1

# Introduction to Differential Geometry

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# Notation

| | |
|---|---|
| $\mathcal{M}_+(X, \mathcal{A})$ | finite measures on $(X, \mathcal{A})$ |
| $v$ | velocity |

# Preface

# Chapter 1

# Review of Fundamentals

## 1.1 Set Theory

**Definition 1.1.0.1.** Let $\{A_i\}_{i\in I}$ be a collection of sets. The **disjoint union of** $\{A_i\}_{i\in I}$, denoted $\coprod\limits_{i\in I} A_i$, is defined by

$$\coprod_{i\in I} A_i = \bigcup_{i\in I} \{i\} \times A_i$$

We define the **natural projection map**, denoted $\pi : \coprod\limits_{i\in I} A_i \to I$, by $\pi(i, a) = i$.

**Definition 1.1.0.2.** Let $E$ and $M$ be sets, $\pi : E \to B$ a surjection and $\sigma : B \to E$. Then $\sigma$ is said to be a section of $(E, M, \pi)$ if $\pi \circ \sigma = \mathrm{id}_M$.

**Note 1.1.0.3.** Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod\limits_{i\in I} A_i$. We will typically be interested in sections $\sigma$ of $\left(\coprod\limits_{i\in I} A_i, I, \pi\right)$.

**Exercise 1.1.0.4.** Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod\limits_{i\in I} A_i$. Then $\sigma$ is a section of $\coprod\limits_{i\in I} A_i$ iff for each $i \in I$, $\sigma(i) \in A_i$

*Proof.* Clear. $\square$

## 1.2 Linear Algebra

**Note 1.2.0.1.** We denote the standard basis on $\mathbb{R}^n$ by $(e_1, \dots, e_n)$.

**Definition 1.2.0.2.** Let $A \in \mathbb{R}^{n\times n}$. Then $A$ is said to be **invertible** if $\det(A) \neq 0$. We denote the set of $n \times n$ invertible matrices by $GL(n, \mathbb{R})$.

$$O(n)$$

**Exercise 1.2.0.3.** Let $A, B \in \mathbb{R}^{n\times n}$. Then $AB = I$ iff $BA = I$.

*Proof.*

- ($\implies$):
  Suppose that $AB = I$. Then

$$\begin{aligned}\ker B &\subset \ker AB \\ &= \ker I \\ &= \{0\}\end{aligned}$$

  so that $\ker B = \{0\}$. Hence $\operatorname{Im} B = \mathbb{R}^n$ and $B$ is surjective. Then

$$\begin{aligned}IB &= BI \\ &= B(AB) \\ &= (BA)B\end{aligned}$$

  Since $B$ is surjective, $I = BA$.

- ($\impliedby$):
  Immediate by the previous part.

□

**Definition 1.2.0.4.** Let $A \in \mathbb{R}^{n\times p}$. Then $A$ is said to be an **orthogonal matrix** if $A^*A = I$. We denote the set of $n \times p$ orthogonal matrices by $O(n, p)$. We write $O(n)$ in place of $O(n, n)$.

$$O(n)$$

**Exercise 1.2.0.5.** Define $\phi : S_n \to GL(n, \mathbb{R})$ by

$$\phi(\sigma) = \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix}$$

Then

1. for each $A \in \mathbb{R}^{n\times p}$,
$$(\phi(\sigma)A)_{i,j} = A_{\sigma(i),j}$$
   i.e. left multiplying $A$ by $\phi(\sigma)$ the the same as permuting the rows of $A$ by $\sigma$
2. $\phi$ is a group homomorphism

*Proof.* 1. Let $A \in \mathbb{R}^{n\times p}$. Then

$$\begin{aligned}(\phi(\sigma)A)_{i,j} &= \langle e^*_{\sigma(i)}, Ae_j\rangle \\ &= A_{\sigma(i),j}\end{aligned}$$

2. Let $\sigma, \tau \in S_n$. Part (1) implies that

$$\begin{aligned}\phi(\sigma\tau) &= \begin{pmatrix} e^*_{\sigma\tau(1)} \\ \vdots \\ e^*_{\sigma\tau(n)} \end{pmatrix} \\ &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e^*_{\tau(1)} \\ \vdots \\ e^*_{\tau(n)} \end{pmatrix} \\ &= \phi(\sigma)\phi(\tau)\end{aligned}$$

Since $\sigma, \tau \in S_n$ are arbitrary, $\phi$ is a group homomorphism.

□

**Definition 1.2.0.6.** Define $\phi : S_n \to GL(n, \mathbb{R})$ as in the previous exercise. Let $P \in GL(n, \mathbb{R})$. Then $P$ is said to be a **permutation matrix** if there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. We denote the set of $n \times n$ permutation matrices by $\mathrm{Perm}(n)$.

**Exercise 1.2.0.7.** We have that

1. $\mathrm{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$
2. $\mathrm{Perm}(n)$ is a subgroup of $O(n)$

*Proof.*

1. By definition, $\mathrm{Perm}(n) = \operatorname{Im}\phi$. Since $\phi : S_n \to GL(n, \mathbb{R})$ is a group homomorphism, $\operatorname{Im}\phi$ is a subgroup of $GL(n, \mathbb{R})$. Hence $\mathrm{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$.
2. Let $P \in \mathrm{Perm}(n)$. Then there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. Then

$$\begin{aligned}PP^* &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix}^* \\ &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e_{\sigma(1)} & \cdots & e_{\sigma(n)} \end{pmatrix} \\ &= (\langle e_{\sigma(i)}, e_{\sigma(j)} \rangle)_{i,j} \\ &= I\end{aligned}$$

A previous exercise implies that $P^*P = I$. Hence $P \in O(n)$. Since $P \in \mathrm{Perm}(n)$ is arbitrary, $\mathrm{Perm}(n) \subset O(n)$. Part (1) implies that $\mathrm{Perm}(n)$ is a group. Hence $\mathrm{Perm}(n)$ is a subgroup of $O(n)$

□

**Note 1.2.0.8.** We will write $P_\sigma$ in place of $\phi(\sigma)$.

**Exercise 1.2.0.9.** Let $Z \in \mathbb{R}^{p \times n}$. If $\operatorname{rank} Z = k$, then there exist $\sigma \in S_n$, $\tau \in S_p$ and $A \in GL(k, \mathbb{R})$, such that for each $i, j \in \{1, \ldots, k\}$,

$$(P_\tau Z P^*_\sigma)_{i,j} = A_{i,j}$$

*Proof.* Suppose that $\operatorname{rank} Z - k$. Then there exist $i_1, \ldots, i_k \in \{1, \ldots, p\}$ such that $i_1 < \cdots < i_k$ and $\{e^*_{i_1} Z, \ldots, e^*_{i_k} Z\}$ is linearly independent. Set

$$Z' = \begin{pmatrix} e^*_{i_1} Z \\ \vdots \\ e^*_{i_k} Z \end{pmatrix}$$

Then $\operatorname{rank} Z' = k$. Hence there exist $j_1, \dots, j_k \in \{1, \dots, n\}$ such that $j_1 < \dots < j_k$, and $\{Z'e_{i_1}, \dots, Z'e_{i_k}\}$ is linearly independent. Set

$$A = \begin{pmatrix} Z'e_{i_1} & \cdots & Z'e_{i_k} \end{pmatrix}$$

Then $A \in \mathbb{R}^{k \times k}$ and $\operatorname{rank} A = k$. Thus $A \in GL(k, \mathbb{R})$. Choose $\sigma \in S_n$ and $\tau \in S_p$ such that $\sigma(1) = j_1, \dots, \sigma(k) = j_k$ and $\tau(1) = i_1, \dots, \tau(k) = i_k$. Let $a, b \in \{1, \dots, k\}$. By construction,

$$\begin{aligned} (P_\tau Z P_\sigma^*)_{a,b} &= Z_{\tau(a),\sigma(b)} \\ &= Z_{i_a, j_b} \\ &= A_{a,b} \end{aligned}$$

□

**Definition 1.2.0.10.** Let $A \in \mathbb{R}^{n \times p}$. Then $A$ is said to be a **diagonal matrix** if for each $i \in [n]$ and $j \in [p]$, $i \neq j$ implies that $A_{i,j} = 0$. We denote the set of $n \times p$ diagonal matrices by $D(n, p, \mathbb{R})$. We write $D(n, \mathbb{R})$ in place of $D(n, n, \mathbb{R})$.

**Definition 1.2.0.11.** For $(n, k)$, $(m, l)$ $\operatorname{diag}_{p,(n \times p)} : \mathbb{R}^p \to \mathbb{R}^{n \times p}$ and $\operatorname{diag}_{n,(n \times p)} : \mathbb{R}^p \to \mathbb{R}^{n \times p}$ by $\operatorname{diag}(v)$ FINISH!!!

**Definition 1.2.0.12.** Let $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \sigma(A)$. Suppose that $A$ is symmetric. We define the **geometric multiplicity** of $\lambda$, denoted $\mu(\lambda)$, by

$$\mu(\lambda) = \dim \ker([\phi_\alpha] - \lambda I)$$

**Definition 1.2.0.13.** Let $V$ be an $n$-dimensional vector space, $U \subset V$ a $k$-dimensional subspace and $(e_j)_{j=1}^n \subset V$ a be a basis. Then $(e_j)_{j=1}^n$ is said to be **adapted to** $U$ if $(e_j)_{j=1}^k$ is a basis for $U$.

## 1.3 Calculus

### 1.3.1 Differentiation

**Definition 1.3.1.1.** Let $n \geq 1$. For $i = 1, \cdots, n$, define $x^i : \mathbb{R}^n \to \mathbb{R}$ by $x^i(a^1, \cdots, a^n) = a^i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on** $\mathbb{R}^n$.

**Definition 1.3.1.2.** Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Then $f$ is said to be **differentiable with respect to** $x^i$ **at** $a$ if

$$\lim_{h \to 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If $f$ is differentiable with respect to $x^i$ at $a$, we define the **partial derivative of** $f$ **with respect to** $x^i$ **at** $a$, denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \frac{\partial}{\partial x^i} f$$

to be the limit above.

**Definition 1.3.1.3.** Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then $f$ is said to be **differentiable with respect to** $x^i$ if for each $a \in U$, $f$ is differentiable with respect to $x^i$ at $a$.

**Exercise 1.3.1.4.** Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i x^j}$ and $\frac{\partial^2 f}{\partial x^j x^i}$ exist and are continuous at $a$. Then

$$\frac{\partial^2 f}{\partial x^i x^j}(a) = \frac{\partial^2 f}{\partial x^j x^i}(a)$$

*Proof.* □

**Definition 1.3.1.5.** Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then $f$ is said to be **smooth** if for each $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $\frac{\partial^k f}{\partial i_1 \cdots i_k}$ exists and is continuous on $U$.

**Definition 1.3.1.6.** Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$. Then $f$ is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \to \mathbb{R}$ such that $U \subset U'$, $U'$ is open, $f'|_U = f$ and $f'$ is smooth. The set of smooth functions on $U$ is denoted $C^\infty(U)$.

**Theorem 1.3.1.7. Taylor's Theorem:**
Let $U \subset \mathbb{R}^n$ be open and convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that for each $x \in U$,

$$f(x) = \sum_{k=0}^{T} \left[ \sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each $|\alpha| = T + 1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

*Proof.* See analysis notes □

**Definition 1.3.1.8.** Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Let $x^1, \cdots, x^n$ be the standard coordinate functions on $\mathbb{R}^n$ and $y_1, \cdots, y_m$ be the standard coordinate functions on $\mathbb{R}^m$. For $i \in \{1, \cdots, m\}$, we define the $i$**th component of** $F$, denoted $F^i : U \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \cdots, F_m)$

**Definition 1.3.1.9.** Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^m$. Then $F$ is said to be **smooth** if for each $i \in \{1, \cdots, m\}$, the $i$th component of $F$, $F^i : U \to \mathbb{R}$, is smooth.

**Definition 1.3.1.10.** Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Then $F$ is said to be **smooth** if for each $x \in U$, there exists $U_x \in \mathcal{N}_x$ and $\tilde{F} : U_x \to \mathbb{R}^m$ such that $U_x$ is open, $\tilde{F}$ is smooth and $\tilde{F}|_{U \cap U_x} = F|_{U \cap U_x}$.

**Definition 1.3.1.11.** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. Then $F$ is said to be a **diffeomorphism** if $F$ is a bijection and $F, F^{-1}$ are smooth.

**Exercise 1.3.1.12.** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. If $F$ is a diffeomorphism, then $F$ is a homeomorphism.

*Proof.* Suppose that $F$ is a diffeomorphism. By definition, $F$ is a bijection and $F$ and $F^{-1}$ are smooth. Thus, $F$ and $F^{-1}$ are continuous and $F$ is a homeomorphism. □

**Definition 1.3.1.13.** Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \to \mathbb{R}^m$. We define the **Jacobian of $F$ at $p$**, denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p)\right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

**Exercise 1.3.1.14. Inverse Function Theorem:**
Let $U, V \subset \mathbb{R}^n$ be open and $F : U \to V$.

**Exercise 1.3.1.15.** Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$. Then $F$ is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of $p$ such that $F|_N : N \to F(N)$ is a diffeomorphism

*Proof.* content... □

**Exercise 1.3.1.16.** Let $\sigma \in S_n$. Define $\phi_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by $\phi(x^1, \dots, x^n) = \phi(x^{\sigma(1), \dots, x^{\sigma(n)}})$. Then $D\phi = P_\sigma$

**Definition 1.3.1.17.** Let $\sigma \in S_n$ and $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. We define $\sigma x \in \mathbb{R}^n$ by

$$\sigma x = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of $S_n$ on $\mathbb{R}^n$ to be the map $S_n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(\sigma, x) \mapsto \sigma x$

**Definition 1.3.1.18.** Let $\sigma \in S_n$, $U$ a set, $V \subset \mathbb{R}^n$ and $\phi : U \to \mathbb{R}^n$ with $\phi = (x^1, \dots, x^m)$. We define $\sigma\phi : U \to \mathbb{R}^n$ by

$$\sigma\phi = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of $S_n$ on $(\mathbb{R}^n)^U$ to be the map $S_n \times (\mathbb{R}^n)^U \to \mathbb{R}^n$ given by $(\sigma, \phi) \mapsto \sigma\phi$.

**Exercise 1.3.1.19.** Let $\sigma \in S_m$. Then for each $p \in \mathbb{R}^n$, $D(\sigma \operatorname{id}_{\mathbb{R}^n})(p) = P_\sigma$.

*Proof.* Note that since $\operatorname{id}_{\mathbb{R}^n} = (\pi_1, \dots, \pi_n)$, we have that $\sigma \operatorname{id}_{\mathbb{R}^n} = (\pi_{\sigma(1)}, \dots, \pi_{\sigma(n)})$. Let $p \in \mathbb{R}^n$. Then

$$\begin{aligned}
D(\sigma \operatorname{id}_{\mathbb{R}^n})(p) &= \left(\frac{\partial \pi_i \circ \sigma \operatorname{id}_{\mathbb{R}^n}}{\partial x^j}(p)\right)_{i,j} \\
&= \left(\frac{\partial \pi_{\sigma(i)}}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma \left(\frac{\partial \pi_i}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma \left(\frac{\partial \pi_i \circ \operatorname{id}_{\mathbb{R}^n}}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma D \operatorname{id}_{\mathbb{R}^n}(p) \\
&= P_\sigma I \\
&= P_\sigma
\end{aligned}$$

□

### 1.3.2 Integration

## 1.4 Topology

**Definition 1.4.0.1.** Let $(X, \mathbb{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then $f$ is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

**Definition 1.4.0.2.** Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then $f$ is said to be a homeomorphism if $f$ is a bijection and $f, f^{-1}$ are continuous.

**Definition 1.4.0.3.** Let $X, Y$ be topological spaces. Then $X$ and $Y$ are said to be **homeomorphic** if there exists $f : X \to Y$ such that $f$ is a homeomorphism. If $X$ and $Y$ are homeomorphic, we write $X \cong Y$.

**Theorem 1.4.0.4.** Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \ncong \mathbb{R}^n$

# Chapter 2

# Multilinear Algebra

## 2.1 $(r,s)$-Tensors

**Definition 2.1.0.1.** Let $V_1,\ldots,V_k,W$ be vector spaces and $\alpha:\prod_{i=1}^n V_i\to W$. Then $\alpha$ is said to be **multilinear** if for each $i\in\{1,\cdots,k\}$, $v\in V$, $c\in\mathbb{R}$ and $v_1,\cdots,v_k\in V$,

$$\alpha(v_1,\cdots,v_i+cv,\cdots,v_k)=\alpha(v_1,\cdots,v_i,\cdots,v_k)+c\alpha(v_1,\cdots,v,\cdots,v_k)$$

We define

$$L(V_1,\ldots,V_k;W)=\left\{\alpha:\prod_{i=1}^n V_i\to W:\alpha\text{ is multilinear}\right\}$$

**Note 2.1.0.2.** For the remainder of this section we let $V$ denote an $n$-dimensional vector space with basis $\{e^1,\cdots,e^n\}$ with dual space $V^*$ and dual basis $\{\epsilon_1,\cdots,\epsilon_n\}$ defined by $\epsilon^i(e^j)=\delta_{i,j}$. We identify $V$ with $V^{**}$ by the isomorphism $V\to V^{**}$ defined by $v\mapsto\hat{v}$ where $\hat{v}(\alpha)=\alpha(v)$ for each $\alpha\in V^*$.

**Definition 2.1.0.3.** Let $\alpha:(V^*)^r\times V^s\to\mathbb{R}$. Then $\alpha$ is said to be an $(r,s)$-tensor on $V$ if $\alpha\in L(\underbrace{V^*,\ldots,V^*}_{r},\underbrace{V,\ldots,V}_{s};\mathbb{R})$. The set of all $(r,s)$-tensors on $V$ is denoted $T^r_s(V)$.
When $r=s=0$, we set $T^r_s=\mathbb{R}$.

**Exercise 2.1.0.4.** We have that $T^r_s(V)$ is a vector space.

*Proof.* Clear. □

**Exercise 2.1.0.5.** Under the identification of $V$ with $V^{**}$ as noted above, we have that $V=T^1_0(V)$.

*Proof.* By definition,

$$\begin{aligned}V&=V^{**}\\&=L(V^*;\mathbb{R})\\&=T^1_0(V)\end{aligned}$$

□

**Definition 2.1.0.6.** Let $\alpha\in T^{r_1}_{s_1}(V)$ and $\beta\in T^{r_2}_{s_2}(V)$. We define the **tensor product of** $\alpha$ **with** $\beta$, denoted $\alpha\otimes\beta\in T^{r_1+r_2}_{s_1+s_2}(V)$, by

$$\alpha\otimes\beta(v^*,w^*,v,w)=\alpha(v^*,v)\beta(w^*,w)$$

for each $v^*\in(V^*)^{r_1}$, $w^*\in(V^*)^{r_2}$, $v\in V^{s_1}$ and $w\in V^{s_2}$.
When $r_1=s_1=r_2=s_2=0$ (so that $\alpha,\beta\in\mathbb{R}$), we set $\alpha\otimes\beta=\alpha\beta$.

**Definition 2.1.0.7.** We define the **tensor product**, denoted $\otimes:T^{r_1}_{s_1}(V)\times T^{r_2}_{s_2}(V)\to T^{r_1+r_2}_{s_1+s_2}(V)$ by

$$(\alpha,\beta)\mapsto\alpha\otimes\beta$$

**Exercise 2.1.0.8.** The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is well defined.

*Proof.* Tedious but straightforward. □

**Exercise 2.1.0.9.** The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is associative.

*Proof.* Let $\alpha \in T^{r_1}_{s_1}(V)$, $\beta \in T^{r_2}_{s_2}(V)$ and $\gamma \in T^{r_3}_{s_3}(V)$. Then for each $u^* \in (V^*)^{r_1}, v^* \in (V^*)^{r_2}, w^* \in (V^*)^{r_3}, u \in V^{s_1}, v \in V^{s_2}, w \in V^{s_3}$,

$$\begin{aligned}
(\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v)\gamma(w^*, w) \\
&= [\alpha(u^*, u)\beta(v^*, v)]\gamma(w^*, w) \\
&= \alpha(u^*, u)[\beta(v^*, v)\gamma(w^*, w)] \\
&= \alpha(u^*, u)(\beta \otimes \gamma)(v^*, w^*, v, w) \\
&= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w)
\end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

**Exercise 2.1.0.10.** The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is bilinear.

*Proof.*

1. Linearity in the first argument:
   Let $\alpha, \beta \in T^{r_1}_{s_1}(V)$, $\gamma \in T^{r_2}_{s_2}(V), \lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $vinV^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned}
[(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v)\gamma(w^*, w) \\
&= [\alpha(v^*, v) + \lambda\beta(v^*, v)]\gamma(w^*, w) \\
&= \alpha(v^*, v)\gamma(w^*, w) + \lambda\beta(v^*, v)\gamma(w^*, w) \\
&= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\
&= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w)
\end{aligned}$$

   So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

2. Linearity in the second argument:
   Similar to (1).

□

**Definition 2.1.0.11.**

1. Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1, \cdots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length** $k$. Recall that $\#\mathcal{I}_{\otimes k} = n^k$.

2. Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length** $k$. Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

**Note 2.1.0.12.** For the remainder of this section we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\otimes k}$.

**Definition 2.1.0.13.** Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.

1. Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by

$$\epsilon^I = (\epsilon^{i_1}, \cdots, \epsilon^{i_k})$$

   and

$$e^I = (e^{i_1}, \cdots, e^{i_k})$$

2. Define $e^{\otimes I} \in T_0^k(V)$ and $\epsilon^{\otimes I} \in T_k^0(V)$ by
$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$
and
$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

**Exercise 2.1.0.14.** Let $\alpha, \beta \in T_s^r(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

*Proof.* Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v_1^*, \dots, v_r^* \in V^*$ and $v_1, \dots, v_s \in V$. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, write
$$v_i^* = \sum_{k_i=1}^{n} a_{i,k_i} \epsilon^{k_i}$$
and
$$v_j = \sum_{l_j=1}^{n} b_{j,l_j} e^{l_j}$$
Then
$$\begin{aligned}
\alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1,\dots,k_r=1}^{n} \sum_{l_1,\dots,l_s=1}^{n} \prod_{i=1}^{r} \prod_{j=1}^{s} a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \sum_{k_1,\dots,k_r=1}^{n} \sum_{l_1,\dots,l_s=1}^{n} \prod_{i=1}^{r} \prod_{j=1}^{s} a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s)
\end{aligned}$$
So that $\alpha = \beta$. □

**Exercise 2.1.0.15.** Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K}\delta_{J,L}$.

*Proof.* Write $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$ and $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$. Then
$$\begin{aligned}
e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K)\epsilon^{\otimes J}(e^L) \\
&= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r})\epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\
&= \left[\prod_{m=1}^{r} e^{i_m}(\epsilon^{k_m})\right]\left[\prod_{n=1}^{s} \epsilon^{j_n}(e^{l_n})\right] \\
&= \left[\prod_{m=1}^{r} \delta_{i_m,k_m}\right]\left[\prod_{n=1}^{s} \delta_{j_n,l_n}\right] \\
&= \delta_{I,K}\delta_{J,L}
\end{aligned}$$
□

**Exercise 2.1.0.16.** The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T_s^r(V)$ and $\dim T_s^r(V) = n^{r+s}$.

*Proof.* Let $(a_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a_J^I e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a_J^I = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T_s^r(V)$. For $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b_J^I = \beta(\epsilon^J, e^I)$. Define $\mu = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b_J^I e^{\otimes I} \otimes \epsilon^{\otimes J} \in T_s^r(V)$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b_J^I = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. □

## 2.2 Covariant $k$-Tensors

### 2.2.1 Symmetric and Alternating Covariant $k$-Tensors

**Definition 2.2.1.1.** Let $\alpha : V^k \to \mathbb{R}$. Then $\alpha$ is said to be a **covariant k-tensor on V** if $\alpha \in T^0_k(V)$. We denote the set of covariant $k$-tensors by $T_k(V)$.

**Definition 2.2.1.2.** For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \to \mathbb{R}$ by

$$\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

We define the **permutation action** of of $S_k$ on $T_k(V)$ to be the map $S_k \times T_k(V) \to T_k(V)$ given by $(\sigma, \alpha) \mapsto \sigma\alpha$

**Exercise 2.2.1.3.** The permutation action of $S_k$ on $T_k(V)$ is a group action.

*Proof.*

1. Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V), \sigma\alpha \in T_k(V)$.
2. Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
3. Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \cdots, v_k \in V$,

$$\begin{aligned}(\tau\sigma)\alpha(v_1, \cdots, v_k) &= \alpha(v_{\tau\sigma(1)}, \cdots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \cdots, v_k)\end{aligned}$$

□

**Exercise 2.2.1.4.** Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \to T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

*Proof.* Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$. Then

$$\begin{aligned}\sigma(c\alpha + \beta)(v_1, \cdots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \cdots, v_k) + \sigma\beta(v_1, \cdots, v_k)\end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

**Definition 2.2.1.5.** Let $\alpha \in T_k(V)$. Then $\alpha$ is said to be

- **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$
- **antisymmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \operatorname{sgn}(\sigma)\alpha$
- **alternating** if for each $v_1, \dots, v_k \in V$, if there exists $i, j \in \{1, \dots, k\}$ such that $v_i = v_j$, then $\alpha(v_1, \cdots, v_k) = 0$.

