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Introduction to Differential Geometry

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Contents

Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

Chapter 1

Review of Fundamentals

1.1 Set Theory

Definition 1.1.0.1. Let $\{A_i\}_{i\in I}$ be a collection of sets. The **disjoint union of** $\{A_i\}_{i\in I}$, denoted $\coprod_{i\in I} A_i$, is defined by

$$\coprod_{i\in I} A_i = \bigcup_{i\in I} \{i\} \times A_i$$

We define the **natural projection map**, denoted $\pi : \coprod_{i\in I} A_i \to I$, by $\pi(i,a) = i$.

Definition 1.1.0.2. Let E and M be sets, $\pi : E \to B$ a surjection and $\sigma : B \to E$. Then σ is said to be a section of (E, M, π) if $\pi \circ \sigma = \mathrm{id}_M$.

Note 1.1.0.3. Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod_{i\in I} A_i$. We will typically be interested in sections σ of $\left(\coprod_{i\in I} A_i, I, \pi\right)$.

Exercise 1.1.0.4. Let $\{A_i\}_{i\in I}$ be a collection of sets and $\sigma : I \to \coprod_{i\in I} A_i$. Then σ is a section of $\coprod_{i\in I} A_i$ iff for each $i \in I$, $\sigma(i) \in A_i$

Proof. Clear. $\square$

1.2 Linear Algebra

Note 1.2.0.1. We denote the standard basis on $\mathbb{R}^n$ by $(e_1, \dots, e_n)$.

Definition 1.2.0.2. Let $A \in \mathbb{R}^{n \times n}$. Then A is said to be **invertible** if $\det(A) \neq 0$. We denote the set of $n \times n$ invertible matrices by $GL(n, \mathbb{R})$.

$$O(n)$$

Exercise 1.2.0.3. Let $A, B \in \mathbb{R}^{n \times n}$. Then $AB = I$ iff $BA = I$.

Proof.

- $(\implies)$:
  Suppose that $AB = I$. Then

$$\begin{aligned}\ker B &\subset \ker AB \\ &= \ker I \\ &= \{0\}\end{aligned}$$

  so that $\ker B = \{0\}$. Hence $\operatorname{Im} B = \mathbb{R}^n$ and B is surjective. Then

$$\begin{aligned}IB &= BI \\ &= B(AB) \\ &= (BA)B\end{aligned}$$

  Since B is surjective, $I = BA$.

- $(\impliedby)$:
  Immediate by the previous part.

□

Definition 1.2.0.4. Let $A \in \mathbb{R}^{n \times p}$. Then A is said to be an **orthogonal matrix** if $A^* A = I$. We denote the set of $n \times p$ orthogonal matrices by $O(n, p)$. We write $O(n)$ in place of $O(n, n)$.

$$O(n)$$

Exercise 1.2.0.5. Define $\phi : S_n \to GL(n, \mathbb{R})$ by

$$\phi(\sigma) = \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix}$$

Then

1. for each $A \in \mathbb{R}^{n \times p}$,
$$(\phi(\sigma)A)_{i,j} = A_{\sigma(i),j}$$
   i.e. left multiplying A by $\phi(\sigma)$ the the same as permuting the rows of A by σ
2. ϕ is a group homomorphism

Proof. 1. Let $A \in \mathbb{R}^{n \times p}$. Then

$$\begin{aligned}(\phi(\sigma)A)_{i,j} &= \langle e^*_{\sigma(i)}, Ae_j \rangle \\ &= A_{\sigma(i),j}\end{aligned}$$

2. Let $\sigma, \tau \in S_n$. Part (1) implies that

$$\begin{aligned}\phi(\sigma\tau) &= \begin{pmatrix} e^*_{\sigma\tau(1)} \\ \vdots \\ e^*_{\sigma\tau(n)} \end{pmatrix} \\ &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e^*_{\tau(1)} \\ \vdots \\ e^*_{\tau(n)} \end{pmatrix} \\ &= \phi(\sigma)\phi(\tau)\end{aligned}$$

Since $\sigma, \tau \in S_n$ are arbitrary, ϕ is a group homomorphism.

□

Definition 1.2.0.6. Define $\phi : S_n \to GL(n, \mathbb{R})$ as in the previous exercise. Let $P \in GL(n, \mathbb{R})$. Then P is said to be a **permutation matrix** if there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. We denote the set of $n \times n$ permutation matrices by $\mathrm{Perm}(n)$.

Exercise 1.2.0.7. We have that

1. $\mathrm{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$
2. $\mathrm{Perm}(n)$ is a subgroup of $O(n)$

Proof.

1. By definition, $\mathrm{Perm}(n) = \mathrm{Im}\,\phi$. Since $\phi : S_n \to GL(n, \mathbb{R})$ is a group homomorphism, $\mathrm{Im}\,\phi$ is a subgroup of $GL(n, \mathbb{R})$. Hence $\mathrm{Perm}(n)$ is a subgroup of $GL(n, \mathbb{R})$.
2. Let $P \in \mathrm{Perm}(n)$. Then there exists $\sigma \in S_n$ such that $P = \phi(\sigma)$. Then

$$\begin{aligned}PP^* &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix}^* \\ &= \begin{pmatrix} e^*_{\sigma(1)} \\ \vdots \\ e^*_{\sigma(n)} \end{pmatrix} \begin{pmatrix} e_{\sigma(1)} & \cdots & e_{\sigma(n)} \end{pmatrix} \\ &= (\langle e_{\sigma(i)}, e_{\sigma(j)} \rangle)_{i,j} \\ &= I\end{aligned}$$

A previous exercise implies that $P^*P = I$. Hence $P \in O(n)$. Since $P \in \mathrm{Perm}(n)$ is arbitrary, $\mathrm{Perm}(n) \subset O(n)$. Part (1) implies that $\mathrm{Perm}(n)$ is a group. Hence $\mathrm{Perm}(n)$ is a subgroup of $O(n)$

□

Note 1.2.0.8. We will write P_σ in place of $\phi(\sigma)$.

Exercise 1.2.0.9. Let $Z \in \mathbb{R}^{p \times n}$. If $\operatorname{rank} Z = k$, then there exist $\sigma \in S_n$, $\tau \in S_p$ and $A \in GL(k, \mathbb{R})$, such that for each $i, j \in \{1, \dots, k\}$,

$$(P_\tau Z P^*_\sigma)_{i,j} = A_{i,j}$$

Proof. Suppose that $\operatorname{rank} Z - k$. Then there exist $i_1, \dots, i_k \in \{1, \dots, p\}$ such that $i_1 < \cdots < i_k$ and $\{e^*_{i_1} Z, \dots, e^*_{i_k} Z\}$ is linearly independent. Set

$$Z' = \begin{pmatrix} e^*_{i_1} Z \\ \vdots \\ e^*_{i_k} Z \end{pmatrix}$$

Then $\operatorname{rank} Z' = k$. Hence there exist $j_1, \dots, j_k \in \{1, \dots, n\}$ such that $j_1 < \dots < j_k$, and $\{Z'e_{i_1}, \dots, Z'e_{i_k}\}$ is linearly independent. Set

$$A = \begin{pmatrix} Z'e_{i_1} & \cdots & Z'e_{i_k} \end{pmatrix}$$

Then $A \in \mathbb{R}^{k\times k}$ and $\operatorname{rank} A = k$. Thus $A \in GL(k, \mathbb{R})$. Choose $\sigma \in S_n$ and $\tau \in S_p$ such that $\sigma(1) = j_1, \dots, \sigma(k) = j_k$ and $\tau(1) = i_1, \dots, \tau(k) = i_k$. Let $a, b \in \{1, \dots, k\}$. By construction,

$$\begin{aligned}(P_\tau Z P_\sigma^*)_{a,b} &= Z_{\tau(a),\sigma(b)} \\ &= Z_{i_a, j_b} \\ &= A_{a,b}\end{aligned}$$

□

Definition 1.2.0.10. Let $A \in \mathbb{R}^{n\times p}$. Then A is said to be a **diagonal matrix** if for each $i \in [n]$ and $j \in [p]$, $i \neq j$ implies that $A_{i,j} = 0$. We denote the set of $n \times p$ diagonal matrices by $D(n, p, \mathbb{R})$. We write $D(n, \mathbb{R})$ in place of $D(n, n, \mathbb{R})$.

Definition 1.2.0.11. For (n, k), (m, l) $\operatorname{diag}_{p,(n\times p)} : \mathbb{R}^p \to \mathbb{R}^{n\times p}$ and $\operatorname{diag}_{n,(n\times p)} : \mathbb{R}^p \to \mathbb{R}^{n\times p}$ by $\operatorname{diag}(v)$ FINISH!!!

Definition 1.2.0.12. Let $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \sigma(A)$. Suppose that A is symmetric. We define the **geometric multiplicity** of λ, denoted $\mu(\lambda)$, by

$$\mu(\lambda) = \dim \ker([\phi_\alpha] - \lambda I)$$

Definition 1.2.0.13. Let V be an n-dimensional vector space, $U \subset V$ a k-dimensional subspace and $(e_j)_{j=1}^n \subset V$ a be a basis. Then $(e_j)_{j=1}^n$ is said to be **adapted to** U if $(e_j)_{j=1}^k$ is a basis for U.

1.3 Calculus

1.3.1 Differentiation

Definition 1.3.1.1. Let $n \geq 1$. For $i = 1, \cdots, n$, define $x^i : \mathbb{R}^n \to \mathbb{R}$ by $x^i(a^1, \cdots, a^n) = a^i$. The functions $(x^i)_{i=1}^n$ are called the **standard coordinate functions on** $\mathbb{R}^n$.

Definition 1.3.1.2. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Then f is said to be **differentiable with respect to** x^i **at** a if

$$\lim_{h \to 0} \frac{f(a + he^i) - f(a)}{h}$$

exists. If f is differentiable with respect to x^i at a, we define the **partial derivative of** f **with respect to** x^i **at** a, denoted

$$\frac{\partial f}{\partial x^i}(a) \text{ or } \frac{\partial}{\partial x^i} f$$

to be the limit above.

Definition 1.3.1.3. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **differentiable with respect to** x^i if for each $a \in U$, f is differentiable with respect to x^i at a.

Exercise 1.3.1.4. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. Suppose that $\frac{\partial^2 f}{\partial x^i x^j}$ and $\frac{\partial^2 f}{\partial x^j x^i}$ exist and are continuous at a. Then

$$\frac{\partial^2 f}{\partial x^i x^j}(a) = \frac{\partial^2 f}{\partial x^j x^i}(a)$$

Proof. □

Definition 1.3.1.5. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$. Then f is said to be **smooth** if for each $i_1, \cdots, i_k \in \{1, \cdots, n\}$, $\frac{\partial^k f}{\partial i_1 \cdots i_k}$ exists and is continuous on U.

Definition 1.3.1.6. Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$. Then f is said to be **smooth** if there exists $U' \subset \mathbb{R}^n$ and $f' : U' \to \mathbb{R}$ such that $U \subset U'$, U' is open, $f'|_U = f$ and f' is smooth. The set of smooth functions on U is denoted $C^\infty(U)$.

Theorem 1.3.1.7. Taylor's Theorem:
Let $U \subset \mathbb{R}^n$ be open and convex, $p \in U$, $f \in C^\infty(U)$ and $T \in \mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1} \subset C^\infty(U)$ such that for each $x \in U$,

$$f(x) = \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x-p)^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x-p)^\alpha g_\alpha(x)$$

and for each $|\alpha| = T + 1$,

$$g_\alpha(p) = \frac{1}{(T+1)!} \partial^\alpha f(p)$$

Proof. See analysis notes □

Definition 1.3.1.8. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Let $x^1, \cdots, x^n$ be the standard coordinate functions on $\mathbb{R}^n$ and $y_1, \cdots, y_m$ be the standard coordinate functions on $\mathbb{R}^m$. For $i \in \{1, \cdots, m\}$, we define the i**th component of** F, denoted $F^i : U \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

Thus $F = (F_1, \cdots, F_m)$

Definition 1.3.1.9. Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^m$. Then F is said to be **smooth** if for each $i \in \{1, \cdots, m\}$, the ith component of F, $F^i : U \to \mathbb{R}$, is smooth.

Definition 1.3.1.10. Let $U \subset \mathbb{R}^n$ and $F : U \to \mathbb{R}^m$. Then F is said to be **smooth** if for each $x \in U$, there exists $U_x \in \mathcal{N}_x$ and $\tilde{F} : U_x \to \mathbb{R}^m$ such that U_x is open, $\tilde{F}$ is smooth and $\tilde{F}|_{U \cap U_x} = F|_{U \cap U_x}$.

Definition 1.3.1.11. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. Then F is said to be a **diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Exercise 1.3.1.12. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and $F : U \to V$. If F is a diffeomorphism, then F is a homeomorphism.

Proof. Suppose that F is a diffeomorphism. By definition, F is a bijection and F and F^{-1} are smooth. Thus, F and F^{-1} are continuous and F is a homeomorphism. □

Definition 1.3.1.13. Let $U \subset \mathbb{R}^n$ be open, $p \in U$ and $F : U \to \mathbb{R}^m$. We define the **Jacobian of F at p**, denoted $\frac{\partial F}{\partial x}(p) \in \mathbb{R}^{m \times n}$, by

$$\left(\frac{\partial F}{\partial x}(p)\right)_{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Exercise 1.3.1.14. Inverse Function Theorem:
Let $U, V \subset \mathbb{R}^n$ be open and $F : U \to V$.

Exercise 1.3.1.15. Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$. Then F is a diffeomorphism iff for each $p \in U$, there exists a relatively open neighborhood $N \subset U$ of p such that $F|_N : N \to F(N)$ is a diffeomorphism

Proof. content... □

Exercise 1.3.1.16. Let $\sigma \in S_n$. Define $\phi_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by $\phi(x^1, \dots, x^n) = \phi(x^{\sigma(1), \dots, x^{\sigma(n)}})$. Then $D\phi = P_\sigma$

Definition 1.3.1.17. Let $\sigma \in S_n$ and $x = (x^1, \dots, x^n) \in \mathbb{R}^n$. We define $\sigma x \in \mathbb{R}^n$ by

$$\sigma x = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of S_n on $\mathbb{R}^n$ to be the map $S_n \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(\sigma, x) \mapsto \sigma x$

Definition 1.3.1.18. Let $\sigma \in S_n$, U a set, $V \subset \mathbb{R}^n$ and $\phi : U \to \mathbb{R}^n$ with $\phi = (x^1, \dots, x^m)$. We define $\sigma\phi : U \to \mathbb{R}^n$ by

$$\sigma\phi = (x^{\sigma(1)}, \dots, x^{\sigma(n)})$$

We define the **permutation action** of S_n on $(\mathbb{R}^n)^U$ to be the map $S_n \times (\mathbb{R}^n)^U \to \mathbb{R}^n$ given by $(\sigma, \phi) \mapsto \sigma\phi$.

Exercise 1.3.1.19. Let $\sigma \in S_m$. Then for each $p \in \mathbb{R}^n$, $D(\sigma \,\mathrm{id}_{\mathbb{R}^n})(p) = P_\sigma$.

Proof. Note that since $\mathrm{id}_{\mathbb{R}^n} = (\pi_1, \dots, \pi_n)$, we have that $\sigma \,\mathrm{id}_{\mathbb{R}^n} = (\pi_{\sigma(1)}, \dots, \pi_{\sigma(n)})$. Let $p \in \mathbb{R}^n$. Then

$$\begin{aligned}
D(\sigma \,\mathrm{id}_{\mathbb{R}^n})(p) &= \left(\frac{\partial \pi_i \circ \sigma \,\mathrm{id}_{\mathbb{R}^n}}{\partial x^j}(p)\right)_{i,j} \\
&= \left(\frac{\partial \pi_{\sigma(i)}}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma \left(\frac{\partial \pi_i}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma \left(\frac{\partial \pi_i \circ \mathrm{id}_{\mathbb{R}^n}}{\partial x^j}(p)\right)_{i,j} \\
&= P_\sigma D \,\mathrm{id}_{\mathbb{R}^n}(p) \\
&= P_\sigma I \\
&= P_\sigma
\end{aligned}$$

□

1.3.2 Differentiation on Subspaces

Definition 1.3.2.1. Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$. Then f is said to be **smooth** if for each $a \in A$, there exists $B \subset \mathbb{R}^m$ and $g : B \to \mathbb{R}^n$ such that $a \in B$, B is open in $\mathbb{R}^m$, g is smooth and $g|_{A\cap B} = f|_{A\cap B}$.

Exercise 1.3.2.2. Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$. If f is smooth, then f is continuous.

Proof. Suppose that f is smooth. Let $a \in A$. Since f is smooth, there exists $B \subset \mathbb{R}^m$ such that $a \in B$, B is open in $\mathbb{R}^m$, g is smooth and $g|_{A\cap B} = f|_{A\cap B}$. Since g is smooth, g is continuous. Let $V \subset \mathbb{R}^n$. Suppose that V is open in $\mathbb{R}^n$ and $f(a) \in V$. Since $f(a) = g(a)$ and g is continuous, there exists $U_g \subset B$ such that U_g is open in B, $a \in U_g$ and $g(U_g) \subset V$. Since B is open in $\mathbb{R}^m$ and U_g is open in B, we have that U_g is open in $\mathbb{R}^m$. Set $U_f = U_g \cap A$. Then $a \in U_f$, U_f is open in A and

$$\begin{aligned} f(U_f) &= f(U_g \cap A) \\ &= g(U_g \cap A) \\ &\subset g(U_g) \\ &\subset V \end{aligned}$$

Since $V \subset \mathbb{R}^n$ such that V is open in $\mathbb{R}^n$ and $f(a) \in V$ is arbitrary, we have that for each $V \subset \mathbb{R}^n$, if V is open in $\mathbb{R}^n$ and $f(a) \in V$, then there exists $U_f \subset A$ such that U_f is open in A, $a \in U_f$ and $f(U_f) \subset V$. Thus f is continuous at a. Since $a \in A$ is arbitrary, f is continuous. ☐

Exercise 1.3.2.3. Let $A \subset \mathbb{R}^m$, $B \subset A$ and $f : A \to \mathbb{R}^n$. If f is smooth, then $f|_B$ is smooth.

Proof. Suppose that f is smooth. Let $b \in B$. Since $B \subset A$, $b \in A$. Since $b \in A$ and f is smooth, there exists $U \subset \mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ such that $b \in U$, U is open in $\mathbb{R}^m$, F is smooth and $F|_{U\cap A} = f|_{U\cap A}$. Define $g : B \to \mathbb{R}^n$ by $g := f|_B$. Since $B \subset A$,

$$\begin{aligned} F|_{U\cap B} &= f|_{U\cap B} \\ &= g|_{U\cap B} \end{aligned}$$

Since $b \in B$ is arbitrary, we have that for each $b \in B$, there exists $U \subset \mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ such that $b \in U$, U is open in $\mathbb{R}^m$, F is smooth and $F|_{U\cap B} = g|_{U\cap B}$. Thus g is smooth. ☐

Exercise 1.3.2.4. Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$. Then f is smooth iff for each $a \in A$, there exists $U \subset A$ such that $a \in U$, U is open in A and $f|_U$ is smooth.

Proof.

- $(\implies)$:
  Suppose that f is smooth. Let $a \in A$. Set $U := A$. Then $a \in U$, U is open in A and $f|_U = f$ which is smooth.

- $(\impliedby)$:
  Suppose that for each $a \in A$, there exists $U \subset A$ such that $a \in U$ and $f|_U$ is smooth. Let $a \in A$. By assumption, there exists $U \subset A$ such that $a \in U$, U is open in A and $f|_U$ is smooth. Define $h : U \to \mathbb{R}^n$ by $h := f|_U$. Since $a \in U$ and h is smooth, there exists $U_0 \subset \mathbb{R}^m$ and $g_0 : U_0 \to \mathbb{R}^n$ such that $a \in U_0$, U_0 is open in $\mathbb{R}^m$ and $g_0|_{U\cap U_0} = h|_{U\cap U_0}$. Since U is open in A, there exists $\tilde{U} \subset \mathbb{R}^m$ such that $\tilde{U}$ is open in $\mathbb{R}^m$ and $U = \tilde{U} \cap A$. Define $B \subset \mathbb{R}^m$ and $g : B \to \mathbb{R}^n$ by $B := U_0 \cap \tilde{U}$ and $g = g_0|_B$. Then $a \in B$ and B is open in $\mathbb{R}^m$. The previous exercise implies that g is smooth. Furthermore,

$$\begin{aligned} g|_{B\cap A} &= g|_{U_0\cap\tilde{U}\cap A} \\ &= g|_{U_0\cap U} \\ &= h|_{U_0\cap U} \\ &= f|_{U_0\cap U} \\ &= f|_{U_0\cap\tilde{U}\cap A} \\ &= f|_{B\cap A} \end{aligned}$$

Since $a \in A$ is arbitrary, we have that for each $a \in A$, there exists $B \subset \mathbb{R}^m$ and $g : B \to \mathbb{R}^n$ such that $a \in B$, B is open in $\mathbb{R}^m$, g is smooth and $g|_{A \cap B} = f|_{A \cap B}$. Hence f is smooth.

□

Exercise 1.3.2.5. Let $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$, $f : A \to B$ and $g : B \to \mathbb{R}^p$. If f and g are smooth, then $g \circ f$ is smooth.

Proof. Suppose that f and g are smooth. Let $a \in A$. Set $b = f(a)$. Then $b \in B$. Since f is smooth, there exists $U \subset \mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ such that $a \in U$, U is open in $\mathbb{R}^m$, F is smooth and $F|_{U \cap A} = f|_{U \cap A}$. Since g is smooth, there exists $V \subset \mathbb{R}^n$ and $G : V \to \mathbb{R}^p$ such that $b \in V$, V is open in $\mathbb{R}^n$, G is smooth and $G|_{V \cap B} = g|_{V \cap B}$. We define $W \subset \mathbb{R}^m$ and $H : W \to \mathbb{R}^p$ by $W := U \cap F^{-1}(V)$ and $H := G \circ F|_W$.

- By construction, $a \in W$.
- Since F is smooth, F is continuous. Thus $F^{-1}(V)$ is open in $\mathbb{R}^m$ which implies that W is open in $\mathbb{R}^m$.
- Since F is smooth, an exercise in the section on differentiation implies that $F|_W$ is smooth. Since $F|_W$ and G are smooth, a previous exercise in the section on differentiation implies that H is smooth.
- Let $x \in W \cap A$. Since $W \cap A \subset A \cap U$, $f(x) = F(x)$. Since $f(x) \in B$ and $W \subset F^{-1}(V)$, we have that $F(x) \in V \cap B$. Thus

$$\begin{aligned} g \circ f(x) &= g(F(x)) \\ &= G(F(x)) \\ &= H(x) \end{aligned}$$

  Since $x \in W \cap A$ is arbitrary, we have that $H|_{W \cap A} = (g \circ f)|_{W \cap A}$.

Thus $g \circ f$ is smooth. □

1.3.3 Integration

1.4 Topology

Definition 1.4.0.1. Let $(X, \mathbb{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be **continuous** if for each $U \in \mathcal{T}$, $f^{-1}(U) \in \mathcal{T}_X$.

Definition 1.4.0.2. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$. Then f is said to be a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Definition 1.4.0.3. Let X, Y be topological spaces. Then X and Y are said to be **homeomorphic** if there exists $f : X \to Y$ such that f is a homeomorphism. If X and Y are homeomorphic, we write $X \cong Y$.

Theorem 1.4.0.4. Let $m, n \in \mathbb{N}$. If $m \neq n$, then $\mathbb{R}^m \ncong \mathbb{R}^n$

Chapter 2

Multilinear Algebra

2.1 Tensor Products

Let V and W be vector spaces.

2.2 (r, s)-Tensors

Definition 2.2.0.1. Let $V_1, \dots, V_k, W$ be vector spaces and $\alpha : \prod_{i=1}^n V_i \to W$. Then α is said to be **multilinear** if for each $i \in \{1, \cdots, k\}$, $v \in V$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$,

$$\alpha(v_1, \cdots, v_i + cv, \cdots, v_k) = \alpha(v_1, \cdots, v_i, \cdots, v_k) + c\alpha(v_1, \cdots, v, \cdots, v_k)$$

We define

$$L(V_1, \dots, V_k; W) = \left\{\alpha : \prod_{i=1}^n V_i \to W : \alpha \text{ is multilinear}\right\}$$

Note 2.2.0.2. For the remainder of this section we let V denote an n-dimensional vector space with basis $\{e^1, \cdots, e^n\}$ with dual space V^* and dual basis $\{\epsilon_1, \cdots, \epsilon_n\}$ defined by $\epsilon^i(e^j) = \delta_{i,j}$. We identify V with V^{**} by the isomorphism $V \to V^{**}$ defined by $v \mapsto \hat{v}$ where $\hat{v}(\alpha) = \alpha(v)$ for each $\alpha \in V^*$.

Definition 2.2.0.3. Let $\alpha : (V^*)^r \times V^s \to \mathbb{R}$. Then α is said to be an (r, s)-tensor on V if $\alpha \in L(\underbrace{V^*, \dots, V^*}_{r}, \underbrace{V, \dots, V}_{s}; \mathbb{R})$. The set of all (r, s)-tensors on V is denoted $T^r_s(V)$.
When $r = s = 0$, we set $T^r_s = \mathbb{R}$.

Exercise 2.2.0.4. We have that $T^r_s(V)$ is a vector space.

Proof. Clear. $\square$

Exercise 2.2.0.5. Under the identification of V with V^{**} as noted above, we have that $V = T^1_0(V)$.

Proof. By definition,

$$\begin{aligned} V &= V^{**} \\ &= L(V^*; \mathbb{R}) \\ &= T^1_0(V) \end{aligned}$$

$\square$

Definition 2.2.0.6. Let $\alpha \in T^{r_1}_{s_1}(V)$ and $\beta \in T^{r_2}_{s_2}(V)$. We define the **tensor product of** α **with** β, denoted $\alpha \otimes \beta \in T^{r_1+r_2}_{s_1+s_2}(V)$, by

$$\alpha \otimes \beta(v^*, w^*, v, w) = \alpha(v^*, v)\beta(w^*, w)$$

for each $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $v \in V^{s_1}$ and $w \in V^{s_2}$.
When $r_1 = s_1 = r_2 = s_2 = 0$ (so that $\alpha, \beta \in \mathbb{R}$), we set $\alpha \otimes \beta = \alpha\beta$.

Definition 2.2.0.7. We define the **tensor product**, denoted $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ by

$$(\alpha, \beta) \mapsto \alpha \otimes \beta$$

Exercise 2.2.0.8. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is well defined.

Proof. Tedious but straightforward. □

Exercise 2.2.0.9. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is associative.

Proof. Let $\alpha \in T^{r_1}_{s_1}(V)$, $\beta \in T^{r_2}_{s_2}(V)$ and $\gamma \in T^{r_3}_{s_3}(V)$. Then for each $u^* \in (V^*)^{r_1}, v^* \in (V^*)^{r_2}, w^* \in (V^*)^{r_3}, u \in V^{s_1}, v \in V^{s_2}, w \in V^{s_3}$,

$$\begin{aligned}
(\alpha \otimes \beta) \otimes \gamma(u^*, v^*, w^*, u, v, w) &= (\alpha \otimes \beta)(u^*, v^*, u, v)\gamma(w^*, w) \\
&= [\alpha(u^*, u)\beta(v^*, v)]\gamma(w^*, w) \\
&= \alpha(u^*, u)[\beta(v^*, v)\gamma(w^*, w)] \\
&= \alpha(u^*, u)(\beta \otimes \gamma)(v^*, w^*, v, w) \\
&= \alpha \otimes (\beta \otimes \gamma)(u^*, v^*, w^*, u, v, w)
\end{aligned}$$

So that

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

□

Exercise 2.2.0.10. The tensor product $\otimes : T^{r_1}_{s_1}(V) \times T^{r_2}_{s_2}(V) \to T^{r_1+r_2}_{s_1+s_2}(V)$ is bilinear.

Proof.

1. Linearity in the first argument:
   Let $\alpha, \beta \in T^{r_1}_{s_1}(V)$, $\gamma \in T^{r_2}_{s_2}(V), \lambda \in \mathbb{R}$, $v^* \in (V^*)^{r_1}$, $w^* \in (V^*)^{r_2}$, $vinV^{s_1}$ and $w \in V^{s_2}$. To see that the tensor product is linear in the first argument, we note that

   $$\begin{aligned}
   [(\alpha + \lambda\beta) \otimes \gamma](v^*, w^*, v, w) &= (\alpha + \lambda\beta)(v^*, v)\gamma(w^*, w) \\
   &= [\alpha(v^*, v) + \lambda\beta(v^*, v)]\gamma(w^*, w) \\
   &= \alpha(v^*, v)\gamma(w^*, w) + \lambda\beta(v^*, v)\gamma(w^*, w) \\
   &= \alpha \otimes \gamma(v^*, w^*, v, w) + \lambda(\beta \otimes \gamma)(v^*, w^*, v, w) \\
   &= [\alpha \otimes \gamma + \lambda(\beta \otimes \gamma)](v^*, w^*, v, w)
   \end{aligned}$$

   So that

   $$(\alpha + \lambda\beta) \otimes \gamma = \alpha \otimes \gamma + \lambda(\beta \otimes \gamma)$$

2. Linearity in the second argument:
   Similar to (1).

□

Definition 2.2.0.11.

1. Define $\mathcal{I}_{\otimes k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1, \cdots, i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **unordered multi-index of length** k. Recall that $\#\mathcal{I}_{\otimes k} = n^k$.

2. Define $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \leq n\}$. Each element $I \in \mathcal{I}_k$ is called an **ordered multi-index of length** k. Recall that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$.

Note 2.2.0.12. For the remainder of this section we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\otimes k}$.

Definition 2.2.0.13. Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.

1. Define $\epsilon^I \in (V^*)^k$ and $e_I \in V^k$ by
$$\epsilon^I = (\epsilon^{i_1}, \cdots, \epsilon^{i_k})$$
and
$$e^I = (e^{i_1}, \cdots, e^{i_k})$$

2. Define $e^{\otimes I} \in T^k_0(V)$ and $\epsilon^{\otimes I} \in T^0_k(V)$ by
$$e^{\otimes I} = e^{i_1} \otimes \cdots \otimes e^{i_k}$$
and
$$\epsilon^{\otimes I} = \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$$

Exercise 2.2.0.14. Let $\alpha, \beta \in T^r_s(V)$. If for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_r, J \in \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = \beta(\epsilon^I, e^J)$. Let $v_1^*, \dots, v_r^* \in V^*$ and $v_1, \dots, v_s \in V$. For each $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$, write
$$v_i^* = \sum_{k_i=1}^n a_{i,k_i} \epsilon^{k_i}$$
and
$$v_j = \sum_{l_j=1}^n b_{j,l_j} e^{l_j}$$
Then
$$\begin{aligned}
\alpha(v_1^*, \dots, v_r^*, v_1, \dots, v_s) &= \sum_{k_1,\dots,k_r=1}^n \sum_{l_1,\dots,l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \alpha(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \sum_{k_1,\dots,k_r=1}^n \sum_{l_1,\dots,l_s=1}^n \prod_{i=1}^r \prod_{j=1}^s a_{i,k_i} b_{j,l_j} \beta(\epsilon^{k_1}, \dots, \epsilon^{k_r}, e^{l_1}, \cdots, e^{l_s}) \\
&= \beta(v_1^*, \dots, v_r^*, v_1, \dots, v_s)
\end{aligned}$$
So that $\alpha = \beta$. □

Exercise 2.2.0.15. Let $I, K \in \mathcal{I}_r$ and $J, L \in \mathcal{I}_s$. Then $e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) = \delta_{I,K}\delta_{J,L}$.

Proof. Write $I = (i_1, \dots, i_r), K = (k_1, \dots, k_r)$ and $J = (j_1, \dots, j_s), L = (l_1, \dots, l_s)$. Then
$$\begin{aligned}
e^{\otimes I} \otimes \epsilon^{\otimes J}(\epsilon^K, e^L) &= e^{\otimes I}(\epsilon^K)\epsilon^{\otimes J}(e^L) \\
&= e^{i_1} \otimes \cdots \otimes e^{i_r}(\epsilon^{k_1}, \dots, \epsilon^{k_r})\epsilon^{j_1} \otimes \cdots \otimes \epsilon^{j_s}(e^{l_1}, \dots, e^{l_s}) \\
&= \left[\prod_{m=1}^r e^{i_m}(\epsilon^{k_m})\right]\left[\prod_{n=1}^s \epsilon^{j_n}(e^{l_n})\right] \\
&= \left[\prod_{m=1}^r \delta_{i_m,k_m}\right]\left[\prod_{n=1}^s \delta_{j_n,l_n}\right] \\
&= \delta_{I,K}\delta_{J,L}
\end{aligned}$$
□

Exercise 2.2.0.16. The set $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is a basis for $T^r_s(V)$ and $\dim T^r_s(V) = n^{r+s}$.

Proof. Let $(a^I_J)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$. Let $\alpha = \sum\limits_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} a^I_J e^{\otimes I} \otimes \epsilon^{\otimes J}$. Suppose that $\alpha = 0$. Then for each $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\alpha(\epsilon^I, e^J) = a^I_J = 0$. Thus $\{e^{\otimes I} \otimes \epsilon^{\otimes J} : I \in \mathcal{I}_r, J \in \mathcal{I}_s\}$ is linearly independent. Let $\beta \in T^r_s(V)$. For $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, put $b^I_J = \beta(\epsilon^J, e^I)$. Define $\mu = \sum\limits_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} b^I_J e^{\otimes I} \otimes \epsilon^{\otimes J} \in T^r_s(V)$. Then for each $(I,J) \in \mathcal{I}_r \times \mathcal{I}_s$, $\mu(\epsilon^I, e^J) = b^I_J = \beta(\epsilon^I, e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{e^{\otimes I} \otimes \epsilon^{\otimes J}\}$. □

2.3 Covariant k-Tensors

2.3.1 Symmetric and Alternating Covariant k-Tensors

Definition 2.3.1.1. Let $\alpha : V^k \to \mathbb{R}$. Then α is said to be a **covariant k-tensor on V** if $\alpha \in T^0_k(V)$. We denote the set of covariant k-tensors by $T_k(V)$.

Definition 2.3.1.2. For $\sigma \in S_k$ and $\alpha \in T_k(V)$, define the $\sigma\alpha : V^k \to \mathbb{R}$ by

$$\sigma\alpha(v_1, \cdots, v_k) = \alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

We define the **permutation action** of of S_k on $T_k(V)$ to be the map $S_k \times T_k(V) \to T_k(V)$ given by $(\sigma, \alpha) \mapsto \sigma\alpha$

Exercise 2.3.1.3. The permutation action of S_k on $T_k(V)$ is a group action.

Proof.

1. Clearly for each $\sigma \in S_k$ and $\alpha \in T_k(V), \sigma\alpha \in T_k(V)$.
2. Clearly for each $\alpha \in T_k(V)$, $e\alpha = \alpha$.
3. Let $\tau, \sigma \in S_k$ and $\alpha \in T_k(V)$. Then for each $v_1, \cdots, v_k \in V$,

$$\begin{aligned}(\tau\sigma)\alpha(v_1, \cdots, v_k) &= \alpha(v_{\tau\sigma(1)}, \cdots, v_{\tau\sigma(k)}) \\ &= \tau\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= \tau(\sigma\alpha)(v_1, \cdots, v_k)\end{aligned}$$

□

Exercise 2.3.1.4. Let $\sigma \in S_k$. Then $L_\sigma : T_k(V) \to T_k(V)$ given by $L_\sigma(\alpha) = \sigma\alpha$ is a linear transformation.

Proof. Let $\alpha, \beta \in T_k(V)$, $c \in \mathbb{R}$ and $v_1, \cdots, v_k \in V$. Then

$$\begin{aligned}\sigma(c\alpha + \beta)(v_1, \cdots, v_k) &= (c\alpha + \beta)(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) + \beta(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \\ &= c\sigma\alpha(v_1, \cdots, v_k) + \sigma\beta(v_1, \cdots, v_k)\end{aligned}$$

So $\sigma(c\alpha + \beta) = c\sigma\alpha + \sigma\beta$. □

Definition 2.3.1.5. Let $\alpha \in T_k(V)$. Then α is said to be

- **symmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \alpha$
- **antisymmetric** if for each $\sigma \in S_k$, $\sigma\alpha = \operatorname{sgn}(\sigma)\alpha$
- **alternating** if for each $v_1, \ldots, v_k \in V$, if there exists $i, j \in \{1, \ldots, k\}$ such that $v_i = v_j$, then $\alpha(v_1, \cdots, v_k) = 0$.

We denote the set of symmetric k-tensors on V by $\Sigma^k(V)$. We denote the set of alternating k-tensors on V by $\Lambda^k(V)$.

Exercise 2.3.1.6. Let $\alpha \in T_k(V)$. Then α is antisymmetric iff α is alternating.

Proof. Suppose that α is antisymmetric. Let $v_1, \ldots, v_k \in V$. Suppose that there exists $i, j \in \{1, \ldots, k\}$ such that $v_i = v_j$. Define $\sigma \in S_k$ by $\sigma = (i, j)$. Then

$$\begin{aligned}\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) &= \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \\ &= \sigma(\alpha)(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &= \operatorname{sgn}(\sigma)\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\ &= -\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k)\end{aligned}$$

Therefore $2\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = 0$ which implies that $\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = 0$. Hence α is alternating.
Conversely, suppose that α is alternating. Let $i, j \in \{1, \ldots, k\}$ and $v_1, \ldots, v_k \in V$. Then

$$\begin{aligned} 0 &= \alpha(v_1, \cdots, v_i + v_j, \cdots, v_i + v_j, \cdots, v_k) \\ &= \alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) + \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) \end{aligned}$$

Since $i, j \in \{1, \ldots, k\}$ and $v_1, \ldots, v_k \in V$ are arbitrary, we have that for each $\tau \in S_k$, τ is a transposition implies that

$$\begin{aligned} \tau\alpha &= -\alpha \\ &= \operatorname{sgn}(\tau)\alpha \end{aligned}$$

Let $n \in \mathbb{N}$. Suppose that for each $\tau_1, \ldots, \tau_{n-1} \in S_k$ if for each $j \in \{1, \ldots, n-1\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_{n-1})\alpha = \sigma(\tau_1 \cdots \tau_{n-1})\alpha$. Let $\tau_1, \ldots, \tau_n \in S_k$. Suppose that for each $j \in \{1, \ldots, n\}$, τ_j is a transposition. Then

$$\begin{aligned} (\tau_1 \cdots \tau_n)\alpha &= (\tau_1 \cdots \tau_{n-1})(\tau_n\alpha) \\ &= (\tau_1 \cdots \tau_{n-1})(\operatorname{sgn}(\tau_n)\alpha) \\ &= (\operatorname{sgn}(\tau_n)(\tau_1 \cdots \tau_{n-1})\alpha \\ &= (\operatorname{sgn}(\tau_n) \operatorname{sgn}((\tau_1 \cdots \tau_{n-1})\alpha \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha \end{aligned}$$

By induction, for each $n \in \mathbb{N}$ and $\tau_1, \ldots, \tau_n \in S_k$, if for each $j \in \{1, \ldots, n\}$, τ_j is a transposition, then $(\tau_1 \cdots \tau_n)\alpha = \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha$. Now let $\sigma \in S_k$. Then there exist $n \in \mathbb{N}$ and $\tau_1, \ldots, \tau_n \in S_k$ such that $\sigma = \tau_1 \cdots \tau_n$ and for each $j \in \{1, \ldots n\}$, τ_j is a transposition. Hence

$$\begin{aligned} \sigma\alpha &= (\tau_1 \cdots \tau_n)\alpha \\ &= \operatorname{sgn}(\tau_1 \cdots \tau_n)\alpha \\ &= \operatorname{sgn}(\sigma)\alpha \end{aligned}$$

Therefore α is antisymmetric. □

Definition 2.3.1.7. Define the **symmetric operator** $S : T_k(V) \to \Sigma^k(V)$ by

$$\operatorname{Sym}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma\alpha$$

Define the **alternating operator** $A : T_k(V) \to \Lambda^k(V)$ by

$$\operatorname{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha$$

Exercise 2.3.1.8.

1. For $\alpha \in T_k(V)$, $\operatorname{Sym}(\alpha)$ is symmetric.
2. For $\alpha \in T_k(V)$, $\operatorname{Alt}(\alpha)$ is alternating.

Proof.

1. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned} \sigma \operatorname{Sym}(\alpha) &= \sigma\left[\frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha\right] \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sigma\tau\alpha \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \tau\alpha \\ &= \operatorname{Sym}(\alpha) \end{aligned}$$

2. Let $\alpha \in T_k(V)$ and $\sigma \in S_k$. Then

$$\begin{aligned}
\sigma \operatorname{Alt}(\alpha) &= \sigma\left[\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha\right] \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\sigma\tau\alpha \\
&= \frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\sigma\tau)\sigma\tau\alpha \\
&= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\tau \in S_k} \operatorname{sgn}(\tau)\tau\alpha \\
&= \operatorname{sgn}(\sigma)\operatorname{Alt}(\alpha)
\end{aligned}$$

$\square$

Exercise 2.3.1.9.

1. For $\alpha \in \Sigma^k(V)$, $\operatorname{Sym}(\alpha) = \alpha$.
2. For $\alpha \in \Lambda^k(V)$, $\operatorname{Alt}(\alpha) = \alpha$.

Proof.

1. Let $\alpha \in \Sigma^k(V)$. Then

$$\begin{aligned}
\operatorname{Sym}(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \sigma\alpha \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} \alpha \\
&= \alpha
\end{aligned}$$

2. Let $\alpha \in \Lambda^k(V)$. Then

$$\begin{aligned}
\operatorname{Alt}(\alpha) &= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\sigma\alpha \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)^2\alpha \\
&= \alpha
\end{aligned}$$

$\square$

Exercise 2.3.1.10. The symmetric operator $S : T_k(V) \to \Sigma^k(V)$ and the alternating operator $A : T_k(V) \to \Lambda^k(V)$ are linear.

Proof. Clear. $\square$

Exercise 2.3.1.11. Let $\alpha \in T_k(V)$ and $\beta \in T_l(V)$. Then

1. $\operatorname{Alt}(\operatorname{Alt}(\alpha) \otimes \beta) = \operatorname{Alt}(\alpha \otimes \beta)$
2. $\operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta)) = \operatorname{Alt}(\alpha \otimes \beta)$

Proof. First note that if we fix $\mu \in S_{k+1}$, then for each $\tau \in S_k$, choosing $\sigma = \mu\tau^{-1}$ yields $\sigma\tau = \mu$. For each $\mu \in S_{k+l}$, the map $\phi_\mu : S_k \to S_{k+l}$ given by $\phi_\mu(\tau) = \mu\tau^{-1}$ is injective. Thus for each $\mu \in S_{k+l}$, we have that $\#\{(\sigma, \tau) \in S_{k+l} \times S_k : \mu = \sigma\tau\} = k!$

1. Then

$$\begin{aligned}
\mathrm{Alt}(\mathrm{Alt}(\alpha) \otimes \beta) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \Big[\mathrm{Alt}(\alpha) \otimes \beta \Big] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\left(\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)\tau\alpha \right) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)(\tau\alpha) \otimes \beta \right] \\
&= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma \left[\frac{1}{k!} \sum_{\tau \in S_k} \mathrm{sgn}(\tau)\tau(\alpha \otimes \beta) \right] \\
&= \frac{1}{k!(k+l)!} \sum_{\sigma \in S_{k+l}} \sum_{\tau \in S_k} \mathrm{sgn}(\sigma\tau)\sigma\tau(\alpha \otimes \beta) \\
&= \frac{k!}{k!(k+l)!} \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \frac{1}{(k+l)!} \sum_{\mu \in S_{k+l}} \mathrm{sgn}(\mu)\mu(\alpha \otimes \beta) \\
&= \mathrm{Alt}(\alpha \otimes \beta)
\end{aligned}$$

2. Similar to (1).

$\square$

2.3.2 Exterior Product

Definition 2.3.2.1. Let $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$. The **exterior product** of α and β is defined to be the map $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ given by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha \otimes \beta)$$

Thus $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$.

Exercise 2.3.2.2. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is bilinear.

Proof. Clear. $\square$

Exercise 2.3.2.3. The exterior product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is associative.

Proof. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$ and $\gamma \in \Lambda^m(V)$. Then

$$\begin{aligned}
(\alpha \wedge \beta) \wedge \gamma &= \left[\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)\right] \wedge \gamma \\
&= \frac{(k+l+m)!}{(k+l)!m!} \operatorname{Alt}\left(\left[\frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta)\right] \otimes \gamma\right) \\
&= \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} \operatorname{Alt}(\operatorname{Alt}(\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{m!} \frac{1}{k!l!} \operatorname{Alt}((\alpha \otimes \beta) \otimes \gamma) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \operatorname{Alt}(\alpha \otimes (\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \frac{(l+m)!}{l!m!} \operatorname{Alt}(\alpha \otimes \operatorname{Alt}(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \operatorname{Alt}(\alpha \otimes \frac{(l+m)!}{l!m!} \operatorname{Alt}(\beta \otimes \gamma)) \\
&= \frac{(k+l+m)!}{k!(l+m)!} \operatorname{Alt}(\alpha \otimes (\beta \wedge \gamma)) \\
&= \alpha \wedge (\beta \wedge \gamma))
\end{aligned}$$

□

Exercise 2.3.2.4. Let $\alpha_i \in \Lambda^{k_i}(V)$ for $i = 1, \cdots, m$. Then

$$\bigwedge_{i=1}^m \alpha_i = \frac{(\sum_{i=1}^m k_i)!}{\prod_{i=1}^m k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^m \alpha_i\right)$$

Proof. To see that the statment is true in the case $m = 3$, the proof of the previous exercise tells us that indeed

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 = \frac{(k_1 + k_2 + k_3)!}{k_1!k_2!k_3!} \operatorname{Alt}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$

Now, suppose that the statement is true for each $3 \leq m \leq m_0$. Then the proof of the previous exercise tells us the

$$\begin{aligned}
\bigwedge_{i=1}^{m_0+1} \alpha_i &= \left(\bigwedge_{i=1}^{m_0-1} \alpha_i\right) \wedge \alpha_{m_0} \wedge \alpha_{m_0+1} \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \operatorname{Alt}\left(\left[\bigwedge_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0-1} k_i + k_{m_0} + k_{m_0+1})!}{(\sum_{i=1}^{m_0-1} k_i)!k_{m_0}!k_{m_0+1}!} \operatorname{Alt}\left(\left[\frac{(\sum_{i=1}^{m_0-1} k_i)!}{\prod_{i=1}^{m_0-1} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m_0-1} \alpha_i\right)\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\operatorname{Alt}\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\left[\bigotimes_{i=1}^{m_0-1} \alpha_i\right] \otimes \alpha_{m_0} \otimes \alpha_{m_0+1}\right) \\
&= \frac{(\sum_{i=1}^{m_0+1} k_i)!}{\prod_{i=1}^{m_0+1} k_i!} \operatorname{Alt}\left(\bigotimes_{i=1}^{m_0+1} \alpha_i\right)
\end{aligned}$$

□

Exercise 2.3.2.5. Define $\tau \in S_{k+l}$ by

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & l & l+1 & l+2 & \cdots & l+k \\ 1+k & 2+k & \cdots & l+k & 1 & 2 & \cdots & k \end{pmatrix}$$

Then the inversion number of τ is kl. (Hint: inversion number)

Proof.

$$\begin{aligned} N(\tau) &= \sum_{i=1}^{l} k \\ &= kl \end{aligned}$$

Since $\text{sgn}(\tau) = (-1)^{N(\tau)}$ we know that $\text{sgn}(\tau) = (-1)^{kl}$. □

Exercise 2.3.2.6. Let $\alpha \in \Lambda^k(V)$, $\beta \in \Lambda^l(V)$. Then

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

Proof. Define $\tau \in S_{k+l}$ as in the previous exercise. Note that For $\sigma \in S_{k+l}$ and $v_1, \cdots, v_{k+l} \in V$, we have that

$$\begin{aligned} \sigma\tau(\beta \otimes \alpha)(v_1, \cdots, v_l, v_{l+1}, \cdots v_{l+k}) &= \beta \otimes \alpha(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)}, v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma\tau(1)}, \cdots, v_{\sigma\tau(l)})\alpha(v_{\sigma\tau(l+1)}, \cdots v_{\sigma\tau(l+k)}) \\ &= \beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)})\alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)}) \\ &= \alpha(v_{\sigma(1)}, \cdots v_{\sigma(k)})\beta(v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\ &= \alpha \otimes \beta(v_{\sigma(1)}, \cdots v_{\sigma(k)}, v_{\sigma(1+k)}, \cdots, v_{\sigma(l+k)}) \\ &= \sigma(\alpha \otimes \beta)(v_1, \cdots, v_k, v_{1+k}, \cdots v_{l+k}) \end{aligned}$$

Thus $\sigma\tau(\beta \otimes \alpha) = \sigma(\alpha \otimes \beta)$. Then

$$\begin{aligned} \beta \wedge \alpha &= \frac{(k+l)!}{k!l!} \text{Alt}(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)\sigma(\beta \otimes \alpha) \\ &= \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma\tau)\sigma\tau(\beta \otimes \alpha) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma)\sigma(\alpha \otimes \beta) \\ &= \text{sgn}(\tau) \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta) \\ &= \text{sgn}(\tau)\alpha \wedge \beta \\ &= (-1)^{kl} \alpha \wedge \beta \end{aligned}$$

□

Exercise 2.3.2.7. Let $\alpha \in \Lambda^k(V)$. If k is odd, then $\alpha \wedge \alpha = 0$.

Proof. Suppose that k is odd. The previous exercise tells us that

$$\begin{aligned} \alpha \wedge \alpha &= (-1)^{k^2} \alpha \wedge \alpha \\ &= -\alpha \wedge \alpha \end{aligned}$$

Thus $\alpha \wedge \alpha = 0$. □

Exercise 2.3.2.8. Fundamental Example:
Let $\alpha_1, \cdots, \alpha_m \in \Lambda^1(V)$ and $v_1, \cdots, v_m \in V$. Then

$$\left(\bigwedge_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) = \det(\alpha_i(v_j))$$

Proof. The previous exercises tell us that

$$\begin{aligned}
\left(\bigwedge_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) &= m! \operatorname{Alt}\left(\bigotimes_{i=1}^m \alpha_i\right)(v_1, \cdots, v_m) \\
&= m!\left[\frac{1}{m!}\sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\sigma\left(\bigotimes_{i=1}^m \alpha_i\right)\right](v_1, \cdots, v_m) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\left(\bigotimes_{i=1}^m \alpha_i\right)(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) \\
&= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\prod_{i=1}^m \alpha_i(v_{\sigma(i)}) \\
&= \det(\alpha_i(v_j))
\end{aligned}$$

□

Note 2.3.2.9. Recall that $\mathcal{I}_{\wedge k} = \{(i_1, i_2, \cdots, i_k) \in \mathbb{N}^k : i_1 < i_2 < \cdots < i_k \le n\}$ and that $\#\mathcal{I}_{\wedge k} = \binom{n}{k}$. For the remainder of this section, we will write $\mathcal{I}_k$ in place of $\mathcal{I}_{\wedge k}$.

Definition 2.3.2.10. Let $I = \{(i_1, i_2, \cdots, i_k) \in \mathcal{I}_k$.
Define $\epsilon^{\wedge I} \in \Lambda^k(V)$ by

$$\epsilon^{\wedge I} = \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k}$$

Exercise 2.3.2.11. Let $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_k) \in \mathcal{I}_k$. Then $\epsilon^{\wedge I}(e^J) = \delta_{I,J}$.

Proof. Put $A = \begin{pmatrix} \epsilon^{i_1}(e^{j_1}) & \cdots & \epsilon^{i_1}(e^{j_k}) \\ & \vdots & \\ \epsilon^{i_k}(e^{j_1}) & \cdots & \epsilon^{i_k}(e^{j_k}) \end{pmatrix}$. A previous exercise tells us that $\epsilon^{\wedge I}(e^J) = \det A$. If $I = J$, then $A = I_{k \times k}$ and therefore $\epsilon^I(e^J) = 1$. Suppose that $I \neq J$. Put $l_0 = \min\{l : 1 \le l \le k, i_l \neq j_l\}$. If $i_{l_0} < j_{l_0}$, then all entries on the l_0-th row of A are 0. If $i_{l_0} > j_{l_0}$, then all entries on the l_0-th column of A are 0. □

Exercise 2.3.2.12. Let $\alpha, \beta \in \Lambda^k(V)$. If for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$, then $\alpha = \beta$.

Proof. Suppose that for each $I \in \mathcal{I}_k$, $\alpha(e^I) = \beta(e^I)$. Let $v_1, \cdots, v_k \in V$. For $i = 1, \cdots, k$, write $v_i =$

$\sum_{j_i=1}^n a_{i,j_i} e^{j_i}$. Then

$$\begin{aligned}
\alpha(v_1, \cdots, v_k) &= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right)\alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{j_1\neq\cdots\neq j_k}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right)\alpha(e^{j_1}, \cdots, e^{j_k}) \\
&= \sum_{J\in\mathcal{I}_k}\left[\sum_{\sigma\in S_J} \operatorname{sgn}(\sigma)\left(\prod_{i=1}^{k} a_{i,\sigma(j_i)}\right)\right]\alpha(e^J) \\
&= \sum_{J\in\mathcal{I}_k}\left[\sum_{\sigma\in S_J} \operatorname{sgn}(\sigma)\left(\prod_{i=1}^{k} a_{i,\sigma(j_i)}\right)\right]\beta(e^J) \\
&= \sum_{j_1,\cdots,j_k=1}^{n} \left(\prod_{i=1}^{k} a_{i,j_i}\right)\beta(e^{j_1}, \cdots, e^{j_k}) \\
&= \beta(v_1, \cdots, v_k)
\end{aligned}$$

□

Exercise 2.3.2.13. The set $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(V)$ and $\dim \Lambda^k(V) = \binom{n}{k}$.

Proof. Let $(a_I)_{I\in\mathcal{I}_k} \subset \mathbb{R}$. Let $\alpha = \sum\limits_{I\in\mathcal{I}_k} a_I \epsilon^{\wedge I}$. Suppose that $\alpha = 0$. Then for each $J \in \mathcal{I}_k$, $\alpha(e^J) = a_J = 0$. Thus $\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$ is linearly independent. Let $\beta \in \Lambda^k(V)$. For $I \in \mathcal{I}_k$, put $b_I = \beta(e^I)$. Define $\mu = \sum\limits_{I\in\mathcal{I}_k} b_I \epsilon^{\wedge I} \in \Lambda^k(V)$. Then for each $J \in \mathcal{I}_k$, $\mu(e^J) = b_J = \beta(e^J)$. Hence $\mu = \beta$ and therefore $\beta \in \operatorname{span}\{\epsilon^{\wedge I} : I \in \mathcal{I}_k\}$. □

2.3.3 Interior Product

Definition 2.3.3.1. Let V be a finite dimensional vector space and $v \in V$. We define **interior multiplication by** v, denoted $\iota_v : T_k \to T_{k-1}$, by

$$\iota_v\alpha(w_1, \ldots, w_{k-1}) = \alpha(v, w_1, \ldots, w_{k-1})$$

Exercise 2.3.3.2. Let V be a finite dimensional vector space and $v \in V$. Then $\iota_v|_{\Lambda^k(V)} : \Lambda^k(V) \to \Lambda^{k-1}(V)$.

Proof. Let $\alpha \in \Lambda^k(V)$. Define $\beta \in \Lambda^k(V)$ by $\beta(w_1, \ldots, w_k) = \alpha(w_k, w_1, \ldots, w_{k-1})$. Let $\sigma \in S_{k-1}$. Define $\tau \in S_k$ by $\tau(j) = \begin{cases} 1 & j = k \\ \sigma(j) & j \neq k \end{cases}$. Let $w_1, \ldots, w_{k-1} \in V$. Set $w_k = v$. Then

$$\begin{aligned}
\sigma(\iota_v\alpha)(w_1, \ldots, w_{k-1}) &= \iota_v\alpha(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}) \\
&= \alpha(v, w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}) \\
&= \beta(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}, v) \\
&= \beta(w_{\sigma(1)}, \ldots, w_{\sigma(k-1)}, w_k) \\
&= \beta(w_{\tau(1)}, \ldots, w_{\tau(k-1)}, w_{\tau(k)}) \\
&= \operatorname{sgn}(\tau)\beta(w_1, \ldots, w_{k-1}, w_k) \\
&= \operatorname{sgn}(\sigma)\beta(w_1, \ldots, w_{k-1}, v) \\
&= \operatorname{sgn}(\sigma)\alpha(v, w_1, \ldots, w_{k-1}) \\
&= \operatorname{sgn}(\sigma)(\iota_v\alpha)(w_1, \ldots, w_{k-1})
\end{aligned}$$

Since $w_1, \ldots, w_{k-1} \in V$ are arbitrary, $\sigma(\iota_v\alpha) = \operatorname{sgn}(\sigma)\iota_v\alpha$. Hence $\iota_v\alpha \in \Lambda^{k-1}(V)$. □

2.4 $(0, 2)$-Tensors

Definition 2.4.0.1. Let V be a finite dimensional vector space, $v \in V$ and $\alpha \in T^0_2(V)$. Then α is said to be **degenerate** if there exists $v \in V$ such that for each $w \in V$, $\alpha(v, w) = 0$ and $v \neq 0$.

Definition 2.4.0.2. Let V be a finite dimensional vector space, $\alpha \in T^0_2(V)$. We define $\phi_\alpha : V \to V^*$ by

$$\phi_\alpha(v) = \iota_v \alpha$$

Exercise 2.4.0.3. Let V be a finite dimensional vector space, $\alpha \in T^0_2(V)$. Then $\phi_\alpha \in L(V; V^*)$.

Proof. Let $v_1, v_2 \in V$ and $\lambda \in \mathbb{R}$. Then for each $w \in V$,

$$\begin{aligned}
\phi_\alpha(v_1 + \lambda v_2)(w) &= (\iota_{v_1+\lambda v_2}\alpha)(w) \\
&= \alpha(v_1 + \lambda v_2, w) \\
&= \alpha(v_1, w) + \lambda\alpha(v_2, w) \\
&= (\iota_{v_1}\alpha)(w) + \lambda(\iota_{v_2}\alpha)(w) \\
&= \phi_\alpha(v_1)(w) + \lambda\phi_\alpha(v_2)(w) \\
&= [\phi_\alpha(v_1) + \lambda\phi_\alpha(v_2)](w)
\end{aligned}$$

Therefore, $\phi_\alpha(v_1 + \lambda v_2) = \phi_\alpha(v_1) + \lambda\phi_\alpha(v_2)$. Thus $\phi_\alpha \in L(V; V^*)$. □

Exercise 2.4.0.4. Let V be a finite dimensional vector space and $\alpha \in T^0_2(V)$. Then α is nondegenerate iff ϕ_α is an isomorphism.

Proof.

- $(\implies:)$
  Suppose that α is nondegenerate. Let $v \in \ker \phi_\alpha$. Then for each $w \in V$,

  $$\begin{aligned}
  \alpha(v, w) &= (\iota_v \alpha)(w) \\
  &= \phi_\alpha(v)(w) \\
  &= 0
  \end{aligned}$$

  Since α is nondegenerate, $v = 0$. Since $v \in \ker \phi_\alpha$ is arbitrary, $\ker \phi_\alpha = \{0\}$. Hence ϕ_α is injective. Since $\dim V = \dim V^*$, ϕ_α is surjective. Hence ϕ_α is an isomorphism.

- $(\impliedby:)$
  Suppose that ϕ_α is an isomorphism. Let $v \in V$. Suppose that for each $w \in V$, $\alpha(v, w) = 0$. Then for each $w \in V$,

  $$\begin{aligned}
  \phi_\alpha(v)(w) &= (\iota_v \alpha)(w) \\
  &= \alpha(v, w) \\
  &= 0
  \end{aligned}$$

  Thus $\phi_\alpha(v) = 0$ which implies that $v \in \ker \phi_\alpha$. Since ϕ_α is an isomorphism, $v = 0$. Hence α is nondegenerate.

□

Exercise 2.4.0.5. Let V be a finite dimensional vector space and $\alpha \in T^0_2(V)$. Then

1. $[\phi_\alpha]_{i,j} = \alpha(e_j, e_i)$
2. for each $v, w \in V$,
   $$\alpha(v, w) = [w]^*[\phi_\alpha][v]$$

Proof. 1. Set $A = [\phi_\alpha]$. Let $i, j \in \{1, \dots, n\}$. By definition,

$$\phi_\alpha(e_j) = \sum_{k=1}^{n} A_{k,j} \epsilon^k$$

Then

$$\begin{aligned} \phi_\alpha(e_j)(e_i) &= \sum_{k=1}^{n} A_{k,j} \epsilon^k(e_i) \\ &= \sum_{k=1}^{n} A_{k,j} \delta_{k,i} \\ &= A_{i,j} \end{aligned}$$

2. Let $v, w \in V$. Then there exist $(v^i)_{i=1}^n, (w^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{i=1}^{n} v^i e_i$ and $w = \sum_{j=1}^{n} v^j e_i$. Part (1) implies that

$$\begin{aligned} \alpha(v, w) &= \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j \alpha(e_i, e_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j [\phi_\alpha]_{j,i} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} [v]_i [w]_j [\phi_\alpha]_{j,i} \\ &= [w]^* [\phi_\alpha][v] \end{aligned}$$

□

2.4.1 Scalar Product Spaces

Definition 2.4.1.1. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be

- **positive semidefinite** if for each $v \in V$, $\alpha(v, v) \geq 0$
- **positive definite** if for each $v \in V$, $v \neq 0$ implies that $\alpha(v, v) > 0$
- **negative semidefinite** if $-\alpha$ is positive semidefinite
- **negative definite** if $-\alpha$ is positive definite

Exercise 2.4.1.2. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then

1. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda > 0$
2. α is positive definite iff for each $\lambda \in \sigma([\phi_\alpha])$, $\lambda \geq 0$

Proof.

1. Suppose that α is positive definite. Write $\sigma(\phi_\alpha) = \{\lambda_1, \dots, \lambda_n\}$. Define $\Lambda \in \mathbb{R}^{n \times n}$ by $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$. Since α is symmetric, $[\phi_\alpha]$ is symmetric. There exists $U \in O(n)$ such that $[\phi_\alpha] = U \Lambda U^*$. FINISH!!!

□

Definition 2.4.1.3. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Then α is said to be a **scalar product** if α is nondegenerate. In this case, (V, α) is said to be a **scalar product space**.

Definition 2.4.1.4. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$ a scalar product on V. We define the **index** of α, denoted $\operatorname{ind}\alpha$ by

$$\operatorname{ind}\alpha = \max\{\dim W : W \text{ is a subspace of } V \text{ and } \alpha|_{W\times W} \text{ is negative definite}\}$$

Definition 2.4.1.5. Let (V, α) be a scalar product space.

- Let $v_1, v_2 \in V$. Then v_1 and v_2 are said to be **orthogonal** if $\alpha(v_1, v_2) = 0$.
- Let $U \subset V$ be a subspace. We define the **orthogonal subspace of** U, denoted by $U^\perp$, by

$$U^\perp = \{v \in V : \text{ for each } u \in U,\ \alpha(u, v) = 0\}$$

Exercise 2.4.1.6. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then $U^\perp$ is a subspace of V.

Proof. We note that since $U^\perp = \bigcap_{u\in U} \ker \phi_\alpha(u)$, $U^\perp$ is a subspace of V. □

Exercise 2.4.1.7. Let (V, α) be an n-dimensional scalar product space, $U \subset V$ a k-dimensional subspace and $(e_j)_{j=1}^n \subset V$ a basis for V. Suppose that $(e_j)_{j=1}^k$ is a basis for U. Then for each $v \in V$, $v \in U^\perp$ iff for each $j \in [k]$, $\alpha(v, e_j) = 0$.

Proof. Let $v \in V$.

- ($\Longrightarrow$): Suppose that $v \in U^\perp$. Since $(e_j)_{j=1}^k \subset U$, we have that for each $j \in [k]$, $\alpha(v, e_j) = 0$.
- ($\Longleftarrow$): Suppose that for each $j \in [k]$, $\alpha(v, e_j) = 0$. Let $u \in U$. Then there exist $(a^j)_{j=1}^k \subset \mathbb{R}$ such that $u = \sum\limits_{j=1}^k a^j u_j$. This implies that

$$\begin{aligned}\alpha(v, u) &= \sum_{j=1}^k a^j \alpha(v, u_j) \\ &= 0\end{aligned}$$

  Since $u \in U$ is arbitrary, we have that $v \in U^\perp$.

□

Exercise 2.4.1.8. Let (V, α) be a scalar product space and $U \subset V$ a subspace. Then

1. $\dim V = \dim U + \dim U^\perp$
2. $(U^\perp)^\perp = U$

Proof.
1. Set $n = \dim V$ and $k = \dim U$. Choose a basis $(e_j)_{j=1}^n$ such that $(e_j)_{j=1}^k$ is a basis for U.
2.

□

Exercise 2.4.1.9. Let V be a finite dimensional vector space and $\alpha \in \Sigma^2(V)$. Set $\sigma([\phi_\alpha])^- = \{\lambda \in \sigma([\phi_\alpha]) : \lambda < 0\}$. Then

$$\operatorname{ind}\alpha = \sum_{\lambda\in\sigma([\phi_\alpha])^-} \mu(\lambda)$$

Proof. Since α is symmetric, there exist $U \in O(n)$ and $\Lambda \in D(n,\mathbb{R})$ such that $[\phi_\alpha] = U\Lambda U^*$. Define $(u_j)_{j=1}^n \subset V$ by $u_j = \sum\limits_{i=1}^n U_{i,j} e_j$. Define $J^- = \{j \in [n] : \Lambda_{j,j} < 0\}$, $n^- = \#J^-$ and $V^- = \operatorname{span}\{u_j : j \in J^-\}$. Let $v \in V^-$. Then there exist $(a^j)_{j \in J^-}$ such that $v = \sum\limits_{j \in J^-} a^j u_j$. We note that

$$\begin{aligned} U^*[\phi_\alpha]U &= U^*(U\Lambda U^*)U \\ &= (U^*U)\Lambda(U^*U) \\ &= I\Lambda I \\ &= \Lambda \end{aligned}$$

A previous exercise implies that

$$\begin{aligned} \alpha(v,v) &= \sum_{j\in J^-}\sum_{k\in J^-} a^j a^k \alpha(u_j,u_k) \\ &= \sum_{j\in J^-}\sum_{k\in J^-} a^j a^k [u_j]^*[\phi_\alpha][u_k] \\ &= \sum_{j\in J^-}\sum_{k\in J^-} a^j a^k ([e_j]^*U^*)[\phi_\alpha](U[e_k]) \\ &= \sum_{j\in J^-}\sum_{k\in J^-} a^j a^k (U^*[\phi_\alpha]U)_{j,k} \\ &= \sum_{j\in J^-}\sum_{k\in J^-} a^j a^k (\Lambda)_{j,k} \\ &= \sum_{j\in J^-} |a^j|^2 \Lambda_{j,j} \\ &< 0 \end{aligned}$$

Since $v \in V^-$ is arbitrary, $\alpha|_{V^-\times V^-}$ is negative definite. Thus

$$\begin{aligned} \operatorname{ind}\alpha &\geq \dim V^- \\ &= n^- \end{aligned}$$

Set $J^+ = (J^-)^c$. Let $W \subset V$ be a subspace. Suppose that $\alpha|_{W\times W}$ is negative definite. For the sake of contradiction, suppose that there exists $j_0 \in J^+$ such that $u_{j_0} \in W$. Then

$$\begin{aligned} \alpha(u_{j_0},u_{j_0}) &= [u_{j_0}]^*[\phi_\alpha][u_{j_0}] \\ &= [u_{j_0}]^*U\Lambda U^*[u_{j_0}] \\ &= \Lambda_{j_0,j_0} \\ &\geq 0 \end{aligned}$$

which is a contradiction since $\alpha|_{W\times W}$ is negative definite. Thus for each $j \in J^+$, $u_j \notin W$. □

2.4.2 Symplectic Vector Spaces

Definition 2.4.2.1. Let V be a finite dimensional vector space and $\omega \in \Lambda^2(V)$. Then ω is said to be a **symplectic form** if ω is nondegenerate. In this case (V,ω) is said to be a **symplectic space**.

Exercise 2.4.2.2. Let V be a $2n$-dimensional vector space with basis $(a_j,b_j)_{j=1}^n$ and corresponding dual basis $(\alpha^j,\beta^j)_{j=1}^n$. Define $\omega \in \Lambda^2(V)$ by

$$\omega = \sum_{j=1}^n \alpha^j \wedge \beta^j$$

Then

1. for each $j, k \in \{1, \dots, n\}$,

   (a) $\omega(a_j, a_k) = 0$

   (b) $\omega(b_j, b_k) = 0$

   (c) $\omega(a_j, b_k) = \delta_{j,k}$

2. (V, ω) is a symplectic space

Proof.

1. Let $j, k \in \{1, \dots, n\}$.

   (a)

$$\begin{aligned}
\omega(a_j, a_k) &= \sum_{l=1}^{n} \alpha^l \wedge \beta^l(a_j, a_k) \\
&= \sum_{l=1}^{n} [\alpha^l(a_j)\beta^l(a_k) - \alpha^l(a_k)\beta^l(a_j)] \\
&= 0
\end{aligned}$$

   (b) Similar to (a)

   (c)

$$\begin{aligned}
\omega(a_j, b_k) &= \sum_{l=1}^{n} \alpha^l \wedge \beta^l(a_j, b_k) \\
&= \sum_{l=1}^{n} [\alpha^l(a_j)\beta^l(b_k) - \alpha^l(b_k)\beta^l(a_j)] \\
&= \sum_{l=1}^{n} \alpha^l(a_j)\beta^l(b_k) \\
&= \sum_{l=1}^{n} \delta_{j,l}\delta_{l,k} \\
&= \delta_{j,k}
\end{aligned}$$

2. Let $v \in V$. Then there exist $(q^j, p^j)_{j=1}^n \subset \mathbb{R}$ such that $v = \sum_{j=1}^{n} q^j a_j + p^j b_j$. Suppose that for each $w \in V$, $\omega(v, w) = 0$. Let $k \in \{1, \dots, n\}$. Then

$$\begin{aligned}
0 &= \omega(v, a_k) \\
&= \sum_{j=1}^{n} q^j \omega(a_j, a_k) + p^j \omega(b_j, a_k) \\
&= \sum_{j=1}^{n} p^j \delta_{j,k} \\
&= p^k
\end{aligned}$$

Similarly,

$$\begin{aligned}
0 &= \omega(v, b_k) \\
&= \sum_{j=1}^{n} q^j \omega(a_j, b_k) + p^j \omega(b_j, b_k) \\
&= \sum_{j=1}^{n} q^j \delta_{j,k} \\
&= q^k
\end{aligned}$$

Since $k \in \{1, \dots, n\}$ is arbitrary, $v = 0$. Hence ω is nondegenerate. Therefore (V, ω) is symplectic.

□

Exercise 2.4.2.3. Let (V, ω) be a symplectic space. Then $\dim V$ is even.

Proof. Set $n = \dim V$. Let $(e_j)_{j=1}^n$ be a basis for V. Define $[\omega] \in \mathbb{R}^{n \times n}$ by $[\omega]_{i,j} = \omega(e_i, e_j)$. Since $\omega \in \Lambda^2(V)$, $[\omega]^* = -[\omega]$. Therefore

$$\begin{aligned}
\det[\omega] &= \det[\omega]^* \\
&= \det(-[\omega]) \\
&= (-1)^n \det[\omega]
\end{aligned}$$

For the sake of contradiction, suppose that n is odd. Then $\det[\omega] = -\det[\omega]$ which implies that $\det[\omega] = 0$. Since ω is nondegenerate, $[\omega] \in GL(n, \mathbb{R})$. This is a contradiction. Hence n is even. □

Definition 2.4.2.4. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. We define the **symplectic complement of** V, denoted $S^\perp$, by

$$S^\perp = \{v \in V : \text{ for each } w \in S,\ \omega(v, w) = 0\}$$

Exercise 2.4.2.5. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then $S^\perp$ is a subspace.

Proof. We note that

$$S^\perp = \bigcap_{v \in S} \ker \iota_v \omega$$

Hence $S^\perp$ is a subspace. □

Exercise 2.4.2.6. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then

$$\dim V = \dim S + \dim S^\perp$$

Proof. □

Exercise 2.4.2.7. Let (V, ω) be a symplectic space and $S \subset V$ a subspace. Then $(S^\perp)^\perp = S$.

Proof. Let $v \in (S^\perp)^\perp$. Then for each $w \in S^\perp$, $\omega(v, w) = 0$. □

Chapter 3

Topological Manifolds

3.1 Introduction

Exercise 3.1.0.1. We have that $\mathbb{R}$ is homeomorphic to $(0, \infty)$

Proof. Define $f : \mathbb{R} \to (0, \infty)$ by $f(x) = e^x$. Then f is a homeomorphism. $\square$

Definition 3.1.0.2. Let $n \in \mathbb{N}$. We define the **upper half space** of $\mathbb{R}^n$, denoted $\mathbb{H}^n$, by

$$\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and we define

$$\partial\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n = 0\}$$
$$\operatorname{Int}\mathbb{H}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

We endow $\mathbb{H}^n$, $\partial\mathbb{H}^n$ and $\operatorname{Int}\mathbb{H}^n$ with the subspace topology inherited from $\mathbb{R}^n$.
We define the projection map $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ by

$$\pi(x_1, \ldots, x_{n-1}, 0) = (x_1, \ldots, x_{n-1})$$

Definition 3.1.0.3. We define $\mathbb{R}^0 = \{0\}$ and $\mathbb{H}^0 = \varnothing$ endowed with the discrete topology.

Exercise 3.1.0.4. Let $n \in \mathbb{N}$.

1. $\partial\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n-1}$
2. $\operatorname{Int}\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^n$

Proof.

1. Let $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ be the projection map given by

$$\pi(x_1, \ldots, x_{n-1}, 0) = (x_1, \ldots, x_{n-1})$$

   Then π is a homeomorphism.

2. Define $f : \mathbb{R}^n \to \operatorname{Int}\mathbb{H}^n$ by $f(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, e^{x_n})$. Then f is a homeomorphism.

$\square$

Exercise 3.1.0.5. Let $A \subset \mathbb{H}^n$. Suppose that A is open in $\mathbb{H}^n$. Then A is open in $\mathbb{R}^n$ iff $A \cap \partial\mathbb{H}^n = \varnothing$.
Hint: simply connected? FINISH!!!

Proof. Suppose that A is open in $\mathbb{R}^n$. For the sake of contradiction, suppose that $A \cap \partial\mathbb{H}^n \neq \varnothing$. Then there exists $x \in A$ such that $x \in \partial\mathbb{H}^n$. Since A is open in $\mathbb{R}^n$, there exists $B \subset A$ such that B is open in $\mathbb{R}^n$, $p \in B$ and B is simply connected. Set $B' := B \setminus \{x\}$. Then B' is not simply connected. get a simply connected ball around p FINISH!!! $\square$

Definition 3.1.0.6. Let $(M, \mathcal{T})$ be a topological space and $n \in \mathbb{N}_0$. Let $U \subset M$ and $V \subset \mathbb{H}^n$ and $\phi : U \to V$. Then (U, ϕ) is said to be a n**-coordinate chart on** $(M, \mathcal{T})$ if

- $U \in \mathcal{T}$
- $V \in \mathcal{T}_{\mathbb{H}^n}$
- ϕ is a $(\mathcal{T} \cap U, \mathcal{T}_{\mathbb{H}^n} \cap V)$-homeomorphism

We denote the set of all n-coordinate charts on M by $X^n(M, \mathcal{T})$.

Note 3.1.0.7. We will write $X^n(M)$ in place of $X^n(M, \mathcal{T})$ when the topology is not ambiguous.

Definition 3.1.0.8. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be **locally Euclidean of dimension** n if for each $p \in M$, there exists $(U, \phi) \in X^n(M)$ such that $p \in U$.

Definition 3.1.0.9. Let M be a topological space and $n \in \mathbb{N}$. Then M is said to be an n**-dimensional topological manifold** if

1. M is Hausdorff
2. M is second-countable
3. M is locally Euclidean of dimension n

Theorem 3.1.0.10. Topological Invariance of Dimension:
Let M be an n-dimensional toplogical manifold and N a p-dimensional toplogical manifold. If M and N are homeomorphic, then $n = p$.

Note 3.1.0.11. In light of the previous theorem, we write $X(M)$ in place of $X^n(M)$ and refer to n-coordinate charts as coordinate charts when the context is clear.

Definition 3.1.0.12. Let M be an n-dimensional topological manifold and $(U, \phi) \in X(M)$. Then (U, ϕ) is said to be an

- **interior chart** if $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$
- **boundary chart** if $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$

We denote the set of all interior charts on M and the set of all boundary charts on M by $X_{\text{Int}}(M)$ and $X_{\partial}(M)$ respectively.

Exercise 3.1.0.13. Let M be an n-dimensional topological manifold. Then

1. $X(M) = X_{\text{Int}}(M) \cup X_{\partial}(M)$
2. $X_{\text{Int}}(M) \cap X_{\partial}(M) = \varnothing$

Proof.

1. By definition, $X_{\text{Int}}(M) \cup X_{\partial}(M) \subset X(M)$. Let $(U, \phi) \in X(M)$. Since (U, ϕ) is a coordinate chart on M, $\phi(U)$ is open in $\mathbb{H}^n$. If $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$, then
$$\begin{aligned}(U, \phi) &\in X_{\text{Int}}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M)\end{aligned}$$
If $\phi(U) \cap \partial\mathbb{H}^n \neq \varnothing$, then
$$\begin{aligned}(U, \phi) &\in X_{\partial}(M) \\ &\subset X_{\text{Int}}(M) \cup X_{\partial}(M)\end{aligned}$$
Since $(U, \phi) \in X(M)$ is arbitrary, $X(M) \subset X_{\text{Int}}(M) \cup X_{\partial}(M)$. Therefore $X(M) = X_{\text{Int}}(M) \cup X_{\partial}(M)$.

2. For the sake of contradiction, suppose that $X_{\text{Int}}(M) \cup X_{\partial}(M) \neq \varnothing$. Then there exists $(U, \phi) \in X(M)$ such that $(U, \phi) \in X_{\text{Int}}(M)$ and $(U, \phi) \in X_{\partial}(M)$. Therefore $\phi(U) \cap \partial \mathbb{H}^n = \varnothing$ and $\phi(U) \cap \partial \mathbb{H}^n \neq \varnothing$. This is a contradiction. Hence $X_{\text{Int}}(M) \cup X_{\partial}(M) = \varnothing$.

□

Definition 3.1.0.14. Let M be an n-dimensional topological manifold. We define the

- **interior** of M, denoted $\operatorname{Int} M$, by

$$\operatorname{Int} M = \{p \in M : \text{there exists } (U, \phi) \in X_{\text{Int}}(M) \text{ such that } p \in U\}$$

- **boundary** of M, denoted ∂M, by

$$\partial M = \{p \in M : \text{there exists } (V, \psi) \in X_{\partial}(M) \text{ such that } p \in V \text{ and } \psi(p) \in \partial \mathbb{H}^n\}$$

Exercise 3.1.0.15. Let M be an n-dimensional topological manifold. Let $(U, \phi) \in X_{\text{Int}}(M)$. Then $U \subset \operatorname{Int} M$.

Proof. Let $p \in U$. Since $(U, \phi) \in X_{\text{Int}}(M)$ and $p \in U$, by definition, $p \in \operatorname{Int} M$. Since $p \in U$ is arbitrary, $U \subset \operatorname{Int} M$. □

Exercise 3.1.0.16. Let M be an n-dimensional topological manifold and $(U, \phi) \in X(M)$. Then $(U, \phi) \in X_{\text{Int}}(M)$ iff $\phi(U)$ is open in $\mathbb{R}^n$.

Proof. Suppose that $(U, \phi) \in X_{\text{Int}}(M)$. Then $\phi(U) \cap \partial \mathbb{H}^n = \varnothing$. Since $\phi(U)$ is open in $\mathbb{H}^n$, a previous exercise implies that $\phi(U)$ is open in $\mathbb{R}^n$. Conversely, suppose that $\phi(U)$ is open in $\mathbb{R}^n$. Since $\phi(U)$ is open in $\mathbb{H}^n$, a previous exercise implies that $\phi(U) \cap \partial \mathbb{H}^n = \varnothing$. Thus $(U, \phi) \in X_{\text{Int}}(M)$. □

Exercise 3.1.0.17. Let M be an n-dimensional topological manifold, $(U, \phi) \in X_{\partial}(M)$ and $p \in U$. If $\phi(p) \notin \partial \mathbb{H}^n$, then $p \in \operatorname{Int} M$.

Proof. Suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then U' is open in M and $\phi' : U' \to B'$ is a homeomorphism. Hence $(U', \phi') \in X_{\text{Int}}(M)$. Since $\phi(p) \in B'$, we have that $p \in U'$. By definition, $p \in \operatorname{Int} M$. □

Exercise 3.1.0.18. Let M be an n-dimensional topological manifold. Then

1. $M = \operatorname{Int} M \cup \partial M$
2. $\operatorname{Int} M \cap \partial M = \varnothing$
   Hint: simply connected

Proof.

1. By definition, $\operatorname{Int} M \cup \partial M \subset M$. Let $p \in M$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. A previous exercise implies that $(U, \phi) \in X_{\text{Int}}(M) \cup X_{\partial}(M)$. If $(U, \phi) \in X_{\text{Int}}(M)$, then by definition,

$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

   Suppose that $(U, \phi) \in X_{\partial}(M)$. If $\phi(p) \in \partial \mathbb{H}^n$, then by definition,

$$\begin{aligned} p &\in \partial M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

   Suppose that $\phi(p) \notin \partial \mathbb{H}^n$. The previous exercise implies that $p \in \operatorname{Int} M$. Therefore,

$$\begin{aligned} p &\in \operatorname{Int} M \\ &\subset \operatorname{Int} M \cup \partial M \end{aligned}$$

   Since $p \in M$ is arbitrary, $M \subset \operatorname{Int} M \cup \partial M$. Therefore $M = \operatorname{Int} M \cup \partial M$.

2. For the sake of contradiction, suppose that $\operatorname{Int} M \cap \partial M \neq \varnothing$. Then there exists $p \in M$ such that $p \in \operatorname{Int} M \cap \partial M$. By definition, there exists $(U, \phi) \in X_{\text{Int}}(M)$, $(V, \psi) \in X_{\partial}(M)$ such that $p \in U \cap V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in $\mathbb{H}^n$, $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi|_{U\cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.
   Since $\psi(U \cap V)$ is open in $\mathbb{H}^n$, there exists an $B_\psi \subset \psi(U \cap V)$ such that B_ψ is open in $\mathbb{H}^n$, B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$. Since $\phi(U \cap V)$ is open in $\mathbb{R}^n$, B_ϕ is open in $\mathbb{R}^n$. Since B_ψ is simply connected and $\phi|_{U\cap V} \circ (\psi|_{U \cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism, B_ϕ is simply connected.
   Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Then $\phi \circ \psi^{-1} : B'_\psi \to B'_\phi$ is a homeomorphism. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_ψ is simply connected. Since B_ϕ is open in $\mathbb{R}^n$, B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ. So $\partial M \cap \operatorname{Int} M = \varnothing$.

□

Exercise 3.1.0.19. Let M be an n-dimensional topological manifold. Then

1. $\operatorname{Int} M$ is open
2. ∂M is closed

Proof.

1. Let $p \in \operatorname{Int} M$. Then there exists $(U, \phi) \in X_{\text{Int}}(M)$ such that $p \in U$. By definition, U is open and a previous exercise implies that $U \subset \operatorname{Int} M$. Since $p \in \operatorname{Int} M$ is arbitrary, we have that for each $p \in \operatorname{Int} M$, there exists $U \subset \operatorname{Int} M$ such that U is open. Hence $\operatorname{Int} M$ is open.
2. Since $\partial M = (\operatorname{Int} M)^c$, and $\operatorname{Int} M$ is open, we have that ∂M is closed.

□

Exercise 3.1.0.20. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $p \in U$. If $p \in \partial M$, then $(U, \phi) \in X_\partial(M)$.
Hint: simply connected

Proof. Suppose that $p \in \partial M$. Then there exists a $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial \mathbb{H}^n$. Note that $\psi(U \cap V)$ is open in $\mathbb{H}^n$, $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and $\phi|_{U\cap V} \circ (\psi|_{U\cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism.
Since $\psi(U \cap V)$ is open in $\mathbb{H}^n$, there exists $B_\psi \subset \psi(U \cap V)$ such B_ψ is open in $\mathbb{H}^n$, B_ψ is simply connected and $\psi(p) \in B_\psi$. Set $B_\phi = \phi \circ \psi^{-1}(B_\psi)$.
For the sake of contradiction, suppose that $(U, \phi) \in X_{\text{Int}}(M)$. Then $\phi(U)$ is open in $\mathbb{R}^n$. Hence $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and B_ϕ is open in $\mathbb{R}^n$. Since $\phi|_{U\cap V} \circ (\psi|_{U\cap V})^{-1} : \psi^{-1}(U \cap V) \to \phi(U \cap V)$ is a homeomorphism, B_ϕ is simply connected. Set $B'_\phi = B_\phi \setminus \{\phi(p)\}$ and $B'_\psi = B_\psi \setminus \{\psi(p)\}$. Since $\psi(p) \in \partial \mathbb{H}^n$, B'_ψ is simply connected. Since B_ϕ is open in $\mathbb{R}^n$, B'_ϕ is not simply connected. This is a contradiction since B'_ϕ is homeomorphic to B'_ψ. So $(U, \phi) \notin X_{\text{Int}}(M)$. Since $(X_{\text{Int}}(M))^c = X_\partial(M)$, we have that $(U, \phi) \in X_\partial(M)$.

□

Exercise 3.1.0.21. Let M be an n-dimensional topological manifold, $(U, \phi) \in X_\partial(M)$ and $p \in U$. Then

1. $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$
2. $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n$

Proof.

1. Suppose that $p \in \partial M$. For the sake of contradiction, suppose that $\phi(p) \notin \partial \mathbb{H}^n$. Then $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence there exists $B' \subset \phi(U)$ such that B' is open in $\mathbb{R}^n$ and $\phi(p) \in B'$. Set $U' = \phi^{-1}(B')$ and $\phi' = \phi|_{U'}$. Then $p \in U'$ and $(U', \phi') \in X_{\text{Int}}(M)$. Since $p \in U'$, the previous exercise implies that $(U', \phi') \in X_\partial(M)$. This is a contradiction since $X_{\text{Int}}(M) \cap X_\partial(M) = \varnothing$. So $\phi(p) \in \partial \mathbb{H}^n$.
   Conversely, suppose that $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$.

2. A previous exercise implies that $\operatorname{Int} M = (\partial M)^c$. Part (1) implies that

$$\begin{aligned} p &\in (\partial M)^c \\ &= \operatorname{Int} M \end{aligned}$$

if and only if

$$\begin{aligned} \phi(p) &\in (\partial \mathbb{H}^n)^c \\ &= \operatorname{Int} \mathbb{H}^n \end{aligned}$$

□

Exercise 3.1.0.22. Let M be an n-dimensional topological manifold and $p \in M$. Then $p \in \partial M$ iff for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.

Proof. Suppose that $p \in \partial M$. Let $(U, \phi) \in X(M)$. Suppose that $p \in U$. The previous two exercises imply that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$.
Conversely, suppose that for each $(U, \phi) \in X(M)$, $p \in U$ implies that $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. Since M is a manifold, there exists $(U, \phi) \in X(M)$ such that $p \in U$. By assumption, $(U, \phi) \in X_\partial(M)$ and $\phi(p) \in \partial \mathbb{H}^n$. By definition, $p \in \partial M$. □

Exercise 3.1.0.23. Let M be an n-dimensional topological manifold. Let $(U, \phi) \in X_\partial(M)$. Then

1. $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$
2. $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int} \mathbb{H}^n$

Proof.

1. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \partial M$ iff $\phi(p) \in \partial \mathbb{H}^n$. Let $q \in \phi(U \cap \partial M)$. Then there exists $p \in U \cap \partial M$ such that $\phi(p) = q$. Since $p \in \partial M$, $\phi(p) \in \partial \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \partial \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \partial M)$ is arbitrary, $\phi(U \cap \partial M) \subset \phi(U) \cap \partial \mathbb{H}^n$.
Let $q \in \phi(U) \cap \partial \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \partial \mathbb{H}^n$, we have that $p \in \partial M$. Hence $p \in U \cap \partial M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \partial M)$. Thus $\phi(U \cap \partial M) = \phi(U) \cap \partial \mathbb{H}^n$.

2. Since $(U, \phi) \in X_\partial(M)$, a previous exercise implies that for each $p \in U$, $p \in \operatorname{Int} M$ iff $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Let $q \in \phi(U \cap \operatorname{Int} M)$. Then there exists $p \in U \cap \operatorname{Int} M$ such that $\phi(p) = q$. Since $p \in \operatorname{Int} M$, $\phi(p) \in \operatorname{Int} \mathbb{H}^n$. Hence

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U) \cap \operatorname{Int} \mathbb{H}^n \end{aligned}$$

Since $q \in \phi(U \cap \operatorname{Int} M)$ is arbitrary, $\phi(U \cap \operatorname{Int} M) \subset \phi(U) \cap \operatorname{Int} \mathbb{H}^n$.
Let $q \in \phi(U) \cap \operatorname{Int} \mathbb{H}^n$. Then there exists $p \in U$ such that $q = \phi(p)$. Since $\phi(p) \in \operatorname{Int} \mathbb{H}^n$, we have that $p \in \operatorname{Int} M$. Hence $p \in U \cap \operatorname{Int} M$ and

$$\begin{aligned} q &= \phi(p) \\ &\in \phi(U \cap \partial M) \end{aligned}$$

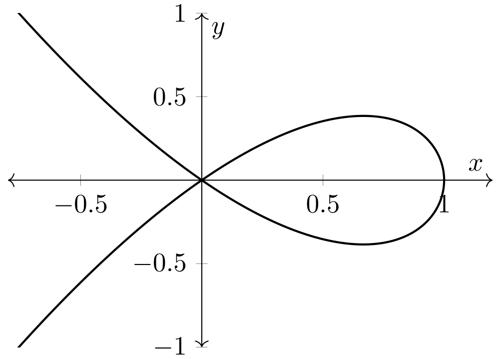
Since $q \in \phi(U) \cap \partial \mathbb{H}^n$ is arbitrary, $\phi(U) \cap \partial \mathbb{H}^n \subset \phi(U \cap \operatorname{Int} M)$. Thus $\phi(U \cap \operatorname{Int} M) = \phi(U) \cap \operatorname{Int} \mathbb{H}^n$.

□

Exercise 3.1.0.24. Graph of Continuous Function:
Let $f \in C(\mathbb{R})$. Set $M = \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$ (i.e. the graph of f). Then M is a 1-dimensional manifold.

Proof. Set $U = \mathbb{R}$ and define $\phi : U \to M$ by $\phi(x) = (x, f(x))$. Then $\phi^{-1} = \pi_1$. Since f is continuous, ϕ is continous. Since π_1 is continiuous, ϕ is a homeomorphism. □

Exercise 3.1.0.25. Nodal Cubic:
Let $M = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2 - x^3\}$. We equip M with the subspace topology.



Then M is not a 1-dimensional topological manifold.
Hint: connected components

Proof. Suppose that M is a 1-dimensional manifold. Set $p = (0, 0)$. Then there exists $(U, \phi) \in X(M)$ such that $p \in U$. Since $\phi(U)$ is open (in $\mathbb{R}$ or $\mathbb{H}$), there exists a $B \subset \phi(U)$ such that B is open (in $\mathbb{R}$ or $\mathbb{H}$), B is connected and $\phi(p) \in B$. Set $V = \phi^{-1}(B)$, $V' = V \setminus \{p\}$ and $B' = B \setminus \{\phi(p)\}$. Then $\phi : V \to B$ and $\phi' : V' \to B'$ are homeomorphisms. Since B is open (in $\mathbb{R}$ or $\mathbb{H}$) and connected, B' has at most two connected components. Then V' This is a contradiction since V' has four connected components and B' and V' are homeomorphic. □

Exercise 3.1.0.26. Topological Manifold Chart Lemma:
Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

- for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection
- for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous
- there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$
- for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : V \in \mathcal{T}_{\mathbb{H}^n} \text{ and } \alpha \in \Gamma\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$

Then

1. $\mathcal{B}$ is a basis for $\mathcal{T}_M$
   Hint: For $B_1, B_2 \subset \mathbb{H}^n$, $\phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) = \phi_{\alpha_1}^{-1}(B_1 \cap [\phi_{\alpha_1}|_{U_{\alpha_1} \cap U_{\alpha_2}} \circ (\phi_{\alpha_2}|_{U_{\alpha_1} \cap U_{\alpha_2}})^{-1}(B_2)])$

2. $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold

3. $\mathcal{T}_M$ is the unique topology $\mathcal{T}$ on M such that $(U_\alpha, \phi_\alpha)_{\alpha\in\Gamma} \subset X^n(M, \mathcal{T})$

Proof.

1.
- By assumption, $M \subset \bigcup_{\alpha\in\Gamma} U_\alpha$
- Let $A_1, A_2 \in \mathcal{B}$ and $p \in A_1 \cap A_2$. By definition, there exist $\alpha_1, \alpha_2 \in \Gamma$ and $B_1, B_2 \subset \mathbb{H}^n$ such that B_1, B_2 are open in $\mathbb{H}^n$ and

$$\begin{aligned} A_1 &= \phi_{\alpha_1}^{-1}(B_1) \\ &\subset U_{\alpha_1} \end{aligned} \qquad\qquad \begin{aligned} A_2 &= \phi_{\alpha_2}^{-1}(B_2) \\ &\subset U_{\alpha_2} \end{aligned}$$

Set $\psi_1 = \phi_{\alpha_1}|_{U_{\alpha_1}\cap U_{\alpha_2}}$ and $\psi_2 = \phi_{\alpha_2}|_{U_{\alpha_1}\cap U_{\alpha_2}}$. We note that

$$\begin{aligned} \psi_1^{-1}(B_1) &= U_{\alpha_2} \cap \phi_{\alpha_1}^{-1}(B_1) \\ &= U_{\alpha_2} \cap A_1 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned} \qquad\qquad \begin{aligned} \psi_2^{-1}(B_2) &= U_{\alpha_1} \cap \phi_{\alpha_2}^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \\ &\subset U_{\alpha_1} \cap U_{\alpha_2} \end{aligned}$$

Let $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Then $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. Hence $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_1}(q) \in \psi_1 \circ \psi_2^{-1}(B_2)$. This implies that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(B_1) \\ &= A_1 \end{aligned}$$

and since $\psi_2^{-1}(B_2) \subset U_{\alpha_1} \cap U_{\alpha_2}$ and $\phi_{\alpha_1} : U_{\alpha_1} \to \phi_{\alpha_1}(U_{\alpha_1})$ is a bijection, we have that

$$\begin{aligned} q &\in \phi_{\alpha_1}^{-1}(\psi_1 \circ \psi_2^{-1}(B_2)) \\ &= \psi_2^{-1}(B_2) \\ &= U_{\alpha_1} \cap A_2 \end{aligned}$$

Thus

$$\begin{aligned} q &\in A_1 \cap (U_{\alpha_1} \cap A_2) \\ &= A_1 \cap A_2 \end{aligned}$$

Since $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$ is arbitrary, we have that $\phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \subset A_1 \cap A_2$. Conversely, let

$$\begin{aligned} q &\in A_1 \cap A_2 \\ &= \phi_{\alpha_1}^{-1}(B_1) \cap \phi_{\alpha_2}^{-1}(B_2) \end{aligned}$$

Then $\phi_{\alpha_1}(q) \in B_1$ and $\phi_{\alpha_2}(q) \in B_2$. Since $A_1 \cap A_2 \subset U_{\alpha_1} \cap U_{\alpha_2}$, we have that

$$\begin{aligned} \psi_2(q) &= \phi_{\alpha_2}(q) \\ &\in B_2 \end{aligned}$$

which implies that $q \in \psi_2^{-1}(B_2)$. Therefore

$$\begin{aligned} \phi_{\alpha_1}(q) &= \psi_1(q) \\ &\in \psi_1(\psi_2^{-1}(B_2)) \\ &= \psi_1 \circ \psi_2^{-1}(B_2) \end{aligned}$$

Hence $\phi_{\alpha_1}(q) \in B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]$. This implies that $q \in \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Since $q \in A_1 \cap A_2$ is arbitrary, we have that $A_1 \cap A_2 \subset \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)])$. Thus

$$\begin{aligned} A_1 \cap A_2 &= \phi_{\alpha_1}^{-1}(B_1 \cap [\psi_1 \circ \psi_2^{-1}(B_2)]) \\ &\in \mathcal{B} \end{aligned}$$

Thus $\mathcal{B}$ is a basis for $\mathcal{T}_M$.

2. (a) **(locally Euclidean of dimension n):**
Let $\alpha \in \Gamma$. By definition, for each $B \subset \mathcal{T}_{\mathbb{H}^n}$,

$$\begin{aligned} \phi_\alpha^{-1}(B) &\in \mathcal{B} \\ &\subset \mathcal{T}_M \end{aligned}$$

Hence ϕ_α is continuous.
Let $A \in \mathcal{T}_{U_\alpha}$. Then there exists $U \subset \mathcal{T}_M$ such that $A = U \cap U_\alpha$. Since $\mathcal{B}$ is a basis for $\mathcal{T}_M$, there exists $\Gamma' \subset \Gamma$, $(V_\beta)_{\beta \in \Gamma'} \subset \mathcal{T}_{\mathbb{H}^n}$ such that $U = \bigcup_{\beta \in \Gamma'} \phi_\beta^{-1}(V_\beta)$. Thus

$$\begin{aligned} A &= U \cap U_\alpha \\ &= \left[\bigcup_{\beta \in \Gamma'} \phi_\beta^{-1}(V_\beta)\right] \cap U_\alpha \\ &= \bigcup_{\beta \in \Gamma'} [\phi_\beta^{-1}(V_\beta) \cap U_\alpha] \end{aligned}$$

Let $\beta \in \Gamma'$. Since $\phi_\alpha(U_\alpha \cap U_\beta) \subset \phi_\alpha(U_\alpha)$ and $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$, we have that

$$\begin{aligned} \phi_\alpha(U_\alpha \cap U_\beta) &= \phi_\alpha(U_\alpha) \cap \phi_\alpha(U_\alpha \cap U_\beta) \\ &\in \mathcal{T}_{\phi_\alpha(U_\alpha)} \end{aligned}$$

Therefore $\mathcal{T}_{\phi_\alpha(U_\alpha \cap U_\beta)} \subset \mathcal{T}_{\phi_\alpha(U_\alpha)}$. Since $(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous, we have that $(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{H}^n$ is continuous and therefore

$$\begin{aligned} [(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1}]^{-1}(V_\beta) &\in \mathcal{T}_{\phi_\alpha(U_\alpha \cap U_\beta)} \\ &\subset \mathcal{T}_{\phi_\alpha(U_\alpha)} \end{aligned}$$

Since $\beta \in \Gamma'$ is arbitrary, we have that

$$\begin{aligned} \phi_\alpha(A) &= \phi_\alpha\left(\bigcup_{\beta \in \Gamma'} [\phi_\beta^{-1}(V_\beta) \cap U_\alpha]\right) \\ &= \bigcup_{\beta \in \Gamma'} \phi_\alpha(\phi_\beta^{-1}(V_\beta) \cap U_\alpha) \\ &= \bigcup_{\beta \in \Gamma'} (\phi_\alpha|_{U_\alpha \cap U_\beta}) \circ (\phi_\beta|_{U_\alpha \cap U_\beta})^{-1}(V_\beta) \\ &= \bigcup_{\beta \in \Gamma'} [(\phi_\beta|_{U_\alpha \cap U_\beta}) \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1}]^{-1}(V_\beta) \\ &\in \mathcal{T}_{\phi_\alpha(U_\alpha)} \end{aligned}$$

Since $A \in \mathcal{T}_{U_\alpha}$ is arbitrary, $\phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to U_\alpha$ is continuous. Hence $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism and $(U_\alpha, \phi_\alpha) \in X^n(M)$. Since $M = \bigcup_{\alpha \in \Gamma} U_\alpha$, we have that M is locally Euclidean of dimension n.

(b) **(Hausdorff):**
Let $p, q \in M$. Suppose that $p \neq q$. Then there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$.

- Suppose that there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$. Since $p \neq q$, $\phi_\alpha(p) \neq \phi_\alpha(q)$. Since $\mathbb{H}^n$ is Hausdorff, there exist $V_p, V_q \subset \phi(U_\alpha)$ such that V_p and V_q are open in $\mathbb{H}^n$, $p \in V_p$, $q \in V_q$ and $V_p \cap V_q = \varnothing$. Set $U_p = \phi_\alpha^{-1}(V_p)$ and $U_q = \phi_\alpha^{-1} V_q$. Then U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$.
- Suppose that there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$. Set $U_p = U_\alpha$ and $U_q = U_\beta$. Then U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$.

Thus for each $p, q \in M$ there exist $U_p, U_q \subset M$ such that U_p, U_q are open, $p \in U_p$, $q \in U_q$ and $U_q \cap U_p = \varnothing$. Hence

(c) **(second-countable):**
By assumption, there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$. Let $\alpha \in \Gamma'$. Since $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$ and $\mathbb{H}^n$ is second-countable, we have that $\phi_\alpha(U_\alpha)$ is second-countable. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism, we have that U_α is second-countable. Since $M = \bigcup_{\alpha \in \Gamma'} U_\alpha$, an exercise in topology cite implies that M is second-countable.

3. Let $\mathcal{T}$ be a topology on M. Suppose that $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T})$. Then for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}$ and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism.
Let $U \in \mathcal{B}$. By definition, there exists $\alpha \in \Gamma$ and $V \in \mathcal{T}_{\mathbb{H}^n}$ such that $U = \phi_\alpha^{-1}(V)$. Since $U_\alpha \in \mathcal{T}$, we have that $\mathcal{T} \cap U_\alpha \subset \mathcal{T}$. Since $V \cap \phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$, and ϕ_α is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that

$$\begin{aligned} U &= \phi_\alpha^{-1}(V) \\ &= \phi_\alpha^{-1}(V \cap \phi_\alpha(U_\alpha)) \\ &\in \mathcal{T} \cap U_\alpha \\ &\subset \mathcal{T} \end{aligned}$$

Since $U \in \mathcal{B}$ is arbitrary, $\mathcal{B} \subset \mathcal{T}$. Therefore

$$\begin{aligned} \mathcal{T}_M &= \tau(\mathcal{B}) \\ &\subset \tau(\mathcal{T}) \\ &= \mathcal{T} \end{aligned}$$

Conversely, Let $U \in \mathcal{T}$ and $\alpha \in \Gamma$. Then $U \cap U_\alpha \in \mathcal{T} \cap U_\alpha$. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T} \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that $\phi_\alpha(U \cap U_\alpha) \in \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha)$. Since $U_\alpha \in \mathcal{T}_M$, $\mathcal{T}_M \cap U_\alpha \subset \mathcal{T}_M$. Since $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a $(\mathcal{T}_M \cap U_\alpha, \mathcal{T}_{\mathbb{H}^n} \cap \phi_\alpha(U_\alpha))$-homeomorphism, we have that

$$\begin{aligned} U \cap U_\alpha &= \phi_\alpha^{-1}(\phi_\alpha(U \cap U_\alpha)) \\ &\in \mathcal{T}_M \cap U_\alpha \\ &\subset \mathcal{T}_M \end{aligned}$$

Then

$$\begin{aligned} U &= U \cap M \\ &= U \cap \left(\bigcup_{\alpha \in \Gamma} U_\alpha \right) \\ &= \bigcup_{\alpha \in \Gamma} (U \cap U_\alpha) \\ &\in \mathcal{T}_M \end{aligned}$$

Since $U \in \mathcal{T}$ is arbitrary, $\mathcal{T} \subset \mathcal{T}_M$. Thus $\mathcal{T} = \mathcal{T}_M$.

□

Exercise 3.1.0.27. Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

- for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$
- for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection
- for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous
- there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup_{\alpha \in \Gamma'} U_\alpha$
- for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Then there exists a unique topology $\mathcal{T}_M$ on M such that $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset X^n(M, \mathcal{T}_M)$.

Proof. Immediate by previous exercise. □

3.2 Open and Boundary Submanifolds

Definition 3.2.0.1. Let M be an n-dimensional topological manifold and $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ the projection map. For $(U, \phi) \in X_\partial(M)$, we define $\bar{U} \subset \partial M$ and $\bar{\phi} : \bar{U} \to \pi(\phi(\bar{U}))$ by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi \circ \phi|_{\bar{U}}$ respectively.

Exercise 3.2.0.2. Let M be an n-dimensional topological manifold, and $\lambda : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ a homeomorphism. Then $\{(\bar{U}, \bar{\phi}) : (U, \phi) \in X_\partial(M)\} \subset X^{n-1}_{\text{Int}}(\partial M)$.

Proof. Let $(U, \phi) \in X_\partial(M)$.

1. Since U is open in M, $\bar{U} = U \cap \partial M$ is open in ∂M.

2. Since $(U, \phi) \in X_\partial(M)$, $\phi(U)$ is open in $\mathbb{H}^n$. A previous exercise implies that $\phi(\bar{U}) = \phi(U) \cap \partial\mathbb{H}^n$ which is open in $\partial\mathbb{H}^n$. Since $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ is a homeomorphism, we have that $\pi(\phi(\bar{U}))$ is open in $\mathbb{R}^{n-1}$.

3. Since $\phi|_{\bar{U}} : \bar{U} \to \phi(U) \cap \partial\mathbb{H}^n$ and $\pi|_{\phi(\bar{U})} : \phi(\bar{U}) \to \lambda(\phi(\bar{U}))$ are homeomorphisms, we have that $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$ is a homeomorphism.

Hence $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$. □

Exercise 3.2.0.3. Topological Boundary Submanifold:
Let M be an n-dimensional topological manifold. Then

1. ∂M is an $(n-1)$-dimensional topological manifold

2. $\partial(\partial M) = \varnothing$

Proof.

1. (a) Since M is Hausdorff, ∂M is Hausdorff.
   (b) Since M is second-countable, ∂M is second countable.
   (c) Let $p \in \partial M$. Then there exists $(U, \phi) \in X_\partial(M)$ such that $\phi(p) \in \partial\mathbb{H}^n$. Then $p \in \bar{U}$ and the previous exercise implies that $(\bar{U}, \bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$. Thus ∂M is locally Euclidean of dimension $n-1$.

   Hence ∂M is an $(n-1)$-dimensional topological manifold.

2. Let $p \in \partial M$. Part (1) implies that there exists $(U, \phi) \in X^{n-1}_{\text{Int}}(\partial M)$ such that $p \in U$. Thus $p \in \operatorname{Int}\partial M$. Since $p \in \partial M$ is arbitrary, $\operatorname{Int}\partial M = \partial M$. Hence

$$\begin{aligned}\partial(\partial M) &= (\operatorname{Int}(\partial M))^c \\ &= (\partial M)^c \\ &= \varnothing\end{aligned}$$

□

Exercise 3.2.0.4. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M, then $(U', \phi|_{U'}) \in X^n(M)$.

Proof. Suppose that U' is open in M. Set $\phi' = \phi|_{U'}$.

- By assumption U' is open in M.

- Since U' is open in M, we have that $U' = U' \cap U$ is open in U. Since ϕ is a homeomorphism and U' is open in U, we have that $\phi(U')$ is open in $\phi(U)$. By assumption $\phi(U)$ is open in $\mathbb{R}^n$ or $\phi(U)$ is open in $\mathbb{H}^n$. Therefore $\phi'(U')$ is open in $\mathbb{R}^n$ or $\phi'(U')$ is open in $\mathbb{H}^n$.

- Since $\phi : U \to V$ is a homeomorphism, $\phi' : U' \to \phi'(U')$ is a homeomorphism.

So $(U', \phi') \in X^n(M)$. □

Note 3.2.0.5. Since U is open in M, U' being open in U is equivalent to U' being open in M, so we could have also assumed that U' is open in U.

Exercise 3.2.0.6. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then

$$X^n(U) = \{(V, \psi) \in X^n(M) : V \subset U\}$$

Proof. Suppose that U is open and set $A = \{(V, \psi) \in X^n(M) : V \subset U\}$. Let $(V, \psi) \in X^n(U)$. By definition of $X^n(U)$, V is open in U. Thus, there exists $W \subset M$ such that W is open in M and $V = U \cap W$. Since U is open in M, we have that $V = U \cap W$ is open in M. Hence $(V, \psi) \in X^n(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X^n(U)$ is arbitary, $X^n(U) \subset A$.
Conversely, suppose that $(V, \psi) \in A$. Then $(V, \psi) \in X^n(M)$ and $V \subset U$. By definition of $X^n(M)$, V is open in M. Since $V \subset U$, we have that $V = V \cap U$ is open in U. Hence $(V, \psi) \in X^n(U)$. Since $(V, \psi) \in X^n(U)$ is arbitary, $A \subset X^n(U)$. Hence $X^n(A) = A$. □

Exercise 3.2.0.7. Let M be an n-dimensional topological manifold, $(U, \phi) \in X(M)$ and $U' \subset U$. If U' is open in M, then $(U', \phi|_{U'}) \in X^n(U)$.

Proof. Suppose that U' is open in M. A previous exercise implies that $(U', \phi') \in X^n(M)$. The previous exercise implies that $(U', \phi') \in X^n(U)$. □

Exercise 3.2.0.8. Topological Open Submanifolds:
Let M be an n-dimensional topological manifold and $U \subset M$ open. Then U is an n-dimensional topological manifold.

Proof.

1. Since M is Hausdorff, U is Hausdorff.
2. M is second-countable, U is second countable.
3. Let $p \in U$. Since then there exists $(V, \psi) \in X^n(M)$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{U \cap V}$. The previous exercise implies that $(V', \psi') \in X^n(U)$. Therefore U is locally Euclidean of dimension n.

Hence U is an n-dimensional topological manifold. □

Exercise 3.2.0.9. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then

1. $X_{\text{Int}}(U) = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$
2. $X_\partial(U) = \{(V, \psi) \in X_\partial(M) : V \subset U\}$

Proof. Suppose that U is open in M.

1. Set $A = \{(V, \psi) \in X_{\text{Int}}(M) : V \subset U\}$. Let $(V, \psi) \in X_{\text{Int}}(U)$. By definition of $X_{\text{Int}}(U)$, V is open in U and $\phi(V)$ is open in $\mathbb{R}^n$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_{\text{Int}}(M)$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_{\text{Int}}(M)$ which implies that $(V, \psi) \in A$. Since $(V, \psi) \in X_{\text{Int}}(U)$ is arbitrary, $X_{\text{Int}}(U) \subset A$.
   Conversely, let $(V, \psi) \in A$. Then $(V, \psi) \in X_{\text{Int}}(M)$ and $V \subset U$. By definition of $X_{\text{Int}}(M)$, V is open in M and $\phi(V)$ is open in $\mathbb{R}^n$. Thus $V = V \cap U$ is open in U. So $(V, \psi) \in X_{\text{Int}}(U)$. Since $(V, \psi) \in A$ is arbitrary, $A \subset X_{\text{Int}}(U)$. Thus $X_{\text{Int}}(U) = A$.

2. Set $B = \{(V, \psi) \in X_\partial(M) : V \subset U\}$. Let $(V, \psi) \in X_\partial(U)$. By definition of $X_\partial(U)$, V is open in U, $\phi(V)$ is open in $\mathbb{H}^n$ and $\partial\mathbb{H}^n \cap \phi(V) \neq \varnothing$. Since U is open in M, V is open in M. Hence $(V, \psi) \in X_\partial(M)$, which implies that $(V, \psi) \in B$. Since $(V, \psi) \in X_\partial(U)$ is arbitrary, $X_\partial(U) \subset B$.
   Conversely, let $(V, \psi) \in B$. Then $(V, \psi) \in X_\partial(M)$ and $V \subset U$. By definition of $X_\partial(M)$, V is open in M, $\phi(V)$ is open in $\mathbb{H}^n$ and $\partial\mathbb{H}^n \cap \phi(V) \neq \varnothing$. Thus $V = V \cap U$ is open in U. So $(V, \psi) \in X_\partial(U)$. Since $(V, \psi) \in B$ is arbitrary, $B \subset X_\partial(U)$. Thus $X_\partial(U) = B$.

□

Exercise 3.2.0.10. Let M be an n-dimensional topological manifold and $U \subset M$. If U is open, then $\partial U = \partial M \cap U$.

Proof. Suppose that U is open. Let $p \in \partial U$. Then there exists $(V, \psi) \in X_\partial(U)$ such that $p \in V$ and $\psi(p) \in \partial\mathbb{H}^n$. Since U is open, the previous exercise implies that $(V, \psi) \in X_\partial(M)$. Thus $p \in \partial M$. Since $p \in \partial U$ is arbitrary, $\partial U \subset \partial M$. Since $\partial U \subset U$, we have that $\partial U \subset \partial M \cap U$.
Conversely, let $p \in \partial M \cap U$. Since $p \in \partial M$, there exists $(V, \psi) \in X_\partial(M)$ such that $p \in V$ and $\psi(p) \in \partial\mathbb{H}^n$. Set $V' = V \cap U$ and $\psi' = \psi|_{V'}$. Then $p \in V'$ since V and U are open in M, V' is open in M. A previous exercise implies that $(V', \psi') \in X(M)$. Since $p \in \partial M$, a previous exercise implies that $(V', \psi') \in X_\partial(M)$. The previous exercise implies that $(V', \psi') \in X_\partial(U)$. Since $\psi'(p) \in \partial\mathbb{H}^n$, $p \in \partial U$. Since $p \in \partial M \cap U$ is arbitrary, $\partial M \cap U \subset \partial U$. Hence $\partial U = \partial M \cap U$.
label exercises and reference them!!! □

3.3 Product Manifolds

Exercise 3.3.0.1. show $\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \in \mathbf{Man}^0$.

Exercise 3.3.0.2. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then there exists $\lambda : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^{m+n}$ such that

1. λ is a $(\mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\operatorname{Int}\mathbb{H}^n}, \mathbb{H}^{m+n})$-homeomorphism
2. $\lambda(\partial\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n) = \partial\mathbb{H}^{m+n}$

Proof. Define $\lambda : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^{m+n}$ by

$$\lambda((x^1, \dots, x^{m-1}, x^m), (y^1, \dots, y^n)) = (x^1, \dots, x^{m-1}, \log y^1, \dots, \log y^n, x^m)$$

1. Clearly λ is a homeomorphism.
2. Clearly $\lambda(\partial\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n) = \partial\mathbb{H}^{m+n}$

$\square$

Definition 3.3.0.3. We define

$\operatorname{Aut}_0^{m,n} = \{\lambda : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^{m+n} : \lambda \in \operatorname{Aut}_{\mathbf{Man}^0}(\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n, \mathbb{H}^{m+n}) \text{ and } \lambda(\partial\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n) = \partial\mathbb{H}^{m+n}\}$

Exercise 3.3.0.4. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then $\operatorname{Aut}_0^{m,n} \neq \varnothing$.

Proof. Immediate by previous exercise. $\square$

Exercise 3.3.0.5. Let $(M, \mathcal{T}_M)$, $(N, \mathcal{T}_N)$ be topological manifolds. Set $m = \dim M$ and $n = \dim N$. Suppose that $\partial N = \varnothing$. Then for each $\lambda \in \operatorname{Aut}_0^{m,n}$, $(U, \phi) \in X^m(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$,

$$(U \times V, \lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$$

Proof. Let $\lambda \in \operatorname{Aut}_0^{m,n}$, $(U, \phi) \in X^m(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$.

- Since $U \in \mathcal{T}_M$ and $V \in \mathcal{T}_N$, $U \times V \in \mathcal{T}_M \otimes \mathcal{T}_N$.
- Since $\phi(U) \in \mathcal{T}_{\mathbb{H}^m}$ and $\psi(V) \in \mathcal{T}_{\mathbb{H}^n}$, $\phi(U) \times \psi(V) \in \mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\mathbb{H}^n}$. Since $\partial N = \varnothing$, $(V, \psi) \in X^n_{\operatorname{Int}}(N, \mathcal{T}_N)$ and therefore $\psi(V) \subset \operatorname{Int}\mathbb{H}^n$. Since $\lambda : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^{m+n}$ is a homeomorphism,
$$\begin{aligned}\lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi](U \times V) &= \lambda(\phi(U) \times \psi(V)) \\ &\in \mathcal{T}_{\mathbb{H}^{m+n}}\end{aligned}$$
- Since $\phi : U \to \phi(U)$ is a $(\mathcal{T}_M \cap U, \mathcal{T}_{\mathbb{H}^m} \cap \phi(U))$-homeomorphism and $\psi : V \to \psi(V)$ is a $(\mathcal{T}_N \cap V, \mathcal{T}_{\mathbb{H}^n} \cap \psi(V))$-homeomorphism, an exercise in the section on product topologies in the analysis notes implies that $\phi \times \psi : U \times V \to \phi(U) \times \phi(V)$ is a $([\mathcal{T}_M \otimes \mathcal{T}_N] \cap [U \times V], [\mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\mathbb{H}^n}] \cap [\phi(U) \times \psi(V)])$-homeomorphism. Since $\lambda|_{\phi(U)\times\psi(V)} : \phi(U) \times \psi(V) \to \lambda(\phi(U) \times \psi(V))$ is a $([\mathcal{T}_{\mathbb{H}^m} \otimes \mathcal{T}_{\operatorname{Int}\mathbb{H}^n}] \cap [\phi(U) \times \psi(V)], \mathcal{T}_{\mathbb{H}^{m+n}} \cap \lambda(\phi(U) \times \psi(V)))$-homeomorphism, $\lambda|_{\phi(U)\times\psi(V)} \circ (\phi \times \psi)$ is a $([\mathcal{T}_M \otimes \mathcal{T}_N] \cap [U \times V], \mathcal{T}_{\mathbb{H}^{m+n}} \cap \lambda(U \times V))$-homeomorphism.

Hence $(U \times V, \lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$. Since $\lambda \in \operatorname{Aut}_0^{m,n}$, $(U, \phi) \in X^m(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$ are arbitrary, we have that for each $\lambda \in \operatorname{Aut}_0^{m,n}$, $(U, \phi) \in X^m(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$,

$$(U \times V, \lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$$

$\square$

Exercise 3.3.0.6. Let M, N be topological manifolds. Set $m = \dim M$ and $n = \dim N$. Suppose that $\partial N = \varnothing$. Then for each $\lambda \in \operatorname{Aut}_0^{m,n}$, $(U, \phi) \in X_p^m(M, \mathcal{T}_M)$ and $(V, \psi) \in X^n(N, \mathcal{T}_N)$,

$$(U \times V, \lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X_\partial^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$$

Proof. Let $\lambda \in \mathrm{Aut}_0^{m,n}$, $(U,\phi) \in X_\partial^m(M)$ and $(V,\psi) \in X^n(N)$. Define $\eta : U \times V \to \lambda(\phi(U) \times \psi(V))$ by

$$\eta := \lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]$$

Since $(U,\phi) \in X_\partial^m(M)$, $\phi(U) \cap \partial\mathbb{H}^m \neq \varnothing$. Then there exists $p \in U$ such that $\phi(p) \in \partial\mathbb{H}^m$. So $\eta(p,q) \in \partial\mathbb{H}^{m+n}$. Thus $\eta(U \times V) \cap \partial\mathbb{H}^{m+n} \neq \varnothing$ and $(U \times V, \eta) \in X_\partial^{m+n}(M \times N)$. Since $\lambda \in \mathrm{Aut}_0^{m,n}$, $(U,\phi) \in X_\partial^m(M)$ and $(V,\psi) \in X^n(N, \mathcal{T}_N)$ are arbitrary, we have that for each $\lambda \in \mathrm{Aut}_0^{m,n}$, $(U,\phi) \in X_p^m(M, \mathcal{T}_M)$ and $(V,\psi) \in X^n(N, \mathcal{T}_N)$,

$$(U \times V, \lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]) \in X_\partial^{m+n}(M \times N, \mathcal{T}_M \otimes \mathcal{T}_N)$$

□

Exercise 3.3.0.7. Let M, N be topological manifolds. Suppose that $\partial N = \varnothing$. Then

1. $M \times N$ is a topological manifold
2. $\partial(M \times N) = \partial M \times N$

Proof. Set $m = \dim M$ and $n = \dim N$.

1. Let $\lambda \in \mathrm{Aut}_0^{m,n}$.
   - Since M and N are Hausdorff, $M \times N$ is Hausdorff.
   - Since M and N are second-countable, $M \times N$ is second-countable.
   - Let $a \in M \times N$. Then there exist $p \in M$ and $q \in N$ such that $a = (p,q)$. Since M and and N are locally Euclidean, there exist $(U,\phi) \in X^m(M)$ and $(V,\psi) \in X^n(N)$ such that $p \in U$ and $q \in V$. Then $(p,q) \in U \times V$. The previous exercise implies that $(U \times V, \lambda \circ [\phi \times \psi]) \in X^{m+n}(M \times N)$. Since $a \in M \times N$ is arbitrary, $M \times N$ is locally Euclidean of dimension $m+n$.

   Thus $M \times N$ is an $(m+n)$-dimensional topological manifold.

2. - Let $a \in \partial(M \times N)$. Then there exists $p \in M$ and $q \in N$ such that $a = (p,q)$. Since $(M, \mathcal{T}_M)$ and and (N) are locally Euclidean, there exist $(U,\phi) \in X^m(M)$ and $(V,\psi) \in X^n(N)$ such that $p \in U$ and $q \in V$. Define $\eta : U \times V \to \lambda(\phi(U) \times \psi(V))$ by

     $$\eta := \lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]$$

     The previous exercise implies that $\eta \in X^{m+n}(M \times N)$. Since $(p,q) \in \partial(M \times N)$, a previous exercise implies that $\eta \in X_\partial^{m+n}(M \times N)$ and $\eta(p,q) \in \partial\mathbb{H}^{m+n}$. Therefore

     $$\begin{aligned} \phi \times \psi(p,q) &= \lambda|_{\phi(U)\times\psi(V)}^{-1} \circ \eta \\ &\in \partial\mathbb{H}^m \times \mathrm{Int}\,\mathbb{H}^n \end{aligned}$$

     Hence $\phi(p) \in \partial\mathbb{H}^m$ and $\psi(q) \in \mathrm{Int}\,\mathbb{H}^n$. Thus $(U,\phi) \in X_\partial^m(M)$ and $p \in \partial M$. Therefore

     $$\begin{aligned} a &= (p,q) \\ &\in \partial M \times N \end{aligned}$$

     Since $a \in \partial(M \times N)$ is arbitrary, we have that $\partial(M \times N) \subset \partial M \times N$.
   - Let $a \in \partial M \times N$. Then there exists $p \in \partial M$ and $q \in N$ such that $a = (p,q)$. By definition, there exists $(U,\phi) \in X_\partial^m(M)$ and $(V,\psi) \in X^n(N)$ such that $p \in U$, $q \in V$ and $\phi(p) \in \partial\mathbb{H}^m$. Since $\partial N = \varnothing$, $\psi(q) \in \mathrm{Int}\,\mathbb{H}^n$. Define $\eta : U \times V \to \lambda(\phi(U) \times \psi(V))$ by

     $$\eta := \lambda|_{\phi(U)\times\psi(V)} \circ [\phi \times \psi]$$

     A previous exercise implies that $(U \times V, \eta) \in X^{m+n}(M \times N \otimes \mathcal{T}_N)$. Then

     $$\begin{aligned} \eta(a) &= \eta(p,q) \\ &= \lambda(\phi(p), \psi(q)) \\ &\in \partial\mathbb{H}^{m+n} \end{aligned}$$

     Thus $\eta \in X_\partial^{m+n}(M \times N)$ and $a \in \partial(M \times N)$. Since $a \in \partial M \times N$ is arbitrary, $\partial M \times N \subset \partial(M \times N)$.

   Thus $\partial(M \times N) = \partial M \times N$.

□

Chapter 4

Smooth Manifolds

4.1 Introduction

Definition 4.1.0.1. Let M be an n-dimensional topological manifold and $(U,\phi),(V,\psi) \in X(M)$. Then (U,ϕ) and (V,ψ) are said to be **smoothly compatible** if

$$\psi|_{U\cap V} \circ (\phi|_{U\cap V})^{-1} : \phi(U\cap V) \to \psi(U\cap V) \text{ is a diffeomorphism}$$

Definition 4.1.0.2. Let M be an n-dimensional topological manifold.

- Let $\mathcal{A} \subset X(M)$. Then $\mathcal{A}$ is said to be an **atlas on** M if $M \subset \bigcup_{(U,\phi)\in\mathcal{A}} U$.
- Let $\mathcal{A}$ be an atlas on M. Then $\mathcal{A}$ is said to be **smooth** if for each $(U,\phi),(V,\psi) \in \mathcal{A}$, (U,ϕ) and (V,ψ) are smoothly compatible.
- Let $\mathcal{A}$ be a smooth atlas on M. Then $\mathcal{A}$ is said to be **maximal** if for each smooth atlas $\mathcal{B}$ on M, $\mathcal{A} \subset \mathcal{B}$ implies that $\mathcal{A} = \mathcal{B}$. A maximal smooth atlas on M is called a **smooth structure on** M.
- Let $\mathcal{A}$ be an atlas on M. Then $(M,\mathcal{A})$ is said to be an n**-dimensional smooth manifold** if $\mathcal{A}$ is a smooth structure on M.

Definition 4.1.0.3. Let M be a topological manifold and $\mathcal{B}$ a smooth atlas on M. We define the **smooth structure on** M **generated by** $\mathcal{B}$, denoted $\alpha_M(\mathcal{B})$, by

$$\alpha_M(\mathcal{B}) = \{(U,\phi) \in X(M) : \text{ for each } (V,\psi) \in \mathcal{B},\ (U,\phi) \text{ and } (V,\psi) \text{ are smoothly compatible}\}$$

Note 4.1.0.4. When the context is clear, we write $\alpha(\mathcal{B})$ in place of $\alpha_M(\mathcal{B})$.

Exercise 4.1.0.5. Let M be an n-dimensional topological manifold and $\mathcal{B}$ a smooth atlas on M. Then $\alpha(\mathcal{B})$ is the unique smooth structure $\mathcal{A}$ on M such that $\mathcal{B} \subset \mathcal{A}$.

Proof. Clearly $\mathcal{B} \subset \alpha(\mathcal{B})$. Let (U,ϕ) and $(V,\psi) \in \alpha(\mathcal{B})$. Define $F : \phi(U\cap V) \to \psi(U\cap V)$ by

$$F = \psi|_{U\cap V} \circ (\phi|_{U\cap V})^{-1}$$

Let $q \in \phi(U\cap V)$. Set $p = \phi^{-1}(q)$. Since $p \in U\cap V \subset M$, there exists $(W,\chi) \in \mathcal{B}$ such that $p \in W$. By definition of $\alpha(\mathcal{B})$, $\psi|_{W\cap V} \circ (\chi|_{W\cap V})^{-1} : \chi(W\cap V) \to \psi(W\cap V)$ and $\chi|_{U\cap W} \circ (\phi|_{U\cap W})^{-1} : \phi(U\cap W) \to \chi(U\cap W)$ are diffeomorphisms. Set $N = U\cap W\cap V$. Then $q \in \phi(N) \subset \phi(U\cap V)$ and

$$\begin{aligned} F|_{\phi(N)} &= \psi|_N \circ (\phi|_N)^{-1} \\ &= [\psi|_N \circ (\chi|_N)^{-1}] \circ [\chi|_N \circ (\phi|_N)^{-1}] \end{aligned}$$

is a diffeomorphism. Thus, for each $q \in \phi(U\cap V)$, there exists $N' \subset \phi(U\cap V)$ such that $F|_{N'}$ is a diffeomorphism. Hence F is a diffeomorphism and (U,ϕ), (V,ψ) are smoothly compatible. Therefore $\alpha(\mathcal{B})$

is a smooth atlas.
To see that $\alpha(\mathcal{B})$ is maximal, let $\mathcal{B}'$ be a smooth atlas on M. Suppose that $\alpha(\mathcal{B}) \subset \mathcal{B}'$ and let $(U,\phi) \in \mathcal{B}'$. By definition, for each chart $(V,\psi) \in \mathcal{B}'$, (U,ϕ) and (V,ψ) are smoothly compatible. Since $\mathcal{B} \subset \alpha(\mathcal{B}) \subset \mathcal{B}'$, we have that $(U,\phi) \in \alpha(\mathcal{B})$. So $\alpha(\mathcal{B}) = \mathcal{B}'$ and $\alpha(\mathcal{B})$ is a maximal smooth atlas on M. $\square$

Definition 4.1.0.6. Let $n \in \mathbb{N}_0$. We define the **standard smooth structure** on $\mathbb{R}^n$, denoted $\mathcal{A}_{\mathbb{R}^n}$, by $\mathcal{A}_{\mathbb{R}^n} = \alpha_{\mathbb{R}^n}(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$.

Exercise 4.1.0.7. Define $U \subset \mathbb{R}$ and $\phi : U \to \mathbb{R}$ by $U := \mathbb{R}$ and $\phi(x) := x^3$. Then

1. $(U,\phi) \in X^1(\mathbb{R})$
2. $(U,\phi) \notin \mathcal{A}_{\mathbb{R}}$

Proof.

1. - Trivially, U is open in $\mathbb{R}$.
   - Trivially, $\mathbb{R}$ is open in $\mathbb{R}$
   - Clearly ϕ is continuous. Also, ϕ is a bijection. and since for each $x \in \mathbb{R}$, $\phi^{-1}(x) = x^{1/3}$, ϕ^{-1} is continuous. Hence ϕ is a homeomorphism.

   So $(U,\phi) \in X^1(\mathbb{R})$.
2. Define $V \subset M$ and $\psi : V \to \mathbb{R}$ by $V := \mathbb{R}$ and $\psi := \mathrm{id}_{\mathbb{R}}$. By defintion, $(V,\psi) \in \mathcal{A}_{\mathbb{R}}$. Since ϕ^{-1} is not differentiable at $x = 0$ and $\psi \circ \phi^{-1} = \phi^{-1}$, we have that $\psi \circ \phi^{-1}$ is not smooth and therefore $\psi \circ \phi^{-1}$ is not a diffeomorphism. Hence (U,ϕ) and (V,ψ) are not smoothly compatible. Thus $(U,\phi) \notin \mathcal{A}_{\mathbb{R}}$.

$\square$

Exercise 4.1.0.8. Let $(M,\mathcal{A})$ be a smooth manifold and $\mathcal{A}_0 \subset \mathcal{A}$. Suppose that $\mathcal{A}_0$ is an atlas on M. Let $(U,\phi) \in X(M)$. Then $(U,\phi) \in \mathcal{A}$ iff for each $(V,\psi) \in \mathcal{A}_0$, (U,ϕ) and (V,ψ) are smoothly compatible.

Proof. Set $n := \dim M$.

- ($\Longrightarrow$):
  Suppose that $(U,\phi) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, for each $(V,\psi) \in \mathcal{A}$, (U,ϕ) and (V,ψ) are smoothly compatible. Since $\mathcal{A}_0 \subset \mathcal{A}$, we have that for each $(V,\psi) \in \mathcal{A}_0$, (U,ϕ) and (V,ψ) are smoothly compatible.

- ($\Longleftarrow$):
  Suppose that for each $(V,\psi) \in \mathcal{A}_0$, (U,ϕ) and (V,ψ) are smoothly compatible. Let $(V,\psi) \in \mathcal{A}$ and $a \in \phi(U \cap V)$. Set $p := \phi^{-1}(a)$. Since $\mathcal{A}_0$ is an atlas on M, there exists $(W_0,\alpha_0) \in \mathcal{A}_0$ such that $p \in W_0$. Define $f : \phi(U \cap W_0) \to \alpha_0(U \cap W_0)$, $g : \alpha_0(W_0 \cap V) \to \psi(W_0 \cap V)$ and $h : \phi(U \cap V) \to \psi(U \cap V)$ by $f := \alpha_0|_{U \cap W_0} \circ \phi|^{-1}_{U \cap W_0}$, $g := \psi|_{W_0 \cap V} \circ \alpha_0|^{-1}_{W_0 \cap V}$ and $h := \psi|_{U \cap V} \circ \phi|^{-1}_{U \cap V}$. By assumption, (U,ϕ) and (W_0,α_0) are smoothly compatible. Thus f is a diffeomorphism and therefore f is smooth. Since $(W_0,\alpha_0), (V,\psi) \in \mathcal{A}$, we have that (W_0,α_0) and (V,ψ) are smoothly compatible. Thus g is a diffeomorphism and therefore g is smooth. Define $A \subset M$ and $A' \subset \mathbb{R}^n$ by $A := U \cap V \cap W_0$ and $A' = \phi(A)$. Since $p \in A$, $a \in A'$. Since A is open in $U \cap V$ and ϕ is a homeomorphism, A' is open in $\phi(U \cap V)$. An exercise in the section on differentiation on subspaces implies that $f|_{A'}$ is smooth. Since $h|_{A'} = g \circ f|_{A'}$, $h|_{A'}$ is smooth. Since $a \in \phi(U \cap V)$ is arbitrary, we have that for each $a \in \phi(U \cap V)$, there exists $A' \subset \phi(U \cap V)$ such that $a \in A'$, A' is open in $\phi(U \cap V)$ and $h|_{A'}$ is smooth. An exercise in the section on differentiation on subspaces implies that h is smooth. Thus (U,ϕ) and (V,ψ) are smoothly compatible. Since $(V,\psi) \in \mathcal{A}$ is arbitrary, we have that $\mathcal{A} \cup \{(U,\phi)\}$ is a smooth atlas on M. Since $\mathcal{A}$ is maximal, $\mathcal{A} \cup \{(U,\phi)\} = \mathcal{A}$. Thus $(U,\phi) \in \mathcal{A}$.

$\square$

Exercise 4.1.0.9. Smooth Manifold Chart Lemma:
Let M be a set, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{H}^n$. Suppose that

(a) for each $\alpha \in \Gamma$, $\phi_\alpha(U_\alpha) \in \mathcal{T}_{\mathbb{H}^n}$

(b) for each $\alpha, \beta \in \Gamma$, $\phi_\alpha(U_\alpha \cap U_\beta) \in \mathcal{T}_{\mathbb{H}^n}$

(c) for each $\alpha \in \Gamma$, $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a bijection

(d) for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

(e) there exists $\Gamma' \subset \Gamma$ such that Γ' is countable and $M \subset \bigcup\limits_{\alpha \in \Gamma'} U_\alpha$

(f) for each $p, q \in M$, there exists $\alpha \in \Gamma$ such that $p, q \in U_\alpha$ or there exist $\alpha, \beta \in \Gamma$ such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \varnothing$

Then there exists a unique smooth structure $\mathcal{A}_M$ on M such that $(M, \mathcal{A}_M)$ is an n-dimensional smooth manifold and $(U_\alpha, \phi_\alpha)_{\alpha \in \Gamma} \subset \mathcal{A}_M$.

Proof. Define

- $\mathcal{B} = \{\phi_\alpha^{-1}(V) : \alpha \in \Gamma \text{ and } V \in \mathcal{T}_{\mathbb{H}^n}\}$
- $\mathcal{T}_M = \tau(\mathcal{B})$
- $\mathcal{A}' = \{(U_\alpha, \phi_\alpha) : \alpha \in \Gamma\}$.

The topological manifold chart lemma implies that $(M, \mathcal{T}_M)$ is an n-dimensional topological manifold and $\mathcal{A}' \subset X^n(M, \mathcal{T}_M)$. Since $M = \bigcup_{\alpha \in \Gamma} U_\alpha$, $\mathcal{A}'$ is an atlas on M. Since for each $\alpha, \beta \in \Gamma$, $\phi_\beta|_{U_\alpha \cap U_\beta} \circ (\phi_\alpha|_{U_\alpha \cap U_\beta})^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth, we have that $\mathcal{A}'$ is smooth. Set $\mathcal{A}_M = \alpha(\mathcal{A}')$. A previous exercise implies that $\mathcal{A}_M$ is the unique smooth structure $\mathcal{A}$ on M such that $\mathcal{A}' \subset \mathcal{A}$. Hence $(M, \mathcal{A}_M)$ is an n-dimensional smooth manifold and $\mathcal{A}' \subset \mathcal{A}_M$. $\square$

4.2 Open and Boundary Submanifolds

4.2.1 Open Submanifolds

in order for a smooth structure on a subspace to make sense, we need to have a notion of smoothness of a map $A \to \mathbb{R}^k$ to make sense when $A \subset \mathbb{R}^n$. But we already have this, think of them as banach spaces, use the definitions of differentiation at a point in A in terms of locally extend to an open neighborhood containing the point.

Exercise 4.2.1.1. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold, $(U, \phi) \in \mathcal{A}$ and $U' \subset U$. If U' is open, then $(U', \phi|_{U'}) \in \mathcal{A}$.

Proof. Set $\phi' = \phi|_{U'}$. A previous exercise implies that $(U', \phi') \in X(U)$. Define $\mathcal{B} = \mathcal{A} \cup \{(U', \phi')\}$. Let $(V, \psi) \in \mathcal{B}$. If $(V, \psi) = (U', \phi')$, then

$$\phi' \circ \psi^{-1} = \mathrm{id}_{U'}$$

which is a diffeomorphism. Thus (U', ϕ'), (V, ψ) are smoothly compatible. Suppose that $(V, \psi) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, $\psi|_{U \cap V} \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism. Therefore $\psi|_{U' \cap V} \circ (\phi'|_{U' \cap V})^{-1} : \phi'(U' \cap V) \to \psi(U' \cap V)$ is a diffeomorphism and (U', ϕ'), (V, ψ) are smoothly compatible. Since $(V, \psi) \in \mathcal{B}$ is arbitrary, $\mathcal{B}$ is smooth. Since $\mathcal{A}$ is maximal and $\mathcal{A} \subset \mathcal{B}$, we have that $\mathcal{A} = \mathcal{B}$ and $(U', \phi') \in \mathcal{A}$. □

Exercise 4.2.1.2. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold and $U \subset M$ open. Set $\mathcal{B} = \{(V, \psi) \in \mathcal{A} : V \subset U\}$. Then $\mathcal{B}$ is a smooth atlas on U.

Proof.

- Some previous exercises imply that U is an n-dimensional topological manifold and $X(U) = \{(V, \psi) \in X(M) : V \subset U\}$. Since

$$\begin{aligned} \mathcal{B} &\subset \mathcal{A} \\ &\subset X(M) \end{aligned}$$

  we have that $\mathcal{B} \subset X(U)$. Let $p \in U$. Then there exists $(V, \psi) \in \mathcal{A}$ such that $p \in V$. Set $V' = U \cap V$ and $\psi' = \psi|_{V'}$. The previous exercise implies that $(V', \psi') \in \mathcal{A}$. By definition, $(V', \psi') \in \mathcal{B}$. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $(V', \psi') \in \mathcal{B}$ such that $p \in V'$. Hence $\mathcal{B}$ is an atlas on U.

- Let $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$. Then $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{A}$. Since $\mathcal{A}$ is smooth, (V_1, ψ_1) and (V_2, ψ_2) are smoothly compatible. Since $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ are arbitrary, $\mathcal{B}$ is smooth.

□

Definition 4.2.1.3. Smooth Open Submanifold:
Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $U \subset M$ open. A previous exercise implies that U is an n-dimensional topological manifold. We define the **induced smooth structure on** U, denoted $\mathcal{A}|_U \subset X(U)$, by

$$\mathcal{A}|_U = \alpha_U(\{(V, \psi) \in \mathcal{A} : V \subset U\})$$

Then $(U, \mathcal{A}|_U)$ is said to be a **smooth open submanifold of** $(M, \mathcal{A})$.

Exercise 4.2.1.4. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $U \subset M$ open. Then

1. $\mathcal{A}|_U \subset \mathcal{A}$.
2. $\mathcal{A}|_U = \{(V, \psi) \in \mathcal{A} : V \subset U\}$.

Proof.

1. Set $\mathcal{B} = \{(V,\psi) \in \mathcal{A} : V \subset U\}$. Let $(U',\phi) \in \mathcal{A}|_U$, $(V,\psi) \in \mathcal{A}$ and $a \in \phi(U' \cap V)$. Set $p = \phi^{-1}(a)$. A previous exercise implies that $\mathcal{B}$ is a smooth atlas on U. Thus there exists $(W,\alpha) \in \mathcal{B}$ such that $p \in W$. Set $A := W \cap U' \cap V$ and $A_0 := \phi(A)$. Then $p \in A$, $a \in A_0$, A is open in M and A_0 is open in $\phi(U' \cap V)$. Define $f : \phi(W \cap U') \to \alpha(W \cap U')$, $g : \alpha(W \cap V) \to \psi(W \cap V)$ and $h : \phi(U' \cap V) \to \psi(U' \cap V)$ by $f := \alpha|_{W\cap U'} \circ \phi|^{-1}_{W \cap U'}$, $g := \psi|_{W \cap V} \circ \alpha|^{-1}_{W \cap V}$ and $h := \psi_{U' \cap V} \circ \phi|^{-1}_{U' \cap V}$. Since $\mathcal{B} \subset \mathcal{A}$, g is smooth. Since $\mathcal{B} \subset \mathcal{A}|_U$, f is smooth. An exercise in the section on differentiability on subspaces implies that $f|_{A_0}$ is smooth. Since $h|_{A_0} = g \circ f|_{A_0}$, an exercise in the section on differentiability implies that $h|_{A_0}$ is smooth. Since $a \in \phi(U' \cap V)$ is arbitrary, we have that for each $a \in \phi(U' \cap V)$, there exists $A_0 \subset \phi(U' \cap V)$ such that $a \in A_0$, A_0 is open in $\phi(U' \cap V)$ and $h|_{A_0}$ is smooth. An exercise in the section on differentiability on subspaces implies that h is smooth. Similarly h^{-1} is smooth. Thus h is a diffeomorphism. Therefore (V,ψ) and (U',ϕ) are smoothly compatible. Since $(V,\psi) \in \mathcal{A}$ is arbitrary, we have that $\{(U',\phi)\} \cup \mathcal{A}$ is a smooth atlas. Since $\mathcal{A}$ is maximal, $\{(U',\phi)\} \cup \mathcal{A} = \mathcal{A}$. Thus $(U',\phi) \in \mathcal{A}$. Since $(U',\phi) \in \mathcal{A}|_U$ is arbitrary, we have that $\mathcal{A}|_U \subset \mathcal{A}$.

2. By definition,
$$\begin{aligned} \mathcal{B} &\subset \alpha_U(\mathcal{B}) \\ &= \mathcal{A}|_U \end{aligned}$$
Since $\mathcal{A}|_U \subset \mathcal{A}$, the definition of $\mathcal{B}$ implies that $\mathcal{A}|_U \subset \mathcal{B}$. Hence $\mathcal{A}|_U = \mathcal{B}$.

□

4.2.2 Boundary Submanifolds

Exercise 4.2.2.1. Let $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ be the projection map given by $\pi(x_1,\dots,x_{n-1},0) = (x_1,\dots,x_{n-1})$. Then π is a diffeomorphism.

Proof. Define projection map $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ by $\pi'(x_1,\dots,x_{n-1},x_n) = (x_1,\dots,x_{n-1})$. Then $\mathbb{R}^n$ is an open neighborhood of $\partial\mathbb{H}^n$, $\pi'|_{\partial H^n} = \pi$ and π' is smooth. Then by definition, π is smooth. Clearly, π^{-1} is smooth. So π is a diffeomorphism. □

Definition 4.2.2.2. Let $(M,\mathcal{A})$ be a n-dimensional smooth manifold and $\pi : \partial\mathbb{H}^n \to \mathbb{R}^{n-1}$ the projection map. Recall that for $(U,\phi) \in X^n_\partial(M)$, the $(n-1)$-coordinate chart $(\bar{U},\bar{\phi}) \in X^{n-1}_{\text{Int}}(\partial M)$ is defined by $\bar{U} = U \cap \partial M$ and $\bar{\phi} = \pi|_{\phi(\bar{U})} \circ \phi|_{\bar{U}}$.
We define
$$\bar{\mathcal{A}} = \{(\bar{U},\bar{\phi}) \in X^{n-1}_\partial(M) : (U,\phi) \in \mathcal{A}\}$$

Exercise 4.2.2.3. Let $(M,\mathcal{A})$ be a n-dimensional smooth manifold. Then $\bar{\mathcal{A}}$ is a smooth atlas on ∂M.

Proof.

- A previous exercise implies that ∂M is an $(n-1)$-dimensional topological manifold. Let $p \in \partial M$. Then there exists $(U,\phi) \in \mathcal{A}$ such that $p \in U$. Since $\mathcal{A} \subset X^n(M)$ and $p \in \partial M$, we have that $p \in \bar{U}$ and a previous exercise implies that $(U,\phi) \in X^n_\partial(M)$. By definition of $\bar{\mathcal{A}}$, $(\bar{U},\bar{\phi}) \in \bar{\mathcal{A}}$. Since $p \in \partial M$ is arbitrary, $\bar{\mathcal{A}}$ is an atlas on ∂M.

- Let $(\bar{U},\bar{\phi})$, $(\bar{V},\bar{\psi}) \in \bar{\mathcal{A}}$. Since (U,ϕ) and (V,ψ) are smoothly compatible, $\psi|_{U\cap V} \circ (\phi|_{U \cap V})^{-1}$ is a diffeomorphism. Thus $\psi|_{\bar{U}\cap\bar{V}} \circ (\phi|_{\bar{U}\cap\bar{V}})^{-1}$ is a diffeomorphism. Since $\pi|_{\phi(U\cap V)}$ and $\pi|_{\psi(U \cap V)}$ are diffeomorphisms, $\pi|_{\phi(\bar{U}\cap\bar{V})}$ and $\pi|_{\psi(\bar{U}\cap\bar{V})}$ are diffeomorphisms. Then
$$\begin{aligned} \bar{\psi}|_{\bar{U}\cap\bar{V}} \circ (\bar{\phi}|_{\bar{U}\cap\bar{V}})^{-1} &= \left[\pi|_{\psi(\bar{U}\cap\bar{V})} \circ \psi|_{\bar{U}\cap\bar{V}}\right] \circ \left[(\phi|_{\bar{U}\cap\bar{V}})^{-1} \circ (\pi|_{\phi(\bar{U}\cap\bar{V})})^{-1}\right] \\ &= \pi|_{\psi(\bar{U}\cap\bar{V})} \circ \left[\psi|_{\bar{U}\cap\bar{V}} \circ (\phi|_{\bar{U}\cap\bar{V}})^{-1}\right] \circ (\pi|_{\phi(\bar{U}\cap\bar{V})})^{-1} \end{aligned}$$
is a diffeomorphism. Therefore $(\bar{U},\bar{\phi})$ and $(\bar{V},\bar{\psi})$ are smoothly compatible. Since $(\bar{U},\bar{\phi}), (\bar{V},\bar{\psi}) \in \bar{\mathcal{A}}$ are arbitrary, $\mathcal{A}$ is smooth.

□

Definition 4.2.2.4. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold. We define the **induced smooth structure on the boundary**, denoted $\mathcal{A}|_{\partial M}$, by

$$\mathcal{A}|_{\partial M} = \alpha(\bar{\mathcal{A}})$$

We define the **smooth boundary submanifold of** M to be $(\partial M, \mathcal{A}|_{\partial M})$.

4.3 Product Manifolds

4.4 To Do

Define a $\mathbb{E}$-space as an open subset of a real finite-dim vector space and define differentiation on these and show that products of $\mathbb{E}$-spaces are $\mathbb{E}$-spaces, then define differentiation on subspaces of euclidean spaces and define $\mathbb{H}$-spaces as half-spaces of $\mathbb{E}$-spaces

Exercise 4.4.0.1. show $\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n \in \mathbf{Man}^\infty$.

Exercise 4.4.0.2. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then there exists $\lambda : \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n \to \mathbb{H}^{m+n}$ such that

1. λ is a diffeomorphism (need to define differentiation on product spaces)
2. $\lambda(\partial \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n) = \partial \mathbb{H}^{m+n}$

Proof. Define $\lambda : \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n \to \mathbb{H}^{m+n}$ by

$$\lambda((x^1, \dots, x^{m-1}, x^m), (y^1, \dots, y^n)) = (x^1, \dots, x^{m-1}, \log y^1, \dots, \log y^n, x^m)$$

1. Clearly λ is a diffeomorphism (show in detail that locally can be extended to smooth function on open neighborhood for each boundary point).
2. Clearly $\lambda(\partial \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n) = \partial \mathbb{H}^{m+n}$

$\square$

Definition 4.4.0.3. We define

$$\operatorname{Aut}_\infty^{m,n} = \{\lambda : \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n \to \mathbb{H}^{m+n} : \lambda \in \operatorname{Aut}_{\mathbf{Man}^\infty}(\mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n, \mathbb{H}^{m+n}) \text{ and } \lambda(\partial \mathbb{H}^m \times \operatorname{Int} \mathbb{H}^n) = \partial \mathbb{H}^{m+n}\}$$

Exercise 4.4.0.4. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then $\lambda \in \operatorname{Aut}_\infty^{m,n} \neq \varnothing$.

Proof. Immediate by previous exercise. $\square$

Definition 4.4.0.5. Let M, N be smooth manifolds of dimension m and n respectively, $\mathcal{A} \subset X^m(M)$ and $\mathcal{B} \subset X^n(N)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are smooth atlases on M and N respectively and $\partial N = \varnothing$. We define the **product atlas of $\mathcal{A}$ and $\mathcal{B}$ on** $M \times N$, denoted $\mathcal{A} \otimes' \mathcal{B}$, by

$$\mathcal{A} \otimes' \mathcal{B} = \{(U \times V, \lambda|_{\phi(U) \times \psi(V)} \circ [\phi \times \psi]) : \lambda \in \operatorname{Aut}_\infty^{m,n}, (U, \phi) \in \mathcal{A} \text{ and } (V, \psi) \in \mathcal{B}\}$$

Exercise 4.4.0.6. Let M, N be smooth manifolds of dimension m and n respectively, $\mathcal{A} \subset X^m(M)$ and $\mathcal{B} \subset X^n(N)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are smooth atlases on M and N respectively and $\partial N = \varnothing$. Then $\mathcal{A} \otimes' \mathcal{B}$ is a smooth atlas on $M \times N$.

Proof.

- An exercise in the section on products of topological manifolds implies that $\mathcal{A} \otimes' \mathcal{B}$ is an atlas on $M \times N$.
- Let $(W_1, \eta_1), (W_2, \eta_2) \in \mathcal{A} \otimes' \mathcal{B}$. Then there exist $\lambda_1, \lambda_2 \in \operatorname{Aut}_\infty^{m,n}$, $(U_1, \phi_1), (U_2, \phi_2) \in \mathcal{A}$ and $(V_1, \psi_1), (V_2, \psi_2) \in \mathcal{B}$ such that $W_1 = U_1 \times V_1$, $W_2 = U_2 \times V_2$, $\eta_1 = \lambda_1|_{\phi_1(U_1) \times \psi_1(V_1)} \circ [\phi_1 \times \psi_1]$ and $\eta_2 = \lambda_2|_{\phi_2(U_2) \times \psi_2(V_2)} \circ [\phi_2 \times \psi_2]$. For notational convenience, set $U := U_1 \cap U_2$ and $V := V_1 \cap V_2$. Then $W_1 \cap W_2 = U \cap V$ and

$$\begin{aligned}
\eta_2|_{W_1 \cap W_2} \circ \eta_1|_{W_1 \cap W_2}^{-1} &= \eta_2|_{U \cap V} \circ \eta_1|_{U \cap V}^{-1} \\
&= \lambda_2|_{\phi_2(U) \times \psi_2(V)} \circ [\phi_2 \times \psi_2]|_{U \times V} \circ [\phi_1 \times \psi_1]|_{U \times V}^{-1} \circ \lambda_1|_{\phi_1(U) \times \psi_1(V)}^{-1} \\
&= \lambda_2|_{\phi_2(U) \times \psi_2(V)} \circ [\phi_2|_U \times \psi_2|_V] \circ [\phi_1|_U^{-1} \times \psi_1|_V^{-1}] \circ \lambda_1|_{\phi_1(U) \times \psi_1(V)}^{-1} \\
&= \lambda_2|_{\phi_2(U) \times \psi_2(V)} \circ [(\phi_2|_U \circ \phi_1|_U^{-1}) \times (\psi_2|_V \circ \psi_1|_V^{-1})] \circ \lambda_1|_{\phi_1(U) \times \psi_1(V)}^{-1}
\end{aligned}$$

Since (U_1, ϕ_1) and (U_2, ϕ_2) are smoothly compatible, $\phi_2|_U \circ \phi_1|_U^{-1}$ is a diffeomorphism. Similarly, $\psi_2|_V \circ \psi_1|_V^{-1}$ is a diffeomorphism. Thus $(\phi_2|_U \circ \phi_1|_U^{-1}) \times (\psi_2|_V \circ \psi_1|_V^{-1})$ is a diffeomorphism. Since $\lambda_1|_{\phi_1(U)\times\psi_1(V)}^{-1}$ and $\lambda_2|_{\phi_2(U)\times\psi_2(V)}$ are diffeomorphisms, we have that $\eta_2|_{W_1\cap W_2} \circ \eta_1|_{W_1\cap W_2}^{-1}$ is a diffeomorphism. Hence (W_1, η_1) and (W_2, η_2) are smoothly compatible. Since $(W_1, \eta_1), (W_2, \eta_2) \in \mathcal{A} \otimes' \mathcal{B}$ are arbitrary, we have that $\mathcal{A} \otimes' \mathcal{B}$ is smooth.

$\square$

Definition 4.4.0.7. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ be smooth manifolds. Suppose that $\partial N = \varnothing$. We define the **product smooth structure**, denoted $\mathcal{A} \otimes \mathcal{B}$, by

$$\mathcal{A} \otimes \mathcal{B} = \alpha_{M\times N}(\mathcal{A} \otimes' \mathcal{B})$$

We define the **smooth product manifold of** $(M, \mathcal{A})$ **and** $(N, \mathcal{B})$ to be $(M \times N, \mathcal{A} \otimes \mathcal{B})$.

Chapter 5

Smooth Maps

5.1 Smooth Maps between Manifolds

Definition 5.1.0.1. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then F is said to be

- $(\mathcal{A}, \mathcal{B})$**-smooth** if for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is smooth.
- a $(\mathcal{A}, \mathcal{B})$**-diffeomorphism** if F is a bijection and F, F^{-1} are smooth.

Note 5.1.0.2. When the context is clear, we write "smooth" in place of "$(\mathcal{A}, \mathcal{B})$-smooth".

Exercise 5.1.0.3. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifold and $F : M \to N$. If F is smooth, then F is continuous.

Proof. Suppose that F is smooth. Let $p \in M$. By defintion, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is smooth. Define $F_0 : \phi(U) \to \psi(V)$ by

$$F_0 = \psi \circ F \circ \phi^{-1}$$

By definition, F_0 is smooth. An exercise in the section on differentiation in subspaces implies that F_0 is continuous. Since ϕ and ψ are homeomorphisms and $F|_U = \psi^{-1} \circ F_0 \circ \phi$, we have that $F|_U$ is continuous. In particular, F is continuous at p. Since $p \in M$ is arbitrary, F is continuous. $\square$

Exercise 5.1.0.4. Equivalence of Smoothness:
Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then the following are equivalent:

1. $F : M \to N$ is smooth
2. for each $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, if $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N, then for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.
3. for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.
4. F is continuous and there exist $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ such that $\mathcal{A}_0$ is an atlas on $\mathcal{A}$, $\mathcal{B}_0$ is an atlas on N and for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth

Proof. Set $m := \dim M$ and $n := \dim N$.

1. (1) $\implies$ (2):
Suppose that F is smooth. Let $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$. Suppose that $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N. Let $(U_0, \phi_0) \in \mathcal{A}_0$ and $(V_0, \psi_0) \in \mathcal{B}_0$. Since $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, we have that $(U_0, \phi_0) \in \mathcal{A}$ and $(V_0, \psi_0) \in \mathcal{B}$. Since F is smooth, F is continuous and therefore $U_0 \cap F^{-1}(V_0)$ is open in M.

Define $F_0 : \phi_0(U_0 \cap F^{-1}(V_0)) \to \psi_0(V_0)$ by $F_0 := \psi_0 \circ F \circ \phi_0|^{-1}_{U_0 \cap F^{-1}(V_0)}$. Let $a \in \phi_0(U_0 \cap F^{-1}(V_0))$. Define $p \in M$ by $p := \phi_0^{-1}(a)$. Since F is smooth, by definition there exists $(U_1, \phi_1) \in \mathcal{A}$ and $(V_1, \psi_1) \in \mathcal{B}$ such that $p \in U_1$, $F(p) \in V_1$, $F(U_1) \subset V_1$ and $\psi_1 \circ F \circ \phi_1^{-1}$ is smooth. Define $U \subset M$, $\alpha : \phi_1(U_0 \cap U_1) \to \phi_0(U_0 \cap U_1)$, $\beta : \psi_1(V_0 \cap V_1) \to \psi_0(V_0 \cap V_1)$ and $F_1 : \phi_1(U_1) \to \psi_1(V_1)$ by $U = U_0 \cap U_1 \cap F^{-1}(V_0 \cap V_1)$, $\alpha = \phi_0|_{U_0 \cap U_1} \circ \phi_1|^{-1}_{U_0 \cap U_1}$, $\beta = \psi_0|_{V_0 \cap V_1} \circ \psi_1|^{-1}_{V_0 \cap V_1}$ and $F_1 = \psi_1 \circ F \circ \phi_1^{-1}$.

We note the following:

- since $p \in U$ and $a = \phi_0(p)$, we have that $a \in \phi_0(U)$
- $\phi_0(U)$ is open in $\phi_0(U_0 \cap F^{-1}(V_0))$
- since $(U_0, \phi_0), (U_1, \phi_1) \in \mathcal{A}$, (U_0, ϕ_0) and (U_1, ϕ_1) are smoothly compatible and α is a diffeomorphism
- since $(V_0, \psi_0), (V_1, \psi_1) \in \mathcal{B}$, (V_0, ψ_0) and (V_1, ψ_1) are smoothly compatible and β is a diffeomorphism
- since F_1 is smooth, an exercise in the section on differentiation on subspaces implies that $F_1|_{\phi_1(U)}$ is smooth
- since α^{-1} is smooth, an exercise in the section on differentiation on subspaces implies that $\alpha|^{-1}_{\phi_1(U)}$ is smooth
- since $F_0|_{\phi_0(U)} = \beta \circ F_1 \circ \alpha|^{-1}_{\phi_1(U)}$, we have that $F_0|_{\phi_0(U)}$ is smooth

Since $a \in \phi_0(U_0 \cap F^{-1}(V_0))$ is arbitrary, we have that for each $a \in \phi_0(U_0 \cap F^{-1}(V_0))$, there exists $A \subset \phi_0(U_0 \cap F^{-1}(V_0))$ such that $a \in A$, A is open in $\phi_0(U_0 \cap F^{-1}(V_0))$ and $F_0|_A$ is smooth. An exercise in the section on differentiation on subspaces implies that F_0 is smooth.

Since $(U_0, \phi_0) \in \mathcal{A}_0$ and $(V_0, \psi_0) \in \mathcal{B}_0$ are arbitrary, we have that for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

Since $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ such that $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N are arbitrary, we have that for each $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, if $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N, then for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

2. (2) $\implies$ (3):
   Suppose that for each $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$, if $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N, then for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth. Let $p \in M$. Since $\mathcal{A}$ is an atlas on M and $\mathcal{B}$ is an atlas on N, there exists $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$ and $F(p) \in V$. By assumption, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

3. (3) $\implies$ (4):
   Suppose that for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

   - Let $p \in M$. By assumption, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth. Define $A \subset M$, $A_1 \subset \mathbb{H}^m$ and $F_1 : A_1 \to \mathbb{R}^n$ by $A := U \cap F^{-1}(V)$, $A_1 := \phi(A)$ and $F_1 := \psi \circ F \circ \phi|^{-1}_A$. Since F_1 is smooth, an exercise in the section on differentiability on subspaces implies that $F_1 : A_1 \to \mathbb{R}^n$ is continuous. Since $\phi|_A$ and ψ are homeomorphisms,

   $$\begin{aligned} F|_A &= \psi^{-1} \circ (\psi \circ F \circ \phi|_A) \circ \phi|^{-1}_A \\ &= \psi^{-1} \circ F_1 \circ \phi^{-1}_A \end{aligned}$$

   which is continuous. We note that $p \in A$ and A is open in M. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exists $A \subset M$ such that $p \in A$, A is open in M and $F|_A$ is continuous. Thus F is continuous.

- – By assumption, for each $p \in M$, there exists $(U_p, \phi_p) \in \mathcal{A}$ and $(V_p, \psi_p) \in \mathcal{B}$ such that $p \in U_p$, $F(p) \in V_p$, $U_p \cap F^{-1}(V_p)$ is open in M and $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(p)}$ is smooth. The axiom of choice implies that there exist $(U_p, \phi_p)_{p \in M} \subset \mathcal{A}$ and $(V_p, \psi_p)_{p \in M} \subset \mathcal{B}$ such that for each $p \in M$, $p \in U_p$, $F(p) \in V_p$, $U_p \cap F^{-1}(V_p)$ is open in M and $\psi_p \circ F \circ \phi_p|^{-1}_{U_p \cap F^{-1}(V_p)}$ is smooth. Define $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ by $\mathcal{A}_0 := (U_p, \phi_p)_{p \in M}$ and $\mathcal{B}_0 := (B_p, \psi_p)_{p \in M}$ respectively. By construction, $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N.
  – Let $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$. Define $\tilde{A} \subset \mathbb{H}^m$ and $\tilde{F} : \tilde{A} \to \mathbb{R}^n$ by $\tilde{A} = \phi(U \cap F^{-1}(V))$ and $\tilde{F} = \psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$. Since F is continuous, $U \cap F^{-1}(V)$ is open in M. Since ϕ is a homeomorphism, $\tilde{A}$ is open in $\mathbb{H}^n$. Let $a \in \tilde{A}$. Set $p := \phi^{-1}(a)$. Define $A \subset M$ by $A := U \cap U_p \cap F^{-1}(V \cap V_p)$. We note that $p \in A$ and since F is continuous, A is open in M. Define $A_0 \subset \mathbb{H}^m$ and $F_0 : A_0 \to \mathbb{R}^n$ by $A_0 = \phi_p(A)$ and $F_0 = \psi_p \circ F \circ \phi_p|^{-1}_A$. By construction, $\psi_p \circ F \circ \phi_p|^{-1}_{U_p \cap F^{-1}(V_p)}$ is smooth. An exercise about restriction in the section on differentation on subspaces implies that F_0 is smooth. We define $\alpha : \phi_p(U \cap U_p) \to \phi(U \cap U_p)$ and $\beta : \psi_p(V \cap V_p) \to \psi(V \cap V_p)$ by

$$\alpha := \phi|_{U \cap U_p} \circ \phi_p|^{-1}_{U \cap U_p}, \quad \beta := \psi|_{V \cap V_p} \circ \psi_p|^{-1}_{V \cap V_p}$$

  Since $\phi, \phi_p \in \mathcal{A}$, we know that ϕ and ϕ_p are smoothly compatible. Therefore α is a diffeomorphism. Similarly, β is a diffeomorphism. the restriction exercise again implies that $\alpha|_{A_0}$ is a diffeomorphism. Since $\tilde{F}|_{\phi(A)} = \beta \circ F_0 \circ \alpha|^{-1}_{A_0}$, we have that $\tilde{F}|_{\phi(A)}$ is smooth. We note that $a \in \phi(A)$, $\phi(A)$ is open in $\tilde{A}$. Since $a \in \tilde{A}$ is arbitrary, we have that for each $a \in \tilde{A}$, there exists $E \subset \tilde{A}$ such that $a \in E$, E is open in $\tilde{A}$ and $\tilde{F}|_E$ is smooth. An exercise in the section on differentiation on subspaces implies that $\tilde{F}$ is smooth. Since $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$ are arbitrary, we have that for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth.

4. (4) $\implies$ (1):
Suppose that F is continuous and there exist $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{B}_0 \subset \mathcal{B}$ such that $\mathcal{A}_0$ is an atlas on $\mathcal{A}$, $\mathcal{B}_0$ is an atlas on N and for each $(U, \phi) \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$, $\psi \circ F \circ \phi|^{-1}_{U \cap F^{-1}(V)}$ is smooth. Let $p \in M$. Since $\mathcal{A}_0$ is an atlas on M and $\mathcal{B}_0$ is an atlas on N, there exists $(U', \phi') \in \mathcal{A}_0$ and $(V, \psi) \in \mathcal{B}_0$ such that $p \in U'$ and $F(p) \in V$. Define $A_0 \subset \mathbb{H}^m$ and $F_0 : A_0 \to \mathbb{R}^n$ by $A_0 = \phi'(U' \cap F^{-1}(V))$ and $F_0 = \psi \circ F \circ \phi'|^{-1}_{U' \cap F^{-1}(V)}$. By assumption F_0 is smooth. Since F is continuous, $F(p) \in V$ and V is open in N, we have that there exists $U_0 \subset M$ such that $p \in U_0$, U_0 is open in M and $F(U_0) \subset V$. Define $U \subset M$ and $\phi : U \to \phi'(U)$ by $U := U' \cap U_0$ and $\phi = \phi'|_U$. Then $p \in U$, U is open in M and

$$\begin{aligned} F(U) &= F(U' \cap U_0) \\ &\subset F(U_0) \\ &\subset V \end{aligned}$$

An exercise in the section on smooth manifolds implies that $(U, \phi) \in \mathcal{A}$. Since F_0 is smooth, an exercise in the section on subspace differentiation implies that $F_0|_{\phi(U)}$ is smooth. Since $\psi \circ F \circ \phi^{-1} = F_0|_{\phi(U)}$, we have that $\psi \circ F \circ \phi^{-1}$. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$, $F(U) \subset V$ and $\psi \circ F \circ \phi^{-1}$ is smooth. Hence F is smooth.

□

Exercise 5.1.0.5. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ $(E, \mathcal{C})$be smooth manifolds and $F : M \to N$, $G : N \to E$. If F and G are smooth, then $G \circ F : M \to E$ is smooth.

Proof. Set $m = \dim M$, $n = \dim N$ and $e = \dim E$. Suppose that F and G are smooth. Let $p_0 \in M$. Since F is smooth, there exists $(U_0, \phi_0) \in \mathcal{A}$ and $(V_0, \psi_0) \in \mathcal{B}$ such that $p_0 \in U_0$, $F(p_0) \in V_0$, $F(U_0) \subset V_0$ and $\psi_0 \circ F \circ \phi_0^{-1}$ is smooth. Set $p_1 = F(p_0)$. Since G is smooth, there exists $(U_1, \phi_1) \in \mathcal{B}$ and $(V_1, \psi_1) \in \mathcal{C}$ such that $p_1 \in U_1$, $G(p_1) \in V_1$, $G(U_1) \subset V_1$ and $\psi_1 \circ F \circ \phi_1^{-1}$ is smooth. Define $f : \phi_0(U_0) \to \mathbb{H}^n$ and $g : \phi_1(U_1) \to \mathbb{H}^e$ by $f = \psi_0 \circ F \circ \phi_0^{-1}$ and $g = \psi_1 \circ G \circ \phi_1^{-1}$ respectively. Set $W_1 = U_1 \cap V_0$ and $W_0 = F^{-1}(W_1)$. Since W_1 is open in N and F is continuous, W_0 is open in M. An exercise in the section on open submanifolds implies that

$$\begin{aligned}(W_0, \phi_0|_{W_0}) &\in \mathcal{A}|_{W_0} \\ &\subset \mathcal{A}\end{aligned}$$

Since $p_1 \in W_1$, $p_0 \in W_0$. Furthermore,

$$\begin{aligned}G \circ F(p_0) &= G(p_1) \\ &\in V_1\end{aligned}$$

and

$$\begin{aligned}G \circ F(W_0) &= G(F(W_0)) \\ &\subset G(W_1) \\ &\subset G(U_1) \\ &\subset V_1\end{aligned}$$

Since $(U_1, \phi_1), (V_0, \psi_0) \in \mathcal{B}$, (U_1, ϕ_1) and (V_0, ψ_0) are smoothly-compatible. Thus $\phi_1|_{W_1} \circ \psi_0|_{W_1}^{-1} : \psi_0(W_1) \to \phi_1(W_1)$ is smooth. Since f and g are smooth, we have that $f|_{\phi_0(W_0)}$ is smooth and therefore

$$\begin{aligned}\psi_1 \circ (G \circ F) \circ \phi_0|_{W_0}^{-1} &= (\psi_1 \circ G \circ \phi_1|_{W_1}^{-1}) \circ (\phi_1|_{W_1} \circ \psi_0|_{W_1}^{-1}) \circ (\psi_0 \circ F \circ \phi_0|_{W_0}^{-1}) \\ &= g \circ (\phi_1|_{W_1} \circ \psi_0|_{W_1}^{-1}) \circ f|_{\phi_0(W_0)}\end{aligned}$$

is smooth. Since $p_0 \in M$ is arbitrary, we have that for each $p_0 \in M$, there exists $(W_0, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{C}$ such that $p_0 \in W_0$, $G \circ F(p_0) \in V$, $G \circ F(W_0) \subset V$ and $\psi \circ (G \circ F) \circ \phi^{-1}$ is smooth. Thus $G \circ F$ is smooth. $\square$

5.2 Smooth Maps on Open and Boundary Submanifolds

Exercise 5.2.0.1. Locality of Smoothness:
Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then the following are equivalent:

1. F is smooth
2. for each $U \subset M$, if U is open in M, then $F|_U : U \to N$ is smooth.
3. for each $p \in M$, there exists $U \subset M$ such that $p \in U$, U is open in M and $F|_U : U \to N$ is smooth.

Proof.

- (1) $\implies$ (2):
  Suppose that F is smooth. Let $U \subset M$. Suppose that U is open in M. Let $p \in U$. Since $\mathcal{A}|_U$ is an atlas on U and $\mathcal{B}$ is an atlas on N, there exist $(U_0, \phi_0) \in \mathcal{A}|_U$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U_0$ and $F(p) \in V$. Since $p \in U$, we have that

$$\begin{aligned} F|_U(p) &= F(p) \\ &\in V \end{aligned}$$

  An exercise in the section on open submanifolds implies that $\mathcal{A}|_U \subset \mathcal{A}$. Thus $(U_0, \phi_0) \in \mathcal{A}$. Since F is smooth a previous exercise implies that $U_0 \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \phi_0|_{U_0 \cap F^{-1}(V)}$ is smooth. Since $U_0 \subset U$, we have that

$$\begin{aligned} U_0 \cap F|_U^{-1}(V) &= U_0 \cap (U \cap F^{-1}(V)) \\ &= U_0 \cap F^{-1}(V) \end{aligned}$$

  and $\psi \circ F|_U \circ \phi_0|^{-1}_{U_0 \cap F|_U^{-1}(V)} = \psi \circ F \circ \phi_0|^{-1}_{U_0 \cap F^{-1}(V)}$. Thus $U_0 \cap F|_U^{-1}(V)$ is open in U and $\psi \circ F|_U \circ \phi_0|^{-1}_{U_0 \cap F|_U^{-1}(V)}$ is smooth. Since $p \in U$ is arbitrary, we have that for each $p \in U$, there exists $(U_0, \phi_0) \in \mathcal{A}|_U$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U_0$, $F|_U(p) \in V$, $U_0 \cap F|_U^{-1}(V)$ is open in U and $\psi \circ F|_U \circ \phi_0|^{-1}_{U_0 \cap F|_U^{-1}(V)}$ is smooth. (3) in smooth equivalence implies that $F|_U$ is smooth. Since $U \subset M$ with U open in M is arbitrary, we have that for each $U \subset M$, if U is open in M, then $F|_U : U \to N$ is smooth.

- (2) $\implies$ (3):
  Suppose that for each $U \subset M$, if U is open in M, then $F|_U : U \to N$ is smooth. Let $p \in M$. Since $\mathcal{A}$ is an atlas on M, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Since $(U, \phi) \in X(M)$, U is open in M. By assumption, $F|_U : U \to N$ is smooth. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exists $U \subset M$ such that $p \in U$, U is open in M and $F|_U : U \to N$ is smooth.

- (3) $\implies$ (1):
  Suppose that for each $p \in M$, there exists $U \subset M$ such that $p \in U$, U is open in M and $F|_U : U \to N$ is smooth. Let $p \in M$. Let $p \in M$. By assumption, there exists $U \subset M$ such that $p \in U$, U is open in M and $F|_U : U \to N$ is smooth. Since $F|_U$ is smooth, there exist $(U', \phi) \in \mathcal{A}|_U$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U'$, $F(p) \in V$, $F|_U(U') \subset V$ and $\psi \circ F|_U \circ \phi^{-1}$ is smooth. An exercise in the section on open submanifolds implies that $\mathcal{A}|_U \subset \mathcal{A}$. Thus $(U', \phi) \in \mathcal{A}$. Since $p \in M$ is arbitrary, we have that for each $p \in M$, there exists $(U', \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U'$, $F(p) \in V$, $F(U') \subset V$ and $\psi \circ F \circ \phi^{-1}$ is smooth. Thus F is smooth.

$\square$

Exercise 5.2.0.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $U \subset M$ and $F : M \to N$. Suppose that U is open in M. If F is a diffeomorphism, then $F|_U : U \to F(U)$ is a diffeomorphism.

Proof. Suppose that F is a diffeomorphism. Then F and F^{-1} are smooth. Hence F is a homeomorphism and $F(U)$ is open in N., By definition, F and F^{-1} are smooth. A previous exercise about locality of smoothness implies that $F|_U$ and $F^{-1}|_{F(U)}$ are smooth. Since $F|_U^{-1} = F^{-1}|_{F(U)}$, $F|_U$ is a diffeomorphism. $\square$

Exercise 5.2.0.3. Let $(M, \mathcal{A})$ be a smooth manifold and $(U, \phi) \in \mathcal{A}$. Then $\phi : U \to \phi(U)$ is a diffeomorphism.

Proof. Set $n := \dim M$. Let $(V, \psi) \in \mathcal{A}$. By definition, ϕ is continuous. Since $(U, \phi), (V, \psi) \in \mathcal{A}$, we have that (U, ϕ) and (V, ψ) are smoothly compatible. Hence $\phi|_{U\cap V} \circ \psi|^{-1}_{U\cap V}$ is a diffeomorphism. Define $\alpha : \psi(U\cap V) \to \phi(U \cap V)$ by $\alpha = \phi|_{U\cap V} \circ \psi|^{-1}_{U\cap V}$. Since $V \cap \phi^{-1}(\phi(U)) = U \cap V$ and $\phi(U) \cap (\phi^{-1})^{-1}(V) = \phi(U \cap V)$, we have that $V \cap \phi^{-1}(\phi(U))$ and $\phi(U) \cap (\phi^{-1})^{-1}(V)$ are open. Furthermore,

$$\begin{aligned} \mathrm{id}_{\phi(U)} \circ \phi \circ \psi|^{-1}_{V\cap\phi^{-1}(\phi(U))} &= \mathrm{id}_{\phi(U)} \circ \phi \circ \psi|^{-1}_{V\cap U} \\ &= \mathrm{id}_{\phi(U)} \circ \alpha \\ &= \alpha \end{aligned}$$

and

$$\begin{aligned} \psi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} |_{\phi(U)\cap(\phi^{-1})^{-1}(V)} &= \psi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} |_{\phi(U\cap V)} \\ &= \alpha^{-1} \circ \mathrm{id}_{\phi(U\cap V)} \\ &= \alpha^{-1} \end{aligned}$$

Since α is a diffeomorphism, we have that $\mathrm{id}_{\phi(U)} \circ \phi \circ \psi|^{-1}_{V\cap\phi^{-1}(\phi(U))}$ and $\psi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} |_{\phi(U)\cap(\phi^{-1})^{-1}(V)}$ are smooth. Since $\mathcal{A}_{\phi(U)} = \alpha(\mathrm{id}_{\phi(U)})$, $\mathcal{A} = \alpha(\mathcal{A})$ and $(V, \psi) \in \mathcal{A}$ is arbitrary, a previous exercise about smoothness depending on a smooth atlas implies that ϕ and ϕ^{-1} are smooth. Hence ϕ is a diffeomorphism. □

Exercise 5.2.0.4. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$ a diffeomorphism. Then

1. for each $(V, \psi) \in \mathcal{B}$, $(F^{-1}(V), \psi \circ F|_{F^{-1}(V)}) \in \mathcal{A}$
2. for each $(U, \phi) \in \mathcal{A}$, $(F(U), \phi \circ F|^{-1}_{F(U)}) \in \mathcal{B}$

Proof. Set $n := \dim M$.

1. Let $(V, \psi) \in \mathcal{B}$. Since $F^{-1}(V)$ is open in M, a previous exercise implies that $F|^{-1}_{F^{-1}(V)}$ is a diffeomorphism. A previous exercise implies that ψ is a diffeomorphism. Therefore $\psi \circ F|^{-1}_{F^{-1}(V)}$ is a diffeomorphism.
   (a) Since $(V, \psi) \in \mathcal{B}$ and $F|^{-1}_{F^{-1}(V)}$ is a homeomorphism, we have that
   - $F^{-1}(V)$ is open in M.
   - $\psi(V)$ is open in $\mathbb{H}^n$
   - $\psi \circ F|_{F^{-1}(V)} : F^{-1}(V) \to \psi(V)$ is a homeomorphism

   So $(F^{-1}(V), \psi \circ F|_{F^{-1}(V)}) \in X^n(M)$.

   (b) Let $(U, \phi) \in \mathcal{A}$. A previous exercise implies that ψ is a diffeomorphism. A previous exercise implies that $\phi|_{U\cap F^{-1}(V)}$ and $\psi \circ F|_{U\cap F^1(V)}$ are diffeomorphisms. Hence $(\psi \circ F|^{-1}_F(V))|_{U\cap F^{-1}(V)} \circ \phi|^{-1}_{U\cap F^{-1}(V)}$ is a diffeomorphism. Therefore $(F(U), \psi \circ F|^{-1}_{F^{-1}(V)})$ and (V, ψ) are smoothly compatible. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{A}$, (U, ϕ) and $(F^{-1}(V), \psi \circ F|_{F^{-1}(V)})$ are smoothly compatible. Since $\mathcal{A}$ is maximal, $(F^{-1}(V), \psi \circ F^{-1}) \in \mathcal{A}$.
2. Similar to (1).

□

Exercise 5.2.0.5. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then F is smooth iff for each $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$, $y^i \circ F$ is smooth.

Proof. Suppose that F is smooth. Let $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. Then for each $i \in \{1, \dots, n\}$, F^i is smooth.
Conversely, suppose that for each $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$ and $i \in \{1, \dots, n\}$, $y^i \circ F$ is smooth. □

Definition 5.2.0.6. Let $(N, \mathcal{B})$ be a smooth n-dimensional manifold, $F : M \to N$ smooth and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \dots, y^n)$. For $i \in \{1, \dots, n\}$, We define the ***i*-th component of F with respect to** (V, ψ), denoted $F^i : V \to \mathbb{R}$, by

$$F^i = y^i \circ F$$

Exercise 5.2.0.7. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $p \in U$ and $f \in C^\infty(M, \mathcal{A})$. Then $f|_U \in C^\infty(U, \mathcal{A}|_U)$.

Proof. Let $\square$

5.3 Smooth Maps and Product Manifolds

Exercise 5.3.0.1. Let $(M,\mathcal{A})$, $(N,\mathcal{B})$, $(E,\mathcal{C})$ be smooth manifolds, $\mathcal{A}_0 \subset \mathcal{A}$, $\mathcal{B}_0 \subset \mathcal{B}$, $\mathcal{C}_0 \subset \mathcal{C}$ and $F : M \times N \to E$. Suppose that $\partial N = \varnothing$, $\mathcal{A}_0$ is an atlas on M, $\mathcal{B}_0$ is an atlas on N and $\mathcal{C}_0$ is an atlas on E. Then F is smooth iff for each $(U,\phi) \in \mathcal{A}_0$, $(V,\psi) \in \mathcal{B}_0$, $(W,\chi) \in \mathcal{C}_0$, $\chi \circ F \circ (\phi \times \psi)|^{-1}_{(U\times V)\cap F^{-1}(W)}$ is smooth.

Proof. Set $m := \dim M$, $n = \dim N$ and $e = \dim E$.

- $(\implies)$:
  Suppose that F is smooth. Let $(U,\phi) \in \mathcal{A}_0$, $(V,\psi) \in \mathcal{B}_0$ and $(W,\chi) \in \mathcal{C}_0$. An exercise in the section on product manifolds implies that there exists $\lambda : \mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n \to \mathbb{H}^{m+n}$ such that λ is a diffeomorphism and $\lambda(\partial\mathbb{H}^m \times \operatorname{Int}\mathbb{H}^n) = \partial\mathbb{H}^{m+n}$. Set $\eta := \lambda|_{\phi(U)\times\psi(V)} \circ (\phi \times \psi)$. An exercise in the section on product manifolds implies that $\eta \in \mathcal{A} \otimes \mathcal{B}$. Since F is smooth and exercise in the section on smooth maps implies that $\chi \circ F \circ \eta|^{-1}_{(U\times V)\cap F^{-1}(W)}$ is smooth. Since

  $$\chi \circ F \circ (\phi \times \psi)|^{-1}_{(U\times V)\cap F^{-1}(W)} = \chi \circ F \circ \eta|^{-1}_{(U\times V)\cap F^{-1}(W)} \circ \lambda|_{(\phi(U)\times\phi(V))\cap(\phi\times\psi)(F^{-1}(W))}$$

  we have that $\chi \circ F \circ (\phi \times \psi)|^{-1}_{(U\times V)\cap F^{-1}(W)}$ is smooth.

  Since $(U,\phi) \in \mathcal{A}_0$, $(V,\psi) \in \mathcal{B}_0$ and $(W,\chi) \in \mathcal{C}_0$ are arbitrary, we have that for each $(U,\phi) \in \mathcal{A}_0$, $(V,\psi) \in \mathcal{B}_0$, $(W,\chi) \in \mathcal{C}_0$, $\chi \circ F \circ (\phi \times \psi)|^{-1}_{(U\times V)\cap F^{-1}(W)}$ is smooth.

- $(\impliedby)$:
  Suppose that for each $(U,\phi) \in \mathcal{A}_0$, $(V,\psi) \in \mathcal{B}_0$, $(W,\chi) \in \mathcal{C}_0$, $\chi \circ F \circ (\phi \times \psi)|^{-1}_{(U\times V)\cap F^{-1}(W)}$ is smooth. Let $(U,\phi) \in \mathcal{A}_0$, $(V,\psi) \in \mathcal{B}_0$, $(W,\chi) \in \mathcal{C}_0$ and $\lambda \in \operatorname{Aut}^{m,n}_\infty$.

  Set $G := \chi \circ F \circ (\phi \times \psi)|^{-1}_{(U\times V)\cap F^{-1}(W)}$. By assumption, G is smooth. Since λ is a diffeomorphism, $G \circ \lambda|^{-1}_{(\phi(U)\times\psi(V))\cap(\phi\times\psi)(F^{-1}(V))}$ is smooth. Since

  $$\chi \circ F \circ [\lambda|_{\phi(U)\times\psi(V)} \circ (\phi \times \psi)]|^{-1}_{(U\times V)\cap F^{-1}(W)} = G \circ \lambda|^{-1}_{(\phi(U)\times\psi(V))\cap(\phi\times\psi)(F^{-1}(V))}$$

  we have that $\chi \circ F \circ [\lambda|_{\phi(U)\times\psi(V)} \circ (\phi \times \psi)]|^{-1}_{(U\times V)\cap F^{-1}(W)}$ is smooth.

  An exercise in the section on products of manifolds implies that $\{(U \times V, \lambda|_{\phi(U)\times\psi(V)} \circ (\phi \times \psi)) : \lambda \in \operatorname{Aut}^{m,n}_\infty, (U,\phi) \in \mathcal{A}_0 \text{ and } (V,\psi) \in \mathcal{B}_0\}$ is an atlas on $M \times N$.

  Since $(U,\phi) \in \mathcal{A}_0$, $(V,\psi) \in \mathcal{B}_0$, $(W,\chi) \in \mathcal{C}_0$ and $\lambda \in \operatorname{Aut}^{m,n}_\infty$ are arbitrary, an exercise in the section on smooth maps implies that F is smooth.

$\square$

Definition 5.3.0.2. Let $(M,\mathcal{A})$ and $(N,\mathcal{B})$ be smooth manifolds. We define the **projection maps onto M and N**, denoted by $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ respectively, by

- $\pi_M(p,q) = p$
- $\pi_N(p,q) = q$

Exercise 5.3.0.3. Let M and N be smooth manifolds. Then $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are smooth.

Proof. Set $m = \dim M$ and $n = \dim N$. Let (U,ϕ), $(U',\phi') \in \mathcal{A}_M$ and $(V,\psi) \in \mathcal{A}_N$. Then for each $(a,b) \in \phi(U) \times \psi(V)$

$$\begin{aligned}\phi'|_{U'\cap U} \circ \pi_M \circ [\phi \times \psi]^{-1}|_{\phi(U)\times\psi(V)}(a,b) &= \phi'|_{U'\cap U} \circ \pi_M \circ [\phi|^{-1}_{\phi(U)} \times \psi|^{-1}_{\psi(V)}](a,b) \\ &= \phi' \circ \phi^{-1}(a)\end{aligned}$$

Since $(a,b) \in \phi(U) \times \psi(V)$ is arbitrary,

$$\phi'|_{U'\cap U} \circ \pi_M \circ [\phi \times \psi]^{-1}|_{\phi(U\cap U')\times\psi(V)} = \phi'|_{U'\cap U} \circ \phi|^{-1}_{U'\cap U} \circ \operatorname{proj}_1|_{\phi(U\cap U')\times\psi(V)}$$

where $\text{proj}_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the usual projection map. Since (U, ϕ), $(U', \phi') \in \mathcal{A}_M$, (U, ϕ) and (U', ϕ') are smoothly compatible. Hence $\phi'|_{U \cap U'} \circ \phi|^{-1}_{U \cap U'}$ is smooth and therefore $\phi'|_{U' \cap U} \circ \pi_M \circ [\phi \times \psi]^{-1}|_{\phi(U) \times \psi(V)}$ is smooth. Since fix here and $(V, \psi) \in \mathcal{A}_N$ are arbitrary, we have that $\pi_M : M \times N \to M$ is smooth. we have that (U, ϕ) and (U', ϕ') are smoothly compatible. Thus $\phi'|_{U \cap U'} \circ \phi^{-1}|^{-1}_{U \cap U'}$ is smooth. □

Definition 5.3.0.4. Let M and N be smooth manifolds and $(p, q) \in M \times N$. We define the **slice maps at** q **and** p, denoted by $\iota^M_q : M \to M \times N$ and $\iota^N_p : N \to M \times N$ respectively, by

- $\iota^M_q(a) = (a, q)$
- $\iota^N_p(b) = (p, b)$

Exercise 5.3.0.5. Let M and N be smooth manifolds and $(p, q) \in M \times N$. Then $\iota^M_q : M \to M \times N$ and $\iota^N_p : N \to M \times N$ are smooth

1. $\iota^M_q : M \to M \times N$ and $\iota^N_p : N \to M \times N$ are smooth
2.

Proof. Let () □

Definition 5.3.0.6. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. We define the **canonical projection maps**, denoted $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$, by $\pi_M(p, q) = p$ and $\pi_N(p, q) = q$ respectively.

Exercise 5.3.0.7. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. Then $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are smooth.

Proof. Let $(U_0, \phi_0) \in \mathcal{A}$, $(V_0, \psi_0), (V_1, \psi_1) \in \mathcal{B}$ and $\lambda \in \text{Aut}_\infty^{m,n}$. Then for each $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$, we have that

$$\begin{aligned}
\psi_1 \circ \pi_M \circ (\lambda \circ [\phi_0 \times \psi_0])^{-1}(a, b) &= \psi_1 \circ \pi_M \circ [\phi_0^{-1} \times \psi_0^{-1}] \circ \lambda^{-1}(a, b) \\
&= \psi_1 \circ \phi_0^{-1} \circ \lambda^{-1}(a) \\
&= \psi_1 \circ \phi_0^{-1} \circ \lambda^{-1} \circ \pi_{\mathbb{R}^m}(a, b)
\end{aligned}$$

□

Exercise 5.3.0.8. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ and $(E, \mathcal{C})$ be smooth manifolds and $F : E \to M \times N$. Then F is smooth iff $\pi_M \circ F$ is smooth and $\pi_N \circ F$ is smooth.

Proof.

- ($\Longrightarrow$):
  Suppose that F is smooth.
- ($\Longleftarrow$):

□

Definition 5.3.0.9. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ be smooth manifolds and $(p_0, q_0) \in M \times N$. We define the **product inclusion maps**, denoted $\iota_{q_0}^M : M \to M \times N$ and $\iota_{p_0}^N : N \to M \times N$, by $\iota_{q_0}^M(p) = (p, q_0)$ and $\iota_{p_0}^N(q) = (p_0, q)$ respectively.

Exercise 5.3.0.10. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ be smooth manifolds and $(p_0, q_0) \in M \times N$. Then $\iota_{q_0}^M$ and $\iota_{p_0}^N$ are smooth.

Proof. □

5.4 Partitions of Unity

Definition 5.4.0.1. Let $p \in M$, $U \in \mathcal{N}_a$ open and $\rho \in C_c^\infty(M)$. Then ρ is said to be a **bump function at p supported in** U if

1. $\rho \geq 0$
2. there exists $V \in \mathcal{N}_p$ such that V is open and $\rho|_V = 1$
3. $\operatorname{supp} \rho \subset U$

Exercise 5.4.0.2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & t \in (-1,1) \\ 0 & t \notin (-1,1) \end{cases}$$

Then $f \in C_c^\infty(\mathbb{R})$.

Proof. □

5.5 Smooth Functions on Manifolds

Definition 5.5.0.1. Let $(M, \mathcal{A})$ be a smooth manifold and $f : M \to \mathbb{R}$. Then f is said to be **smooth** if for each $(U, \phi) \in \mathcal{A}$, $f \circ \phi^{-1}$ is smooth. The set of all smooth functions on M is denoted $C^\infty(M, \mathcal{A})$.

Note 5.5.0.2. When the context is clear, we write $C^\infty(M)$ in place of $C^\infty(M, \mathcal{A})$.

Exercise 5.5.0.3. Let $(M, \mathcal{A})$ be a smooth manifold and $f : M \to \mathbb{R}$. Then f is smooth iff f is $(\mathcal{A}, \mathcal{A}_\mathbb{R})$-smooth.

Proof.

- $(\implies)$:
Suppose that f is smooth. Let $(U, \phi) \in \mathcal{A}$. Since $\text{id}_\mathbb{R} \circ f \circ \phi^{-1} = f \circ \phi^{-1}$ and $f \circ \phi^{-1}$ is smooth, we have that $\text{id}_\mathbb{R} \circ f \circ \phi^{-1}$ is smooth. Since $\mathcal{A} = \alpha(\mathcal{A})$ and $\mathcal{A}_\mathbb{R} = \alpha((\mathbb{R}, \text{id}_\mathbb{R}))$, an exercise in the section on smooth maps implies that f is $(\mathcal{A}, \mathcal{A}_\mathbb{R})$-smooth.

- $(\impliedby)$:
Suppose that f is $(\mathcal{A}, \mathcal{A}_\mathbb{R})$-smooth. Let $(U, \phi) \in \mathcal{A}$. Since $(\mathbb{R}, \text{id}_\mathbb{R}) \in \mathcal{A}_\mathbb{R}$ and $f \circ \phi^{-1} = \text{id}_\mathbb{R} \circ f \circ \phi^{-1}$, we have that $f \circ \phi^{-1}$ is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, we have that f is smooth.

□

Note 5.5.0.4. When the context is clear, we write $C^\infty(M, \mathcal{A})$ in place of $C^\infty(M)$.

Exercise 5.5.0.5. Let $(M, \mathcal{A})$ be a smooth manifold, $\mathcal{A}_0 \subset \mathcal{A}$. Suppose that $\mathcal{A}_0$ is an atlas on M and $f : M \to \mathbb{R}$. Then f is smooth iff for each $(U, \phi) \in \mathcal{A}_0$, $f \circ \phi^{-1}$ is smooth.

Proof.

- $(\implies)$:
Suppose that f is smooth. Let $(U, \phi) \in \mathcal{A}_0$. Since $\mathcal{A}_0 \subset \mathcal{A}$, $(U, \phi) \in \mathcal{A}$. Since f is smooth, $f \circ \phi^{-1}$ is smooth. Since $(U, \phi) \in \mathcal{A}_0$ is arbitrary, we have that for each $(U, \phi) \in \mathcal{A}_0$, $f \circ \phi^{-1}$ is smooth.

- $(\impliedby)$:
Suppose that for each $(U, \phi) \in \mathcal{A}_0$, $f \circ \phi^{-1}$ is smooth. Then for each $(U, \phi) \in \mathcal{A}_0$, $\text{id}_\mathbb{R} \circ f \circ \phi^{-1}$ is smooth. Since $\mathcal{A} = \alpha(\mathcal{A}_0)$ and $\mathcal{A}_\mathbb{R} = \alpha(\mathbb{R}, \text{id}_\mathbb{R})$, an exercise in the section on smooth maps implies that f is $(\mathcal{A}, \mathcal{A}_\mathbb{R})$-smooth. A previous exercise implies that f is smooth.

□

Exercise 5.5.0.6. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$. Then F is smooth iff F is continuous and for each $g \in C^\infty(N)$, $g \circ F$ is smooth.

Proof.

- $(\implies)$:
Suppose that F is smooth. Then F is continuous. Let $g \in C^\infty(N)$. Then $g \circ F$ is smooth. Since $g \in C^\infty(N)$ is arbitrary, we have that for each $g \in C^\infty(N)$, $g \circ F$ is smooth.

- $(\impliedby)$:
Suppose that F is continuous and for each $g \in C^\infty(N)$, $g \circ F$ is smooth. Let $p \in U$.

  Let $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$. Set $W = U \cap F^{-1}(V)$. Since F is continuous, W is open in M. Define $G : W \to V$ by $G := F|_W$. FINISH!!!, maybe use bump functions to go from a smooth g on V to N

□

Exercise 5.5.0.7. Let M be a smooth manifold. Then $C^\infty(M)$ is a vector space.

Proof. Let $f, g \in C^\infty(M)$, $\lambda \in \mathbb{R}$ and $(U, \phi) \in \mathcal{A}$. By assumption, $f \circ \phi^{-1}$ and $g \circ \phi^{-1}$ are smooth. Hence

$$(f + \lambda g) \circ \phi^{-1} = f \circ \phi^{-1} + \lambda g \circ \phi^{-1}$$

is smooth. Since $(U, \phi) \in \mathcal{A}$ is arbitrary, $f + \lambda g \in C^\infty(M)$. Since $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$ are arbitrary, $C^\infty(M)$ is a vector space. □

Definition 5.5.0.8. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. We define the **partial derivative of f with respect to x^i**, denoted

$$\partial f / \partial x^i : U \to \mathbb{R} \text{ or } \partial_i f : U \to \mathbb{R}$$

by

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi(p))$$

or equivalently,

$$\frac{\partial f}{\partial x^i} = \left(\frac{\partial}{\partial u^i}[f \circ \phi^{-1}]\right) \circ \phi$$

Exercise 5.5.0.9. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i \in \{1, \cdots, n\}$. Then $\partial/\partial x^i : C^\infty(U) \to C^\infty(U)$ is linear.

Proof. **FINISH!!!** □

Exercise 5.5.0.10. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$, $f \in C^\infty(U)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f = \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f &= \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial x^j} f\right) \\
&= \frac{\partial}{\partial x^i}\left(\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right] \circ \phi\right) \\
&= \left(\frac{\partial}{\partial u^i}\left[\left(\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right] \circ \phi\right) \circ \phi^{-1}\right]\right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i}\left[\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right]\right) \circ \phi \\
&= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}]\right) \circ \phi
\end{aligned}$$

□

Exercise 5.5.0.11. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $i, j \in \{1, \cdots, n\}$. Then

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}$$

Proof. Let $f \in C^\infty(U)$. Since $f \circ \phi^{-1}$ is smooth,

$$\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f \circ \phi^{-1}] = \frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f \circ \phi^{-1}]$$

The previous exercise implies that

$$\begin{aligned}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}f &= \left(\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j}[f\circ\phi^{-1}]\right)\circ\phi \\ &= \left(\frac{\partial}{\partial u^j}\frac{\partial}{\partial u^i}[f\circ\phi^{-1}]\right)\circ\phi \\ &= \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}f\end{aligned}$$

□

Exercise 5.5.0.12. Let $(M,\mathcal{A})$ be a smooth manifold, $(U,\phi)\in\mathcal{A}$ with $\phi=(x^1,\cdots,x^n)$ and $f\in C^\infty(U)$. Then for each $\alpha\in\mathbb{N}_0^n$,

$$\partial^\alpha f = (\partial^\alpha[f\circ\phi^{-1}])\circ\phi$$

Proof. The claim is clearly true when $|\alpha|=0$ or by definition if $|\alpha|=1$. Let $n\in\mathbb{N}$ and suppose the claim is true for each $|\alpha|\in\{1,\ldots,n-1\}$. Then there exists $i\in\{1,\ldots,n\}$ such that $\alpha_i\geq 1$. Hence

$$\begin{aligned}\partial^\alpha f &= \partial^{e^i}(\partial^{\alpha-e^i}f) \\ &= \partial^{e^i}(\partial^{\alpha-e^i}[f\circ\phi^{-1}]\circ\phi) \\ &= (\partial^{e^i}[(\partial^{\alpha-e^i}[f\circ\phi^{-1}]\circ\phi)\circ\phi^{-1}])\circ\phi \\ &= (\partial^{e^i}[\partial^{\alpha-e^i}[f\circ\phi^{-1}]])\circ\phi \\ &= (\partial^\alpha[f\circ\phi^{-1}])\circ\phi\end{aligned}$$

□

Exercise 5.5.0.13. Taylor's Theorem:
Let $(M,\mathcal{A})$ be a smooth manifold, $(U,\phi)\in\mathcal{A}$ with $\phi=(x^1,\ldots,x^n)$ and $\phi(U)$ convex, $p\in U$, $f\in C^\infty(U)$ and $T\in\mathbb{N}$. Then there exist $(g_\alpha)_{|\alpha|=T+1}\subset C^\infty(U)$ such that

$$f=\sum_{k=0}^{T}\left[\sum_{|\alpha|=k}(x-p)^\alpha\partial^\alpha f(x_0)\right]+\sum_{|\alpha|=T+1}(x^i-x^i(p))^\alpha g_\alpha$$

and for each $|\alpha|=T+1$,

$$g_\alpha(p)=\frac{1}{(T+1)!}\partial^\alpha f(p)$$

Proof. Since $\phi(U)$ is open and convex and $f\circ\phi^{-1}\in C^\infty(\phi(U))$, Taylors thorem in section 2.1 implies that there exist $(\tilde g_\alpha)_{|\alpha|=T+1}\subset C^\infty(\phi(U))$ such that for each $q\in U$,

$$f\circ\phi^{-1}(\phi(q))=\sum_{k=0}^{T}\left[\sum_{|\alpha|=k}(x^i(q)-x^i(p))^\alpha\partial^\alpha[f\circ\phi^{-1}](\phi(p))\right]+\sum_{|\alpha|=T+1}(x^i(q)-x^i(p))^\alpha\tilde g_\alpha(\phi(q))$$

and for each $|\alpha|=T+1$,

$$\begin{aligned}\tilde g_\alpha(\phi(p)) &= \frac{1}{(T+1)!}\partial^\alpha[f\circ\phi^{-1}](\phi(p)) \\ &= \frac{1}{(T+1)!}\partial^\alpha f(p)\end{aligned}$$

For $|\alpha| = T+1$, set $g_\alpha = \tilde{g} \circ \phi$. Then

$$\begin{aligned} f(q) &= f \circ \phi^{-1}(\phi(q)) \\ &= \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha [f \circ \phi^{-1}](\phi(p)) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha \tilde{g}_\alpha(\phi(q)) \\ &= \sum_{k=0}^{T} \left[\sum_{|\alpha|=k} (x^i(q) - x^i(p))^\alpha \partial^\alpha f(p) \right] + \sum_{|\alpha|=T+1} (x^i(q) - x^i(p))^\alpha g_\alpha(q) \end{aligned}$$

□

Chapter 6

The Tangent and Cotangent Spaces

6.1 The Tangent Space

Definition 6.1.0.1. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. For $i \in \{1, \cdots, n\}$, define the partial derivative with respect to x^i at p, denoted

$$\frac{\partial}{\partial x^i}\bigg|_p : C^\infty(M) \to \mathbb{R}, \text{ or } \partial_i|_p : C^\infty(M) \to \mathbb{R}$$

by

$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}(p)$$

Exercise 6.1.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$, we have that

$$\frac{\partial}{\partial x^i} x^j(p) = \delta_{i,j}$$

Proof. Let $i, j \in \{1, \cdots, n\}$. Then

$$\begin{aligned}
\frac{\partial}{\partial x^i}\bigg|_p x^i &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} x^i \circ \phi^{-1} \\
&= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} u^i \circ \phi \circ \phi^{-1} \\
&= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} u^i \\
&= \delta_{i,j}
\end{aligned}$$

$\square$

Exercise 6.1.0.3. Change of Coordinates:
Let $(U, \phi), (V, \psi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $\psi = (y^1, \cdots, y^n)$, $p \in U \cap V$ and $f \in C^\infty(M)$. Then for each $i \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial y^i}\bigg|_p = \sum_{j=1}^n \frac{\partial}{\partial x^j} y^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

Proof. Put $h = \phi \circ \psi^{-1}$ and write $h = (h_1, \cdots, h_n)$. Then $\phi = h \circ \psi$ and $\psi^{-1} = \phi^{-1} \circ h$. By definition and the chain rule, we have that

$$\begin{aligned}
\frac{\partial}{\partial y^i}\bigg|_p f &= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \psi^{-1} \\
&= \frac{\partial}{\partial u^i}\bigg|_{\psi(p)} f \circ \phi^{-1} \circ h \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial u^j}\bigg|_{h\circ\psi(p)} f \circ \phi^{-1}\right)\left(\frac{\partial}{\partial u^i}\bigg|_{\psi(p)} h_j\right) \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial u^j}\bigg|_{\phi(p)} f \circ \phi^{-1}\right)\left(\frac{\partial}{\partial u^i}\bigg|_{\psi(p)} x^j \circ \psi^{-1}\right) \\
&= \sum_{j=1}^n \left(\frac{\partial}{\partial x^i}\bigg|_p f\right)\left(\frac{\partial}{\partial y^i}\bigg|_p x^j\right)
\end{aligned}$$

□

Definition 6.1.0.4. Let $p \in M$ and $v : C^\infty(M) \to \mathbb{R}$. Then v is said to be **Leibnizian** if for each $f, g \in C^\infty(M)$,

$$v(fg) = v(f)g(p) + f(p)v(g)$$

and v is said to be a **derivation at** p if for each $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$,

1. v is linear
2. v is Leibnizian

We define the **tangent space of** M **at** p, denoted T_pM, by

$$T_pM = \{v : C^\infty(M) \to \mathbb{R} : v \text{ is a derivation at } p\}$$

Exercise 6.1.0.5. Let $f \in C^\infty(M)$ and $v \in T_pM$. If f is constant, then $vf = 0$.

Proof. Suppose that $f = 1$. Then $f^2 = f$ and $v(f^2) = 2v(f)$. So $v(f) = 2v(f)$ which implies that $v(f) = 0$. If $f \neq 1$, then there exists $c \in \mathbb{R}$ such that $f = c$. Since v is linear, $v(f) = cv(1) = 0$. □

Exercise 6.1.0.6. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$\left\{\frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p\right\}$$

is a basis for T_pM and $\dim T_pM = n$.

Proof. Clearly $\frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \in T_pM$. Let $a_1, \cdots, a_n \in \mathbb{R}$. Suppose that

$$v = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p = 0$$

Then

$$\begin{aligned}
0 &= vx^j \\
&= \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_p x^j \\
&= a_j
\end{aligned}$$

Hence $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is independent.
Now, let $v \in T_pM$ and $f \in \mathbb{C}^\infty(M)$. By Taylor's theorem, there exist $g_1, \cdots g_n \in C_p^\infty(M)$ such that

$$f = f(p) + \sum_{i=1}^n (x^i - x^i(p))g_i$$

and for each $i \in \{1, \cdots, n\}$,

$$g_i(p) = \frac{\partial}{\partial x^i}\bigg|_p f$$

Then

$$\begin{aligned} v(f) &= \sum_{i=1}^n v(x^i - x^i(p))g_i(p) + \sum_{i=1}^n (x^i(p) - x^i(p))v(g_i) \\ &= \sum_{i=1}^n v(x^i)g_i(p) \\ &= \sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p f \\ &= \left[\sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p\right] f \end{aligned}$$

So

$$v = \sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}\bigg|_p$$

and

$$v \in \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

$\square$

Definition 6.1.0.7. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. We define the **differential of** F **at** p, denoted $DF_p : T_pM \to T_{F(p)}N$, by

$$\Big[DF_p(v)\Big](f) = v(f \circ F)$$

for $v \in T_pM$ and $f \in C^\infty(N)$.

Exercise 6.1.0.8. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. Then for each $v \in T_pM$, $DF_p(v)$ is a derivation.

Proof. Let $v \in T_pM$, $f, g \in C^\infty_{F(p)}(N)$ and $c \in \mathbb{R}$. Then

1.

$$\begin{aligned} DF_p(v)(f + cg) &= v((f + cg) \circ F) \\ &= v(f \circ F + cg \circ F) \\ &= v(f \circ F) + cv(g \circ F) \\ &= DF_p(v)(f) + cDF_p(v)(g) \end{aligned}$$

So $DF_p(v)$ is linear.

2.

$$\begin{aligned} DF_p(v)(fg) &= v(fg \circ F) \\ &= v((f \circ F) * (g \circ F)) \\ &= v(f \circ F) * (g \circ F)(p) + (f \circ F)(p) * v(g \circ F) \\ &= DF_p(v)(f) * g(F(p)) + f(F(p)) * DF_p(v)(g) \end{aligned}$$

So $DF_p(v)$ is Leibnizian and hence $DF_p(v) \in T_{F(p)}N$ □

Exercise 6.1.0.9. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ smooth and $p \in M$. If F is a diffeomorphism, then DF_p is an isomorphism.

Proof. Suppose that F is a diffeomorphism. Since F is a homeomorphism, $\dim N = n$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. A previous exercise tells us that $(F(U), \phi \circ F^{-1}) \in \mathcal{B}$. Write $\phi = (x^1, \cdots, x^n)$ and $\phi \circ F^{-1} = (y^1, \cdots, y^n)$. Let $f \in C^\infty(N)$ Then

$$\begin{aligned} \left.\frac{\partial}{\partial y^i}\right|_{F(p)} f &= \left.\frac{\partial}{\partial u^i}\right|_{\phi \circ F^{-1}(F(p))} f \circ (\phi \circ F^{-1})^{-1} \\ &= \left.\frac{\partial}{\partial u^i}\right|_{\phi(p)} f \circ F \circ \phi^{-1} \\ &= \left.\frac{\partial}{\partial x^i}\right|_{p} f \circ F \end{aligned}$$

Therefore

$$\begin{aligned} \left[DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)\right](f) &= \left.\frac{\partial}{\partial x^i}\right|_p f \circ F \\ &= \left.\frac{\partial}{\partial y^i}\right|_{F(p)} f \end{aligned}$$

Hence

$$DF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial y^i}\right|_{F(p)}$$

Since $\left\{\left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^n}\right|_p\right\}$ is a basis for T_pM and $\left\{\left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \cdots, \left.\frac{\partial}{\partial y^n}\right|_{F(p)}\right\}$ is a basis for $T_{F(p)}N$, DF_p is an isomorphism. □

Exercise 6.1.0.10. Let $(M, \mathcal{A})$ be a smooth m-dimensional manifold, $(N, \mathcal{B})$ a n-dimensional smooth manifold, $F : M \to N$ smooth, $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \ldots, x^m)$ and $(V, \psi) \in \mathcal{B}$ with $\psi = (y^1, \ldots, y^n)$. Suppose that $p \in U$ and $F(p) \in V$. Define the ordered bases $B_\phi = \left\{\left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^m}\right|_p\right\}$ and $B_\psi = \left\{\left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \cdots, \left.\frac{\partial}{\partial y^n}\right|_{F(p)}\right\}$. Then the matrix representation of DF_p with respect to the bases B_ϕ and B_ψ is

$$DF_p^{i,j} = \frac{\partial F^i}{\partial x^j}(p)$$

Proof. Let $(DF_p)_{B_\phi, B_\psi} = (a_{i,j})_{i,j} \in \mathbb{R}^{n \times m}$. Then for each $j \in \{1, \ldots, m\}$,

$$DF_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{i=1}^n a_{i,j} \left.\frac{\partial}{\partial y^i}\right|_{F(p)}$$

This implies that

$$\begin{aligned}
DF_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)(y^k) &= \sum_{i=1}^{n} a_{i,j} \left.\frac{\partial}{\partial y^i}\right|_{F(p)} (y^k) \\
&= \sum_{i=1}^{n} a_{i,j}\delta_{i,k} \\
&= a_{k,j}
\end{aligned}$$

By definition,

$$\begin{aligned}
DF_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)(y^k) &= \left.\frac{\partial}{\partial x^j}\right|_p y^k \circ F \\
&= \left.\frac{\partial}{\partial x^j}\right|_p F^k \\
&= \frac{\partial F^k}{\partial x^j}(p)
\end{aligned}$$

$\square$

Note 6.1.0.11. Since $\operatorname{rank} DF_p$ is independent of basis, it is independent of coordinate charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$.

6.2 The Cotangent Space

Definition 6.2.0.1. Let $p \in M$. We define the **cotangent space of** M **at** p, denoted T_p^*M, by

$$T_p^*M = (T_pM)^*$$

Definition 6.2.0.2. Let $f \in C^\infty(M)$. We define the **differential of** f **at** p, denoted $df_p : T_pM \to \mathbb{R}$, by

$$df_p(v) = vf$$

Exercise 6.2.0.3. Let $f \in C^\infty(M)$ and $p \in M$. Then $df_p \in T_p^*M$.

Proof. Let $v_1, v_2 \in T_pM$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} df_p(v_1 + \lambda v_2) &= (v_1 + \lambda v_2)f \\ &= v_1 f + \lambda v_2 f \\ &= df_p(v_1) + \lambda df_p(v_2) \end{aligned}$$

So that df_p is linear and hence $df_p \in T_p^*M$. □

Exercise 6.2.0.4. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then for each $i, j \in \{1, \cdots, n\}$,

$$dx_p^i \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \delta_{i,j}$$

In particular, $\{dx_p^1, \cdots, dx_p^n\}$ is the dual basis to $\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$ and $T_p^*M = \operatorname{span}\{dx_p^1, \cdots, dx_p^n\}$.

Proof. Let $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned} \left[dx_p^i \left(\frac{\partial}{\partial x^i}\bigg|_p \right) \right]_p &= \frac{\partial}{\partial x^i}\bigg|_p x^i \\ &= \delta_{i,j} \end{aligned}$$

□

Exercise 6.2.0.5. Let $f \in C^\infty(M)$, (U, ϕ) a chart on M with $\phi = (x^1, \cdots, x^n)$ and $p \in U$. Then

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i{}_p$$

Proof. Since $\{dx_p^1, \cdots, dx_p^n\}$ is a basis for T_p^*M, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum_{i=1}^n a_i(p) dx_p^i$. Therefore, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= \sum_{i=1}^n a_i(p) dx_p^i \left(\frac{\partial}{\partial x^i}\bigg|_p \right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) &= \frac{\partial}{\partial x^i}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p) dx_p^i$$

$\square$

Chapter 7

Submersions and Immersions

7.1 Maps of Constant Rank

Definition 7.1.0.1. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. We define the **rank map of** F, denoted $\operatorname{rank} F : M \to \mathbb{N}_0$ by

$$\operatorname{rank}_p F = \dim \operatorname{Im} DF(p)$$

and F is said to have **constant rank** if for each $p, q \in M$, $\operatorname{rank}_p F = \operatorname{rank}_q F$. If F has constant rank, we define the **rank of** F, denoted $\operatorname{rank} F$, by $\operatorname{rank} F = \operatorname{rank}_p F$ for $p \in M$.

Exercise 7.1.0.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$ and $p \in M$. Suppose that $\operatorname{rank}_p F = k$. Then there exist $(U, \phi) \in \mathcal{A}_M$, $(V, \psi) \in \mathcal{A}_N$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \ldots, k\}$,

$$([DF(p)]_{\phi,\psi})_{i,j} = A_{i,j}$$

Proof. Define $q \in V$ by $q = F(p)$. Choose $(U', \phi') \in \mathcal{A}$ and $(V', \psi') \in \mathcal{B}$ such that $p \in U'$ and $q \in V'$. Set $Z = [DF(p)]_{\phi',\psi'}$. By assumption, $\operatorname{rank} Z = k$. An exercise in the subsection on linear algebra implies that there exist $\sigma \in S_m$, $\tau \in S_n$ and $A \in GL(k, \mathbb{R})$ such that for each $i, j \in \{1, \ldots, k\}$,

$$(P_\tau Z P_\sigma^*)_{i,j} = A_{i,j}$$

Define $\phi : U \to \sigma\phi(U)$ and $\psi : V \to \tau\psi(V)$ by

$$\phi = \sigma\phi', \quad \psi = \tau\psi'$$

A previous exercise implies that

$$[DF(p)]_{\phi,\psi} = P_\tau Z P_\sigma^*$$

□

Exercise 7.1.0.3. Constant Rank Theorem:
Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds of dimensions m and n respectively, $F \in C^\infty(M, N)$. Suppose that F has constant rank and $\operatorname{rank} F = k$. Then for each $p \in M$, there exist $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that $p \in U$, $F(p) \in V$ and

$$\psi \circ F \circ \phi^{-1}(x^1, \ldots, x^k, x^{k+1}, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

Hint: Needs a hint

Proof. Let $p \in M$. The previous exercise implies that there exist $(U_0, \phi_0) \in \mathcal{A}$, $(V_0, \psi_0) \in \mathcal{B}$ and $L \in GL(k, \mathbb{R})$ such that $p \in U$, $F(p) \in V_0$ and for each $i, j \in \{1, \ldots, k\}$,

$$([DF(p)]_{\phi_0,\psi_0})_{i,j} = L_{i,j}$$

Define $\hat{M} \subset \mathbb{R}^m$, $\hat{N} \subset \mathbb{R}^n$ and $\hat{F} : \hat{M} \to \hat{N}$ by $\hat{M} := \phi_0(U_0)$, $\hat{N} := \psi_0(V_0)$ and $\hat{F} := \psi_0 \circ F \circ \phi_0^{-1}$. Set $\hat{p} := \phi_0(p)$. Let (x, y) be the standard coordinates on $\mathbb{R}^m$, with $\pi_x : \mathbb{R}^m \to \mathbb{R}^k$ and $\pi_y : \mathbb{R}^m \to \mathbb{R}^{m-k}$ the standard projection maps. Write $\hat{p} = (x_0, y_0)$. There exist $Q : \hat{M} \to \mathbb{R}^k$ and $R : \hat{M} \to \mathbb{R}^{n-k}$ such that $\hat{F} = (Q, R)$. By construction, $[D_x Q(x_0, y_0)] = L$. Define $G : \hat{M} \to \mathbb{R}^m$ by $G(x, y) := (Q(x, y), y)$. Then

$$\begin{aligned} [DG(x_0, y_0)] &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_x Q(x_0, y_0)] \\ [D_x \pi_y(x_0, y_0)] & [D_y \pi_y(x_0, y_0)] \end{pmatrix} \\ &= \begin{pmatrix} [D_x Q(x_0, y_0)] & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} L & [D_y Q(x_0, y_0)] \\ 0 & I \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \det([DG(x_0, y_0)]) &= \det(L) \det(I) \\ &= \det(L) \\ &\neq 0 \end{aligned}$$

The inverse function theorem implies that there exist $\hat{U} \subset \hat{M}$ such that $\hat{U}$ is open, $\hat{p} \in \hat{U}$ and $G|_{\hat{U}} : \hat{U} \to G(\hat{U})$ is a diffeomorphism. Since

$$\{U_1 \times U_2 : U_1 \subset \mathbb{R}^k, U_2 \subset \mathbb{R}^{m-k} \text{ and } U_1,\, U_2 \text{ are open}\}$$

is a basis for the topology on $\mathbb{R}^m$, there exist $\hat{U}_1 \subset \mathbb{R}^k$ and $\hat{U}_2 \subset \mathbb{R}^{m-k}$ such that $\hat{U}_1$, $\hat{U}_2$ are open, $\hat{p} \in \hat{U}_1 \times \hat{U}_2$ and $\hat{U}_1 \times \hat{U}_2 \subset \hat{U}$. Set $\hat{U}_{12} := \hat{U}_1 \times \hat{U}_2$ and define $G_{12} : \hat{U}_{12} \to Q(\hat{U}_{12}) \times \hat{U}_2$ by $G_{12} := G|_{\hat{U}_{12}}$. Since $G|_{\hat{U}} : \hat{U} \to G(\hat{U})$ is a diffeomorphism, $\hat{U}_{12} \subset \hat{U}$ and

$$\begin{aligned} G(\hat{U}_{12}) &= G(\hat{U}_1 \times \hat{U}_2) \\ &= Q(\hat{U}_{12}) \times \hat{U}_2 \end{aligned}$$

we have that $G_{12} : \hat{U}_{12} \to Q(\hat{U}_{12}) \times \hat{U}_2$ is a diffeomorphism. Since G_{12} is a homeomorphism and π_x is open, $Q(\hat{U}_{12})$ is open. Since $G_{12}^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_{12}$, there exist $A : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_1$ and $B : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_2$ such that A, B are smooth and $G_{12}^{-1} = (A, B)$. Define $\tilde{R} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \mathbb{R}^{n-k}$ by $\tilde{R}(x, y) := R(A(x, y), y)$. Then $\tilde{R}$ is smooth. Let $(x, y) \in Q(\hat{U}_{12}) \times \hat{U}_2$. Then

$$\begin{aligned} (x, y) &= G_{12} \circ G_{12}^{-1}(x, y) \\ &= G(A(x, y), B(x, y)) \\ &= (Q(A(x, y), B(x, y)), B(x, y)) \end{aligned}$$

This implies that $B(x, y) = y$,

$$\begin{aligned} x &= Q(A(x, y), B(x, y)) \\ &= Q(A(x, y), y) \end{aligned}$$

and

$$\begin{aligned} G_{12}^{-1}(x, y) &= (A(x, y), B(x, y)) \\ &= (A(x, y), y) \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{F} \circ G_{12}^{-1}(x, y) &= \hat{F}(A(x, y), y) \\ &= (Q(A(x, y), y), R(A(x, y), y)) \\ &= (x, R(A(x, y), y)) \\ &= (x, \tilde{R}(x, y)) \end{aligned}$$

We note that

$$\begin{aligned}[D(\hat{F} \circ G_{12}^{-1})(x,y)] &= \begin{pmatrix} [D_x \pi_x(x,y)] & [D_y \pi_x(x,y)] \\ [D_x \tilde{R}(x,y)] & [D_y \tilde{R}(x,y)] \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ [D_x \tilde{R}(x,y)] & [D_y \tilde{R}(x,y)] \end{pmatrix}\end{aligned}$$

Since $G_{12}^{-1} : Q(\hat{U}_{12}) \times \hat{U}_2 \to \hat{U}_{12}$ is a diffeomorphism, we have that $[DG^{-1}(x,y)] \in GL(m, \mathbb{R})$. Since $\hat{F}$ has constant rank and $\operatorname{rank} \hat{F} = k$, we have that

$$\begin{aligned}\operatorname{rank}[D(\hat{F} \circ G_{12}^{-1})(x,y)] &= \operatorname{rank}([D\hat{F}(G_{12}^{-1}(x,y))][DG_{12}^{-1}(x,y)]) \\ &= \operatorname{rank}[D\hat{F}(G_{12}^{-1}(x,y))] \\ &= k\end{aligned}$$

Since $\operatorname{rank} \begin{pmatrix} I \\ [D_x \tilde{R}(x,y)] \end{pmatrix} = k$, we have that $\operatorname{rank} \begin{pmatrix} 0 \\ [D_y \tilde{R}(x,y)] \end{pmatrix} = 0$. Thus $[D_y \tilde{R}(x,y)] = 0$. Since $(x,y) \in Q(\hat{U}_{12}) \times \hat{U}_2$ is arbitrary, for each $(x,y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\tilde{R}(x,y) = \tilde{R}(x,y_0)$$

Define $\tilde{S} : Q(\hat{U}_{12}) \to \mathbb{R}^{n-k}$ by $\tilde{S}(x) := \tilde{R}(x,y_0)$. Then $\tilde{S}$ is smooth and for each $(x,y) \in Q(\hat{U}_{12}) \times \hat{U}_2$,

$$\hat{F} \circ G_{12}^{-1}(x,y) = (x, \tilde{S}(x))$$

Let (a,b) be the standard coordinates on $\mathbb{R}^n$, with $\pi_a : \mathbb{R}^n \to \mathbb{R}^k$ and $\pi_b : \mathbb{R}^n \to \mathbb{R}^{n-k}$ the standard projection maps. Write $\hat{F}(\hat{p}) = (a_0, b_0)$. Set

$$\begin{aligned}\hat{V}_{12} &:= \pi_a|_{\hat{N}}^{-1}(Q(\hat{U}_{12})) \\ &= \pi_a^{-1}(Q(\hat{U}_{12})) \cap \hat{N}\end{aligned}$$

Since $Q(\hat{U}_{12})$ is open, $\hat{N}$ is open and π_a is continuous, we have that $\hat{V}_{12}$ is open. Since

$$\begin{aligned}Q(\hat{U}_{12}) &= \pi_a|_{\hat{N}} \circ \hat{F} \circ G^{-1}(Q(\hat{U}_{12}) \times \hat{U}_2) \\ &= \pi_a|_{\hat{N}} \circ \hat{F}(\hat{U}_{12})\end{aligned}$$

we have that

$$\begin{aligned}\hat{F}(\hat{U}_{12}) &\subset \pi_a|_{\hat{N}}^{-1}(Q(\hat{U}_{12})) \\ &\subset \hat{V}_{12}\end{aligned}$$

In particular, $\hat{F}(\hat{p}) \in \hat{V}_{12}$. Define $H : Q(\hat{U}_{12}) \times \mathbb{R}^{n-k} \to Q(\hat{U}_{12}) \times \mathbb{R}^{n-k}$ by $H := (\pi_a, \pi_b - \tilde{S} \circ \pi_a)$, i.e. for each $(a,b) \in Q(\hat{U}_{12}) \times \mathbb{R}^{n-k}$, $H(a,b) = (a, b - \tilde{S}(a))$. Then H is a bijection and $H^{-1}(a,b) = (\pi_a, \pi_b + \tilde{S} \circ \pi_a)$. Thus H and H^{-1} are smooth and therefore H is a diffeomorphism. Define $H_{12} : \hat{V}_{12} \to H(\hat{V}_{12})$ by $H_{12} = H|_{\hat{V}_{12}}$. Then H_{12} is a diffeomorphism and for each $x, y \in Q(\hat{U}_{12} \times \hat{U}_2)$, $H_{12} \circ \hat{F} \circ G_{12}^{-1}(x,y) = (x,0)$. Define $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ by $U := \phi_0^{-1}(\hat{U}_{12})$, $V := \psi_0^{-1}(\hat{V}_{12})$, $\phi := G_{12} \circ \phi_0|_U$ and $\psi := H_{12} \circ \psi_0|_V$. Then for each $(x,y) \in \phi(U)$,

$$\begin{aligned}\psi \circ F \circ \phi^{-1}(x,y) &= H_{12} \circ \psi_0|_V \circ F \circ \phi_0|_U^{-1} \circ G_{12}^{-1}(x,y) \\ &= H_{12} \circ \hat{F} \circ G_{12}^{-1}(x,y) \\ &= (x,0)\end{aligned}$$

□

Definition 7.1.0.4. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map. Then F is said to be

- an **immersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is injective
- a **submersion** if for each $p \in M$, $DF(p) : T_pM \to T_{F(p)}N$ is surjective

Exercise 7.1.0.5. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds, $F : M \to N$ a smooth map.

Definition 7.1.0.6. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $F : M \to N$ smooth. Then F is said to be an **embedding** if

1. F is an immersion
2. $F : M \to F(M)$.

Note 7.1.0.7. Here the topology on $F(M)$ is the subspace topology.

7.2 Submanifolds

Exercise 7.2.0.1. Let $(M, \mathcal{A})$ be a smooth manifold and $S \subset M$ open. For $(U, \phi) \in \mathcal{A}$, define $\tilde{U} \subset S$ and $\tilde{\phi} : \tilde{U} \to \phi(\tilde{U})$ by $\tilde{U} = U \cap S$ and $\tilde{\phi} = \phi|_{U \cap S}$. Set $\mathcal{B} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \in \mathcal{A}\}$. Then $\mathcal{B}$ is a smooth structure on S.

Proof. $\square$

Definition 7.2.0.2. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. Suppose that $M \subset N$. Then $(M, \mathcal{A})$ is said to be

1. an **immersed submanifold** of $(N, \mathcal{B})$ if $\mathrm{id} : M \to N$ is a smooth immersion
2. an **embedded submanifold** of $(N, \mathcal{B})$ if $\mathrm{id} : M \to N$ is a smooth embedding

Note 7.2.0.3. Essentially, embedded submanifolds are immersed submanifolds with the subspace topology.

Note 7.2.0.4. For the remainder of this section, we assume that $k \leq n$.

Definition 7.2.0.5. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Then S is said to be a k**-slice** of U if $S = \{u \in U : u^{k+1}, \dots, u^n = 0\}$.

Exercise 7.2.0.6. Let $U \subset \mathbb{R}^n$ and $S \subset U$. Suppose that S is a k-slice of U. Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ by

$$\pi(u^1, \dots, u^k, \dots, u^n) = (u^1, \dots, u^k)$$

Then $\pi|_S \to \pi(S)$ is a diffeomorphism.

Proof. Clear. □

Definition 7.2.0.7. Let $(M, \mathcal{A})$ be a smooth manifold, $(U, \phi) \in \mathcal{A}$ and $S \subset U$. Then S is said to be a k**-slice** of U if $\phi(S)$ is a k-slice of $\phi(U)$.

Definition 7.2.0.8. Let $(M, \mathcal{A})$ be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$. Then (U, ϕ) is said to be a k**-slice chart for** S if $U \cap S$ is a k-slice of U.

Exercise 7.2.0.9. Let $(M, \mathcal{A})$ be a smooth manifold, $S \subset M$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$. If (U, ϕ) is a k-slice chart for S, then $\phi|_S = (x^1|_S, \dots, x^k|_S, 0, \dots, 0)$.

Proof. Clear. □

Definition 7.2.0.10. Let $(M, \mathcal{A})$ be a smooth manifold and $S \subset M$. Then S is said to satisfy the **local** k**-slice condition** if for each $p \in S$, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k-slice chart of S.

Exercise 7.2.0.11. Let $(M, \mathcal{A})$ be a n-dimensional smooth manifold and $S \subset M$ a subspace. If S satisfies the local k-slice condition, then there exists a smooth structure $\tilde{\mathcal{A}}$ on S such that $(S, \tilde{\mathcal{A}})$ is an embedded submanifold of M.

Proof. Suppose that S satisfies the local k-slice condition. Define $\pi : \mathbb{R}^n \to \mathbb{R}^k$ as above Let $(U, \phi) \in \mathcal{A}$. Suppose that (U, ϕ) is a k-slice chart for S. Define $\tilde{U} = U \cap S$ and $\tilde{\phi} : \tilde{U} \to \pi \circ \phi(\tilde{U})$ by

$$\tilde{\phi} = \pi \circ \phi|_{\tilde{U}}$$

By definition, $\phi(\tilde{U})$ is a k-slice of $\phi(U)$. A previous exercise implies that $\pi|_{\phi(\tilde{U})} \to \pi \circ \phi(\tilde{U})$ is a diffeomorphism and hence a homeomorphism. Thus $\tilde{\phi}$ is a homeomorphism.
Define

$$\tilde{\mathcal{B}} = \{(\tilde{U}, \tilde{\phi}) : (U, \phi) \text{ is a } k\text{-slice for } S\}$$

Let $p \in S$. By assumption, there exists $(U, \phi) \in \mathcal{A}$ such that $p \in U$ and (U, ϕ) is a k-slice chart of S. Then $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{B}}$ and $\mathcal{A}$ is an atlas on S. By construction of $\tilde{\mathcal{B}}$, S is locally half Euclidean of dimension k. Since M is second countable Hausdorff, so is S in the subspace topology. Thus $(S, \tilde{\mathcal{B}})$ is a k-dimensional manifold. Let $(\tilde{U}, \tilde{\phi})$, $(\tilde{V}, \tilde{\psi}) \in \tilde{\mathcal{B}}$. Then

$$\tilde{\phi} \circ \tilde{\psi}^{-1}|_{\tilde{U} \cap \tilde{V}} = \pi|_{\phi(\tilde{U} \cap \tilde{V})} \circ \phi|_{\tilde{U} \cap \tilde{V}} \circ \psi|^{-1}_{\tilde{U} \cap \tilde{V}} \circ \pi|^{-1}_{\psi(\tilde{U} \cap \tilde{V})}$$

which is a diffeomorphism. So $(\tilde{U}, \tilde{\phi})$ and $(\tilde{V}, \tilde{\psi})$ smoothly compatible. Hence $\tilde{\mathcal{B}}$ is smooth. An exercise in section 4.1 implies that there exists a unique smooth structure $\tilde{\mathcal{A}}$ on S such that $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$. So $(S, \tilde{\mathcal{A}})$ is a smooth k-dimensional manifold.
Clearly $\mathrm{id} : S \to S$ is a homeomorphism. Let $(V, \psi) \in \mathcal{A}$ and $(\tilde{U}, \tilde{\phi}) \in \tilde{\mathcal{A}}$.

Finish!! □

Definition 7.2.0.12.

Exercise 7.2.0.13.

Chapter 8

Bundles and Sections

8.1 Fiber Bundles

8.1.1 Local Trivializations

Note 8.1.1.1. Let M, F be sets, we write $\text{proj}_1 : M \times F \to M$ to denote the projection onto M.

Definition 8.1.1.2. Let $E, M, F \in \text{Obj}(\textbf{Set})$, $\pi \in \text{Hom}_{\textbf{Set}}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **local trivialization with respect to** π **of** E **over** U **with fiber** F if

1. Φ is a bijection
2. $\text{proj}_1 \circ \Phi = \pi|_{\pi^{-1}(U)}$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\text{proj}_1"] \\
& U
\end{tikzcd}$$

Exercise 8.1.1.3. Let $E, M, F \in \text{Obj}(\textbf{Set})$ and $\pi \in \text{Hom}_{\textbf{Set}}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$ a local trivialization with respect to π of E over U with fiber F. Then for each $A \subset U$,

$$\Phi(\pi^{-1}(A)) = A \times F$$

Hint: consider $\Phi^{-1}(A \times F)$

Proof. Let $A \subset U$. Since $\text{proj}_1^{-1}(A) = A \times F$, we have that

$$\begin{aligned}
\Phi^{-1}(A \times F) &= \Phi^{-1}(\text{proj}_1^{-1}(A)) \\
&= (\text{proj}_1 \circ \Phi)^{-1}(A) \\
&= (\pi|_{\pi^{-1}(U)})^{-1}(A) \\
&= \pi^{-1}(A) \cap \pi^{-1}(U) \\
&\pi^{-1}(A \cap U) \\
&= \pi^{-1}(A)
\end{aligned}$$

Since Φ is a bijection, we have that

$$\begin{aligned}
\Phi(\pi^{-1}(A)) &= \Phi \circ \Phi^{-1}(A \times F) \\
&= A \times F
\end{aligned}$$

$\square$

8.1.2 $\mathbf{Man}^0$ Fiber Bundles

Definition 8.1.2.1. Let $E, M, F \in \text{Obj}(\mathbf{Man}^0)$ and $\pi \in \text{Hom}_{\mathbf{Man}^0}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **continuous local trivialization with respect to** π **of** E **over** U **with fiber** F if

1. U is open
2. (U, Φ) is a local trivialization with respect to π of E over U with fiber F
3. Φ is a homeomorphism

Definition 8.1.2.2. Let $E, M, F \in \text{Obj}(\mathbf{Man}^0)$ and $\pi \in \text{Hom}_{\mathbf{Man}^0}(E, M)$ a surjection. Then (E, M, π, F) is said to be a $\mathbf{Man}^0$ **fiber bundle with total space** E**, base space** M**, fiber** F **and projection** π if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that (U, Φ) is a continuous local trivialization with respect to π of E over U with fiber F. For $p \in M$, we define the **fiber over** p, denoted E_p, by $E_p = \pi^{-1}(\{p\})$.

Exercise 8.1.2.3. $\mathbf{Man}^0$ Fiber Bundle Chart Lemma:
Let $E \in \text{Obj}(\mathbf{Set})$, $M, F \in \text{Obj}(\mathbf{Man}^0)$, $\pi : E \to M$ a surjection, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. Set $n = \dim M$ and $k = \dim F$. Suppose that

- for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}_M$
- $M \subset \bigcup\limits_{\alpha \in \Gamma} U_\alpha$
- for each $\alpha \in \Gamma$, (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
- for each $\alpha, \beta \in \Gamma$, $\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is continuous.

Then there exist a unique topology, $\mathcal{T}_E$, on E such that

1. $(E, \mathcal{T}_E)$ is a $n + k$-dimensional topological manifold
2. for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism
3. $\pi : E \to M$ is continuous
4. (E, M, π, F) is an $\mathbf{Man}^0$ fiber bundle

Proof.

1. For $\alpha \in \Gamma$, we define $X^n_\alpha(M, \mathcal{T}_M) \subset X^n(M, \mathcal{T}_M)$ by

$$X^n_\alpha(M, \mathcal{T}_M) = \{(V^M, \psi^M) \in X^n(M, \mathcal{T}_M) : V^M \subset U_\alpha\}$$

Choose index sets $(\Pi^M_\alpha)_{\alpha \in \Gamma}$ and Π^F such that for each $\alpha \in \Gamma$, $X^n_\alpha(M, \mathcal{T}_M) = (V^M_{\alpha,\mu}, \psi^M_{\alpha,\mu})_{\mu \in \Pi^M_\alpha}$ and $X^k(F, \mathcal{T}_F) = (V^F_\nu, \psi^F_\nu)_{\nu \in \Pi^F}$. Set $\Pi^M = \coprod_{\alpha \in \Gamma} \Pi^M_\alpha$ and $\Pi^E = \Pi^M \times \Pi^F$. For $(\alpha, \mu, \nu) \in \Pi^E$, we define $V^E_{\alpha,\mu,\nu} \subset E$ and $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ by

- $V^E_{\alpha,\mu,\nu} = \Phi^{-1}_\alpha(V^M_{\alpha,\mu} \times V^F_\nu)$
- $\psi^E_{\alpha,\mu,\nu} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu) \circ \Phi_\alpha|_{V^E_{\alpha,\mu,\nu}}$

We have the following:

- For each $(\alpha, \mu, \nu) \in \Pi^E$, $\psi^E_{\alpha,\mu,\nu}(V^E_{\alpha,\mu,\nu}) = \psi^M_\mu(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ and thus $\psi^E_{\alpha,\mu,\nu}(V^E_{\alpha,\mu,\nu}) \in \mathcal{T}_{\mathbb{H}^{n+k}}$

- For each $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$,

$$\begin{aligned}\psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}) &= (\psi^M_{\alpha_1,\mu_1} \times \psi^F_{\nu_1}) \circ \Phi_{\alpha_1}|_{V^E_{\alpha_1,\mu_1,\nu_1}}(\Phi^{-1}_{\alpha_1}([V^M_{\alpha_1,\mu_1} \times V^F_{\nu_1}] \cap [V^M_{\alpha_2,\mu_2} \times V^F_{\nu_2}])) \\ &= (\psi^M_{\alpha_1,\mu_1} \times \psi^F_{\nu_1})([V^M_{\alpha_1,\mu_1} \times V^F_{\nu_1}] \cap [V^M_{\alpha_2,\mu_2} \times V^F_{\nu_2}]) \\ &= (\psi^M_{\alpha_1,\mu_1} \times \psi^F_{\nu_1})([V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}] \times [V^F_{\nu_1} \cap V^F_{q_2}]) \\ &= \psi^M_{\alpha_1,\mu_1}(V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}) \times \psi^F_{\nu_1}(V^F_{\nu_1} \cap V^F_{\nu_2}) \\ &\in \mathcal{T}_{\mathbb{H}^{n+k}}\end{aligned}$$

- For each $(\alpha, \mu, \nu) \in \Pi^E$, $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a bijection
- Let $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$. For notational convenience, set $\psi^E_1 = \psi^E_{\alpha_1,\mu_1,\nu_1}$, $\psi^E_2 = \psi_{\alpha_2,\mu_2,\nu_2}$, $V^E = V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}$, $V^M = V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}$ and $V^F = V^F_{\nu_1} \cap V^F_{\nu_2}$. Then $\psi_2|_{V^E} \circ (\psi_1|_{V^E})^{-1} : \psi_1(V^E) \to \psi_2(V^E)$ is given by

$$\begin{aligned}\psi^E_2|_{V^E} \circ (\psi^E_1|_{V^E})^{-1} &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F}) \circ \Phi_{\alpha_1}|_{V^E}]^{-1} \\ &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\Phi_{\alpha_1}|_{V^E})^{-1} \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}] \\ &= (\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ [\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}] \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}\end{aligned}$$

  Since $\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}$ is continuous, we have that $\psi^E_{\alpha_2,\mu_2,\nu_2}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}} \circ (\psi^E_{\alpha_1,\mu_1,\nu_1}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}})^{-1} : \psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}) \to \psi^E_{\alpha_2,\mu_2,\nu_2}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2})$ is continuous.
- A previous exercise in the section on topological manifolds implies that $(V^M_{\alpha,\mu})_{(\alpha,\mu)\in\Pi^M}$ is an open cover of M and $(V^F_\nu)_{\nu\in\Pi^F}$ is an open cover of F. Since M, F are second-countable M, F are Lindelöf and there exists $S^M \subset \Pi^M$, $S^F \subset \Pi^F$ such that S^M, S^F are countable, $(V^M_{\alpha,\mu})_{(\alpha,\mu)\in S^M}$ is an open cover of M and $(V^F_\nu)_{\nu\in\Pi^F}$ is an open cover of F. Then $S^M \times S^F$ is countable and $(V^M_{\alpha,\mu} \times V^F_\nu)_{(\alpha,\mu,\nu)\in S^M\times S^F}$ is an open cover of $M \times F$.
  Let $a \in E$. Set $p = \pi(a)$. Choose $(\alpha, \mu) \in S^M$ such that $p \in V^M_{\alpha,\mu}$. Since $V^M_{\alpha,\mu} \subset U_\alpha$, $a \in \pi^{-1}(U_\alpha)$ which implies that

$$\begin{aligned}p &= \pi(a) \\ &= \mathrm{proj}_1 \circ \Phi_\alpha(a)\end{aligned}$$

  Set $q = \mathrm{proj}_2 \circ \Phi_\alpha(a)$. Choose $\nu \in S^F$ such that $q \in V^F_\nu$. Then

$$\begin{aligned}\Phi_\alpha(a) &= (\mathrm{proj}_1 \circ \Phi_\alpha(a), \mathrm{proj}_2 \circ \Phi_\alpha(a)) \\ &= (p, q) \\ &\in V^M_{\alpha,\mu} \times V^F_\nu\end{aligned}$$

  Thus

$$\begin{aligned}a &\in \Phi^{-1}_\alpha(V^M_{\alpha,\mu} \times V^F_\nu) \\ &= V^E_{\alpha,\mu,\nu}\end{aligned}$$

  Since $a \in E$ is arbitrary, we have that for each $a \in E$, there exists $(\alpha, \mu, \nu) \in S^M \times S^F \subset \Pi^E$ such that $a \in V^E_{\alpha,\mu,\nu}$. Thus

$$E \subset \bigcup_{(\alpha,\mu,\nu)\in S^M\times S^F} V_{\alpha,\mu,\nu}$$

- Let $a_1, a_2 \in E$.
  For now, suppose that $\pi(a_1) \neq \pi(a_2)$. Set $p_1 = \pi(a_1)$ and $p_2 = \pi(a_2)$. Since M is Hausdorff, there exist $(\alpha_1, \mu_1), (\alpha_2, \mu_2) \in \Pi^M$ such that $p_1 \in V^M_{\alpha_1,\mu_1}$, $p_2 \in V^M_{\alpha_2,\mu_2}$ and $V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2} = \varnothing$.

Set $q_1 = \text{proj}_2 \circ \Phi_{\alpha_1}(a_1)$ and $q_2 = \text{proj}_2 \circ \Phi_{\alpha_2}(a_2)$. Choose $\nu_1, \nu_2 \in \Pi^F$ such that $q_1 \in V^F_{\nu_1}$ and $q_2 \in V^F_{\nu_2}$. Then similarly to the previous part, $a_1 \in V^E_{\alpha_1,\mu_1,\nu_1}$ and $a_2 \in V^E_{\alpha_2,\mu_2,\nu_2}$ and therefore

$$\begin{aligned}
V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2} &= \Phi^{-1}_{\alpha_1}(V^M_{\alpha_1,\mu_1} \times V^F_{\nu_1}) \cap \Phi^{-1}_{\alpha_2}(V^M_{\alpha_2,\mu_2} \times V^F_{\nu_2}) \\
&\subset \pi^{-1}(V^M_{\alpha_1,\mu_1}) \cap \pi^{-1}(V^M_{\alpha_2,\mu_2}) \\
&= \pi^{-1}(V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}) \\
&= \pi^{-1}(\varnothing) \\
&= \varnothing
\end{aligned}$$

Now suppose that $\pi(a_1) = \pi(a_2)$. Set $p = \pi(a_1)$. Then there exists $(\alpha, \mu) \in \Pi^M$ such that $p \in V^M_{\alpha,\mu} \subset U_\alpha$.
For now, suppose that $\text{proj}_2 \circ \Phi_\alpha(a_1) \neq \text{proj}_2 \circ \Phi_\alpha(a_2)$. Set $q_1 = \text{proj}_2 \circ \Phi_\alpha(a_1)$ and $q_2 = \text{proj}_2 \circ \Phi_\alpha(a_2)$. Since F is Hausdorff, there exist $\nu_1, \nu_2 \in \Pi^F$ such that $q_1 \in V^F_{\nu_1}$ and $q_2 \in V^F_{\nu_2}$ and $V^F_{\nu_1} \cap V^F_{\nu_2} = \varnothing$. Then $a_1 \in V^E_{\alpha,\mu,\nu_1}$, $a_2 \in V^E_{\alpha,\mu,\nu_2}$ and

$$\begin{aligned}
V^E_{\alpha,\mu,\nu_1} \cap V^E_{\alpha,\mu,\nu_2} &= \Phi^{-1}_{\alpha}(V^M_{\alpha,\mu} \times V^F_{\nu_1}) \cap \Phi^{-1}_{\alpha}(V^M_{\alpha,\mu} \times V^F_{\nu_2}) \\
&= \Phi^{-1}_{\alpha}([V^M_{\alpha,\mu} \times V^F_{\nu_1}] \cap [V^M_{\alpha,\mu} \times V^F_{\nu_2}]) \\
&= \Phi^{-1}_{\alpha}([V^M_{\alpha,\mu} \cap V^M_{\alpha,\mu}] \times [V^F_{\nu_1} \cap V^F_{\nu_2}]) \\
&= \Phi^{-1}_{\alpha}(V^M_{\alpha,\mu} \times [V^F_{\nu_1} \cap V^F_{\nu_2}]) \\
&= \Phi^{-1}_{\alpha}(V^M_{\alpha,\mu} \times \varnothing) \\
&= \Phi^{-1}_{\alpha}(\varnothing) \\
&= \varnothing
\end{aligned}$$

Now, suppose that $\text{proj}_2 \circ \Phi_\alpha(a_1) = \text{proj}_2 \circ \Phi_\alpha(a_2)$. Set $q = \text{proj}_2 \circ \Phi_\alpha(a_1)$. Choose $\nu \in \Pi^F$ such that $q \in V^F_\nu$. Since

$$\begin{aligned}
\Phi_\alpha(a_1) &= (\text{proj}_1 \circ \Phi_\alpha(a_1), \text{proj}_2 \circ \Phi_\alpha(a_1)) \\
&= (p, q) \\
&= (\text{proj}_1 \circ \Phi_\alpha(a_2), \text{proj}_2 \circ \Phi_\alpha(a_2)) \\
&= \Phi_\alpha(a_2)
\end{aligned}$$

we have that $a_1 = a_2$ and $a_1, a_2 \in V^E_{\alpha,\mu,\nu}$. Therefore, for each $a_1, a_2 \in E$, there exists $(\alpha, \mu, \nu) \in \Pi^E$ such that $p, q \in V^E_{\alpha,\mu,\nu}$ or there exist $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$ such that $a_1 \in V^E_{\alpha_1,\mu_1,\nu_1}$, $a_2 \in V^E_{\alpha_2,\mu_2,\nu_2}$ and $V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2} = \varnothing$.

The topological manifold chart lemma implies that there exists a unique topology $\mathcal{T}_E$ on E such that $(E, \mathcal{T}_E)$ is an $n+k$-dimensional topological manifold and $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E} \subset X^{n+k}(E, \mathcal{T}_E)$.

2. Let $\alpha \in \Gamma$. By assumption $U_\alpha \in \mathcal{T}_M$. Let $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$. Then $(\alpha, \mu, \nu) \in \Pi^E$. Since

- $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a homeomorphism
- $\psi^M_{\alpha,\mu} \times \psi^F_\nu : V^M_{\alpha,\mu} \times V^F_\nu \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a homeomorphism
- $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is given by $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu)^{-1} \circ \psi^E_{\alpha,\mu,\nu}$,

we have that $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a homeomorphism. Since $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$ are arbitrary we have that for each $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$, $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a homeomorphism. Since $(V^M_{\alpha,\mu})_{\mu\in\Pi^M_\alpha}$ is an open cover of U_α and $(V^M_{\alpha,\mu} \times V^F_\nu)_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F}$ is an open cover of $U_\alpha \times F$, we have that

$$\begin{aligned}\pi^{-1}(U_\alpha) &= \pi^{-1}\bigg(\bigcup_{\mu \in \Pi^M_\alpha} V^M_{\alpha,\mu}\bigg) \\ &= \bigcup_{\mu \in \Pi^M_\alpha} \pi^{-1}(V^M_{\alpha,\mu}) \\ &= \bigcup_{\mu \in \Pi^M_\alpha} \Phi_\alpha^{-1}(V^M_{\alpha,\mu} \times F) \\ &= \bigcup_{\mu \in \Pi^M_\alpha} \Phi_\alpha^{-1}\bigg(V^M_{\alpha,\mu} \times \bigg[\bigcup_{\nu \in \Pi^F} V^F_\nu\bigg]\bigg) \\ &= \bigcup_{\mu \in \Pi^M_\alpha} \Phi_\alpha^{-1}\bigg(\bigcup_{\nu \in \Pi^F} [V^M_{\alpha,\mu} \times V^F_\nu]\bigg) \\ &= \bigcup_{\mu \in \Pi^M_\alpha} \bigg[\bigcup_{\nu \in \Pi^F} \Phi_\alpha^{-1}(V^M_{\alpha,\mu} \times V^F_\nu)\bigg] \\ &= \bigcup_{(\mu,\nu) \in \Pi^M_\alpha \times \Pi^F} V^E_{\alpha,\mu,\nu}\end{aligned}$$

Hence $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$, $(V^E_{\alpha,\mu,\nu})_{(\mu,\nu)\in\Pi^M_\alpha \times \Pi^F}$ is an open cover of $\pi^{-1}(U_\alpha)$ and Φ_α is a local homeomorphism. Since Φ_α is a bijection, Φ_α is a homeomorphism. Since $\alpha \in \Gamma$ is arbitrary, we have that for each $\alpha \in \Gamma$, $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism.

3. Let $(\alpha, \mu, \nu) \in \Pi^E$. Since

   - $V^E_{\alpha,\mu,\nu} \subset \pi^{-1}(U_\alpha)$
   - $\text{proj}_1 : M \times F \to M$ is continuous
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is continuous
   - $\pi|_{V^E_{\alpha,\mu,\nu}} = \text{proj}_1 \circ \Phi|_{V^E_{\alpha,\mu,\nu}}$

   we have that $\pi|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu}$ is continuous. Since $(\alpha,\mu,\nu) \in \Pi^E$ is arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is an open cover of E, we have that $\pi : E \to M$ is continuous.

4. Let $p \in M$. By assumption, there exists $\alpha \in \Gamma$ such that $p \in U_\alpha$, $U_\alpha \in \mathcal{T}_M$. Since $E, M, F \in \text{Obj}(\mathbf{Man}^0)$, $\pi \in \text{Hom}_{\mathbf{Man}^0}(E, M)$ is a surjection, and

   - U_α is open
   - (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism

   we have that (U_α, Φ_α) is a continuous local trivialization with respect to π of E over U_α with fiber F. Since $p \in M$ is arbitrary, (E, M, π, F) is a $\mathbf{Man}^0$ fiber bundle.

□

8.1.3 Man$^\infty$ Fiber Bundles

Definition 8.1.3.1. Let $E, M, F \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection, $U \subset M$ and $\Phi : \pi^{-1}(U) \to U \times F$. Then (U, Φ) is said to be a **smooth local trivialization of E over U with fiber F** if

1. U is open
2. (U, Φ) is a local trivialization of E over U with fiber F

3. Φ is a diffeomorphism

Definition 8.1.3.2. Let $E, M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then (E, M, π, F) is said to be a **$\mathbf{Man}^\infty$ fiber bundle with total space E, base space M, fiber F and projection π** if for each $p \in M$, there exist $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that U is open and (U, Φ) is a smooth local trivialization of E over U with fiber F. For $p \in M$, we define the **fiber over** p, denoted E_p, by $E_p = \pi^{-1}(\{p\})$.

Exercise 8.1.3.3. $\mathbf{Man}^\infty$ Fiber Bundle Chart Lemma:
Let $E \in \mathrm{Obj}(\mathbf{Set})$, $M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $\pi : E \to M$ a surjection, Γ an index set and for each $\alpha \in \Gamma$, $U_\alpha \subset M$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. Set $n = \dim M$ and $k = \dim F$. Suppose that

- for each $\alpha \in \Gamma$, $U_\alpha \in \mathcal{T}_M$
- $M \subset \bigcup\limits_{\alpha \in \Gamma} U_\alpha$
- for each $\alpha \in \Gamma$, (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
- for each $\alpha, \beta \in \Gamma$, $\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ is smooth.

Then there exist a unique topology $\mathcal{T}_E$ on E and smooth structure $\mathcal{A}_E \subset X^{n+k}(M, \mathcal{T}_E)$ on E such that

1. $(E, \mathcal{A}_E)$ is an $n+k$-dimensional smooth manifold
2. for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism
3. $\pi : E \to M$ is smooth
4. (E, M, π, F) is an $\mathbf{Man}^\infty$ fiber bundle

Proof. The $\mathbf{Man}^0$ fiber bundle chart lemma implies that there exists a unique topology $\mathcal{T}_E$ on E such that

- $(E, \mathcal{T}_E)$ is a $n+k$-dimensional topological manifold
- for each $\alpha \in \Gamma$, $\pi^{-1}(U_\alpha) \in \mathcal{T}_E$ and $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a homeomorphism
- $\pi : E \to M$ is continuous
- (E, M, π, F) is an $\mathbf{Man}^0$ fiber bundle

1. Define $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E} \subset X^{n+k}(E, \mathcal{T}_E)$ as in the proof of the $\mathbf{Man}^0$ fiber bundle chart lemma. Let $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$. For notational convenience, set $\psi^E_1 = \psi^E_{\alpha_1,\mu_1,\nu_1}$, $\psi^E_2 = \psi_{\alpha_2,\mu_2,\nu_2}$, $V^E = V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}$, $V^M = V^M_{\alpha_1,\mu_1} \cap V^M_{\alpha_2,\mu_2}$ and $V^F = V^F_{\nu_1} \cap V^F_{\nu_2}$. Then $\psi_2|_{V^E} \circ (\psi_1|_{V^E})^{-1} : \psi_1(V^E) \to \psi_2(V^E)$ is given by

$$\begin{aligned}\psi^E_2|_{V^E} \circ (\psi^E_1|_{V^E})^{-1} &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F}) \circ \Phi_{\alpha_1}|_{V^E}]^{-1} \\ &= [(\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ \Phi_{\alpha_2}|_{V^E}] \circ [(\Phi_{\alpha_1}|_{V^E})^{-1} \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}] \\ &= (\psi^M_{\alpha_2,\mu_2}|_{V^M} \times \psi^F_{\nu_2}|_{V^F}) \circ [\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}] \circ (\psi^M_{\alpha_1,\mu_1}|_{V^M} \times \psi^F_{\nu_1}|_{V^F})^{-1}\end{aligned}$$

   Since $\Phi_{\alpha_2}|_{V^E} \circ (\Phi_{\alpha_1}|_{V^E})^{-1}$ is smooth, we have that $\psi^E_{\alpha_2,\mu_2,\nu_2}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}} \circ (\psi^E_{\alpha_1,\mu_1,\nu_1}|_{V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}})^{-1} : \psi^E_{\alpha_1,\mu_1,\nu_1}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2}) \to \psi^E_{\alpha_2,\mu_2,\nu_2}(V^E_{\alpha_1,\mu_1,\nu_1} \cap V^E_{\alpha_2,\mu_2,\nu_2})$ is smooth. Since $(\alpha_1, \mu_1, \nu_1), (\alpha_2, \mu_2, \nu_2) \in \Pi^E$ are arbitrary, we have that $(V^E_{\alpha,\mu,\nu}, \psi^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is a smooth atlas on E. An exercise in the section on smooth manifolds implies that there exists a unique smooth structure $\mathcal{A}_E$ on E such that $(E, \mathcal{A}_E)$ is an $n+k$-dimensional smooth manifold.

2. Let $\alpha \in \Gamma$. By assumption $U_\alpha \in \mathcal{T}_M$. Let $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$. Then $(\alpha, \mu, \nu) \in \Pi^E$. Since
   - $\psi^E_{\alpha,\mu,\nu} : V^E_{\alpha,\mu,\nu} \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a diffeomorphism
   - $\psi^M_{\alpha,\mu} \times \psi^F_\nu : V^M_{\alpha,\mu} \times V^F_\nu \to \psi^M_{\alpha,\mu}(V^M_{\alpha,\mu}) \times \psi^F_\nu(V^F_\nu)$ is a diffeomorphism

- $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is given by $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} = (\psi^M_{\alpha,\mu} \times \psi^F_\nu)^{-1} \circ \psi^E_{\alpha,\mu,\nu}$,

  we have that $\Phi_\alpha|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu} \times V^F_\nu$ is a diffeomorphism. Since $\mu \in \Pi^M_\alpha$ and $\nu \in \Pi^F$ are arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\mu,\nu)\in\Pi^M_\alpha\times\Pi^F}$ is an open cover of $\pi^{-1}(U_\alpha)$, we have that $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a local diffeomorphism. Since Φ_α is a bijection, Φ_α is a diffeomorphism. Since $\alpha \in \Gamma$ is arbitrary, we have that for each $\alpha \in \Gamma$, $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism.

3. Let $(\alpha, \mu, \nu) \in \Pi^E$. Since

   - $V^E_{\alpha,\mu,\nu} \subset \pi^{-1}(U_\alpha)$
   - $\mathrm{proj}_1 : M \times F \to M$ is smooth
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is smooth
   - $\pi|_{V^E_{\alpha,\mu,\nu}} = \mathrm{proj}_1 \circ \Phi|_{V^E_{\alpha,\mu,\nu}}$

   we have that $\pi|_{V^E_{\alpha,\mu,\nu}} : V^E_{\alpha,\mu,\nu} \to V^M_{\alpha,\mu}$ is smooth. Since $(\alpha, \mu, \nu) \in \Pi^E$ is arbitrary and $(V^E_{\alpha,\mu,\nu})_{(\alpha,\mu,\nu)\in\Pi^E}$ is an open cover of E, we have that $\pi : E \to M$ is smooth.

4. Let $p \in M$. By assumption, there exists $\alpha \in \Gamma$ such that $p \in U_\alpha$, $U_\alpha \in \mathcal{T}_M$. Since $E, M, F \in \mathrm{Obj}(\mathbf{Man}^\infty)$, $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ is a surjection, and

   - U_α is open
   - (U_α, Φ_α) is a local trivialization with respect to π of E over U_α with fiber F
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism

   we have that (U_α, Φ_α) is a smooth local trivialization with respect to π of E over U_α with fiber F. Since $p \in M$ is arbitrary, (E, M, π, F) is a $\mathbf{Man}^\infty$ fiber bundle.

$\square$

Definition 8.1.3.4. Let (E_1, M_1, π_1, F_1) and (E_2, M_2, π_2, F_2) be $\mathbf{Man}^\infty$ fiber bundles, $\Phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E_1, E_2)$ and $\phi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(M_1, M_2)$. Then (Φ, ϕ) is said to be a **smooth bundle morphism** from (E_1, M_1, π_1, F_1) to (E_2, M_2, π_2, F_2) if $\pi_2 \circ \Phi = \phi \circ \pi_1$, i.e. the following diagram commutes:

$$\begin{tikzcd}
E_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\
M_1 \arrow[r, "\phi"'] & M_2
\end{tikzcd}$$

Definition 8.1.3.5. We define the category of $\mathbf{Man}^\infty$ fiber bundles, denoted $\mathbf{Bun}^\infty$, by

- $\mathrm{Obj}(\mathbf{Bun}^\infty) = \{(E, M, \pi, F) : (E, M, \pi, F) \text{ is a } \mathbf{Man}^\infty \text{ fiber bundle}\}$
- For $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$,

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) = \\ &\{(\Phi, \phi) : (\Phi, \phi) \text{ is a smooth bundle morphism from } (E_1, M_1, \pi_1, F_1) \text{ to } (E_2, M_2, \pi_2, F_2)\}\end{aligned}$$

- For

  – $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$
  – $(\Phi_{12}, \phi_{12}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$
  – $(\Phi_{23}, \phi_{23}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_2, M_2, \pi_2, F_2), (E_3, M_3, \pi_3))$

  we define $(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) \in \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_3, M_3, \pi_3))$ by

$$(\Phi_{23}, \phi_{23}) \circ (\Phi_{12}, \phi_{12}) = (\Phi_{23} \circ \Phi_{12}, \phi_{23} \circ \phi_{12})$$

Exercise 8.1.3.6. We have that $\mathbf{Bun}^\infty$ is a full subcategory of $(\mathrm{id}_{\mathbf{Man}^\infty} \downarrow \mathrm{id}_{\mathbf{Man}^\infty})$.

Proof. Set $\mathcal{C} = (\mathrm{id}_{\mathbf{Man}^\infty} \downarrow \mathrm{id}_{\mathbf{Man}^\infty})$. We note that

- $\mathrm{Obj}(\mathbf{Bun}^\infty) \subset \mathrm{Obj}(\mathcal{C})$
- for each $(E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$,

$$\mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2)) = \mathrm{Hom}_{\mathcal{C}}((E_1, M_1, \pi_1, F_1), (E_2, M_2, \pi_2, F_2))$$

So $\mathbf{Bun}^\infty$ is a full subcategory of $\mathcal{C}$. $\square$

Exercise 8.1.3.7. Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$ and (U, Φ) a local trivialization of E over U and (V, Ψ) a local trvialization of E over V. Then

1. $\mathrm{proj}_{U\cap V} \circ \Psi|_{\pi^{-1}(U\cap V)} \circ (\Phi|_{\pi^{-1}(U\cap V)})^{-1} = \mathrm{proj}_1$
2. there exists $\sigma \in \mathrm{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that for each $p \in U \cap V$, $\sigma(p, \cdot) : F \to F$ is a diffeomorphism.

Proof.

1. By definition, the following diagram commutes:

$$\begin{tikzcd}
(U\cap V)\times F \arrow[dr, "\mathrm{proj}_1"'] & \pi^{-1}(U\cap V) \arrow[l, "\Phi"'] \arrow[r, "\Psi"] \arrow[d, "\pi"] & (U\cap V)\times F \arrow[dl, "\mathrm{proj}_1"] \\
 & N &
\end{tikzcd}$$

$\mathrm{proj}_1 \circ \Psi|_{\pi^{-1}(U\cap V)} \circ (\Phi|_{\pi^{-1}(U\cap V)})^{-1} = \mathrm{proj}_1$

2. there exists $\sigma \in \mathrm{Hom}_{\mathbf{Man}^\infty}((U \cap V) \times F, F)$ such that for each $p \in U \cap V$ and $x \in F$,

$$\Psi|_{\pi^{-1}(U\cap V)} \circ (\Phi|_{\pi^{-1}(U\cap V)})^{-1}(p, x) = (p, \sigma(p, x))$$

and $\sigma(p, \cdot) : F \to F$ is a diffeomorphism.

$\square$

Definition 8.1.3.8. Let $(E, M, \pi, F) \in \mathbf{Bun}^\infty$ and $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ a collection of smooth local trivializations of E. Then $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ is said to be a **fiber bundle atlas** if for each $p \in M$, there exists $\alpha \in A$ such that $p \in U_\alpha$. For $\alpha, \beta \in A$, we define ϕ

8.2 Subbundles

Definition 8.2.0.1.

8.3 G-Bundles

Definition 8.3.0.1. Let (E, M, π, F) be a $\mathbf{Man}^\infty$ fiber bundle, Γ an index set and for each $\alpha \in \Gamma$, $(U_\alpha, \Phi_\alpha)_{\alpha \in \Gamma}$ a local trivializations with respect to π of E over U_α. Then $(U_\alpha, \Phi_\alpha)_{\alpha \in \Gamma}$ is said to be a $\mathbf{Man}^\infty$ **bundle atlas on** E if $(U_\alpha)_\alpha$ is an open cover of E.

Definition 8.3.0.2. Let (E, M, π, F) be a $\mathbf{Man}^\infty$ fiber bundle, Γ an index set and $(U_\alpha, \Phi_\alpha)_{\alpha \in \Gamma}$ a $\mathbf{Man}^\infty$ bundle atlas on E. For each $\alpha, \beta \in \Gamma$, we define $U_{\alpha,\beta} \subset M$ and $\Phi_{\alpha,\beta} : U_{\alpha,\beta} \times F \to U_{\alpha,\beta} \times F$ by

- $U_{\alpha,\beta} = U_\alpha \cap U_\beta$
- $\Phi_{\alpha,\beta} = \Phi_\alpha|_{U_{\alpha,\beta}} \circ \Phi_\beta|_{U_{\alpha,\beta}}^{-1}$

Exercise 8.3.0.3. Let (E, M, π, F) be a $\mathbf{Man}^\infty$ fiber bundle, Γ an index set and $(U_\alpha, \Phi_\alpha)_{\alpha \in \Gamma}$ a $\mathbf{Man}^\infty$ bundle atlas on E. Then for each $\alpha, \beta \in \Gamma$ and $p \in U_{\alpha,\beta}$, $\Phi_{\alpha,\beta}(p, \cdot) \in \mathrm{Aut}(F)$.

Proof. Let $\alpha, \beta \in \Gamma$ and $p \in U_{\alpha,\beta}$. Since □

Exercise 8.3.0.4. Cocycle Condition:
Let (E, M, π, F) be a $\mathbf{Man}^\infty$ fiber bundle, Γ an index set and $(U_\alpha, \Phi_\alpha)_{\alpha \in \Gamma}$ a $\mathbf{Man}^\infty$ bundle atlas on E. Then for each $\alpha, \beta \in \Gamma$ and $p \in U_{\alpha,\beta}$, $\Phi_{\alpha,\beta}(p, \cdot) \in \mathrm{Aut}(F)$.

Proof. Let $\alpha, \beta \in \Gamma$ and $p \in U_{\alpha,\beta}$. Since □

Definition 8.3.0.5.

8.4 Product Bundles

Definition 8.4.0.1.

8.5 Vertical and Horizontal Subbundles

Definition 8.5.0.1. Let $(E, M, \pi_M) \in \text{Obj}(\mathbf{Bun}^\infty)$. We define the **vertical bundle associated to** (E, M, π_M), denoted $(VE, M, \pi_V) \in \mathbf{Bun}^\infty$, by

$$VE = \coprod_{q \in E} \ker D\pi(q)$$

relocate this to after tangent bundle is introduced

Exercise 8.5.0.2. Let $(M, \mathcal{A})$ be an n-dimensional smooth manifold and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \dots, x^n)$, $(\pi^{-1}(U), \Phi_\phi) \in \mathcal{A}_{TM}$ the induced chart on TM with $\Phi_\phi = (\tilde{x}^1, \dots, \tilde{x}^n, \tilde{y}^1, \dots, \tilde{y}^n)$. Then

$$V(TM)|_{\pi^{-1}(U)} = \coprod_{(p,\xi) \in \pi^{-1}(U)} \text{span}\left\{ \frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \dots, n\} \right\}$$

Split into smaller exercises

Proof. Let $f \in C^\infty(M)$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$ the standard coordinates on $\mathbb{R}^n \times \mathbb{R}^n$. We note that by definition, $\Phi_\phi(p, \xi) = (\phi(p), \psi(\xi))$ where $\psi : \bigcup_{p \in U} T_pM \to \mathbb{R}^n$ is given by

$$\psi\left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p \right) = (\xi^1, \dots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{(\phi(p),\psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial v^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i}\bigg|_{(\phi(p),\psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial v^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= 0 \end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned} D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} f \circ \pi \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi))\frac{\partial x^k \circ \pi}{\partial \tilde{x}^i}(p,\xi) \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)\delta_{i,k} \\ &= \frac{\partial f}{\partial x^i}(p) \end{aligned}$$

and

$$\begin{aligned} D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} f \circ \pi \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(\pi(p,\xi))\frac{\partial x^k \circ \pi}{\partial \tilde{y}^i}(p,\xi) \\ &= \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(p)0 \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \ker D\pi(p,\xi) \\ &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \operatorname{span}\left\{\frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \dots, n\}\right\} \end{aligned}$$

$\square$

Chapter 9

G-Bundles

Definition 9.0.0.1. Let G be a Lie group and $(E, M, \pi, F) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$. Then

Chapter 10

Vector Bundles

Note 10.0.0.1. Let M be a set and $p \in M$. We endow $\{p\} \times \mathbb{R}^n$ with the natural vector space structure such that $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

Definition 10.0.0.2. Let $E, M \in \mathrm{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \mathrm{Hom}_{\mathbf{Man}^\infty}(E, M)$ a surjection. Then (E, M, π) is said to be a **rank k smooth vector bundle** if

1. $(E, M, \pi, \mathbb{R}^k) \in \mathrm{Obj}(\mathbf{Bun}^\infty)$
2. for each $p \in M$, E_p is a k-dimensional real vector space
3. for each smooth local trivialization (U, Φ) of E over U with fiber $\mathbb{R}^k$ and $p \in U$,

$$\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$$

   is a vector space isomorphism

In this case we define the **rank of** (E, M, π), denoted $\mathrm{rank}(E, M, \pi)$, by $\mathrm{rank}(E, M, \pi) = k$.

Definition 10.0.0.3. We define the category of smooth vector bundles, denoted $\mathbf{VecBun}^\infty$, by

- $\mathrm{Obj}(\mathbf{VecBun}^\infty) = \{(E, M, \pi) : (E, M, \pi) \text{ is a smooth vector bundle}\}$
- For $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ with $\mathrm{rank}(E_1, M_1, \pi_1) = k_1$ and $\mathrm{rank}(E_2, M_2, \pi_2) = k_2$,

$$\mathrm{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

Exercise 10.0.0.4. We have that $\mathbf{VecBun}^\infty$ is a full subcategory of $\mathbf{Bun}^\infty$.

Proof. We note that

- $\mathrm{Obj}(\mathbf{VecBun}^\infty) \subset \mathrm{Obj}(\mathbf{Bun}^\infty)$
- for each $(E_1, M_1, \pi_1), (E_2, M_2, \pi_2) \in \mathrm{Obj}(\mathbf{VecBun}^\infty)$ with $\mathrm{rank}(E_1, M_1, \pi_1) = k_1$ and $\mathrm{rank}(E_2, M_2, \pi_2) = k_2$,

$$\mathrm{Hom}_{\mathbf{VecBun}^\infty}((E_1, M_1, \pi_1), (E_2, M_2, \pi_2)) = \mathrm{Hom}_{\mathbf{Bun}^\infty}((E_1, M_1, \pi_1, \mathbb{R}^{k_1}), (E_2, M_2, \pi_2, \mathbb{R}^{k_2}))$$

So $\mathbf{Bun}^\infty$ is a full subcategory of $\mathcal{C}$. $\square$

Exercise 10.0.0.5. Let $M \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Set $n = \dim M$, $E = M \times \mathbb{R}^k$ and define $\pi : E \to M$ by $\pi(p, x) = p$. Then (E, M, π) is a rank k smooth vector bundle.

Proof.

1. For each $p \in M$, $\pi_1^{-1}(\{p\}) = \{p\} \times \mathbb{R}^k$ is an n-dimensional real vector space.
2. Let $p \in M$. Set $U = M$. Then $\pi^{-1}(U) = E$. Define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ by $\Phi = \mathrm{id}_E$. Then (U, Φ) is a smooth local trivialization of E over U.
3. Let $p \in M$. Then $\Phi|_{\pi^{-1}(\{p\})} : \pi^{-1}(\{p\}) \to \{p\} \times \mathbb{R}^k$ is clearly an isomorphism.

$\square$

Exercise 10.0.0.6. Smooth Vector Bundle Chart Lemma:
Let $M \in \mathrm{Obj}(\mathbf{Man}^\infty)$. Denote the topology on M by $\mathcal{T}_M$. Suppose that for each $p \in M$, there exists $E_p \in \mathrm{Obj}(\mathbf{Vect}_\mathbb{R})$ such that $\dim E_p = k$. We define $E \in \mathrm{Obj}(\mathbf{Set})$ and $\pi \in \mathrm{Hom}_{\mathbf{Set}}(E, M)$ by

$$E = \coprod_{p \in M} E_p$$

and $\pi(p, v) = p$. Let Γ be an index set and $(U_\alpha)_{\alpha \in \Gamma} \subset \mathcal{T}_M$. Suppose that

1. $M \subset \bigcup_{\alpha \in \Gamma} U_\alpha$
2. for each $\alpha \in \Gamma$, there exists $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that
   - $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ is a bijection
   - $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a vector space isomorphism
3. for each $\alpha, \beta \in \Gamma$, there exists $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that
   - $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ is smooth
   - $\Phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)} \circ (\Phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)})^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is given by

10.1 The Tangent Bundle

Definition 10.1.0.1. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p \in M} T_pM$$

We denote the natrual projection map by $\pi : TM \to M$.

Definition 10.1.0.2. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \ldots, x^n)$. Define $\tilde{U} \subset TM$ and $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}^n$ by

- $\tilde{U} = \pi^{-1}(U)$
-

$$\begin{aligned}\tilde{\phi}\bigg(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p\bigg) &= (\phi(p), v) \\ &= (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n)\end{aligned}$$

,

Exercise 10.1.0.3. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \ldots, x^n)$. Then $\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}$ is a bijection.

10.2 The cotangent Bundle

Definition 10.2.0.1. We define the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p \in M} T_p^*M$$

10.3 The (r, s)-Tensor Bundle

Definition 10.3.0.1.

1. the **cotangent bundle of** M, denoted T^*M, by

$$T^*M = \coprod_{p \in M} T_p^*M$$

2. the (r, s)**-tensor bundle of** M, denoted $T^r_s M$, by

$$T^r_s M = \coprod_{p \in M} T^r_s(T_p M)$$

3. the k**-alternating tensor bundle of** M, denoted $\Lambda^k(M)$, by

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k(T_p M)$$

10.4 Vector Fields

Definition 10.4.0.1. Let $X : M \to TM$. Then X is said to be a **vector field on** M if for each $p \in M$, $X_p \in T_pM$.
For $f \in \mathbb{C}^\infty(M)$, we define $Xf : M \to \mathbb{R}$ by

$$(Xf)_p = X_p(f)$$

and X is said to be **smooth** if for each $f \in \mathbb{C}^\infty(M)$, Xf is smooth.
We denote the set of smooth vector fields on M by $\Gamma^1(M)$.

Definition 10.4.0.2. Let $f \in C^\infty(M)$ and $X, Y \in \Gamma^1(M)$. We define

- $fX \in \Gamma^1(M)$ by
$$(fX)_p = f(p)X_p$$
- $X + Y \in \Gamma^1(M)$ by
$$(X+Y)_p = X_p + Y_p$$

Exercise 10.4.0.3. The set $\Gamma^1(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Exercise 10.4.0.4. Let $X \in \Gamma^1(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then

$$X|_U = \sum_{i=1}^n (Xx^i)\frac{\partial}{\partial x^i}$$

Proof. Let $p \in M$. Then $X_p \in T_pM$ and $\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$ is a basis of T_pM. So there exist $f_1(p), \cdots, f_n(p) \in \mathbb{R}$ such that $X_p = \sum\limits_{i=1}^n f^i(p) \frac{\partial}{\partial x^i}\bigg|_p$. Let $j \in \{1, \cdots, n\}$. Then,

$$\begin{aligned} X_p(x^j) &= \sum_{i=1}^n f^i(p) \frac{\partial}{\partial x^j} x^i(p) \\ &= f_j(p) \end{aligned}$$

Hence $Xx^j = f_j$ and $X|_U = \sum\limits_{i=1}^n (Xx^i)\frac{\partial}{\partial x^i}$. □

Exercise 10.4.0.5. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then for each $i \in \{1, \cdots, n\}$,

$$\frac{\partial}{\partial x^i} \in \Gamma(U)$$

Proof. Let $i \in \{1, \cdots, n\}$ and $f \in C^\infty(M)$. Define $g : M \to \mathbb{R}$ by $g = \frac{\partial}{\partial x^i} f$. Let $(V, \psi) \in \mathcal{A}$. Then for each $x \in \psi(U \cap V)$,

$$\begin{aligned} g \circ \psi^{-1}(x) &= \frac{\partial}{\partial x^i}\bigg|_{\psi^{-1}(x)} f \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi \circ \psi^{-1}(x)} f \circ \phi^{-1} \\ &= \frac{\partial}{\partial u^i}[f \circ \phi^{-1}](\phi \circ \psi^{-1}(x)) \end{aligned}$$

Since $f \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth, $g \circ \psi^{-1}$ is smooth and hence g is smooth. Since $f \in C^\infty(M)$ was arbitrary, by definition, $\frac{\partial}{\partial x^i}$ is smooth. □

10.5 1-Forms

Definition 10.5.0.1. Let $\omega : M \to T^*M$. Then ω is said to be a **1-form on** M if for each $p \in M$, $\omega_p \in T_p^*M$.
For each $X \in \Gamma^1(M)$, we define $\omega(X) : M \to \mathbb{R}$ by

$$\omega(X)_p = \omega_p(X_p)$$

and ω is said to be **smooth** if for each $X \in \Gamma^1(M)$, $\omega(X)$ is smooth.
The set of smooth 1-forms on M is denoted $\Gamma_1(M)$.

Definition 10.5.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in \Gamma^1(M)$. We define

- $f\alpha \in \Gamma_1(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta \in \Gamma^1(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 10.5.0.3. The set $\Gamma_1(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Exercise 10.5.0.4.

10.6 (r, s)-Tensor Fields

Definition 10.6.0.1. Let $\alpha : M \to T^r_s M$. Then α is said to be an (r, s)**-tensor field on** M if for each $p \in M$, $\alpha_p \in T^r_s(T_p M)$.
For each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, we define $\alpha(\omega, X) : M \to \mathbb{R}$ by

$$\alpha(\omega, X)_p = \alpha_p(\omega_p, X_p)$$

and α is said to be **smooth** if for each $\omega \in \Gamma_1(M)^r$ and $X \in \Gamma^1(M)^s$, $\alpha(\omega, X)$ is smooth.
The set of smooth (r, s)-tensor fields on M is denoted $T^r_s(M)$.

Definition 10.6.0.2. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T^r_s(M)$. We define

- $f\alpha : M \to T^r_s M$ by
$$(f\omega)_p = f(p)\omega_p$$
- $\alpha + \beta : M \to T^r_s M$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Exercise 10.6.0.3. Let $f \in C^\infty(M)$ and $\alpha, \beta \in T^r_s(M)$. Then

1. $f\alpha \in T^r_s(M)$ by
$$(f\omega)_p = f(p)\omega_p$$
2. $\alpha + \beta \in T^r_s(M)$ by
$$(\alpha + \beta)_p = \alpha_p + \beta_p$$

Proof. Clear. □

Exercise 10.6.0.4. The set $T^r_s(M)$ is a $C^\infty(M)$-module.

Proof. Clear. □

Definition 10.6.0.5. Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. We define the **tensor product of** α **with** β, denoted $\alpha \otimes \beta : M \to T^{r_1+r_2}_{s_1+s_2} M$, by
$$(\alpha \otimes \beta)_p = \alpha_p \otimes \beta_p$$

Exercise 10.6.0.6. Let $\alpha_1 \in \Gamma^{r_1}_{s_1}(M)$ and $\alpha_2 \in \Gamma^{r_2}_{s_2}(M)$. Then $\alpha_1 \otimes \alpha_2 \in \Gamma^{r_1+r_2}_{s_1+s_2}(M)$

Proof. Let $\omega_1 \in \Gamma_1(M)^{r_1}$, $\omega_2 \in \Gamma_1(M)^{r_2}$, $X_1 \in \Gamma^1(M)^{s_1}$ and $X_2 \in \Gamma^1(M)^{s_2}$. By definition,

$$\alpha_1 \otimes \alpha_2(\omega_1, \omega_2, X_1, X_2) = \alpha_1(\omega_1, X_1)\alpha_2(\omega_2, X_2)$$

This implies that $\alpha_1 \otimes \alpha_2$ is smooth since α_1 and α_2 are smooth by assumption. □

Definition 10.6.0.7. We define the **tensor product**, denoted $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ by

$$(\alpha_1, \alpha_2) \mapsto \alpha_1 \otimes \alpha_2$$

Exercise 10.6.0.8. The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is associative.

Proof. Clear. □

Exercise 10.6.0.9. The tensor product $\otimes : \Gamma^{r_1}_{s_1}(M) \times \Gamma^{r_2}_{s_2}(M) \to \Gamma^{r_1+r_2}_{s_1+s_2}(M)$ is $C^\infty(M)$-bilinear.

Proof. Clear. □

Definition 10.6.0.10. Let $(N, \mathcal{B})$ be a smooth manifold, $F : M \to N$ a smooth map and $\alpha \in \Gamma^0_k(N)$. We define the **pullback of** α **by** F, denoted $F^*\alpha \in \Gamma^0_k(M)$, by

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k))$$

for $p \in M$ and $v_1, \dots, v_k \in T_pM$

Exercise 10.6.0.11. Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ and $(L, \mathcal{C})$ be smooth manifolds, $F : M \to N$ and $G : N \to L$ smooth maps, $\alpha \in \Gamma^0_k(N)$, $\beta \in \Gamma^0_l(N)$, $\gamma \in \Gamma^0_k(L)$ and $f \in C^\infty(N)$. Then

1. $F^*(f\alpha) = (f \circ F)F^*\alpha$
2. $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$
3. $F^*(\alpha + \beta) = F^*\alpha + F^*\beta$
4. $(G \circ F)^*\gamma = F^*(G^*\gamma)$
5. $id_N^*\alpha = \alpha$

Proof.

1.

$$\begin{aligned}
[F^*(f\alpha)]_p(v_1, \dots, v_k) &= (f\alpha)_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\
&= f(F(p))\alpha_{F(p)}(DF_p(v_1), \dots, DF_p(v_k)) \\
&= (f \circ F)(p)(F^*\alpha)_p(v_1, \dots, v_k)
\end{aligned}$$

So that $F^*(f\alpha) = (f \circ F)F^*\alpha$

2.

$$F^*$$

□

Definition 10.6.0.12.

Exercise 10.6.0.13.

Proof. □

Exercise 10.6.0.14. Let $\alpha \in T^r_s(M)$ and $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$. Then there exist $(f^I_J)_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset C^\infty(M)$ such that

$$\alpha|_U = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J \partial_{x^{\otimes I}} \otimes dx^{\otimes J}$$

Proof. Let $p \in M$. Then $\omega_p \in T^r_s(T_pM)$ and $\left\{\partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}\right\}$ is a basis of $T^r_s(T_pM)$. So there exist $(f^I_J(p))_{I \in \mathcal{I}_r, J \in \mathcal{I}_s} \subset \mathbb{R}$ such that

$$\omega_p = \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}$$

Let $(K, L) \in \mathcal{I}_r \times \mathcal{I}_s$. Then

$$\begin{aligned}
\alpha_p(dx_p^K, \partial_{x^L}|_p) &= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p \otimes dx_p^{\otimes J}(dx_p^K, \partial_{x^L}|_p) \\
&= \sum_{(I,J) \in \mathcal{I}_r \times \mathcal{I}_s} f^I_J(p) \partial_{x^{\otimes I}}|_p(dx_p^K) dx_p^{\otimes J}(\partial_{x^L}|_p) \\
&= f^K_L(p)
\end{aligned}$$

By assumption, the map $p \mapsto \alpha(dx^K, \partial_{x^L})_p$ is smooth, so that $f^K_L \in C^\infty(U)$.

□

Definition 10.6.0.15.

10.7 Differential Forms

Definition 10.7.0.1. We define
$$\Lambda^k(TM) = \coprod_{p \in M} \Lambda^k(T_pM)$$

Definition 10.7.0.2. Let $\omega : M \to \Lambda^k(TM)$. Then ω is said to be a **k-form on** M if for each $p \in M$, $\omega_p \in \Lambda^k(T_pM)$.
For each $X \in \Gamma^1(M)^k$, we define $\omega(X) : M \to \mathbb{R}$ by
$$\omega(X)_p = \omega_p(X_p)$$
and ω is said to be **smooth** if for each $X \in \Gamma^1(M)^k$, $\omega(X)$ is smooth.
The set of smooth k-forms on M is denoted $\Omega^k(M)$.

Note 10.7.0.3. Observe that

1. $\Omega^k(M) \subset \Gamma^0_k(M)$
2. $\Omega^0(M) = C^\infty(M)$

Exercise 10.7.0.4. The set $\Omega^k(M)$ is a $C^\infty(M)$-submodule of $\Gamma^0_k(M)$.

Proof. Clear. □

Definition 10.7.0.5. Define the **exterior product**
$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$
by
$$(\alpha \wedge \beta)_p = (\alpha)_p \wedge (\beta)_p$$

Note 10.7.0.6. For $f \in \Omega^0(M)$ and $\alpha \in \Omega^k(M)$, we have that $f \wedge \alpha = f\alpha$.

Exercise 10.7.0.7. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is well defined.

Proof. Let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$, $(x^i)_{i=1}^k \subset \Gamma^1(M)$, $(y^j)_{i=1}^l \subset \Gamma^1(M)$ and $p \in M$. Then
$$\begin{aligned}
\alpha \wedge \beta(X_1, \dots, X_{k+l})_p &= (\alpha \wedge \beta)_p(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\sigma(\alpha_p \otimes \beta_p)(X_1(p), \cdots, X_{k+l}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)(\alpha_p \otimes \beta_p)(X_{\sigma(1)}(p), \cdots, X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p)) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\alpha_p(X_{\sigma(1)}(p), \dots, X_{\sigma(k)}(p))\beta(X_{\sigma(k+1)(p)}, \dots X_{\sigma(k+l)}(p))
\end{aligned}$$
□

Exercise 10.7.0.8. The exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is $C^\infty(M)$-bilinear.

Proof.

1. $C^\infty(M)$-linearity in the first argument:
   Let $\alpha \in \Omega^k(M)$, $\beta, \gamma \in \Omega^l(M)$, $f \in C^\infty(M)$ and $p \in M$. Bilinearity of $\wedge : \Lambda^k(T_pM) \times \Lambda^l(T_pM) \to \Lambda^{k+l}(T_pM)$ implies that

$$\begin{aligned}[(\beta + f\gamma) \wedge \alpha]_p &= (\beta + f\gamma)_p \wedge \alpha_p \\ &= (\beta_p + f(p)\gamma_p) \wedge \alpha_p \\ &= \beta_p \wedge \alpha_p + f(p)(\gamma_p \wedge \alpha_p) \\ &= [\beta \wedge \alpha + f(\gamma \wedge \alpha)]_p\end{aligned}$$

   So that

$$(\beta + f\gamma) \wedge \alpha = \beta \wedge \alpha + f(\gamma \wedge \alpha)$$

   and $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ is $C^\infty(M)$-linear in the first argument.

2. $C^\infty(M)$-linearity in the second argument:
   Similar to (1).

□

Note 10.7.0.9. All of the results from multilinear algebra apply here.

Definition 10.7.0.10. We define the **exterior derivative** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ inductively by

1. $d(d\alpha) = 0$ for $\alpha \in \Omega^p(M)$
2. $df(X) = Xf$ for $f \in \Omega^0(M)$
3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$
4. extending linearly

Exercise 10.7.0.11. Let (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then on U, for each $i, j \in \{1, \cdots, n\}$,

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{i,j}$$

In particular, for each $p \in U$, $\{dx^1_p, \cdots, dx^n_p\}$ is the dual basis to $\left\{\frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p\right\}$ and $T^*_pM = \operatorname{span}\{dx^1_p, \cdots, dx^n_p\}$.

Proof. Let $p \in U$ and $i, j \in \{1, \cdots, n\}$. Then by defintion,

$$\begin{aligned}\left[dx^i\left(\frac{\partial}{\partial x^j}\right)\right]_p &= \left(\frac{\partial}{\partial x^j}x^i\right)_p \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^i \\ &= \delta_{i,j}\end{aligned}$$

□

Exercise 10.7.0.12. Let $f \in C^\infty(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then

$$df|_U = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$$

Proof. Let $p \in U$. Since $\{dx^1, \cdots, dx^n\}$ is a basis for $\Lambda(T_pM)$, for each there exist $a_1(p), \cdots, a_n(p) \in \mathbb{R}$ such that $df_p = \sum\limits_{i=1}^{n} a^i(p)dx^i_p$. Therefore, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) &= \sum_{i=1}^{n} a^i(p)dx^i_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) \\ &= a_j(p) \end{aligned}$$

By definition, we have that

$$\begin{aligned} df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) &= \frac{\partial}{\partial x^i}\bigg|_p f \\ &= \frac{\partial f}{\partial x^j}(p) \end{aligned}$$

So $a_j(p) = \frac{\partial f}{\partial x^j}(p)$ and

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p)dx^i_p$$

Therefore

$$df|_U = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}dx^i$$

□

Exercise 10.7.0.13. Let $f \in \Omega^0(M)$. If f is constant, then $df = 0$.

Proof. Suppose that f is constant. Let $p \in M$. Choose $(U, \phi) \in \mathcal{A}$ such that $p \in U$. Write $\phi = (x_1, \ldots, x_n)$. Then for each $i \in \{1, \ldots, n\}$,

$$\frac{\partial}{\partial x^i}\bigg|_p f = 0$$

This implies that

$$\begin{aligned} df_p &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^j}(p)dx^i_p \\ &= 0 \end{aligned}$$

□

Exercise 10.7.0.14.

Definition 10.7.0.15. Let $(U, \phi) \in \mathcal{A}$ with $\phi = (x^1, \cdots, x^n)$ and $I = (i_1, \cdots, i_k) \in \mathcal{I}_k$. We define

$$dx^i = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k(M)$$

and we define

$$\frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_k}}\right)$$

Note 10.7.0.16. We have that

1. $$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{I,J}$$

2. Since $\frac{\partial}{\partial x^i} \in \Gamma(U)^k$, by definition, for each $\omega \in \Omega^k(U)$,

$$\omega\left(\frac{\partial}{\partial x^i}\right) \in C^\infty(U)$$

Exercise 10.7.0.17. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. Then

$$\omega = \sum_{I \in \mathcal{I}_k} \omega\left(\frac{\partial}{\partial x^i}\right) dx^i$$

Proof. Let $p \in U$. Since $\{dx^i_p : I \in \mathcal{I}_k\}$ is a basis for $\Lambda^k(T_pM)$, there exists $(f_I(p))_{I \in \mathcal{I}} \subset \mathbb{R}$ such that $\omega_p = \sum\limits_{I \in \mathcal{I}_k} f_I(p) dx^i_p$. So for each $J \in \mathcal{I}_k$,

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial x^j}\right) &= \sum_{I \in \mathcal{I}_k} f_I dx^i \left(\frac{\partial}{\partial x^j}\right) \\
&= f_J
\end{aligned}$$

□

Exercise 10.7.0.18. Let $\omega \in \Omega^k(M)$ and (U, ϕ) be a chart on M with $\phi = (x^1, \cdots, x^n)$. If $\omega = \sum\limits_{I \in \mathcal{I}_k} f_I dx^i$, then

$$d\omega = \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i$$

.

Proof. First we note that

$$\begin{aligned}
d(f_I dx^i) &= df_I \wedge dx^i + (-1)^0 f d(dx^i) \\
&= df_I \wedge dx^i \\
&= \left(\sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i\right) \wedge dx^i \\
&= \sum_{i=1}^n \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^i
\end{aligned}$$

Then we extend linearly. □

Definition 10.7.0.19. Let $(N, \mathcal{B})$ be a smooth manifold and $F : M \to N$ be a diffeomorphism. Define the **pullback of** F, denoted $F^* : \Omega^k(N) \to \Omega^k(M)$ by

$$(F^*\omega)_p(v_1, \cdots, v_k) = \omega_{F(p)}(DF_p(v_1), \cdots, DF_p(v_k))$$

for $\omega \in \Omega^k(N)$, $p \in M$ and $v_1, \cdots, v_k \in T_pM$

Chapter 11

Vector Fields

11.1 The Tangent Bundle

Definition 11.1.0.1. Let $(M, \mathcal{A}_M)$ be an n-dimensional smooth manifold. We define the **tangent bundle of** M, denoted TM, by

$$TM = \coprod_{p \in M} T_pM$$

and we define the **tangent bundle projection**, denoted $\pi : TM \to M$, by

$$\pi(p, v) = p$$

Let $(U, \phi) \in \mathcal{A}_M$ with $\phi = (x^1, \ldots, x^n)$. We define $\Phi_\phi : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\Phi_\phi\left(p, \sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p\right) = (\phi(p), \xi^1, \ldots, \xi^n)$$

We define $\mathcal{T}_{TM} = \tau_{TM}(\iota_p : p \in M)$.

Exercise 11.1.0.2. $\psi : \bigcup_{p \in U} T_pM \to \mathbb{R}^n$ is given by

$$\psi\left(\sum_{j=1}^n \xi^j \frac{\partial}{\partial x^j}\bigg|_p\right) = (\xi^1, \ldots, \xi^n)$$

$$\begin{aligned} x^k \circ \pi \circ \Phi_\phi^{-1}(u, v) &= x^k \circ \pi(\phi^{-1}(u), \psi^{-1}(v)) \\ &= x^k \circ \phi^{-1}(u) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} [x^k \circ \pi] &= \frac{\partial}{\partial u^i}\bigg|_{\Phi_\phi(p,\xi)} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{(\phi(p), \psi(\xi))} [x^k \circ \pi \circ \Phi_\phi^{-1}] \\ &= \frac{\partial}{\partial u^i}\bigg|_{\phi(p)} [x^k \circ \phi^{-1}] \\ &= \frac{\partial}{\partial x^i}\bigg|_p x^k \\ &= \delta_{i,k} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}[x^k \circ \pi] &= \frac{\partial}{\partial v^i}\bigg|_{\Phi_\phi(p,\xi)}[x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \frac{\partial}{\partial v^i}\bigg|_{(\phi(p),\psi(\xi))}[x^k \circ \pi \circ \Phi_\phi^{-1}] \\
&= \frac{\partial}{\partial v^i}\bigg|_{\phi(p)}[x^k \circ \phi^{-1}] \\
&= 0
\end{aligned}$$

This implies that for each $i \in \{1, \dots, n\}$, we have that

$$\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{x}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(\pi(p,\xi))\frac{\partial x^k \circ \pi}{\partial \tilde{x}^i}(p,\xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(p)\delta_{i,k} \\
&= \frac{\partial f}{\partial x^i}(p)
\end{aligned}$$

and

$$\begin{aligned}
D\pi(p,\xi)\left(\frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)}\right)(f) &= \frac{\partial}{\partial \tilde{y}^i}\bigg|_{(p,\xi)} f \circ \pi \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(\pi(p,\xi))\frac{\partial x^k \circ \pi}{\partial \tilde{y}^i}(p,\xi) \\
&= \sum_{k=1}^n \frac{\partial f}{\partial x^k}(p)0 \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
V(TM)|_{\pi^{-1}(U)} &= \coprod_{(p,\xi)\in\pi^{-1}(U)} \ker D\pi(p,\xi) \\
&= \coprod_{(p,\xi)\in\pi^{-1}(U)} \operatorname{span}\left\{\frac{\partial}{\partial \tilde{y}^j}\bigg|_{(p,\xi)} : j \in \{1, \dots, n\}\right\}
\end{aligned}$$

Chapter 12

Lie Theory

12.1 Lie Groups

Definition 12.1.0.1. Let G be a smooth manifold and group. Then G is said to be a **Lie group** if

- multiplication $G \times G \to G$ given by $(g, h) \mapsto gh$ is smooth
- inversion $G \to G$ given by $g \mapsto g^{-1}$ is smooth

Definition 12.1.0.2. Let $\mathfrak{g}$ be a vector space and $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $[\cdot, \cdot]$ is said to be a **Lie bracket** on $\mathfrak{g}$ if

1. $[\cdot, \cdot]$ is bilinear
2. $[\cdot, \cdot]$ is antisymmetric
3. $[\cdot, \cdot]$ satisfies the Jacobi identity:
   for each $x, w, y \in \mathcal{F}g$,
   $$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

In this case, $(\mathfrak{g}, [\cdot, \cdot])$ is said to be a **Lie algebra**.

Definition 12.1.0.3. Let $X \in$

Chapter 13

de Rham Cohomology

13.1 TO DO

1. de Rham cohomology
2. de Rham homology
3. in de Rham homology, measures on the manifold can be identified with the 0th Homology, group
4. think about how the other homology groups can be used in statistics

13.2 Introduction

Note 13.2.0.1. We recall that $d : \Omega^*(M) \to \Omega^*(M)$ satisfies the properties:

1. $d^2 = 0$
2.
3.

Definition 13.2.0.2. Let M be an n-dimensional smooth manifold. For $k \in \{1, \dots, n\}$, we define the

- **k-th coboundary operator**, denoted $d^k : \Omega^k(M) \to \Omega^{k+1}(M)$, by $d^k = d|_{\Omega^k(M)}$
-
-

Chapter 14

Jet Bundles

14.1 Fibered Manifolds

Definition 14.1.0.1. Let $E, M \in \text{Obj}(\mathbf{Man}^\infty)$ and $\pi \in \text{Hom}_{\mathbf{Man}^\infty}(E, M)$. Then (E, M, π) is said to be a **smooth fibered manifold** if π is a surjective submersion.

Note 14.1.0.2. We write $\text{proj}_1^n : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ to denote the projection onto M.

Definition 14.1.0.3. Let (E, M, π) be a smooth fibered manifold and $(V, \psi) \in \mathcal{A}_E$. Set $n := \dim M$ and $k := \dim E - n$. Then (V, ψ) is said to be a **π-fibered chart on** E if there exists $(U, \phi) \in \mathcal{A}_M$ such that

1. $U = \pi(V)$
2. $\phi \circ \pi|_V = \text{proj}_1^n \circ \psi$

i.e. if $\psi = (y^1, \dots, y^{n+k})$ and $\phi = (x^1, \dots, x^n)$, then $\psi = (x^1 \circ \pi, \dots, x^n \circ \pi, y^{n+1}, \dots, y^{n+k})$.

Exercise 14.1.0.4. Let (E, M, π) be a smooth fibered manifold. Then for each $a \in E$, there exists $(V, \psi) \in \mathcal{A}_E$ such that $a \in V$ and (V, ψ) is a π-fibered chart on E.
Hint: Constant rank theorem

Proof. Set $n := \dim M$, $k := \dim E - n$. Let $a \in E$. Set $p := \pi(a)$. Since $\pi : E \to M$ is a submersion, π has constant rank and $\text{rank}\, \pi = n$. The constant rank theorem implies that there exist $(V_0, \psi_0) \in \mathcal{A}_E$, $(U_0, \phi_0) \in \mathcal{A}_M$ such that $a \in V_0$, $p \in U$ and $\phi_0 \circ \pi \circ \psi_0^{-1} = \text{proj}_1^n |_{\psi_0(V \cap \pi^{-1}(U))}$. Hence $\phi_0 \circ \pi = \text{proj}_1^n \circ \psi_0$. Define $V := V_0 \cap \pi^{-1}(U_0)$, $U = U_0 \cap \pi(V_0)$, $\psi = \psi_0|_V$ and $\phi = \phi_0|_U$. Then

1.
$$\begin{aligned} \pi(V) &= \pi(\pi^{-1}(U_0) \cap V_0) \\ &= U_0 \cap \pi(V_0) \\ &= U \end{aligned}$$

2.
$$\begin{aligned} \phi \circ \pi|_V &= \phi_0|_U \circ \pi|_V \\ &= \text{proj}_1^n \circ \psi_0|_V \\ &= \text{proj}_1^n \circ \psi \end{aligned}$$

So that (V, ψ) is a π-fibered chart on E. $\square$

Exercise 14.1.0.5. Let (E, M, π, F) be a $\mathbf{Man}^\infty$ fiber bundle with total space E, base space M, fiber F and projection π. Then (E, M, π) is a smooth fibered manifold.

Proof. Let $a \in E$. Set $p = \pi(a)$. Then there exists $U \in \mathcal{N}_p$ and $\Phi : \pi^{-1}(U) \to U \times F$ such that U is open and (U, Φ) is a smooth local trivialization of E over U with fiber F. Then Φ is a diffeomorphim and

$$\begin{aligned}\operatorname{rank}_a \pi &= \operatorname{rank} D\pi(a) \\ &= \operatorname{rank} D \operatorname{proj}_1(\Phi(a)) \\ &= \dim M\end{aligned}$$

Since $a \in E$ is arbitrary, π has constant rank. Thus π is a submersion. Hence (E, M, π) is a smooth fibered manifold. $\square$

Chapter 15

Connections

15.1 Koszul Connections

Definition 15.1.0.1. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$ and $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$. Then ∇ is said to be a **Koszul connection on E in the first representation** if

1. for each $\sigma \in \Gamma(E)$, $\nabla(\cdot, \sigma)$ is $C^\infty(M)$-linear
2. for each $X \in \mathfrak{X}(M)$, $\nabla(X, \cdot)$ is $\mathbb{R}$-linear
3. for each $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\nabla(X, f\sigma) = f\,\nabla(X, \sigma) + X(f)\sigma$$

Definition 15.1.0.2. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$ be a smooth vector bundle and $\nabla : \Gamma(E) \to T^*M \otimes \Gamma(E)$. Then ∇ is said to be a **Koszul connection on E in the second representation** if

1. ∇ is $\mathbb{R}$-linear
2. for each $\sigma \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\nabla(f\sigma) = f\,\nabla\,\sigma + df \otimes \sigma$$

Note 15.1.0.3. When the context is clear, we will write $\nabla_X Y$ in place of $\nabla(X, Y)$ and we will refer to ∇ as a connection.

Exercise 15.1.0.4. Define $\phi : \Gamma(E)^{\mathfrak{X}(M) \times \Gamma(E)} \to [T^*M \otimes \Gamma(E)]^{\Gamma(E)}$ by

$$\phi(\nabla)(X) = \nabla_X\,\sigma$$

Then ∇ is a Koszul connection on E in the first representation iff $\phi(\nabla)$ Koszul connection on E in the second representation.

Proof. □

Exercise 15.1.0.5. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$, ∇ a connection on E, $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$. If $X = 0$ or $Y = 0$, then $\nabla_X\,Y = 0$.

Proof.

- If $X = 0$, then

$$\begin{aligned}\nabla_X\,Y &= \nabla_{0X}\,Y \\ &= 0\,\nabla_X\,Y \\ &= 0\end{aligned}$$

- Similarly, if $Y = 0$, then $\nabla_X Y = 0$.

$\square$

Exercise 15.1.0.6. Let (E, M, π) be a smooth vector bundle, ∇ a connection on E, $X \in \mathfrak{X}(M)$, $Y \in \Gamma(E)$ and $p \in M$. If $X \sim_p 0$ or $Y \sim_p 0$, then $[\nabla_X Y]_p = 0$.

Proof.

- Suppose that $X \sim_p 0$. Then there exists $U \subset M$ such that U is open and $X|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\operatorname{supp} \phi \subset U$ and $\phi \sim_p 1$. Then $\phi X = 0$. The previous exercise implies that $\nabla_{\phi X} Y = 0$. Therefore

$$\begin{aligned}
\nabla_X Y &= \nabla_{\phi X + (1-\phi) X} Y \\
&= \nabla_{\phi X} Y + \nabla_{(1-\phi)X} Y \\
&= 0 + (1-\phi) \nabla_X Y \\
&= (1-\phi) \nabla_X Y
\end{aligned}$$

  Hence

$$\begin{aligned}
[\nabla_X Y]_p &= [(1-\phi) \nabla_X Y]_p \\
&= (1-\phi(p))[\nabla_X Y]_p \\
&= 0
\end{aligned}$$

- Suppose that $Y \sim_p 0$. Then there exists $U \subset M$ such that U is open and $Y|_U = 0$. Choose $\phi \in C^\infty(M)$ such that $\operatorname{supp} \phi \subset U$ and $\phi \sim_p= 1$. Then $\phi Y = 0$. The previous exercise implies that $\nabla_X \phi Y = 0$. Since $\phi \sim_p 1$, we have that $1 - \phi \sim_p 0$. Thus $X(1-\phi) \sim_p 0$ and

$$\begin{aligned}
\nabla_X Y &= \nabla_X[\phi Y + (1-\phi)Y] \\
&= \nabla_X[\phi Y] + \nabla_X[(1-\phi)Y] \\
&= \nabla_X[(1-\phi)Y] \\
&= (1-\phi) \nabla_X Y + [X(1-\phi)] \nabla_X Y
\end{aligned}$$

  Hence

$$\begin{aligned}
[\nabla_X Y]_p &= (1-\phi(p))[\nabla_X Y]_p + [X(1-\phi)](p)[\nabla_X Y]_p \\
&= 0
\end{aligned}$$

$\square$

Exercise 15.1.0.7. Let (E, M, π) be a smooth vector bundle and ∇ a connection on E. Then for each $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$, $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$ implies that $[\nabla_{X_1} Y_1]_p = [\nabla_{X_2} Y_2]_p$.

Proof. Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \Gamma(E)$. Suppose that $X_1 \sim_p X_2$ and $Y_1 \sim_p Y_2$. Define $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$ by $X = X_2 - X_1$ and $Y = Y_2 - Y_1$. Then $X \sim_p 0$ and $Y \sim_p 0$. The previous exercise implies that $[\nabla_X Y_1]_p = 0$ and $[\nabla_{X_2} Y]_p = 0$. Therefore

$$\begin{aligned}
[\nabla_{X_1} Y_1]_p &= [\nabla_{X_1} Y_1]_p + [\nabla_X Y_1]_p \\
&= [\nabla_{X_1} Y_1 + \nabla_X Y_1]_p \\
&= [\nabla_{X_1+X} Y_1]_p \\
&= [\nabla_{X_2} Y_1]_p \\
&= [\nabla_{X_2} Y_1]_p + [\nabla_{X_2} Y]_p \\
&= [\nabla_{X_2} Y_1 + \nabla_{X_2} Y]_p \\
&= [\nabla_{X_2}(Y_1 + Y)]_p \\
&= [\nabla_{X_2} Y_2]_p
\end{aligned}$$

□

Exercise 15.1.0.8. Let (E, M, π) be a smooth vector bundle, ∇ a connection on E and $U \subset M$. If U is open, then there exists a unique connection $\nabla^U : \mathfrak{X}(U) \times \Gamma(E|_U) \to \Gamma(E|_U)$ such that for each $X \in \mathfrak{X}(M)$ and $Y \in \Gamma(E)$,

$$\nabla^U_{X|_U} Y|_U = (\nabla_X Y)|_U$$

Chapter 16

Semi-Riemannian Geometry

Definition 16.0.0.1. Let M be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then g is said to be nondegenerate if for each $p \in M$, g_p is nondegenerate.

Definition 16.0.0.2. Let M be a manifold and $g \in \Gamma(\Sigma^2 M)$. Then g is said to be a **metric tensor field** on M if

1. g is nondegenerate
2. g has constant index

In this case (M, g) is said to be a **semi-Riemannian manifold**

Definition 16.0.0.3. Define Interval
FINISH!!!

Definition 16.0.0.4. Let $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ and $\gamma \in \text{Hom}_{\mathbf{Man}^\infty}(I, E)$. Then γ is said to be a **section of** E **over** α if $\pi \circ \gamma = \alpha$. We denote the set of sections of E over α by $\Gamma(E, \alpha)$.

Definition 16.0.0.5. Let $(E, M, \pi) \in \text{Obj}(\mathbf{Bun}^\infty)$, $I \subset \mathbb{R}$ an interval, $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$ and $\gamma \in \Gamma(E, \alpha)$. Then γ is said to be said to be **extendible** if there exists $U \in \mathcal{N}_{\alpha(I)}$ and $\tilde{\gamma} \in \Gamma(E|_U)$ such that U is open and $\tilde{\gamma} \circ \alpha = \gamma$.

Exercise 16.0.0.6. figure 8 not extendible FINISH!!!

Exercise 16.0.0.7. Let $(E, M, \pi) \in \text{Obj}(\mathbf{VecBun}^\infty)$, ∇ a connection on E, $I \subset \mathbb{R}$ an interval and $\alpha \in \text{Hom}_{\mathbf{Man}^\infty}(I, M)$. There exists a unique $D_\alpha : \Gamma(E, \alpha) \to \Gamma(E, \alpha)$ such that

1. for each $\lambda \in \mathbb{R}$ and $\gamma, \sigma \in \Gamma(E, \alpha)$,
$$D_\alpha(\gamma + \lambda\sigma) = D_\alpha\gamma + \lambda D_\alpha\sigma$$
2. for each $f \in C^\infty(I)$ and $\gamma \in \Gamma(E, \alpha)$,
$$D_\alpha(f\gamma) = f'\gamma + f D_\alpha\gamma$$
3. for each $\gamma \in \Gamma(E)$, if $\tilde{\gamma}$ extends γ, then
$$D_\alpha\gamma = \nabla_{\alpha'}\gamma$$

Proof. $\square$

Chapter 17

Riemannian Geometry

Definition 17.0.0.1. Let M be a smooth manifold and $g \in T^0_2(M)$ a metric tensor on M. We define $\hat{g} \in T^2_0(M)$ by $\hat{g}(\omega, \eta) = g(\phi_g^{-1}(\omega), \phi_g^{-1}(\eta))$.

Exercise 17.0.0.2. content...

Exercise 17.0.0.3. Let (M, g) be a semi-Riemannian manifold and $(U, \phi) \in \mathcal{A}$. Then the induced metric $\langle\rangle_{T^*M \otimes TM}$ on $T^*M \otimes TM$ is given by

$$\left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} = g^{i,j} g_{kl}$$

Proof. We have that

$$\begin{aligned} \left\langle dx^i \otimes \frac{\partial}{\partial x^k}, dx^j \otimes \frac{\partial}{\partial x^l} \right\rangle_{T^*M \otimes TM} &= \langle dx^i, dx^j \rangle_{T^*M} \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} \right\rangle_{TM} \\ &= g^{i,j} g_{k,l} \end{aligned}$$

□

Exercise 17.0.0.4. Let (M, g) be an n-dimensional Riemannian manifold.

1. There exists $\lambda \in \Omega^n(M)$ such that for each orthonormal frame $e_1, \dots, e_n$,

$$\lambda(e_1, \dots, e_n) = 1$$

   Hint: Choose a frame $z_1, \dots, z_n$ on M with corresponding dual frame $\zeta^1, \dots, \zeta^n$. Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n$$

2. Let $N \in \mathfrak{X}(M)$ be the outward pointing normal to ∂M and $X \in \mathfrak{X}(M)$. Then

$$\int_M \operatorname{div} X \lambda = \int_{\partial M} g(X, N) \tilde{\lambda}$$

3. For each $u \in \mathbb{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, we have that

$$\operatorname{div}(uX) = u \operatorname{div}(X) + du(X)$$

   and therefore

$$\int_M du(X) \lambda = \int_{\partial M} u g(X, N) \tilde{\lambda} - \int_M u \operatorname{div}(X) \lambda$$

Proof.

1. Let $z_1, \dots, z_n$ be a frame on M and $\zeta^1, \dots, \zeta^n$ with corresponding dual frame $\zeta^1, \dots, \zeta^n$. Define

$$\lambda = \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n$$

Let $e_1, \dots, e_n$, be an orthonormal frame on M with corresponding dual coframe $\epsilon^1, \dots, \epsilon^n$. Let $i, j \in \{1, \dots, n\}$. Then there exist $(a_{k,i}) \subset \mathbb{R}$ such that $\zeta^i = \sum_{k=1}^n a_{k,i} \epsilon^k$. Then

$$\begin{aligned}
\hat{g}(\epsilon^j, \zeta^i) &= \sum_{k=1}^n a_{k,i} \hat{g}(\epsilon^j, \epsilon^k) \\
&= \sum_{k=1}^n a_{k,i} g(\phi_g^{-1}(\epsilon^j), \phi_g^{-1}(\epsilon^k)) \\
&= \sum_{k=1}^n a_{k,i} g(e_j, e_k) \\
&= \sum_{k=1}^n a_{k,i} \delta_{j,k} \\
&= a_{j,i}
\end{aligned}$$

which implies that

$$\begin{aligned}
\delta_{i,j} &= \zeta^i(z_j) \\
&= \sum_{k=1}^n a_{k,i} \epsilon^k(z_j) \\
&= \sum_{k=1}^n a_{k,i} g(e_k, z_j) \\
&= \sum_{k=1}^n \hat{g}(\epsilon^k, \zeta^i) g(e_k, z_j)
\end{aligned}$$

Define $U, V \in \mathbb{R}^{n \times n}$ by $U_{i,k} = \hat{g}(\zeta^i, \epsilon^k)$ and $V_{k,j} = g(e_k, z_j)$. Then from above, we have that $UV = I$. Since $U, V \in \mathbb{R}^{n \times n}$, $VU = I$. Hence $U = V^{-1}$. Since

$$\begin{aligned}
\zeta^i(e_j) &= \sum_{k=1}^n a_{k,i} \epsilon^k(e_j) \\
&= \sum_{k=1}^n a_{k,i} \delta_{k,j} \\
&= a_{j,i} \\
&= \hat{g}(\epsilon^j, \zeta^i) \\
&= U_{i,j}
\end{aligned}$$

and

$$\begin{aligned} g(z_i, z_j) &= \left(\sum_{k=1}^{n} g(e_k, z_i) e_k, \sum_{l=1}^{n} g(e_l, z_j) e_l \right) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} g(e_k, z_i) g(e_l, z_j) g(e_k, e_l) \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} g(e_k, z_i) g(e_l, z_j) \delta_{k,l} \\ &= \sum_{k=1}^{n} g(e_k, z_i) g(e_k, z_j) \\ &= (V^* V)_{i,j} \end{aligned}$$

we have that

$$\begin{aligned} \lambda(e_1, \ldots, e_n) &= \det[g(z_i, z_j)]^{1/2} \zeta^1 \wedge \cdots \wedge \zeta^n (e_1, \ldots, e_n) \\ &= \det[g(z_i, z_j)]^{1/2} \det[\zeta^i(e_j)] \\ &= \det(V^* V)^{1/2} \det U \\ &= \det V (\det V)^{-1} \\ &= 1 \end{aligned}$$

2. Choose an orthonormal frame $e_1, \ldots, e_{n-1} \in \mathfrak{X}(\partial M)$ with dual coframe $\epsilon^1, \ldots, \epsilon^{n-1}$. Define $\nu \in \Omega^1(M)$ to be the dual covector to N. We note that $N, e_1, \ldots, e_{n-1}$ is an orthonormal frame on $\mathfrak{X}(M)$. Let $X_1, \ldots, X_{n-1} \in \mathfrak{X}(\partial M)$. Since for each $j \in \{1, \ldots, n-1\}$, $X_j \in \mathfrak{X}(\partial M)$ and for each $p \in \partial M$, $N_p \in (T_p \partial M)^\perp$, we have that for each $j \in \{1, \ldots, n-1\}$, $g(X_j, N) = 0$. This implies that

$$\begin{aligned} \iota^* \iota_X \lambda(X_1, \ldots, X_{n-1}) &= \lambda(X, X_1, \ldots, X_{n-1}) \\ &= \nu \wedge \epsilon^1 \wedge \cdots \wedge \epsilon^{n-1}(X, X_1, \ldots, X_{n-1}) \\ &= \det \begin{pmatrix} \nu(X) & \nu(X_1) & \cdots & \nu(X_{n-1}) \\ \epsilon^1(X) & \epsilon^1(X_1) & \cdots & \epsilon^1(X_{n-1}) \\ \vdots & & & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \cdots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\ &= \det \begin{pmatrix} g(X, N) & g(X_1, N) & \cdots & g(X_{n-1}, N) \\ \epsilon^1(X) & \epsilon^1(X_1) & \cdots & \epsilon^1(X_{n-1}) \\ \vdots & & & \vdots \\ \epsilon^{n-1}(X) & \epsilon^{n-1}(X_1) & \cdots & \epsilon^{n-1}(X_{n-1}) \end{pmatrix} \\ &= g(X, N) \det(\epsilon^i(X_j)) \\ &= g(X, N) \epsilon^1 \wedge \ldots \wedge \epsilon^{n-1}(X_1, \ldots, X_{n-1}) \\ &= g(X, N) \tilde{\lambda}(X_1, \ldots, X_n) \end{aligned}$$

Therefore $\iota^* \iota_X \lambda = g(X, N) \tilde{\lambda}$ and

$$\begin{aligned} \int_M \operatorname{div} X \lambda &= \int_M d(\iota_X \lambda) \\ &= \int_{\partial M} \iota^*(\iota_X \lambda) \\ &= \int_{\partial M} g(X, N) \tilde{\lambda} \end{aligned}$$

3. We note that

$$\begin{aligned}0 &= \iota_X(du \wedge \lambda)\\ &= \iota_X(du) \wedge \lambda - du \wedge (\iota_X \lambda)\\ &= du(X)\lambda - du \wedge (\iota_X \lambda)\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{div}(uX)\lambda &= d(\iota_{uX}\lambda)\\ &= d(u\iota_X\lambda)\\ &= du \wedge (\iota_X\lambda) + ud(\iota_X\lambda)\\ &= du(X)\lambda + u\mathrm{div}(X)\lambda\\ &= [du(X) + u\mathrm{div}(X)]\lambda\end{aligned}$$

This implies that $\mathrm{div}(uX) = du(X) + u\mathrm{div}(X)$. From before, we have that

$$\begin{aligned}\int_M du(X)\lambda &= \int_M \mathrm{div}(uX)\lambda - \int_M u\mathrm{div}(X)\lambda\\ &= \int_{\partial M} g(uX, N)\tilde{\lambda} - \int_M u\mathrm{div}(X)\lambda\\ &= \int_{\partial M} ug(X, N)\tilde{\lambda} - \int_M u\mathrm{div}(X)\lambda\end{aligned}$$

□

Exercise 17.0.0.5.

$$\mathrm{div}(X) = \sum_{j=1}^{n} (\nabla_{\partial_j} X)^j$$

Proof. We have that

$$\begin{aligned}\nabla_{\partial_i}(X) &= \sum_{j=1}^{n} \nabla_{\partial_i}(X^j\partial_j)\\ &= \sum_{j=1}^{n} \left[X^j \, \nabla_{\partial_i} \partial_j + \partial_i(X^j)\partial_j\right]\\ &= \sum_{j=1}^{n} \left[X^j(\sum_{k=1}^{n} \Gamma^k_{i,j}\partial_k) + \partial_i(X^j)\partial_j\right]\\ &= \sum_{j=1}^{n} \left[X^j\left(\sum_{k=1}^{n} \Gamma^k_{i,j}\partial_k\right) + \partial_i(X^j)\partial_j\right]\\ &= \sum_{j=1}^{n} X^j\left(\sum_{k=1}^{n} \Gamma^k_{i,j}\partial_k\right) + \sum_{j=1}^{n} \partial_i(X^j)\partial_j\\ &= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} X^j\Gamma^k_{i,j}\right)\partial_k + \sum_{k=1}^{n} \partial_i(X^k)\partial_k\\ &= \sum_{k=1}^{n} \left[\left(\sum_{j=1}^{n} X^j\Gamma^k_{i,j}\right) + \partial_i(X^k)\right]\partial_k\end{aligned}$$

so that $(\nabla_{\partial_i}(X))^i = \left(\sum_{j=1}^n X^j \Gamma^i_{i,j}\right) + \partial_i(X^i)$. We note that

$$\begin{aligned}
\operatorname{div}(X) &= \sum_{i=1}^n \operatorname{div}(X^i \partial_i) \\
&= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + dx^i(\partial_i)] \\
&= \sum_{i=1}^n [X^i \operatorname{div}(\partial_i) + 1]
\end{aligned}$$

Since $\lambda = [\det g(\partial_i, \partial_j)]^{1/2} dx^1 \wedge \cdots \wedge dx^n = (\det g)^{1/2} dx$, we have that

$$\begin{aligned}
d(\iota_{\partial_i}\lambda) &= d((\det g)^{1/2}\iota_{\partial_i} dx) \\
&= d[(\det g)^{1/2}]\iota_{\partial_i} dx + (\det g)^{1/2} d(\iota_{\partial_i} dx) \\
&= d[(\det(g)^{1/2}] \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots dx^n + (\det g)^{1/2} \sum_{k=1}^n (-1)^{k-1} dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots dx^n)
\end{aligned}$$

FINISH!!! □

Exercise 17.0.0.6. Let (M, g) be a Riemannian manifold.

1. For each $u, v \in C^\infty(M)$. Then

   (a)
   $$\int_M u\Delta v\lambda + \int_M g(\nabla u, \nabla v)\lambda = \int_{\partial M} uN(v)\tilde{\lambda}$$

   (b)
   $$\int_M [u\Delta v - v\Delta u]\lambda = \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}$$

2. (a) If $\partial M \neq \varnothing$, then for each $u, v \in C^{\infty(M)}$, u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$ implies that $u = v$.

   (b) If $\partial M = \varnothing$, then for each $u \in C^\infty(M)$, u is harmonic implies that u is constant.

Proof.

1. Let $u, v \in C^\infty(M)$. Then

   (a)

$$\begin{aligned}
\int_M u\Delta v\lambda &= \int_M u\operatorname{div}(\nabla v)\lambda \\
&= \int_{\partial M} ug(\nabla v, N)\tilde{\lambda} - \int_M du(\nabla v)\lambda \\
&= \int_{\partial M} udv(N)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda \\
&= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda
\end{aligned}$$

(b) From above, we have that

$$\begin{aligned}\int_M [u\Delta v - v\Delta u]\lambda &= \int_M u\Delta v\lambda - \int_M v\Delta u\lambda \\ &= \int_{\partial M} uN(v)\tilde{\lambda} - \int_M g(\nabla u, \nabla v)\lambda - \left(\int_{\partial M} vN(u)\tilde{\lambda} - \int_M g(\nabla v, \nabla u)\lambda\right) \\ &= \int_{\partial M} uN(v)\tilde{\lambda} - \int_{\partial M} vN(u)\tilde{\lambda} \\ &= \int_{\partial M} [uN(v) - vN(u)]\tilde{\lambda}\end{aligned}$$

2. (a) Suppose that $\partial M \neq \varnothing$. Let $u, v \in C^{\infty(M)}$. Suppose that u and v are harmonic and $u|_{\partial M} = v|_{\partial M}$. Then $u - v$ is harmonic and

$$\begin{aligned}\int_M \| \nabla(u-v)\|_g^2 \lambda &= \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\ &= 0 + \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\ &= \int_M (u-v)\Delta(u-v)\lambda + \int_M g(\nabla(u-v), \nabla(u-v))\lambda \\ &= \int_{\partial M} (u-v)N(u-v)\tilde{\lambda} \\ &= 0\end{aligned}$$

Thus $\nabla(u-v) = 0$ and $u - v$ is constant. Since $u|_{\partial M} = v|_{\partial M}$, we have that $u - v = 0$ and thus $u = v$.

(b) Suppose that $\partial M = \varnothing$. Let $u \in C^\infty(M)$. Suppose that u is harmonic. Then

$$\begin{aligned}\int_M \| \nabla u\|_g^2 \lambda &= \int_M g(\nabla u, \nabla u)\lambda \\ &= 0 + \int_M g(\nabla u, \nabla u)\lambda \\ &= \int_M u\Delta u\lambda + \int_M g(\nabla u, \nabla u)\lambda \\ &= \int_{\partial M} (u-v)g(\nabla(u-v), N)\tilde{\lambda} \\ &= 0\end{aligned}$$

Therefore $\nabla u - 0$ and u is constant.

$\square$

Chapter 18

Symplectic Geometry

18.1 Symplectic Manifolds

Definition 18.1.0.1. Let $M \in \mathrm{Obj}(\mathbf{Man}^{\infty})$ and $\omega \in \Omega^2(M)$. Then ω is said to be **symplectic** if

1. ω is nondegenerate
2. ω is closed

Chapter 19

Extra

Definition 19.0.0.1. When working in $\mathbb{R}^n$, we introduce the formal objects $dx^1, dx_2, \cdots, dx^n$. Let $I = (i_1, i_2, \cdots, i_k) \in \mathcal{I}_{k,n}$ and $\phi : \mathbb{R}^k \to \mathbb{R}^n$. Write $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$. We formally define $dx^i = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and $\phi_I = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_k})$.

Definition 19.0.0.2. Let $k \in \{0, 1, \cdots, n\}$. We define a $C^\infty(\mathbb{R}^n)$-module of dimension $\binom{n}{k}$, denoted $\Gamma^k(\mathbb{R}^n)$ to be

$$\Phi_k(\mathbb{R}^n) = \begin{cases} C^\infty(\mathbb{R}^n) & k = 0 \\ \operatorname{span}\{dx^i : I \in \mathcal{I}_{k,n}\} & k \geq 1 \end{cases}$$

For each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, we may form their **exterior product**, denoted by $\omega \wedge \chi \in \Gamma^{k+l}(\mathbb{R}^n)$. Thus the exterior product is a map $\wedge : \Phi_k(\mathbb{R}^n) \times \Gamma^l(\mathbb{R}^n) \to \Gamma^{k+l}(\mathbb{R}^n)$. The exterior product is characterized by the following properties:

1. the exterior product is bilinear

2. for each $\omega \in \Phi_k(\mathbb{R}^n)$ and $\chi \in \Gamma^l(\mathbb{R}^n)$, $\omega \wedge \chi = -\chi \wedge \omega$

3. for each $\omega \in \Phi_k(\mathbb{R}^n)$, $\omega \wedge \omega = 0$

4. for each $f \in C^\infty(\mathbb{R}^n)$ and $\omega \in \Phi_k(\mathbb{R}^n)$, $f \wedge \omega = f\omega$

We call $\Phi_k(\mathbb{R}^n)$ the differential k-forms on $\mathbb{R}^n$. Let ω be a k-form on $\mathbb{R}^n$. If $k \geq 1$, then for each $I \in \mathcal{I}_{k,n}$, there exists $f_I \in C^\infty(\mathbb{R}^n)$ such that $\omega = \sum\limits_{I \in \mathcal{I}_{k,n}} f_I dx^i$

Note 19.0.0.3. The terms $dx^1, dx_2, \cdots, dx^n$ are are a sort of place holder for the coordinates of a point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. When we work with functions $\phi : \mathbb{R}^k \to \mathbb{R}^n$, we will have different coordinates and to avoid confusion, we will write $\{du^1, du_2, \cdots, du_k\}$ when referencing the coordinates on $\mathbb{R}^k$ and $\{dx^1, dx_2, \cdots, dx^n\}$ when referencing the coordinates on $\mathbb{R}^n$.

Exercise 19.0.0.4. Let $B_{n\times n} = (b_{i,j}) \in [C^\infty(M)]^{n\times n}$ be an $n \times n$ matrix. Then

$$\bigwedge_{i=1}^{n} \left(\sum_{j=1}^{n} b_{i,j} dx^j \right) = (\det B) dx^1 \wedge dx_2 \wedge \cdots \wedge dx^n$$

Proof. Bilinearity of the exterior product implies that

$$\begin{aligned}\bigwedge_{i=1}^{n}\left(\sum_{j=1}^{n} b_{i,j}dx^j\right) &= \left(\sum_{j=1}^{n} b_{1,j}dx^j\right)\wedge\left(\sum_{j=1}^{n} b_{2,j}dx^j\right)\wedge\cdots\wedge\left(\sum_{j=1}^{n} b_{n,j}dx^j\right)\\ &= \sum_{j_1,\cdots,j_n=1}^{n}\left(\prod_{i=1}^{n} b_{i,j_i}\right)dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}\\ &= \sum_{j_1\neq\cdots\neq j_n}\left(\prod_{i=1}^{n} b_{i,j_i}\right)dx_{j_1}\wedge dx_{j_2}\wedge\cdots\wedge dx_{j_n}\\ &= \left[\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\left(\prod_{i=1}^{n} b_{i,\sigma(i)}\right)\right]dx_1\wedge dx_2\wedge\cdots\wedge dx_n\\ &= (\det B)dx_1\wedge dx_2\wedge\cdots\wedge dx_n\end{aligned}$$

□

Definition 19.0.0.5. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a 0-form on $\mathbb{R}^n$. We define a 1-form, denoted df, on $\mathbb{R}^n$ by

$$df=\sum_{i=1}^{n}\frac{\partial f}{\partial x^i}dx^i$$

Let $\omega=\sum_{I\in\mathcal{I}_{k,n}} f_I dx^i$ be a k-form on $\mathbb{R}^n$. We can define a differential $k+1$-form, denoted $d\omega$, on $\mathbb{R}^n$ by

$$d\omega=\sum_{I\in\mathcal{I}_{k,n}} df_I\wedge dx^i$$

Exercise 19.0.0.6. On $\mathbb{R}^3$, put

1. $\omega_0=f_0$,
2. $\omega_1=f_1dx^1+f2dx_2+f_2dx_3$,
3. $\omega_2=f_1dx_2\wedge dx_3-f_2dx^1\wedge dx_3+f_3dx^1\wedge dx_2$

Show that

1. $d\omega_0=\frac{\partial f_0}{\partial x^1}dx^1+\frac{\partial f_0}{\partial x^2}dx_2+\frac{\partial f_0}{\partial x^3}dx_3$
2. $d\omega_1=\left(\frac{\partial f_3}{\partial x^2}-\frac{\partial f_2}{\partial x^3}\right)dx_2\wedge dx_3+\left(\frac{\partial f_3}{\partial x^1}-\frac{\partial f_1}{\partial x^3}\right)dx^1\wedge dx_3+\left(\frac{\partial f_2}{\partial x^1}-\frac{\partial f_1}{\partial x^2}\right)dx^1\wedge dx_2$
3. $d\omega_2=\left(\frac{\partial f_1}{\partial x^1}+\frac{\partial f_2}{\partial x^2}+\frac{\partial f_3}{\partial x^3}\right)dx^1\wedge dx_2\wedge dx_3$

Proof. Straightforward. □

Exercise 19.0.0.7. Let $I\in\mathcal{I}_{k,n}$. Then there is a unique $I_*\in\mathcal{I}_{n-k,n}$ such that $dx^i\wedge dx_{I_*}=dx^1\wedge dx_2\wedge\cdots\wedge dx^n$.

Definition 19.0.0.8. We define a linear map $*:\Phi_k(\mathbb{R}^n)\to\Gamma^{n-k}(\mathbb{R}^n)$ called the **Hodge $*$-operator** by

$$*\sum_{I\in\mathcal{I}_{k,n}} f_I dx^i=\sum_{I\in\mathcal{I}_{k,n}} f_I dx_{I_*}$$

Definition 19.0.0.9. Let $\phi:\mathbb{R}^k\to\mathbb{R}^n$ be smooth. Write $\phi=(\phi_1,\phi_2,\cdots,\phi_n)$. We define $\phi^*:\Phi_k(\mathbb{R}^n)\to\Phi_k(\mathbb{R}^k)$ via the following properties:

1. for each 0-form f on $\mathbb{R}^n$, $\phi^* f = f \circ \phi$
2. for $i = 1, \cdots, n$, $\phi^* dx^i = d\phi_i$
3. for an s-form ω, and a t-form χ on $\mathbb{R}^n$, $\phi^*(\omega \wedge \chi) = (\phi^*\omega) \wedge (\phi^*\chi)$
4. for l-forms ω, χ on $\mathbb{R}^n$, $\phi^*(\omega + \chi) = \phi^*\omega + \phi^*\chi$

Exercise 19.0.0.10. Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth submanifold of $\mathbb{R}^n$, $\phi : U \to V$ a smooth parametrization of M, $\omega = \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i$ an k-form on $\mathbb{R}^n$. Then

$$\phi^*\omega = \bigg(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \bigg) du^1 \wedge du_2 \wedge \cdots \wedge du_k$$

Proof. By definition,

$$\begin{aligned}
\phi^*\omega &= \phi^* \sum_{I \in \mathcal{I}_{k,n}} f_I dx^i \\
&= \sum_{I \in \mathcal{I}_{k,n}} (\phi^* f_I) \phi^* dx^i \\
&= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I
\end{aligned}$$

A previous exercise tells us that for each $I \in \mathcal{I}_{k,n}$,

$$\begin{aligned}
d\phi_I &= d\phi_{i_1} \wedge d\phi_{i_2} \wedge \cdots \wedge d\phi_{i_n} \\
&= (\sum_{j=1}^{n} \frac{\partial \phi_{i_1}}{\partial u^j} du^j) \wedge (\sum_{j=1}^{n} \frac{\partial \phi_{i_2}}{\partial u^j} du^j) \wedge \cdots \wedge (\sum_{j=1}^{n} \frac{\partial \phi_{i_k}}{\partial u^j} du^j) \\
&= (\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi^*\omega &= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi) d\phi_I \\
&= \sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) du^1 \wedge du_2 \wedge \cdots \wedge du_k \\
&= \bigg(\sum_{I \in \mathcal{I}_{k,n}} (f_I \circ \phi)(\det v\phi_I) \bigg) du^1 \wedge du_2 \wedge \cdots \wedge du_k
\end{aligned}$$

□

19.1 Integration of Differential Forms

Definition 19.1.0.1. Let $U \subset \mathbb{R}^k$ be open and $\omega = f dx^1 \wedge dx_2 \wedge \cdots \wedge dx_k$ a k-form on $\mathbb{R}^k$. Define

$$\int_U \omega = \int_U f dx$$

Definition 19.1.0.2. Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$, ω a k-form on $\mathbb{R}^n$ and $\phi : U \to V$ a local smooth, orientation-preserving parametrization of M. Define

$$\int_V \omega = \int_U \phi^*\omega$$

Exercise 19.1.0.3.

Theorem 19.1.0.4. Stokes Theorem:
Let $M \subset \mathbb{R}^n$ be a k-dimensional oriented smooth submanifold of $\mathbb{R}^n$ and ω a $k-1$-form on $\mathbb{R}^n$. Then

$$\int_{\partial M} \omega = \int_M d\omega$$

Appendix A

Summation

Appendix B

Asymptotic Notation

Bibliography

[1] Introduction to Algebra

[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration