

# INTRODUCTION TO FOURIER ANALYSIS

CARSON JAMES

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1. FOURIER ANALYSIS ON  $\mathbb{R}^n$ 

## 1.1. Schwartz Space.

**Definition 1.1.1.** Let  $\alpha \in \mathbb{N}_0^n$  and  $x, y \in \mathbb{R}^n$ . We define

- (1)  $\langle x, y \rangle = \sum_j x_j y_j$
- (2)  $|x| = \langle x, x \rangle^{1/2}$
- (3)  $|\alpha| = \alpha_1 + \cdots + \alpha_n$
- (4)  $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
- (5)  $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$

**Definition 1.1.2.** Let  $f \in C^\infty(\mathbb{R}^n)$ ,  $\alpha \in \mathbb{N}_0^n$  and  $N \in \mathbb{N}_0$ . We define

$$\|f\|_{\alpha, N} = \sup_{x \in \mathbb{R}^n} (1 + |x|^N) |\partial^\alpha f(x)|$$

We define Schwartz space, denoted  $\mathcal{S}$ , by

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \text{for each } \alpha \in \mathbb{N}_0^n, N \in \mathbb{N}_0, \|f\|_{\alpha, N} < \infty\}$$

**Exercise 1.1.3.** For each  $f \in \mathcal{S}$  and  $\alpha \in \mathbb{N}_0^n$ ,  $\partial^\alpha f \in L^1(\mathbb{R}^n)$ .

*Proof.* Let  $f \in \mathcal{S}$ ,  $\alpha \in \mathbb{N}_0^n$ . Then there exists  $C \geq 0$  such that for each  $x \in \mathbb{R}^n$ ,

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^{-1}$$

Define  $g : \mathbb{R}^n \rightarrow [0, \infty)$  defined by  $g(x) = (1 + |x|^2)^{-1}$ . Then  $g \in L^1(\mathbb{R}^n)$  which implies that  $\partial^\alpha f \in L^1(\mathbb{R}^n)$ .  $\square$

**Definition 1.1.4.**

## 1.2. The Convolution.

**Definition 1.2.1.** Let  $f, g \in L^0(\mathbb{R}^n)$ . If for a.e.  $x \in \mathbb{R}^n$ ,

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) < \infty$$

we define the **convolution of  $f$  with  $g$** , denoted  $f * g : \mathbb{R}^n \rightarrow \mathbb{C}$ , by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dm(y)$$

**Exercise 1.2.2.** Let  $f, g \in L^1(\mathbb{R}^n)$ . Then  $f * g \in L^1(\mathbb{R}^n)$  and  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ .

*Proof.* Define  $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$  by  $h(x, y) = f(x-y)g(y)$ . Tonelli's theorem implies that,

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |h|dm^2 &= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |f(x-y)g(y)|dm(y) \right] dm(x) \\ &= \int_{\mathbb{R}^n} |g(y)| \left[ \int_{\mathbb{R}^n} |f(x-y)|dm(y) \right] dm(x) \\ &= \|f\|_1 \int_{\mathbb{R}^n} |g(y)|dm(x) \\ &= \|f\|_1 \|g\|_1 \\ &< \infty \end{aligned}$$

Then  $h \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ . Fubini's theorem implies that  $f * g \in L^1(\mathbb{R}^n)$ . Clearly

$$\begin{aligned} \|f * g\|_1 &\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |h|dm^2 \\ &\leq \|f\|_1 \|g\|_1 \end{aligned}$$

□

**Exercise 1.2.3.** Let  $f, g, h \in L^1(\mathbb{R}^n)$ . Then  $(f * g) * h = f * (g * h)$ .

**Hint:** use the substitution  $z \mapsto z - y$

*Proof.* Let  $x \in \mathbb{R}^n$ . Then using the substitution  $z \mapsto z - y$  and Fubini's theorem, we obtain

$$\begin{aligned}
 (f * g) * h(x) &= \int f * g(x - y)h(y)dm(y) \\
 &= \int \left[ \int f(x - y - z)g(z)dm(z) \right] h(y)dm(y) \\
 &= \int \left[ \int f(x - z)g(z - y)dm(z) \right] h(y)dm(y) \\
 &= \int \left[ \int f(x - z)g(z - y)h(y)dm(z) \right] dm(y) \\
 &= \int \left[ \int f(x - z)g(z - y)h(y)dm(y) \right] dm(z) \\
 &= \int f(x - z) \left[ \int g(z - y)h(y)dm(y) \right] dm(z) \\
 &= \int f(x - z)g * h(z)dm(z) \\
 &= f * (g * h)(x)
 \end{aligned}$$

So  $(f * g) * h = f * (g * h)$ . □

**Exercise 1.2.4.** Let  $f, g \in L^1(\mathbb{R}^n)$ . Then  $f * g = g * f$ .

*Proof.* Let  $x \in \mathbb{R}^n$ . Using the transformation  $y \mapsto x - y$ , we obtain that

$$\begin{aligned}
 f * g(x) &= \int f(x - y)g(y)dm(y) \\
 &= \int f(y)g(x - y)dm(y) \\
 &= \int g(x - y)f(y)dm(y) \\
 &= g * f(x)
 \end{aligned}$$

So  $f * g = g * f$ . □

**Note 1.2.5.** To summarize,  $(L^1(\mathbb{R}^n), *)$  is a commutative Banach algebra.

**Exercise 1.2.6. Young's Inequality:**

Let  $p \in [1, \infty]$ ,  $f \in L^1$  and  $g \in L^p$ . Then  $f * g \in L^p$  and  $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ .

*Proof.* Define  $K \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$  by  $K(x, y) = f(x - y)$ . Since for each  $x, y \in \mathbb{R}^n$ ,

$$\begin{aligned}
 \int |K(x, y)|dm(x) &= \int |K(x, y)|dm(y) \\
 &= \|f\|_1
 \end{aligned}$$

an exercise in section 5.1 of [Introduction to Measure and Integration](#) implies that  $f * g \in L^p$  and  $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ . □

**Exercise 1.2.7.** Let  $p, q \in [1, \infty]$  be conjugate,  $f \in L^p(\mathbb{R}^n)$  and  $g \in L^q(\mathbb{R}^n)$ . Then

(1) for each  $x \in \mathbb{R}^n$ ,  $f * g(x)$  exists.

$$(2) \|f * g\|_u \leq \|f\|_p \|g\|_q$$

(3)

*Proof.* (1) Let  $x \in \mathbb{R}^n$ . Holder's inequality implies that

$$\int_{\mathbb{R}^n} |f(x-y)g(y)| dm(y) \leq \|f\|_p \|g\|_q$$

Then  $f * g(x)$  exists.

(2) Let  $x \in \mathbb{R}^n$ . Then in part (1) we showed that

$$\begin{aligned} |f * g(x)| &= \left| \int_{\mathbb{R}^n} f(x-y)g(y) dm(y) \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y)g(y)| dm(y) \\ &\leq \|f\|_p \|g\|_q \end{aligned}$$

Since  $x \in \mathbb{R}^n$  is arbitrary,  $\|f * g\|_u \leq \|f\|_p \|g\|_q$ .

(3)

□

**Exercise 1.2.8.** Let  $f \in L^1(\mathbb{R}^n)$ ,  $k \in \mathbb{N}$  and  $g \in C^k(\mathbb{R}^n)$ . Suppose that for each  $\alpha \in \mathbb{N}_0^n$ ,  $|\alpha| \leq k$  implies that  $\partial^\alpha g \in L^\infty$ . Then for each  $\alpha \in \mathbb{N}_0^n$ ,  $|\alpha| \leq k$  implies that  $f * g \in C^k$  and

$$\partial^\alpha(f * g) = f * \partial^\alpha g$$

*Proof.* Let  $\alpha \in \mathbb{N}_0^n$ . Suppose that  $|\alpha| = 1$ . Define  $h \in L^0(\mathbb{R}^n \times \mathbb{R}^n)$  by  $h(x, y) = g(x-y)f(y)$ . Young's inequality implies that for a.e.  $x \in \mathbb{R}^n$ ,  $h(x, \cdot) \in L^1(m)$ . For each  $y \in \mathbb{R}^n$ ,  $\partial^\alpha h(\cdot, y) = \partial^\alpha g(\cdot - y)f(y)$  and for each  $x, y \in \mathbb{R}^n$ ,  $|\partial^\alpha h(x, y)| \leq \|\partial^\alpha g\|_\infty |f(y)| \in L^1(\mathbb{R}^n)$ . An exercise in section 3.3 of [Introduction to Measure and Integration](#) implies that for a.e.  $x \in \mathbb{R}^n$ ,  $\partial^\alpha(g * f)(x)$  exists and

$$\begin{aligned} \partial^\alpha(f * g)(x) &= \partial^\alpha(g * f)(x) \\ &= \partial^\alpha \int_{\mathbb{R}^n} h(x, y) dm(y) \\ &= \int_{\mathbb{R}^n} \partial^\alpha g(x-y) f(y) dm(y) \\ &= (\partial^\alpha g) * f(x) \\ &= f * (\partial^\alpha g)(x) \end{aligned}$$

Now proceed by induction on  $|\alpha|$ .

□

### 1.3. The Fourier Transform on $L^1(\mathbb{R}^n)$ .

#### Definition 1.3.1.

**Exercise 1.3.2.** Let  $\phi : \mathbb{R} \rightarrow S^1$  be a measurable homomorphism.

- (1) Then  $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$  and there exists  $a > 0$  such that

$$\int_{(0,a]} \phi dm \neq 0$$

- (2) Define

$$c = \left[ \int_{(0,a]} \phi dm \right]^{-1}$$

Then For each  $x \in \mathbb{R}$ ,

$$\phi(x) = c \int_{(x,x+a]} \phi dm$$

- (3)  $\phi \in C^\infty(\mathbb{R})$  and  $\phi' = c(\phi(a) - 1)\phi$   
 (4) Define  $b = c(\phi(a) - 1)$  and  $g \in C^\infty(\mathbb{R})$  by  $g(x) = e^{-bx}\phi(x)$ . Then  $g$  is constant and there exists  $\xi \in \mathbb{R}$  such that for each  $x \in \mathbb{R}$ ,  $\phi(x) = e^{2\pi i \xi x}$ .

*Proof.*

- (1) Let  $K \subset \mathbb{R}$  be compact. Then

$$\int_K |\phi| dm = m(K) < \infty$$

So  $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ . For the sake of contradiction, suppose that for each  $a > 0$ ,

$$\int_{(0,a]} \phi dm = 0$$

Then the FTC implies that  $\phi = 0$  a.e. on  $[0, \infty)$ , which is a contradiction. So there exists  $a > 0$  such that

$$\int_{(0,a]} \phi dm \neq 0$$

- (2) For  $x \in \mathbb{R}$ ,

$$\begin{aligned} \phi(x) &= c \int_{(0,a]} \phi(x)\phi(t) dm(t) \\ &= c \int_{(0,a]} \phi(x+t) dm(t) \\ &= c \int_{(x,x+a]} \phi dm \end{aligned}$$

- (3) Part (2) and the FTC imply that  $\phi$  is continuous. Let  $d \in \mathbb{R}$ . Define  $f_d \in C((d, \infty))$  by

$$f_d(x) = \int_{(d,x]} \phi dm$$

Since  $\phi$  is continuous, the FTC implies that  $f_d$  is differentiable and for each  $x > d$   $f'_d(x) = \phi(x)$ . Part (2) implies that for each  $x > d$ ,

$$\begin{aligned}\phi(x) &= c \int_{(x, x+a]} \phi dm \\ &= c(f_d(x+a) - f_d(x))\end{aligned}$$

So for each  $x > d$ ,  $\phi$  is differentiable at  $x$  and

$$\begin{aligned}\phi'(x) &= c(\phi(x+a) - \phi(x)) \\ &= c(\phi(a) - 1)\phi(x)\end{aligned}$$

Since  $d \in \mathbb{R}$  is arbitrary,  $\phi$  is differentiable and  $\phi' = c(\phi(a) - 1)\phi$ . This implies that  $\phi \in C^\infty(\mathbb{R})$ .

(4) Let  $x \in \mathbb{R}$ . Then

$$\begin{aligned}g'(x) &= e^{-bx}\phi'(x) - be^{-bx}\phi(x) \\ &= be^{-bx}\phi(x) - be^{-bx}\phi(x) \\ &= 0\end{aligned}$$

So  $g' = 0$  and  $g$  is constant. Hence there exists  $k \in \mathbb{R}$  such that for each  $x \in \mathbb{R}$ ,  $\phi(x) = ke^{bx}$ . Since  $\phi(0) = 1$ ,  $k = 1$ . Since  $|\phi| = 1$ , there exists  $\xi \in \mathbb{R}$  such that  $b = 2\pi i\xi$ .

□

**Note 1.3.3.** To summarize, for each measurable homomorphism  $\phi : \mathbb{R} \rightarrow S^1$ , there exists  $\xi \in \mathbb{R}$  such that for each  $x \in \mathbb{R}$ ,  $\phi(x) = e^{2\pi i\xi x}$ .

**Exercise 1.3.4.** Let  $\phi : \mathbb{R}^n \rightarrow S^1$  be a measurable homomorphism. Then there  $\xi \in \mathbb{R}^n$  such that for each  $x \in \mathbb{R}^n$ ,  $\phi(x) = e^{2\pi i\langle \xi, x \rangle}$ .

*Proof.* When done in the category of measurable groups, an exercise in the section on direct products of groups of [Introduction to Group Theory](#) implies that there exist measurable homomorphism  $(\phi_j)_{j=1}^n \subset (S^1)^\mathbb{R}$  such that  $\phi = \bigotimes_{j=1}^n \phi_j$ . The previous exercise implies that there exist  $\xi \in \mathbb{R}^n$  such that for each  $x \in \mathbb{R}^n$ ,  $\phi_j(x_j) = e^{2\pi i\xi_j x_j}$ . Then for each  $x \in \mathbb{R}^n$ ,

$$\begin{aligned}\phi(x) &= \prod_{j=1}^n \phi_j(x_j) \\ &= \prod_{j=1}^n e^{2\pi i\xi_j x_j} \\ &= e^{2\pi i \sum_{j=1}^n \xi_j x_j} \\ &= e^{2\pi i\langle \xi, x \rangle}\end{aligned}$$

□

**Definition 1.3.5.** Let  $f \in L^1(\mathbb{R}^n)$ . We define the **Fourier transform of  $f$** , denoted  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$  by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-2\pi i\langle \xi, x \rangle} dm(x)$$

## 2. FOURIER ANALYSIS ON LCA GROUPS