

INTRODUCTION TO DIFFERENTIAL GEOMETRY

CARSON JAMES

CONTENTS

1. Review of Fundamentals	1
1.1. Set Theory	1
1.2. Smooth Maps	2
1.3. Topology	4
2. Multilinear Algebra	5
2.1. (r, s) -Tensors	5
2.2. k -Tensors	9
3. Manifolds	17
3.1. Smooth Manifolds	17
3.2. Smooth Maps	20
3.3. Partitions of Unity	24
3.4. The Tangent Space	25
3.5. The Cotangent Space	31
3.6. Maps of Full Rank	33
3.7. Submanifolds	34
4. Vector Bundles and Tensor Fields	37
4.1. The Vector Bundle	37
4.2. The cotangent Bundle	39
4.3. The (r, s) -Tensor Bundle	39
4.4. Vector Fields	40
4.5. 1-Forms	42
4.6. (r, s) -Tensor Fields	43
4.7. Differential Forms	46
5. Extra	51
5.1. Integration of Differential Forms	53

1. REVIEW OF FUNDAMENTALS

1.1. Set Theory.

Definition 1.1.1. Let $\{A_i\}_{i \in I}$ be a collection of sets. The **disjoint union** of $\{A_i\}_{i \in I}$, denoted $\coprod_{i \in I} A_i$, is defined by

$$\coprod_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$$

We define the **natural projection map**, denoted $\pi : \coprod_{i \in I} A_i \rightarrow I$, by $\pi(i, a) = i$.

Definition 1.1.2. Let $\{A_i\}_{i \in I}$ be a collection of sets and $\sigma : I \rightarrow \prod_{i \in I} A_i$. Then σ is said to be a **section of $\prod_{i \in I} A_i$** if

$$\pi \circ \sigma = \text{id}_I$$

Note 1.1.3. In these notes, we will identify $\{i\} \times A_i$ and A_i .

Exercise 1.1.4. Let $\{A_i\}_{i \in I}$ be a collection of sets and $\sigma : I \rightarrow \prod_{i \in I} A_i$. Then σ is a section of $\prod_{i \in I} A_i$ iff for each $i \in I$, $\sigma(i) \in A_i$

Proof. Clear. □

1.2. Smooth Maps.

Definition 1.2.1. Let $n \geq 1$. For $i = 1, \dots, n$, define $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$ by $x^i(a^1, \dots, a^n) = a^i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on \mathbb{R}^n** .

Definition 1.2.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to x^i at a** if

$$\lim_{h \rightarrow 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If f is differentiable with respect to x^i at a , we define the **partial derivative of f with respect to x^i at a** , denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \left. \frac{\partial}{\partial x^i} \right|_a f$$

to be the limit above.

Definition 1.2.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$. Then f is said to be **differentiable with respect to x^i** if for each $a \in U$, f is differentiable with respect to x^i at a .

Exercise 1.2.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i \partial x^j}$ and $\frac{\partial^2 f}{\partial x^j \partial x^i}$ exist and are continuous at a . Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(a) = \frac{\partial^2 f}{\partial x^j \partial x^i}(a)$$

Proof. □

Definition 1.2.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \dots, i_k \in \{1, \dots, n\}$, $\frac{\partial^k f}{\partial i_1 \dots i_k}$ exists and is continuous on U .

Definition 1.2.6. Let $U \subset \mathbb{R}^n$, $f : U \rightarrow \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \rightarrow \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Theorem 1.2.7. Taylor's Theorem:

Let $U \subset \mathbb{R}^n$ be open and convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that for each $x \in U$,

$$f(x) = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each $|\alpha| = T + 1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. See analysis notes □

Definition 1.2.8. Let $U \subset \mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^m$. Let x^1, \dots, x^n be the standard coordinate functions on \mathbb{R}^n and y_1, \dots, y_m be the standard coordinate functions on \mathbb{R}^m . For $i \in \{1, \dots, m\}$, we define the ***i*th component of F** , denoted $F^i : U \rightarrow \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \dots, F_m)$

Definition 1.2.9. Let $U \subset \mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \dots, m\}$, the *i*th component of F , $F^i : U \rightarrow \mathbb{R}$, is smooth.

Definition 1.2.10. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \rightarrow V$. Then F is said to be a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 1.2.11. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \rightarrow V$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F is a bijection and F and F^{-1} are smooth. Thus, F and F^{-1} are continuous and F is a homeomorphism. □

Definition 1.2.12. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \rightarrow \mathbb{R}^m$. We define the **Jacobian of F at p** , denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p) \right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Exercise 1.2.13. Inverse Function Theorem:

Let $U, V \subset \mathbb{R}^n$ be open and $F : U \rightarrow V$.

Exercise 1.2.14. Let $U, V \subset \mathbb{R}^n$ and $F : U \rightarrow V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \rightarrow F(N)$ is a diffeomorphism

Proof. content... □

1.3. Topology.

Definition 1.3.1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \rightarrow Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 1.3.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \rightarrow Y$. Then f is said to be a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Definition 1.3.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \rightarrow Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 1.3.4. Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \not\cong \mathbb{R}^n$

2. MULTILINEAR ALGEBRA

2.1. (r, s) -Tensors.

Definition 2.1.1. Let V_1, \dots, V_k, W be vector spaces and $\alpha : \prod_{i=1}^n V_i \rightarrow W$. Then α is said to be **multilinear** if for each $i \in \{1, \dots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \dots, v_k \in V$,

$$\alpha(v_1, \dots, v_i + cv, \dots, v_k) = \alpha(v_1, \dots, v_i, \dots, v_k) + c\alpha(v_1, \dots, v, \dots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{ \alpha : \prod_{i=1}^n V_i \rightarrow W : \alpha \text{ is multilinear} \right\}$$

Note 2.1.2. For the remainder of this section we let V denote an n -dimensional vector space with basis $\{e^1, \dots, e^n\}$ with dual space V^* and dual basis $\{\epsilon_1, \dots, \epsilon_n\}$ defined by $\epsilon^i(e^j) = \delta_{i,j}$. We identify V with V^{**} by the isomorphism $V \rightarrow V^{**}$ defined by $v \mapsto \hat{v}$ where $\hat{v}(\alpha) = \alpha(v)$ for each $\alpha \in V^*$.

Definition 2.1.3. Let $\alpha : (V^*)^r \times V^s \rightarrow \mathbb{R}$. Then α is said to be an (r, s) -tensor on V if $\alpha \in L(\underbrace{V^*, \dots, V^*}_r, \underbrace{V, \dots, V}_s; \mathbb{R})$. The set of all (r, s) -tensors on V is denoted $T_s^r(V)$.

When $r = s = 0$, we set $T_s^r = \mathbb{R}$.

Exercise 2.1.4. We have that $T_s^r(V)$ is a vector space.

Proof. Clear. □

Exercise 2.1.5. Under the identification of V with V^{**} as noted above, we have that $V = T_0^1(V)$.

Proof. By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T_0^1(V) \end{aligned}$$

□

Definition 2.1.6. Let $\alpha \in T_{s_1}^{r_1}(V)$ and $\beta \in T_{s_2}^{r_2}(V)$. We define the **tensor product of α with β** , denoted $\alpha \otimes \beta \in T_{s_1+s_2}^{r_1+r_2}(V)$, by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$.

When $r_1 = s_1 = r_2 = s_2 = 0$ (so that $\alpha, \beta \in \mathbb{R}$), we set $\alpha \otimes \beta = \alpha\beta$.

Definition 2.1.7. We define the **tensor product**, denoted $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

Exercise 2.1.8. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is well defined.

Proof. Tedious but straightforward. □

Exercise 2.1.9. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is associative.

Proof. Let $\alpha \in T_{s_1}^{r_1}(V)$, $\beta \in T_{s_2}^{r_2}(V)$ and $\gamma \in T_{s_3}^{r_3}(V)$. Then for each $u^* \in (V^*)^{r_1}$, $v^* \in (V^*)^{r_2}$, $w^* \in (V^*)^{r_3}$, $u \in V^{s_1}$, $v \in V^{s_2}$, $w \in V^{s_3}$,

$$\begin{aligned} (\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v) \gamma(w^*, w) \\ &= [\alpha(u^*, u) \beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(u^*, u) [\beta(v^*, v) \gamma(w^*, w)] \\ &= \alpha(u^*, u) (\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w) \end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

Exercise 2.1.10. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is bilinear.

Proof.

(1) Linearity in the first argument:

Let $\alpha, \beta \in T_{s_1}^{r_1}(V)$, $\gamma \in T_{s_2}^{r_2}(V)$, $\lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned} [(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v) \gamma(w^*, w) \\ &= [\alpha(v^*, v) + \lambda\beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(v^*, v) \gamma(w^*, w) + \lambda\beta(v^*, v) \gamma(w^*, w) \\ &= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w) \end{aligned}$$

So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

(2) Linearity in the second argument:

Similar to (1).

□

Definition 2.1.11.

- (1) Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1, \dots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length k** . Recall that $\#\mathcal{I}_{\otimes k} = n^k$.
- (2) Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length k** . Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

Note 2.1.12. For the remainder of this section we will write \mathcal{I}_k in place of $\mathcal{I}_{\otimes k}$.

Definition 2.1.13. Let $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k\}$.

- (1) Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by

$$\epsilon^I = (\epsilon^{i_1}, \dots, \epsilon^{i_k})$$

and

$$e^I = (e^{i_1}, \dots, e^{i_k})$$

(2) Define $e^{\otimes I} \in T_0^k(V)$ and $\epsilon^{\otimes I} \in T_k^0(V)$ by

$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$

and

$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

Exercise 2.1.14. Let $\alpha, \beta \in T_s^r(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v_1^*, \dots, v_r^* \in V^*$ and $v_1, \dots, v_s \in V$. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, write

$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$

and

$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$

Then

$$\begin{aligned} \alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s) \end{aligned}$$

So that $\alpha = \beta$. □

Exercise 2.1.15. Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K} \delta_{J,L}$.

Proof. Write $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$ and $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$. Then

$$\begin{aligned} e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K) \epsilon^{\otimes J}(e^L) \\ &= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r}) \epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\ &= \left[\prod_{m=1}^r e^{i_m}(\epsilon^{k_m}) \right] \left[\prod_{n=1}^s \epsilon^{j_n}(e^{l_n}) \right] \\ &= \left[\prod_{m=1}^r \delta_{i_m, k_m} \right] \left[\prod_{n=1}^s \delta_{j_n, l_n} \right] \\ &= \delta_{I,K} \delta_{J,L} \end{aligned}$$

□

Exercise 2.1.16. The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T_s^r(V)$ and $\dim T_s^r(V) = n^{r+s}$.

Proof. Let $(a_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a_J^I e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a_J^I = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T_s^r(V)$. For $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b_J^I = \beta(\epsilon^J, e^I)$. Define

$\mu = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b_J^I e^{\otimes I} \otimes \epsilon^{\otimes J} \in T_s^r(V)$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b_J^I = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. \square

2.2. k -Tensors.

Definition 2.2.1. Let $\alpha : V^k \rightarrow \mathbb{R}$. Then α is said to be a **k -tensor on V** if $\alpha \in T_k^0(V)$. We will write $T_k(V)$ in place of $T_k^0(V)$.

Definition 2.2.2. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \rightarrow \mathbb{R}$ by

$$\sigma\alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

The map $\alpha \mapsto \sigma\alpha$ is called the **permutation action** of S_k on $T_k(V)$

Exercise 2.2.3. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

- (1) Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V)$, $\sigma\alpha \in T_k(V)$.
- (2) Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
- (3) Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \dots, v_k \in V$,

$$\begin{aligned} (\tau\sigma)\alpha(v_1, \dots, v_k) &= \alpha(v_{\tau\sigma(1)}, \dots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \dots, v_k) \end{aligned}$$

□

Exercise 2.2.4. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \rightarrow T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \dots, v_k \in V$. Then

$$\begin{aligned} \sigma(c\alpha + \beta)(v_1, \dots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \dots, v_k) + \sigma\beta(v_1, \dots, v_k) \end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

Definition 2.2.5. Let $\alpha \in T_k(V)$. Then α is said to be **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$. and α is said to be **alternating** if for each $\sigma \in S_k$, $\sigma\alpha = \text{sgn}(\sigma)\alpha$. The set of symmetric k -tensors on V is denoted $\Xi_k(V)$ and the set of alternating k -tensors on V is denoted $\Lambda_k(V)$.

Definition 2.2.6. Define the **symmetric operator** $S : T_k(V) \rightarrow \Xi_k(V)$ by

$$S(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \rightarrow \Lambda_k(V)$ by

$$A(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma\alpha$$

Exercise 2.2.7.

- (1) For $\alpha \in T_k(V)$, $S(\alpha)$ is symmetric.
- (2) For $\alpha \in T_k(V)$, $A(\alpha)$ is alternating.

Proof.

(1) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}\sigma S(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \tau \alpha \right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma \tau \alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \tau \alpha \\ &= S(\alpha)\end{aligned}$$

(2) Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}\sigma A(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \sigma \tau \alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\sigma) \text{sgn}(\sigma \tau) \sigma \tau \alpha \\ &= \text{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\sigma \tau) \sigma \tau \alpha \\ &= \text{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \\ &= \text{sgn}(\sigma) A(\alpha)\end{aligned}$$

□

Exercise 2.2.8.

- (1) For $\alpha \in \Xi_k(V)$, $S(\alpha) = \alpha$.
- (2) For $\alpha \in \Lambda_k(V)$, $A(\alpha) = \alpha$.

Proof.

(1) Let $\alpha \in \Xi_k(V)$. Then

$$\begin{aligned}S(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \alpha \\ &= \alpha\end{aligned}$$

(2) Let $\alpha \in \Lambda_k(V)$. Then

$$\begin{aligned} A(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma \alpha \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)^2 \alpha \\ &= \alpha \end{aligned}$$

□

Exercise 2.2.9. The symmetric operator $S : T_k(V) \rightarrow \Xi_k(V)$ and the alternating operator $A : T_k(V) \rightarrow \Lambda_k(V)$ are linear.

Proof. Clear.

□

Definition 2.2.10. Let $\alpha \in \Lambda_k(V)$ and $\beta \in \Lambda_l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda_{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$.

Exercise 2.2.11. The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$ is bilinear.

Proof. Clear.

□

Exercise 2.2.12. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

- (1) $A(A(\alpha) \otimes \beta) = A(\alpha \otimes \beta)$
- (2) $A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+l}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \rightarrow S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

(1) Then

$$\begin{aligned}
A(A(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[A(\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) (\tau \alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau (\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \text{sgn}(\sigma\tau) \sigma\tau (\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= A(\alpha \otimes \beta)
\end{aligned}$$

(2) Similar to (1).

□

Exercise 2.2.13. The exterior product $\wedge : \Lambda_k(V) \times \Lambda_l(V) \rightarrow \Lambda_{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$ and $\gamma \in \Lambda_m(V)$. Then

$$\begin{aligned}
 (\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \right] \wedge \gamma \\
 &= \frac{(k+l+m)!}{(k+l)!m!} A \left(\left[\frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \right] \otimes \gamma \right) \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} A(A(\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} A((\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes (\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes A(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes \frac{(l+m)!}{l!m!} A(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes (\beta \wedge \gamma)) \\
 &= \alpha \wedge (\beta \wedge \gamma)
 \end{aligned}$$

□

Exercise 2.2.14. Let $\alpha_i \in \Lambda_{k_i}(V)$ for $i = 1, \dots, m$. Then

$$\bigwedge_{i=1}^m \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} A \left(\bigotimes_{i=1}^m \alpha_i \right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} A(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i \right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A \left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)! k_{m_0}! k_{m_0+1}!} A \left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} A \left(\bigotimes_{i=1}^{m_0-1} \alpha_i \right) \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left(A \left[\bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} A \left(\bigotimes_{i=1}^{m_0+1} \alpha_i \right)
\end{aligned}$$

□

Exercise 2.2.15. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl . (Hint: inversion number)

Proof.

$$\begin{aligned}
N(\tau) &= \sum_{i=1}^l k \\
&= kl
\end{aligned}$$

Since $\text{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\text{sgn}(\tau) = (-1)^{kl}$.

□

Exercise 2.2.16. Let $\alpha \in \Lambda_k(V)$, $\beta \in \Lambda_l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \dots, v_{k+l} \in V$, we have that

$$\begin{aligned}
\sigma\tau(\beta \otimes \alpha)(v_1, \dots, v_l, v_{l+1}, \dots, v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}) \alpha(v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\
&= \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\
&= \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\
&= \alpha \otimes \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}, v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\
&= \sigma(\alpha \otimes \beta)(v_1, \dots, v_k, v_{1+k}, \dots, v_{l+k})
\end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned}
 \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} A(\beta \otimes \alpha) \\
 &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\beta \otimes \alpha) \\
 &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau) \sigma\tau(\beta \otimes \alpha) \\
 &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha \otimes \beta) \\
 &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \\
 &= \text{sgn}(\tau) \alpha \wedge \beta \\
 &= (-1)^{kl} \alpha \wedge \beta
 \end{aligned}$$

□

Exercise 2.2.17. Let $\alpha \in \Lambda_k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned}
 \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\
 &= -\alpha \wedge \alpha
 \end{aligned}$$

Thus $\alpha \wedge \alpha = 0$.

□

Exercise 2.2.18. Fundamental Example:

Let $\alpha_1, \dots, \alpha_m \in \Lambda_1(V)$ and $v_1, \dots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}
 \left(\bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) &= m! A \left(\bigotimes_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) \\
 &= m! \left[\frac{1}{m!} \sum_{\sigma \in S_m} \text{sgn}(\sigma) \sigma \left(\bigotimes_{i=1}^m \alpha_i \right) \right] (v_1, \dots, v_m) \\
 &= \sum_{\sigma \in S_m} \text{sgn}(\sigma) \left(\bigotimes_{i=1}^m \alpha_i \right) (v_{\sigma(1)}, \dots, v_{\sigma(m)}) \\
 &= \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
 &= \det(\alpha_i(v_j))
 \end{aligned}$$

□

Note 2.2.19. Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write \mathcal{I}_k in place of $\mathcal{I}_{\wedge k}$.

Definition 2.2.20. Let $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_k$.