We denote the set of symmetric $k$-tensors on $V$ by $\Sigma^k(V)$. We denote the set of alternating $k$-tensors on $V$ by $\Lambda^k(V)$.

**Exercise 2.2.1.6.** Let $\alpha \in T_k(V)$. Then $\alpha$ is antisymmetric iff $\alpha$ is alternating.

*Proof.* Suppose that $\alpha$ is antisymmetric. Let $v_1, \dots, v_k \in V$. Suppose that there exists $i, j \in \{1, \dots, k\}$ such that $v_i = v_j$. Define $\sigma \in S_k$ by $\sigma = (i, j)$. Then

$$\begin{aligned}\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) &= \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \\ &= \sigma(\alpha)(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &= \operatorname{sgn}(\sigma)\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &= -\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k)\end{aligned}$$

Therefore $2\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = 0$ which implies that $\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = 0$. Hence $\alpha$ is alternating.
Conversely, suppose that $\alpha$ is alternating. Let $i, j \in \{1, \ldots, k\}$ and $v_1, \ldots, v_k \in V$. Then

$$\begin{aligned} 0 &= \alpha(v_1, \cdots, v_i + v_j, \cdots, v_i + v_j, \cdots, v_k) \\ &= \alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) + \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \end{aligned}$$

Since $i, j \in \{1, \ldots, k\}$ and $v_1, \ldots, v_k \in V$ are arbitrary, we have that for each $\tau \in S_k$, $\tau$ is a transposition implies that

$$\begin{aligned} \tau\alpha &= -\alpha \\ &= \operatorname{sgn}(\tau)\alpha \end{aligned}$$

Let $n \in \mathbb{N}$. Suppose that for each $\tau_1, \ldots, \tau_{n-1} \in S_k$ if for each $j \in \{1, \ldots, n-1\}$, $\tau_j$ is a transposition, then $(\tau_1 \cdots \tau_{n-1})\alpha = \sigma(\tau_1 \cdots \tau_{n-1})\alpha$. Let $\tau_1, \ldots, \tau_n \in S_k$. Suppose that for each $j \in \{1, \ldots, n\}$, $\tau_j$ is a transposition. Then

$$\begin{aligned} (\tau_1 \cdots \tau_n)\alpha &= (\tau_1 \cdots \tau_{n-1})(\tau_n\alpha) \\ &= (\tau_1 \cdots \tau_{n-1})(\operatorname{sgn}(\tau_n)\alpha) \\ &= (\operatorname{sgn}(\tau_n)(\tau_1 \cdots \tau_{n-1})\alpha \\ &= (\operatorname{sgn}(\tau_n)\operatorname{sgn}((\tau_1 \cdots \tau_{n-1})\alpha \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha \end{aligned}$$

By induction, for each $n \in \mathbb{N}$ and $\tau_1, \ldots, \tau_n \in S_k$, if for each $j \in \{1, \ldots, n\}$, $\tau_j$ is a transposition, then $(\tau_1 \cdots \tau_n)\alpha = \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha$. Now let $\sigma \in S_k$. Then there exist $n \in \mathbb{N}$ and $\tau_1, \ldots, \tau_n \in S_k$ such that $\sigma = \tau_1 \cdots \tau_n$ and for each $j \in \{1, \ldots n\}$, $\tau_j$ is a transposition. Hence

$$\begin{aligned} \sigma\alpha &= (\tau_1 \cdots \tau_n)\alpha \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha \\ &= \operatorname{sgn}(\sigma)\alpha \end{aligned}$$

Therefore $\alpha$ is antisymmetric. □

**Definition 2.2.1.7.** Define the **symmetric operator** $S : T_k(V) \to \Sigma^k(V)$ by

$$\operatorname{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \to \Lambda^k(V)$ by

$$\operatorname{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha$$

**Exercise 2.2.1.8.**

1. For $\alpha \in T_k(V)$, $\operatorname{Sym}(\alpha)$ is symmetric.
2. For $\alpha \in T_k(V)$, $\operatorname{Alt}(\alpha)$ is alternating.

*Proof.*

1. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned} \sigma \operatorname{Sym}(\alpha) &= \sigma\left[\frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha\right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \\ &= \operatorname{Sym}(\alpha) \end{aligned}$$

2. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
\sigma \operatorname{Alt}(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right] \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\sigma\tau\alpha \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \\
&= \operatorname{sgn}(\sigma)\operatorname{Alt}(\alpha)
\end{aligned}$$

□

**Exercise 2.2.1.9.**

1. For $\alpha \in \Sigma^k(V)$, $\operatorname{Sym}(\alpha) = \alpha$.
2. For $\alpha \in \Lambda^k(V)$, $\operatorname{Alt}(\alpha) = \alpha$.

*Proof.*

1. Let $\alpha \in \Sigma^k(V)$. Then

$$\begin{aligned}
\operatorname{Sym}(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} \alpha \\
&= \alpha
\end{aligned}$$

2. Let $\alpha \in \Lambda^k(V)$. Then

$$\begin{aligned}
\operatorname{Alt}(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2\alpha \\
&= \alpha
\end{aligned}$$

□

**Exercise 2.2.1.10.** The symmetric operator $S : T_k(V) \to \Sigma^k(V)$ and the alternating operator $A : T_k(V) \to \Lambda^k(V)$ are linear.

*Proof.* Clear. □

**Exercise 2.2.1.11.** Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

1. $\operatorname{Alt}(\operatorname{Alt}(\alpha) \otimes \beta) = \operatorname{Alt}(\alpha \otimes \beta)$
2. $\operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta)) = \operatorname{Alt}(\alpha \otimes \beta)$

*Proof.* First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \to S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma,\tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

1. Then

$$\begin{aligned}
\mathrm{Alt}(\mathrm{Alt}(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \Big[ \mathrm{Alt}(\alpha) \otimes \beta \Big] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[ \left( \frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)\tau\alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)(\tau\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)\tau(\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \mathrm{sgn}(\sigma\tau)\sigma\tau(\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \mathrm{Alt}(\alpha \otimes \beta)
\end{aligned}$$

2. Similar to (1).

□

### 2.2.2 Exterior Product

**Definition 2.2.2.1.** Let $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$. The **exterior product** of $\alpha$ and $\beta$ is defined to be the map $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$.

**Exercise 2.2.2.2.** The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is bilinear.

*Proof.* Clear. □

**Exercise 2.2.2.3.** The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is associative.

*Proof.* Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$ and $\gamma \in \Lambda^m(V)$. Then

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)\right] \wedge \gamma \\
&= \frac{(k+l+m)!}{(k+l)!m!} \operatorname{Alt}\left(\left[\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)\right] \otimes \gamma\right) \\
&= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} \operatorname{Alt}(\operatorname{Alt}(\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} \operatorname{Alt}((\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \operatorname{Alt}(\alpha \otimes (\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \operatorname{Alt}(\alpha \otimes \frac{(l+m)!}{l!m!} \operatorname{Alt}(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \operatorname{Alt}(\alpha \otimes (\beta \wedge \gamma)) \\
&= \alpha \wedge (\beta \wedge \gamma))
\end{aligned}$$

$\square$

**Exercise 2.2.2.4.** Let $\alpha_i \in \Lambda^{k_i}(V)$ for $i = 1, \cdots, m$. Then

$$\bigwedge_{i=1}^m \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^m \alpha_i\right)$$

*Proof.* To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1+k_2+k_3)!}{k_1!k_2!k_3!} \operatorname{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \le m \le m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i\right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \operatorname{Alt}\left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \operatorname{Alt}\left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m_0-1} \alpha_i\right)\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\operatorname{Alt}\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m_0+1} \alpha_i\right)
\end{aligned}$$

$\square$

**Exercise 2.2.2.5.** Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of $\tau$ is $kl$. (Hint: inversion number)

*Proof.*

$$\begin{aligned} N(\tau) &= \sum_{i=1}^{l} k \\ &= kl \end{aligned}$$

Since $\operatorname{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\operatorname{sgn}(\tau) = (-1)^{kl}$. □

**Exercise 2.2.2.6.** Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

*Proof.* Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \cdots, v_{k+l} \in V$, we have that

$$\begin{aligned} \sigma\tau(\beta \otimes \alpha)(v_1, \cdots, v_l, v_{l+1}, \cdots v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)})\alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \\ &= \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)})\beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\ &= \alpha \otimes \beta(v_{\sigma(1)}, \cdots v_{\sigma(k)}, v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\ &= \sigma(\alpha \otimes \beta)(v_1, \cdots, v_k, v_{1+k}, \cdots v_{l+k}) \end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned} \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} \operatorname{Alt}(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma\tau)\sigma\tau(\beta \otimes \alpha) \\ &= \operatorname{sgn}(\tau)\frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha \otimes \beta) \\ &= \operatorname{sgn}(\tau)\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta) \\ &= \operatorname{sgn}(\tau)\alpha \wedge \beta \\ &= (-1)^{kl}\alpha \wedge \beta \end{aligned}$$

□

**Exercise 2.2.2.7.** Let $\alpha \in \Lambda^k(V)$. If $k$ is odd, then $\alpha \wedge \alpha = 0$.

*Proof.* Suppose that $k$ is odd. The previous exercise tells us that

$$\begin{aligned} \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\ &= -\alpha \wedge \alpha \end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

**Exercise 2.2.2.8. Fundamental Example:**
Let $\alpha_1, \cdots, \alpha_m \in \Lambda^1(V)$ and $v_1, \cdots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^{m} \alpha_i\right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

*Proof.* The previous exercises tell us that

$$\begin{aligned}\left(\bigwedge_{i=1}^{m} \alpha_i\right)(v_1, \cdots, v_m) &= m! \operatorname{Alt}\left(\bigotimes_{i=1}^{m} \alpha_i\right)(v_1, \cdots, v_m) \\ &= m!\left[\frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\sigma\left(\bigotimes_{i=1}^{m} \alpha_i\right)\right](v_1, \cdots, v_m) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\left(\bigotimes_{i=1}^{m} \alpha_i\right)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} \alpha_i(v_{\sigma(i)}) \\ &= \det(\alpha_i(v_j))\end{aligned}$$

□

**Note 2.2.2.9.** Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\wedge k}$.

**Definition 2.2.2.10.** Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.
Define $\epsilon^{\wedge I} \in \Lambda^k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$$

**Exercise 2.2.2.11.** Let $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

*Proof.* Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \cdots & \epsilon^{i_1}(e^{j_k}) \\ & \vdots & \\ \epsilon^{i_k}(e^{j_1}) & \cdots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$. If $I = J$, then $A = I_{k \times k}$ and therefore $\epsilon^I(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the $l_0$-th row of $A$ are 0. If $i_{l_0} > j_{l_0}$, then all entries on the $l_0$-th column of $A$ are 0. □

**Exercise 2.2.2.12.** Let $\alpha, \beta \in \Lambda^k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

*Proof.* Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \cdots, v_k \in V$. For $i = 1, \cdots, k$, write $v_i =$

$\sum_{j_i=1}^n a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned}
\alpha(v_1, \cdots, v_k) &= \sum_{j_1, \cdots, j_k=1}^{n} \left( \prod_{i=1}^{k} a_{i,j_i} \right) \alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{j_1 \neq \cdots \neq j_k}^{n} \left( \prod_{i=1}^{k} a_{i,j_i} \right) \alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{J \in \mathcal{I}_k} \left[ \sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left( \prod_{i=1}^{k} a_{i,\sigma(j_i)} \right) \right] \alpha(e^J) \\
&= \sum_{J \in \mathcal{I}_k} \left[ \sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left( \prod_{i=1}^{k} a_{i,\sigma(j_i)} \right) \right] \beta(e^J) \\
&= \sum_{j_1, \cdots, j_k=1}^{n} \left( \prod_{i=1}^{k} a_{i,j_i} \right) \beta(e^{j_1}, \cdots, e^{j_k}) \\
&= \beta(v_1, \cdots, v_k)
\end{aligned}$$

□

**Exercise 2.2.2.13.** The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(V)$ and $\dim \Lambda^k(V) = \binom{n}{k}$.

*Proof.* Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum\limits_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$. Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda^k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum\limits_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda^k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$. □

### 2.2.3 Interior Product

**Definition 2.2.3.1.** Let $V$ be a finite dimensional vector space and $v \in V$. We define **interior multiplication by** $v$, denoted $\iota_v : T_k \to T_{k-1}$, by

$$\iota_v \alpha(w_1, \ldots, w_{k-1}) = \alpha(v, w_1, \ldots, w_{k-1})$$

**Exercise 2.2.3.2.** Let $V$ be a finite dimensional vector space and $v \in V$. Then $\iota_v|_{\Lambda^k(V)} : \Lambda^k(V) \to \Lambda^{k-1}(V)$.

*Proof.* Let $\alpha \in \Lambda^k(V)$. Define $\beta \in \Lambda^k(V)$ by $\beta(w_1, \ldots, w_k) = \alpha(w_k, w_1, \ldots, w_{k-1})$ . Let $\sigma \in S_{k-1}$. Define $\tau \in S_k$ by $\tau(j) = \begin{cases} 1 & j = k \\ \sigma(j) & j \neq k \end{cases}$. Let $w_1, \ldots, w_{k-1} \in V$. Set $w_k = v$. Then

$$\begin{aligned}
\sigma(\iota_v \alpha)(w_1, \ldots, w_{k-1}) &= \iota_v \alpha(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}) \\
&= \alpha(v, w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}) \\
&= \beta(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}, v) \\
&= \beta(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}, w_k) \\
&= \beta(w_{\tau(1)}, \ldots, w_{\tau(k-1)}, w_{\tau(k)}) \\
&= \operatorname{sgn}(\tau) \beta(w_1, \ldots, w_{k-1}, w_k) \\
&= \operatorname{sgn}(\sigma) \beta(w_1, \ldots, w_{k-1}, v) \\
&= \operatorname{sgn}(\sigma) \alpha(v, w_1, \ldots, w_{k-1}) \\
&= \operatorname{sgn}(\sigma)(\iota_v \alpha)(w_1, \ldots, w_{k-1})
\end{aligned}$$

Since $w_1, \ldots, w_{k-1} \in V$ are arbitrary, $\sigma(\iota_v \alpha) = \operatorname{sgn}(\sigma) \iota_v \alpha$. Hence $\iota_v \alpha \in \Lambda^{k-1}(V)$. □

## 2.3 $(0, 2)$-Tensors

**Definition 2.3.0.1.** Let $V$ be a finite dimensional vector space, $v \in V$ and $\alpha \in T^0_2(V)$. Then $\alpha$ is said to be **degenerate** if there exists $v \in V$ such that for each $w \in V$, $\alpha(v, w) = 0$ and $v \neq 0$.

**Definition 2.3.0.2.** Let $V$ be a finite dimensional vector space, $\alpha \in T^0_2(V)$. We define $\phi_\alpha : V \to V^*$ by

$$\phi_\alpha(v) = \iota_v \alpha$$

**Exercise 2.3.0.3.** Let $V$ be a finite dimensional vector space, $\alpha \in T^0_2(V)$. Then $\phi_\alpha \in L(V; V^*)$.

*Proof.* Let $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$. Then for each $w \in V$,

$$\begin{aligned}
\phi_\alpha(v_1 + \lambda v_2)(w) &= (\iota_{v_1+\lambda v_2}\alpha)(w) \\
&= \alpha(v_1 + \lambda v_2, w) \\
&= \alpha(v_1, w) + \lambda\alpha(v_2, w) \\
&= (\iota_{v_1}\alpha)(w) + \lambda(\iota_{v_2}\alpha)(w) \\
&= \phi_\alpha(v_1)(w) + \lambda\phi_\alpha(v_2)(w) \\
&= [\phi_\alpha(v_1) + \lambda\phi_\alpha(v_2)](w)
\end{aligned}$$

Therefore, $\phi_\alpha(v_1 + \lambda v_2) = \phi_\alpha(v_1) + \lambda\phi_\alpha(v_2)$. Thus $\phi_\alpha \in L(V; V^*)$. □

**Exercise 2.3.0.4.** Let $V$ be a finite dimensional vector space and $\alpha \in T^0_2(V)$. Then $\alpha$ is nondegenerate iff $\phi_\alpha$ is an isomorphism.

*Proof.*

- ($\implies$ :)
  Suppose that $\alpha$ is nondegenerate. Let $v \in \ker \phi_\alpha$. Then for each $w \in V$,

  $$\begin{aligned}
  \alpha(v, w) &= (\iota_v\alpha)(w) \\
  &= \phi_\alpha(v)(w) \\
  &= 0
  \end{aligned}$$

  Since $\alpha$ is nondegenerate, $v = 0$. Since $v \in \ker \phi_\alpha$ is arbitrary, $\ker \phi_\alpha = \{0\}$. Hence $\phi_\alpha$ is injective. Since $\dim V = \dim V^*$, $\phi_\alpha$ is surjective. Hence $\phi_\alpha$ is an isomorphism.

- ($\impliedby$ :)
  Suppose that $\phi_\alpha$ is an isomorphism. Let $v \in V$. Suppose that for each $w \in V$, $\alpha(v, w) = 0$. Then for each $w \in V$,

  $$\begin{aligned}
  \phi_\alpha(v)(w) &= (\iota_v\alpha)(w) \\
  &= \alpha(v, w) \\
  &= 0
  \end{aligned}$$

  Thus $\phi_\alpha(v) = 0$ which implies that $v \in \ker \phi_\alpha$. Since $\phi_\alpha$ is an isomorphism, $v = 0$. Hence $\alpha$ is nondegenerate.

□

**Exercise 2.3.0.5.** Let $V$ be a finite dimensional vector space and $\alpha \in T^0_2(V)$. Then

1. $[\phi_\alpha]_{i,j} = \alpha(e_j, e_i)$
2. for each $v, w \in V$,

$$\alpha(v, w) = [w]^*[\phi_\alpha][v]$$

*Proof.* 1. Set $A = [\phi_\alpha]$. Let $i, j \in \{1, \dots, n\}$. By definition,

$$\phi_\alpha(e_j) = \sum_{k=1}^{n} A_{k,j} \epsilon^k$$

Then

$$\begin{aligned}\phi_\alpha(e_j)(e_i) &= \sum_{k=1}^{n} A_{k,j} \epsilon^k(e_i) \\ &= \sum_{k=1}^{n} A_{k,j} \delta_{k,i} \\ &= A_{i,j}\end{aligned}$$

2. Let $v, w \in V$. Then there exist $(v^i)_{i=1}^n, (w^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum\limits_{i=1}^{n} v^i e_i$ and $w = \sum\limits_{j=1}^{n} v^j e_i$. Part (1) implies that

$$\begin{aligned}\alpha(v, w) &= \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j \alpha(e_i, e_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j [\phi_\alpha]_{j,i} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} [v]_i [w]_j [\phi_\alpha]_{j,i} \\ &= [w]^* [\phi_\alpha][v]\end{aligned}$$

□

### 2.3.1 Scalar Product Spaces

**Definition 2.3.1.1.** Let $V$ be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then $\alpha$ is said to be

- **positive semidefinite** if for each $v \in V$, $\alpha(v, v) \geq 0$
- **positive definite** if for each $v \in V$, $v \neq 0$ implies that $\alpha(v, v) > 0$
- **negative semidefinite** if $-\alpha$ is positive semidefinite
- **negative definite** if $-\alpha$ is positive definite

**Exercise 2.3.1.2.** Let $V$ be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then

1. $\alpha$ is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda > 0$
2. $\alpha$ is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda \geq 0$

*Proof.*

1. Suppose that $\alpha$ is positive definite. Write $\sigma(\phi_\alpha) = \{\lambda_1, \dots, \lambda_n\}$. Define $\Lambda \in \mathbb{R}^{n \times n}$ by $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. Since $\alpha$ is symmetric, $[\phi_\alpha]$ is symmetric. There exists $U \in O(n)$ such that $[\phi_\alpha] = U \Lambda U^*$. FINISH!!!

□

**Definition 2.3.1.3.** Let $V$ be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then $\alpha$ is said to be a **scalar product** if $\alpha$ is nondegenerate. In this case, $(V, \alpha)$ is said to be a **scalar product space**.

**Definition 2.3.1.4.** Let $V$ be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$ a scalar product on $V$. We define the **index** of $\alpha$, denoted $\operatorname{ind} \alpha$ by

$$\operatorname{ind} \alpha = \max\{\dim W : W \text{ is a subspace of } V \text{ and } \alpha|_{W \times W} \text{ is negative definite}\}$$

**Definition 2.3.1.5.** Let $(V, \alpha)$ be a scalar product space.

- Let $v_1, v_2 \in V$. Then $v_1$ and $v_2$ are said to be **orthogonal** if $\alpha(v_1, v_2) = 0$.
- Let $U \subset V$ be a subspace. We define the **orthogonal subspace of** $U$, denoted by $U^\perp$, by

$$U^\perp = \{v \in V : \text{ for each } u \in U, \, \alpha(u, v) = 0\}$$

**Exercise 2.3.1.6.** Let $(V, \alpha)$ be a scalar product space and $U \subset V$ a subspace. Then $U^\perp$ is a subspace of $V$.

*Proof.* We note that since $U^\perp = \bigcap_{u \in U} \ker \phi_\alpha(u)$, $U^\perp$ is a subspace of $V$. □

**Exercise 2.3.1.7.** Let $(V, \alpha)$ be an $n$-dimensional scalar product space, $U \subset V$ a $k$-dimensional subspace and $(e_j)_{j=1}^n \subset V$ a basis for $V$. Suppose that $(e_j)_{j=1}^k$ is a basis for $U$. Then for each $v \in V$, $v \in U^\perp$ iff for each $j \in [k]$, $\alpha(v, e_j) = 0$.

*Proof.* Let $v \in V$.

- $(\implies)$: Suppose that $v \in U^\perp$. Since $(e_j)_{j=1}^k \subset U$, we have that for each $j \in [k]$, $\alpha(v, e_j) = 0$.
- $(\impliedby)$: Suppose that for each $j \in [k]$, $\alpha(v, e_j) = 0$. Let $u \in U$. Then there exist $(a^j)_{j=1}^k \subset \mathbb{R}$ such that $u = \sum_{j=1}^k a^j u_j$. This implies that

$$\begin{aligned}\alpha(v, u) &= \sum_{j=1}^k a^j \alpha(v, u_j) \\ &= 0\end{aligned}$$

  Since $u \in U$ is arbitrary, we have that $v \in U^\perp$.

□

**Exercise 2.3.1.8.** Let $(V, \alpha)$ be a scalar product space and $U \subset V$ a subspace. Then

1. $\dim V = \dim U + \dim U^\perp$
2. $(U^\perp)^\perp = U$

*Proof.*

1. Set $n = \dim V$ and $k = \dim U$. Choose a basis $(e_j)_{j=1}^n$ such that $(e_j)_{j=1}^k$ is a basis for $U$.
2. 

□

**Exercise 2.3.1.9.** Let $V$ be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Set $\sigma([\phi_\alpha])^- = \{\lambda \in \sigma([\phi_\alpha]) : \lambda < 0\}$. Then

$$\operatorname{ind} \alpha = \sum_{\lambda \in \sigma([\phi_\alpha])^-} \mu(\lambda)$$

*Proof.* Since $\alpha$ is symmetric, there exist $U \in O(n)$ and $\Lambda \in D(n, \mathbb{R})$ such that $[\phi_\alpha] = U\Lambda U^*$. Define $(u_j)_{j=1}^n \subset V$ by $u_j = \sum_{i=1}^n U_{i,j} e_j$. Define $J^- = \{j \in [n] : \Lambda_{j,j} < 0\}$, $n^- = \# J^-$ and $V^- = \operatorname{span}\{u_j : j \in J^-\}$. Let $v \in V^-$. Then there exist $(a^j)_{j \in J^-}$ such that $v = \sum_{j \in J^-} a^j u_j$. We note that

$$\begin{aligned} U^*[\phi_\alpha]U &= U^*(U\Lambda U^*)U \\ &= (U^*U)\Lambda(U^*U) \\ &= I\Lambda I \\ &= \Lambda \end{aligned}$$

A previous exercise implies that

$$\begin{aligned} \alpha(v,v) &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k \alpha(u_j, u_k) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k [u_j]^* [\phi_\alpha][u_k] \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k ([e_j]^* U^*)[\phi_\alpha](U[e_k]) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (U^*[\phi_\alpha]U)_{j,k} \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (\Lambda)_{j,k} \\ &= \sum_{j \in J^-} |a^j|^2 \Lambda_{j,j} \\ &< 0 \end{aligned}$$

Since $v \in V^-$ is arbitrary, $\alpha|_{V^- \times V^-}$ is negative definite. Thus

$$\begin{aligned} \operatorname{ind} \alpha &\geq \dim V^- \\ &= n^- \end{aligned}$$

Set $J^+ = (J^-)^c$. Let $W \subset V$ be a subspace. Suppose that $\alpha|_{W \times W}$ is negative definite. For the sake of contradiction, suppose that there exists $j_0 \in J^+$ such that $u_{j_0} \in W$. Then

$$\begin{aligned} \alpha(u_{j_0}, u_{j_0}) &= [u_{j_0}]^*[\phi_\alpha][u_{j_0}] \\ &= [u_{j_0}]^* U\Lambda U^* [u_{j_0}] \\ &= \Lambda_{j_0, j_0} \\ &\geq 0 \end{aligned}$$

which is a contradiction since $\alpha|_{W \times W}$ is negative definite. Thus for each $j \in J^+$, $u_j \notin W$. □

### 2.3.2 Symplectic Vector Spaces

**Definition 2.3.2.1.** Let $V$ be a finite dimensional vector space and $\omega \in \Lambda^2(V)$. Then $\omega$ is said to be a **symplectic form** if $\omega$ is nondegenerate. In this case $(V, \omega)$ is said to be a **symplectic space**.

**Exercise 2.3.2.2.** Let $V$ be a $2n$-dimensional vector space with basis $(a_j, b_j)_{j=1}^n$ and corresponding dual basis $(\alpha^j, \beta^j)_{j=1}^n$. Define $\omega \in \Lambda^2(V)$ by

$$\omega = \sum_{j=1}^n \alpha^j \wedge \beta^j$$

Then

1. for each $j, k \in \{1, \dots, n\}$,
   (a) $\omega(a_j, a_k) = 0$
   (b) $\omega(b_j, b_k) = 0$
   (c) $\omega(a_j, b_k) = \delta_{j,k}$
2. $(V, \omega)$ is a symplectic space

*Proof.*

1. Let $j, k \in \{1, \dots, n\}$.
   (a)
   $$\begin{aligned}\omega(a_j, a_k) &= \sum_{l=1}^{n} \alpha^l \wedge \beta^l(a_j, a_k) \\ &= \sum_{l=1}^{n} [\alpha^l(a_j)\beta^l(a_k) - \alpha^l(a_k)\beta^l(a_j)] \\ &= 0\end{aligned}$$
   (b) Similar to (a)
   (c)
   $$\begin{aligned}\omega(a_j, b_k) &= \sum_{l=1}^{n} \alpha^l \wedge \beta^l(a_j, b_k) \\ &= \sum_{l=1}^{n} [\alpha^l(a_j)\beta^l(b_k) - \alpha^l(b_k)\beta^l(a_j)] \\ &= \sum_{l=1}^{n} \alpha^l(a_j)\beta^l(b_k) \\ &= \sum_{l=1}^{n} \delta_{j,l}\delta_{l,k} \\ &= \delta_{j,k}\end{aligned}$$
2. Let $v \in V$. Then there exist $(q^j, p^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{j=1}^{n} q^j a_j + p^j b_j$. Suppose that for each $w \in V$, $\omega(v, w) = 0$. Let $k \in \{1, \dots, n\}$. Then
   $$\begin{aligned}0 &= \omega(v, a_k) \\ &= \sum_{j=1}^{n} q^j \omega(a_j, a_k) + p^j \omega(b_j, a_k) \\ &= \sum_{j=1}^{n} p^j \delta_{j,k} \\ &= p^k\end{aligned}$$

Similarly,

$$\begin{aligned}0 &= \omega(v, b_k) \\ &= \sum_{j=1}^{n} q^j \omega(a_j, b_k) + p^j \omega(b_j, b_k) \\ &= \sum_{j=1}^{n} q^j \delta_{j,k} \\ &= q^k\end{aligned}$$

Since $k \in \{1, \dots, n\}$ is arbitrary, $v = 0$. Hence $\omega$ is nondegenerate. Therefore $(V, \omega)$ is symplectic.

□

**Exercise 2.3.2.3.** Let $(V, \omega)$ be a symplectic space. Then $\dim V$ is even.

*Proof.* Set $n = \dim V$. Let $(e_j)_{j=1}^n$ be a basis for $V$. Define $[\omega] \in \mathbb{R}^{n \times n}$ by $[\omega]_{i,j} = \omega(e_i, e_j)$. Since $\omega \in \Lambda^2(V)$, $[\omega]^* = -[\omega]$. Therefore

$$\begin{aligned}\det[\omega] &= \det[\omega]^* \\ &= \det(-[\omega]) \\ &= (-1)^n \det[\omega]\end{aligned}$$

For the sake of contradiction, suppose that $n$ is odd. Then $\det[\omega] = -\det[\omega]$ which implies that $\det[\omega] = 0$. Since $\omega$ is nondegenerate, $[\omega] \in GL(n, \mathbb{R})$. This is a contradiction. Hence $n$ is even. □

**Definition 2.3.2.4.** Let $(V, \omega)$ be a symplectic space and $S \subset V$ a subspace. We define the **symplectic complement of** $V$, denoted $S^\perp$, by

$$S^\perp = \{v \in V : \text{ for each } w \in S, \, \omega(v, w) = 0\}$$

**Exercise 2.3.2.5.** Let $(V, \omega)$ be a symplectic space and $S \subset V$ a subspace. Then $S^\perp$ is a subspace.

*Proof.* We note that

$$S^\perp = \bigcap_{v \in S} \ker \iota_v \omega$$

Hence $S^\perp$ is a subspace. □

**Exercise 2.3.2.6.** Let $(V, \omega)$ be a symplectic space and $S \subset V$ a subspace. Then

$$\dim V = \dim S + \dim S^\perp$$

*Proof.* □

**Exercise 2.3.2.7.** Let $(V, \omega)$ be a symplectic space and $S \subset V$ a subspace. Then $(S^\perp)^\perp = S$.

*Proof.* Let $v \in (S^\perp)^\perp$. Then for each $w \in S^\perp$, $\omega(v, w) = 0$. □

# Chapter 3

# Smooth Manifolds

## 3.1 Topological Manifolds

**Exercise 3.1.0.1.** We have that $\mathbb{R}$ is homeomorphic to $(0, \infty)$

*Proof.* Define $f : \mathbb{R} \to (0, \infty)$ by $f(x) = e^x$. Then $f$ is a homeomorphism. □

**Definition 3.1.0.2.** Let $n \in \mathbb{N}$. We define the **upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}^n$, by

$$\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and we define

$$\partial\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$$
$$\operatorname{Int}\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

We endow $\mathbb{H}^n$, $\partial\mathbb{H}^n$ and $\operatorname{Int}\mathbb{H}^n$ with the subspace topology inherited from $\mathbb{R}^n$.
We define the projection map $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ by

$$\pi(x_1, \ldots, x_{n-1}, 0) = (x_1, \ldots, x_{n-1})$$

**Definition 3.1.0.3.** We define $\mathbb{R}^0 = \{0\}$ and $\mathbb{H}^0 = \varnothing$ endowed with the discrete topology.

**Exercise 3.1.0.4.** Let $n \in \mathbb{N}$.

1. $\partial\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n-1}$
2. $\operatorname{Int}\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^n$

*Proof.*

1. Let $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ be the projection map given by

   $$\pi(x_1, \ldots, x_{n-1}, 0) = (x_1, \ldots, x_{n-1})$$

   Then $\pi$ is a homeomorphism.

2. Define $f : \mathbb{R}^n \to \operatorname{Int}\mathbb{H}^n$ by $f(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, e^{x_n})$. Then $f$ is a homeomorphism.

□

**Definition 3.1.0.5.** Let $(M, \mathcal{T})$ be a topological space and $n \in \mathbb{N}_0$. Let $U \subset M$ and $V \subset \mathbb{H}^n$ and $\phi : U \to V$. Then $(U, \phi)$ is said to be a $n$**-coordinate chart on** $(M, \mathcal{T})$ if

- $U \in \mathcal{T}$
- $V \in \mathcal{T}_{\mathbb{H}^n}$

- $\phi$ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n} \cap V)$-homeomorphism

We denote the set of all $n$-coordinate charts on $M$ by $X^n(M, \mathcal{T})$.

**Note 3.1.0.6.** We will write $X^n(M)$ in place of $X^n(M, \mathcal{T})$ when the topology is not ambiguous.

**Definition 3.1.0.7.** Let $M$ be a topological space and $n \in \mathbb{N}$. Then $M$ is said to be **locally Euclidean of dimension** $n$ if for each $p \in M$, there exists $(U, \phi) \in X^n(M)$ such that $p \in U$.

**Definition 3.1.0.8.** Let $M$ be a topological space and $n \in \mathbb{N}$. Then $M$ is said to be an $n$**-dimensional topological manifold** if

1. $M$ is Hausdorff
2. $M$ is second-countable
3. $M$ is locally Euclidean of dimension $n$

**Theorem 3.1.0.9. Topological Invariance of Dimension:**
Let $M$ be an $n$-dimensional toplogical manifold and $N$ a $p$-dimensional toplogical manifold. If $M$ and $N$ are homeomorphic, then $n = p$.

**Note 3.1.0.10.** In light of the previous theorem, we write $X(M)$ in place of $X^n(M)$ and refer to $n$-coordinate charts as coordinate charts when the context is clear.

**Definition 3.1.0.11.** Let $M$ be an $n$-dimensional topological manifold and $(U, \phi) \in X(M)$. Then $(U, \phi)$ is said to be an

- **interior chart** if $\phi(U)$ is open in $\mathbb{R}^n$
- **boundary chart** if $\phi(U)$ is open in $\mathbb{H}^n$ and $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$

We denote the set of all interior charts on $M$ and the set of all boundary charts on $M$ by $X_{\text{Int}}(M)$ and $X_\partial(M)$ respectively.

**Exercise 3.1.0.12.** Let $M$ be an $n$-dimensional topological manifold. Then

1. $X(M) = X_{\text{Int}}(M) \cup X_\partial(M)$
2. $X_{\text{Int}}(M) \cap X_\partial(M) = \varnothing$

*Proof.*

1. By definition, $X_{\text{Int}}(M) \cup X_\partial(M) \subset X(M)$. Let $(U, \phi) \in X(M)$. Since $(U, \phi)$ is a coordinate chart on $M$, $\phi(U)$ is open in $\mathbb{R}^n$ or $\phi(U)$ is open in $\mathbb{H}^n$. If $\phi(U)$ is open in $\mathbb{R}^n$, then

$$\begin{aligned}(U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M)\end{aligned}$$

Suppose that $\phi(U)$ is open in $\mathbb{H}^n$. If $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$, then $\phi(U)$ is open in $\mathbb{R}^n$ and

$$\begin{aligned}(U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M)\end{aligned}$$

Suppose that $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$. Then

$$\begin{aligned}(U, \phi) &\in X_\partial(M) \\ &\subset X_{\text{Int}}(M) \cup X_\partial(M)\end{aligned}$$

Since $(U, \phi) \in X(M)$ is arbitrary, $X(M) \subset X_{\text{Int}}(M) \cup X_\partial(M)$. Therefore $X(M) = X_{\text{Int}}(M) \cup X_\partial(M)$.

2. For the sake of contradiction, suppose that $X_{\text{Int}}(M) \cup X_{\partial}(M) \neq \varnothing$. Then there exists $(U, \phi) \in X(M)$ such that $(U, \phi) \in X_{\text{Int}}(M)$ and $(U, \phi) \in X_{\partial}(M)$. Therefore $\phi(U)$ is open in $\mathbb{R}^n$, $\phi(U)$ is open in $\mathbb{H}^n$ and $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$. Since $\phi(U)$ is open in $\mathbb{R}^n$ and $\phi(U) \subset \mathbb{H}^n$, $\phi(U) \subset \operatorname{Int}\mathbb{H}^n$ and therefore $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$ which is a contradiction.

□

**Definition 3.1.0.13.** Let $M$ be an $n$-dimensional topological manifold. We define the

- **interior** of $M$, denoted $\operatorname{Int} M$, by

$$\operatorname{Int} M = \{p \in M : \text{there exists } (U, \phi) \in X_{\text{Int}}(M) \text{ such that } p \in U\}$$

- **boundary** of $M$, denoted $\partial M$, by

$$\partial M = \{p \in M : \text{there exists } (V, \psi) \in X_{\partial}(M) \text{ such that } p \in V \text{ and } \psi(p) \in \partial\mathbb{H}^n\}$$

**Exercise 3.1.0.14.** Let $M$ be an $n$-dimensional topological manifold. Let $(U, \phi) \in X_{\text{Int}}(M)$. Then $U \subset \operatorname{Int} M$.

*Proof.* Let $p \in U$. Since $(U, \phi) \in X_{\text{Int}}(M)$ and $p \in U$, by definition, $p \in \operatorname{Int} M$. Since $p \in U$ is arbitrary, $U \subset \operatorname{Int} M$. □

**Exercise 3.1.0.15.** Let $M$ be an $n$-dimensional topological manifold, $(U, \phi) \in X_{\partial}(M)$ and $p \in U$. If $\phi(p) \notin \partial\mathbb{H}^n$, then $p \in \operatorname{Int} M$.

*Proof.* Suppose that $\phi(p) \notin \partial\mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int}\mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that $B'$ is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then $U'$ is open in $M$ and $\phi' : U' \to B'$ is a homeomorphism. Hence $(U', \phi') \in X_{\text{Int}}(M)$. Since $\phi(p) \in B'$, we have that $p \in U'$. By definition, $p \in \operatorname{Int} M$. □

**Exercise 3.1.0.16.** Let $M$ be an $n$-dimensional topological manifold. Then

1. $M = \operatorname{Int} M \cup \partial M$
2. $\operatorname{Int} M \cap \partial M = \varnothing$
   **Hint:** simply connected

*Proof.*

1. By definition, $\operatorname{Int} M \cup \partial M \subset M$. Let $p \in M$. Since $M$ is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. A previous exercise implies that $(U, \phi) \in X_{\text{Int}}(M) \cup X_{\partial}(M)$. If $(U, \phi) \in X_{\text{Int}}(M)$, then by definition,

$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Suppose that $(U, \phi) \in X_{\partial}(M)$. If $\phi(p) \in \partial\mathbb{H}^n$, then by definition,

$$\begin{aligned} p &\in \partial M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Suppose that $\phi(p) \notin \partial\mathbb{H}^n$. The previous exercise implies that $p \in \operatorname{Int} M$. Therefore,

$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

Since $p \in M$ is arbitrary, $M \subset \operatorname{Int} M \cup \partial M$. Therefore $M = \operatorname{Int} M \cup \partial M$.

2. For the sake of contradiction, suppose that $\operatorname{Int} M \cap \partial M \neq \varnothing$. Then there exists $p \in M$ such that $p \in \operatorname{Int} M \cap \partial M$. By definition, there exists $(U, \phi) \in X_{\operatorname{Int}}(M)$, $(V, \psi) \in X_{\partial}(M)$ such that $p \in U \cap V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in $\mathbb{H}^n$, $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.
   Since $\psi(U \cap V)$ is open in $\mathbb{H}^n$, there exists an $B_\psi \subset \psi(U \cap V)$ such that $B_\psi$ is open in $\mathbb{H}^n$, $B_\psi$ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$. Since $\phi(U \cap V)$ is open in $\mathbb{R}^n$, $B_\phi$ is open in $\mathbb{R}^n$. Since $B_\psi$ is simply connected and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism, $B_\phi$ is simply connected.
   Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Then $\phi \circ \psi^{-1} : B'_\psi \to B'_\phi$ is a homeomorphism. Since $\psi(p) \in \partial \mathbb{H}^n$, $B'_\psi$ is simply connected. Since $B_\phi$ is open in $\mathbb{R}^n$, $B'_\phi$ is not simply connected. This is a contradiction since $B'_\phi$ is homeomorphic to $B'_\psi$. So $\partial M \cap \operatorname{Int} M = \varnothing$.

□

**Exercise 3.1.0.17.** Let $M$ be an $n$-dimensional topological manifold. Then

1. $\operatorname{Int} M$ is open
2. $\partial M$ is closed

*Proof.*

1. Let $p \in \operatorname{Int} M$. Then there exists $(U, \phi) \in X_{\operatorname{Int}}(M)$ such that $p \in U$. By definition, $U$ is open and a previous exercise implies that $U \subset \operatorname{Int} M$. Since $p \in \operatorname{Int} M$ is arbitrary, we have that for each $p \in \operatorname{Int} M$, there exists $U \subset \operatorname{Int} M$ such that $U$ is open. Hence $\operatorname{Int} M$ is open.
2. Since $\partial M = (\operatorname{Int} M)^c$, and $\operatorname{Int} M$ is open, we have that $\partial M$ is closed.

□

**Exercise 3.1.0.18.** Let $M$ be an $n$-dimensional topological manifold, $(U, \phi) \in X(M)$ and $p \in U$. If $p \in \partial M$, then $(U, \phi) \in X_\partial(M)$.
**Hint:** simply connected

*Proof.* Suppose that $p \in \partial M$. Then there exists a $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in $\mathbb{H}^n$, $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.
Since $\psi(U \cap V)$ is open in $\mathbb{H}^n$, there exists $B_\psi \subset \psi(U \cap V)$ such $B_\psi$ is open in $\mathbb{H}^n$, $B_\psi$ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$.
For the sake of contradiction, suppose that $(U, \phi) \in X_{\operatorname{Int}}(M)$. Then $\phi(U)$ is open in $\mathbb{R}^n$. Hence $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $B_\phi$ is open in $\mathbb{R}^n$. Since $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism, $B_\phi$ is simply connected. Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Since $\psi(p) \in \partial \mathbb{H}^n$, $B'_\psi$ is simply connected. Since $B_\phi$ is open in $\mathbb{R}^n$, $B'_\phi$ is not simply connected. This is a contradiction since $B'_\phi$ is homeomorphic to $B'_\psi$. So $(U, \phi) \notin X_{\operatorname{Int}}(M)$. Since $(X_{\operatorname{Int}}(M))^c = X_\partial(M)$, we have that $(U, \phi) \in X_\partial(M)$.

□

**Exercise 3.1.0.19.** Let $M$ be an $n$-dimensional topological manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. Then

1. $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$
2. $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n$

*Proof.*

1. Suppose that $p \in \partial M$. For the sake of contradiction, suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that $B'$ is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then $p \in U'$ and $(U', \phi') \in X_{\operatorname{Int}}(M)$. Since $p \in U'$, the previous exercise implies that $(U', \phi') \in X_\partial(M)$. This is a contradiction since $X_{\operatorname{Int}}(M) \cap X_\partial(M) = \varnothing$. So $\phi(p) \in \partial \mathbb{H}^n$.
   Conversely, suppose that $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$.

2. A previous exercise implies that $\operatorname{Int} M = (\partial M)^c$. Part (1) implies that

$$\begin{aligned} p &\in (\partial M)^c \\ &= \operatorname{Int} M \end{aligned}$$

if and only if

$$\begin{aligned} \phi(p) &\in (\partial \mathbb{H}^n)^c \\ &= \operatorname{Int} \mathbb{H}^n \end{aligned}$$

□

**Exercise 3.1.0.20.** Let $M$ be an $n$-dimensional topological manifold and $p \in M$. Then $p \in \partial M$ iff for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.

*Proof.* Suppose that $p \in \partial M$. Let $(U, \phi) \in X(M)$. Suppose that $p \in U$. The previous two exercises imply that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.
Conversely, suppose that for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. Since $M$ is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. By assumption, $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$. □

**Exercise 3.1.0.21.** Let $M$ be an $n$-dimensional topological manifold. Let $(U, \phi) \in X_\partial(M)$. Then

1. $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$
2. $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int} \mathbb{H}^n$

*Proof.*

1. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$. Let $q \in \phi(U \cap \partial M)$. Then there exists $p \in U \cap \partial M$ such that $\phi(p) = q$. Since $p \in \partial M$, $\phi(p) \in \partial \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \partial \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \partial M)$ is arbitrary, $\phi(U \cap \partial M) \subset \phi(U) \cap \partial \mathbb{H}^n$.
Let $q \in \phi(U) \cap \partial \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \partial \mathbb{H}^n$, we have that $p \in \partial M$. Hence $p \in U \cap \partial M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \partial M)$. Thus $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$.

2. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Let $q \in \phi(U \cap \operatorname{Int} M)$. Then there exists $p \in U \cap \operatorname{Int} M$ such that $\phi(p) = q$. Since $p \in \operatorname{Int} M$, $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \operatorname{Int} \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \operatorname{Int} M)$ is arbitrary, $\phi(U \cap \operatorname{Int} M) \subset \phi(U) \cap \operatorname{Int} \mathbb{H}^n$.
Let $q \in \phi(U) \cap \operatorname{Int} \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \operatorname{Int} \mathbb{H}^n$, we have that $p \in \operatorname{Int} M$. Hence $p \in U \cap \operatorname{Int} M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \operatorname{Int} M)$. Thus $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int} \mathbb{H}^n$.

□

**Definition 3.1.0.22.** Let $M$ be an $n$-dimensional topological manifold and $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ the projection map. For $(U, \phi) \in X_\partial(M)$, we define $\bar{U} \subset \partial M$ and $\bar{\phi} : \bar{U} \to \pi(\phi(\bar{U}))$ by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi \circ \phi|_{\bar{U}}$ respectively.

**Exercise 3.1.0.23.** Let $M$ be an $n$-dimensional topological manifold, and $\lambda : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ a homeomorphism. Then $\{(\bar{U}, \bar{\phi}) : (U, \phi) \in X_\partial(M)\} \subset X^{n-1}_{\text{Int}}(\partial M)$.

*Proof.* Let $(U, \phi) \in X_\partial(M)$.

1. Since $U$ is open in $M$, $\bar{U} = U \cap \partial M$ is open in $\partial M$.
2. Since $(U, \phi) \in X_\partial(M)$, $\phi(U)$ is open in $\mathbb{H}^n$. A previous exercise implies that $\phi(\bar{U}) = \phi(U) \cap \partial\mathbb{H}^n$ which is open in $\partial\mathbb{H}^n$. Since $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ is a homeomorphism, we have that $\pi(\phi(\bar{U}))$ is open in $\mathbb{R}^{n-1}$.
3. Since $\phi|_{\bar{U}} : \bar{U} \to \phi(U) \cap \partial\mathbb{H}^n$ and $\pi|_{\phi(\bar{U})} : \phi(\bar{U}) \to \lambda(\phi(\bar{U}))$ are homeomorphisms, we have that $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$ is a homeomorphism.

Hence $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$. □

**Exercise 3.1.0.24.** Let $M$ be an $n$-dimensional topological manifold. Then

1. $\partial M$ is an $(n-1)$-dimensional topological manifold
2. $\partial(\partial M) = \varnothing$

*Proof.*

1. (a) Since $M$ is Hausdorff, $\partial M$ is Hausdorff.
   (b) Since $M$ is second-countable, $\partial M$ is second countable.
   (c) Let $p \in \partial M$. Then there exists $(U, \phi) \in X_\partial(M)$ such that $\phi(p) \in \partial\mathbb{H}^n$. Then $p \in \bar{U}$ and the previous exercise implies that $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$. Thus $\partial M$ is locally Euclidean of dimension $n-1$.

   Hence $\partial M$ is an $(n-1)$-dimensional topological manifold.
2. Let $p \in \partial M$. Part (1) implies that there exists $(U, \phi) \in X^{n-1}_{\text{Int}}(\partial M)$ such that $p \in U$. Thus $p \in \text{Int}\,\partial M$. Since $p \in \partial M$ is arbitrary, $\text{Int}\,\partial M = \partial M$. Hence

$$\begin{aligned} \partial(\partial M) &= (\text{Int}(\partial M))^c \\ &= (\partial M)^c \\ &= \varnothing \end{aligned}$$

□

**Exercise 3.1.0.25.** Let $M$ be an $n$-dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If $U'$ is open in $M$, then $(U', \phi|_{U'}) \in X^n(M)$.

*Proof.* Suppose that $U'$ is open in $M$. Set $\phi' = \phi|_{U'}$.

- By assumption $U'$ is open in $M$.
- Since $U'$ is open in $M$, we have that $U' = U' \cap U$ is open in $U$. Since $\phi$ is a homeomorphism and $U'$ is open in $U$, we have that $\phi(U')$ is open in $\phi(U)$. By assumption $\phi(U)$ is open in $\mathbb{R}^n$ or $\phi(U)$ is open in $\mathbb{H}^n$. Therefore $\phi'(U')$ is open in $\mathbb{R}^n$ or $\phi'(U')$ is open in $\mathbb{H}^n$.
- Since $\phi : U \to V$ is a homeomorphism, $\phi' : U' \to \phi'(U')$ is a homeomorphism.

So $(U', \phi') \in X^n(M)$. □

**Note 3.1.0.26.** Since $U$ is open in $M$, $U'$ being open in $U$ is equivalent to $U'$ being open in $M$, so we could have also assumed that $U'$ is open in $U$.

**Exercise 3.1.0.27.** Let $M$ be an $n$-dimensional topological manifold and $U \subset M$. If $U$ is open, then

$$X^n(U) = \{(V, \psi) \in X^n(M) : V \subset U\}$$

*Proof.* Suppose that $U$ is open and set $A = \{(V, \psi) \in X^n(M) : V \subset U\}$. Let $(V, \psi) \in X^n(U)$. By definition of $X^n(U)$, $V$ is open in $U$. Thus, there exists $W \subset M$ such that $W$ is open in $M$ and $V = U \cap W$. Since $U$ is open in $M$, we have that $V = U \cap W$ is open in $M$. Hence $(V, \psi) \in X^n(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X^n(U)$ is arbitary, $X^n(U) \subset A$.
Conversely, suppose that $(V, \psi) \in A$. Then $(V, \psi) \in X^n(M)$ and $V \subset U$. By definition of $X^n(M)$, $V$ is open in $M$. Since $V \subset U$, we have that $V = V \cap U$ is open in $U$. Hence $(V, \psi) \in X^n(U)$. Since $(V, \psi) \in X^n(U)$ is arbitary, $A \subset X^n(U)$. Hence $X^n(A) = A$. □

**Exercise 3.1.0.28.** Let $M$ be an $n$-dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If $U'$ is open in $M$, then $(U', \phi|_{U'}) \in X^n(U)$.

