### INTRODUCTION TO DIFFERENTIAL GEOMETRY

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#### 1. Review of Basic Definitions and Results

### 1.1. Set Theory.

**Definition 1.1.1.** Let  $\{A_i\}_{i\in I}$  be a collection of sets. The **disjoint union of**  $\{A_i\}_{i\in I}$ , denoted  $\coprod_{i\in I} A_i$ , is defined by

$$\coprod_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$$

Note 1.1.2. In these notes, we will identify  $\{i\} \times A_i$  and  $A_i$ .

**Definition 1.1.3.** Let  $\{A_i\}_{i\in I}$  be a collection of sets and  $\sigma: I \to \coprod_{i\in I} A_i$ . Then  $\sigma$  is said to be a **section of**  $\coprod_{i\in I} A_i$  if for each  $i\in I$ ,  $\sigma(i)\in A_i$ .

### 1.2. Differentiation.

**Definition 1.2.1.** Let  $n \ge 1$ . For  $i = 1, \dots, n$ , define  $x_i : \mathbb{R}^n \to \mathbb{R}$  by  $x_i(a_1, \dots, a_n) = a_i$ . The functions  $(x_i)_{i=1}^n$  are called the **standard coordinate functions on**  $\mathbb{R}^n$ .

**Definition 1.2.2.** Let  $U \subset \mathbb{R}^n$  be open,  $f: U \to \mathbb{R}$  and  $a \in U$ . Then f is said to be differentiable with respect to  $x_i$  at a if

$$\lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}$$

exists. If f is differentiable with respect to  $x_i$  at a, we define the **partial derivative of** f with respect to  $x_i$  at a, denoted

$$\frac{\partial f}{\partial x_i}(a), \ \frac{\partial}{\partial x_i}\Big|_a f, \ \partial_{x_i} f(a) \ or \ \partial_{x_i}\Big|_a f$$

to be the limit above.

**Definition 1.2.3.** Let  $U \subset \mathbb{R}^n$  be open and  $f: U \to \mathbb{R}$ . Then f is said to be **differentiable** with respect to  $x_i$  if for each  $a \in U$ , f is differentiable with respect to  $x_i$  at a.

**Exercise 1.2.4.** Let  $U \subset \mathbb{R}^n$  be open,  $f: U \to \mathbb{R}$  and  $a \in U$ . Suppose that  $\frac{\partial^2 f}{\partial x_i x_j}$  and  $\frac{\partial^2 f}{\partial x_j x_i}$  exist and are continuous at a. Then

$$\frac{\partial^2 f}{\partial x_i x_j}(a) = \frac{\partial^2 f}{\partial x_j x_i}(a)$$

Proof.

**Definition 1.2.5.** Let  $U \subset \mathbb{R}^n$  be open and  $f: U \to \mathbb{R}$ . Then f is said to be **smooth** if for each  $i_1, \dots, i_k \in \{1, \dots, n\}$ ,  $\frac{\partial^k f}{\partial i_1 \dots i_k}$  exists and is continuous on U.

**Definition 1.2.6.** Let  $U \subset \mathbb{R}^n$ ,  $f: U \to \mathbb{R}$ . Then f is said to be **smooth** if there exists  $U' \subset \mathbb{R}^n$  and  $f': U' \to \mathbb{R}$  such that  $U \subset U'$ , U' is open,  $f'|_U = f$  and f' is smooth. The set of smooth functions on U is denoted  $C^{\infty}(U)$ .

**Definition 1.2.7.** Let  $U, V \subset \mathbb{R}^n$  and  $F : U \to V$ . Then F is said to be a **diffeomorphism** if F is a homeomorphism and  $F, F^{-1}$  are smooth.

**Exercise 1.2.8.** Let  $U, V \subset \mathbb{R}^n$  and  $F: U \to V$ . Then F is a diffeomorphism iff for each  $p \in U$ , there exists a relatively open neighborhood  $N \subset U$  of p such that  $F|_N: N \to F(N)$  is a diffeomorphism

Proof. content...

**Definition 1.2.9.** Let  $U \subset \mathbb{R}^n$  and  $p \in U$ . Then U is said to be **star-shaped** if for each  $q \in U$ ,  $\{p + t(q - p) : 0 \le t \le 1\} \subset U$ .

**Theorem 1.2.10.** (Taylor's Theorem) Let  $U \subset \mathbb{R}^n$  be open,  $p \in U$  and  $f \in C^{\infty}(U)$ . Suppose that U is star-shaped with respect to p. Then there exist  $g_1, \dots, g_n \in C^{\infty}(U)$  such that for each  $x \in U$ ,

$$f(x) = f(p) + \sum_{i=1}^{n} (x_i - p_i)g_i(x)$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \frac{\partial f}{\partial x_i}(p)$$

*Proof.* Since U is star-shaped,  $\{p+t(x-p):0\leq t\leq 1\}\subset U$ . By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}f(p+t(x-p)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p+t(x-p))(x_i-p_i)$$

Integrating both sides with respect to t from 0 to 1, we obtain

$$f(x) - f(p) = \sum_{i=1}^{n} (x_i - p_i) \int_0^1 \frac{\partial f}{\partial x_i} (p + t(x - p)) dt$$

For  $i \in \{1, \dots, n\}$ , define  $g_i \in C^{\infty}(U)$  by

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x - p))dt$$

Then for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \frac{\partial f}{\partial x_i}(p)$$

#### 2. Multilinear Algebra

Note 2.0.1. For the remainder of this section we let V denote an n-dimensional vector space with basis  $\{e_1, \dots, e_n\}$  with dual space  $V^*$  and dual basis  $\{\epsilon_1, \dots, \epsilon_n\}$  defined by  $\epsilon_i(e_j) = \delta_{i,j}$ .

2.1. k-Tensors.

**Definition 2.1.1.** Let  $\alpha: V^k \to \mathbb{R}$ . Then  $\alpha$  is said to be **multilinear** or a **k-tensor on** V if for  $i \in \{1, \dots, k\}$ ,  $w \in V$ ,  $c \in \mathbb{R}$  and  $v_1, \dots, v_k \in V$ ,

$$\alpha(v_1, \dots, v_i + cw, \dots, v_k) = \alpha(v_1, \dots, v_i, \dots, v_k) + c\alpha(v_1, \dots, w, \dots, v_k)$$

The set of all k-tensors on V is denoted by  $T_k(V)$ . Define  $L_0(V) = \mathbb{R}$ .

**Exercise 2.1.2.** We have that  $T_k(V)$  is a vector space.

Proof. Clear. 
$$\Box$$

**Definition 2.1.3.** For  $\sigma \in S_k$  and  $\alpha \in T_k(V)$ , define the  $\sigma \alpha : V^k \to \mathbb{R}$  by

$$\sigma\alpha(v_1,\cdots,v_k)=\alpha(v_{\sigma(1)},\cdots,v_{\sigma(k)})$$

The map  $\alpha \mapsto \sigma \alpha$  is called the **permutation action** of  $S_k$  on  $T_k(V)$ 

**Exercise 2.1.4.** The permutation action of  $S_k$  on  $T_k(V)$  is a group action.

Proof.

- (1) Clearly for each  $\sigma \in S_k$  and  $\alpha \in T_k(V)$ ,  $\sigma \alpha \in T_k(V)$ .
- (2) Clearly for each  $\alpha \in T_k(V)$ ,  $e\alpha = \alpha$ .
- (3) Let  $\tau, \sigma \in S_k$  and  $\alpha \in T_k(V)$ . Then for each  $v_1, \dots, v_k \in V$ ,

$$(\tau\sigma)\alpha(v_1,\dots,v_k) = \alpha(v_{\tau\sigma(1)},\dots,v_{\tau\sigma(k)})$$
$$= \tau\alpha(v_{\sigma(1)},\dots,v_{\sigma(k)})$$
$$= \tau(\sigma\alpha)(v_1,\dots,v_k)$$

**Exercise 2.1.5.** Let  $\sigma \in S_k$ . Then  $L_{\sigma} : T_k(V) \to T_k(V)$  given by  $L_{\sigma}(\alpha) = \sigma \alpha$  is a linear transformation.

*Proof.* Let  $\alpha, \beta \in T_k(V)$ ,  $c \in \mathbb{R}$  and  $v_1, \dots, v_k \in V$ . Then

$$\sigma(c\alpha + \beta)(v_1, \dots, v_k) = (c\alpha + \beta)(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

$$= c\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

$$= c\sigma\alpha(v_1, \dots, v_k) + \sigma\beta(v_1, \dots, v_k)$$

So  $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$ .

**Definition 2.1.6.** Let  $\alpha \in T_k(V)$ . Then  $\alpha$  is said to be **symmetric** if for each  $\sigma \in S_k$ ,  $\sigma \alpha = \alpha$ . and  $\alpha$  is said to be **alternating** if for each  $\sigma \in S_k$ ,  $\sigma \alpha = \operatorname{sgn}(\sigma)\alpha$ . The set of symmetric k-tensors on V is denoted  $\Xi_k(V)$  and the set of alternating k-tensors on V is denoted  $\Lambda_k(V)$ .

**Definition 2.1.7.** Define the symmetric operator  $S: T_k(V) \to \Xi_k(V)$  by

$$S(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha$$

Define the alternating operator  $A: T_k(V) \to \Lambda_k(V)$  by

$$A(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma \alpha$$

# Exercise 2.1.8.

- (1) For  $\alpha \in T_k(V)$ ,  $S(\alpha)$  is symmetric.
- (2) For  $\alpha \in T_k(V)$ ,  $A(\alpha)$  is alternating.

Proof.

(1) Let  $\alpha \in T_k(V)$  and  $\sigma \in S_k$ . Then

$$\sigma S(\alpha) = \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \tau \alpha \right]$$
$$= \frac{1}{k!} \sum_{\tau \in S_k} \sigma \tau \alpha$$
$$= \frac{1}{k!} \sum_{\tau \in S_k} \tau \alpha$$
$$= S(\alpha)$$

(2) Let  $\alpha \in T_k(V)$  and  $\sigma \in S_k$ . Then

$$\sigma A(\alpha) = \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \right]$$

$$= \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \sigma \tau \alpha$$

$$= \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma \tau) \sigma \tau \alpha$$

$$= \operatorname{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma \tau) \sigma \tau \alpha$$

$$= \operatorname{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha$$

$$= \operatorname{sgn}(\sigma) A(\alpha)$$

## Exercise 2.1.9

- (1) For  $\alpha \in \Xi_k(V)$ ,  $S(\alpha) = \alpha$ .
- (2) For  $\alpha \in \Lambda_k(V)$ ,  $A(\alpha) = \alpha$ .

Proof.

(1) Let  $\alpha \in \Xi_k(V)$ . Then

$$S(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha$$
$$= \frac{1}{k!} \sum_{\sigma \in S_k} \alpha$$
$$= \alpha$$

(2) Let  $\alpha \in \Lambda_k(V)$ . Then

$$A(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma \alpha$$
$$= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2 \alpha$$
$$= \alpha$$

**Exercise 2.1.10.** The symmetric operator  $S: T_k(V) \to \Xi_k(V)$  and the alternating operator  $A: T_k(V) \to \Lambda_k(V)$  are linear.

Proof. Clear. 
$$\Box$$

**Definition 2.1.11.** Let  $\alpha \in T_k(V)$  and  $\beta \in T_l(V)$ . The **tensor product** of  $\alpha$  and  $\beta$  is defined to be the map  $\alpha \otimes \beta \in T_{k+l}(V)$  given by

$$\alpha \otimes \beta(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = \alpha(v_1, \cdots, v_k)\beta(v_{k+1}, \cdots, v_{k+l})$$

 $Thus \otimes : T_k(V) \times T_l(V) \to T_{k+l}(V).$ 

**Exercise 2.1.12.** The tensor product  $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$  is associative.

Proof. Clear. 
$$\Box$$

**Exercise 2.1.13.** The tensor product  $\otimes : T_k(V) \times T_l(V) \to T_{k+l}(V)$  is bilinear.

Proof. Clear. 
$$\Box$$

**Definition 2.1.14.** Let  $\alpha \in \Lambda_k(V)$  and  $\beta \in \Lambda_l(V)$ . The **exterior product** of  $\alpha$  and  $\beta$  is defined to be the map  $\alpha \wedge \beta \in \Lambda_{k+l}(V)$  given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)$$

Thus  $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to \Lambda_{k+l}(V)$ .

**Exercise 2.1.15.** The exterior product  $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to T_{k+l}(V)$  is bilinear.

Proof. Clear. 
$$\Box$$

**Exercise 2.1.16.** Let  $\alpha \in T_k(V)$  and  $\beta \in T_l(V)$ . Then

- $(1) \ A(A(\alpha) \otimes \beta) = A(\alpha \otimes \beta)$
- $(2) \ A(\alpha \otimes A(\beta)) = A(\alpha \otimes \beta)$

*Proof.* First note that if we fix  $\mu \in S_{k+1}$ , then for each  $\tau \in S_k$ , choosing  $\sigma = \mu \tau^{-1}$  yields  $\sigma \tau = \mu$ . For each  $\mu \in S_{k+l}$ , the map  $\phi_{\mu} : S_k \to S_{k+l}$  given by  $\phi_{\mu}(\tau) = \mu \tau^{-1}$  is injective. Thus for each  $\mu \in S_{k+l}$ , we have that  $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma \tau\} = k!$ 

(1) Then

$$A(A(\alpha) \otimes \beta) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma \left[ A(\alpha) \otimes \beta \right]$$

$$= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma \left[ \left( \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \right) \otimes \beta \right]$$

$$= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) (\tau \alpha) \otimes \beta \right]$$

$$= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma \left[ \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau (\alpha \otimes \beta) \right]$$

$$= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma \tau) \sigma \tau (\alpha \otimes \beta)$$

$$= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \operatorname{sgn}(\mu) \mu(\alpha \otimes \beta)$$

$$= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \operatorname{sgn}(\mu) \mu(\alpha \otimes \beta)$$

$$= A(\alpha \otimes \beta)$$

(2) Similar to (1).

**Exercise 2.1.17.** The exterior product  $\wedge : \Lambda_k(V) \times \Lambda_l(V) \to \Lambda_{k+l}(V)$  is associative.

*Proof.* Let  $\alpha \in \Lambda_k(V)$ ,  $\beta \in \Lambda_l(V)$  and  $\gamma \in \Lambda_m(V)$ . Then

$$(\alpha \wedge \beta) \wedge \gamma = \left[ \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \right] \wedge \gamma$$

$$= \frac{(k+l+m)!}{(k+l)!m!} A \left( \left[ \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta) \right] \otimes \gamma \right)$$

$$= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} A(A(\alpha \otimes \beta) \otimes \gamma)$$

$$= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} A((\alpha \otimes \beta) \otimes \gamma)$$

$$= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes (\beta \otimes \gamma))$$

$$= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} A(\alpha \otimes A(\beta \otimes \gamma))$$

$$= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes \frac{(l+m)!}{l!m!} A(\beta \otimes \gamma))$$

$$= \frac{(k+l+m)!}{k!(l+m)!} A(\alpha \otimes (\beta \wedge \gamma))$$

$$= \alpha \wedge (\beta \wedge \gamma)$$

**Exercise 2.1.18.** Let  $\alpha_i \in \Lambda_{k_i}(V)$  for  $i = 1, \dots, m$ . Then

$$\bigwedge_{i=1}^{m} \alpha_i = \frac{\left(\sum_{i=1}^{m} k_i\right)!}{\prod_{i=1}^{m} k_i!} A\left(\bigotimes_{i=1}^{m} \alpha_i\right)$$

*Proof.* To see that the statment is true in the case m=3, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1! k_2! k_3!} A(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each  $3 \leq m \leq m_0$ . Then the proof of the previous exercise tells us the

$$\bigwedge_{i=1}^{m_0+1} \alpha_i = \left( \bigwedge_{i=1}^{m_0-1} \alpha_i \right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} 
= \frac{\left( \sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1} \right)!}{\left( \sum_{i=1}^{m_0-1} k_i \right)! k_{m_0}! k_{m_0+1}!} A \left( \left[ \bigwedge_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) 
= \frac{\left( \sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1} \right)!}{\left( \sum_{i=1}^{m_0-1} k_i \right)! k_{m_0}! k_{m_0+1}!} A \left( \left[ \left( \sum_{i=1}^{m_0-1} k_i \right)! \right] A \left( \left( \bigotimes_{i=1}^{m_0-1} \alpha_i \right) \right) \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) 
= \frac{\left( \sum_{i=1}^{m_0+1} k_i \right)!}{\prod_{i=1}^{m_0+1} k_i !} A \left( A \left[ \bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) 
= \frac{\left( \sum_{i=1}^{m_0+1} k_i \right)!}{\prod_{i=1}^{m_0+1} k_i !} A \left( \left[ \bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) 
= \frac{\left( \sum_{i=1}^{m_0+1} k_i \right)!}{\prod_{i=1}^{m_0+1} k_i !} A \left( \left( \bigotimes_{i=1}^{m_0+1} \alpha_i \right) \right)$$

Exercise 2.1.19. Define  $\tau \in S_{k+l}$  by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of  $\tau$  is kl. (Hint: inversion number)

Proof.

$$N(\tau) = \sum_{i=1}^{l} k^{i}$$
$$= kl$$

Since  $\operatorname{sgn}(\tau) = (-1)^{N(\tau)}$  we know that  $\operatorname{sgn}(\tau) = (-1)^{kl}$ .

**Exercise 2.1.20.** Let  $\alpha \in \Lambda_k(V)$ ,  $\beta \in \Lambda_l(V)$ . Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

*Proof.* Note that For  $\sigma \in S_{k+l}$  and  $v_1, \dots, v_{k+l} \in V$ , we have that

$$\sigma\tau(\beta\otimes\alpha)(v_{1},\cdots,v_{l},v_{l+1},\cdots v_{l+k}) = \beta\otimes\alpha(v_{\sigma\tau(1)},\cdots,v_{\sigma\tau(l)},v_{\sigma\tau(l+1)},\cdots v_{\sigma\tau(l+k)})$$

$$= \beta(v_{\sigma\tau(1)},\cdots,v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)},\cdots v_{\sigma\tau(l+k)})$$

$$= \beta(v_{\sigma(1+k)},\cdots,v_{\sigma(l+k)})\alpha(v_{\sigma(1)},\cdots v_{\sigma(k)})$$

$$= \alpha(v_{\sigma(1)},\cdots v_{\sigma(k)})\beta(v_{\sigma(1+k)},\cdots,v_{\sigma(l+k)})$$

$$= \alpha\otimes\beta(v_{\sigma(1)},\cdots v_{\sigma(k)},v_{\sigma(1+k)},\cdots,v_{\sigma(l+k)})$$

$$= \sigma(\alpha\otimes\beta)(v_{1},\cdots,v_{k},v_{1+k},\cdots v_{l+k})$$

Thus  $\sigma \tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$ . Define  $\tau$  as in the previous exercise. Then

$$\beta \wedge \alpha = \frac{(k+l)!}{k!l!} A(\beta \otimes \alpha)$$

$$= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma(\beta \otimes \alpha)$$

$$= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma\tau) \sigma\tau(\beta \otimes \alpha)$$

$$= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \sigma(\alpha \otimes \beta)$$

$$= \operatorname{sgn}(\tau) \frac{(k+l)!}{k!l!} A(\alpha \otimes \beta)$$

$$= \operatorname{sgn}(\tau) \alpha \wedge \beta$$

$$= (-1)^{kl} \alpha \wedge \beta$$

**Exercise 2.1.21.** Let  $\alpha \in \Lambda_k(V)$ . If k is odd, then  $\alpha \wedge \alpha = 0$ .

*Proof.* Suppose that k is odd. The previous exercise tells us that

$$\alpha \wedge \alpha = (-1)^{k^2} \alpha \wedge \alpha$$
$$= -\alpha \wedge \alpha$$

Thus  $\alpha \wedge \alpha = 0$ .

Exercise 2.1.22. (Fundamental Example) Let  $\alpha_1, \dots, \alpha_m \in \Lambda_1(V)$  and  $v_1, \dots, v_m \in V$ . Then

$$\left(\bigwedge_{i=1}^{m} \alpha_i\right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

*Proof.* The previous exercises tell us that

$$\left(\bigwedge_{i=1}^{m} \alpha_{i}\right)(v_{1}, \dots, v_{m}) = m! A\left(\bigotimes_{i=1}^{m} \alpha_{i}\right)(v_{1}, \dots, v_{m})$$

$$= m! \left[\frac{1}{m!} \sum_{\sigma \in S_{m}} \operatorname{sgn}(\sigma) \sigma\left(\bigotimes_{i=1}^{m} \alpha_{i}\right)\right](v_{1}, \dots, v_{m})$$

$$= \sum_{\sigma \in S_{m}} \operatorname{sgn}(\sigma) \left(\bigotimes_{i=1}^{m} \alpha_{i}\right)(v_{\sigma(1)}, \dots, v_{\sigma(m)})$$

$$= \sum_{\sigma \in S_{m}} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} \alpha_{i}(v_{\sigma(i)})$$

$$= \det(\alpha_{i}(v_{i}))$$

**Definition 2.1.23.** Define  $\mathcal{I}_k = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$ . Each element  $I \in \mathcal{I}_k$  is called a **multi-index**. Recall that  $\#\mathcal{I}_k = \binom{n}{k}$ .

**Definition 2.1.24.** Let  $I = \{(i_1, i_2, \dots, i_k) \in I_k\}$ 

Define  $e_I \in V^k$  by

$$e_I = (e_{i_1}, \cdots, e_{i_k})$$

Define  $\epsilon_I \in \Lambda_k(V)$  by

$$\epsilon_I = \epsilon_{i_1} \wedge \cdots, \wedge \epsilon_{i_k}$$

**Exercise 2.1.25.** Let  $I=(i_1,\cdots,i_k)$  and  $J=(j_1,\cdots,j_k)\in\mathcal{I}_k$ . Then  $\epsilon_I(e_J)=\delta_{I,J}$ .

*Proof.* Put  $A = \begin{pmatrix} \epsilon_{i_1}(e_{j_1}) & \cdots & \epsilon_{i_1}(e_{j_k}) \\ & \vdots & \\ \epsilon_{i_k}(e_{j_1}) & \cdots & \epsilon_{i_k}(e_{j_k}) \end{pmatrix}$ . A previous exercise tells us that  $\epsilon_I(e_J) = \det A$ .

If I = J, then  $A = I_{k \times k}$  and therefore  $\epsilon_I(e_J) = 1$ . Suppose that  $I \neq J$ . Put  $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$ . If  $i_{l_0} < j_{l_0}$ , then all entries on the  $l_0th$  row of A are 0. If  $i_{l_0} > j_{l_0}$ , then all entries on the  $l_0th$  column of A are 0.

**Exercise 2.1.26.** Let  $\alpha, \beta \in \Lambda_k(V)$ . If for each  $I \in \mathcal{I}_k$ ,  $\alpha(e_I) = \beta(e_I)$ , then  $\alpha = \beta$ .

*Proof.* Suppose that for each  $I \in \mathcal{I}_k$ ,  $\alpha(e_I) = \beta(e_I)$ . Let  $v_1, \dots, v_k \in V$ . For  $i = 1, \dots, k$ , write  $v_i = \sum_{j_i=1}^n a_{i,j_i} e_{j_i}$ . Then

$$\alpha(v_1, \dots, v_k) = \sum_{j_1, \dots, j_k=1}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \alpha(e_{j_1}, \dots, e_{j_k})$$

$$= \sum_{j_1 \neq \dots \neq j_k}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \alpha(e_{j_1}, \dots, e_{j_k})$$

$$= \sum_{J \in \mathcal{I}_k} \left[ \sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left( \prod_{i=1}^k a_{i,\sigma(j_i)} \right) \right] \alpha(e_J)$$

$$= \sum_{J \in \mathcal{I}_k} \left[ \sum_{\sigma \in S_J} \operatorname{sgn}(\sigma) \left( \prod_{i=1}^k a_{i,\sigma(j_i)} \right) \right] \beta(e_J)$$

$$= \sum_{j_1, \dots, j_k=1}^n \left( \prod_{i=1}^k a_{i,j_i} \right) \beta(e_{j_1}, \dots, e_{j_k})$$

$$= \beta(v_1, \dots, v_k)$$

**Exercise 2.1.27.** The set  $\{\epsilon_I : I \in \mathcal{I}_k\}$  is a basis for  $\Lambda_k(V)$  and  $\dim \Lambda_k(V) = \binom{n}{k}$ .

Proof. Let  $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$ . Let  $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon_I$ . Suppose that  $\alpha = 0$ . Then for each  $J \in \mathcal{I}_k$ ,  $\alpha(e_J) = a_J = 0$ . Thus  $\{e_I : I \in \mathcal{I}_k\}$  is linearly independent. Let  $\beta \in \Lambda_k(V)$ . For  $I \in \mathcal{I}_k$ , put  $b_I = \beta(e_I)$ . define  $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon_I$ . Then for each  $J \in \mathcal{I}_k$ ,  $\mu(e_J) = b_J = \beta(e_J)$ . Hence  $\mu = \beta$  and therefore  $\beta \in \text{span}\{\epsilon_I : I \in \mathcal{I}_k\}$ .

2.2. (r, s)-Tensors.

#### 3. Manifolds

## 3.1. Smooth Manifolds.

**Definition 3.1.1.** Define the upper half space of  $\mathbb{R}^n$ , denoted  $\mathbb{H}_n$ , by

$$\mathbb{H}_n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \ge 0\}$$

and define

$$\partial \mathbb{H}_n = \{ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0 \}$$
  
$$(\mathbb{H}^n)^\circ = \{ (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n > 0 \}$$

**Definition 3.1.2.** Let M be a topological space.

- (1) Let  $n \geq 1$ ,  $U \subset M$ ,  $V \subset \mathbb{H}^n$  open and  $\phi : U \to V$ . Then  $(U, \phi)$  is said to be a **coordinate chart** on M if  $\phi$  is a homeomorphism.
- (2) Let  $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$  be a collection of coordinate charts on M. Then  $\mathcal{A}$  is said to be an **atlas** on M if  $\bigcup_{i} U_a = M$ .
- (3) Let  $n \geq 1$ . Then M is said to be **locally half Euclidean of dimension** n if there exists an atlas  $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$  on M such that for each  $a \in A$ ,  $\phi_a(U_a) \subset \mathbb{H}^n$ .
- (4) The space M is said to be an n-dimensional manifold if M is Hausdorff, second countable and locally half Euclidean of dimension n.

Note 3.1.3. For the remainder of this section, we assume M is an n-dimensional manifold.

### Definition 3.1.4.

(1) Define the **boundary** of M, denoted  $\partial M$ , by

 $\partial M = \{ p \in M : \text{ there exists a chart } (U, \phi) \text{ on } M \text{ such that } p \in U \text{ and } \phi(p) \in \partial \mathbb{H}^n \}$ 

(2) Define the **interior** of M, denoted  $M^{\circ}$ , by

$$M^\circ = M \setminus \partial M$$

**Exercise 3.1.5.** Let  $p \in M$ . Then  $p \in \partial M$  iff for each chart  $(U, \phi)$  on M,  $p \in U$  implies that  $\phi(p) \in \partial \mathbb{H}^n$ . (Hint: simply connected)

Proof. Supposet that  $p \in \partial M$ . Then there exists a coordinate chart  $(V, \psi)$  on M such that  $\psi(p) \in \partial \mathbb{H}^n$ . Let  $(U, \phi)$  be a coordinate chart on M. Suppose that  $p \in U$ . Note that  $\phi \circ \psi : \psi(V \cap U) \to \phi(V \cap U)$  is a homeomorphism. Choose open n-balls  $B_{\phi}$ ,  $B_{\psi} \subset \mathbb{H}^n$  such that  $B_{\phi} \subset \phi(V \cap U)$ ,  $B_{\psi} \subset \psi(V \cap U)$ ,  $\phi(p) \in B_{\phi}$  and  $\psi(p) \in B_{\psi}$ . For the sake of contradiction, suppose that  $\phi(p) \notin \partial \mathbb{H}^n$ . Put  $U' = B_{\phi} \setminus \{\phi(p)\}$  and  $V' = B_{\psi} \setminus \{\psi(p)\}$ . Define  $\lambda : V' \to U'$  by  $\lambda = \phi \circ \psi|_{B_{\psi}}$ . Then  $\lambda$  is a homeomorphism. Note that V' is simply connected and U' is not. This is a contradiction.

Exercise 3.1.6. If  $\partial M \neq \emptyset$ , then

- (1)  $\partial M$  is an n-1-dimensional manifold
- (2)  $\partial(\partial M) = \varnothing$ .

Proof. (1) Since subspaces of Hausdorff, second countable spaces are Hausdorff and second countable, we need only show that  $\partial M$  is locally half euclidean of dimension n-1. Let  $p \in \partial M$ . Then there exists a coordinate chart  $(U, \phi)$  on M such that  $p \in U$  and  $\phi(p) \in \partial \mathbb{H}^n$ .

Put  $U' = U \cap \partial M$ . Note that U' is open in  $\partial M$  and  $\phi(U) \cap \partial \mathbb{H}^n$  is open in  $\partial \mathbb{H}^n$ .

Define  $\phi': U' \to \phi(U) \cap \partial \mathbb{H}^n$  by  $\phi' = \phi|_{U'}$ . Then  $\phi'$  is a homeomorphism.

Since  $\partial \mathbb{H}^n$  is homeomorphic to  $\mathbb{R}^{n-1}$  which is homeomorphic to  $(\mathbb{H}^{n-1})^{\circ}$  there exists  $\psi: \partial \mathbb{H}^n \to (\mathbb{H}^{n-1})^{\circ}$  such that  $\psi$  is a homeomorphism.

Define  $V' = \psi(\phi(U) \cap \partial \mathbb{H}^n)$  and  $\psi' : \phi(U) \cap \partial \mathbb{H}^n \to V'$  by and  $\psi' = \psi|_{\phi(U) \cap \partial \mathbb{H}^n}$ . Then V' is open in  $(\mathbb{H}^{n-1})^{\circ}$  and  $\psi'$  is a homeomrophism.

Define  $\lambda: U' \to V'$  by  $\lambda = \psi' \circ \phi'$ . Then  $\lambda$  is a homeomorhism and  $(U', \lambda)$  is a cooridnate chart on  $\partial M$ . So  $\partial M$  is locally Euclidean of dimension n-1.

(2) Let  $p \in \partial M$ . Define  $(U \cap \partial M, \lambda \circ \psi)$  as in (1). Since  $\lambda \circ \psi(p) \in (\mathbb{H}^{n-1})^{\circ}$ , we have that  $p \in M^{\circ}$ . Thus  $\partial M = (\partial M)^{\circ}$  and  $\partial(\partial M) = \emptyset$ .

### Definition 3.1.7.

(1) Let  $(U, \phi), (V, \psi)$  be coordinate charts on M. Then  $(U, \phi)$  and  $(V, \psi)$  are said to be **smoothly compatible** if

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$$
 is a diffeomorphism

- (2) Let  $\mathcal{A} = ((U_a, \phi_a))_{a \in A}$  be an atlas on M. Then  $\mathcal{A}$  is said to be **smooth** if for each  $a, b \in A$ ,  $(U_a, \phi_a)$  and  $(U_b, \phi_b)$  are smoothly compatible.
- (3) Let  $\mathcal{A}$  be a smooth atlas on M. Then  $\mathcal{A}$  is said to be **maximal** if for each smooth atlas  $\mathcal{B}$  on M,  $\mathcal{A} \subset \mathcal{B}$  implies that  $\mathcal{A} = \mathcal{B}$ . A maximal smooth atlas on M is called a **smooth structure on** M.
- (4) Let A be a smooth structure on M. Then (M, A) is said to be a **smooth** n-dimensional manifold.

**Exercise 3.1.8.** Let  $\mathcal{B}$  be a smooth atlas on M. Then there exists a unique smooth structure  $\mathcal{A}$  on M such that  $\mathcal{B} \subset \mathcal{A}$ .

*Proof.* Define  $\mathcal{A}$  to be the set of all coordinate charts  $(U, \phi)$  on M such that for each coordinate chart  $(V, \psi) \in \mathcal{B}$ ,  $(U, \phi)$  and  $(V, \psi)$  are smoothly compatible. Clearly  $\mathcal{B} \subset \mathcal{A}$ .

Let  $(U,\phi),(V,\psi)\in\mathcal{A}$  and  $p\in U\cap V$ . Then there exists  $(W,\chi)\in\mathcal{B}$  such that  $p\in W$ . By assumption,  $\phi\circ\chi^{-1}:\chi(U\cap W)\to\phi(U\cap W)$  and  $\chi\circ\psi^{-1}:\psi(W\cap V)\to\chi(W\cap V)$  are diffeomorphisms. Then  $(\phi\circ\chi^{-1})\circ(\chi\circ\psi^{-1})=\phi\circ\psi^{-1}:\psi(U\cap W\cap V)\to\phi(U\cap W\cap V)$  is a diffeomorphism. Since for each  $q\in\psi(U\cap V)$ , there exits an open neighborhood  $N\subset\psi(U\cap V)$  of q on which  $\phi\circ\psi^{-1}$  are diffeomorphic, we have that  $\phi\circ\psi^{-1}$  is a diffeomorphism on  $\psi(U\cap V)$  and therefore  $(U,\phi)$  and  $(V,\psi)$  are smoothly compatible. Hence  $\mathcal A$  is a smooth atlas.

To see that  $\mathcal{A}$  is maximal, let  $\mathcal{B}'$  be a smooth atlas on M. Suppose that  $\mathcal{A} \subset \mathcal{B}'$  and let  $(U,\phi) \in \mathcal{B}'$ . By definition, for each chart  $(V,\psi) \in \mathcal{B}'$ ,  $(U,\phi)$  and  $(V,\psi)$  are smoothly compatible. Since  $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$ , we have that  $(U,\phi) \in \mathcal{A}$ . So  $\mathcal{A} = \mathcal{B}'$  and  $\mathcal{A}$  is a maximal smooth atlas on M.

**Exercise 3.1.9.** Let  $\mathcal{A}$  be a smooth atlas on M. Define  $\lambda : \partial \mathbb{H}^n \to \mathbb{R}^{n-1}$  by  $\lambda(x_1, \dots, x_{n-1,0}) = (x_1, \dots, x_{n-1})$ . Put  $\mathcal{A}|_{\partial M} = \{(U \cap \partial M, \lambda \circ \phi_{U \cap \partial M}) : (U, \phi) \in \mathcal{A}\}$ . Then

- (1)  $\mathcal{A}|_{\partial M}$  is a smooth atlas on  $\partial M$ .
- (2) if  $\mathcal{A}$  is maximal, then  $\mathcal{A}|_{\partial M}$  is maximal.

Proof.

**Note 3.1.10.** For the rest of this section, we assume that (M, A) is a smooth n-dimensional manifold and we denote the standard coordinate functions on  $\mathbb{R}^n$  by  $u_1, \dots, u_n$ . For a coordinate chart  $(U, \phi) \in A$  and  $i \in \{1, \dots, n\}$ , we will typically denote the ith coordinate of  $\phi$  by  $x_i$ , that is,  $x_i = u_i(\phi)$ .

**Definition 3.1.11.** Let  $f: M \to \mathbb{R}$ . Then f is said to be smooth if for each coordinate chart  $(U, \phi) \in \mathcal{A}$ ,  $f \circ \phi^{-1}$  is smooth. The set of all smooth functions on M is denoted  $C^{\infty}(M)$ .

**Exercise 3.1.12.** We have that  $C^{\infty}(M)$  is a vector space.

Proof. Clear. 
$$\Box$$

**Definition 3.1.13.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F: M \to N$ . Then F is said to be **smooth** if for each  $(U, \phi) \in \mathcal{A}$  and  $(V, \psi) \in \mathcal{B}$ ,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(F(U) \cap V)$$

is smooth and F is said to be a **diffeomorphism** if F is a homeomorphism and  $F, F^{-1}$  are smooth.

**Exercise 3.1.14.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F: M \to N$  a diffeomorphism. Then for each  $(U, \phi) \in \mathcal{A}$ ,  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ .

*Proof.* Let  $(V, \psi) \in \mathcal{B}$ . Since F is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \to \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \psi(V \cap F(U))$$

are smooth. Therefore  $(F(U), \phi \circ F^{-1})$  and  $(V, \psi)$  are smoothly compatible. Since  $\mathcal{B}$  is maximal,  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ .

## 3.2. The Tangent Space.

**Definition 3.2.1.** Let  $p \in M$ . Define the relation  $\sim_p$  on  $C^{\infty}(M)$  by  $f \sim_p g$  iff there exists an open  $U \subset M$  such that  $f|_U = g|_U$ . Clearly  $\sim_p$  is an equivalence relation on  $C^{\infty}(M)$ . We denote  $C^{\infty}(M)/\sim_p$  by  $C_p^{\infty}(M)$ . For  $f \in C^{\infty}(M)$ , we define the **germ of** f **at** p to be the equivalence class of f under  $\sim_p$ .

**Exercise 3.2.2.** Let  $p \in We$  have that  $C_p^{\infty}(M)$  is a vector space.

Proof. Clear. 
$$\Box$$

**Definition 3.2.3.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$ ,  $p \in U$  and  $f \in C_p^{\infty}(M)$ . For  $i \in \{1, \dots, n\}$ , define the partial derivative of f with respect to  $x_i$  at p, denoted

$$\frac{\partial f}{\partial x_i}(p), \frac{\partial}{\partial x_i}\Big|_{p} f, \partial_{x_i} f(p) \text{ or } \partial_{x_i}\Big|_{p} f$$

by

$$\frac{\partial}{\partial x_i}\Big|_p f = \frac{\partial}{\partial u_i}\Big|_{\phi(p)} f \circ \phi^{-1}$$

**Exercise 3.2.4.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$  and  $p \in U$ . Then for each  $i, j \in \{1, \dots, n\}$ , we have that

$$\frac{\partial x_i}{\partial x_j}(p) = \delta_{i,j}$$

*Proof.* Let  $i, j \in \{1, \dots, n\}$ . Then

$$\frac{\partial}{\partial x_j} \bigg|_p x_i = \frac{\partial}{\partial u_j} \bigg|_{\phi(p)} x_i \circ \phi^{-1}$$

$$= \frac{\partial}{\partial u_j} \bigg|_{\phi(p)} u_i \circ \phi \circ \phi^{-1}$$

$$= \frac{\partial}{\partial u_j} \bigg|_{\phi(p)} u_i$$

$$= \delta_{i,j}$$

Exercise 3.2.5. (Change of Coordinates): Let  $(U, \phi), (V, \psi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$  and  $\psi = (y_1, \dots, y_n), p \in U \cap V$  and  $f \in C_p^{\infty}(M)$ . Then for each  $i \in \{1, \dots, n\}$ , we have

$$\frac{\partial f}{\partial y_i}(p) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p) \frac{\partial x_j}{\partial y_i}(p)$$

*Proof.* Put  $h = \phi \circ \psi^{-1}$  and write  $h = (h_1, \dots, h_n)$ . Then  $\phi = h \circ \psi$  and  $\psi^{-1} = \phi^{-1} \circ h$ . By definition and the chain rule, we have that

$$\frac{\partial}{\partial y_{i_{p}}} \left| f = \frac{\partial}{\partial u_{i}} \right|_{\psi(p)} f \circ \psi^{-1} 
= \frac{\partial}{\partial u_{i}} \Big|_{\psi(p)} f \circ \phi^{-1} \circ h 
= \sum_{j=1}^{n} \left( \frac{\partial}{\partial u_{j}} \Big|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left( \frac{\partial}{\partial u_{i}} \Big|_{\psi(p)} h_{j} \right) 
= \sum_{j=1}^{n} \left( \frac{\partial}{\partial u_{j}} \Big|_{\phi(p)} f \circ \phi^{-1} \right) \left( \frac{\partial}{\partial u_{i}} \Big|_{\psi(p)} h_{j} \circ \psi \circ \psi^{-1} \right) 
= \sum_{j=1}^{n} \left( \frac{\partial}{\partial u_{j}} \Big|_{\phi(p)} f \circ \phi^{-1} \right) \left( \frac{\partial}{\partial u_{i}} \Big|_{\psi(p)} x_{j} \circ \psi^{-1} \right) 
= \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_{j}} \Big|_{p} f \right) \left( \frac{\partial}{\partial y_{i}} \Big|_{p} x_{j} \right)$$

**Exercise 3.2.6.** (Taylor's Theorem) Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$ ,  $p \in U$  and  $f \in C_p^{\infty}(M)$ . Then there exist  $g_1, \dots g_n \in C_p^{\infty}(M)$  such that

$$f = f(p) + \sum_{i=1}^{n} (x_i - x_i(p))g_i$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \frac{\partial}{\partial x_i} \Big|_{p} f$$

*Proof.* Since we are interested in the germ of f at p, we may assume that  $\phi(U)$  is star-shaped with respect to  $\phi(p)$ . Let  $q \in U$ . From Taylor's theorem in section 1, we know that there exist  $g'_1, \dots, g'_n \in C^{\infty}(\phi(U))$  such that for each  $q \in U$ ,

$$f \circ \phi^{-1}(\phi(q)) = f \circ \phi^{-1}(\phi(p)) + \sum_{i=1}^{n} [u_i \circ \phi(q) - u_i \circ \phi(p)] g_i'(\phi(q))$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g'_i(\phi(p)) = \frac{\partial}{\partial u_i}\Big|_{\phi(p)} f \circ \phi^{-1}$$

For each  $i \in \{1, \dots, n\}$ , define  $g_i = g_i' \circ \phi$ . Then for each  $q \in U$ ,

$$f(q) = f(p) + \sum_{i=1}^{n} [x_i(q) - x_i(p)]g_i(q)$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \frac{\partial}{\partial x_i} \bigg|_p f$$

**Definition 3.2.7.** Let  $D: C_p^{\infty}(M) \to \mathbb{R}$  and  $p \in M$ . Then D is said to be a **derivation** at p if for each  $f, g \in C_p^{\infty}(M)$  and  $a \in \mathbb{R}$ ,

- (1) D(f+cg) = D(f) + cD(g) (D is linear)
- (2) D(fg) = D(f)g(p) + f(p)D(g) (D is Leibnizian)

**Definition 3.2.8.** Let  $p \in M$ . The set of derivations at p, denoted  $T_pM$  is called the tangent space of M at p.

**Exercise 3.2.9.** Let  $f \in C_n^{\infty}(M)$  and  $D \in T_pM$ . If f is constant, then Df = 0.

Proof. Suppose that  $f \equiv 1$ . Then  $f^2 = f$  and  $D(f^2) = 2D(f)$ . So D(f) = 2D(f) which implies that D(f) = 0. If  $f \not\equiv 1$ , then there exists  $c \in \mathbb{R}$  such that  $f \equiv c$ . Since D is linear, D(f) = cD(1) = 0.

**Exercise 3.2.10.** Let  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$  and  $p \in U$ . Then

$$\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \cdots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$$

is a basis for  $T_pM$  and dim  $T_pM = n$ .

*Proof.* Clearly  $\frac{\partial}{\partial x_1}\Big|_p$ ,  $\cdots$ ,  $\frac{\partial}{\partial x_n}\Big|_p \in T_pM$ . Let  $a_1, \cdots, a_n \in \mathbb{R}$ . Suppose that

$$D = \sum_{i=1}^{n} a_i \left. \frac{\partial}{\partial x_i} \right|_p = 0$$

Then

$$0 = Dx_j$$

$$= \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} \Big|_p x_j$$

$$= a_j$$

Hence  $\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \cdots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$  is independent.

Now, let  $D \in T_pM$  and  $f \in \mathbb{C}_p^{\infty}(M)$ . By Taylor's theorem, there exist  $g_1, \dots, g_n \in C_p^{\infty}(M)$  such that

$$f = f(p) + \sum_{i=1}^{n} (x_i - x_i(p))g_i$$

and for each  $i \in \{1, \dots, n\}$ ,

$$g_i(p) = \left. \frac{\partial}{\partial x_i} \right|_p f$$

Then

$$D(f) = \sum_{i=1}^{n} D(x_i - x_i(p))g_i(p) + \sum_{i=1}^{n} (x_i(p) - x_i(p))D(g_i)$$

$$= \sum_{i=1}^{n} D(x_i)g_i(p)$$

$$= \sum_{i=1}^{n} D(x_i) \frac{\partial}{\partial x_i} \Big|_{p} f$$

$$= \left[ \sum_{i=1}^{n} D(x_i) \frac{\partial}{\partial x_i} \Big|_{p} \right] f$$

So

$$D = \sum_{i=1}^{n} D(x_i) \left. \frac{\partial}{\partial x_i} \right|_p$$

and

$$D \in \operatorname{span} \left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \cdots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$$

**Definition 3.2.11.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F: M \to N$  a diffeomorphism and  $p \in M$ . We define the **push forward of** F **at** p, denoted  $(F_*)_p: T_pM \to T_{F(p)}N$ , by

$$\left[ (F_*)_p(D) \right] (f) = D(f \circ F)$$

for  $D \in T_pM$  and  $f \in C^{\infty}_{F(p)}(N)$ .

**Exercise 3.2.12.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F: M \to N$  a diffeomorphism and  $p \in M$ . Then  $(F_*)_p$  is an isomorphism.

*Proof.* Choose  $(U, \phi) \in \mathcal{A}$  such that  $p \in U$ . A previous exercise tells us that  $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$ . Write  $\phi = (x_1, \dots, x_n)$  and  $\phi \circ F^{-1} = (y_1, \dots, y_n)$ . Let  $f \in C^{\infty}_{F(p)}(N)$  Then

$$\frac{\partial}{\partial y_i}\Big|_{F(p)} f = \frac{\partial}{\partial u_i}\Big|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1}$$

$$= \frac{\partial}{\partial u_i}\Big|_{\phi(p)} f \circ F \circ \phi^{-1}$$

$$= \frac{\partial}{\partial x_i}\Big|_p f \circ F$$

Therefore

$$\left[ (F_*)_p \left( \left. \frac{\partial}{\partial x_i} \right|_p \right) \right] (f) = \left. \frac{\partial}{\partial x_i} \right|_p f \circ F$$

$$= \left. \frac{\partial}{\partial y_i} \right|_{F(p)} f$$

Hence

$$(F_*)_p \left( \left. \frac{\partial}{\partial x_i} \right|_p \right) = \left. \frac{\partial}{\partial y_i} \right|_{F(p)}$$

Since  $\left\{ \frac{\partial}{\partial x_1} \Big|_p, \cdots, \frac{\partial}{\partial x_n} \Big|_p \right\}$  is a basis for  $T_p M$  and  $\left\{ \frac{\partial}{\partial y_1} \Big|_{F(p)}, \cdots, \frac{\partial}{\partial y_n} \Big|_{F(p)} \right\}$  is a basis for  $T_{F(p)} N$ ,  $(F_*)_p$  is an isomorphism.

**Definition 3.2.13.** Let  $(N, \mathcal{B})$  be a smooth manifold,  $F: M \to N$  a diffeomorphism. Define the **push forward of** F, denoted

$$F_*: M \to \coprod_{p \in M} \operatorname{Iso}(T_p M, T_{F(p)} N)$$

by

$$p \mapsto (F_*)_p$$

**Definition 3.2.14.** We define the tangent bundle of M, denoted TM, by

$$TM = \coprod_{p \in M} T_p M$$

and we define the **cotangent bundle of** M, denoted  $T^*M$ , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

**Definition 3.2.15.** Let  $X: M \to TM$ . Then X is said to be a **vector field on** M if for each  $p \in M$ ,  $X_p \in T_pM$ .

For  $\hat{f} \in \mathbb{C}^{\infty}(M)$  we define  $Xf: M \to \mathbb{R}$  by

$$(Xf)_p = X_p(f)$$

Finally, X is said to be **smooth** if for each  $f \in \mathbb{C}^{\infty}(M)$ , Xf is smooth. We denote the set of smooth vector fields on M by  $\Gamma(M)$ .

**Exercise 3.2.16.** Let  $X \in \Gamma(M)$  and  $(U, \phi) \in \mathcal{A}$  with  $\phi = (x_1, \dots, x_n)$ . Then there exist  $f_1, \dots, f_n \in C^{\infty}(U)$  such that for each  $p \in U$ ,

$$X_p = \sum_{i=1}^n f_i(p) \left. \frac{\partial}{\partial x_i} \right|_p$$

Proof. Let  $p \in M$ . Then  $X_p \in T_pM$  and  $\left\{ \left. \frac{\partial}{\partial x_1} \right|_p, \dots, \left. \frac{\partial}{\partial x_n} \right|_p \right\}$  is a basis of  $T_pM$ . So there exist  $f_1(p), \dots, f_n(p) \in \mathbb{R}$  such that  $X_p = \sum_{i=1}^n f_i(p) \left. \frac{\partial}{\partial x_i} \right|_p$ . Let  $j \in \{1, \dots, n\}$ . Since X is smooth, the map

$$p \mapsto X_p(x_j) = \sum_{i=1}^n f_i(p) \frac{\partial x_j}{\partial x_i}(p)$$
$$= f_j(p)$$

is smooth.

# 3.3. Integration on Manifolds.

**Definition 3.3.1.** We define

$$\Lambda_k(TM) = \coprod_{p \in M} \Lambda_k(T_pM)$$

**Definition 3.3.2.** Let  $\omega : M \to \Lambda_k(TM)$ . Then  $\omega$  is said to be a k-form on M if for each  $p \in M$ ,  $\omega_p \in \Lambda_k(T_pM)$ .

For each  $X_1, \dots, X_k \in \Gamma(M)$ , we define  $\omega(X_1, \dots, X_k) : M \to \mathbb{R}$  by

$$\omega(X_1,\cdots,X_k)_p=\omega_p(X_{1p},\cdots,X_{kp})$$

Finally,  $\omega$  is said to be **smooth** if for each  $X_1, \dots, X_k \in \Gamma(M)$ ,  $\omega(X_1, \dots, X_k)$  is smooth. The set of smooth k-forms on M is denoted  $\Omega_k(M)$ .

Note 3.3.3. Observe that  $\Omega_0(M) = C^{\infty}(M)$ .

Definition 3.3.4. Define the exterior product

$$\wedge: \Omega_k(M) \times \Omega_l(M) \to \Omega_{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Define the **permutation action of**  $S_k$  **on**  $\Omega_k(M)$  by

$$(\sigma\omega)_p = \sigma\omega_p$$

Note 3.3.5. All of the results from multilinear algebra apply here.

**Note 3.3.6.** For  $f \in \Omega_0(M)$  and  $\alpha \in \Omega_k(M)$ , we have that  $f \wedge \alpha = f\alpha$ .

**Definition 3.3.7.** We define the **exterior derivative**  $d: \Omega_k(M) \to \Omega_{k+1}(M)$  inductively by

- (1)  $df(X) = Xf \text{ for } f \in \Omega_0(M)$
- (2)  $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \text{ for } al \in \Omega_p(M) \text{ and } \beta \in \Omega_q(M)$
- (3) extending linearly

**Exercise 3.3.8.** Let  $(U, \phi)$  be a chart on M with  $\phi = (x_1, \dots, x_n)$ . Then on U, for each  $i, j \in \{1, \dots, n\}$ ,

$$dx_i \left( \frac{\partial}{\partial x_j} \right) \equiv \delta_{i,j}$$

*Proof.* Let  $p \in U$  and  $i, j \in \{1, \dots, n\}$ . Then

$$(dx_i)_p \left( \left. \frac{\partial}{\partial x_j} \right|_p \right) = \left. \frac{\partial}{\partial x_j} \right|_p x_i$$
$$= \delta_{i,j}$$

**Note 3.3.9.** The previous exercise tells us that for each  $p \in U$ ,  $\{(dx_1)_p, \dots, (dx_n)_p\}$  is the dual basis to  $\left\{\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p\right\}$ .

**Exercise 3.3.10.** Let  $f \in C^{\infty}(M)$  and  $(U, \phi)$  be a chart on M with  $\phi = (x_1, \dots, x_n)$ . Then on U,  $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$ 

Proof. Let  $p \in U$ . Since  $\{dx_1, \dots, dx_n\}$  is a basis for  $\Lambda(T_pM)$ , for each there exist  $a_1(p), \dots, a_n(p) \in \mathbb{R}$  such that  $(df)_p = \sum_{i=1}^n a_i(p)(dx_i)_p$ . Therefore, we have that

$$(df)_p \left( \left. \frac{\partial}{\partial x_j} \right|_p \right) = \sum_{i=1}^n a_i(p) (dx_i)_p \left( \left. \frac{\partial}{\partial x_j} \right|_p \right)$$

$$= a_j(p)$$

By definition, we have that

$$(df)_p \left( \left. \frac{\partial}{\partial x_j} \right|_p \right) = \left. \frac{\partial}{\partial x_j} \right|_p f$$
$$= \frac{\partial f}{\partial x_j}(p)$$

So

$$(df)_p = \sum_{i=1}^n \frac{\partial f}{\partial x_j}(p)(dx_i)_p$$

and therefore on U, we have that

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

**Definition 3.3.11.** Let  $(U, \phi)$  be a chart on M with  $\phi = (x_1, \dots, x_n)$  and  $I = (i_1, \dots, i_k) \in \mathcal{I}_k$ . We define

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega_k(M)$$

and we define

$$\frac{\partial}{\partial x_I} = \left(\frac{\partial}{\partial x_{i_1}}, \cdots, \frac{\partial}{\partial x_{i_k}}\right)$$

**Exercise 3.3.12.** Let  $\omega \in \Omega_k(M)$  and  $(U, \phi)$  be a chart on M with  $\phi = (x_1, \dots, x_n)$ . Then there exists  $(f_I)_{I \in \mathcal{I}_k} \subset C^{\infty}(U)$  such that for each  $p \in U$ ,

$$\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p) (dx_I)_p$$

*Proof.* Let  $p \in U$ . For each  $I \in \mathcal{I}_k$ , put

$$f_I(p) = \omega_p \left( \left. \frac{\partial}{\partial x_I} \right|_p \right) \in \mathbb{R}$$

Since  $\{(dx_I)_p : I \in \mathcal{I}_k\}$  is a basis for  $\Lambda_k(T_pM)$ , we have that  $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p)(dx_I)_p$ . Since  $\omega$  is smooth, we have that for each  $J \in \mathcal{I}_k$ ,

$$\omega\left(\frac{\partial}{\partial x_J}\right) = \sum_{I \in \mathcal{I}_k} f_I dx_I \left(\frac{\partial}{\partial x_J}\right)$$
$$= f_J$$

is smooth.

**Exercise 3.3.13.** Let  $\omega \in \Omega_k(M)$  and  $(U,\phi)$  be a chart on M with  $\phi = (x_1, \dots, x_n)$ . If  $\omega = \sum_{I \in \mathcal{I}_k} f_I dx_I$ , then

$$d\omega = \sum_{I \in \mathcal{I}_L} \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I$$

.

*Proof.* First we note that

$$d(f_I dx_I) = df_I \wedge dx_I + (-1)^0 f d(dx_I)$$

$$= df_I \wedge dx_I$$

$$= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i\right) \wedge dx_I$$

$$= \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I$$

Then we extend linearly.

**Definition 3.3.14.** Let  $(N, \mathcal{B})$  be a smooth manifold and  $F: M \to N$  be a diffeomorphism. Define the **pullback of** F, denoted  $F^*: \Omega_k(N) \to \Omega_k(M)$  by

$$(F^*\omega)_p(D_1,\cdots,D_k) = \omega_{F(p)}((F_*)_p(D_1),\cdots,(F_*)_p(D_k))$$

for  $\omega \in \Omega_k(N)$ ,  $p \in M$  and  $D_1, \dots, D_k \in T_pM$ 

•

**Definition 3.3.15.** When working in  $\mathbb{R}^n$ , we introduce the formal objects  $dx_1, dx_2, \dots, dx_n$ . Let  $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_{k,n}$  and  $\phi : \mathbb{R}^k \to \mathbb{R}^n$ . Write  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ . We formally define  $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$  and  $\phi_I = (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$ .

**Definition 3.3.16.** Let  $k \in \{0, 1, \dots, n\}$ . We define a  $C^{\infty}(\mathbb{R}^n)$ -module of dimension  $\binom{n}{k}$ , denoted  $\Gamma^k(\mathbb{R}^n)$  to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^{\infty}(\mathbb{R}^n) & k = 0\\ \operatorname{span}\{dx_I : I \in \mathcal{I}_{k,n}\} & k \ge 1 \end{cases}$$

For each  $\omega \in \Phi_k(\mathbb{R}^n)$  and  $\chi \in \Gamma^l(\mathbb{R}^n)$ , we may form their **exterior product**, denoted by  $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$ . Thus the exterior product is a map  $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$ . The exterior product is characterized by the following properties:

- (1) the exterior product is bilinear
- (2) for each  $\omega \in \Phi_k(\mathbb{R}^n)$  and  $\chi \in \Gamma^l(\mathbb{R}^n)$ ,  $\omega \wedge \chi = -\chi \wedge \omega$
- (3) for each  $\omega \in \Phi_k(\mathbb{R}^n)$ ,  $\omega \wedge \omega = 0$
- (4) for each  $f \in C^{\infty}(\mathbb{R}^n)$  and  $\omega \in \Phi_k(\mathbb{R}^n)$ ,  $f \wedge \omega = f\omega$

We call  $\Phi_k(\mathbb{R}^n)$  the differential k-forms on  $\mathbb{R}^n$ . Let  $\omega$  be a k-form on  $\mathbb{R}^n$ . If  $k \geq 1$ , then for each  $I \in \mathcal{I}_{k,n}$ , there exists  $f_I \in C^{\infty}(\mathbb{R}^n)$  such that  $\omega = \sum_{I \in \mathcal{I}_k} f_I dx_I$ 

**Note 3.3.17.** The terms  $dx_1, dx_2, \dots, dx_n$  are are a sort of place holder for the coordinates of a point  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ . When we work with functions  $\phi : \mathbb{R}^k \to \mathbb{R}^n$ , we will have different coordinates and to avoid confusion, we will write  $\{du_1, du_2, \dots, du_k\}$  when referencing the coordinates on  $\mathbb{R}^k$  and  $\{dx_1, dx_2, \dots, dx_n\}$  when referencing the coordinates on  $\mathbb{R}^n$ .

**Exercise 3.3.18.** Let  $B_{n\times n}=(b_{i,j})\in [C^{\infty}(M)]^{n\times n}$  be an  $n\times n$  matrix. Then

$$\bigwedge_{i=1}^{n} \left( \sum_{j=1}^{n} b_{i,j} dx_j \right) = (\det B) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

*Proof.* Bilinearity of the exterior product implies that

$$\bigwedge_{i=1}^{n} \left( \sum_{j=1}^{n} b_{i,j} dx_{j} \right) = \left( \sum_{j=1}^{n} b_{1,j} dx_{j} \right) \wedge \left( \sum_{j=1}^{n} b_{2,j} dx_{j} \right) \wedge \dots \wedge \left( \sum_{j=1}^{n} b_{n,j} dx_{j} \right)$$

$$= \sum_{j_{1}, \dots, j_{n}=1}^{n} \left( \prod_{i=1}^{n} b_{i,j_{i}} \right) dx_{j_{1}} \wedge dx_{j_{2}} \wedge \dots \wedge dx_{j_{n}}$$

$$= \sum_{j_{1} \neq \dots \neq j_{n}} \left( \prod_{i=1}^{n} b_{i,j_{i}} \right) dx_{j_{1}} \wedge dx_{j_{2}} \wedge \dots \wedge dx_{j_{n}}$$

$$= \left[ \sum_{\sigma \in S_{n}} \operatorname{sgn}(\sigma) \left( \prod_{i=1}^{n} b_{i,\sigma(i)} \right) \right] dx_{1} \wedge dx_{2} \wedge \dots \wedge dx_{n}$$

$$= (\det B) dx_{1} \wedge dx_{2} \wedge \dots \wedge dx_{n}$$

**Definition 3.3.19.** Let  $f: \mathbb{R}^n \to \mathbb{R}$  be a 0-form on  $\mathbb{R}^n$ . We define a 1-form, denoted df, on  $\mathbb{R}^n$  by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Let  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$  be a k-form on  $\mathbb{R}^n$ . We can define a differential k+1-form, denoted  $d\omega$ , on  $\mathbb{R}^n$  by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx_I$$

Exercise 3.3.20. On  $\mathbb{R}^3$ , put

- (1)  $\omega_0 = f_0$ ,
- (2)  $\omega_1 = f_1 dx_1 + f_2 dx_2 + f_2 dx_3$ ,
- (3)  $\omega_2 = f_1 dx_2 \wedge dx_3 f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2$

Show that

(1) 
$$d\omega_0 = \frac{\partial f_0}{\partial x_1} dx_1 + \frac{\partial f_0}{\partial x_2} dx_2 + \frac{\partial f_0}{\partial x_3} dx_3$$

$$(1) \ d\omega_0 = \frac{\partial f_0}{\partial x_1} dx_1 + \frac{\partial f_0}{\partial x_2} dx_2 + \frac{\partial f_0}{\partial x_3} dx_3$$

$$(2) \ d\omega_1 = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right) dx_1 \wedge dx_3 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

(3) 
$$d\omega_2 = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$$

*Proof.* Straightforward.

**Exercise 3.3.21.** Let  $I \in \mathcal{I}_{k,n}$ . Then there is a unique  $I_* \in \mathcal{I}_{n-k,n}$  such that  $dx_I \wedge dx_{I_*} =$  $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ .

**Definition 3.3.22.** We define a linear map  $*: \Phi_k(\mathbb{R}^n) \to \Gamma^{n-k}(\mathbb{R}^n)$  called the **Hodge** \*-operator by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

**Definition 3.3.23.** Let  $\phi: \mathbb{R}^k \to \mathbb{R}^n$  be smooth. Write  $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ . We define  $\phi^*: \Phi_k(\mathbb{R}^n) \to \Phi_k(\mathbb{R}^k)$  via the following properties:

- (1) for each 0-form f on  $\mathbb{R}^n$ ,  $\phi^* f = f \circ \phi$
- (2) for  $i = 1, \dots, n, \ \phi^* dx_i = d\phi_i$
- (3) for an s-form  $\omega$ , and a t-form  $\chi$  on  $\mathbb{R}^n$ ,  $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
- (4) for l-forms  $\omega, \chi$  on  $\mathbb{R}^n$ ,  $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

**Exercise 3.3.24.** Let  $M \subset \mathbb{R}^n$  be a k-dimensional smooth submanifold of  $\mathbb{R}^n$ ,  $\phi: U \to V$  a smooth parametrization of M,  $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$  an k-form on  $\mathbb{R}^n$ . Then

$$\phi^*\omega = \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det D\phi_I)\right) du_1 \wedge du_2 \wedge \dots \wedge du_k$$

*Proof.* Using the definitions, we see that

$$\phi^* \omega = \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx_I$$
$$= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx_I$$
$$= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I$$

A previous exercise tells us that for each  $I \in \mathcal{I}_{k,n}$ ,

$$d\phi_{I} = d\phi_{i_{1}} \wedge d\phi_{i_{2}} \wedge \dots \wedge d\phi_{i_{n}}$$

$$= \left(\sum_{j=1}^{n} \frac{\partial \phi_{i_{1}}}{\partial u_{j}} du_{j}\right) \wedge \left(\sum_{j=1}^{n} \frac{\partial \phi_{i_{2}}}{\partial u_{j}} du_{j}\right) \wedge \dots \wedge \left(\sum_{j=1}^{n} \frac{\partial \phi_{i_{k}}}{\partial u_{j}} du_{j}\right)$$

$$= \left(\det D\phi_{I}\right) du_{1} \wedge du_{2} \wedge \dots \wedge du_{k}$$

Therefore

$$\phi^* \omega = \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I$$

$$= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det D\phi_I) du_1 \wedge du_2 \wedge \dots \wedge du_k$$

$$= \left( \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) (\det D\phi_I) \right) du_1 \wedge du_2 \wedge \dots \wedge du_k$$

## 3.4. Integration of Differential Forms.

**Definition 3.4.1.** Let  $U \subset \mathbb{R}^k$  be open and  $\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$  a k-form on  $\mathbb{R}^k$ . Define

$$\int_{U} \omega = \int_{U} f dx$$

**Definition 3.4.2.** Let  $M \subset \mathbb{R}^n$  be a k-dimensional oriented smooth submanifold of  $\mathbb{R}^n$ ,  $\omega$  a k-form on  $\mathbb{R}^n$  and  $\phi: U \to V$  a local smooth, orientation-preserving parametrization of M. Define

$$\int_{V} \omega = \int_{U} \phi^* \omega$$

Exercise 3.4.3.

**Theorem 3.4.4.** (Stokes Theorem) Let  $M \subset \mathbb{R}^n$  be a k-dimensional oriented smooth submanifold of  $\mathbb{R}^n$  and  $\omega$  a k-1-form on  $\mathbb{R}^n$ . Then

$$\int_{\partial M} \omega = \int_{M} d\omega$$