INTRODUCTION TO NETWORKS

CARSON JAMES

Contents

1. Setup

Definition 1.0.1. Let (M, d) be a metric space, (G, τ) a topological group, and $\cdot : G \times M \to M$ a group action. Suppose that for each $g \in G$, the map $x \mapsto g \cdot x$ is an isometry. We define $\bar{d} : M/G \to [0, \infty)$ by

$$\begin{aligned} \bar{d}(o_x, o_y) &= \inf_{\substack{a \in o_x \\ b \in o_y}} d(a, b) \\ &= \inf_{g \in G} d(g \cdot x, y) \end{aligned}$$

Exercise 1.0.2. If for each $x \in M$, o_x is closed, then $\bar{d}$ is a metric.

Proof. Suppose that for each $x \in M$, o_x is closed. We need only show that for each $x, y \in M$, $\bar{d}(o_x, o_y) = 0$ implies that $o_x = o_y$. Suppose that $\bar{d}(o_x, o_y) = 0$. Then $\inf_{g \in G} d(g \cdot x, y) = 0$. Hence there exists $(\tau_n)_{n \in N} \subset G$ such that $\tau_n \cdot x \to y$. Since $(\tau_n \cdot x)_{n \in \mathbb{N}} \subset o_x$ and o_x is closed, $y \in o_x$. Thus $o_x = o_y$. □

Example 1.0.3. Consider the metric space $(\mathbb{C}, |\cdot|)$, topological group $(S^1, |\cdot|)$ and the (right) action $x \cdot u = xu$. Then the orbits are concentric cirles, which are closed.

Example 1.0.4. Consider the metric space $(\mathbb{C}^{n \times d}, \|\cdot\|_F)$, topological group $(U(d), \|\cdot\|_F)$ and the (right) action $X \cdot U = XU$

Definition 1.0.5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $\|\cdot\|_* : L^1(X, \mathcal{A}, \mu) \to [0, \infty)$ by

$$\|f\|_* = \sup_{A \in \mathcal{A}} \left| \int_A f d\mu \right|$$

Exercise 1.0.6. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $\|\cdot\|_*$ is a norm on $L^1(X, \mathcal{A}, \mu)$.

Proof. Clear. □

Definition 1.0.7. Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose that X is a compact metric space. Put $\text{Aut}(X) = \{\sigma : X \to X : \sigma \text{ is a homeomorphism}\}$. We metrize $\text{Aut}(X)$ with uniform convergence d_u. It is known that this topology is equivalent to the compact-open topology.

Exercise 1.0.8. With the setup as above, $(\text{Aut}(X), d_u)$ is a topological group.

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}, (\tau_n)_{n\in\mathbb{N}} \subset \mathrm{Aut}(X)$ and $\sigma, \tau \in \mathrm{Aut}(X)$. Suppose that $\sigma_n \xrightarrow{\mathrm{u}} \sigma$ and $\tau_n \xrightarrow{\mathrm{u}} \tau$.

(1) Let $\epsilon > 0$. Since X is compact and σ is continuous, σ is uniformly continuous. Then there exists $\delta > 0$ such that for each $x, y \in X$, $d(x, y) < \delta$ implies that $d(\sigma(x), \sigma(y)) \leq \epsilon/2$. Choose $N_\sigma \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq \mathbb{N}$ implies that $d_u(\sigma_n, \sigma) < \epsilon/2$. Choose $N_\tau \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq \mathbb{N}$ implies that $d_u(\tau_n, \tau) < \delta$. Put $N = \max(N_\sigma, N_\tau)$. Let $n \in \mathbb{N}$ and $x \in X$. Suppose that $n \geq N$. Then

$$\begin{aligned} d(\sigma_n \circ \tau_n(x), \sigma \circ \tau(x)) &\leq d(\sigma_n(\tau_n(x)), \sigma(\tau_n(x))) + d(\sigma(\tau_n(x)), \sigma(\tau(x))) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon \end{aligned}$$

So $d_u(\sigma_n \circ \tau_n, \sigma \circ \tau) \leq \epsilon$ and $\circ : \mathrm{Aut}(X)^2 \to \mathrm{Aut}(X)$ is continuous.

(2) Suppose that $\sigma = \mathrm{id}_X$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $d_u(\sigma_n, \mathrm{id}_X) < \epsilon$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Then

$$\begin{aligned} \sup_{x\in X} d(\sigma_n^{-1}(x), x) &= \sup_{x\in\sigma_n(X)} d(\sigma_n^{-1}(x), x) \\ &= \sup_{x\in X} d(\sigma_n^{-1}(\sigma_n(x)), \sigma_n(x)) \\ &= \sup_{x\in X} d(x, \sigma_n(x)) \\ &< \epsilon \end{aligned}$$

So $\sigma_n^{-1} \xrightarrow{\mathrm{u}} \mathrm{id}_X$. Now suppose that $\sigma \neq \mathrm{id}_X$. Since $\sigma_n \xrightarrow{\mathrm{u}} \sigma$, part (1) implies that $\sigma^{-1} \circ \sigma_n \xrightarrow{\mathrm{u}} \mathrm{id}_X$. Applying the result from above, we get that $\sigma_n^{-1} \circ \sigma \xrightarrow{\mathrm{u}} \mathrm{id}_X$. Applying part (1) again implies that $\sigma_n^{-1} \xrightarrow{\mathrm{u}} \sigma^{-1}$. So the map $\sigma \mapsto \sigma^{-1}$ is continuous.

Hence $\mathrm{Aut}(X)$ is a topological group. □

Definition 1.0.9. Define

$$\mathrm{Aut}(X, \mathcal{A}, \mu) = \{\sigma \in \mathrm{Aut}(X) : \sigma_*\mu = \mu\}$$

So that $(\mathrm{Aut}(X, \mathcal{A}, \mu), d_u)$ is a subspace of $(\mathrm{Aut}(X), d_u)$.

Exercise 1.0.10. We have that $\mathrm{Aut}(X, \mathcal{A}, \mu)$ is a closed subgroup of $\mathrm{Aut}(X)$.

Proof. Still working on this. It is clearly a subgroup. I think μ needs to be a Radon measure to work well with uniform convergence of f_n. □

Example 1.0.11. With the setup as before, define the (right) group action $\cdot : (L^1(X, \mathcal{A}, \mu), \|\cdot\|_*) \times \mathrm{Aut}(X, \mathcal{A}, \mu) \to (L^1(X, \mathcal{A}, \mu), \|\cdot\|_*)$ by $f \cdot \sigma = f \circ \sigma$. Then for each $\sigma \in \mathrm{Aut}(X, \mathcal{A}, \mu)$, the map $f \mapsto f \cdot \sigma$ is an isometry.

Proof. Clear. □

Exercise 1.0.12. With the setup from above, the orbits are closed

Proof. IDK, would like to show. I think I can show $\mathrm{Aut}(X, \mathcal{A}, \mu)$ is compact, then since the action is continuous, for fixed f, the map $\sigma \mapsto f \circ \sigma$ is continuous and hence o_f is compact?? □