Obj}(\mathbf{Vect}_{\mathbb{C}})$. □

Definition 2.1.1.6. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H_\pi, \pi), (H_\rho, \rho) \in \mathbf{URep}(G)$. Then (H_π, π) is said to be **unitarily equivalent** to (H_ρ, ρ), denoted $(H_\pi, \pi) \equiv (H_\rho, \rho)$, if $\mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho)) \cap U(H_\pi, H_\rho) \neq \varnothing$.

Note 2.1.1.7. Let $\pi \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(H))$. Since $U(H)$ is equipped with the strong operator topology, we have that for each $u \in H$, the map $g \mapsto \pi(g)u$ is continuous.

Definition 2.1.1.8. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. We define the **induced group action of G on H**, denoted $\phi_{(H,\pi)} : G \times H \to H$, by

$$\phi_{(H,\pi)}(g, v) = \pi(g)v$$

Note 2.1.1.9. When the context is clear, we write $g \cdot v$ in place of $\phi_{(H,\pi)}(g, v)$.

Exercise 2.1.1.10. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then

1. $\phi_{(H,\pi)}$ is a linear group action.
2. G is locally compact implies that $\phi_{(H,\pi)}$ is continuous

Proof.

1. - Let $g, h \in G$ and $v \in H$.
     (a) Since $\pi \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(H))$,
     $$\begin{aligned} e \cdot v &= \pi(e)v \\ &= \mathrm{id}_H v \\ &= v \end{aligned}$$
     (b) Since $\pi \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(H))$,
     $$\begin{aligned} g \cdot (h \cdot v) &= \pi(g)[\pi(h)v] \\ &= [\pi(g)\pi(h)]v \\ &= \pi(gh)v \\ &= (gh) \cdot v \end{aligned}$$
     Since $g, h \in G$ and $v \in H$ are arbitrary, $\phi_{(H,\pi)}$ is a group action of G on H.
   - Let $g \in G$, $\lambda \in \mathbb{C}$ and $v, w \in H$. Then
     $$\begin{aligned} &g \cdot (\lambda v + w) \\ &\quad = \pi(g)(\lambda v + w) \\ &\quad = \lambda\pi(g)v + \pi(g)w \\ &\quad = \lambda g \cdot v + g \cdot w \end{aligned}$$
     Since $g \in G$, $\lambda \in \mathbb{C}$ and $v, w \in H$ are arbitrary, $\phi_{(H,\pi)}$ is a linear action.
2. Suppose that G is locally compact. Let $(g_0, v_0) \in G \times H$ and $\epsilon > 0$. Since G is locally compact, there exists $K \subset G$ such that $g_0 \in \mathrm{Int}\, K$ and K is compact. Let $v \in H$. Define $f_v : G \to H$ by $f_v(g) = g \cdot v$. Since $\pi : G \to U(H)$ is continuous, f_v is continuous. Thus $\|f_v\|$ is continuous. Since K is compact, $\|f_v\|(K)$ is compact. Thus
   $$\begin{aligned} \sup_{g \in K} \|g \cdot v\| &= \sup_{g \in K} \|f_v(g)\| \\ &< \infty \end{aligned}$$
   Since $v \in H$ is arbitrary, we have that for each $v \in H$, $\sup\limits_{g \in K} \|g \cdot v\| < \infty$. The uniform boundedness principle implies that there exists $M > 0$ such that $\sup\limits_{g \in K} \|\pi(g)\| \leq M$. Since f_{v_0} is continuous, there

exists $U \subset K$ such that U is open, $g_0 \in U$, and $f_{v_0}(U) \subset B(f_{v_0}(g_0), \epsilon/2)$. Let $(g_1, v_1) \in U \times B(v_0, (2M)^{-1}\epsilon)$. Then

$$\begin{aligned}
\|\phi_{(H,\pi)}(g_0, v_0) - \phi_{(H,\pi)}(g_1, v_1)\| &= \|g_0 \cdot v_0 - g_1 \cdot v_1\| \\
&\leq \|g_0 \cdot v_0 - g_1 \cdot v_0\| + \|g_1 \cdot v_0 - g_1 \cdot v_1\| \\
&= \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + \|\pi(g_1)(v_0 - v_1)\| \\
&\leq \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + \|\pi(g_1)\|\|v_0 - v_1\| \\
&\leq \|f_{v_0}(g_0) - f_{v_0}(g_1)\| + M\|v_0 - v_1\| \\
&\leq \frac{\epsilon}{2} + M\frac{\epsilon}{2M} \\
&= \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\phi_{(H,\pi)}$ is continuous at (g_0, v_0). Since $(g_0, v_0) \in G \times H$ is arbitrary, we have that $\phi_{(H,\pi)} : G \times H \to H$ is continuous.

□

2.1.2 Subrepresentations

Definition 2.1.2.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. Then E is said to be

- **nontrivial** if $E \neq H, \varnothing$
- (H, π)**-invariant** if for each $g \in G$, $\pi(g)(E) \subset E$

Exercise 2.1.2.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. Suppose that E is (H, π)-invariant. Then for each $g, h \in G$,

1. $\pi(g)|_E \in \mathrm{Aut}_{\mathbf{Hilb}}(E)$, $\pi(g)|_E^{-1} = \pi(g^{-1})|_E$ and $\pi(g)(E) = E$,
2. $\pi(g)|_E \in U(E)$ and $\pi(g)|_E^* = \pi(g^{-1})|_E$,
3. $\pi(gh)|_E = \pi(g)|_E \circ \pi(h)|_E$.

Proof. Let $g, h \in G$.

1. Let $x \in E$. Since E is (H, π)-invariant, we have that $\pi(g)(x) \in E$.

$$\begin{aligned}
[\pi(g^{-1})|_E \circ \pi(g)|_E](x) &= \pi(g^{-1})|_E[\pi(g)|_E(x)] \\
&= \pi(g^{-1})|_E[\pi(g)(x)] \\
&= \pi(g^{-1})[\pi(g)(x)] \\
&= [\pi(g^{-1}) \circ \pi(g)](x) \\
&= \pi(g^{-1}g)(x) \\
&= \pi(e)(x) \\
&= I(x) \\
&= I_E(x).
\end{aligned}$$

Similarly, $\pi(g^{-1})(x) \in E$ and $[\pi(g)|_E \circ \pi(g^{-1})|_E](x) = I|_E(x)$. Since $x \in E$ is arbitrary, we have that $\pi(g)|_E \in \mathrm{Aut}_{\mathbf{Hilb}}(E)$ and $\pi(g^{-1})|_E = \pi(g)|_E^{-1}$. Since $\pi(g)|_E \in \mathrm{Aut}_{\mathbf{Hilb}}(E)$, we have that

$$\begin{aligned}
\pi(g)(E) &= \pi(g)|_E(E) \\
&= E.
\end{aligned}$$

2. Let $x, y \in E$. Then

$$\begin{aligned}\langle \pi(g)|_E x, y\rangle &= \langle \pi(g)x, y\rangle \\ &= \langle x, \pi(g)^* y\rangle \\ &= \langle x, \pi(g)^*|_E y\rangle\end{aligned}$$

   Since $x, y \in E$ are arbitrary, we have that $\pi(g)|_E^* = \pi(g)^*|_E$. The previous part then implies that

$$\begin{aligned}\pi(g)|_E^* &= \pi(g)^*|_E \\ &= \pi(g)^{-1}|_E \\ &= \pi(g^{-1})|_E \\ &= \pi(g)|_E^{-1}.\end{aligned}$$

   Since $\pi(g)|_E^* = \pi(g)|_E^{-1}$, we have that $\pi(g)|_E \in U(E)$.

3. Let $x \in E$. Since E is (H, π)-invariant, we have that $\pi(h)(x) \in E$ and therefore

$$\begin{aligned}\pi(gh)|_E(x) &= \pi(gh)(x) \\ &= [\pi(g) \circ \pi(g)](x) \\ &= \pi(g)[\pi(h)(x)] \\ &= \pi(g)|_E[\pi(h)(x)] \\ &= \pi(g)|_E[\pi(h)|_E(x)] \\ &= [\pi(g)|_E \circ \pi(g)|_E](x).\end{aligned}$$

   Since $x \in E$ is arbitrary, we have that $\pi(gh)|_E = \pi(g)|_E \circ \pi(g)|_E$.

□

Definition 2.1.2.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $\mathbb{K} \in \mathrm{Obj}(\mathbf{Field})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then

- (H, π) is said to be **reducible** if there exists a closed subspace $E \subset H$ such that E is not trivial and E is (H, π)-invariant
- (H, π) is said to be **irreducible** if (H, π) is not reducible.

