INTRODUCTION TO CATEGORY THEORY

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Contents

Preface		1
1.	Categories and Functors	2
1.1.	Categories	2
1.2.	Functors	5
1.3.	Natural Transformations	6

Preface

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1. Categories and Functors

1.1. Categories.

Definition 1.1.1. Let C_0 , C_1 be classes and dom, cod : $C_1 \to C_0$. Set $C = (C_0, C_1, \text{dom}, \text{cod})$. Then C is said to be a **category** if

- (1) (composition): for each $f, g \in C_1$, if $\operatorname{cod}(f) = \operatorname{dom}(g)$, then there exists $g \circ f \in C_1$ such that $\operatorname{dom}(g \circ f) = \operatorname{dom}(f)$ and $\operatorname{cod}(g \circ f) = \operatorname{cod}(g)$
- (2) (associativity): for each $f, g, h \in C_1$, if cod(f) = dom(g) and cod(g) = dom(h), then $(h \circ g) \circ f = h \circ (g \circ f)$
- (3) (identity): for each $X \in \mathcal{C}_0$, there exists $1_X \in \mathcal{C}_1$ such that $dom(1_X) = cod(1_X) = X$ and for each $f, g \in \mathcal{C}_1$, if dom(f) = X and cod(g) = X, then

$$f \circ 1_X = f$$
 and $1_X \circ g = g$

We define the

- objects of \mathcal{C} , denoted $\mathrm{Obj}(\mathcal{C})$, by $\mathrm{Obj}(\mathcal{C}) = C_0$
- morphisms of C, denoted Hom_C , by $Hom_C = C_1$

For $X, Y \in \text{Obj}(\mathcal{C})$, we define the **morphisms from** X **to** Y, denoted $\text{Hom}_{\mathcal{C}}(X, Y)$, by $\text{Hom}_{\mathcal{C}}(X, Y) = \{ f \in \text{Hom}(\mathcal{C}) : \text{dom}(f) = X \text{ and } \text{cod}(f) = Y \}.$

Note 1.1.2. We typically define a category \mathcal{C} by specifying

- Obj(C)
- for $X, Y \in \text{Obj}(\mathcal{C})$, the class $\text{Hom}_{\mathcal{C}}(X, Y)$
- for $X, Y, Z \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$, the composite morphism $g \circ f \in \text{Hom}_{\mathcal{C}}(X, Z)$.

and then show

- composition is well defined
- associativity of composition
- existence of identities

Definition 1.1.3. Let \mathcal{C} be a category, we define the dual of \mathcal{C} or the **opposite of** \mathcal{C} , denoted \mathcal{C}^{op} , by

- $Obj(\mathcal{C}^{op}) = Obj(\mathcal{C})$
- for $X, Y \in \text{Obj}(\mathcal{C}^{\text{op}})$, $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$
- for $f \in \operatorname{Hom}_{\mathcal{C}^{\operatorname{op}}}(X,Y), q \in \operatorname{Hom}_{\mathcal{C}^{\operatorname{op}}}(Y,Z), q \circ_{\mathcal{C}^{\operatorname{op}}} f = f \circ_{\mathcal{C}} q$

Exercise 1.1.4. Let \mathcal{C} be a category. Then \mathcal{C}^{op} is a category.

Proof.

• for $W, X, Y, Z \in \text{Obj}(\mathcal{C})$, $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(W, X)$ and $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$ and $h \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$. Then

$$(h \circ_{\mathcal{C}^{\mathrm{op}}} g) \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ_{\mathcal{C}} (h \circ_{\mathcal{C}^{\mathrm{op}}} g)$$

$$= f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h)$$

$$= (f \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} h$$

$$= h \circ_{\mathcal{C}^{\mathrm{op}}} (f \circ_{\mathcal{C}} g)$$

$$= h \circ_{\mathcal{C}^{\mathrm{op}}} (g \circ_{\mathcal{C}^{\mathrm{op}}} f)$$

So composition is associative.

• Let $X \in \text{Obj}(\mathcal{C})$ and $f, g \in \text{Hom}_{\mathcal{C}^{\text{op}}}$. Suppose that dom(f) = X and cod(g) = XThen

$$f \circ_{\mathcal{C}^{\mathrm{op}}} 1_X = 1_X \circ_{\mathcal{C}} f$$
$$= f$$

and

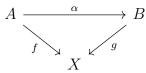
$$1_X \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} 1_X$$
$$= g$$

So
$$(1_X)_{\mathcal{C}^{op}} = (1_X)_{\mathcal{C}}$$
.

Definition 1.1.5. Let \mathcal{C} be a category and $X \in \text{Obj}(\mathcal{C})$. We define the **slice category of** \mathcal{C} **over** X, denoted \mathcal{C}/X , by

- $\operatorname{Obj}(\mathcal{C}/X) = \{ f \in \operatorname{Hom}_{\mathcal{C}} : \operatorname{cod}(f) = X \}$
- for $f, g \in \text{Obj}(\mathcal{C}/X)$,

 $\operatorname{Hom}_{\mathcal{C}/X}(f,g) = \{\alpha \in \operatorname{Hom}_{\mathcal{C}} : \operatorname{dom}(\alpha) = \operatorname{dom}(f), \operatorname{cod}(\alpha) = \operatorname{dom}(g) \text{ and } f = g \circ \alpha \}$ i.e. for $f \in \operatorname{Hom}_{\mathcal{C}}(A,X)$ and $g \in \operatorname{Hom}_{\mathcal{C}}(B,X)$, $\alpha \in \operatorname{Hom}_{\mathcal{C}/X}(f,g)$ iff the following diagram commutes:



• for $f, g, h \in \text{Obj}(\mathcal{C}/X)$, $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \text{Hom}_{\mathcal{C}/X}(g, h)$, $\beta \circ_{\mathcal{C}/X} \alpha = \beta \circ_{\mathcal{C}} \alpha$ **Exercise 1.1.6.** Let \mathcal{C} be a category and $X \in \text{Obj}(\mathcal{C})$. Then \mathcal{C}/X is a category.

Proof.

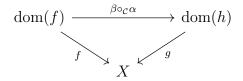
• $f, g, h \in \text{Obj}(\mathcal{C}/X)$, $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$ and $\beta \in \text{Hom}_{\mathcal{C}/X}(g, h)$. Then $f = g \circ_{\mathcal{C}} \alpha$ and $g = h \circ_{\mathcal{C}} \beta$, i.e. the following diagrams commute:



Therefore, we have that

$$f = g \circ_{\mathcal{C}} \alpha$$
$$= (h \circ_{\mathcal{C}} \beta) \circ_{\mathcal{C}} \alpha$$
$$= h \circ_{\mathcal{C}} (\beta \circ_{\mathcal{C}} \alpha)$$

i.e. the following diagram commutes:



which implies that

$$\beta \circ_{\mathcal{C}/X} \alpha = \beta \circ_{\mathcal{C}} \alpha$$
$$\in \operatorname{Hom}_{\mathcal{C}/X}(f, h)$$

and composition is well defined.

- Associativity of $\circ_{\mathcal{C}/X}$ follows from associativity of $\circ_{\mathcal{C}}$.
- Let $f \in \text{Obj}(\mathcal{C}/X)$ and $\alpha, \beta \in \text{Hom}_{\mathcal{C}/X}$. Since $f \circ 1_{\text{dom}_{\mathcal{C}}(f)} = f$, i.e. the following diagram commutes:

$$\operatorname{dom}(f) \xrightarrow{1_{\operatorname{dom}(f)}} \operatorname{dom}(f)$$

$$f \xrightarrow{X} X$$

we have that $1_{\text{dom}(f)} \in \text{Hom}_{\mathcal{C}/X}(f, f)$. Suppose that $\text{dom}_{\mathcal{C}/X}(\alpha) = f$ and $\text{cod}_{\mathcal{C}/X}(\beta) = f$. Then

$$\alpha \circ_{\mathcal{C}/X} 1_{\text{dom}(f)} = \alpha \circ_{\mathcal{C}} 1_{\text{dom}(f)}$$
$$= \alpha$$

and

$$1_{\text{dom}(f)} \circ_{\mathcal{C}/X} \beta = 1_{\text{dom}(f)} \circ_{\mathcal{C}} \beta$$
$$= \beta$$

So
$$(1_f)_{\mathcal{C}/X} = (1_{\text{dom}(f)})_{\mathcal{C}}$$
.

1.2. Functors.

Definition 1.2.1. Let \mathcal{C} and \mathcal{D} be categories, $F_0: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and $F_1: \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}$. Set $F = (F_0, F_1)$. Then F is said to be a functor from \mathcal{C} to \mathcal{D} if

- (1) for each $A, B \in \text{Obj}(C)$ and $f \in \text{Hom}_{\mathcal{C}}(A, B), F_1(f) \in \text{Hom}_{\mathcal{D}}(F_0(A), F_0(B))$
- (2) for each $A, B, C \in \text{Obj}(\mathcal{C}), f \in \text{Hom}_{\mathcal{C}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}}(B, C), F_1(g \circ f) = F_1(g) \circ F_1(f)$
- (3) for each $A \in \mathrm{Obj}(\mathcal{C}), F_1(\mathrm{id}_A) = \mathrm{id}_{F_0(A)}$

Note 1.2.2. For $A \in \text{Obj}(C)$ and $f \in \text{Hom}_{\mathcal{C}}$, we typically write F(A) and F(f) instead of $F_0(A)$ and $F_1(f)$ respectively.

1.3. Natural Transformations.