

INTRODUCTION TO PROBABILITY

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1. BASIC PROBABILITY

2. PROBABILITY

2.1. Distributions.

Definition 2.1.1. Let Ω be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then \mathcal{P} is said to be a π -**system** on Ω if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Definition 2.1.2. Let Ω be a set and $\mathcal{L} \subset \mathcal{P}(\Omega)$. Then \mathcal{L} is said to be a λ -**system** on Ω if

- (1) $\mathcal{L} \neq \emptyset$
- (2) for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$
- (3) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}$, if $(A_n)_{n \in \mathbb{N}}$ is disjoint, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}$

Exercise 2.1.3. Let Ω be a set and \mathcal{L} a λ -system on Ω . Then

- (1) $\Omega, \emptyset \in \mathcal{L}$

Proof. Straightforward. □

Definition 2.1.4. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. Put

$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(\Omega) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the λ -**system on Ω generated by \mathcal{C}** , $\lambda(\mathcal{C})$, to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L} \in \mathcal{S}} \mathcal{L}$$

Exercise 2.1.5. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. If \mathcal{C} is a λ -system and \mathcal{C} is a π -system, then \mathcal{C} is a σ -algebra.

Proof. Suppose that \mathcal{C} is a λ -system and \mathcal{C} is a π -system. Then we need only verify the third axiom in the definition of a σ -algebra. Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$. Define $B_1 = A_1$ and for $n \geq 2$, define $B_n = A_n \cap \left(\bigcup_{k=1}^{n-1} A_k \right)^c = A_n \cap \left(\bigcap_{k=1}^{n-1} A_k^c \right) \in \mathcal{C}$. Then $(B_n)_{n \in \mathbb{N}}$ is disjoint and therefore $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{C}$. □

Theorem 2.1.6. (Dynkin's Theorem)

Let Ω be a set.

- (1) Let \mathcal{P} be a π -system on Ω and \mathcal{L} a λ -system on Ω . If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
- (2) Let \mathcal{P} be a π -system on Ω . Then $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

Exercise 2.1.7. Let (Ω, \mathcal{F}) be a measurable space and μ, ν probability measures on (Ω, \mathcal{F}) . Put $\mathcal{L}_{\mu, \nu} = \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$. Then $\mathcal{L}_{\mu, \nu}$ is a λ -system on Ω .

Proof.

- (1) $\emptyset \in \mathcal{L}_{\mu, \nu}$.
- (2) Let $A \in \mathcal{L}_{\mu, \nu}$. Then $\mu(A) = \nu(A)$. Thus

$$\begin{aligned} \mu(A^c) &= 1 - \mu(A) \\ &= 1 - \nu(A) \\ &= \nu(A^c) \end{aligned}$$

So $A^c \in \mathcal{L}_{\mu, \nu}$.

- (3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}_{\mu, \nu}$. So for each $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$. Suppose that $(A_n)_{n \in \mathbb{N}}$ is disjoint. Then

$$\begin{aligned} \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \sum_{n \in \mathbb{N}} \mu(A_n) \\ &= \sum_{n \in \mathbb{N}} \nu(A_n) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \end{aligned}$$

Hence $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}_{\mu, \nu}$.

□

Exercise 2.1.8. Let (Ω, \mathcal{F}) be a measurable space, μ, ν probability measures on (Ω, \mathcal{F}) and $\mathcal{P} \subset \mathcal{F}$ a π -system on Ω . Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Proof. Using the previous exercise, we see that $\mathcal{P} \subset \mathcal{L}_{\mu, \nu}$. Dynkin's theorem implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu, \nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$. □

Definition 2.1.9. Let $F : \mathbb{R} \rightarrow \mathbb{R}$. Then F is said to be a **probability distribution function** if

- (1) F is right continuous
- (2) F is increasing
- (3) $F(-\infty) = 0$ and $F(\infty) = 1$

Definition 2.1.10. Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We define $F_P : \mathbb{R} \rightarrow \mathbb{R}$, by

$$F_P(x) = P((-\infty, x])$$

We call F_P the **probability distribution function of P** .

Exercise 2.1.11. Let (Ω, \mathcal{F}, P) be a probability measure. Then F_P is a probability distribution function.

Proof. (1) Let $x \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}} \subset [x, \infty)$. Suppose that $x_n \rightarrow x$. Then $(x, x_n] \rightarrow \emptyset$ because $\limsup_{n \rightarrow \infty} (x, x_n] = \emptyset$. Thus

$$F(x_n) - F(x) = P((x, x_n]) \rightarrow P(\emptyset) = 0$$

This implies that

$$F(x_n) \rightarrow F(x)$$

So F is right continuous.

- (2) Clearly F_P is increasing.
- (3) Continuity from below tells us that

$$F(-\infty) = \lim_{n \rightarrow -\infty} F(n) = \lim_{n \rightarrow -\infty} P((-\infty, n]) = 0$$

and continuity from above tell us that

$$F(\infty) = \lim_{n \rightarrow \infty} F(n) = \lim_{n \rightarrow \infty} P((-\infty, n]) = 1$$

□

Exercise 2.1.12. Let μ, ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $F_\mu = F_\nu$ iff $\mu = \nu$.

Proof. Clearly if $\mu = \nu$, then $F_\mu = F_\nu$. Conversely, suppose that $F_\mu = F_\nu$. Then for each $x \in \mathbb{R}$,

$$\begin{aligned}\mu((-\infty, x]) &= F_\mu(x) \\ &= F_\nu(x) \\ &= \nu((-\infty, x])\end{aligned}$$

Put $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then \mathcal{C} is a π -system and for each $A \in \mathcal{C}$, $\mu(A) = \nu(A)$. Hence for each $A \in \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, $\mu(A) = \nu(A)$. So $\mu = \nu$. \square

Definition 2.1.13. Let (Ω, \mathcal{F}, P) be a probability space and $X : \Omega \rightarrow \mathbb{R}^n$. Then X is said to be a **random vector** on (Ω, \mathcal{F}) if X is \mathcal{F} - $\mathcal{B}(\mathbb{R}^n)$ measurable. If $n = 1$, then X is said to be a **random variable**. We define

$$L_n^0(\Omega, \mathcal{F}, P) = \{X : \Omega \rightarrow \mathbb{R}^n : X \text{ is a random vector}\}$$

and

$$L_n^p(\Omega, \mathcal{F}, P) = \left\{ X \in L_n^0 : \int \|X\|^p dP < \infty \right\}$$

Definition 2.1.14. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . We define the **probability distribution** of X , $P_X : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$, to be the measure

$$P_X = X_*P$$

That is, for each $A \in \mathcal{B}(\mathbb{R})$,

$$P_X(A) = P(X^{-1}(A))$$

We define the **probability distribution function** of X , $F_X : \mathbb{R} \rightarrow [0, 1]$, to be

$$F_X = F_{P_X}$$

Definition 2.1.15. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . If $P_X \ll m$, we define the **probability density** of X , $f_X : \mathbb{R} \rightarrow \mathbb{R}$, by

$$f_X = \frac{dP_X}{dm}$$

Exercise 2.1.16. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on (Ω, \mathcal{F}) . Then for each $x \in \mathbb{R}$,

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} X_n > x\right) \leq \liminf_{n \rightarrow \infty} P(X_n > x)$$

Proof. Let $\omega \in \left\{ \liminf_{n \rightarrow \infty} X_n > x \right\}$. Then $x < \liminf_{n \rightarrow \infty} X_n(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} X_k(\omega) \right)$. So there exists $n^* \in \mathbb{N}$ such that $x < \inf_{k \geq n^*} X_k(\omega)$. Then for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x < X_k(\omega)$.

So there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Hence

$\inf_{k \geq n^*} \mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Thus $\liminf_{n \rightarrow \infty} \mathbf{1}_{\{X_n > x\}}(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \mathbf{1}_{\{X_k > x\}}(\omega) \right) = 1$. Therefore

$\omega \in \liminf_{n \rightarrow \infty} \{X_k > x\}$ and we have shown that

$$\left\{ \liminf_{n \rightarrow \infty} X_n > x \right\} \subset \liminf_{n \rightarrow \infty} \{X_k > x\}$$

Then

$$\begin{aligned} P\left(\liminf_{n \rightarrow \infty} X_n > x\right) &\leq P\left(\liminf_{n \rightarrow \infty} \{X_k > x\}\right) \\ &\leq \liminf_{n \rightarrow \infty} P(\{X_k > x\}) \end{aligned}$$

□

Definition 2.1.17. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+(\Omega) \cup L^1$. Define the **expectation of X**, $E(X)$, to be

$$E(X) = \int X dP$$

2.2. Independence.

Definition 2.2.1. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C} \subset \mathcal{F}$. Then \mathcal{C} is said to be **independent** if for each $(A_i)_{i=1}^n \subset \mathcal{C}$,

$$P\left(\bigcap_{k=1}^n A_k\right) = \prod_{k=1}^n P(A_k)$$

Definition 2.2.2. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Then $\mathcal{C}_1, \dots, \mathcal{C}_n$ are said to be **independent** if for each $A_1 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$, A_1, \dots, A_n are independent.

Note 2.2.3. We will explicitly say that for each $i = 1, \dots, n$, \mathcal{C}_i is independent when talking about the independence of the elements of \mathcal{C}_i to avoid ambiguity.

Definition 2.2.4. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are said to be **independent** if for each $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent.

Exercise 2.2.5. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff $\sigma(X_1), \dots, \sigma(X_n)$ are independent.

Proof. Suppose that X_1, \dots, X_n are independent. Let $A_1 \in \sigma(X_1), \dots, A_n \in \sigma(X_n)$. Then for each $i = 1, \dots, n$, there exists $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X_i^{-1}(B_i)$. Then A_1, \dots, A_n are independent. Hence $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Conversely, suppose that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$. Then for each $i = 1, \dots, n$, $X_i^{-1}B_i \in \sigma(X_i)$. Then $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent. Hence X_1, \dots, X_n are independent. □

Exercise 2.2.6. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{F}$ a collection of σ -algebras on Ω . Suppose that for each $i = 1, \dots, n$, X_i is \mathcal{F}_i -measurable. If $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent, then X_1, \dots, X_n are independent.

Proof. For each $i = 1, \dots, n$, $\sigma(X_i) \subset \mathcal{F}_i$. So $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Hence X_1, \dots, X_n are independent. □

Exercise 2.2.7. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Suppose that for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent, then $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent.

Proof. Let $A_2 \in \mathcal{C}_2$. Define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap A_2) = P(A)P(A_2)\}$. Then

- (1) $\Omega \in \mathcal{L}$
- (2) If $A \in \mathcal{L}$, then

$$\begin{aligned} P(A^c \cap A_2) &= P(A_2) - P(A_2 \cap A) \\ &= P(A_2) - P(A_2)P(A) \\ &= (1 - P(A))P(A_2) \\ &= P(A^c)P(A_2) \end{aligned}$$

So $A^c \in \mathcal{L}$

- (3) If $(B_n)_{n \in \mathbb{N}} \subset \mathcal{L}$ is disjoint, then

$$\begin{aligned} P\left(\left[\bigcup_{n \in \mathbb{N}} B_n\right] \cap A_2\right) &= P\left(\bigcup_{n \in \mathbb{N}} B_n \cap A_2\right) \\ &= \sum_{n \in \mathbb{N}} P(B_n \cap A_2) \\ &= \sum_{n \in \mathbb{N}} P(B_n)P(A_2) \\ &= \left[\sum_{n \in \mathbb{N}} P(B_n)\right]P(A_2) \\ &= P\left(\bigcup_{n \in \mathbb{N}} B_n\right)P(A_2) \end{aligned}$$

So $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{L}$.

Thus \mathcal{L} is a λ -system. Since $\mathcal{C}_1 \subset \mathcal{L}$ is a π -system, Dynkin's theorem tells us that $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Since $A_2 \in \mathcal{C}_2$ is arbitrary $\sigma(\mathcal{C}_1)$ and \mathcal{C}_2 are independent. The same reasoning implies that $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent. Let $A_2 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$. We may do the same process with

$$\mathcal{L} = \left\{A \in \mathcal{F} : P\left(A \cap \left(\bigcap_{i=2}^n A_i\right)\right) = P(A) \prod_{i=2}^n P(A_i)\right\}$$

and conclude that $\sigma(\mathcal{C}_1), \mathcal{C}_2, \dots, \mathcal{C}_n$ are independent. Which, using the same reasoning would imply that $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent. \square

Exercise 2.2.8. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff for each $x_1, \dots, x_n \in \mathbb{R}$,

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$$

Proof. Suppose that X_1, \dots, X_n are independent. Then $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $x_1, \dots, x_n \in \mathbb{R}$. Then for each $i = 1, \dots, n$, $\{X_i \leq x_i\} \in \sigma(X_i)$. Hence

$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Conversely, suppose that for each $x_1, \dots, x_n \in \mathbb{R}$, $P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Define $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$. For each $i = 1, \dots, n$, define $\mathcal{C}_i = X_i^{-1}\mathcal{C}$. Then for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and

$$\begin{aligned}\sigma(\mathcal{C}_i) &= \sigma(X_i^{-1}(\mathcal{C})) \\ &= X_i^{-1}(\sigma(\mathcal{C})) \\ &= X_i^{-1}(\mathcal{B}(\mathbb{R})) \\ &= \sigma(X_i)\end{aligned}$$

By assumption, $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent. The previous exercise tells us that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Then X_1, \dots, X_n are independent. \square

Exercise 2.2.9. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Define $X = (X_1, \dots, X_n)$. If X_1, \dots, X_n are independent, then

$$P_X = \prod_{i=1}^n P_{X_i}$$

.

Proof. Let $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}P_X(A_1 \times \dots \times A_n) &= P(X \in A_1 \times \dots \times A_n) \\ &= P(X_1 \in A_1, \dots, X_n \in A_n) \\ &= P(X_1 \in A_1) \cdots P(X_n \in A_n) \\ &= P_{X_1}(A_1) \cdots P_{X_n}(A_n) \\ &= \prod_{i=1}^n P_{X_i}(A_1 \times \dots \times A_n)\end{aligned}$$

Put

$$\mathcal{P} = \{A_1 \times \dots \times A_n : A_1 \in \mathcal{B}(\mathbb{R}), \dots, A_n \in \mathcal{B}(\mathbb{R})\}$$

Then \mathcal{P} is a π -system and

$$\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R}) \otimes \dots \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^n)$$

A previous exercise then tells us that $P_X = \prod_{i=1}^n P_{X_i}$ \square

Exercise 2.2.10. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R} \in L^0$. Suppose that $f_1 \circ X_1, \dots, f_n \circ X_n \in L^+(\Omega)$ or $f_1 \circ X_1, \dots, f_n \circ X_n \in L^1(\Omega)$. If X_1, \dots, X_n are independent, then

$$E(f_1(X_1) \cdots f_n(X_n)) = \prod_{i=1}^n E(f_i(X_i))$$

Proof. Define the random vector $X : \Omega \rightarrow \mathbb{R}^n$ by $X = (X_1, \dots, X_n)$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ by $g(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$. Suppose that for each $i = 1, \dots, n$, $f_i \in L^+(\mathbb{R})$. Then $g \in L^+(\mathbb{R}^n)$ and by change of variables,

$$\begin{aligned}
 E(f_1(X_1) \cdots f_n(X_n)) &= E(g(X)) \\
 &= \int_{\Omega} g \circ X \, dP \\
 &= \int_{\mathbb{R}^n} g(x) \, dP_X(x) \\
 &= \int_{\mathbb{R}^n} g(x) \, d \prod_{i=1}^n P_{X_i}(x) \\
 &= \prod_{i=1}^n \int_{\mathbb{R}} f_i(x) \, dP_{X_i}(x) \\
 &= \prod_{i=1}^n \int_{\Omega} f_i \circ X \, dP \\
 &= \prod_{i=1}^n E(f_i(X_i))
 \end{aligned}$$

If for each $i = 1, \dots, n$, $f_i \in L^1(\mathbb{R}, P_{X_i})$, then following the above reasoning with $|g|$ tells us that $g \in L^1(\mathbb{R}^n, P_X)$ and we use change of variables and Fubini's theorem to get the same result. \square

2.3. L^p Spaces for Probability.

Note 2.3.1. Recall that for a probability space (Ω, \mathcal{F}, P) and $1 \leq p \leq q \leq \infty$ we have $L^q \subset L^p$ and for each $X \in L^q$, $\|X\|_p \leq \|X\|_q$. Also recall that for $X, Y \in L^2$, we have that $\|XY\|_1 \leq \|X\|_2 \|Y\|_2$.

Definition 2.3.2. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Define the **variance of X**, $Var(X)$, to be

$$Var(X) = E([(X - E(X))^2])$$

.

Definition 2.3.3. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the

Definition 2.3.4. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the **covariance of X and Y**, $Cov(X, Y)$, to be

$$Cov(X, Y) = E([X - E(X)][Y - E(Y)])$$

Exercise 2.3.5. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Then the covariance is well defined and $Cov(X, Y)^2 \leq Var(X)Var(Y)$

Proof. By Holder's inequality,

$$\begin{aligned}
 |Cov(X, Y)| &= \left| \int (X - E(X))(Y - E(Y)) dP \right| \\
 &\leq \int |(X - E(X))(Y - E(Y))| dP \\
 &= \|(X - E(X))(Y - E(Y))\|_1 \\
 &\leq \|X - E(X)\|_2 \|Y - E(Y)\|_2 \\
 &= \left(\int |X - E(X)|^2 dP \right)^{\frac{1}{2}} \left(\int |Y - E(Y)|^2 dP \right)^{\frac{1}{2}} \\
 &= Var(X)^{\frac{1}{2}} Var(Y)^{\frac{1}{2}}
 \end{aligned}$$

So $Cov(X, Y)^2 \leq Var(X)Var(Y)$. □

Exercise 2.3.6. Let (Ω, \mathcal{F}, P) be a measure space and $X, Y \in L^2$. Then

- (1) $Cov(X, Y) = E(XY) - E(X)E(Y)$
- (2) If X, Y are independent, then $Cov(X, Y) = 0$
- (3) $Var(X) = E(X^2) - E(X)^2$
- (4) for each $a, b \in \mathbb{R}$, $Var(aX + b) = a^2 Var(X)$.
- (5) $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Proof.

- (1) We have that

$$\begin{aligned}
 Cov(X, Y) &= E\left[(X - E(X))(Y - E(Y))\right] \\
 &= E(XY - E(Y)X - E(X)Y + E(X)E(Y)) \\
 &= E(XY) - E(X)E(Y) - E(X)E(Y) + E(X)E(Y) \\
 &= E(XY) - E(X)E(Y)
 \end{aligned}$$

- (2) Suppose that X, Y are independent. Then $E(XY) = E(X)E(Y)$. Hence

$$\begin{aligned}
 Cov(X, Y) &= E(XY) - E(X)E(Y) \\
 &= E(X)E(Y) - E(X)E(Y) \\
 &= 0
 \end{aligned}$$

- (3) Part (1) implies that

$$\begin{aligned}
 Var(X) &= Cov(X, X) \\
 &= E(X^2) - E(X)^2
 \end{aligned}$$

(4) Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
 \text{Var}(aX + b) &= E[(aX + b)^2] - E(aX + b)^2 \\
 &= E[a^2X^2 + 2abX + b^2] - (aE(X) + b)^2 \\
 &= a^2E(X^2) + 2abE(X) + b^2 - (a^2E(X)^2 + 2abE(X) + b^2) \\
 &= a^2(E(X^2) - E(X)^2) \\
 &= a^2\text{Var}(X)
 \end{aligned}$$

(5) We have that

$$\begin{aligned}
 \text{Var}(X + Y) &= E[(X + Y)^2] - E[X + Y]^2 \\
 &= E[X^2 + 2XY + Y^2] - (E(X) + E(Y))^2 \\
 &= E(X^2) + 2E[XY] + E(Y^2) - (E(X)^2 + 2E(X)E(Y) + E(Y)^2) \\
 &= (E(X^2) - E(X)^2) + (E(Y^2) - E(Y)^2) + 2(E[XY] - E(X)E(Y)) \\
 &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)
 \end{aligned}$$

□

Definition 2.3.7. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. The **correlation of X and Y**, $\text{Cor}(X, Y)$, is defined to be

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Exercise 2.3.8.

Exercise 2.3.9. Jensen's Inequality:

Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^1$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$. If ϕ is convex, then

$$\phi(E(X)) \leq E[\phi(X)]$$

Proof. Put $x_0 = E(X)$. Since ϕ is convex, there exist $a, b \in \mathbb{R}$ such that $\phi(x_0) = ax_0 + b$ and for each $x \in \mathbb{R}$, $\phi(x) \geq ax + b$. Then

$$\begin{aligned}
 E[\phi(X)] &= \int \phi(X) dP \\
 &\geq \int [aX + b] dP \\
 &= a \int X dP + b \\
 &= aE(X) + b \\
 &= ax_0 + b \\
 &= \phi(x_0) \\
 &= \phi(E(X))
 \end{aligned}$$

□

Exercise 2.3.10. Markov's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+$. Then for each $a \in (0, \infty)$,

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Proof. Let $a \in (0, \infty)$. Then $a\mathbf{1}_{\{X \geq a\}} \leq X\mathbf{1}_{\{X \geq a\}}$. Thus

$$\begin{aligned} aP(X \geq a) &= \int a\mathbf{1}_{\{X \geq a\}} dP \\ &= \int X\mathbf{1}_{\{X \geq a\}} dP \\ &\leq \int X dP \\ &= E(X) \end{aligned}$$

Therefore

$$P(X \geq a) \leq \frac{E(X)}{a}$$

□

Exercise 2.3.11. Chebychev's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a \in (0, \infty)$,

$$P(|X - E(X)| \geq a) \leq \frac{Var(X)}{a^2}$$

Proof. Let $a \in (0, \infty)$. Then

$$\begin{aligned} P(|X - E(X)| \geq a) &= P((X - E(X))^2 \geq a^2) \\ &\leq \frac{E[(X - E(X))^2]}{a^2} \\ &= \frac{Var(X)}{a^2} \end{aligned}$$

□

Exercise 2.3.12. Chernoff's Bound: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a, t \in (0, \infty)$,

$$P(X \geq a) \leq e^{-ta} E[e^{tX}]$$

Proof. Let $a, t \in (0, \infty)$. Then

$$\begin{aligned} P(X \geq a) &= P(tX \geq ta) \\ &= P(e^{tX} \geq e^{ta}) \\ &\leq e^{-ta} E[e^{tX}] \end{aligned}$$

□

Exercise 2.3.13. Weak Law of Large Numbers: Let (Ω, \mathcal{F}, P) be a probability space $(X_i)_{i \in \mathbb{N}} \subset L^2$. Suppose that $(X_i)_{i \in \mathbb{N}}$ are iid. Then

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

Proof. Put $\mu = E[X_1]$ and $\sigma^2 = \text{Var}(X_1)$. Then

$$\begin{aligned} E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] &= \frac{1}{n} \sum_{i=1}^n E[X_i] \\ &= \frac{1}{n} \sum_{i=1}^n \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) &= \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) \\ &= \frac{1}{n^2} \sum_{i=1}^n \sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Let $\epsilon > 0$. Then

$$\begin{aligned} P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - E[X_1]\right| \geq \epsilon\right) &= P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) \\ &= P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - E\left[\frac{1}{n} \sum_{i=1}^n X_i\right]\right| \geq \epsilon\right) \\ &\leq \frac{\text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right)}{\epsilon^2} \\ &= \frac{\sigma^2/n}{\epsilon^2} \\ &= \frac{\sigma^2}{n\epsilon^2} \rightarrow 0 \end{aligned}$$

So

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

□

2.4. Borel Cantelli Lemma.

Exercise 2.4.1. Borel Cantelli Lemma:

Let (Ω, \mathcal{F}, P) be a probability space and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{F}$.

- (1) If $\sum_{n \in \mathbb{N}} P(A_n) < \infty$, then $P(\limsup_{n \rightarrow \infty} A_n) = 0$.
- (2) If $(A_n)_{n \in \mathbb{N}}$ are independent and $\sum_{n \in \mathbb{N}} P(A_n) = \infty$, then $P(\limsup_{n \rightarrow \infty} A_n) = 1$.

Proof.

- (1) Suppose that $\sum_{n \in \mathbb{N}} P(A_n) < \infty$. Recall that

$$\limsup_{n \rightarrow \infty} A_n = \left\{ \omega \in \Omega : \sum_{n \in \mathbb{N}} 1_{A_n}(\omega) = \infty \right\}$$

Then

$$\begin{aligned} \infty &> \sum_{n \in \mathbb{N}} P(A_n) \\ &= \sum_{n \in \mathbb{N}} \int 1_{A_n} dP \\ &= \int \sum_{n \in \mathbb{N}} 1_{A_n} dP \end{aligned}$$

Thus $\sum_{n \in \mathbb{N}} 1_{A_n} < \infty$ a.e. and $P(\limsup_{n \rightarrow \infty} A_n) = 0$.

- (2) Suppose that $(A_n)_{n \in \mathbb{N}}$ are independent and $\sum_{n \in \mathbb{N}} P(A_n) = \infty$.

□

Exercise 2.4.2. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}} \subset L^0$ and $X \in L^0$.

- (1) If for each $\epsilon > 0$, $\sum_{n \in \mathbb{N}} P(|X_n - X| \geq \epsilon) < \infty$, then $X_n \rightarrow X$ a.e.
- (2) If $(X_n)_{n \in \mathbb{N}}$ are independent and there exists $\epsilon > 0$ such that $\sum_{n \in \mathbb{N}} P(|X_n - X| \geq \epsilon) = \infty$, then $X_n \not\rightarrow X$ a.e.

Proof.

- (1) For $\epsilon > 0$ and $n \in \mathbb{N}$, set $A_n(\epsilon) = \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}$. Suppose that for each $\epsilon > 0$, $\sum_{n \in \mathbb{N}} P(|X_n - X| \geq \epsilon) < \infty$. The Borel-Cantelli lemma implies that for each $m \in \mathbb{N}$,

$$P(\limsup_{n \rightarrow \infty} A_n(1/m)) = 0$$

Let $\omega \in \Omega$. Then $X_n(\omega) \not\rightarrow X(\omega)$ iff

$$\omega \in \bigcup_{m \in \mathbb{N}} \limsup_{n \rightarrow \infty} A_n(1/m)$$

So

$$\begin{aligned}
 P(X_n \not\rightarrow X) &= P\left(\bigcup_{m \in \mathbb{N}} \limsup_{n \rightarrow \infty} A_n(1/m)\right) \\
 &\leq \sum_{m \in \mathbb{N}} P(\limsup_{n \rightarrow \infty} A_n(1/m)) \\
 &= 0
 \end{aligned}$$

Hence $X_n \rightarrow X$ a.e.

(2)

□

3. PROBABILITY ON LOCALLY COMPACT GROUPS

Note 3.0.1. In this section, familiarity with Haar measure will be assumed. This section is intended as a continuation of section 7 of [3].

3.1. Action on Probability Measures.

Note 3.1.1. We recall some notation from section 7.1 of [3].

- $l_g \in \text{Homeo}(G)$, $l_g(x) = gx$
- $L_g \in \text{Sym}(L_0(G))$, $L_g f = f \circ l_g^{-1}$ We continue from section 7

Note 3.1.2. The next exercise generalizes the notion of a scale-family.

Exercise 3.1.3. Let (Ω, \mathcal{F}, P) be a probability space, G a locally compact group, μ a left Haar measure on G , $X \in L_G^0$ and $g \in G$. If $P_X \ll \mu$, then $f_{gX} = L_g f_X$.

Proof. Suppose that $P_X \ll \mu$. Let $A \in \mathcal{B}(G)$. Then

$$\begin{aligned}
 P_{gX}(A) &= P(gX \in A) \\
 &= P(X \in g^{-1}A) \\
 &= P_X(g^{-1}A) \\
 &= P_X(l_g^{-1}(A)) \\
 &= l_{g*}P_X(A) \\
 &= g \cdot P_X(A)
 \end{aligned}$$

The previous exercise tells us that $f_{gX} = L_g f_X$. □

4. WEAK CONVERGENCE OF MEASURES

5. CONCENTRATION INEQUALITIES

5.1. Introduction.

Exercise 5.1.1. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$. Then for each $s, t \in \mathbb{R}$,

$$P(X + Y \geq s + t) \leq P(X \geq s) + P(Y \geq t)$$

Proof. For $Z \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ and $t \in \mathbb{R}$, define $A_Z^t \in \mathcal{F}$ by

$$A_Z^t = \{\omega \in \Omega : Z(\omega) \geq t\}$$

Let $s, t \in \mathbb{R}$. Since $(A_X^s)^c \cap (A_Y^t)^c \subset (A_{X+Y}^{s+t})^c$, we have that $A_{X+Y}^{s+t} \subset A_X^s \cup A_Y^t$. Then

$$\begin{aligned} P(X + Y \geq s + t) &= P(A_{X+Y}^{s+t}) \\ &\leq P(A_X^s) + P(A_Y^t) \\ &= P(X \geq s) + P(Y \geq t) \end{aligned}$$

□

Exercise 5.1.2. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^+(\Omega, \mathcal{F}, P)$. Then for each $s, t \geq 0$,

$$P(XY \geq st) \leq P(X \geq s) + P(Y \geq t)$$

Proof. For $Z \in L^0_{\mathbb{R}}(\Omega, \mathcal{F}, P)$ and $t \in \mathbb{R}$, define $A_Z^t \in \mathcal{F}$ by

$$A_Z^t = \{\omega \in \Omega : Z(\omega) \geq t\}$$

Let $s, t \in \mathbb{R}$. Since $(A_X^s)^c \cap (A_Y^t)^c \subset (A_{XY}^{st})^c$, we have that $A_{XY}^{st} \subset A_X^s \cup A_Y^t$. Then

$$\begin{aligned} P(XY \geq st) &= P(A_{XY}^{st}) \\ &\leq P(A_X^s) + P(A_Y^t) \\ &= P(X \geq s) + P(Y \geq t) \end{aligned}$$

□

5.2. Sub α -Exponential Random Variables.

Definition 5.2.1. Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then X is said to be **sub α -exponential** if there exist $M, K > 0$ such that for each $t \geq 0$,

$$P(|X| \geq t) \leq Me^{-Kt^\alpha}$$

Exercise 5.2.2. Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then the following are equivalent:

- (1) X is sub α -exponential
- (2) there exists $K > 0$ such that for each $p \geq 1$, $\|X\|_p \leq Kp^{1/\alpha}$
- (3)
- (4)

Proof.

- (1) \implies (2):

Choose $C_\alpha > 0$ such that for each $x \geq \alpha^{-1}$, $\Gamma(x) \leq C_\alpha x^x$. Since X is sub α -exponential, there exist $M, K_0 > 0$ such that for each $t \geq 0$, $P(|X| \geq t) \leq Me^{-Kt^\alpha}$. Set $K = \max(M\alpha^{-1}C_\alpha, 1)2K_0^{-1/\alpha}\alpha^{-1/\alpha}$. Let $p \geq 1$. Then $p\alpha^{-1} \geq \alpha^{-1}$

$$\begin{aligned}
\|X\|_p^p &= E(|X|^p) \\
&= \int_0^\infty P(|X|^p \geq t) dt \\
&= \int_0^\infty P(|X| \geq t^{1/p}) dt \\
&\leq \int_0^\infty Me^{-K_0 t^{\alpha/p}} dt \\
&= Mp\alpha^{-1} \int_0^\infty u^{p/\alpha-1} e^{-K_0 u} du \\
&= Mp\alpha^{-1} \Gamma(p/\alpha) K_0^{-p/\alpha} \\
&\leq Mp\alpha^{-1} C_\alpha (p\alpha^{-1})^{p/\alpha} K_0^{-p/\alpha}
\end{aligned}$$

Therefore

$$\begin{aligned}
\|X\|_p &\leq (M\alpha^{-1}C_\alpha)^{1/p} p^{1/p} K_0^{-1/\alpha} \alpha^{-1/\alpha} p^{1/\alpha} \\
&\leq \max(M\alpha^{-1}C_\alpha, 1) 2K_0^{-1/\alpha} \alpha^{-1/\alpha} p^{1/\alpha} \\
&= K p^{1/\alpha}
\end{aligned}$$

- (2) \implies (3):

□

Definition 5.2.3. Let $\psi : [0, \infty) \rightarrow [0, \infty)$. Then ψ is said to be an **Orlicz function** if

- (1) ψ is convex
- (2) ψ is increasing
- (3) $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$
- (4) $\psi(0) = 0$

Definition 5.2.4. Let (Ω, \mathcal{F}, P) be a probability space and $\psi : [0, \infty) \rightarrow [0, \infty)$ and Orlicz function. We define the **Orlicz ψ -norm**, denoted $\|\cdot\|_\psi : L^0(\Omega, \mathcal{F}, P) \rightarrow [0, \infty]$, by

$$\|X\|_\psi = \inf\{t > 0 : E[\psi(|X|/t)] \leq 1\}$$

We define the **Orlicz ψ -space**, denoted $L^\psi(\Omega, \mathcal{F}, P)$, by

$$L^\psi(\Omega, \mathcal{F}, P) = \{X \in L^1(\Omega, \mathcal{F}, P) : \|X\|_\psi < \infty\}$$

Exercise 5.2.5. Let (Ω, \mathcal{F}, P) be a probability space and $\psi : [0, \infty) \rightarrow [0, \infty)$ an Orlicz function. Then L^ψ is a vector space and $\|\cdot\|_\psi : L^\psi \rightarrow [0, \infty)$ is a norm.

Hint: note that

- for $s, t > 0$,

$$\frac{|X|}{s+t} + \frac{|Y|}{s+t} = \frac{s}{s+t} \frac{|X|}{s} + \frac{t}{s+t} \frac{|Y|}{t}$$

- ψ is star-shaped, i.e. for each $x, t \geq 0$ and $f(tx) \leq tf(x)$.

Proof. For $X \in L^0(\Omega, \mathcal{F}, P)$, define $A_X \in \mathcal{F}$ by

$$A_X = \{t > 0 : E[\psi(|X|/t)] \leq 1\}$$

Let $X, Y \in L^\psi(\Omega, \mathcal{F}, P)$ and $\lambda \in \mathbb{C}$. Since $\|X\|_\psi < \infty$ and $\|Y\|_\psi < \infty$, we have that $A_X \neq \emptyset$ and $A_Y \neq \emptyset$,

- (1) **subadditivity:** Let $\epsilon > 0$. Then there exists $s \in A_X$ and $t \in A_Y$ such that $s < \inf A_X + \epsilon/2$ and $t < \inf A_Y + \epsilon/2$. Since ψ is convex and increasing, we have that

$$\begin{aligned} \psi\left(\frac{|X+Y|}{s+t}\right) &\leq \psi\left(\frac{|X|}{s+t} + \frac{|Y|}{s+t}\right) \\ &= \psi\left(\frac{s}{s+t} \frac{|X|}{s} + \frac{t}{s+t} \frac{|Y|}{t}\right) \\ &\leq \frac{s}{s+t} \psi\left(\frac{|X|}{s}\right) + \frac{t}{s+t} \psi\left(\frac{|Y|}{t}\right) \end{aligned}$$

Therefore,

$$\begin{aligned} E\left[\psi\left(\frac{|X+Y|}{s+t}\right)\right] &\leq \frac{s}{s+t} E\left[\psi\left(\frac{|X|}{s}\right)\right] + \frac{t}{s+t} E\left[\psi\left(\frac{|Y|}{t}\right)\right] \\ &\leq \frac{s}{s+t} + \frac{t}{s+t} \\ &\leq 1 \end{aligned}$$

Hence $s+t \in A_{X+Y}$. Thus $A_{X+Y} \neq \emptyset$. Since $s < \inf A_X + \epsilon/2$ and $t < \inf A_Y + \epsilon/2$, we have that

$$\begin{aligned} \|X+Y\|_\psi &= \inf A_{X+Y} \\ &\leq s+t \\ &< \inf A_X + \inf A_Y + \epsilon \\ &= \|X\|_\psi + \|Y\|_\psi + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|X+Y\|_\psi \leq \|X\|_\psi + \|Y\|_\psi$. So $X+Y \in L^\psi(\Omega, \mathcal{F}, P)$

- (2) **absolute homogeneity:** Suppose that $\lambda = 0$. Then for each $t > 0$,

$$\begin{aligned} E[\psi(|\lambda X|/t)] &= E[\psi(0)] \\ &= E[0] \\ &= 0 \\ &\leq 1 \end{aligned}$$

Thus

$$\begin{aligned} \|\lambda X\|_\psi &= 0 \\ &= |\lambda| \|X\|_\psi \end{aligned}$$

Suppose that $\lambda \neq 0$. Since for each $t > 0$,

$$\psi\left(\frac{|X|}{t}\right) = \psi\left(\frac{|\lambda X|}{|\lambda|t}\right)$$

we have that for each $t > 0$, $t \in A_X$ iff $|\lambda|t \in A_{\lambda X}$. Therefore $A_{\lambda X} = |\lambda|A_X$ and

$$\begin{aligned}\|\lambda X\|_\psi &= \inf A_{\lambda X} \\ &= |\lambda| \inf A_X \\ &= |\lambda| \|X\|_\psi\end{aligned}$$

So $\|\lambda X\| \in L^\psi(\Omega, \mathcal{F}, P)$.

- (3) **positive definiteness:** Note that since ψ is increasing, for each $t_0 > 0$, $t_0 \in A_X$ implies that $[t_0, \infty) \subset A_X$. Suppose that $\|X\|_\psi = 0$. Then $(0, \infty) \subset A_X$. Jensen's inequality implies that for each $t > 0$,

$$\begin{aligned}\psi\left(\frac{E|X|}{t}\right) &\leq E\left[\psi\left(\frac{|X|}{t}\right)\right] \\ &\leq 1\end{aligned}$$

For the sake of contradiction, suppose that $E|X| > 0$. Since $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$, Choose $x > 0$ such that $\psi(x) > 1$. Set $t = E|X|/x$. Then $t > 0$ and

$$\begin{aligned}1 &< \psi(x) \\ &= \psi\left(\frac{E|X|}{t}\right) \\ &\leq 1\end{aligned}$$

which is a contradiction. Hence $E|X| = 0$. Thus $X = 0$ a.s. □

Exercise 5.2.6. Let (Ω, \mathcal{F}, P) be a probability space and $\psi : [0, \infty) \rightarrow [0, \infty)$ an Orlicz function. Then $L^\psi(\Omega, \mathcal{F}, P)$ is a Banach space.

Proof. □

6. CONDITIONAL EXPECTATION AND PROBABILITY

6.1. Conditional Expectation.

Exercise 6.1.1. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -algebra of \mathcal{F} and $X \in L^1(\Omega, \mathcal{F}, P)$. Define $P_{\mathcal{G}} = P|_{\mathcal{G}}$ and $Q : \mathcal{G} \rightarrow [0, \infty)$ by $Q(G) = \int_G X dP$. Then $Q \ll P_{\mathcal{G}}$.

Proof. Let $G \in \mathcal{G}$. Suppose that $P_{\mathcal{G}}(G) = 0$. By definition, $P(G) = 0$. So $Q(G) = 0$ and $Q \ll P_{\mathcal{G}}$. \square

Definition 6.1.2. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -algebra of \mathcal{F} and $X, Y \in L^1(\Omega, \mathcal{F}, P)$. Then Y is said to be a **conditional expectation of X given \mathcal{G}** if

- (1) Y is \mathcal{G} -measurable
- (2) for each $G \in \mathcal{G}$,

$$\int_G Y dP = \int_G X dP$$

Since (2) implies that conditional expectations of X given \mathcal{G} are equal $P_{\mathcal{G}}$ -a.e., we write $Y = E(X|\mathcal{G})$.

Note 6.1.3. Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. We typically write $E(X|Y)$ instead of $E(X|Y^*\mathcal{S})$.

Exercise 6.1.4. Existence of Conditional Expectation:

Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -algebra of \mathcal{F} and $X \in L^1(\Omega, \mathcal{F}, P)$. Define Q and $P_{\mathcal{G}}$ as in the previous exercise. Define $Y = dQ/dP_{\mathcal{G}}$. Then Y is a conditional expectation of X given \mathcal{G} .

Proof. The Radon-Nikodym theorem implies that Y is \mathcal{G} -measurable. Since Q is finite, so is $|Q|$. Since $d|Q| = |Y| dP_{\mathcal{G}}$, we have that $Y \in L^1(\Omega, \mathcal{G}, P_{\mathcal{G}})$. An exercise in section 3.3 of [3], implies that for each $G \in \mathcal{G}$

$$\begin{aligned} \int_G Y dP &= \int_G Y dP_{\mathcal{G}} \\ &= Q(G) \\ &= \int_G X dP \end{aligned}$$

\square

Definition 6.1.5. Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. Let $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$. Then ϕ is said to be a **conditional expectation function of X given Y** if for each $B \in \mathcal{S} \cap Y(\Omega)$,

$$\int_{Y^{-1}(B)} X dP = \int_B \phi dP_Y$$

To denote this, we write $\phi(y) = E[X|Y = y]$.

Exercise 6.1.6. Existence of Conditional Expectation Function:

Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L^0_S(\Omega, \mathcal{F})$. Suppose that for each $y \in S$, $\{y\} \in \mathcal{S}$. Then there exists $\phi \in L^0(Y(S), \mathcal{S} \cap Y(\Omega))$ such that ϕ is a conditional expectation function of X given Y .

Hint: Doob-Dynkin lemma

Proof. Since $E[X|Y] \in L^0(\Omega, Y^*\mathcal{S})$, the Doob-Dynkin lemma implies that there exists $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$ such that $\phi \circ Y = E(X|Y)$. Let $B \in \mathcal{S} \cap Y(\Omega)$. Then

$$\begin{aligned} \int_B \phi dP_Y &= \int_{Y^{-1}(B)} \phi \circ Y dP \\ &= \int_{Y^{-1}(B)} E(X|Y) dP \\ &= \int_{Y^{-1}(B)} X dP \end{aligned}$$

□

6.2. Conditional Probability.

Definition 6.2.1. Let (A, \mathcal{A}) be a measurable space, (B, \mathcal{B}, P_Y) a probability space and $Q : B \times \mathcal{A} \rightarrow [0, 1]$. Then Q is said to be a **stochastic transition kernel from (B, \mathcal{B}, P) to (A, \mathcal{A})** if

- (1) for each $E \in \mathcal{A}$, $Q(\cdot, E)$ is \mathcal{B} -measurable
- (2) for P -a.e. $b \in B$, $Q(b, \cdot)$ is a probability measure on (A, \mathcal{A})

Definition 6.2.2. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0(\Omega, \mathcal{F}, P)$ and $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$. Then Q is said to be a **conditional probability distribution of X given Y** if

- (1) Q is a stochastic transition kernel from $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_Y)$ to $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$
- (2) for each $A, B \in \mathcal{F}$,

$$\int_B Q(y, A) dP_Y(y) = P(X \in A, Y \in B)$$

Note 6.2.3. It is helpful to connect this notion of conditional probability with the elementary one by writing $Q(y, A) = P(X \in A | Y = y)$. If $P_Y \ll \mu$, then property (2) in the definition becomes

$$\begin{aligned} P(X \in A, Y \in B) &= \int_B Q(y, A) dP_Y(y) \\ &= \int_B P(X \in A | Y = y) f_Y(y) d\mu(y) \end{aligned}$$

as in a first course on probability.

Exercise 6.2.4. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$. Suppose that for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable, for P_Y -a.e. $y \in \mathbb{R}^n$, $P_{X|Y}(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) and $Q(Y, A) = P(X \in A | Y)$ a.e. Then Q is a conditional probability of X given Y .

Proof. By assumption, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable and for P_Y -a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) . Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned}
 \int_B Q(y, A) dP_Y(y) &= \int_{Y^{-1}(B)} Q(Y(\omega), A) dP(\omega) \\
 &= \int_{Y^{-1}(B)} P(X \in A | Y) dP \\
 &= \int_{Y^{-1}(B)} E[1_{X^{-1}(A)} | Y] dP \\
 &= \int_{Y^{-1}(B)} 1_{X^{-1}(A)} dP \\
 &= \int 1_{X^{-1}(A)} 1_{Y^{-1}(B)} dP \\
 &= \int 1_{X^{-1}(A) \cap Y^{-1}(B)} dP \\
 &= P(X \in A, Y \in B)
 \end{aligned}$$

So Q is a conditional probability distribution of X given Y . \square

Definition 6.2.5. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and μ a σ -finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Then $P_{X,Y} \ll \mu^2$. Let $f_X = dP_X/d\mu$, $f_Y = dP_Y/d\mu$ and $f_{X,Y} = dP_{X,Y}/d\mu^2$. Define $f_{X|Y} : \mathbb{R}^n \times \mathbb{R}^n$ by

$$f_{X|Y}(x, y) = \begin{cases} \frac{f_{X,Y}(x, y)}{f_Y(y)}, & y \in \text{supp } Y \\ 0, & y \notin \text{supp } Y \end{cases}$$

Then $f_{X|Y}$ is called the **conditional probability density of X given Y** .

Exercise 6.2.6. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and μ a σ -finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Define $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$ by

$$Q(y, A) = \int_A f_{X|Y}(x, y) d\mu(x)$$

Then Q is a conditional probability distribution of X given Y .

Proof. By the Fubini-Tonelli Theorem, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable and for P_Y -a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) . Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned}
 \int_B Q(y, A) dP_Y(y) &= \int_B \left[\int_A f_{X|Y}(x, y) d\mu(x) \right] dP_Y(y) \\
 &= \int_{B \cap \text{supp } Y} \left[\int_{A \cap \text{supp } Y} \frac{f_{X,Y}(x, y)}{f_Y(y)} d\mu(x) f_Y(y) \right] d\mu(y) \\
 &= \int_{B \cap \text{supp } Y} \left[\int_A f_{X,Y}(x, y) d\mu(x) \right] d\mu(y) \\
 &= P(X \in A, Y \in B \cap \text{supp } Y) \\
 &= P(X \in A, Y \in B)
 \end{aligned}$$

□

Theorem 6.2.7. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^1(\Omega, \mathcal{F}, P)$. Suppose that $\text{Im } X \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a conditional probability distribution of Y given X .

7. MARKOV CHAINS

Definition 7.0.1. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}_0} \in L_n^0$. Then $(X_n)_{n \in \mathbb{N}_0}$ is said to be a **homogeneous Markov chain** if for each $A \in \mathcal{F}$ and $n \in \mathbb{N}$, $P(X_n \in A | X_1, \dots, X_{n-1}) = P(X_1 \in A | X_0)$ a.e.

8. PROBABILITIES INDUCED BY ISOMETRIC GROUP ACTIONS

8.1. Applications to Bayesian Statistics.

Exercise 8.1.1. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \rightarrow \Theta$ an isometric group action. Suppose that \bar{d} is a metric on Θ/G . Let

- H_p^Θ be the Hausdorff measure on Θ , $\mu_{\mathcal{X}}$ a measure on \mathcal{X} ,
- p a density on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on \mathcal{X} .
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose that μ_Θ is G -invariant, p is G -invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G -invariant and continuous on Θ . For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \dots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \rightarrow [0, 1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)}) dH_p^\Theta(\theta)$$

Then there exists a density $\bar{p}(\cdot|X^{(n)})$ on Θ/G such that

$$d\bar{P}_{\Theta|X^{(n)}}(\theta) = \bar{p}(\theta|X^{(n)}) d\bar{H}^\Theta(\theta)$$

Proof. Clear from previous work. □

Exercise 8.1.2. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \rightarrow \Theta$ an isometric group action. Suppose that \bar{d} is a metric on Θ/G . Let

- H_p^Θ be the Hausdorff measure on Θ , $\mu_{\mathcal{X}}$ a measure on \mathcal{X} ,
- p a density on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on \mathcal{X} .
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose p is G -invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G -invariant and continuous on Θ . For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \dots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \rightarrow [0, 1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)}) dH_p^\Theta(\theta)$$

Suppose that $(P_{\Theta|X^{(n)}})_{n \in \mathbb{N}}$ concentrates on $\bar{\theta}_0 \subset \Theta$ a.s. or in probability. Then $(\bar{P}_{\Theta|X^{(n)}})_{n \in \mathbb{N}}$ concentrates a.s. or in probability on $\{\bar{\theta}_0\} \subset \Theta/G$ (i.e. is consistent a.s. or in probability)

Proof. Let $V \in \mathcal{N}_{\bar{\theta}_0}$. Then $\pi^{-1}(V) \in \mathcal{N}_{\bar{\theta}_0}$. By definition,

$$\begin{aligned} \bar{P}_{\Theta|X^{(n)}}(V^c) &= P_{\Theta|X^{(n)}}(\pi^{-1}(V^c)) \\ &= P_{\Theta|X^{(n)}}(\pi^{-1}(V)^c) \\ &\xrightarrow{\text{a.s./}P} 0 \end{aligned}$$

□

Note 8.1.3. Some examples of G -invariant priors would be the uniform distribution, or $N_n(0, \sigma^2 I)$ on \mathbb{R}^n when acted on by $O(n)$. An example of a G -invariant likelihood would be $f(A|Z) \sim \text{Ber}(ZZ^T)$ as in a latent position random graph model where $Z \in \mathbb{R}^{n \times d}$ is the parameter is invariant under right multiplication by $U \in O_d$.

9. STOCHASTIC INTEGRATION

Exercise 9.0.1. Let (Ω, \mathcal{F}, P) be a probability space, X a set \mathcal{A}_0 an algebra, $\mu_0 : \mathcal{A}_0 \rightarrow \mathbb{C}$ and $B : \mathcal{A}_0 \rightarrow L^2(\Omega, \mathcal{F}, P)$. Suppose that

- (1) $B(\emptyset) = 0$
- (2) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $B(E \cup F) = B(E) + B(F)$
- (3) $E(B(E)B(F)^*) = \mu_0(E \cap F)$

Then

- (1) for each $E \in \mathcal{A}_0$, $\mu_0(E) = E(|B(E)|^2)$.
- (2) for each $E \in \mathcal{A}_0$, $0 \leq \mu_0(E) < \infty$
- (3) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$

Proof.

- (1) Clear
- (2) Clear
- (3) Let $E, F \in \mathcal{A}_0$. Suppose that $E \cap F = \emptyset$. Then

$$\begin{aligned}
 E(B(E)B(F)^*) &= \mu_0(E \cap F) \\
 &= \mu_0(\emptyset) \\
 &= E(|B(\emptyset)|^2) \\
 &= E(0) \\
 &= 0
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \mu_0(E \cup F) &= E(|B(E \cup F)|^2) \\
 &= E(|B(E) + B(F)|^2) \\
 &= E(|B(E)|^2) + E(|B(F)|^2) + 2\operatorname{Re}E(B(E)B(F)^*) \\
 &= \mu_0(E) + \mu_0(F) + 0 \\
 &= \mu_0(E) + \mu_0(F)
 \end{aligned}$$

□

Definition 9.0.2. Let (Ω, \mathcal{F}, P) be a probability space, X a set \mathcal{A}_0 an algebra, $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty)$ a premeasure and $B : \mathcal{A}_0 \rightarrow L^2(\Omega, \mathcal{F}, P)$. Suppose that

- (1) $B(\emptyset) = 0$
- (2) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $B(E \cup F) = B(E) + B(F)$
- (3) $E(B(E)B(F)^*) = \mu_0(E \cap F)$

Then B is said to be a **stochastic premeasure with structure** μ_0

10. TODO

Incorporate vector measures to have conditional expectations of Banach space valued functions given a σ -algebra

REFERENCES

- [1] [Introduction to Analysis](#)
- [2] [Introduction to Group Theory](#)
- [3] [Introduction to Measure and Integration](#)