Definition 2.1.2.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. Suppose that E is (H, π)-invariant.

- We define $\pi^E \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(E))$ by $\pi^E(g) := \pi(g)|_E$
- We define the **restriction** (H, π) **to** E, denoted $(H, \pi)|_E$, by $(H, \pi)|_E := (E, \pi^E)$

Exercise 2.1.2.5. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace.

1. If E is nontrivial, then $E^\perp$ is nontrivial.
2. If E is (H, π)-invariant, then $E^\perp$ is (H, π)-invariant.

Proof.

1. Suppose that E is nontrivial. Then $E \neq \{0\}, H$. Then $E^\perp \neq \{0\}, H$. Thus $E^\perp$ is nontrivial.
2. Suppose that E is (H, π)-invariant. Let $g \in G$. Since $\pi(g) \in U(H)$ and $\pi(g)(E) = E$, An exercise in the analysis notes section on Hilbert spaces implies that $\pi(g)(E^\perp) = E^\perp$. Since $g \in G$ is arbitrary, $E^\perp$ is (H, π)-invariant.

□

Definition 2.1.2.6. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $u \in H$. We define the **cyclic subspace of H generated by u under** (H, π), denoted $\mathrm{cyc}_{(H,\pi)}(u)$, by

$$\mathrm{cyc}_{(H,\pi)}(u) := \mathrm{cl}\,\mathrm{span}(\phi_{(H,\pi)}(G, u))$$

Note 2.1.2.7. When the context is clear, we write $\mathrm{cyc}(u)$ in place of $\mathrm{cyc}_{(H,\pi)}(u)$.

Exercise 2.1.2.8. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $u \in H$. Then $\mathrm{cyc}(u)$ is (H, π)-invariant. this should largely be a result about linear group actions.

Proof. Let $g \in G$. Since G acts linearly and homeomorphically on H,

$$\begin{aligned} g \cdot \mathrm{cyc}(u) &= g \cdot \mathrm{cl}\,\mathrm{span}(G \cdot u) \\ &= \mathrm{cl}\, g \cdot \mathrm{span}(G \cdot u) \\ &= \mathrm{cl}\,\mathrm{span}[g \cdot (G \cdot u)] \\ &= \mathrm{cl}\,\mathrm{span}(G \cdot u) \\ &= \mathrm{cyc}(u) \end{aligned}$$

Since $g \in G$ is arbitrary, $\mathrm{cyc}(u)$ is G-invariant. $\square$

Definition 2.1.2.9. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$.

- Let $u \in H$. Then u is said to be (H, π)**-cyclic** if $\mathrm{cyc}(u) = H$.
- Then (H, π) is said to be **cyclic** if there exists $u \in H$ such that u is (H, π)-cyclic.

2.1.3 Direct Sum of Representations

Definition 2.1.3.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H_\alpha, \pi_\alpha)_{\alpha \in A} \subset \mathrm{Obj}(\mathbf{URep}(G))$.

- We define $\bigoplus\limits_{\alpha \in A} \pi_\alpha \in \mathrm{Hom}_{\mathbf{TopGrp}}(G, U(\bigoplus\limits_{\alpha \in A} H_\alpha))$ by

$$\left[\bigoplus_{\alpha \in A} \pi_\alpha\right](g) = \bigoplus_{\alpha \in A} \pi_\alpha(g)$$

- We define the **direct sum** of $(H_\alpha, \pi_\alpha)_{\alpha \in A}$, denoted $\bigoplus\limits_{\alpha \in A} (H_\alpha, \pi_\alpha)$, by

$$\bigoplus_{\alpha \in A} (H_\alpha, \pi_\alpha) = \left(\bigoplus_{\alpha \in A} H_\alpha, \bigoplus_{\alpha \in A} \pi_\alpha\right)$$

Note 2.1.3.2. FINISH!!! the last definition works for internal or external direct sum, just need to define inner or external sum of H_α and π_α in either case.

Exercise 2.1.3.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$, $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $E \subset H$ a closed subspace. If E is (H, π)-invariant, then $(H, \pi) = (E \oplus E^\perp, \pi^E \oplus \pi^{E^\perp})$.

Proof. Suppose that E is (H, π)-invariant. A previous exercise implies that $E^\perp$ is (H, π)-invariant. Since $H = E \oplus E^\perp$. Let $g \in G$ and $u \in H$. Since $H = E \oplus E^\perp$, there exists $v \in E$ and $w \in E^\perp$ such that $u = v + w$. Then

$$\begin{aligned} \pi(g)(u) &= \pi(g)(v + w) \\ &= \pi(g)(v) + \pi(g)(w) \\ &= \pi(g)|_E(v) + \pi(g)|_{E^\perp}(w) \\ &= \pi^E(g)(v) + \pi^{E^\perp}(g)(w) \\ &= [\pi^E(g) \oplus \pi^{E^\perp}(g)](v + w) \\ &= [\pi^E \oplus \pi^{E^\perp}](g)(v + w) \\ &= [\pi^E \oplus \pi^{E^\perp}](g)(u) \end{aligned}$$

Since $u \in H$ is arbitrary, $\pi(g) = [\pi^E \oplus \pi^{E^\perp}](g)$. Since $g \in G$ is arbitrary, $\pi = \pi^E \oplus \pi^{E^\perp}$. □

Definition 2.1.3.4. Let $G \in \text{Obj}(\textbf{TopGrp})$, $(H, \pi) \in \text{Obj}(\textbf{URep}(G))$ and $\mathcal{E} \subset \mathcal{P}(H)$. Then $\mathcal{E}$ is said to be an (H, π)**-orthocyclic system** if for each $E, F \in \mathcal{E}$,

1. E is a closed subspace of H
2. $(H, \pi)|_E$ is cyclic
3. if $E \neq F$, then $E \perp F$

Exercise 2.1.3.5. Let $G \in \text{Obj}(\textbf{TopGrp})$ and $(H, \pi) \in \text{Obj}(\textbf{URep}(G))$. Then there exists $\mathcal{E} \subset \mathcal{P}(H)$ such that $\mathcal{E}$ is an (H, π)-orthocyclic system and $(H, \pi) = \bigoplus\limits_{E \in \mathcal{E}} (H, \pi)|_E$.

Hint: Zorn's lemma

Proof. Define $\mathcal{P} = \{\mathcal{E} : \mathcal{E} \text{ is an } (H, \pi)\text{-orthocyclic system}\}$. We partially order $\mathcal{P}$ by inclusion. Let $\mathcal{C} \subset \mathcal{P}$ be a chain. Set $\mathcal{E}_0 = \bigcup\limits_{\mathcal{E} \in \mathcal{C}} \mathcal{E}$. Let $E_1, E_2 \in \mathcal{E}_0$. Then there exist $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}$ such that $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$. Since $\mathcal{C}$ is a chain, $\mathcal{E}_1 \subset \mathcal{E}_2$ or $\mathcal{E}_2 \subset \mathcal{E}_1$. Suppose that $\mathcal{E}_1 \subset \mathcal{E}_2$. Then $E_1 \in \mathcal{E}_2$. Since $\mathcal{E}_2$ is an (H, π)-orthocyclic system, we have that E_1 is a closed subspaces of H, $(H, \pi)|_{E_1}$ is cyclic and if $E_1 \neq E_2$, then $E_1 \perp E_2$. Similarly, $\mathcal{E}_2 \subset \mathcal{E}_1$ implies the same conclusion. Since $E_1, E_2 \in \mathcal{E}_0$ are arbitrary, we have that for each $E_1, E_2 \in \mathcal{E}_0$

1. E_1 is a closed subspaces of H and E_1 is (H, π)-invariant
2. $(H, \pi)|_{E_1}$ is cyclic
3. if $E_1 \neq E_2$, then $E_1 \perp E_2$

Thus $\mathcal{E}_0$ is an (H, π)-orthocyclic system. Hence $\mathcal{E}_0 \in \mathcal{P}$. By construction, for each $\mathcal{E} \in \mathcal{C}$, $\mathcal{E} \subset \mathcal{E}_0$. So $\mathcal{E}_0$ is an upper bound of $\mathcal{C}$. Since $\mathcal{C} \subset \mathcal{P}$ such that $\mathcal{C}$ is a chain is arbitrary, we have that for each $\mathcal{C} \subset \mathcal{P}$, if $\mathcal{C}$ is a chain, then there exists $\mathcal{E}_0 \in \mathcal{P}$ such that $\mathcal{E}_0$ is an upper bound of $\mathcal{C}$. Zorn's lemma implies that there exists $\mathcal{E} \in \mathcal{P}$ such that $\mathcal{E}$ is maximal. Set $E = \bigoplus\limits_{E_0 \in \mathcal{E}} E_0$. For the sake of contradiction, suppose that $H \neq E$. Then $E^\perp \neq \{0\}$. Thus there exists $u \in E^\perp$ such that $u \neq 0$. Therefore $\text{cyc}(u) \neq 0$ and $\text{cyc}(u) \subset E^\perp$. Let $E_0 \in \mathcal{E}$. By construction, $E_0 \subset E$. Thus

$$\begin{aligned} \text{cyc}(u) &\subset E^\perp \\ &\subset E_0^\perp \end{aligned}$$

Since $E_0 \in \mathcal{E}$ is arbitrary, we have that for each $E_0 \in \mathcal{E}$, $\text{cyc}(u) \subset E_0^\perp$. Set $\mathcal{E}' = \mathcal{E} \cup \{\text{cyc}(u)\}$. Then for each $E, F \in \mathcal{E}'$,

1. E is a closed subspaces of H and E is (H, π)-invariant
2. $(H, \pi)|_E$ is cyclic
3. if $E \neq F$, then $E \perp F$

Hence $\mathcal{E}' \in \mathcal{P}$. Since $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{E}$ □

Note 2.1.3.6. Let H be a Hilbert space and $E \subset H$ a closed subspace. We denote the orthogonal projection onto E by P_E.

Exercise 2.1.3.7. Let $G \in \text{Obj}(\textbf{TopGrp})$, $(H, \pi) \in \text{Obj}(\textbf{URep}(G))$ and $E \subset H$ a closed subspace. Then E is (H, π)-invariant iff $P_E \in \text{End}_{\textbf{URep}(G)}((H, \pi))$.

Proof.

- $(\implies)$:
  Suppose that E is (H,π)-invariant. Let $g \in G$ and $z \in H$. Then there exists $x \in E$ and $y \in E^{\perp}$ such that $z = x + y$. Since E is (H,π) invariant, $\pi(g)(x) \in E$. Thus

  $$\begin{aligned} \pi(g) \circ P_E(x) &= \pi(g)(x) \\ &= P_E \circ \pi(g)(x). \end{aligned}$$

  Since E is (H,π)-invariant, ref previous ex here implies that $E^{\perp}$ is (H,π)-invariant. Therefore $\pi(g)(y) \in E^{\perp}$ and

  $$\begin{aligned} \pi(g) \circ P_E(x) &= \pi(g)(0) \\ &= 0 \\ &= P_E \circ \pi(g)(y). \end{aligned}$$

  Hence

  $$\begin{aligned} \pi(g) \circ P_E(z) &= \pi(g) \circ P_E(x+y) \\ &= \pi(g) \circ P_E(x) + \pi(g) \circ P_E(y) \\ &= P_E \circ \pi(g)(x) + P_E \circ \pi(g)(y) \\ &= P_E \circ \pi(g)(x+y) \\ &= P_E \circ \pi(g)(z). \end{aligned}$$

  Since $z \in H$ is arbitrary, we have that $\pi(g) \circ P_E = P_E \circ \pi(g)$. Since $g \in G$ is arbitrary, $P_E \in \text{End}_{\mathbf{URep}(G)}(H,\pi)$.

- $(\impliedby)$:
  Conversely, suppose that $P_E \in \text{End}_{\mathbf{URep}(G)}((H,\pi))$. Let $g \in G$ and $x \in E$. Then

  $$\begin{aligned} &\pi(g)(x) \\ &\quad= \pi(g) \circ P_E(x) \\ &\quad= P_E \circ \pi(g)(x) \\ &\quad\in E. \end{aligned}$$

  Since $x \in E$ is arbitrary, $\pi(g)(E) \subset E$. Since $g \in G$ is arbitrary, E is (H,π)-invariant.

$\square$

2.2 Tannaka Duality

Definition 2.2.0.1. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. We define the **forgetful functor from $\mathbf{URep}(G)$ to $\mathbf{Hilb}$**, denoted $U : \mathbf{URep}(G) \to \mathbf{Hilb}$, by

- $U(H, \pi) = H, \quad (H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$
- $U(T) = T, \quad T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$.