*Proof.* Suppose that $U'$ is open in $M$. A previous exercise implies that $(U', \phi') \in X^n(M)$. The previous exercise implies that $(U', \phi') \in X^n(U)$. □

**Exercise 3.1.0.29. Topological Open Submanifolds:**
Let $M$ be an $n$-dimensional topological manifold and $U \subset M$ open. Then $U$ is an $n$-dimensional topological manifold.

*Proof.*

1. Since $M$ is Hausdorff, $U$ is Hausdorff.

2. $M$ is second-countable, $U$ is second countable.

3. Let $p \in U$. Since then there exists $(V, \psi) \in X^n(M)$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{U \cap V}$. The previous exercise implies that $(V', \psi') \in X^n(U)$. Therefore $U$ is locally Euclidean of dimension $n$.

Hence $U$ is an $n$-dimensional topological manifold. □

**Exercise 3.1.0.30.** Let $M$ be an $n$-dimensional topological manifold and $U \subset M$. If $U$ is open, then

1. $X_{\text{Int}}(U) = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$

2. $X_{\partial}(U) = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$

*Proof.* Suppose that $U$ is open in $M$.

1. Set $A = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\text{Int}}(U)$. By definition of $X_{\text{Int}}(U)$, $V$ is open in $U$ and $\phi(V)$ is open in $\mathbb{R}^n$. Since $U$ is open in $M$, $V$ is open in $M$. Hence $(V, \psi) \in X_{\text{Int}}(M)$. Since $U$ is open in $M$, $V$ is open in $M$. Hence $(V, \psi) \in X_{\text{Int}}(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X_{\text{Int}}(U)$ is arbitrary, $X_{\text{Int}}(U) \subset A$.
   Conversely, let $(V, \psi) \in A$. Then $(V, \psi) \in X_{\text{Int}}(M)$ and $V \subset U$. By definition of $X_{\text{Int}}(M)$, $V$ is open in $M$ and $\phi(V)$ is open in $\mathbb{R}^n$. Thus $V = V \cap U$ is open in $U$. So $(V, \psi) \in X_{\text{Int}}(U)$. Since $(V, \psi) \in A$ is arbitrary, $A \subset X_{\text{Int}}(U)$. Thus $X_{\text{Int}}(U) = A$.

2. Set $B = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\partial}(U)$. By definition of $X_{\partial}(U)$, $V$ is open in $U$, $\phi(V)$ is open in $\mathbb{H}^n$ and $\partial\mathbb{H}^n \cap \phi(V) \neq \varnothing$. Since $U$ is open in $M$, $V$ is open in $M$. Hence $(V, \psi) \in X_{\partial}(M)$, which implies that $(V, \psi) \in B$. Since $(V, \psi) \in X_{\partial}(U)$ is arbitrary, $X_{\partial}(U) \subset B$.
   Conversely, let $(V, \psi) \in B$. Then $(V, \psi) \in X_{\partial}(M)$ and $V \subset U$. By definition of $X_{\partial}(M)$, $V$ is open in $M$, $\phi(V)$ is open in $\mathbb{H}^n$ and $\partial\mathbb{H}^n \cap \phi(V) \neq \varnothing$. Thus $V = V \cap U$ is open in $U$. So $(V, \psi) \in X_{\partial}(U)$. Since $(V, \psi) \in B$ is arbitrary, $B \subset X_{\partial}(U)$. Thus $X_{\partial}(U) = B$.

□

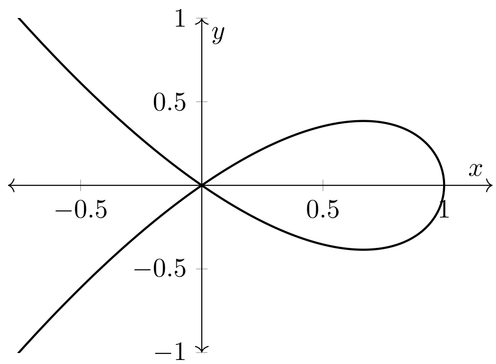
**Exercise 3.1.0.31.** Let $M$ be an $n$-dimensional topological manifold and $U \subset M$. If $U$ is open, then $\partial U = \partial M \cap U$.

*Proof.* Suppose that $U$ is open. Let $p \in \partial U$. Then there exists $(V, \psi) \in X_\partial(U)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Since $U$ is open, the previous exercise implies that $(V, \psi) \in X_\partial(M)$. Thus $p \in \partial M$. Since $p \in \partial U$ is arbitrary, $\partial U \subset \partial M$. Since $\partial U \subset U$, we have that $\partial U \subset \partial M \cap U$.
Conversely, let $p \in \partial M \cap U$. Since $p \in \partial M$, there exists $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Set $V' = V \cap U$ and $\psi' = \psi|_{V'}$. Then $p \in V'$ since $V$ and $U$ are open in $M$, $V'$ is open in $M$. A previous exercise implies that $(V', \psi') \in X(M)$. Since $p \in \partial M$, a previous exercise implies that $(V', \psi') \in X_\partial(M)$. The previous exercise implies that $(V', \psi') \in X_\partial(U)$. Since $\psi'(p) \in \partial \mathbb{H}^n$, $p \in \partial U$. Since $p \in \partial M \cap U$ is arbitrary, $\partial M \cap U \subset \partial U$. Hence $\partial U = \partial M \cap U$.
label exercises and reference them!!! □

**Exercise 3.1.0.32. Graph of Continuous Function:**
Let $f \in C(\mathbb{R})$. Set $M = \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$ (i.e. the graph of $f$). Then $M$ is a 1-dimensional manifold.

*Proof.* Set $U = \mathbb{R}$ and define $\phi : U \to M$ by $\phi(x) = (x, f(x))$. Then $\phi^{-1} = \pi_1$. Since $f$ is continuous, $\phi$ is continous. Since $\pi_1$ is continiuous, $\phi$ is a homeomorphism. □

**Exercise 3.1.0.33. Nodal Cubic:**
Let $M = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2 - x^3\}$. We equip $M$ with the subspace topology.




Then $M$ is not a 1-dimensional topological manifold.
**Hint:** connected components

*Proof.* Suppose that $M$ is a 1-dimensional manifold. Set $p = (0, 0)$. Then there exists $(U, \phi) \in X(M)$ such that $p \in U$. Since $\phi(U)$ is open (in $\mathbb{R}$ or $\mathbb{H}$), there exists a $B \subset \phi(U)$ such that $B$ is open (in $\mathbb{R}$ or $\mathbb{H}$), $B$ is connected and $\phi(p) \in B$. Set $V = \phi^{-1}(B)$, $V' = V \setminus \{p\}$ and $B' = B \setminus \{\phi(p)\}$. Then $\phi : V \to B$ and $\phi' : V' \to B'$ are homeomorphisms. Since $B$ is open (in $\mathbb{R}$ or $\mathbb{H}$) and connected, $B'$ has at most two connected components. Then $V'$ This is a contradiction since $V'$ has four connected components and $B'$ and $V'$ are homeomorphic. □

**Exercise 3.1.0.34. Topological Manifold Chart Lemma:**
Let $M$ be a set, $\Gamma$ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

(a) for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha)$ is open in $\mathbb{H}^n$

(b) for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection

(c) for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{H}^n$

(d) for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous

(e) there exists $\Gamma' \subset \Gamma$ such that $\Gamma'$ is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$

(f) for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : V \subset \mathbb{H}^n \text{ is open in } \mathbb{H}^n \text{ and } \alpha \in \Gamma\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$

Then

1. $\mathcal{B}$ is a basis for $\mathcal{T}_M$
   **Hint:** For $B_1, B_2 \subset \mathbb{H}^n$, $\phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) = \phi_{\alpha_1}^{-1}(B_1 \cap [\phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}} \circ (\phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}})^{-1}(B_2)])$
2. $(M, \mathcal{T}_M)$ is an $n$-dimensional topological manifold
3. $\mathcal{T}_M$ is the unique topology $\mathcal{T}$ on $M$ such that $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T})$

*Proof.*

1.
   - By assumption, $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
   - Let $A_1, A_2 \in \mathcal{B}$ and $p \in A_1 \cap A_2$. By definition, there exist $\alpha_1, \alpha_2 \in \Gamma$ and $B_1, B_2 \subset \mathbb{H}^n$ such that $B_1$, $B_2$ are open in $\mathbb{H}^n$ and

   $$\begin{aligned} A_1 &= \phi_{\alpha_1}^{-1}(B_1) \\ &\subset U_{\alpha_1} \end{aligned} \qquad\qquad \begin{aligned} A_2 &= \phi_{\alpha_2}^{-1}(B_2) \\ &\subset U_{\alpha_2} \end{aligned}$$

   Set $\psi_1 = \phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}}$ and $\psi_2 = \phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}}$. We note that

   $$\begin{aligned} \psi_1^{-1}(B_1) &= U_{\alpha_2} \cap \phi_{\alpha_1}^{-1}(B_1) \\ &= U_{\alpha_2} \cap A_1 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned} \qquad\qquad \begin{aligned} \psi_2^{-1}(B_2) &= U_{\alpha_1} \cap \phi_{\alpha_2}^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned}$$

   Let $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Then $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. Hence $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_1}(q) \in \psi_1 \circ \psi_2^{-1}(B_2)$. This implies that

   $$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(B_1) \\ &= A_1 \end{aligned}$$

   and since $\psi_2^{-1}(B_2) \subset U_{\alpha_1} \cap U_{\alpha_2}$ and $\phi_{\alpha_1} : U_{\alpha_1} \to \phi_{\alpha_1}(U_{\alpha_1})$ is a bijection, we have that

   $$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(\psi_1 \circ \psi_2^{-1}(B_2)) \\ &= \psi_2^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \end{aligned}$$

   Thus

   $$\begin{aligned} q &\in A_1 \cap (U_{\alpha_1} \cap A_2) \\ &= A_1 \cap A_2 \end{aligned}$$

   Since $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$ is arbitrary, we have that $\phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \subset A_1 \cap A_2$. Conversely, let

   $$\begin{aligned} q &\in A_1 \cap A_2 \\ &= \phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) \end{aligned}$$

Then $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_2}(q) \in B_2$. Since $A_1 \cap A_2 \subset U_{\alpha_1} \cap U_{\alpha_2}$, we have that

$$\begin{aligned} \psi_2(q) &= \phi_{\alpha_2}(q) \\ &\in B_2 \end{aligned}$$

which implies that $q \in \psi_2^{-1}(B_2)$. Therefore

$$\begin{aligned} \phi_{\alpha_1}(q) &= \psi_1(q) \\ &\in \psi_1(\psi_2^{-1}(B_2)) \\ &= \psi_1 \circ \psi_2^{-1}(B_2) \end{aligned}$$

Hence $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. This implies that $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Since $q \in A_1 \cap A_2$ is arbitrary, we have that $A_1 \cap A_2 \subset \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Thus

$$\begin{aligned} A_1 \cap A_2 &= \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \\ &\in \mathcal{B} \end{aligned}$$

Thus $\mathcal{B}$ is a basis for $\mathcal{T}_M$.

2. (a) **(locally Euclidean of dimension $n$):**
Let $\alpha \in \Gamma$. By definition, for each $B \subset \mathcal{T}_{\mathbb{H}^n}$,

$$\begin{aligned} \phi_\alpha^{-1}(B) &\in \mathcal{B} \\ &\subset \mathcal{T}_M \end{aligned}$$

Hence $\phi_\alpha$ is continuous.
Let $A \in \mathcal{T}_{U_\alpha}$. Then there exists $U \subset \mathcal{T}_M$ such that $A = U \cap U_\alpha$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_M$, there exists $\Gamma' \subset \Gamma$, $(V_\beta)_{\beta \in \Gamma'} \subset \mathcal{T}_{\mathbb{H}^n}$ such that $U = \bigcup_{\beta \in \Gamma'} \phi_\beta^{-1}(V_\beta)$. Thus

$$\begin{aligned} A &= U \cap U_\alpha \\ &= \left[\bigcup_{\beta \in \Gamma'} \phi_\beta^{-1}(V_\beta)\right] \cap U_\alpha \\ &= \bigcup_{\beta \in \Gamma'} [\phi_\beta^{-1}(V_\beta) \cap U_\alpha] \end{aligned}$$

Let $\beta \in \Gamma'$. Since $\phi_\alpha(U_\alpha \cap U_\beta) \subset \phi_\alpha(U_\alpha)$ and $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$, we have that

$$\begin{aligned} \phi_\alpha(U_\alpha \cap U_\beta) &= \phi_\alpha(U_\alpha) \cap \phi_\alpha(U_\alpha \cap U_\beta) \\ &\in \mathcal{T}_{\phi_\alpha(U_\alpha)} \end{aligned}$$

Therefore $\mathcal{T}_{\phi_\alpha(U_\alpha \cap U_\beta)} \subset \mathcal{T}_{\phi_\alpha(U_\alpha)}$. Since $(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous, we have that $(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{H}^n$ is continuous and therefore

$$\begin{aligned} [(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1}]^{-1}(V_\beta) &\in \mathcal{T}_{\phi_\alpha(U_\alpha \cap U_\beta)} \\ &\subset \mathcal{T}_{\phi_\alpha(U_\alpha)} \end{aligned}$$

Since $\beta \in \Gamma'$ is arbitrary, we have that

$$\begin{aligned}\phi_\alpha(A) &= \phi_\alpha\bigg(\bigcup_{\beta\in\Gamma'}[\phi_\beta^{-1}(V_\beta)\cap U_\alpha]\bigg)\\ &= \bigcup_{\beta\in\Gamma'}\phi_\alpha(\phi_\beta^{-1}(V_\beta)\cap U_\alpha)\\ &= \bigcup_{\beta\in\Gamma'}(\phi_\alpha|_{U_\alpha\cap U_\beta})\circ(\phi_\beta|_{U_\alpha\cap U_\beta})^{-1}(V_\beta)\\ &= \bigcup_{\beta\in\Gamma'}[(\phi_\beta|_{U_\alpha\cap U_\beta})\circ(\phi_\alpha|_{U_\alpha\cap U_\beta})^{-1}]^{-1}(V_\beta)\\ &\in \mathcal{T}_{\phi_\alpha(U_\alpha)}\end{aligned}$$

Since $A \in \mathcal{T}_{U_\alpha}$ is arbitrary, $\phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to U_\alpha$ is continuous. Hence $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism and $(U_\alpha, \phi_\alpha) \in X^n(M)$. Since $M = \bigcup_{\alpha\in\Gamma} U_\alpha$, we have that $M$ is locally Euclidean of dimension $n$.

(b) **(Hausdorff):**
Let $p, q \in M$. Suppose that $p \neq q$. Then there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$.

- Suppose that there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$. Since $p \neq q$, $\phi_\alpha(p) \neq \phi_\alpha(q)$. Since $\mathbb{H}^n$ is Hausdorff, there exist $V_p, V_q \subset \phi(U_\alpha)$ such that $V_p$ and $V_q$ are open in $\mathbb{H}^n$, $p \in V_p$, $q \in V_q$ and $V_p \cap V_q = \varnothing$. Set $U_p = \phi_\alpha^{-1}(V_p)$ and $U_q = \phi_\alpha^{-1} V_q$. Then $U_p, U_q$ are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$.
- Suppose that there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$. Set $U_p = U_\alpha$ and $U_q = U_\beta$. Then $U_p, U_q$ are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$.

Thus for each $p, q \in M$ there exist $U_p, U_q \subset M$ such that $U_p, U_q$ are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$. Hence

(c) **(second-countable):**
By assumption, there exists $\Gamma' \subset \Gamma$ such that $\Gamma'$ is countable and $M \subset \bigcup_{\alpha\in\Gamma'} U_\alpha$. Let $\alpha \in \Gamma'$. Since $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$ and $\mathbb{H}^n$ is second-countable, we have that $\phi_\alpha(U_\alpha)$ is second-countable. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism, we have that $U_\alpha$ is second-countable. Since $M = \bigcup_{\alpha\in\Gamma'} U_\alpha$, an exercise in topology cite implies that $M$ is second-countable.

3. Let $\mathcal{T}$ be a topology on $M$. Suppose that $(U_\alpha, \phi_\alpha)_{\alpha\in\Gamma} \subset X^n(M, \mathcal{T})$. Then for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}$ and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism.
Let $U \in \mathcal{B}$. By definition, there exists $\alpha \in \Gamma$ and $V \in \mathcal{T}_{\mathbb{H}^n}$ such that $U = \phi_\alpha^{-1}(V)$. Since $U_\alpha \in \mathcal{T}$, we have that $\mathcal{T} \cap U_\alpha \subset \mathcal{T}$. Since $V \cap \phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$, and $\phi_\alpha$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that

$$\begin{aligned}U &= \phi_\alpha^{-1}(V)\\ &= \phi_\alpha^{-1}(V \cap \phi_\alpha(U_\alpha))\\ &\in \mathcal{T} \cap U_\alpha\\ &\subset \mathcal{T}\end{aligned}$$

Since $U \in \mathcal{B}$ is arbitrary, $\mathcal{B} \subset \mathcal{T}$. Therefore

$$\begin{aligned}\mathcal{T}_M &= \tau(\mathcal{B})\\ &\subset \tau(\mathcal{T})\\ &= \mathcal{T}\end{aligned}$$

Conversely, Let $U \in \mathcal{T}$ and $\alpha \in \Gamma$. Then $U \cap U_\alpha \in \mathcal{T} \cap U_\alpha$. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that $\phi_\alpha(U \cap U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$. Since $U_\alpha \in \mathcal{T}_M$, $\mathcal{T}_M \cap U_\alpha \subset \mathcal{T}_M$. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T}_M \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that

$$\begin{aligned} U \cap U_\alpha &= \phi_\alpha^{-1}(\phi_\alpha(U \cap U_\alpha)) \\ &\in \mathcal{T}_M \cap U_\alpha \\ &\subset \mathcal{T}_M \end{aligned}$$

Then

$$\begin{aligned} U &= U \cap M \\ &= U \cap \left( \bigcup_{\alpha \in \Gamma} U_\alpha \right) \\ &= \bigcup_{\alpha \in \Gamma} (U \cap U_\alpha) \\ &\in \mathcal{T}_M \end{aligned}$$

Since $U \in \mathcal{T}$ is arbitrary, $\mathcal{T} \subset \mathcal{T}_M$. Thus $\mathcal{T} = \mathcal{T}_M$.

□

**Exercise 3.1.0.35.** Let $M$ be a set, $\Gamma$ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

(a) for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha)$ is open in $\mathbb{H}^n$

(b) for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection

(c) for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{H}^n$

(d) for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous

(e) there exists $\Gamma' \subset \Gamma$ such that $\Gamma'$ is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$

(f) for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Then there exists a unique topology $\mathcal{T}_M$ on $M$ such that $(M, \mathcal{T}_M)$ is an $n$-dimensional topological manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T}_M)$.

*Proof.* Immediate by previous exercise. □

## 3.2 Smooth Manifolds

**Definition 3.2.0.1.** Let $M$ be an $n$-dimensional topological manifold and $(U,\phi),(V,\psi) \in X(M)$. Then $(U,\phi)$ and $(V,\psi)$ are said to be **smoothly compatible** if

$$\psi|_{U\cap V} \circ (\phi|_{U\cap V})^{-1} : \phi(U\cap V) \to \psi(U\cap V) \text{ is a diffeomorphism}$$

**Definition 3.2.0.2.** Let $M$ be an $n$-dimensional topological manifold.

- Let $\mathcal{A} \subset X(M)$. Then $\mathcal{A}$ is said to be an **atlas on** $M$ if $M \subset \bigcup\limits_{(U,\phi)\in\mathcal{A}} U$.
- Let $\mathcal{A}$ be an atlas on $M$. Then $\mathcal{A}$ is said to be **smooth** if for each $(U,\phi),(V,\psi) \in \mathcal{A}$, $(U,\phi)$ and $(V,\psi)$ are smoothly compatible.
- Let $\mathcal{A}$ be a smooth atlas on $M$. Then $\mathcal{A}$ is said to be **maximal** if for each smooth atlas $\mathcal{B}$ on $M$, $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on $M$ is called a **smooth structure on** $M$.
- Let $\mathcal{A}$ be an atlas on $M$. Then $(M,\mathcal{A})$ is said to be an **$n$-dimensional smooth manifold** if $\mathcal{A}$ is a smooth structure on $M$.

**Exercise 3.2.0.3.** Let $M$ be an $n$-dimensional topological manifold and $\mathcal{B}$ a smooth atlas on $M$. Then there exists a unique smooth structure $\mathcal{A}$ on $M$ such that $\mathcal{B} \subset \mathcal{A}$.

*Proof.* Define

$$\mathcal{A} = \{(U,\phi) \in X(M) : \text{ for each } (V,\psi) \in \mathcal{B},\ (U,\phi) \text{ and } (V,\psi) \text{ are smoothly compatible}\}$$

Clearly $\mathcal{B} \subset \mathcal{A}$. Let $(U,\phi)$ and $(V,\psi) \in \mathcal{A}$. Define $F : \phi(U\cap V) \to \psi(U\cap V)$ by

$$F = \psi|_{U\cap V} \circ (\phi|_{U\cap V})^{-1}$$

Let $q \in \phi(U\cap V)$. Set $p = \phi^{-1}(q)$. Since $p \in U\cap V \subset M$, there exists $(W,\chi) \in \mathcal{B}$ such that $p \in W$. By definition of $\mathcal{A}$, $\psi|_{W\cap V} \circ (\chi|_{W\cap V})^{-1} : \chi(W\cap V) \to \psi(W\cap V)$ and $\chi|_{U\cap W} \circ (\phi|_{U\cap W})^{-1} : \phi(U\cap W) \to \chi(U\cap W)$ are diffeomorphisms. Set $N = U\cap W\cap V$. Then $q \in \phi(N) \subset \phi(U\cap V)$ and

$$\begin{aligned} F|_{\phi(N)} &= \psi|_N \circ (\phi|_N)^{-1} \\ &= [\psi|_N \circ (\chi|_N)^{-1}] \circ [\chi|_N \circ (\phi|_N)^{-1}] \end{aligned}$$

is a diffeomorphism. Thus, for each $q \in \phi(U\cap V)$, there exists $N' \subset \phi(U\cap V)$ such that $F|_{N'}$ is a diffeomorphism. Hence $F$ is a diffeomorphism and $(U,\phi)$, $(V,\psi)$ are smoothly compatible. Therefore $\mathcal{A}$ is a smooth atlas.
To see that $\mathcal{A}$ is maximal, let $\mathcal{B}'$ be a smooth atlas on $M$. Suppose that $\mathcal{A} \subset \mathcal{B}'$ and let $(U,\phi) \in \mathcal{B}'$. By definition, for each chart $(V,\psi) \in \mathcal{B}'$, $(U,\phi)$ and $(V,\psi)$ are smoothly compatible. Since $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$, we have that $(U,\phi) \in \mathcal{A}$. So $\mathcal{A} = \mathcal{B}'$ and $\mathcal{A}$ is a maximal smooth atlas on $M$. ☐

**Exercise 3.2.0.4.** Let $(M,\mathcal{A})$ be an $n$-dimensional smooth manifold, $(U,\phi) \in \mathcal{A}$ and $U' \subset U$. If $U'$ is open, then $(U', \phi|_{U'}) \in \mathcal{A}$.

*Proof.* Set $\phi' = \phi|_{U'}$. A previous exercise implies that $(U',\phi') \in X(U)$. Define $\mathcal{B} = \mathcal{A} \cup \{(U',\phi')\}$. Let $(V,\psi) \in \mathcal{B}$. If $(V,\psi) = (U',\phi')$, then

$$\phi' \circ \psi^{-1} = \mathrm{id}_{U'}$$

which is a diffeomorphism. Thus $(U',\phi')$, $(V,\psi)$ are smoothly compatible. Suppose that $(V,\psi) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, $\psi|_{U\cap V} \circ (\phi|_{U\cap V})^{-1} : \phi(U\cap V) \to \psi(U\cap V)$ is a diffeomorphism. Therefore $\psi|_{U'\cap V} \circ (\phi'|_{U'\cap V})^{-1} : \phi'(U'\cap V) \to \psi(U'\cap V)$ is a diffeomorphism and $(U',\phi')$, $(V,\psi)$ are smoothly compatible. Since $(V,\psi) \in \mathcal{A}$ is arbitrary, $\mathcal{B}$ is smooth. Since $\mathcal{A}$ is maximal and $\mathcal{A} \subset \mathcal{B}$, we have that $\mathcal{A} = \mathcal{B}$ and $(U',\phi') \in \mathcal{A}$. ☐

**Exercise 3.2.0.5.** Let $(M,\mathcal{A})$ be a $n$-dimensional smooth manifold and $U \subset M$ open. Set $\mathcal{B} = \{(V,\psi) \in \mathcal{A} : V \subset U\}$. Then $\mathcal{B}$ is a smooth atlas on $U$.

*Proof.*

- Some previous exercises imply that $U$ is an $n$-dimensional topological manifold and $X(U) = \{(V, \psi) \in X(M) : V \subset U\}$. Since

$$\begin{aligned} \mathcal{B} &\subset \mathcal{A} \\ &\subset X(M) \end{aligned}$$

  we have that $\mathcal{B} \subset X(U)$. Let $p \in U$. Then there exists $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{V'}$. The previous exercise implies that $(V', \psi') \in \mathcal{A}$. By definition, $(V', \psi') \in \mathcal{B}$. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $(V', \psi') \in \mathcal{B}$ such that $p \in V'$. Hence $\mathcal{B}$ is an atlas on $U$.

- Let $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$. Then $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, $(V_1, \psi_1)$ and $(V_2, \psi_2)$ are smoothly compatible. Since $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ are arbitrary, $\mathcal{B}$ is smooth.

□

**Definition 3.2.0.6. Smooth Open Submanifold:**
Let $(M, \mathcal{A})$ be an $n$-dimensional smooth manifold and $U \subset M$ open. A previous exercise implies that $U$ is an $n$-dimensional topological manifold. We define $\mathcal{A}|_U \subset X(U)$ to be the unique smooth structure on $U$ such that $\{(V, \psi) \in \mathcal{A} : V \subset U\} \subset \mathcal{A}|_{\mathcal{U}}$. Then $(U, \mathcal{A}|_U)$ is said to be a **smooth open submanifold of** $(M, \mathcal{A})$.

**Exercise 3.2.0.7.** Let $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ be the projection map given by $\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$. Then $\pi$ is a diffeomorphism.

*Proof.* Define projection map $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by $\pi'(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1})$. Then $\mathbb{R}^n$ is an open neighborhood of $\partial H^n$, $\pi'|_{\partial H^n} = \pi$ and $\pi'$ is smooth. Then by definition, $\pi$ is smooth. Clearly, $\pi^{-1}$ is smooth. So $\pi$ is a diffeomorphism. □

**Definition 3.2.0.8.** Let $(M, \mathcal{A})$ be a $n$-dimensional smooth manifold and $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ the projection map. Recall that for $(U, \phi) \in X^n_\partial(M)$, the $(n-1)$-coordinate chart $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$ is defined by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$.
We define

$$\overline{\mathcal{A}} = \{(\bar{U}, \bar{\phi}) : (U, \phi) \in \mathcal{A} \cap X^n_\partial(M)\}$$

**Exercise 3.2.0.9.** Let $(M, \mathcal{A})$ be a $n$-dimensional smooth manifold. Then $\overline{\mathcal{A}}$ is a smooth atlas on $\partial M$.

*Proof.*

- A previous exercise implies that $\partial M$ is an $(n-1)$-dimensional topological manifold. Let $p \in \partial M$. Then there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Since $\mathcal{A} \subset X^n(M)$ and $p \in \partial M$, we have that $p \in \bar{U}$ and a previous exercise implies that $(U, \phi) \in X^n_\partial(M)$. By definition of $\overline{\mathcal{A}}$, $(\bar{U}, \bar{\phi}) \in \overline{\mathcal{A}}$. Since $p \in \partial M$ is arbitrary, $\overline{\mathcal{A}}$ is an atlas on $\partial M$.

- Let $(\bar{U}, \bar{\phi})$, $(\bar{V}, \bar{\psi}) \in \overline{\mathcal{A}}$. Since $(U, \phi)$ and $(V, \psi)$ are smoothly compatible, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$ is a diffeomorphism. Thus $\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}$ is a diffeomorphism. Since $\pi|_{\phi(U \cap V)}$ and $\pi|_{\psi(U \cap V)}$ are diffeomorphisms, $\pi|_{\phi(\bar{U} \cap \bar{V})}$ and $\pi|_{\psi(\bar{U} \cap \bar{V})}$ are diffeomorphisms. Then

$$\begin{aligned} \bar{\psi}|_{\bar{U} \cap \bar{V}} \circ (\bar{\phi}|_{\bar{U} \cap \bar{V}})^{-1} &= \left[\pi|_{\psi(\bar{U} \cap \bar{V})} \circ \psi|_{\bar{U} \cap \bar{V}}\right] \circ \left[(\phi|_{\bar{U} \cap \bar{V}})^{-1} \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1}\right] \\ &= \pi|_{\psi(\bar{U} \cap \bar{V})} \circ \left[\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}\right] \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \end{aligned}$$

  is a diffeomorphism. Therefore $(\bar{U}, \bar{\phi})$ and $(\bar{V}, \bar{\psi})$ are smoothly compatible. Since $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \overline{\mathcal{A}}$ are arbitrary, $\mathcal{A}$ is smooth.

□

**Definition 3.2.0.10.** Let $(M, \mathcal{A})$ be a $n$-dimensional smooth manifold. We define $\mathcal{A}|_{\partial M}$ to be the unique smooth structure on $\partial M$ such that $\overline{\mathcal{A}} \subset \mathcal{A}|_{\partial M}$. We define the **smooth boundary submanifold of** $M$ to be $(\partial M, \mathcal{A}|_{\partial M})$.

**Exercise 3.2.0.11. Smooth Manifold Chart Lemma:**
Let $M$ be a set, $\Gamma$ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

(a) for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha)$ is open in $\mathbb{H}^n$

(b) for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection

(c) for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{H}^n$

(d) for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

(e) there exists $\Gamma' \subset \Gamma$ such that $\Gamma'$ is countable and $M \subset \bigcup\limits_{\alpha \in \Gamma'} U_\alpha$

(f) for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Then there exists a unique smooth structure $\mathcal{A}$ on $M$ such that $(M, \mathcal{A})$ is an $n$-dimensional smooth manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset \mathcal{A}$.

*Proof.* Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : \alpha \in \Gamma \text{ and } V \in \mathcal{T}_{\mathbb{H}^n}\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$
- $\mathcal{A}' = \{(U_\alpha, \phi_\alpha) : \alpha \in \Gamma\}$.

The topological manifold chart lemma implies that $(M, \mathcal{T}_M)$ is an $n$-dimensional topological manifold and $\mathcal{A}' \subset X^n(M, \mathcal{T}_M)$. Since $M = \bigcup_{\alpha \in \Gamma} U_\alpha$, $\mathcal{A}'$ is an atlas on $M$. Since for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth, we have that $\mathcal{A}'$ is smooth. A previous exercise implies that there exists a unique smooth structure $\mathcal{A}$ on $M$ such that $\mathcal{A}' \subset \mathcal{A}$. $\square$

## 3.3 Smooth Maps

**Definition 3.3.0.1.** Let $(M, \mathcal{A})$ be a smooth manifold and $f : M \to \mathbb{R}$. Then $f$ is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. The set of all smooth functions on $M$ is denoted $C^\infty(M)$.

**Exercise 3.3.0.2.** Let $(M, \mathcal{A})$ be a smooth manifold. Then $C^\infty(M)$ is a vector space.

*Proof.* Let $f, g \in C^\infty(M)$, $\lambda \in \mathbb{R}$ and $(U, \phi) \in \mathcal{A}$. By assumption, $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are smooth. Hence

$$(f + \lambda g) \circ \phi^{-1} = f \circ \phi^{-1} + \lambda g \circ \phi^{-1}$$

is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, $f + \lambda g \in C^\infty(M)$. Since $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$ are arbitrary, $C^\infty(M)$ is a vector space. □

**Exercise 3.3.0.3.** Let $(M, \mathcal{A})$ be a smooth manifold, $\mathcal{B}$ an atlas on $M$ and $f : M \to \mathbb{R}$. Suppose that $\mathcal{B} \subset \mathcal{A}$. Then $f$ is smooth iff for each $(U, \phi) \in \mathcal{B}$, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth.

*Proof.*

- ( $\Longrightarrow$ ):
  Suppose that $f$ is smooth. Let $(U, \phi) \in \mathcal{B}$. Since $\mathcal{B} \subset \mathcal{A}$, $(U, \phi) \in \mathcal{A}$. Since $f$ is smooth, $f \circ \phi^{-1}$ is smooth. Since $(U, \phi) \in \mathcal{B}$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{B}$, $f \circ \phi^{-1}$ is smooth.

- ( $\Longleftarrow$ ):
  Suppose that for each $(V, \psi) \in \mathcal{B}$, $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}$ is smooth. Let $(U, \phi) \in \mathcal{A}$ and $q \in \phi(U)$. Set $p = \phi^{-1}(q)$. Since $\mathcal{B}$ is an atlas, there exists $(V, \psi) \in \mathcal{B}$ such that $p \in V$. Since $\mathcal{B} \subset \mathcal{A}$, $(V, \psi) \in \mathcal{A}$. Set $W = U \cap V$ and $\tilde{\phi} = \phi|_W$ and $\tilde{\psi} = \psi|_W$. We note that $\phi(W) \in \mathcal{N}_q$ and $\phi(W)$ is open. An exercise in the section on smooth manifolds implies that $(W, \tilde{\phi}), (W, \tilde{\psi}) \in \mathcal{A}$. Therefore $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(W) \to \psi(W)$ is smooth. By assumption, $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}$ is smooth. This implies that $(f \circ \psi^{-1})|_{\psi(W)} : \psi(W) \to \mathbb{R}$ is smooth. Hence

  $$\begin{aligned} (f \circ \phi^{-1})|_{\phi(W)} &= f \circ \tilde{\phi}^{-1} \\ &= f \circ (\tilde{\psi}^{-1} \circ \tilde{\psi}) \circ \tilde{\phi}^{-1} \\ &= (f \circ \tilde{\psi}^{-1}) \circ (\tilde{\psi} \circ \tilde{\phi}^{-1}) \end{aligned}$$

  is smooth. Since $q \in \phi(U)$ is arbitrary, for each $q \in \phi(U)$, there exists $A \in \mathcal{N}_q$ such that $A$ is open and $(f \circ \phi^{-1})|_A : A \to \mathbb{R}$ is smooth. This implies that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, $f$ is smooth.

□

**Exercise 3.3.0.4.** Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $p \in U$ and $f \in C^\infty(M)$. Then $f|_U \in C^\infty(U)$.

*Proof.* Let □

**Definition 3.3.0.5.** Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. We define the **partial derivative of $f$ with respect to $x^i$**, denoted

$$\partial f / \partial x^i : U \to \mathbb{R} \quad \text{or} \quad \partial_i f : U \to \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left( \frac{\partial}{\partial u^i}[f \circ \phi^{-1}] \right) \circ \phi$$

**Exercise 3.3.0.6.** Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. Then $\partial/\partial x^i : C^\infty(U) \to C^\infty(U)$ is linear.

*Proof.* **FINISH!!!** □

**Exercise 3.3.0.7.** Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f = \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi$$

*Proof.*

$$\begin{aligned}
\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f &= \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j} f\right) \\
&= \frac{\partial}{\partial x^i}\left(\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right] \circ \phi\right) \\
&= \left(\frac{\partial}{\partial u^i}\left[\left(\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right] \circ \phi\right) \circ \phi^{-1}\right]\right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i}\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right]\right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi
\end{aligned}$$

□

**Exercise 3.3.0.8.** Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}$$

*Proof.* Let $f \in C^\infty(U)$. Since $f \circ \phi^{-1}$ is smooth,

$$\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}] = \frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f \circ \phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned}
\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f &= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f \circ \phi^{-1}]\right) \circ \phi \\
&= \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} f
\end{aligned}$$

□

**Exercise 3.3.0.9.** Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $f \in C^\infty(U)$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha f = (\partial^\alpha[f \circ \phi^{-1}]) \circ \phi$$

*Proof.* The claim is clearly true when $|\alpha| = 0$ or by definition if $|\alpha| = 1$. Let $n \in \mathbb{N}$ and suppose the claim is true for each $|\alpha| \in \{1, \dots, n-1\}$. Then there exists $i \in \{1, \dots, n\}$ such that $\alpha_i \geq 1$. Hence

$$\begin{aligned}
\partial^\alpha f &= \partial^{e^i}(\partial^{\alpha - e^i} f) \\
&= \partial^{e^i}(\partial^{\alpha - e^i}[f \circ \phi^{-1}] \circ \phi) \\
&= (\partial^{e^i}[(\partial^{\alpha - e^i}[f \circ \phi^{-1}] \circ \phi) \circ \phi^{-1}]) \circ \phi \\
&= (\partial^{e^i}[\partial^{\alpha - e^i}[f \circ \phi^{-1}]]) \circ \phi \\
&= (\partial^\alpha[f \circ \phi^{-1}]) \circ \phi
\end{aligned}$$

□

**Exercise 3.3.0.10. Taylor's Theorem:**
Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\phi(U)$ convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that

$$f = \sum_{k=0}^{T} \left[ \sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(x_0) \right] + \sum_{|\alpha|=T+1} (x^i - x^i(p))^\alpha g_\alpha$$

and for each $|\alpha| = T+1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

*Proof.* Since $\phi(U)$ is open and convex and $f \circ \phi^{-1} \in C^\infty(\phi(U))$, Taylors thorem in section 2.1 implies that there exist $(\tilde{g}_\alpha)_{|\alpha|=T+1} \subset C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = \sum_{k=0}^{T} \left[ \sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q))$$

and for each $|\alpha| = T+1$,

$$\begin{aligned}
\tilde{g}_\alpha(\phi(p)) &= \frac{1}{(T+1)!} \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \\
&= \frac{1}{(T+1)!} \partial^\alpha f(p)
\end{aligned}$$

For $|\alpha| = T+1$, set $g_\alpha = \tilde{g} \circ \phi$. Then

$$\begin{aligned}
f(q) &= f \circ \phi^{-1}(\phi(q)) \\
&= \sum_{k=0}^{T} \left[ \sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\
&= \sum_{k=0}^{T} \left[ \sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q)
\end{aligned}$$

□

**Definition 3.3.0.11.** Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$. Then $F$ is said to be

- **smooth** if for each $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$,

  $$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(F(U) \cap V)$$

  is smooth

- a **diffeomorphism** if $F$ is a bijection and $F, F^{-1}$ are smooth.

**Exercise 3.3.0.12.** Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifold and $F : M \to N$. If $F$ is smooth, then $F$ is continuous.

*Proof.* Suppose that $F$ is smooth. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $F(p) \in V$. Put $\tilde{U} = U \cap F^{-1}(V)$ and $\tilde{V} = F(U) \cap V$.
Define $\tilde{\phi} : \tilde{U} \to \phi(\tilde{U})$ and $\tilde{\psi} : \tilde{V} \to \psi(\tilde{V})$ by

$$\tilde{\phi} = \phi|_{\tilde{U}}, \quad \tilde{\phi} = \psi|_{\tilde{V}}$$

Then $\tilde{\phi}$ and $\tilde{\psi}$ are homeomorphisms, $p \in \tilde{U}$ and $F(\tilde{U}) \subset \tilde{V}$. Define $\tilde{F} : \phi(\tilde{U}) \to \psi(\tilde{V})$ by

$$\tilde{F} = \tilde{\psi} \circ F \circ \tilde{\phi}^{-1}$$

By definition, $\tilde{F}$ is smooth and therefore continuous. Since $\phi$ and $\psi$ are homeomorphisms and $F|_{\tilde{U}} = \tilde{\psi}^{-1} \circ \tilde{F} \circ \tilde{\phi}$, we have that $F|_{\tilde{U}}$ is continuous. In particular, $F$ is continuous at $p$ and since $p \in M$ is arbitrary, $F$ is continuous. $\square$

**Exercise 3.3.0.13.** Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifold and $F : M \to N$. If $F$ is a diffeomorphism, then $F$ is a homeomorphism.

*Proof.* Suppose that $F$ is a diffeomorphism. By definition, $F$ and $F^{-1}$ are smooth. The previous exercise implies that $F$ and $F^{-1}$ are continuous. Hence $F$ is a homeomorphism. $\square$

**Exercise 3.3.0.14.** Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ a diffeomorphism. Then for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

*Proof.* Let $(V, \psi) \in \mathcal{B}$.

1. Since $\phi$ and $F^{-1}$ are homeomorphisms, $\phi \circ F^{-1} : F(U) \cap V \to \phi(U \cap F^{-1}(V))$ is a homeomorphism

2. Since $F$ is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \to \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \psi(V \cap F(U))$$

are smooth.

Therefore $(F(U), \phi \circ F^{-1})$ and $(V, \psi)$ are smoothly compatible. Since $\mathcal{B}$ is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. $\square$

**Definition 3.3.0.15.** Let $(N, \mathcal{B})$ be a smooth $n$-dimensional manifold, $F : M \to N$ smooth and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. For $i \in \{1, \dots, n\}$, We define the ***i*-th component of $F$ with respect to** $(V, \psi)$, denoted $F^i : V \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

## 3.4 Partitions of Unity

**Definition 3.4.0.1.** Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then $\rho$ is said to be a **bump function at p supported in** $U$ if

1. $\rho \geq 0$
2. there exists $V \in \mathcal{N}_p$ such that $V$ is open and $\rho|_V = 1$
3. $\operatorname{supp} \rho \subset U$

**Exercise 3.4.0.2.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

*Proof.* $\square$

## 3.5 The Tangent Space

**Definition 3.5.0.1.** Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. For $i \in \{1, \cdots, n\}$, define the partial derivative with respect to $x^i$ at $p$, denoted

$$\frac{\partial}{\partial x^i}\bigg|_p : C^\infty(M) \to \mathbb{R}, \text{ or } \partial_i|_p : C^\infty(M) \to \mathbb{R}$$

by

$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}(p)$$

**Exercise 3.5.0.2.** Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$, we have that

$$\frac{\partial}{\partial x^i} x^j(p) = \delta_{i,j}$$

*Proof.* Let $i, j \in \{1, \cdots, n\}$. Then

$$\begin{aligned} \frac{\partial}{\partial x^i}\bigg|_p x^i &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} x^i \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} u^i \circ \phi \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} u^i \\ &= \delta_{i,j} \end{aligned}$$

□

**Exercise 3.5.0.3. Change of Coordinates:**
Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $\psi = (y^1, \cdots, y^n)$, $p \in U \cap V$ and $f \in C^\infty(M)$. Then for each $i \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial y^i}\bigg|_p = \sum_{j=1}^n \frac{\partial}{\partial x^j} y^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

*Proof.* Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \cdots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned} \frac{\partial}{\partial y^i}\bigg|_p f &= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \psi^{-1} \\ &= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \phi^{-1} \circ h \\ &= \sum_{j=1}^n \left( \frac{\partial}{\partial u^j}\bigg|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left( \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} h_j \right) \\ &= \sum_{j=1}^n \left( \frac{\partial}{\partial u^j}\bigg|_{\phi(p)} f \circ \phi^{-1} \right) \left( \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} x^j \circ \psi^{-1} \right) \\ &= \sum_{j=1}^n \left( \frac{\partial}{\partial x^i}\bigg|_p f \right) \left( \frac{\partial}{\partial y^i}\bigg|_p x^j \right) \end{aligned}$$

□

**Definition 3.5.0.4.** Let $p \in M$ and $v : C^\infty(M) \to \mathbb{R}$. Then $v$ is said to be **Leibnizian** if for each $f, g \in C^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and $v$ is said to be a **derivation at** $p$ if for each $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,

1. $v$ is linear
2. $v$ is Leibnizian

We define the **tangent space of** $M$ **at** $p$, denoted $T_pM$, by

$$T_pM = \{v : C^\infty(M) \to \mathbb{R} : v \text{ is a derivation at } p\}$$

**Exercise 3.5.0.5.** Let $f \in C^\infty(M)$ and $v \in T_pM$. If $f$ is constant, then $vf = 0$.

*Proof.* Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since $v$ is linear, $v(f) = cv(1) = 0$. □

**Exercise 3.5.0.6.** Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

is a basis for $T_pM$ and $\dim T_pM = n$.

*Proof.* Clearly $\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \in T_pM$. Let $a_1, \cdots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p x^j \\ &= a_j \end{aligned}$$

Hence $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is independent.

Now, let $v \in T_pM$ and $f \in \mathbb{C}^\infty(M)$. By Taylor's theorem, there exist $g_1, \cdots g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x^i}\bigg|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p f \\ &= \left[\sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p\right] f \end{aligned}$$

So

$$v = \sum_{i=1}^{n} v(x^i) \frac{\partial}{\partial x^i}\bigg|_p$$

and

$$v \in \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

□

**Definition 3.5.0.7.** Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. We define the **differential of** $F$ **at** $p$, denoted $DF_p : T_pM \to T_{F(p)}N$, by

$$\Big[DF_p(v)\Big](f) = v(f \circ F)$$

for $v \in T_pM$ and $f \in C^\infty(N)$.

**Exercise 3.5.0.8.** Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. Then for each $v \in T_pM$, $DF_p(v)$ is a derivation.

*Proof.* Let $v \in T_pM$, $f, g \in C^\infty_{F(p)}(N)$ and $c \in \mathbb{R}$. Then

1.

$$\begin{aligned} DF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= DF_p(v)(f) + cDF_p(v)(g) \end{aligned}$$

So $DF_p(v)$ is linear.

2.

$$\begin{aligned} DF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= DF_p(v)(f) * g(F(p)) + f(F(p)) * DF_p(v)(g) \end{aligned}$$

So $DF_p(v)$ is Leibnizian and hence $DF_p(v) \in T_{F(p)}N$ □

**Exercise 3.5.0.9.** Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. If $F$ is a diffeomorphism, then $DF_p$ is an isomorphism.

*Proof.* Suppose that $F$ is a diffeomorphism. Since $F$ is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \cdots, x^n)$ and $\phi \circ F^{-1} = (y^1, \cdots, y^n)$. Let $f \in C^\infty(N)$ Then

$$\begin{aligned} \frac{\partial}{\partial y^i}\bigg|_{F(p)} f &= \frac{\partial}{\partial u^i}\bigg|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \frac{\partial}{\partial x^i}\bigg|_p f \circ F \end{aligned}$$

Therefore

$$\begin{aligned}\left[DF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right)\right](f) &= \frac{\partial}{\partial x^i}\bigg|_p f\circ F\\ &= \frac{\partial}{\partial y^i}\bigg|_{F(p)} f\end{aligned}$$

Hence

$$DF_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial y^i}\bigg|_{F(p)}$$

Since $\left\{\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ is a basis for $T_pM$ and $\left\{\frac{\partial}{\partial y^1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}\Big|_{F(p)}\right\}$ is a basis for $T_{F(p)}N$, $DF_p$ is an isomorphism. □

**Exercise 3.5.0.10.** Let $(M,\mathcal{A})$ be a smooth $m$-dimensional manifold, $(N,\mathcal{B})$ a $n$-dimensional smooth manifold, $F: M\to N$ smooth, $(U,\phi)\in\mathcal{A}$ with $\phi=(x^1,\dots,x^m)$ and $(V,\psi)\in\mathcal{B}$ with $\psi=(y^1,\dots,y^n)$. Suppose that $p\in U$ and $F(p)\in V$. Define the ordered bases $B_\phi = \left\{\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^m}\Big|_p\right\}$ and $B_\psi = \left\{\frac{\partial}{\partial y^1}\Big|_{F(p)}, \cdots, \frac{\partial}{\partial y^n}\Big|_{F(p)}\right\}$. Then the matrix representation of $DF_p$ with respect to the bases $B_\phi$ and $B_\psi$ is

$$DF_p^{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

*Proof.* Let $(DF_p)_{B_\phi,B_\psi} = (a_{i,j})_{i,j}\in\mathbb{R}^{n\times m}$. Then for each $j\in\{1,\dots,m\}$,

$$DF_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{i=1}^n a_{i,j}\frac{\partial}{\partial y^i}\bigg|_{F(p)}$$

This implies that

$$\begin{aligned}DF_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right)(y^k) &= \sum_{i=1}^n a_{i,j}\frac{\partial}{\partial y^i}\bigg|_{F(p)}(y^k)\\ &= \sum_{i=1}^n a_{i,j}\delta_{i,k}\\ &= a_{k,j}\end{aligned}$$

By definition,

$$\begin{aligned}DF_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right)(y^k) &= \frac{\partial}{\partial x^j}\bigg|_p y^k\circ F\\ &= \frac{\partial}{\partial x^j}\bigg|_p F^k\\ &= \frac{\partial F^k}{\partial x^j}(p)\end{aligned}$$

□

**Note 3.5.0.11.** Since $\operatorname{rank} DF_p$ is independent of basis, it is independent of coordinate charts $(U,\phi)\in\mathcal{A}$ and $(V,\psi)\in\mathcal{B}$.

## 3.6 The Cotangent Space

**Definition 3.6.0.1.** Let $p \in M$. We define the **cotangent space of** $M$ **at** $p$, denoted $T_p^*M$, by

$$T_p^*M = (T_pM)^*$$

**Definition 3.6.0.2.** Let $f \in C^\infty(M)$. We define the **differential of** $f$ **at** $p$, denoted $df_p : T_pM \to \mathbb{R}$, by

$$df_p(v) = vf$$

**Exercise 3.6.0.3.** Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

*Proof.* Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1 f + \lambda v_2 f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that $df_p$ is linear and hence $df_p \in T_p^*M$. $\square$

**Exercise 3.6.0.4.** Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$,

$$dx_p^i \left( \frac{\partial}{\partial x^j}\bigg|_p \right) = \delta_{i,j}$$

In particular, $\{dx_p^1, \cdots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ and $T_p^*M = \operatorname{span}\{dx_p^1, \cdots, dx_p^n\}$.

