INTRODUCTION TO NETWORKS

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Contents

1. Setup

Definition 1.0.1. Let (M, d) be a metric space, (G, τ) a topological group, and $\cdot : G \times M \to M$ a group action. Suppose that for each $g \in G$, the map $x \mapsto g \cdot x$ is an isometry. We define $\bar{d} : M/G \to [0, \infty)$ by

$$\begin{aligned} \bar{d}(o_x, o_y) &= \inf_{\substack{a \in o_x \\ b \in o_y}} d(a, b) \\ &= \inf_{g \in G} d(g \cdot x, y) \end{aligned}$$

Exercise 1.0.2. If for each $x \in M$, o_x is closed, then $\bar{d}$ is a metric.

Proof. Suppose that for each $x \in M$, o_x is closed. We need only show that for each $x, y \in M$, $\bar{d}(o_x, o_y) = 0$ implies that $o_x = o_y$. Suppose that $\bar{d}(o_x, o_y) = 0$. Then $\inf\limits_{g \in G} d(g \cdot x, y) = 0$. Hence there exists $(\tau_n)_{n \in N} \subset G$ such that $\tau_n \cdot x \to y$. Since $(\tau_n \cdot x)_{n \in \mathbb{N}} \subset o_x$ and o_x is closed, $y \in o_x$. Thus $o_x = o_y$. $\square$

Example 1.0.3. Consider the metric space $(\mathbb{C}, |\cdot|)$, topological group $(S^1, |\cdot|)$ and the (right) action $x \cdot u = xu$. Then the orbits are concentric cirles, which are closed.

Example 1.0.4. Consider the metric space $(\mathbb{C}^{n \times d}, \|\cdot\|_F)$, topological group $(U_d, \|\cdot\|_F)$ and the (right) action $X \cdot U = XU$

Definition 1.0.5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $\|\cdot\|_* : L^1(X, \mathcal{A}, \mu) \to [0, \infty)$ by

$$\|f\|_* = \sup_{A \in \mathcal{A}} \left| \int_A f d\mu \right|$$

Exercise 1.0.6. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $\|\cdot\|_*$ is a norm on $L^1(X, \mathcal{A}, \mu)$.

Proof. Clear. $\square$

Definition 1.0.7. Let (X, d) be a compact space. Define

$$\mathrm{Aut}(X) = \{\sigma : X \to X : \sigma \text{ is a homeomorphism}\}$$

We metrize $\mathrm{Aut}(X)$ with uniform convergence d_u. It is known that this topology is equivalent to the compact-open topology.

Exercise 1.0.8. With the setup as above, $(\mathrm{Aut}(X), d_u)$ is a topological group.

Proof. Please see section on topological groups: Analysis Notes ☐

Definition 1.0.9. Let (X,d) be a compact metric space and $\mu:\mathcal{B}(X)\to\mathbb{R}$ a Borel measure. Define

$$\mathrm{Aut}(X,\mathcal{B}(X),\mu)=\{\sigma\in\mathrm{Aut}(X):\sigma_*\mu=\mu\}$$

So that $(\mathrm{Aut}(X,\mathcal{B}(X),\mu),d_u)$ is a subspace of $(\mathrm{Aut}(X),d_u)$.

Exercise 1.0.10. Let (X,d) be a compact metric space and $\mu:\mathcal{B}(X)\to\mathbb{R}$ an outer-regular Borel measure. Then $\mathrm{Aut}(X,\mathcal{B}(X),\mu)$ is a closed subgroup of $\mathrm{Aut}(X)$.

Proof. Please see section on topological groups: Analysis Notes ☐

Example 1.0.11. With the setup as before, define the (right) group action $\cdot:(L^1(X,\mathcal{B}(X),\mu),\|\cdot\|_*)\times\mathrm{Aut}(X,\mathcal{B}(X),\mu)\to(L^1(X,\mathcal{B}(X),\mu),\|\cdot\|_*)$ by $f\cdot\sigma=f\circ\sigma$. Then for each $\sigma\in\mathrm{Aut}(X,\mathcal{B}(X),\mu)$, the map $f\mapsto f\cdot\sigma$ is an isometry.

Proof. Clear. ☐

Exercise 1.0.12. With the setup from above, the orbits are closed

Proof. IDK, would like to show. I dont think $\mathrm{Aut}(X,\mathcal{B}(X),\mu)$ is compact. So still thinking about how to show this. ☐