1

Introduction to Measure and Integration

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Notation

$\mathcal{M}_+(X, \mathcal{A})$	finite measures on $(X, \mathcal{A})$
v	velocity

Preface

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Preliminaries

2.1 The Darboux Integral

Definition 2.1.0.1. Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Define

$$B([a,b]) = \{f : [a,b] \to \mathbb{R} : f \text{ is bounded}\}$$

Definition 2.1.0.2. Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Let $x_0, \cdots, x_n \in [a,b]$. Suppose that $a = x_0 < x_1 < \cdots < x_n = b$. Put $\mathcal{P} = \{x_0, \cdots, x_n\}$. Then $\mathcal{P}$ is said to be a **partion** of $[a,b]$.

Definition 2.1.0.3. Let $f \in B([a,b])$ and $\mathcal{P} = \{x_0, \cdots, x_n\}$ a partion of $[a,b]$. Suppose that f is bounded. For $i = 1, \cdots, n$, put

$$M_i^f = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

and

$$m_i^f = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

We define the **upper Darboux sum** of f with respect to $\mathcal{P}$, denoted $U_{\mathcal{P}} f$, to be

$$U_{\mathcal{P}} f = \sum_{i=1}^{n} M_i^f (x_i - x_{i-1})$$

and we define the **lower Darboux sum** of f with respect to $\mathcal{P}$, denoted $L_{\mathcal{P}} f$, to be

$$L_{\mathcal{P}} f = \sum_{i=1}^{n} m_i^f (x_i - x_{i-1})$$

Exercise 2.1.0.4. Let $f \in B([a,b])$ and $\mathcal{P}$ a partition of $[a,b]$. Then

$$\left[\inf_{x \in [a,b]} f(x)\right](b-a) \leq L_{\mathcal{P}} f \leq U_{\mathcal{P}} f \leq \left[\sup_{x \in [a,b]} f(x)\right](b-a)$$

Proof. Clear. □

Exercise 2.1.0.5. Let $f \in B([a,b])$ and $\mathcal{P}$, $\mathcal{P}'$ partitions of $[a,b]$. If $\mathcal{P} \subset \mathcal{P}'$, then

1. $U_{\mathcal{P}'} f \leq U_{\mathcal{P}} f$
2. $L_{\mathcal{P}} f \leq L_{\mathcal{P}'} f$

Proof.

1. Assume that $\mathcal{P} = \{x_0, \cdots, x_n\}$ and $\mathcal{P}' = \mathcal{P} \cup \{x'\}$. Then there exists $j \in \{1, \cdots, n\}$ such that $x_{j-1} < x' < x_j$. Define

$$M_1' = \sup_{x \in [x_{j-1}, x']} f(x), \quad M_2' = \sup_{x \in [x', x_j]} f(x)$$

Since $[x_{j-1}, x'], [x', x_j] \subset [x_{j-1}, x_j]$, we have that $M_1', M_2' \leq M_j^f$. Then

$$\begin{aligned} U_{P'} f &= \sum_{i=1}^{j-1} M_i^f (x_i - x_{i-1}) + M_1'(x' - x_{j-1}) + M_2'(x_j - x') + \sum_{i=j+1}^{n} M_i^f (x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} M_i^f (x_i - x_{i-1}) \\ &= U_P f \end{aligned}$$

By induction, this is true for general partitions $P \subset \mathcal{P}'$.

2. Similar to (1).

$\square$

Exercise 2.1.0.6. Let $f, g \in B([a,b])$ and $\mathcal{P} = \{x_0, \cdots, x_n\}$ a partition of $[a,b]$. Then

1. $U_{\mathcal{P}}(f+g) \leq U_{\mathcal{P}} f + U_{\mathcal{P}} g$
2. $L_{\mathcal{P}}(f+g) \geq L_{\mathcal{P}} f + L_{\mathcal{P}} g$

Proof.

1. For each $i \in \{1, \cdots, n\}$, $M_i^{f+g} \leq M_i^f + M_i^g$. So

$$\begin{aligned} U_{\mathcal{P}}(f+g) &= \sum_{i=1}^{n} M_i^{f+g}(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i^f + M_i^g)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} M_i^f (x_i - x_{i-1}) + \sum_{i=1}^{n} M_i^g (x_i - x_{i-1}) \\ &= U_{\mathcal{P}} f + U_{\mathcal{P}} g \end{aligned}$$

2. Similar to (1)

$\square$

Exercise 2.1.0.7. Let $f \in B([a,b])$ and $\mathcal{P} = \{x_0, \cdots, x_n\}$ a partition of $[a,b]$. Then

1. $U_{\mathcal{P}}(-f) = -L_P f$
2. $L_{\mathcal{P}}(-f) = -U_{\mathcal{P}} f$

Proof.

1. Since for $i \in \{1, \cdots, n\}$, $M_i^{-f} = -m_i^f$ we see that

$$\begin{aligned} U_{\mathcal{P}}(-f) &= \sum_{i=1}^{n} M_i^{-f} (x_i - x_{i-1}) \\ &= -\sum_{i=1}^{n} m_i^f (x_i - x_{i-1}) \\ &= -L_{\mathcal{P}} f \end{aligned}$$

2. Similar to (1).

□

Exercise 2.1.0.8. Let $f \in B([a,b])$, $c > 0$ and $\mathcal{P} = \{x_0, \cdots, x_n\}$ a partition of $[a,b]$. Then

1. $U_{\mathcal{P}}(cf) = cU_{\mathcal{P}}f$
2. $L_{\mathcal{P}}(cf) = cL_{\mathcal{P}}f$

Proof.

1. Since for $i \in \{1, \cdots, n\}$, $M_i^{cf} = cM_i^f$, we see that

$$\begin{aligned} U_{\mathcal{P}}(cf) &= \sum_{i=1}^{n} M_i^{cf}(x_i - x_{i-1}) \\ &= c\sum_{i=1}^{n} M_i^{f}(x_i - x_{i-1}) \\ &= cU_{\mathcal{P}}f \end{aligned}$$

2. Similar to (1)

□

Definition 2.1.0.9. Let $f \in B([a,b])$. We define the **upper Darboux integral** of f, denoted Uf, to be

$$Uf = \inf\{U_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a,b]\}$$

and we define the **lower Darboux integral** of f, denoted Lf, to be

$$Lf = \sup\{L_{\mathcal{P}}f : \mathcal{P} \text{ is a partition of } [a,b]\}$$

Exercise 2.1.0.10. Let $f \in B([a,b])$. Then

$$\left[\inf_{x\in[a,b]} f(x)\right](b-a) \leq Lf \leq Uf \leq \left[\sup_{x\in[a,b]} f(x)\right](b-a)$$

Proof. Clearly

$$\left[\inf_{x\in[a,b]} f(x)\right](b-a) \leq Lf \quad \text{and} \quad Uf \leq \left[\sup_{x\in[a,b]} f(x)\right](b-a)$$

Let $\epsilon > 0$. Then there exist partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a,b]$ such that $U_{\mathcal{P}_1}f < Uf + \epsilon/2$ and $L_{\mathcal{P}_2}f > Lf - \epsilon/2$. Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then

$$\begin{aligned} Uf &\geq U_{\mathcal{P}_1} - \epsilon/2 \\ &> U_{\mathcal{P}}f - \epsilon/2 \\ &\geq L_{\mathcal{P}}f - \epsilon/2 \\ &\geq L_{\mathcal{P}_2}f - \epsilon/2 \\ &> Lf - \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $Uf \geq Lf$. □

Exercise 2.1.0.11. Let $f, g \in B([a,b])$. Then

1. $U(f+g) \leq Uf + Ug$
2. $L(f+g) \geq Lf + Lg$

Proof.

1. Let $\epsilon > 0$. Then there exists a partitions $\mathcal{P}_1$ of $[a, b]$ such that $U_{\mathcal{P}_1} f < Uf + \epsilon/2$ and $U_{\mathcal{P}_2} g < Uf + \epsilon/2$. Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then

$$\begin{aligned} U_{\mathcal{P}}(f+g) &\leq U_{\mathcal{P}} f + U_{\mathcal{P}} g \\ &\leq U_{\mathcal{P}_1} f + U_{\mathcal{P}_2} g \\ &< Uf + \epsilon/2 + Ug + \epsilon/2 \\ &= Uf + Ug + \epsilon \end{aligned}$$

   Since $\epsilon > 0$ is arbitrary, $U_{\mathcal{P}}(f+g) \leq Uf + Ug$.

2. Similar to (1).

□

Exercise 2.1.0.12. Let $f \in B([a, b])$. Then

1. $U(-f) = -Lf$
2. $L(-f) = -Uf$

Proof.

1. Using a previous exercise, we have that

$$\begin{aligned} U(-f) &= \inf\{U_{\mathcal{P}}(-f) : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= \inf\{-L_{\mathcal{P}} f : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= -\sup\{L_{\mathcal{P}} f : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= -Lf \end{aligned}$$

2. Similar to (1)

□

Exercise 2.1.0.13. Let $f \in B([a, b])$ and $c \geq 0$. Then

1. $U(cf) = cUf$
2. $L(cf) = cLf$

Proof.

1. Using a previous exercise, we have that

$$\begin{aligned} U(cf) &= \inf\{U_{\mathcal{P}}(cf) : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= \inf\{cU_{\mathcal{P}} f : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= c \inf\{U_{\mathcal{P}} f : \mathcal{P} \text{ is a partition of } [a, b]\} \\ &= cUf \end{aligned}$$

2. Similar to (1)

□

Definition 2.1.0.14. Let $f \in B([a,b])$. Then f is said to be **Darboux integrable** if $Uf = Lf$. If f is Darboux integrable, we define the **Darboux integral** of f, denoted by

$$\int f \quad \text{or} \quad \int f(x)dx$$

to be

$$\int f = Uf = Lf$$

The set of bounded, Darboux integrable functions is denoted by $D([a,b])$.

Exercise 2.1.0.15. Let $f \in B([a,b])$. Then $f \in D([a,b])$ iff for each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$.

Proof. Suppose that $f \in D([a,b])$. Let $\epsilon > 0$. Then there exist partions $\mathcal{P}_1$, $\mathcal{P}_2$ of $[a,b]$ such that $U_{\mathcal{P}_1}f < Uf + \epsilon/2$ and $L_{\mathcal{P}_2}f > Lf - \epsilon/2$. Define $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $U_{\mathcal{P}}f \leq U_{\mathcal{P}_1}f$ and $L_P f \geq L_{\mathcal{P}_2}f$. So

$$\begin{aligned} U_{\mathcal{P}}f - L_{\mathcal{P}}f &< Uf - Lf + \epsilon \\ &= \epsilon \end{aligned}$$

Conversely, suppose that for each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$. For the sake of contradiction, suppose that $Uf - Lf > 0$. Choose $\epsilon = Uf - Lf$. Then there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$. Since $Uf \leq U_{\mathcal{P}}f$ and $Lf \geq L_{\mathcal{P}}f$, we have that

$$\begin{aligned} \epsilon &> U_{\mathcal{P}}f - L_{\mathcal{P}}f \\ &\geq Uf - Lf \\ &= \epsilon \end{aligned}$$

which is a contradiction. Hence $Uf = Lf$ and $f \in D([a,b])$. □

Exercise 2.1.0.16. Let $f, g \in D([a,b])$. Then $f + g \in D([a,b])$ and

$$\int (f+g) = \int f + \int g$$

Proof. Clearly $f + g \in B([a,b])$. Using some previous results, we have that

$$\begin{aligned} \int f + \int g &= Lf + Lg \\ &\leq L(f+g) \\ &\leq U(f+g) \\ &\leq Uf + Ug \\ &= \int f + \int g \end{aligned}$$

So $U(f+g) = L(f+g) = \int f + \int g$. Therefore $f + g \in D([a,b])$ and

$$\int (f+g) = \int f + \int g$$

. □

Exercise 2.1.0.17. Let $f \in D([a,b])$ and $c \in \mathbb{R}$. Then $cf \in D([a,b])$ and

$$\int (cf) = c \int f$$

Proof. Clearly $cf \in B([a,b])$. If $c \geq 0$, then

$$\begin{aligned} L(cf) &= cLf \\ &= c\int f \\ &= cUf \\ &= U(cf) \end{aligned}$$

So

$$L(cf) = U(cf) = c\int f$$

If $c < 0$, then

$$\begin{aligned} L(cf) &= L(-|c|f) \\ &= -U(|c|f) \\ &= -|c|Uf \\ &= c\int f \\ &= -|c|Lf \\ &= -L(|c|f) \\ &= U(-|c|f) \\ &= U(cf) \end{aligned}$$

So

$$L(cf) = U(cf) = c\int f$$

Therefore $cf \in D([a,b])$ and

$$\int (cf) = c\int f$$

□

Corollary 2.1.0.18. We have that $D([a,b])$ is a vector space and the map $I : D([a,b]) \to \mathbb{R}$ given by $If = \int f$ is linear.

Proof. Clear. □

Exercise 2.1.0.19. Let $f : [a,b] \to \mathbb{R}$. If f is continuous, then $f \in D([a,b])$.

Proof. Suppose that f is continuous. Then f is uniformly continuous. Let $\epsilon > 0$. Uniform continuity implies that there exists $\delta > 0$ such that for each $x, y \in [a,b]$, $|x-y| < \delta$ implies that $|f(x) - f(Y)| < \epsilon/(b-a)$. Choose $n \in \mathbb{N}$ such that $(b-a)/n < \delta$. For $i \in \{0, \cdots, n\}$, define $x_i = a + i(b-a)/n$. Put $\mathcal{P} = \{x_0, \cdots, x_n\}$. Continuity implies that for each $i \in \{1, \cdots, n\}$, there exists $x_i^M, x_i^m \in [x_{i-1}, x_i]$ such that $f(x_i^M) = M_i^f$ and $f(x_i^m) = m_i^f$. Then

$$\begin{aligned} U_{\mathcal{P}}f - L_{\mathcal{P}}f &= \sum_{i=1}^n M_i^f(x_i - x_{i-1}) - \sum_{i=1}^n m_i^f(x_i - x_{i-1}) \\ &= \sum_{i=1}^n (M_i^f - m_i^f)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n [f(x_i^M) - f(x_i^m)](x_i - x_{i-1}) \\ &< \sum_{i=1}^n \frac{\epsilon}{b-a}(x_i - x_{i-1}) \\ &= \epsilon \end{aligned}$$

So for each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$. Hence $f \in D([a,b])$. $\square$

Exercise 2.1.0.20. Let $f : [a,b] \to \mathbb{R}$. If f is monotonic, then $f \in D([a,b])$.

Proof. Suppose that f is increasing. Let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $(b-a)[f(b)-f(a)]/n < \epsilon$. For $i \in \{0, \cdots, n\}$, define $x_i = a + i(b-a)/n$. Put $\mathcal{P} = \{x_0, \cdots, x_n\}$. Then

$$\begin{aligned}
U_{\mathcal{P}}f - L_{\mathcal{P}}f &= \sum_{i=1}^{n} M_i^f(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i^f(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n}(M_i^f - m_i^f)(x_i - x_{i-1}) \\
&= \frac{b-a}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] \\
&= \frac{b-a}{n}[f(b) - f(a)] \\
&< \epsilon
\end{aligned}$$

So for each $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U_{\mathcal{P}}f - L_{\mathcal{P}}f < \epsilon$. Hence $f \in D([a,b])$. The case is similar if f is decreasing. $\square$

Exercise 2.1.0.21. Define $\chi_{\mathbb{Q}} : [0,1] \to \mathbb{R}$ by

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Then $\chi_{\mathbb{Q}} \notin D([a,b])$.

Proof. Let $\mathcal{P} = \{x_0, \cdots, x_n\}$ be a partition of $[0,1]$. Then for each $i \in \{1, \cdots, n\}$, $M_i^{\chi_{\mathbb{Q}}} = 1$ and $m_i^{\chi_{\mathbb{Q}}} = 0$. So $U_{\mathcal{P}}\chi_{\mathbb{Q}} = 1$ and $L_{\mathcal{P}}\chi_{\mathbb{Q}} = 0$. Since $\mathcal{P}$ is arbitrary, we have that $U\chi_{\mathbb{Q}} = 1$ and $L\chi_{\mathbb{Q}} = 0$ $\square$

2.2 The Extended Real Line

- talk about $\overline{\mathbb{R}}$, define addition, multiplication, talk about inversion as order reversing, talk about the topology (i.e. that it is homeomorphic to a closed interval)

Chapter 3

Measurable Spaces

3.1 Elementary Families and Algebras

Definition 3.1.0.1. Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$. Then X is said to be an **elementary family on X** if

1. $\varnothing \in \mathcal{E}$
2. for each $A, B \in \mathcal{E}$, $A \cap B \in \mathcal{E}$
3. for each $A \in \mathcal{E}$, there exist $(A_j)_{j=1}^n \subset \mathcal{E}$ such that $(A_j)_{j=1}^n$ is disjoint and $A^c = \bigcup\limits_{j=1}^n A_j$

Exercise 3.1.0.2. Define

$$\mathcal{E} = \{(a, b] : a, b \in \overline{\mathbb{R}}\}$$

where we take $(a, \infty] = (a, \infty)$. Then $\mathcal{E}$ is an elementary family on $\mathbb{R}$

Proof.

1. $\varnothing = (0, 0] \in \mathcal{E}$
2. Let $a_1, a_2, b_1, b_2 \in \overline{\mathbb{R}}$. Then

$$(a_1, b_1] \cap (a_2, b_2] = \begin{cases} \varnothing & b_1 \leq a_2 \\ (a_2, b_1] & b_1 > a_2 \end{cases}$$

   So $(a_1, b_1] \cap (a_2, b_2] \in \mathcal{E}$.

3. Let $a, b \in \mathbb{R}$. Suppose that $a < b$. Then $(a, b]^c = (-\infty, a] \cup (b, \infty) \in \mathcal{E}$.

$\square$

Definition 3.1.0.3. Let X be a set and $\mathcal{A}_0 \subset \mathcal{P}(X)$. Then $\mathcal{A}_0$ is said to be an **algebra** on X if

1. $\mathcal{A}_0 \neq \varnothing$
2. for each $A \in \mathcal{A}_0$, $A^c \in \mathcal{A}_0$
3. for each $A, B \in \mathcal{A}_0$, $A \cup B \in \mathcal{A}_0$

Exercise 3.1.0.4. Let X be a set and $\mathcal{E}$ an elementary family on X. Define

$$\mathcal{A}_0^{\mathcal{E}} = \left\{ \bigcup_{j=1}^n A_j : (A_j)_{j=1}^n \text{ is disjoint and } (A_j)_{j=1}^n \subset \mathcal{E} \right\}$$

Then $\mathcal{A}_0^{\mathcal{E}}$ is an algebra on X.

Proof.

1. By definition, $\varnothing \in \mathcal{E} \subset \mathcal{A}_0^{\mathcal{E}}$. So $\mathcal{A}_0^{\mathcal{E}} \neq \varnothing$.

2. Let $A \in \mathcal{A}_0^{\mathcal{E}}$, there exists $(A_j)_{j=1}^n \subset \mathcal{E}$ such that $(A_j)_{j=1}^n$ is disjoint and $A = \bigcup_{j=1}^n A_j$. By definition of $\mathcal{E}$, for each $j \in \{1, \ldots, n\}$, there exist $(B_{j,k})_{k=1}^{n_j} \subset \mathcal{E}$ such that $(B_{j,k})_{k=1}^{n_j}$ is disjoint and $A_j^c = \bigcup_{k=1}^{n_j} B_{j,k}$. Then
$$\begin{aligned} A^c &= \bigcap_{j=1}^n A_j^c \\ &= \bigcap_{j=1}^n \left(\bigcup_{k=1}^{n_j} B_{j,k} \right) \\ &= \bigcup \end{aligned}$$

3. Let $A, B \in \mathcal{A}_0^{\mathcal{E}}$. Then there exist $(A_j)_{j=1}^n, (B_j)_{j=1}^m \subset \mathcal{E}$ such that $A = \bigcup_{j=1}^n A_j$ and $B = \bigcup_{j=1}^m B_j$. Then
$$A \cup B = \left(\bigcup_{j=1}^n A_j \right) \cup \left(\bigcup_{j=1}^m B_j \right)$$

FINISH!!! □

3.2 Sigma Algebras

Definition 3.2.0.1. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. Then $\mathcal{A}$ is said to be a **σ-algebra** on X if

1. $\mathcal{A} \neq \varnothing$
2. for each $A \in \mathcal{A}$, $A^c \in \mathcal{A}$
3. for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}$

Exercise 3.2.0.2. Let X be a set and $\mathcal{A}$ a σ-algebra on X. Then

1. $X, \varnothing \in \mathcal{A}$
2. for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, $\bigcap\limits_{n\in\mathbb{N}} \in \mathcal{A}$
3. For each $A, B \in \mathcal{A}$, $A \setminus B \in \mathcal{A}$

Proof.

1. Since $\mathcal{A} \neq \varnothing$, there exists $A \in \mathcal{A}$. Then $A^c \in \mathcal{A}$. Hence $X = A \cup A^c \in \mathcal{A}$ and $\varnothing = X^c \in \mathcal{A}$.
2. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Then $(A_n^c)_{n\in\mathbb{N}} \subset MA$. So $\bigcup\limits_{n\in\mathbb{N}} A_n^c \in \mathcal{A}$. Therefore
$$\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n^c\right)^c \in \mathcal{A}$$
3. Let $A, B \in \mathcal{A}$. Then $A \setminus B = A \cap B^c \in \mathcal{A}$.

$\square$

Exercise 3.2.0.3. Let X be a set and $(\mathcal{A}_j)_{j\in J}$ a collection of σ-algebras (resp. algebra) on X. Then $\bigcap\limits_{j\in J} \mathcal{A}_j$ is a σ-algebra (resp. algebra) on X.

Proof.

1. For each $i \in I$, $X \in \mathcal{A}_j$. Thus $X \in \bigcap\limits_{j\in J} \mathcal{A}_j$ and $\bigcap\limits_{j\in J} \mathcal{A}_j \neq \varnothing$.
2. Let $A \in \bigcap\limits_{j\in J} \mathcal{A}_j$. Then for each $j \in J$, $A \in \mathcal{A}_j$. Hence for each $j \in J$, $A^c \in \mathcal{A}_j$. Thus $A^c \in \bigcap\limits_{j\in J} \mathcal{A}_j$.
3. Let $(A_n)_{n\in\mathbb{N}} \subset \bigcap\limits_{j\in J} \mathcal{A}_j$. Then for each $j \in J$, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_j$. Thus for each $j \in J$, $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}_j$. So $\bigcup\limits_{n\in\mathbb{N}} A_n \in \bigcap\limits_{j\in J} \mathcal{A}_j$.

$\square$

Definition 3.2.0.4. Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$. Set
$$\mathcal{S} := \{\mathcal{A} \subset \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X \text{ and } \mathcal{C} \subset \mathcal{L}\}$$
We define the **σ-algebra generated by** $\mathcal{C}$ on X, denoted $\sigma_X(\mathcal{C})$, by
$$\sigma_X(\mathcal{C}) = \bigcap_{\mathcal{A}\in\mathcal{S}} \mathcal{A}$$

Note 3.2.0.5. When the context is clear, we write $\sigma(\mathcal{C})$ in place of $\sigma_X(\mathcal{C})$.

Note 3.2.0.6. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $\mathcal{A}$ a σ-alg on X. By definition, if $\mathcal{C} \subset \mathcal{A}$, then $\sigma(\mathcal{C}) \subset \mathcal{A}$.

Note 3.2.0.7. Let X be a set, $\mathcal{T}$ an ordered set and $(\mathcal{A}_t)_{t\in\mathcal{T}}$ a collection of σ-algebras on X. Suppose that for each $s,t \in \mathcal{T}$, if $s \leq t$, then $\mathcal{A}_s \subset \mathcal{A}_t$. If there exists $t \in \mathcal{T}$ such that $\mathcal{A}_t = \bigcup\limits_{t\in\mathcal{T}} \mathcal{A}_t$, then $\bigcup\limits_{t\in\mathcal{T}} \mathcal{A}_t$ is a σ-algebra on X. So if $\mathcal{T}$ is finite or if $(\mathcal{A}_t)_{t\in\mathcal{T}}$ termintates, the union is σ-algebra.

Definition 3.2.0.8. Let $(X,\mathcal{T})$ be a topological space. We define the **Borel σ-algebra** on X, denoted $\mathcal{B}(X,\mathcal{T})$, by

$$\mathcal{B}(X,\mathcal{T}) = \sigma(\mathcal{T})$$

Let $E \subset X$. Then E is said to be **Borel** if $E \in \mathcal{B}(X,\mathcal{T})$.

Note 3.2.0.9. If the topology $\mathcal{T}$ on X is unambiguous, we write $\mathcal{B}(X)$ in place of $\mathcal{B}(X,\mathcal{T})$.

Exercise 3.2.0.10. Show that $\mathcal{B}(\overline{\mathbb{R}}) = \sigma((\alpha,\infty] : \alpha \in \mathbb{R})$ and similar

Proof. FINISH!!! ∎

Exercise 3.2.0.11. The Borel σ-algebra on $\mathbb{R}$ with the standard topology is given by

$$\mathcal{B}(\mathbb{R}) = \begin{cases} \sigma(\{(a,b] : a,b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{[a,b] : a,b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{[a,b) : a,b \in \mathbb{R} \text{ and } a < b\}) \\ \sigma(\{(a,b) : a,b \in \mathbb{R} \text{ and } a < b\}) \end{cases}$$

Proof. Define

1. $\mathcal{C}_{lo} = \{(a,b] : a,b \in \mathbb{R} \text{ and } a < b\}$
2. $\mathcal{C}_{c} = \{[a,b] : a,b \in \mathbb{R} \text{ and } a < b\}$
3. $\mathcal{C}_{ro} = \{[a,b) : a,b \in \mathbb{R} \text{ and } a < b\}$
4. $\mathcal{C}_{o} = \{(a,b) : a,b \in \mathbb{R} \text{ and } a < b\}$

Recall that for each open set $A \subset \mathbb{R}$, there exist $(a_i)_{n\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ such that for each $i \in \mathbb{N}$, $a_i < b_i$, for each $i,j \in \mathbb{N}$, if $i \neq j$, then $(a_i,b_i) \cap (a_j,b_j) = \varnothing$ and $A = \bigcup\limits_{i\in\mathbb{N}} (a_i,b_i)$. This implies that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o)$.

Now, let $a,b \in \mathbb{R}$. Suppose that $a < b$. Then

1. $[a,b] = \bigcap\limits_{n\in\mathbb{N}} (a - \frac{1}{n}, b]$, so $\sigma(\mathcal{C}_c) \subset \sigma(\mathcal{C}_{lo})$
2. $[a,b) = \bigcup\limits_{n\in\mathbb{N}} [a, b - \frac{1}{n}]$, so $\sigma(\mathcal{C}_{ro}) \subset \sigma(\mathcal{C}_c)$
3. $(a,b) = \bigcup\limits_{n\in\mathbb{N}} [a + \frac{1}{n}, b)$, so $\sigma(\mathcal{C}_o) \subset \sigma(\mathcal{C}_{ro})$
4. $(a,b] = \bigcap\limits_{n\in\mathbb{N}} (a, b + \frac{1}{n})$, so $\sigma(\mathcal{C}_{lo}) \subset \sigma(\mathcal{C}_o)$

Hence $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_o) = \sigma(\mathcal{C}_{ro}) = \sigma(\mathcal{C}_c) = \sigma(\mathcal{C}_{lo}) = \sigma(\mathcal{C}_o)$. ∎

Exercise 3.2.0.12. Let $(X,\mathcal{T})$ be a topologcial space and $\mathcal{E} \subset \mathcal{T}$ a basis for $\mathcal{T}$. If $\mathcal{E}$ is countable, then $\mathcal{B}(X) = \sigma(\mathcal{E})$.

Proof. Since $\mathcal{E} \subset \mathcal{T}$,

$$\begin{aligned}\sigma(\mathcal{E}) &\subset \sigma(\mathcal{T}) \\ &= \mathcal{B}(X)\end{aligned}$$

Let $U \in \mathcal{T}$. Since $\mathcal{E}$ is a countable basis, there exists $\mathcal{C}_U \subset \mathcal{E}$ such that $\mathcal{C}_U$ is countable and $U = \bigcup_{C \in \mathcal{C}_U} C$. Hence $U \in \sigma(\mathcal{E})$. Since $U \in \mathcal{T}$ is arbitary, $\mathcal{T} \subset \sigma(\mathcal{E})$. Thus

$$\begin{aligned}\mathcal{B}(X) &= \sigma(\mathcal{T}) \\ &\subset \sigma(\mathcal{E})\end{aligned}$$

Therefore $\mathcal{B}(X) = \sigma(\mathcal{E})$. □

Exercise 3.2.0.13. Let X be a set. Define $\mathcal{A} = \{A \in \mathcal{A} : A \text{ is countable or } A^c \text{ is countable}\}$. Then $\mathcal{A}$ is a σ-algebra on X.

Proof.

1. Since $X^c = \varnothing$ is countable, $X \in \mathcal{A}$.
2. Let $A \in \mathcal{A}$. Suppose that A^c is uncountable. Then by assumption, $A = (A^c)^c$ is countable. Hence $A^c \in \mathcal{A}$.
3. Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Then for each $n \in \mathbb{N}$, A_n is countable or A_n^c is countable. Suppose that $\bigcup_{n \in \mathbb{N}} A_n$ is uncountable. Then there exists $N \in \mathbb{N}$ such that A_N is uncountable. Hence A_N^c is countable. Thus

$$\begin{aligned}\left(\bigcup_{n \in \mathbb{N}} A_n\right)^c &= \bigcap_{n \in \mathbb{N}} A_n^c \\ &\subset A_N^c\end{aligned}$$

So $\left(\bigcup_{n \in \mathbb{N}} A_n\right)^c$ is countable and $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

□

Definition 3.2.0.14. Let X be a set and $\mathcal{A}$ be a σ-algebra on X. Then $(X, \mathcal{A})$ is called a **measurable space**.

3.3 Measurable Functions

Definition 3.3.0.1. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then f is said to be $(\mathcal{A}, \mathcal{B})$**-measurable** if for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. When $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we say that f is $\mathcal{A}$**-measurable**. If $(Y, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(X, \mathcal{A}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}, \mathcal{L})$, then we say that f is **Borel measurable** or **Lebsgue measurbale** respectively.

Definition 3.3.0.2. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Define

- $L^+(X, \mathcal{A}) = \{f : X \to [0, \infty] : f \text{ is measurable}\}$
- $L(X, \mathcal{A}) = \{f : X \to \mathbb{C} : f \text{ is measurable}\}$

Definition 3.3.0.3. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $\phi : X \to Y$. Then ϕ is said to be $(\mathcal{A}, \mathcal{B})$**-bimeasurable** if

1. ϕ is $(\mathcal{A}, \mathcal{B})$-measurable
2. for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$

Definition 3.3.0.4. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $\phi : X \to Y$. Then ϕ is said to be a **isomorphism** if

1. ϕ is a bijection
2. ϕ is $(\mathcal{A}, \mathcal{B})$-bimeasurable

Definition 3.3.0.5. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Then $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are said to be **isomorphic** if there exists $\phi : X \to Y$ such that ϕ is an isomorphism.

Definition 3.3.0.6. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. We define the

1. **pushforward of** $\mathcal{A}$, denoted $f_*\mathcal{A}$, by
$$f_*\mathcal{A} = \{B \subset Y : f^{-1}(B) \in \mathcal{A}\}$$
2. **pullback of** $\mathcal{B}$, denoted $f^*\mathcal{B}$, by
$$f^*\mathcal{B} = \{f^{-1}(B) : B \in \mathcal{B}\}$$

Note 3.3.0.7. It is also common to write $\sigma(f)$ or $f^{-1}(\mathcal{B})$ in place of $f^*\mathcal{B}$.

Exercise 3.3.0.8. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then

1. $f_*\mathcal{A}$ is a σ-algebra on Y
2. $f^*\mathcal{B}$ is a σ-algebra on X

Proof.

1. - Since $f^{-1}(Y) = X \in \mathcal{A}$, $Y \in f_*\mathcal{A}$ and $f_*\mathcal{A} \neq \varnothing$.
   - Let $B \in f_*\mathcal{A}$. Then $f^{-1}(B) \in \mathcal{A}$. Hence
   $$f^{-1}(B^c) = (f^{-1}(B))^c \in \mathcal{A}$$
   Thus $B^c \in f_*\mathcal{A}$.
   - Now, let $(B_n)_{n\in\mathbb{N}} \subset f_*\mathcal{A}$. Then for each $n \in \mathbb{N}$, $f^{-1}(B_n) \in \mathcal{A}$. Thus
   $$f^{-1}\bigg(\bigcup_{n\in\mathbb{N}} B_n\bigg) = \bigcup_{n\in\mathbb{N}} f^{-1}(B_n) \in \mathcal{A}$$
   Hence $\bigcup\limits_{n\in\mathbb{N}} B_n \in f_*\mathcal{A}$.
2. Similar to (1).

□

Exercise 3.3.0.9. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. If f is an isomorphism, then

1. $f^*(\mathcal{B}) = \mathcal{A}$
2. $f_*(\mathcal{A}) = \mathcal{B}$

Proof. Suppose that f is an isomorphism.

1. Since f is $(\mathcal{A}, \mathcal{B})$-measurable, $f^*(\mathcal{B}) \subset \mathcal{A}$. Let $A \in \mathcal{A}$. Set $B = f(A)$. Since f^{-1} is $(\mathcal{B}, \mathcal{A})$-measurable, $B \in \mathcal{B}$. By definition,
$$\begin{aligned} A &= f^{-1}(B) \\ &\in f^*(\mathcal{B}) \end{aligned}$$
Since $A \in \mathcal{A}$ is arbitrary, $\mathcal{A} \subset f^*(\mathcal{B})$. Hence $f^*(\mathcal{B}) = \mathcal{A}$.
2. Since f is $(\mathcal{A}, \mathcal{B})$-measurable, $\mathcal{B} \subset f_*(\mathcal{A})$. Let $B \in f_*(\mathcal{A})$. By definition, $f^{-1}(B) \in \mathcal{A}$. Set $A = f^{-1}(B)$. Since f^{-1} is $(\mathcal{B}, \mathcal{A})$-measurable,
$$\begin{aligned} B &= f(A) \\ &\in \mathcal{B} \end{aligned}$$
Since $B \in f_*(\mathcal{A})$ is arbitrary, $f_*(\mathcal{A}) \subset \mathcal{B}$. Hence $f_*(\mathcal{A}) = \mathcal{B}$.

□

Exercise 3.3.0.10. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. If f is constant, then

1. $f^*(\mathcal{B}) = \{\varnothing, X\}$
2. $f_*(\mathcal{A}) = \mathcal{P}(Y)$

Proof. Suppose that f is constant. Then there exists $y \in Y$ such that for each $x \in X$, $f(x) = y$. Then for each $B \subset Y$,

$$f^{-1}(B) = \begin{cases} X, & y \in B \\ \varnothing, & y \notin B \end{cases}$$

1. Clearly $\{\varnothing, X\} \subset f^*(\mathcal{B})$. Let $A \in f^*(\mathcal{B})$. Then there exists $B \in \mathcal{B}$ such that $A = f^{-1}(B)$. Then
$$\begin{aligned} A &= f^{-1}(B) \\ &\in \{\varnothing, X\} \end{aligned}$$
Since $A \in f^*(\mathcal{B})$ is arbitrary, $f^*(\mathcal{B}) \subset \{\varnothing, X\}$. Hence $f^*(\mathcal{B}) = \{\varnothing, X\}$.
2. Clearly $f_*(\mathcal{A}) \subset \mathcal{P}(Y)$. Let $B \in \mathcal{P}(Y)$. Since $\{\varnothing, X\} \subset \mathcal{A}$, we have that
$$\begin{aligned} f^{-1}(B) &= X \\ &\in \{\varnothing, X\} \\ &\subset \mathcal{A} \end{aligned}$$
Hence $B \in f_*(\mathcal{A})$. Since $B \in \mathcal{P}(Y)$ is arbitrary, $\mathcal{P}(Y) \subset f_*(\mathcal{A})$. Hence $f_*(\mathcal{A}) = \mathcal{P}(Y)$.

□

Exercise 3.3.0.11. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. Suppose that there exists $\mathcal{E} \subset Y$ such that $\sigma(\mathcal{E}) = \mathcal{B}$. Let $f : X \to Y$. Then f is $(\mathcal{A}, \mathcal{B})$ measurable iff for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$.

Proof. By definition, if f is $\mathcal{A}$-$\mathcal{B}$ measurable, then for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. Conversely, suppose that for each $B \in \mathcal{E}$, $f^{-1}(B) \in \mathcal{A}$. The previous exercise tells us that $f_*\mathcal{A}$ is a σ-algebra on Y. Since $\mathcal{E} \subset f_*\mathcal{A}$, we have that $\mathcal{B} = \sigma(\mathcal{E}) \subset f_*\mathcal{A}$. So f is $(\mathcal{A}, \mathcal{B})$ measurable. $\square$

Exercise 3.3.0.12. Let X, Y be sets, $f : X \to Y$ and $\mathcal{E} \subset \mathcal{P}(Y)$. Then $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$.

Proof. Clealy $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$. Since $f^{-1}(\sigma(\mathcal{E}))$ is a σ-algebra, we have that $\sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. Since $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$, the previous exercise tells us that f is $f^{-1}(\sigma(\mathcal{E}))$-$\sigma(\mathcal{E})$ measurable. Then $f^{-1}(\sigma(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))$. So $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$.
FINISH!!! $\square$

Definition 3.3.0.13. Let X be a set, $(Y_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ a collection of measurable spaces and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y_\alpha^X$ (i.e. $\mathcal{F} = (f_\alpha)_{\alpha \in A}$ where for each $\alpha \in A$, $f_\alpha : X \to Y_\alpha$). We define the **initial σ-algebra generated by** $\mathcal{F}$ on X, denoted $\sigma_X(\mathcal{F})$, by

$$\sigma_X(\mathcal{F}) = \sigma(\{f_\alpha^{-1}(B) : B \in \mathcal{A}_\alpha \text{ and } \alpha \in A\})$$

Note 3.3.0.14. If $\mathcal{F} = \{f\}$, then $\sigma_X(\mathcal{F}) = f^*\mathcal{A}$.

Exercise 3.3.0.15. Let X be a set, $(Y_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ a collection of measurable spaces and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y_\alpha^X$ (i.e. $\mathcal{F} = (f_\alpha)_{\alpha \in A}$ where for each $\alpha \in A$, $f_\alpha : X \to Y_\alpha$). Then for each $\mathcal{A} \subset \mathcal{P}(X)$ if $\mathcal{A}$ is a σ-algebra on X and for each $\alpha \in A$, f_α is $(\mathcal{A}, \mathcal{A}_\alpha)$-measurable, then $\sigma_X(\mathcal{F}) \subset \mathcal{A}$.

Proof. Let $\mathcal{A} \subset \mathcal{P}(X)$. Suppose that $\mathcal{A}$ is a topology on X and for each $\alpha \in A$, f_α is $(\mathcal{A}, \mathcal{A}_\alpha)$-measurable. Set $\mathcal{V} := \{f_\alpha^{-1}(V) : V \in \mathcal{A}_\alpha \text{ and } \alpha \in A\ \}$. By definition, $\sigma_X(\mathcal{F}) = \sigma_X(\mathcal{V})$. Since for each $\alpha \in A$, f_α is $(\mathcal{A}, \mathcal{A}_\alpha)$-measurable, we have that for each $\alpha \in A$ and $V \in \mathcal{A}_\alpha$, $f_\alpha^{-1}(V) \in \mathcal{A}$. Hence $\mathcal{V} \subset \mathcal{A}$. Therefore

$$\begin{aligned} \sigma_X(\mathcal{F}) &= \sigma_X(\mathcal{V}) \\ &\subset \mathcal{A}. \end{aligned}$$

$\square$

Note 3.3.0.16. Essentially, $\sigma_X(\mathcal{F})$ is the smallest σ-algebra on X such that for each $\alpha \in A$, $f_\alpha : X \to Y_\alpha$ is measurable.

Exercise 3.3.0.17. Let $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$ be a a collection of measurable spaces, X a set, $(Z, \mathcal{C})$ a measurable space, $\mathcal{F} = (f_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} Y_\alpha^X$ and $g : Z \to X$. Then g is $(\mathcal{C}, \sigma_X(\mathcal{F}))$-measurable iff for each $\alpha \in A$, $f_\alpha \circ g$ is $(\mathcal{C}, \mathcal{B}_\alpha)$-measurable:

$$\begin{tikzcd} Y_\alpha & X \arrow[l, "f_\alpha"'] \\ & Z \arrow[u, "g"'] \arrow[ul, "g\circ f_\alpha"] \end{tikzcd}$$

Proof. If g is $(\mathcal{C}, \sigma_X(\mathcal{F}))$-measurable, then clearly for each $\alpha \in A$, $f_\alpha \circ g$ is $(\mathcal{C}, \mathcal{B}_\alpha)$-measurable.
Conversely, suppose that for each $\alpha \in A$, $f_\alpha \circ g$ is $(\mathcal{C}, \mathcal{B}_\alpha)$-measurable. Let $\alpha \in A$ and $V \in \mathcal{B}_\alpha$. Measurability implies that,

$$\begin{aligned} g^{-1}(f_\alpha^{-1}(V)) &= (f_\alpha \circ g)^{-1}(V) \\ &\in \mathcal{C} \end{aligned}$$

Since $\alpha \in A$ and $V \in \mathcal{B}_\alpha$ are arbitrary, we have that for each $\alpha \in A$ and $V \in \mathcal{B}_\alpha$, $g^{-1}(f_\alpha^{-1}(V)) \in \mathcal{C}$. Since $\sigma_X(\mathcal{F}) = \sigma(\{f_\alpha^{-1}(V) : \alpha \in A \text{ and } V \in \mathcal{B}_\alpha)$, a previous exercise implies that g is $(\mathcal{C}, \sigma_X(\mathcal{F}))$-measurable. $\square$

Definition 3.3.0.18. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a a collection of measurable spaces, Y a set and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y^{X^\alpha}$ (i.e. $\mathcal{F} = (f_\alpha)_{\alpha \in A}$ where for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y$). We define the **final σ-algebra generated by** $\mathcal{F}$ on X, denoted $\sigma_Y(\mathcal{F})$, by

$$\sigma_Y(\mathcal{F}) = \sigma(\{V \subset Y : \text{ for each } \alpha \in A,\ f_\alpha^{-1}(V) \in \mathcal{A}_\alpha\})$$

Note 3.3.0.19. If $\mathcal{F} = \{f\}$, then $\sigma_Y(\mathcal{F}) = f_*\mathcal{A}$.

Exercise 3.3.0.20. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a a collection of topological spaces, Y a set and $\mathcal{F} \in \prod\limits_{\alpha \in A} Y^{X^\alpha}$ (i.e. $\mathcal{F} = (f_\alpha)_{\alpha \in A}$ where for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y$). Then for each $\mathcal{A} \subset \mathcal{P}(Y)$ if $\mathcal{A}$ is a topology on Y and for each $\alpha \in A$, f_α is $(\mathcal{A}_\alpha, \mathcal{A})$-continuous, then $\mathcal{A} \subset \sigma_Y(\mathcal{F})$.

Proof. Let $\mathcal{A} \subset \mathcal{P}(Y)$. Suppose that $\mathcal{A}$ is a topology on Y and for each $\alpha \in A$, f_α is $(\mathcal{A}_\alpha, \mathcal{A})$-continuous. Set $\mathcal{V} := \{V \subset Y : \text{ for each } \alpha \in A, f_\alpha^{-1}(V) \in \mathcal{A}_\alpha\}$. By definition, $\sigma_Y(\mathcal{F}) = \sigma_Y(\mathcal{V})$. Let $V \in \mathcal{A}$. By assumption, for each $\alpha \in A$, f_α is $(\mathcal{A}_\alpha, \mathcal{A})$-measurable. Thus for each $\alpha \in A$, $f_\alpha^{-1}(V) \in \mathcal{A}_\alpha$. Therefore $V \in \mathcal{V}$. Since $V \in \mathcal{A}$ is arbitrary, we have that

$$\begin{aligned}\mathcal{A} &\subset \mathcal{V} \\ &\subset \sigma_Y(\mathcal{V}) \\ &= \sigma_Y(\mathcal{F}).\end{aligned}$$

□

Note 3.3.0.21. Essentially, $\sigma_X(\mathcal{F})$ is the largest σ-algebra on X such that for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y$ is measurable.

Exercise 3.3.0.22. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a a collection of measurable spaces, Y a set, $(Z, \mathcal{C})$ a measurable space, $\mathcal{F} = (f_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} Y^{X_\alpha}$ and $g : Y \to Z$. Then g is $(\sigma_Y(\mathcal{F}), \mathcal{C})$-measurable iff for each $\alpha \in A$, $g \circ f_\alpha$ is $(X_\alpha, \mathcal{C})$-measurable, i.e. for each $\alpha \in A$, the following diagram commutes in the category of measurable spaces:

$$\begin{tikzcd} X_\alpha \arrow[r, "f_\alpha"] \arrow[dr, "g \circ f_\alpha"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Proof. If g is $(\sigma_Y(\mathcal{F})$-$\mathcal{C})$ measurable, then clearly for each $\alpha \in A$, $g \circ f_\alpha$ is $(X_\alpha, \mathcal{C})$-measurable.
Conversely, suppose that for each $\alpha \in A$, $g \circ f_\alpha$ is $(X_\alpha, \mathcal{C})$-measurable. Let $V \in \mathcal{C}$. Measurability implies that for each $\alpha \in A$, $f_\alpha^{-1}(g^{-1}(V)) \in \mathcal{A}_\alpha$. By definition, $g^{-1}(V) \in \sigma_Y(\mathcal{F})$. So g is $(\sigma_Y(\mathcal{F}), \mathcal{C})$-measurable. □

Exercise 3.3.0.23. Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces and $f : X \to Y$. If f is continuous, then f is $\mathcal{B}(X)$-$\mathcal{B}(Y)$ measurable.

Proof. Recall that $\mathcal{B}(Y) = \sigma(\mathcal{T}_2)$ and continuity tells us that for each $U \in \mathcal{T}_2$, $f^{-1}(U) \in \mathcal{T}_1 \subset \mathcal{B}(X)$. □

Exercise 3.3.0.24. Let $(X, \mathcal{T})$ be a topological space and $f : X \to \overline{\mathbb{R}}$.

1. If f is lower semicontinuous, then f is $\mathcal{B}(X)$-measurable.
2. If f is upper semicontinuous, then f is $\mathcal{B}(X)$-measurable.

Proof.

1. Suppose that f is lower semicontinuous. An exercise in the section on semicontinuity in the analysis notes implies that for each $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty]) \in \mathcal{T}$. (need to show that) Exercise 6.1.0.6 implies that $\mathcal{B}(\overline{\mathbb{R}}) = \sigma((\alpha, \infty])$. Exercise 3.3.0.11 implies that f is $\mathcal{B}(X)$ measurable.
2. Similar to (1).

□

Exercise 3.3.0.25. pointwise convergece implies measurability

Definition 3.3.0.26. Let $(X, \mathcal{A})$ be a measurable space and $f : X \to \mathbb{C}$. Then f is said to be **simple** if $f(X)$ is finite. We define

$$S^+(X, \mathcal{A}) := \{f : X \to [0, \infty) : f \text{ is simple and } (\mathcal{A}, \mathcal{B}_{[0,\infty]})\text{-measurable}\}$$

and

$$S(X, \mathcal{A}) := \{f : X \to \mathbb{C} : f \text{ is simple and } (\mathcal{A}, \mathcal{B}_{\mathbb{C}})\text{-measurable}\}$$

Exercise 3.3.0.27. Let $(X, \mathcal{A})$ be a measurable space. Then $S(X, \mathcal{A})$ is a subspace of $L^0(X, \mathcal{A})$.

Proof. FINISH!!! $\square$

Definition 3.3.0.28. Let $(X, \mathcal{A})$ be a measurable space and $f \in S(X, \mathcal{A})$. Set $(a_j)_{j=1}^n := f(X)$ and for each $j \in [n]$, set $E_j := f^{-1}(y_j)$. We define the **standard representation of** f to be the sum

$$f = \sum_{j=1}^{n} a_j \chi_{E_j}.$$

Exercise 3.3.0.29. Let $(X, \mathcal{A})$ be a measurable space. Let $f, g \in S(X, \mathcal{A})$ and $\lambda \in \mathbb{C}$. Suppose that the standard representations of f and g are

$$f = \sum_{j=1}^{n} a_j \chi_{A_j}$$

and

$$g = \sum_{k=1}^{m} b_k \chi_{B_k}$$

respectively.

1. Then $(A_j)_{j=1}^n$ is disjoint and $\bigcup_{j=1}^{n} A_j = X$.

2. Set
$$L := \{(j,k) \in \mathbb{N}^2 : j \in [n], k \in [m], \text{ and } A_j \cap B_k \neq \varnothing\}$$
Then the standard representation of $f + \lambda g$ is
$$f + \lambda g = \sum_{(j,k) \in L} (a_j + \lambda b_k) \chi_{A_j \cap B_k}.$$

Proof.

1.

2.

FINISH!!! $\square$

Exercise 3.3.0.30. Let $(X, \mathcal{A})$ be a measurable space. Then

1. If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.

2. If $f : X \to \mathbb{C}$ is measurable, then there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset S$ such that for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$ and $\phi_n \to f$ pointwise and $\phi_n \to f$ uniformly on any set on which f is bounded.

Proof. FINISH!!! $\square$

Exercise 3.3.0.31. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. If f is $\mathcal{A}$-$\mathcal{B}$ measurable iff f is $\mathcal{A}$-$\mathcal{B} \cap f(X)$ measurable.

Proof. Suppose that f is $\mathcal{A}$-$\mathcal{B}$ measurable. Let $E \in \mathcal{B} \cap f(X)$. Then there exists $B \in \mathcal{B}$ such that $E = B \cap f(X)$. Then

$$\begin{aligned}
f^{-1}(E) &= f^{-1}(B \cap f(X)) \\
&= f^{-1}(B) \cap f^{-1}(f(X)) \\
&= f^{-1}(B) \cap X \\
&= f^{-1}(B) \\
&\in \mathcal{A}
\end{aligned}$$

Conversely, suppose that f is $\mathcal{A}$-$\mathcal{B}\cap f(X)$ measurable. Let $B \in \mathcal{B}$. Then as before,

$$\begin{aligned} f^{-1}(B) &= f^{-1}(B \cap f(X)) \\ &\in \mathcal{A} \end{aligned}$$

□

Exercise 3.3.0.32. Doob-Dynkin Lemma:
Let $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2)$ and $(X_3, \mathcal{A}_3)$ be measurable spaces and $f : X_1 \to X_2$ and $g : X_1 \to X_3$. Suppose that f is surjective and $(\mathcal{A}_1, \mathcal{A}_2)$-measurable and g is $(\mathcal{A}_1, \mathcal{A}_3)$-measurable and for each $t \in X_3$, $\{t\} \in \mathcal{A}_3$. Then g is $(f^*\mathcal{A}_2, \mathcal{A}_3)$ measurable iff there exists a unique $\phi : X_2 \to X_3$ such that ϕ is $(\mathcal{A}_2, \mathcal{A}_3)$-measurable and $g = \phi \circ f$.
Hint: For each $t \in X_3$, set $A_t = g^{-1}(\{t\}) \in f^*\mathcal{A}_2$ and choose $B_t \in \mathcal{A}_2$ such that $A_t = f^{-1}(B_t)$. Set $\phi(y) = t$ for $y \in B_t \cap f(X_1)$ and $t \in g(X_1)$.

Proof. Suppose that there exists a unique $\phi : X_2 \to X_3$ such that ϕ is $(\mathcal{A}_2, \mathcal{A}_3)$-measurable and $g = \phi \circ f$. Since f is $f^*\mathcal{A}_2$ - $\mathcal{A}_2$ measurable, we have that $g = \phi \circ f$ is $(f^*\mathcal{A}_2, \mathcal{A}_3)$-measurable.
Conversely, suppose that g is $(f^*\mathcal{A}_2, \mathcal{A}_3)$-measurable.

- **(Existence)**
  For each $t \in X_3$, set $A_t = g^{-1}(\{t\}) \in f^*\mathcal{A}_2$ and choose $B_t \in \mathcal{A}_2$ such that $A_t = f^{-1}(B_t)$.
  Note that
  - for each $t \in g(X_1)$, there exists $x \in A_t$ such that $g(x) = t$. Hence $f(x) \in B_t$.
  - for $t_1, t_2 \in g(X_1)$, $t_1 \neq t_2$ implies that

    $$\begin{aligned} f^{-1}(B_{t_1} \cap B_{t_2}) &= A_{t_1} \cap A_{t_2} \\ &= g^{-1}(\{t_1\} \cap \{t_2\}) \\ &= \varnothing \end{aligned}$$

    and since f is surjective,

    $$\begin{aligned} B_{t_1} \cap B_{t_2} &= f(f^{-1}(B_{t_1} \cap B_{t_2})) \\ &= f(\varnothing) \\ &= \varnothing \end{aligned}$$

  - we have that

    $$\begin{aligned} f^{-1}\bigg(\bigcup_{t \in g(X_1)} B_t\bigg) &= \bigcup_{t \in g(X_1)} A_t \\ &= \bigcup_{t \in g(X_1)} g^{-1}(\{t\}) \\ &= g^{-1}(g(X_1)) \\ &= X_1 \end{aligned}$$

    Since f is surjective, we have that

    $$\begin{aligned} X_2 &= f(X_1) \\ &= f\bigg(f^{-1}\bigg(\bigcup_{t \in g(X_1)} B_t\bigg)\bigg) \\ &= \bigcup_{t \in g(X_1)} B_t \end{aligned}$$

Therefore,

- for each $t \in g(X_1)$, $B_t \neq \varnothing$
- $(A_t)_{t \in g(X_1)}$ is a partion of X_1
- $(B_t)_{t \in g(X_1)}$ is a partition of X_2

Define $\phi : X_2 \to X_3$ by $\phi(y) = t$ for $t \in g(X_1)$ and $y \in B_t$. Then the previous observations imply that ϕ is well defined and $\phi(X_2) = g(X_1)$. Since for each $t \in g(X_1)$ and $x \in A_t$, $f(x) \in B_t$ and $g(x) = t$, we have that $\phi \circ f(x) = t = g(x)$. So $\phi \circ f = g$.

To show that ϕ is measurable, let $C \in \mathcal{A}_3$. Choose $B \in \mathcal{A}_2$ such that $g^{-1}(C) = f^{-1}(B)$. Let $y \in \phi^{-1}(C) \subset X_2$. Set $t = \phi(y) \in C$ and choose $x \in X_1$ such that $y = f(x)$. Since

$$\begin{aligned} g(x) &= \phi \circ f(x) \\ &= \phi(y) \\ &= t \\ &\in C \end{aligned}$$

$x \in g^{-1}(C) = f^{-1}(B)$. Therefore, $y = f(x) \in B$. So $\phi^{-1}(C) \subset B$.
Let $y \in B$. Choose $x \in X_1$ such that $f(x) = y$. Then $x \in f^{-1}(B) = g^{-1}(C)$. So

$$\begin{aligned} \phi(y) &= \phi \circ f(x) \\ &= g(x) \\ &\in C \end{aligned}$$

and $y \in \phi^{-1}(C)$. So $B \subset \phi^{-1}(C)$. Hence $\phi^{-1}(C) = B \in \mathcal{A}_2$ and ϕ is $(\mathcal{A}_2, \mathcal{A}_3)$-measurable.

- **(Uniqueness)**
Let $\psi : X_2 \to X_3$. Suppose that ψ is $(\mathcal{A}_2, \mathcal{A}_3)$-measurable and $g = \psi \circ f$. Let $y \in X_2$. Then there exists $x \in X_1$ such that $y = f(x)$. Then

$$\begin{aligned} \psi(y) &= \psi \circ f(x) \\ &= g(x) \\ &= \phi \circ f(x) \\ &= \phi(y) \end{aligned}$$

So $\psi = \phi$.

□

Note 3.3.0.33. discuss the information interpretation where we think of maps f as processing and possibly deleting information in $(X_1, \mathcal{A}_1)$ and $f^*\mathcal{A}_2$ represents the information in $\mathcal{A}_1$ preserved by f and f preserves at least as much information as g iff $g = \phi \circ f$ so that g is basically f, but with possibly more data processing and loss. This creates a partial order on the set of measurable maps to $(X_3, \mathcal{A}_3)$. Relate this to category theory with the corresponding poset and the coslice category of maps. Is there a dual notion? dual theorem?

Exercise 3.3.0.34. Let $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2)$ and $(X_3, \mathcal{A}_3)$ be measurable spaces and $f : X_1 \to X_2$ and $g : X_1 \to X_3$. Suppose that f is $(\mathcal{A}_1, \mathcal{A}_2)$-measurable and g is $(\mathcal{A}_1, \mathcal{A}_3)$-measurable and for each $t \in X_3$, $\{t\} \in \mathcal{A}_3$. Then g is $(f^*\mathcal{A}_2, \mathcal{A}_3)$-measurable iff there exists a unique $\phi : f(X_1) \to X_3$ such that ϕ is $(\mathcal{A}_2 \cap f(X_1), \mathcal{A}_3)$-measurable and $g = \phi \circ f$.

Proof. A previous exercise implies that $f : X_1 \to f(X_1)$ is $(\mathcal{A}_1, \mathcal{A}_2 \cap f(X_1))$-measurable. Now apply the previous exercise. FINISH!!! □

3.4 Subspace Sigma Algebras

Definition 3.4.0.1. Let X be a set and $E \subset X$. We define the **inclusion map from** E **to** B, denoted $\iota_E : E \to X$, by $\iota_E(x) = x$.

Definition 3.4.0.2. Let $(X, \mathcal{A})$ be a measurable space and $E \subset X$. We define the **subspace σ-algebra on** E, denoted $\mathcal{A} \cap E$, by

$$\mathcal{A} \cap E = \iota_E^* \mathcal{A}.$$

Note 3.4.0.3. Exercise 3.3.0.9 implies that $\mathcal{A} \cap E$ is a σ-algebra on E.

Exercise 3.4.0.4. Let $(X, \mathcal{A})$ be a measurable space and $E \subset X$. Then

1. $\mathcal{A} \cap E = \{A \cap E : A \in \mathcal{A}\}$,
2. for each $F \subset E$, $F \in \mathcal{A} \cap E$ iff there exists $A \in \mathcal{A}$ such that $F = A \cap E$.

Proof.

1. Since for each $F \subset X$, $\iota_E^{-1}(F) = F \cap E$, we have that

$$\begin{aligned} \mathcal{A} \cap E &= \iota_E^* \mathcal{A} \\ &= \{\iota_E^{-1}(A) : A \in \mathcal{A}\} \\ &= \{A \cap E : A \in \mathcal{A}\} \end{aligned}$$

2. Clear.

□

Exercise 3.4.0.5. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $E \subset X$. Then

$$\sigma_X(\mathcal{C}) \cap E = \sigma_E(\mathcal{C} \cap E).$$

Hint: $\sigma_X(\mathcal{C}) \subset (\iota_E)_* \sigma_E(\mathcal{C} \cap E)$

Proof.

- Clearly $\mathcal{C} \cap E \subset \sigma_X(\mathcal{C}) \cap E$. Since $\sigma_X(\mathcal{C}) \cap E$ is a σ-algebra on E, we have that

$$\begin{aligned} \sigma_E(\mathcal{C} \cap E) &\subset \sigma_E[\sigma_X(\mathcal{C}) \cap E] \\ &= \sigma_X(\mathcal{C}) \cap E. \end{aligned}$$

- Let $A_0 \in \mathcal{C}$. Then

$$\begin{aligned} \iota_E^{-1}(A_0) &= A_0 \cap E \\ &\in \mathcal{C} \cap E \\ &\subset \sigma_E(\mathcal{C} \cap E). \end{aligned}$$

  Hence $A_0 \in (\iota_E)_* \sigma_E(\mathcal{C} \cap E)$. Since $A_0 \in \mathcal{C}$ is arbitrary, we have that $\mathcal{C} \subset (\iota_E)_* \sigma_E(\mathcal{C} \cap E)$. Hence

$$\begin{aligned} \sigma_X(\mathcal{C}) &\subset \sigma_X[(\iota_E)_* \sigma_E(\mathcal{C} \cap E)] \\ &= (\iota_E)_* \sigma_E(\mathcal{C} \cap E). \end{aligned}$$

  Let $A \in \sigma_X(\mathcal{C}) \cap E$. Then there exists $A_0 \in \sigma_X(\mathcal{C})$ such that $A = A_0 \cap E$. Therefore

$$\begin{aligned} A_0 &\in \sigma_X(\mathcal{C}) \\ &\subset (\iota_E)_* \sigma_E(\mathcal{C} \cap E). \end{aligned}$$

By definition of $(\iota_E)_* \sigma_E(\mathcal{C} \cap E)$, we have that

$$\begin{aligned} A &= A_0 \cap E \\ &= \iota_E^{-1}(A_0) \\ &\in \sigma_E(\mathcal{C} \cap E). \end{aligned}$$

Since $A \in \sigma_X(\mathcal{C}) \cap E$ is arbitrary, we have that $\sigma_X(\mathcal{C}) \cap E \subset \sigma_E(\mathcal{C} \cap E)$.

Since $\sigma_E(\mathcal{C} \cap E) \subset \sigma_X(\mathcal{C}) \cap E$ and $\sigma_X(\mathcal{C}) \cap E \subset \sigma_E(\mathcal{C} \cap E)$, we have that $\sigma_X(\mathcal{C}) \cap E = \sigma_E(\mathcal{C} \cap E)$. □

Exercise 3.4.0.6. Let $(X, \mathcal{A})$ be a measurable space, $E \subset X$ and $F \subset E$. Then $\mathcal{A} \cap F = (\mathcal{A} \cap E) \cap F$.

Proof.

- Let $U \in (\mathcal{A} \cap E) \cap F$. Then there exists $U_E \in \mathcal{A} \cap E$ such that $U = U_E \cap F$. Similarly, there exists $U_X \in \mathcal{A}$ such that $U_E = U_X \cap E$. Therefore

$$\begin{aligned} U &= U_E \cap F \\ &= (U_X \cap E) \cap F \\ & U_X \cap (E \cap F) \\ &= U_X \cap F \\ &\in \mathcal{A} \cap F \end{aligned}$$

  Since $U \in (\mathcal{A} \cap E) \cap F$ is arbitrary, we have that $(\mathcal{A} \cap E) \cap F \subset \mathcal{A} \cap F$.

- Conversely, let $U \in \mathcal{A} \cap F$. Then there exists $U_X \in \mathcal{A}$ such that $U = U_X \cap F$. Then

$$\begin{aligned} U &= U_X \cap F \\ &= U_X \cap (E \cap F) \\ &= (U_X \cap E) \cap F \\ &\in (\mathcal{A} \cap E) \cap F \end{aligned}$$

  Since $U \in U_X \cap F$ is arbitrary, we have that $U_X \cap F \subset (\mathcal{A} \cap E) \cap F$.

Therefore $\mathcal{A} \cap F = (\mathcal{A} \cap E) \cap F$. □

Exercise 3.4.0.7. Let $(X, \mathcal{A})$ be a measurable space and $E \subset X$. If $E \in \mathcal{A}$, then $\mathcal{A} \cap E \subset \mathcal{A}$.

Proof. Suppose that $E \in \mathcal{A}$. Then for each $A \in \mathcal{A}$, $A \cap E \in \mathcal{A}$. Hence $\mathcal{A} \cap E \subset \mathcal{A}$. □

Exercise 3.4.0.8. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then $\mathcal{B}(A, \mathcal{T} \cap A) = \mathcal{B}(X, \mathcal{T}) \cap A$.

Proof. Exercise 3.4.0.5 implies that

$$\begin{aligned} \mathcal{B}(A, \mathcal{T} \cap A) &= \sigma_A(\mathcal{T} \cap A) \\ &= \sigma_X(\mathcal{T}) \cap A \\ &= \mathcal{B}(X, \mathcal{T}) \cap A \end{aligned}$$

□

Exercise 3.4.0.9. Let $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then ι_E is $(\mathcal{A} \cap E, \mathcal{A})$-bimeasurable.

Proof.

- By Definition 3.4.0.2, ι_E is $(\mathcal{A} \cap E, \mathcal{A})$-measurable.

- Let $A \in \mathcal{A} \cap E$. Since $E \in \mathcal{A}$, $\mathcal{A} \cap E \subset \mathcal{A}$. Therefore

$$\begin{aligned} \iota_E(A) &= A \\ &\in \mathcal{A} \cap E \\ &\subset \mathcal{A}. \end{aligned}$$

  Since $A \in \mathcal{A} \cap E$ is arbitrary, we have that for each $A \in \mathcal{A} \cap E$, $\iota_E(A) \in \mathcal{A}$. Hence ι_E is $(\mathcal{A} \cap E, \mathcal{A})$-bimeasurable.

$\square$

Exercise 3.4.0.10. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and $f : X \to Y$. Then f is $(\mathcal{A}, \mathcal{B})$-measurable iff $f|^{f(X)}$ is $(\mathcal{A}, \mathcal{B} \cap f(X))$-measurable.

Proof. Set $g := f|^{f(X)}$.

- $(\implies)$:
  Suppose that f is $(\mathcal{A}, \mathcal{B})$-measurable. Let $B \in \mathcal{B} \cap f(X)$. Then there exists $V \in \mathcal{B}$ such that $B = V \cap f(X)$. Then

$$\begin{aligned} g^{-1}(B) &= f^{-1}(B) \\ &= f^{-1}(V \cap f(X)) \\ &= f^{-1}(V) \cap f^{-1}(f(X)) \\ &= f^{-1}(V) \cap X \\ &= f^{-1}(V) \\ &\in \mathcal{A} \end{aligned}$$

  Since $B \in \mathcal{B} \cap f(X)$ is arbitrary, g is $(\mathcal{A}, \mathcal{B} \cap f(X))$-measurable.

- $(\impliedby)$:
  Conversely, suppose that g is $(\mathcal{A}, \mathcal{B} \cap f(X))$-measurable. Let $V \in \mathcal{B}$. Then $V \cap f(X) \in \mathcal{B} \cap f(X)$ and

$$\begin{aligned} f^{-1}(V) &= f^{-1}(V \cap f(X)) \\ &= g^{-1}(V \cap f(X)) \\ &\in \mathcal{A} \end{aligned}$$

  Since $V \in \mathcal{B}$ is arbitrary, f is $(\mathcal{A}, \mathcal{B})$-measurable.

$\square$

Exercise 3.4.0.11. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, $E \subset X$ and $f : X \to Y$. If f is $(\mathcal{A}, \mathcal{B})$-measurable, then $f|_E$ is $(\mathcal{A} \cap E, \mathcal{B})$-measurable.

Proof. Suppose that f is $(\mathcal{A}, \mathcal{B})$-measurable. We note that $f|_E = f \circ \iota_E$. Since ι_E is $(\mathcal{A} \cap E, \mathcal{A})$-measurable and f is $(\mathcal{A}, \mathcal{B})$-measurable, we have that $f|_E$ is $(\mathcal{A} \cap E, \mathcal{B})$-measurable. $\square$

Exercise 3.4.0.12. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ and $f : X \to Y$. Suppose that $X = \bigcup_{j \in \mathbb{N}} A_j$. Then f is $(\mathcal{A}, \mathcal{B})$-measurable iff for each $j \in \mathbb{N}$, $f|_{A_j}$ is $(\mathcal{A} \cap A_j, \mathcal{B})$-measurable.

Proof.

- $(\implies)$:
  Suppose that f is $(\mathcal{A}, \mathcal{B})$-measurable. Exercise 3.4.0.11 implies that for each $j \in \mathbb{N}$, $f|_{A_j}$ is $(\mathcal{A} \cap A_j, \mathcal{B})$-measurable.

- $(\Longleftarrow)$:
  Suppose that for each $j \in \mathbb{N}$, $f|_{A_j}$ is $(\mathcal{A} \cap A_j, \mathcal{B})$-measurable. Let $B \in \mathcal{B}$. Since $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$, we have that for each $j \in \mathbb{N}$,

$$\begin{aligned} f^{-1}(B) \cap A_j &= f|_{A_j}^{-1}(B) \\ &\in \mathcal{A} \cap A_j \\ &\subset \mathcal{A}. \end{aligned}$$

  Therefore

$$\begin{aligned} f^{-1}(B) &= f^{-1}(B) \cap X \\ &= f^{-1}(B) \cap \left[\bigcup_{j\in\mathbb{N}} A_j\right] \\ &= \bigcup_{j\in\mathbb{N}} (f^{-1}(B) \cap A_j) \\ &\in \mathcal{A}. \end{aligned}$$

  Since $B \in \mathcal{B}$ is arbitrary, we have that for each $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. Hence f is $(\mathcal{A}, \mathcal{B})$-measurable.

□

Exercise 3.4.0.13. Schroder-Bernstein Theorem:
Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. If there exists $f : X \to Y$ and $g : Y \to X$ such that f, g are injective, f is $(\mathcal{A}, \mathcal{B})$-bimeasurable and g is $(\mathcal{B}, \mathcal{A})$-bimeasurable, then there exists $h : X \to Y$ such that h is a $(\mathcal{A}, \mathcal{B})$-measurable isomorphism.
Hint:

1. Define $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ by $A_1 := \varnothing$ and for $n \geq 2$, $A_n := g(f(A_{n-1})^c)^c$. Define $A \subset \mathcal{P}(X)$ by $A := \bigcup_{n\in\mathbb{N}} A_n$. Then
   - $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$,
   - $A \in \mathcal{A}$,
   - $A = g(f(A)^c)^c$ and $A^c \subset g(Y)$,
2. Define $h : X \to Y$ by

$$h(x) := \begin{cases} f(x), & x \in A \\ (g|^{g(Y)})^{-1}(x), & x \in A^c. \end{cases}$$

Proof. Suppose that there exists $f : X \to Y$ and $g : Y \to X$ such that f, g are injective and f, g are bimeasurable.

1. Define $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ by $A_1 := \varnothing$ and for $n \geq 2$, $A_n := g(f(A_{n-1})^c)^c$. Define $A \subset \mathcal{P}(X)$ by $A := \bigcup_{n\in\mathbb{N}} A_n$.
   - – **Base Case:**
       Clearly $A_1 \in \mathcal{A}$.
     – **Induction Step:**
       Let $n \in \mathbb{N}$. Suppose that $A_n \in \mathcal{A}$. Since f is bimeasurable, $f(A_n) \in \mathcal{B}$ and thus $f(A_n)^c \in \mathcal{B}$. Since g is bimeasurable, $g(f(A_n)^c) \in \mathcal{A}$ and thus

$$\begin{aligned} A_{n+1} &= g(f(A_n)^c)^c \\ &\in \mathcal{A}. \end{aligned}$$

     Since $A_1 \in \mathcal{A}$ and for each $n \in \mathbb{N}$, $A_n \in \mathcal{A}$ implies that $A_{n+1} \in \mathcal{A}$, by induction we have that $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$.
   - Since $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, we have that

$$\begin{aligned} A &= \bigcup_{n\in\mathbb{N}} A_n \\ &\in \mathcal{A}. \end{aligned}$$

- Since g is injective, an exercise in the analysis notes section on set theory subsection on bijections implies that

$$\begin{aligned} g(f(A)^c)^c &= g\bigg(f\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)^c\bigg)^c \\ &= g\bigg(\bigg[\bigcup_{n\in\mathbb{N}} f(A_n)\bigg]^c\bigg)^c \\ &= g\bigg(\bigcap_{n\in\mathbb{N}} f(A_n)^c\bigg)^c \\ &= \bigg[\bigcap_{n\in\mathbb{N}} g(f(A_n)^c)\bigg)\bigg]^c \\ &= \bigcup_{n\in\mathbb{N}} g(f(A_n)^c)^c \\ &= \bigcup_{n\in\mathbb{N}} A_{n+1} \\ &= \bigg[\bigcup_{n\in\mathbb{N}} A_{n+1}\bigg] \cup \varnothing \\ &= \bigg[\bigcup_{n\in\mathbb{N}} A_{n+1}\bigg] \cup A_1 \\ &= \bigcup_{n\in\mathbb{N}} A_n \\ &= A. \end{aligned}$$

  Since $A^c = g(f(A)^c)$, we have that

$$\begin{aligned} A^c &= g(f(A)^c) \\ &\subset g(Y). \end{aligned}$$

2. Define $B \in \mathcal{P}(Y)$ by $B := f(A)^c$. Since f is bimeasurable and $A \in \mathcal{A}$, we have that

$$\begin{aligned} B &= f(A)^c \\ &\in \mathcal{B}. \end{aligned}$$

Since $B^c = f(A)$, we have that

$$\begin{aligned} B^c &= f(A) \\ &\subset f(X). \end{aligned}$$

Define $f_0 : f(X) \to X$ and $g_0 : g(Y) \to Y$ by $f_0 := (f|^{f(X)})^{-1}$ and $g_0 := (g|^{g(Y)})^{-1}$. Since $B^c = f(A)$, $A = g(f(A)^c)^c$ and f, g are injective, an exercise in the analysis notes section on set theory:bijections implies that

$$\begin{aligned} f_0(B^c) &= (f|^{f(X)})^{-1}(f(A)) \\ &= f^{-1}(f(A)) \\ &= A, \end{aligned}$$

and

$$\begin{aligned} g_0(A^c) &= (g|^{g(Y)})^{-1}(A^c) \\ &= g^{-1}(A^c) \\ &= g^{-1}(g(f(A)^c)) \\ &= f(A)^c \\ &= B. \end{aligned}$$

Define $h : X \to Y$ and $\beta : Y \to X$ by

$$h(x) := \begin{cases} f(x), & x \in A \\ g_0(x), & x \in A^c. \end{cases}$$

and

$$\beta(x) := \begin{cases} f_0(x), & y \in B^c \\ g(x), & y \in B. \end{cases}$$

Then $h \circ \beta = \mathrm{id}_Y$ and $\beta \circ h = \mathrm{id}_X$. Hence h is a bijection and $\beta = h^{-1}$. Exercise 3.4.0.11 implies that $f|_A$ is $(\mathcal{A} \cap A, \mathcal{B})$-measurable. Let $F \in \mathcal{B}$. Since g is bimeasurable, $g(F) \in \mathcal{A}$. Therefore

$$\begin{aligned}
(g_0|_{A^c})^{-1}(F) &= g_0^{-1}(F) \cap A^c \\
&= [(g|^{g(Y)})^{-1}]^{-1}(F) \cap A^c \\
&= g|^{g(Y)}(F) \cap A^c \\
&= g(F) \cap A^c \\
&\in \mathcal{A} \cap A^c.
\end{aligned}$$

Since $F \in \mathcal{B}$ is arbitrary, we have that for each $F \in \mathcal{B}$, $(g_0|_{A^c})^{-1}(F) \in \mathcal{A} \cap A^c$. Hence $g_0|_{A^c}$ is $(\mathcal{A} \cap A^c, \mathcal{B})$-measurable. Since $h|_A = f|_A$ and $h|_{A^c} = g_0|_{A^c}$, Exercise 3.4.0.12 implies that h is $(\mathcal{A}, \mathcal{B})$-measurable. Similarly, β is $(\mathcal{B}, \mathcal{A})$-measurable. Thus h is a measurable isomorphism.

$\square$

3.5 Product Sigma Algebras

Definition 3.5.0.1. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces. We define the **product σ-algebra** on $\prod_{\alpha \in A} X_\alpha$, denoted by $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha$, by

$$\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\pi_\alpha : \alpha \in A)$$

Exercise 3.5.0.2. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces and for each $\alpha \in A$, $\mathcal{E}_\alpha \subset \mathcal{A}_\alpha$. Suppose that for each $\alpha \in A$, $\mathcal{A}_\alpha = \sigma(\mathcal{E}_\alpha)$. Then

$$\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{E}_\alpha)$$

Hint: set $\mathcal{G} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{E}_\alpha\}$ and for $\alpha \in A$, consider the pushforward σ-algebra on X_α, $(\pi_\alpha)_* \sigma(\mathcal{G})$

Proof. Set

- $\mathcal{F} = \{\pi_\alpha^{-1}(V_\alpha) : \alpha \in A \text{ and } V_\alpha \in \mathcal{A}_\alpha\}$
- $\mathcal{G} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{E}_\alpha\}$

Clearly, $\mathcal{G} \subset \mathcal{F}$. By definition, $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{F})$. Therefore,

$$\begin{aligned} \sigma(\mathcal{G}) &\subset \sigma(\mathcal{F}) \\ &= \bigotimes_{\alpha \in A} \mathcal{A}_\alpha \end{aligned}$$

Let $\alpha \in A$. By definition, for each $V \subset X_\alpha$, $V \in \pi_{\alpha *}\sigma(\mathcal{G})$ iff $\pi_\alpha^{-1}(V) \in \sigma(\mathcal{G})$. Thus $\mathcal{E}_\alpha \subset \pi_\alpha^* \sigma(\mathcal{G})$ which implies that

$$\begin{aligned} \mathcal{A}_\alpha &= \sigma(\mathcal{E}_\alpha) \\ &\subset \pi_\alpha^* \sigma(\mathcal{G}) \end{aligned}$$

Since $\alpha \in A$ is arbitrary, $\mathcal{F} \subset \sigma(\mathcal{G})$. Hence

$$\begin{aligned} \bigotimes_{\alpha \in A} \mathcal{A}_\alpha &= \sigma(\mathcal{F}) \\ &\subset \sigma(\mathcal{G}) \end{aligned}$$

Thus $\sigma(\mathcal{G}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$. □

Exercise 3.5.0.3. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces. Define

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} B_\alpha : \text{ for each } \alpha \in A,\, B_\alpha \in \mathcal{A}_\alpha \right\}$$

If A is countable, then $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{B})$.

Proof. Suppose that A is countable. Set $\mathcal{C} = \{\pi_\alpha^{-1}(B_\alpha) : \alpha \in A, B_\alpha \in \mathcal{A}_\alpha\}$. By definition, $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{C})$. Let $\alpha \in A$ and $B_\alpha \in \mathcal{A}_\alpha$. For $\beta \in A$, set

$$C_\beta = \begin{cases} B_\beta & \beta = \alpha \\ X_\beta & \beta \neq \alpha \end{cases}$$

Then

$$\begin{aligned} \pi_\alpha^{-1}(B_\alpha) &= \prod_{\beta \in A} C_\beta \\ &\in \mathcal{B} \end{aligned}$$

So $\mathcal{C} \subset \mathcal{B}$ and

$$\begin{aligned} \bigotimes_{\alpha \in A} \mathcal{A}_\alpha &= \sigma(\mathcal{C}) \\ &\subset \sigma(\mathcal{B}) \end{aligned}$$

For each $\alpha \in A$, let $B_\alpha \in \mathcal{A}_\alpha$. Since A is countable, we have that

$$\begin{aligned} \prod_{\alpha \in A} B_\alpha &= \bigcap_{a \in A} \pi_\alpha^{-1}(B_\alpha) \\ &\in \sigma(\mathcal{C}) \end{aligned}$$

Thus $\mathcal{B} \subset \sigma(\mathcal{C})$ and

$$\begin{aligned} \sigma(\mathcal{B}) &\subset \sigma(\mathcal{C}) \\ &= \bigotimes_{\alpha \in A} \mathcal{A}_\alpha \end{aligned}$$

Hence $\sigma(\mathcal{B}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$. □

Exercise 3.5.0.4. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces and for each $\alpha \in A$, $\mathcal{E}_\alpha \subset \mathcal{A}_\alpha$. Suppose that for each $\alpha \in A$, $X_\alpha \in \mathcal{E}_\alpha$ and $\mathcal{A}_\alpha = \sigma(\mathcal{E}_\alpha)$. Set

$$\mathcal{B} = \left\{ \prod_{\alpha \in A} E_\alpha : \text{for each } \alpha \in A,\ E_\alpha \in \mathcal{E}_\alpha \right\}$$

If A is countable, then $\bigotimes_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\mathcal{B})$.

Proof. Suppose that A is countable. Set $\mathcal{C} = \left\{(\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{E}_\alpha\right\}$. A previous exercise implies that $\sigma(\mathcal{C}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$. Let $\alpha \in A$ and $E_\alpha \in \mathcal{E}_\alpha$. For $\beta \in A$, set

$$C_\beta = \begin{cases} E_\beta & \beta = \alpha \\ X_\beta & \beta \neq \alpha \end{cases}$$

Then for each $\beta \in A$, $C_\beta \in \mathcal{E}_\beta$ and

$$\begin{aligned} \pi_\alpha^{-1}(E_\alpha) &= \prod_{\beta \in A} C_\beta \\ &\in \mathcal{B} \end{aligned}$$

So $\mathcal{C} \subset \mathcal{B}$ and

$$\begin{aligned} \bigotimes_{\alpha \in A} \mathcal{A}_\alpha &= \sigma(\mathcal{C}) \\ &\subset \sigma(\mathcal{B}) \end{aligned}$$

For each $\alpha \in A$, let $E_\alpha \in \mathcal{E}_\alpha$. Since A is countable, we have that

$$\begin{aligned} \prod_{\alpha \in A} E_\alpha &= \bigcap_{a \in A} \pi_\alpha^{-1}(E_\alpha) \\ &\in \sigma(\mathcal{C}) \end{aligned}$$

Thus $\mathcal{B} \subset \sigma(\mathcal{C})$ and

$$\begin{aligned}\sigma(\mathcal{B}) &\subset \sigma(\mathcal{C}) \\ &\subset \bigotimes_{\alpha \in A} \mathcal{A}_\alpha\end{aligned}$$

Hence $\sigma(\mathcal{B}) = \bigotimes_{\alpha \in A} \mathcal{A}_\alpha$. □

Exercise 3.5.0.5. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection of topological spaces. Then

1. $$\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha, \mathcal{T}_\alpha) \subset \mathcal{B}\bigg(\prod_{\alpha \in A} X_\alpha, \bigotimes_{\alpha \in A} \mathcal{T}_\alpha\bigg)$$
2. if A is countable and for each $\alpha \in A$, X_α is second-countable, then $$\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha, \mathcal{T}_\alpha) = \mathcal{B}\bigg(\prod_{\alpha \in A} X_\alpha, \bigotimes_{\alpha \in A} \mathcal{T}_\alpha\bigg)$$

Proof. Set $X := \prod_{j=1}^n X_j$ and $\mathcal{T} := \bigotimes_{\alpha \in A} \mathcal{T}_\alpha$.

1. By definition, $\mathcal{B}(X, \mathcal{T}) = \sigma(\mathcal{T})$, and for each $\alpha \in A$, $X_\alpha \in \mathcal{T}_\alpha$, $\mathcal{B}(X_\alpha, \mathcal{T}_\alpha) = \sigma(\mathcal{T}_\alpha)$. Set $$\mathcal{E} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A \text{ and } E_\alpha \in \mathcal{T}_\alpha\}$$ A previous exercise implies that $\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha, \mathcal{T}_\alpha) = \sigma(\mathcal{E})$. Since $\mathcal{E} \subset \mathcal{T}$, we have that $$\begin{aligned}\bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha) &= \sigma(\mathcal{E}) \\ &\subset \sigma(\mathcal{T}) \\ &= \mathcal{B}(X, \mathcal{T})\end{aligned}$$
2. Suppose that A is countable and for each $\alpha \in A$, X_α, is second-countable. Then for each $\alpha \in A$, there exists $\mathcal{B}_\alpha \subset \mathcal{T}_\alpha$ such that $\mathcal{B}_\alpha$ is a countable basis for $\mathcal{T}_\alpha$. Set $$\mathcal{B} = \bigg\{ \prod_{\alpha \in A} U_\alpha : \text{there exists } J \subset A \text{ such that } \#J < \infty, \\ \text{for each } \alpha \in J,\ U_\alpha \in \mathcal{B}_\alpha \text{ and for each } \alpha \in J^c,\ U_\alpha = X_\alpha \bigg\}$$ Since A is countable, $\mathcal{B}$ is a countable basis for $\mathcal{T}$. An exercise in the section on σ-algebras implies that $\mathcal{B}(X, \mathcal{T}) = \sigma(\mathcal{B})$. The previous exercise implies that $\mathcal{B} \subset \bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha, \mathcal{T}_\alpha)$. Hence $$\begin{aligned}\mathcal{B}(X, \mathcal{T}) &= \sigma(\mathcal{B}) \\ &\subset \bigotimes_{\alpha \in A} \mathcal{B}(X_\alpha, \mathcal{T}_\alpha)\end{aligned}$$

□

Exercise 3.5.0.6. Let $(X, \mathcal{A})$ be a measurable space, $(Y_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ a collection of measurable spaces and $f : X \to \prod_{\alpha \in A} Y_\alpha$. Then f is $(\mathcal{A}, \bigotimes_{\alpha \in A} \mathcal{A}_\alpha)$-measurable iff for each $\alpha \in A$, $\pi_\alpha \circ f$ is $(\mathcal{A}, \mathcal{A}_\alpha)$-measurable.

Proof. Immediate by a previous exercise about the initial σ-algebra. □

Exercise 3.5.0.7. Let $(X, \mathcal{A})$ be a measurable space, $(Y_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ a collection of measurable spaces and $(f_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} Y_\alpha^X$, i.e. for each $\alpha \in A$, $f_\alpha : X \to Y_\alpha$. Set $Y = \prod\limits_{\alpha \in A} Y_\alpha$. Then $(f_\alpha)_{\alpha \in A}$ is $(\mathcal{A}, \bigotimes\limits_{\alpha \in A} \mathcal{A}_\alpha)$-measurable iff for each $\alpha \in A$, f_α is $(\mathcal{A}, \mathcal{A}_\alpha)$-measurable.

Proof.

- $(\implies)$:
  Suppose that $(f_\alpha)_{\alpha \in A}$ is $(\mathcal{A}, \bigotimes\limits_{\alpha \in A} \mathcal{A}_\alpha)$-measurable. Let $\beta \in A$. Since π_β^Y is $(\bigotimes\limits_{\alpha \in A} \mathcal{A}_\alpha, \mathcal{A}_\beta)$-measurable and $f_\beta = \pi_\beta^Y \circ (f_\alpha)_{\alpha \in A}$, we have that f_β is $(\mathcal{A}, \mathcal{A}_\beta)$-measurable. Since $\beta \in A$ is arbitrary, we have that for each $\beta \in A$, f_β is $(\mathcal{A}, \mathcal{A}_\beta)$-measurable.

- $(\impliedby)$:
  Suppose that for each $\alpha \in A$, f_α is $(\mathcal{A}, \mathcal{A}_\alpha)$-measurable. Since for each $\beta \in A$, $\pi_\beta^Y \circ (f_\alpha)_{\alpha \in A} = f_\beta$ and f_β is $(\mathcal{A}, \mathcal{A}_\beta)$-measurable, Exercise 3.5.0.6 implies that $(f_\alpha)_{\alpha \in A}$ is $(\mathcal{A}, \bigotimes\limits_{\alpha \in A} \mathcal{A}_\alpha)$-measurable.

$\square$

Definition 3.5.0.8. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ and $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$ be collections of measurable spaces and $(f_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. Set $X = \prod\limits_{\alpha \in A} X_\alpha$ and $Y = \prod\limits_{\alpha \in A} Y_\alpha$. We define the **product of** $(f_\alpha)_{\alpha \in A}$, denoted $\prod_{\alpha \in A} f_\alpha : X \to Y$, by $\prod_{\alpha \in A} f_\alpha((x_\alpha)_{\alpha \in A}) = (f_\alpha(x_\alpha))_{\alpha \in A}$. eventually delete, reference set theory section instead

Exercise 3.5.0.9. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ and $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$ be collections of measurable spaces and $(f_\alpha)_{\alpha \in A} \in \prod\limits_{\alpha \in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. Set $X := \prod\limits_{\alpha \in A} X_\alpha$, $Y := \prod\limits_{\alpha \in A} Y_\alpha$ and $f := \prod_{\alpha \in A} f_\alpha$. If for each $\alpha \in A$, f_α is $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$-measurable, then f is $(\bigotimes\limits_{\alpha \in A} \mathcal{A}_\alpha, \bigotimes\limits_{\alpha \in A} \mathcal{B}_\alpha)$-measurable.

Proof. Suppose that for each $\alpha \in A$, f_α is $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$-measurable. Let $\alpha \in A$. Denote the α-th projection maps on X and Y by π_α^X and π_α^Y respectively. An exercise in the intro-math/analysis notes chapter on set theory implies that $\pi_\alpha^Y \circ f = f_\alpha \circ \pi_\alpha^X$. Since $f_\alpha \circ \pi_\alpha^X$ is $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$-measurable, we have that $\pi_\alpha^Y \circ f$ is $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$-measurable. Since $\alpha \in A$ is arbitrary, a previous exercise implies that f is $(\bigotimes\limits_{\alpha \in A} \mathcal{A}_\alpha, \bigotimes\limits_{\alpha \in A} \mathcal{B}_\alpha)$-measurable. $\square$

Definition 3.5.0.10. Let X, Y be sets, $x \in X$ and $y \in Y$. We define the **slice maps at x and y**, denoted $\iota_X^y : X \to X \times Y$ and $\iota_Y^x : Y \to X \times Y$ respectively, by $\iota_X^y(\cdot) = (\cdot, y)$ and $\iota_Y^x(\cdot) = (x, \cdot)$ respectively.

Exercise 3.5.0.11. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, $x \in X$ and $y \in Y$. Then ι_X^y is $(\mathcal{A}, \mathcal{A} \otimes \mathcal{B})$-measurable and ι_Y^x is $(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$-measurable.

Proof. Since $\pi_1 \circ \iota_X^y = \mathrm{id}_X$ and $\pi_2 \circ \iota_X^y$ is constant, we have that $\pi_1 \circ \iota_X^y = \mathrm{id}_X$ is $(\mathcal{A}, \mathcal{A})$-measurable and $\pi_2 \circ \iota_X^y$ is $(\mathcal{B}, \mathcal{B})$-measurable. Since $\mathcal{A} \otimes \mathcal{B} = \sigma_{X \times Y}(\pi_1, \pi_2)$, an exercise in the section on measurable functions implies that ι_X^y is $(\mathcal{A}, \mathcal{A} \otimes \mathcal{B})$-measurable. Similarly, ι_Y^x is $(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$-measurable. $\square$

Definition 3.5.0.12. Let X, Y, and Z be sets, $E \subset X \times Y$, $f : X \times Y \to Z$, $x \in X$ and $y \in Y$. Then

- we define the **sections of E at x and y**, denoted E_x and E^y respectively, by $E_x = \{y \in Y : (x, y) \in E\}$ and $E^y = \{x \in X : (x, y) \in E\}$ respectively

- we define the **sections of f at x and y**, denoted $f_x : Y \to Z$ and $f^y : X \to Z$ respectively, by $f_x(\cdot) = f(x, \cdot)$ and$f^y(\cdot) = f(\cdot, y)$ respectively

Exercise 3.5.0.13. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces, $E \in \mathcal{A} \otimes \mathcal{B}$, $x \in X$ and $y \in Y$. Then $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}$.

Proof. Since ι_Y^x is $(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$-measurable, we have that

$$\begin{aligned} E_x &= (\iota_Y^x)^{-1}(E) \\ &\in \mathcal{B} \end{aligned}$$

Similarly, $E^y \in \mathcal{A}$. $\square$

Exercise 3.5.0.14. Let $(X,\mathcal{A}),(Y,\mathcal{B})$, $(Z,\mathcal{C})$ be measurable spaces, $f:X\times Y\to Z$, $x\in X$ and $y\in Y$. Suppose that f is $(\mathcal{A}\otimes\mathcal{B},\mathcal{C})$-measurable. Then f_x is $(\mathcal{B},\mathcal{C})$-measurable and f^y is $(\mathcal{A},\mathcal{C})$-measurable.

Proof. Since ι^x_Y is $(\mathcal{B},\mathcal{A}\otimes\mathcal{B})$-measurable, f is $(\mathcal{A}\otimes\mathcal{B},\mathcal{C})$-measurable and $f_x=f\circ\iota^x_Y$, we have that f_x is $(\mathcal{B},\mathcal{C})$-measurable. Similarly, f^y is $(\mathcal{A},\mathcal{C})$-measurable. □

Exercise 3.5.0.15. Let X_1,X_2,Y_1,Y_2 be topological spaces and $f_1:X_1\to Y_1$, $f_2:X_2\to Y_2$. If f_1 and f_2 are open, then $f_1\times f_1$ is open.

Proof. Let $A_1\subset X_1, A_2\subset X_2$ be open. Then $f_1\times f_2(A_1\times A_2)=f_1(A_1)\times f_2(A_2)$ which is open in $Y_1\times Y_2$. Since $\mathcal{B}=\{A_1\times A_2 : A_1\subset X_1 \text{ and } A_2\subset X_2 \text{ are open}\}$ is a basis for the product topology on $X_1\times X_2$, an exercise in the section on continuous maps implies that $f_1\times f_2$ is open. □

Exercise 3.5.0.16. Let X and Y be topological spaces and $U\subset X\times Y$ open. Then for each $(x_0,y_0)\in U$, U^{x_0} and U^{y_0} are open.

Proof. Let $(x_0,y_0)\in U$. Define $\phi:X\to X\times Y$ by $\phi(x)=(x,y_0)$. Since $\pi_X\circ\phi=\mathrm{id}_X$ and $\pi_Y\circ\phi$ is constant, $\pi_X\circ\phi$ and $\pi_Y\circ\phi$ are continous. Therefore, ϕ is continuous. Then U^{y_0} is open since U is open and $\phi^{-1}(U)=U^{y_0}$. Similarly, U_{x_0} is open. □

Exercise 3.5.0.17. Let X, Y and Z be topological spaces, $U\subset X\times Y$ open and $f:U\to Z$. Equip U with the subspace topology. Suppose that f is continuous. Let $(x_0,y_0)\in U$. Equip U_{x_0} and U^{y_0} with the subspace topology. Then $f_{x_0}:U_{x_0}\to Z$ and $f^{y_0}:U^{y_0}\to Z$ are continuous.

Proof. Let $(x_0,y_0)\in U$. Let $V\subset Z$. Suppose that V is open. Continuity of f implies that $f^{-1}(V)$ is open in U. Since U is open in $X\times Y$, $f^{-1}(V)$ is open in $X\times Y$. A previous exercise in the section on product sets implies that $(f^{y_0})^{-1}(V)=(f^{-1}(V))^{y_0}$. The previous exercise implies that $(f^{-1}(V))^{y_0}$ is open in X. So $(f^{y_0})^{-1}(V)$ is open in X. Since $(f^{y_0})^{-1}(V)\subset U^{y_0}$, $(f^{y_0})^{-1}(V)$ is open in U^{y^0}. Thus $f^{y_0}:U^{y_0}\to Z$ is continuous. Similarly, $f_{x_0}:U_{x_0}\to Z$ is continuous. □

3.6 Coproduct Sigma Algebra

Definition 3.6.0.1. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces. We define the **coproduct σ-algebra** on $\coprod\limits_{\alpha \in A} X_\alpha$, denoted $\bigoplus\limits_{\alpha \in A} \mathcal{A}_\alpha$, by

$$\bigoplus_{\alpha \in A} \mathcal{A}_\alpha = \sigma(\iota_\alpha : \alpha \in A)$$

Exercise 3.6.0.2. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces. Then

$$\bigoplus_{\alpha \in A} \mathcal{A}_\alpha = \left\{ V \subset \coprod_{\alpha \in A} X_\alpha : \text{ for each } \alpha \in A, \, \iota_\alpha^{-1}(V) \in \mathcal{A}_\alpha \right\}$$

Proof. Set $X := \coprod\limits_{\alpha \in A} X_\alpha$ and $\mathcal{A} := \left\{ V \subset \coprod\limits_{\alpha \in A} X_\alpha : \text{ for each } \alpha \in A, \, \iota_\alpha^{-1}(V) \in \mathcal{A}_\alpha \right\}$.

- 1. Clearly $\varnothing \in \mathcal{A}$ and $\mathcal{A} \neq \varnothing$.
  2. Let $V \in \mathcal{A}$. Then by definition, for each $\alpha \in A$,
  $$\begin{aligned} \iota_\alpha^{-1}(V^c) &= (\iota_\alpha^{-1}(V))^c \\ &\in \mathcal{A}_\alpha \end{aligned}$$
  3. Let $(V_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Then by definition, for each $\alpha \in A$,
  $$\begin{aligned} \iota_\alpha^{-1}\left(\bigcup_{n \in \mathbb{N}} V_n \right) &= \bigcup_{n \in \mathbb{N}} \iota_\alpha^{-1}(V_n) \\ &\in \mathcal{A}_\alpha \end{aligned}$$

  So $\mathcal{A}$ is a σ-algebra.
- Since $\mathcal{A}$ is a σ-algebra on X, we have that $\sigma_X(\mathcal{A}) = \mathcal{A}$. By Definition 3.3.0.18,
  $$\begin{aligned} \mathcal{A} &= \sigma_X(\mathcal{A}) \\ &= \sigma_X(\iota_\alpha : \alpha \in A) \\ &= \bigoplus_{\alpha \in A} \mathcal{A}_\alpha. \end{aligned}$$

$\square$

Exercise 3.6.0.3. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces. Then

$$\bigoplus_{\alpha \in A} \mathcal{A}_\alpha = \left\{ \coprod_{\alpha \in A} B_\alpha : B_\alpha \in \mathcal{A}_\alpha \right\}$$

Proof. Set

- $\mathcal{F} = \{V \subset \coprod_{\alpha \in A} X_\alpha : \text{ for each } \alpha \in A, \, \iota_\alpha^{-1}(V) \in \mathcal{A}_\alpha\}$
- $\mathcal{G} = \left\{ \coprod\limits_{\alpha \in A} B_\alpha : \text{ for each } \alpha \in A, \, B_\alpha \in \mathcal{A}_\alpha \right\}$

Let $V \in \mathcal{G}$. Then for each $\alpha \in A$, there exists $B_\alpha \in \mathcal{A}_\alpha$ such that $V = \coprod\limits_{\alpha \in A} B_\alpha$. Therefore, for each $\alpha \in A$,

$$\begin{aligned} \iota_\alpha^{-1}(V) &= \iota_\alpha^{-1}\left(\coprod_{\alpha \in A} B_\alpha \right) \\ &= B_\alpha \\ &\in \mathcal{A}_\alpha \end{aligned}$$

Hence $V \in \mathcal{F}$. Since $V \in \mathcal{G}$ is arbitrary, $\mathcal{G} \subset \mathcal{F}$.
Conversely, let $V \in \mathcal{F}$. Then for each $\alpha \in A$, $\iota_\alpha^{-1}(V) \in \mathcal{A}_\alpha$. For each $\alpha \in A$, define $B_\alpha \in \mathcal{A}_\alpha$ by $B_\alpha = \iota_\alpha^{-1}(V)$. Then

$$\begin{aligned} V &= \coprod_{\alpha \in A} B_\alpha \\ &\in \mathcal{G} \end{aligned}$$

Since $V \in \mathcal{F}$ is arbitrary, $\mathcal{F} \subset \mathcal{G}$. The previous exercise implies that

$$\begin{aligned} \mathcal{G} &= \mathcal{F} \\ &= \bigoplus_{\alpha \in A} \mathcal{A}_\alpha \end{aligned}$$

$\square$

Exercise 3.6.0.4. Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in A}$ be a collection oftopological spaces. Then

$$\mathcal{B}\bigg(\coprod_{\alpha \in A} X_\alpha, \bigoplus_{\alpha \in A} \mathcal{T}_\alpha\bigg) = \bigoplus_{\alpha \in A} \mathcal{B}(X_\alpha, \mathcal{T}_\alpha)$$

Proof. FINISH!!! $\square$

Exercise 3.6.0.5. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ be a collection of measurable spaces, $(Y, \mathcal{B})$ a measurable space and $f : \coprod_{\alpha \in A} X_\alpha \to Y$. Then f is $\bigg(\bigoplus_{\alpha \in A} \mathcal{A}_\alpha, \mathcal{B}\bigg)$-measurable iff for each $\alpha \in A$, $f \circ \iota_\alpha$ is $(\mathcal{A}_\alpha, \mathcal{B})$-measurable.

Proof. Clear by Exercise 3.3.0.22. add more details $\square$

Exercise 3.6.0.6. Let $(X_\alpha, \mathcal{A}_\alpha)_{\alpha \in A}$ and $(Y_\alpha, \mathcal{B}_\alpha)_{\alpha \in A}$ be collections of measurable spaces and $(f_\alpha)_{\alpha \in A} \in \coprod_{\alpha \in A} Y_\alpha^{X_\alpha}$, i.e. for each $\alpha \in A$, $f_\alpha : X_\alpha \to Y_\alpha$. If for each $\alpha \in A$, f_α is $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$-measurable, then $\coprod_{\alpha \in A} f_\alpha$ is $(\bigoplus_{\alpha \in A} \mathcal{A}_\alpha, \bigoplus_{\alpha \in A} \mathcal{B}_\alpha)$-measurable.

Proof. Set $X := \coprod_{\alpha \in A} X_\alpha$, $Y := \coprod_{\alpha \in A} Y_\alpha$, $\mathcal{A} := \bigoplus_{\alpha \in A} \mathcal{A}_\alpha$ and $\mathcal{B} := \bigoplus_{\alpha \in A} \mathcal{B}_\alpha$. Suppose that for each $\alpha \in A$, f_α is $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$-measurable. Set $f = \coprod_{\alpha \in A} f_\alpha$. Denote the α-th embedding maps on X and Y by ι_α^X and ι_α^Y respectively. Let $\alpha \in A$. An exercise in the intro-math/analysis notes chapter on set theory implies that $f \circ \iota_\alpha^X = \iota_\alpha^Y \circ f_\alpha$. Since $\iota_\alpha^Y \circ f_\alpha$ is $(\mathcal{A}_\alpha, \mathcal{B})$-measurable, we have that $f \circ \iota_\alpha^X$ is $(\mathcal{A}_\alpha, \mathcal{B})$-measurable. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $f \circ \iota_\alpha^X$ is $(\mathcal{A}_\alpha, \mathcal{B})$-measurable. Exercise 3.6.0.5 implies that f is $(\mathcal{A}, \mathcal{B})$-measurable. $\square$

Exercise 3.6.0.7. Let $(X, \mathcal{A})$ be a measurable space and $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Define $\phi : \coprod_{\alpha \in A} E_\alpha \to \bigcup_{\alpha \in A} E_\alpha$ by $\phi(\alpha, x) := x$. If $(E_n)_{n \in \mathbb{N}}$ is disjoint, then ϕ is a $\bigg(\bigoplus_{n \in \mathbb{N}} [\mathcal{A} \cap E_n], \mathcal{A} \cap \bigg[\bigcup_{n \in \mathbb{N}} E_n\bigg]\bigg)$-isomorphism.

Proof. Suppose that $(E_n)_{n \in \mathbb{N}}$ is disjoint. Define $\phi : \coprod_{n \in \mathbb{N}} E_n \to \bigcup_{n \in \mathbb{N}} E_n$ by $\phi(n, x) = x$.

- **(bijectivity) :**
  - **(injectivity) :**
    Let $(n, x), (\beta, y) \in \coprod_{n \in \mathbb{N}} E_n$. Suppose that $\phi(n, x) = \phi(m, y)$. Then $x = y$. Thus $x \in E_n \cap E_m$ and therefore $E_n \cap E_m \neq \varnothing$. Since $(E_{n'})_{n' \in \mathbb{N}}$ is disjoint, we have that $n = m$. Hence $(n, x) = (m, y)$. Since $(n, x), (m, y) \in \coprod_{n \in \mathbb{N}} E_n$ are arbitrary, we have that for each $(n, x), (m, y) \in \coprod_{n \in \mathbb{N}} E_n$, $\phi(n, x) = \phi(m, y)$ implies that $(n, x) = (m, y)$. Thus ϕ is injective.

- **(surjectivity) :**
Let $x \in \bigcup_{n\in\mathbb{N}} E_n$. Then there exists $n \in \mathbb{N}$ such that $x \in E_n$. Then $(n, x) \in \coprod_{n\in\mathbb{N}} E_n$ and $\phi(n, x) = x$. Since $x \in \bigcup_{n\in\mathbb{N}} E_n$ is arbitrary, we have that for each $x \in \bigcup_{n\in\mathbb{N}} E_n$, there exists $a \in \coprod_{n\in\mathbb{N}} E_n$ such that $\phi(a) = x$. Hence ϕ is surjective.

So ϕ is a bijection.

- **(measurability) :**
For each $n \in \mathbb{N}$, define $\iota_{E_n} : E_n \to \bigcup_{n\in\mathbb{N}} E_n$ by $\iota_{E_n}(x) = x$.
  - Let $n \in \mathbb{N}$. Since $\phi \circ \iota_n = \iota_{E_n}$ and ι_{E_n} is $\left(\mathcal{A}\cap E_n, \mathcal{A}\cap\left[\bigcup_{n\in\mathbb{N}} E_n\right]\right)$-measurable (maybe give more details), we have that $\phi \circ \iota_n$ is $\left(\mathcal{A}\cap E_n, \mathcal{A}\cap\left[\bigcup_{n\in\mathbb{N}} E_n\right]\right)$-measurable. Since $n \in \mathbb{N}$ is arbitrary, we have that for each $n \in \mathbb{N}$, $\phi \circ \iota_n$ is $\left(\mathcal{A}\cap E_n, \mathcal{A}\cap\left[\bigcup_{n\in\mathbb{N}} E_n\right]\right)$-measurable. Exercise 3.6.0.5 implies that ϕ is $\left(\bigoplus_{n\in\mathbb{N}}(\mathcal{A}\cap E_n), \mathcal{A}\cap\left[\bigcup_{n\in\mathbb{N}} E_n\right]\right)$-measurable.
  - Let $B \in \bigoplus_{n\in\mathbb{N}}(\mathcal{A}\cap E_n)$. Exercise 3.6.0.3 implies that for each $n \in \mathbb{N}$, there exist $B_n \in \mathcal{A}\cap E_n$ such that $B = \coprod_{n\in\mathbb{N}} B_n$. Then for each $n \in \mathbb{N}$, there exists $C_n \in \mathcal{A}$ such that $B_n = C_n \cap E_n$. Since $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, we have that for each $n \in \mathbb{N}$, $C_n \cap E_n \in \mathcal{A}$. Hence for each $n \in \mathbb{N}$,
$$\begin{aligned} B_n &= C_n \cap E_n \\ &= (C_n \cap E_n) \cap \left[\bigcup_{n'\in\mathbb{N}} E'_n\right] \\ &\in \mathcal{A} \cap \left[\bigcup_{n'\in\mathbb{N}} E'_n\right]. \end{aligned}$$
Therefore
$$\begin{aligned} \phi(B) &= \phi\left(\coprod_{n\in\mathbb{N}} B_n\right) \\ &= \phi\left(\bigcup_{n\in\mathbb{N}} \iota_n(B_n)\right) \\ &= \bigcup_{n\in\mathbb{N}} \phi\circ\iota_n(B_n) \\ &= \bigcup_{n\in\mathbb{N}} \iota_{E_n}(B_n) \\ &= \bigcup_{n\in\mathbb{N}} B_n \\ &\in \mathcal{A}\cap\left[\bigcup_{n\in\mathbb{N}} E_n\right]. \end{aligned}$$
Since $B \in \bigoplus_{n\in\mathbb{N}}(\mathcal{A}\cap E_n)$ is arbitrary, we have that for each $B \in \bigoplus_{n\in\mathbb{N}}(\mathcal{A}\cap E_n)$, $f(B) \in \mathcal{A}$.

Hence ϕ is a measurable isomorphism.

$\square$

3.7 Quotient Sigma Algebras

Definition 3.7.0.1. Let X, Y be sets, $\sim$ an equivalence relation on X and $f : X \to Y$. Then f is said to be **invariant under** $\sim$ if for each $a, b \in X$, $\bar{a} = \bar{b}$ implies that $f(a) = f(b)$.

Exercise 3.7.0.2. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be topological spaces, $\sim$ an eqivalence relation on X, $\pi : X \to X/\sim$ the projection map and $f : X \to Y$ measurable. If f is invariant under $\sim$, then there exists a unique $\bar{f} : X/\sim \to Y$ such that

1. $\bar{f} \circ \pi = f$
2. $\bar{f}$ is $\mathcal{A}$-$\pi_*\mathcal{A}$ measurable

Proof. Suppose that f is invariant under $\sim$. Define $\bar{f} : X/\sim \to Y$ by $\bar{f}(\bar{x}) = f(x)$. By assumption, for each $a, b \in X$, $\bar{a} = \bar{b}$ implies that $f(a) = f(b)$. Thus $\bar{f}$ is well defined. By construction, $f = \bar{f} \circ \pi$. Let $V \in \mathcal{B}$. Measurability of f implies that $f^{-1}(V) \in \mathcal{A}$. Since

$$\begin{aligned} f^{-1}(V) &= \pi^{-1}(\bar{f}^{-1}(V)) \\ &\in \mathcal{A} \end{aligned}$$

by definition of $\pi_*\mathcal{A}$, $\bar{f}^{-1}(V) \in \pi_*\mathcal{A}$. So $\bar{f}$ is $\mathcal{A}$-$\pi_*\mathcal{A}$ measurable. □

3.8 Projective Limits of Measurable Spaces

Definition 3.8.0.1. Let $(J, \leq)$ be a directed set, $(X_j, \mathcal{A}_j)_{j \in J}$ a collection of measurable spaces and for each $(j,k) \in \leq$, $\pi_{j,k} : X_k \to X_j$ a $(\mathcal{A}_k, \mathcal{A}_j)$-measurable map. Suppose that $((X_j, \mathcal{A}_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ is a **Meas**-projective system. We define the **projective σ-algebra** of $(\mathcal{A}_j)_{j \in J}$ on $\varprojlim_{j \in J} X_j$, denoted $\varprojlim_{j \in J} \mathcal{A}_j$, by

$$\varprojlim_{j \in J} \mathcal{A}_j := \sigma(\pi_j : j \in J).$$

Exercise 3.8.0.2. Let $(J, \leq)$ be a directed set, $(X_j, \mathcal{A}_j)_{j \in J}$ a collection of measurable spaces and for each $(j,k) \in \leq$, $\pi_{j,k} : X_k \to X_j$ a $(\mathcal{A}_k, \mathcal{A}_j)$-measurable map. Suppose that $((X_j, \mathcal{A}_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ is a **Meas**-projective system. Then

$$\varprojlim_{j \in J} \mathcal{A}_j = \left[\bigotimes_{j \in J} \mathcal{A}_j\right] \cap \varprojlim_{j \in J} X_j.$$

Proof.

- Define $\mathcal{E} \subset \varprojlim_{j \in J} \mathcal{A}_j$ by $\mathcal{E} := \{\pi_j^{-1}(A) : j \in J \text{ and } A \in \mathcal{A}_j\}$. Let $E \in \mathcal{E}$. Then there exists $j \in J$ and $A \in \mathcal{A}_j$ such that $E = \pi_j^{-1}(A_j)$. This implies that

$$\begin{aligned} E &= \pi_j^{-1}(A_j) \\ &= (\mathrm{proj}_j |_{\varprojlim_{j \in J} X_j})^{-1}(A) \\ &= \mathrm{proj}_j^{-1}(A) \cap \varprojlim_{j \in J} X_j \\ &\in \left[\bigotimes_{j \in J} \mathcal{A}_j\right] \cap \varprojlim_{j \in J} X_j. \end{aligned}$$

  Since $E \in \mathcal{E}$ is arbitrary, we have that $\mathcal{E} \subset \left[\bigotimes_{j \in J} \mathcal{A}_j\right] \cap \varprojlim_{j \in J} X_j$. Since by definition, $\varprojlim_{j \in J} \mathcal{A}_j = \sigma(\mathcal{E})$, we have that

$$\begin{aligned} \varprojlim_{j \in J} \mathcal{A}_j &= \sigma(\mathcal{E}) \\ &\subset \sigma\left(\left[\bigotimes_{j \in J} \mathcal{A}_j\right] \cap \varprojlim_{j \in J} X_j\right) \\ &= \left[\bigotimes_{j \in J} \mathcal{A}_j\right] \cap \varprojlim_{j \in J} X_j. \end{aligned}$$

- Define $\mathcal{F} \subset \bigotimes_{j \in J} \mathcal{A}_j$ by $\mathcal{F} := \{\mathrm{proj}_j^{-1}(A) : j \in J \text{ and } A \in \mathcal{A}_j\}$. Let $F \in \mathcal{F} \cap \varprojlim_{j \in J} X_j$. Then there exists $F_0 \in \mathcal{F}$ such that $F = F_0 \cap \varprojlim_{j \in J} X_j$. Thus there exists $j \in J$ and $A \in \mathcal{A}_j$ such that $F_0 = \mathrm{proj}_j^{-1}(A)$. Therefore

$$\begin{aligned} F &= F_0 \cap \varprojlim_{j \in J} X_j \\ &= \mathrm{proj}_j^{-1}(A) \cap \varprojlim_{j \in J} X_j \\ &= (\mathrm{proj}_j |_{\varprojlim_{j \in J} X_j})^{-1}(A) \\ &= \pi_j^{-1}(A) \\ &\in \mathcal{E}. \end{aligned}$$

Since $F \in \mathcal{F} \cap \varprojlim_{j\in J} X_j$ is arbitrary, we have that $\mathcal{F} \cap \varprojlim_{j\in J} X_j \subset \mathcal{E}$. Exercise 3.4.0.6 implies that

$$\begin{aligned}\left[\bigotimes_{j\in J} \mathcal{A}_j\right] \cap \varprojlim_{j\in J} X_j &= \sigma(\mathcal{F}) \cap \varprojlim_{j\in J} X_j \\ &= \sigma(\mathcal{F} \cap \varprojlim_{j\in J} X_j) \\ &\subset \sigma(\mathcal{E}) \\ &= \varprojlim_{j\in J} \mathcal{A}_j.\end{aligned}$$

□

Exercise 3.8.0.3. Let $(J, \leq)$ be a directed set and $((X_j, \mathcal{T}_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$ is a **Top**-projective system. Suppose that J is countable and for each $j \in J$, X_j is second countable. Then

$$\mathcal{B}\left(\varprojlim_{j\in J} X_j, \varprojlim_{j\in J} \mathcal{T}_j\right) = \varprojlim_{j\in J} \mathcal{B}(X_j, \mathcal{T}_j).$$

Proof. Since J is countable and for each $j \in J$, X_j is second countable, Exercise 3.5.0.5 implies that $\mathcal{B}\left(\prod_{j\in J} X_j, \bigotimes_{j\in J} \mathcal{T}_j\right) = \bigotimes_{j\in J} \mathcal{B}(X_j, \mathcal{T}_j)$. Exercise 3.4.0.8 and Exercise 3.8.0.2 then imply that

$$\begin{aligned}\mathcal{B}\left(\varprojlim_{j\in J} X_j, \varprojlim_{j\in J} \mathcal{T}_j\right) &= \mathcal{B}\left(\varprojlim_{j\in J} X_j, \left[\bigotimes_{j\in J} \mathcal{T}_j\right] \cap \varprojlim_{j\in J} X_j\right) \\ &= \mathcal{B}\left(\prod_{j\in J} X_j, \bigotimes_{j\in J} \mathcal{T}_j\right) \cap \varprojlim_{j\in J} X_j \\ &= \left[\bigotimes_{j\in J} \mathcal{B}(X_j, \mathcal{T}_j)\right] \cap \varprojlim_{j\in J} X_j \\ &= \varprojlim_{j\in J} \mathcal{B}(X_j, \mathcal{T}_j)\end{aligned}$$

□

Exercise 3.8.0.4. Let $(J, \leq)$ be a directed set and $((X_j, \mathcal{A}_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$ a **Meas**-projective system. Set $X := \varprojlim_{j\in J} X_j$ and $\mathcal{A} := \varprojlim_{j\in J} \mathcal{A}_j$. Define $\mathcal{A}_0 \subset \mathcal{A}$ by $\mathcal{A}_0 = \bigcup_{j\in J} \pi_j^* \mathcal{A}_j$. Then $\mathcal{A}_0$ is an algebra on X.

Proof.

1. Since J is a directed set, $J \neq \varnothing$. Thus there exists $j \in J$. Since $\mathcal{A}_j$ is an algebra on X, $\mathcal{A}_j \neq \varnothing$. Hence there exists $A_j \in \mathcal{A}_j$. Then

$$\begin{aligned}\pi_j^{-1}(A_j) &\in \pi_j^* \mathcal{A}_j \\ &\subset \bigcup_{j\in J} \pi_j^* \mathcal{A}_j \\ &= \mathcal{A}_0.\end{aligned}$$

Thus $\mathcal{A}_0 \neq \varnothing$.

2. Let $A \in \mathcal{A}_0$. Since $\mathcal{A}_0 = \bigcup_{j \in J} \pi_j^* \mathcal{A}_j$, there exists $j \in J$ such that $A \in \pi_j^* \mathcal{A}_j$. By definition of $\pi_j^* \mathcal{A}_j$ (Definition 3.3.0.6), there exists $A_j \in \mathcal{A}_j$ such that $A = \pi_j^{-1}(A_j)$. Since $\mathcal{A}_j$ is a σ-algebra on X_j, $A_j^c \in \mathcal{A}_j$. Hence

$$\begin{aligned} A^c &= \pi_j^{-1}(A_j)^c \\ &= \pi_j^{-1}(A_j^c) \\ &\in \pi_j^{-1}\mathcal{A}_j \\ &\subset \bigcup_{j \in J} \pi_j^* \mathcal{A}_j \\ &= \mathcal{A}_0. \end{aligned}$$

   Since $A \in \mathcal{A}_0$ is arbitrary, we have that for each $A \in \mathcal{A}_0$, $A^c \in \mathcal{A}_0$.

3. Let $A_1, A_2 \in \mathcal{A}_0$. Then there exist $j_1, j_2 \in J$ such that $A_1 \in \pi_{j_1}^{-1}(\mathcal{A}_{j_1})$ and $A_2 \in \pi_{j_2}^{-1}(\mathcal{A}_{j_2})$. Therefore there exists $E_1 \in \mathcal{A}_{j_1}$ and $E_2 \in \mathcal{A}_{j_2}$ such that $A_1 = \pi_{j_1}^{-1}(E_1)$ and $A_2 = \pi_{j_2}^{-1}(E_2)$. Since J is a directed set, there exists $j_0 \in J$ such that $j_1, j_2 \leq j_0$. Since for each $(j,k) \in \leq$, $p_{j,k}$ is $(\mathcal{A}_k, \mathcal{A}_j)$-measurable, we have that $\pi_{j_1,j_0}^{-1}(E_1), \pi_{j_2,j_0}^{-1}(E_2) \in \mathcal{A}_{j_0}$. Since $\mathcal{A}_0$ is a σ-algebra on X_{j_0} we have that $\pi_{j_1,j_0}^{-1}(E_1) \cup \pi_{j_2,j_0}^{-1}(E_2) \in \mathcal{A}_{j_0}$. Hence

$$\begin{aligned} A_1 \cup A_2 &= \pi_{j_1}^{-1}(E_1) \cup \pi_{j_2}^{-1}(E_2) \\ &= (\pi_{j_1,j_0} \circ \pi_{j_0})^{-1}(E_1) \cup (\pi_{j_2,j_0} \circ \pi_{j_0})^{-1}(E_2) \\ &= \pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1)) \cup \pi_{j_0}^{-1}(\pi_{j_2,j_0}^{-1}(E_2)) \\ &= \pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1) \cup \pi_{j_2,j_0}^{-1}(E_2)) \\ &\in \pi_{j_0}^* \mathcal{A}_{j_0} \\ &\subset \bigcup_{j \in J} \pi_j^* \mathcal{A}_j \\ &= \mathcal{A}_0. \end{aligned}$$

   Since $A_1, A_2 \in \mathcal{A}_0$ are arbitrary, we have that for each $A_1, A_2 \in \mathcal{A}_0$, $A_1 \cup A_2 \in \mathcal{A}_0$.

Thus $\mathcal{A}_0$ is an algebra on X. $\square$

3.9 Dynkin's Lemma

Definition 3.9.0.1. Let X be a set and $\mathcal{P} \subset \mathcal{P}(X)$. Then $\mathcal{P}$ is said to be a **π-system** on X if for each $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$.

Definition 3.9.0.2. Let X be a set and $\mathcal{L} \subset \mathcal{P}(X)$. Then $\mathcal{L}$ is said to be a **λ-system** on X if

1. $\mathcal{L} \neq \varnothing$
2. for each $A \in \mathcal{L}$, $A^c \in \mathcal{L}$
3. for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}$, if $(A_n)_{n\in\mathbb{N}}$ is disjoint, then $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{L}$

Exercise 3.9.0.3. Let X be a set and $\mathcal{L}$ a λ-system on X. Then

1. $X, \varnothing \in \mathcal{L}$

Proof. Straightforward. □

Definition 3.9.0.4. Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$. Put

$$\mathcal{S} = \{\mathcal{L} \subset \mathcal{P}(X) : \mathcal{L} \text{ is a } \lambda\text{-system on } \Omega \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the **λ-system on X generated by** $\mathcal{C}$, $\lambda(\mathcal{C})$, to be

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{L}\in\mathcal{S}} \mathcal{L}$$

Exercise 3.9.0.5. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. If $\mathcal{A}$ is a λ-system and $\mathcal{A}$ is a π-system, then $\mathcal{A}$ is a σ-algebra.

Proof. Suppose that $\mathcal{A}$ is a λ-system and $\mathcal{A}$ is a π-system. Then we need only verify the third axiom in the definition of a σ-algebra. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Define $B_1 = A_1$ and for $n \geq 2$, define $B_n = A_n \cap \left(\bigcup\limits_{k=1}^{n-1} A_k\right)^c = A_n \cap \left(\bigcap\limits_{k=1}^{n-1} A_k^c\right) \in \mathcal{A}$. Then $(B_n)_{n\in\mathbb{N}}$ is disjoint and therefore $\bigcup\limits_{n\in\mathbb{N}} A_n = \bigcup\limits_{n\in\mathbb{N}} B_n \in \mathcal{A}$. □

Theorem 3.9.0.6. Dynkin's Lemma:
Let X be a set, $\mathcal{P}$ be a π-system on X and $\mathcal{L}$ a λ-system on X. Then

1. $\mathcal{P} \subset \mathcal{L}$ implies that $\sigma(\mathcal{P}) \subset \mathcal{L}$
2. $\sigma(\mathcal{P}) = \lambda(\mathcal{P})$

Exercise 3.9.0.7. Define $\mathcal{P} \subset \mathcal{B}(\mathbb{R})$ by

$$\mathcal{P} = \{(a, b] : a, b \in \mathbb{R}\} \cup \{\varnothing, X\}$$

Then $\mathcal{P}$ is a π-system on X.

Proof. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Then

$$\begin{aligned}(a_1, b_1] \cap (a_2, b_2] &= (a_2, b_1] \\ &\in \mathcal{P}\end{aligned}$$

□

3.10 Limits of Sets

Definition 3.10.0.1. Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$. We define

$$\inf \mathcal{A} = \bigcap_{A \in \mathcal{A}} A, \quad \sup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

Definition 3.10.0.2. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. We define

$$\liminf_{n \to \infty} A_n = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} A_k \right), \quad \limsup_{n \to \infty} A_n = \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} A_k \right)$$

Note 3.10.0.3.

1. $\liminf_{n \to \infty} A_n$ is the set of elements that are in all A_n except for finitely many.
2. $\limsup_{n \to \infty} A_n$ is the set of elements that are in infinitely many A_n.

Exercise 3.10.0.4. Let X be a set and $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then

1. $\liminf_{n \to \infty} A_n = \left\{ x \in X : \liminf_{n \to \infty} \chi_{A_n}(x) = 1 \right\}$
2. $\limsup_{n \to \infty} A_n = \left\{ x \in X : \limsup_{n \to \infty} \chi_{A_n}(x) = 1 \right\}$

Proof.

1. Let $x \in \liminf_{n \to \infty} A_n$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $x \in A_k$. So for each $k \in \mathbb{N}$, $k \geq n^*$ implies that $\chi_{A_k}(x) = 1$. Then $\inf_{k \geq n^*} \chi_{A_k}(x) = 1$ and thus

$$1 = \sup_{n \in \mathbb{N}} \left(\inf_{k \geq n} \chi_{A_k}(x) \right) = \liminf_{n \to \infty} \chi_{A_n}(x)$$

   Conversely, if $1 = \liminf_{n \to \infty} \chi_{A_n}(x)$, then choosing $\epsilon = \frac{1}{2}$, there exists $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq n$ implies that $\chi_{A_k}(x) > 1 - \epsilon$. Hence for each $k \in \mathbb{N}$, $k \geq n$ implies that $\chi_{A_k}(x) = 1$. So for each for each $k \in \mathbb{N}$, $k \geq n$ implies that $x \in A_k$. So $x \in \liminf_{n \to \infty} A_n$.

2. Similar to (1).

□

Exercise 3.10.0.5. Let $A_k = [0, \frac{k}{k+1})$. Then

1. $\inf_{k \geq n} A_k = [0, \frac{n}{n+1})$
2. $\sup_{k \geq n} A_k = [0, 1)$
3. $\liminf_{n \to \infty} A_n = [0, 1)$
4. $\liminf_{n \to \infty} A_n = [0, 1)$

Proof. Straightforward. □

Exercise 3.10.0.6. Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$$

Proof. Let $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq n*$, then $x \in A_k$. Let $n \in \mathbb{N}$. Choose $k = \max\{n^*, n\} \geq n^*$. Then $x \in A_k$. Hence for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x \in A_k$. So $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Thus $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$. □

Definition 3.10.0.7. Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. If

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

then we define

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

Exercise 3.10.0.8. Let X be a set and $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Suppose that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$ and $B_{n+1} \subset B_n$. Then

1. $\lim_{n\to\infty} A_n = \sup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$
2. $\lim_{n\to\infty} B_n = \inf_{n\in\mathbb{N}} B_n = \bigcap_{n=1}^{\infty} B_n$

Proof.

1. Let $n \in \mathbb{N}$. Then

$$\begin{aligned}\inf_{k\geq n} A_k &= \bigcap_{k=n}^{\infty} A_k \\ &= A_n\end{aligned}$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} A_n &= \bigcup_{n=1}^{\infty} \inf_{k\geq n} A_k \\ &= \bigcup_{n=1}^{\infty} A_n\end{aligned}$$

In addition,

$$\begin{aligned}\sup_{n\geq k} A_k &= \bigcup_{k=n}^{\infty} A_k \\ &= \bigcup_{k=1}^{\infty} A_k\end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{n\to\infty} A_n &= \bigcap_{n=1}^{\infty} \inf_{k\geq n} A_k \\ &= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_k \\ &= \bigcup_{n=1}^{\infty} A_n\end{aligned}$$

So

$$\lim_{n\to\infty} A_n = \sup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n$$

2. Similar

□

Exercise 3.10.0.9. Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets and $(A_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(A_n)_{n\in\mathbb{N}}$. Then

1. $\limsup\limits_{k\to\infty} A_{n_k} \subset \limsup\limits_{n\to\infty}(A_n)$
2. $\liminf\limits_{n\to\infty} A_n \subset \liminf\limits_{k\to\infty}(A_{n_k})$

Proof.

1. The elements that are in A_{n_k} for infinitely many k are in A_n for infinitely many n.
2. Similar.

□

Exercise 3.10.0.10. Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets, $(A_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(A_n)_{n\in\mathbb{N}}$ and $A \subset X$. If $A_{n_k} \to A$, then

$$\liminf_{n\to\infty} A_n \subset A \subset \limsup_{n\to\infty} A_n$$

Proof. The previous exercises tells us that

$$\begin{aligned}\liminf_{n\to\infty} A_n &\subset \liminf_{k\to\infty} A_{n_k} \\ &= A \\ &= \limsup_{k\to\infty} A_{n_k} \\ &\subset \limsup_{n\to\infty} A_n\end{aligned}$$

□

Exercise 3.10.0.11. Let X be a set and $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Suppose that for each $n \in \mathbb{N}$, $A_n \subset B_n$. Then

1. $\limsup\limits_{n\to\infty} A_n \subset \limsup\limits_{n\to\infty} B_n$
2. $\liminf\limits_{n\to\infty} A_n \subset \liminf\limits_{n\to\infty} B_n$

Proof.

1. Let $x \in \limsup\limits_{n\to\infty} A_n$. Then for infinitely many $n \in \mathbb{N}$, $x \in A_n \subset B_n$. So for infinitely many $n \in \mathbb{N}$, $x \in B_n$. Hence $x \in \limsup\limits_{n\to\infty} B_n$. Therefore $\limsup\limits_{n\to\infty} A_n \subset \limsup\limits_{n\to\infty} B_n$.
2. Similar.

□

Exercise 3.10.0.12. Let X be a set and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ a sequence of subsets. Then

1. $\limsup\limits_{n\to\infty} A_n = \left(\liminf\limits_{n\to\infty} A_n^c\right)^c$

2. $\liminf_{n\to\infty} A_n = \left(\limsup_{n\to\infty} A_n^c\right)^c$

Proof.

1.
$$\begin{aligned}\left(\liminf_{n\to\infty} A_n^c\right)^c &= \left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^c\right)^c \\ &= \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \\ &= \limsup_{n\to\infty} A_n\end{aligned}$$

2. Similar.

□

Exercise 3.10.0.13. For $n \in \mathbb{N}$, define

$$A_n = \left\{\frac{m}{n} : m \in \mathbb{N}\right\}$$

Then

1. $\liminf_{n\to\infty} A_n = \mathbb{N}$

2. $\limsup_{n\to\infty} A_n = \mathbb{Q} \cap (0, \infty)$

Proof.

1. For each $x \in \mathbb{N}$ and $n \in \mathbb{N}$, $x = \frac{nx}{n} \in A_n$ Hence $\mathbb{N} \subset \liminf_{n\to\infty} A_n$. Conversely, let $x \in \liminf_{n\to\infty} A_n$. Then there exists $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq n$, then $x \in A_k$. In particular, $x \in A_n$. Hence there exists $m_n \in \mathbb{N}$ such that $x = \frac{m_n}{n}$. Choose $s, t \in \mathbb{N}$ such that $x = \frac{s}{t}$ and $\gcd(s,t) = 1$. Choose a prime $p > n$. By assumption, $x \in A_p$. Then there exist $m_p \in \mathbb{N}$ such that $x = \frac{m_p}{p}$. Hence $\frac{s}{t} = \frac{m_p}{p}$ and $tm_p = sp$. Since $t|sp$ and $\gcd(s,t) = 1$, we see that $t|p$. If $t > 1$, then p is not prime, which is a contradiction. So $t = 1$. Hence $x \in \mathbb{N}$. Thus $\liminf_{n\to\infty} A_n \subset \mathbb{N}$.

2. Let $x \in \mathbb{Q} \cap (0, \infty)$. Then there exist $s, t \in \mathbb{N}$ such that $x = \frac{s}{t}$. Define the subsequence $(A_{n_k})_{k\in\mathbb{N}}$ by $A_{n_k} = A_{tk}$. Then for each $k \in \mathbb{N}$, $x = \frac{sk}{tk} \in A_{tk} = A_{n_k}$. Thus
$$\begin{aligned}x &\in \inf_{k\in\mathbb{N}} A_{n_k} \\ &\subset \liminf_{n\to\infty} A_{n_k} \\ &\subset \limsup_{n\to\infty} A_{n_k} \\ &\subset \limsup_{n\to\infty} A_n\end{aligned}$$
Conversely, clearly $\limsup_{n\to\infty} A_n \subset \mathbb{Q} \cap (0, \infty)$

□

Exercise 3.10.0.14. Let X be a set and $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Then

$$\limsup_{n\to\infty} A_n \cup B_n = \limsup_{n\to\infty} A_n \cup \limsup_{n\to\infty} B_n$$

Proof. Let $x \in \limsup\limits_{n \to \infty} A_n \cup B_n$. Suppose that $x \notin \limsup\limits_{n \to \infty} A_n$. Then there exists $n^* \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ if $k \geq n^*$, then $x \notin A_k$. Let $n \in \mathbb{N}$. Then there exists k such that $k \geq \max\{n, n^*\}$ and $x \in A_k \cup B_k$. Since $k \geq n^*$, $x \notin A_k$ Thus $x \in B_k$. So for each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k \geq n$ and $x \in B_k$. Therefore $x \in \limsup\limits_{n \to \infty} B_n$ and

$$\limsup_{n \to \infty} A_n \cup B_n \subset \limsup_{n \to \infty} A_n \cup \limsup_{n \to \infty} B_n$$

Conversely, a previous exercise tells us that $\limsup\limits_{n \to \infty} A_n \subset \limsup\limits_{n \to \infty} A_n \cup B_n$ and $\limsup\limits_{n \to \infty} B_n \subset \limsup\limits_{n \to \infty} A_n \cup B_n$. Thus

$$\limsup_{n \to \infty} A_n \cup \limsup_{n \to \infty} B_n \subset \limsup_{n \to \infty} A_n \cup B_n$$

□

Exercise 3.10.0.15. Let X be a set and $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ sequences of subsets. Then

$$\liminf_{n \to \infty} A_n \cap B_n = \liminf_{n \to \infty} A_n \cap \liminf_{n \to \infty} B_n$$

Proof. A previous exercise tells us that

$$\begin{aligned}
\liminf_{n \to \infty} A_n \cap B_n &= \left(\limsup_{n \to \infty} A_n^c \cup B_n^c \right)^c \\
&= \left(\limsup_{n \to \infty} A_n^c \cup \limsup_{n \to \infty} B_n^c \right)^c \\
&= \left(\limsup_{n \to \infty} A_n^c \right)^c \cap \left(\limsup_{n \to \infty} B_n^c \right)^c \\
&= \liminf_{n \to \infty} A_n \cap \liminf_{n \to \infty} B_n
\end{aligned}$$

□

Chapter 4

Standard Borel Spaces

4.1 Introduction

Definition 4.1.0.1. We define the

- **Cantor space**, denoted $\mathcal{C}$, by $\mathcal{C} := \{0,1\}^{\mathbb{N}}$,
- **Baire space**, denoted $\mathcal{N}$, by $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$,
- **Hilbert Cube**, denoted $\mathbb{H}$, by $\mathbb{H} := [0,1]^{\mathbb{N}}$.

We equip each of $\mathcal{C}$, $\mathcal{N}$ and $\mathbb{H}$ with the product topology.

Note 4.1.0.2. We note that exercises from analysis notes $\mathcal{C}, \mathcal{N}\mathbb{H}$ are Polish spaces and $\mathcal{C}, \mathcal{N}$ are zero-dimensional spaces.

4.2 The Cantor Space

Definition 4.2.0.1. We recall from the analysis notes section on cantor space the following definitions of $\mathcal{C}, Z, \phi : Z \to [0,1]$ and for $n \in \mathbb{N}$, $l \in \{0,1\}$, Z_n^l:

- We define $\mathcal{C} := \{0,1\}^{\mathbb{N}}$
- We define $Z \subset \mathcal{C}$ by
$$Z := \left\{(x_n)_{n\in\mathbb{N}} \in \mathcal{C} : \#\{n \in \mathbb{N} : x_n = 0\} = \infty\right\} \cup \{(1,1,1,\dots)\}$$
- We define $\phi : Z \to [0,1]$ by
$$\phi(x) = \sum_{n\in\mathbb{N}} x_n 2^{-n}$$
- For $n \in \mathbb{N}$ and $l \in \{0,1\}$ we define $Z_n^l \subset Z$ by $Z_n^l := \{\pi_n^{-1}(\{l\})\} \cap Z$ where $\pi_n : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ is the projection onto the n-th coordinate.

Exercise 4.2.0.2. We have that $\mathcal{C}$ and Z are zero-dimensional Polish spaces and ϕ is a continuous bijection.
Hint: cite Analysis notes section on zero-dimensional metric spaces subsection on Cantor space.

Proof. Analysis notes section on zero-dimensional metric spaces subsection on Cantor space □

Exercise 4.2.0.3. We have that $Z \in \mathcal{B}(\mathcal{C})$.
Hint: Analysis notes section on Cantor space.

Proof. An exercise in the analysis notes section on zero-dimensional polish spaces subsection cantor space implies that Z is a G_δ-set. Thus $Z \in \mathcal{B}(\mathcal{C})$. □

Definition 4.2.0.4. We define $(\theta_n)_{n\in\mathbb{N}_0} \subset Z^Z$ by

- $\theta_0 = \mathrm{id}_Z$
- $\theta_1(z) = \phi^{-1}(2\phi(z) - z_1)$
- for $n \geq 2$, $\theta_n = \theta_1 \circ \theta_{n-1}$

Exercise 4.2.0.5. For each $n \in \mathbb{N}$ and $z \in Z$, $\theta_n(z) = (z_{j+n})_{j\in\mathbb{N}}$.

Proof. Let $z \in Z$. Since

$$\begin{aligned}
\theta_1(z) &= \phi^{-1}(2\phi(z) - z_1) \\
&= \phi^{-1}(2\sum_{j\in\mathbb{N}} z_j 2^{-j} - z_1) \\
&= \phi^{-1}(\sum_{j\in\mathbb{N}} z_j 2^{-j+1} - z_1) \\
&= \phi^{-1}(\sum_{j\in\mathbb{N}} z_{j+1} 2^{-j}) \\
&= (z_{j+1})_{j\in\mathbb{N}}
\end{aligned}$$

The claim is true for $n = 1$. Let $n \in \mathbb{N}$. Suppose that the claim is true for $n-1$. Let $z \in Z$. Set $w = \theta_{n-1}(z)$. Then $(w_j)_{j\in\mathbb{N}} = (z_{j+n-1})_{j\in\mathbb{N}}$ and therefore

$$\begin{aligned}
\theta_n(z) &= \theta_1 \circ \theta_{n-1}(z) \\
&= \theta_1(w) \\
&= (w_{j+1})_{j\in\mathbb{N}} \\
&= (z_{(j+1)+n-1})_{j\in\mathbb{N}} \\
&= (z_{j+n})_{j\in\mathbb{N}}
\end{aligned}$$

□

Exercise 4.2.0.6. For each $n \in \mathbb{N}$,

1.
$$\phi(Z_n^0) \subset \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)$$

2.
$$\phi(Z_n^1) \subset \left[\bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k+1}{2^n}, \frac{2(k+1)}{2^n}\right)\right] \cup \{1\}$$

Hint: Induction

Proof.

1. The claim is clearly true for $n = 1$. Let $n \geq 2$. Suppose the claim is true for $n-1$. Let $z \in Z_n^0$. Set $w = \theta_1(z)$. Then

$$\begin{aligned} w_{n-1} &= z_n \\ &= 0 \end{aligned}$$

Hence $w \in Z_{n-1}^0$. Our induction hypothesis implies that $\phi(w) \in \bigcup_{k=0}^{2^{n-2}-1} \left[\frac{2k}{2^{n-1}}, \frac{2k+1}{2^{n-1}}\right)$. Therefore, there exists $k \in \{0, \dots, 2^{n-2}-1\}$ such that

$$\phi(w) \in \left[\frac{2k}{2^{n-1}}, \frac{2k+1}{2^{n-1}}\right)$$

Since

$$\begin{aligned} \phi(w) &= \phi(\theta_1(z)) \\ &= 2\phi(z) - z_1 \end{aligned}$$

We have that

$$\begin{aligned} \phi(z) &= 2^{-1}\phi(w) + 2^{-1}z_1 \\ &\in \left[\frac{2k}{2^n} + 2^{-1}z_1, \frac{2k+1}{2^n} + 2^{-1}z_1\right) \\ &= \left[\frac{2(k + 2^{n-2}z_1)}{2^n}, \frac{2(k + 2^{n-2}z_1) + 1}{2^n}\right) \end{aligned}$$

Since $k \in \{0, \dots, 2^{n-2}-1\}$ and $1 + z_1 \leq 2$, we have that

$$\begin{aligned} k + 2^{n-2}z_1 &\leq 2^{n-2} - 1 + 2^{n-2}z_1 \\ &= 2^{n-2}(1 + z_1) - 1 \\ &\leq 2^{n-1} - 1 \end{aligned}$$

Therefore $k + 2^{n-2}z_1 \in \{0, \dots, 2^{n-1}-1\}$ which implies that $\phi(z) \in \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)$. Since $z \in \phi(Z_n^0)$ is arbitrary, we have that

$$\phi(Z_n^0) \subset \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)$$

2. The claim is clearly true for $n = 1$. Let $n \geq 2$. Suppose that the claim is true for $n-1$. Let $z \in Z_n^1$. If for each $j \in \mathbb{N}$, $z_j = 1$, then $\phi(z) = 1$ and the claim is true. Suppse that there exists $j \in \mathbb{N}$ such that $z_j \neq 1$. Set $w = \theta_1(z)$. Then

$$\begin{aligned} w_{n-1} &= z_n \\ &= 1 \end{aligned}$$

Thus $w \in Z^1_{n-1}$. Our induction hypothesis implies that $\phi(w) \in \bigcup_{k=0}^{2^{n-2}-1} \left[\frac{2k+1}{2^{n-1}}, \frac{2(k+1)}{2^{n-1}}\right)$. Therefore, there exists $k \in \{0, \dots, 2^{n-2}-1\}$ such that

$$\phi(w) \in \left[\frac{2k+1}{2^{n-1}}, \frac{2(k+1)}{2^{n-1}}\right)$$

Since

$$\begin{aligned}\phi(w) &= \phi(\theta_1(z)) \\ &= 2\phi(z) - z_1\end{aligned}$$

We have that

$$\begin{aligned}\phi(z) &= 2^{-1}\phi(w) + 2^{-1}z_1 \\ &\in \left[\frac{2k+1}{2^n} + 2^{-1}z_1, \frac{2(k+1)}{2^n} + 2^{-1}z_1\right) \\ &= \left[\frac{2(k+2^{n-2}z_1)+1}{2^n}, \frac{2[(k+2^{n-2}z_1)+1]}{2^n}\right)\end{aligned}$$

Since $k \in \{0, \dots, 2^{n-2}-1\}$ and $1 + z_1 \leq 2$, we have that

$$\begin{aligned}k + 2^{n-2}z_1 &\leq 2^{n-2} - 1 + 2^{n-2}z_1 \\ &= 2^{n-2}(1+z_1) - 1 \\ &\leq 2^{n-1} - 1\end{aligned}$$

Therefore $k + 2^{n-2}z_1 \in \{0, \dots, 2^{n-1}-1\}$ which implies that $\phi(z) \in \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k+1}{2^n}, \frac{2(k+1)}{2^n}\right)$. Since $z \in \phi(Z^1_n) \setminus \{\phi^{-1}(1)\}$ is arbitrary, we have that

$$\phi(Z^1_n) \subset \left[\bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k+1}{2^n}, \frac{2(k+1)}{2^n}\right)\right] \cup \{1\}$$

□

Exercise 4.2.0.7. For each $n \in \mathbb{N}$,

1. $$\phi(Z^0_n) = \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)$$

2. $$\phi(Z^1_n) = \left[\bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k+1}{2^n}, \frac{2(k+1)}{2^n}\right)\right] \cup \{1\}$$

Proof.

1. Let $n \in \mathbb{N}$. Set

$$A = \bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k}{2^n}, \frac{2k+1}{2^n}\right)$$

and

$$B = \left[\bigcup_{k=0}^{2^{n-1}-1} \left[\frac{2k+1}{2^n}, \frac{2(k+1)}{2^n}\right)\right] \cup \{1\}$$

Part (1) of Exercise 4.2.0.6 implies that $\phi(Z_n^0) \subset A$. Since $A \cap B = \varnothing$, part (2) of Exercise 4.2.0.6 implies that

$$\begin{aligned}\phi(Z_n^0)^c &= \phi(Z_n^1)\\ &\subset B\\ &\subset A^c\end{aligned}$$

Therefore $A \subset \phi(Z_n^0)$. Hence $\phi(Z_n^0) = A$.

2. Similar to part (1)

$\square$

Exercise 4.2.0.8. We have that

1. ϕ is $(\mathcal{B}(Z), \mathcal{B}([0,1]))$-measurable
2. ϕ^{-1} is $(\mathcal{B}([0,1]), \mathcal{B}(Z))$-measurable
   Hint: Recall that $\mathcal{B}(Z) = Z \cap \mathcal{B}(\mathcal{C})$ and $\mathcal{B}(\{0,1\})^{\otimes \mathbb{N}} = \sigma_{\mathcal{C}}(\pi_j : j \in \mathbb{N})$
3. ϕ is a $(\mathcal{B}(Z), \mathcal{B}([0,1]))$-measurable isomorphism.

Proof.

1. Since ϕ is continuous, ϕ is $(\mathcal{B}(Z), \mathcal{B}([0,1]))$-measurable.
2. Since

$$\begin{aligned}\mathcal{B}(Z) &= \mathcal{B}(\{0,1\}^{\mathbb{N}}) \cap Z\\ &= \left[\mathcal{B}(\{0,1\})^{\otimes \mathbb{N}}\right] \cap Z\\ &= \sigma(\{\pi_n^{-1}(\{0\}) : n \in \mathbb{N}\}) \cap Z\\ &= \sigma(\{\pi_n^{-1}(\{0\}) \cap Z : n \in \mathbb{N}\})\\ &= \sigma(\{Z_n^0 : n \in \mathbb{N}\})\end{aligned}$$

Exercise 4.2.0.7 implies that for each $n \in \mathbb{N}$,

$$\begin{aligned}(\phi^{-1})^{-1}(Z_n^0) &= \phi(Z_n^0)\\ &\in \mathcal{B}([0,1])\end{aligned}$$

and therefore ϕ^{-1} is $(\mathcal{B}([0,1]), \mathcal{B}(Z))$-measurable.

3. Clear by definition.

$\square$

4.3 Analytic Sets

Definition 4.3.0.1. Let X be a topological space and $\mathcal{D} \subset \mathcal{P}(X)$. Then $\mathcal{D}$ is said to be a **δ-system** if for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}$,

1. $\bigcap\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}$
2. $(A_n)_{n\in\mathbb{N}}$ is disjoint implies that $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}$

Exercise 4.3.0.2. Let X be a set and $(\mathcal{D}_\alpha)_{\alpha\in A}$ a collection of δ-systems on X. Then $\bigcap\limits_{\alpha\in A} \mathcal{D}_\alpha$ is a δ-system on X.

Proof. Set $\mathcal{D} := \bigcap\limits_{\alpha\in A} \mathcal{D}_\alpha$. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}$.

1. Let $\alpha \in A$. Since $\mathcal{D} \subset \mathcal{D}_\alpha$, we have that $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}_\alpha$. Since $\mathcal{D}_\alpha$ is a δ-system, $\bigcap\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}_\alpha$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $\bigcap\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}_\alpha$. Therefore $\bigcap\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}$.
2. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Let $\alpha \in A$. Since $\mathcal{D} \subset \mathcal{D}_\alpha$, we have that $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}_\alpha$. Since $\mathcal{D}_\alpha$ is a δ-system and $(A_n)_{n\in\mathbb{N}}$ is disjoint, we have that $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}_\alpha$. Since $\alpha \in A$ is arbitrary, we have that for each $\alpha \in A$, $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}_\alpha$. Therefore $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

Thus $\mathcal{D}$ is a δ-system. □

Definition 4.3.0.3. Let X be a set and $\mathcal{C} \subset \mathcal{P}(X)$. Set

$$\mathcal{S} := \{\mathcal{D} \subset \mathcal{P}(X) : \mathcal{D} \text{ is a } \delta\text{-system on } X \text{ and } \mathcal{C} \subset \mathcal{L}\}$$

We define the **δ-system generated by** $\mathcal{C}$ on X, denoted $\delta_X(\mathcal{C})$, by

$$\delta(\mathcal{C}) = \bigcap_{\mathcal{D}\in\mathcal{S}} \mathcal{A}$$

Note 4.3.0.4. Let X be a set, $\mathcal{C} \subset \mathcal{P}(X)$ and $\mathcal{D}$ a δ-system on X. By definition, if $\mathcal{C} \subset \mathcal{D}$, then $\delta(\mathcal{C}) \subset \mathcal{D}$.

Exercise 4.3.0.5. Let $(X, \mathcal{T})$ be a topological space. Define $\mathcal{D} := \{B \in \delta(\mathcal{T}) : B^c \in \delta(\mathcal{T})\}$. Then

1. $\mathcal{D}$ is a δ-system on X.
2. $\mathcal{D} \neq \varnothing$ implies that $\mathcal{D}$ a σ-algebra on X.

Proof.

1.
   (a) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D}$. By definition, $(A_n^c)_{n\in\mathbb{N}} \subset \mathcal{D}$. Since $\mathcal{D} \subset \delta(\mathcal{T})$, we have that $(A_n)_{n\in\mathbb{N}} \subset \delta(\mathcal{T})$ and $(A_n^c)_{n\in\mathbb{N}} \subset \delta(\mathcal{T})$.
   - Since $(A_n)_{n\in\mathbb{N}} \subset \delta(\mathcal{T})$ and $\delta(\mathcal{T})$ is a δ-system, we have that $\bigcap_{n\in\mathbb{N}} A_n \in \delta(\mathcal{T})$.
   - Define $(B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ by

     $$B_n := \begin{cases} A_1^c, & n = 1 \\ A_n^c \setminus \left(\bigcup\limits_{k=1}^{n-1} A_n^c\right), & n \geq 2 \end{cases}$$

     Since $\delta(\mathcal{T})$ is a δ-system, for each $n \geq 2$,

     $$\begin{aligned} B_n &= A_n^c \cap \left(\bigcap_{k=1}^{n-1} A_n\right) \\ &\in \delta(\mathcal{T}) \end{aligned}$$

Since $A_1^c \in \delta(\mathcal{T})$, $B_1 \in \delta(\mathcal{T})$. Hence $(B_n)_{n\in\mathbb{N}} \subset \delta(\mathcal{T})$. Since $(B_n)_{n\in\mathbb{N}}$ is disjoint and $\delta(\mathcal{T})$ is a δ-system, we have that

$$\begin{aligned}\left(\bigcap_{n\in\mathbb{N}} A_n\right)^c &= \bigcup_{n\in\mathbb{N}} A_n^c \\ &= \bigcup_{n\in\mathbb{N}} B_n \\ &\in \delta(\mathcal{T})\end{aligned}$$

Since $\bigcap_{n\in\mathbb{N}} A_n \in \delta(\mathcal{T})$ and $\left(\bigcap\limits_{n\in\mathbb{N}} A_n\right)^c \in \delta(\mathcal{T})$, we have that $\bigcap\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

(b)
- Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Since $(A_n)_{n\in\mathbb{N}} \subset \delta(\mathcal{T})$ and $\delta(\mathcal{T})$ is a δ-system, we have that $\bigcup\limits_{n\in\mathbb{N}} A_n \in \delta(\mathcal{T})$.
- Since $(A_n^c)_{n\in\mathbb{N}} \subset \delta(\mathcal{T})$ and $\delta(\mathcal{T})$ is a δ-system, we have that

$$\begin{aligned}\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c &= \bigcap_{n\in\mathbb{N}} A_n^c \\ &\in \delta(\mathcal{T}).\end{aligned}$$

Since $\bigcup\limits_{n\in\mathbb{N}} A_n \in \delta(\mathcal{T})$ and $\left(\bigcup\limits_{n\in\mathbb{N}} A_n\right)^c \in \delta(\mathcal{T})$, we have that $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

Thus $\mathcal{D}$ is a δ-system on X.

2. Suppose that $\mathcal{D} \neq \varnothing$.

(a) By assumption $\mathcal{D} \neq \varnothing$.

(b) Let $A \in \mathcal{D}$. By definition, $A^c \in \mathcal{D}$. Since $A \in \mathcal{D}$ is arbitrary, we have that for each $A \in \mathcal{D}$, $A^c \in \mathcal{D}$.

(c) Let $(A_n)_{n\in\mathbb{N}} \in \mathcal{D}$. By definition, $(A_n^c)_{n\in\mathbb{N}} \subset \mathcal{D}$. Since $\mathcal{D}$ is a δ-system on X, we have that $\bigcap\limits_{n\in\mathbb{N}} A_n^c \in \mathcal{D}$. Therefore the previous part implies that

$$\begin{aligned}\bigcup_{n\in\mathbb{N}} A_n &= \left(\bigcap_{n\in\mathbb{N}} A_n^c\right)^c \\ &\in \mathcal{D}.\end{aligned}$$

Since $(A_n)_{n\in\mathbb{N}} \in \mathcal{D}$ is arbitrary, we have that for each $(A_n)_{n\in\mathbb{N}} \in \mathcal{D}$, $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{D}$.

Hence $\mathcal{D}$ is a σ-algebra on X.

$\square$

Exercise 4.3.0.6. Let $(X, \mathcal{T})$ be a topological space. Then

1. $\delta(\mathcal{T}) \subset \mathcal{B}(X)$,
2. If $(X, \mathcal{T})$ is metrizable, then $\delta(\mathcal{T}) = \mathcal{B}(X)$.
   Hint: Define $\mathcal{D}$ as in Exercise 4.3.0.5, then $\mathcal{T} \subset \mathcal{D}$.

Proof.

1. Since $\mathcal{B}(X)$ is a δ-system and $\mathcal{T} \subset \mathcal{B}(X)$, we have that $\delta(\mathcal{T}) \subset \mathcal{B}(X)$.

2. Suppose that $(X, \mathcal{T})$ is metrizable. Define $\mathcal{D} := \{B \in \delta(\mathcal{T}) : B^c \in \delta(\mathcal{T})\}$ as in Exercise 4.3.0.5. By definition, $\mathcal{D} \subset \delta(\mathcal{T})$ and $\mathcal{T} \subset \delta(\mathcal{T})$. Let $U \in \mathcal{T}$. Then U^c is closed in X. Since $(X, \mathcal{T})$ is metrizable, an exercise in the analysis notes chapter on metric spaces implies that U^c is a G_δ-set. Thus there exists $(U_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ such that $U^c = \bigcap_{n\in\mathbb{N}} U_n$. Since
$$\begin{aligned}(U_n)_{n\in\mathbb{N}} &\subset \mathcal{T}\\ &\subset \delta(\mathcal{T})\end{aligned}$$
and $\delta(\mathcal{T})$ is a δ-system, we have that
$$\begin{aligned}U^c &= \bigcap_{n\in\mathbb{N}} U_n\\ &\in \delta(\mathcal{T}).\end{aligned}$$
Since $U \in \delta(\mathcal{T})$ and $U^c \in \delta(\mathcal{T})$, we have that $U \in \mathcal{D}$. Since $U \in \mathcal{T}$ is arbitrary, we have that $\mathcal{T} \subset \mathcal{D}$. Exercise 4.3.0.5 implies that $\mathcal{D}$ is a δ-system. Therefore $\delta(\mathcal{T}) \subset \mathcal{D}$. Hence $\delta(\mathcal{T}) = \mathcal{D}$. Exercise 4.3.0.5 implies that $\mathcal{D}$ is a σ-algebra on X. Since $\mathcal{T} \subset \mathcal{D}$, we have that
$$\begin{aligned}\mathcal{B}(X) &\subset \mathcal{D}\\ &= \delta(\mathcal{T}).\end{aligned}$$
Hence $\delta(\mathcal{T}) = \mathcal{B}(X)$.

□

Exercise 4.3.0.7. Let X be a Polish space, $(Z_n)_{n\in\mathbb{N}}$ a collection of Polish spaces, $(B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ and $(f_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} B_n^{Z_n}$.

1. Suppose that for each $n \in \mathbb{N}$, f_n is continuous.
   (a) Define Z_0 and $f_0 : Z_0 \to X^{\mathbb{N}}$ by $Z_0 := \prod_{n\in\mathbb{N}} Z_n$ and $f_0 := \prod_{n\in\mathbb{N}} f_n$. Then f_0 is continuous.
   (b) Define $\Delta_{X^{\mathbb{N}}} \subset X^{\mathbb{N}}$ and $Z \subset Z_0$ by $\Delta_{X^{\mathbb{N}}} := \{x \in X^{\mathbb{N}} :$ for each $m, n \in \mathbb{N}$, $\pi_m(x) = \pi_n(x)\}$ and $Z := f_0^{-1}(\Delta_{X^{\mathbb{N}}})$. Then for each $m, n \in \mathbb{N}$, $f_m \circ \pi_m|_Z = f_n \circ \pi_n|_Z$.
   (c) For each $n \in \mathbb{N}$, $f_n \circ \pi_n(Z) = \bigcap_{n\in\mathbb{N}} B_n$.
   (d) Set $B := \bigcap_{n\in\mathbb{N}} B_n$ and define $f : Z \to B$ by $f := f_1 \circ \pi_1|_Z$. Then f is continous
2. If for each $n \in \mathbb{N}$, f_n is surjective, then f is surjective.
3. If for each $n \in \mathbb{N}$, f_n is injective, then f is injective.
4. If for each $n \in \mathbb{N}$, f_n is a measurable isomorphism, then f is a measurable isomorphism.
5. We have that $\Delta_{X^{\mathbb{N}}}$, Z_0 and Z are Polish spaces.
6. If for each $n \in \mathbb{N}$, Z_n is a zero-dimensional, then Z is zero-dimensional

Proof.

1. (a) Since for each $n \in \mathbb{N}$, $f_n : Z_n \to B_n$ is continuous, an exercise in the analysis notes section on the product topology implies that f_0 is continuous.
   (b) An exercise in the analysis notes section on set theory implies that for each $m, n \in \mathbb{N}$, $f_m \circ \pi_m|_Z = f_n \circ \pi_n|_Z$.
   (c) An exercise in the analysis notes section on set theory implies that for each $n \in \mathbb{N}$, $f_n \circ \pi_n(Z) = \bigcap_{n\in\mathbb{N}} B_n$.
   (d) Since π_m is continuous, $\pi_m|_Z$ is continuous. Since f_1 is continuous, $f_1 \circ \pi_1|_Z$ is continous. Thus f is continuous.

2. Suppose that for each $n \in \mathbb{N}$, f_n is surjective. Since $f = f_1 \circ \pi_1|_Z$, the previous part implies that

$$\begin{aligned} f(Z) &= f_1 \circ \pi_1|_Z(Z) \\ &= f_1 \circ \pi_1(Z) \\ &= B \end{aligned}$$

Hence f is surjective.

3. Suppose that for each $n \in \mathbb{N}$, f_n is injective. Let $x, y \in Z$. Suppose that $f(x) = f(y)$. Let $n \in \mathbb{N}$. Then the previous part implies that

$$\begin{aligned} f_n(\pi_n(x)) &= f_1(\pi_1(x)) \\ &= f(x) \\ &= f(y) \\ &= f_1(\pi_1(y)) \\ &= f_n(\pi_n(y)). \end{aligned}$$

Since f_n is injective, we have that $\pi_n(x) = \pi_n(x)$. Since $n \in \mathbb{N}$ is arbitrary, we have that for each $n \in \mathbb{N}$, $\pi_n(x) = \pi_n(y)$. Hence $x = y$. Since $x, y \in Z$ are arbitrary, we have that for each $x, y \in Z$, $f(x) = f(y)$ implies that $x = y$. Hence f is injective.

4. Suppose that for each $n \in \mathbb{N}$, f_n is a measurable isomorphism. The previous parts implies that f is a bijection. Define $g : B \to Z$ by $g := (f_n^{-1}|_B)_{n \in \mathbb{N}}$. Then for each $x \in B$, $g(x)_m := f_m^{-1}(x)$. Therefore for each $x \in B$,

$$\begin{aligned} f \circ g(x) &= f_1 \circ \pi_1(g(x)) \\ &= f_m \circ \pi_m(g(x)) \\ &= f_m(f_m^{-1}(x)) \\ &= x, \end{aligned}$$

so that $f \circ g = \mathrm{id}_B$. We note that for each $z \in Z$ and $m \in \mathbb{N}$,

$$\begin{aligned} &\pi_m \circ g \circ f(z) \\ &\qquad = \pi_m \circ g \circ (f_1 \circ \pi_1)(z) \\ &\qquad = \pi_m \circ g \circ (f_m \circ \pi_m)(z) \\ &\qquad = g(f_m(\pi_m(z)))_m \\ &\qquad = f_m^{-1}(f_m(\pi_m(z))) \\ &\qquad = \pi_m(z), \end{aligned}$$

so that $g \circ f = \mathrm{id}_Z$. Hence $f^{-1} = g$.

   - Since f is continuous, f is measurable.
   - Since for each $n \in \mathbb{N}$, f_n^{-1} is measurable, we have that for each $n \in \mathbb{N}$, $f_n^{-1}|_B$ is measurable. Exercise 3.5.0.7 then implies that f^{-1} is measurable. Hence f is a measurable isomorphism.

5. Since X is a Polish space, $X^{\mathbb{N}}$ is a Polish space. Since X is a Polish space, X is Hausdorff. An exercise in the analysis notes section on separation and topology and an exercise in the analysis notes section on Polish spaces imply that $\Delta_{X^{\mathbb{N}}}$ is closed in $X^{\mathbb{N}}$ and $\Delta_{X^{\mathbb{N}}}$ is a Polish space. Since for each $n \in \mathbb{N}$, Z_n is a Polish space, an exercise in the analysis notes section on Polish spaces implies that Z_0 is a Polish space. Since f_0 is continuous and $\Delta_{X^{\mathbb{N}}}$ is closed in $X^{\mathbb{N}}$, we have that Z is closed in Z_0. Since Z_0 is a Polish space and Z is closed in Z_0, an exercise in the analysis notes section on Polish spaces implies that Z is a Polish space.

6. Suppose that for each $n \in \mathbb{N}$, Z_n is zero-dimensional. An exercise in the analysis notes topology section on products of zero dimensional spaces implies that Z_0 is zero-dimensional. Since $Z \subset Z_0$, An exercise in the analysis notes topology section on subsapces of zero dimensional spaces implies that Z is zero-dimensional.

□

Exercise 4.3.0.8. Let X be a Polish space, $(Z_n)_{n\in\mathbb{N}}$ a collection of Polish spaces, $(B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ and $(f_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} B_n^{Z_n}$. Suppose that $(B_n)_{n\in\mathbb{N}}$ is disjoint.

1. If for each $n \in \mathbb{N}$, f_n is continuous, then there exists $f : \coprod_{n\in\mathbb{N}} Z_n \to \bigcup_{n\in\mathbb{N}} B_n$ such that f is continuous.
2. If for each $n \in \mathbb{N}$, f_n is surjective, then f is surjective.
3. If for each $n \in \mathbb{N}$, f_n is injective, then f is injective.
4. If for each $n \in \mathbb{N}$, f_n is a measurable isomorphism, then f is a measurable isomorphism.
5. We have that $\coprod_{n\in\mathbb{N}} Z_n$ is a Polish space.
6. If for each $n \in \mathbb{N}$, Z_n is zero-dimensional, then $\coprod_{n\in\mathbb{N}} Z_n$ is zero-dimensional.

Proof. Define $f_0 : \coprod_{n\in\mathbb{N}} Z_n \to \coprod_{n\in\mathbb{N}} B_n$ by $f_0 := \coprod_{n\in\mathbb{N}} f_n$.

1.
   - Since for each $n \in \mathbb{N}$, f_n is continuous, an exercise in the analysis notes section on coproducts implies that f_0 is continuous.
   - Define $\phi : \coprod_{n\in\mathbb{N}} B_n \to \bigcup_{n\in\mathbb{N}} B_n$ by $\phi(n, x) := x$. An exercise in the analysis notes section on coproduct topologies implies that ϕ is a homeomorphism.
   - Define $f : \coprod_{n\in\mathbb{N}} Z_n \to \bigcup_{n\in\mathbb{N}} B_n$ by $f := \phi \circ f_0$. Since ϕ and f_0 are continuous, f is continuous.
2. Suppose that for each $n \in \mathbb{N}$, f_n is surjective. Let $x \in \bigcup_{n\in\mathbb{N}} B_n$. Then there exists $n \in \mathbb{N}$ such that $x \in B_n$. Since f_n is surjective, there exists $z_n \in Z_n$ such that $f_n(z_n) = x$. Define $z \in \coprod_{n\in\mathbb{N}} Z_n$ by $z := (n, z_n)$. Then
$$\begin{aligned} f(z) &= \phi \circ f_0(n, z_n) \\ &= \phi(n, f_n(z_n)) \\ &= \phi(n, x) \\ &= x. \end{aligned}$$
   Since $x \in \bigcup_{n\in\mathbb{N}} B_n$ is arbitrary, we have that for each $x \in \bigcup_{n\in\mathbb{N}} B_n$, there exists $z \in \coprod_{n\in\mathbb{N}} Z_n$ such that $f(z) = x$. Hence f is surjective.
3. Suppose that for each $n \in \mathbb{N}$, f_n is injective. Let $x, y \in \coprod_{n\in\mathbb{N}} Z_n$. Suppose that $f(x) = f(y)$. By definition, there exists $n, m \in \mathbb{N}$, $z_x \in Z_n$ and $z_y \in Z_m$ such that $x = (n, z_x)$ and $y = (m, z_y)$. Then
$$\begin{aligned} \phi(n, f_n(z_x)) &= \phi \circ f_0(n, z_x) \\ &= f(x) \\ &= f(y) \\ &= \phi \circ f_0(m, z_y) \\ &= \phi(m, f_m(z_y)). \end{aligned}$$
   Since ϕ is a bijection, $(n, f_n(z_x)) = (m, f_m(z_y))$. Thus $n = m$ and therefore $f_n(z_x) = f_n(z_y)$. Since f_n is injective, $z_x = z_y$. Thus
$$\begin{aligned} x &= (n, z_x) \\ &= (n, z_y) \\ &= y. \end{aligned}$$

Since $x, y \in \coprod_{n\in\mathbb{N}} Z_n$ are arbitrary, we have that for each $x, y \in \coprod_{n\in\mathbb{N}} Z_n$, $f(x) = f(y)$ implies that $x = y$. Thus f is injective.

4. Suppose that for each $n \in \mathbb{N}$, f_n is a measurable isomorphism. Since $f_0^{-1} = \coprod_{n\in\mathbb{N}} f_n^{-1}$, Exercise 3.6.0.6 implies that f_0 is a measurable isomorphism. Since ϕ is a homeomorphism, ϕ is a measurable isomorphism. Since ϕ and f_0 are measurable isomorphisms, f is a measurable isomorphism.

5. an exercise in the analysis notes topology section on Polish spaces implies that $\coprod_{n\in\mathbb{N}} Z_n$ is a Polish space.

6. Since for each $n \in \mathbb{N}$, Z_n is zero-dimensional, an exercise in the analysis notes topology section on zero dimensional spaces implies that $\coprod_{n\in\mathbb{N}} Z_n$ is zero-dimensional.

□

Exercise 4.3.0.9. Let X be a Polish space. Then

1. there exists a Polish space Z and $f : Z \to X$ such that Z is zero-dimensional, f is a continuous and f is a measurable isomorphism.
   Hint: Exercise 4.2.0.2

2. for each $B \in \mathcal{B}(X)$, there exists a Polish space Z and $f : Z \to B$ such that Z is zero-dimensional, f is continuous and f is a measurable isomorphism.
   Hint: Define $\mathcal{D} := \{B \in \mathcal{B}(X, \mathcal{T}_X) :$ there exists a Polish space Z and $f : Z \to B$ such that Z is zero-dimensional, f is continuous and f is a measurable isomorphism$\}$ and consider Exercise 4.3.0.7, Exercise 4.3.0.8 and Exercise 4.3.0.6

Proof.

1. Exercise 4.2.0.2 implies that there exists $E \in \mathcal{B}(\mathcal{C})$ and $\phi : E \to [0, 1]$ such that E is a zero-dimensional Polish space and ϕ is a continuous bijection. Exercise 4.2.0.8 implies that ϕ is a measurable isomorphism. Since ϕ is continuous, an exercise in analysis notes section on product topology implies that $\phi^{\mathbb{N}}$ is continuous and ϕ is a measurable isomorphism, Exercise 3.5.0.9 implies that $\phi^{\mathbb{N}}$ is a measurable isomorphism. Define $Z \subset E^{\mathbb{N}}$ by $Z := (\phi^{\mathbb{N}})^{-1}(B)$. Since $\phi^{\mathbb{N}}$ is continuous, an exercise in the analysis notes section on subspace topology implies that $\phi^{\mathbb{N}}|_Z^B$ is continuous. Since $\phi^{\mathbb{N}}$ is a measurable isomorphism, Exercise 3.4.0.10 and Exercise 3.4.0.11 imply that $\phi^{\mathbb{N}}|_Z^B$ is a measurable isomorphism. An exercise in the analysis notes section on Polish spaces implies that there exists $B \subset [0, 1]^{\mathbb{N}}$ and $h : X \to B$ such that B is a G_δ-set and h is a homeomorphism. Thus h^{-1} is continuous and h^{-1} is a measurable isomorphism. Define $f : Z \to X$ by $f := h^{-1} \circ \phi^{\mathbb{N}}|_Z^B$. Then f is continuous and f is a measurable isomorphism. Since B is a G_δ-set and $\phi^{\mathbb{N}}$ is continuous, we have that Z is a G_δ-set. Since E is a Polish space, $E^{\mathbb{N}}$ is a Polish space. Since $Z \subset E^{\mathbb{N}}$ and Z is a G_δ-set, an exercise in the analysis notes section on Polish spaces implies that Z is Polish space. Since E is zero-dimensional, $E^{\mathbb{N}}$ is zero-dimensional. Since $Z \subset E^{\mathbb{N}}$, we have that Z is zero-dimensional.

2. Denote the topology on X by $\mathcal{T}_X$.
   Define $\mathcal{D} := \{B \in \mathcal{B}(X, \mathcal{T}_X) :$ there exists a Polish space Z and $f : Z \to B$ such that Z is zero-dimensional, f is continuous and f is a measurable isomorphism$\}$.

   - Let $U \in \mathcal{T}_X$. An exercise in the analysis notes on polish spaces implies that U is a Polish space. The previous part implies that there exists a Polish space Z and $f : Z \to U$ such that Z is zero-dimensional, f is continuous and f is a measurable isomorphism. Since $U \in \mathcal{T}_X$ is arbitrary, we have that $\mathcal{T}_X \subset \mathcal{D}$.
   - Let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{D}$. Then $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ and for each $n \in \mathbb{N}$, there exists a Polish space Z_n and $f_n : Z_n \to B_n$ such that Z_n is zero-dimensional, f_n is continuous and f_n is a measurable isomorphism. Define $Z \subset \prod_{n\in\mathbb{N}} Z_n$ by $Z := (\prod_{n\in\mathbb{N}} f_n)^{-1}(\Delta_{X^{\mathbb{N}}})$. Then Exercise 4.3.0.7 implies that Z is a Polish space, Z is zero-dimensional and there exists $f : Z \to \bigcap_{n\in\mathbb{N}} B_n$ such that f is continuous and f is a measurable isomorphism. Thus $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{D}$. Since $(B_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ is arbitrary, we have that for each $(B_n)_{n\in\mathbb{N}} \subset \mathcal{D}$, $\bigcap_{n\in\mathbb{N}} B_n \in \mathcal{D}$.

- Let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{D}$. Suppose that $(B_n)_{n\in\mathbb{N}}$ is disjoint. Set $Z := \coprod_{n\in\mathbb{N}} Z_n$. Exercise 4.3.0.8 implies that Z is a Polish space, Z is zero-dimensional and there exists $f : Z \to \bigcup_{n\in\mathbb{N}} B_n$ such that f is continuous and f is a measurable isomorphism. Thus $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{D}$. Since $(B_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ such that $(B_n)_{n\in\mathbb{N}}$ is disjoint is arbitrary, we have that for each $(B_n)_{n\in\mathbb{N}} \subset \mathcal{D}$, if $(B_n)_{n\in\mathbb{N}}$ is disjoint, then $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{D}$.

Thus $\mathcal{D}$ is a δ-system on X. Since X is a Polish space, X metrizable. Since $\mathcal{T}_X \subset \mathcal{D}$, Exercise 4.3.0.6 implies that

$$\begin{aligned}\mathcal{D} &\subset \mathcal{B}(X, \mathcal{T}_X)\\ &= \delta(\mathcal{T}_X)\\ &\subset \delta(\mathcal{D})\\ &= \mathcal{D}\end{aligned}$$

Hence $\mathcal{D} = \mathcal{B}(X, \mathcal{T}_X)$.

□

Definition 4.3.0.10. Let X be a Polish space and $A \subset X$. Then A is said to be **analytic** if there exists a Polish space Z and $f : Z \to A$ such that f is continuous and surjective.

Exercise 4.3.0.11. Let X be a Polish space and $(A_n) \subset \mathcal{P}(X)$. If for each $n \in \mathbb{N}$, A_n is analytic, then

1. $\bigcap_{n\in\mathbb{N}} A_n$ is analytic,
2. $\bigcup_{n\in\mathbb{N}} A_n$ is analytic,
3. $\prod_{n\in\mathbb{N}} A_n$ is analytic.

Proof. Suppose that for each $n \in \mathbb{N}$, A_n is analytic. Then for each $n \in \mathbb{N}$, there exists a Polish space Z_n and $f_n : Z_n \to A_n$ such that f_n is continuous and surjective.

1. Exercise 4.3.0.7 implies that there exists a Polish space Z and $f : Z \to \bigcap_{n\in\mathbb{N}} A_n$ such that f is continuous and surjective. Thus $\bigcap_{n\in\mathbb{N}} A_n$ is analytic.
2. Exercise 4.3.0.8 implies that there exists a Polish space Z and $f : Z \to \coprod_{n\in\mathbb{N}} A_n$ such that f is continuous and surjective. Define $\phi : \coprod_{n\in\mathbb{N}} A_n \to \bigcup_{n\in\mathbb{N}} A_n$ by $\phi(n, x) := x$. An excercise in the analysis notes section on coproduct topology implies that ϕ is a homeomorphism. Define $g : Z \to \bigcup_{n\in\mathbb{N}} A_n$ by $g := \phi \circ f$. Since ϕ and f are continuous, g is continuous. Since ϕ and f are surjective, g is surjective. Thus $\bigcup_{n\in\mathbb{N}} A_n$ is analytic.
3. Define $f : \prod_{n\in\mathbb{N}} Z_n \to \prod_{n\in\mathbb{N}} A_n$ by $f := \prod_{n\in\mathbb{N}} f_n$. Since for each $n \in \mathbb{N}$, Z_n is a Polish space, an exercise in the analysis notes section on Polish spaces implies that $\prod_{n\in\mathbb{N}} Z_n$ is a Polish space. Since for each $n \in \mathbb{N}$, f_n is continuous, an exercise in the analysis notes section on product topology implies that f is continuous. Since for each $n \in \mathbb{N}$, f_n is surjective, $\prod_{n\in\mathbb{N}} f_n : \prod_{n\in\mathbb{N}} Z_n \to \prod_{n\in\mathbb{N}} A_n$ is surjective (maybe reference exercise)

□

Exercise 4.3.0.12. Let X be a Polish space. Then for each $B \in \mathcal{B}(X)$, B is analytic.

Proof. Clear by Exercise 4.3.0.9. □

Definition 4.3.0.13.

- for $n \in \mathbb{N}$ and $\gamma \in \mathbb{N}^n$, we define the **length** of γ, denoted $|\gamma|$, by $|\gamma| := n$,
- for $n \in \mathbb{N}$, we define the **projection of** $\mathbb{N}^{n+1}$ **onto** $\mathbb{N}^n$, denoted $\pi^{[n+1]}_{[n]} : \mathbb{N}^{n+1} \to \mathbb{N}^n$, by $\pi^{[n+1]}_{[n]}(\gamma_1, \dots, \gamma_{n+1}) := (\gamma_1, \dots, \gamma_n)$,
- for $n \in \mathbb{N}$, we define the **projection of** $\mathcal{N}$ **onto** $\mathbb{N}^n$, denoted $\pi_{[n]} : \mathcal{N} \to \mathbb{N}^n$, by $\pi_{[n]}(\gamma) := (\gamma_1, \dots, \gamma_n)$.
  maybe put this definition inside the following exercise, wait to see if it is needed elsewhere first

Exercise 4.3.0.14. Let X be a Polish space. Suppose that $X \neq \varnothing$. Then there exists $f : \mathcal{N} \to X$ such that f is continuous and surjective.
Hint: Define $\Gamma := \{\mathbb{N}^n : n \in \mathbb{N}\}$. An exercise in the analysis notes section on countability of metric spaces implies that there exists $(C_\gamma)_{\gamma \in \Gamma} \subset \mathcal{P}(X)$ such that

1. for each $\gamma \in \Gamma$,
   (a) C_γ is closed and $C_\gamma \neq \varnothing$,
   (b) $\operatorname{diam} C_\gamma \leq |\gamma|^{-1}$,
   (c) for each $n \in \mathbb{N}$, $|\gamma| = n$ implies that $C_\gamma = \bigcup\limits_{\gamma' \in (\pi^{[n+1]}_{[n]})^{-1}(\{\gamma\})} C_{\gamma'}$
2. $X = \bigcup\limits_{|\gamma|=1} C_\gamma$

and use Cantor's nested set theorem.

Proof.

- Since X is a Polish space, there exists a metric d on X such that (X, d) is a complete separable metric space. Define $\Gamma := \{\mathbb{N}^n : n \in \mathbb{N}\}$. An exercise in the analysis notes section on countability of metric spaces implies that there exists $(C_\gamma)_{\gamma \in \Gamma} \subset \mathcal{P}(X)$ such that
  1. for each $\gamma \in \Gamma$,
     (a) C_γ is closed and $C_\gamma \neq \varnothing$,
     (b) $\operatorname{diam} C_\gamma \leq |\gamma|^{-1}$,
     (c) for each $n \in \mathbb{N}$, $|\gamma| = n$ implies that $C_\gamma = \bigcup\limits_{\gamma' \in (\pi^{[n+1]}_{[n]})^{-1}(\{\gamma\})} C_{\gamma'}$
  2. $X = \bigcup\limits_{|\gamma|=1} C_\gamma$.

  We note that Cantor's nested set theorem in the analysis notes section on completeness implies that for each $\omega \in \mathcal{N}$, that there exists $x \in X$ such that $\bigcap\limits_{n \in \mathbb{N}} C_{\pi_{[n]}(\omega)} = \{x\}$. The axiom of choice implies that there exists $f : \mathcal{N} \to X$ such that for each $\omega \in \mathcal{N}$, $\bigcap\limits_{n \in \mathbb{N}} C_{\pi_{[n]}(\omega)} = \{f(\omega)\}$.

- Let $(\omega_k)_{k \in \mathbb{N}} \subset \mathcal{N}$ and $\omega \in \mathcal{N}$. Suppose that $\omega_k \to \omega$. Then for each $n \in \mathbb{N}$, $\pi_{[n]}(\omega_k) \to \pi_{[n]}(\omega)$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N^{-1} < \epsilon$. We note that

$$\begin{aligned} f(\omega) &\in \bigcap_{n \in \mathbb{N}} C_{\pi_{[n]}(\omega)} \\ &\subset C_{\pi_{[N]}(\omega)}. \end{aligned}$$

  Since $\pi_{[N]}(\omega_k) \to \pi_{[N]}(\omega)$ and $\mathbb{N}^N$ is discrete, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq K$ implies that $\pi_{[N]}(\omega_k) = \pi_{[N]}(\omega)$. Let $k \in \mathbb{N}$. Suppose that $k \geq K$. Then

$$\begin{aligned} f(\omega_k) &\in \bigcap_{n \in \mathbb{N}} C_{\pi_{[n]}(\omega_k)} \\ &\subset C_{\pi_{[N]}(\omega_k)} \\ &= C_{\pi_{[N]}(\omega)}. \end{aligned}$$

and therefore

$$\begin{aligned} d(f(\omega_k), f(\omega)) &\leq \operatorname{diam} C_{\pi_{[N]}(\omega)} \\ &\leq N^{-1} \\ &< \epsilon. \end{aligned}$$

Since $k \in \mathbb{N}$ with $k \geq K$ is arbitrary, we have that for each $k \in \mathbb{N}$, $k \geq K$ implies that $d(f(\omega_k), f(\omega)) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq K$ implies that $d(f(\omega_k), f(\omega)) < \epsilon$. Hence $d(f(\omega_k), f(\omega)) \to 0$ and $f(\omega_k) \to f(\omega)$. Since $(\omega_k)_{k\in\mathbb{N}} \subset \mathcal{N}$ and $\omega \in \mathcal{N}$ are arbitrary, we have that for each $(\omega_k)_{k\in\mathbb{N}} \subset \mathcal{N}$ and $\omega \in \mathcal{N}$, $\omega_k \to \omega$ implies that $f(\omega_k) \to f(\omega)$. Thus f is continuous.

- Let $x \in X$. By construction, there exists $\gamma \in \Gamma$ such that $|\gamma| = 1$ and $x \in C_\gamma$. Continuing inductively, for each $n \geq 2$, there exists $\gamma' \in (\pi^{[n+1]}_{[n]})^{-1}(\{\gamma\})$ such that $x \in C_{\gamma'}$. The axiom of choice implies that there exists $(\gamma_n)_{n\in\mathbb{N}} \subset \Gamma$ such that for each $n \in \mathbb{N}$, $|\gamma_n| = n$, $\pi^{[n+1]}_{[n]}(\gamma_{n+1}) = \gamma_n$ and $x \in C_{\gamma_n}$. We define $\omega \in \mathcal{N}$ by $\omega_n := \pi_n(\gamma_n)$. By construction, for each $n \in \mathbb{N}$, $\pi_{[n]}(\omega) = \gamma_n$ and $x \in C_{\pi_{[n]}(\omega)}$. Since $\{f(\omega)\} = \bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega)}$ and $x \in \bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega)}$, we have that $f(\omega) = x$. Since $x \in X$ is arbitrary, we have that for each $x \in X$, there exists $\omega \in \mathcal{N}$ such that $f(\omega) = x$. Hence f is surjective.

□

Definition 4.3.0.15. Let X be a polish space and $A \subset X$. Then A is said to be **z-analytic** if there exists a Polish space Z and $f : Z \to A$ such that Z is zero-dimensional and f is continuous and f is a measurable isomorphism. (this definition is not standard and should probably be changed)

Exercise 4.3.0.16. Let X be a Polish space and $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$. If for each $n \in \mathbb{N}$, A_n is z-analytic, then

1. $\bigcap_{n\in\mathbb{N}} A_n$ is z-analytic,
2. $(A_n)_{n\in\mathbb{N}}$ is disjoint implies that $\bigcup_{n\in\mathbb{N}} A_n$ is z-analytic,
3. $\prod_{n\in\mathbb{N}} A_n$ is z-analytic.

Proof. Suppose that for each $n \in \mathbb{N}$, A_n is z-analytic. Then for each $n \in \mathbb{N}$, there exists a Polish space Z_n and $f_n : Z_n \to A_n$ such that Z_n is zero-dimensional and f_n is a continuous bijection.

1. Exercise 4.3.0.7 implies that there exists a Polish space Z and $f : Z \to \bigcap_{n\in\mathbb{N}} A_n$ such that Z is zero-dimensional and f is a continuous bijection. Hence $\bigcap_{n\in\mathbb{N}} A_n$ is z-analytic.
2. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Exercise 4.3.0.8 implies that there exists $f : \coprod_{n\in\mathbb{N}} Z_n \to \bigcup_{n\in\mathbb{N}} A_n$ such that f is a continuous bijection. An exercise in the analysis notes section on zero-dimensional topological spaces implies that $\coprod_{n\in\mathbb{N}} Z_n$ is zero-dimensional. Hence $\bigcup_{n\in\mathbb{N}} A_n$ is z-analytic.
3. FINISH!!!

□

Exercise 4.3.0.17. Let X be a Polish space. Then for each $B \in \mathcal{B}(X)$, B is z-analytic.

Proof. Clear by Exercise 4.3.0.19. □

Exercise 4.3.0.18. Let Z be a Polish space. Suppose that Z is zero-dimensional and Z is uncountable. Then there exists $f : \mathcal{N} \to Z$ such that

1. f is continuous
2. f is injective

3. $f(\mathcal{N})^c$ is countable.

Hint: Define $\Gamma := \bigcup\limits_{n\in\mathbb{N}} N^n$ and $Z_0 := \{z \in Z : z \text{ is a condensation point of } Z\}$. An exercise in the analysis notes section on countability of metric spaces implies that there exists $(C_\gamma)_{\gamma\in\Gamma} \subset \mathcal{T}_d$ and $(z_\gamma)_{\gamma\in\Gamma} \in \prod\limits_{\gamma\in\Gamma} C_\gamma$ such that

1. • there exists $z_0 \in Z_0$ such that $Z_0 \setminus \{z_0\} = \bigcup\limits_{|\gamma|=1} C_\gamma$,
   • $(C_\gamma)_{\gamma\in\mathbb{N}}$ is disjoint
2. for each $\gamma \in \Gamma$,
   (a) C_γ is closed and $C_\gamma \neq \varnothing$,
   (b) $\operatorname{diam} C_\gamma \leq |\gamma|^{-1}$,
   (c) for each $n \in \mathbb{N}$, $|\gamma| = n$ implies that
      • $C_\gamma \setminus \{z_\gamma\} = \bigcup\limits_{\gamma' \in (\pi^{[n+1]}_{[n]})^{-1}(\{\gamma\})} C_{\gamma'}$
      • $(C_{\gamma'})_{\gamma' \in (\pi^{[n+1]}_{[n]})^{-1}(\{\gamma\})}$ is disjoint

(possibly reformat exercise from analysis notes to say the same more efficiently, maybe index by Γ including length 0 indices?)and use Cantor's nested set theorem.

Proof. Define $Z_0 \subset Z$ by $Z_0 := \{z \in Z : z \text{ is a condensation point of } Z\}$. Since Z is Polish and zero-dimensional, An exercise in the analysis notes metric space chapter in section on zero dimensional polish spaces implies that Z_0 is Polish, Z_0 is zero dimensional and for each $z \in Z_0$, z is a condensation point of Z_0. Since Z is Polish, Z is second-countable and An exercise in the analysis notes topology section on second countability implies that Z_0^c is countable. Since Z is uncountable, we have that Z_0 is uncountable. In particular, $Z_0 \neq \varnothing$. Since Z_0 is a Polish space, Z_0 is separable and there exists a metric d on Z_0 such that (Z_0, d) is a complete metric space. Define $\Gamma := \{\mathbb{N}^n : n \in \mathbb{N}\}$. Since Z_0 is separable, Z_0 is zero-dimensional, $Z_0 \neq \varnothing$, and for each $z \in Z_0$, z_0 is a condensation point of Z_0, an exercise in the analysis notes section on countability of metric spaces implies that there exists $(C_\gamma)_{\gamma\in\Gamma} \subset \mathcal{T}_d$ and $(z_\gamma)_{\gamma\in\Gamma} \in \prod\limits_{\gamma\in\Gamma} C_\gamma$ such that

1. • there exists $z_0 \in Z_0$ such that $Z_0 \setminus \{z_0\} = \bigcup\limits_{|\gamma|=1} C_\gamma$,
   • $(C_\gamma)_{\gamma\in\mathbb{N}}$ is disjoint
2. for each $\gamma \in \Gamma$,
   (a) C_γ is closed and $C_\gamma \neq \varnothing$,
   (b) $\operatorname{diam} C_\gamma \leq |\gamma|^{-1}$,
   (c) for each $n \in \mathbb{N}$, $|\gamma| = n$ implies that
      • $C_\gamma \setminus \{z_\gamma\} = \bigcup\limits_{\gamma' \in (\pi^{[n+1]}_{[n]})^{-1}(\{\gamma\})} C_{\gamma'}$
      • $(C_{\gamma'})_{\gamma' \in (\pi^{[n+1]}_{[n]})^{-1}(\{\gamma\})}$ is disjoint

We note that Cantor's nested set theorem in the analysis notes section on completeness implies that for each $\omega \in \mathcal{N}$, that there exists $z \in Z_0$ such that $\bigcap\limits_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega)} = \{z_0\}$. The axiom of choice implies that there exists $f_0 : \mathcal{N} \to Z_0$ such that for each $\omega \in \mathcal{N}$, $\bigcap\limits_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega)} = \{f_0(\omega)\}$. Define $f : \mathcal{N} \to Z$ by $f := \iota_{Z_0} \circ f_0$.

1. Let $(\omega_k)_{k\in\mathbb{N}} \subset \mathcal{N}$ and $\omega \in \mathcal{N}$. Suppose that $\omega_k \to \omega$. Then for each $n \in \mathbb{N}$, $\pi_{[n]}(\omega_k) \to \pi_{[n]}(\omega)$. Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N^{-1} < \epsilon$. We note that

$$\begin{aligned} f_0(\omega) &\in \bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega)} \\ &\subset C_{\pi_{[N]}(\omega)}. \end{aligned}$$

Since $\pi_{[N]}(\omega_k) \to \pi_{[N]}(\omega)$ and $\mathbb{N}^N$ is discrete, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq K$ implies that $\pi_{[N]}(\omega_k) = \pi_{[N]}(\omega)$. Let $k \in \mathbb{N}$. Suppose that $k \geq K$. Then

$$\begin{aligned}f_0(\omega_k) &\in \bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega_k)} \\ &\subset C_{\pi_{[N]}(\omega_k)} \\ &= C_{\pi_{[N]}(\omega)}.\end{aligned}$$

and therefore

$$\begin{aligned}d(f_0(\omega_k), f_0(\omega)) &\leq \operatorname{diam} C_{\pi_{[N]}(\omega)} \\ &\leq N^{-1} \\ &< \epsilon.\end{aligned}$$

Since $k \in \mathbb{N}$ with $k \geq K$ is arbitrary, we have that for each $k \in \mathbb{N}$, $k \geq K$ implies that $d(f_0(\omega_k), f_0(\omega)) < \epsilon$. Since $\epsilon > 0$ is arbitrary, we have that for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq K$ implies that $d(f_0(\omega_k), f_0(\omega)) < \epsilon$. Hence $d(f_0(\omega_k), f_0(\omega)) \to 0$ and $f_0(\omega_k) \to f_0(\omega)$. Since $(\omega_k)_{k\in\mathbb{N}} \subset \mathcal{N}$ and $\omega \in \mathcal{N}$ are arbitrary, we have that for each $(\omega_k)_{k\in\mathbb{N}} \subset \mathcal{N}$ and $\omega \in \mathcal{N}$, $\omega_k \to \omega$ implies that $f_0(\omega_k) \to f_0(\omega)$. Thus f_0 is continuous. Since ι_{Z_0} is continuous, f is continuous.

2. Let $\omega_1, \omega_2 \in \mathcal{N}$. Suppose that $f(\omega_1) = f(\omega_2)$. Therefore

$$\begin{aligned}\iota_{Z_0}(f_0(\omega_1)) &= f(\omega_1) \\ &= f(\omega_2) \\ &= \iota_{Z_0}(f_0(\omega_2)).\end{aligned}$$

Since ι_{Z_0} is injective, we have that $f_0(\omega_1) = f_0(\omega_2)$. Therefore,

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega_1)} &= \{f_0(\omega_1)\} \\ &= \{f_0(\omega_2)\} \\ &= \bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega_2)}.\end{aligned}$$

Define $A \subset \mathbb{N}$ by $A := \{n \in \mathbb{N} : \pi_n(\omega_1) \neq \pi_n(\omega_2)\}$. For the sake of contradiction, suppose that $A \neq \varnothing$. Define $n_0 \in \mathbb{N}$ by $n_0 := \min A$.

- Suppose that $n_0 = 1$. By definition, $\pi_1(\omega_1) \neq \pi_1(\omega_2)$. Then

$$\begin{aligned}\{f_0(\omega_1)\} &= \left[\bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega_1)}\right] \cap \left[\bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega_1)}\right] \\ &\subset C_{\pi_1(\omega_1)} \cap C_{\pi_1(\omega_2)} \\ &= \varnothing\end{aligned}$$

  which is a contradiction. Hence $n_0 \neq 1$.

- Suppose that $n_0 \geq 2$. By definition of n_0, $\pi_{[n_0-1]}(\omega_1) = \pi_{[n_0-1]}(\omega_2)$ Define $\tilde{\gamma} \in \mathbb{N}^{n_0-1}$ and $\gamma_1, \gamma_2 \in \mathbb{N}^{n_0}$ by $\tilde{\gamma} := \pi_{[n_0-1]}(\omega_1)$ and $\gamma_1 := (\tilde{\gamma}, \pi_{n_0}(\omega_1))$ and $\gamma_2 := (\tilde{\gamma}, \pi_{n_0}(\omega_2))$.By construction, $\gamma_1, \gamma_2 \in (\pi^{[n_0]}_{[n_0-1]})^{-1}(\{\tilde{\gamma}\})$ and $\gamma_1 \neq \gamma_2$. Thus $C_{\gamma_1} \cap C_{\gamma_2} = \varnothing$. Therefore

$$\begin{aligned}\{f_0(\omega_1)\} &= \left[\bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega_1)}\right] \cap \left[\bigcap_{n\in\mathbb{N}} C_{\pi_{[n]}(\omega_1)}\right] \\ &\subset C_{\pi_{[n_0]}(\omega_1)} \cap C_{\pi_{[n_0]}(\omega_2)} \\ &= C_{\gamma_1} \cap C_{\gamma_2} \\ &= \varnothing\end{aligned}$$

  which is a contradiction.

Thus $A = \varnothing$ and for each $n \in \mathbb{N}$, $\pi_n(\omega_1) \neq \pi_n(\omega_2)$. Thus $\omega_1 = \omega_2$. Since $\omega_1, \omega_2 \in \mathcal{N}$ are arbitrary, we have that for each $\omega_1, \omega_2 \in \mathcal{N}$, $f(\omega_1) = f(\omega_2)$ implies that $\omega_1 = \omega_2$. Thus f is injective.

3. $f(\mathcal{N})^c = Z_0^c \cup \{z_0\} \cup \{z_\gamma\}_{\gamma \in \Gamma}$. Since Z_0^c is countable, and Γ is countable, we have that $f(\mathcal{N})^c$ is countable.

$\square$

Exercise 4.3.0.19. Let X be a Polish space and $B \in \mathcal{B}(X)$. Suppose that B is uncountable. Then there exists $f : \mathcal{N} \to B$ such that f is a continuous injection and $f(\mathcal{N})^c$ is countable. replace $f(\mathcal{N})^c$ with $B \setminus f(\mathcal{N})$ because $f(\mathcal{N})^c$ may be interpreted as $X \setminus f(\mathcal{N})$.

Proof. Exercise 4.3.0.9 implies that there exists a Polish space Z and $g : Z \to B$ such that Z is zero-dimensional, g is continuous and g is a measurable isomorphism. Since B is uncountable and g is a bijection, Z is uncountable. Exercise 4.3.0.18 then implies that implies that there exists $h : \mathcal{N} \to Z$ such that h is continuous, h is injective and $h(\mathcal{N})^c$ is countable. Define $f : \mathcal{N} \to B$ by $f := g \circ h$. Then f is continuous and f is a injective. Since g is a bijection,

$$\begin{aligned} f(\mathcal{N})^c &= g(h(\mathbb{N}))^c \\ &= g(h(\mathcal{N})^c). \end{aligned}$$

Since $h(\mathcal{N})^c$ is countable and g is a bijection, $f(\mathcal{N})^c$ is countable. $\square$

Exercise 4.3.0.20. Let X be a Polish space and $B \in \mathcal{B}(X)$. Suppose that B is uncountable. Then there exists $A \subset B$ and $f : \mathcal{C} \to A$ such that f is a homeomorphism. replace A^c with $B \setminus A$ because A^c may be interpreted as $X \setminus A$.

Proof. Denote the topology on X by $\mathcal{T}_X$. Exercise 4.3.0.19 implies that there exists $f_0 : \mathcal{N} \to B$ such that f_0 is a continuous injection and $B \setminus f_0(\mathcal{N})$ is countable. Define $A \subset B$ and $f : \mathcal{C} \to A$ by $A := f_0(\mathcal{N})$ and $f(x) := f_0(x)$. Since $B \setminus A$ is countable, and $(B, \mathcal{T}_X \cap B)$ is T_1, $B \setminus A$ is an F_σ-set in $(B, \mathcal{T}_X \cap B)$. Thus A is a G_δ-set in $(B, \mathcal{T}_X \cap B)$ and

$$\begin{aligned} A &\in \mathcal{B}(B) \\ &= \mathcal{B}(X) \cap B \\ &\subset \mathcal{B}(X). \end{aligned}$$

Since f_0 is a continuous injection, f is a continuous bijection. Since $\mathcal{C}$ is compact and A is Hausdorff, the closed map lemma (an exercise in the analysis notes subsection on compactness and continuity) implies that f is a homeomorphism. $\square$

4.4 The Borel Isomorphism Theorem

Definition 4.4.0.1. Let $(X, \mathcal{A})$ be a measurable space. Then $(X, \mathcal{A})$ is said to be a **standard Borel space** if there exists a Polish space Z, $B \in \mathcal{B}(Z)$ and $f : B \to Z$ such that f is a $(\mathcal{B}(B), \mathcal{A})$-measurable isomorphism.

Exercise 4.4.0.2. Let X be a Polish space and $B \in \mathcal{B}(X)$. Then $(B, \mathcal{B}(B))$ is a standard Borel space.

Proof. Since id_B is a $(\mathcal{B}(B), \mathcal{B}(B))$-measurable isomorphism, $(B, \mathcal{B}(B))$ is a standard Borel space. $\square$

Exercise 4.4.0.3. Define Z and $\phi : Z \to [0, 1]$ as in Definition 4.2.0.1. Then $(Z, \mathcal{B}(Z))$ is a standard Borel space.

Proof. Since $[0, 1]$ is a Polish space, Exercise 4.2.0.8 implies that $(Z, \mathcal{B}(Z))$ is a standard Borel space. $\square$

Exercise 4.4.0.4. Let X be a Polish space and $B \in \mathcal{B}(X)$.

1. FINISH!!! (finite)
2. FINISH!!! (countably infinite)
3. If B is uncountable, then there exists $f : B \to \mathcal{C}$ such that f is a $(\mathcal{B}(X) \cap B, \mathcal{B}(\mathcal{C}))$-measurable isomorphism.
   Hint: Consider Exercise 4.3.0.20, Exercise 3.7.0.2 (an exercise in the analysis notes section on Polish spaces, or metrizable spaces to get $f : X \to G \subset \mathbb{H}$ a homeomorphism) and Exercise 3.4.0.13.

Proof.

1. FINISH!!!
2. FINISH!!!
3. Suppose that B is uncountable.
   - Exercise 4.3.0.20 implies that there exists $A \in \mathcal{B}(B)$ and $f_0 : \mathcal{C} \to A$ such that f_0 is a homeomorphism. Then f is $(\mathcal{B}(\mathcal{C}), \mathcal{B}(B) \cap A)$-bimeasurable. Define $\iota : A \to B$ by $\iota(x) := x$. Since $\mathcal{B}(A) = \mathcal{B}(B) \cap A$ and $A \in \mathcal{B}(B)$, Exercise 3.4.0.9 implies that ι is $(\mathcal{B}(B) \cap A, \mathcal{B}(B))$-bimeasurable. Define $f : \mathcal{C} \to A$ by $f := \iota \circ f_0$. Therefore f is $(\mathcal{B}(\mathcal{C}), \mathcal{B}(B))$-bimeasurable. Since f_0 is a bijective and ι is injective, f is injective.
   - An exercise in the analysis notes chapter on metric spaces section on polish spaces implies that there exists $E \subset \mathbb{H}$ and $g_0 : X \to E$ such that E is a G_δ-set and g_0 is a homeomorphism. Since g_0 is a homeomorphism and $B \in \mathcal{B}(X)$, we have that g_0 is a $(\mathcal{B}(X), \mathcal{B}(E))$-measurable isomorphism and $g_0(B) \in \mathcal{B}(E)$. Since E is a G_δ-set, $E \in \mathcal{B}(\mathbb{H})$. Therefore
   $$\begin{aligned} g_0(B) &\in \mathcal{B}(E) \\ &= \mathcal{B}(\mathbb{H}) \cap E \\ &\subset \mathcal{B}(\mathbb{H}). \end{aligned}$$
   Define $g' : B \to g_0(B)$ by $g'(x) := g_0(x)$. Since g_0 is a $(\mathcal{B}(X), \mathcal{B}(E))$-measurable isomorphism, g' is a $(\mathcal{B}(B), \mathcal{B}(\mathbb{H}) \cap g_0(B))$-measurable isomorphism. Define $Z \subset \mathcal{C}$ and $\phi : Z \to [0, 1]$ as in Definition 4.2.0.1. Exercise 4.2.0.8 implies that ϕ is a $(\mathcal{B}(Z), \mathcal{B}([0, 1]))$-measurable isomorphism. Exercise 3.7.0.2 (cite exercise) then implies that $\phi^{\mathbb{N}}$ is a $(\mathcal{B}(Z^{\mathbb{N}}), \mathcal{B}(\mathbb{H}))$-measurable isomorphism. Since Exercise 3.7.0.2 (cite exercise) implies that $\mathcal{B}(Z^{\mathbb{N}}) = \mathcal{B}(\mathcal{C}^{\mathbb{N}}) \cap Z^{\mathbb{N}}$, we have that $(\phi^{\mathbb{N}})^{-1}$ is a $(\mathcal{B}(\mathbb{H}), \mathcal{B}(\mathcal{C}^{\mathbb{N}}) \cap Z^{\mathbb{N}})$-measurable isomorphism. An exercise in the analysis notes chapter on metric spaces, section on zero-dimensional metric spaces, subsection on cantor space implies that there exists $H : \mathcal{C}^{\mathbb{N}} \to \mathcal{C}$ such that H is a homeomorphism. Therefore H is a $(\mathcal{B}(\mathcal{C}^{\mathbb{N}}), \mathcal{B}(\mathcal{C}))$-measurable isomorphism. Exercise 4.2.0.3 implies that $Z \in \mathcal{B}(\mathcal{C})$. Therefore
   $$\begin{aligned} Z^{\mathbb{N}} &= \bigcap_{n \in \mathbb{N}} \pi_n^{-1}(Z) \\ &\in \mathcal{B}(\mathbb{C}^{\mathbb{N}}). \end{aligned}$$
   Since $g_0(B) \in \mathcal{B}(\mathbb{H})$ and $Z^{\mathbb{N}} \in \mathcal{B}(\mathcal{C}^{\mathbb{N}})$, Exercise 3.4.0.9 implies that $\iota_{g_0(B)}$ is $(\mathcal{B}(\mathbb{H}) \cap g_0(B), \mathcal{B}(\mathbb{H}))$-bimeasurable and $\iota_{Z^{\mathbb{N}}}$ is $(\mathcal{B}(\mathcal{C}^{\mathbb{N}}) \cap Z^{\mathbb{N}}, \mathcal{B}(\mathcal{C}^{\mathbb{N}}))$-bimeasurable. Define $g : B \to \mathcal{C}$ by $g := H \circ \iota_{Z^{\mathbb{N}}} \circ (\phi^{\mathbb{N}})^{-1} \circ \iota_{g_0(B)} \circ g'$. Since H, $\iota_{Z^{\mathbb{N}}}$, $(\phi^{\mathbb{N}})^{-1}$, $\iota_{g_0(B)}$ and g' are injective and bimeasurable, we have that g is injective and $(\mathcal{B}(B), \mathcal{B}(\mathcal{C}))$-bimeasurable.

Thus there exist $f : \mathcal{C} \to B$ and $g : B \to \mathcal{C}$ such that f, g are injective, f is $(\mathcal{B}(\mathcal{C}), \mathcal{B}(B))$-bimeasurable and g is $(\mathcal{B}(B), \mathcal{B}(\mathcal{C}))$-bimeasurable. Exercise 3.4.0.13 then implies that there exists $h : B \to \mathcal{C}$ such that h is a $(\mathcal{B}(B), \mathcal{B}(\mathcal{C}))$-isomorphism.

□

Exercise 4.4.0.5. Borel Isomorphism Theorem:
Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be a standard Borel spaces. Then $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable isomorphic iff $|X| = |Y|$.

Proof. Exercise 4.4.0.4 FINISH!!! □

Chapter 5

Measures

5.1 Introduction

Definition 5.1.0.1. (maybe relocate or delete this) Let X be a set, $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$. Then ρ is said to be

- **finitely-additive** if for each $A, B \in \mathcal{E}$, $A \cap B = \varnothing$ and $A \cup B \in \mathcal{E}$ implies that $\rho(A \cup B) = \rho(A) + \rho(B)$,
- **σ-additive** if for each $(A_j)_{j \in \mathbb{N}} \subset \mathcal{E}$, $(A_j)_{j \in \mathbb{N}}$ is disjoint and $\bigcup\limits_{j \in \mathbb{N}} A_j \in \mathcal{E}$ implies that $\rho\left(\bigcup\limits_{j \in \mathbb{N}} A_j\right) = \sum_{j \in \mathbb{N}} \rho(A_j)$,
- **increasing** if for each $A, B \in \mathcal{E}$, $A \subset B$ implies that $\rho(A) \leq \rho(B)$.

Definition 5.1.0.2. Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [0, \infty]$. Then μ is said to be a **measure** on $(X, \mathcal{A})$ if

1. there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$
2. for each $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. If $(A_n)_{n \in \mathbb{N}}$ is disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

Definition 5.1.0.3. Let $(X, \mathcal{A})$ be a measurable space.

- Let μ a measure on $(X, \mathcal{A})$. Then $(X, \mathcal{A}, \mu)$ is called a **measure space**.
- We define

$$M_+(X, \mathcal{A}) := \{\mu : \mathcal{A} \to [0, \infty] : \mu \text{ is a measure}\}.$$

  When X is a topological space, we write $M_+(X)$ in place of $M_+(X, \mathcal{B}(X))$.

Exercise 5.1.0.4. Let $(X, \mathcal{A})$ be a measurable space and $\mu \in M_+(X, \mathcal{A})$. Then

1. for each $A, B \in \mathcal{A}$, $A \subset B$ implies that $\mu(A) \leq \mu(B)$,
2. $\mu(\varnothing) = 0$.

Proof.

1. Let $A, B \in \mathcal{A}$. Suppose that $A \subset B$. Then

$$\begin{aligned}
\mu(B) &= \mu[(B \cap A) \cup (B \cap A^c)] \\
&= \mu(B \cap A) + \mu(B \cap A^c) \\
&= \mu(A) + \mu(B \cap A^c) \\
&\geq \mu(A).
\end{aligned}$$

2. Since μ is a measure, there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$. Since $\varnothing \subset A$, the previous part implies that

$$\begin{aligned}\mu(\varnothing) &\leq \mu(A) \\ &< \infty.\end{aligned}$$

Since $\varnothing \cap \varnothing = \varnothing$, we have that

$$\begin{aligned}\mu(\varnothing) &= \mu(\varnothing \cup \varnothing) \\ &= \mu(\varnothing) + \mu(\varnothing) \\ &= 2\mu(\varnothing).\end{aligned}$$

Thus $\mu(\varnothing) = 0$.

$\square$

Exercise 5.1.0.5. Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [0, \infty]$. Suppose that for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, $(A_n)_{n\in\mathbb{N}}$ is disjoint implies that

$$\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

Then $\mu \in M_+(X, \mathcal{A})$ iff $\mu(\varnothing) = 0$.

Proof.

- $(\implies)$:
Suppose that $\mu \in M_+(X, \mathcal{A})$. Exercise 5.1.0.4 implies that $\mu(\varnothing) = 0$.

- $(\impliedby)$:
Suppose that $\mu(\varnothing) = 0$. Then there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$. Hence $\mu \in M_+(X, \mathcal{A})$.

$\square$

Exercise 5.1.0.6. Let $(X, \mathcal{A})$ be a measurable space and $(\mu_j)_{j\in\mathbb{N}} \subset M_+(X, \mathcal{A})$ and $(\lambda_j) \subset [0, \infty)$. Then $\sum\limits_{j\in\mathbb{N}} \lambda_j \mu_j \in M_+(X, \mathcal{A})$.

Proof. Define $\mu : \mathcal{A} \to [0, \infty]$ by $\mu := \sum\limits_{j\in\mathbb{N}} \lambda_j \mu_j$.

1. We note that

$$\begin{aligned}\mu(\varnothing) &= \bigg[\sum_{j\in\mathbb{N}} \lambda_j \mu_j\bigg](\varnothing) \\ &= \sum_{j\in\mathbb{N}} \lambda_j \mu_j(\varnothing) \\ &= 0.\end{aligned}$$

2. Let $(E_k)_{k\in\mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_k)_{k\in\mathbb{N}}$ is disjoint. Then

$$\begin{aligned}
\mu\bigg(\bigcup_{k\in\mathbb{N}} E_k\bigg) &= \bigg[\sum_{j\in\mathbb{N}} \lambda_j\mu_j\bigg]\bigg(\bigcup_{k\in\mathbb{N}} E_k\bigg) \\
&= \sum_{j\in\mathbb{N}} \lambda_j\mu_j\bigg(\bigcup_{k\in\mathbb{N}} E_k\bigg) \\
&= \sum_{j\in\mathbb{N}} \lambda_j\bigg[\sum_{k\in\mathbb{N}} \mu_j(E_k)\bigg] \\
&= \sum_{j\in\mathbb{N}} \bigg[\sum_{k\in\mathbb{N}} \lambda_j\mu_j(E_k)\bigg] \\
&= \sum_{k\in\mathbb{N}} \bigg[\sum_{j\in\mathbb{N}} \lambda_j\mu_j(E_k)\bigg] \\
&= \sum_{k\in\mathbb{N}} \mu(E_k).
\end{aligned}$$

Thus $\mu \in M_+(X, \mathcal{A})$. □

Exercise 5.1.0.7. Let $(X, \mathcal{A}, \mu)$ be a measure space, A and index set and $(E_\alpha)_{\alpha\in A} \subset \mathcal{A}$. Suppose that $\mu(X) < \infty$ and $(E_\alpha)_{\alpha\in A}$ is disjoint. Then $\{\alpha \in A : \mu(E_\alpha) > 0\}$ is countable.
Hint: set $A_n = \{\alpha \in A : \mu(E_\alpha) \geq 1/n\}$

Proof. For $n \in \mathbb{N}$, set $A_n = \{\alpha \in A : \mu(E_\alpha) \geq 1/n\}$ and define $A_> = \{\alpha \in A : \mu(E_\alpha) > 0\}$. Then

$$A_> = \bigcup_{n\in\mathbb{N}} A_n$$

For the sake of contradiction, suppose that $A_>$ is uncountable. Then there exists $N \in \mathbb{N}$ such that A_N is uncountable. So there exists a sequence $(\alpha_j)_{j\in\mathbb{N}} \subset A_N$. Then

$$\begin{aligned}
\infty &> \mu(X) \\
&\geq \mu\bigg(\bigcup_{j\in\mathbb{N}} E_{\alpha_j}\bigg) \\
&= \sum_{j\in\mathbb{N}} \mu(E_{\alpha_j}) \\
&\geq \sum_{j\in\mathbb{N}} \frac{1}{N} \\
&= \infty
\end{aligned}$$

which is a contradiction. So $A_>$ is countable. □

Exercise 5.1.0.8. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then

1. (monotonicity): for each $A, B \in \mathcal{A}$, if $A \subset B$, then $\mu(A) \leq \mu(B)$.
2. (subadditivity): for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$,
$$\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$$
3. (continuity from below): for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$, then
$$\mu\bigg(\sup_{n\in\mathbb{N}} A_n\bigg) = \sup_{n\in\mathbb{N}} \mu(A_n)$$

4. (continuity from above): for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if for each $n \in \mathbb{N}$, $A_{n+1} \subset A_n$ and $\mu(A_1) < \infty$, then

$$\mu\bigg(\inf_{n\in\mathbb{N}} A_n\bigg) = \inf_{n\in\mathbb{N}} \mu(A_n)$$

Proof.

1. Let $A, B \in \mathcal{A}$. Suppose that $A \subset B$. Then

$$\begin{aligned}\mu(B) &= \mu\Big((B\cap A)\cup(B\cap A^c)\Big)\\ &= \mu(B\cap A) + \mu(B\cap A^c)\\ &= \mu(A) + \mu(B\cap A^c)\\ &\geq \mu(A)\end{aligned}$$

2. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Define $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus \bigg(\bigcup_{k=1}^{n-1} A_k\bigg)$. Then $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, $\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n$, $(B_n)_{n\in\mathbb{N}}$ disjoint and for each $n \in \mathbb{N}$, $B_n \subset A_n$. Thus

$$\begin{aligned}\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) &= \mu\bigg(\bigcup_{n\in\mathbb{N}} B_n\bigg)\\ &= \sum_{n\in\mathbb{N}} \mu(B_n)\\ &\leq \sum_{n\in\mathbb{N}} \mu(A_n)\end{aligned}$$

3. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Suppose that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$. Then for each $n \in \mathbb{N}$, $\mu(A_n) \leq \mu(A_{n+1})$ and $\lim_{n\to\infty} \mu(A_n) = \sup_{n\in\mathbb{N}} \mu(A_n)$. Recall that $\sup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} A_n$. Define $B_1 = A_1$ and for $n \geq 2$, $B_n = A_n \setminus A_{n-1}$. Then $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, $(B_n)_{n\in\mathbb{N}}$ is disjoint, $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$ and for each $n \in \mathbb{N}$, $\bigcup_{n=1}^{k} B_n = A_k$. Then

$$\begin{aligned}\mu\bigg(\sup_{n\in\mathbb{N}} A_n\bigg) &= \mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)\\ &= \mu\bigg(\bigcup_{n\in\mathbb{N}} B_n\bigg)\\ &= \sum_{n\in\mathbb{N}} \mu(B_n)\\ &= \lim_{k\to\infty} \sum_{n=1}^{k} \mu(B_n)\\ &= \lim_{k\to\infty} \mu\bigg(\bigcup_{n=1}^{k} B_n\bigg)\\ &= \lim_{k\to\infty} \mu(A_k)\\ &= \sup_{n\in\mathbb{N}} \mu(A_n)\end{aligned}$$

4. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Suppose that for each $n \in \mathbb{N}$, $A_{n+1} \subset A_n$ and $\mu(A_1) < \infty$. Then for each $n \in \mathbb{N}$ $\mu(A_{n+1}) \leq \mu(A_n) \leq \mu(A_1) < \infty$ and the arithmetic that follows is well defined. Recall that $\inf_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} A_n$. For each $n \in \mathbb{N}$, define

$B_n = A_1 \cap A_n$. Then for each $n \in \mathbb{N}$, $B_n \subset B_{n+1}$ and

$$\begin{aligned}\sup_{n\in\mathbb{N}} B_n &= \bigcup_{n\in\mathbb{N}} B_n \\ &= A_1 \setminus \bigcap_{n\in\mathbb{N}} A_n \\ &= A_1 \setminus \inf_{n\in\mathbb{N}} A_n\end{aligned}$$

So (3) implies that

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mu(B_n) &= \mu\bigg(\sup_{n\in\mathbb{N}} B_n\bigg) \\ &= \mu\bigg(A_1 \setminus \inf_{n\in\mathbb{N}} A_n\bigg) \\ &= \mu(A_1) - \mu\bigg(\inf_{n\in\mathbb{N}} A_n\bigg)\end{aligned}$$

On the other hand,

$$\begin{aligned}\sup_{n\in\mathbb{N}} \mu(B_n) &= \sup_{n\in\mathbb{N}} \mu(A_1 \setminus A_n) \\ &= \sup_{n\in\mathbb{N}} \bigg[\mu(A_1) - \mu(A_n)\bigg] \\ &= \mu(A_1) - \inf_{n\in\mathbb{N}} \mu(A_n)\end{aligned}$$

Therefore

$$\mu\bigg(\inf_{n\in\mathbb{N}} A_n\bigg) = \inf_{n\in\mathbb{N}} \mu(A_n)$$

□

Exercise 5.1.0.9. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$. Then

1. $\mu\bigg(\liminf_{n\to\infty} A_n\bigg) \le \liminf_{n\to\infty} \mu(A_n)$
2. If $\mu\bigg(\sup_{n\in\mathbb{N}} A_n\bigg) < \infty$, then $\limsup_{n\to\infty} \mu(A_n) \le \mu\bigg(\limsup_{n\to\infty} A_n\bigg)$

Proof.

1. Since $\bigg(\inf_{k\ge n} A_k\bigg)_{n\in\mathbb{N}}$ is an increasing sequence and for each $n \in \mathbb{N}$ $\inf_{k\ge n} A_k \subset A_n$, we have that

$$\begin{aligned}\mu\bigg(\liminf_{n\to\infty} A_n\bigg) &= \mu\bigg[\sup_{n\in\mathbb{N}}\bigg(\inf_{k\ge n} A_k\bigg)\bigg] \\ &= \sup_{n\in\mathbb{N}} \mu\bigg(\inf_{k\ge n} A_k\bigg) \\ &= \liminf_{n\to\infty} \mu\bigg(\inf_{k\ge n} A_k\bigg) \\ &\le \liminf_{n\to\infty} \mu(A_n)\end{aligned}$$

2. Since $\mu\bigg(\sup_{\geq 1} A_k\bigg) < \infty$, $\bigg(\sup_{k \geq n}\bigg)_{n \in \mathbb{N}}$ is a decreasing and for each $n \in \mathbb{N}$, $A_n \subset \sup_{k \geq n} A_n$, we have that

$$\begin{aligned}
\mu\bigg(\limsup_{n \to \infty} A_n\bigg) &= \mu\bigg[\inf_{n \in \mathbb{N}}\bigg(\sup_{k \geq n} A_k\bigg)\bigg] \\
&= \inf_{n \in \mathbb{N}} \mu\bigg(\sup_{k \geq n} A_k\bigg) \\
&= \limsup_{n \to \infty} \mu\bigg(\sup_{k \geq n} A_k\bigg) \\
&\geq \limsup_{n \to \infty} \mu(A_n)
\end{aligned}$$

□

Exercise 5.1.0.10. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$. Suppose that $\mu\bigg(\sup_{n \in \mathbb{N}} A_n\bigg) < \infty$. Then $A_n \to A$ implies that $\mu(A_n) \to \mu(A)$.

Proof. Suppose that $A_n \to A$. Then the previous exercise tells us that

$$\begin{aligned}
\mu(A) &= \mu\bigg(\liminf_{n \to \infty} A_n\bigg) \\
&\leq \liminf_{n \to \infty} \mu(A_n) \\
&\leq \limsup_{n \to \infty} \mu(A_n) \\
&\leq \mu(\limsup_{n \to \infty} A_n) \\
&= \mu(A)
\end{aligned}$$

Thus $\mu(A) = \limsup_{n \to \infty} \mu(A_n) = \liminf_{n \to \infty} \mu(A_n)$ and $\mu(A_n) \to \mu(A)$ □

Definition 5.1.0.11. Let $(X, \mathcal{A})$ be a measurable space and $\mu \in M_+(X, \mathcal{A})$. Then μ is said to be

- **finite** if $\mu(X) < \infty$
- **σ-finite** if there exists $(E_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ such that
  1. $X = \bigcup_{j \in \mathbb{N}} E_j$
  2. for each $j \in \mathbb{N}$, $\mu(E_j) < \infty$
- **semifinite** if for each $F \in \mathcal{A}$, $\mu(F) = \infty$ implies that there exists $E \in \mathcal{A}$ such that $E \subset F$ and $\mu(E) < \infty$.

Exercise 5.1.0.12. Let $(X, \mathcal{A})$ be a measurable space and $\mu \in M_+(X, \mathcal{A})$.

1. If μ is finite, then μ is σ-finite.
2. If μ is σ-finite, then μ is semifinite.

Proof.

- Suppose that μ is finite. Define $(E_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ by

$$E_j = \begin{cases} X & j = 1 \\ \varnothing & j > 1 \end{cases}$$

  Then $X = \bigcup_{j \in \mathbb{N}} E_j$ and for each $j \in \mathbb{N}$, $0 < \mu(E_j) < \infty$.

- Suppose that μ is σ-finite. Then there exists $(E_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $X = \bigcup\limits_{j\in\mathbb{N}} E_j$ and for each $j \in \mathbb{N}$, $\mu(E_j) < \infty$. Let $F \in \mathcal{A}$. Suppose that $\mu(F) = \infty$. Define $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ by

$$A_n = \bigcup_{j=1}^{n} E_j$$

Note that $X = \bigcup\limits_{n\in\mathbb{N}} A_n$ and for each $n \in \mathbb{N}$, $F \cap A_n \subset F \cap A_{n+1}$ and

$$\begin{aligned} \mu\bigg(F \cap A_n\bigg) &= \mu\bigg(F \cap \bigg[\bigcup_{j=1}^{n} E_j\bigg]\bigg) \\ &\leq \mu\bigg(\bigcup_{j=1}^{n} E_j\bigg) \\ &\leq \sum_{j=1}^{n} \mu(E_j) \\ &< \infty \end{aligned}$$

For the sake of contradiction, suppose that for each $n \in \mathbb{N}$, $\mu(F \cap A_n) = 0$. Then

$$\begin{aligned} \infty &= \mu(F) \\ &= \mu(F \cap X) \\ &= \mu\bigg(F \cap \bigg[\bigcup_{n\in\mathbb{N}} A_n\bigg]\bigg) \\ &= \mu\bigg(\bigcup_{n\in\mathbb{N}} [F \cap A_n]\bigg) \\ &= \sup_{n\in\mathbb{N}} \mu(F \cap A_n) \\ &= 0 \end{aligned}$$

which is a contradiction. So there exists $N \in \mathbb{N}$ such that $\mu(F \cap A_N) > 0$. Set $E = F \cap A_N$. Then $E \subset F$ and $0 < \mu(E) < \infty$. Hence μ is semifinite.

□

Exercise 5.1.0.13. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Then there exists $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

1. $X = \bigcup\limits_{n\in\mathbb{N}} E_n$
2. for each $n \in \mathbb{N}$, $\mu(E_n) < \infty$
3. for each $n \in \mathbb{N}$, $E_n \subset E_{n+1}$

Proof. Since $(X, \mathcal{A}, \mu)$ is σ-finite, there exists $(F_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that

1. $X = \bigcup\limits_{j\in\mathbb{N}} F_j$
2. for each $j \in \mathbb{N}$, $\mu(F_j) < \infty$

For $n \in \mathbb{N}$, define $E_n \in \mathcal{A}$ by $E_n = \bigcup\limits_{j=1}^{n} F_j$.

1. Since for each $n \in \mathbb{N}$, $F_n \subset E_n$, we have that

$$\begin{aligned} X &= \bigcup_{n\in\mathbb{N}} F_n \\ &\subset \bigcup_{n\in\mathbb{N}} E_n \\ &\subset X \end{aligned}$$

   Hence $\bigcup_{n\in\mathbb{N}} E_n = X$.

2. for each $n \in \mathbb{N}$,

$$\begin{aligned} \mu(E_n) &= \mu\bigg(\bigcup_{j=1}^{n} F_j\bigg) \\ &\leq \sum_{j=1}^{n} \mu(F_j) \\ &< \infty \end{aligned}$$

3. for each $n \in \mathbb{N}$,

$$\begin{aligned} E_n &= \bigcup_{j=1}^{n} F_j \\ &\subset \bigcup_{j=1}^{n+1} F_j \\ &= E_{n+1} \end{aligned}$$

□

Exercise 5.1.0.14. Let $(X, \mathcal{A})$ be a measurable space and $\mu, \nu \in M_+(X, \mathcal{A})$. If μ and ν are σ-finite, then there exists $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

1. $X = \bigcup_{n\in\mathbb{N}} E_n$

2. for each $n \in \mathbb{N}$,

   - $E_n \subset E_{n+1}$
   - $\nu(E_n), \mu(E_n) < \infty$

Proof. Suppose that μ and ν are σ-finite. By definition, there exist $(F_j^\nu)_{j\in\mathbb{N}}, (F_k^\mu)_{k\in\mathbb{N}} \subset \mathcal{A}$ such that $X = \bigcup_{j\in\mathbb{N}} F_j^\nu$, $X = \bigcup_{k\in\mathbb{N}} F_k^\mu$ and for each $j, k \in \mathbb{N}$, $\nu(F_j^\nu), \mu(F_k^\nu) < \infty$. Define $(F_{j,k})_{j,k\in\mathbb{N}} \subset \mathcal{A}$ and $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ by $F_{j,k} = F_j^\nu \cap F_k^\mu$ and $E_n = \bigcup_{j,k\leq n} F_{j,k}$.

1. We see that

$$\begin{aligned}
X &= X \cap X \\
&= \left[\bigcup_{j\in\mathbb{N}} F_j^{\nu}\right] \cap \left[\bigcup_{k\in\mathbb{N}} F_k^{\mu}\right] \\
&= \bigcup_{j\in\mathbb{N}} \left[F_j^{\nu} \cap \left(\bigcup_{k\in\mathbb{N}} F_k^{\mu}\right)\right] \\
&= \bigcup_{j\in\mathbb{N}} \left[\bigcup_{k\in\mathbb{N}} (F_j^{\nu} \cap F_k^{\mu})\right] \\
&= \bigcup_{(j,k)\in\mathbb{N}^2} F_{j,k} \\
&= \bigcup_{n\in\mathbb{N}} \left[\bigcup_{j,k\leq n} F_{j,k}\right] \\
&= \bigcup_{n\in\mathbb{N}} E_n
\end{aligned}$$

2. Let $n \in \mathbb{N}$

   - We have that

$$\begin{aligned}
E_n &= \bigcup_{j,k\leq n} F_{j,k} \\
&\subset \left[\bigcup_{j,k\leq n} F_{j,k}\right] \cup \left[\bigcup_{j=1}^{n} F_{j,k+1}\right] \cup \left[\bigcup_{k=1}^{n} F_{j+1,k}\right] \cup F_{j+1,K+1} \\
&= E_{n+1}
\end{aligned}$$

   - Let $j,k \in \mathbb{N}$. Suppose that $j,k \leq n$. Since $\nu(F_j^{\nu}), \mu(F_k^{\nu}) < \infty$, we have that

$$\begin{aligned}
\nu(F_{j,k}) &= \nu(F_j^{\nu} \cap F_k^{\mu}) \\
&\leq \nu(F_j^{\nu}) \\
&< \infty
\end{aligned}$$

   and

$$\begin{aligned}
\mu(F_{j,k}) &= \mu(F_j^{\nu} \cap F_k^{\mu}) \\
&\leq \mu(F_k^{\mu}) \\
&< \infty
\end{aligned}$$

   Since $j,k \in \mathbb{N}$ with $j,k \leq n$ are arbitrary, we have that

$$\begin{aligned}
\nu(E_n) &= \nu\left[\bigcup_{j,k\leq n} F_{j,k}\right] \\
&\leq \sum_{j,k\leq n} \nu(F_{j,k}) \\
&< \infty
\end{aligned}$$

and

$$\begin{aligned}\mu(E_n) &= \mu\bigg[\bigcup_{j,k\leq n} F_{j,k}\bigg] \\ &\leq \sum_{j,k\leq n} \mu(F_{j,k}) \\ &< \infty\end{aligned}$$

□

Exercise 5.1.0.15. Let $(X, \mathcal{A})$ be a measurable space and $\mu, \nu \in M_+(X, \mathcal{A})$. If μ and ν are σ-finite, then there exists $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

1. $X = \bigcup_{n\in\mathbb{N}} E_n$
2. for each $m, n \in \mathbb{N}$,
   - $m \neq n$ implies that $E_n \cap E_m = \varnothing$
   - $\nu(E_n), \mu(E_n) < \infty$

Proof. Suppose that μ and ν are σ-finite. Exercise 5.1.0.14 implies that there exists $(F_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that

1. $X = \bigcup_{n\in\mathbb{N}} F_n$
2. for each $n \in \mathbb{N}$,
   - $F_n \subset F_{n+1}$
   - $\nu(F_n), \mu(F_n) < \infty$

Define $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ by $E_1 := F_1$ and for each $n \geq 2$, $E_n := F_n \setminus F_{n-1}$. Then

- for each $m, n \in \mathbb{N}$, $m \neq n$ implies that $E_n \cap E_m = \varnothing$
- $X = \bigcup_{n\in\mathbb{N}} E_n$

□

Definition 5.1.0.16. Let $(X, \mathcal{A})$ be a measurable space, $\mu \in M_+(X, \mathcal{A})$ and $E \subset X$. Then E is said to be **μ-null** if there exists $N \in \mathcal{A}$ such that $E \subset N$ and $\mu(N) = 0$.

Exercise 5.1.0.17. content...

Definition 5.1.0.18. Let $(X, \mathcal{A})$

Definition 5.1.0.19. Let $(X, \mathcal{A}, \mu)$ be a measure space and $(f_\alpha)_{\alpha\in A} \subset L^0(X, \mathcal{A})$ a net. Suppose that for each $\alpha \in A$, $f_\alpha : X \to \mathbb{R}$. For each $\alpha, \beta \in A$, define $M_{\alpha,\beta}, N_{\alpha,\beta} \in \mathcal{A}$ by

$$M_{\alpha,\beta} = \{x \in X : f_\alpha(x) \leq f_\beta(x)\}$$

and

$$N_{\alpha,\beta} = \{x \in X : f_\alpha(x) \geq f_\beta(x)\}$$

respectively. Define $M, N \subset X$ by $M = \bigcap_{\substack{(\alpha,\beta)\in A^2 \\ \alpha\leq\beta}} M_{\alpha,\beta}$ and $N = \bigcap_{\substack{(\alpha,\beta)\in A^2 \\ \alpha\leq\beta}} N_{\alpha,\beta}$ respectively. Then $(f_\alpha)_{\alpha\in A}$ is said to be

- **increasing μ-a.e.** if M^c is a μ-null set
- **decreasing μ-a.e.** if N^c is a μ-null set

- **monotonic μ-a.e.** if $(f_n)_{n\in\mathbb{N}}$ is increasing μ-a.e. or $(f_n)_{n\in\mathbb{N}}$ is decreasing μ-a.e.

Exercise 5.1.0.20. Let $(X, \mathcal{A}, \mu)$ be a measure space and $(f_\alpha)_{\alpha\in A} \subset L^0(X, \mathcal{A})$ a net. Suppose that for each $\alpha \in A$, $f_\alpha : X \to \mathbb{R}$. If A is countable, then

1. $(f_\alpha)_{\alpha\in A}$ is increasing μ-a.e. iff for each $\alpha, \beta \in A$, $\alpha \leq \beta$ implies that $f_\alpha \leq f_\beta$ μ-a.e.
2. $(f_\alpha)_{\alpha\in A}$ is decreasing μ-a.e. iff for each $\alpha, \beta \in A$, $\alpha \leq \beta$ implies that $f_\alpha \geq f_\beta$ μ-a.e.

Proof. Suppose that A is countable. For each $\alpha, \beta \in A$, define $M_{\alpha,\beta}, N_{\alpha,\beta}, M, N \in \mathcal{A}$ as in the previous definition. Since A is countable, $M, N \in \mathcal{A}$.

1. Suppose that $(f_\alpha)_{n\in\mathbb{N}}$ is increasing μ-a.e. By definition, M^c is a μ-null set. Since $M^c \in \mathcal{A}$, $\mu(M^c) = 0$. Let $\alpha, \beta \in A$. Suppose that $\alpha \leq \beta$. Since $M \subset M_{\alpha,\beta}$, $M^c_{\alpha,\beta} \subset M^c$. Hence $\mu(M^c_{\alpha,\beta}) = 0$. By definition, $f_\alpha \leq f_\beta$ μ-a.e.
   Conversely, suppose that for each $\alpha, \beta \in A$, $\alpha \leq \beta$ implies that $f_\alpha \leq f_\beta$ μ-a.e. Then for each $\alpha, \beta \in A$, $\mu(M^c_{\alpha,\beta}) = 0$. Since A is countable, we have that
   $$\begin{aligned}\mu(M^c) &= \mu\bigg(\bigcup_{\substack{(\alpha,\beta)\in A^2\\ \alpha\leq\beta}} M^c_{\alpha,\beta}\bigg)\\ &\leq \sum_{\substack{(\alpha,\beta)\in A^2\\ \alpha\leq\beta}} \mu(M^c_{\alpha,\beta})\\ &= 0\end{aligned}$$
   Thus $(f_n)_{n\in\mathbb{N}}$ is increasing μ-a.e.
2. Similar to (1).

□

Definition 5.1.0.21. Let X be a set. We define **counting measure** on X, denoted $\# : \mathcal{P}(X) \to [0, \infty]$, by

$$\#(E) = |E|$$

where $|\cdot| : \mathcal{P}(X) \to [0, \infty]$ denotes the cardinality of E.

Exercise 5.1.0.22. Let X be a set. Then $\# : \mathcal{P}(X) \to [0, \infty]$ is a measure.
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FINISH!!!

Proof.

1. Clearly $\#(\varnothing) = 0$
2. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Set $A = \bigcup\limits_{n\in\mathbb{N}} A_n$ and $J = \{n \in \mathbb{N} : A_n \neq \varnothing\}$. We note that $A = \bigcup\limits_{n\in J} A_n$. Suppose that $|J| = \infty$. Since $(A_n)_{n\in\mathbb{N}}$ is disjoint, we have that
   $$\begin{aligned}\infty &= \sum_{n\in J} 1\\ &\leq \sum_{n\in J} |A_n|\\ &= \sum_{n\in\mathbb{N}} |A_n|\\ &= \sum_{n\in\mathbb{N}} \#(A_n)\end{aligned}$$

and

$$\begin{aligned}\infty &= \left|\bigcup_{n\in J} A_n\right| \\ &\leq \left|A\right| \\ &= \#(A)\end{aligned}$$

Thus

$$\begin{aligned}\#(A) &= \infty \\ &= \sum_{n\in J} \#(A_n)\end{aligned}$$

Suppose that $|J| < \infty$. Then there exists $N \in \mathbb{N}$ such that $A = \bigcup\limits_{n=1}^{N} A_n$. Then the principal of inclusion-exclusion implies that

$$A =$$

$\square$

5.2 Pushforward Measures

Definition 5.2.0.1. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be a measurable spaces and $f : X \to Y$. Suppose that f is $(\mathcal{A}, \mathcal{B})$ measurable. Let $\mu \in M_+(X, \mathcal{A})$. We define the **pushforward of** μ **by** f **on** $(Y, \mathcal{B})$, denoted $f_*\mu : \mathcal{B} \to [0, \infty]$, by

$$f_*\mu(B) = \mu(f^{-1}(B))$$

Exercise 5.2.0.2. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be a measurable spaces and $f : X \to Y$. Suppose that f is $(\mathcal{A}, \mathcal{B})$ measurable. For each $\mu \in M_+(X, \mathcal{A})$, $f_*\mu \in M_+(Y, \mathcal{B})$.

Proof.

1. Since $f^{-1}(\varnothing) = \varnothing$,

$$\begin{aligned} f_*\mu(\varnothing) &= \mu(f^{-1}(\varnothing)) \\ &= \mu(\varnothing) \\ &= 0 \end{aligned}$$

2. Let $(B_j)_{j\in\mathbb{N}} \subset \mathcal{B}$. Suppose that $(B_j)_{j\in\mathbb{N}}$ is disjoint. Then $(f^{-1}(B_j))_{j\in\mathbb{N}}$ is disjoint. Hence

$$\begin{aligned} f_*\mu\bigg(\bigcup_{j\in\mathbb{N}} B_j\bigg) &= \mu\bigg(\bigcup_{j\in\mathbb{N}} f^{-1}(B_j)\bigg) \\ &= \sum_{j\in\mathbb{N}} \mu(f^{-1}(B_j)) \\ &= \sum_{j\in\mathbb{N}} f_*\mu(B_j) \end{aligned}$$

Hence $f_*\mu$ is a measure. □

Exercise 5.2.0.3. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be a measurable spaces and $f : X \to Y$. Suppose that f is $(\mathcal{A}, \mathcal{B})$ measurable. Let $\mu \in M_+(X, \mathcal{A})$. If $f_*\mu$ is σ-finite, then μ is σ-finite.

Proof. Suppose that $f_*\mu$ is σ-finite. Then there exists $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $Y = \bigcup\limits_{n\in\mathbb{N}} B_n$ and for each $n \in \mathbb{N}$, $f_*\mu(B_n) < \infty$. Define $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ by $A_n := f^{-1}(B_n)$. Then

$$\begin{aligned} X &= f^{-1}(Y) \\ &= f^{-1}\bigg(\bigcup_{n\in\mathbb{N}} B_n\bigg) \\ &= \bigcup_{n\in\mathbb{N}} f^{-1}(B_n) \\ &= \bigcup_{n\in\mathbb{N}} A_n \end{aligned}$$

and for each $n \in \mathbb{N}$,

$$\begin{aligned} \mu(A_n) &= \mu(f^{-1}(B_n)) \\ &= f_*\mu(B_n) \\ &< \infty \end{aligned}$$

Thus μ is σ-finite. □

Exercise 5.2.0.4. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be a measurable spaces and $f : X \to Y$. Suppose that f is $(\mathcal{A}, \mathcal{B})$ measurable. Let $\mu \in M_+(X, \mathcal{A})$. Suppose that

- f is injective
- for each $A \in \mathcal{A}$, $f(A) \in \mathcal{B}$.

Then μ is σ-finite iff $f_*\mu$ is σ-finite.

Proof.

- $(\implies)$:
Suppose that μ is σ-finite. Then there exists $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $X = \bigcup\limits_{n\in\mathbb{N}} A_n$ and for each $n \in \mathbb{N}$, $\mu(A_n) < \infty$. Define $(B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(Y)$ by
$$B_n = \begin{cases} f(X)^c, & n = 1 \\ f(A_{n-1}), & n \geq 2 \end{cases}$$
By assumption $f(X) \in \mathcal{B}$ and for each $n \in \mathbb{N}$, $f(A_n) \in \mathcal{B}$. Therefore $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}$. Furthermore,
$$\begin{aligned} Y &= f(X)^c \cup f(X) \\ &= f(X)^c \cup f\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \\ &= f(X)^c \cup \bigg(\bigcup_{n\in\mathbb{N}} f(A_n)\bigg) \\ &= \bigcup_{n\in\mathbb{N}} B_n \end{aligned}$$
We note that
$$\begin{aligned} f_*\mu(B_1) &= \mu(f^{-1}(B_1)) \\ &= \mu(f^{-1}(f(X)^c)) \\ &= \mu(f^{-1}(f(X))^c) \\ &= \mu(X^c) \\ &= \mu(\varnothing) \\ &= 0 \end{aligned}$$
and for each $n \geq 2$,
$$\begin{aligned} f_*\mu(B_n) &= \mu(f^{-1}(B_n)) \\ &= \mu(f^{-1}(f(A_{n-1}))) \\ &= \mu(A_{n-1}) \end{aligned}$$
Therefore for each $n \in \mathbb{N}$, $f_*\mu(B_n) < \infty$. Therefore $f_*\mu$ is σ-finite.
- $(\impliedby)$:
Suppose that $f_*\mu$ is σ-finite. Exercise 5.2.0.3 implies that μ is σ-finite.