*Proof.* Let $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned} \left[ dx_p^i \left( \frac{\partial}{\partial x^i}\bigg|_p \right) \right]_p &= \frac{\partial}{\partial x^i}\bigg|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

$\square$

**Exercise 3.6.0.5.** Let $f \in C^\infty(M)$, $(U, \phi)$ a chart on $M$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i{}_p$$

*Proof.* Since $\{dx_p^1, \cdots, dx_p^n\}$ is a basis for $T_p^*M$, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left( \frac{\partial}{\partial x^i}\bigg|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left( \frac{\partial}{\partial x^i}\bigg|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left( \frac{\partial}{\partial x^i}\bigg|_p \right) &= \frac{\partial}{\partial x^i}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p) dx_p^i$$

$\square$

# Chapter 4

# Submersions and Immersions

## 4.1 Maps of Constant Rank

**Definition 4.1.0.1.** Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. We define the **rank map of** $F$, denoted $\operatorname{rank} F : M \to \mathbb{N}_0$ by

$$\operatorname{rank}_p F = \dim \operatorname{Im} DF(p)$$

and $F$ is said to have **constant rank** if for each $p, q \in M$, $\operatorname{rank}_p F = \operatorname{rank}_q F$. If $F$ has constant rank, we define the **rank of** $F$, denoted $\operatorname{rank} F$, by $\operatorname{rank} F = \operatorname{rank}_p F$ for $p \in M$.

**Exercise 4.1.0.2.** Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimensions $m$ and $n$ respectively, $F \in C^\infty(M, N)$ and $p \in M$. Suppose that $\operatorname{rank}_p F = k$. Then there exist $(U, \phi) \in \mathcal{A}_M$, $(V, \psi) \in \mathcal{A}_N$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi,\psi})_{i,j} = A_{i,j}$$

*Proof.* Define $q \in V$ by $q = F(p)$. Choose $(U', \phi') \in \mathcal{A}$ and $(V', \psi') \in \mathcal{B}$ such that $p \in U'$ and $q \in V'$. Set $Z = [DF(p)]_{\phi',\psi'}$. By assumption, $\operatorname{rank} Z = k$. An exercise in the subsection on linear algebra implies that there exist $\sigma \in S_m$, $\tau \in S_n$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Define $\phi : U \to \sigma\phi(U)$ and $\psi : V \to \tau\psi(V)$ by

$$\phi = \sigma\phi', \quad \psi = \tau\psi'$$

A previous exercise implies that

$$[DF(p)]_{\phi,\psi} = P_\tau Z P_\sigma^*$$

$\square$

**Exercise 4.1.0.3. Constant Rank Theorem:**
Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimensions $m$ and $n$ respectively, $F \in C^\infty(M, N)$. Suppose that $F$ has constant rank and $\operatorname{rank} F = k$. Then for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$ and

$$\psi \circ F \circ \phi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

*Proof.* Let $p \in M$. The previous exercise implies that there exist $(U_0, \phi_0) \in \mathcal{A}$, $(V_0, \psi_0) \in \mathcal{B}$ and $L \in GL(k, \mathbb{R})$ such that $p \in U$, $F(p) \in V_0$ and for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi_0,\psi_0})_{i,j} = L_{i,j}$$

Define $\hat{M} \subset \mathbb{R}^m$, $\hat{N} \subset \mathbb{R}^n$ and $\hat{F} : \hat{M} \to \hat{N}$ by $\hat{M} = \phi_0(U_0)$, $\hat{N} = \psi_0(V_0)$ and $\hat{F} = \psi_0 \circ F \circ \phi_0^{-1}$. Set $\hat{p} = \phi_0(p)$. Let $(x, y)$ be the standard coordinates on $\mathbb{R}^m$, with $\pi_x : \mathbb{R}^m \to \mathbb{R}^k$ and $\pi_y : \mathbb{R}^m \to \mathbb{R}^{m-k}$ the standard projection maps. Write $\hat{p} = (x_0, y_0)$. There exist $Q : \hat{M} \to \mathbb{R}^k$ and $R : \hat{M} \to \mathbb{R}^{n-k}$ such that $\hat{F} = (Q, R)$. By construction, $[D_x Q(x_0, y_0)] = A$. Define $G : \hat{M} \to \mathbb{R}^m$ by $G(x, y) = (Q(x, y), y)$. Then

$$\begin{aligned}
[DG(x_0, y_0)] &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_x Q(x_0, y_0)] \\ [D_x \pi_y(x_0, y_0)] & [D_y \pi_y(x_0, y_0)] \end{pmatrix} \\
&= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} L & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\det([DG(x_0, y_0)]) &= \det(L)\det(I) \\
&= \det(L) \\
&\neq 0
\end{aligned}$$

The inverse function theorem implies that there exist $\hat{U} \subset \hat{M}$ such that $\hat{U}$ is open, $\hat{p} \in \hat{U}$ and $G|_{\hat{U}} : \hat{U} \to G(\hat{U})$ is a diffeomorphism. Since

$$\{U_1 \times U_2 : U_1 \subset \mathbb{R}^k, U_2 \subset \mathbb{R}^{m-k} \text{ and } U_1, U_2 \text{ are open}\}$$

is a basis for the topology on $\mathbb{R}^m$, there exist $\hat{U}_1 \subset \mathbb{R}^k$ and $\hat{U}_2 \subset \mathbb{R}^{m-k}$ such that $\hat{U}_1$, $\hat{U}_2$ are open, $\hat{p} \in \hat{U}_1 \times \hat{U}_2$ and $\hat{U}_1 \times \hat{U}_2 \subset \hat{U}$. Set $\hat{U}_{12} = \hat{U}_1 \times \hat{U}_2$. Since $G(\hat{U}_1 \times \hat{U}_2) = Q(\hat{U}_{12}) \times \hat{U}_2$, we have that $G|_{\hat{U}_{12}} : \hat{U}_{12} \to Q(\hat{U}_{12}) \times \hat{U}_2$ is a diffeomorphism. Since $\pi_x$ is open, $Q(\hat{U}_{12})$ is open. There exist $A : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_1$ and $B : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_2$ such that $G^{-1} = (A, B)$. Define $\tilde{R} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \mathbb{R}^{n-k}$ by $\tilde{R}(x, y) = R(A(x, y), y)$. Let $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$. Then

$$\begin{aligned}
(x, y) &= G \circ G^{-1}(x, y) \\
&= G(A(x, y), B(x, y)) \\
&= (Q(A(x, y), B(x, y)), B(x, y))
\end{aligned}$$

This implies that $B(x, y) = y$,

$$\begin{aligned}
x &= Q(A(x, y), B(x, y)) \\
&= Q(A(x, y), y)
\end{aligned}$$

Hand

$$\begin{aligned}
G^{-1}(x, y) &= (A(x, y), B(x, y)) \\
&= (A(x, y), y)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{F} \circ G^{-1}(x, y) &= \hat{F}(A(x, y), y) \\
&= (Q(A(x, y), y), R(A(x, y), y)) \\
&= (x, R(A(x, y), y)) \\
&= (x, \tilde{R}(x, y))
\end{aligned}$$

We note that

$$\begin{aligned}
[D(\hat{F} \circ G^{-1})(x, y)] &= \begin{pmatrix} [D_x \pi_x(x, y)] & [D_y \pi_x(x, y)] \\ [D_x \tilde{R}(x, y)] & [D_y \tilde{R}(x, y)] \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ [D_x \tilde{R}(x, y)] & [D_y \tilde{R}(x, y)] \end{pmatrix}
\end{aligned}$$

Since $G^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_{12}$ is a diffeomorphism, we have that $[DG^{-1}(x, y)] \in GL(m, \mathbb{R})$. Since $\hat{F}$ has constant rank and rank $\hat{F} = k$, we have that

$$\begin{aligned}\operatorname{rank}[D(\hat{F} \circ G^{-1})(x, y)] &= \operatorname{rank}([D\hat{F}(G^{-1}(x, y))][DG^{-1}(x, y)]) \\ &= \operatorname{rank}[D\hat{F}(G^{-1}(x, y))] \\ &= k\end{aligned}$$

Since rank $\begin{pmatrix} I \\ [D_x\tilde{R}(x, y)] \end{pmatrix} = k$, we have that rank $\begin{pmatrix} 0 \\ [D_y\tilde{R}(x, y)] \end{pmatrix} = 0$. Thus $[D_y\tilde{R}(x, y)] = 0$. Since $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$ is arbitrary, for each $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\tilde{R}(x, y) = \tilde{R}(x, y_0)$$

Define $\tilde{S} : Q(\hat{U}_{12}) \to \mathbb{R}^{n-k}$ by $\tilde{S}(x) = \tilde{R}(x, y_0)$. Then for each $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\hat{F} \circ G^{-1}(x, y) = (x, \tilde{S}(x))$$

Let $(a, b)$ be the standard coordinates on $\mathbb{R}^n$, with $\pi_a : \mathbb{R}^n \to \mathbb{R}^k$ and $\pi_b : \mathbb{R}^n \to \mathbb{R}^{n-k}$ the standard projection maps. Write $\hat{F}(\hat{p}) = (a_0, b_0)$. Set

$$\begin{aligned}\hat{V} &= [(\pi_a)|_{\hat{N}}]^{-1}(Q(\hat{U}_{12})) \\ &= \pi_a^{-1}(Q(\hat{U}_{12})) \cap \hat{N}\end{aligned}$$

Since $Q(\hat{U}_{12})$ is open, $\hat{N}$ is open and $\pi_a$ is continuous, we have that $\hat{V}$ is open. Since

$$\begin{aligned}Q(\hat{U}_{12}) &= (\pi_a)|_{\hat{N}} \circ \hat{F} \circ G^{-1}(Q(\hat{U}_{12}) \times \hat{U}_2) \\ &= (\pi_a)|_{\hat{N}} \circ \hat{F}(\hat{U}_{12})\end{aligned}$$

we have that $\hat{F}(\hat{U}_{12}) \subset \hat{V}$. In particular, $\hat{F}(\hat{p}) \in \hat{V}$. Define $H : \hat{V} \to \mathbb{R}^n$ by $H(a, b) = (a, b - \tilde{S}(a))$. Then $H \circ \hat{F} \circ G^{-1}(x, y) = (x, 0)$. Define $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{N}$ by $U = \phi_0^{-1}(\hat{U}_{12})$, $V = \psi_0^{-1}(\hat{V})$, $\phi = G \circ \phi_0$ and $\psi = H \circ \psi_0$. Then for each $(x, y) \in \phi(U)$,

$$\begin{aligned}\psi \circ F \circ \phi^{-1}(x, y) &= H \circ \psi_0 \circ F \circ \phi_0^{-1} \circ G^{-1}(x, y) \\ &= H \circ \hat{F} \circ G^{-1}(x, y) \\ &= (x, 0)\end{aligned}$$

□

**Definition 4.1.0.4.** Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. Then $F$ is said to be

- an **immersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is injective
- a **submersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is surjective

**Exercise 4.1.0.5.** Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map.

**Definition 4.1.0.6.** Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$ smooth. Then $F$ is said to be an **embedding** if

1. $F$ is an immersion
2. $F : M \to F(M)$.

**Note 4.1.0.7.** Here the topology on $F(M)$ is the subspace topology.

## 4.2 Submanifolds

**Exercise 4.2.0.1.** Let $(M, \mathcal{A})$ be a smooth manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $\tilde{U} \subset S$ and $\tilde{\phi} : \tilde{U} \to \phi(\tilde{U})$ by $\tilde{U} = U \cap S$ and $\tilde{\phi} = \phi|_{U \cap S}$. Set $\mathcal{B} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \in \mathcal{A}\}$. Then $\mathcal{B}$ is a smooth structure on $S$.

*Proof.* □

**Definition 4.2.0.2.** Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. Suppose that $M \subset N$. Then $(M, \mathcal{A})$ is said to be

1. an **immersed submanifold** of $(N, \mathcal{B})$ if $\mathrm{id} : M \to N$ is a smooth immersion
2. an **embedded submanifold** of $(N, \mathcal{B})$ if $\mathrm{id} : M \to N$ is a smooth embedding

**Note 4.2.0.3.** Essentially, embedded submanifolds are immersed submanifolds with the subspace topology.

**Note 4.2.0.4.** For the remainder of this section, we assume that $k \leq n$.

**Definition 4.2.0.5.** Let $U \subset \mathbb{R}^n$ and $S \subset U$. Then $S$ is said to be a $k$**-slice** of $U$ if $S = \{u \in U : u^{k+1}, \dots, u^n = 0\}$.

**Exercise 4.2.0.6.** Let $U \subset \mathbb{R}^n$ and $S \subset U$. Suppose that $S$ is a $k$-slice of $U$. Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then $\pi|_S \to \pi(S)$ is a diffeomorphism.

*Proof.* Clear. $\square$

**Definition 4.2.0.7.** Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ and $S \subset U$. Then $S$ is said to be a $k$**-slice** of $U$ if $\phi(S)$ is a $k$-slice of $\phi(U)$.

**Definition 4.2.0.8.** Let $(M, \mathcal{A})$ be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$. Then $(U, \phi)$ is said to be a $k$**-slice chart for** $S$ if $U \cap S$ is a $k$-slice of $U$.

**Exercise 4.2.0.9.** Let $(M, \mathcal{A})$ be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. If $(U, \phi)$ is a $k$-slice chart for $S$, then $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$.

*Proof.* Clear. $\square$

**Definition 4.2.0.10.** Let $(M, \mathcal{A})$ be a smooth manifold and $S \subset M$. Then $S$ is said to satisfy the **local** $k$**-slice condition** if for each $p \in S$, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and $(U, \phi)$ is a $k$-slice chart of $S$.

**Exercise 4.2.0.11.** Let $(M, \mathcal{A})$ be a $n$-dimensional smooth manifold and $S \subset M$ a subspace. If $S$ satisfies the local $k$-slice condition, then there exists a smooth structure $\tilde{\mathcal{A}}$ on $S$ such that $(S, \tilde{\mathcal{A}})$ is an embedded submanifold of $M$.

*Proof.* Suppose that $S$ satisfies the local $k$-slice condition. Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ as above Let $(U, \phi) \in \mathcal{A}$. Suppose that $(U, \phi)$ is a $k$-slice chart for $S$. Define $\tilde{U} = U \cap S$ and $\tilde{\phi} : \tilde{U} \to \pi \circ \phi(\tilde{U})$ by

$$\tilde{\phi} = \pi \circ \phi|_{\tilde{U}}$$

By definition, $\phi(\tilde{U})$ is a $k$-slice of $\phi(U)$. A previous exercise implies that $\pi|_{\phi(\tilde{U})} \to \pi \circ \phi(\tilde{U})$ is a diffeomorphism and hence a homeomorphism. Thus $\tilde{\phi}$ is a homeomorphism.
Define

$$\tilde{\mathcal{B}} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \text{ is a } k\text{-slice for } S\}$$

Let $p \in S$. By assumption, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and $(U, \phi)$ is a $k$-slice chart of $S$. Then $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{B}}$ and $\mathcal{A}$ is an atlas on $S$. By construction of $\tilde{\mathcal{B}}$, $S$ is locally half Euclidean of dimension $k$. Since $M$ is second countable Hausdorff, so is $S$ in the subspace topology. Thus $(S, \tilde{\mathcal{B}})$ is a $k$-dimensional manifold.
Let $(\tilde{U}, \tilde{\phi})$, $(\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$. Then

$$\tilde{\phi} \circ \tilde{\psi}^{-1}|_{\tilde{U} \cap \tilde{V}} = \pi|_{\phi(\tilde{U} \cap \tilde{V})} \circ \phi|_{\tilde{U} \cap \tilde{V}} \circ \psi|^{-1}_{\tilde{U} \cap \tilde{V}} \circ \pi|^{-1}_{\psi(\tilde{U} \cap \tilde{V})}$$

which is a diffeomorphism. So $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ smoothly compatible. Hence $\tilde{\mathcal{B}}$ is smooth. An exercise in section 4.1 implies that there exists a unique smooth structure $\tilde{\mathcal{A}}$ on $S$ such that $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$. So $(S, \tilde{\mathcal{A}})$ is a smooth $k$-dimensional manifold.
Clearly $\mathrm{id} : S \to S$ is a homeomorphism. Let $(V, \psi) \in \mathcal{A}$ and $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$.

**Finish!!** $\square$

**Definition 4.2.0.12.**

**Exercise 4.2.0.13.**

# Chapter 5

# Vector Fields

## 5.1 The Tangent Bundle

**Definition 5.1.0.1.** Let $(M, \mathcal{A}_M)$ be an $n$-dimensional smooth manifold. We define the **tangent bundle of** $M$, denoted $TM$, by

$$TM = \coprod_{p \in M} T_pM$$

and we define the **tangent bundle projection**, denoted $\pi : TM \to M$, by

$$\pi(p, v) = p$$

Let $(U, \phi) \in \mathcal{A}_M$ with $\phi = (x^1, \dots, x^n)$. We define $\Phi_\phi : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\Phi_\phi\left(p, \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p\right) = (\phi(p), \xi^1, \dots, \xi^n)$$

We define $\mathcal{T}_{TM} = \tau_{TM}(\iota_p : p \in M)$.

**Exercise 5.1.0.2.** $\psi : \bigcup_{p \in U} T_pM \to \mathbb{R}^n$ is given by

$$\psi\left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p\right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{(\phi(p),\psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial v^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \frac{\partial}{\partial v^i}\bigg|_{(\phi(p),\psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \frac{\partial}{\partial v^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\
&= 0
\end{aligned}
$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$
\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(\pi(p,\xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i}(p,\xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(p)\delta_{i,k} \\
&= \frac{\partial f}{\partial x^i}(p)
\end{aligned}
$$

and

$$
\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(\pi(p,\xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i}(p,\xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(p)0 \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \ker D\pi(p,\xi) \\
&= \coprod_{(p,\xi)\in\pi^{-1}(U)} \operatorname{span}\left\{ \frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \dots, n\} \right\}
\end{aligned}
$$

# Chapter 6

# Lie Theory

## 6.1 Lie Groups

**Definition 6.1.0.1.** Let $G$ be a smooth manifold and group. Then $G$ is said to be a **Lie group** if

- multiplication $G \times G \to G$ given by $(g, h) \mapsto gh$ is smooth
- inversion $G \to G$ given by $g \mapsto g^{-1}$ is smooth

**Definition 6.1.0.2.** Let $\mathfrak{g}$ be a vector space and $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $[\cdot, \cdot]$ is said to be a **Lie bracket** on $\mathfrak{g}$ if

1. $[\cdot, \cdot]$ is bilinear
2. $[\cdot, \cdot]$ is antisymmetric
3. $[\cdot, \cdot]$ satisfies the Jacobi identity:
   for each $x, w, y \in \mathcal{F}g$,
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

In this case, $(\mathfrak{g}, [\cdot, \cdot])$ is said to be a **Lie algebra**.

**Definition 6.1.0.3.** Let $X \in$

# Chapter 7

# Bundles and Sections

## 7.1 Fiber Bundles

**Note 7.1.0.1.** Let $U, F$ be sets, we write $\text{proj}_1 : U \times F \to U$ to denote the projection onto $U$.

**Definition 7.1.0.2.** Let $E, M, F \in \text{Obj}(\mathbf{Set})$ and $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then $(U, \Phi)$ is said to be a **local trivialization of $E$ over $U$ with fiber $F$** if

1. $\Phi$ is a bijection
2. $\text{proj}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$, i.e. the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\text{proj}_1"] \\ & U \end{tikzcd}$$

**Exercise 7.1.0.3.** Let $E, M, F \in \text{Obj}(\mathbf{Set})$ and $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$ a local trivialization of $E$ over $U$ with fiber $F$. Then for each $A \subset U$,

$$\Phi(\pi^{-1}(A)) = A \times F$$

**Hint:** consider $\Phi^{-1}(A \times F)$

*Proof.* Let $A \subset U$. Since $\text{proj}_1^{-1}(A) = A \times F$, we have that

$$\begin{aligned} \Phi^{-1}(A \times F) &= \Phi^{-1}(\text{proj}_1^{-1}(A)) \\ &= (\text{proj}_1 \circ \Phi)^{-1}(A) \\ &= (\pi|_{\pi^{-1}(U)})^{-1}(A) \\ &= \pi^{-1}(A) \cap \pi^{-1}(U) \\ & \pi^{-1}(A \cap U) \\ &= \pi^{-1}(A) \end{aligned}$$

Since $\Phi$ is a bijection, we have that

$$\begin{aligned} \Phi(\pi^{-1}(A)) &= \Phi \circ \Phi^{-1}(A \times F) \\ &= A \times F \end{aligned}$$

□

**Definition 7.1.0.4.** Let $E, M, F \in \mathbf{Man}^\infty$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then $(U, \Phi)$ is said to be a **smooth local trivialization of $E$ over $U$ with fiber $F$** if

1. $(U, \Phi)$ is a local trivialization of $E$ over $U$ with fiber $F$
2. $U$ is open
3. $\Phi$ is a diffeomorphism

**Definition 7.1.0.5.** Let $E, M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then $(E, M, \pi, F)$ is said to be a **smooth fiber bundle with total space $E$, base space $M$, fiber $F$ and projection $\pi$** if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that $U$ is open and $(U, \Phi)$ is a smooth local trivialization of $E$ over $U$ with fiber $F$. For $p \in M$, we define the **fiber over** $p$, denoted $E_p$, by $E_p = \pi^{-1}(\{p\})$.

**Note 7.1.0.6.** When the context is clear, we will supress the fiber manifold $F$.

**Definition 7.1.0.7.** Let $(E_1, M_1, \pi_1, F_1)$ and $(E_2, M_2, \pi_2, F_2)$ be smooth fiber bundles, $\Phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E_1, E_2)$ and $\phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M_1, M_2)$. Then $(\Phi, \phi)$ is said to be a **smooth bundle morphism** from $(E_1, M_1, \pi_1, F_1)$ to $(E_2, M_2, \pi_2, F_2)$ if $\pi_2 \circ \Phi = \phi \circ \pi_1$, i.e. the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\phi"'] & M_2 \end{tikzcd}$$

**Definition 7.1.0.8.** We define the category of smooth fiber bundles, denoted $\mathbf{Bun}^\infty$, by

- $\mathrm{Obj}(\mathbf{Bun}^\infty) = \{(E, M, \pi, F) : (E, M, \pi, F) \text{ is a smooth fiber bundle}\}$
- For $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$,

$$\mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) =$$
$$\{(\Phi, \phi) : (\Phi, \phi) \text{ is a smooth bundle morphism from } (E_1, M_1, \pi_1, F_1) \text{ to } (E_2, M_2, \pi_2, F_2)\}$$

- For
  - $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$
  - $(\Phi_{12}, \phi_{12}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$
  - $(\Phi_{23}, \phi_{23}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3))$

  we define $(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_3, M_3, \pi_3))$ by

$$(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) = (\Phi_{23} \circ \Phi_{12}, \phi_{23} \circ \phi_{12})$$

**Exercise 7.1.0.9.** We have that $\mathbf{Bun}^\infty$ is a full subcategory of $(\mathrm{id}_{\mathbf{Man}^\infty} \downarrow \mathrm{id}_{\mathbf{Man}^\infty})$.

*Proof.* Set $\mathcal{C} = (\mathrm{id}_{\mathbf{Man}^\infty} \downarrow \mathrm{id}_{\mathbf{Man}^\infty})$. We note that

- $\mathrm{Obj}(\mathbf{Bun}^\infty) \subset \mathrm{Obj}(\mathcal{C})$
- for each $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$,

$$\mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) = \mathrm{Hom}_{\mathcal{C}}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$$

So $\mathbf{Bun}^\infty$ is a full subcategory of $\mathcal{C}$. □

**Exercise 7.1.0.10.** Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$ and $(U, \Phi)$ a local trivialization of $E$ over $U$ and $(V, \Psi)$ a local trvialization of $E$ over $V$. Then

1. $\mathrm{proj}_{U \cap V} \circ \Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1} = \mathrm{proj}_1$

2. there exists $\sigma \in \mathrm{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that for each $p \in U \cap V$, $\sigma(p, \cdot) : F \to F$ is a diffeomorphism.

*Proof.*

1. By definition, the following diagram commutes:

$$\begin{tikzcd}
(U \cap V) \times F \arrow[dr, "\mathrm{proj}_1"'] & \pi^{-1}(U \cap V) \arrow[l, "\Phi"'] \arrow[r, "\Psi"] \arrow[d, "\pi"'] & (U \cap V) \times F \arrow[dl, "\mathrm{proj}_1"] \\
& N &
\end{tikzcd}$$

$\mathrm{proj}_1 \circ \Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1} = \mathrm{proj}_1$

2. there exists $\sigma \in \mathrm{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that for each $p \in U \cap V$ and $x \in F$,

$$\Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1}(p, x) = (p, \sigma(p, x))$$

and $\sigma(p, \cdot) : F \to F$ is a diffeomorphism.

$\square$

**Definition 7.1.0.11.** Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$ and $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ a collection of smooth local trivializations of $E$. Then $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ is said to be a **fiber bundle atlas** if for each $p \in M$, there exists $\alpha \in A$ such that $p \in U_\alpha$. For $\alpha, \beta \in A$, we define $\phi$

## 7.2 $G$-Bundles

**Definition 7.2.0.1.** Let $G$ be a Lie group and $(E, M, \pi, F) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$. Then

## 7.3 Vector Bundles

**Note 7.3.0.1.** Let $M$ be a set and $p \in M$. We endow $\{p\} \times \mathbb{R}^n$ with the natural vector space structure such that $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

**Definition 7.3.0.2.** Let $E, M \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then $(E, M, \pi)$ is said to be a **rank $k$ smooth vector bundle** if

1. $(E, M, \pi, \mathbb{R}^k) \in \text{Obj}(\mathbf{Bun}^\infty)$
2. for each $p \in M$, $E_p$ is a $k$-dimensional real vector space
3. for each smooth local trivialization $(U, \Phi)$ of $E$ over $U$ with fiber $\mathbb{R}^k$ and $p \in U$,
$$\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$$
is a vector space isomorphism

In this case we define the **rank of** $(E, M, \pi)$, denoted $\text{rank}(E, M, \pi)$, by $\text{rank}(E, M, \pi) = k$.

**Definition 7.3.0.3.** We define the category of smooth vector bundles, denoted $\mathbf{VecBun}^\infty$, by

- $\text{Obj}(\mathbf{VecBun}^\infty) = \{(E, M, \pi) : (E, M, \pi) \text{ is a smooth vector bundle}\}$
- For $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{VecBun}^\infty)$ with $\text{rank}(E_1, M_1, \pi_1) = k_1$ and $\text{rank}(E_2, M_2, \pi_2) = k_2$,
$$\text{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

**Exercise 7.3.0.4.** We have that $\mathbf{VecBun}^\infty$ is a full subcategory of $\mathbf{Bun}^\infty$.

*Proof.* We note that

- $\text{Obj}(\mathbf{VecBun}^\infty) \subset \text{Obj}(\mathbf{Bun}^\infty)$
- for each $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{Bun}^\infty)$ with $\text{rank}(E_1, M_1, \pi_1) = k_1$ and $\text{rank}(E_2, M_2, \pi_2) = k_2$,
$$\text{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

So $\mathbf{Bun}^\infty$ is a full subcategory of $\mathcal{C}$. □

**Exercise 7.3.0.5.** Let $M \in \mathbf{Man}^\infty$. Set $n = \dim M$, $E = M \times \mathbb{R}^k$ and define $\pi : E \to M$ by $\pi(p, x) = p$. Then $(E, M, \pi)$ is a rank $k$ smooth vector bundle.

*Proof.*

1. For each $p \in M$, $\pi_1^{-1}(\{p\}) = \{p\} \times \mathbb{R}^k$ is an $n$-dimensional real vector space.
2. Let $p \in M$. Set $U = M$. Then $\pi^{-1}(U) = E$. Define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ by $\Phi = \text{id}_E$. Then $(U, \Phi)$ is a smooth local trivialization of $E$ over $U$.
3. Let $p \in M$. Then $\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \to \{p\} \times \mathbb{R}^k$ is clearly an isomorphism.

□

**Exercise 7.3.0.6. Smooth Vector Bundle Chart Lemma:**
Let $M \in \text{Obj}(\mathbf{Man}^\infty)$. Denote the topology on $M$ by $\mathcal{T}_M$. Suppose that for each $p \in M$, there exists $E_p \in \text{Obj}(\mathbf{Vect}_\mathbb{R})$ such that $\dim E_p = k$. We define $E \in \text{Obj}(\mathbf{Set})$ and $\pi \in \text{Hom}_{\mathbf{Set}}(E, M)$ by
$$E = \coprod_{p \in M} E_p$$
and $\pi(p, v) = p$. Let $\Gamma$ be an index set and $(U_\alpha)_{\alpha \in \Gamma} \subset \mathcal{T}_M$. Suppose that

1. $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
2. for each $\alpha \in \Gamma$, there exists $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ is a bijection
   - $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a vector space isomorphism
3. for each $\alpha, \beta \in \Gamma$, there exists $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that
   - $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ is smooth
   - $\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is given by

## 7.4 Bundle Morphisms

**Definition 7.4.0.1.** Let $(E, M, \pi_E)$ and $(F, N, \pi_F)$ be smooth fiber bundles and $\Phi : E \to F$ and $\phi : M \to N$. Then $(\Phi, \phi)$ is said to be a **smooth fiber bundle morphism** from $(E, M, \pi_E)$ to $(F, N, \pi_F)$ if $\Phi$ is smooth, $\phi$ is smooth and $\pi_F \circ \Phi = \phi \circ \pi_E$, i.e. the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

and we write $(\Phi, \phi) : (E, M, \pi_E) \to (F, N, \pi_F)$.

**Exercise 7.4.0.2.** Let $(E, M, \pi_E)$ and $(F, N, \pi_F)$ be smooth fiber bundles and $(\Phi, \phi) : (E, M, \pi_E) \to (F, N, \pi_F)$. Suppose that $(\Phi, \phi)$ is smooth. Then for each $p \in M$,

$$\Phi^{-1}(F_{\phi(p)}) = E_p$$

*Proof.* Let $p \in M$. Set $q = \phi(p)$. Then

$$\begin{aligned} \Phi^{-1}(F_q) &= \Phi^{-1}(\pi_F^{-1}(\{q\})) \\ &= (\pi_F \circ \Phi)^{-1}(\{q\}) \\ &= (\phi \circ \pi_E)^{-1}(\{q\}) \\ &= \pi_E^{-1}(\phi^{-1}(\{\phi(p)\})) \end{aligned}$$

FINISH!!!, multiple fibers get mapped to same fiber $\square$

## 7.5 Subbundles

## 7.6 Vertical and Horizontal Subbundles

**Definition 7.6.0.1.** Let $(E, M, \pi_M) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$. We define the **vertical bundle associated to** $(E, M, \pi_M)$ , denoted $(VE, M, \pi_V) \in \mathbf{Bun}^\infty$, by

$$VE = \coprod_{q \in E} \ker D\pi(q)$$

relocate this to after tangent bundle is introduced

**Exercise 7.6.0.2.** Let $(M, \mathcal{A})$ be an $n$-dimensional smooth manifold and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $(\pi^{-1}(U), \Phi_\phi) \in \mathcal{A}_{TM}$ the induced chart on $TM$ with $\Phi_\phi = (\tilde{x}^1, \dots, \tilde{x}^n, \tilde{y}^1, \dots, \tilde{y}^n)$. Then

$$V(TM)|_{\pi^{-1}(U)} = \coprod_{(p,\xi) \in \pi^{-1}(U)} \mathrm{span}\left\{ \frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \dots, n\} \right\}$$

Split into smaller exercises

*Proof.* Let $f \in C^\infty(M)$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$ the standard coordinates on $\mathbb{R}^n \times \mathbb{R}^n$. We note that by definition, $\Phi_\phi(p, \xi) = (\phi(p), \psi(\xi))$ where $\psi : \bigcup_{p \in U} T_pM \to \mathbb{R}^n$ is given by

$$\psi\left( \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial v^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i}\bigg|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= 0 \end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i}(p,\xi) \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)\delta_{i,k} \\
&= \frac{\partial f}{\partial x^i}(p)
\end{aligned}$$

and

$$\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i}(p,\xi) \\
&= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)0 \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \ker D\pi(p,\xi) \\
&= \coprod_{(p,\xi)\in\pi^{-1}(U)} \operatorname{span}\left\{\frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1,\dots,n\}\right\}
\end{aligned}$$

$\square$

## 7.7 The Tangent Bundle

**Definition 7.7.0.1.** We define the **tangent bundle of** $M$, denoted $TM$, by

$$TM = \coprod_{p \in M} T_pM$$

We denote the natrual projection map by $\pi : TM \to M$.

**Definition 7.7.0.2.** Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Define $\tilde{U} \subset TM$ and $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}^n$ by

- $\tilde{U} = \pi^{-1}(U)$
- $$\begin{aligned} \tilde{\phi}\left( \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p \right) &= (\phi(p), v) \\ &= (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \end{aligned}$$
  ,

**Exercise 7.7.0.3.** Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}$ is a bijection.

## 7.8 The cotangent Bundle

**Definition 7.8.0.1.** We define the **cotangent bundle of** $M$, denoted $T^*M$, by

$$T^*M = \coprod_{p \in M} T_p^* M$$

## 7.9 The $(r, s)$-Tensor Bundle

**Definition 7.9.0.1.**

1. the **cotangent bundle of** $M$, denoted $T^*M$, by

$$T^*M = \coprod_{p \in M} T_p^* M$$

2. the $(r, s)$**-tensor bundle of** $M$, denoted $T_s^r M$, by

$$T_s^r M = \coprod_{p \in M} T_s^r (T_p M)$$

3. the $k$**-alternating tensor bundle of** $M$, denoted $\Lambda^k(M)$, by

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k (T_p M)$$

## 7.10 Vector Fields

**Definition 7.10.0.1.** Let $X : M \to TM$. Then $X$ is said to be a **vector field on** $M$ if for each $p \in M$, $X_p \in T_pM$.
For $f \in \mathbb{C}^\infty(M)$, we define $Xf : M \to \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and $X$ is said to be **smooth** if for each $f \in \mathbb{C}^\infty(M)$, $Xf$ is smooth.
We denote the set of smooth vector fields on $M$ by $\Gamma^1(M)$.

**Definition 7.10.0.2.** Let $f \in C^\infty(M)$ and $X, Y \in \Gamma^1(M)$. We define

- $fX \in \Gamma^1(M)$ by
$$(fX)_p = f(p)X_p$$
- $X + Y \in \Gamma^1(M)$ by
$$(X+Y)_p = X_p + Y_p$$

**Exercise 7.10.0.3.** The set $\Gamma^1(M)$ is a $C^\infty(M)$-module.

*Proof.* Clear. □

**Exercise 7.10.0.4.** Let $X \in \Gamma^1(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then

$$X|_U = \sum_{i=1}^{n} (Xx^i) \frac{\partial}{\partial x^i}$$

*Proof.* Let $p \in M$. Then $X_p \in T_pM$ and $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis of $T_pM$. So there exist $f_1(p), \cdots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum\limits_{i=1}^{n} f^i(p) \frac{\partial}{\partial x^i}\Big|_p$. Let $j \in \{1, \cdots, n\}$. Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^{n} f^i(p) \frac{\partial}{\partial x^j} x^i(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx^j = f_j$ and $X|_U = \sum\limits_{i=1}^{n} (Xx^i) \frac{\partial}{\partial x^i}$. □

**Exercise 7.10.0.5.** Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then for each $i \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

*Proof.* Let $i \in \{1, \cdots, n\}$ and $f \in C^\infty(M)$. Define $g : M \to \mathbb{R}$ by $g = \frac{\partial}{\partial x^i} f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i}\Big|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i}\Big|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i} [f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence $g$ is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x^i}$ is smooth. □

## 7.11 1-Forms

**Definition 7.11.0.1.** Let $\omega : M \to T^*M$. Then $\omega$ is said to be a **1-form on** $M$ if for each $p \in M$, $\omega_p \in T_p^*M$.
For each $X \in \Gamma^1(M)$, we define $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and $\omega$ is said to be **smooth** if for each $X \in \Gamma^1(M)$, $\omega(X)$ is smooth.
The set of smooth 1-forms on $M$ is denoted $\Gamma_1(M)$.

**Definition 7.11.0.2.** Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma^1(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta \in \Gamma^1(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

**Exercise 7.11.0.3.** The set $\Gamma_1(M)$ is a $C^\infty(M)$-module.

*Proof.* Clear. $\square$

**Exercise 7.11.0.4.**

## 7.12 $(r,s)$-Tensor Fields

**Definition 7.12.0.1.** Let $\alpha : M \to T^r_s M$. Then $\alpha$ is said to be an **$(r,s)$-tensor field on** $M$ if for each $p \in M$, $\alpha_p \in T^r_s(T_pM)$.
For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \to \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and $\alpha$ is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth.
The set of smooth $(r,s)$-tensor fields on $M$ is denoted $T^r_s(M)$.

**Definition 7.12.0.2.** Let $f \in C^\infty(M)$ and $\alpha, \beta \in T^r_s(M)$. We define

- $f\alpha : M \to T^r_s M$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta : M \to T^r_s M$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

**Exercise 7.12.0.3.** Let $f \in C^\infty(M)$ and $\alpha, \beta \in T^r_s(M)$. Then

1. $f\alpha \in T^r_s(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
2. $\alpha + \beta \in T^r_s(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

*Proof.* Clear. □

**Exercise 7.12.0.4.** The set $T^r_s(M)$ is a $C^\infty(M)$-module.

*Proof.* Clear. □

**Definition 7.12.0.5.** Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. We define the **tensor product of $\alpha$ with $\beta$**, denoted $\alpha \otimes \beta : M \to T^{r_1+r_2}_{s_1+s_2} M$, by
$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

**Exercise 7.12.0.6.** Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma^{r_1+r_2}_{s_1+s_2}(M)$

*Proof.* Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since $\alpha_1$ and $\alpha_2$ are smooth by assumption. □

**Definition 7.12.0.7.** We define the **tensor product**, denoted $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

**Exercise 7.12.0.8.** The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is associative.

*Proof.* Clear. □

**Exercise 7.12.0.9.** The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is $C^\infty(M)$-bilinear.

*Proof.* Clear. □

**Definition 7.12.0.10.** Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ a smooth map and $\alpha \in \Gamma^0_k(N)$. We define the **pullback of** $\alpha$ **by** $F$, denoted $F^*\alpha \in \Gamma^0_k(M)$, by

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k))$$

for $p \in M$ and $v_1, \dots, v_k \in T_pM$

**Exercise 7.12.0.11.** Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ and $(L, \mathcal{C})$ be smooth manifolds, $F : M \to N$ and $G : N \to L$ smooth maps, $\alpha \in \Gamma^0_k(N)$, $\beta \in \Gamma^0_l(N)$, $\gamma \in \Gamma^0_k(L)$ and $f \in C^\infty(N)$. Then

1. $F^*(f\alpha) = (f \circ F)F^*\alpha$
2. $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
3. $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
4. $(G \circ F)^*\gamma = F^*(G^*\gamma)$
5. $id_N^*\alpha = \alpha$

*Proof.*

1.
$$\begin{aligned}[F^*(f\alpha)]_p(v_1, \dots, v_k) &= (f\alpha)_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\ &= f(F(p))\alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\ &= (f \circ F)(p)(F^*\alpha)_p(v_1, \dots, v_k)\end{aligned}$$

   So that $F^*(f\alpha) = (f \circ F)F^*\alpha$

2.
$$F^*$$

□

**Definition 7.12.0.12.**

**Exercise 7.12.0.13.**

*Proof.* □

**Exercise 7.12.0.14.** Let $\alpha \in T^r_s(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then there exist $(f^I_J)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

*Proof.* Let $p \in M$. Then $\omega_p \in T^r_s(T_pM)$ and $\left\{\partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}\right\}$ is a basis of $T^r_s(T_pM)$. So there exist $(f^I_J(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned}\alpha_p(dx^K_p, \partial_{x^L}|_p) &= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}(dx^K_p, \partial_{x^L}|_p) \\ &= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p(dx^K_p) dx_p^{\otimes J}(\partial_{x^L}|_p) \\ &= f^K_L(p)\end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx^K, \partial_{x^L})_p$ is smooth, so that $f^K_L \in C^\infty(U)$.

□

**Definition 7.12.0.15.**

## 7.13 Differential Forms

**Definition 7.13.0.1.** We define

$$\Lambda^k(TM) = \coprod_{p \in M} \Lambda^k(T_pM)$$

**Definition 7.13.0.2.** Let $\omega : M \to \Lambda^k(TM)$. Then $\omega$ is said to be a **$k$-form on** $M$ if for each $p \in M$, $\omega_p \in \Lambda^k(T_pM)$.
For each $X \in \Gamma^1(M)^k$, we define $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and $\omega$ is said to be **smooth** if for each $X \in \Gamma^1(M)^k$, $\omega(X)$ is smooth.
The set of smooth $k$-forms on $M$ is denoted $\Omega^k(M)$.

**Note 7.13.0.3.** Observe that

1. $\Omega^k(M) \subset \Gamma^0_k(M)$
2. $\Omega^0(M) = C^\infty(M)$

**Exercise 7.13.0.4.** The set $\Omega^k(M)$ is a $C^\infty(M)$-submodule of $\Gamma^0_k(M)$.

*Proof.* Clear. □

**Definition 7.13.0.5.** Define the **exterior product**

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

**Note 7.13.0.6.** For $f \in \Omega^0(M)$ and $\alpha \in \Omega^k(M)$, we have that $f \wedge \alpha = f\alpha$.

**Exercise 7.13.0.7.** The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is well defined.

*Proof.* Let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{i=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then

$$\begin{aligned}
\alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\sigma(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)(\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \cdots, X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p))
\end{aligned}$$

□

**Exercise 7.13.0.8.** The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is $C^\infty(M)$-bilinear.

*Proof.*

1. $C^\infty(M)$-linearity in the first argument:
   Let $\alpha \in \Omega^k(M)$, $\beta, \gamma \in \Omega^l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda^k(T_pM) \times \Lambda^l(T_pM) \to \Lambda^{k+l}(T_pM)$ implies that

$$\begin{aligned}[(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p\end{aligned}$$

   So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

   and $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is $C^\infty(M)$-linear in the first argument.

2. $C^\infty(M)$-linearity in the second argument:
   Similar to (1).

□

**Note 7.13.0.9.** All of the results from multilinear algebra apply here.

**Definition 7.13.0.10.** We define the **exterior derivative** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ inductively by

1. $d(d\alpha) = 0$ for $\alpha \in \Omega^p(M)$
2. $df(X) = Xf$ for $f \in \Omega^0(M)$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$
4. extending linearly

**Exercise 7.13.0.11.** Let $(U, \phi)$ be a chart on $M$ with $\phi = (x^1, \cdots, x^n)$. Then on $U$, for each $i, j \in \{1, \cdots, n\}$,

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx^1_p, \cdots, dx^n_p\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$ and $T^*_pM = \operatorname{span}\{dx^1_p, \cdots, dx^n_p\}$.

*Proof.* Let $p \in U$ and $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned}\left[dx^i\left(\frac{\partial}{\partial x^j}\right)\right]_p &= \left(\frac{\partial}{\partial x^j} x^i\right)_p \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^i \\ &= \delta_{i,j}\end{aligned}$$

□

**Exercise 7.13.0.12.** Let $f \in C^\infty(M)$ and $(U, \phi)$ be a chart on $M$ with $\phi = (x^1, \cdots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

*Proof.* Let $p \in U$. Since $\{dx^1, \cdots, dx^n\}$ is a basis for $\Lambda(T_pM)$, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a^i(p)dx^i_p$. Therefore, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) &= \sum_{i=1}^n a^i(p)dx^i_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) &= \frac{\partial}{\partial x^i}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p)dx^i_p$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i}dx^i$$

□

**Exercise 7.13.0.13.** Let $f \in \Omega^0(M)$. If $f$ is constant, then $df = 0$.

*Proof.* Suppose that $f$ is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \ldots, x_n)$. Then for each $i \in \{1, \ldots, n\}$,

$$\frac{\partial}{\partial x^i}\bigg|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p)dx^i_p \\ &= 0 \end{aligned}$$

□

**Exercise 7.13.0.14.**

**Definition 7.13.0.15.** Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$. We define

$$dx^i = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_k}}\right)$$

**Note 7.13.0.16.** We have that

1.
$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{I,J}$$

2. Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega^k(U)$,

$$\omega\left(\frac{\partial}{\partial x^i}\right) \in C^\infty(U)$$

**Exercise 7.13.0.17.** Let $\omega \in \Omega^k(M)$ and $(U, \phi)$ be a chart on $M$ with $\phi = (x^1, \cdots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega\left(\frac{\partial}{\partial x^i}\right) dx^i$$

*Proof.* Let $p \in U$. Since $\{dx^i_p : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(T_pM)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum\limits_{I \in \mathcal{I}_k} f_I(p) dx^i_p$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial x^j}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j}\right) \\
&= f_J
\end{aligned}$$

□

**Exercise 7.13.0.18.** Let $\omega \in \Omega^k(M)$ and $(U, \phi)$ be a chart on $M$ with $\phi = (x^1, \cdots, x^n)$. If $\omega = \sum\limits_{I \in \mathcal{I}_k} f_I dx^i$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

.

*Proof.* First we note that

$$\begin{aligned}
d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f d(dx^i) \\
&= df_I \wedge dx^i \\
&= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i\right) \wedge dx^i \\
&= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i
\end{aligned}$$

Then we extend linearly. □

**Definition 7.13.0.19.** Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ be a diffeomorphism. Define the **pullback of** $F$, denoted $F^* : \Omega^k(N) \to \Omega^k(M)$ by

$$(F^*\omega)_p(v_1, \cdots, v_k) = \omega_{F(p)}(DF_p(v_1), \cdots, DF_p(v_k))$$

for $\omega \in \Omega^k(N)$, $p \in M$ and $v_1, \cdots, v_k \in T_pM$

# Chapter 8

# Connections

## 8.1 Koszul Connections

**Definition 8.1.0.1.** Let $(E, M, \pi) \in \text{Obj}(\textbf{VecBun}^{\infty})$ and $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then $\nabla$ is said to be a **Koszul connection on $E$ in the first representation** if

1. for each $\sigma \in \Gamma(E)$, $\nabla(\cdot, \sigma)$ is $C^{\infty}(M)$-linear
2. for each $X \in \mathfrak{X}(M)$, $\nabla(X, \cdot)$ is $\mathbb{R}$-linear
3. for each $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$ and $f \in C^{\infty}(M)$,

$$\nabla(X, f\sigma) = f\,\nabla(X, \sigma) + X(f)\sigma$$

**Definition 8.1.0.2.** Let $(E, M, \pi) \in \text{Obj}(\textbf{VecBun}^{\infty})$ be a smooth vector bundle and $\nabla : \Gamma(E) \to T^*M \otimes \Gamma(E)$. Then $\nabla$ is said to be a **Koszul connection on $E$ in the second representation** if

1. $\nabla$ is $\mathbb{R}$-linear
2. for each $\sigma \in \Gamma(E)$ and $f \in C^{\infty}(M)$,

$$\nabla(f\sigma) = f\,\nabla\,\sigma + df \otimes \sigma$$

**Note 8.1.0.3.** When the context is clear, we will write $\nabla_X\, Y$ in place of $\nabla(X, Y)$ and we will refer to $\nabla$ as a connection.

**Exercise 8.1.0.4.** Define $\phi : \Gamma(E)^{\mathfrak{X}(M)\times\Gamma(E)} \to [T^*M \otimes \Gamma(E)]^{\Gamma(E)}$ by

$$\phi(\nabla)(X) = \nabla_X\,\sigma$$

Then $\nabla$ is a Koszul connection on $E$ in the first representation iff $\phi(\nabla)$ Koszul connection on $E$ in the second representation.

*Proof.* □

**Exercise 8.1.0.5.** Let $(E, M, \pi) \in \text{Obj}(\textbf{VecBun}^{\infty})$, $\nabla$ a connection on $E$, $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$. If $X = 0$ or $Y = 0$, then $\nabla_X\, Y = 0$.

*Proof.*

- If $X = 0$, then

$$\begin{aligned}\nabla_X\, Y &= \nabla_{0X}\, Y \\ &= 0\,\nabla_X\, Y \\ &= 0\end{aligned}$$

- Similarly, if $Y = 0$, then $\nabla_X Y = 0$.

$\square$

**Exercise 8.1.0.6.** Let $(E, M, \pi)$ be a smooth vector bundle, $\nabla$ a connection on $E$, $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$ and $p \in M$. If $X \sim_p 0$ or $Y \sim_p 0$, then $[\nabla_X Y]_p = 0$.

*Proof.*

- Suppose that $X \sim_p 0$. Then there exists $U \subset M$ such that $U$ is open and $X|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\operatorname{supp} \phi \subset U$ and $\phi \sim_p 1$. Then $\phi X = 0$. The previous exercise implies that $\nabla_{\phi X} Y = 0$. Therefore

$$\begin{aligned} \nabla_X Y &= \nabla_{\phi X + (1-\phi)X} Y \\ &= \nabla_{\phi X} Y + \nabla_{(1-\phi)X} Y \\ &= 0 + (1-\phi)\nabla_X Y \\ &= (1-\phi)\nabla_X Y \end{aligned}$$

Hence

$$\begin{aligned} [\nabla_X Y]_p &= [(1-\phi)\nabla_X Y]_p \\ &= (1-\phi(p))[\nabla_X Y]_p \\ &= 0 \end{aligned}$$

- Suppose that $Y \sim_p 0$. Then there exists $U \subset M$ such that $U$ is open and $Y|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\operatorname{supp} \phi \subset U$ and $\phi \sim_p = 1$. Then $\phi Y = 0$. The previous exercise implies that $\nabla_X \phi Y = 0$. Since $\phi \sim_p 1$, we have that $1 - \phi \sim_p 0$. Thus $X(1-\phi) \sim_p 0$ and

$$\begin{aligned} \nabla_X Y &= \nabla_X[\phi Y + (1-\phi)Y] \\ &= \nabla_X[\phi Y] + \nabla_X[(1-\phi)Y] \\ &= \nabla_X[(1-\phi)Y] \\ &= (1-\phi)\nabla_X Y + [X(1-\phi)]\nabla_X Y \end{aligned}$$

Hence

$$\begin{aligned} [\nabla_X Y]_p &= (1-\phi(p))[\nabla_X Y]_p + [X(1-\phi)](p)[\nabla_X Y]_p \\ &= 0 \end{aligned}$$

$\square$

**Exercise 8.1.0.7.** Let $(E, M, \pi)$ be a smooth vector bundle and $\nabla$ a connection on $E$. Then for each $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$, $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$ implies that $[\nabla_{X_1} Y_1]_p = [\nabla_{X_2} Y_2]_p$.

*Proof.* Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$. Suppose that $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$. Define $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$ by $X = X_2 - X_1$ and $Y = Y_2 - Y_1$. Then $X \sim_p 0$ and $Y \sim_p 0$. The previous exercise implies that $[\nabla_X Y_1]_p = 0$ and $[\nabla_{X_2} Y]_p = 0$. Therefore

$$\begin{aligned}
[\nabla_{X_1} Y_1]_p &= [\nabla_{X_1} Y_1]_p + [\nabla_X Y_1]_p \\
&= [\nabla_{X_1} Y_1 + \nabla_X Y_1]_p \\
&= [\nabla_{X_1+X} Y_1]_p \\
&= [\nabla_{X_2} Y_1]_p \\
&= [\nabla_{X_2} Y_1]_p + [\nabla_{X_2} Y]_p \\
&= [\nabla_{X_2} Y_1 + \nabla_{X_2} Y]_p \\
&= [\nabla_{X_2}(Y_1 + Y)]_p \\
&= [\nabla_{X_2} Y_2]_p
\end{aligned}$$

□

**Exercise 8.1.0.8.** Let $(E, M, \pi)$ be a smooth vector bundle, $\nabla$ a connection on $E$ and $U \subset M$. If $U$ is open, then there exists a unique connection $\nabla^U : \mathfrak{X}(U) \times \Gamma(E|_U) \to \Gamma(E|_U)$ such that for each $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$,

$$\nabla^U_{X|_U} Y|_U = (\nabla_X Y)|_U$$

# Chapter 9

# Semi-Riemannian Geometry

**Definition 9.0.0.1.** Let $M$ be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then $g$ is said to be nondegenerate if for each $p \in M$, $g_p$ is nondegenerate.

**Definition 9.0.0.2.** Let $M$ be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then $g$ is said to be a **metric tensor field** on $M$ if

1. $g$ is nondegenerate
2. $g$ has constant index

In this case $(M, g)$ is said to be a **semi-Riemannian manifold**

**Definition 9.0.0.3.** Define Interval
FINISH!!!

**Definition 9.0.0.4.** Let $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ and $\gamma \in \text{Hom}_{\mathbf{Man}^\infty}(I, E)$. Then $\gamma$ is said to be a **section of** $E$ **over** $\alpha$ if $\pi \circ \gamma = \alpha$. We denote the set of sections of $E$ over $\alpha$ by $\Gamma(E, \alpha)$.

**Definition 9.0.0.5.** Let $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ and $\gamma \in \Gamma(E, \alpha)$. Then $\gamma$ is said to be said to be **extendible** if there exists $U \in \mathcal{N}_{\alpha(I)}$ and $\tilde{\gamma} \in \Gamma(E|_U)$ such that $U$ is open and $\tilde{\gamma} \circ \alpha = \gamma$.

**Exercise 9.0.0.6.** figure 8 not extendible FINISH!!!

**Exercise 9.0.0.7.** Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$, $\nabla$ a connection on $E$, $I \subset \mathbb{R}$ an interval and $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$. There exists a unique $D_\alpha : \Gamma(E, \alpha) \to \Gamma(E, \alpha)$ such that

1. for each $\lambda \in \mathbb{R}$ and $\gamma, \sigma \in \Gamma(E, \alpha)$,
$$D_\alpha(\gamma + \lambda\sigma) = D_\alpha\gamma + \lambda D_\alpha\sigma$$
2. for each $f \in C^\infty(I)$ and $\gamma \in \Gamma(E, \alpha)$,
$$D_\alpha(f\gamma) = f'\gamma + f D_\alpha\gamma$$
3. for each $\gamma \in \Gamma(E)$, if $\tilde{\gamma}$ extends $\gamma$, then
$$D_\alpha\gamma = \nabla_{\alpha'}\gamma$$

*Proof.* □

# Chapter 10

# Riemannian Geometry

**Definition 10.0.0.1.** Let $M$ be a smooth manifold and $g \in T^0_2(M)$ a metric tensor on $M$. We define $\hat{g} \in T^2_0(M)$ by $\hat{g}(\omega, \eta) = g(\phi_g^{-1}(\omega), \phi_g^{-1}(\eta))$.

**Exercise 10.0.0.2.** content...

**Exercise 10.0.0.3.** Let $(M, g)$ be a semi-Riemannian manifold and $(U, \phi) \in \mathcal{A}$. Then the induced metric $\langle\rangle_{T^*M \otimes TM}$ on $T^*M \otimes TM$ is given by

$$\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} = g^{i,j} g_{kl}$$

*Proof.* We have that

$$\begin{aligned}
\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} &= \langle dx^i, dx^j \rangle_{T^*M} \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle_{TM} \\
&= g^{i,j} g_{k,l}
\end{aligned}$$

□

**Exercise 10.0.0.4.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold.

1. There exists $\lambda \in \Omega^n(M)$ such that for each orthonormal frame $e_1, \ldots, e_n$,

   $$\lambda(e_1, \ldots, e_n) = 1$$

   **Hint:** Choose a frame $z_1, \ldots, z_n$ on $M$ with corresponding dual frame $\zeta^1, \ldots, \zeta^n$. Define

   $$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n$$

2. Let $N \in \mathfrak{X}(M)$ be the outward pointing normal to $\partial M$ and $X \in \mathfrak{X}(M)$. Then

   $$\int_M \text{div} X \lambda = \int_{\partial M} g(X, N) \tilde{\lambda}$$

3. For each $u \in \mathbb{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, we have that

   $$\text{div}(uX) = u\text{div}(X) + du(X)$$

   and therefore

   $$\int_M du(X) \lambda = \int_{\partial M} u g(X, N) \tilde{\lambda} - \int_M u \text{div}(X) \lambda$$

*Proof.*

1. Let $z_1, \dots, z_n$ be a frame on $M$ and $\zeta^1, \dots, \zeta^n$ with corresponding dual frame $\zeta^1, \dots, \zeta^n$. Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n$$

Let $e_1, \dots, e_n$, be an orthonormal frame on $M$ with corresponding dual coframe $\epsilon^1, \dots, \epsilon^n$. Let $i, j \in \{1, \dots, n\}$. Then there exist $(a_{k,i}) \subset \mathbb{R}$ such that $\zeta^i = \sum_{k=1}^n a_{k,i}\epsilon^k$. Then

$$\begin{aligned}
\hat{g}(\epsilon^j, \zeta^i) &= \sum_{k=1}^n a_{k,i} \hat{g}(\epsilon^j, \epsilon^k) \\
&= \sum_{k=1}^n a_{k,i} g(\phi_g^{-1}(\epsilon^j), \phi_g^{-1}(\epsilon^k)) \\
&= \sum_{k=1}^n a_{k,i} g(e_j, e_k) \\
&= \sum_{k=1}^n a_{k,i} \delta_{j,k} \\
&= a_{j,i}
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta_{i,j} &= \zeta^i(z_j) \\
&= \sum_{k=1}^n a_{k,i} \epsilon^k(z_j) \\
&= \sum_{k=1}^n a_{k,i} g(e_k, z_j) \\
&= \sum_{k=1}^n \hat{g}(\epsilon^k, \zeta^i) g(e_k, z_j)
\end{aligned}$$

Define $U, V \in \mathbb{R}^{n \times n}$ by $U_{i,k} = \hat{g}(\zeta^i, \epsilon^k)$ and $V_{k,j} = g(e_k, z_j)$. Then from above, we have that $UV = I$. Since $U, V \in \mathbb{R}^{n \times n}$, $VU = I$. Hence $U = V^{-1}$. Since

$$\begin{aligned}
\zeta^i(e_j) &= \sum_{k=1}^n a_{k,i} \epsilon^k(e_j) \\
&= \sum_{k=1}^n a_{k,i} \delta_{k,j} \\
&= a_{j,i} \\
&= \hat{g}(\epsilon^j, \zeta^i) \\
&= U_{i,j}
\end{aligned}$$

and

$$\begin{aligned}
g(z_i, z_j) &= \left( \sum_{k=1}^{n} g(e_k, z_i) e_k, \sum_{l=1}^{n} g(e_l, z_j) e_l \right) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} g(e_k, z_i) g(e_l, z_j) g(e_k, e_l) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{n} g(e_k, z_i) g(e_l, z_j) \delta_{k,l} \\
&= \sum_{k=1}^{n} g(e_k, z_i) g(e_k, z_j) \\
&= (V^* V)_{i,j}
\end{aligned}$$

we have that

$$\begin{aligned}
\lambda(e_1, \dots, e_n) &= \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n (e_1, \dots, e_n) \\
&= \det[g(z_i, z_j)]^{1/2} \det[\zeta^i(e_j)] \\
&= \det(V^* V)^{1/2} \det U \\
&= \det V (\det V)^{-1} \\
&= 1
\end{aligned}$$

2. Choose an orthonormal frame $e_1, \dots, e_{n-1} \in \mathfrak{X}(\partial M)$ with dual coframe $\epsilon^1, \dots, \epsilon^{n-1}$. Define $\nu \in \Omega^1(M)$ to be the dual covector to $N$. We note that $N, e_1, \dots, e_{n-1}$ is an orthonormal frame on $\mathfrak{X}(M)$. Let $X_1, \dots, X_{n-1} \in \mathfrak{X}(\partial M)$. Since for each $j \in \{1, \dots, n-1\}$, $X_j \in \mathfrak{X}(\partial M)$ and for each $p \in \partial M$, $N_p \in (T_p \partial M)^\perp$, we have that for each $j \in \{1, \dots, n-1\}$, $g(X_j, N) = 0$. This implies that

$$\begin{aligned}
\iota^* \iota_X \lambda(X_1, \dots, X_{n-1}) &= \lambda(X, X_1, \dots, X_{n-1}) \\
&= \nu \wedge \epsilon^1 \wedge \dots \wedge \epsilon^{n-1} (X, X_1, \dots, X_{n-1}) \\
&= \det \begin{pmatrix} \nu(X) & \nu(X_1) & \cdots & \nu(X_{n-1}) \\ \epsilon^1(X) & \epsilon^1(X_1) & \cdots & \epsilon^1(X_{n-1}) \\ \vdots & & & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \cdots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= \det \begin{pmatrix} g(X, N) & g(X_1, N) & \cdots & g(X_{n-1}, N) \\ \epsilon^1(X) & \epsilon^1(X_1) & \cdots & \epsilon^1(X_{n-1}) \\ \vdots & & & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \cdots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= g(X, N) \det(\epsilon^i(X_j)) \\
&= g(X, N) \epsilon^1 \wedge \ldots \wedge \epsilon^{n-1} (X_1, \dots, X_{n-1}) \\
&= g(X, N) \tilde{\lambda}(X_1, \dots, X_n)
\end{aligned}$$

Therefore $\iota^* \iota_X \lambda = g(X, N) \tilde{\lambda}$ and

$$\begin{aligned}
\int_M \mathrm{div} X \lambda &= \int_M d(\iota_X \lambda) \\
&= \int_{\partial M} \iota^*(\iota_X \lambda) \\
&= \int_{\partial M} g(X, N) \tilde{\lambda}
\end{aligned}$$

3. We note that

$$\begin{aligned}0 &= \iota_X(du \wedge \lambda)\\ &= \iota_X(du) \wedge \lambda - du \wedge (\iota_X \lambda)\\ &= du(X)\lambda - du \wedge (\iota_X \lambda)\end{aligned}$$

which implies that

$$\begin{aligned}\operatorname{div}(uX)\lambda &= d(\iota_{uX}\lambda)\\ &= d(u\iota_X\lambda)\\ &= du \wedge (\iota_X\lambda) + ud(\iota_X\lambda)\\ &= du(X)\lambda + u\operatorname{div}(X)\lambda\\ &= [du(X) + u\operatorname{div}(X)]\lambda\end{aligned}$$

This implies that $\operatorname{div}(uX) = du(X) + u\operatorname{div}(X)$. From before, we have that

$$\begin{aligned}\int_M du(X)\lambda &= \int_M \operatorname{div}(uX)\lambda - \int_M u\operatorname{div}(X)\lambda\\ &= \int_{\partial M} g(uX, N)\tilde{\lambda} - \int_M u\operatorname{div}(X)\lambda\\ &= \int_{\partial M} ug(X, N)\tilde{\lambda} - \int_M u\operatorname{div}(X)\lambda\end{aligned}$$

□

**Exercise 10.0.0.5.**

$$\operatorname{div}(X) = \sum_{j=1}^{n} (\nabla_{\partial_j} X)^j$$

*Proof.* We have that

$$\begin{aligned}\nabla_{\partial_i}(X) &= \sum_{j=1}^{n} \nabla_{\partial_i}(X^j \partial_j)\\ &= \sum_{j=1}^{n} \left[X^j \nabla_{\partial_i} \partial_j + \partial_i(X^j)\partial_j\right]\\ &= \sum_{j=1}^{n} \left[X^j (\sum_{k=1}^{n} \Gamma^k_{i,j}\partial_k) + \partial_i(X^j)\partial_j\right]\\ &= \sum_{j=1}^{n} \left[X^j \left(\sum_{k=1}^{n} \Gamma^k_{i,j}\partial_k\right) + \partial_i(X^j)\partial_j\right]\\ &= \sum_{j=1}^{n} X^j \left(\sum_{k=1}^{n} \Gamma^k_{i,j}\partial_k\right) + \sum_{j=1}^{n} \partial_i(X^j)\partial_j\\ &= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} X^j\Gamma^k_{i,j}\right)\partial_k + \sum_{k=1}^{n} \partial_i(X^k)\partial_k\\ &= \sum_{k=1}^{n} \left[\left(\sum_{j=1}^{n} X^j\Gamma^k_{i,j}\right) + \partial_i(X^k)\right]\partial_k\end{aligned}$$

so that $(\nabla_{\partial_i}(X))^i = \left(\sum_{j=1}^n X^j \Gamma^i_{i,j}\right) + \partial_i(X^i)$. We note that

$$\begin{aligned}\operatorname{div}(X) &= \sum_{i=1}^n \operatorname{div}(X^i \partial_i) \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + dx^i(\partial_i)] \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + 1]\end{aligned}$$

Since $\lambda = [\det g(\partial_i, \partial_j)]^{1/2} dx^1 \wedge \cdots \wedge dx^n = (\det g)^{1/2} dx$, we have that

$$\begin{aligned}d(\iota_{\partial_i}\lambda) &= d((\det g)^{1/2} \iota_{\partial_i} dx) \\ &= d[(\det g)^{1/2}]\iota_{\partial_i} dx + (\det g)^{1/2} d(\iota_{\partial_i} dx) \\ &= d[(\det(g)^{1/2}] \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots dx^n + (\det g)^{1/2} \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots dx^n)\end{aligned}$$

FINISH!!! $\square$

**Exercise 10.0.0.6.** Let $(M, g)$ be a Riemannian manifold.

1. For each $u, v \in C^\infty(M)$. Then
   (a)
   $$\int_M u \Delta v \lambda + \int_M g(\nabla u, \nabla v)\lambda = \int_{\partial M} u N(v)\tilde{\lambda}$$
   (b)
   $$\int_M [u\Delta v - v \Delta u]\lambda = \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}$$
2. (a) If $\partial M \neq \varnothing$, then for each $u, v \in C^{\infty(M)}$, $u$ and $v$ are harmonic and $u|_{\partial M} = v|_{\partial M}$ implies that $u = v$.
   (b) If $\partial M = \varnothing$, then for each $u \in C^\infty(M)$, $u$ is harmonic implies that $u$ is constant.

*Proof.*

1. Let $u, v \in C^\infty(M)$. Then
   (a)
   $$\begin{aligned}\int_M u\Delta v \lambda &= \int_M u \operatorname{div}(\nabla v)\lambda \\ &= \int_{\partial M} u g(\nabla v, N)\tilde{\lambda} - \int_M du(\nabla v)\lambda \\ &= \int_{\partial M} u dv(N)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda \\ &= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda\end{aligned}$$

(b) From above, we have that

$$\begin{aligned}
\int_M [u\Delta v - v\Delta u]\lambda &= \int_M u\Delta v\lambda - \int_M v\Delta u\lambda \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda - \left(\int_{\partial M} vN(u)\tilde{\lambda} - \int_M g(\nabla v, \nabla u)\lambda\right) \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_{\partial M} vN(u)\tilde{\lambda} \\
&= \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}
\end{aligned}$$

2. (a) Suppose that $\partial M \neq \varnothing$. Let $u, v \in C^{\infty(M)}$. Suppose that $u$ and $v$ are harmonic and $u|_{\partial M} = v|_{\partial M}$. Then $u - v$ is harmonic and

$$\begin{aligned}
\int_M \| \nabla(u-v)\|_g^2 \lambda &= \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= 0 + \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= \int_M (u-v)\Delta(u-v)\lambda + \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\
&= \int_{\partial M} (u-v)N(u-v)\tilde{\lambda} \\
&= 0
\end{aligned}$$

Thus $\nabla(u-v) = 0$ and $u - v$ is constant. Since $u|_{\partial M} = v|_{\partial M}$, we have that $u - v = 0$ and thus $u = v$.

(b) Suppose that $\partial M = \varnothing$. Let $u \in C^\infty(M)$. Suppose that $u$ is harmonic. Then

$$\begin{aligned}
\int_M \| \nabla u\|_g^2 \lambda &= \int_M g(\nabla u, \nabla u)\lambda \\
&= 0 + \int_M g(\nabla u, \nabla u)\lambda \\
&= \int_M u\Delta u\lambda + \int_M g(\nabla u, \nabla u)\lambda \\
&= \int_{\partial M} (u-v)g(\nabla(u-v), N)\tilde{\lambda} \\
&= 0
\end{aligned}$$

Therefore $\nabla u - 0$ and $u$ is constant.

$\square$

# Chapter 11

# Symplectic Geometry

## 11.1 Symplectic Manifolds

**Definition 11.1.0.1.** Let $M \in \mathbf{Man}^\infty$ and $\omega \in \Omega^2(M)$. Then $\omega$ is said to be **symplectic** if

1. $\omega$ is nondegenerate
2. $\omega$ is closed

# Chapter 12

# Extra

**Definition 12.0.0.1.** When working in $\mathbb{R}^n$, we introduce the formal objects $dx^1, dx_2, \cdots, dx^n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx^i = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.

**Definition 12.0.0.2.** Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx^i : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

1. the exterior product is bilinear

2. for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$

3. for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$

4. for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential $k$-forms on $\mathbb{R}^n$. Let $\omega$ be a $k$-form on $\mathbb{R}^n$. If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx^i$

**Note 12.0.0.3.** The terms $dx^1, dx_2, \cdots, dx^n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx^1, dx_2, \cdots, dx^n\}$ when referencing the coordinates on $\mathbb{R}^n$.

**Exercise 12.0.0.4.** Let $B_{n\times n} = (b_{i,j}) \in [C^\infty(M)]^{n\times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^{n} \left( \sum_{j=1}^{n} b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$$

*Proof.* Bilinearity of the exterior product implies that

$$\begin{aligned}
\bigwedge_{i=1}^{n}\left(\sum_{j=1}^{n} b_{i,j}dx^j\right) &= \left(\sum_{j=1}^{n} b_{1,j}dx^j\right)\wedge\left(\sum_{j=1}^{n} b_{2,j}dx^j\right)\wedge\cdots\wedge\left(\sum_{j=1}^{n} b_{n,j}dx^j\right)\\
&= \sum_{j_1,\cdots,j_n=1}^{n}\left(\prod_{i=1}^{n} b_{i,j_i}\right)dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}\\
&= \sum_{j_1\neq\cdots\neq j_n}\left(\prod_{i=1}^{n} b_{i,j_i}\right)dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}\\
&= \left[\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\left(\prod_{i=1}^{n} b_{i,\sigma(i)}\right)\right]dx_1\wedge dx_2\wedge\cdots\wedge dx_n\\
&= (\det B)dx_1\wedge dx_2\wedge\cdots\wedge dx_n
\end{aligned}$$

□

**Definition 12.0.0.5.** Let $f:\mathbb{R}^n\to\mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted $df$, on $\mathbb{R}^n$ by

$$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x^i}dx^i$$

Let $\omega = \sum_{I\in\mathcal{I}_{k,n}} f_I dx^i$ be a $k$-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega = \sum_{I\in\mathcal{I}_{k,n}} df_I\wedge dx^i$$

**Exercise 12.0.0.6.** On $\mathbb{R}^3$, put

1. $\omega_0 = f_0$,
2. $\omega_1 = f_1dx^1 + f2dx_2 + f_2dx_3$,
3. $\omega_2 = f_1dx_2\wedge dx_3 - f_2dx^1\wedge dx_3 + f_3dx^1\wedge dx_2$

Show that

1. $d\omega_0 = \frac{\partial f_0}{\partial x^1}dx^1 + \frac{\partial f_0}{\partial x^2}dx_2 + \frac{\partial f_0}{\partial x^3}dx_3$
2. $d\omega_1 = \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3}\right)dx_2\wedge dx_3 + \left(\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3}\right)dx^1\wedge dx_3 + \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2}\right)dx^1\wedge dx_2$
3. $d\omega_2 = \left(\frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3}\right)dx^1\wedge dx_2\wedge dx_3$

*Proof.* Straightforward. □

**Exercise 12.0.0.7.** Let $I\in\mathcal{I}_{k,n}$. Then there is a unique $I_*\in\mathcal{I}_{n-k,n}$ such that $dx^i\wedge dx_{I_*} = dx^1\wedge dx_2\wedge\cdots\wedge dx^n$.

**Definition 12.0.0.8.** We define a linear map $*:\Phi_k(\mathbb{R}^n)\to\Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge $*$-operator** by

$$*\sum_{I\in\mathcal{I}_{k,n}} f_I dx^i = \sum_{I\in\mathcal{I}_{k,n}} f_I dx_{I_*}$$

**Definition 12.0.0.9.** Let $\phi:\mathbb{R}^k\to\mathbb{R}^n$ be smooth. Write $\phi=(\phi_1,\phi_2,\cdots,\phi_n)$. We define $\phi^*:\Phi_k(\mathbb{R}^n)\to\Phi_k(\mathbb{R}^k)$ via the following properties:

1. for each 0-form $f$ on $\mathbb{R}^n$, $\phi^* f = f \circ \phi$
2. for $i = 1, \cdots, n$, $\phi^* dx^i = d\phi_i$
3. for an $s$-form $\omega$, and a $t$-form $\chi$ on $\mathbb{R}^n$, $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
4. for $l$-forms $\omega, \chi$ on $\mathbb{R}^n$, $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

**Exercise 12.0.0.10.** Let $M \subset \mathbb{R}^n$ be a $k$-dimensional smooth submanifold of $\mathbb{R}^n$, $\phi : U \to V$ a smooth parametrization of $M$, $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$ an $k$-form on $\mathbb{R}^n$. Then

$$\phi^*\omega = \bigg( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \bigg) du^1 \wedge du_2 \wedge \cdots \wedge du_k$$

*Proof.* By definition,

$$\begin{aligned}
\phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i \\
&= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx^i \\
&= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I
\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}
d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\
&= (\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u^j} du^j) \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u^j} du^j) \wedge \cdots \wedge (\sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u^j} du^j) \\
&= (\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\
&= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k \\
&= \bigg( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \bigg) du^1 \wedge du_2 \wedge \cdots \wedge du_k
\end{aligned}$$

□

## 12.1 Integration of Differential Forms

**Definition 12.1.0.1.** Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \cdots \wedge dx_k$ a $k$-form on $\mathbb{R}^k$. Define

$$\int_U \omega = \int_U f dx$$

**Definition 12.1.0.2.** Let $M \subset \mathbb{R}^n$ be a $k$-dimensional oriented smooth submanifold of $\mathbb{R}^n$, $\omega$ a $k$-form on $\mathbb{R}^n$ and $\phi : U \to V$ a local smooth, orientation-preserving parametrization of $M$. Define

$$\int_V \omega = \int_U \phi^*\omega$$

**Exercise 12.1.0.3.**

**Theorem 12.1.0.4. Stokes Theorem:**
Let $M \subset \mathbb{R}^n$ be a $k$-dimensional oriented smooth submanifold of $\mathbb{R}^n$ and $\omega$ a $k-1$-form on $\mathbb{R}^n$. Then

$$\int_{\partial M} \omega = \int_M d\omega$$

# Appendix A

# Summation

# Appendix B

# Asymptotic Notation

# Bibliography


[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration