INTRODUCTION TO GROUP THEORY

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Contents

0.1. **Direct Products.**

Definition 0.1.1. Let G, H be groups. Define a product $* : (G \times H) \times (G \times H) \to G \times H$ by

$$(x_1, y_1) * (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

Then $(G \times H, *)$ is called the **direct product of G and H**.

Exercise 0.1.2. Let G, H be groups. Then the direct product $G \times H$ is a group.

Proof. Clear. □

Definition 0.1.3. Let G, H be groups. Define $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ by $\pi_G(x, y) = x$ and $\pi_H(x, y) = y$. Then π_G and π_H are respectively called the **projection maps onto G and H**.

Exercise 0.1.4. Let G, H be groups. Then

(1) $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ are homomorphisms
(2) $\ker \pi_G \cong H$ and $\ker \pi_H \cong G$

Proof.

(1) Clear
(2) Define $\iota_G : G \to \ker \pi_H$ by

$$\iota_G(x) = (x, e_H)$$

Then ι_G is an isomorphism. Similarly, we can define $\iota_H : H \to \ker \pi_G$ and show that it is an isomorphism.

□

Definition 0.1.5. Let G, H, K be groups, $\phi \in \mathrm{Hom}(G, K)$ and $\psi \in \mathrm{Hom}(H, K)$. We define $\phi \times \psi : G \times H \to K$ by $\phi \times \psi(x, y) = \phi(x)\psi(y)$

Exercise 0.1.6. Let G, H, K be groups, $\phi \in \mathrm{Hom}(G, K)$ and $\psi \in \mathrm{Hom}(H, K)$. If K is abelian, then $\phi \times \psi \in Hom(G \times H, K)$.

Proof. Let $x_1, x_2 \in G$ and $y_1, y_2 \in H$. Then

$$\begin{aligned}
\phi \times \psi[(x_1, y_1)(x_2, y_2)] &= \phi \times \psi(x_1 x_2, y_1 y_2) \\
&= \phi(x_1 x_2)\psi(y_1 y_2) \\
&= \phi(x_1)\phi(x_2)\psi(y_1)\psi(y_2) \\
&= \phi(x_1)\psi(y_1)\phi(x_2)\psi(y_2) \\
&= [\phi \times \psi(x_1, y_1)][\phi \times \psi(x_2, y_2)]
\end{aligned}$$

□

Exercise 0.1.7. Let G, H, K be groups and $\phi \in \mathrm{Hom}(G \times H, K)$. Then there exist $\phi_G \in \mathrm{Hom}(G, K)$, $\phi_H \in \mathrm{Hom}(H, K)$ such that $\phi_G \times \phi_H = \phi$.

Proof. Suppose that K is abelian. Define $\iota_G \in \mathrm{Hom}(G, \ker \pi_H)$ and $\iota_H \in \mathrm{Hom}(H, \ker \pi_G)$ as in part (2) of Exercise 0.1.4 Define $\phi_G \in \mathrm{Hom}(G, K)$ and $\phi_H \in \mathrm{Hom}(H, K)$ by $\phi_G = \phi \circ \iota_G$

and $\phi_H = \phi \circ \iota_H$. Let $(x, y) \in G \times H$. Then

$$\begin{aligned} \phi_G \times \phi_H(x, y) &= \phi_G(x)\phi_H(y) \\ &= \phi \circ \iota_G(x)\phi \circ \iota_H(y) \\ &= \phi(x, e_H)\phi(e_G, y) \\ &= \phi(x, y) \end{aligned}$$

So $\phi = \phi_G \times \phi_H$ $\square$