Define $\epsilon^{\wedge I} \in \Lambda_k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \dots \wedge \epsilon^{i_k}$$

Exercise 2.2.21. Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \dots & \epsilon^{i_1}(e^{j_k}) \\ \vdots & & \vdots \\ \epsilon^{i_k}(e^{j_1}) & \dots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$.

If $I = J$, then $A = I_{k \times k}$ and therefore $\epsilon^{\wedge I}(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the l_0 th row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the l_0 th column of A are 0. \square

Exercise 2.2.22. Let $\alpha, \beta \in \Lambda_k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \dots, v_k \in V$. For $i = 1, \dots, k$, write $v_i = \sum_{j_i=1}^n a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned} \alpha(v_1, \dots, v_k) &= \sum_{j_1, \dots, j_k=1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\ &= \sum_{j_1 \neq \dots \neq j_k}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\ &= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \text{sgn}(\sigma) \left(\prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \alpha(e^J) \\ &= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \text{sgn}(\sigma) \left(\prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \beta(e^J) \\ &= \sum_{j_1, \dots, j_k=1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \beta(e^{j_1}, \dots, e^{j_k}) \\ &= \beta(v_1, \dots, v_k) \end{aligned}$$

\square

Exercise 2.2.23. The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(V)$ and $\dim \Lambda_k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$. Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda_k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda_k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$. \square

3. MANIFOLDS

3.1. Smooth Manifolds.

Definition 3.1.1. Define the **upper half space** of \mathbb{R}^n , denoted \mathbb{H}_n , by

$$\mathbb{H}_n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and define

$$\begin{aligned}\partial\mathbb{H}_n &= \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n = 0\} \\ (\mathbb{H}_n)^\circ &= \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}\end{aligned}$$

Definition 3.1.2. Let M be a topological space and $n \geq 1$.

- (1) Let $U \subset M$ and $V \subset \mathbb{H}^n$ be open and $\phi : U \rightarrow V$. Then (U, ϕ) is said to be a **coordinate chart** on M if ϕ is a homeomorphism.
- (2) Let \mathcal{A} be a collection of coordinate charts on M . Then \mathcal{A} is said to be an **atlas** on M if $\bigcup_{(U, \phi) \in \mathcal{A}} U = M$.
- (3) The space M is said to be **locally half Euclidean of dimension n** if there exists an atlas \mathcal{A} on M such that for each $(U, \phi) \in \mathcal{A}$, $\phi(U) \subset \mathbb{H}^n$.
- (4) The space M is said to be an **n -dimensional manifold** if M is Hausdorff, second countable and locally half Euclidean of dimension n .

Note 3.1.3. For the remainder of this section, we assume M is an n -dimensional manifold.

Definition 3.1.4.

- (1) Define the **boundary** of M , denoted ∂M , by
$$\partial M = \{p \in M : \text{there exists a chart } (U, \phi) \text{ on } M \text{ such that } p \in U \text{ and } \phi(p) \in \partial\mathbb{H}^n\}$$
- (2) Define the **interior** of M , denoted M° , by

$$M^\circ = M \setminus \partial M$$

Exercise 3.1.5. Let $p \in M$. Then $p \in \partial M$ iff for each chart (U, ϕ) on M , $p \in U$ implies that $\phi(p) \in \partial\mathbb{H}^n$. (Hint: simply connected)

Proof. Suppose that $p \in \partial M$. Then there exists a coordinate chart (V, ψ) on M such that $\psi(p) \in \partial\mathbb{H}^n$. Let (U, ϕ) be a coordinate chart on M . Suppose that $p \in U$. Note that $\phi \circ \psi^{-1} : \psi(V \cap U) \rightarrow \phi(V \cap U)$ is a homeomorphism. Choose open n -balls $B_\phi, B_\psi \subset \mathbb{H}^n$ such that $B_\phi \subset \phi(V \cap U)$, $B_\psi \subset \psi(V \cap U)$, $\phi(p) \in B_\phi$ and $\psi(p) \in B_\psi$. For the sake of contradiction, suppose that $\phi(p) \notin \partial\mathbb{H}^n$. Put $U' = B_\phi \setminus \{\phi(p)\}$ and $V' = B_\psi \setminus \{\psi(p)\}$. Define $\lambda : V' \rightarrow U'$ by $\lambda = \phi \circ \psi|_{V'}$. Then λ is a homeomorphism. Note that V' is simply connected and U' is not. This is a contradiction. \square

Exercise 3.1.6. If $\partial M \neq \emptyset$, then

- (1) ∂M is an $n - 1$ -dimensional manifold
- (2) $\partial(\partial M) = \emptyset$.

Proof.

- (1) Since subspaces of Hausdorff, second countable spaces are Hausdorff and second countable, we need only show that ∂M is locally half euclidean of dimension $n - 1$. Let $p \in \partial M$. Then there exists a coordinate chart (U, ϕ) on M such that $p \in U$ and $\phi(p) \in \partial\mathbb{H}^n$.

Put $U' = U \cap \partial M$. Note that U' is open in ∂M and $\phi(U) \cap \partial \mathbb{H}^n$ is open in $\partial \mathbb{H}^n$. Define $\phi' : U' \rightarrow \phi(U) \cap \partial \mathbb{H}^n$ by $\phi' = \phi|_{U'}$. Then ϕ' is a homeomorphism.

Since $\partial \mathbb{H}^n$ is homeomorphic to \mathbb{R}^{n-1} which is homeomorphic to $(\mathbb{H}^{n-1})^\circ$ there exists $\psi : \partial \mathbb{H}^n \rightarrow (\mathbb{H}^{n-1})^\circ$ such that ψ is a homeomorphism.

Define $V' = \psi(\phi(U) \cap \partial \mathbb{H}^n)$ and $\psi' : \phi(U) \cap \partial \mathbb{H}^n \rightarrow V'$ by $\psi' = \psi|_{\phi(U) \cap \partial \mathbb{H}^n}$. Then V' is open in $(\mathbb{H}^{n-1})^\circ$ and ψ' is a homeomorphism.

Define $\lambda : U' \rightarrow V'$ by $\lambda = \psi' \circ \phi'$. Then λ is a homeomorphism and (U', λ) is a coordinate chart on ∂M . So ∂M is locally Euclidean of dimension $n - 1$.

- (2) Let $p \in \partial M$. Define $(U \cap \partial M, \lambda \circ \psi)$ as in (1). Since $\lambda \circ \psi(p) \in (\mathbb{H}^{n-1})^\circ$, we have that $p \in M^\circ$. Thus $\partial M = (\partial M)^\circ$ and $\partial(\partial M) = \emptyset$.

□

Theorem 3.1.7. Let (M, \mathcal{A}) be an m -dimensional manifold, (N, \mathcal{B}) a n -dimensional manifold and $F : M \rightarrow N$. If F is a homeomorphism, then $m = n$.

Definition 3.1.8.

- (1) Let $(U, \phi), (V, \psi)$ be coordinate charts on M . Then (U, ϕ) and (V, ψ) are said to be **smoothly compatible** if

$$\phi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \phi(U \cap V) \text{ is a diffeomorphism}$$

- (2) Let $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$ be an atlas on M . Then \mathcal{A} is said to be **smooth** if for each $a, b \in A$, (U_a, ϕ_a) and (U_b, ϕ_b) are smoothly compatible.
- (3) Let \mathcal{A} be a smooth atlas on M . Then \mathcal{A} is said to be **maximal** if for each smooth atlas \mathcal{B} on M , $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on M** .
- (4) Let \mathcal{A} be a smooth structure on M . Then (M, \mathcal{A}) is said to be a **smooth n -dimensional manifold**.

Exercise 3.1.9. Let \mathcal{B} be a smooth atlas on M . Then there exists a unique smooth structure \mathcal{A} on M such that $\mathcal{B} \subset \mathcal{A}$.

Proof. Define \mathcal{A} to be the set of all coordinate charts (U, ϕ) on M such that for each coordinate chart $(V, \psi) \in \mathcal{B}$, (U, ϕ) and (V, ψ) are smoothly compatible.

Clearly $\mathcal{B} \subset \mathcal{A}$.

Let $(U, \phi), (V, \psi) \in \mathcal{A}$ and $p \in U \cap V$. Then there exists $(W, \chi) \in \mathcal{B}$ such that $p \in W$. By assumption, $\phi \circ \chi^{-1} : \chi(U \cap W) \rightarrow \phi(U \cap W)$ and $\chi \circ \psi^{-1} : \psi(W \cap V) \rightarrow \chi(W \cap V)$ are diffeomorphisms. Then $(\phi \circ \chi^{-1}) \circ (\chi \circ \psi^{-1}) = \phi \circ \psi^{-1} : \psi(U \cap W \cap V) \rightarrow \phi(U \cap W \cap V)$ is a diffeomorphism. Since for each $q \in \psi(U \cap V)$, there exists an open neighborhood $N \subset \psi(U \cap V)$ of q on which $\phi \circ \psi^{-1}$ are diffeomorphic, we have that $\phi \circ \psi^{-1}$ is a diffeomorphism on $\psi(U \cap V)$ and therefore (U, ϕ) and (V, ψ) are smoothly compatible. Hence \mathcal{A} is a smooth atlas.