Need to find out if quotienting by equivalence of isomorphism makes $\mathbf{URep}(G)$ a small category so that we can talk about the functor category $\mathbf{Hilb}^{\mathbf{URep}(G)}$ containing the forgetful functor as an object.

Definition 2.2.0.2. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $g \in G$. We define $\hat{g} : U \Rightarrow U$ by

$$\hat{g}_{(H,\pi)} = \pi(g)$$

Exercise 2.2.0.3. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $g \in G$. Then

1. $\hat{g} : U \Rightarrow U$ is a natural transformation.
2. $\hat{g} \in \mathrm{Aut}_{\mathbf{Hilb}^{\mathbf{URep}(G)}}(U)$

Proof.

1. (a) Let $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. By definition,

$$\begin{aligned}\hat{g}_{(H,\pi)} &= \pi(g) \\ &\in U(H) \\ &\subset \mathrm{Aut}_{\mathbf{Hilb}}(U(H, \pi))\end{aligned}$$

   (b) Let $(H_\pi, \pi), (H_\rho, \rho) \in \mathrm{Obj}(\mathbf{URep}(G))$ and $T \in \mathrm{Hom}_{\mathbf{URep}(G)}((H_\pi, \pi), (H_\rho, \rho))$. By definition, $T \in \mathrm{Hom}_{\mathbf{Hilb}}(H_\pi, H_\rho)$ and T is (π, ρ)-equivariant. Therefore

$$\begin{aligned}U(T) \circ \hat{g}_{(H_\pi,\pi)} &= T \circ \pi(g) \\ &= \rho(g) \circ T \\ &= \hat{g}_{(H_\rho,\rho)} \circ U(T)\end{aligned}$$

   i.e. the following diagram commutes:

$$\begin{array}{ccc} U(H_\pi, \pi) & \xrightarrow{\hat{g}_{(H_\pi,\pi)}} & U(H_\pi, \pi) \\ {\scriptstyle U(T)}\downarrow & & \downarrow{\scriptstyle U(T)} \\ U(H_\rho, \rho) & \xrightarrow{\hat{g}_{(H_\rho,\rho)}} & U(H_\rho, \rho) \end{array} = \begin{array}{ccc} H_\pi & \xrightarrow{\pi(g)} & H_\pi \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ H_\rho & \xrightarrow{\rho(g)} & H_\rho \end{array}$$

   Thus $\hat{g} : U \Rightarrow U$ is a natural transformation.

2. Set $h = g^{-1}$. Part (1) implies that $\hat{g}, \hat{h} \in \mathrm{End}_{\mathbf{Hilb}^{\mathbf{URep}(G)}}(U)$. Let $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. Then

$$(\hat{g} \circ \hat{h})_{(H,\pi)} = \hat{g}_{(H,\pi)}$$

   The previous part implies that

$$\begin{aligned}\hat{g} &\in \mathrm{Hom}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U, U) \\ &= \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U)\end{aligned}$$

□

Definition 2.2.0.4. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$ and $(H, \pi) \in \mathrm{Obj}(\mathbf{URep}(G))$. We define the (H, π)**-projection**, denoted $\pi_{(H,\pi)} : \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U) \to \mathrm{End}_{\mathbf{TopVect}_{\mathbb{C}}}(V)$, by $\pi_{(H,\pi)}(\alpha) = \alpha_{(H,\pi)}$. We define the **topology of endomorphisms of** U, denoted $\mathcal{T}_{\mathcal{E}(U)}$, by

$$\mathcal{T}_{\mathcal{E}(U)} = \tau(\pi_{(H,\pi)} : (H, \pi) \in \mathbf{URep}(G))$$

Definition 2.2.0.5. define addition of endomorphisms of U pointwise

Exercise 2.2.0.6. Let $G \in \mathrm{Obj}(\mathbf{TopGrp})$. Then $(\mathrm{Aut}_{\mathbf{TopVect}_{\mathbb{C}}^{\mathbf{URep}(G)}}(U), \mathcal{T}_{\mathcal{E}(U)})$ is a topological unital algebra.

Proof. $\square$

Chapter 3

Groupoids

Definition 3.0.0.1.

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration