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# Introduction to Category Theory



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# Preface

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# Chapter 1

## Basic Concepts

### 1.1 von Neumann–Bernays–Gödel Set Theory

**Definition 1.1.0.1.** Let  $x$  be a class. Then  $x$  is said to be a set iff there exists a class  $A$  such that  $x \in A$ .

**Note 1.1.0.2.** We can define cartesian products, relations, and functions for classes just like for sets.

**Axiom 1.1.0.3. Axiom of Replacement:**

Let  $A, B$  be classes and  $f : A \rightarrow B$ . If  $A$  is a set, then  $f(A)$  is a set.

**Axiom 1.1.0.4. Schema of Specification:**

Let  $\phi$  a propositional function on sets. Then there exists a class  $A$  such that for each set  $x$ ,  $x \in A$  iff  $\phi(x)$ .

**Exercise 1.1.0.5.** There exists a class  $A$  such that for each class  $x$ ,  $x \in A$  iff  $x$  is a set.

*Proof.* Define  $\phi$  by

$$\phi(x) : x = x$$

Axiom 1.1.0.4 implies that there exists a class  $A$  such that for each set  $x$ ,  $x \in A$  iff  $x = x$ . Let  $x$  be a class. If  $x \in A$ , then by definition,  $x$  is a set.

Conversely, if  $x$  is a set, then by construction,  $x \in A$ . □

**Exercise 1.1.0.6.** There exists a class  $A$  such that for each class  $G$  and  $*$  :  $G \times G \rightarrow G$ ,  $(G, *) \in A$  iff  $(G, *)$  is a group.

*Proof.* Define  $\phi_1, \phi_2$  and  $\phi_3$  by

- $\phi_1(G, *) : * : G \times G \rightarrow G$  is associative
- $\phi_2(G, *) : \text{there exists } e \in G \text{ such that for each } g \in G, e * g = g * e = g$
- $\phi_3(G, *) : \text{for each } g \in G \text{ there exists } h \in G \text{ such that } g * h = h * g = e$

Define  $\phi$  by

$$\phi(G, *) : \phi_1(G, *) \text{ and } \phi_2(G, *) \text{ and } \phi_3(G, *)$$

Then there exists a class  $A$  such that for each set  $G$  and  $*$  :  $G \times G \rightarrow G$ ,  $(G, *) \in A$  iff  $\phi(G, *)$  “is a group”. Therefore, for each group  $(G, *)$ ,  $(G, *) \in A$ . **FINISH!!!** □

## 1.2 Categories

### 1.2.1 Introduction

**Definition 1.2.1.1.** Let  $\mathcal{C}_0, \mathcal{C}_1$  be classes and  $\text{dom}, \text{cod} : \mathcal{C}_1 \rightarrow \mathcal{C}_0$  class functions. Set  $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \text{dom}, \text{cod})$ . Then  $\mathcal{C}$  is said to be a **category** if

1. (composition): for each  $f, g \in \mathcal{C}_1$ , if  $\text{cod}(f) = \text{dom}(g)$ , then there exists  $g \circ f \in \mathcal{C}_1$  such that  $\text{dom}(g \circ f) = \text{dom}(f)$  and  $\text{cod}(g \circ f) = \text{cod}(g)$
2. (associativity): for each  $f, g, h \in \mathcal{C}_1$ , if  $\text{cod}(f) = \text{dom}(g)$  and  $\text{cod}(g) = \text{dom}(h)$ , then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

3. (identity): for each  $X \in \mathcal{C}_0$ , there exists  $\text{id}_X \in \mathcal{C}_1$  such that  $\text{dom}(\text{id}_X) = \text{cod}(\text{id}_X) = X$  and for each  $f, g \in \mathcal{C}_1$ , if  $\text{dom}(f) = X$  and  $\text{cod}(g) = X$ , then

$$f \circ \text{id}_X = f \text{ and } \text{id}_X \circ g = g$$

We define the

- **objects of  $\mathcal{C}$** , denoted  $\text{Obj}(\mathcal{C})$ , by  $\text{Obj}(\mathcal{C}) = \mathcal{C}_0$
- **morphisms of  $\mathcal{C}$** , denoted  $\text{Hom}_{\mathcal{C}}$ , by  $\text{Hom}_{\mathcal{C}} = \mathcal{C}_1$

For  $X, Y \in \text{Obj}(\mathcal{C})$ , we define the **morphisms from  $X$  to  $Y$** , denoted  $\text{Hom}_{\mathcal{C}}(X, Y)$ , by  $\text{Hom}_{\mathcal{C}}(X, Y) = \{f \in \text{Hom}(\mathcal{C}) : \text{dom}(f) = X \text{ and } \text{cod}(f) = Y\}$ .

**Note 1.2.1.2.** We typically define a category  $\mathcal{C}$  by specifying

- $\text{Obj}(\mathcal{C})$
- for  $X, Y \in \text{Obj}(\mathcal{C})$ , the class  $\text{Hom}_{\mathcal{C}}(X, Y)$
- for  $X, Y, Z \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$  and  $g \in \text{Hom}_{\mathcal{C}}(Y, Z)$ , the composite morphism  $g \circ f \in \text{Hom}_{\mathcal{C}}(X, Z)$ .

and then show

- well-definedness of composition
- associativity of composition
- existence of identities

**Definition 1.2.1.3.** We define the **empty category**, denoted  $\mathbf{0}$ , by

- $\text{Obj}(\mathbf{0}) = \emptyset$
- $\text{Hom}_{\mathbf{0}} = \emptyset$

**Exercise 1.2.1.4.** We have that  $\mathbf{0}$  is a category

*Proof.* Vacuously true. □

**Definition 1.2.1.5.** We define the **trivial category**, denoted  $\mathbf{1}$ , by

- $\text{Obj}(\mathbf{1}) = \{*\}$
- $\text{Hom}_{\mathbf{1}} = \{\text{id}_*\}$

**Exercise 1.2.1.6.** We have that  $\mathbf{1}$  is a category.

*Proof.* Clear. □

**Definition 1.2.1.7.** We define **Set** by

- $\text{Obj}(\mathbf{Set}) = \{A : A \text{ is a set}\}$
- for each  $A, B \in \text{Obj}(\mathbf{Set})$ ,  $\text{Hom}_{\mathbf{Set}}(A, B) = \{f : f : A \rightarrow B\}$
- for  $A, B, C \in \mathbf{Set}$ ,  $f \in \text{Hom}_{\mathbf{Set}}(A, B)$  and  $g \in \text{Hom}_{\mathbf{Set}}(B, C)$ ,  $g \circ_{\mathbf{Set}} f = g \circ f$ .

**Exercise 1.2.1.8.** We have that **Set** is a category.

*Proof.*

- **well-definedness of composition:**
- **associativity of composition:**
- **existence of identities:**

**FINISH!!!** □

**Definition 1.2.1.9.** Let  $\mathcal{C}$  be a category. Then  $\mathcal{C}$  is said to be

- **small** if  $\text{Obj}(\mathcal{C})$  and  $\text{Hom}_{\mathcal{C}}$  are sets
- **locally small** if for each  $A, B \in \text{Obj}(\mathcal{C})$ ,  $\text{Hom}_{\mathcal{C}}(A, B)$  is a set

**Exercise 1.2.1.10.** Let  $\mathcal{C}$  be a category. If  $\mathcal{C}$  is small, then  $\mathcal{C}$  is a set.

*Proof.* Suppose that  $\mathcal{C}$  is small. Then  $\text{Obj}(\mathcal{C})$  and  $\text{Hom}_{\mathcal{C}}$  are sets. Then  $\mathcal{P}(\text{Obj}(\mathcal{C}))$ ,  $\mathcal{P}(\text{Hom}_{\mathcal{C}})$  and  $\text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}}$  are sets. Hence  $\text{Obj}(\mathcal{C}) \times \text{Hom}_{\mathcal{C}} \times \text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}} \times \text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}}$  is a set. By definition,  $\mathcal{C} = (\text{Obj}(\mathcal{C}), \text{Hom}_{\mathcal{C}}, \text{dom}, \text{cod}) \in \text{Obj}(\mathcal{C}) \times \text{Hom}_{\mathcal{C}} \times \text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}} \times \text{Obj}(\mathcal{C})^{\text{Hom}_{\mathcal{C}}}$ . By definition,  $\mathcal{C}$  is a set. □

**Exercise 1.2.1.11.** There exists a class  $A$  such that  $\mathcal{C} \in A$  iff  $\mathcal{C}$  is a small category.

*Proof.* Exercise 1.2.1.10 implies that for each category  $\mathcal{C}$ ,  $\mathcal{C}$  is small implies that  $\mathcal{C}$  is a set. Define  $\phi$  by

$$\phi(\mathcal{C}) : \mathcal{C} \text{ is a small category}$$

Then Axiom 1.1.0.4 implies that there exists a class  $A$  such that  $\mathcal{C} \in A$  iff  $\mathcal{C}$  is a small category. □

## 1.2.2 Opposite Category

**Definition 1.2.2.1.** Let  $\mathcal{C}$  be a category, we define the dual of  $\mathcal{C}$  or the **opposite of  $\mathcal{C}$** , denoted  $\mathcal{C}^{\text{op}}$ , by

- $\text{Obj}(\mathcal{C}^{\text{op}}) = \text{Obj}(\mathcal{C})$
- for  $X, Y \in \text{Obj}(\mathcal{C}^{\text{op}})$ ,  $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$
- for  $X, Y, Z \in \text{Obj}(\mathcal{C}^{\text{op}})$  and  $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$ ,  $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$ ,  $g \circ_{\mathcal{C}^{\text{op}}} f = f \circ_{\mathcal{C}} g$

**Exercise 1.2.2.2.** Let  $\mathcal{C}$  be a category. Then  $\mathcal{C}^{\text{op}}$  is a category.

*Proof.*

- for  $W, X, Y, Z \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(W, X)$  and  $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$  and  $h \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$ . Then

$$\begin{aligned} (h \circ_{\mathcal{C}^{\text{op}}} g) \circ_{\mathcal{C}^{\text{op}}} f &= f \circ_{\mathcal{C}} (h \circ_{\mathcal{C}^{\text{op}}} g) \\ &= f \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} h) \\ &= (f \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} h \\ &= h \circ_{\mathcal{C}^{\text{op}}} (f \circ_{\mathcal{C}} g) \\ &= h \circ_{\mathcal{C}^{\text{op}}} (g \circ_{\mathcal{C}^{\text{op}}} f) \end{aligned}$$

So composition is associative.

- Let  $X \in \text{Obj}(\mathcal{C})$  and  $f, g \in \text{Hom}_{\mathcal{C}^{\text{op}}}$ . Suppose that  $\text{dom}(f) = X$  and  $\text{cod}(g) = X$ . Then

$$\begin{aligned} f \circ_{\mathcal{C}^{\text{op}}} \text{id}_X &= \text{id}_X \circ_{\mathcal{C}} f \\ &= f \end{aligned}$$

and

$$\begin{aligned} \text{id}_X \circ_{\mathcal{C}^{\text{op}}} g &= g \circ_{\mathcal{C}} \text{id}_X \\ &= g \end{aligned}$$

So  $(\text{id}_X)_{\mathcal{C}^{\text{op}}} = (\text{id}_X)_{\mathcal{C}}$ .

□

### 1.2.3 Slice Category

**Definition 1.2.3.1.** Let  $\mathcal{C}$  be a category and  $X \in \text{Obj}(\mathcal{C})$ . We define the **slice category of  $\mathcal{C}$  over  $X$** , denoted  $\mathcal{C}/X$ , by

- $\text{Obj}(\mathcal{C}/X) = \{f \in \text{Hom}_{\mathcal{C}} : \text{cod}(f) = X\}$
- for  $f, g \in \text{Obj}(\mathcal{C}/X)$ ,

$$\text{Hom}_{\mathcal{C}/X}(f, g) = \{\alpha \in \text{Hom}_{\mathcal{C}} : \text{dom}(\alpha) = \text{dom}(f), \text{cod}(\alpha) = \text{dom}(g) \text{ and } f = g \circ \alpha\}$$

i.e. for  $f \in \text{Hom}_{\mathcal{C}}(A, X)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, X)$ ,  $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$  iff the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

- for  $f, g, h \in \text{Obj}(\mathcal{C}/X)$ ,  $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$  and  $\beta \in \text{Hom}_{\mathcal{C}/X}(g, h)$ ,

$$\beta \circ_{\mathcal{C}/X} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

**Exercise 1.2.3.2.** Let  $\mathcal{C}$  be a category and  $X \in \text{Obj}(\mathcal{C})$ . Then  $\mathcal{C}/X$  is a category.

*Proof.*

- $f, g, h \in \text{Obj}(\mathcal{C}/X)$ ,  $\alpha \in \text{Hom}_{\mathcal{C}/X}(f, g)$  and  $\beta \in \text{Hom}_{\mathcal{C}/X}(g, h)$ . Then  $f = g \circ_{\mathcal{C}} \alpha$  and  $g = h \circ_{\mathcal{C}} \beta$ , i.e. the following diagrams commute:

$$\begin{array}{ccc} \text{dom}(f) & \xrightarrow{\alpha} & \text{dom}(g) \\ & \searrow f & \swarrow g \\ & X & \end{array} \qquad \begin{array}{ccc} \text{dom}(g) & \xrightarrow{\beta} & \text{dom}(h) \\ & \searrow g & \swarrow h \\ & X & \end{array}$$

Therefore, we have that

$$\begin{aligned} f &= g \circ_{\mathcal{C}} \alpha \\ &= (h \circ_{\mathcal{C}} \beta) \circ_{\mathcal{C}} \alpha \\ &= h \circ_{\mathcal{C}} (\beta \circ_{\mathcal{C}} \alpha) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} \text{dom}(f) & \xrightarrow{\beta \circ_{\mathcal{C}} \alpha} & \text{dom}(h) \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

which implies that

$$\begin{aligned}\beta \circ_{\mathcal{C}/X} \alpha &= \beta \circ_{\mathcal{C}} \alpha \\ &\in \text{Hom}_{\mathcal{C}/X}(f, h)\end{aligned}$$

and composition is well defined.

- Associativity of  $\circ_{\mathcal{C}/X}$  follows from associativity of  $\circ_{\mathcal{C}}$ .
- Let  $f \in \text{Obj}(\mathcal{C}/X)$  and  $\alpha, \beta \in \text{Hom}_{\mathcal{C}/X}$ . Since  $f \circ \text{id}_{\text{dom}_{\mathcal{C}}(f)} = f$ , i.e. the following diagram commutes:

$$\begin{array}{ccc}\text{dom}_{\mathcal{C}}(f) & \xrightarrow{\text{id}_{\text{dom}_{\mathcal{C}}(f)}} & \text{dom}_{\mathcal{C}}(f) \\ & \searrow f \quad \swarrow f & \\ & X & \end{array}$$

we have that  $\text{id}_{\text{dom}_{\mathcal{C}}(f)} \in \text{Hom}_{\mathcal{C}/X}(f, f)$ . Suppose that  $\text{dom}_{\mathcal{C}/X}(\alpha) = f$  and  $\text{cod}_{\mathcal{C}/X}(\beta) = f$ . Then

$$\begin{aligned}\alpha \circ_{\mathcal{C}/X} \text{id}_{\text{dom}_{\mathcal{C}}(f)} &= \alpha \circ_{\mathcal{C}} \text{id}_{\text{dom}_{\mathcal{C}}(f)} \\ &= \alpha\end{aligned}$$

and

$$\begin{aligned}\text{id}_{\text{dom}_{\mathcal{C}}(f)} \circ_{\mathcal{C}/X} \beta &= \text{id}_{\text{dom}_{\mathcal{C}}(f)} \circ_{\mathcal{C}} \beta \\ &= \beta\end{aligned}$$

So  $\text{id}_f = \text{id}_{\text{dom}_{\mathcal{C}}(f)}$ .

□

### 1.2.4 Product Category

**Definition 1.2.4.1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. We define the **product category of  $\mathcal{C}$  and  $\mathcal{D}$** , denoted  $\mathcal{C} \times \mathcal{D}$  by

- $\text{Obj}(\mathcal{C} \times \mathcal{D}) = \{(A, B) : A \in \text{Obj}(\mathcal{C}) \text{ and } B \in \text{Obj}(\mathcal{D})\}$
- for each  $(A, A'), (B, B') \in \text{Obj}(\mathcal{C} \times \mathcal{D})$ ,  $\text{Hom}_{\mathcal{C} \times \mathcal{D}}((A, A'), (B, B')) = \{(f, g) : f \in \text{Hom}_{\mathcal{C}}(A, B) \text{ and } g \in \text{Hom}_{\mathcal{D}}(A', B')\}$
- for each  $(A, A'), (B, B'), (C, C') \in \text{Obj}(\mathcal{C} \times \mathcal{D})$ ,  $(f, f') \in \text{Hom}_{\mathcal{C} \times \mathcal{D}}((A, A'), (B, B'))$  and  $(g, g') \in \text{Hom}_{\mathcal{C} \times \mathcal{D}}((B, B'), (C, C'))$ ,

$$(g, g') \circ_{\mathcal{C} \times \mathcal{D}} (f, f') = (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f')$$

**Exercise 1.2.4.2.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Then  $\mathcal{C} \times \mathcal{D}$  is a category.

*Proof.*

- **well-definedness of composition:**

Let  $(A, A'), (B, B'), (C, C') \in \text{Obj}(\mathcal{C} \times \mathcal{D})$ ,  $(f, f') \in \text{Hom}_{\mathcal{C} \times \mathcal{D}}((A, A'), (B, B'))$  and  $(g, g') \in \text{Hom}_{\mathcal{C} \times \mathcal{D}}((B, B'), (C, C'))$ . Then  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ ,  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ ,  $f' \in \text{Hom}_{\mathcal{D}}(A', B')$ , and  $g' \in \text{Hom}_{\mathcal{D}}(B', C')$ . Hence  $g \circ_{\mathcal{C}} f \in \text{Hom}_{\mathcal{C}}(A, C)$  and  $g' \circ_{\mathcal{D}} f' \in \text{Hom}_{\mathcal{D}}(A', C')$ . Thus

$$\begin{aligned}(g, g') \circ_{\mathcal{C} \times \mathcal{D}} (f, f') &= (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f') \\ &\in \text{Hom}_{\mathcal{C} \times \mathcal{D}}((A, A'), (C, C'))\end{aligned}$$

Thus, composition is well defined.

- **associativity of composition:**

Let  $(A, A'), (B, B'), (C, C'), (D, D') \in \text{Obj}(\mathcal{C} \times \mathcal{D})$ ,  $(f, f') \in \text{Hom}_{\mathcal{C} \times \mathcal{D}}((A, A'), (B, B'))$ ,  $(g, g') \in \text{Hom}_{\mathcal{C} \times \mathcal{D}}((B, B'), (C, C'))$ , and  $(h, h') \in \text{Hom}_{\mathcal{C} \times \mathcal{D}}((C, C'), (D, D'))$ . Then

$$\begin{aligned}
 [(h, h') \circ_{\mathcal{C} \times \mathcal{D}} (g, g')] \circ_{\mathcal{C} \times \mathcal{D}} (f, f') &= (h \circ_{\mathcal{C}} g, h' \circ_{\mathcal{D}} g') \circ_{\mathcal{C} \times \mathcal{D}} (f, f') \\
 &= ((h \circ_{\mathcal{C}} g) \circ_{\mathcal{C}} f, (h' \circ_{\mathcal{D}} g') \circ_{\mathcal{D}} f') \\
 &= (h \circ_{\mathcal{C}} (g \circ_{\mathcal{C}} f), h' \circ_{\mathcal{D}} (g' \circ_{\mathcal{D}} f')) \\
 &= (h, h') \circ_{\mathcal{C} \times \mathcal{D}} (g \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} f') \\
 &= (h, h') \circ_{\mathcal{C} \times \mathcal{D}} [(g, g') \circ_{\mathcal{C} \times \mathcal{D}} (f, f')]
 \end{aligned}$$

Thus composition is associative.

- **existence of identities:**

Let  $(A, B) \in \text{Obj}(\mathcal{C} \times \mathcal{D})$ ,  $(f, f'), (g, g') \in \text{Hom}_{\mathcal{C} \times \mathcal{D}}$ . Suppose that  $\text{dom}_{\mathcal{C} \times \mathcal{D}}(f, f') = (A, B)$  and  $\text{cod}_{\mathcal{C} \times \mathcal{D}}(g, g') = (A, B)$ . Then  $\text{dom}_{\mathcal{C}}(f) = A$ ,  $\text{dom}_{\mathcal{D}}(f') = B$ ,  $\text{cod}_{\mathcal{C}}(g) = A$  and  $\text{cod}_{\mathcal{D}}(g') = B$ . Hence

$$\begin{aligned}
 (f, f') \circ_{\mathcal{C} \times \mathcal{D}} (\text{id}_A, \text{id}_B) &= (f \circ_{\mathcal{C}} \text{id}_A, f' \circ_{\mathcal{D}} \text{id}_B) \\
 &= (f, f')
 \end{aligned}$$

and

$$\begin{aligned}
 (\text{id}_A, \text{id}_B) \circ_{\mathcal{C} \times \mathcal{D}} (g, g') &= (\text{id}_A \circ_{\mathcal{C}} g, \text{id}_B \circ_{\mathcal{D}} g') \\
 &= (g, g')
 \end{aligned}$$

Therefore  $(\text{id}_{(A,B)})_{\mathcal{C} \times \mathcal{D}} = (\text{id}_A, \text{id}_B)$ .

□

## 1.3 Functors

### 1.3.1 Introduction

**Definition 1.3.1.1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F_0 : \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$ ,  $F_1 : \text{Hom}_{\mathcal{C}} \rightarrow \text{Hom}_{\mathcal{D}}$  class functions. Set  $F = (F_0, F_1)$ . Then  $F$  is said to be a functor from  $\mathcal{C}$  to  $\mathcal{D}$ , denoted  $F : \mathcal{C} \rightarrow \mathcal{D}$ , if

1. for each  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ ,  $F_1(f) \in \text{Hom}_{\mathcal{D}}(F_0(A), F_0(B))$
2. for each  $A, B, C \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ ,  $F_1(g \circ f) = F_1(g) \circ F_1(f)$
3. for each  $A \in \text{Obj}(\mathcal{C})$ ,  $F_1(\text{id}_A) = \text{id}_{F_0(A)}$

**Note 1.3.1.2.** For  $A \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}$ , we typically write  $F(A)$  and  $F(f)$  instead of  $F_0(A)$  and  $F_1(f)$  respectively.

**Definition 1.3.1.3.** Let  $\mathcal{C}$  be a category. We define the **empty functor** from  $\mathbf{0}$  to  $\mathcal{C}$ , denoted  $E_{\mathcal{C}} : \mathbf{0} \rightarrow \mathcal{C}$  by  $(E_{\mathcal{C}})_0 = (E_{\mathcal{C}})_1 = \emptyset$ .

**Exercise 1.3.1.4.** Let  $\mathcal{C}$  be a category. Then  $E_{\mathcal{C}} : \mathbf{0} \rightarrow \mathcal{C}$  is a functor.

*Proof.* Since  $\text{Obj}(\mathbf{0}) = \emptyset$  and  $\text{Hom}_{\mathbf{0}} = \emptyset$ , this is vacuously true. □

**Definition 1.3.1.5.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $X \in \mathcal{D}$ . We define the **constant functor** from  $\mathcal{C}$  onto  $X$ , denoted  $\Delta_X^{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{D}$  by

- $\Delta_X^{\mathcal{C}}(A) = X$
- $\Delta_X^{\mathcal{C}}(f) = \text{id}_X$

**Exercise 1.3.1.6.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $X \in \text{Obj}(\mathcal{D})$ . Then  $\Delta_X^{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{D}$  is a functor.

*Proof.*

1. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Then

$$\begin{aligned} \Delta_X^{\mathcal{C}}(f) &= \text{id}_X \\ &\in \text{Hom}_{\mathcal{D}}(X, X) \\ &= \text{Hom}_{\mathcal{D}}(\Delta_X^{\mathcal{C}}(A), \Delta_X^{\mathcal{C}}(B)) \end{aligned}$$

2. Let  $A, B, C \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ . Then

$$\begin{aligned} \Delta_X^{\mathcal{C}}(g \circ f) &= \text{id}_X \\ &= \text{id}_X \circ \text{id}_X \\ &= \Delta_X^{\mathcal{C}}(g) \circ \Delta_X^{\mathcal{C}}(f) \end{aligned}$$

3. Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} \Delta_X^{\mathcal{C}}(\text{id}_A) &= \text{id}_X \\ &= \text{id}_{\Delta_X^{\mathcal{C}}(A)} \end{aligned}$$

So  $\Delta_X^{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{D}$  is a functor. □

### 1.3.2 Category of Small Categories

**Definition 1.3.2.1.** Let  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{E}$  functors. We define the **composition of  $G$  with  $F$** , denoted  $G \circ F : \mathcal{C} \rightarrow \mathcal{E}$ , by

- $G \circ F(A) = G(F(A))$
- $G \circ F(f) = G(F(f))$

**Exercise 1.3.2.2.** Let  $\mathcal{C}$ ,  $\mathcal{D}$  and  $\mathcal{E}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{E}$  functors. Then  $G \circ F : \mathcal{C} \rightarrow \mathcal{E}$  is a functor.

*Proof.*

1. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Since  $F(f) \in \text{Hom}_{\mathcal{D}}(F(A), F(B))$ , we have that  $G(F(f)) \in \text{Hom}_{\mathcal{E}}(G(F(A)), G(F(B)))$ . Then

$$\begin{aligned} G \circ F(f) &= G(F(f)) \\ &\in \text{Hom}_{\mathcal{E}}(G(F(A)), G(F(B))) \\ &= \text{Hom}_{\mathcal{E}}(G \circ F(A), G \circ F(B)) \end{aligned}$$

2. Let  $A, B, C \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ . Then

$$\begin{aligned} G \circ F(g \circ f) &= G(F(g \circ f)) \\ &= G(F(g) \circ F(f)) \\ &= G(F(g)) \circ G(F(f)) \\ &= G \circ F(g) \circ G \circ F(f) \end{aligned}$$

3. Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} G \circ F(\text{id}_A) &= G(F(\text{id}_A)) \\ &= G(\text{id}_{F(A)}) \\ &= \text{id}_{G(F(A))} \\ &= \text{id}_{G \circ F(A)} \end{aligned}$$

So  $G \circ F : \mathcal{C} \rightarrow \mathcal{E}$  is a functor. □

**Exercise 1.3.2.3.** Let  $\mathcal{C}$ ,  $\mathcal{D}$ ,  $\mathcal{E}$ ,  $\mathcal{F}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{E}$ ,  $H : \mathcal{E} \rightarrow \mathcal{F}$  functors. Then  $(H \circ G) \circ F = H \circ (G \circ F)$ .

*Proof.* Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Then

•

$$\begin{aligned} (H \circ G) \circ F(A) &= H \circ G(F(A)) \\ &= H(G(F(A))) \\ &= H(G \circ F(A)) \\ &= H \circ (G \circ F)(A) \end{aligned}$$

•

$$\begin{aligned} (H \circ G) \circ F(f) &= H \circ G(F(f)) \\ &= H(G(F(f))) \\ &= H(G \circ F(f)) \\ &= H \circ (G \circ F)(f) \end{aligned}$$



Hence  $(H \circ G) \circ F = H \circ (G \circ F)$ .  $\square$

**Definition 1.3.2.4.** Let  $\mathcal{C}$  be a category. We define the **identity functor from  $\mathcal{C}$  to  $\mathcal{C}$** , denoted  $\text{id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ , by

- $\text{id}_{\mathcal{C}}(A) = A, (A \in \text{Obj}(\mathcal{C}))$
- $\text{id}_{\mathcal{C}}(f) = f, (f \in \text{Hom}_{\mathcal{C}})$

**Exercise 1.3.2.5.** Let  $\mathcal{C}$  be a category. Then  $\text{id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$  is a functor.

*Proof.*

1. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Then

$$\begin{aligned} \text{id}_{\mathcal{C}}(f) &= f \\ &\in \text{Hom}_{\mathcal{C}}(A, B) \\ &= \text{Hom}_{\mathcal{C}}(\text{id}_{\mathcal{C}}(A), \text{id}_{\mathcal{C}}(B)) \end{aligned}$$

2. Let  $A, B, C \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ . Then

$$\begin{aligned} \text{id}_{\mathcal{C}}(g \circ f) &= g \circ f \\ &= \text{id}_{\mathcal{C}}(g) \circ \text{id}_{\mathcal{C}}(f) \end{aligned}$$

3. Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} \text{id}_{\mathcal{C}}(\text{id}_A) &= \text{id}_A \\ &= \text{id}_{\text{id}_{\mathcal{C}}(A)} \end{aligned}$$

$\square$

**Exercise 1.3.2.6.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Then

1.  $\text{id}_{\mathcal{D}} \circ F = F$
2.  $F \circ \text{id}_{\mathcal{C}} = F$

*Proof.*

1. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Then

$$\begin{aligned} \text{id}_{\mathcal{D}} \circ F(A) &= \text{id}_{\mathcal{D}}(F(A)) \\ &= F(A) \end{aligned}$$

and

$$\begin{aligned} \text{id}_{\mathcal{D}} \circ F(f) &= \text{id}_{\mathcal{D}}(F(f)) \\ &= F(f) \end{aligned}$$

Since  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  are arbitrary,  $\text{id}_{\mathcal{D}} \circ F = F$ .

2. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Then

$$\begin{aligned} F \circ \text{id}_{\mathcal{C}}(A) &= F(\text{id}_{\mathcal{C}}(A)) \\ &= F(A) \end{aligned}$$

and

$$\begin{aligned} F \circ \text{id}_{\mathcal{C}}(f) &= F(\text{id}_{\mathcal{C}}(f)) \\ &= F(f) \end{aligned}$$

Since  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  are arbitrary,  $F \circ \text{id}_{\mathcal{C}} = F$ .

□

**Exercise 1.3.2.7.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . If  $\mathcal{C}$  is small, then  $F$  is a set.

*Proof.* Suppose that  $\mathcal{C}$  is small. Then  $\text{Obj}(\mathcal{C})$  and  $\text{Hom}_{\mathcal{C}}$  are sets. By definition, there exist  $F_0 : \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{D})$  and  $F_1 : \text{Hom}_{\mathcal{C}} \rightarrow \text{Hom}_{\mathcal{D}}$  such that  $F = (F_0, F_1)$ . Axiom 1.1.0.3 implies that  $F_0(\text{Obj}(\mathcal{C}))$  and  $F_1(\text{Hom}_{\mathcal{C}})$  are sets. Therefore,  $\text{Obj}(\mathcal{C}) \times F_0(\text{Obj}(\mathcal{C}))$  and  $\text{Hom}_{\mathcal{C}} \times F_1(\text{Hom}_{\mathcal{C}})$  are sets. Hence  $\mathcal{P}(\text{Obj}(\mathcal{C}) \times F_0(\text{Obj}(\mathcal{C})))$  and  $\mathcal{P}(\text{Hom}_{\mathcal{C}} \times F_1(\text{Hom}_{\mathcal{C}}))$  are sets. Since  $F_0 \subset \text{Obj}(\mathcal{C}) \times F_0(\text{Obj}(\mathcal{C}))$  and  $F_1 \subset \text{Hom}_{\mathcal{C}} \times F_1(\text{Hom}_{\mathcal{C}})$ , we have that  $F_0 \in \mathcal{P}(\text{Obj}(\mathcal{C}) \times F_0(\text{Obj}(\mathcal{C})))$  and  $F_1 \in \mathcal{P}(\text{Hom}_{\mathcal{C}} \times F_1(\text{Hom}_{\mathcal{C}}))$ . Hence  $F_0$  and  $F_1$  are sets. Thus  $F = (F_0, F_1)$  is a set. □

**Exercise 1.3.2.8.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Suppose that  $\mathcal{C}$  is small. Then there exists a class  $A$  such that for each class  $F$ ,  $F \in A$  iff  $F : \mathcal{C} \rightarrow \mathcal{D}$ .

*Proof.* Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Suppose that  $\mathcal{C}$  is small. Define  $\phi$  by

$$\phi(F) : F : \mathcal{C} \rightarrow \mathcal{D}$$

Then there exists a class  $A$  such that for each set  $F$ ,  $F \in A$  iff  $\phi(F)$ . Let  $F$  be a class. Suppose that  $F \in A$ . By Definition 1.1.0.1,  $F$  is a set. Since  $F$  is a set and  $F \in A$ , we have that  $\phi(F)$ . Hence  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Conversely, suppose that  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Exercise 1.3.2.7 implies that  $F$  is a set. Since  $F$  is a set and  $\phi(F)$  is true, we have that  $F \in A$ . □

**Definition 1.3.2.9.** We define **Cat** by

- $\text{Obj}(\mathbf{Cat}) = \{\mathcal{C} : \mathcal{C} \text{ is a small category}\}.$
  - for  $\mathcal{C}, \mathcal{D} \in \text{Obj}(\mathbf{Cat})$ ,
- $$\text{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) = \{F : F : \mathcal{C} \rightarrow \mathcal{D}\}$$
- for  $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \text{Obj}(\mathbf{Cat})$ ,  $F \in \text{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$  and  $G \in \text{Hom}_{\mathbf{Cat}}(\mathcal{D}, \mathcal{E})$ ,

$$G \circ_{\mathbf{Cat}} F = G \circ F$$

**Exercise 1.3.2.10.** We have that **Cat** is

1. a category
2. locally small

*Proof.*

1. Exercise 1.3.2.2 implies that composition is well defined. Exercise 1.3.2.3 implies that composition is associative. Exercise 1.3.2.5 and Exercise 1.3.2.6 imply the existence of identities.
2. Let  $\mathcal{C}, \mathcal{D} \in \text{Obj}(\mathbf{Cat})$  and  $F \in \text{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ . Definition 1.2.1.9 implies that  $\text{Obj}(\mathcal{C})$ ,  $\text{Obj}(\mathcal{D})$ ,  $\text{Hom}_{\mathcal{C}}$  and  $\text{Hom}_{\mathcal{D}}$  are sets. Then  $\text{Obj}(\mathcal{D})^{\text{Obj}(\mathcal{C})}$  and  $\text{Hom}_{\mathcal{D}}^{\text{Hom}_{\mathcal{C}}}$  are sets. Hence  $\text{Obj}(\mathcal{D})^{\text{Obj}(\mathcal{C})} \times \text{Hom}_{\mathcal{D}}^{\text{Hom}_{\mathcal{C}}}$  is a set. Let  $F \in \text{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ . Then there exist  $F_0 \in \text{Obj}(\mathcal{D})^{\text{Obj}(\mathcal{C})}$  and  $F_1 \in \text{Hom}_{\mathcal{D}}^{\text{Hom}_{\mathcal{C}}}$  such that  $F = (F_0, F_1)$ . Therefore  $F \in \text{Obj}(\mathcal{D})^{\text{Obj}(\mathcal{C})} \times \text{Hom}_{\mathcal{D}}^{\text{Hom}_{\mathcal{C}}}$ . Since  $F \in \text{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$  is arbitrary,

$$\text{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \subset \text{Obj}(\mathcal{D})^{\text{Obj}(\mathcal{C})} \times \text{Hom}_{\mathcal{D}}^{\text{Hom}_{\mathcal{C}}}$$

which implies that  $\text{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$  is a set. Therefore, **Cat** is locally small. □

### 1.3.3 Comma Categories

**Definition 1.3.3.1.** Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be categories and  $S : \mathcal{A} \rightarrow \mathcal{C}, T : \mathcal{B} \rightarrow \mathcal{C}$  functors. We define the **comma category of  $S$  to  $T$** , denoted  $(S \downarrow T)$ , by

- $\text{Obj}(S \downarrow T) = \{(A, B, h) : A \in \text{Obj}(\mathcal{A}), B \in \text{Obj}(\mathcal{B}), \text{ and } h \in \text{Hom}_{\mathcal{C}}(S(A), T(B))\}$
- For  $(A_1, B_1, h_1), (A_2, B_2, h_2) \in \text{Obj}(S \downarrow T)$ ,

$$\begin{aligned} \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2)) = \\ \{(\alpha, \beta) : \alpha \in \text{Hom}_{\mathcal{A}}(A_1, A_2), \beta \in \text{Hom}_{\mathcal{B}}(B_1, B_2) \text{ and } T(\beta) \circ_C h_1 = h_2 \circ_C S(\alpha)\} \end{aligned}$$

i.e. for  $(A_1, B_1, h_1), (A_2, B_2, h_2) \in \text{Obj}(S \downarrow T)$ ,  $\alpha \in \text{Hom}_{\mathcal{A}}(A_1, A_2)$  and  $\beta \in \text{Hom}_{\mathcal{B}}(B_1, B_2)$ ,  $(\alpha, \beta) \in \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$  iff the following diagram commutes:

$$\begin{array}{ccc} S(A_1) & \xrightarrow{S(\alpha)} & S(A_2) \\ h_1 \downarrow & & \downarrow h_2 \\ T(B_1) & \xrightarrow{T(\beta)} & T(B_2) \end{array}$$

- For
  - $(A_1, B_1, h_1), (A_2, B_2, h_2), (A_3, B_3, h_3) \in \text{Obj}(S \downarrow T)$
  - $(\alpha_{12}, \beta_{12}) \in \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$
  - $(\alpha_{23}, \beta_{23}) \in \text{Hom}_{(S \downarrow T)}((A_2, B_2, h_2), (A_3, B_3, h_3))$

we define

$$(\alpha_{23}, \beta_{23}) \circ_{(S \downarrow T)} (\alpha_{12}, \beta_{12}) = (\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}, \beta_{23} \circ_{\mathcal{B}} \beta_{12})$$

**Exercise 1.3.3.2.** Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be categories and  $S : \mathcal{A} \rightarrow \mathcal{C}, T : \mathcal{B} \rightarrow \mathcal{C}$  functors. Then  $(S \downarrow T)$  is a category.

*Proof.*

- **well-definedness of composition:**

Let

- $(A_1, B_1, h_1), (A_2, B_2, h_2), (A_3, B_3, h_3) \in \text{Obj}(S \downarrow T)$
- $(\alpha_{12}, \beta_{12}) \in \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$
- $(\alpha_{23}, \beta_{23}) \in \text{Hom}_{(S \downarrow T)}((A_2, B_2, h_2), (A_3, B_3, h_3))$

By definition,  $\alpha_{12} \in \text{Hom}_{\mathcal{A}}(A_1, A_2)$ ,  $\alpha_{23} \in \text{Hom}_{\mathcal{A}}(A_2, A_3)$ ,  $\beta_{12} \in \text{Hom}_{\mathcal{B}}(B_1, B_2)$ ,  $\beta_{23} \in \text{Hom}_{\mathcal{B}}(B_2, B_3)$ ,  $T(\beta_{12}) \circ_C h_1 = h_2 \circ_C S(\alpha_{12})$  and  $T(\beta_{23}) \circ_C h_2 = h_3 \circ_C S(\alpha_{23})$ , i.e. the following diagram commutes:

$$\begin{array}{ccccc} S(A_1) & \xrightarrow{S(\alpha_{12})} & S(A_2) & \xrightarrow{S(\alpha_{23})} & S(A_3) \\ h_1 \downarrow & & \downarrow h_2 & & \downarrow h_3 \\ T(B_1) & \xrightarrow{T(\beta_{12})} & T(B_2) & \xrightarrow{T(\beta_{23})} & T(B_3) \end{array}$$

Then  $\alpha_{23} \circ_{\mathcal{A}} \alpha_{12} \in \text{Hom}_{\mathcal{A}}(A_1, A_3)$ ,  $\beta_{23} \circ_{\mathcal{B}} \beta_{12} \in \text{Hom}_{\mathcal{B}}(B_1, B_3)$  and

$$\begin{aligned} T(\beta_{23} \circ_{\mathcal{B}} \beta_{12}) \circ_C h_1 &= (T(\beta_{23}) \circ_C T(\beta_{12})) \circ_C h_1 \\ &= T(\beta_{23}) \circ_C (T(\beta_{12}) \circ_C h_1) \\ &= T(\beta_{23}) \circ_C (h_2 \circ_C S(\alpha_{12})) \\ &= (T(\beta_{23}) \circ_C h_2) \circ_C S(\alpha_{12}) \\ &= (h_3 \circ_C S(\alpha_{23})) \circ_C S(\alpha_{12}) \\ &= h_3 \circ_C (S(\alpha_{23}) \circ_C S(\alpha_{12})) \\ &= h_3 \circ_C S(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} S(A_1) & \xrightarrow{S(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12})} & S(A_3) \\ h_1 \downarrow & & \downarrow h_3 \\ T(B_1) & \xrightarrow{T(\beta_{23} \circ_{\mathcal{B}} \beta_{12})} & T(B_3) \end{array}$$

Hence  $(\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}, \beta_{23} \circ_{\mathcal{B}} \beta_{12}) \in \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_3, B_3, h_3))$  and composition is well defined.

• **associativity of composition:**

Let

- $(A_1, B_1, h_1), (A_2, B_2, h_2), (A_3, B_3, h_3), (A_4, B_4, h_4) \in \text{Obj}(S \downarrow T)$
- $(\alpha_{12}, \beta_{12}) \in \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$
- $(\alpha_{23}, \beta_{23}) \in \text{Hom}_{(S \downarrow T)}((A_2, B_2, h_2), (A_3, B_3, h_3))$
- $(\alpha_{34}, \beta_{34}) \in \text{Hom}_{(S \downarrow T)}((A_3, B_3, h_3), (A_4, B_4, h_4))$

Then

$$\begin{aligned} [(\alpha_{34}, \beta_{34}) \circ_{(S \downarrow T)} (\alpha_{23}, \beta_{23})] \circ_{(S \downarrow T)} (\alpha_{12}, \beta_{12}) &= (\alpha_{34} \circ_{\mathcal{A}} \alpha_{23}, \beta_{34} \circ_{\mathcal{B}} \beta_{23}) \circ_{(S \downarrow T)} (\alpha_{12}, \beta_{12}) \\ &= ([\alpha_{34} \circ_{\mathcal{A}} \alpha_{23}] \circ_{\mathcal{A}} \alpha_{12}, [\beta_{34} \circ_{\mathcal{B}} \beta_{23}] \circ_{\mathcal{B}} \beta_{12}) \\ &= (\alpha_{34} \circ_{\mathcal{A}} [\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}], \beta_{34} \circ_{\mathcal{B}} [\beta_{23} \circ_{\mathcal{B}} \beta_{12}]) \\ &= (\alpha_{34}, \beta_{34}) \circ_{(S \downarrow T)} (\alpha_{23} \circ_{\mathcal{A}} \alpha_{12}, \beta_{23} \circ_{\mathcal{B}} \beta_{12}) \\ &= (\alpha_{34}, \beta_{34}) \circ_{(S \downarrow T)} [(\alpha_{23}, \beta_{23}) \circ_{(S \downarrow T)} (\alpha_{12}, \beta_{12})] \end{aligned}$$

So composition is associative.

• **existence of identities:**

Let

- $(A_1, B_1, h_1), (A_2, B_2, h_2) \in \text{Obj}(S \downarrow T)$
- $(\alpha, \beta) \in \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$

By definition,

- $\alpha \in \text{Hom}_{\mathcal{A}}(A_1, A_2), \beta \in \text{Hom}_{\mathcal{B}}(B_1, B_2)$
- $h_1 \in \text{Hom}_{\mathcal{C}}(S(A_1), T(B_1)), h_2 \in \text{Hom}_{\mathcal{C}}(S(A_2), T(B_2))$
- $T(\beta) \circ h_1 = h_2 \circ S(\alpha)$

Since  $\text{id}_{A_1} \in \text{Hom}_{\mathcal{A}}(A_1, A_1)$ ,  $\text{id}_{B_1} \in \text{Hom}_{\mathcal{B}}(B_1, B_1)$ , and

$$\begin{aligned} T(\text{id}_{B_1}) \circ_{\mathcal{C}} h_1 &= \text{id}_{T(B_1)} \circ_{\mathcal{C}} h_1 \\ &= h_1 \\ &= h_1 \circ_{\mathcal{C}} \text{id}_{S(A_1)} \\ &= h_1 \circ_{\mathcal{C}} S(\text{id}_{A_1}) \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} S(A_1) & \xrightarrow{S(\text{id}_{A_1})} & S(A_1) \\ h_1 \downarrow & & \downarrow h_1 \\ T(B_1) & \xrightarrow{T(\text{id}_{B_1})} & T(B_1) \end{array}$$

we have that  $(\text{id}_{A_1}, \text{id}_{B_1}) \in \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_1, B_1, h_1))$ . Similarly  $(\text{id}_{A_2}, \text{id}_{B_2}) \in \text{Hom}_{(S \downarrow T)}((A_2, B_2, h_2), (A_2, B_2, h_2))$ . Therefore

$$\begin{aligned} (\alpha, \beta) \circ_{(S \downarrow T)} (\text{id}_{A_1}, \text{id}_{B_1}) &= (\alpha \circ_{\mathcal{A}} \text{id}_{A_1}, \beta \circ_{\mathcal{B}} \text{id}_{B_1}) \\ &= (\alpha, \beta) \end{aligned}$$

and

$$\begin{aligned} (\text{id}_{A_2}, \text{id}_{B_2}) \circ_{(S \downarrow T)} (\alpha, \beta) &= (\text{id}_{A_2} \circ_{\mathcal{A}} \alpha, \text{id}_{B_2} \circ_{\mathcal{B}} \beta) \\ &= (\alpha, \beta) \end{aligned}$$

Since  $(A_1, B_1, h_1), (A_2, B_2, h_2) \in \text{Obj}(S \downarrow T)$  and  $(\alpha, \beta) \in \text{Hom}_{(S \downarrow T)}((A_1, B_1, h_1), (A_2, B_2, h_2))$  are arbitrary, we have that for each  $(A, B, h) \in \text{Obj}(S \downarrow T)$ ,  $\text{id}_{(A, B, h)} = (\text{id}_A, \text{id}_B)$ .

□

**Definition 1.3.3.3.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X \in \text{Obj}(\mathcal{D})$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . We define the **comma category from  $X$  to  $F$** , denoted  $(X \downarrow F)$ , by  $(X \downarrow F) = (\Delta_X^1 \downarrow F)$ .

We may make the following identification:

- $\text{Obj}(X \downarrow F) = \{(A, f) : A \in \text{Obj}(\mathcal{C}) \text{ and } f \in \text{Hom}_{\mathcal{D}}(X, F(A))\}$
- For  $(A_1, f_1), (A_2, f_2) \in \text{Obj}(X \downarrow F)$ ,

$$\text{Hom}_{(X \downarrow F)}((A_1, f_1), (A_2, f_2)) = \{\alpha \in \text{Hom}_{\mathcal{C}}(A_1, A_2) \text{ and } F(\alpha) \circ f_1 = f_2\}$$

i.e. for  $(A_1, f_1), (A_2, f_2) \in \text{Obj}(X \downarrow F)$  and  $\alpha \in \text{Hom}_{A_1, A_2}$ ,  $\alpha \in \text{Hom}_{(X \downarrow F)}((A_1, f_1), (A_2, f_2))$  iff the following diagram commutes:

$$\begin{array}{ccc} & X & \\ f_1 \swarrow & & \searrow f_2 \\ F(A_1) & \xrightarrow{F(\alpha)} & F(A_2) \end{array}$$

- For
  - $(A_1, f_1), (A_2, f_2), (A_3, f_3) \in \text{Obj}(X \downarrow F)$
  - $\alpha \in \text{Hom}_{(X \downarrow F)}((A_1, f_1), (A_2, f_2))$
  - $\beta \in \text{Hom}_{(X \downarrow F)}((A_2, f_2), (A_3, f_3))$

we define

$$\beta \circ_{(X \downarrow F)} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

**Definition 1.3.3.4.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X \in \text{Obj}(\mathcal{D})$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . We define the **comma category from  $F$  to  $X$** , denoted  $(F \downarrow X)$ , by  $(F \downarrow X) = (F \downarrow \Delta_X^1)$ .

We may make the following identification:

- $\text{Obj}(F \downarrow X) = \{(A, f) : A \in \text{Obj}(\mathcal{C}) \text{ and } f \in \text{Hom}_{\mathcal{D}}(F(A), X)\}$
- For  $(A_1, f_1), (A_2, f_2) \in \text{Obj}(F \downarrow X)$ ,

$$\text{Hom}_{(F \downarrow X)}((A_1, f_1), (A_2, f_2)) = \{\alpha \in \text{Hom}_{\mathcal{C}}(A_1, A_2) \text{ and } f_2 \circ F(\alpha) = f_1\}$$

i.e. for  $(A_1, f_1), (A_2, f_2) \in \text{Obj}(F \downarrow X)$  and  $\alpha \in \text{Hom}_{A_1, A_2}$ ,  $\alpha \in \text{Hom}_{(F \downarrow X)}((A_1, f_1), (A_2, f_2))$  iff the following diagram commutes:

$$\begin{array}{ccc} F(A_1) & \xrightarrow{F(\alpha)} & F(A_2) \\ & \searrow f_1 & \swarrow f_2 \\ & X & \end{array}$$

• For

- $(A_1, f_1), (A_2, f_2), (A_3, f_3) \in \text{Obj}(F \downarrow X)$
- $\alpha \in \text{Hom}_{(F \downarrow X)}((A_1, f_1), (A_2, f_2))$
- $\beta \in \text{Hom}_{(F \downarrow X)}((A_2, f_2), (A_3, f_3))$

we define

$$\beta \circ_{(F \downarrow X)} \alpha = \beta \circ_{\mathcal{C}} \alpha$$

## 1.4 Natural Transformations

### 1.4.1 Introduction

**Definition 1.4.1.1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories,  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  and  $\alpha : \text{Obj}(\mathcal{C}) \rightarrow \text{Hom}_{\mathcal{D}}$ . Then  $\alpha$  is said to be a **natural transformation from  $F$  to  $G$** , denoted  $\alpha : F \Rightarrow G$ , if

1. for each  $A \in \text{Obj}(\mathcal{C})$ ,  $\alpha_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$
2. for each  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ ,  $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

### 1.4.2 Category of Functors

**Definition 1.4.2.1.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$  functors and  $\alpha : F \Rightarrow G$ ,  $\beta : G \Rightarrow H$  natural transformations. We define the **composition of  $\beta$  with  $\alpha$** , denoted  $\beta \circ \alpha : F \Rightarrow H$ , by

$$(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$$

**Exercise 1.4.2.2.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$  functors and  $\alpha : F \Rightarrow G$ ,  $\beta : G \Rightarrow H$  natural transformations. Then  $\beta \circ \alpha : F \Rightarrow H$  is a natural transformation.

*Proof.*

1. Let  $A \in \text{Obj}(\mathcal{C})$ . Since  $\alpha_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$  and  $\beta_A \in \text{Hom}_{\mathcal{D}}(G(A), H(A))$ , we have that

$$\begin{aligned} (\beta \circ \alpha)_A &= \beta_A \circ \alpha_A \\ &\in \text{Hom}_{\mathcal{D}}(F(A), H(A)) \end{aligned}$$

2. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Since  $\alpha : F \Rightarrow G$  and  $\beta : G \Rightarrow H$ ,  $G(f) \circ \alpha_A = \alpha_B \circ F(f)$  and  $H(f) \circ \beta_A = \beta_B \circ G(f)$ . Therefore

$$\begin{aligned} H(f) \circ (\beta \circ \alpha)_A &= H(f) \circ (\beta_A \circ \alpha_A) \\ &= (H(f) \circ \beta_A) \circ \alpha_A \\ &= (\beta_B \circ G(f)) \circ \alpha_A \\ &= \beta_B \circ (G(f) \circ \alpha_A) \\ &= \beta_B \circ (\alpha_B \circ F(f)) \\ &= (\beta_B \circ \alpha_B) \circ F(f) \\ &= (\beta \circ \alpha)_B \circ F(f) \end{aligned}$$

So  $\beta \circ \alpha : F \Rightarrow H$  is a natural transformation. □

**Exercise 1.4.2.3.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $F, G, H, I : \mathcal{C} \rightarrow \mathcal{D}$  functors and  $\alpha : F \Rightarrow G$ ,  $\beta : G \Rightarrow H$  and  $\gamma : H \Rightarrow I$  natural transformations. Then

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

*Proof.* Let  $A \in \text{Obj}(\mathcal{C})$ . By definition,

$$\begin{aligned} [(\gamma \circ \beta) \circ \alpha]_A &= (\gamma \circ \beta)_A \circ \alpha_A \\ &= (\gamma_A \circ \beta_A) \circ \alpha_A \\ &= \gamma_A \circ (\beta_A \circ \alpha_A) \\ &= \gamma_A \circ (\beta \circ \alpha)_A \\ &= [\gamma \circ (\beta \circ \alpha)]_A \end{aligned}$$

Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary,

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$$

□

**Definition 1.4.2.4.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . We define the **identity natural transformation from  $F$  to  $F$** , denoted  $\text{id}_F : F \Rightarrow F$ , by

$$(\text{id}_F)_A = \text{id}_{F(A)}$$

**Exercise 1.4.2.5.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Then  $\text{id}_F : F \Rightarrow F$  is a natural transformation from  $F$  to  $F$ .

*Proof.*

1. Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} (\text{id}_F)_A &= \text{id}_{F(A)} \\ &\in \text{Hom}_{\mathcal{D}}(F(A), F(A)) \end{aligned}$$

2. Let  $A, B \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Then

$$\begin{aligned} F(f) \circ (\text{id}_F)_A &= F(f) \circ \text{id}_{F(A)} \\ &= F(f) \\ &= \text{id}_{F(B)} \circ F(f) \\ &= (\text{id}_F)_B \circ F(f) \end{aligned}$$

□

**Exercise 1.4.2.6.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  and  $\alpha : F \Rightarrow G$ . Then

1.  $\text{id}_G \circ \alpha = \alpha$
2.  $\alpha \circ \text{id}_F = \alpha$

*Proof.*

1. Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} (\text{id}_G \circ \alpha)_A &= (\text{id}_G)_A \circ \alpha_A \\ &= \text{id}_{G(A)} \circ \alpha_A \\ &= \alpha_A \end{aligned}$$

Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary,  $\text{id}_G \circ \alpha = \alpha$



2. Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} (\alpha \circ \text{id}_F)_A &= \alpha_A \circ (\text{id}_F)_A \\ &= \alpha_A \circ \text{id}_{F(A)} \\ &= \alpha_A \end{aligned}$$

Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary,  $\alpha \circ \text{id}_F = \alpha$ .

□

**Exercise 1.4.2.7.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories,  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  and  $\alpha : F \Rightarrow G$ . If  $\mathcal{C}$  is small, then  $\alpha$  is a set.

*Proof.* Suppose that  $\mathcal{C}$  is small. Then  $\text{Obj}(\mathcal{C})$  is a set. Since  $\alpha : \text{Obj}(\mathcal{C}) \rightarrow \text{Hom}_{\mathcal{D}}$ , Axiom 1.1.0.3 implies that  $\alpha(\text{Obj}(\mathcal{C}))$  is a set. Then  $\text{Obj}(\mathcal{C}) \times \alpha(\text{Obj}(\mathcal{C}))$  is a set. Therefore  $\mathcal{P}(\text{Obj}(\mathcal{C}) \times \alpha(\text{Obj}(\mathcal{C})))$  is a set. Since  $\alpha \subset \text{Obj}(\mathcal{C}) \times \alpha(\text{Obj}(\mathcal{C}))$ , we have that  $\alpha \in \mathcal{P}(\text{Obj}(\mathcal{C}) \times \alpha(\text{Obj}(\mathcal{C})))$  which implies that  $\alpha$  is a set. □

**Exercise 1.4.2.8.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ . If  $\mathcal{C}$  is small, then there exists a class  $A$  such that for each class  $\alpha$ ,  $\alpha \in A$  iff  $\alpha : F \Rightarrow G$ .

*Proof.* Suppose that  $\mathcal{C}$  is small. Define  $\phi$  by

$$\phi(\alpha) : \alpha : F \Rightarrow G$$

Axiom 1.1.0.4 implies that there exists a class  $A$  such that for each set  $\alpha$ ,  $\alpha \in A$  iff  $\phi(\alpha)$ . Let  $\alpha$  be a class. Suppose that  $\alpha \in A$ . By Definition 1.1.0.1,  $\alpha$  is a set. Since  $\alpha$  is a set and  $\alpha \in A$ , we have that  $\phi(\alpha)$ . Hence  $\alpha : F \Rightarrow G$ .

Conversely, suppose that  $\alpha : F \Rightarrow G$ . Since  $\mathcal{C}$  is small, Exercise 1.4.2.7 implies that  $\alpha$  is a set. Since  $\phi(\alpha)$ , we have that  $\alpha \in A$ . □

**Definition 1.4.2.9.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Suppose that  $\mathcal{C}$  is small. We define the **functor category from  $\mathcal{C}$  to  $\mathcal{D}$** , denoted  $\mathcal{D}^{\mathcal{C}}$ , by

- $\text{Obj}(\mathcal{D}^{\mathcal{C}}) = \{F : \mathcal{C} \rightarrow \mathcal{D}\}$
- For  $F, G \in \text{Obj}(\mathcal{D}^{\mathcal{C}})$ ,  $\text{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G) = \{\alpha : \alpha : F \Rightarrow G\}$
- For  $F, G, H \in \text{Obj}(\mathcal{D}^{\mathcal{C}})$ ,  $\alpha \in \text{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$  and  $\beta \in \text{Hom}_{\mathcal{D}^{\mathcal{C}}}(G, H)$ ,  $\beta \circ_{\mathcal{D}^{\mathcal{C}}} \alpha = \beta \circ \alpha$

**Exercise 1.4.2.10.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Suppose that  $\mathcal{C}$  is small. Then  $\mathcal{D}^{\mathcal{C}}$  is a category.

*Proof.* Exercise 1.4.2.2 implies that composition is well-defined. Exercise 1.4.2.3 implies that composition is associative. Exercise 1.4.2.5 and Exercise 1.4.2.6 imply the existence of identities. □

### 1.4.3 Diagonal Functor

**Definition 1.4.3.1.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X, Y \in \text{Obj}(\mathcal{D})$  and  $f \in \text{Hom}_{\mathcal{D}}(X, Y)$ . We define the **constant natural transformation on  $\mathcal{C}$  at  $f$** , denoted  $\delta_f^{\mathcal{C}} : \Delta_X^{\mathcal{C}} \Rightarrow \Delta_Y^{\mathcal{C}}$ , by

$$(\delta_f^{\mathcal{C}})_A = f$$

**Exercise 1.4.3.2.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X, Y \in \text{Obj}(\mathcal{D})$  and  $f \in \text{Hom}_{\mathcal{D}}(X, Y)$ . Then  $\delta_f^{\mathcal{C}} : \Delta_X^{\mathcal{C}} \Rightarrow \Delta_Y^{\mathcal{C}}$  is a natural transformation.

*Proof.*

1. By definition, for each  $A \in \text{Obj}(\mathcal{C})$   $(\delta_f^{\mathcal{C}})_A \in \text{Hom}_{\mathcal{D}}(\Delta_X^{\mathcal{C}}(A), \Delta_Y^{\mathcal{C}}(A))$ .

2. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $g \in \text{Hom}_{\mathcal{C}}(A, B)$ . Then

$$\begin{aligned}\Delta_Y^{\mathcal{C}}(g) \circ (\delta_f^{\mathcal{C}})_A &= \text{id}_Y \circ f \\ &= f \\ &= f \circ \text{id}_X \\ &= (\delta_f^{\mathcal{C}})_B \circ \Delta_X^{\mathcal{C}}(g)\end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} \Delta_X^{\mathcal{C}}(A) & \xrightarrow{(\delta_f^{\mathcal{C}})_A} & \Delta_Y^{\mathcal{C}}(A) \\ \Delta_X^{\mathcal{C}}(g) \downarrow & & \downarrow \Delta_Y^{\mathcal{C}}(g) \\ \Delta_X^{\mathcal{C}}(B) & \xrightarrow{(\delta_f^{\mathcal{C}})_B} & \Delta_Y^{\mathcal{C}}(B) \end{array} = \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{id}_X \downarrow & & \downarrow \text{id}_Y \\ X & \xrightarrow{f} & Y \end{array}$$

So  $\delta_f^{\mathcal{C}} : \Delta_X^{\mathcal{C}} \Rightarrow \Delta_Y^{\mathcal{C}}$  is a natural transformation. □

**Exercise 1.4.3.3.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X, Y, Z \in \text{Obj}(\mathcal{D})$ ,  $f \in \text{Hom}_{\mathcal{D}}(X, Y)$  and  $g \in \text{Hom}_{\mathcal{D}}(Y, Z)$ . Then  $\delta_{g \circ f}^{\mathcal{C}} = \delta_g^{\mathcal{C}} \circ \delta_f^{\mathcal{C}}$ .

*Proof.* Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned}(\delta_{g \circ f}^{\mathcal{C}})_A &= g \circ f \\ &= (\delta_g^{\mathcal{C}})_A \circ (\delta_f^{\mathcal{C}})_A \\ &= (\delta_g^{\mathcal{C}} \circ \delta_f^{\mathcal{C}})_A\end{aligned}$$

Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary,  $\delta_{g \circ f}^{\mathcal{C}} = \delta_g^{\mathcal{C}} \circ \delta_f^{\mathcal{C}}$ . □

**Exercise 1.4.3.4.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $X \in \text{Obj}(\mathcal{D})$ . Then  $\delta_{\text{id}_X}^{\mathcal{C}} = \text{id}_{\Delta_X^{\mathcal{C}}}$ .

*Proof.* Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned}(\delta_{\text{id}_X}^{\mathcal{C}})_A &= \text{id}_X \\ &= \text{id}_{\Delta_X^{\mathcal{C}}(A)} \\ &= (\text{id}_{\Delta_X^{\mathcal{C}}})_A\end{aligned}$$

Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary,  $\delta_{\text{id}_X}^{\mathcal{C}} = \text{id}_{\Delta_X^{\mathcal{C}}}$ . □

**Definition 1.4.3.5.** Let  $\mathcal{C}, \mathcal{D}$  be categories. Suppose that  $\mathcal{C}$  is small. We define the  **$\mathcal{C}$ -ary diagonal functor** on  $\mathcal{D}$ , denoted by  $\Delta^{\mathcal{C}} : \mathcal{D} \rightarrow \mathcal{D}^{\mathcal{C}}$ , by

- $\Delta^{\mathcal{C}}(X) = \Delta_X^{\mathcal{C}}$
- $\Delta^{\mathcal{C}}(f) = \delta_f^{\mathcal{C}}$

**Exercise 1.4.3.6.** Let  $\mathcal{C}, \mathcal{D}$  be categories. Suppose that  $\mathcal{C}$  is small. Then  $\Delta^{\mathcal{C}} : \mathcal{D} \rightarrow \mathcal{D}^{\mathcal{C}}$  is a functor.

*Proof.*

1. Exercise 1.4.3.2 implies that for each  $X, Y \in \text{Obj}(\mathcal{D})$  and  $f \in \text{Hom}_{\mathcal{D}}(X, Y)$ ,  $\Delta^{\mathcal{C}}(f) \in \text{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta^{\mathcal{C}}(X), \Delta^{\mathcal{C}}(Y))$
2. Exercise 1.4.3.3 implies that for each  $X, Y, Z \in \text{Obj}(\mathcal{D})$ ,  $f \in \text{Hom}_{\mathcal{D}}(X, Y)$  and  $g \in \text{Hom}_{\mathcal{D}}(Y, Z)$ ,  $\Delta^{\mathcal{C}}(g \circ f) = \Delta^{\mathcal{C}}(g) \circ \Delta^{\mathcal{C}}(f)$
3. Exercise 1.4.3.4 implies that for each  $X \in \text{Obj}(\mathcal{D})$ ,  $\Delta^{\mathcal{C}}(\text{id}_X) = \text{id}_{\Delta^{\mathcal{C}}(X)}$

So  $\Delta^{\mathcal{C}} : \mathcal{D} \rightarrow \mathcal{D}^{\mathcal{C}}$  is a functor. □

## 1.5 Algebra of Morphisms

### 1.5.1 Isomorphisms

#### Exercise 1.5.1.1. Uniqueness of Identities:

Let  $\mathcal{C}$  be a category. Then for each  $A \in \text{Obj}(\mathcal{C})$ , there exists a unique  $e_A \in \text{Hom}_{\mathcal{C}}(A, A)$  such that for each  $B \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, A)$ ,  $f \circ e_A = f$  and  $e_A \circ g = g$ .

*Proof.* Let  $A \in \text{Obj}(\mathcal{C})$ . Let  $e_A \in \text{Hom}_{\mathcal{C}}(A, A)$ . Suppose that for each  $B \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, A)$ ,  $f \circ e_A = f$  and  $e_A \circ g = g$ . Then

$$\begin{aligned} e_A &= e_A \circ \text{id}_A \\ &= \text{id}_A \end{aligned}$$

□

**Definition 1.5.1.2.** Let  $\mathcal{C}$  be a category,  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Then  $f$  is said to be an **isomorphism** if there exists  $g \in \text{Hom}_{\mathcal{C}}(B, A)$  such that  $g \circ f = \text{id}_A$  and  $f \circ g = \text{id}_B$ .

#### Exercise 1.5.1.3. Uniqueness of Inverses:

Let  $\mathcal{C}$  be a category,  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . If  $f$  is an isomorphism, then there exists a unique  $g \in \text{Hom}_{\mathcal{C}}(B, A)$  such that  $g \circ f = \text{id}_A$  and  $f \circ g = \text{id}_B$ .

*Proof.* Suppose that  $f$  is an isomorphism. Let  $g, h \in \text{Hom}_{\mathcal{C}}(B, A)$ . Suppose that  $g \circ f = \text{id}_A$ ,  $f \circ g = \text{id}_B$  and  $h \circ f = \text{id}_A$ ,  $f \circ h = \text{id}_B$ . Then

$$\begin{aligned} g &= g \circ \text{id}_B \\ &= g \circ (f \circ h) \\ &= (g \circ f) \circ h \\ &= \text{id}_A \circ h \\ &= h \end{aligned}$$

□

**Definition 1.5.1.4.** Let  $\mathcal{C}$  be a category,  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Suppose that  $f$  is an isomorphism. We define the **inverse of  $f$** , denoted  $f^{-1}$ , to be the unique  $g \in \text{Hom}_{\mathcal{C}}(B, A)$  such that  $g \circ f = \text{id}_A$  and  $f \circ g = \text{id}_B$ .

**Exercise 1.5.1.5.** Let  $\mathcal{C}$  be a category and  $A \in \text{Obj}(\mathcal{C})$ . Then  $\text{id}_A$  is an isomorphism and  $(\text{id}_A)^{-1} = \text{id}_A$ .

*Proof.* Since  $\text{id}_A \circ \text{id}_A = \text{id}_A$ , we have that  $\text{id}_A$  is an isomorphism and  $(\text{id}_A)^{-1} = \text{id}_A$ . □

**Exercise 1.5.1.6.** Let  $\mathcal{C}$  be a category and  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . If  $f$  is an isomorphism, then  $f^{-1}$  is an isomorphism and  $(f^{-1})^{-1} = f$ .

*Proof.* Suppose that  $f$  is an isomorphism. By definition,  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ . Hence  $f^{-1}$  is an isomorphism and  $(f^{-1})^{-1} = f$ . □

**Exercise 1.5.1.7.** Let  $\mathcal{C}$  be a category,  $A, B, C \in \text{Obj}(\mathcal{C})$ ,  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$ . If  $f$  and  $g$  are isomorphisms, then  $g \circ f$  is an isomorphism and  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .

*Proof.* Suppose that  $f$  and  $g$  are isomorphisms. Then

$$\begin{aligned} (f^{-1} \circ g^{-1}) \circ (g \circ f) &= ((f^{-1} \circ g^{-1}) \circ g) \circ f \\ &= (f^{-1} \circ (g^{-1} \circ g)) \circ f \\ &= (f^{-1} \circ \text{id}_B) \circ f \\ &= f^{-1} \circ f \\ &= \text{id}_A \end{aligned}$$

and

$$\begin{aligned}
 (g \circ f) \circ (f^{-1} \circ g^{-1}) &= ((g \circ f) \circ f^{-1}) \circ g^{-1} \\
 &= (g \circ (f \circ f^{-1})) \circ g^{-1} \\
 &= (g \circ \text{id}_B) \circ g^{-1} \\
 &= g \circ g^{-1} \\
 &= \text{id}_C
 \end{aligned}$$

So  $g \circ f$  is an isomorphism and  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .  $\square$

**Definition 1.5.1.8.** Let  $\mathcal{C}$  be a category and  $A, B \in \text{Obj}(\mathcal{C})$ . Then  $A$  is said to be **isomorphic** to  $B$  if there exists  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  such that  $f$  is an isomorphism.

**Exercise 1.5.1.9.** Let  $\mathcal{C}$  be a category. We define the relation  $\cong$  on  $\text{Obj}(\mathcal{C})$  by  $A \cong B$  iff  $A$  is isomorphic to  $B$ . Then  $\cong$  is an equivalence relation on  $\text{Obj}(\mathcal{C})$ .

*Proof.*

1. **reflexivity:**

Let  $A \in \text{Obj}(\mathcal{C})$ . Exercise 1.5.1.5 implies that  $\text{id}_A$  is an isomorphism. So  $A \cong A$ . Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary, we have that for each  $A \in \text{Obj}(\mathcal{C})$ ,  $A \cong A$  and thus  $\cong$  is reflexive.

2. **symmetry:**

Let  $A, B \in \text{Obj}(\mathcal{C})$ . Suppose that  $A \cong B$ . Then there exists  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  such that  $f$  is an isomorphism. Exercise 1.5.1.6 implies that  $f^{-1}$  is an isomorphism. Since  $f^{-1} \in \text{Hom}_{\mathcal{C}}(B, A)$ ,  $B \cong A$ . Since  $A, B \in \text{Obj}(\mathcal{C})$  are arbitrary, we have that for each  $A, B \in \text{Obj}(\mathcal{C})$ ,  $A \cong B$  implies that  $B \cong A$  and thus  $\cong$  is reflexive.

3. **transitivity:** Let  $A, B, C \in \text{Obj}(\mathcal{C})$ . Suppose that  $A \cong B$  and  $B \cong C$ . Then there exist  $f \in \text{Hom}_{\mathcal{C}}(A, B)$  and  $g \in \text{Hom}_{\mathcal{C}}(B, C)$  such that that  $f$  and  $g$  are isomorphisms. Exercise 1.5.1.7 implies that  $g \circ f$  is an isomorphism. Since  $g \circ f \in \text{Hom}_{\mathcal{C}}(A, C)$ ,  $A \cong C$ . Since  $A, B, C \in \text{Obj}(\mathcal{C})$  are arbitrary, we have that for each  $A, B, C \in \text{Obj}(\mathcal{C})$ ,  $A \cong B$  and  $B \cong C$  implies that  $A \cong C$  and thus  $\cong$  is transitive.

Since  $\cong$  is reflexive, symmetric and transitive,  $\cong$  is an equivalence relation on  $\text{Obj}(\mathcal{C})$ .  $\square$

**Definition 1.5.1.10.** Let  $\mathcal{C}$  be a category,  $A, B \in \text{Obj}(\mathcal{C})$  and  $f : A \rightarrow B$ . Then

- $f$  is said to be a **monomorphism** if for each  $C \in \text{Obj}(\mathcal{C})$  and  $g, h \in \text{Hom}_{\mathcal{C}}(C, A)$ ,  $f \circ g = f \circ h$  implies that  $g = h$ , i.e. we have the following implication of commutative diagrams:

$$\begin{array}{ccc}
 C & \xrightarrow{g} & A \\
 h \downarrow & & \downarrow f \\
 A & \xrightarrow{f} & B
 \end{array} \implies \begin{array}{ccc}
 & \xrightarrow{g} & \\
 C & & A \\
 & \xleftarrow{h} &
 \end{array}$$

- $f$  is said to be an **epimorphism** if for each  $C \in \text{Obj}(\mathcal{C})$  and  $g, h \in \text{Hom}_{\mathcal{C}}(B, C)$ ,  $g \circ f = h \circ f$  implies that  $g = h$ , i.e. we have the following implication of commutative diagrams:

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 f \downarrow & & \downarrow g \\
 B & \xrightarrow{h} & C
 \end{array} \implies \begin{array}{ccc}
 & \xrightarrow{g} & \\
 B & & C \\
 & \xleftarrow{h} &
 \end{array}$$

**Exercise 1.5.1.11.** Let  $A, B \in \text{Obj}(\text{Set})$  and  $f \in \text{Hom}_{\text{Set}}(A, B)$ . Then

1.  $f$  is a monomorphism iff  $f$  is injective
2.  $f$  is an epimorphism iff  $f$  is surjective

**Hint:** consider  $C = \{0\}$  and  $C = \{0, 1\}$ .

*Proof.*

1. Suppose that  $f$  is injective. Let  $C \in \text{Obj}(\mathbf{Set})$  and  $g, h \in \text{Hom}_{\mathbf{Set}}(C, A)$ . Suppose that  $f \circ g = f \circ h$ . Let  $x \in C$ . Then  $f(g(x)) = f(h(x))$ . Injectivity of  $f$  implies that  $g(x) = h(x)$ . Since  $x \in C$  is arbitrary,  $g = h$ . Hence  $f$  is a monomorphism.  
Conversely, suppose that  $f$  is a monomorphism. Let  $a, b \in A$ . Suppose that  $f(a) = f(b)$ . Set  $C = \{0\}$  and define  $g, h : C \rightarrow A$  by  $g(0) = a$  and  $h(0) = b$ . Then

$$\begin{aligned} f \circ g(0) &= f(g(0)) \\ &= f(a) \\ &= f(b) \\ &= f(h(0)) \\ &= f \circ h(0) \end{aligned}$$

Therefore  $f \circ g = f \circ h$ . Since  $f$  is a monomorphism, we have that  $g = h$ . Hence

$$\begin{aligned} a &= g(0) \\ &= h(0) \\ &= b \end{aligned}$$

2. Suppose that  $f$  is surjective. Let  $C \in \text{Obj}(\mathbf{C})$  and  $g, h \in \text{Hom}_{\mathbf{Set}}(B, C)$ . Suppose that  $g \circ f = h \circ f$ . Let  $y \in B$ . Surjective of  $f$  implies that there exists  $x \in A$  such that  $y = f(x)$ . Then

$$\begin{aligned} g(y) &= g(f(x)) \\ &= g \circ f(x) \\ &= h \circ f(x) \\ &= h(f(x)) \\ &= h(y) \end{aligned}$$

Since  $y \in B$  is arbitrary,  $g = h$ . Hence  $f$  is an epimorphism.

Conversely, suppose that  $f$  is an epimorphism. Set  $C = \{0, 1\}$  and define  $g, h : B \rightarrow C$  by  $g = \chi_{f(A)}$  and  $h = \chi_B$ . Then  $g \circ f = h \circ f$ . Since  $f$  is an epimorphism,  $g = h$  and  $f(A) = B$ . Hence  $f$  is surjective.

□

**Exercise 1.5.1.12.** Let  $\mathcal{C}$  be a category,  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . If  $f$  is an isomorphism, then  $f$  is a monomorphism and  $f$  is an epimorphism.

*Proof.* Suppose that  $f$  is an isomorphism.

- (monomorphism)

Let  $C \in \text{Obj}(\mathcal{C})$  and  $g, h \in \text{Hom}_{\mathcal{C}}(C, A)$ . Suppose that  $f \circ g = f \circ h$ . Then

$$\begin{aligned} g &= \text{id}_A \circ g \\ &= (f^{-1} \circ f) \circ g \\ &= f^{-1} \circ (f \circ g) \\ &= f^{-1} \circ (f \circ h) \\ &= (f^{-1} \circ f) \circ h \\ &= \text{id}_A \circ h \\ &= h \end{aligned}$$

So  $f$  is a monomorphism.

- (epimorphism)

Let  $C \in \text{Obj}(\mathcal{C})$  and  $g, h \in \text{Hom}_{\mathcal{C}}(B, C)$ . Suppose that  $g \circ f = h \circ f$ . Then

$$\begin{aligned}
 g &= g \circ \text{id}_B \\
 &= g \circ (f \circ f^{-1}) \\
 &= (g \circ f) \circ f^{-1} \\
 &= (h \circ f) \circ f^{-1} \\
 &= h \circ (f \circ f^{-1}) \\
 &= h \circ \text{id}_B \\
 &= h
 \end{aligned}$$

So  $f$  is an epimorphism. □

**Definition 1.5.1.13.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories,  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  and  $\alpha : F \Rightarrow G$ . Then  $\alpha$  is said to be a **natural isomorphism** if for each  $A \in \text{Obj}(\mathcal{C})$ ,  $\alpha_A$  is an isomorphism.

**Definition 1.5.1.14.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories,  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  and  $\alpha : F \Rightarrow G$ . Suppose that  $\alpha$  is a natural isomorphism. We define  $\alpha^{-1} : G \Rightarrow F$  by  $(\alpha^{-1})_A = \alpha_A^{-1}$ .

**Exercise 1.5.1.15.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories,  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  and  $\alpha : F \Rightarrow G$ . Suppose that  $\alpha$  is a natural isomorphism. Then  $\alpha^{-1} : G \Rightarrow F$  is a natural transformation.

*Proof.*

1. Let  $A \in \text{Obj}(\mathcal{C})$ . Since  $\alpha_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$ , we have that

$$\begin{aligned}
 (\alpha^{-1})_A &= \alpha_A^{-1} \\
 &\in \text{Hom}_{\mathcal{D}}(G(A), F(A))
 \end{aligned}$$

2. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Since  $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ , i.e. the following diagram commutes:

$$\begin{array}{ccc}
 F(A) & \xrightarrow{\alpha_A} & G(A) \\
 F(f) \downarrow & & \downarrow G(f) \\
 F(B) & \xrightarrow{\alpha_B} & G(B)
 \end{array}$$

we have that

$$\begin{aligned}
 F(f) \circ (\alpha^{-1})_A &= F(f) \circ \alpha_A^{-1} \\
 &= \text{id}_{F(B)} \circ (F(f) \circ \alpha_A^{-1}) \\
 &= (\alpha_B^{-1} \circ \alpha_B) \circ (F(f) \circ \alpha_A^{-1}) \\
 &= \alpha_B^{-1} \circ (\alpha_B \circ (F(f) \circ \alpha_A^{-1})) \\
 &= \alpha_B^{-1} \circ ((\alpha_B \circ F(f)) \circ \alpha_A^{-1}) \\
 &= \alpha_B^{-1} \circ ((G(f) \circ \alpha_A) \circ \alpha_A^{-1}) \\
 &= \alpha_B^{-1} \circ (G(f) \circ (\alpha_A \circ \alpha_A^{-1})) \\
 &= \alpha_B^{-1} \circ (G(f) \circ \text{id}_{G(A)}) \\
 &= \alpha_B^{-1} \circ G(f) \\
 &= (\alpha^{-1})_B \circ G(f)
 \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} G(A) & \xrightarrow{(\alpha^{-1})_A} & F(A) \\ G(f) \downarrow & & \downarrow F(f) \\ G(B) & \xrightarrow{(\alpha^{-1})_B} & F(B) \end{array}$$

So  $\alpha^{-1} : G \Rightarrow F$ .

□

**Exercise 1.5.1.16.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories,  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  and  $\alpha : F \Rightarrow G$ . Suppose that  $\alpha$  is a natural isomorphism. Then  $\alpha^{-1} \circ \alpha = \text{id}_F$  and  $\alpha \circ \alpha^{-1} = \text{id}_G$ .

*Proof.* Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} (\alpha^{-1} \circ \alpha)_A &= (\alpha^{-1})_A \circ \alpha_A \\ &= \alpha_A^{-1} \circ \alpha_A \\ &= \text{id}_{F(A)} \\ &= (\text{id}_F)_A \end{aligned}$$

and

$$\begin{aligned} (\alpha \circ \alpha^{-1})_A &= \alpha_A \circ (\alpha^{-1})_A \\ &= \alpha_A \circ \alpha_A^{-1} \\ &= \text{id}_{G(A)} \\ &= (\text{id}_G)_A \end{aligned}$$

Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary,  $\alpha^{-1} \circ \alpha = \text{id}_F$  and  $\alpha \circ \alpha^{-1} = \text{id}_G$ . □

**Exercise 1.5.1.17.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Suppose that  $\mathcal{C}$  is small. Let  $F, G \in \mathcal{D}^{\mathcal{C}}$  and  $\alpha \in \text{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$ . Then  $\alpha$  is an isomorphism iff  $\alpha$  is a natural isomorphism.

*Proof.* Suppose that  $\alpha$  is an isomorphism. Then there exists  $\beta \in \text{Hom}_{\mathcal{D}^{\mathcal{C}}}(G, F)$  such that  $\beta \circ \alpha = \text{id}_F$  and  $\alpha \circ \beta = \text{id}_G$ . Let  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} \beta_A \circ \alpha_A &= (\beta \circ \alpha)_A \\ &= (\text{id}_F)_A \\ &= \text{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} \alpha_A \circ \beta_A &= (\alpha \circ \beta)_A \\ &= (\text{id}_G)_A \\ &= \text{id}_{G(A)} \end{aligned}$$

Hence  $\alpha_A$  is an isomorphism. Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary,  $\alpha$  is a natural isomorphism.

Conversely, suppose that  $\alpha$  is a natural isomorphism. Exercise 1.5.1.15 and Exercise 1.5.1.16 imply that  $\alpha$  is an isomorphism. □

### 1.5.2 Initial and Final Objects

**Definition 1.5.2.1.** Let  $\mathcal{C}$  be a category and  $0 \in \text{Obj}(\mathcal{C})$ . Then  $0$  is said to be **initial** if for each  $A \in \text{Obj}(\mathcal{C})$ , there exists  $f \in \text{Hom}_{\mathcal{C}}(0, A)$  such that  $\text{Hom}_{\mathcal{C}}(0, A) = \{f\}$ .

**Definition 1.5.2.2.** Let  $\mathcal{C}$  be a category and  $1 \in \text{Obj}(\mathcal{C})$ . Then  $1$  is said to be **final** if for each  $A \in \text{Obj}(\mathcal{C})$ , there exists  $f \in \text{Hom}_{\mathcal{C}}(A, 1)$  such that  $\text{Hom}_{\mathcal{C}}(A, 1) = \{f\}$ .

**Exercise 1.5.2.3.** Let  $\mathcal{C}$  be a category and  $0 \in \text{Obj}(\mathcal{C})$ . If  $0$  is initial, then  $\text{Hom}_{\mathcal{C}}(0, 0) = \{\text{id}_0\}$ .

*Proof.* Suppose that  $0$  is initial. Then there exists a  $f \in \text{Hom}_{\mathcal{C}}(0, 0)$  such that  $\text{Hom}_{\mathcal{C}}(0, 0) = \{f\}$ . Since  $\text{id}_0 \in \text{Hom}_{\mathcal{C}}(0, 0)$ ,  $f = \text{id}_0$  and therefore  $\text{Hom}_{\mathcal{C}}(0, 0) = \{\text{id}_0\}$ .  $\square$

**Exercise 1.5.2.4.** Let  $\mathcal{C}$  be a category and  $1 \in \text{Obj}(\mathcal{C})$ . If  $1$  is final, then  $\text{Hom}_{\mathcal{C}}(1, 1) = \{\text{id}_1\}$ .

*Proof.* Similar to Exercise 1.5.2.3  $\square$

**Exercise 1.5.2.5.** Let  $\mathcal{C}$  be a category and  $0, 0' \in \text{Obj}(\mathcal{C})$ . If  $0$  and  $0'$  are initial, then  $0$  and  $0'$  are isomorphic.

*Proof.* Suppose that  $0$  and  $0'$  are initial. By definition, there exist  $f \in \text{Hom}_{\mathcal{C}}(0, 0')$  and  $f' \in \text{Hom}_{\mathcal{C}}(0', 0)$  such that  $\text{Hom}_{\mathcal{C}}(0, 0') = \{f\}$  and  $\text{Hom}_{\mathcal{C}}(0', 0) = \{f'\}$ , i.e. we have the following commutative diagram:

$$\begin{array}{ccc} & f' & \\ & \curvearrowright & \\ f' \circ f & \hookrightarrow 0 & \hookrightarrow 0' \\ & \curvearrowleft & \\ & f & \end{array}$$

Exercise 1.5.2.3 implies that  $f' \circ f = \text{id}_0$  and  $f \circ f' = \text{id}_{0'}$ . Hence  $f$  is an isomorphism. Since  $f \in \text{Hom}_{\mathcal{C}}(0, 0')$ , we have that  $0 \cong 0'$ .  $\square$

**Exercise 1.5.2.6.** Let  $\mathcal{C}$  be a category and  $1, 1' \in \text{Obj}(\mathcal{C})$ . If  $1$  and  $1'$  are final, then  $1$  and  $1'$  are isomorphic.

*Proof.* Similar to Exercise 1.5.2.5  $\square$

**Exercise 1.5.2.7.** We have that  $\emptyset$  is initial in **Set**.

*Proof.* Let  $A \in \text{Obj}(\mathbf{Set})$ . Define  $f \in \text{Hom}_{\mathbf{Set}}(\emptyset, A)$  by  $f = \emptyset$ . Let  $g \in \text{Hom}_{\mathbf{Set}}(\emptyset, A)$ . Then  $g = f$ . Since  $g \in \text{Hom}_{\mathbf{Set}}(\emptyset, A)$  is arbitrary,  $\text{Hom}_{\mathbf{Set}}(\emptyset, A) = \{f\}$ . Hence  $\emptyset$  is initial.  $\square$

**Exercise 1.5.2.8.** We have that  $\{\emptyset\}$  is terminal in **Set**.

*Proof.* Let  $A \in \text{Obj}(\mathbf{Set})$ . Define  $f \in \text{Hom}_{\mathbf{Set}}(A, \{\emptyset\})$  by  $f(x) = \emptyset$ . Let  $g \in \text{Hom}_{\mathbf{Set}}(A, \{\emptyset\})$ . Then  $g = f$ . Since  $g \in \text{Hom}_{\mathbf{Set}}(A, \{\emptyset\})$  is arbitrary,  $\text{Hom}_{\mathbf{Set}}(A, \{\emptyset\}) = \{f\}$ . Hence  $\{\emptyset\}$  is final.  $\square$

**Exercise 1.5.2.9.** We have that  $\mathbf{0}$  is initial in **Cat**.

*Proof.* Let  $\mathcal{C} \in \text{Obj}(\mathbf{Cat})$ . It is clear that  $\text{Hom}_{\mathbf{Cat}}(\mathbf{0}, \mathcal{C}) = \{E_{\mathcal{C}}\}$ . Hence  $\mathbf{0}$  is initial in **Cat**.  $\square$

**Exercise 1.5.2.10.** We have that  $\mathbf{1}$  is final in **Cat**.

*Proof.* Let  $\mathcal{C} \in \text{Obj}(\mathbf{Cat})$ . It is clear that  $\text{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathbf{1}) = \{\Delta_{\mathcal{C}}^{\mathcal{C}}\}$ . Hence  $\mathbf{1}$  is final in **Cat**.  $\square$

**Definition 1.5.2.11.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $0 \in \text{Obj}(\mathcal{D})$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Suppose that  $0$  is initial in  $\mathcal{D}$ . Then for each  $A \in \text{Obj}(\mathcal{C})$ , there exists  $f_A \in \text{Hom}_{\mathcal{D}}(0, F(A))$  such that  $\text{Hom}_{\mathcal{D}}(0, F(A)) = \{f_A\}$ . We define the **initial natural transformation induced by**  $0$  from  $\Delta_0^{\mathcal{C}}$  to  $F$ , denoted  $\zeta_0 : \Delta_0^{\mathcal{C}} \Rightarrow F$ , by  $(\eta_0)_A = f_A$ .

**Definition 1.5.2.12.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $1 \in \text{Obj}(\mathcal{D})$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Suppose that  $1$  is final in  $\mathcal{D}$ . Then for each  $A \in \text{Obj}(\mathcal{C})$ , there exists  $f_A \in \text{Hom}_{\mathcal{D}}(F(A), 1)$  such that  $\text{Hom}_{\mathcal{D}}(F(A), 1) = \{f_A\}$ . We define the **final natural transformation induced by**  $1$  from  $F$  to  $\Delta_1^{\mathcal{C}}$ , denoted  $\phi_1 : F \Rightarrow \Delta_1^{\mathcal{C}}$ , by  $(\phi_1)_A = f_A$ .

**Exercise 1.5.2.13.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $0 \in \text{Obj}(\mathcal{D})$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Suppose that  $0$  is initial in  $\mathcal{D}$ . Then  $\eta_0 : \Delta_0^{\mathcal{C}} \Rightarrow F$  is a natural transformation.



*Proof.*

1. By definition, for each  $A \in \text{Obj}(\mathcal{C})$ ,  $(\eta_0)_A \in \text{Hom}_{\mathcal{D}}(\Delta_0^{\mathcal{C}}(A), F(A))$
2. Let  $A, B \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{C}}(A, B)$ . Since

$$\begin{aligned} F(f) \circ (\eta_0)_A &\in \text{Hom}_{\mathcal{D}}(0, F(B)) \\ &= \{(\eta_0)_B\} \end{aligned}$$

we have that

$$\begin{aligned} F(f) \circ (\eta_0)_A &= (\eta_0)_B \\ &= (\eta_0)_B \circ \text{id}_0 \end{aligned}$$

i.e. the following diagram commutes:

$$\begin{array}{ccc} \Delta_0^{\mathcal{C}}(A) & \xrightarrow{(\eta_0)_A} & F(A) \\ \Delta_0^{\mathcal{C}}(f) \downarrow & & \downarrow F(f) \\ \Delta_0^{\mathcal{C}}(B) & \xrightarrow{(\eta_0)_B} & F(B) \end{array} = \begin{array}{ccc} 0 & \xrightarrow{(\eta_0)_A} & F(A) \\ \text{id}_0 \downarrow & & \downarrow F(f) \\ 0 & \xrightarrow{(\eta_0)_B} & F(B) \end{array}$$

So  $\eta_0 : \Delta_0^{\mathcal{C}} \Rightarrow F$  is a natural transformation. □

**Exercise 1.5.2.14.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $1 \in \text{Obj}(\mathcal{D})$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$ . Suppose that  $1$  is final in  $\mathcal{D}$ . Then  $\phi_1 : F \Rightarrow \Delta_0^{\mathcal{C}}$  is a natural transformation.

*Proof.* Similar to Exercise 1.5.2.13 □

**Exercise 1.5.2.15.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $0 \in \text{Obj}(\mathcal{D})$ . Suppose that  $\mathcal{C}$  is small. If  $0$  is initial in  $\mathcal{D}$ , then  $\Delta_0^{\mathcal{C}}$  is initial in  $\mathcal{D}^{\mathcal{C}}$ .

*Proof.* Suppose that  $0$  is initial in  $\mathcal{D}$ . Let  $F \in \text{Obj}(\mathcal{D}^{\mathcal{C}})$ ,  $\alpha \in \text{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F)$  and  $A \in \text{Obj}(\mathcal{C})$ . Then

$$\begin{aligned} \alpha_A &\in \text{Hom}_{\mathcal{D}}(\Delta_0^{\mathcal{C}}(A), F(A)) \\ &= \text{Hom}_{\mathcal{D}}(0, F(A)) \\ &= \{(\eta_0)_A\} \end{aligned}$$

Hence  $\alpha_A = (\eta_0)_A$ . Since  $A \in \text{Obj}(\mathcal{C})$  is arbitrary,  $\alpha = \eta_0$ . Since  $\alpha \in \text{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F)$  is arbitrary,  $\text{Hom}_{\mathcal{D}^{\mathcal{C}}}(\Delta_0^{\mathcal{C}}, F) = \{\eta_0\}$ . Therefore  $\Delta_0^{\mathcal{C}}$  is initial in  $\mathcal{D}^{\mathcal{C}}$ . □

**Exercise 1.5.2.16.** Let  $\mathcal{C}, \mathcal{D}$  be categories and  $1 \in \text{Obj}(\mathcal{D})$ . Suppose that  $\mathcal{C}$  is small. If  $1$  is final in  $\mathcal{D}$ , then  $\Delta_1^{\mathcal{C}}$  is final in  $\mathcal{D}^{\mathcal{C}}$ .

*Proof.* Similar to Exercise 1.5.2.15. □



## Chapter 2

# Universal Morphisms and Limits

### 2.0.1 Universal Morphisms

**Definition 2.0.1.1.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X \in \text{Obj}(\mathcal{D})$ ,  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $A \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{D}}(X, F(A))$ . Then  $(A, f)$  is said to be a **universal morphism** from  $X$  to  $F$  if for each  $A' \in \text{Obj}(\mathcal{C})$   $f' \in \text{Hom}_{\mathcal{D}}(X, F(A'))$ , there exists a unique  $\alpha \in \text{Hom}_{\mathcal{C}}(A, A')$  such that  $f' = F(\alpha) \circ f$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & F(A) \\ & \searrow f' & \downarrow F(\alpha) \\ & & F(A') \end{array} \quad \begin{array}{c} A \\ \downarrow \alpha \\ A' \end{array}$$

**Definition 2.0.1.2.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X \in \text{Obj}(\mathcal{D})$ ,  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $A \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{D}}(F(A), X)$ . Then  $(A, f)$  is said to be a **universal morphism** from  $F$  to  $X$  if for each  $A' \in \text{Obj}(\mathcal{C})$   $f' \in \text{Hom}_{\mathcal{D}}(F(A'), X)$ , there exists a unique  $\alpha \in \text{Hom}_{\mathcal{C}}(A', A)$  such that  $f' = f \circ F(\alpha)$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} X & \xleftarrow{f} & F(A) \\ & \swarrow f' & \uparrow F(\alpha) \\ & & F(A') \end{array} \quad \begin{array}{c} A \\ \uparrow \alpha \\ A' \end{array}$$

**Exercise 2.0.1.3.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X \in \text{Obj}(\mathcal{D})$ ,  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $A \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{D}}(X, F(A))$ . Then  $(A, f)$  is a universal morphism from  $X$  to  $F$  iff  $(A, f)$  is initial in  $(X \downarrow F)$ .

*Proof.*

□

**Exercise 2.0.1.4.** Let  $\mathcal{C}, \mathcal{D}$  be categories,  $X \in \text{Obj}(\mathcal{D})$ ,  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $A \in \text{Obj}(\mathcal{C})$  and  $f \in \text{Hom}_{\mathcal{D}}(F(A), X)$ . Then  $(A, f)$  is a universal morphism from  $F$  to  $X$  iff  $(A, f)$  is terminal in  $(F \downarrow X)$ .

*Proof.*

□

## 2.1 Limits

**Definition 2.1.0.1.** Let  $\mathcal{J}, \mathcal{C}$  be categories and  $D : \mathcal{J} \rightarrow \mathcal{C}$ . Then  $D$  is said to be a **diagram of type  $\mathcal{J}$  in  $\mathcal{C}$** .

**Note 2.1.0.2.** We are usually interested in the case that  $\mathcal{J}$  is small. We will identify a diagram  $D$  with its image.

**Example 2.1.0.3.** Define  $\mathcal{J}$  by

- $\text{Obj}(\mathcal{J}) = \{1, 2, 3\}$  and for  $i, j \in \text{Obj}(\mathcal{J})$ ,  $\text{Hom}_{\mathcal{J}}(i, j) = \{a_{i,j}\}$ ,
- for  $i, j \in \text{Obj}(\mathcal{J})$ ,  $\text{Hom}_{\mathcal{J}}(i, j) = \{a_{ij}\}$ .

Let  $\mathcal{C}$  be a category and  $D : \mathcal{J} \rightarrow \mathcal{C}$ . Without including the identity morphisms or compositions, we can visualize  $D$  as follows:

$$\begin{array}{ccc} 1 & \xrightarrow{b} & 2 \\ a \swarrow & & \searrow c \\ 3 & \xrightarrow{d} & 4 \end{array} \xrightarrow{D} \begin{array}{ccc} D_1 & \longrightarrow & D_2 \\ & \swarrow & \searrow \\ D_3 & \longrightarrow & D_4 \end{array}$$

**Definition 2.1.0.4.** Let  $\mathcal{J}, \mathcal{C}$  be categories. Suppose that  $\mathcal{J}$  is small. Let  $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$ . We define the **category of cones to  $D$** , denoted  $\mathbf{Cone}(D)$ , by  $\mathbf{Cone}(D) = (\Delta^{\mathcal{J}} \downarrow D)$ .

**Example 2.1.0.5.** Let  $\mathcal{J}$

$$\begin{array}{ccccc} & & X, & & \\ & \phi_3 \swarrow & \downarrow \phi_1 & \searrow \phi_2 & \\ & & D_1 & \longrightarrow & D_2 \\ & \swarrow \phi_4 & & \searrow & \\ D_3 & \longrightarrow & D_4 & & \end{array}$$

**Definition 2.1.0.6.** Let  $\mathcal{J}, \mathcal{C}$  be categories. Suppose that  $\mathcal{J}$  is small. Let  $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$ . We define the **category of cocones from  $D$** , denoted  $\mathbf{Cocone}(D)$ , by  $\mathbf{Cocone}(D) = (D \downarrow \Delta^{\mathcal{J}})$ .

**Definition 2.1.0.7.** Let  $\mathcal{J}, \mathcal{C}$  be categories. Suppose that  $\mathcal{J}$  is small. Let  $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$  and  $(X, \phi) \in \mathbf{Cone}(D)$ . Then  $(X, \phi)$  is said to be a **limit of  $D$**  if  $(X, \phi)$  is a universal morphism from  $\Delta^{\mathcal{J}}$  to  $D$ .

**Note 2.1.0.8.** Let  $\mathcal{J}, \mathcal{C}$  be categories. Suppose that  $\mathcal{J}$  is small. Let  $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$  and  $(X, \phi) \in \mathbf{Cone}(D)$ . Then

$$\begin{aligned} (X, \phi) \text{ is a limit of } D &\iff (X, \phi) \text{ is terminal in } \mathbf{Cone}(D) \\ &\iff \text{for each } (Y, \psi) \in \mathbf{Cone}(D), \text{ there exists a unique} \\ &\quad f \in \text{Hom}_{\mathcal{C}}(Y, X) \text{ such that for each } j \in \mathcal{J}, \psi_j = \phi_j \circ f \end{aligned}$$

**Definition 2.1.0.9.** Let  $\mathcal{J}, \mathcal{C}$  be categories. Suppose that  $\mathcal{J}$  is small. Let  $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$  and  $(X, \phi) \in \mathbf{Cocone}(D)$ . Then  $(X, \phi)$  is said to be a **colimit of  $D$**  if  $(X, \phi)$  is a universal morphism from  $D$  to  $\Delta^{\mathcal{J}}$ .

**Note 2.1.0.10.** Let  $\mathcal{J}, \mathcal{C}$  be categories. Suppose that  $\mathcal{J}$  is small. Let  $D \in \text{Obj}(\mathcal{C}^{\mathcal{J}})$  and  $(X, \phi) \in \mathbf{Cone}(D)$ . Then

$$\begin{aligned} (X, \phi) \text{ is a colimit of } D &\iff (X, \phi) \text{ is initial in } \mathbf{Cocone}(D) \\ &\iff \text{for each } (Y, \psi) \in \mathbf{Cocone}(D), \text{ there exists a unique} \\ &\quad f \in \text{Hom}_{\mathcal{C}}(X, Y) \text{ such that for each } j \in \mathcal{J}, \psi_j = f \circ \phi_j \end{aligned}$$

**2.1.1 Products and Coproducts**

**2.1.2 Equalizers and Coequalizers**