To see that \mathcal{A} is maximal, let \mathcal{B}' be a smooth atlas on M . Suppose that $\mathcal{A} \subset \mathcal{B}'$ and let $(U, \phi) \in \mathcal{B}'$. By definition, for each chart $(V, \psi) \in \mathcal{B}'$, (U, ϕ) and (V, ψ) are smoothly compatible. Since $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$, we have that $(U, \phi) \in \mathcal{A}$. So $\mathcal{A} = \mathcal{B}'$ and \mathcal{A} is a maximal smooth atlas on M . □

Exercise 3.1.10. Let \mathcal{A} be a smooth atlas on M . Define $\lambda : \partial \mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ by $\lambda(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$. Put $\mathcal{A}|_{\partial M} = \{(U \cap \partial M, \lambda \circ \phi|_{U \cap \partial M}) : (U, \phi) \in \mathcal{A}\}$. Then

- (1) $\mathcal{A}|_{\partial M}$ is a smooth atlas on ∂M .
- (2) if \mathcal{A} is maximal, then $\mathcal{A}|_{\partial M}$ is maximal.

Proof. □

Note 3.1.11. For the rest of this section, we assume that (M, \mathcal{A}) is a smooth n -dimensional manifold and we denote the standard coordinate functions on \mathbb{R}^n by u^1, \dots, u^n . For a coordinate chart $(U, \phi) \in \mathcal{A}$ and $i \in \{1, \dots, n\}$, we will typically denote the i th coordinate of ϕ by x^i , that is, $x^i = u^i(\phi)$.

Exercise 3.1.12. Let (M, \mathcal{A}) be a smooth n -dimensional manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $U' \subset S$ and $\phi' : \tilde{U} \rightarrow \phi(U')$ by $U' = U \cap S$ and $\phi' = \phi|_{U \cap S}$. Set $\mathcal{A}' = \{(U', \phi') : (U, \phi) \in \mathcal{A}\}$. Then (S, \mathcal{A}') is a smooth n -dimensional manifold.

Proof. Since M is Hausdorff and second countable, so is S . Clearly $S = \bigcup_{(U', \phi') \in \mathcal{A}'} U'$ and for each $(U', \phi') \in \mathcal{A}'$, ϕ' is a homeomorphism and $\phi'(U') \subset \mathbb{H}^n$. Let $(U', \phi'), (V', \psi') \in \mathcal{A}'$. Then there exist $(U, \phi), (V, \psi) \in \mathcal{A}$ such that $U' = U \cap S$, $V' = V \cap S$. Then $\psi' \circ \phi'^{-1} : \phi(U \cap V \cap S) \rightarrow \psi(U \cap V \cap S)$ is given by

$$\psi' \circ \phi'^{-1} : \phi(U' \cap V') \rightarrow \psi(U' \cap V') = \psi \circ \phi^{-1} : \phi(U \cap V \cap S) \rightarrow \psi(U \cap V \cap S)$$

which is smooth. So \mathcal{A}' is a smooth atlas. Let \mathcal{B}' be a smooth atlas on S . Suppose that $\mathcal{A}' \subset \mathcal{B}'$. □

3.2. Smooth Maps.

Definition 3.2.1. Let $f : M \rightarrow \mathbb{R}$. Then f is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M)$.

Exercise 3.2.2. We have that $C^\infty(M)$ is a vector space.

Proof. Clear. □

Definition 3.2.3. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$. We define

$$C^\infty(U) = \{f : U \rightarrow \mathbb{R} : f \circ \phi^{-1} \in C^\infty(\phi(U))\}$$

Note 3.2.4. Later we will give a subset $S \subset M$ the structure of a smooth manifold such that Definition 3.2.1 and Definition 3.2.3 coincide.

Exercise 3.2.5. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $p \in U$ and $f \in C^\infty(M)$. Then $f|_U \in C^\infty(U)$.

Proof. **FINISH!!!** □

Definition 3.2.6. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \dots, n\}$. We define the **partial derivative of f with respect to x^i** , denoted

$$\partial f / \partial x^i : U \rightarrow \mathbb{R} \text{ or } \partial_i f : U \rightarrow \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left(\frac{\partial}{\partial u^i}[f \circ \phi^{-1}] \right) \circ \phi$$

Exercise 3.2.7. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \dots, n\}$. Then $\partial / \partial x^i : C^\infty(U) \rightarrow C^\infty(U)$ is linear.

Proof. **FINISH!!!** □

Exercise 3.2.8. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i, j \in \{1, \dots, n\}$. Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j}[f \circ \phi^{-1}] \right) \circ \phi$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} f \right) \\
&= \frac{\partial}{\partial x^i} \left(\left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \circ \phi \right) \\
&= \left(\frac{\partial}{\partial u^i} \left[\left(\left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \circ \phi \right) \circ \phi^{-1} \right] \right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i} \left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi
\end{aligned}$$

□

Exercise 3.2.9. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $i, j \in \{1, \dots, n\}$. Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i}$$

Proof. Let $f \in C^\infty(U)$. Since $f \circ \phi^{-1}$ is smooth,

$$\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] = \frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i} [f \circ \phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned}
\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i} [f \circ \phi^{-1}] \right) \circ \phi \\
&= \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f
\end{aligned}$$

□

Exercise 3.2.10. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $f \in C^\infty(U)$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha f = (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi$$

Proof. The claim is clearly true when $|\alpha| = 0$ or by definition if $|\alpha| = 1$. Let $n \in \mathbb{N}$ and suppose the claim is true for each $|\alpha| \in \{1, \dots, n-1\}$. Then there exists $i \in \{1, \dots, n\}$ such

that $\alpha_i \geq 1$. Hence

$$\begin{aligned}
\partial^\alpha f &= \partial^{e^i}(\partial^{\alpha-e^i} f) \\
&= \partial^{e^i}(\partial^{\alpha-e^i}[f \circ \phi^{-1}] \circ \phi) \\
&= (\partial^{e^i}[(\partial^{\alpha-e^i}[f \circ \phi^{-1}] \circ \phi) \circ \phi^{-1}]) \circ \phi \\
&= (\partial^{e^i}[\partial^{\alpha-e^i}[f \circ \phi^{-1}]]) \circ \phi \\
&= (\partial^\alpha[f \circ \phi^{-1}]) \circ \phi
\end{aligned}$$

□

Exercise 3.2.11. Taylor's Theorem:

Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\phi(U)$ convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that

$$f = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(x_0) \right] + \sum_{|\alpha|=T+1} (x^i - x^i(p))^\alpha g_\alpha$$

and for each $|\alpha| = T+1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. Since $\phi(U)$ is open and convex and $f \circ \phi^{-1} \in C^\infty(\phi(U))$, Taylors thorem in section 2.1 implies that there exist $(\tilde{g}_\alpha)_{|\alpha|=T+1} \subset C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q))$$

and for each $|\alpha| = T+1$,

$$\begin{aligned}
\tilde{g}_\alpha(\phi(p)) &= \frac{1}{(T+1)!} \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \\
&= \frac{1}{(T+1)!} \partial^\alpha f(p)
\end{aligned}$$

For $|\alpha| = T+1$, set $g_\alpha = \tilde{g}_\alpha \circ \phi$. Then

$$\begin{aligned}
f(q) &= f \circ \phi^{-1}(\phi(q)) \\
&= \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\
&= \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q)
\end{aligned}$$

□

Definition 3.2.12. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$. Then F is said to be

- **smooth** if for each $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \psi(F(U) \cap V)$$

is smooth

- a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 3.2.13. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifold and $F : M \rightarrow N$. If F is smooth, then F is continuous.

Proof. Suppose that F is smooth. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $F(p) \in V$. Put $\tilde{U} = U \cap F^{-1}(V)$ and $\tilde{V} = F(U) \cap V$.

Define $\tilde{\phi} : \tilde{U} \rightarrow \phi(\tilde{U})$ and $\tilde{\psi} : \tilde{V} \rightarrow \psi(\tilde{V})$ by

$$\tilde{\phi} = \phi|_{\tilde{U}}, \quad \tilde{\psi} = \psi|_{\tilde{V}}$$

Then $\tilde{\phi}$ and $\tilde{\psi}$ are homeomorphisms, $p \in \tilde{U}$ and $F(\tilde{U}) \subset \tilde{V}$. Define $\tilde{F} : \phi(\tilde{U}) \rightarrow \psi(\tilde{V})$ by

$$\tilde{F} = \tilde{\psi} \circ F \circ \tilde{\phi}^{-1}$$

By definition, \tilde{F} is smooth and therefore continuous. Since ϕ and ψ are homeomorphisms and $F|_{\tilde{U}} = \tilde{\psi}^{-1} \circ \tilde{F} \circ \tilde{\phi}$, we have that $F|_{\tilde{U}}$ is continuous. In particular, F is continuous at p and since $p \in M$ is arbitrary, F is continuous. \square

Exercise 3.2.14. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifold and $F : M \rightarrow N$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F and F^{-1} are smooth. The previous exercise implies that F and F^{-1} are continuous. Hence F is a homeomorphism. \square

Exercise 3.2.15. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$ a diffeomorphism. Then for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

Proof. Let $(V, \psi) \in \mathcal{B}$.

- (1) Since ϕ and F^{-1} are homeomorphisms, $\phi \circ F^{-1} : F(U) \cap V \rightarrow \phi(U \cap F^{-1}(V))$ is a homeomorphism
- (2) Since F is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \rightarrow \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \psi(V \cap F(U))$$

are smooth.

Therefore $(F(U), \phi \circ F^{-1})$ and (V, ψ) are smoothly compatible. Since \mathcal{B} is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. \square

Definition 3.2.16. Let (N, \mathcal{B}) be a smooth n -dimensional manifold, $F : M \rightarrow N$ smooth and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. For $i \in \{1, \dots, n\}$, We define the **i -th component of F with respect to (V, ψ)** , denoted $F^i : V \rightarrow \mathbb{R}$, by

$$F^i = y^i \circ F$$

3.3. Partitions of Unity.

Definition 3.3.1. Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then ρ is said to be a **bump function at p supported in U** if

- (1) $\rho \geq 0$
- (2) there exists $V \in \mathcal{N}_p$ such that V is open and $\rho|_V = 1$
- (3) $\text{supp } \rho \subset U$

Exercise 3.3.2. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

Proof.

□

3.4. The Tangent Space.

Definition 3.4.1. Let $p \in M$. Define the relation \sim_p on $C^\infty(M)$ by $f \sim_p g$ iff there exists $U \in \mathcal{N}_p$ such that U is open and $f|_U = g|_U$. Clearly \sim_p is an equivalence relation on $C^\infty(M)$. We denote $C^\infty(M)/\sim_p$ by $C_p^\infty(M)$. For $f \in C^\infty(M)$, we define the **germ of f at p** to be the equivalence class of f under \sim_p .

Exercise 3.4.2. Let $p \in M$. We have that $C_p^\infty(M)$ is a vector space.

Proof. Clear. □

Definition 3.4.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. For $i \in \{1, \dots, n\}$, define the partial derivative with respect to x^i at p , denoted

$$\left. \frac{\partial}{\partial x^i} \right|_p : C_p^\infty(M) \rightarrow \mathbb{R}, \text{ or } \partial_i|_p : C_p^\infty(M) \rightarrow \mathbb{R}$$

by

$$\left. \frac{\partial}{\partial x^i} \right|_p f = \frac{\partial f}{\partial x^i}(p)$$

Exercise 3.4.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \dots, n\}$, we have that

$$\frac{\partial x^i}{\partial x^j}(p) = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \dots, n\}$. Then

$$\begin{aligned} \left. \frac{\partial}{\partial x^j} \right|_p x^i &= \left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} x^i \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} u^i \circ \phi \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} u^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 3.4.5. Change of Coordinates:

Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$, $p \in U \cap V$ and $f \in C_p^\infty(M)$. Then for each $i \in \{1, \dots, n\}$,

$$\left. \frac{\partial}{\partial y^i} \right|_p = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i}(p) \left. \frac{\partial}{\partial x^j} \right|_p$$

Proof. Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \dots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned} \left. \frac{\partial}{\partial y^i} \right|_p f &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \psi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \phi^{-1} \circ h \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial u^j} \right|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left(\left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} h_j \right) \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} f \circ \phi^{-1} \right) \left(\left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} x^j \circ \psi^{-1} \right) \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial x^j} \right|_p f \right) \left(\left. \frac{\partial}{\partial y^i} \right|_p x^j \right) \end{aligned}$$

□

Definition 3.4.6. Let $p \in M$ and $v : C_p^\infty(M) \rightarrow \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C_p^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation at p** if for each $f, g \in C_p^\infty(M)$ and $a \in \mathbb{R}$,

- (1) v is linear
- (2) v is Leibnizian

We define the **tangent space of M at p** , denoted $T_p M$, by

$$T_p M = \{v : C_p^\infty(M) \rightarrow \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 3.4.7. Let $f \in C_p^\infty(M)$ and $v \in T_p M$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. □

Exercise 3.4.8. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then

$$\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right\}$$

is a basis for $T_p M$ and $\dim T_p M = n$.

Proof. Clearly $\left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \in T_p M$. Let $a_1, \dots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \left. \frac{\partial}{\partial x^i} \right|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p x^j \\ &= a_j \end{aligned}$$

Hence $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is independent.

Now, let $v \in T_p M$ and $f \in C_p^\infty(M)$. By Taylor's theorem, there exist $g_1, \dots, g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \dots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x^i} \Big|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p f \\ &= \left[\sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p \right] f \end{aligned}$$

So

$$v = \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p$$

and

$$v \in \text{span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

□

Definition 3.4.9. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. We define the **differential of F at p** , denoted $dF_p : T_p M \rightarrow T_{F(p)} N$, by

$$\left[dF_p(v) \right] (f) = v(f \circ F)$$

for $v \in T_p M$ and $f \in C_{F(p)}^\infty(N)$.

Exercise 3.4.10. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. Then dF_p is a derivation.

Proof. Let $v \in T_p M$, $f, g \in C_{F(p)}^\infty(N)$ and $c \in \mathbb{R}$. Then

(1)

$$\begin{aligned} dF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= dF_p(v)(f) + cdF_p(v)(g) \end{aligned}$$

So $dF_p(v)$ is linear.

(2)

$$\begin{aligned} dF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= dF_p(v)(f) * g(F(p)) + f(F(p)) * dF_p(v)(g) \end{aligned}$$

So $dF_p(v)$ is Leibnizian and hence $dF_p(v) \in T_{F(p)}N$ □

Exercise 3.4.11. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. If F is a diffeomorphism, then dF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \dots, x^n)$ and $\phi \circ F^{-1} = (y^1, \dots, y^n)$. Let $f \in C_{F(p)}^\infty(N)$. Then

$$\begin{aligned} \left. \frac{\partial}{\partial y^i} \right|_{F(p)} f &= \left. \frac{\partial}{\partial u^i} \right|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial x^i} \right|_p f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) \right] (f) &= \left. \frac{\partial}{\partial x^i} \right|_p f \circ F \\ &= \left. \frac{\partial}{\partial y^i} \right|_{F(p)} f \end{aligned}$$

Hence

$$dF_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \left. \frac{\partial}{\partial y^i} \right|_{F(p)}$$

Since $\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right\}$ is a basis for $T_p M$ and $\left\{ \left. \frac{\partial}{\partial y^1} \right|_{F(p)}, \dots, \left. \frac{\partial}{\partial y^n} \right|_{F(p)} \right\}$ is a basis for $T_{F(p)}N$, dF_p is an isomorphism. □

Exercise 3.4.12. Let (M, \mathcal{A}) be a smooth m -dimensional manifold, (N, \mathcal{B}) a smooth n -dimensional manifold, $F : M \rightarrow N$ smooth, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^m)$, $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$ and $p \in U$. Define the ordered bases $B_\phi = \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^m} \Big|_p \right\}$ and $B_\psi = \left\{ \frac{\partial}{\partial y^1} \Big|_{F(p)}, \dots, \frac{\partial}{\partial y^n} \Big|_{F(p)} \right\}$. Then the matrix representation of dF_p with respect to the bases B_ϕ and B_ψ is

$$dF_p^{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Proof. Let $(dF_p)_{B_\phi, B_\psi} = (a_{i,j})_{i,j} \in \mathbb{R}^{n \times m}$. Then for each $j \in \{1, \dots, m\}$,

$$dF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i} \Big|_{F(p)}$$

This implies that

$$\begin{aligned} dF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) (y^k) &= \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i} \Big|_{F(p)} (y^k) \\ &= \sum_{i=1}^n a_{i,j} \delta_{i,k} \\ &= a_{k,j} \end{aligned}$$

By definition,

$$\begin{aligned} dF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) (y^k) &= \frac{\partial}{\partial x^j} \Big|_p y^k \circ F \\ &= \frac{\partial}{\partial x^j} \Big|_p F^k \\ &= \frac{\partial F^k}{\partial x^j}(p) \end{aligned}$$

□

Note 3.4.13. Since $\text{rank } dF_p$ is independent of basis, it is independent of coordinate charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$.

Definition 3.4.14. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ a diffeomorphism. Define the **push forward of F** , denoted

$$F_* : M \rightarrow \coprod_{p \in M} \text{Iso}(T_p M, T_{F(p)} N)$$

by

$$p \mapsto dF_p$$

3.5. The Cotangent Space.

Definition 3.5.1. Let $p \in M$. We define the **cotangent space of M at p** , denoted T_p^*M , by

$$T_p^*M = (T_pM)^*$$

Definition 3.5.2. Let $f \in C^\infty(M)$. We define the **differential of f at p** , denoted $df_p : T_pM \rightarrow \mathbb{R}$, by

$$df_p(v) = vf$$

Exercise 3.5.3. Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

Proof. Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1f + \lambda v_2f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that df_p is linear and hence $df_p \in T_p^*M$. □

Exercise 3.5.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \dots, n\}$,

$$dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \delta_{i,j}$$

In particular, $\{dx_p^1, \dots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ and $T_p^*M = \text{span}\{dx_p^1, \dots, dx_p^n\}$.

Proof. Let $i, j \in \{1, \dots, n\}$. Then by definition,

$$\begin{aligned} \left[dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) \right]_p &= \frac{\partial}{\partial x^j} \Big|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 3.5.5. Let $f \in C^\infty(M)$, (U, ϕ) a chart on M with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

Proof. Since $\{dx_p^1, \dots, dx_p^n\}$ is a basis for T_p^*M , for each there exist $a_1(p), \dots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) &= \left.\frac{\partial}{\partial x^j}\right|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx_p^i$$

□

3.6. Maps of Full Rank.

Definition 3.6.1. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map and $p \in M$. We define the **rank of F at p** , denoted $\text{rank}_p F$, by $\text{rank}_p F = \text{rank } dF_p$. We say that F has **constant rank** if for each $p, q \in M$, $\text{rank}_p F = \text{rank}_q F$. If F has constant rank, we define the **rank of F** , denoted $\text{rank } F$, by $\text{rank } F = \text{rank}_p F$.

Definition 3.6.2. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map. Then F is said to be

- an **immersion** if for each $p \in M$, $dF_p : T_p M \rightarrow T_{F(p)} N$ is injective
- a **submersion** if for each $p \in M$, $dF_p : T_p M \rightarrow T_{F(p)} N$ is surjective

Exercise 3.6.3. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map.

Definition 3.6.4. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds and $F : M \rightarrow N$ smooth. Then F is said to be an **embedding** if

- (1) F is an immersion
- (2) $F : M \rightarrow F(M)$.

Note 3.6.5. Here the topology on $F(M)$ is the subspace topology.

3.7. Submanifolds.

Definition 3.7.1. Let (M, \mathcal{A}) and (S, \mathcal{B}) be smooth manifolds. Suppose that $S \subset M$. Then (S, \mathcal{B}) is said to be a **submanifold** of (M, \mathcal{A}) if

- S is a subspace of M
- the inclusion map $\iota : M \rightarrow N$ is smooth

Exercise 3.7.2. Let (M, \mathcal{A}) be a smooth manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $\tilde{U} \subset S$ and $\tilde{\phi} : \tilde{U} \rightarrow \phi(\tilde{U})$ by $\tilde{U} = U \cap S$ and $\tilde{\phi} = \phi|_{U \cap S}$. Set $\mathcal{B} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \in \mathcal{A}\}$. Then \mathcal{B} is a smooth structure on S .

Proof.

□

Definition 3.7.3. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds. Suppose that $M \subset N$. Then (M, \mathcal{A}) is said to be

- (1) an **immersed submanifold** of (N, \mathcal{B}) if $\text{id} : M \rightarrow N$ is a smooth immersion
- (2) an **embedded submanifold** of (N, \mathcal{B}) if $\text{id} : M \rightarrow N$ is a smooth embedding

Note 3.7.4. Essentially, embedded submanifolds are immersed submanifolds with the subspace topology.

Note 3.7.5. For the remainder of this section, we assume that $k \leq n$.

Definition 3.7.6. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Then S is said to be a **k -slice** of U if $S = \{u \in U : u^{k+1}, \dots, u^n = 0\}$.

Exercise 3.7.7. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Suppose that S is a k -slice of U . Define $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then $\pi|_S \rightarrow \pi(S)$ is a diffeomorphism.

Proof. Clear. □

Definition 3.7.8. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ and $S \subset U$. Then S is said to be a **k -slice** of U if $\phi(S)$ is a k -slice of $\phi(U)$.

Definition 3.7.9. Let (M, \mathcal{A}) be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$. Then (U, ϕ) is said to be a **k -slice chart for S** if $U \cap S$ is a k -slice of U .

Exercise 3.7.10. Let (M, \mathcal{A}) be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. If (U, ϕ) is a k -slice chart for S , then $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$.

Proof. Clear. □

Definition 3.7.11. Let (M, \mathcal{A}) be a smooth manifold and $S \subset M$. Then S is said to satisfy the **local k -slice condition** if for each $p \in S$, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k -slice chart of S .

Exercise 3.7.12. Let (M, \mathcal{A}) be a smooth n -dimensional manifold and $S \subset M$ a subspace. If S satisfies the local k -slice condition, then there exists a smooth structure $\tilde{\mathcal{A}}$ on S such that $(S, \tilde{\mathcal{A}})$ is an embedded submanifold of M .

Proof. Suppose that S satisfies the local k -slice condition. Define $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ as above. Let $(U, \phi) \in \mathcal{A}$. Suppose that (U, ϕ) is a k -slice chart for S . Define $\tilde{U} = U \cap S$ and $\tilde{\phi} : \tilde{U} \rightarrow \pi \circ \phi(\tilde{U})$ by

$$\tilde{\phi} = \pi \circ \phi|_{\tilde{U}}$$

By definition, $\phi(\tilde{U})$ is a k -slice of $\phi(U)$. A previous exercise implies that $\pi|_{\phi(\tilde{U})} \rightarrow \pi \circ \phi(\tilde{U})$ is a diffeomorphism and hence a homeomorphism. Thus $\tilde{\phi}$ is a homeomorphism.

Define

$$\tilde{\mathcal{B}} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \text{ is a } k\text{-slice for } S\}$$

Let $p \in S$. By assumption, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k -slice chart of S . Then $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{B}}$ and $\tilde{\mathcal{A}}$ is an atlas on S . By construction of $\tilde{\mathcal{B}}$, S is locally half

Euclidean of dimension k . Since M is second countable Hausdorff, so is S in the subspace topology. Thus $(S, \tilde{\mathcal{B}})$ is a k -dimensional manifold. Let $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$. Then

$$\tilde{\phi} \circ \tilde{\psi}^{-1}|_{\tilde{U} \cap \tilde{V}} = \pi|_{\phi(\tilde{U} \cap \tilde{V})} \circ \phi|_{\tilde{U} \cap \tilde{V}} \circ \psi|_{\tilde{U} \cap \tilde{V}}^{-1} \circ \pi|_{\psi(\tilde{U} \cap \tilde{V})}^{-1}$$

which is a diffeomorphism. So $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ smoothly compatible. Hence $\tilde{\mathcal{B}}$ is smooth. An exercise in section 4.1 implies that there exists a unique smooth structure $\tilde{\mathcal{A}}$ on S such that $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$. So $(S, \tilde{\mathcal{A}})$ is a smooth k -dimensional manifold.

Clearly $\text{id} : S \rightarrow S$ is a homeomorphism. Let $(V, \psi) \in \mathcal{A}$ and $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$.

Finish!!

□

Definition 3.7.13.

Exercise 3.7.14.

4. VECTOR BUNDLES AND TENSOR FIELDS

4.1. The Vector Bundle.

Definition 4.1.1. Let E , M and F be smooth manifolds and $\pi : E \rightarrow M$ a smooth surjection, $U \subset M$ open and $\Phi : \pi^{-1}(U) \rightarrow U \times F$. Then (U, Φ) is said to be a **smooth local trivialization of E over U** if

- (1) Φ is a diffeomorphism
- (2) $\pi_U \circ \Phi = \pi|_{\pi^{-1}(U)}$ (where $\pi_U : U \times F \rightarrow U$ denotes projection onto U)

Exercise 4.1.2. Let E , M and F be topological spaces and $\pi : E \rightarrow M$ a continuous surjection and (U, Φ) a local trivialization of E over U . Then for each $A \subset U$,

$$\Phi(\pi^{-1}(A)) = A \times F$$

Hint: show that $\pi^{-1}(A) = (\pi_U \circ \Phi)^{-1}(A)$

Proof. Let $A \subset U$. Since $\pi^{-1}(A) \subset \pi^{-1}(U)$, property (2) implies that $\pi^{-1}(A) = (\pi_U \circ \Phi)^{-1}(A)$. Since Φ is a bijection,

$$\begin{aligned} \Phi(\pi^{-1}(A)) &= \Phi \circ (\pi_U \circ \Phi)^{-1}(A) \\ &= \Phi \circ \Phi^{-1}(\pi_U^{-1}(A)) \\ &= \pi_U^{-1}(A) \\ &= A \times F \end{aligned}$$

□

Definition 4.1.3. Let E and M be topological spaces and $\pi : E \rightarrow M$ a continuous surjection. Then (E, M, π) is said to be a **smooth vector bundle of rank n** if

- (1) for each $p \in M$, $\pi^{-1}(\{p\})$ is a n -dimensional real vector space.
- (2) for each $p \in M$, there exist open $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ such that (U, Φ) is a smooth local trivialization of E over U .
- (3) for each $p \in M$,

$$\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \rightarrow \{p\} \times \mathbb{R}^n$$

is an isomorphism.

Exercise 4.1.4. Let M be a smooth n -dimensional manifold. Set $E = M \times \mathbb{R}^n$ and define $\pi : E \rightarrow M$ by $\pi(p, x) = p$. Then (E, M, π) is a smooth vector bundle of rank n .

Proof.

- (1) For each $p \in M$, $\pi^{-1}(\{p\}) = \{p\} \times \mathbb{R}^n$ which may be given the obvious vector space structure.
- (2) Let $p \in M$. Set $U = M$. Then $\pi^{-1}(U) = E$. Define $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ by $\Phi = \text{id}_E$. Then (U, Φ) is a smooth local trivialization of E over U .
- (3) Let $p \in M$. Then $\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \rightarrow \{p\} \times \mathbb{R}^n$ is clearly an isomorphism.

□

Theorem 4.1.5. Let E and M be smooth manifolds and $\pi : E \rightarrow M$ a smooth surjection.

Definition 4.1.6. We define the **tangent bundle of** M , denoted TM , by

$$TM = \coprod_{p \in M} T_p M$$

We denote the natural projection map by $\pi : TM \rightarrow M$.

Definition 4.1.7. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Define $\tilde{U} \subset TM$ and $\tilde{\phi} : \tilde{U} \rightarrow \phi(U) \times \mathbb{R}^n$ by

- $\tilde{U} = \pi^{-1}(U)$
-

$$\begin{aligned} \tilde{\phi} \left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p \right) &= (\phi(p), v) \\ &= (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \end{aligned}$$

,

Exercise 4.1.8. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then $\tilde{\phi} : \tilde{U} \rightarrow \phi(U) \times \mathbb{R}^n$ is a bijection.

4.2. The cotangent Bundle.

Definition 4.2.1. We define the **cotangent bundle of M** , denoted T^*M , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

4.3. The (r, s) -Tensor Bundle.

Definition 4.3.1. (1) the **cotangent bundle of M** , denoted T^*M , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

(2) the **(r, s) -tensor bundle of M** , denoted $T_s^r M$, by

$$T_s^r M = \coprod_{p \in M} T_s^r(T_p M)$$

(3) the **k -alternating tensor bundle of M** , denoted $\Lambda_k(M)$, by

$$\Lambda_k M = \coprod_{p \in M} \Lambda_k(T_p M)$$

4.4. Vector Fields.

Definition 4.4.1. Let $X : M \rightarrow TM$. Then X is said to be a **vector field on M** if for each $p \in M$, $X_p \in T_p M$.

For $f \in C^\infty(M)$, we define $Xf : M \rightarrow \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in C^\infty(M)$, Xf is smooth.

We denote the set of smooth vector fields on M by $\Gamma^1(M)$.

Definition 4.4.2. Let $f \in C^\infty(M)$ and $X, Y \in \Gamma^1(M)$. We define

- $fX \in \Gamma^1(M)$ by

$$(fX)_p = f(p)X_p$$

- $X + Y \in \Gamma^1(M)$ by

$$(X + Y)_p = X_p + Y_p$$

Exercise 4.4.3. The set $\Gamma^1(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Exercise 4.4.4. Let $X \in \Gamma^1(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then

$$X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$$

Proof. Let $p \in M$. Then $X_p \in T_p M$ and $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is a basis of $T_p M$. So there exist $f_1(p), \dots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i} \Big|_p$. Let $j \in \{1, \dots, n\}$. Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^n f_i(p) \frac{\partial x^j}{\partial x^i}(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx^j = f_j$ and $X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$. □

Exercise 4.4.5. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then for each $i \in \{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

Proof. Let $i \in \{1, \dots, n\}$ and $f \in C^\infty(M)$. Define $g : M \rightarrow \mathbb{R}$ by $g = \frac{\partial}{\partial x^i} f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i} \Big|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i} [f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence g is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x^i}$ is smooth. \square

4.5. 1-Forms.

Definition 4.5.1. Let $\omega : M \rightarrow T^*M$. Then ω is said to be a **1-form on M** if for each $p \in M$, $\omega_p \in T_p^*M$.

For each $X \in \Gamma^1(M)$, we define $\omega(X) : M \rightarrow \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)$, $\omega(X)$ is smooth.

The set of smooth 1-forms on M is denoted $\Gamma_1(M)$.

Definition 4.5.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma_1(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by

$$(f\omega)_p = f(p)\omega_p$$

- $\alpha + \beta \in \Gamma_1(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 4.5.3. The set $\Gamma_1(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Exercise 4.5.4.

4.6. (r, s) -Tensor Fields.

Definition 4.6.1. Let $\alpha : M \rightarrow T_s^r M$. Then α is said to be a (r, s) -**tensor field on M** if for each $p \in M$, $\alpha_p \in T_p^r(T_p M)$.

For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \rightarrow \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and α is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth. The set of smooth (r, s) -tensor fields on M is denoted $\Gamma_s^r(M)$.

Definition 4.6.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma_s^r(M)$. We define

- $f\alpha : M \rightarrow T_s^r M$ by

$$(f\alpha)_p = f(p)\alpha_p$$

- $\alpha + \beta : M \rightarrow T_s^r M$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 4.6.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma_s^r(M)$. Then

- (1) $f\alpha \in \Gamma_s^r(M)$ by

$$(f\alpha)_p = f(p)\alpha_p$$

- (2) $\alpha + \beta \in \Gamma_s^r(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Proof. Clear. □

Exercise 4.6.4. The set $\Gamma_s^r(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Definition 4.6.5. Let $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$ and $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$. We define the **tensor product of α with β** , denoted $\alpha \otimes \beta : M \rightarrow T_{s_1+s_2}^{r_1+r_2} M$, by

$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

Exercise 4.6.6. Let $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$ and $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma_{s_1+s_2}^{r_1+r_2}(M)$

Proof. Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since α_1 and α_2 are smooth by assumption. □

Definition 4.6.7. We define the **tensor product**, denoted $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

Exercise 4.6.8. The tensor product $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ is associative.

Proof. Clear. □

Exercise 4.6.9. The tensor product $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ is $C^\infty(M)$ -bilinear.

Proof. Clear. □

Definition 4.6.10. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ a smooth map and $\alpha \in \Gamma_k^0(N)$. We define the **pullback of α by F** , denoted $F^*\alpha \in \Gamma_k^0(M)$, by

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))$$

for $p \in M$ and $v_1, \dots, v_k \in T_p M$

Exercise 4.6.11. Let (M, \mathcal{A}) , (N, \mathcal{B}) and (L, \mathcal{C}) be smooth manifolds, $F : M \rightarrow N$ and $G : N \rightarrow L$ smooth maps, $\alpha \in \Gamma_k^0(N)$, $\beta \in \Gamma_l^0(N)$, $\gamma \in \Gamma_k^0(L)$ and $f \in C^\infty(N)$. Then

- (1) $F^*(f\alpha) = (f \circ F)F^*\alpha$
- (2) $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
- (3) $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
- (4) $(G \circ F)^*\gamma = F^*(G^*\gamma)$
- (5) $id_N^*\alpha = \alpha$

Proof.

(1)

$$\begin{aligned} [F^*(f\alpha)]_p(v_1, \dots, v_k) &= (f\alpha)_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)) \\ &= f(F(p))\alpha_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)) \\ &= (f \circ F)(p)(F^*\alpha)_p(v_1, \dots, v_k) \end{aligned}$$

So that $F^*(f\alpha) = (f \circ F)F^*\alpha$

(2)

$$F^*$$

□

Definition 4.6.12.

Exercise 4.6.13.

Proof.

□

Exercise 4.6.14. Let $\alpha \in \Gamma_s^r(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then there exist $(f_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

Proof. Let $p \in M$. Then $\omega_p \in T_s^r(T_p M)$ and $\left\{ \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J} \right\}$ is a basis of $T_s^r(T_p M)$. So there exist $(f_J^I(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned} \alpha_p(dx_p^K, \partial_{x^L}|_p) &= \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}(dx_p^K, \partial_{x^L}|_p) \\ &= \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p(dx_p^K) dx_p^{\otimes J}(\partial_{x^L}|_p) \\ &= f_L^K(p) \end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx^K, \partial_{x^L})_p$ is smooth, so that $f_L^K \in C^\infty(U)$.

□

Definition 4.6.15.

4.7. Differential Forms.

Definition 4.7.1. We define

$$\Lambda_k(TM) = \coprod_{p \in M} \Lambda_k(T_p M)$$

Definition 4.7.2. Let $\omega : M \rightarrow \Lambda_k(TM)$. Then ω is said to be a **k -form on M** if for each $p \in M$, $\omega_p \in \Lambda_k(T_p M)$.

For each $X \in \Gamma^1(M)^k$, we define $\omega(X) : M \rightarrow \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)^k$, $\omega(X)$ is smooth.

The set of smooth k -forms on M is denoted $\Omega_k(M)$.

Note 4.7.3. Observe that

- (1) $\Omega_k(M) \subset \Gamma_k^0(M)$
- (2) $\Omega_0(M) = C^\infty(M)$

Exercise 4.7.4. The set $\Omega_k(M)$ is a $C^\infty(M)$ -submodule of $\Gamma_k^0(M)$.

Proof. Clear. □

Definition 4.7.5. Define the **exterior product**

$$\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Note 4.7.6. For $f \in \Omega_0(M)$ and $\alpha \in \Omega_k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 4.7.7. The exterior product $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$ is well defined.

Proof. Let $\alpha \in \Omega_k(M)$, $\beta \in \Omega_l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{j=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then

$$\begin{aligned} \alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{(k+l)!}{k!l!} A(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) (\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \end{aligned}$$

□

Exercise 4.7.8. The exterior product $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$ is $C^\infty(M)$ -bilinear.

Proof.

(1) $C^\infty(M)$ -linearity in the first argument:

Let $\alpha \in \Omega_k(M)$, $\beta, \gamma \in \Omega_l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda_k(T_p M) \times \Lambda_l(T_p M) \rightarrow \Lambda_{k+l}(T_p M)$ implies that

$$\begin{aligned} [(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p \end{aligned}$$

So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

and $\wedge : \Omega_k(M) \times \Omega_l(M) \rightarrow \Omega_{k+l}(M)$ is $C^\infty(M)$ -linear in the first argument.

(2) $C^\infty(M)$ -linearity in the second argument:

Similar to (1).

□

Note 4.7.9. All of the results from multilinear algebra apply here.

Definition 4.7.10. We define the **exterior derivative** $d : \Omega_k(M) \rightarrow \Omega_{k+1}(M)$ inductively by

- (1) $d(d\alpha) = 0$ for $\alpha \in \Omega_p(M)$
- (2) $df(X) = Xf$ for $f \in \Omega_0(M)$
- (3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega_p(M)$ and $\beta \in \Omega_q(M)$
- (4) extending linearly

Exercise 4.7.11. Let (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then on U , for each $i, j \in \{1, \dots, n\}$,

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx_p^1, \dots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ and $T_p^* M = \text{span}\{dx_p^1, \dots, dx_p^n\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \dots, n\}$. Then by definition,

$$\begin{aligned} \left[dx^i \left(\frac{\partial}{\partial x^j} \right) \right]_p &= \left(\frac{\partial}{\partial x^j} x^i \right)_p \\ &= \frac{\partial}{\partial x^j} \Big|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 4.7.12. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Proof. Let $p \in U$. Since $\{dx^1, \dots, dx^n\}$ is a basis for $\Lambda(T_p M)$, for each there exist $a_1(p), \dots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a^i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) &= \sum_{i=1}^n a^i(p) dx_p^i \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) &= \left. \frac{\partial}{\partial x^j} \right|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

□

Exercise 4.7.13. Let $f \in \Omega_0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \dots, x_n)$. Then for each $i \in \{1, \dots, n\}$,

$$\left. \frac{\partial}{\partial x^i} \right|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i \\ &= 0 \end{aligned}$$

□

Exercise 4.7.14.

Definition 4.7.15. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $I = (i_1, \dots, i_k) \in \mathcal{I}_k$. We define

$$dx^I = dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Omega_k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}} \right)$$

Note 4.7.16. We have that

(1)

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{I,J}$$

(2) Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega_k(U)$,

$$\omega \left(\frac{\partial}{\partial x^i} \right) \in C^\infty(U)$$

Exercise 4.7.17. Let $\omega \in \Omega_k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega \left(\frac{\partial}{\partial x^I} \right) dx^I$$

Proof. Let $p \in U$. Since $\{dx_p^i : I \in \mathcal{I}_k\}$ is a basis for $\Lambda_k(T_p M)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p) dx_p^I$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned} \omega \left(\frac{\partial}{\partial x^J} \right) &= \sum_{I \in \mathcal{I}_k} f_I dx^I \left(\frac{\partial}{\partial x^J} \right) \\ &= f_J \end{aligned}$$

□

Exercise 4.7.18. Let $\omega \in \Omega_k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. If $\omega = \sum_{I \in \mathcal{I}_k} f_I dx^I$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^I$$

.

Proof. First we note that

$$\begin{aligned} d(f_I dx^I) &= df_I \wedge dx^I + (-1)^0 f_I d(dx^I) \\ &= df_I \wedge dx^I \\ &= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \right) \wedge dx^I \\ &= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^I \end{aligned}$$

Then we extend linearly.

□

Definition 4.7.19. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$ be a diffeomorphism. Define the **pullback of F** , denoted $F^* : \Omega_k(N) \rightarrow \Omega_k(M)$ by

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k))$$

for $\omega \in \Omega_k(N)$, $p \in M$ and $v_1, \dots, v_k \in T_pM$

5. EXTRA

Definition 5.0.1. When working in \mathbb{R}^n , we introduce the formal objects dx^1, dx_2, \dots, dx^n . Let $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. We formally define $dx^I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$.

Definition 5.0.2. Let $k \in \{0, 1, \dots, n\}$. We define a $C^\infty(\mathbb{R}^n)$ -module of dimension $\binom{n}{k}$, denoted $\Phi_k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx^I : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \rightarrow \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

- (1) the exterior product is bilinear
- (2) for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$
- (3) for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$
- (4) for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k -forms on \mathbb{R}^n . Let ω be a k -form on \mathbb{R}^n . If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$

Note 5.0.3. The terms dx^1, dx_2, \dots, dx^n are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \dots, du_k\}$ when referencing the coordinates on \mathbb{R}^k and $\{dx^1, dx_2, \dots, dx^n\}$ when referencing the coordinates on \mathbb{R}^n .

Exercise 5.0.4. Let $B_{n \times n} = (b_{i,j}) \in [C^\infty(M)]^{n \times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \dots \wedge dx^n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned} \bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) &= \left(\sum_{j=1}^n b_{1,j} dx^j \right) \wedge \left(\sum_{j=1}^n b_{2,j} dx^j \right) \wedge \dots \wedge \left(\sum_{j=1}^n b_{n,j} dx^j \right) \\ &= \sum_{j_1, \dots, j_n=1}^n \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_n} \\ &= \sum_{j_1 \neq \dots \neq j_n} \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_n} \\ &= \left[\sum_{\sigma \in S_n} \text{sgn}(\sigma) \left(\prod_{i=1}^n b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \\ &= (\det B) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \end{aligned}$$

□

Definition 5.0.5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a 0-form on \mathbb{R}^n . We define a 1-form, denoted df , on \mathbb{R}^n by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Let $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$ be a k -form on \mathbb{R}^n . We can define a differential $k+1$ -form, denoted $d\omega$, on \mathbb{R}^n by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx^I$$

Exercise 5.0.6. On \mathbb{R}^3 , put

- (1) $\omega_0 = f_0$,
- (2) $\omega_1 = f_1 dx^1 + f_2 dx^2 + f_3 dx^3$,
- (3) $\omega_2 = f_1 dx^2 \wedge dx^3 - f_2 dx^1 \wedge dx^3 + f_3 dx^1 \wedge dx^2$

Show that

- (1) $d\omega_0 = \frac{\partial f_0}{\partial x^1} dx^1 + \frac{\partial f_0}{\partial x^2} dx^2 + \frac{\partial f_0}{\partial x^3} dx^3$
- (2) $d\omega_1 = \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) dx^2 \wedge dx^3 + \left(\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3} \right) dx^1 \wedge dx^3 + \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) dx^1 \wedge dx^2$
- (3) $d\omega_2 = \left(\frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3$

Proof. Straightforward. □

Exercise 5.0.7. Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx^I \wedge dx^{I_*} = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$.

Definition 5.0.8. We define a linear map $*$: $\Phi_k(\mathbb{R}^n) \rightarrow \Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge *-operator** by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^{I_*}$$

Definition 5.0.9. Let $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. We define $\phi^* : \Phi_k(\mathbb{R}^n) \rightarrow \Phi_k(\mathbb{R}^k)$ via the following properties:

- (1) for each 0-form f on \mathbb{R}^n , $\phi^* f = f \circ \phi$
- (2) for $i = 1, \dots, n$, $\phi^* dx^i = d\phi_i$
- (3) for an s -form ω , and a t -form χ on \mathbb{R}^n , $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
- (4) for l -forms ω, χ on \mathbb{R}^n , $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 5.0.10. Let $M \subset \mathbb{R}^n$ be a k -dimensional smooth submanifold of \mathbb{R}^n , $\phi : U \rightarrow V$ a smooth parametrization of M , $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$ an k -form on \mathbb{R}^n . Then

$$\phi^*\omega = \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du^2 \wedge \cdots \wedge du^k$$

Proof. By definition,

$$\begin{aligned}\phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx^i \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\ &= \left(\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u^j} du^j \right) \wedge \left(\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u^j} du^j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n \frac{\partial \phi_{i_n}}{\partial u^j} du^j \right) \\ &= (\det v\phi_I) du^1 \wedge du^2 \wedge \cdots \wedge du^n\end{aligned}$$

Therefore

$$\begin{aligned}\phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det v\phi_I) du^1 \wedge du^2 \wedge \cdots \wedge du^n \\ &= \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det v\phi_I) \right) du^1 \wedge du^2 \wedge \cdots \wedge du^n\end{aligned}$$

□

5.1. Integration of Differential Forms.

Definition 5.1.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \cdots \wedge dx_k$ a k -form on \mathbb{R}^k . Define

$$\int_U \omega = \int_U f dx$$

Definition 5.1.2. Let $M \subset \mathbb{R}^n$ be a k -dimensional oriented smooth submanifold of \mathbb{R}^n , ω a k -form on \mathbb{R}^n and $\phi : U \rightarrow V$ a local smooth, orientation-preserving parametrization of M . Define

$$\int_V \omega = \int_U \phi^* \omega$$

Exercise 5.1.3.

Theorem 5.1.4. Stokes Theorem:

Let $M \subset \mathbb{R}^n$ be a k -dimensional oriented smooth submanifold of \mathbb{R}^n and ω a $k-1$ -form on \mathbb{R}^n . Then

$$\int_{\partial M} \omega = \int_M d\omega$$