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Introduction to Dynamical Systems

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Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Basic Concepts

1.1 Measure Preserving Transformations

Definition 1.1.0.1. We define **Meas** by

- $\text{Obj}(\mathbf{Meas}) := \{(X, \mathcal{A}) : (X, \mathcal{A}) \text{ is a measurable space}\}$.
- for $(X, \mathcal{A}), (Y, \mathcal{B}) \in \text{Obj}(\mathbf{Meas})$,
$$\text{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B})) := \{f : X \to Y : f \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable}\}$$
- for $(X, \mathcal{A}), (Y, \mathcal{B}), (Z, \mathcal{C}) \in \text{Obj}(\mathbf{Meas})$, $f \in \text{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B}))$ and $g \in \text{Hom}_{\mathbf{Meas}}((Y, \mathcal{B}), (Z, \mathcal{C}))$,
$$g \circ_{\mathbf{Meas}} f := g \circ_{\mathbf{Set}} f$$

Exercise 1.1.0.2. We have that **Meas** is a category.

Proof. □

Exercise 1.1.0.3. We have that **Meas** is a Cartesian monoidal category.

Definition 1.1.0.4. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be probability spaces and $f \in \text{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B}))$. Then T is said to be **measure preserving** if $f_*\mu = \nu$.

Exercise 1.1.0.5. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be probability spaces and $f \in \text{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B}))$. Then f is measure preserving iff for each $\phi \in L^1(Y, \mathcal{B}, \nu)$, $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and

$$\int_Y \phi \, d\nu = \int_X \phi \circ f \, d\mu$$

Proof.

- $(\implies)$:
Suppose that f is measure preserving. $\phi \in L^1(Y, \mathcal{B}, \nu)$. Then the a basic result on the change of variables implies that $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and
$$\begin{aligned}\int_Y \phi \, d\nu &= \int_Y \phi d \, f_* mu \\ &= \int_X \phi \, d\mu\end{aligned}$$

- ($\Longleftarrow$):
  Suppose that for each $\phi \in L^1(Y, \mathcal{B}, \nu)$, $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and

$$\int_Y \phi \, d\nu = \int_X \phi \circ f \, d\mu$$

  Let $B \in \mathcal{B}$. Since ν is a probability measure, $\chi_B \in L^1(Y, \mathcal{B}, \nu)$. Thus

$$\begin{aligned}
\nu(B) &= \int_Y \chi_B \, d\nu \\
&= \int_X \chi_B \circ f \, d\mu \\
&= \int_X \chi_{f^{-1}(B)} \, d\mu \\
&= \mu(f^{-1}(B)) \\
&= f_*\mu(B)
\end{aligned}$$

  Since $B \in \mathcal{B}$ is arbitrary, $f_*\mu = \nu$.

$\square$

Definition 1.1.0.6. We define **Prob** by

- $\mathrm{Obj}(\mathbf{Prob}) = \{(X, \mathcal{A}, \mu) : (X, \mathcal{A}, \mu) \text{ is a probability space}\}$.
- for $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu) \in \mathrm{Obj}(\mathbf{Prob})$,

$$\mathrm{Hom}_{\mathbf{Prob}}((X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)) = \{f \in \mathrm{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B})) : f \text{ is measure preserving}\}$$

- for $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu), (Z, \mathcal{C}, \lambda) \in \mathrm{Obj}(\mathbf{Prob})$, $f \in \mathrm{Hom}_{\mathbf{Prob}}((X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu))$ and $g \in \mathrm{Hom}_{\mathbf{Prob}}((Y, \mathcal{B}, \nu), (Z, \mathcal{C}, \lambda))$,

$$g \circ_{\mathbf{Prob}} f := g \circ_{\mathbf{Set}} f$$

Exercise 1.1.0.7. We have that **Prob** is a category.

Proof. $\square$

Exercise 1.1.0.8. We have that **Prob** is not a Cartesian monoidal category.

Proof. content... $\square$

Even though **Prob** does not have products, when applying the forgetful functor $U : \mathbf{Prob} \to \mathbf{Meas}$, we get a category with products **Meas**, so in some sense, an object in **Meas** is an equivalence class of objects in **Prob** where we ignore our notions of size/interaction of sub-objects. After applying the U to a potential product $(Z, \mathcal{C}, \lambda) \in \mathrm{Obj}(\mathbf{Prob})$ (i.e. there are associated measure preserving maps $f_X : Z \to X$ and $f_Y : Z \to Y$) to get $(Z, \mathcal{C}) \in \mathrm{Obj}(\mathbf{Meas})$, then $(Z, \mathcal{C}) \in \mathrm{Obj}(\mathbf{Meas})$ is a potential product with the same associated maps and we get the unique map $h : Z \to X \times Y$ in **Meas** yielding the typical commutative diagram for products in **Meas** (i.e. $h = f_X, f_Y$). In general h does not preserve measure unless λ can be written as a tensor product. We can quantify how far off a potential product $(Z, \mathcal{C}, \lambda) \in \mathrm{Obj}(\mathbf{Prob})$ (i.e. an element of the equivalence class) is from being a product by looking at the information loss (relative entropy) across h

1.2 Measure Preserving Systems

Definition 1.2.0.1. Let $(X, \mathcal{A}) \in \mathrm{Obj}(\mathbf{Meas})$, $f \in \mathrm{End}_{\mathbf{Meas}}(X, \mathcal{A})$ and $\mu \in \mathcal{M}(X, \mathcal{A})$. Then μ is said to be **f-invariant** if $f_* \mu = \mu$.

Exercise 1.2.0.2. Let X be a compact metric space and $f \in \mathrm{End}_{\mathbf{Top}}(X)$. Then there exists $\mu \in \mathcal{P}(X, \mathcal{A})$ such that μ is f-invariant.
Hint:

Proof. □

Definition 1.2.0.3. Let $(X, \mathcal{A}, \mu) \in \mathbf{Prob}$ and $f \in \mathrm{End}_{\mathbf{Prob}}(X, \mathcal{A}, \mu)$. Then $(X, \mathcal{A}, \mu, f)$ is said to be a **measure-preserving dynamical system**.

Exercise 1.2.0.4.

Chapter 2

Thoughts

Exercise 2.0.0.1. Try showing classical and quantum versions of Khintchine's recurrence theorem for observables in the Heisenberg picture.

First, classically on phase space $X = \mathbb{R}^{2n}$ with Hamiltonian flow $(\phi_t)_{t\in\mathbb{R}} \subset Iso(X)$, the schrodinger picture is where the state evolves $(p(t), q(t)) = \phi_t(p(0), q(0))$ and satisfies hamilton's equations and the Heisenberg picture is where the observables evolve $f_t = f_0 \circ \phi_t$ and satisfies Louiville's theorem.

In the quantum case on phase space $H = L^2(\mathbb{R}^n)$ with unitary flow $(U_t)_{t\in\mathbb{R}} \subset \mathrm{Iso}(H)$, the schodinger picture is where the state evolves $\psi(t) = U_t\psi(0)$ and satisfies the schrodinger equation and the heisenberg picture is where the observable evolves $A(t) = U_t^* A(0) U_t$ and satisfies von Neumann's equation.

Classically in ergodic theory, we care about Khintchine's recurrence theorem: Given measure space $(X, \mathcal{A}, \mu)$ and T which is μ-invariant, then for each $A \in \mathcal{A}$, $\epsilon > 0$, there is a relatively dense $\mathcal{N} \subset \mathbb{N}$ such that for each $n \in \mathcal{N}$, $\mu(T^{-n}(A) \cap A) > \mu(A)^2 - \epsilon$. Here relatively dense means that for some $L > 0$, every interval in $\mathbb{N}$ of length $\geq L$, contains an element of $\mathcal{N}$.

This should generalize to the Heisenberg picture $\int_X f \circ \phi_t f \, d\mu > \left(\int_X f \, d\mu\right)^2 - \epsilon$, i.e. $\int_X f_t f \, d\mu > \left(\int_X f_0 \, d\mu\right)^2 - \epsilon$. This is strong if we recall cauchy schwarz for L^2 and the fact that μ is ϕ_t-invariant, so that if $f \geq 0$, then

$$\begin{aligned}
\left(\int_X f_0 \, d\mu\right)^2 &= \int_X f_t \, d\mu \int_X f_0 \, d\mu \\
&\geq \int_X f_t f_0 \, d\mu \\
&> \left(\int_X f_0 \, d\mu\right)^2 - \epsilon
\end{aligned}$$

. If μ is a probability measure we interpret this to mean that for an observable f, recurrently after enough time, we expect to measure the same value (in some weak sense) as in the outset, or at least the observable at the outset and at time t overlap and are indistinguishable. Then follwoing the idea in nestruevs smooth manifold book about how knowing the state space is equivalent to knowing the observables, as we can determine one from the other, we would like to know if for sufficiently many observables to determine the state well enough $(f_t^{(j)})_{j\in[J]} \in C^\infty(X; [0,\infty)^J)$, whether or not we get enough overlap between $(f_t^{(j)})_{j\in[J]}$ and $(f_0^{(j)})_{j\in[J]}$ recurrently in the l_2-sense. I cant believe its not random

This should generalize to the quantum case as in
quantum ergodic theorems
quantum ergodic phd thesis

Appendix A

App

A.1 Reading Diagrams and associated digraphs of diagrams

Definition A.1.0.1. Let

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "h"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \implies \begin{tikzcd} C \arrow[rr, bend left, "g"] \arrow[rr, bend right, "h"'] & & A \end{tikzcd}$$

see an intro to the language of category theory by roman for description

Definition A.1.0.2. A diagram is said to be **commutative** if for each path of length ≥ 2, in the associated digraph gives the same morphism.