

Introduction to Differential Geometry

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Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on (X, \mathcal{A})
v	velocity

Preface

cc-by-nc-sa

Chapter 1

Review of Fundamentals

1.1 Set Theory

Definition 1.1.0.1. Let $\{A_i\}_{i \in I}$ be a collection of sets. The **disjoint union** of $\{A_i\}_{i \in I}$, denoted $\coprod_{i \in I} A_i$, is defined by

$$\coprod_{i \in I} A_i = \bigcup_{i \in I} \{i\} \times A_i$$

We define the **natural projection map**, denoted $\pi : \coprod_{i \in I} A_i \rightarrow I$, by $\pi(i, a) = i$.

Definition 1.1.0.2. Let E and M be sets, $\pi : E \rightarrow B$ a surjection and $\sigma : B \rightarrow E$. Then σ is said to be a section of (E, M, π) if $\pi \circ \sigma = \text{id}_M$.

Note 1.1.0.3. Let $\{A_i\}_{i \in I}$ be a collection of sets and $\sigma : I \rightarrow \coprod_{i \in I} A_i$. We will typically be interested in sections σ of $\left(\coprod_{i \in I} A_i, I, \pi \right)$.

Exercise 1.1.0.4. Let $\{A_i\}_{i \in I}$ be a collection of sets and $\sigma : I \rightarrow \coprod_{i \in I} A_i$. Then σ is a section of $\coprod_{i \in I} A_i$ iff for each $i \in I$, $\sigma(i) \in A_i$

Proof. Clear. □

1.2 Linear Algebra

Note 1.2.0.1. We denote the standard basis on \mathbb{R}^n by (e_1, \dots, e_n) .

Definition 1.2.0.2. Let $A \in \mathbb{R}^{n \times n}$. Then A is said to be **invertible** if $\det(A) \neq 0$. We denote the set of $n \times n$ invertible matrices by $GL(n, \mathbb{R})$.

$$O(n)$$

Exercise 1.2.0.3. Let $A, B \in \mathbb{R}^{n \times n}$. Then $AB = I$ iff $BA = I$.

Proof.

- (\implies):
Suppose that $AB = I$. Then

$$\begin{aligned} \ker B &\subset \ker AB \\ &= \ker I \\ &= \{0\} \end{aligned}$$

so that $\ker B = \{0\}$. Hence $\text{Im } B = \mathbb{R}^n$ and B is surjective. Then

$$\begin{aligned} IB &= BI \\ &= B(AB) \\ &= (BA)B \end{aligned}$$

Since B is surjective, $I = BA$.

- (\impliedby):
Immediate by the previous part.

□

Definition 1.2.0.4. Let $A \in \mathbb{R}^{n \times p}$. Then A is said to be an **orthogonal matrix** if $A^*A = I$. We denote the set of $n \times p$ orthogonal matrices by $O(n, p)$. We write $O(n)$ in place of $O(n, n)$.

$$O(n)$$

Exercise 1.2.0.5. Define $\phi : S_n \rightarrow GL(n, \mathbb{R})$ by

$$\phi(\sigma) = \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix}$$

Then

1. for each $A \in \mathbb{R}^{n \times p}$,

$$(\phi(\sigma)A)_{i,j} = A_{\sigma(i),j}$$

i.e. left multiplying A by $\phi(\sigma)$ the the same as permuting the rows of A by σ

2. ϕ is a group homomorphism

Proof. 1. Let $A \in \mathbb{R}^{n \times p}$. Then

$$\begin{aligned} (\phi(\sigma)A)_{i,j} &= \langle e_{\sigma(i)}^*, Ae_j \rangle \\ &= A_{\sigma(i),j} \end{aligned}$$

2. Let $\sigma, \tau \in S_n$. Part (1) implies that

$$\begin{aligned}\phi(\sigma\tau) &= \begin{pmatrix} e_{\sigma\tau(1)}^* \\ \vdots \\ e_{\sigma\tau(n)}^* \end{pmatrix} \\ &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} \begin{pmatrix} e_{\tau(1)}^* \\ \vdots \\ e_{\tau(n)}^* \end{pmatrix} \\ &= \phi(\sigma)\phi(\tau)\end{aligned}$$

Since $\sigma, \tau \in S_n$ are arbitrary, ϕ is a group homomorphism. □

Definition 1.2.0.6. Define $\phi : S_n \rightarrow GL(n, \mathbb{R})$ as in the previous exercise. Let $P \in GL(n, \mathbb{R})$. Then P is said to be a **permutation matrix** if there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. We denote the set of $n \times n$ permutation matrices by $\text{Perm}(n)$.

Exercise 1.2.0.7. We have that

1. $\text{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$
2. $\text{Perm}(n)$ is a subgroup of $O(n)$

Proof.

1. By definition, $\text{Perm}(n) = \text{Im } \phi$. Since $\phi : S_n \rightarrow GL(n, \mathbb{R})$ is a group homomorphism, $\text{Im } \phi$ is a subgroup of $GL(n, \mathbb{R})$. Hence $\text{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$.
2. Let $P \in \text{Perm}(n)$. Then there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. Then

$$\begin{aligned}PP^* &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix}^* \\ &= \begin{pmatrix} e_{\sigma(1)}^* \\ \vdots \\ e_{\sigma(n)}^* \end{pmatrix} (e_{\sigma(1)} \quad \cdots \quad e_{\sigma(n)}) \\ &= (\langle e_{\sigma(i)}, e_{\sigma(j)} \rangle)_{i,j} \\ &= I\end{aligned}$$

A previous exercise implies that $P^*P = I$. Hence $P \in O(n)$. Since $P \in \text{Perm}(n)$ is arbitrary, $\text{Perm}(n) \subset O(n)$. Part (1) implies that $\text{Perm}(n)$ is a group. Hence $\text{Perm}(n)$ is a subgroup of $O(n)$ □

Note 1.2.0.8. We will write P_σ in place of $\phi(\sigma)$.

Exercise 1.2.0.9. Let $Z \in \mathbb{R}^{p \times n}$. If $\text{rank } Z = k$, then there exist $\sigma \in S_n$, $\tau \in S_p$ and $A \in GL(k, \mathbb{R})$, such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Proof. Suppose that $\text{rank } Z = k$. Then there exist $i_1, \dots, i_k \in \{1, \dots, p\}$ such that $i_1 < \dots < i_k$ and $\{e_{i_1}^* Z, \dots, e_{i_k}^* Z\}$ is linearly independent. Set

$$Z' = \begin{pmatrix} e_{i_1}^* Z \\ \vdots \\ e_{i_k}^* Z \end{pmatrix}$$

Then $\text{rank } Z' = k$. Hence there exist $j_1, \dots, j_k \in \{1, \dots, n\}$ such that $j_1 < \dots < j_k$, and $\{Z'e_{i_1}, \dots, Z'e_{i_k}\}$ is linearly independent. Set

$$A = (Z'e_{i_1} \quad \dots \quad Z'e_{i_k})$$

Then $A \in \mathbb{R}^{k \times k}$ and $\text{rank } A = k$. Thus $A \in GL(k, \mathbb{R})$. Choose $\sigma \in S_n$ and $\tau \in S_p$ such that $\sigma(1) = j_1, \dots, \sigma(k) = j_k$ and $\tau(1) = i_1, \dots, \tau(k) = i_k$. Let $a, b \in \{1, \dots, k\}$. By construction,

$$\begin{aligned} (P_\tau Z P_\sigma^*)_{a,b} &= Z_{\tau(a), \sigma(b)} \\ &= Z_{i_a, j_b} \\ &= A_{a,b} \end{aligned}$$

□

Definition 1.2.0.10. Let $A \in \mathbb{R}^{n \times p}$. Then A is said to be a **diagonal matrix** if for each $i \in [n]$ and $j \in [p]$, $i \neq j$ implies that $A_{i,j} = 0$. We denote the set of $n \times p$ diagonal matrices by $D(n, p, \mathbb{R})$. We write $D(n, \mathbb{R})$ in place of $D(n, n, \mathbb{R})$.

Definition 1.2.0.11. For $(n, k), (m, l)$ $\text{diag}_{p, (n \times p)} : \mathbb{R}^p \rightarrow \mathbb{R}^{n \times p}$ and $\text{diag}_{n, (n \times p)} : \mathbb{R}^p \rightarrow \mathbb{R}^{n \times p}$ by $\text{diag}(v)$
FINISH!!!

Definition 1.2.0.12. Let $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \sigma(A)$. Suppose that A is symmetric. We define the **geometric multiplicity** of λ , denoted $\mu(\lambda)$, by

$$\mu(\lambda) = \dim \ker([\phi_\alpha] - \lambda I)$$

Definition 1.2.0.13. Let V be an n -dimensional vector space, $U \subset V$ a k -dimensional subspace and $(e_j)_{j=1}^n \subset V$ a basis. Then $(e_j)_{j=1}^n$ is said to be **adapted to** U if $(e_j)_{j=1}^k$ is a basis for U .

1.3 Calculus

1.3.1 Differentiation

Definition 1.3.1.1. Let $n \geq 1$. For $i = 1, \dots, n$, define $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$ by $x^i(a^1, \dots, a^n) = a^i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on \mathbb{R}^n** .

Definition 1.3.1.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to x^i at a** if

$$\lim_{h \rightarrow 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If f is differentiable with respect to x^i at a , we define the **partial derivative of f with respect to x^i at a** , denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \frac{\partial}{\partial x^i} f$$

to be the limit above.

Definition 1.3.1.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$. Then f is said to be **differentiable with respect to x^i** if for each $a \in U$, f is differentiable with respect to x^i at a .

Exercise 1.3.1.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i \partial x^j}$ and $\frac{\partial^2 f}{\partial x^j \partial x^i}$ exist and are continuous at a . Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(a) = \frac{\partial^2 f}{\partial x^j \partial x^i}(a)$$

Proof. □

Definition 1.3.1.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \dots, i_k \in \{1, \dots, n\}$, $\frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_k}}$ exists and is continuous on U .

Definition 1.3.1.6. Let $U \subset \mathbb{R}^n$, $f : U \rightarrow \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \rightarrow \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Theorem 1.3.1.7. Taylor's Theorem:

Let $U \subset \mathbb{R}^n$ be open and convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that for each $x \in U$,

$$f(x) = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each $|\alpha| = T+1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. See analysis notes □

Definition 1.3.1.8. Let $U \subset \mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^m$. Let x^1, \dots, x^n be the standard coordinate functions on \mathbb{R}^n and y_1, \dots, y_m be the standard coordinate functions on \mathbb{R}^m . For $i \in \{1, \dots, m\}$, we define the **i th component of F** , denoted $F^i : U \rightarrow \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \dots, F_m)$

Definition 1.3.1.9. Let $U \subset \mathbb{R}^n$ be open and $F : U \rightarrow \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \dots, m\}$, the i th component of F , $F^i : U \rightarrow \mathbb{R}$, is smooth.

Definition 1.3.1.10. Let $U \subset \mathbb{R}^n$ and $F : U \rightarrow \mathbb{R}^m$. Then F is said to be **smooth** if for each $x \in U$, there exists $U_x \in \mathcal{N}_x$ and $\tilde{F} : U_x \rightarrow \mathbb{R}^m$ such that U_x is open, \tilde{F} is smooth and $\tilde{F}|_{U \cap U_x} = F|_{U \cap U_x}$.

Definition 1.3.1.11. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \rightarrow V$. Then F is said to be a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 1.3.1.12. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \rightarrow V$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F is a bijection and F and F^{-1} are smooth. Thus, F and F^{-1} are continuous and F is a homeomorphism. \square

Definition 1.3.1.13. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \rightarrow \mathbb{R}^m$. We define the **Jacobian of F at p** , denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p) \right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Exercise 1.3.1.14. Inverse Function Theorem:

Let $U, V \subset \mathbb{R}^n$ be open and $F : U \rightarrow V$.

Exercise 1.3.1.15. Let $U, V \subset \mathbb{R}^n$ and $F : U \rightarrow V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \rightarrow F(N)$ is a diffeomorphism

Proof. content... \square

Exercise 1.3.1.16. Let $\sigma \in S_n$. Define $\phi_\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\phi(x^1, \dots, x^n) = \phi(x^{\sigma(1)}, \dots, x^{\sigma(n)})$. Then $D\phi = P_\sigma$

Definition 1.3.1.17. Let $\sigma \in S_n$ and $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. We define $\sigma x \in \mathbb{R}^n$ by

$$\sigma x = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of S_n on \mathbb{R}^n to be the map $S_n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $(\sigma, x) \mapsto \sigma x$

Definition 1.3.1.18. Let $\sigma \in S_n$, U a set, $V \subset \mathbb{R}^n$ and $\phi : U \rightarrow \mathbb{R}^n$ with $\phi = (x^1, \dots, x^n)$. We define $\sigma\phi : U \rightarrow \mathbb{R}^n$ by

$$\sigma\phi = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of S_n on $(\mathbb{R}^n)^U$ to be the map $S_n \times (\mathbb{R}^n)^U \rightarrow (\mathbb{R}^n)^U$ given by $(\sigma, \phi) \mapsto \sigma\phi$.

Exercise 1.3.1.19. Let $\sigma \in S_m$. Then for each $p \in \mathbb{R}^n$, $D(\sigma \text{id}_{\mathbb{R}^n})(p) = P_\sigma$.

Proof. Note that since $\text{id}_{\mathbb{R}^n} = (\pi_1, \dots, \pi_n)$, we have that $\sigma \text{id}_{\mathbb{R}^n} = (\pi_{\sigma(1)}, \dots, \pi_{\sigma(n)})$. Let $p \in \mathbb{R}^n$. Then

$$\begin{aligned} D(\sigma \text{id}_{\mathbb{R}^n})(p) &= \left(\frac{\partial \pi_i \circ \sigma \text{id}_{\mathbb{R}^n}}{\partial x^j}(p) \right)_{i,j} \\ &= \left(\frac{\partial \pi_{\sigma(i)}}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma \left(\frac{\partial \pi_i}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma \left(\frac{\partial \pi_i \circ \text{id}_{\mathbb{R}^n}}{\partial x^j}(p) \right)_{i,j} \\ &= P_\sigma D \text{id}_{\mathbb{R}^n}(p) \\ &= P_\sigma I \\ &= P_\sigma \end{aligned}$$

\square

1.3.2 Integration

1.4 Topology

Definition 1.4.0.1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \rightarrow Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 1.4.0.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \rightarrow Y$. Then f is said to be a **homeomorphism** if f is a bijection and f, f^{-1} are continuous.

Definition 1.4.0.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \rightarrow Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 1.4.0.4. Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \not\cong \mathbb{R}^n$

Chapter 2

Multilinear Algebra

2.1 (r, s) -Tensors

Definition 2.1.0.1. Let V_1, \dots, V_k, W be vector spaces and $\alpha : \prod_{i=1}^n V_i \rightarrow W$. Then α is said to be **multilinear** if for each $i \in \{1, \dots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \dots, v_k \in V$,

$$\alpha(v_1, \dots, v_i + cv, \dots, v_k) = \alpha(v_1, \dots, v_i, \dots, v_k) + c\alpha(v_1, \dots, v, \dots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{ \alpha : \prod_{i=1}^n V_i \rightarrow W : \alpha \text{ is multilinear} \right\}$$

Note 2.1.0.2. For the remainder of this section we let V denote an n -dimensional vector space with basis $\{e^1, \dots, e^n\}$ with dual space V^* and dual basis $\{\epsilon_1, \dots, \epsilon_n\}$ defined by $\epsilon^i(e^j) = \delta_{i,j}$. We identify V with V^{**} by the isomorphism $V \rightarrow V^{**}$ defined by $v \mapsto \hat{v}$ where $\hat{v}(\alpha) = \alpha(v)$ for each $\alpha \in V^*$.

Definition 2.1.0.3. Let $\alpha : (V^*)^r \times V^s \rightarrow \mathbb{R}$. Then α is said to be an (r, s) -tensor on V if $\alpha \in L(\underbrace{V^*, \dots, V^*}_r, \underbrace{V, \dots, V}_s; \mathbb{R})$. The set of all (r, s) -tensors on V is denoted $T_s^r(V)$.

When $r = s = 0$, we set $T_s^r = \mathbb{R}$.

Exercise 2.1.0.4. We have that $T_s^r(V)$ is a vector space.

Proof. Clear. □

Exercise 2.1.0.5. Under the identification of V with V^{**} as noted above, we have that $V = T_0^1(V)$.

Proof. By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T_0^1(V) \end{aligned}$$

□

Definition 2.1.0.6. Let $\alpha \in T_{s_1}^{r_1}(V)$ and $\beta \in T_{s_2}^{r_2}(V)$. We define the **tensor product of α with β** , denoted $\alpha \otimes \beta \in T_{s_1+s_2}^{r_1+r_2}(V)$, by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$.

When $r_1 = s_1 = r_2 = s_2 = 0$ (so that $\alpha, \beta \in \mathbb{R}$), we set $\alpha \otimes \beta = \alpha\beta$.

Definition 2.1.0.7. We define the **tensor product**, denoted $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

Exercise 2.1.0.8. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is well defined.

Proof. Tedious but straightforward. □

Exercise 2.1.0.9. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is associative.

Proof. Let $\alpha \in T_{s_1}^{r_1}(V)$, $\beta \in T_{s_2}^{r_2}(V)$ and $\gamma \in T_{s_3}^{r_3}(V)$. Then for each $u^* \in (V^*)^{r_1}$, $v^* \in (V^*)^{r_2}$, $w^* \in (V^*)^{r_3}$, $u \in V^{s_1}$, $v \in V^{s_2}$, $w \in V^{s_3}$,

$$\begin{aligned} (\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v) \gamma(w^*, w) \\ &= [\alpha(u^*, u) \beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(u^*, u) [\beta(v^*, v) \gamma(w^*, w)] \\ &= \alpha(u^*, u) (\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w) \end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

Exercise 2.1.0.10. The tensor product $\otimes : T_{s_1}^{r_1}(V) \times T_{s_2}^{r_2}(V) \rightarrow T_{s_1+s_2}^{r_1+r_2}(V)$ is bilinear.

Proof.

1. Linearity in the first argument:

Let $\alpha, \beta \in T_{s_1}^{r_1}(V)$, $\gamma \in T_{s_2}^{r_2}(V)$, $\lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

$$\begin{aligned} [(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v) \gamma(w^*, w) \\ &= [\alpha(v^*, v) + \lambda\beta(v^*, v)] \gamma(w^*, w) \\ &= \alpha(v^*, v) \gamma(w^*, w) + \lambda\beta(v^*, v) \gamma(w^*, w) \\ &= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\ &= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w) \end{aligned}$$

So that

$$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

2. Linearity in the second argument:

Similar to (1).

□

Definition 2.1.0.11.

1. Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1, \dots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length k** . Recall that $\#\mathcal{I}_{\otimes k} = n^k$.
2. Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length k** . Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

Note 2.1.0.12. For the remainder of this section we will write \mathcal{I}_k in place of $\mathcal{I}_{\otimes k}$.

Definition 2.1.0.13. Let $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k$.

1. Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by

$$\epsilon^I = (\epsilon^{i_1}, \dots, \epsilon^{i_k})$$

and

$$e^I = (e^{i_1}, \dots, e^{i_k})$$

2. Define $e^{\otimes I} \in T_0^k(V)$ and $\epsilon^{\otimes I} \in T_k^0(V)$ by

$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$

and

$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

Exercise 2.1.0.14. Let $\alpha, \beta \in T_s^r(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v_1^*, \dots, v_r^* \in V^*$ and $v_1, \dots, v_s \in V$. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, write

$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$

and

$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$

Then

$$\begin{aligned} \alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \sum_{k_1, \dots, k_r=1}^n \sum_{l_1, \dots, l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \dots, e^{l_s}) \\ &= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s) \end{aligned}$$

So that $\alpha = \beta$. □

Exercise 2.1.0.15. Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K} \delta_{J,L}$.

Proof. Write $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$ and $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$. Then

$$\begin{aligned} e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K) \epsilon^{\otimes J}(e^L) \\ &= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r}) \epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\ &= \left[\prod_{m=1}^r e^{i_m}(\epsilon^{k_m}) \right] \left[\prod_{n=1}^s \epsilon^{j_n}(e^{l_n}) \right] \\ &= \left[\prod_{m=1}^r \delta_{i_m, k_m} \right] \left[\prod_{n=1}^s \delta_{j_n, l_n} \right] \\ &= \delta_{I,K} \delta_{J,L} \end{aligned}$$

□

Exercise 2.1.0.16. The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T_s^r(V)$ and $\dim T_s^r(V) = n^{r+s}$.

Proof. Let $(a_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a_J^I e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a_J^I = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T_s^r(V)$. For $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b_J^I = \beta(\epsilon^I, e^J)$. Define $\mu = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b_J^I e^{\otimes I} \otimes \epsilon^{\otimes J} \in T_s^r(V)$. Then for each $(I, J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b_J^I = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. □

2.2 Covariant k -Tensors

2.2.1 Symmetric and Alternating Covariant k -Tensors

Definition 2.2.1.1. Let $\alpha : V^k \rightarrow \mathbb{R}$. Then α is said to be a **covariant k -tensor on V** if $\alpha \in T_k^0(V)$. We denote the set of covariant k -tensors by $T_k(V)$.

Definition 2.2.1.2. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \rightarrow \mathbb{R}$ by

$$\sigma\alpha(v_1, \dots, v_k) = \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

We define the **permutation action** of S_k on $T_k(V)$ to be the map $S_k \times T_k(V) \rightarrow T_k(V)$ given by $(\sigma, \alpha) \mapsto \sigma\alpha$

Exercise 2.2.1.3. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

1. Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V)$, $\sigma\alpha \in T_k(V)$.
2. Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
3. Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \dots, v_k \in V$,

$$\begin{aligned} (\tau\sigma)\alpha(v_1, \dots, v_k) &= \alpha(v_{\tau\sigma(1)}, \dots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \dots, v_k) \end{aligned}$$

□

Exercise 2.2.1.4. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \rightarrow T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \dots, v_k \in V$. Then

$$\begin{aligned} \sigma(c\alpha + \beta)(v_1, \dots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \dots, v_k) + \sigma\beta(v_1, \dots, v_k) \end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$.

□

Definition 2.2.1.5. Let $\alpha \in T_k(V)$. Then α is said to be

- **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$
- **antisymmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \text{sgn}(\sigma)\alpha$
- **alternating** if for each $v_1, \dots, v_k \in V$, if there exists $i, j \in \{1, \dots, k\}$ such that $v_i = v_j$, then $\alpha(v_1, \dots, v_k) = 0$.

We denote the set of symmetric k -tensors on V by $\Sigma^k(V)$. We denote the set of alternating k -tensors on V by $\Lambda^k(V)$.

Exercise 2.2.1.6. Let $\alpha \in T_k(V)$. Then α is antisymmetric iff α is alternating.

Proof. Suppose that α is antisymmetric. Let $v_1, \dots, v_k \in V$. Suppose that there exists $i, j \in \{1, \dots, k\}$ such that $v_i = v_j$. Define $\sigma \in S_k$ by $\sigma = (i, j)$. Then

$$\begin{aligned} \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) &= \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ &= \sigma(\alpha)(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\ &= \text{sgn}(\sigma)\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\ &= -\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \end{aligned}$$

Therefore $2\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = 0$ which implies that $\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = 0$. Hence α is alternating.

Conversely, suppose that α is alternating. Let $i, j \in \{1, \dots, k\}$ and $v_1, \dots, v_k \in V$. Then

$$\begin{aligned} 0 &= \alpha(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) \\ &= \alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + \alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \end{aligned}$$

Since $i, j \in \{1, \dots, k\}$ and $v_1, \dots, v_k \in V$ are arbitrary, we have that for each $\tau \in S_k$, τ is a transposition implies that

$$\begin{aligned} \tau\alpha &= -\alpha \\ &= \text{sgn}(\tau)\alpha \end{aligned}$$

Let $n \in \mathbb{N}$. Suppose that for each $\tau_1, \dots, \tau_{n-1} \in S_k$ if for each $j \in \{1, \dots, n-1\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_{n-1})\alpha = \text{sgn}(\tau_1 \cdots \tau_{n-1})\alpha$. Let $\tau_1, \dots, \tau_n \in S_k$. Suppose that for each $j \in \{1, \dots, n\}$, τ_j is a transposition. Then

$$\begin{aligned} (\tau_1 \cdots \tau_n)\alpha &= (\tau_1 \cdots \tau_{n-1})(\tau_n\alpha) \\ &= (\tau_1 \cdots \tau_{n-1})(\text{sgn}(\tau_n)\alpha) \\ &= (\text{sgn}(\tau_n)(\tau_1 \cdots \tau_{n-1})\alpha) \\ &= (\text{sgn}(\tau_n) \text{sgn}(\tau_1 \cdots \tau_{n-1})\alpha) \\ &= \text{sgn}(\tau_1 \cdots \tau_n)\alpha \end{aligned}$$

By induction, for each $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in S_k$, if for each $j \in \{1, \dots, n\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_n)\alpha = \text{sgn}(\tau_1 \cdots \tau_n)\alpha$. Now let $\sigma \in S_k$. Then there exist $n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n \in S_k$ such that $\sigma = \tau_1 \cdots \tau_n$ and for each $j \in \{1, \dots, n\}$, τ_j is a transposition. Hence

$$\begin{aligned} \sigma\alpha &= (\tau_1 \cdots \tau_n)\alpha \\ &= \text{sgn}(\tau_1 \cdots \tau_n)\alpha \\ &= \text{sgn}(\sigma)\alpha \end{aligned}$$

Therefore α is antisymmetric. □

Definition 2.2.1.7. Define the **symmetric operator** $S : T_k(V) \rightarrow \Sigma^k(V)$ by

$$\text{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \rightarrow \Lambda^k(V)$ by

$$\text{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma\alpha$$

Exercise 2.2.1.8.

1. For $\alpha \in T_k(V)$, $\text{Sym}(\alpha)$ is symmetric.
2. For $\alpha \in T_k(V)$, $\text{Alt}(\alpha)$ is alternating.

Proof.

1. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned} \sigma \text{Sym}(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \\ &= \text{Sym}(\alpha) \end{aligned}$$

2. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
 \sigma \operatorname{Alt}(\alpha) &= \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \right] \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \sigma \tau \alpha \\
 &= \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\sigma \tau) \sigma \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \tau \alpha \\
 &= \operatorname{sgn}(\sigma) \operatorname{Alt}(\alpha)
 \end{aligned}$$

□

Exercise 2.2.1.9.

1. For $\alpha \in \Sigma^k(V)$, $\operatorname{Sym}(\alpha) = \alpha$.
2. For $\alpha \in \Lambda^k(V)$, $\operatorname{Alt}(\alpha) = \alpha$.

Proof.

1. Let $\alpha \in \Sigma^k(V)$. Then

$$\begin{aligned}
 \operatorname{Sym}(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \alpha \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} \alpha \\
 &= \alpha
 \end{aligned}$$

2. Let $\alpha \in \Lambda^k(V)$. Then

$$\begin{aligned}
 \operatorname{Alt}(\alpha) &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \sigma \alpha \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2 \alpha \\
 &= \alpha
 \end{aligned}$$

□

Exercise 2.2.1.10. The symmetric operator $S : T_k(V) \rightarrow \Sigma^k(V)$ and the alternating operator $A : T_k(V) \rightarrow \Lambda^k(V)$ are linear.

Proof. Clear.

□

Exercise 2.2.1.11. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

1. $\operatorname{Alt}(\operatorname{Alt}(\alpha) \otimes \beta) = \operatorname{Alt}(\alpha \otimes \beta)$
2. $\operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta)) = \operatorname{Alt}(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \rightarrow S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

1. Then

$$\begin{aligned}
\text{Alt}(\text{Alt}(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\text{Alt}(\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau \alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) (\tau \alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \text{sgn}(\tau) \tau (\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \text{sgn}(\sigma\tau) \sigma\tau (\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \text{sgn}(\mu) \mu (\alpha \otimes \beta) \\
&= \text{Alt}(\alpha \otimes \beta)
\end{aligned}$$

2. Similar to (1).

□

2.2.2 Exterior Product

Definition 2.2.2.1. Let $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$.

Exercise 2.2.2.2. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$ is bilinear.

Proof. Clear.

□

Exercise 2.2.2.3. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \rightarrow \Lambda^{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$ and $\gamma \in \Lambda^m(V)$. Then

$$\begin{aligned}
 (\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \right] \wedge \gamma \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \text{Alt} \left(\left[\frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \right] \otimes \gamma \right) \\
 &= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} \text{Alt}(\text{Alt}(\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} \text{Alt}((\alpha \otimes \beta) \otimes \gamma) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \text{Alt}(\alpha \otimes (\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \text{Alt}(\alpha \otimes \text{Alt}(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \text{Alt}(\alpha \otimes \frac{(l+m)!}{l!m!} \text{Alt}(\beta \otimes \gamma)) \\
 &= \frac{(k+l+m)!}{k!(l+m)!} \text{Alt}(\alpha \otimes (\beta \wedge \gamma)) \\
 &= \alpha \wedge (\beta \wedge \gamma)
 \end{aligned}$$

□

Exercise 2.2.2.4. Let $\alpha_i \in \Lambda^{k_i}(V)$ for $i = 1, \dots, m$. Then

$$\bigwedge_{i=1}^m \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} \text{Alt} \left(\bigotimes_{i=1}^m \alpha_i \right)$$

Proof. To see that the statement is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} \text{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
 \bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i \right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
 &= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \text{Alt} \left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \text{Alt} \left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} \text{Alt} \left(\bigotimes_{i=1}^{m_0-1} \alpha_i \right) \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left(\text{Alt} \left[\bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i \right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1} \right) \\
 &= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \text{Alt} \left(\bigotimes_{i=1}^{m_0+1} \alpha_i \right)
 \end{aligned}$$

□

Exercise 2.2.2.5. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl . (Hint: inversion number)

Proof.

$$\begin{aligned} N(\tau) &= \sum_{i=1}^l k \\ &= kl \end{aligned}$$

Since $\text{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\text{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.2.2.6. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \dots, v_{k+l} \in V$, we have that

$$\begin{aligned} \sigma\tau(\beta \otimes \alpha)(v_1, \dots, v_l, v_{l+1}, \dots, v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)}, \dots, v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)})\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\beta(v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\ &= \alpha \otimes \beta(v_{\sigma(1)}, \dots, v_{\sigma(k)}, v_{\sigma(1+k)}, \dots, v_{\sigma(l+k)}) \\ &= \sigma(\alpha \otimes \beta)(v_1, \dots, v_k, v_{1+k}, \dots, v_{l+k}) \end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned} \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} \text{Alt}(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau) \sigma\tau(\beta \otimes \alpha) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha \otimes \beta) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \\ &= \text{sgn}(\tau) \alpha \wedge \beta \\ &= (-1)^{kl} \alpha \wedge \beta \end{aligned}$$

□

Exercise 2.2.2.7. Let $\alpha \in \Lambda^k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned} \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\ &= -\alpha \wedge \alpha \end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.2.2.8. Fundamental Example:

Let $\alpha_1, \dots, \alpha_m \in \Lambda^1(V)$ and $v_1, \dots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned} \left(\bigwedge_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) &= m! \operatorname{Alt} \left(\bigotimes_{i=1}^m \alpha_i \right) (v_1, \dots, v_m) \\ &= m! \left[\frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \sigma \left(\bigotimes_{i=1}^m \alpha_i \right) \right] (v_1, \dots, v_m) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \left(\bigotimes_{i=1}^m \alpha_i \right) (v_{\sigma(1)}, \dots, v_{\sigma(m)}) \\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\ &= \det(\alpha_i(v_j)) \end{aligned}$$

□

Note 2.2.2.9. Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \dots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \dots < i_k \leq n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write \mathcal{I}_k in place of $\mathcal{I}_{\wedge k}$.

Definition 2.2.2.10. Let $I = \{(i_1, i_2, \dots, i_k) \in \mathcal{I}_k\}$. Define $\epsilon^{\wedge I} \in \Lambda^k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \dots \wedge \epsilon^{i_k}$$

Exercise 2.2.2.11. Let $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \dots & \epsilon^{i_1}(e^{j_k}) \\ \vdots & & \vdots \\ \epsilon^{i_k}(e^{j_1}) & \dots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$. If $I = J$, then

$A = I_{k \times k}$ and therefore $\epsilon^{\wedge I}(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \leq l \leq k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the l_0 -th row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the l_0 -th column of A are 0. □

Exercise 2.2.2.12. Let $\alpha, \beta \in \Lambda^k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \dots, v_k \in V$. For $i = 1, \dots, k$, write $v_i =$

$\sum_{j_i=1}^n a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned}
 \alpha(v_1, \dots, v_k) &= \sum_{j_1, \dots, j_k=1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\
 &= \sum_{j_1 \neq \dots \neq j_k}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \alpha(e^{j_1}, \dots, e^{j_k}) \\
 &= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \text{sgn}(\sigma) \left(\prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \alpha(e^J) \\
 &= \sum_{J \in \mathcal{I}_k} \left[\sum_{\sigma \in S_J} \text{sgn}(\sigma) \left(\prod_{i=1}^k a_{i, \sigma(j_i)} \right) \right] \beta(e^J) \\
 &= \sum_{j_1, \dots, j_k=1}^n \left(\prod_{i=1}^k a_{i,j_i} \right) \beta(e^{j_1}, \dots, e^{j_k}) \\
 &= \beta(v_1, \dots, v_k)
 \end{aligned}$$

□

Exercise 2.2.2.13. The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(V)$ and $\dim \Lambda^k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I \in \mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum_{I \in \mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$.

Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda^k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum_{I \in \mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda^k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \text{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$.

□

2.2.3 Interior Product

Definition 2.2.3.1. Let V be a finite dimensional vector space and $v \in V$. We define **interior multiplication by v** , denoted $\iota_v : T_k \rightarrow T_{k-1}$, by

$$\iota_v \alpha(w_1, \dots, w_{k-1}) = \alpha(v, w_1, \dots, w_{k-1})$$

Exercise 2.2.3.2. Let V be a finite dimensional vector space and $v \in V$. Then $\iota_v|_{\Lambda^k(V)} : \Lambda^k(V) \rightarrow \Lambda^{k-1}(V)$.

Proof. Let $\alpha \in \Lambda^k(V)$. Define $\beta \in \Lambda^k(V)$ by $\beta(w_1, \dots, w_k) = \alpha(w_k, w_1, \dots, w_{k-1})$. Let $\sigma \in S_{k-1}$. Define $\tau \in S_k$ by $\tau(j) = \begin{cases} 1 & j = k \\ \sigma(j) & j \neq k \end{cases}$. Let $w_1, \dots, w_{k-1} \in V$. Set $w_k = v$. Then

$$\begin{aligned}
 \sigma(\iota_v \alpha)(w_1, \dots, w_{k-1}) &= \iota_v \alpha(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}) \\
 &= \alpha(v, w_{\sigma(1)}, \dots, w_{\sigma(k-1)}) \\
 &= \beta(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}, v) \\
 &= \beta(w_{\sigma(1)}, \dots, w_{\sigma(k-1)}, w_k) \\
 &= \beta(w_{\tau(1)}, \dots, w_{\tau(k-1)}, w_{\tau(k)}) \\
 &= \text{sgn}(\tau) \beta(w_1, \dots, w_{k-1}, w_k) \\
 &= \text{sgn}(\sigma) \beta(w_1, \dots, w_{k-1}, v) \\
 &= \text{sgn}(\sigma) \alpha(v, w_1, \dots, w_{k-1}) \\
 &= \text{sgn}(\sigma) (\iota_v \alpha)(w_1, \dots, w_{k-1})
 \end{aligned}$$

Since $w_1, \dots, w_{k-1} \in V$ are arbitrary, $\sigma(\iota_v \alpha) = \text{sgn}(\sigma) \iota_v \alpha$. Hence $\iota_v \alpha \in \Lambda^{k-1}(V)$.

□

2.3 $(0, 2)$ -Tensors

Definition 2.3.0.1. Let V be a finite dimensional vector space, $v \in V$ and $\alpha \in T_2^0(V)$. Then α is said to be **degenerate** if there exists $v \in V$ such that for each $w \in V$, $\alpha(v, w) = 0$ and $v \neq 0$.

Definition 2.3.0.2. Let V be a finite dimensional vector space, $\alpha \in T_2^0(V)$. We define $\phi_\alpha : V \rightarrow V^*$ by

$$\phi_\alpha(v) = \iota_v \alpha$$

Exercise 2.3.0.3. Let V be a finite dimensional vector space, $\alpha \in T_2^0(V)$. Then $\phi_\alpha \in L(V; V^*)$.

Proof. Let $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$. Then for each $w \in V$,

$$\begin{aligned} \phi_\alpha(v_1 + \lambda v_2)(w) &= (\iota_{v_1 + \lambda v_2} \alpha)(w) \\ &= \alpha(v_1 + \lambda v_2, w) \\ &= \alpha(v_1, w) + \lambda \alpha(v_2, w) \\ &= (\iota_{v_1} \alpha)(w) + \lambda (\iota_{v_2} \alpha)(w) \\ &= \phi_\alpha(v_1)(w) + \lambda \phi_\alpha(v_2)(w) \\ &= [\phi_\alpha(v_1) + \lambda \phi_\alpha(v_2)](w) \end{aligned}$$

Therefore, $\phi_\alpha(v_1 + \lambda v_2) = \phi_\alpha(v_1) + \lambda \phi_\alpha(v_2)$. Thus $\phi_\alpha \in L(V; V^*)$. \square

Exercise 2.3.0.4. Let V be a finite dimensional vector space and $\alpha \in T_2^0(V)$. Then α is nondegenerate iff ϕ_α is an isomorphism.

Proof.

- (\implies :)

Suppose that α is nondegenerate. Let $v \in \ker \phi_\alpha$. Then for each $w \in V$,

$$\begin{aligned} \alpha(v, w) &= (\iota_v \alpha)(w) \\ &= \phi_\alpha(v)(w) \\ &= 0 \end{aligned}$$

Since α is nondegenerate, $v = 0$. Since $v \in \ker \phi_\alpha$ is arbitrary, $\ker \phi_\alpha = \{0\}$. Hence ϕ_α is injective. Since $\dim V = \dim V^*$, ϕ_α is surjective. Hence ϕ_α is an isomorphism.

- (\impliedby :)

Suppose that ϕ_α is an isomorphism. Let $v \in V$. Suppose that for each $w \in V$, $\alpha(v, w) = 0$. Then for each $w \in V$,

$$\begin{aligned} \phi_\alpha(v)(w) &= (\iota_v \alpha)(w) \\ &= \alpha(v, w) \\ &= 0 \end{aligned}$$

Thus $\phi_\alpha(v) = 0$ which implies that $v \in \ker \phi_\alpha$. Since ϕ_α is an isomorphism, $v = 0$. Hence α is nondegenerate. \square

Exercise 2.3.0.5. Let V be a finite dimensional vector space and $\alpha \in T_2^0(V)$. Then

1. $[\phi_\alpha]_{i,j} = \alpha(e_j, e_i)$

2. for each $v, w \in V$,

$$\alpha(v, w) = [w]^* [\phi_\alpha] [v]$$

Proof. 1. Set $A = [\phi_\alpha]$. Let $i, j \in \{1, \dots, n\}$. By definition,

$$\phi_\alpha(e_j) = \sum_{k=1}^n A_{k,j} \epsilon^k$$

Then

$$\begin{aligned} \phi_\alpha(e_j)(e_i) &= \sum_{k=1}^n A_{k,j} \epsilon^k(e_i) \\ &= \sum_{k=1}^n A_{k,j} \delta_{k,i} \\ &= A_{i,j} \end{aligned}$$

2. Let $v, w \in V$. Then there exist $(v^i)_{i=1}^n, (w^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{i=1}^n v^i e_i$ and $w = \sum_{j=1}^n w^j e_j$. Part (1) implies that

$$\begin{aligned} \alpha(v, w) &= \sum_{i=1}^n \sum_{j=1}^n v^i w^j \alpha(e_i, e_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n v^i w^j [\phi_\alpha]_{j,i} \\ &= \sum_{i=1}^n \sum_{j=1}^n [v]_i [w]_j [\phi_\alpha]_{j,i} \\ &= [w]^* [\phi_\alpha] [v] \end{aligned}$$

□

2.3.1 Scalar Product Spaces

Definition 2.3.1.1. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be

- **positive semidefinite** if for each $v \in V$, $\alpha(v, v) \geq 0$
- **positive definite** if for each $v \in V$, $v \neq 0$ implies that $\alpha(v, v) > 0$
- **negative semidefinite** if $-\alpha$ is positive semidefinite
- **negative definite** if $-\alpha$ is positive definite

Exercise 2.3.1.2. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then

1. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda > 0$
2. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda \geq 0$

Proof.

1. Suppose that α is positive definite. Write $\sigma(\phi_\alpha) = \{\lambda_1, \dots, \lambda_n\}$. Define $\Lambda \in \mathbb{R}^{n \times n}$ by $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. Since α is symmetric, $[\phi_\alpha]$ is symmetric. There exists $U \in O(n)$ such that $[\phi_\alpha] = U \Lambda U^*$. **FINISH!!!**

□

Definition 2.3.1.3. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be a **scalar product** if α is nondegenerate. In this case, (V, α) is said to be a **scalar product space**.

Definition 2.3.1.4. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$ a scalar product on V . We define the **index** of α , denoted $\text{ind } \alpha$ by

$$\text{ind } \alpha = \max\{\dim W : W \text{ is a subspace of } V \text{ and } \alpha|_{W \times W} \text{ is negative definite}\}$$

Definition 2.3.1.5. Let (V, α) be a scalar product space.

- Let $v_1, v_2 \in V$. Then v_1 and v_2 are said to be **orthogonal** if $\alpha(v_1, v_2) = 0$.
- Let $U \subset V$ be a subspace. We define the **orthogonal subspace of U** , denoted by U^\perp , by

$$U^\perp = \{v \in V : \text{for each } u \in U, \alpha(u, v) = 0\}$$

Exercise 2.3.1.6. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then U^\perp is a subspace of V .

Proof. We note that since $U^\perp = \bigcap_{u \in U} \ker \phi_\alpha(u)$, U^\perp is a subspace of V . □

Exercise 2.3.1.7. Let (V, α) be an n -dimensional scalar product space, $U \subset V$ a k -dimensional subspace and $(e_j)_{j=1}^n \subset V$ a basis for V . Suppose that $(e_j)_{j=1}^k$ is a basis for U . Then for each $v \in V$, $v \in U^\perp$ iff for each $j \in [k]$, $\alpha(v, e_j) = 0$.

Proof. Let $v \in V$.

- (\implies): Suppose that $v \in U^\perp$. Since $(e_j)_{j=1}^k \subset U$, we have that for each $j \in [k]$, $\alpha(v, e_j) = 0$.
- (\impliedby): Suppose that for each $j \in [k]$, $\alpha(v, e_j) = 0$. Let $u \in U$. Then there exist $(a^j)_{j=1}^k \subset \mathbb{R}$ such that $u = \sum_{j=1}^k a^j u_j$. This implies that

$$\begin{aligned} \alpha(v, u) &= \sum_{j=1}^k a^j \alpha(v, u_j) \\ &= 0 \end{aligned}$$

Since $u \in U$ is arbitrary, we have that $v \in U^\perp$. □

Exercise 2.3.1.8. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then

1. $\dim V = \dim U + \dim U^\perp$
2. $(U^\perp)^\perp = U$

Proof. 1. Set $n = \dim V$ and $k = \dim U$. Choose a basis $(e_j)_{j=1}^n$ such that $(e_j)_{j=1}^k$ is a basis for U .

2. □

Exercise 2.3.1.9. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Set $\sigma([\phi_\alpha])^- = \{\lambda \in \sigma([\phi_\alpha]) : \lambda < 0\}$. Then

$$\text{ind } \alpha = \sum_{\lambda \in \sigma([\phi_\alpha])^-} \mu(\lambda)$$

Proof. Since α is symmetric, there exist $U \in O(n)$ and $\Lambda \in D(n, \mathbb{R})$ such that $[\phi_\alpha] = U\Lambda U^*$. Define $(u_j)_{j=1}^n \subset V$ by $u_j = \sum_{i=1}^n U_{i,j} e_i$. Define $J^- = \{j \in [n] : \Lambda_{j,j} < 0\}$, $n^- = \#J^-$ and $V^- = \text{span}\{u_j : j \in J^-\}$. Let $v \in V^-$. Then there exist $(a^j)_{j \in J^-}$ such that $v = \sum_{j \in J^-} a^j u_j$. We note that

$$\begin{aligned} U^*[\phi_\alpha]U &= U^*(U\Lambda U^*)U \\ &= (U^*U)\Lambda(U^*U) \\ &= I\Lambda I \\ &= \Lambda \end{aligned}$$

A previous exercise implies that

$$\begin{aligned} \alpha(v, v) &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k \alpha(u_j, u_k) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k [u_j]^* [\phi_\alpha] [u_k] \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k ([e_j]^* U^*) [\phi_\alpha] (U[e_k]) \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (U^*[\phi_\alpha]U)_{j,k} \\ &= \sum_{j \in J^-} \sum_{k \in J^-} a^j a^k (\Lambda)_{j,k} \\ &= \sum_{j \in J^-} |a^j|^2 \Lambda_{j,j} \\ &< 0 \end{aligned}$$

Since $v \in V^-$ is arbitrary, $\alpha|_{V^- \times V^-}$ is negative definite. Thus

$$\begin{aligned} \text{ind } \alpha &\geq \dim V^- \\ &= n^- \end{aligned}$$

Set $J^+ = (J^-)^c$. Let $W \subset V$ be a subspace. Suppose that $\alpha|_{W \times W}$ is negative definite. For the sake of contradiction, suppose that there exists $j_0 \in J^+$ such that $u_{j_0} \in W$. Then

$$\begin{aligned} \alpha(u_{j_0}, u_{j_0}) &= [u_{j_0}]^* [\phi_\alpha] [u_{j_0}] \\ &= [u_{j_0}]^* U\Lambda U^* [u_{j_0}] \\ &= \Lambda_{j_0, j_0} \\ &\geq 0 \end{aligned}$$

which is a contradiction since $\alpha|_{W \times W}$ is negative definite. Thus for each $j \in J^+$, $u_j \notin W$. □

2.3.2 Symplectic Vector Spaces

Definition 2.3.2.1. Let V be a finite dimensional vector space and $\omega \in \Lambda^2(V)$. Then ω is said to be a **symplectic form** if ω is nondegenerate. In this case (V, ω) is said to be a **symplectic space**.

Exercise 2.3.2.2. Let V be a $2n$ -dimensional vector space with basis $(a_j, b_j)_{j=1}^n$ and corresponding dual basis $(\alpha^j, \beta^j)_{j=1}^n$. Define $\omega \in \Lambda^2(V)$ by

$$\omega = \sum_{j=1}^n \alpha^j \wedge \beta^j$$

Then

1. for each $j, k \in \{1, \dots, n\}$,

(a) $\omega(a_j, a_k) = 0$

(b) $\omega(b_j, b_k) = 0$

(c) $\omega(a_j, b_k) = \delta_{j,k}$

2. (V, ω) is a symplectic space

Proof.

1. Let $j, k \in \{1, \dots, n\}$.

(a)

$$\begin{aligned}\omega(a_j, a_k) &= \sum_{l=1}^n \alpha^l \wedge \beta^l(a_j, a_k) \\ &= \sum_{l=1}^n [\alpha^l(a_j)\beta^l(a_k) - \alpha^l(a_k)\beta^l(a_j)] \\ &= 0\end{aligned}$$

(b) Similar to (a)

(c)

$$\begin{aligned}\omega(a_j, b_k) &= \sum_{l=1}^n \alpha^l \wedge \beta^l(a_j, b_k) \\ &= \sum_{l=1}^n [\alpha^l(a_j)\beta^l(b_k) - \alpha^l(b_k)\beta^l(a_j)] \\ &= \sum_{l=1}^n \alpha^l(a_j)\beta^l(b_k) \\ &= \sum_{l=1}^n \delta_{j,l}\delta_{l,k} \\ &= \delta_{j,k}\end{aligned}$$

2. Let $v \in V$. Then there exist $(q^j, p^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{j=1}^n q^j a_j + p^j b_j$. Suppose that for each $w \in V$, $\omega(v, w) = 0$. Let $k \in \{1, \dots, n\}$. Then

$$\begin{aligned}0 &= \omega(v, a_k) \\ &= \sum_{j=1}^n q^j \omega(a_j, a_k) + p^j \omega(b_j, a_k) \\ &= \sum_{j=1}^n p^j \delta_{j,k} \\ &= p^k\end{aligned}$$

Similarly,

$$\begin{aligned}
 0 &= \omega(v, b_k) \\
 &= \sum_{j=1}^n q^j \omega(a_j, b_k) + p^j \omega(b_j, b_k) \\
 &= \sum_{j=1}^n q^j \delta_{j,k} \\
 &= q^k
 \end{aligned}$$

Since $k \in \{1, \dots, n\}$ is arbitrary, $v = 0$. Hence ω is nondegenerate. Therefore (V, ω) is symplectic. \square

Exercise 2.3.2.3. Let (V, ω) be a symplectic space. Then $\dim V$ is even.

Proof. Set $n = \dim V$. Let $(e_j)_{j=1}^n$ be a basis for V . Define $[\omega] \in \mathbb{R}^{n \times n}$ by $[\omega]_{i,j} = \omega(e_i, e_j)$. Since $\omega \in \Lambda^2(V)$, $[\omega]^* = -[\omega]$. Therefore

$$\begin{aligned}
 \det[\omega] &= \det[\omega]^* \\
 &= \det(-[\omega]) \\
 &= (-1)^n \det[\omega]
 \end{aligned}$$

For the sake of contradiction, suppose that n is odd. Then $\det[\omega] = -\det[\omega]$ which implies that $\det[\omega] = 0$. Since ω is nondegenerate, $[\omega] \in GL(n, \mathbb{R})$. This is a contradiction. Hence n is even. \square

Definition 2.3.2.4. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. We define the **symplectic complement of V** , denoted S^\perp , by

$$S^\perp = \{v \in V : \text{for each } w \in S, \omega(v, w) = 0\}$$

Exercise 2.3.2.5. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then S^\perp is a subspace.

Proof. We note that

$$S^\perp = \bigcap_{v \in S} \ker \iota_v \omega$$

Hence S^\perp is a subspace. \square

Exercise 2.3.2.6. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then

$$\dim V = \dim S + \dim S^\perp$$

Proof. \square

Exercise 2.3.2.7. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then $(S^\perp)^\perp = S$.

Proof. Let $v \in (S^\perp)^\perp$. Then for each $w \in S^\perp$, $\omega(v, w) = 0$. \square

Chapter 3

Smooth Manifolds

3.1 Topological Manifolds

Exercise 3.1.0.1. We have that \mathbb{R} is homeomorphic to $(0, \infty)$

Proof. Define $f : \mathbb{R} \rightarrow (0, \infty)$ by $f(x) = e^x$. Then f is a homeomorphism. □

Definition 3.1.0.2. Let $n \in \mathbb{N}$. We define the **upper half space** of \mathbb{R}^n , denoted \mathbb{H}^n , by

$$\mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and we define

$$\partial\mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}$$

$$\text{Int } \mathbb{H}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

We endow \mathbb{H}^n , $\partial\mathbb{H}^n$ and $\text{Int } \mathbb{H}^n$ with the subspace topology inherited from \mathbb{R}^n .

We define the projection map $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ by

$$\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$$

Definition 3.1.0.3. We define $\mathbb{R}^0 = \{0\}$ and $\mathbb{H}^0 = \emptyset$ endowed with the discrete topology.

Exercise 3.1.0.4. Let $n \in \mathbb{N}$.

1. $\partial\mathbb{H}^n$ is homeomorphic to \mathbb{R}^{n-1}
2. $\text{Int } \mathbb{H}^n$ is homeomorphic to \mathbb{R}^n

Proof.

1. Let $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ be the projection map given by

$$\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$$

Then π is a homeomorphism.

2. Define $f : \mathbb{R}^n \rightarrow \text{Int } \mathbb{H}^n$ by $f(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, e^{x_n})$. Then f is a homeomorphism. □

Definition 3.1.0.5. Let M be a topological space and $n \in \mathbb{N}_0$. Let $U \subset M$ and $V \subset \mathbb{H}^n$ and $\phi : U \rightarrow V$. Then (U, ϕ) is said to be a **n -coordinate chart on M** if

- U is open in M
- V is open in \mathbb{H}^n

- ϕ is a homeomorphism

We denote the set of all n -coordinate charts on M by $X^n(M)$.

Definition 3.1.0.6. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be **locally Euclidean of dimension n** if for each $p \in M$, there exists $(U, \phi) \in X^n(M)$ such that $p \in U$.

Definition 3.1.0.7. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be an **n -dimensional topological manifold** if

1. M is Hausdorff
2. M is second-countable
3. M is locally Euclidean of dimension n

Theorem 3.1.0.8. Topological Invariance of Dimension:

Let M be an n -dimensional topological manifold and N a p -dimensional topological manifold. If M and N are homeomorphic, then $n = p$.

Note 3.1.0.9. In light of the previous theorem, we write $X(M)$ in place of $X^n(M)$ and refer to n -coordinate charts as coordinate charts when the context is clear.

Definition 3.1.0.10. Let M be an n -dimensional topological manifold and $(U, \phi) \in X(M)$. Then (U, ϕ) is said to be an

- **interior chart** if $\phi(U)$ is open in \mathbb{R}^n
- **boundary chart** if $\phi(U)$ is open in \mathbb{H}^n and $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$

We denote the set of all interior charts on M and the set of all boundary charts on M by $X_{\text{Int}}(M)$ and $X_{\partial}(M)$ respectively.

Exercise 3.1.0.11. Let M be an n -dimensional topological manifold. Then

1. $X(M) = X_{\text{Int}}(M) \cup X_{\partial}(M)$
2. $X_{\text{Int}}(M) \cap X_{\partial}(M) = \emptyset$

Proof.

1. By definition, $X_{\text{Int}}(M) \cup X_{\partial}(M) \subset X(M)$. Let $(U, \phi) \in X(M)$. Since (U, ϕ) is a coordinate chart on M , $\phi(U)$ is open in \mathbb{R}^n or $\phi(U)$ is open in \mathbb{H}^n . If $\phi(U)$ is open in \mathbb{R}^n , then

$$\begin{aligned} (U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

Suppose that $\phi(U)$ is open in \mathbb{H}^n . If $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$, then $\phi(U)$ is open in \mathbb{R}^n and

$$\begin{aligned} (U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

Suppose that $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$. Then

$$\begin{aligned} (U, \phi) &\in X_{\partial}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M) \end{aligned}$$

Since $(U, \phi) \in X(M)$ is arbitrary, $X(M) \subset X_{\text{Int}}(M) \cup X_{\partial}(M)$. Therefore $X(M) = X_{\text{Int}}(M) \cup X_{\partial}(M)$.

2. For the sake of contradiction, suppose that $X_{\text{Int}}(M) \cup X_{\partial}(M) \neq \emptyset$. Then there exists $(U, \phi) \in X(M)$ such that $(U, \phi) \in X_{\text{Int}}(M)$ and $(U, \phi) \in X_{\partial}(M)$. Therefore $\phi(U)$ is open in \mathbb{R}^n , $\phi(U)$ is open in \mathbb{H}^n and $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$. Since $\phi(U)$ is open in \mathbb{R}^n and $\phi(U) \subset \mathbb{H}^n$, $\phi(U) \subset \text{Int } \mathbb{H}^n$ and therefore $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$ which is a contradiction.

□

Definition 3.1.0.12. Let M be an n -dimensional topological manifold. We define the

- **interior** of M , denoted $\text{Int } M$, by

$$\text{Int } M = \{p \in M : \text{there exists } (U, \phi) \in X_{\text{Int}}(M) \text{ such that } p \in U\}$$

- **boundary** of M , denoted ∂M , by

$$\partial M = \{p \in M : \text{there exists } (V, \psi) \in X_{\partial}(M) \text{ such that } p \in V \text{ and } \psi(p) \in \partial \mathbb{H}^n\}$$

Exercise 3.1.0.13. Let M be an n -dimensional topological manifold. Let $(U, \phi) \in X_{\text{Int}}(M)$. Then $U \subset \text{Int } M$.

Proof. Let $p \in U$. Since $(U, \phi) \in X_{\text{Int}}(M)$ and $p \in U$, by definition, $p \in \text{Int } M$. Since $p \in U$ is arbitrary, $U \subset \text{Int } M$. □

Exercise 3.1.0.14. Let M be an n -dimensional topological manifold, $(U, \phi) \in X_{\partial}(M)$ and $p \in U$. If $\phi(p) \notin \partial \mathbb{H}^n$, then $p \in \text{Int } M$.

Proof. Suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \text{Int } \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in \mathbb{R}^n and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then U' is open in M and $\phi' : U' \rightarrow B'$ is a homeomorphism. Hence $(U', \phi') \in X_{\text{Int}}(M)$. Since $\phi(p) \in B'$, we have that $p \in U'$. By definition, $p \in \text{Int } M$. □

Exercise 3.1.0.15. Let M be an n -dimensional topological manifold. Then

$$1. M = \text{Int } M \cup \partial M$$

$$2. \text{Int } M \cap \partial M = \emptyset$$

Hint: simply connected

Proof.

1. By definition, $\text{Int } M \cup \partial M \subset M$. Let $p \in M$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. A previous exercise implies that $(U, \phi) \in X_{\text{Int}}(M) \cup X_{\partial}(M)$. If $(U, \phi) \in X_{\text{Int}}(M)$, then by definition,

$$\begin{aligned} p &\in \text{Int } M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Suppose that $(U, \phi) \in X_{\partial}(M)$. If $\phi(p) \in \partial \mathbb{H}^n$, then by definition,

$$\begin{aligned} p &\in \partial M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Suppose that $\phi(p) \notin \partial \mathbb{H}^n$. The previous exercise implies that $p \in \text{Int } M$. Therefore,

$$\begin{aligned} p &\in \text{Int } M \\ &\subset \text{Int } M \cup \partial M \end{aligned}$$

Since $p \in M$ is arbitrary, $M \subset \text{Int } M \cup \partial M$. Therefore $M = \text{Int } M \cup \partial M$.

2. For the sake of contradiction, suppose that $\text{Int } M \cap \partial M \neq \emptyset$. Then there exists $p \in M$ such that $p \in \text{Int } M \cap \partial M$. By definition, there exists $(U, \phi) \in X_{\text{Int}}(M)$, $(V, \psi) \in X_{\partial}(M)$ such that $p \in U \cap V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in \mathbb{H}^n , $\phi(U \cap V)$ is open in \mathbb{R}^n and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$ is a homeomorphism. Since $\psi(U \cap V)$ is open in \mathbb{H}^n , there exists an $B_{\psi} \subset \psi(U \cap V)$ such that B_{ψ} is open in \mathbb{H}^n , B_{ψ} is simply

connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$. Since $\phi(U \cap V)$ is open in \mathbb{R}^n , B_ϕ is open in \mathbb{R}^n . Since B_ψ is simply connected and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$ is a homeomorphism, B_ϕ is simply connected.

Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Then $\phi \circ \psi^{-1} : B'_\psi \rightarrow B'_\phi$ is a homeomorphism. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_ψ is simply connected. Since B_ϕ is open in \mathbb{R}^n , B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ . So $\partial M \cap \text{Int } M = \emptyset$.

□

Exercise 3.1.0.16. Let M be an n -dimensional topological manifold. Then

1. $\text{Int } M$ is open
2. ∂M is closed

Proof.

1. Let $p \in \text{Int } M$. Then there exists $(U, \phi) \in X_{\text{Int}}(M)$ such that $p \in U$. By definition, U is open and a previous exercise implies that $U \subset \text{Int } M$. Since $p \in \text{Int } M$ is arbitrary, we have that for each $p \in \text{Int } M$, there exists $U \subset \text{Int } M$ such that U is open. Hence $\text{Int } M$ is open.
2. Since $\partial M = (\text{Int } M)^c$, and $\text{Int } M$ is open, we have that ∂M is closed.

□

Exercise 3.1.0.17. Let M be an n -dimensional topological manifold, $(U, \phi) \in X(M)$ and $p \in U$. If $p \in \partial M$, then $(U, \phi) \in X_\partial(M)$.

Hint: simply connected

Proof. Suppose that $p \in \partial M$. Then there exists a $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in \mathbb{H}^n , $\phi(U \cap V)$ is open in \mathbb{R}^n and $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$ is a homeomorphism.

Since $\psi(U \cap V)$ is open in \mathbb{H}^n , there exists $B_\psi \subset \psi(U \cap V)$ such B_ψ is open in \mathbb{H}^n , B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$.

For the sake of contradiction, suppose that $(U, \phi) \in X_{\text{Int}}(M)$. Then $\phi(U)$ is open in \mathbb{R}^n . Hence $\phi(U \cap V)$ is open in \mathbb{R}^n and B_ϕ is open in \mathbb{R}^n . Since $\phi|_{U \cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \rightarrow \phi(U \cap V)$ is a homeomorphism, B_ϕ is simply connected. Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_ψ is simply connected. Since B_ϕ is open in \mathbb{R}^n , B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ . So $(U, \phi) \notin X_{\text{Int}}(M)$. Since $(X_{\text{Int}}(M))^c = X_\partial(M)$, we have that $(U, \phi) \in X_\partial(M)$.

□

Exercise 3.1.0.18. Let M be an n -dimensional topological manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. Then

1. $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$
2. $p \in \text{Int } M$ iff $\phi(p) \in \text{Int } \mathbb{H}^n$

Proof.

1. Suppose that $p \in \partial M$. For the sake of contradiction, suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \text{Int } \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in \mathbb{R}^n and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then $p \in U'$ and $(U', \phi') \in X_{\text{Int}}(M)$. Since $p \in U'$, the previous exercise implies that $(U', \phi') \in X_\partial(M)$. This is a contradiction since $X_{\text{Int}}(M) \cap X_\partial(M) = \emptyset$. So $\phi(p) \in \partial \mathbb{H}^n$. Conversely, suppose that $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$.

2. A previous exercise implies that $\text{Int } M = (\partial M)^c$. Part (1) implies that

$$\begin{aligned} p &\in (\partial M)^c \\ &= \text{Int } M \end{aligned}$$

if and only if

$$\begin{aligned}\phi(p) &\in (\partial\mathbb{H}^n)^c \\ &= \text{Int } \mathbb{H}^n\end{aligned}$$

□

Exercise 3.1.0.19. Let M be an n -dimensional topological manifold and $p \in M$. Then $p \in \partial M$ iff for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_{\partial}(M)$ and $\phi(p) \in \partial\mathbb{H}^n$.

Proof. Suppose that $p \in \partial M$. Let $(U, \phi) \in X(M)$. Suppose that $p \in U$. The previous two exercises imply that $(U, \phi) \in X_{\partial}(M)$ and $\phi(p) \in \partial\mathbb{H}^n$.

Conversely, suppose that for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_{\partial}(M)$ and $\phi(p) \in \partial\mathbb{H}^n$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. By assumption, $(U, \phi) \in X_{\partial}(M)$ and $\phi(p) \in \partial\mathbb{H}^n$. By definition, $p \in \partial M$. □

Exercise 3.1.0.20. Let M be an n -dimensional topological manifold. Let $(U, \phi) \in X_{\partial}(M)$. Then

1. $\phi(U \cap \partial M) = \phi(U) \cap \partial\mathbb{H}^n$
2. $\phi(U \cap \text{Int } M) = \phi(U) \cap \text{Int } \mathbb{H}^n$

Proof.

1. Since $(U, \phi) \in X_{\partial}(M)$, a previous exercise implies that for each $p \in U$, $p \in \partial M$ iff $\phi(p) \in \partial\mathbb{H}^n$. Let $q \in \phi(U \cap \partial M)$. Then there exists $p \in U \cap \partial M$ such that $\phi(p) = q$. Since $p \in \partial M$, $\phi(p) \in \partial\mathbb{H}^n$. Hence

$$\begin{aligned}q &= \phi(p) \\ &\in \phi(U) \cap \partial\mathbb{H}^n\end{aligned}$$

Since $q \in \phi(U \cap \partial M)$ is arbitrary, $\phi(U \cap \partial M) \subset \phi(U) \cap \partial\mathbb{H}^n$.

Let $q \in \phi(U) \cap \partial\mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \partial\mathbb{H}^n$, we have that $p \in \partial M$. Hence $p \in U \cap \partial M$ and

$$\begin{aligned}q &= \phi(p) \\ &\in \phi(U \cap \partial M)\end{aligned}$$

Since $q \in \phi(U) \cap \partial\mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial\mathbb{H}^n \subset \phi(U \cap \partial M)$. Thus $\phi(U \cap \partial M) = \phi(U) \cap \partial\mathbb{H}^n$.

2. Since $(U, \phi) \in X_{\partial}(M)$, a previous exercise implies that for each $p \in U$, $p \in \text{Int } M$ iff $\phi(p) \in \text{Int } \mathbb{H}^n$. Let $q \in \phi(U \cap \text{Int } M)$. Then there exists $p \in U \cap \text{Int } M$ such that $\phi(p) = q$. Since $p \in \text{Int } M$, $\phi(p) \in \text{Int } \mathbb{H}^n$. Hence

$$\begin{aligned}q &= \phi(p) \\ &\in \phi(U) \cap \text{Int } \mathbb{H}^n\end{aligned}$$

Since $q \in \phi(U \cap \text{Int } M)$ is arbitrary, $\phi(U \cap \text{Int } M) \subset \phi(U) \cap \text{Int } \mathbb{H}^n$.

Let $q \in \phi(U) \cap \text{Int } \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \text{Int } \mathbb{H}^n$, we have that $p \in \text{Int } M$. Hence $p \in U \cap \text{Int } M$ and

$$\begin{aligned}q &= \phi(p) \\ &\in \phi(U \cap \text{Int } M)\end{aligned}$$

Since $q \in \phi(U) \cap \text{Int } \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \text{Int } \mathbb{H}^n \subset \phi(U \cap \text{Int } M)$. Thus $\phi(U \cap \text{Int } M) = \phi(U) \cap \text{Int } \mathbb{H}^n$. □

Definition 3.1.0.21. Let M be an n -dimensional topological manifold and $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ the projection map. For $(U, \phi) \in X_\partial(M)$, we define $\bar{U} \subset \partial M$ and $\bar{\phi} : \bar{U} \rightarrow \pi(\phi(\bar{U}))$ by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi \circ \phi|_{\bar{U}}$ respectively.

Exercise 3.1.0.22. Let M be an n -dimensional topological manifold, and $\lambda : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ a homeomorphism. Then $\{(\bar{U}, \bar{\phi}) : (U, \phi) \in X_\partial(M)\} \subset X_{\text{Int}}^{n-1}(\partial M)$.

Proof. Let $(U, \phi) \in X_\partial(M)$.

1. Since U is open in M , $\bar{U} = U \cap \partial M$ is open in ∂M .
2. Since $(U, \phi) \in X_\partial(M)$, $\phi(U)$ is open in \mathbb{H}^n . A previous exercise implies that $\phi(\bar{U}) = \phi(U) \cap \partial\mathbb{H}^n$ which is open in $\partial\mathbb{H}^n$. Since $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ is a homeomorphism, we have that $\pi(\phi(\bar{U}))$ is open in \mathbb{R}^{n-1} .
3. Since $\phi|_{\bar{U}} : \bar{U} \rightarrow \phi(U) \cap \partial\mathbb{H}^n$ and $\pi|_{\phi(\bar{U})} : \phi(\bar{U}) \rightarrow \pi(\phi(\bar{U}))$ are homeomorphisms, we have that $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$ is a homeomorphism.

Hence $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$. □

Exercise 3.1.0.23. Let M be an n -dimensional topological manifold. Then

1. ∂M is an $(n-1)$ -dimensional topological manifold
2. $\partial(\partial M) = \emptyset$

Proof.

1. (a) Since M is Hausdorff, ∂M is Hausdorff.
 (b) Since M is second-countable, ∂M is second countable.
 (c) Let $p \in \partial M$. Then there exists $(U, \phi) \in X_\partial(M)$ such that $\phi(p) \in \partial\mathbb{H}^n$. Then $p \in \bar{U}$ and the previous exercise implies that $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$. Thus ∂M is locally Euclidean of dimension $n-1$.

Hence ∂M is an $(n-1)$ -dimensional topological manifold.

2. Let $p \in \partial M$. Part (1) implies that there exists $(U, \phi) \in X_{\text{Int}}^{n-1}(\partial M)$ such that $p \in U$. Thus $p \in \text{Int } \partial M$. Since $p \in \partial M$ is arbitrary, $\text{Int } \partial M = \partial M$. Hence

$$\begin{aligned} \partial(\partial M) &= (\text{Int}(\partial M))^c \\ &= (\partial M)^c \\ &= \emptyset \end{aligned}$$

□

Exercise 3.1.0.24. Let M be an n -dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M , then $(U', \phi|_{U'}) \in X^n(M)$.

Proof. Suppose that U' is open in M . Set $\phi' = \phi|_{U'}$.

- By assumption U' is open in M .
- Since U' is open in M , we have that $U' = U' \cap U$ is open in U . Since ϕ is a homeomorphism and U' is open in U , we have that $\phi(U')$ is open in $\phi(U)$. By assumption $\phi(U)$ is open in \mathbb{R}^n or $\phi(U)$ is open in \mathbb{H}^n . Therefore $\phi'(U')$ is open in \mathbb{R}^n or $\phi'(U')$ is open in \mathbb{H}^n .
- Since $\phi : U \rightarrow V$ is a homeomorphism, $\phi' : U' \rightarrow \phi'(U')$ is a homeomorphism.

So $(U', \phi') \in X^n(M)$. □

Note 3.1.0.25. Since U is open in M , U' being open in U is equivalent to U' being open in M , so we could have also assumed that U' is open in U .

Exercise 3.1.0.26. Let M be an n -dimensional topological manifold and $U \subset M$. If U is open, then

$$X^n(U) = \{(V, \psi) \in X^n(M) : V \subset U\}$$

Proof. Suppose that U is open and set $A = \{(V, \psi) \in X^n(M) : V \subset U\}$. Let $(V, \psi) \in X^n(U)$. By definition of $X^n(U)$, V is open in U . Thus, there exists $W \subset M$ such that W is open in M and $V = U \cap W$. Since U is open in M , we have that $V = U \cap W$ is open in M . Hence $(V, \psi) \in X^n(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X^n(U)$ is arbitrary, $X^n(U) \subset A$.

Conversely, suppose that $(V, \psi) \in A$. Then $(V, \psi) \in X^n(M)$ and $V \subset U$. By definition of $X^n(M)$, V is open in M . Since $V \subset U$, we have that $V = V \cap U$ is open in U . Hence $(V, \psi) \in X^n(U)$. Since $(V, \psi) \in X^n(U)$ is arbitrary, $A \subset X^n(U)$. Hence $X^n(A) = A$. \square

Exercise 3.1.0.27. Let M be an n -dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M , then $(U', \phi|_{U'}) \in X^n(U)$.

Proof. Suppose that U' is open in M . A previous exercise implies that $(U', \phi') \in X^n(M)$. The previous exercise implies that $(U', \phi') \in X^n(U)$. \square

Exercise 3.1.0.28. Topological Open Submanifolds:

Let M be an n -dimensional topological manifold and $U \subset M$ open. Then U is an n -dimensional topological manifold.

Proof.

1. Since M is Hausdorff, U is Hausdorff.
2. M is second-countable, U is second countable.
3. Let $p \in U$. Since then there exists $(V, \psi) \in X^n(M)$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{U \cap V}$. The previous exercise implies that $(V', \psi') \in X^n(U)$. Therefore U is locally Euclidean of dimension n .

