

INTRODUCTION TO PROBABILITY

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CONTENTS

1. Basic Probability	2
2. Probability	2
2.1. Distributions	2
2.2. Independence	5
2.3. L^p Spaces for Probability	8
2.4. Borel Cantelli Lemma	13
3. Probability on locally compact Groups	15
3.1. Action on Probability Measures	15
4. Weak Convergence of Measures	16
5. Concentration Inequalities	17
5.1. Sub α -Exponential Random Variables	17
6. Conditional Expectation and Probability	19
6.1. Conditional Expectation	19
6.2. Conditional Probability	20
7. Markov Chains	23
8. Probabilities Induced by Isometric Group Actions	24
8.1. Applications to Bayesian Statistics	24
9. Stochastic Integration	25
References	26

1. BASIC PROBABILITY

2. PROBABILITY

2.1. Distributions.

Definition 2.1.1. Let Ω be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then \mathcal{P} is said to be a π -**system** on Ω if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Definition 2.1.2. Let Ω be a set and $\mathcal{L} \subset \mathcal{P}(\Omega)$. Then \mathcal{L} is said to be a λ -**system** on Ω if

- (1) $\mathcal{L} \neq \emptyset$
- (2) for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$
- (3) for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}$, if $(A_n)_{n \in \mathbb{N}}$ is disjoint, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}$

Exercise 2.1.3. Let Ω be a set and \mathcal{L} a λ -system on Ω . Then

- (1) $\Omega, \emptyset \in \mathcal{L}$

Proof. Straightforward. □

Definition 2.1.4. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. Put

$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(\Omega) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the λ -**system on Ω generated by \mathcal{C}** , $\lambda(\mathcal{C})$, to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L} \in \mathcal{S}} \mathcal{L}$$

Exercise 2.1.5. Let Ω be a set and $\mathcal{C} \subset \mathcal{P}(\Omega)$. If \mathcal{C} is a λ -system and \mathcal{C} is a π -system, then \mathcal{C} is a σ -algebra.

Proof. Suppose that \mathcal{C} is a λ -system and \mathcal{C} is a π -system. Then we need only verify the third axiom in the definition of a σ -algebra. Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{C}$. Define $B_1 = A_1$ and for $n \geq 2$, define $B_n = A_n \cap \left(\bigcup_{k=1}^{n-1} A_k \right)^c = A_n \cap \left(\bigcap_{k=1}^{n-1} A_k^c \right) \in \mathcal{C}$. Then $(B_n)_{n \in \mathbb{N}}$ is disjoint and therefore $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n \in \mathcal{C}$. □

Theorem 2.1.6. (Dynkin's Theorem)

Let Ω be a set.

- (1) Let \mathcal{P} be a π -system on Ω and \mathcal{L} a λ -system on Ω . If $\mathcal{P} \subset \mathcal{L}$, then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
- (2) Let \mathcal{P} be a π -system on Ω . Then $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

Exercise 2.1.7. Let (Ω, \mathcal{F}) be a measurable space and μ, ν probability measures on (Ω, \mathcal{F}) . Put $\mathcal{L}_{\mu, \nu} = \{A \in \mathcal{F} : \mu(A) = \nu(A)\}$. Then $\mathcal{L}_{\mu, \nu}$ is a λ -system on Ω .

Proof.

- (1) $\emptyset \in \mathcal{L}_{\mu, \nu}$.
- (2) Let $A \in \mathcal{L}_{\mu, \nu}$. Then $\mu(A) = \nu(A)$. Thus

$$\begin{aligned} \mu(A^c) &= 1 - \mu(A) \\ &= 1 - \nu(A) \\ &= \nu(A^c) \end{aligned}$$

So $A^c \in \mathcal{L}_{\mu, \nu}$.

- (3) Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}_{\mu, \nu}$. So for each $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$. Suppose that $(A_n)_{n \in \mathbb{N}}$ is disjoint. Then

$$\begin{aligned} \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) &= \sum_{n \in \mathbb{N}} \mu(A_n) \\ &= \sum_{n \in \mathbb{N}} \nu(A_n) \\ &= \nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \end{aligned}$$

Hence $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{L}_{\mu, \nu}$.

□

Exercise 2.1.8. Let (Ω, \mathcal{F}) be a measurable space, μ, ν probability measures on (Ω, \mathcal{F}) and $\mathcal{P} \subset \mathcal{F}$ a π -system on Ω . Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Proof. Using the previous exercise, we see that $\mathcal{P} \subset \mathcal{L}_{\mu, \nu}$. Dynkin's theorem implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu, \nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$. □

Definition 2.1.9. Let $F : \mathbb{R} \rightarrow \mathbb{R}$. Then F is said to be a **probability distribution function** if

- (1) F is right continuous
- (2) F is increasing
- (3) $F(-\infty) = 0$ and $F(\infty) = 1$

Definition 2.1.10. Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We define $F_P : \mathbb{R} \rightarrow \mathbb{R}$, by

$$F_P(x) = P((-\infty, x])$$

We call F_P the **probability distribution function of P** .

Exercise 2.1.11. Let (Ω, \mathcal{F}, P) be a probability measure. Then F_P is a probability distribution function.

Proof. (1) Let $x \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}} \subset [x, \infty)$. Suppose that $x_n \rightarrow x$. Then $(x, x_n] \rightarrow \emptyset$ because $\limsup_{n \rightarrow \infty} (x, x_n] = \emptyset$. Thus

$$F(x_n) - F(x) = P((x, x_n]) \rightarrow P(\emptyset) = 0$$

This implies that

$$F(x_n) \rightarrow F(x)$$

So F is right continuous.

- (2) Clearly F_P is increasing.
- (3) Continuity from below tells us that

$$F(-\infty) = \lim_{n \rightarrow -\infty} F(n) = \lim_{n \rightarrow -\infty} P((-\infty, n]) = 0$$

and continuity from above tell us that

$$F(\infty) = \lim_{n \rightarrow \infty} F(n) = \lim_{n \rightarrow \infty} P((-\infty, n]) = 1$$

□

Exercise 2.1.12. Let μ, ν be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $F_\mu = F_\nu$ iff $\mu = \nu$.

Proof. Clearly if $\mu = \nu$, then $F_\mu = F_\nu$. Conversely, suppose that $F_\mu = F_\nu$. Then for each $x \in \mathbb{R}$,

$$\begin{aligned}\mu((-\infty, x]) &= F_\mu(x) \\ &= F_\nu(x) \\ &= \nu((-\infty, x])\end{aligned}$$

Put $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then \mathcal{C} is a π -system and for each $A \in \mathcal{C}$, $\mu(A) = \nu(A)$. Hence for each $A \in \sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, $\mu(A) = \nu(A)$. So $\mu = \nu$. \square

Definition 2.1.13. Let (Ω, \mathcal{F}, P) be a probability space and $X : \Omega \rightarrow \mathbb{R}^n$. Then X is said to be a **random vector** on (Ω, \mathcal{F}) if X is \mathcal{F} - $\mathcal{B}(\mathbb{R}^n)$ measurable. If $n = 1$, then X is said to be a **random variable**. We define

$$L_n^0(\Omega, \mathcal{F}, P) = \{X : \Omega \rightarrow \mathbb{R}^n : X \text{ is a random vector}\}$$

and

$$L_n^p(\Omega, \mathcal{F}, P) = \left\{ X \in L_n^0 : \int \|X\|^p dP < \infty \right\}$$

Definition 2.1.14. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . We define the **probability distribution** of X , $P_X : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$, to be the measure

$$P_X = X_*P$$

That is, for each $A \in \mathcal{B}(\mathbb{R})$,

$$P_X(A) = P(X^{-1}(A))$$

We define the **probability distribution function** of X , $F_X : \mathbb{R} \rightarrow [0, 1]$, to be

$$F_X = F_{P_X}$$

Definition 2.1.15. Let (Ω, \mathcal{F}, P) be a probability space and X a random variable on (Ω, \mathcal{F}) . If $P_X \ll m$, we define the **probability density** of X , $f_X : \mathbb{R} \rightarrow \mathbb{R}$, by

$$f_X = \frac{dP_X}{dm}$$

Exercise 2.1.16. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on (Ω, \mathcal{F}) . Then for each $x \in \mathbb{R}$,

$$\mathbb{P}\left(\liminf_{n \rightarrow \infty} X_n > x\right) \leq \liminf_{n \rightarrow \infty} P(X_n > x)$$

Proof. Let $\omega \in \left\{ \liminf_{n \rightarrow \infty} X_n > x \right\}$. Then $x < \liminf_{n \rightarrow \infty} X_n(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} X_k(\omega) \right)$. So there exists $n^* \in \mathbb{N}$ such that $x < \inf_{k \geq n^*} X_k(\omega)$. Then for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x < X_k(\omega)$.

So there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Hence

$\inf_{k \geq n^*} \mathbf{1}_{\{X_k > x\}}(\omega) = 1$. Thus $\liminf_{n \rightarrow \infty} \mathbf{1}_{\{X_n > x\}}(\omega) = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \mathbf{1}_{\{X_k > x\}}(\omega) \right) = 1$. Therefore

$\omega \in \liminf_{n \rightarrow \infty} \{X_k > x\}$ and we have shown that

$$\left\{ \liminf_{n \rightarrow \infty} X_n > x \right\} \subset \liminf_{n \rightarrow \infty} \{X_k > x\}$$

Then

$$\begin{aligned} P\left(\liminf_{n \rightarrow \infty} X_n > x\right) &\leq P\left(\liminf_{n \rightarrow \infty} \{X_k > x\}\right) \\ &\leq \liminf_{n \rightarrow \infty} P(\{X_k > x\}) \end{aligned}$$

□

Definition 2.1.17. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+(\Omega) \cup L^1$. Define the **expectation of X**, $E[X]$, to be

$$E[X] = \int X dP$$

2.2. Independence.

Definition 2.2.1. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C} \subset \mathcal{F}$. Then \mathcal{C} is said to be **independent** if for each $(A_i)_{i=1}^n \subset \mathcal{C}$,

$$P\left(\bigcap_{k=1}^n A_k\right) = \prod_{k=1}^n P(A_k)$$

Definition 2.2.2. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Then $\mathcal{C}_1, \dots, \mathcal{C}_n$ are said to be **independent** if for each $A_1 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$, A_1, \dots, A_n are independent.

Note 2.2.3. We will explicitly say that for each $i = 1, \dots, n$, \mathcal{C}_i is independent when talking about the independence of the elements of \mathcal{C}_i to avoid ambiguity.

Definition 2.2.4. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are said to be **independent** if for each $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent.

Exercise 2.2.5. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff $\sigma(X_1), \dots, \sigma(X_n)$ are independent.

Proof. Suppose that X_1, \dots, X_n are independent. Let $A_1 \in \sigma(X_1), \dots, A_n \in \sigma(X_n)$. Then for each $i = 1, \dots, n$, there exists $B_i \in \mathcal{B}(\mathbb{R})$ such that $A_i = X_i^{-1}(B_i)$. Then A_1, \dots, A_n are independent. Hence $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Conversely, suppose that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$. Then for each $i = 1, \dots, n$, $X_i^{-1}B_i \in \sigma(X_i)$. Then $X_1^{-1}B_1, \dots, X_n^{-1}B_n$ are independent. Hence X_1, \dots, X_n are independent. □

Exercise 2.2.6. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{F}$ a collection of σ -algebras on Ω . Suppose that for each $i = 1, \dots, n$, X_i is \mathcal{F}_i -measurable. If $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent, then X_1, \dots, X_n are independent.

Proof. For each $i = 1, \dots, n$, $\sigma(X_i) \subset \mathcal{F}_i$. So $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Hence X_1, \dots, X_n are independent. □

Exercise 2.2.7. Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{F}$. Suppose that for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent, then $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent.

Proof. Let $A_2 \in \mathcal{C}_2$. Define $\mathcal{L} = \{A \in \mathcal{F} : P(A \cap A_2) = P(A)P(A_2)\}$. Then

- (1) $\Omega \in \mathcal{L}$
- (2) If $A \in \mathcal{L}$, then

$$\begin{aligned} P(A^c \cap A_2) &= P(A_2) - P(A_2 \cap A) \\ &= P(A_2) - P(A_2)P(A) \\ &= (1 - P(A))P(A_2) \\ &= P(A^c)P(A_2) \end{aligned}$$

So $A^c \in \mathcal{L}$

- (3) If $(B_n)_{n \in \mathbb{N}} \subset \mathcal{L}$ is disjoint, then

$$\begin{aligned} P\left(\left[\bigcup_{n \in \mathbb{N}} B_n\right] \cap A_2\right) &= P\left(\bigcup_{n \in \mathbb{N}} B_n \cap A_2\right) \\ &= \sum_{n \in \mathbb{N}} P(B_n \cap A_2) \\ &= \sum_{n \in \mathbb{N}} P(B_n)P(A_2) \\ &= \left[\sum_{n \in \mathbb{N}} P(B_n)\right]P(A_2) \\ &= P\left(\bigcup_{n \in \mathbb{N}} B_n\right)P(A_2) \end{aligned}$$

So $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{L}$.

Thus \mathcal{L} is a λ -system. Since $\mathcal{C}_1 \subset \mathcal{L}$ is a π -system, Dynkin's theorem tells us that $\sigma(\mathcal{C}_1) \subset \mathcal{L}$. Since $A_2 \in \mathcal{C}_2$ is arbitrary $\sigma(\mathcal{C}_1)$ and \mathcal{C}_2 are independent. The same reasoning implies that $\sigma(\mathcal{C}_1)$ and $\sigma(\mathcal{C}_2)$ are independent. Let $A_2 \in \mathcal{C}_1, \dots, A_n \in \mathcal{C}_n$. We may do the same process with

$$\mathcal{L} = \left\{A \in \mathcal{F} : P\left(A \cap \left(\bigcap_{i=2}^n A_i\right)\right) = P(A) \prod_{i=2}^n P(A_i)\right\}$$

and conclude that $\sigma(\mathcal{C}_1), \mathcal{C}_2, \dots, \mathcal{C}_n$ are independent. Which, using the same reasoning would imply that $\sigma(\mathcal{C}_1), \dots, \sigma(\mathcal{C}_n)$ are independent. \square

Exercise 2.2.8. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Then X_1, \dots, X_n are independent iff for each $x_1, \dots, x_n \in \mathbb{R}$,

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$$

Proof. Suppose that X_1, \dots, X_n are independent. Then $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Let $x_1, \dots, x_n \in \mathbb{R}$. Then for each $i = 1, \dots, n$, $\{X_i \leq x_i\} \in \sigma(X_i)$. Hence

$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Conversely, suppose that for each $x_1, \dots, x_n \in \mathbb{R}$, $P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$. Define $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$. Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$. For each $i = 1, \dots, n$, define $\mathcal{C}_i = X_i^{-1}\mathcal{C}$. Then for each $i = 1, \dots, n$, \mathcal{C}_i is a π -system and

$$\begin{aligned}\sigma(\mathcal{C}_i) &= \sigma(X_i^{-1}(\mathcal{C})) \\ &= X_i^{-1}(\sigma(\mathcal{C})) \\ &= X_i^{-1}(\mathcal{B}(\mathbb{R})) \\ &= \sigma(X_i)\end{aligned}$$

By assumption, $\mathcal{C}_1, \dots, \mathcal{C}_n$ are independent. The previous exercise tells us that $\sigma(X_1), \dots, \sigma(X_n)$ are independent. Then X_1, \dots, X_n are independent. \square

Exercise 2.2.9. Let (Ω, \mathcal{F}, P) be a probability space and X_1, \dots, X_n random variables on (Ω, \mathcal{F}) . Define $X = (X_1, \dots, X_n)$. If X_1, \dots, X_n are independent, then

$$P_X = \prod_{i=1}^n P_{X_i}$$

.

Proof. Let $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}P_X(A_1 \times \dots \times A_n) &= P(X \in A_1 \times \dots \times A_n) \\ &= P(X_1 \in A_1, \dots, X_n \in A_n) \\ &= P(X_1 \in A_1) \dots P(X_n \in A_n) \\ &= P_{X_1}(A_1) \dots P_{X_n}(A_n) \\ &= \prod_{i=1}^n P_{X_i}(A_1 \times \dots \times A_n)\end{aligned}$$

Put

$$\mathcal{P} = \{A_1 \times \dots \times A_n : A_1 \in \mathcal{B}(\mathbb{R}), \dots, A_n \in \mathcal{B}(\mathbb{R})\}$$

Then \mathcal{P} is a π -system and

$$\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R}) \otimes \dots \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^n)$$

A previous exercise then tells us that $P_X = \prod_{i=1}^n P_{X_i}$ \square

Exercise 2.2.10. Let (Ω, \mathcal{F}, P) be a probability space, X_1, \dots, X_n random variables on (Ω, \mathcal{F}) and $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R} \in L^0$. Suppose that $f_1 \circ X_1, \dots, f_n \circ X_n \in L^+(\Omega)$ or $f_1 \circ X_1, \dots, f_n \circ X_n \in L^1(\Omega)$. If X_1, \dots, X_n are independent, then

$$E[f_1(X_1) \dots f_n(X_n)] = \prod_{i=1}^n E[f_i(X_i)]$$

Proof. Define the random vector $X : \Omega \rightarrow \mathbb{R}^n$ by $X = (X_1, \dots, X_n)$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ by $g(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$. Suppose that for each $i = 1, \dots, n$, $f_i \in L^+(\mathbb{R})$. Then $g \in L^+(\mathbb{R}^n)$ and by change of variables,

$$\begin{aligned}
 E[f_1(X_1) \cdots f_n(X_n)] &= E[g(X)] \\
 &= \int_{\Omega} g \circ X dP \\
 &= \int_{\mathbb{R}^n} g(x) dP_X(x) \\
 &= \int_{\mathbb{R}^n} g(x) d \prod_{i=1}^n P_{X_i}(x) \\
 &= \prod_{i=1}^n \int_{\mathbb{R}} f_i(x) dP_{X_i}(x) \\
 &= \prod_{i=1}^n \int_{\Omega} f_i \circ X dP \\
 &= \prod_{i=1}^n E[f_i(X_i)]
 \end{aligned}$$

If for each $i = 1, \dots, n$, $f_i \in L^1(\mathbb{R}, P_{X_i})$, then following the above reasoning with $|g|$ tells us that $g \in L^1(\mathbb{R}^n, P_X)$ and we use change of variables and Fubini's theorem to get the same result. \square

2.3. L^p Spaces for Probability.

Note 2.3.1. Recall that for a probability space (Ω, \mathcal{F}, P) and $1 \leq p \leq q \leq \infty$ we have $L^q \subset L^p$ and for each $X \in L^q$, $\|X\|_p \leq \|X\|_q$. Also recall that for $X, Y \in L^2$, we have that $\|XY\|_1 \leq \|X\|_2 \|Y\|_2$.

Definition 2.3.2. Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Define the **variance of X**, $Var(X)$, to be

$$Var(X) = E[(X - E[X])^2]$$

.

Definition 2.3.3. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the

Definition 2.3.4. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Define the **covariance of X and Y**, $Cov(X, Y)$, to be

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Exercise 2.3.5. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. Then the covariance is well defined and $Cov(X, Y)^2 \leq Var(X)Var(Y)$

Proof. By Holder's inequality,

$$\begin{aligned}
 |Cov(X, Y)| &= \left| \int (X - E[X])(Y - E[Y])dP \right| \\
 &\leq \int |(X - E[X])(Y - E[Y])|dP \\
 &= \|(X - E[X])(Y - E[Y])\|_1 \\
 &\leq \|X - E[X]\|_2 \|Y - E[Y]\|_2 \\
 &= \left(\int |X - E[X]|^2 dP \right)^{\frac{1}{2}} \left(\int |Y - E[Y]|^2 dP \right)^{\frac{1}{2}} \\
 &= Var(X)^{\frac{1}{2}} Var(Y)^{\frac{1}{2}}
 \end{aligned}$$

So $Cov(X, Y)^2 \leq Var(X)Var(Y)$. □

Exercise 2.3.6. Let (Ω, \mathcal{F}, P) be a measure space and $X, Y \in L^2$. Then

- (1) $Cov(X, Y) = E[XY] - E[X]E[Y]$
- (2) If X, Y are independent, then $Cov(X, Y) = 0$
- (3) $Var(X) = E[X^2] - E[X]^2$
- (4) for each $a, b \in \mathbb{R}$, $Var(aX + b) = a^2 Var(X)$.
- (5) $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

Proof.

- (1) We have that

$$\begin{aligned}
 Cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
 &= E[XY - E[Y]X - E[X]Y + E[X]E[Y]] \\
 &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
 &= E[XY] - E[X]E[Y]
 \end{aligned}$$

- (2) Suppose that X, Y are independent. Then $E[XY] = E[X]E[Y]$. Hence

$$\begin{aligned}
 Cov(X, Y) &= E[XY] - E[X]E[Y] \\
 &= E[X]E[Y] - E[X]E[Y] \\
 &= 0
 \end{aligned}$$

- (3) Part (1) implies that

$$\begin{aligned}
 Var(X) &= Cov(X, X) \\
 &= E[X^2] - E[X]^2
 \end{aligned}$$

- (4) Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
 Var(aX + b) &= E[(aX + b)^2] - E[aX + b]^2 \\
 &= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\
 &= a^2E[X^2] + 2abE[X] + b^2 - (a^2E[X]^2 + 2abE[X] + b^2) \\
 &= a^2(E[X^2] - E[X]^2) \\
 &= a^2Var(X)
 \end{aligned}$$

(5) We have that

$$\begin{aligned}
 \text{Var}(X + Y) &= E[(X + Y)^2] - E[X + Y]^2 \\
 &= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
 &= E[X^2] + 2E[XY] + E[Y^2] - (E[X]^2 + 2E[X]E[Y] + E[Y]^2) \\
 &= (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2) + 2(E[XY] - E[X]E[Y]) \\
 &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)
 \end{aligned}$$

□

Definition 2.3.7. Let (Ω, \mathcal{F}, P) be a probability space and $X, Y \in L^2$. The **correlation of X and Y**, $\text{Cor}(X, Y)$, is defined to be

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Exercise 2.3.8.

Exercise 2.3.9. Jensen's Inequality Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^1$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$. If ϕ is convex, then

$$\phi(E[X]) \leq E[\phi(X)]$$

Proof. Put $x_0 = E[X]$. Since ϕ is convex, there exist $a, b \in \mathbb{R}$ such that $\phi(x_0) = ax_0 + b$ and for each $x \in \mathbb{R}$, $\phi(x) \geq ax + b$. Then

$$\begin{aligned}
 E[\phi(X)] &= \int \phi(X) dP \\
 &\geq \int [aX + b] dP \\
 &= a \int X dP + b \\
 &= aE[X] + b \\
 &= ax_0 + b \\
 &= \phi(x_0) \\
 &= \phi(E[X])
 \end{aligned}$$

□

Exercise 2.3.10. Markov's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^+$. Then for each $a \in (0, \infty)$,

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Proof. Let $a \in (0, \infty)$. Then $a\mathbf{1}_{\{X \geq a\}} \leq X\mathbf{1}_{\{X \geq a\}}$. Thus

$$\begin{aligned} aP(X \geq a) &= \int a\mathbf{1}_{\{X \geq a\}} dP \\ &= \int X\mathbf{1}_{\{X \geq a\}} dP \\ &\leq \int X dP \\ &= E[X] \end{aligned}$$

Therefore

$$P(X \geq a) \leq \frac{E[X]}{a}$$

□

Exercise 2.3.11. Chebychev's Inequality: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a \in (0, \infty)$,

$$P(|X - E[X]| \geq a) \leq \frac{\text{Var}(X)}{a^2}$$

Proof. Let $a \in (0, \infty)$. Then

$$\begin{aligned} P(|X - E[X]| \geq a) &= P((X - E[X])^2 \geq a^2) \\ &\leq \frac{E[(X - E[X])^2]}{a^2} \\ &= \frac{\text{Var}(X)}{a^2} \end{aligned}$$

□

Exercise 2.3.12. Chernoff's Bound: Let (Ω, \mathcal{F}, P) be a probability space and $X \in L^2$. Then for each $a, t \in (0, \infty)$,

$$P(X \geq a) \leq e^{-ta} E[e^{tX}]$$

Proof. Let $a, t \in (0, \infty)$. Then

$$\begin{aligned} P(X \geq a) &= P(tX \geq ta) \\ &= P(e^{tX} \geq e^{ta}) \\ &\leq e^{-ta} E[e^{tX}] \end{aligned}$$

□

Exercise 2.3.13. Weak Law of Large Numbers: Let (Ω, \mathcal{F}, P) be a probability space $(X_i)_{i \in \mathbb{N}} \subset L^2$. Suppose that $(X_i)_{i \in \mathbb{N}}$ are iid. Then

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

Proof. Put $\mu = E[X_1]$ and $\sigma^2 = Var(X_1)$. Then

$$\begin{aligned} E\left[\frac{1}{n} \sum_{i=1}^n X_i\right] &= \frac{1}{n} \sum_{i=1}^n E[X_i] \\ &= \frac{1}{n} \sum_{i=1}^n \mu \\ &= \mu \end{aligned}$$

and

$$\begin{aligned} Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right) &= \frac{1}{n^2} Var\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n^2} \sum_{i=1}^n Var(X_i) \\ &= \frac{1}{n^2} \sum_{i=1}^n \sigma^2 \\ &= \frac{\sigma^2}{n} \end{aligned}$$

Let $\epsilon > 0$. Then

$$\begin{aligned} P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - E[X_1]\right| \geq \epsilon\right) &= P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) \\ &= P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - E\left[\frac{1}{n} \sum_{i=1}^n X_i\right]\right| \geq \epsilon\right) \\ &\leq \frac{Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right)}{\epsilon^2} \\ &= \frac{\sigma^2/n}{\epsilon^2} \\ &= \frac{\sigma^2}{n\epsilon^2} \rightarrow 0 \end{aligned}$$

So

$$\frac{1}{n} \sum_{i=1}^n X_i \xrightarrow{P} E[X_1]$$

□

2.4. Borel Cantelli Lemma.

Exercise 2.4.1. Borel Cantelli Lemma:

Let (Ω, \mathcal{F}, P) be a probability space and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{F}$.

- (1) If $\sum_{n \in \mathbb{N}} P(A_n) < \infty$, then $P(\limsup_{n \rightarrow \infty} A_n) = 0$.
- (2) If $(A_n)_{n \in \mathbb{N}}$ are independent and $\sum_{n \in \mathbb{N}} P(A_n) = \infty$, then $P(\limsup_{n \rightarrow \infty} A_n) = 1$.

Proof.

- (1) Suppose that $\sum_{n \in \mathbb{N}} P(A_n) < \infty$. Recall that

$$\limsup_{n \rightarrow \infty} A_n = \left\{ \omega \in \Omega : \sum_{n \in \mathbb{N}} 1_{A_n}(\omega) = \infty \right\}$$

Then

$$\begin{aligned} \infty &> \sum_{n \in \mathbb{N}} P(A_n) \\ &= \sum_{n \in \mathbb{N}} \int 1_{A_n} dP \\ &= \int \sum_{n \in \mathbb{N}} 1_{A_n} dP \end{aligned}$$

Thus $\sum_{n \in \mathbb{N}} 1_{A_n} < \infty$ a.e. and $P(\limsup_{n \rightarrow \infty} A_n) = 0$.

- (2) Suppose that $(A_n)_{n \in \mathbb{N}}$ are independent and $\sum_{n \in \mathbb{N}} P(A_n) = \infty$.

□

Exercise 2.4.2. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}} \subset L^0$ and $X \in L^0$.

- (1) If for each $\epsilon > 0$, $\sum_{n \in \mathbb{N}} P(|X_n - X| \geq \epsilon) < \infty$, then $X_n \rightarrow X$ a.e.
- (2) If $(X_n)_{n \in \mathbb{N}}$ are independent and there exists $\epsilon > 0$ such that $\sum_{n \in \mathbb{N}} P(|X_n - X| \geq \epsilon) = \infty$, then $X_n \not\rightarrow X$ a.e.

Proof.

- (1) For $\epsilon > 0$ and $n \in \mathbb{N}$, set $A_n(\epsilon) = \{\omega \in \Omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}$. Suppose that for each $\epsilon > 0$, $\sum_{n \in \mathbb{N}} P(|X_n - X| \geq \epsilon) < \infty$. The Borel-Cantelli lemma implies that for each $m \in \mathbb{N}$,

$$P(\limsup_{n \rightarrow \infty} A_n(1/m)) = 0$$

Let $\omega \in \Omega$. Then $X_n(\omega) \not\rightarrow X(\omega)$ iff

$$\omega \in \bigcup_{m \in \mathbb{N}} \limsup_{n \rightarrow \infty} A_n(1/m)$$

So

$$\begin{aligned}
 P(X_n \not\rightarrow X) &= P\left(\bigcup_{m \in \mathbb{N}} \limsup_{n \rightarrow \infty} A_n(1/m)\right) \\
 &\leq \sum_{m \in \mathbb{N}} P(\limsup_{n \rightarrow \infty} A_n(1/m)) \\
 &= 0
 \end{aligned}$$

(2) Hence $X_n \rightarrow X$ a.e.

□

3. PROBABILITY ON LOCALLY COMPACT GROUPS

Note 3.0.1. In this section, familiarity with Haar measure will be assumed. This section is intended as a continuation of section 7 of [3].

3.1. Action on Probability Measures.

Note 3.1.1. We recall some notation from section 7.1 of [3].

- $l_g \in \text{Homeo}(G)$, $l_g(x) = gx$
- $L_g \in \text{Sym}(L_0(G))$, $L_g f = f \circ l_g^{-1}$ We continue from section 7

Note 3.1.2. The next exercise generalizes the notion of a scale-family.

Exercise 3.1.3. Let (Ω, \mathcal{F}, P) be a probability space, G a locally compact group, μ a left Haar measure on G , $X \in L_G^0$ and $g \in G$. If $P_X \ll \mu$, then $f_{gX} = L_g f_X$.

Proof. Suppose that $P_X \ll \mu$. Let $A \in \mathcal{B}(G)$. Then

$$\begin{aligned}
 P_{gX}(A) &= P(gX \in A) \\
 &= P(X \in g^{-1}A) \\
 &= P_X(g^{-1}A) \\
 &= P_X(l_g^{-1}(A)) \\
 &= l_{g*}P_X(A) \\
 &= g \cdot P_X(A)
 \end{aligned}$$

The previous exercise tells us that $f_{gX} = L_g f_X$. □

4. WEAK CONVERGENCE OF MEASURES

5. CONCENTRATION INEQUALITIES

5.1. Sub α -Exponential Random Variables.

Definition 5.1.1. Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then X is said to be **sub α -exponential** if there exist $M, K > 0$ such that for each $t \geq 0$,

$$P(|X| \geq t) \leq Me^{-Kt^\alpha}$$

Exercise 5.1.2. Let (Ω, \mathcal{F}, P) be a probability space, $X \in L^0(\Omega, \mathcal{F}, P)$ and $\alpha > 0$. Then the following are equivalent:

- (1) X is sub α -exponential
- (2) there exists $K > 0$ such that for each $p \geq 1$, $\|X\|_p \leq Kp^{1/\alpha}$
- (3)
- (4)

Proof.

- (1) \implies (2):
Choose $C_\alpha > 0$ such that for each $x \geq \alpha^{-1}$, $\Gamma(x) \leq C_\alpha x^\alpha$. Since X is sub α -exponential, there exist $M, K_0 > 0$ such that for each $t \geq 0$, $P(|X| \geq t) \leq Me^{-Kt^\alpha}$. Set $K = \max(M\alpha^{-1}C_\alpha, 1)2K_0^{-1/\alpha}\alpha^{-1/\alpha}$. Let $p \geq 1$. Then $p\alpha^{-1} \geq \alpha^{-1}$

$$\begin{aligned} \|X\|_p^p &= E(|X|^p) \\ &= \int_0^\infty P(|X|^p \geq t) dt \\ &= \int_0^\infty P(|X| \geq t^{1/p}) dt \\ &\leq \int_0^\infty Me^{-K_0 t^{\alpha/p}} dt \\ &= Mp\alpha^{-1} \int_0^\infty u^{p/\alpha-1} e^{-K_0 u} du \\ &= Mp\alpha^{-1} \Gamma(p/\alpha) K_0^{-p/\alpha} \\ &\leq Mp\alpha^{-1} C_\alpha (p\alpha^{-1})^{p/\alpha} K_0^{-p/\alpha} \end{aligned}$$

Therefore

$$\begin{aligned} \|X\|_p &\leq (M\alpha^{-1}C_\alpha)^{1/p} p^{1/p} K_0^{-1/\alpha} \alpha^{-1/\alpha} p^{1/\alpha} \\ &\leq \max(M\alpha^{-1}C_\alpha, 1) 2K_0^{-1/\alpha} \alpha^{-1/\alpha} p^{1/\alpha} \\ &= Kp^{1/\alpha} \end{aligned}$$

- (2) \implies (3):

□

Definition 5.1.3. Let $\psi : [0, \infty) \rightarrow [0, \infty)$. Then ψ is said to be an **Orlicz function** if

- (1) ψ is convex
- (2) ψ is increasing

- (3) $\psi(x) \rightarrow \infty$ as $x \rightarrow \infty$
- (4) $\psi(0) = 0$

Definition 5.1.4. Let (Ω, \mathcal{F}, P) be a probability space and $\psi : [0, \infty) \rightarrow [0, \infty)$ and Orlicz function. We define the **Orlicz ψ -norm**, denoted $\|\cdot\|_\psi : L^0(\Omega, \mathcal{F}, P) \rightarrow [0, \infty]$, by

$$\|X\|_\psi = \inf\{t > 0 : E[\psi(|X|/t)] \leq 1\}$$

We define the **Orlicz ψ -space**, denoted $L^\psi(\Omega, \mathcal{F}, P)$, by

$$L^\psi(\Omega, \mathcal{F}, P) = \{X \in L^0(\Omega, \mathcal{F}, P) : \|X\|_\psi < \infty\}$$

Exercise 5.1.5. Let (Ω, \mathcal{F}, P) be a probability space and $\psi : [0, \infty) \rightarrow [0, \infty)$ and Orlicz function. Then L^ψ is a vector space and $\|\cdot\|_\psi : L^\psi \rightarrow [0, \infty)$ is a norm.

Hint:

Proof. Let $X, Y \in L^\psi(\Omega, \mathcal{F}, P)$ and $\lambda \in \mathbb{C}$.

- (1) Let
- (2)
- (3)

□

6. CONDITIONAL EXPECTATION AND PROBABILITY

6.1. Conditional Expectation.

Definition 6.1.1. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -alg of \mathcal{F} and $X, Y \in L^1(\Omega, \mathcal{F}, P)$. Then Y is said to be a **conditional expectation of X given \mathcal{G}** if

- (1) Y is \mathcal{G} -measurable
- (2) for each $G \in \mathcal{G}$,

$$\int_G Y dP = \int_G X dP$$

To denote this, we write $Y = E[X|\mathcal{G}]$

Note 6.1.2. Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L_S^0(\Omega, \mathcal{F})$. We typically write $E[X|Y]$ instead of $E[X|Y^*\mathcal{S}]$.

Exercise 6.1.3. Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -alg of \mathcal{F} and $X \in L^1(\Omega, \mathcal{F}, P)$. Define $P_G = P|_{\mathcal{G}}$ and $Q : \mathcal{G} \rightarrow [0, \infty)$ by $Q(G) = \int_G X dP$. Then $Q \ll P_G$.

Proof. Let $G \in \mathcal{G}$. Suppose that $P_G(G) = 0$. By definition, $P(G) = 0$. So $Q(G) = 0$ and $Q \ll P_G$. \square

Exercise 6.1.4. Existence of Conditional Expectation:

Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{G} a sub σ -alg of \mathcal{F} and $X \in L^1(\Omega, \mathcal{F}, P)$. Define Q and P_G as in the previous exercise. Define $Y = dQ/dP_G$. Then Y is a conditional expectation of X given \mathcal{G} .

Proof. The Radon-Nikodym theorem implies that Y is \mathcal{G} -measurable. Since Q is finite, so is $|Q|$. Since $d|Q| = |Y| dP$, we have that $Y \in L^1(\Omega, \mathcal{G}, P_G)$. An exercise in section 3.3 of [3], implies that for each $G \in \mathcal{G}$

$$\begin{aligned} \int_G Y dP &= \int_G Y dP_G \\ &= Q(G) \\ &= \int_G X dP \end{aligned}$$

\square

Definition 6.1.5. Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L_S^0(\Omega, \mathcal{F})$. Let $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$. Then ϕ is said to be a **conditional expectation function of X given Y** if for each $B \in \mathcal{S} \cap Y(\Omega)$,

$$\int_{Y^{-1}(B)} X dP = \int_B \phi dP_Y$$

To denote this, we write $\phi(y) = E[X|Y = y]$.

Exercise 6.1.6. Existence of Conditional Expectation Function:

Let (Ω, \mathcal{F}, P) be a probability space, (S, \mathcal{S}) a measurable space, $X \in L^1(\Omega, \mathcal{F}, P)$ and $Y \in L_S^0(\Omega, \mathcal{F})$. Suppose that for each $y \in S$, $\{y\} \in \mathcal{S}$. Then there exists $\phi \in L^0(Y(S), \mathcal{S} \cap Y(\Omega))$ such that ϕ is a conditional expectation function of X given Y .

Hint: Doob-Dynkin lemma

Proof. Since $E[X|Y] \in L^0(\Omega, Y^*\mathcal{S})$, the Doob-Dynkin lemma implies that there exists $\phi \in L^0(Y(\Omega), \mathcal{S} \cap Y(\Omega))$ such that $\phi \circ Y = E[X|Y]$. Let $B \in \mathcal{S} \cap Y(\Omega)$. Then

$$\begin{aligned} \int_B \phi dP_Y &= \int_{Y^{-1}(B)} \phi \circ Y dP \\ &= \int_{Y^{-1}(B)} E[X|Y] dP \\ &= \int_{Y^{-1}(B)} X dP \end{aligned}$$

□

6.2. Conditional Probability.

Definition 6.2.1. Let (A, \mathcal{A}) be a measurable space, (B, \mathcal{B}, P_Y) a probability space and $Q : B \times \mathcal{A} \rightarrow [0, 1]$. Then Q is said to be a **stochastic transition kernel from (B, \mathcal{B}, P) to (A, \mathcal{A})** if

- (1) for each $E \in \mathcal{A}$, $Q(\cdot, E)$ is \mathcal{B} -measurable
- (2) for P -a.e. $b \in B$, $Q(b, \cdot)$ is a probability measure on (A, \mathcal{A})

Definition 6.2.2. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0(\Omega, \mathcal{F}, P)$ and $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$. Then Q is said to be a **conditional probability distribution of X given Y** if

- (1) Q is a stochastic transition kernel from $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), P_Y)$ to $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$
- (2) for each $A, B \in \mathcal{F}$,

$$\int_B Q(y, A) dP_Y(y) = P(X \in A, Y \in B)$$

Note 6.2.3. It is helpful to connect this notion of conditional probability with the elementary one by writing $Q(y, A) = P(X \in A | Y = y)$. If $P_Y \ll \mu$, then property (2) in the definition becomes

$$\begin{aligned} P(X \in A, Y \in B) &= \int_B Q(y, A) dP_Y(y) \\ &= \int_B P(X \in A | Y = y) f_Y(y) d\mu(y) \end{aligned}$$

as in a first course on probability.

Exercise 6.2.4. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$. Suppose that for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable, for P_Y -a.e. $y \in \mathbb{R}^n$, $P_{X|Y}(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) and $Q(Y, A) = P(X \in A | Y)$ a.e. Then Q is a conditional probability of X given Y .

Proof. By assumption, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable and for P_Y -a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) . Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned}
 \int_B Q(y, A) dP_Y(y) &= \int_{Y^{-1}(B)} Q(Y(\omega), A) dP(\omega) \\
 &= \int_{Y^{-1}(B)} P(X \in A | Y) dP \\
 &= \int_{Y^{-1}(B)} E[1_{X^{-1}(A)} | Y] dP \\
 &= \int_{Y^{-1}(B)} 1_{X^{-1}(A)} dP \\
 &= \int 1_{X^{-1}(A)} 1_{Y^{-1}(B)} dP \\
 &= \int 1_{X^{-1}(A) \cap Y^{-1}(B)} dP \\
 &= P(X \in A, Y \in B)
 \end{aligned}$$

So Q is a conditional probability distribution of X given Y . □

Definition 6.2.5. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and μ a σ -finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Then $P_{X,Y} \ll \mu^2$. Let $f_X = dP_X/d\mu$, $f_Y = dP_Y/d\mu$ and $f_{X,Y} = dP_{X,Y}/d\mu^2$. Define $f_{X|Y} : \mathbb{R}^n \times \mathbb{R}^n$ by

$$f_{X|Y}(x, y) = \begin{cases} \frac{f_{X,Y}(x, y)}{f_Y(y)}, & y \in \text{supp } Y \\ 0, & y \notin \text{supp } Y \end{cases}$$

Then $f_{X|Y}$ is called the **conditional probability density of X given Y** .

Exercise 6.2.6. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^0$ and μ a σ -finite measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Suppose that $P_X, P_Y \ll \mu$. Define $Q : \mathbb{R}^n \times \mathcal{F} \rightarrow [0, 1]$ by

$$Q(y, A) = \int_A f_{X|Y}(x, y) d\mu(x)$$

Then Q is a conditional probability distribution of X given Y .

Proof. By the Fubini-Tonelli Theorem, for each $A \in \mathcal{F}$, $Q(\cdot, A)$ is $\mathcal{B}(\mathbb{R}^n)$ -measurable and for P_Y -a.e. $y \in \mathbb{R}^n$, $Q(y, \cdot)$ is a probability measure on (Ω, \mathcal{F}) . Let $A, B \in \mathcal{F}$. Then

$$\begin{aligned}
 \int_B Q(y, A) dP_Y(y) &= \int_B \left[\int_A f_{X|Y}(x, y) d\mu(x) \right] dP_Y(y) \\
 &= \int_{B \cap \text{supp } Y} \left[\int_{A \cap \text{supp } Y} \frac{f_{X,Y}(x, y)}{f_Y(y)} d\mu(x) f_Y(y) \right] d\mu(y) \\
 &= \int_{B \cap \text{supp } Y} \left[\int_A f_{X,Y}(x, y) d\mu(x) \right] d\mu(y) \\
 &= P(X \in A, Y \in B \cap \text{supp } Y) \\
 &= P(X \in A, Y \in B)
 \end{aligned}$$

□

Theorem 6.2.7. Let (Ω, \mathcal{F}, P) be a probability space, $X, Y \in L_n^1(\Omega, \mathcal{F}, P)$. Suppose that $\text{Im } X \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a conditional probability distribution of Y given X .

7. MARKOV CHAINS

Definition 7.0.1. Let (Ω, \mathcal{F}, P) be a probability space and $(X_n)_{n \in \mathbb{N}_0} \in L_n^0$. Then $(X_n)_{n \in \mathbb{N}_0}$ is said to be a **homogeneous Markov chain** if for each $A \in \mathcal{F}$ and $n \in \mathbb{N}$, $P(X_n \in A | X_1, \dots, X_{n-1}) = P(X_1 \in A | X_0)$ a.e.

8. PROBABILITIES INDUCED BY ISOMETRIC GROUP ACTIONS

8.1. Applications to Bayesian Statistics.

Exercise 8.1.1. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \rightarrow \Theta$ an isometric group action. Suppose that \bar{d} is a metric on Θ/G . Let

- H_p^Θ be the Hausdorff measure on Θ , $\mu_{\mathcal{X}}$ a measure on \mathcal{X} ,
- p a density on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on \mathcal{X} .
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose that μ_Θ is G -invariant, p is G -invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G -invariant and continuous on Θ . For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \dots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \rightarrow [0, 1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)}) dH_p^\Theta(\theta)$$

Then there exists a density $\bar{p}(\cdot|X^{(n)})$ on Θ/G such that

$$d\bar{P}_{\Theta|X^{(n)}}(\theta) = \bar{p}(\theta|X^{(n)}) d\bar{H}^\Theta(\theta)$$

Proof. Clear from previous work. □

Exercise 8.1.2. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space (Θ, d) a metric space, G a group, $\phi : G \times \Theta \rightarrow \Theta$ an isometric group action. Suppose that \bar{d} is a metric on Θ/G . Let

- H_p^Θ be the Hausdorff measure on Θ , $\mu_{\mathcal{X}}$ a measure on \mathcal{X} ,
- p a density on Θ and for each $\theta \in \Theta$, $p(\cdot|\theta)$ a density on \mathcal{X} .
- $\theta_0 \in \Theta$ and for $j \in \mathbb{N}$, $X_j \sim p(x|\theta_0)$

Suppose p is G -invariant and continuous on Θ and for each $x \in \mathcal{X}$, $p(x|\cdot)$ is G -invariant and continuous on Θ . For $n \in \mathbb{N}$, set $p(\cdot|X^{(n)}) \propto f(X_1, \dots, X_n|\cdot)p(\cdot)$. Define the posterior measure $P_{\Theta|X^{(n)}} : \mathcal{B}(\Theta) \rightarrow [0, 1]$ by

$$dP_{\Theta|X^{(n)}}(\theta) = p(\theta|X^{(n)}) dH_p^\Theta(\theta)$$

Suppose that $(P_{\Theta|X^{(n)}})_{n \in \mathbb{N}}$ concentrates on $\bar{\theta}_0 \subset \Theta$ a.s. or in probability. Then $(\bar{P}_{\Theta|X^{(n)}})_{n \in \mathbb{N}}$ concentrates a.s. or in probability on $\{\bar{\theta}_0\} \subset \Theta/G$ (i.e. is consistent a.s. or in probability)

Proof. Let $V \in \mathcal{N}_{\bar{\theta}_0}$. Then $\pi^{-1}(V) \in \mathcal{N}_{\bar{\theta}_0}$. By definition,

$$\begin{aligned} \bar{P}_{\Theta|X^{(n)}}(V^c) &= P_{\Theta|X^{(n)}}(\pi^{-1}(V^c)) \\ &= P_{\Theta|X^{(n)}}(\pi^{-1}(V)^c) \\ &\xrightarrow{\text{a.s./}P} 0 \end{aligned}$$

□

Note 8.1.3. Some examples of G -invariant priors would be the uniform distribution, or $N_n(0, \sigma^2 I)$ on \mathbb{R}^n when acted on by $O(n)$. An example of a G -invariant likelihood would be $f(A|Z) \sim \text{Ber}(ZZ^T)$ as in a latent position random graph model where $Z \in \mathbb{R}^{n \times d}$ is the parameter is invariant under right multiplication by $U \in O_d$.

9. STOCHASTIC INTEGRATION

Exercise 9.0.1. Let (Ω, \mathcal{F}, P) be a probability space, X a set \mathcal{A}_0 an algebra, $\mu_0 : \mathcal{A}_0 \rightarrow \mathbb{C}$ and $B : \mathcal{A}_0 \rightarrow L^2(\Omega, \mathcal{F}, P)$. Suppose that

- (1) $B(\emptyset) = 0$
- (2) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $B(E \cup F) = B(E) + B(F)$
- (3) $E[B(E)B(F)^*] = \mu_0(E \cap F)$

Then

- (1) for each $E \in \mathcal{A}_0$, $\mu_0(E) = E[|B(E)|^2]$.
- (2) for each $E \in \mathcal{A}_0$, $0 \leq \mu_0(E) < \infty$
- (3) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$

Proof.

- (1) Clear
- (2) Clear
- (3) Let $E, F \in \mathcal{A}_0$. Suppose that $E \cap F = \emptyset$. Then

$$\begin{aligned} E[B(E)B(F)^*] &= \mu_0(E \cap F) \\ &= \mu_0(\emptyset) \\ &= E[|B(\emptyset)|^2] \\ &= E[0] \\ &= 0 \end{aligned}$$

This implies that

$$\begin{aligned} \mu_0(E \cup F) &= E[|B(E \cup F)|^2] \\ &= E[|B(E) + B(F)|^2] \\ &= E[|B(E)|^2] + E[|B(F)|^2] + 2\operatorname{Re}E[B(E)B(F)^*] \\ &= \mu_0(E) + \mu_0(F) + 0 \\ &= \mu_0(E) + \mu_0(F) \end{aligned}$$

□

Definition 9.0.2. Let (Ω, \mathcal{F}, P) be a probability space, X a set \mathcal{A}_0 an algebra, $\mu_0 : \mathcal{A}_0 \rightarrow [0, \infty)$ a premeasure and $B : \mathcal{A}_0 \rightarrow L^2(\Omega, \mathcal{F}, P)$. Suppose that

- (1) $B(\emptyset) = 0$
- (2) for each $E, F \in \mathcal{A}_0$, if $E \cap F = \emptyset$, then $B(E \cup F) = B(E) + B(F)$
- (3) $E[B(E)B(F)^*] = \mu_0(E \cap F)$

Then B is said to be a **stochastic premeasure with structure** μ_0

REFERENCES

- [1] [Introduction to Analysis](#)
- [2] [Introduction to Group Theory](#)
- [3] [Introduction to Measure and Integration](#)