Hence U is an n -dimensional topological manifold. \square

Exercise 3.1.0.29. Let M be an n -dimensional topological manifold and $U \subset M$. If U is open, then

1. $X_{\text{Int}}(U) = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$
2. $X_{\partial}(U) = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$

Proof. Suppose that U is open in M .

1. Set $A = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\text{Int}}(U)$. By definition of $X_{\text{Int}}(U)$, V is open in U and $\phi(V)$ is open in \mathbb{R}^n . Since U is open in M , V is open in M . Hence $(V, \psi) \in X_{\text{Int}}(M)$. Since U is open in M , V is open in M . Hence $(V, \psi) \in X_{\text{Int}}(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X_{\text{Int}}(U)$ is arbitrary, $X_{\text{Int}}(U) \subset A$.
Conversely, let $(V, \psi) \in A$. Then $(V, \psi) \in X_{\text{Int}}(M)$ and $V \subset U$. By definition of $X_{\text{Int}}(M)$, V is open in M and $\phi(V)$ is open in \mathbb{R}^n . Thus $V = V \cap U$ is open in U . So $(V, \psi) \in X_{\text{Int}}(U)$. Since $(V, \psi) \in A$ is arbitrary, $A \subset X_{\text{Int}}(U)$. Thus $X_{\text{Int}}(U) = A$.
2. Set $B = \{(V, \psi) \in X_{\partial}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\partial}(U)$. By definition of $X_{\partial}(U)$, V is open in U , $\phi(V)$ is open in \mathbb{H}^n and $\partial\mathbb{H}^n \cap \phi(V) \neq \emptyset$. Since U is open in M , V is open in M . Hence $(V, \psi) \in X_{\partial}(M)$, which implies that $(V, \psi) \in B$. Since $(V, \psi) \in X_{\partial}(U)$ is arbitrary, $X_{\partial}(U) \subset B$.
Conversely, let $(V, \psi) \in B$. Then $(V, \psi) \in X_{\partial}(M)$ and $V \subset U$. By definition of $X_{\partial}(M)$, V is open in M , $\phi(V)$ is open in \mathbb{H}^n and $\partial\mathbb{H}^n \cap \phi(V) \neq \emptyset$. Thus $V = V \cap U$ is open in U . So $(V, \psi) \in X_{\partial}(U)$. Since $(V, \psi) \in B$ is arbitrary, $B \subset X_{\partial}(U)$. Thus $X_{\partial}(U) = B$.

\square

Exercise 3.1.0.30. Let M be an n -dimensional topological manifold and $U \subset M$. If U is open, then $\partial U = \partial M \cap U$.

Proof. Suppose that U is open. Let $p \in \partial U$. Then there exists $(V, \psi) \in X_\partial(U)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Since U is open, the previous exercise implies that $(V, \psi) \in X_\partial(M)$. Thus $p \in \partial M$. Since $p \in \partial U$ is arbitrary, $\partial U \subset \partial M$. Since $\partial U \subset U$, we have that $\partial U \subset \partial M \cap U$.

Conversely, let $p \in \partial M \cap U$. Since $p \in \partial M$, there exists $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Set $V' = V \cap U$ and $\psi' = \psi|_{V'}$. Then $p \in V'$ since V and U are open in M , V' is open in M . A previous exercise implies that $(V', \psi') \in X(M)$. Since $p \in \partial M$, a previous exercise implies that $(V', \psi') \in X_\partial(M)$. The previous exercise implies that $(V', \psi') \in X_\partial(U)$. Since $\psi'(p) \in \partial \mathbb{H}^n$, $p \in \partial U$. Since $p \in \partial M \cap U$ is arbitrary, $\partial M \cap U \subset \partial U$. Hence $\partial U = \partial M \cap U$.

label exercises and reference them!!! □

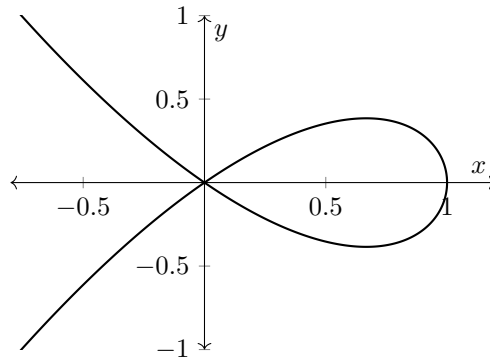
Exercise 3.1.0.31. Graph of Continuous Function:

Let $f \in C(\mathbb{R})$. Set $M = \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$ (i.e. the graph of f). Then M is a 1-dimensional manifold.

Proof. Set $U = \mathbb{R}$ and define $\phi : U \rightarrow M$ by $\phi(x) = (x, f(x))$. Then $\phi^{-1} = \pi_1$. Since f is continuous, ϕ is continuous. Since π_1 is continuous, ϕ is a homeomorphism. □

Exercise 3.1.0.32. Nodal Cubic:

Let $M = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2 - x^3\}$. We equip M with the subspace topology.



Then M is not a 1-dimensional topological manifold.

Hint: connected components

Proof. Suppose that M is a 1-dimensional manifold. Set $p = (0, 0)$. Then there exists $(U, \phi) \in X(M)$ such that $p \in U$. Since $\phi(U)$ is open (in \mathbb{R} or \mathbb{H}), there exists a $B \subset \phi(U)$ such that B is open (in \mathbb{R} or \mathbb{H}), B is connected and $\phi(p) \in B$. Set $V = \phi^{-1}(B)$, $V' = V \setminus \{p\}$ and $B' = B \setminus \{\phi(p)\}$. Then $\phi : V \rightarrow B$ and $\phi' : V' \rightarrow B'$ are homeomorphisms. Since B is open (in \mathbb{R} or \mathbb{H}) and connected, B' has at most two connected components. Then V' This is a contradiction since V' has four connected components and B' and V' are homeomorphic. □

Exercise 3.1.0.33. Topological Manifold Chart Lemma:

Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \rightarrow \mathbb{H}^n$. Suppose that

- (a) for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha)$ is open in \mathbb{H}^n
- (b) for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$ is a bijection
- (c) for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in \mathbb{H}^n
- (d) for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ is continuous
- (e) there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$
- (f) for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$

Define

$$\mathcal{B} = \{\phi_\alpha^{-1}(V) : V \subset \mathbb{H}^n \text{ is open in } \mathbb{H}^n \text{ and } \alpha \in \Gamma\}$$

Then

1. \mathcal{B} is a basis for $\tau(\mathcal{B})$
Hint: For $B_1, B_2 \subset \mathbb{H}^n$, $\phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) = \phi_{\alpha_1}^{-1}(B_1 \cap [\phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}} \circ (\phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}})^{-1}(B_2)])$
2. $(M, \tau(\mathcal{B}))$ is an n -dimensional topological manifold

Proof.

1. • By assumption, $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
- Let $A_1, A_2 \in \mathcal{B}$ and $p \in A_1 \cap A_2$. By definition, there exist $\alpha_1, \alpha_2 \in \Gamma$ and $B_1, B_2 \subset \mathbb{H}^n$ such that B_1, B_2 are open in \mathbb{H}^n and

$$\begin{aligned} A_1 &= \phi_{\alpha_1}^{-1}(B_1) & A_2 &= \phi_{\alpha_2}^{-1}(B_2) \\ &\subset U_{\alpha_1} & &\subset U_{\alpha_2} \end{aligned}$$

Set $\psi_1 = \phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}}$ and $\psi_2 = \phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}}$. We note that

$$\begin{aligned} \psi_1^{-1}(B_1) &= U_{\alpha_2} \cap \phi_{\alpha_1}^{-1}(B_1) & \psi_2^{-1}(B_2) &= U_{\alpha_1} \cap \phi_{\alpha_2}^{-1}(B_2) \\ &= U_{\alpha_2} \cap A_1 & &= U_{\alpha_1} \cap A_2 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} & &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned}$$

Let $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Then $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. Hence $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_1}(q) \in \psi_1 \circ \psi_2^{-1}(B_2)$. This implies that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(B_1) \\ &= A_1 \end{aligned}$$

and since $\psi_2^{-1}(B_2) \subset U_{\alpha_1} \cap U_{\alpha_2}$ and $\phi_{\alpha_1} : U_{\alpha_1} \rightarrow \phi_{\alpha_1}(U_{\alpha_1})$ is a bijection, we have that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(\psi_1 \circ \psi_2^{-1}(B_2)) \\ &= \psi_2^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \end{aligned}$$

Thus

$$\begin{aligned} q &\in A_1 \cap (U_{\alpha_1} \cap A_2) \\ &= A_1 \cap A_2 \end{aligned}$$

Since $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$ is arbitrary, we have that $\phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \subset A_1 \cap A_2$. Conversely, let

$$\begin{aligned} q &\in A_1 \cap A_2 \\ &= \phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) \end{aligned}$$

Then $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_2}(q) \in B_2$. Since $A_1 \cap A_2 \subset U_{\alpha_1} \cap U_{\alpha_2}$, we have that

$$\begin{aligned} \psi_2(q) &= \phi_{\alpha_2}(q) \\ &\in B_2 \end{aligned}$$

which implies that $q \in \psi_2^{-1}(B_2)$. Therefore

$$\begin{aligned}\phi_{\alpha_1}(q) &= \psi_1(q) \\ &\in \psi_1(\psi_2^{-1}(B_2)) \\ &= \psi_1 \circ \psi_2^{-1}(B_2)\end{aligned}$$

Hence $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. This implies that $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Since $q \in A_1 \cap A_2$ is arbitrary, we have that $A_1 \cap A_2 \subset \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Thus

$$\begin{aligned}A_1 \cap A_2 &= \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \\ &\in \mathcal{B}\end{aligned}$$

Thus \mathcal{B} is a basis for $\tau(\mathcal{B})$.

2. (a) **(locally Euclidean of dimension n):**

Set $\mathcal{T}_M = \tau(\mathcal{B})$. Let $\alpha \in \Gamma$. By definition, for each $B \subset \mathcal{T}_{\mathbb{H}^n}$,

$$\begin{aligned}\phi_{\alpha}^{-1}(B) &\in \mathcal{B} \\ &\subset \mathcal{T}_M\end{aligned}$$

Hence ϕ_{α} is continuous.

Let $A \in \mathcal{T}_{U_{\alpha}}$. Then there exists $U \subset \mathcal{T}_M$ such that $A = U \cap U_{\alpha}$. Since \mathcal{B} is a basis for \mathcal{T}_M , there exists $\Gamma' \subset \Gamma$, $(V_{\beta})_{\beta \in \Gamma'} \subset \mathcal{T}_{\mathbb{H}^n}$ such that $U = \bigcup_{\beta \in \Gamma'} \phi_{\beta}^{-1}(V_{\beta})$. Thus

$$\begin{aligned}A &= U \cap U_{\alpha} \\ &= \left[\bigcup_{\beta \in \Gamma'} \phi_{\beta}^{-1}(V_{\beta}) \right] \cap U_{\alpha} \\ &= \bigcup_{\beta \in \Gamma'} [\phi_{\beta}^{-1}(V_{\beta}) \cap U_{\alpha}]\end{aligned}$$

Let $\beta \in \Gamma'$. Since $\phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \subset \phi_{\alpha}(U_{\alpha})$ and $\phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \in \mathcal{T}_{\mathbb{H}^n}$, we have that

$$\begin{aligned}\phi_{\alpha}(U_{\alpha} \cap U_{\beta}) &= \phi_{\alpha}(U_{\alpha}) \cap \phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \\ &\in \mathcal{T}_{\phi_{\alpha}(U_{\alpha})}\end{aligned}$$

Therefore $\mathcal{T}_{\phi_{\alpha}(U_{\alpha} \cap U_{\beta})} \subset \mathcal{T}_{\phi_{\alpha}(U_{\alpha})}$. Since $(\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}}) \circ (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}})^{-1} : \phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow \phi_{\beta}(U_{\alpha} \cap U_{\beta})$ is continuous, we have that $(\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}}) \circ (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}})^{-1} : \phi_{\alpha}(U_{\alpha} \cap U_{\beta}) \rightarrow \mathbb{H}^n$ is continuous and therefore

$$\begin{aligned}[(\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}}) \circ (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}})^{-1}]^{-1}(V_{\beta}) &\in \mathcal{T}_{\phi_{\alpha}(U_{\alpha} \cap U_{\beta})} \\ &\subset \mathcal{T}_{\phi_{\alpha}(U_{\alpha})}\end{aligned}$$

Since $\beta \in \Gamma'$ is arbitrary, we have that

$$\begin{aligned}\phi_{\alpha}(A) &= \phi_{\alpha}\left(\bigcup_{\beta \in \Gamma'} [\phi_{\beta}^{-1}(V_{\beta}) \cap U_{\alpha}]\right) \\ &= \bigcup_{\beta \in \Gamma'} \phi_{\alpha}(\phi_{\beta}^{-1}(V_{\beta}) \cap U_{\alpha}) \\ &= \bigcup_{\beta \in \Gamma'} (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}}) \circ (\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}})^{-1}(V_{\beta}) \\ &= \bigcup_{\beta \in \Gamma'} [(\phi_{\beta}|_{U_{\alpha} \cap U_{\beta}}) \circ (\phi_{\alpha}|_{U_{\alpha} \cap U_{\beta}})^{-1}]^{-1}(V_{\beta}) \\ &\in \mathcal{T}_{\phi_{\alpha}(U_{\alpha})}\end{aligned}$$

Since $A \in \mathcal{T}_{U_\alpha}$ is arbitrary, $\phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \rightarrow U_\alpha$ is continuous. Hence $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$ is a homeomorphism and $(U_\alpha, \phi_\alpha) \in X^n(M)$. Since $M = \bigcup_{\alpha \in \Gamma} U_\alpha$, we have that M is locally Euclidean of dimension n .

(b) **(Hausdorff):**

Let $p, q \in M$. Suppose that $p \neq q$. Then there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha, q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$.

- Suppose that there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$. Since $p \neq q$, $\phi_\alpha(p) \neq \phi_\alpha(q)$. Since \mathbb{H}^n is Hausdorff, there exist $V_p, V_q \subset \phi(U_\alpha)$ such that V_p and V_q are open in \mathbb{H}^n , $p \in V_p, q \in V_q$ and $V_p \cap V_q = \emptyset$. Set $U_p = \phi_\alpha^{-1}(V_p)$ and $U_q = \phi_\alpha^{-1}(V_q)$. Then U_p, U_q are open, $p \in U_p, q \in U_q$ and $U_p \cap U_q = \emptyset$.
- Suppose that there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha, q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$. Set $U_p = U_\alpha$ and $U_q = U_\beta$. Then U_p, U_q are open, $p \in U_p, q \in U_q$ and $U_p \cap U_q = \emptyset$.

Thus for each $p, q \in M$ there exist $U_p, U_q \subset M$ such that U_p, U_q are open, $p \in U_p, q \in U_q$ and $U_p \cap U_q = \emptyset$. Hence

(c) **(second-countable):**

By assumption, there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$. Let $\alpha \in \Gamma'$.

Since $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$ and \mathbb{H}^n is second-countable, we have that $\phi_\alpha(U_\alpha)$ is second-countable. Since $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$ is a homeomorphism, we have that U_α is second-countable. Since $M = \bigcup_{\alpha \in \Gamma'} U_\alpha$,

an exercise in topology [cite](#) implies that M is second-countable.

□

3.2 Smooth Manifolds

Definition 3.2.0.1. Let M be an n -dimensional topological manifold and $(U, \phi), (V, \psi) \in X(M)$. Then (U, ϕ) and (V, ψ) are said to be **smoothly compatible** if

$$\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V) \text{ is a diffeomorphism}$$

Definition 3.2.0.2. Let M be an n -dimensional topological manifold.

- Let $\mathcal{A} \subset X(M)$. Then \mathcal{A} is said to be an **atlas on M** if $M \subset \bigcup_{(U, \phi) \in \mathcal{A}} U$.
- Let \mathcal{A} be an atlas on M . Then \mathcal{A} is said to be **smooth** if for each $(U, \phi), (V, \psi) \in \mathcal{A}$, (U, ϕ) and (V, ψ) are smoothly compatible.
- Let \mathcal{A} be a smooth atlas on M . Then \mathcal{A} is said to be **maximal** if for each smooth atlas \mathcal{B} on M , $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on M** .
- Let \mathcal{A} be an atlas on M . Then (M, \mathcal{A}) is said to be an **n -dimensional smooth manifold** if \mathcal{A} is a smooth structure on M .

Exercise 3.2.0.3. Let M be an n -dimensional topological manifold and \mathcal{B} a smooth atlas on M . Then there exists a unique smooth structure \mathcal{A} on M such that $\mathcal{B} \subset \mathcal{A}$.

Proof. Define

$$\mathcal{A} = \{(U, \phi) \in X(M) : \text{for each } (V, \psi) \in \mathcal{B}, (U, \phi) \text{ and } (V, \psi) \text{ are smoothly compatible}\}$$

Clearly $\mathcal{B} \subset \mathcal{A}$. Let (U, ϕ) and $(V, \psi) \in \mathcal{A}$. Define $F : \phi(U \cap V) \rightarrow \psi(U \cap V)$ by

$$F = \psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$$

Let $q \in \phi(U \cap V)$. Set $p = \phi^{-1}(q)$. Since $p \in U \cap V \subset M$, there exists $(W, \chi) \in \mathcal{B}$ such that $p \in W$. By definition of \mathcal{A} , $\psi|_{W \cap V} \circ (\chi|_{W \cap V})^{-1} : \chi(W \cap V) \rightarrow \psi(W \cap V)$ and $\chi|_{U \cap W} \circ (\phi|_{U \cap W})^{-1} : \phi(U \cap W) \rightarrow \chi(U \cap W)$ are diffeomorphisms. Set $N = U \cap W \cap V$. Then $q \in \phi(N) \subset \phi(U \cap V)$ and

$$\begin{aligned} F|_{\phi(N)} &= \psi|_N \circ (\phi|_N)^{-1} \\ &= [\psi|_N \circ (\chi|_N)^{-1}] \circ [\chi|_N \circ (\phi|_N)^{-1}] \end{aligned}$$

is a diffeomorphism. Thus, for each $q \in \phi(U \cap V)$, there exists $N' \subset \phi(U \cap V)$ such that $F|_{N'}$ is a diffeomorphism. Hence F is a diffeomorphism and $(U, \phi), (V, \psi)$ are smoothly compatible. Therefore \mathcal{A} is a smooth atlas.

To see that \mathcal{A} is maximal, let \mathcal{B}' be a smooth atlas on M . Suppose that $\mathcal{A} \subset \mathcal{B}'$ and let $(U, \phi) \in \mathcal{B}'$. By definition, for each chart $(V, \psi) \in \mathcal{B}'$, (U, ϕ) and (V, ψ) are smoothly compatible. Since $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}'$, we have that $(U, \phi) \in \mathcal{A}$. So $\mathcal{A} = \mathcal{B}'$ and \mathcal{A} is a maximal smooth atlas on M . \square

Exercise 3.2.0.4. Let (M, \mathcal{A}) be an n -dimensional smooth manifold, $(U, \phi) \in \mathcal{A}$ and $U' \subset U$. If U' is open, then $(U', \phi|_{U'}) \in \mathcal{A}$.

Proof. Set $\phi' = \phi|_{U'}$. A previous exercise implies that $(U', \phi') \in X(U)$. Define $\mathcal{B} = \mathcal{A} \cup \{(U', \phi')\}$. Let $(V, \psi) \in \mathcal{B}$. If $(V, \psi) = (U', \phi')$, then

$$\phi' \circ \psi^{-1} = \text{id}_{U'}$$

which is a diffeomorphism. Thus $(U', \phi'), (V, \psi)$ are smoothly compatible. Suppose that $(V, \psi) \in \mathcal{A}$. Since \mathcal{A} is smooth, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ is a diffeomorphism. Therefore $\psi|_{U' \cap V} \circ (\phi'|_{U' \cap V})^{-1} : \phi'(U' \cap V) \rightarrow \psi(U' \cap V)$ is a diffeomorphism and $(U', \phi'), (V, \psi)$ are smoothly compatible. Since $(V, \psi) \in \mathcal{A}$ is arbitrary, \mathcal{B} is smooth. Since \mathcal{A} is maximal and $\mathcal{A} \subset \mathcal{B}$, we have that $\mathcal{A} = \mathcal{B}$ and $(U', \phi') \in \mathcal{A}$. \square

Exercise 3.2.0.5. Let (M, \mathcal{A}) be a n -dimensional smooth manifold and $U \subset M$ open. Set $\mathcal{B} = \{(V, \psi) \in \mathcal{A} : V \subset U\}$. Then \mathcal{B} is a smooth atlas on U .

Proof.

- Some previous exercises imply that U is an n -dimensional topological manifold and $X(U) = \{(V, \psi) \in X(M) : V \subset U\}$. Since

$$\begin{aligned} \mathcal{B} &\subset \mathcal{A} \\ &\subset X(M) \end{aligned}$$

we have that $\mathcal{B} \subset X(U)$. Let $p \in U$. Then there exists $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{V'}$. The previous exercise implies that $(V', \psi') \in \mathcal{A}$. By definition, $(V', \psi') \in \mathcal{B}$. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $(V', \psi') \in \mathcal{B}$ such that $p \in V'$. Hence \mathcal{B} is an atlas on U .

- Let $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$. Then $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{A}$. Since \mathcal{A} is smooth, (V_1, ψ_1) and (V_2, ψ_2) are smoothly compatible. Since $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ are arbitrary, \mathcal{B} is smooth. □

Definition 3.2.0.6. Smooth Open Submanifold:

Let (M, \mathcal{A}) be an n -dimensional smooth manifold and $U \subset M$ open. A previous exercise implies that U is an n -dimensional topological manifold. We define $\mathcal{A}|_U \subset X(U)$ to be the unique smooth structure on U such that $\{(V, \psi) \in \mathcal{A} : V \subset U\} \subset \mathcal{A}|_U$. Then $(U, \mathcal{A}|_U)$ is said to be a **smooth open submanifold of (M, \mathcal{A})** .

Exercise 3.2.0.7. Let $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ be the projection map given by $\pi(x_1, \dots, x_{n-1}, 0) = (x_1, \dots, x_{n-1})$. Then π is a diffeomorphism.

Proof. Define projection map $\pi' : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ by $\pi'(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1})$. Then \mathbb{R}^n is an open neighborhood of $\partial\mathbb{H}^n$, $\pi'|_{\partial\mathbb{H}^n} = \pi$ and π' is smooth. Then by definition, π is smooth. Clearly, π^{-1} is smooth. So π is a diffeomorphism. □

Definition 3.2.0.8. Let (M, \mathcal{A}) be a n -dimensional smooth manifold and $\pi : \partial\mathbb{H}^n \rightarrow \mathbb{R}^{n-1}$ the projection map. Recall that for $(U, \phi) \in X_{\partial}^n(M)$, the $(n-1)$ -coordinate chart $(\bar{U}, \bar{\phi}) \in X_{\text{Int}}^{n-1}(\partial M)$ is defined by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$.

We define

$$\bar{\mathcal{A}} = \{(\bar{U}, \bar{\phi}) : (U, \phi) \in \mathcal{A} \cap X_{\partial}^n(M)\}$$

Exercise 3.2.0.9. Let (M, \mathcal{A}) be a n -dimensional smooth manifold. Then $\bar{\mathcal{A}}$ is a smooth atlas on ∂M .

Proof.

- A previous exercise implies that ∂M is an $(n-1)$ -dimensional topological manifold. Let $p \in \partial M$. Then there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Since $\mathcal{A} \subset X^n(M)$ and $p \in \partial M$, we have that $p \in \bar{U}$ and a previous exercise implies that $(U, \phi) \in X_{\partial}^n(M)$. By definition of $\bar{\mathcal{A}}$, $(\bar{U}, \bar{\phi}) \in \bar{\mathcal{A}}$. Since $p \in \partial M$ is arbitrary, $\bar{\mathcal{A}}$ is an atlas on ∂M .
- Let $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \bar{\mathcal{A}}$. Since (U, ϕ) and (V, ψ) are smoothly compatible, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1}$ is a diffeomorphism. Thus $\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1}$ is a diffeomorphism. Since $\pi|_{\phi(U \cap V)}$ and $\pi|_{\psi(U \cap V)}$ are diffeomorphisms, $\pi|_{\phi(\bar{U} \cap \bar{V})}$ and $\pi|_{\psi(\bar{U} \cap \bar{V})}$ are diffeomorphisms. Then

$$\begin{aligned} \bar{\psi}|_{\bar{U} \cap \bar{V}} \circ (\bar{\phi}|_{\bar{U} \cap \bar{V}})^{-1} &= \left[\pi|_{\psi(\bar{U} \cap \bar{V})} \circ \psi|_{\bar{U} \cap \bar{V}} \right] \circ \left[(\phi|_{\bar{U} \cap \bar{V}})^{-1} \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \right] \\ &= \pi|_{\psi(\bar{U} \cap \bar{V})} \circ \left[\psi|_{\bar{U} \cap \bar{V}} \circ (\phi|_{\bar{U} \cap \bar{V}})^{-1} \right] \circ (\pi|_{\phi(\bar{U} \cap \bar{V})})^{-1} \end{aligned}$$

is a diffeomorphism. Therefore $(\bar{U}, \bar{\phi})$ and $(\bar{V}, \bar{\psi})$ are smoothly compatible. Since $(\bar{U}, \bar{\phi}), (\bar{V}, \bar{\psi}) \in \bar{\mathcal{A}}$ are arbitrary, $\bar{\mathcal{A}}$ is smooth. □

Definition 3.2.0.10. Let (M, \mathcal{A}) be a n -dimensional smooth manifold. We define $\mathcal{A}|_{\partial M}$ to be the unique smooth structure on ∂M such that $\overline{\mathcal{A}} \subset \mathcal{A}|_{\partial M}$. We define the **smooth boundary submanifold of M** to be $(\partial M, \mathcal{A}|_{\partial M})$.

Exercise 3.2.0.11. Smooth Manifold Chart Lemma:

Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \rightarrow \mathbb{H}^n$. Suppose that

- (a) for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha)$ is open in \mathbb{H}^n
- (b) for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \rightarrow \phi_\alpha(U_\alpha)$ is a bijection
- (c) for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta)$ is open in \mathbb{H}^n
- (d) for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ is smooth
- (e) there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$
- (f) for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta \neq \emptyset$

Then there exists a unique smooth structure \mathcal{A} on M such that (M, \mathcal{A}) is an n -dimensional smooth manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset \mathcal{A}$.

Proof. Set $\mathcal{B} = \{\phi_\alpha^{-1}(V) : \alpha \in \Gamma \text{ and } V \in \mathcal{T}_{\mathbb{H}^n}\}$ and $\mathcal{A}' = \{(U_\alpha, \phi_\alpha) : \alpha \in \Gamma\}$. The topological manifold chart lemma implies that $(M, \tau(\mathcal{B}))$ is an n -dimensional topological manifold and $\mathcal{A}' \subset X^n(M)$. Since $M = \bigcup_{\alpha \in \Gamma} U_\alpha$, \mathcal{A}' is an atlas on M . Since for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ is smooth, we have that \mathcal{A}' is smooth. A previous exercise implies that there exists a unique smooth structure \mathcal{A} on M such that $\mathcal{A}' \subset \mathcal{A}$. \square

3.3 Smooth Maps

Definition 3.3.0.1. Let (M, \mathcal{A}) be a smooth manifold and $f : M \rightarrow \mathbb{R}$. Then f is said to be smooth if for each coordinate chart $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M)$.

Exercise 3.3.0.2. Let (M, \mathcal{A}) be a smooth manifold. Then $C^\infty(M)$ is a vector space.

Proof. Let $f, g \in C^\infty(M)$, $\lambda \in \mathbb{R}$ and $(U, \phi) \in \mathcal{A}$. By assumption, $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are smooth. Hence

$$(f + \lambda g) \circ \phi^{-1} = f \circ \phi^{-1} + \lambda g \circ \phi^{-1}$$

is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, $f + \lambda g \in C^\infty(M)$. Since $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$ are arbitrary, $C^\infty(M)$ is a vector space. \square

Exercise 3.3.0.3. Let (M, \mathcal{A}) be a smooth manifold, \mathcal{B} an atlas on M and $f : M \rightarrow \mathbb{R}$. Suppose that $\mathcal{B} \subset \mathcal{A}$. Then f is smooth iff for each $(U, \phi) \in \mathcal{B}$, $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is smooth.

Proof.

- (\implies) :
Suppose that f is smooth. Let $(U, \phi) \in \mathcal{B}$. Since $\mathcal{B} \subset \mathcal{A}$, $(U, \phi) \in \mathcal{A}$. Since f is smooth, $f \circ \phi^{-1}$ is smooth. Since $(U, \phi) \in \mathcal{B}$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{B}$, $f \circ \phi^{-1}$ is smooth.
- (\impliedby) :
Suppose that for each $(V, \psi) \in \mathcal{B}$, $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$ is smooth. Let $(U, \phi) \in \mathcal{A}$ and $q \in \phi(U)$. Set $p = \phi^{-1}(q)$. Since \mathcal{B} is an atlas, there exists $(V, \psi) \in \mathcal{B}$ such that $p \in V$. Since $\mathcal{B} \subset \mathcal{A}$, $(V, \psi) \in \mathcal{A}$. Set $W = U \cap V$ and $\tilde{\phi} = \phi|_W$ and $\tilde{\psi} = \psi|_W$. We note that $\phi(W) \in \mathcal{N}_q$ and $\phi(W)$ is open. An exercise in the section on smooth manifolds implies that $(W, \tilde{\phi}), (W, \tilde{\psi}) \in \mathcal{A}$. Therefore $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(W) \rightarrow \psi(W)$ is smooth. By assumption, $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$ is smooth. This implies that $(f \circ \psi^{-1})|_{\psi(W)} : \psi(W) \rightarrow \mathbb{R}$ is smooth. Hence

$$\begin{aligned} (f \circ \phi^{-1})|_{\phi(W)} &= f \circ \tilde{\phi}^{-1} \\ &= f \circ (\tilde{\psi}^{-1} \circ \tilde{\psi}) \circ \tilde{\phi}^{-1} \\ &= (f \circ \tilde{\psi}^{-1}) \circ (\tilde{\psi} \circ \tilde{\phi}^{-1}) \end{aligned}$$

is smooth. Since $q \in \phi(U)$ is arbitrary, for each $q \in \phi(U)$, there exists $A \in \mathcal{N}_q$ such that A is open and $(f \circ \phi^{-1})|_A : A \rightarrow \mathbb{R}$ is smooth. This implies that $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, f is smooth. \square

Exercise 3.3.0.4. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $p \in U$ and $f \in C^\infty(M)$. Then $f|_U \in C^\infty(U)$.

Proof. Let \square

Definition 3.3.0.5. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \dots, n\}$. We define the **partial derivative of f with respect to x^i** , denoted

$$\partial f / \partial x^i : U \rightarrow \mathbb{R} \quad \text{or} \quad \partial_i f : U \rightarrow \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left(\frac{\partial}{\partial u^i}[f \circ \phi^{-1}] \right) \circ \phi$$

Exercise 3.3.0.6. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \dots, n\}$. Then $\partial/\partial x^i : C^\infty(U) \rightarrow C^\infty(U)$ is linear.

Proof. **FINISH!!!** □

Exercise 3.3.0.7. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $f \in C^\infty(U)$ and $i, j \in \{1, \dots, n\}$. Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} f \right) \\ &= \frac{\partial}{\partial x^i} \left(\left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \circ \phi \right) \\ &= \left(\frac{\partial}{\partial u^i} \left[\left(\left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \circ \phi \right) \circ \phi^{-1} \right] \right) \circ \phi \\ &= \left(\frac{\partial}{\partial u^i} \left[\frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right] \right) \circ \phi \\ &= \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi \end{aligned}$$

□

Exercise 3.3.0.8. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $i, j \in \{1, \dots, n\}$. Then

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i}$$

Proof. Let $f \in C^\infty(U)$. Since $f \circ \phi^{-1}$ is smooth,

$$\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] = \frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i} [f \circ \phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f &= \left(\frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} [f \circ \phi^{-1}] \right) \circ \phi \\ &= \left(\frac{\partial}{\partial u^j} \frac{\partial}{\partial u^i} [f \circ \phi^{-1}] \right) \circ \phi \\ &= \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f \end{aligned}$$

□

Exercise 3.3.0.9. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $f \in C^\infty(U)$. Then for each $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha f = (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi$$

Proof. The claim is clearly true when $|\alpha| = 0$ or by definition if $|\alpha| = 1$. Let $n \in \mathbb{N}$ and suppose the claim is true for each $|\alpha| \in \{1, \dots, n-1\}$. Then there exists $i \in \{1, \dots, n\}$ such that $\alpha_i \geq 1$. Hence

$$\begin{aligned} \partial^\alpha f &= \partial^{e^i} (\partial^{\alpha-e^i} f) \\ &= \partial^{e^i} (\partial^{\alpha-e^i} [f \circ \phi^{-1}] \circ \phi) \\ &= (\partial^{e^i} [(\partial^{\alpha-e^i} [f \circ \phi^{-1}] \circ \phi) \circ \phi^{-1}]) \circ \phi \\ &= (\partial^{e^i} [\partial^{\alpha-e^i} [f \circ \phi^{-1}]]) \circ \phi \\ &= (\partial^\alpha [f \circ \phi^{-1}]) \circ \phi \end{aligned}$$

□

Exercise 3.3.0.10. Taylor's Theorem:

Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\phi(U)$ convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that

$$f = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(x_0) \right] + \sum_{|\alpha|=T+1} (x^i - x^i(p))^\alpha g_\alpha$$

and for each $|\alpha| = T+1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. Since $\phi(U)$ is open and convex and $f \circ \phi^{-1} \in C^\infty(\phi(U))$, Taylors thorem in section 2.1 implies that there exist $(\tilde{g}_\alpha)_{|\alpha|=T+1} \subset C^\infty(\phi(U))$ such that for each $q \in U$,

$$f \circ \phi^{-1}(\phi(q)) = \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q))$$

and for each $|\alpha| = T+1$,

$$\begin{aligned} \tilde{g}_\alpha(\phi(p)) &= \frac{1}{(T+1)!} \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \\ &= \frac{1}{(T+1)!} \partial^\alpha f(p) \end{aligned}$$

For $|\alpha| = T+1$, set $g_\alpha = \tilde{g}_\alpha \circ \phi$. Then

$$\begin{aligned} f(q) &= f \circ \phi^{-1}(\phi(q)) \\ &= \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\ &= \sum_{k=0}^T \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q) \end{aligned}$$

□

Definition 3.3.0.11. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$. Then F is said to be

- **smooth** if for each $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$,

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \psi(F(U) \cap V)$$

is smooth

- a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 3.3.0.12. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifold and $F : M \rightarrow N$. If F is smooth, then F is continuous.

Proof. Suppose that F is smooth. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $F(p) \in V$. Put $\tilde{U} = U \cap F^{-1}(V)$ and $\tilde{V} = F(U) \cap V$.

Define $\tilde{\phi} : \tilde{U} \rightarrow \phi(\tilde{U})$ and $\tilde{\psi} : \tilde{V} \rightarrow \psi(\tilde{V})$ by

$$\tilde{\phi} = \phi|_{\tilde{U}}, \quad \tilde{\psi} = \psi|_{\tilde{V}}$$

Then $\tilde{\phi}$ and $\tilde{\psi}$ are homeomorphisms, $p \in \tilde{U}$ and $F(\tilde{U}) \subset \tilde{V}$. Define $\tilde{F} : \phi(\tilde{U}) \rightarrow \psi(\tilde{V})$ by

$$\tilde{F} = \tilde{\psi} \circ F \circ \tilde{\phi}^{-1}$$

By definition, \tilde{F} is smooth and therefore continuous. Since ϕ and ψ are homeomorphisms and $F|_{\tilde{U}} = \tilde{\psi}^{-1} \circ \tilde{F} \circ \tilde{\phi}$, we have that $F|_{\tilde{U}}$ is continuous. In particular, F is continuous at p and since $p \in M$ is arbitrary, F is continuous. \square

Exercise 3.3.0.13. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifold and $F : M \rightarrow N$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F and F^{-1} are smooth. The previous exercise implies that F and F^{-1} are continuous. Hence F is a homeomorphism. \square

Exercise 3.3.0.14. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$ a diffeomorphism. Then for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$.

Proof. Let $(V, \psi) \in \mathcal{B}$.

1. Since ϕ and F^{-1} are homeomorphisms, $\phi \circ F^{-1} : F(U) \cap V \rightarrow \phi(U \cap F^{-1}(V))$ is a homeomorphism
2. Since F is a diffeomorphism,

$$\phi \circ F^{-1} \circ \psi^{-1} : \psi(F(U) \cap V) \rightarrow \phi(U \cap F^{-1}(V))$$

and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \psi(V \cap F(U))$$

are smooth.

Therefore $(F(U), \phi \circ F^{-1})$ and (V, ψ) are smoothly compatible. Since \mathcal{B} is maximal, $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. \square

Definition 3.3.0.15. Let (N, \mathcal{B}) be a smooth n -dimensional manifold, $F : M \rightarrow N$ smooth and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. For $i \in \{1, \dots, n\}$, We define the **i -th component of F with respect to (V, ψ)** , denoted $F^i : V \rightarrow \mathbb{R}$, by

$$F^i = y^i \circ F$$

3.4 Partitions of Unity

Definition 3.4.0.1. Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then ρ is said to be a **bump function at p supported in U** if

1. $\rho \geq 0$
2. there exists $V \in \mathcal{N}_p$ such that V is open and $\rho|_V = 1$
3. $\text{supp } \rho \subset U$

Exercise 3.4.0.2. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1, 1) \\ 0 & t \notin (-1, 1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

Proof.

□

3.5 The Tangent Space

Definition 3.5.0.1. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. For $i \in \{1, \dots, n\}$, define the partial derivative with respect to x^i at p , denoted

$$\left. \frac{\partial}{\partial x^i} \right|_p : C^\infty(M) \rightarrow \mathbb{R}, \text{ or } \partial_i|_p : C^\infty(M) \rightarrow \mathbb{R}$$

by

$$\left. \frac{\partial}{\partial x^i} \right|_p f = \frac{\partial f}{\partial x^i}(p)$$

Exercise 3.5.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \dots, n\}$, we have that

$$\left. \frac{\partial}{\partial x^i} \right|_p x^j = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \dots, n\}$. Then

$$\begin{aligned} \left. \frac{\partial}{\partial x^i} \right|_p x^j &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} x^j \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} u^j \circ \phi \circ \phi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\phi(p)} u^j \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 3.5.0.3. Change of Coordinates:

Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$, $p \in U \cap V$ and $f \in C^\infty(M)$. Then for each $i \in \{1, \dots, n\}$,

$$\left. \frac{\partial}{\partial y^i} \right|_p = \sum_{j=1}^n \left. \frac{\partial}{\partial x^j} \right|_p y^j(p) \left. \frac{\partial}{\partial x^i} \right|_p$$

Proof. Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \dots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned} \left. \frac{\partial}{\partial y^i} \right|_p f &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \psi^{-1} \\ &= \left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} f \circ \phi^{-1} \circ h \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial u^j} \right|_{h \circ \psi(p)} f \circ \phi^{-1} \right) \left(\left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} h_j \right) \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial u^j} \right|_{\phi(p)} f \circ \phi^{-1} \right) \left(\left. \frac{\partial}{\partial u^i} \right|_{\psi(p)} x^j \circ \psi^{-1} \right) \\ &= \sum_{j=1}^n \left(\left. \frac{\partial}{\partial x^j} \right|_p f \right) \left(\left. \frac{\partial}{\partial y^i} \right|_p x^j \right) \end{aligned}$$

□

Definition 3.5.0.4. Let $p \in M$ and $v : C^\infty(M) \rightarrow \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation at p** if for each $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,

1. v is linear
2. v is Leibnizian

We define the **tangent space of M at p** , denoted $T_p M$, by

$$T_p M = \{v : C^\infty(M) \rightarrow \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 3.5.0.5. Let $f \in C^\infty(M)$ and $v \in T_p M$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. \square

Exercise 3.5.0.6. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

is a basis for $T_p M$ and $\dim T_p M = n$.

Proof. Clearly $\frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \in T_p M$. Let $a_1, \dots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p = 0$$

Then

$$\begin{aligned} 0 &= vx^j \\ &= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i} \Big|_p x^j \\ &= a_j \end{aligned}$$

Hence $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is independent.

Now, let $v \in T_p M$ and $f \in C^\infty(M)$. By Taylor's theorem, there exist $g_1, \dots, g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \dots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x^i} \Big|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p f \\ &= \left[\sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p \right] f \end{aligned}$$

So

$$v = \sum_{i=1}^n v(x^i) \frac{\partial}{\partial x^i} \Big|_p$$

and

$$v \in \text{span} \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

□

Definition 3.5.0.7. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. We define the **differential of F at p** , denoted $DF_p : T_p M \rightarrow T_{F(p)} N$, by

$$\left[DF_p(v) \right] (f) = v(f \circ F)$$

for $v \in T_p M$ and $f \in C^\infty(N)$.

Exercise 3.5.0.8. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. Then for each $v \in T_p M$, $DF_p(v)$ is a derivation.

Proof. Let $v \in T_p M$, $f, g \in C_{F(p)}^\infty(N)$ and $c \in \mathbb{R}$. Then

1.

$$\begin{aligned} DF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= DF_p(v)(f) + cDF_p(v)(g) \end{aligned}$$

So $DF_p(v)$ is linear.

2.

$$\begin{aligned} DF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= DF_p(v)(f) * g(F(p)) + f(F(p)) * DF_p(v)(g) \end{aligned}$$

So $DF_p(v)$ is Leibnizian and hence $DF_p(v) \in T_{F(p)} N$

□

Exercise 3.5.0.9. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ smooth and $p \in M$. If F is a diffeomorphism, then DF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \dots, x^n)$ and $\phi \circ F^{-1} = (y^1, \dots, y^n)$. Let $f \in C^\infty(N)$. Then

$$\begin{aligned} \frac{\partial}{\partial y^i} \Big|_{F(p)} f &= \frac{\partial}{\partial u^i} \Big|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \frac{\partial}{\partial x^i} \Big|_p f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[DF_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) \right] (f) &= \frac{\partial}{\partial x^i} \Big|_p f \circ F \\ &= \frac{\partial}{\partial y^i} \Big|_{F(p)} f \end{aligned}$$

Hence

$$DF_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) = \frac{\partial}{\partial y^i} \Big|_{F(p)}$$

Since $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is a basis for $T_p M$ and $\left\{ \frac{\partial}{\partial y^1} \Big|_{F(p)}, \dots, \frac{\partial}{\partial y^n} \Big|_{F(p)} \right\}$ is a basis for $T_{F(p)} N$, DF_p is an isomorphism. \square

Exercise 3.5.0.10. Let (M, \mathcal{A}) be a smooth m -dimensional manifold, (N, \mathcal{B}) a n -dimensional smooth manifold, $F : M \rightarrow N$ smooth, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^m)$ and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. Suppose that $p \in U$ and $F(p) \in V$. Define the ordered bases $B_\phi = \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^m} \Big|_p \right\}$ and $B_\psi = \left\{ \frac{\partial}{\partial y^1} \Big|_{F(p)}, \dots, \frac{\partial}{\partial y^n} \Big|_{F(p)} \right\}$. Then the matrix representation of DF_p with respect to the bases B_ϕ and B_ψ is

$$DF_p^{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Proof. Let $(DF_p)_{B_\phi, B_\psi} = (a_{i,j})_{i,j} \in \mathbb{R}^{n \times m}$. Then for each $j \in \{1, \dots, m\}$,

$$DF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i} \Big|_{F(p)}$$

This implies that

$$\begin{aligned} DF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) (y^k) &= \sum_{i=1}^n a_{i,j} \frac{\partial}{\partial y^i} \Big|_{F(p)} (y^k) \\ &= \sum_{i=1}^n a_{i,j} \delta_{i,k} \\ &= a_{k,j} \end{aligned}$$

By definition,

$$\begin{aligned} DF_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) (y^k) &= \frac{\partial}{\partial x^j} \Big|_p y^k \circ F \\ &= \frac{\partial}{\partial x^j} \Big|_p F^k \\ &= \frac{\partial F^k}{\partial x^j}(p) \end{aligned}$$

\square

Note 3.5.0.11. Since $\text{rank } DF_p$ is independent of basis, it is independent of coordinate charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$.

3.6 The Cotangent Space

Definition 3.6.0.1. Let $p \in M$. We define the **cotangent space of M at p** , denoted T_p^*M , by

$$T_p^*M = (T_pM)^*$$

Definition 3.6.0.2. Let $f \in C^\infty(M)$. We define the **differential of f at p** , denoted $df_p : T_pM \rightarrow \mathbb{R}$, by

$$df_p(v) = vf$$

Exercise 3.6.0.3. Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

Proof. Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1f + \lambda v_2f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that df_p is linear and hence $df_p \in T_p^*M$. □

Exercise 3.6.0.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \dots, n\}$,

$$dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \delta_{i,j}$$

In particular, $\{dx_p^1, \dots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ and $T_p^*M = \text{span}\{dx_p^1, \dots, dx_p^n\}$.

Proof. Let $i, j \in \{1, \dots, n\}$. Then by definition,

$$\begin{aligned} \left[dx_p^i \left(\frac{\partial}{\partial x^i} \Big|_p \right) \right]_p &= \frac{\partial}{\partial x^i} \Big|_p x^i \\ &= \delta_{i,i} \end{aligned}$$

□

Exercise 3.6.0.5. Let $f \in C^\infty(M)$, (U, ϕ) a chart on M with $\phi = (x^1, \dots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

Proof. Since $\{dx_p^1, \dots, dx_p^n\}$ is a basis for T_p^*M , for each there exist $a_1(p), \dots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left(\frac{\partial}{\partial x^j} \Big|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) &= \frac{\partial f}{\partial x^j} \Big|_p \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx_p^i$$

□

Chapter 4

Submersions and Immersions

4.1 Maps of Constant Rank

Definition 4.1.0.1. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map. We define the **rank map of F** , denoted $\text{rank } F : M \rightarrow \mathbb{N}_0$ by

$$\text{rank}_p F = \dim \text{Im } DF(p)$$

and F is said to have **constant rank** if for each $p, q \in M$, $\text{rank}_p F = \text{rank}_q F$. If F has constant rank, we define the **rank of F** , denoted $\text{rank } F$, by $\text{rank } F = \text{rank}_p F$ for $p \in M$.

Exercise 4.1.0.2. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$ and $p \in M$. Suppose that $\text{rank}_p F = k$. Then there exist $(U, \phi) \in \mathcal{A}_M$, $(V, \psi) \in \mathcal{A}_N$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi, \psi})_{i,j} = A_{i,j}$$

Proof. Define $q \in V$ by $q = F(p)$. Choose $(U', \phi') \in \mathcal{A}$ and $(V', \psi') \in \mathcal{B}$ such that $p \in U'$ and $q \in V'$. Set $Z = [DF(p)]_{\phi', \psi'}$. By assumption, $\text{rank } Z = k$. An exercise in the subsection on linear algebra implies that there exist $\sigma \in S_m$, $\tau \in S_n$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Define $\phi : U \rightarrow \sigma\phi(U)$ and $\psi : V \rightarrow \tau\psi(V)$ by

$$\phi = \sigma\phi', \quad \psi = \tau\psi'$$

A previous exercise implies that

$$[DF(p)]_{\phi, \psi} = P_\tau Z P_\sigma^*$$

□

Exercise 4.1.0.3. Constant Rank Theorem:

Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$. Suppose that F has constant rank and $\text{rank } F = k$. Then for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$ and

$$\psi \circ F \circ \phi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0)$$

Proof. Let $p \in M$. The previous exercise implies that there exist $(U_0, \phi_0) \in \mathcal{A}$, $(V_0, \psi_0) \in \mathcal{B}$ and $L \in GL(k, \mathbb{R})$ such that $p \in U$, $F(p) \in V_0$ and for each $i, j \in \{1, \dots, k\}$,

$$([DF(p)]_{\phi_0, \psi_0})_{i,j} = L_{i,j}$$

Define $\hat{M} \subset \mathbb{R}^m$, $\hat{N} \subset \mathbb{R}^n$ and $\hat{F} : \hat{M} \rightarrow \hat{N}$ by $\hat{M} = \phi_0(U_0)$, $\hat{N} = \psi_0(V_0)$ and $\hat{F} = \psi_0 \circ F \circ \phi_0^{-1}$. Set $\hat{p} = \phi_0(p)$. Let (x, y) be the standard coordinates on \mathbb{R}^m , with $\pi_x : \mathbb{R}^m \rightarrow \mathbb{R}^k$ and $\pi_y : \mathbb{R}^m \rightarrow \mathbb{R}^{m-k}$ the standard projection maps. Write $\hat{p} = (x_0, y_0)$. There exist $Q : \hat{M} \rightarrow \mathbb{R}^k$ and $R : \hat{M} \rightarrow \mathbb{R}^{n-k}$ such that $\hat{F} = (Q, R)$. By construction, $[D_x Q(x_0, y_0)] = A$. Define $G : \hat{M} \rightarrow \mathbb{R}^m$ by $G(x, y) = (Q(x, y), y)$. Then

$$\begin{aligned} [DG(x_0, y_0)] &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_x Q(x_0, y_0)] \\ [D_x \pi_y(x_0, y_0)] & [D_y \pi_y(x_0, y_0)] \end{pmatrix} \\ &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} L & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \det([DG(x_0, y_0)]) &= \det(L) \det(I) \\ &= \det(L) \\ &\neq 0 \end{aligned}$$

The inverse function theorem implies that there exist $\hat{U} \subset \hat{M}$ such that \hat{U} is open, $\hat{p} \in \hat{U}$ and $G|_{\hat{U}} : \hat{U} \rightarrow G(\hat{U})$ is a diffeomorphism. Since

$$\{U_1 \times U_2 : U_1 \subset \mathbb{R}^k, U_2 \subset \mathbb{R}^{m-k} \text{ and } U_1, U_2 \text{ are open}\}$$

is a basis for the topology on \mathbb{R}^m , there exist $\hat{U}_1 \subset \mathbb{R}^k$ and $\hat{U}_2 \subset \mathbb{R}^{m-k}$ such that \hat{U}_1, \hat{U}_2 are open, $\hat{p} \in \hat{U}_1 \times \hat{U}_2$ and $\hat{U}_1 \times \hat{U}_2 \subset \hat{U}$. Set $\hat{U}_{12} = \hat{U}_1 \times \hat{U}_2$. Since $G(\hat{U}_1 \times \hat{U}_2) = Q(\hat{U}_{12}) \times \hat{U}_2$, we have that $G|_{\hat{U}_{12}} : \hat{U}_{12} \rightarrow Q(\hat{U}_{12}) \times \hat{U}_2$ is a diffeomorphism. Since π_x is open, $Q(\hat{U}_{12})$ is open. There exist $A : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_1$ and $B : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_2$ such that $G^{-1} = (A, B)$. Define $\tilde{R} : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \mathbb{R}^{n-k}$ by $\tilde{R}(x, y) = R(A(x, y), y)$. Let $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$. Then

$$\begin{aligned} (x, y) &= G \circ G^{-1}(x, y) \\ &= G(A(x, y), B(x, y)) \\ &= (Q(A(x, y), B(x, y)), B(x, y)) \end{aligned}$$

This implies that $B(x, y) = y$,

$$\begin{aligned} x &= Q(A(x, y), B(x, y)) \\ &= Q(A(x, y), y) \end{aligned}$$

Hand

$$\begin{aligned} G^{-1}(x, y) &= (A(x, y), B(x, y)) \\ &= (A(x, y), y) \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{F} \circ G^{-1}(x, y) &= \hat{F}(A(x, y), y) \\ &= (Q(A(x, y), y), R(A(x, y), y)) \\ &= (x, R(A(x, y), y)) \\ &= (x, \tilde{R}(x, y)) \end{aligned}$$

We note that

$$\begin{aligned} [D(\hat{F} \circ G^{-1})(x, y)] &= \begin{pmatrix} [D_x \pi_x(x, y)] & [D_y \pi_x(x, y)] \\ [D_x \tilde{R}(x, y)] & [D_y \tilde{R}(x, y)] \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ [D_x \tilde{R}(x, y)] & [D_y \tilde{R}(x, y)] \end{pmatrix} \end{aligned}$$

Since $G^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \rightarrow \hat{U}_{12}$ is a diffeomorphism, we have that $[DG^{-1}(x, y)] \in GL(m, \mathbb{R})$. Since \hat{F} has constant rank and $\text{rank } \hat{F} = k$, we have that

$$\begin{aligned} \text{rank}[D(\hat{F} \circ G^{-1})(x, y)] &= \text{rank}([D\hat{F}(G^{-1}(x, y))][DG^{-1}(x, y)]) \\ &= \text{rank}[D\hat{F}(G^{-1}(x, y))] \\ &= k \end{aligned}$$

Since $\text{rank} \begin{pmatrix} I \\ [D_x \tilde{R}(x, y)] \end{pmatrix} = k$, we have that $\text{rank} \begin{pmatrix} 0 \\ [D_y \tilde{R}(x, y)] \end{pmatrix} = 0$. Thus $[D_y \tilde{R}(x, y)] = 0$. Since $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$ is arbitrary, for each $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\tilde{R}(x, y) = \tilde{R}(x, y_0)$$

Define $\tilde{S} : Q(\hat{U}_{12}) \rightarrow \mathbb{R}^{n-k}$ by $\tilde{S}(x) = \tilde{R}(x, y_0)$. Then for each $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\hat{F} \circ G^{-1}(x, y) = (x, \tilde{S}(x))$$

Let (a, b) be the standard coordinates on \mathbb{R}^n , with $\pi_a : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $\pi_b : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ the standard projection maps. Write $\hat{F}(\hat{p}) = (a_0, b_0)$. Set

$$\begin{aligned} \hat{V} &= [(\pi_a)|_{\hat{N}}]^{-1}(Q(\hat{U}_{12})) \\ &= \pi_a^{-1}(Q(\hat{U}_{12})) \cap \hat{N} \end{aligned}$$

Since $Q(\hat{U}_{12})$ is open, \hat{N} is open and π_a is continuous, we have that \hat{V} is open. Since

$$\begin{aligned} Q(\hat{U}_{12}) &= (\pi_a)|_{\hat{N}} \circ \hat{F} \circ G^{-1}(Q(\hat{U}_{12}) \times \hat{U}_2) \\ &= (\pi_a)|_{\hat{N}} \circ \hat{F}(\hat{U}_{12}) \end{aligned}$$

we have that $\hat{F}(\hat{U}_{12}) \subset \hat{V}$. In particular, $\hat{F}(\hat{p}) \in \hat{V}$. Define $H : \hat{V} \rightarrow \mathbb{R}^n$ by $H(a, b) = (a, b - \tilde{S}(a))$. Then $H \circ \hat{F} \circ G^{-1}(x, y) = (x, 0)$. Define $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{N}$ by $U = \phi_0^{-1}(\hat{U}_{12})$, $V = \psi_0^{-1}(\hat{V})$, $\phi = G \circ \phi_0$ and $\psi = H \circ \psi_0$. Then for each $(x, y) \in \phi(U)$,

$$\begin{aligned} \psi \circ F \circ \phi^{-1}(x, y) &= H \circ \psi_0 \circ F \circ \phi_0^{-1} \circ G^{-1}(x, y) \\ &= H \circ \hat{F} \circ G^{-1}(x, y) \\ &= (x, 0) \end{aligned}$$

□

Definition 4.1.0.4. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map. Then F is said to be

- an **immersion** if for each $p \in M$, $DF(p) : T_p M \rightarrow T_{F(p)} N$ is injective
- a **submersion** if for each $p \in M$, $DF(p) : T_p M \rightarrow T_{F(p)} N$ is surjective

Exercise 4.1.0.5. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds, $F : M \rightarrow N$ a smooth map.

Definition 4.1.0.6. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds and $F : M \rightarrow N$ smooth. Then F is said to be an **embedding** if

1. F is an immersion
2. $F : M \rightarrow F(M)$.

Note 4.1.0.7. Here the topology on $F(M)$ is the subspace topology.

4.2 Submanifolds

Exercise 4.2.0.1. Let (M, \mathcal{A}) be a smooth manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $\tilde{U} \subset S$ and $\tilde{\phi} : \tilde{U} \rightarrow \phi(\tilde{U})$ by $\tilde{U} = U \cap S$ and $\tilde{\phi} = \phi|_{U \cap S}$. Set $\mathcal{B} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \in \mathcal{A}\}$. Then \mathcal{B} is a smooth structure on S .

Proof.

□

Definition 4.2.0.2. Let (M, \mathcal{A}) and (N, \mathcal{B}) be smooth manifolds. Suppose that $M \subset N$. Then (M, \mathcal{A}) is said to be

1. an **immersed submanifold** of (N, \mathcal{B}) if $\text{id} : M \rightarrow N$ is a smooth immersion
2. an **embedded submanifold** of (N, \mathcal{B}) if $\text{id} : M \rightarrow N$ is a smooth embedding

Note 4.2.0.3. Essentially, embedded submanifolds are immersed submanifolds with the subspace topology.

Note 4.2.0.4. For the remainder of this section, we assume that $k \leq n$.

Definition 4.2.0.5. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Then S is said to be a **k -slice** of U if $S = \{u \in U : u^{k+1}, \dots, u^n = 0\}$.

Exercise 4.2.0.6. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Suppose that S is a k -slice of U . Define $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then $\pi|_S \rightarrow \pi(S)$ is a diffeomorphism.

Proof. Clear. □

Definition 4.2.0.7. Let (M, \mathcal{A}) be a smooth manifold, $(U, \phi) \in \mathcal{A}$ and $S \subset U$. Then S is said to be a **k -slice** of U if $\phi(S)$ is a k -slice of $\phi(U)$.

Definition 4.2.0.8. Let (M, \mathcal{A}) be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$. Then (U, ϕ) is said to be a **k -slice chart for S** if $U \cap S$ is a k -slice of U .

Exercise 4.2.0.9. Let (M, \mathcal{A}) be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. If (U, ϕ) is a k -slice chart for S , then $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$.

Proof. Clear. □

Definition 4.2.0.10. Let (M, \mathcal{A}) be a smooth manifold and $S \subset M$. Then S is said to satisfy the **local k -slice condition** if for each $p \in S$, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k -slice chart of S .

Exercise 4.2.0.11. Let (M, \mathcal{A}) be a n -dimensional smooth manifold and $S \subset M$ a subspace. If S satisfies the local k -slice condition, then there exists a smooth structure $\tilde{\mathcal{A}}$ on S such that $(S, \tilde{\mathcal{A}})$ is an embedded submanifold of M .

Proof. Suppose that S satisfies the local k -slice condition. Define $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ as above. Let $(U, \phi) \in \mathcal{A}$. Suppose that (U, ϕ) is a k -slice chart for S . Define $\tilde{U} = U \cap S$ and $\tilde{\phi} : \tilde{U} \rightarrow \pi \circ \phi(\tilde{U})$ by

$$\tilde{\phi} = \pi \circ \phi|_{\tilde{U}}$$

By definition, $\phi(\tilde{U})$ is a k -slice of $\phi(U)$. A previous exercise implies that $\pi|_{\phi(\tilde{U})} \rightarrow \pi \circ \phi(\tilde{U})$ is a diffeomorphism and hence a homeomorphism. Thus $\tilde{\phi}$ is a homeomorphism.

Define

$$\tilde{\mathcal{B}} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \text{ is a } k\text{-slice for } S\}$$

Let $p \in S$. By assumption, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k -slice chart of S . Then $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{B}}$ and $\tilde{\mathcal{A}}$ is an atlas on S . By construction of $\tilde{\mathcal{B}}$, S is locally half Euclidean of dimension k . Since M is second countable Hausdorff, so is S in the subspace topology. Thus $(S, \tilde{\mathcal{B}})$ is a k -dimensional manifold. Let $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$. Then

$$\tilde{\phi} \circ \tilde{\psi}^{-1}|_{\tilde{U} \cap \tilde{V}} = \pi|_{\phi(\tilde{U} \cap \tilde{V})} \circ \phi|_{\tilde{U} \cap \tilde{V}} \circ \psi|_{\tilde{U} \cap \tilde{V}}^{-1} \circ \pi|_{\psi(\tilde{U} \cap \tilde{V})}^{-1}$$

which is a diffeomorphism. So $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ smoothly compatible. Hence $\tilde{\mathcal{B}}$ is smooth. An exercise in section 4.1 implies that there exists a unique smooth structure $\tilde{\mathcal{A}}$ on S such that $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$. So $(S, \tilde{\mathcal{A}})$ is a smooth k -dimensional manifold.

Clearly $\text{id} : S \rightarrow S$ is a homeomorphism. Let $(V, \psi) \in \mathcal{A}$ and $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$.

Finish!! □

Definition 4.2.0.12.

Exercise 4.2.0.13.

Chapter 5

Vector Fields

5.1 The Tangent Bundle

Definition 5.1.0.1. Let (M, \mathcal{A}_M) be an n -dimensional smooth manifold. We define the **tangent bundle** of M , denoted TM , by

$$TM = \coprod_{p \in M} T_p M$$

and we define the **tangent bundle projection**, denoted $\pi : TM \rightarrow M$, by

$$\pi(p, v) = p$$

Let $(U, \phi) \in \mathcal{A}_M$ with $\phi = (x^1, \dots, x^n)$. We define $\Phi_\phi : \pi^{-1}(U) \rightarrow \mathbb{R}^{2n}$ by

$$\Phi_\phi \left(p, \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j} \Big|_p \right) = (\phi(p), \xi^1, \dots, \xi^n)$$

We define $\mathcal{T}_{TM} = \tau_{TM}(\iota_p : p \in M)$.

Exercise 5.1.0.2. $\psi : \bigcup_{p \in U} T_p M \rightarrow \mathbb{R}^n$ is given by

$$\psi \left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j} \Big|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i} \Big|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i} \Big|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned}
\left. \frac{\partial}{\partial \tilde{y}^i} \right|_{(p, \xi)} [x^k \circ \pi] &= \left. \frac{\partial}{\partial v^i} \right|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \left. \frac{\partial}{\partial v^i} \right|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \left. \frac{\partial}{\partial v^i} \right|_{\phi(p)} [x^k \circ \phi^{-1}] \\
&= 0
\end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
D\pi(p, \xi) \left(\left. \frac{\partial}{\partial \tilde{x}^i} \right|_{(p, \xi)} \right) (f) &= \left. \frac{\partial}{\partial \tilde{x}^i} \right|_{(p, \xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i} (p, \xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) \delta_{i,k} \\
&= \frac{\partial f}{\partial x^i} (p)
\end{aligned}$$

and

$$\begin{aligned}
D\pi(p, \xi) \left(\left. \frac{\partial}{\partial \tilde{y}^i} \right|_{(p, \xi)} \right) (f) &= \left. \frac{\partial}{\partial \tilde{y}^i} \right|_{(p, \xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i} (p, \xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) 0 \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p, \xi) \in \pi^{-1}(U)} \ker D\pi(p, \xi) \\
&= \coprod_{(p, \xi) \in \pi^{-1}(U)} \text{span} \left\{ \left. \frac{\partial}{\partial \tilde{y}^j} \right|_{(p, \xi)} : j \in \{1, \dots, n\} \right\}
\end{aligned}$$

Chapter 6

Lie Theory

6.1 Lie Groups

Definition 6.1.0.1. Let G be a smooth manifold and group. Then G is said to be a **Lie group** if

- multiplication $G \times G \rightarrow G$ given by $(g, h) \mapsto gh$ is smooth
- inversion $G \rightarrow G$ given by $g \mapsto g^{-1}$ is smooth

Definition 6.1.0.2. Let \mathfrak{g} be a vector space and $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. Then $[\cdot, \cdot]$ is said to be a **Lie bracket** on \mathfrak{g} if

1. $[\cdot, \cdot]$ is bilinear
2. $[\cdot, \cdot]$ is antisymmetric
3. $[\cdot, \cdot]$ satisfies the Jacobi identity:
for each $x, w, y \in \mathcal{F}g$,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

In this case, $(\mathfrak{g}, [\cdot, \cdot])$ is said to be a **Lie algebra**.

Definition 6.1.0.3. Let $X \in$

Chapter 7

Bundles and Sections

7.1 Fiber Bundles

Note 7.1.0.1. Let U, F be sets, we write $\text{proj}_1 : U \times F \rightarrow U$ to denote the projection onto U .

Definition 7.1.0.2. Let $E, M, F \in \mathbf{Man}^\infty$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection, $U \subset M$ open and $\Phi : \pi^{-1}(U) \rightarrow U \times F$. Then (U, Φ) is said to be a **smooth local trivialization of E over U with fiber F** if

1. Φ is a diffeomorphism
2. $\text{proj}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ & \searrow \pi & \downarrow \text{proj}_1 \\ & & U \end{array}$$

Exercise 7.1.0.3. Let E, M and F be sets and $\pi : E \rightarrow M$ and $\Phi : \pi^{-1}(U) \rightarrow U \times F$ a bijection. If $\text{proj}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$, then for each $A \subset U$,

$$\Phi(\pi^{-1}(A)) = A \times F$$

Hint: consider $\Phi^{-1}(A \times F)$

Proof. Let $A \subset U$. Since $\text{proj}_1^{-1}(A) = A \times F$, we have that

$$\begin{aligned} \Phi^{-1}(A \times F) &= \Phi^{-1}(\text{proj}_1^{-1}(A)) \\ &= (\text{proj}_1 \circ \Phi)^{-1}(A) \\ &= (\pi|_{\pi^{-1}(U)})^{-1}(A) \\ &= \pi^{-1}(A) \cap \pi^{-1}(U) \\ &= \pi^{-1}(A \cap U) \\ &= \pi^{-1}(A) \end{aligned}$$

Since Φ is a bijection, we have that

$$\begin{aligned} \Phi(\pi^{-1}(A)) &= \Phi \circ \Phi^{-1}(A \times F) \\ &= A \times F \end{aligned}$$

□

Definition 7.1.0.4. Let $E, M, F \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then (E, M, π, F) is said to be a **smooth fiber bundle with total space E , base space M , fiber F and projection π** if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \rightarrow U \times F$ such that U is open and (U, Φ) is a smooth local trivialization of E over U with fiber F . For $p \in M$, we define the **fiber over p** , denoted E_p , by $E_p = \pi^{-1}(\{p\})$.

Note 7.1.0.5. When the context is clear, we will suppress the fiber manifold F .

Definition 7.1.0.6. Let (E_1, M_1, π_1, F_1) and (E_2, M_2, π_2, F_2) be smooth fiber bundles, $\Phi \in \text{Hom}_{\mathbf{Man}^\infty}(E_1, E_2)$ and $\phi \in \text{Hom}_{\mathbf{Man}^\infty}(M_1, M_2)$. Then (Φ, ϕ) is said to be a **smooth bundle morphism** from (E_1, M_1, π_1, F_1) to (E_2, M_2, π_2, F_2) if $\pi_2 \circ \Phi = \phi \circ \pi_1$, i.e. the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{\phi} & M_2 \end{array}$$

Definition 7.1.0.7. We define the category of smooth fiber bundles, denoted \mathbf{Bun}^∞ , by

- $\text{Obj}(\mathbf{Bun}^\infty) = \{(E, M, \pi, F) : (E, M, \pi, F) \text{ is a smooth fiber bundle}\}$
- For $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \text{Obj}(\mathbf{Bun}^\infty)$,

$$\begin{aligned} \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) = \\ \{(\Phi, \phi) : (\Phi, \phi) \text{ is a smooth bundle morphism from } (E_1, M_1, \pi_1, F_1) \text{ to } (E_2, M_2, \pi_2, F_2)\} \end{aligned}$$

- For
 - $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3) \in \text{Obj}(\mathbf{Bun}^\infty)$
 - $(\Phi_{12}, \phi_{12}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$
 - $(\Phi_{23}, \phi_{23}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3))$

we define $(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) \in \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_3, M_3, \pi_3))$ by

$$(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) = (\Phi_{23} \circ \Phi_{12}, \phi_{23} \circ \phi_{12})$$

Exercise 7.1.0.8. We have that \mathbf{Bun}^∞ is a full subcategory of $(\text{id}_{\mathbf{Man}^\infty} \downarrow \text{id}_{\mathbf{Man}^\infty})$.

Proof. Set $\mathcal{C} = (\text{id}_{\mathbf{Man}^\infty} \downarrow \text{id}_{\mathbf{Man}^\infty})$. We note that

- $\text{Obj}(\mathbf{Bun}^\infty) \subset \text{Obj}(\mathcal{C})$
- for each $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \text{Obj}(\mathbf{Bun}^\infty)$,

$$\text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) = \text{Hom}_{\mathcal{C}}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$$

So \mathbf{Bun}^∞ is a full subcategory of \mathcal{C} . □

Exercise 7.1.0.9. Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$ and (U, Φ) a local trivialization of E over U and (V, Ψ) a local trivialization of E over V . Then

1. $\text{proj}_{U \cap V} \circ \Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1} = \text{proj}_1$
2. there exists $\sigma \in \text{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that for each $p \in U \cap V$, $\sigma(p, \cdot) : F \rightarrow F$ is a diffeomorphism.

Proof.

1. By definition, the following diagram commutes:

$$\begin{array}{ccccc}
 (U \cap V) \times F & \xleftarrow{\Phi} & \pi^{-1}(U \cap V) & \xrightarrow{\Psi} & (U \cap V) \times F \\
 & \searrow \text{proj}_1 & \downarrow \pi & \swarrow \text{proj}_1 & \\
 & & N & &
 \end{array}$$

$$\text{proj}_1 \circ \Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1} = \text{proj}_1$$

2. there exists $\sigma \in \text{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that for each $p \in U \cap V$ and $x \in F$,

$$\Psi|_{\pi^{-1}(U \cap V)} \circ (\Phi|_{\pi^{-1}(U \cap V)})^{-1}(p, x) = (p, \sigma(p, x))$$

and $\sigma(p, \cdot) : F \rightarrow F$ is a diffeomorphism.

□

Definition 7.1.0.10. Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$ and $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ a collection of smooth local trivializations of E . Then $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ is said to be a **fiber bundle atlas** if for each $p \in M$, there exists $\alpha \in A$ such that $p \in U_\alpha$. For $\alpha, \beta \in A$, we define ϕ

7.2 G -Bundles

Definition 7.2.0.1. Let G be a Lie group and $(E, M, \pi, F) \in \text{Obj}(\mathbf{Bun}^\infty)$. Then

7.3 Vector Bundles

Note 7.3.0.1. Let M be a set and $p \in M$. We endow $\{p\} \times \mathbb{R}^n$ with the natural vector space structure such that $\{p\} \times \mathbb{R}^n \cong \mathbb{R}^n$.

Definition 7.3.0.2. Let $E, M \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then (E, M, π) is said to be a **rank n smooth vector bundle** if

1. $(E, M, \pi, \mathbb{R}^n) \in \text{Obj}(\mathbf{Bun}^\infty)$
2. for each $p \in M$, E_p is a n -dimensional real vector space
3. for each smooth local trivialization (U, Φ) of E over U with fiber \mathbb{R}^n and $p \in U$,

$$\Phi|_{E_p} : E_p \rightarrow \{p\} \times \mathbb{R}^n$$

is a vector space isomorphism

In this case we define the **rank of** (E, M, π) , denoted $\text{rank}(E, M, \pi)$ by $\text{rank}(E, M, \pi) = n$.

Definition 7.3.0.3. We define the category of smooth vector bundles, denoted \mathbf{VecBun}^∞ , by

- $\text{Obj}(\mathbf{VecBun}^\infty) = \{(E, M, \pi) : (E, M, \pi) \text{ is a smooth vector bundle}\}$
- For $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{VecBun}^\infty)$ with $\text{rank}(E_1, M_1, \pi_1) = n_1$ and $\text{rank}(E_2, M_2, \pi_2) = n_2$,

$$\text{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{n_1}), (E_2, M_2, \pi_2, \mathbb{R}^{n_2}))$$

Exercise 7.3.0.4. We have that \mathbf{VecBun}^∞ is a full subcategory of \mathbf{Bun}^∞ .

Proof. We note that

- $\text{Obj}(\mathbf{VecBun}^\infty) \subset \text{Obj}(\mathbf{Bun}^\infty)$
- for each $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \text{Obj}(\mathbf{Bun}^\infty)$ with $\text{rank}(E_1, M_1, \pi_1) = n_1$ and $\text{rank}(E_2, M_2, \pi_2) = n_2$,

$$\text{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \text{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{n_1}), (E_2, M_2, \pi_2, \mathbb{R}^{n_2}))$$

So \mathbf{Bun}^∞ is a full subcategory of \mathcal{C} . □

Exercise 7.3.0.5. Let $M \in \mathbf{Man}^\infty$. Set $n = \dim M$, $E = M \times \mathbb{R}^n$ and define $\pi : E \rightarrow M$ by $\pi(p, x) = p$. Then (E, M, π) is a smooth vector bundle of rank n .

Proof.

1. For each $p \in M$, $\pi^{-1}(\{p\}) = \{p\} \times \mathbb{R}^n$ is an n -dimensional real vector space.
2. Let $p \in M$. Set $U = M$. Then $\pi^{-1}(U) = E$. Define $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ by $\Phi = \text{id}_E$. Then (U, Φ) is a smooth local trivialization of E over U .
3. Let $p \in M$. Then $\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \rightarrow \{p\} \times \mathbb{R}^n$ is clearly an isomorphism.

□

Exercise 7.3.0.6. Let $(E, M, \pi) \in \mathbf{VecBun}^\infty$ with $\text{rank}(E, M, \pi) = n$.

Theorem 7.3.0.7. Let $(E,)$ and M be smooth manifolds and $\pi : E \rightarrow M$ a smooth surjection.

7.4 Bundle Morphisms

Definition 7.4.0.1. Let (E, M, π_E) and (F, N, π_F) be smooth fiber bundles and $\Phi : E \rightarrow F$ and $\phi : M \rightarrow N$. Then (Φ, ϕ) is said to be a **smooth fiber bundle morphism** from (E, M, π_E) to (F, N, π_F) if Φ is smooth, ϕ is smooth and $\pi_F \circ \Phi = \phi \circ \pi_E$, i.e. the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ \pi_E \downarrow & & \downarrow \pi_F \\ M & \xrightarrow{\phi} & N \end{array}$$

and we write $(\Phi, \phi) : (E, M, \pi_E) \rightarrow (F, N, \pi_F)$.

Exercise 7.4.0.2. Let (E, M, π_E) and (F, N, π_F) be smooth fiber bundles and $(\Phi, \phi) : (E, M, \pi_E) \rightarrow (F, N, \pi_F)$. Suppose that (Φ, ϕ) is smooth. Then for each $p \in M$,

$$\Phi^{-1}(F_{\phi(p)}) = E_p$$

Proof. Let $p \in M$. Set $q = \phi(p)$. Then

$$\begin{aligned} \Phi^{-1}(F_q) &= \Phi^{-1}(\pi_F^{-1}(\{q\})) \\ &= (\pi_F \circ \Phi)^{-1}(\{q\}) \\ &= (\phi \circ \pi_E)^{-1}(\{q\}) \\ &= \pi_E^{-1}(\phi^{-1}(\{\phi(p)\})) \end{aligned}$$

FINISH!!!, multiple fibers get mapped to same fiber

□

7.5 Subbundles

7.6 Vertical and Horizontal Subbundles

Definition 7.6.0.1. Let $(E, M, \pi_M) \in \text{Obj}(\mathbf{Bun}^\infty)$. We define the **vertical bundle associated to** (E, M, π_M) , denoted $(VE, M, \pi_V) \in \mathbf{Bun}^\infty$, by

$$VE = \coprod_{q \in E} \ker D\pi(q)$$

relocate this to after tangent bundle is introduced

Exercise 7.6.0.2. Let (M, \mathcal{A}) be an n -dimensional smooth manifold and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $(\pi^{-1}(U), \Phi_\phi) \in \mathcal{A}_{TM}$ the induced chart on TM with $\Phi_\phi = (\tilde{x}^1, \dots, \tilde{x}^n, \tilde{y}^1, \dots, \tilde{y}^n)$. Then

$$V(TM)|_{\pi^{-1}(U)} = \coprod_{(p, \xi) \in \pi^{-1}(U)} \text{span} \left\{ \frac{\partial}{\partial \tilde{y}^j} \Big|_{(p, \xi)} : j \in \{1, \dots, n\} \right\}$$

Split into smaller exercises

Proof. Let $f \in C^\infty(M)$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$ the standard coordinates on $\mathbb{R}^n \times \mathbb{R}^n$. We note that by definition, $\Phi_\phi(p, \xi) = (\phi(p), \psi(\xi))$ where $\psi : \bigcup_{p \in U} T_p M \rightarrow \mathbb{R}^n$ is given by

$$\psi \left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j} \Big|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i} \Big|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i} \Big|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} [x^k \circ \pi] &= \frac{\partial}{\partial v^i} \Big|_{\Phi_\phi(p, \xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i} \Big|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i} \Big|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= 0 \end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
 D\pi(p, \xi) \left(\frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} \right) (f) &= \frac{\partial}{\partial \tilde{x}^i} \Big|_{(p, \xi)} f \circ \pi \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{x}^i} (p, \xi) \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) \delta_{i,k} \\
 &= \frac{\partial f}{\partial x^i} (p)
 \end{aligned}$$

and

$$\begin{aligned}
 D\pi(p, \xi) \left(\frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} \right) (f) &= \frac{\partial}{\partial \tilde{y}^i} \Big|_{(p, \xi)} f \circ \pi \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (\pi(p, \xi)) \frac{\partial x^k \circ \pi}{\partial \tilde{y}^i} (p, \xi) \\
 &= \sum_{k=1}^n \frac{\partial f}{\partial x^k} (p) 0 \\
 &= 0
 \end{aligned}$$

Hence

$$\begin{aligned}
 V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p, \xi) \in \pi^{-1}(U)} \ker D\pi(p, \xi) \\
 &= \coprod_{(p, \xi) \in \pi^{-1}(U)} \text{span} \left\{ \frac{\partial}{\partial \tilde{y}^j} \Big|_{(p, \xi)} : j \in \{1, \dots, n\} \right\}
 \end{aligned}$$

□

7.7 The Tangent Bundle

Definition 7.7.0.1. We define the **tangent bundle of M** , denoted TM , by

$$TM = \coprod_{p \in M} T_p M$$

We denote the natural projection map by $\pi : TM \rightarrow M$.

Definition 7.7.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Define $\tilde{U} \subset TM$ and $\tilde{\phi} : \tilde{U} \rightarrow \phi(U) \times \mathbb{R}^n$ by

- $\tilde{U} = \pi^{-1}(U)$
-

$$\begin{aligned} \tilde{\phi} \left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p \right) &= (\phi(p), v) \\ &= (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \end{aligned}$$

Exercise 7.7.0.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then $\tilde{\phi} : \tilde{U} \rightarrow \phi(U) \times \mathbb{R}^n$ is a bijection.

7.8 The cotangent Bundle

Definition 7.8.0.1. We define the **cotangent bundle of M** , denoted T^*M , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

7.9 The (r, s) -Tensor Bundle

Definition 7.9.0.1. 1. the **cotangent bundle of M** , denoted T^*M , by

$$T^*M = \coprod_{p \in M} T_p^*M$$

2. the **(r, s) -tensor bundle of M** , denoted $T_s^r M$, by

$$T_s^r M = \coprod_{p \in M} T_s^r(T_p M)$$

3. the **k -alternating tensor bundle of M** , denoted $\Lambda^k(M)$, by

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k(T_p M)$$

7.10 Vector Fields

Definition 7.10.0.1. Let $X : M \rightarrow TM$. Then X is said to be a **vector field on M** if for each $p \in M$, $X_p \in T_p M$.

For $f \in C^\infty(M)$, we define $Xf : M \rightarrow \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in C^\infty(M)$, Xf is smooth.

We denote the set of smooth vector fields on M by $\Gamma^1(M)$.

Definition 7.10.0.2. Let $f \in C^\infty(M)$ and $X, Y \in \Gamma^1(M)$. We define

- $fX \in \Gamma^1(M)$ by

$$(fX)_p = f(p)X_p$$

- $X + Y \in \Gamma^1(M)$ by

$$(X + Y)_p = X_p + Y_p$$

Exercise 7.10.0.3. The set $\Gamma^1(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Exercise 7.10.0.4. Let $X \in \Gamma^1(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then

$$X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$$

Proof. Let $p \in M$. Then $X_p \in T_p M$ and $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ is a basis of $T_p M$. So there exist $f_1(p), \dots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i} \Big|_p$. Let $j \in \{1, \dots, n\}$. Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x^i} x^j(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx^j = f_j$ and $X|_U = \sum_{i=1}^n (Xx^i) \frac{\partial}{\partial x^i}$. □

Exercise 7.10.0.5. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then for each $i \in \{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

Proof. Let $i \in \{1, \dots, n\}$ and $f \in C^\infty(M)$. Define $g : M \rightarrow \mathbb{R}$ by $g = \frac{\partial}{\partial x^i} f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i} \Big|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i} \Big|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i} [f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence g is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x^i}$ is smooth. □

7.11 1-Forms

Definition 7.11.0.1. Let $\omega : M \rightarrow T^*M$. Then ω is said to be a **1-form on M** if for each $p \in M$, $\omega_p \in T_p^*M$.

For each $X \in \Gamma^1(M)$, we define $\omega(X) : M \rightarrow \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)$, $\omega(X)$ is smooth. The set of smooth 1-forms on M is denoted $\Gamma_1(M)$.

Definition 7.11.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma_1(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by

$$(f\omega)_p = f(p)\omega_p$$

- $\alpha + \beta \in \Gamma_1(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 7.11.0.3. The set $\Gamma_1(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Exercise 7.11.0.4.

7.12 (r, s) -Tensor Fields

Definition 7.12.0.1. Let $\alpha : M \rightarrow T_s^r M$. Then α is said to be an (r, s) -**tensor field on** M if for each $p \in M$, $\alpha_p \in T_p^r(T_p M)$.

For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \rightarrow \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and α is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth.

The set of smooth (r, s) -tensor fields on M is denoted $T_s^r(M)$.

Definition 7.12.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T_s^r(M)$. We define

- $f\alpha : M \rightarrow T_s^r M$ by

$$(f\alpha)_p = f(p)\alpha_p$$

- $\alpha + \beta : M \rightarrow T_s^r M$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 7.12.0.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T_s^r(M)$. Then

1. $f\alpha \in T_s^r(M)$ by

$$(f\alpha)_p = f(p)\alpha_p$$

2. $\alpha + \beta \in T_s^r(M)$ by

$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Proof. Clear. □

Exercise 7.12.0.4. The set $T_s^r(M)$ is a $C^\infty(M)$ -module.

Proof. Clear. □

Definition 7.12.0.5. Let $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$ and $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$. We define the **tensor product of** α **with** β , denoted $\alpha \otimes \beta : M \rightarrow T_{s_1+s_2}^{r_1+r_2} M$, by

$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

Exercise 7.12.0.6. Let $\alpha_1 \in \Gamma_{s_1}^{r_1}(M)$ and $\alpha_2 \in \Gamma_{s_2}^{r_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma_{s_1+s_2}^{r_1+r_2}(M)$

Proof. Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since α_1 and α_2 are smooth by assumption. □

Definition 7.12.0.7. We define the **tensor product**, denoted $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

Exercise 7.12.0.8. The tensor product $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ is associative.

Proof. Clear. □

Exercise 7.12.0.9. The tensor product $\otimes : \Gamma_{s_1}^{r_1}(M) \times \Gamma_{s_2}^{r_2}(M) \rightarrow \Gamma_{s_1+s_2}^{r_1+r_2}(M)$ is $C^\infty(M)$ -bilinear.

Proof. Clear. □

Definition 7.12.0.10. Let (N, \mathcal{B}) be a smooth manifold, $F : M \rightarrow N$ a smooth map and $\alpha \in \Gamma_k^0(N)$. We define the **pullback of α by F** , denoted $F^*\alpha \in \Gamma_k^0(M)$, by

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k))$$

for $p \in M$ and $v_1, \dots, v_k \in T_p M$

Exercise 7.12.0.11. Let (M, \mathcal{A}) , (N, \mathcal{B}) and (L, \mathcal{C}) be smooth manifolds, $F : M \rightarrow N$ and $G : N \rightarrow L$ smooth maps, $\alpha \in \Gamma_k^0(N)$, $\beta \in \Gamma_l^0(N)$, $\gamma \in \Gamma_k^0(L)$ and $f \in C^\infty(N)$. Then

1. $F^*(f\alpha) = (f \circ F)F^*\alpha$
2. $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
3. $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
4. $(G \circ F)^*\gamma = F^*(G^*\gamma)$
5. $id_N^*\alpha = \alpha$

Proof.

1.

$$\begin{aligned} [F^*(f\alpha)]_p(v_1, \dots, v_k) &= (f\alpha)_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\ &= f(F(p))\alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\ &= (f \circ F)(p)(F^*\alpha)_p(v_1, \dots, v_k) \end{aligned}$$

So that $F^*(f\alpha) = (f \circ F)F^*\alpha$

2.

$$F^*$$

□

Definition 7.12.0.12.

Exercise 7.12.0.13.

Proof.

□

Exercise 7.12.0.14. Let $\alpha \in T_s^r(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. Then there exist $(f_J^I)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

Proof. Let $p \in M$. Then $\omega_p \in T_s^r(T_p M)$ and $\left\{ \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J} \right\}$ is a basis of $T_s^r(T_p M)$. So there exist $(f_J^I(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned} \alpha_p(dx_p^K, \partial_{x^L}|_p) &= \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}(dx_p^K, \partial_{x^L}|_p) \\ &= \sum_{(I, J) \in \mathcal{I}_r \times \mathcal{I}_s} f_J^I(p) \partial_{x^{\otimes I}}|_p(dx_p^K) dx_p^{\otimes J}(\partial_{x^L}|_p) \\ &= f_L^K(p) \end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx_p^K, \partial_{x^L}|_p)$ is smooth, so that $f_L^K \in C^\infty(U)$.

□

Definition 7.12.0.15.

7.13 Differential Forms

Definition 7.13.0.1. We define

$$\Lambda^k(TM) = \coprod_{p \in M} \Lambda^k(T_p M)$$

Definition 7.13.0.2. Let $\omega : M \rightarrow \Lambda^k(TM)$. Then ω is said to be a **k -form on M** if for each $p \in M$, $\omega_p \in \Lambda^k(T_p M)$.

For each $X \in \Gamma^1(M)^k$, we define $\omega(X) : M \rightarrow \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)^k$, $\omega(X)$ is smooth.

The set of smooth k -forms on M is denoted $\Omega^k(M)$.

Note 7.13.0.3. Observe that

1. $\Omega^k(M) \subset \Gamma_k^0(M)$
2. $\Omega^0(M) = C^\infty(M)$

Exercise 7.13.0.4. The set $\Omega^k(M)$ is a $C^\infty(M)$ -submodule of $\Gamma_k^0(M)$.

Proof. Clear. □

Definition 7.13.0.5. Define the **exterior product**

$$\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$$

by

$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Note 7.13.0.6. For $f \in \Omega^0(M)$ and $\alpha \in \Omega^k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 7.13.0.7. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$ is well defined.

Proof. Let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{j=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then

$$\begin{aligned} \alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{(k+l)!}{k!l!} \text{Alt}(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \sigma(\alpha_p \otimes \beta_p)(X_1(p), \dots, X_{k+l}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) (\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p)) \beta(X_{\sigma(k+1)}(p), \dots, X_{\sigma(k+l)}(p)) \end{aligned}$$

□

Exercise 7.13.0.8. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$ is $C^\infty(M)$ -bilinear.

Proof.

1. $C^\infty(M)$ -linearity in the first argument:

Let $\alpha \in \Omega^k(M)$, $\beta, \gamma \in \Omega^l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda^k(T_p M) \times \Lambda^l(T_p M) \rightarrow \Lambda^{k+l}(T_p M)$ implies that

$$\begin{aligned} [(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p \end{aligned}$$

So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

and $\wedge : \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M)$ is $C^\infty(M)$ -linear in the first argument.

2. $C^\infty(M)$ -linearity in the second argument:

Similar to (1).

□

Note 7.13.0.9. All of the results from multilinear algebra apply here.

Definition 7.13.0.10. We define the **exterior derivative** $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ inductively by

1. $d(d\alpha) = 0$ for $\alpha \in \Omega^p(M)$
2. $df(X) = Xf$ for $f \in \Omega^0(M)$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$
4. extending linearly

Exercise 7.13.0.11. Let (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then on U , for each $i, j \in \{1, \dots, n\}$,

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx_p^1, \dots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$ and $T_p^* M = \text{span}\{dx_p^1, \dots, dx_p^n\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \dots, n\}$. Then by definition,

$$\begin{aligned} \left[dx^i \left(\frac{\partial}{\partial x^j} \right) \right]_p &= \left(\frac{\partial}{\partial x^j} x^i \right)_p \\ &= \frac{\partial}{\partial x^i} \Big|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 7.13.0.12. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Proof. Let $p \in U$. Since $\{dx^1, \dots, dx^n\}$ is a basis for $\Lambda(T_p M)$, for each there exist $a_1(p), \dots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a^i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) &= \sum_{i=1}^n a^i(p) dx_p^i \left(\frac{\partial}{\partial x^i} \Big|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i} \Big|_p \right) &= \frac{\partial}{\partial x^i} \Big|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx_p^i$$

Therefore

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

□

Exercise 7.13.0.13. Let $f \in \Omega^0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \dots, x_n)$. Then for each $i \in \{1, \dots, n\}$,

$$\frac{\partial}{\partial x^i} \Big|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^n \frac{\partial f}{\partial x^j}(p) dx_p^i \\ &= 0 \end{aligned}$$

□

Exercise 7.13.0.14.

Definition 7.13.0.15. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$ and $I = (i_1, \dots, i_k) \in \mathcal{I}_k$. We define

$$dx^i = dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Omega^k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right)$$

Note 7.13.0.16. We have that

1.

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta_{I,J}$$

2. Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega^k(U)$,

$$\omega\left(\frac{\partial}{\partial x^i}\right) \in C^\infty(U)$$

Exercise 7.13.0.17. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega\left(\frac{\partial}{\partial x^i}\right) dx^i$$

Proof. Let $p \in U$. Since $\{dx_p^i : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(T_p M)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum_{I \in \mathcal{I}_k} f_I(p) dx_p^i$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned} \omega\left(\frac{\partial}{\partial x^j}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j}\right) \\ &= f_J \end{aligned}$$

□

Exercise 7.13.0.18. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \dots, x^n)$. If $\omega = \sum_{I \in \mathcal{I}_k} f_I dx^i$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

Proof. First we note that

$$\begin{aligned} d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f_I d(dx^i) \\ &= df_I \wedge dx^i \\ &= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \right) \wedge dx^i \\ &= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i \end{aligned}$$

Then we extend linearly. □

Definition 7.13.0.19. Let (N, \mathcal{B}) be a smooth manifold and $F : M \rightarrow N$ be a diffeomorphism. Define the **pullback of F** , denoted $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$ by

$$(F^* \omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(DF_p(v_1), \dots, DF_p(v_k))$$

for $\omega \in \Omega^k(N)$, $p \in M$ and $v_1, \dots, v_k \in T_p M$

Chapter 8

Connections

8.1 Koszul Connections

Definition 8.1.0.1. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$ and $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$. Then ∇ is said to be a **Koszul connection on E in the first representation** if

1. for each $\sigma \in \Gamma(E)$, $\nabla(\cdot, \sigma)$ is $C^\infty(M)$ -linear
2. for each $X \in \mathfrak{X}(M)$, $\nabla(X, \cdot)$ is \mathbb{R} -linear
3. for each $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\nabla(X, f\sigma) = f \nabla(X, \sigma) + X(f)\sigma$$

Definition 8.1.0.2. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$ be a smooth vector bundle and $\nabla : \Gamma(E) \rightarrow T^*M \otimes \Gamma(E)$. Then ∇ is said to be a **Koszul connection on E in the second representation** if

1. ∇ is \mathbb{R} -linear
2. for each $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\nabla(f\sigma) = f \nabla \sigma + df \otimes \sigma$$

Note 8.1.0.3. When the context is clear, we will write $\nabla_X Y$ in place of $\nabla(X, Y)$ and we will refer to ∇ as a connection.

Exercise 8.1.0.4. Define $\phi : \Gamma(E)^{\mathfrak{X}(M) \times \Gamma(E)} \rightarrow [T^*M \otimes \Gamma(E)]^{\Gamma(E)}$ by

$$\phi(\nabla)(X) = \nabla_X \sigma$$

Then ∇ is a Koszul connection on E in the first representation iff $\phi(\nabla)$ Koszul connection on E in the second representation.

Proof. □

Exercise 8.1.0.5. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$, ∇ a connection on E , $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$. If $X = 0$ or $Y = 0$, then $\nabla_X Y = 0$.

Proof.

- If $X = 0$, then

$$\begin{aligned} \nabla_X Y &= \nabla_{0X} Y \\ &= 0 \nabla_X Y \\ &= 0 \end{aligned}$$

- Similarly, if $Y = 0$, then $\nabla_X Y = 0$.

□

Exercise 8.1.0.6. Let (E, M, π) be a smooth vector bundle, ∇ a connection on E , $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$ and $p \in M$. If $X \sim_p 0$ or $Y \sim_p 0$, then $[\nabla_X Y]_p = 0$.

Proof.

- Suppose that $X \sim_p 0$. Then there exists $U \subset M$ such that U is open and $X|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\text{supp } \phi \subset U$ and $\phi \sim_p 1$. Then $\phi X = 0$. The previous exercise implies that $\nabla_{\phi X} Y = 0$. Therefore

$$\begin{aligned} \nabla_X Y &= \nabla_{\phi X + (1-\phi)X} Y \\ &= \nabla_{\phi X} Y + \nabla_{(1-\phi)X} Y \\ &= 0 + (1-\phi) \nabla_X Y \\ &= (1-\phi) \nabla_X Y \end{aligned}$$

Hence

$$\begin{aligned} [\nabla_X Y]_p &= [(1-\phi) \nabla_X Y]_p \\ &= (1-\phi(p))[\nabla_X Y]_p \\ &= 0 \end{aligned}$$

- Suppose that $Y \sim_p 0$. Then there exists $U \subset M$ such that U is open and $Y|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\text{supp } \phi \subset U$ and $\phi \sim_p 1$. Then $\phi Y = 0$. The previous exercise implies that $\nabla_X \phi Y = 0$. Since $\phi \sim_p 1$, we have that $1-\phi \sim_p 0$. Thus $X(1-\phi) \sim_p 0$ and

$$\begin{aligned} \nabla_X Y &= \nabla_X [\phi Y + (1-\phi)Y] \\ &= \nabla_X [\phi Y] + \nabla_X [(1-\phi)Y] \\ &= \nabla_X [(1-\phi)Y] \\ &= (1-\phi) \nabla_X Y + [X(1-\phi)] \nabla_X Y \end{aligned}$$

Hence

$$\begin{aligned} [\nabla_X Y]_p &= (1-\phi(p))[\nabla_X Y]_p + [X(1-\phi)](p)[\nabla_X Y]_p \\ &= 0 \end{aligned}$$

□

Exercise 8.1.0.7. Let (E, M, π) be a smooth vector bundle and ∇ a connection on E . Then for each $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$, $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$ implies that $[\nabla_{X_1} Y_1]_p = [\nabla_{X_2} Y_2]_p$.

Proof. Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$. Suppose that $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$. Define $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$ by $X = X_2 - X_1$ and $Y = Y_2 - Y_1$. Then $X \sim_p 0$ and $Y \sim_p 0$. The previous exercise implies

that $[\nabla_X Y_1]_p = 0$ and $[\nabla_{X_2} Y]_p = 0$. Therefore

$$\begin{aligned}
 [\nabla_{X_1} Y_1]_p &= [\nabla_{X_1} Y_1]_p + [\nabla_X Y_1]_p \\
 &= [\nabla_{X_1} Y_1 + \nabla_X Y_1]_p \\
 &= [\nabla_{X_1+X} Y_1]_p \\
 &= [\nabla_{X_2} Y_1]_p \\
 &= [\nabla_{X_2} Y_1]_p + [\nabla_{X_2} Y]_p \\
 &= [\nabla_{X_2} Y_1 + \nabla_{X_2} Y]_p \\
 &= [\nabla_{X_2} (Y_1 + Y)]_p \\
 &= [\nabla_{X_2} Y_2]_p
 \end{aligned}$$

□

Exercise 8.1.0.8. Let (E, M, π) be a smooth vector bundle, ∇ a connection on E and $U \subset M$. If U is open, then there exists a unique connection $\nabla^U : \mathfrak{X}(U) \times \Gamma(E|_U) \rightarrow \Gamma(E|_U)$ such that for each $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$,

$$\nabla^U_{X|_U} Y|_U = (\nabla_X Y)|_U$$

Chapter 9

Semi-Riemannian Geometry

Definition 9.0.0.1. Let M be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then g is said to be nondegenerate if for each $p \in M$, g_p is nondegenerate.

Definition 9.0.0.2. Let M be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then g is said to be a **metric tensor field** on M if

1. g is nondegenerate
2. g has constant index

In this case (M, g) is said to be a **semi-Riemannian manifold**

Definition 9.0.0.3. [Define Interval](#)
FINISH!!!

Definition 9.0.0.4. Let $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ and $\gamma \in \text{Hom}_{\mathbf{Man}^\infty}(I, E)$. Then γ is said to be a **section of E over α** if $\pi \circ \gamma = \alpha$. We denote the set of sections of E over α by $\Gamma(E, \alpha)$.

Definition 9.0.0.5. Let $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ and $\gamma \in \Gamma(E, \alpha)$. Then γ is said to be said to be **extendible** if there exists $U \in \mathcal{N}_{\alpha(I)}$ and $\tilde{\gamma} \in \Gamma(E|_U)$ such that U is open and $\tilde{\gamma} \circ \alpha = \gamma$.

Exercise 9.0.0.6. figure 8 not extendible **FINISH!!!**

Exercise 9.0.0.7. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$, ∇ a connection on E , $I \subset \mathbb{R}$ an interval and $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$. There exists a unique $D_\alpha : \Gamma(E, \alpha) \rightarrow \Gamma(E, \alpha)$ such that

1. for each $\lambda \in \mathbb{R}$ and $\gamma, \sigma \in \Gamma(E, \alpha)$,

$$D_\alpha(\gamma + \lambda\sigma) = D_\alpha\gamma + \lambda D_\alpha\sigma$$

2. for each $f \in C^\infty(I)$ and $\gamma \in \Gamma(E, \alpha)$,

$$D_\alpha(f\gamma) = f'\gamma + fD_\alpha\gamma$$

3. for each $\gamma \in \Gamma(E)$, if $\tilde{\gamma}$ extends γ , then

$$D_\alpha\gamma = \nabla_{\alpha'}\gamma$$

Proof.

□

Chapter 10

Riemannian Geometry

Definition 10.0.0.1. Let M be a smooth manifold and $g \in T_2^0(M)$ a metric tensor on M . We define $\hat{g} \in T_0^2(M)$ by $\hat{g}(\omega, \eta) = g(\phi_g^{-1}(\omega), \phi_g^{-1}(\eta))$.

Exercise 10.0.0.2. content...

Exercise 10.0.0.3. Let (M, g) be a semi-Riemannian manifold and $(U, \phi) \in \mathcal{A}$. Then the induced metric $\langle \cdot, \cdot \rangle_{T^*M \otimes TM}$ on $T^*M \otimes TM$ is given by

$$\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} = g^{i,j} g_{kl}$$

Proof. We have that

$$\begin{aligned} \left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} &= \langle dx^i, dx^j \rangle_{T^*M} \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle_{TM} \\ &= g^{i,j} g_{kl} \end{aligned}$$

□

Exercise 10.0.0.4. Let (M, g) be an n -dimensional Riemannian manifold.

1. There exists $\lambda \in \Omega^n(M)$ such that for each orthonormal frame e_1, \dots, e_n ,

$$\lambda(e_1, \dots, e_n) = 1$$

Hint: Choose a frame z_1, \dots, z_n on M with corresponding dual frame ζ^1, \dots, ζ^n . Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n$$

2. Let $N \in \mathfrak{X}(M)$ be the outward pointing normal to ∂M and $X \in \mathfrak{X}(M)$. Then

$$\int_M \operatorname{div} X \lambda = \int_{\partial M} g(X, N) \tilde{\lambda}$$

3. For each $u \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we have that

$$\operatorname{div}(uX) = u \operatorname{div}(X) + du(X)$$

and therefore

$$\int_M du(X) \lambda = \int_{\partial M} u g(X, N) \tilde{\lambda} - \int_M u \operatorname{div}(X) \lambda$$

Proof.

1. Let z_1, \dots, z_n be a frame on M and ζ^1, \dots, ζ^n with corresponding dual frame ζ^1, \dots, ζ^n . Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n$$

Let e_1, \dots, e_n , be an orthonormal frame on M with corresponding dual coframe $\epsilon^1, \dots, \epsilon^n$. Let $i, j \in \{1, \dots, n\}$. Then there exist $(a_{k,i}) \subset \mathbb{R}$ such that $\zeta^i = \sum_{k=1}^n a_{k,i} \epsilon^k$. Then

$$\begin{aligned} \hat{g}(\epsilon^j, \zeta^i) &= \sum_{k=1}^n a_{k,i} \hat{g}(\epsilon^j, \epsilon^k) \\ &= \sum_{k=1}^n a_{k,i} g(\phi_g^{-1}(\epsilon^j), \phi_g^{-1}(\epsilon^k)) \\ &= \sum_{k=1}^n a_{k,i} g(e_j, e_k) \\ &= \sum_{k=1}^n a_{k,i} \delta_{j,k} \\ &= a_{j,i} \end{aligned}$$

which implies that

$$\begin{aligned} \delta_{i,j} &= \zeta^i(z_j) \\ &= \sum_{k=1}^n a_{k,i} \epsilon^k(z_j) \\ &= \sum_{k=1}^n a_{k,i} g(e_k, z_j) \\ &= \sum_{k=1}^n \hat{g}(\epsilon^k, \zeta^i) g(e_k, z_j) \end{aligned}$$

Define $U, V \in \mathbb{R}^{n \times n}$ by $U_{i,k} = \hat{g}(\zeta^i, \epsilon^k)$ and $V_{k,j} = g(e_k, z_j)$. Then from above, we have that $UV = I$. Since $U, V \in \mathbb{R}^{n \times n}$, $VU = I$. Hence $U = V^{-1}$. Since

$$\begin{aligned} \zeta^i(e_j) &= \sum_{k=1}^n a_{k,i} \epsilon^k(e_j) \\ &= \sum_{k=1}^n a_{k,i} \delta_{k,j} \\ &= a_{j,i} \\ &= \hat{g}(\epsilon^j, \zeta^i) \\ &= U_{i,j} \end{aligned}$$

and

$$\begin{aligned}
g(z_i, z_j) &= \left(\sum_{k=1}^n g(e_k, z_i) e_k, \sum_{l=1}^n g(e_l, z_j) e_l \right) \\
&= \sum_{k=1}^n \sum_{l=1}^n g(e_k, z_i) g(e_l, z_j) g(e_k, e_l) \\
&= \sum_{k=1}^n \sum_{l=1}^n g(e_k, z_i) g(e_l, z_j) \delta_{k,l} \\
&= \sum_{k=1}^n g(e_k, z_i) g(e_k, z_j) \\
&= (V^* V)_{i,j}
\end{aligned}$$

we have that

$$\begin{aligned}
\lambda(e_1, \dots, e_n) &= \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \dots \wedge \zeta^n(e_1, \dots, e_n) \\
&= \det[g(z_i, z_j)]^{1/2} \det[\zeta^i(e_j)] \\
&= \det(V^* V)^{1/2} \det U \\
&= \det V (\det V)^{-1} \\
&= 1
\end{aligned}$$

2. Choose an orthonormal frame $e_1, \dots, e_{n-1} \in \mathfrak{X}(\partial M)$ with dual coframe $\epsilon^1, \dots, \epsilon^{n-1}$. Define $\nu \in \Omega^1(M)$ to be the dual covector to N . We note that N, e_1, \dots, e_{n-1} is an orthonormal frame on $\mathfrak{X}(M)$. Let $X_1, \dots, X_{n-1} \in \mathfrak{X}(\partial M)$. Since for each $j \in \{1, \dots, n-1\}$, $X_j \in \mathfrak{X}(\partial M)$ and for each $p \in \partial M$, $N_p \in (T_p \partial M)^\perp$, we have that for each $j \in \{1, \dots, n-1\}$, $g(X_j, N) = 0$. This implies that

$$\begin{aligned}
\iota^* \iota_X \lambda(X_1, \dots, X_{n-1}) &= \lambda(X, X_1, \dots, X_{n-1}) \\
&= \nu \wedge \epsilon^1 \wedge \dots \wedge \epsilon^{n-1}(X, X_1, \dots, X_{n-1}) \\
&= \det \begin{pmatrix} \nu(X) & \nu(X_1) & \dots & \nu(X_{n-1}) \\ \epsilon^1(X) & \epsilon^1(X_1) & \dots & \epsilon^1(X_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \dots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= \det \begin{pmatrix} g(X, N) & g(X_1, N) & \dots & g(X_{n-1}, N) \\ \epsilon^1(X) & \epsilon^1(X_1) & \dots & \epsilon^1(X_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \dots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\
&= g(X, N) \det(\epsilon^i(X_j)) \\
&= g(X, N) \epsilon^1 \wedge \dots \wedge \epsilon^{n-1}(X_1, \dots, X_{n-1}) \\
&= g(X, N) \tilde{\lambda}(X_1, \dots, X_{n-1})
\end{aligned}$$

Therefore $\iota^* \iota_X \lambda = g(X, N) \tilde{\lambda}$ and

$$\begin{aligned}
\int_M \operatorname{div} X \lambda &= \int_M d(\iota_X \lambda) \\
&= \int_{\partial M} \iota^* (\iota_X \lambda) \\
&= \int_{\partial M} g(X, N) \tilde{\lambda}
\end{aligned}$$

3. We note that

$$\begin{aligned}
 0 &= \iota_X(du \wedge \lambda) \\
 &= \iota_X(du) \wedge \lambda - du \wedge (\iota_X \lambda) \\
 &= du(X)\lambda - du \wedge (\iota_X \lambda)
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \operatorname{div}(uX)\lambda &= d(\iota_{uX}\lambda) \\
 &= d(u\iota_X\lambda) \\
 &= du \wedge (\iota_X\lambda) + u d(\iota_X\lambda) \\
 &= du(X)\lambda + u \operatorname{div}(X)\lambda \\
 &= [du(X) + u \operatorname{div}(X)]\lambda
 \end{aligned}$$

This implies that $\operatorname{div}(uX) = du(X) + u \operatorname{div}(X)$. From before, we have that

$$\begin{aligned}
 \int_M du(X)\lambda &= \int_M \operatorname{div}(uX)\lambda - \int_M u \operatorname{div}(X)\lambda \\
 &= \int_{\partial M} g(uX, N)\tilde{\lambda} - \int_M u \operatorname{div}(X)\lambda \\
 &= \int_{\partial M} u g(X, N)\tilde{\lambda} - \int_M u \operatorname{div}(X)\lambda
 \end{aligned}$$

□

Exercise 10.0.0.5.

$$\operatorname{div}(X) = \sum_{j=1}^n (\nabla_{\partial_j} X)^j$$

Proof. We have that

$$\begin{aligned}
 \nabla_{\partial_i}(X) &= \sum_{j=1}^n \nabla_{\partial_i}(X^j \partial_j) \\
 &= \sum_{j=1}^n \left[X^j \nabla_{\partial_i} \partial_j + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n \left[X^j \left(\sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n \left[X^j \left(\sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \partial_i(X^j) \partial_j \right] \\
 &= \sum_{j=1}^n X^j \left(\sum_{k=1}^n \Gamma_{i,j}^k \partial_k \right) + \sum_{j=1}^n \partial_i(X^j) \partial_j \\
 &= \sum_{k=1}^n \left(\sum_{j=1}^n X^j \Gamma_{i,j}^k \right) \partial_k + \sum_{k=1}^n \partial_i(X^k) \partial_k \\
 &= \sum_{k=1}^n \left[\left(\sum_{j=1}^n X^j \Gamma_{i,j}^k \right) + \partial_i(X^k) \right] \partial_k
 \end{aligned}$$

so that $(\nabla_{\partial_i}(X))^i = \left(\sum_{j=1}^n X^j \Gamma_{i,j}^i \right) + \partial_i(X^i)$. We note that

$$\begin{aligned} \operatorname{div}(X) &= \sum_{i=1}^n \operatorname{div}(X^i \partial_i) \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + dx^i(\partial_i)] \\ &= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + 1] \end{aligned}$$

Since $\lambda = [\det g(\partial_i, \partial_j)]^{1/2} dx^1 \wedge \cdots \wedge dx^n = (\det g)^{1/2} dx$, we have that

$$\begin{aligned} d(\iota_{\partial_i} \lambda) &= d((\det g)^{1/2} \iota_{\partial_i} dx) \\ &= d[(\det g)^{1/2}] \iota_{\partial_i} dx + (\det g)^{1/2} d(\iota_{\partial_i} dx) \\ &= d[(\det g)^{1/2}] \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n + (\det g)^{1/2} \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n \end{aligned}$$

FINISH!!! □

Exercise 10.0.0.6. Let (M, g) be a Riemannian manifold.

1. For each $u, v \in C^\infty(M)$. Then

(a)

$$\int_M u \Delta v \lambda + \int_M g(\nabla u, \nabla v) \lambda = \int_{\partial M} u N(v) \tilde{\lambda}$$

(b)

$$\int_M [u \Delta v - v \Delta u] \lambda = \int_{\partial M} [u N(v) - v N(u)] \tilde{\lambda}$$

2. (a) If $\partial M \neq \emptyset$, then for each $u, v \in C^\infty(M)$, u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$ implies that $u = v$.

(b) If $\partial M = \emptyset$, then for each $u \in C^\infty(M)$, u is harmonic implies that u is constant.

Proof.

1. Let $u, v \in C^\infty(M)$. Then

(a)

$$\begin{aligned} \int_M u \Delta v \lambda &= \int_M u \operatorname{div}(\nabla v) \lambda \\ &= \int_{\partial M} u g(\nabla v, N) \tilde{\lambda} - \int_M du(\nabla v) \lambda \\ &= \int_{\partial M} u dv(N) \tilde{\lambda} - \int_M g(\nabla u, \nabla v) \lambda \\ &= \int_{\partial M} u N(v) \tilde{\lambda} - \int_M g(\nabla u, \nabla v) \lambda \end{aligned}$$

(b) From above, we have that

$$\begin{aligned}
\int_M [u\Delta v - v\Delta u]\lambda &= \int_M u\Delta v\lambda - \int_M v\Delta u\lambda \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda - \left(\int_{\partial M} vN(u)\tilde{\lambda} - \int_M g(\nabla v, \nabla u)\lambda \right) \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_{\partial M} vN(u)\tilde{\lambda} \\
&= \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}
\end{aligned}$$

2. (a) Suppose that $\partial M \neq \emptyset$. Let $u, v \in C^\infty(M)$. Suppose that u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$. Then $u - v$ is harmonic and

$$\begin{aligned}
\int_M \|\nabla(u - v)\|_g^2 \lambda &= \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
&= 0 + \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
&= \int_M (u - v)\Delta(u - v)\lambda + \int_M g(\nabla(u - v), \nabla(u - v))\lambda \\
&= \int_{\partial M} (u - v)N(u - v)\tilde{\lambda} \\
&= 0
\end{aligned}$$

Thus $\nabla(u - v) = 0$ and $u - v$ is constant. Since $u|_{\partial M} = v|_{\partial M}$, we have that $u - v = 0$ and thus $u = v$.

- (b) Suppose that $\partial M = \emptyset$. Let $u \in C^\infty(M)$. Suppose that u is harmonic. Then

$$\begin{aligned}
\int_M \|\nabla u\|_g^2 \lambda &= \int_M g(\nabla u, \nabla u)\lambda \\
&= 0 + \int_M g(\nabla u, \nabla u)\lambda \\
&= \int_M u\Delta u\lambda + \int_M g(\nabla u, \nabla u)\lambda \\
&= \int_{\partial M} (u - v)g(\nabla(u - v), N)\tilde{\lambda} \\
&= 0
\end{aligned}$$

Therefore $\nabla u = 0$ and u is constant.

□

Chapter 11

Symplectic Geometry

11.1 Symplectic Manifolds

Definition 11.1.0.1. Let $M \in \mathbf{Man}^\infty$ and $\omega \in \Omega^2(M)$. Then ω is said to be **symplectic** if

1. ω is nondegenerate
2. ω is closed

Chapter 12

Extra

Definition 12.0.0.1. When working in \mathbb{R}^n , we introduce the formal objects dx^1, dx_2, \dots, dx^n . Let $I = (i_1, i_2, \dots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. We formally define $dx^i = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$.

Definition 12.0.0.2. Let $k \in \{0, 1, \dots, n\}$. We define a $C^\infty(\mathbb{R}^n)$ -module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \text{span}\{dx^i : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \rightarrow \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

1. the exterior product is bilinear
2. for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$
3. for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$
4. for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k -forms on \mathbb{R}^n . Let ω be a k -form on \mathbb{R}^n . If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$

Note 12.0.0.3. The terms dx^1, dx_2, \dots, dx^n are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \dots, du_k\}$ when referencing the coordinates on \mathbb{R}^k and $\{dx^1, dx_2, \dots, dx^n\}$ when referencing the coordinates on \mathbb{R}^n .

Exercise 12.0.0.4. Let $B_{n \times n} = (b_{i,j}) \in [C^\infty(M)]^{n \times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \dots \wedge dx^n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned}
 \bigwedge_{i=1}^n \left(\sum_{j=1}^n b_{i,j} dx^j \right) &= \left(\sum_{j=1}^n b_{1,j} dx^j \right) \wedge \left(\sum_{j=1}^n b_{2,j} dx^j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n b_{n,j} dx^j \right) \\
 &= \sum_{j_1, \dots, j_n=1}^n \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\
 &= \sum_{j_1 \neq \dots \neq j_n} \left(\prod_{i=1}^n b_{i,j_i} \right) dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} \\
 &= \left[\sum_{\sigma \in S_n} \text{sgn}(\sigma) \left(\prod_{i=1}^n b_{i,\sigma(i)} \right) \right] dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \\
 &= (\det B) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n
 \end{aligned}$$

□

Definition 12.0.0.5. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a 0-form on \mathbb{R}^n . We define a 1-form, denoted df , on \mathbb{R}^n by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Let $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I$ be a k -form on \mathbb{R}^n . We can define a differential $k+1$ -form, denoted $d\omega$, on \mathbb{R}^n by

$$d\omega = \sum_{I \in \mathcal{I}_{k,n}} df_I \wedge dx^I$$

Exercise 12.0.0.6. On \mathbb{R}^3 , put

1. $\omega_0 = f_0$,
2. $\omega_1 = f_1 dx^1 + f_2 dx_2 + f_3 dx_3$,
3. $\omega_2 = f_1 dx_2 \wedge dx_3 - f_2 dx^1 \wedge dx_3 + f_3 dx^1 \wedge dx_2$

Show that

1. $d\omega_0 = \frac{\partial f_0}{\partial x^1} dx^1 + \frac{\partial f_0}{\partial x^2} dx_2 + \frac{\partial f_0}{\partial x^3} dx_3$
2. $d\omega_1 = \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) dx_2 \wedge dx_3 + \left(\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3} \right) dx^1 \wedge dx_3 + \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) dx^1 \wedge dx_2$
3. $d\omega_2 = \left(\frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3} \right) dx^1 \wedge dx_2 \wedge dx_3$

Proof. Straightforward. □

Exercise 12.0.0.7. Let $I \in \mathcal{I}_{k,n}$. Then there is a unique $I_* \in \mathcal{I}_{n-k,n}$ such that $dx^I \wedge dx_{I_*} = dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$.

Definition 12.0.0.8. We define a linear map $*$: $\Phi_k(\mathbb{R}^n) \rightarrow \Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge *-operator** by

$$* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^I = \sum_{I \in \mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 12.0.0.9. Let $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be smooth. Write $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. We define $\phi^* : \Phi_k(\mathbb{R}^n) \rightarrow \Phi_k(\mathbb{R}^k)$ via the following properties:

1. for each 0-form f on \mathbb{R}^n , $\phi^* f = f \circ \phi$
2. for $i = 1, \dots, n$, $\phi^* dx^i = d\phi_i$
3. for an s -form ω , and a t -form χ on \mathbb{R}^n , $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
4. for l -forms ω, χ on \mathbb{R}^n , $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 12.0.0.10. Let $M \subset \mathbb{R}^n$ be a k -dimensional smooth submanifold of \mathbb{R}^n , $\phi : U \rightarrow V$ a smooth parametrization of M , $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$ an k -form on \mathbb{R}^n . Then

$$\phi^*\omega = \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \dots \wedge du_k$$

Proof. By definition,

$$\begin{aligned} \phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i \\ &= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx^i \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned} d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \dots \wedge d\phi_{i_n} \\ &= \left(\sum_{j=1}^n \frac{\partial \phi_{i_1}}{\partial u^j} du^j \right) \wedge \left(\sum_{j=1}^n \frac{\partial \phi_{i_2}}{\partial u^j} du^j \right) \wedge \dots \wedge \left(\sum_{j=1}^n \frac{\partial \phi_{i_k}}{\partial u^j} du^j \right) \\ &= (\det v\phi_I) du^1 \wedge du_2 \wedge \dots \wedge du_k \end{aligned}$$

Therefore

$$\begin{aligned} \phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\ &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) du^1 \wedge du_2 \wedge \dots \wedge du_k \\ &= \left(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \right) du^1 \wedge du_2 \wedge \dots \wedge du_k \end{aligned}$$

□

12.1 Integration of Differential Forms

Definition 12.1.0.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \dots \wedge dx_k$ a k -form on \mathbb{R}^k . Define

$$\int_U \omega = \int_U f dx$$

Definition 12.1.0.2. Let $M \subset \mathbb{R}^n$ be a k -dimensional oriented smooth submanifold of \mathbb{R}^n , ω a k -form on \mathbb{R}^n and $\phi : U \rightarrow V$ a local smooth, orientation-preserving parametrization of M . Define

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 12.1.0.3.**Theorem 12.1.0.4. Stokes Theorem:**

Let $M \subset \mathbb{R}^n$ be a k -dimensional oriented smooth submanifold of \mathbb{R}^n and ω a $k-1$ -form on \mathbb{R}^n . Then

$$\int_{\partial M} \omega = \int_M d\omega$$

Appendix A

Summation

Appendix B

Asymptotic Notation

Bibliography

- [1] [Introduction to Algebra](#)
- [2] [Introduction to Analysis](#)
- [3] [Introduction to Fourier Analysis](#)
- [4] [Introduction to Measure and Integration](#)