□

Exercise 5.2.0.5. Let $(X, \mathcal{A}_X)$, $(Y, \mathcal{A}_Y)$, $(Z, \mathcal{A}_Z)$ be a measurable spaces, $f : X \to Y$, $g : Y \to Z$ and $\mu \in M_+(X, \mathcal{A}_X)$. Suppose that f is $(\mathcal{A}_X, \mathcal{A}_Y)$-measurable and g is $(\mathcal{A}_Y, \mathcal{A}_Z)$-measurable. Then $(g \circ f)_*\mu = g_*(f_*\mu)$.

Proof. Let $E \in \mathcal{A}_Z$. Then

$$\begin{aligned}
&(g \circ f)_*\mu(E) \\
&\qquad = \mu((g \circ f)^{-1}(E)) \\
&\qquad = \mu(f^{-1}(g^{-1}(E))) \\
&\qquad = f_*\mu(g^{-1}(E)) \\
&\qquad = g_*(f_*\mu)(E).
\end{aligned}$$

Since $E \in \mathcal{A}_Z$ is arbitrary, we have that for each $(g \circ f)_*\mu(E) = g_*(f_*\mu)$. □

Exercise 5.2.0.6. Let $(X, \mathcal{A}_X)$, $(Y, \mathcal{A}_Y)$ be a measurable spaces, $f : X \to Y$ and $\mu, \nu \in M_+(X, \mathcal{A}_X)$. Suppose that f is $(\mathcal{A}_X, \mathcal{A}_Y)$-measurable. Then $f_*(\mu + \nu) = f_*\mu + f_*\nu$.

Proof. Let $E \in \mathcal{A}_Y$. Then

$$\begin{aligned}
f_*(\mu + \nu)(E) &= (\mu + \nu)(f^{-1}(E)) \\
&= \mu(f^{-1}(E)) + \nu(f^{-1}(E)) \\
&= f_*\mu(E) + f_*\nu(E) \\
&= (f_*\mu + f_*\nu)(E).
\end{aligned}$$

Since $E \in \mathcal{A}_Y$ is arbitrary, we have that $f_*(\mu + \nu) = f_*\mu + f_*\nu$. □

5.3 Outer Measures

Definition 5.3.0.1. Let X be a set and $\nu : \mathcal{P}(X) \to [0, \infty]$. Then ν is said to be an **outer measure on X** if

1. $\nu(\varnothing) = 0$
2. for each $A, B \subset X$, if $A \subset B$, then $\nu(A) \leq \nu(B)$.
3. for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$,
$$\nu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) \leq \sum_{n\in\mathbb{N}} \nu(A_n)$$

Definition 5.3.0.2. Let X be a set, ν an outer measure on X and $A \subset X$. Then

- A is said to be ν**-outer measurable** if for each $E \subset X$,
$$\nu(E) = \nu(E \cap A) + \nu(E \cap A^c)$$
- we define $\mathcal{A}_\nu = \{A \subset X : A \text{ is } \nu\text{-outer measurable}\}$

Exercise 5.3.0.3. Let X be a set, ν an outer measure on X and $A \subset X$. Then $A \in \mathcal{A}_\nu$ iff for each $E \subset X$, $\nu(E) < \infty$ implies that
$$\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c)$$

Proof. Suppose that $A \in \mathcal{A}_\nu$. Let $E \subset X$, Suppose that $\nu(E) < \infty$. By definition $\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c)$. Conversely, suppose that for each $E \subset X$, $\nu(E) < \infty$ implies that $\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c)$. Let $E \subset X$.

- If $\nu(E) < \infty$, then by assumption,
$$\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c)$$
If $\nu(E) = \infty$, then trivially,
$$\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c)$$
So $\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c)$
- Since $E = (E \cap A) \cup (E \cap A^c)$, by definition,
$$\nu(E) \leq \nu(E \cap A) + \nu(E \cap A^c)$$

So $\nu(E) = \nu(E \cap A) + \nu(E \cap A^c)$ and $A \in \mathcal{A}_\nu$. □

Exercise 5.3.0.4. Let X be a set and ν an outer measure on X. Then

1. $\mathcal{A}_\nu$ is a σ-algebra on X
2. $\nu|_{\mathcal{A}_\mu} \in M_+(X, \mathcal{A}_\nu)$
3. $(X, \mathcal{A}_\nu, \nu|_{\mathcal{A}_\mu})$ is complete

Proof. FINISH!!! □

Exercise 5.3.0.5. Let X be a set, ν an outer measure on X and $A \subset X$. If $\nu(A) = 0$, then $A \in \mathcal{A}_\nu$.

Proof. Suppose that $\nu(A) = 0$. Let $E \subset X$. Suppose that $\nu(E) < \infty$. Then
$$\begin{aligned}\nu(E \cap A) &\leq \nu(E)\\ &= 0\end{aligned}$$
so that $\nu(E \cap A) = 0$. Therefore
$$\begin{aligned}\nu(E) &\geq \nu(E \cap A^c)\\ &= 0 + \nu(E \cap A^c)\\ &= \nu(E \cap A) + \nu(E \cap A^c)\end{aligned}$$
Since $E \subset X$ is arbitrary, the previous exercise implies that $A \in \mathcal{A}_\nu$. □

Definition 5.3.0.6. Let X be a set, $\mathcal{A}_0 \subset \mathcal{P}(X)$ an algebra on X and $\mu_0 : \mathcal{A}_0 \to [0, \infty]$. Then μ_0 is said to be a **premeasure on** $(X, \mathcal{A}_0)$ if

1. there exists $A \in \mathcal{A}_0$ such that $\mu_0(A) < \infty$
2. for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$, if $(A_n)_{n\in\mathbb{N}}$ is disjoint and $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}_0$, then

$$\mu_0\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) = \sum_{n\in\mathbb{N}} \mu_0(A_n)$$

Note 5.3.0.7. The same reasoning applied to measures shows that $\mu_0(\varnothing) = 0$.

Exercise 5.3.0.8. Let X be a set, $\mathcal{A}_0 \subset \mathcal{P}(X)$ an algebra on X and $\mu_0 : \mathcal{A}_0 \to [0, \infty)$. Suppose that

- μ_0 is finitely-additive,
- for each $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$, $(B_n)_{n\in\mathbb{N}}$ is decreasing and $\inf\limits_{n\in\mathbb{N}} B_n = \varnothing$ implies that $\mu_0(B_n) \to 0$.

Then μ_0 is a premeasure on $(X, \mathcal{A}_0)$.

Proof.

1. Since $\mu_0 : \mathcal{A}_0 \to [0, \infty)$, we have that $\mu_0(X) < \infty$.
2. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint and $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}_0$. Set $A := \bigcup\limits_{n\in\mathbb{N}} A_n$. Define $(B_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ by $B_n := A \setminus \bigg(\bigcup\limits_{j=1}^{n} A_j\bigg)$. Let $n \in \mathbb{N}$. Since $\mathcal{A}_0$ is an algebra $\bigcup\limits_{j=1}^{n} A_j \in \mathcal{A}_0$ and since $A \in \mathcal{A}_0$, we have that

$$\begin{aligned} B_n &= A \setminus \bigg(\bigcup_{j=1}^{n} A_j\bigg) \\ &\in \mathcal{A}_0. \end{aligned}$$

   Since $n \in \mathbb{N}$ is arbitrary, we have that $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$. By construction, for each $n \in \mathbb{N}$, $B_n \subset B_{n+1}$ and therefore $(B_n)_{n\in\mathbb{N}}$ is decreasing. We note that

$$\begin{aligned} \bigcap_{n\in\mathbb{N}} B_n &= \bigcap_{n\in\mathbb{N}} \left[A \setminus \bigg(\bigcup_{j=1}^{n} A_j\bigg)\right] \\ &= \bigcap_{n\in\mathbb{N}} \left[A \cap \bigg(\bigcup_{j=1}^{n} A_j\bigg)^c\right] \\ &= \bigcap_{n\in\mathbb{N}} \left[A \cap \bigg(\bigcap_{j=1}^{n} A_j^c\bigg)\right] \\ &= A \cap \bigcap_{n\in\mathbb{N}} \bigg(\bigcap_{j=1}^{n} A_j^c\bigg) \\ &= A \cap \bigg(\bigcap_{j=1}^{n} A_j^c\bigg) \\ &= A \cap \bigg(\bigcup_{j\in\mathbb{N}} A_j\bigg)^c \\ &= A \cap A^c \\ &= \varnothing. \end{aligned}$$

Thus $\inf_{n\in\mathbb{N}} B_n = \varnothing$. By assumption, $\mu_0(B_n) \to 0$. Therefore, for each $n \in \mathbb{N}$, we have that

$$\begin{aligned}\mu_0(A) &= \mu_0\bigg(B_n \cup \bigg[\bigcup_{j=1}^n A_j\bigg]\bigg) \\ &= \mu_0(B_n) + \mu_0\bigg(\bigcup_{j=1}^n A_j\bigg) \\ &= \mu_0(B_n) + \sum_{j=1}^n \mu_0(A_j)\end{aligned}$$

Hence

$$\begin{aligned}\mu_0(A) &= \lim_{n\to\infty} \mu_0(A) \\ &= \lim_{n\to\infty} \bigg[\mu_0(B_n) + \sum_{j=1}^n \mu_0(A_j)\bigg] \\ &= \lim_{n\to\infty} \mu_0(B_n) + \lim_{n\to\infty} \sum_{j=1}^n \mu_0(A_j) \\ &= 0 + \sum_{j\in\mathbb{N}} \mu_0(A_j) \\ &= \sum_{j\in\mathbb{N}} \mu_0(A_j).\end{aligned}$$

Since $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$ such that $(A_n)_{n\in\mathbb{N}}$ is disjoint and $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}_0$ is arbitrary, we have that for each $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$, $(A_n)_{n\in\mathbb{N}}$ is disjoint and $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}_0$ implies that $\mu_0\bigg(\bigcup\limits_{n\in\mathbb{N}} A_n\bigg) = \sum\limits_{n\in\mathbb{N}} \mu_0(A_n)$. Hence μ_0 is a premeasure on $(X, \mathcal{A}_0)$.

$\square$

Theorem 5.3.0.9. Caratheodory Extension Theorem:
Let X be a set, $\mathcal{A}_0$ an algebra on X and μ_0 a premeasure on $(X, \mathcal{A}_0)$. Set $\mathcal{A} = \sigma(\mathcal{A}_0)$. If μ_0 is σ-finite (define σ-finite for premeasures), then there exists a unique measure μ on $(X, \mathcal{A})$ such that $\mu|_{\mathcal{A}_0} = \mu_0^*|_{\mathcal{A}_0} = \mu_0$.

Exercise 5.3.0.10. Let $(X, \mathcal{A})$ be a measurable space and $\mu_0 : \mathcal{A} \to [0, \infty)$. Suppose that

- μ_0 is finitely-additive,
- for each $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$, $(B_n)_{n\in\mathbb{N}}$ is decreasing and $\inf\limits_{n\in\mathbb{N}} B_n = \varnothing$ implies that $\mu_0(B_n) \to 0$.

Then $\mu \in M_+(X, \mathcal{A})$.

Proof. Since $\mathcal{A}$ is an alegbra on X, Exercise 5.3.0.8 implies that μ is a premeasure on $(X, \mathcal{A})$. Since $\mathcal{A}$ is a σ-algebra on X, $\sigma(\mathcal{A}) = \mathcal{A}$. Since μ is finite, Exercise 5.3.0.9 implies that there exists a unique $\mu' \in M_+(X, \mathcal{A})$ such that $\mu'|_{\mathcal{A}} = \mu$. Since $\mu'|_{\mathcal{A}} = \mu'$, we have $\mu' = \mu$. Hence $\mu \in M_+(X, \mathcal{A})$. $\square$

Definition 5.3.0.11. Let X be a set and ν an outer measure on X.

- Let $\mathcal{F} \subset \mathcal{P}(X)$ and $A \subset X$. Then $\mathcal{F}$ is said to **ν-cover** A if

$$\nu\bigg(A \setminus \bigg[\bigcup_{F\in\mathcal{F}} F\bigg]\bigg) = 0$$

- Let $E \subset X$. Then E is said to **ν-cover** A if $\{E\}$ ν-covers A.

Exercise 5.3.0.12. Let X be a set, ν an outer measure on X and $A, E \subset X$. If E ν-covers A, then for each $B \subset A$,

1. E ν-covers B
2. $B \cap E$ ν-covers B

Proof. Suppose that E ν-covers A. Let $B \subset A$.

1. We have that

$$\begin{aligned}\nu(B \setminus E) &= \nu(B \cap E^c)\\ &\leq \nu(A \cap E^c)\\ &= \nu(A \setminus E)\\ &= 0\end{aligned}$$

Hence E ν-covers B.

2. By part (1),

$$\begin{aligned}\nu[B \setminus (B \cap E)] &= \nu[B \cap (B \cap E)^c]\\ &= \nu[B \cap (B^c \cup E^c)]\\ &= \nu[(B \cap B^c) \cup (B \cap E^c)]\\ &= \nu[\varnothing \cup (B \cap E^c)]\\ &= \nu(B \cap E^c)\\ &= \nu(B \setminus E)\\ &= 0\end{aligned}$$

Hence $B \cap E$ ν-covers B.

$\square$

Exercise 5.3.0.13. Let X be a set, ν an outer measure on X and $A \in \mathcal{P}(X)$ and $(E_j)_{j \in \mathbb{N}} \subset \mathcal{P}(X)$. If for each $j \in \mathbb{N}$, E_j ν-covers A, then $\bigcap\limits_{j \in \mathbb{N}} E_j$ ν-covers A.

Proof. Suppose that for each $j \in \mathbb{N}$, E_j ν-covers A. Then

$$\begin{aligned}\nu\left[A \setminus \left(\bigcap_{j \in \mathbb{N}} E_j\right)\right] &= \nu\left[A \cap \left(\bigcap_{j \in \mathbb{N}} E_j\right)^c\right]\\ &= \nu\left[A \cap \left(\bigcup_{j \in \mathbb{N}} E_j^c\right)\right]\\ &= \nu\left[\bigcup_{j \in \mathbb{N}} (A \cap E_j^c)\right]\\ &= \nu\left[\bigcup_{j \in \mathbb{N}} (A \setminus E_j)\right]\\ &\leq \sum_{j \in \mathbb{N}} \nu(A \setminus E_j)\\ &= 0\end{aligned}$$

So $\nu\left[A \setminus \left(\bigcap\limits_{j \in \mathbb{N}} E_j\right)\right] = 0$ and $\bigcap\limits_{j \in \mathbb{N}} E_j$ ν-covers A. $\square$

Definition 5.3.0.14. Let X be a set, $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$. Suppose that $\varnothing, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. We define the **outer measure on X induced by** ρ, denoted $\rho^* : \mathcal{P}(X) \to [0, \infty]$, by

$$\rho^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \rho(E_n) : (E_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } A \subset \bigcup_{n \in \mathbb{N}} E_n \right\}$$

Exercise 5.3.0.15. Construction of Outer Measures:
Let X be a set, $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$. Suppose that $\varnothing, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. Then ρ^* is an outer measure on X.

Proof. For $A \subset \mathcal{P}(X)$, set

$$V(A) = \left\{ \sum_{n \in \mathbb{N}} \rho(E_n) : (E_n)_{n \in \mathbb{N}} \subset \mathcal{E} \text{ and } A \subset \bigcup_{n \in \mathbb{N}} E_n \right\}$$

1. Since $\rho(\varnothing) = 0$,

$$\begin{aligned} \rho^*(\varnothing) &= \inf V(\varnothing) \\ &\leq \rho(\varnothing) \\ &= 0 \end{aligned}$$

   So $\rho^*(\varnothing) = 0$.

2. Let $A, B \subset X$. Suppose that $A \subset B$. Let $a \in V(B)$. Then there exist $(E_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ such that $B \subset \bigcup\limits_{n \in \mathbb{N}} E_n$ and $a = \sum\limits_{n \in \mathbb{N}} \rho(E_n)$. Then

$$\begin{aligned} A &\subset B \\ &\subset \bigcup_{n \in \mathbb{N}} E_n \end{aligned}$$

   Hence $a \in V(A)$. Since $a \in V(B)$ is arbitrary, we have that $V(B) \subset V(A)$. Thus

$$\begin{aligned} \rho^*(A) &= \inf V(A) \\ &\leq \inf V(B) \\ &= \rho^*(B) \end{aligned}$$

3. Let $(A_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\rho^*(A_n) = \infty$. Then

$$\begin{aligned} \infty &= \rho^*(A_{n_0}) \\ &\leq \rho^*\left(\bigcup_{n \in \mathbb{N}} A_n \right) \end{aligned}$$

   Therefore

$$\begin{aligned} \rho^*\left(\bigcup_{n \in \mathbb{N}} A_n \right) &= \infty \\ &= \sum_{n \in \mathbb{N}} \rho^*(A_n) \end{aligned}$$

   Suppose that for each $n \in \mathbb{N}$, $\rho^*(A_n) < \infty$. Let $\epsilon > 0$. Then for each $n \in \mathbb{N}$, there exists $(E_{n,j})_{j \in \mathbb{N}} \subset \mathcal{E}$ such that $A_n \subset \bigcup\limits_{j \in \mathbb{N}} E_{n,j}$ and

$$\sum_{j \in \mathbb{N}} \rho(E_{n,j}) < \rho^*(A_n) + \epsilon 2^{-n}$$

Then $(E_{n,j})_{n,j\in\mathbb{N}} \subset \mathcal{E}$ and

$$\begin{aligned}\sum_{n,j\in\mathbb{N}} \rho(E_{n,j}) &= \sum_{n\in\mathbb{N}} \left[\sum_{j\in\mathbb{N}} \rho(E_{n,j})\right] \\ &\leq \sum_{n\in\mathbb{N}} (\rho^*(A_n) + \epsilon 2^{-n}) \\ &= \sum_{n\in\mathbb{N}} \rho^*(A_n) + \epsilon\end{aligned}$$

This implies that

$$\begin{aligned}\rho^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) &= \inf V\left(\bigcup_{n\in\mathbb{N}} A_n\right) \\ &\leq \sum_{n,j\in\mathbb{N}} \rho(E_{n,j}) \\ &\leq \sum_{n\in\mathbb{N}} \rho^*(A_n) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that

$$\rho^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \rho^*(A_n)$$

Hence ρ^* is an outer measure on X. $\square$

Exercise 5.3.0.16. Let $(X, \mathcal{A})$ be a measurable space and $\mu \in M_+(X, \mathcal{A})$. Then $\mu^*|_{\mathcal{A}} = \mu$.

Proof. Let $A \in \mathcal{A}$. Define $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ by

$$E_n = \begin{cases} A & n = 1 \\ \varnothing & n > 1 \end{cases}$$

Then $A \subset \bigcup_{n\in\mathbb{N}} E_n$

$$\begin{aligned}\mu^*(A) &\leq \sum_{j\in\mathbb{N}} \mu(E_n) \\ &= \mu(A)\end{aligned}$$

For the sake of contradiction, suppose that $\mu^*(A) < \mu(A)$. Then $\mu^*(A) < \infty$. Let $\epsilon > 0$. Then there exists $(E_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $A \subset \bigcup_{j\in\mathbb{N}} E_j$ and $\sum_{j\in\mathbb{N}} \mu(E_j) \leq \mu^*(A) + \epsilon$. Therefore

$$\begin{aligned}\mu(A) &\leq \mu\left(\bigcup_{j\in\mathbb{N}} E_j\right) \\ &\leq \sum_{j\in\mathbb{N}} \mu(E_j) \\ &\leq \mu^*(A) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\begin{aligned}\mu(A) &\leq \mu^*(A) \\ &< \mu(A)\end{aligned}$$

This is a contradiction. Hence $\mu(A) \leq \mu^*(A)$. Therefore $\mu^*(A) = \mu(A)$. Since $A \in \mathcal{A}$ is arbitrary, $\mu^*|_{\mathcal{A}} = \mu$. $\square$

Exercise 5.3.0.17. Let $(X, \mathcal{A})$ be a measurable space and $\mu \in M_+(X, \mathcal{A})$. Then for each $A \subset X$, there exists $B \in \mathcal{A}$ such that $A \subset B$ and $\mu^*(A) = \mu(B)$.

Proof. Let $A \subset X$.

- Suppose that $\mu^*(A) = \infty$. Set $B = X$. Then $A \subset B$ and Exercise 5.3.0.16 implies that

$$\begin{aligned} \mu(B) &= \mu(X) \\ &= \mu^*(X) \\ &\geq \mu^*(A) \\ &= \infty \end{aligned}$$

  Thus $\mu(B) = \infty$ and

$$\begin{aligned} \mu^*(A) &= \infty \\ &= \mu(B) \end{aligned}$$

- Suppose that $\mu^*(A) < \infty$. Then for each $n \in \mathbb{N}$, there exists $(E_{n,j})_{j \in \mathbb{N}} \subset \mathcal{A}$ such that $A \subset \bigcup\limits_{j \in \mathbb{N}} E_{n,j}$ and $\sum\limits_{j \in \mathbb{N}} \mu(E_{n,j}) < \mu^*(A) + 1/n$. For each $n \in \mathbb{N}$, set $B_n = \bigcup\limits_{j \in \mathbb{N}} E_{n,j}$ and set $B = \bigcap\limits_{n \in \mathbb{N}} B_n$. Since for each $n \in \mathbb{N}$ $A \subset B_n$, we have that

$$\begin{aligned} A &\subset \bigcap_{n \in \mathbb{N}} B_n \\ &= B \end{aligned}$$

  Exercise 5.3.0.16 implies that

$$\begin{aligned} \mu^*(A) &\leq \mu^*(B) \\ &= \mu(B) \end{aligned}$$

  Let $n \in \mathbb{N}$. Since $B \subset B_n$, we have that

$$\begin{aligned} \mu(B) &\leq \mu(B_n) \\ &\leq \sum_{j \in \mathbb{N}} \mu(E_{n,j}) \\ &< \mu^*(A) + 1/n \end{aligned}$$

  Since $n \in \mathbb{N}$ is arbitrary, we have that $\mu(B) \leq \mu^*(A)$. Hence $\mu^*(A) = \mu(B)$

$\square$

Definition 5.3.0.18. Let $(X, \mathcal{A})$ be a measurable space and μ, ν measures on $(X, \mathcal{A})$. We define $\nu_\mu : \mathcal{P}(X) \to [0, \infty]$ by

$$\nu_\mu(A) = \inf\{\nu(E) : E \in \mathcal{A} \text{ and } E \ \mu^*\text{-covers } A\}$$

Exercise 5.3.0.19. Let $(X, \mathcal{A})$ be a measurable space and $\mu, \nu \in M_+(X, \mathcal{A})$. Then ν_μ is an outer measure on X.

Proof. For each A set $V(A) = \{\nu(E) : E \in \mathcal{A} \text{ and } E \ \mu^*\text{-covers } A\}$.

1. Since $\varnothing \in \mathcal{A}$ and $\varnothing$ μ^*-covers $\varnothing$, we have that

$$\begin{aligned} \nu_\mu(\varnothing) &= \inf V(\varnothing) \\ &\leq \nu(\varnothing) \\ &= 0 \end{aligned}$$

   Hence $\nu_\mu(\varnothing) = 0$.

2. Let $A, B \in \mathcal{P}(X)$. Suppose that $A \subset B$. Let $a \in V(B)$. Then there exists $E \in \mathcal{A}$ such that E μ^*-covers B and $a = \nu(E)$. Since $A \subset B$, we have that $A \cap E^c \subset B \cap E^c$. Therefore

$$\begin{aligned} \mu^*(A \setminus E) &\leq \mu^*(B \setminus E) \\ &= 0 \end{aligned}$$

Hence E μ^*-covers A. Thus

$$\begin{aligned} a &= \nu(E) \\ &\in V(A) \end{aligned}$$

Since $a \in V(B)$ is arbitrary, $V(B) \subset V(A)$. Hence

$$\begin{aligned} \nu_\mu(A) &= \inf V(A) \\ &\leq \inf V(B) \\ &= \nu_\mu(B) \end{aligned}$$

3. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\nu_\mu(A_n) = \infty$. Then

$$\begin{aligned} \infty &= \nu_\mu(A_{n_0}) \\ &\leq \nu_\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \end{aligned}$$

Therefore

$$\begin{aligned} \nu_\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) &= \infty \\ &= \sum_{n\in\mathbb{N}} \nu_\mu(A_n) \end{aligned}$$

Suppose that for each $n \in \mathbb{N}$, $\nu_\mu(A_n) < \infty$. Let $\epsilon > 0$. Then for each $n \in \mathbb{N}$, there exists $E_n \in \mathcal{A}$ such that E_n μ^*-covers A_n and $\nu(E_n) < \nu_\mu(A_n) + \epsilon 2^{-n}$. We observe that $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$ and

$$\begin{aligned} \bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \setminus \bigg(\bigcup_{j\in\mathbb{N}} E_j\bigg) &= \bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \cap \bigg(\bigcup_{j\in\mathbb{N}} E_j\bigg)^c \\ &= \bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \cap \bigg(\bigcap_{j\in\mathbb{N}} E_j^c\bigg) \\ &= \bigcup_{n\in\mathbb{N}} \bigg(A_n \cap \bigg[\bigcap_{j\in\mathbb{N}} E_j^c\bigg]\bigg) \\ &\subset \bigcup_{n\in\mathbb{N}} [A_n \cap E_n^c] \\ &= \bigcup_{n\in\mathbb{N}} [A_n \setminus E_n] \end{aligned}$$

This implies that

$$\begin{aligned} \mu^*\bigg[\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \setminus \bigg(\bigcup_{j\in\mathbb{N}} E_j\bigg)\bigg] &\leq \mu^*\bigg(\bigcup_{n\in\mathbb{N}} [A_n \setminus E_n]\bigg) \\ &\leq \sum_{n\in\mathbb{N}} \mu^*(A_n \setminus E_n) \\ &= 0 \end{aligned}$$

so that $\bigcup_{n\in\mathbb{N}} E_n$ μ^*-covers $\bigcup_{n\in\mathbb{N}} A_n$. Therefore

$$\begin{aligned}\nu_\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) &= \inf V\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \\ &\leq \nu\bigg(\bigcup_{n\in\mathbb{N}} E_n\bigg) \\ &\leq \sum_{n\in\mathbb{N}} \nu(E_n) \\ &\leq \sum_{n\in\mathbb{N}} [\nu_\mu(A_n) + \epsilon 2^{-n}] \\ &= \sum_{n\in\mathbb{N}} \nu_\mu(A_n) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\nu_\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) \leq \sum_{n\in\mathbb{N}} \nu_\mu(A_n)$$

Hence ν_μ is an outer measure on X. □

Exercise 5.3.0.20. Let $(X, \mathcal{A})$ be a measurable space and $\mu, \nu \in M_+(X, \mathcal{A})$. Then

1. $\nu_\mu \leq \nu^*$.
2. $\nu_\mu|_\mathcal{A} \leq \nu$.

Proof.

1. Let $A \subset X$.
   - If $\nu^*(A) = \infty$, then
   $$\begin{aligned}\nu_\mu(A) &\leq \infty \\ &= \nu^*(A)\end{aligned}$$
   - Suppose that $\nu^*(A) \neq \infty$. Let $\epsilon > 0$. Then $\nu^*(A) < \nu^*(A) + \epsilon$. Exercise 5.3.0.17 implies that there exists $E \in \mathcal{A}$ such that $A \subset E$ and $\nu^*(A) = \nu(E)$. Then E ν^*-covers A and therefore
   $$\begin{aligned}\nu_\mu(A) &\leq \nu(E) \\ &= \nu^*(A)\end{aligned}$$
   Since $A \subset X$ is arbitrary, $\nu_\mu \leq \nu^*$.
2. Let $E \in \mathcal{A}$. Part (1) and Exercise 5.3.0.16 imply that
   $$\begin{aligned}\nu_\mu(E) &\leq \nu^*(E) \\ &= \nu(E)\end{aligned}$$
   Since $E \in \mathcal{A}$ is arbitrary, $\nu_\mu|_\mathcal{A} \leq \nu$.

□

Exercise 5.3.0.21. Let $(X, \mathcal{A})$ be a measurable space, μ, ν measures on $(X, \mathcal{A})$ and $A \in \mathcal{A}$. Then

$$\nu_\mu(A) = \inf\{\nu(E) : E \in \mathcal{A},\, E \subset A \text{ and } E\ \mu^*\text{-covers } A\}$$

Proof. Set
$$V(A) = \{\nu(E) : E \in \mathcal{A} \text{ and } E\ \mu^*\text{-covers } A\}$$
and
$$V'(A) = \{\nu(E) : E \in \mathcal{A},\ E \subset A \text{ and } E\ \mu^*\text{-covers } A\}$$
Since $V'(A) \subset V(A)$, we have that
$$\begin{aligned} \nu_\mu(A) &= \inf V(A) \\ &\leq \inf V'(A) \end{aligned}$$

- First, suppose that $\nu_\mu(A) = \infty$. Since $\nu_\mu(A) \leq \inf V'(A)$, we have that
$$\begin{aligned} \nu_\mu(A) &= \infty \\ &= \inf V'(A) \end{aligned}$$

- Now, suppose that $\nu_\mu(A) < \infty$. Let $\epsilon > 0$. Then there exists $E \in \mathcal{A}$ such that E μ^*-covers A and $\nu(E) < \nu_\mu(A) + \epsilon$. Since $A, E \in \mathcal{A}$, we have that $A \cap E \in \mathcal{A}$. Exercise 5.3.0.12 implies that $A \cap E$ μ^*-covers A. Therefore $\nu(A \cap E) \in V'(A)$ and
$$\begin{aligned} \inf V'(A) &\leq \nu(A \cap E) \\ &\leq \nu(E) \\ &< \nu_\mu(A) + \epsilon \end{aligned}$$
Since $\epsilon > 0$ is arbitrary, $\inf V'(A) \leq \nu_\mu(A)$. Hence $\nu_\mu(A) = \inf V'(A)$.

$\square$

Exercise 5.3.0.22. Let $(X, \mathcal{A})$ be a measurable space and $\mu, \nu \in M_+(X, \mathcal{A})$. Then for each $A \in \mathcal{A}$, there exists $E \in \mathcal{A}$ such that $E \subset A$, E μ^*-covers A and $\nu_\mu(A) = \mu(E)$.

Proof. Let $A \in \mathcal{A}$.

- Suppose that $\nu_\mu(A) = \infty$. Define $E \in \mathcal{A}$ by $E := A$. Then $E \subset A$ and E μ^*-covers A. Since $\nu_\mu|_{\mathcal{A}} \leq \nu$,
$$\begin{aligned} \infty &= \nu_\mu(A) \\ &\leq \nu(A) \\ &= \nu(E) \end{aligned}$$
Thus
$$\begin{aligned} \nu(E) &= \infty \\ &= \nu(A) \end{aligned}$$

- Suppose that $\nu_\mu(A) < \infty$. Exercise 5.3.0.21 implies that for each $n \in \mathbb{N}$, there exists $E_n \in \mathcal{A}$ such that $E_n \subset A$, E_n μ^*-covers A and $\nu(E_n) < \nu_\mu(A) + 1/n$. Define $E \in \mathcal{A}$ by $E := \bigcap_{j \in \mathbb{N}} E_j$. By construction, $E \subset A$. Exercise 5.3.0.13 implies that E μ^*-covers A. By definition, $\nu_\mu(A) \leq \nu(E)$. Let $n \in \mathbb{N}$. By construction,
$$\begin{aligned} \nu(E) &\leq \nu(E_n) \\ &\leq \nu_\mu(A) + 1/n \end{aligned}$$
Since $n \in \mathbb{N}$ is arbitrary, $\nu(E) \leq \nu_\mu(A)$. Hence $\nu_\mu(A) = \mu(E)$.

$\square$

Exercise 5.3.0.23. Let $(X, \mathcal{A})$ be a measurable space, $\mu, \nu \in M_+(X, \mathcal{A})$, $A \subset X$ and $E \in \mathcal{A}$. If E μ^*-covers A and $\nu(E) = 0$, then $\nu_\mu(A) = 0$.

Proof. Set $V(A) = \{\nu(F) : F \in \mathcal{A} \text{ and } F\ \mu^*\text{-covers } A\}$. Suppose that E μ^*-covers A and $\nu(E) = 0$. Then $\nu(E) \in V(A)$ and therefore

$$\begin{aligned} \nu_\mu(A) &= \inf V(A) \\ &\leq \nu(E) \\ &= 0 \end{aligned}$$

Hence $\nu_\mu(A) = 0$. □

Exercise 5.3.0.24. Let $(X, \mathcal{A})$ be a measurable space and $\mu, \nu \in M_+(X, \mathcal{A})$. Let $A \subset X$ and $E \in \mathcal{A}$. If E μ^*-covers A, then for each $B \subset A$, $\nu_\mu(B) = \nu_\mu(B \cap E)$.

Proof. For each $B \subset X$, set $V(B) = \{\nu(F) : F \in \mathcal{A} \text{ and } F\ \mu^*\text{-covers } B\}$. Let $B \subset A$. Suppose that E μ^*-covers A. Since ν_μ is an outer measure, $\nu_\mu(B \cap E) \leq \nu_\mu(B)$. Exercise 5.3.0.12 implies that E μ^*-covers B. Therefore

$$\begin{aligned} \mu^*[(B \cap E^c) \setminus \varnothing] &= \mu^*[(B \cap E^c) \cap \varnothing^c] \\ &= \mu^*(B \cap E^c) \\ &= \mu^*(B \setminus E) \\ &= 0 \end{aligned}$$

Hence $\varnothing$ μ^*-covers $B \cap E^c$. Exercise 5.3.0.23 implies that $\nu_\mu(B \cap E^c) = 0$. Since ν_μ is an outer measure, we have that

$$\begin{aligned} \nu_\mu(B) &\leq \nu_\mu(B \cap E) + \nu_\mu(B \cap E^c) \\ &= \nu_\mu(B \cap E) \end{aligned}$$

Thus $\nu_\mu(B) = \nu_\mu(B \cap E)$. □

Definition 5.3.0.25. Let X be a set, ν an outer measure on X, $\mathcal{A} \subset \mathcal{A}_\nu$ a σ-algebra on X and $A, B \subset X$. Then B is said to be a $(\mathcal{A}, \nu)$**-hull of** A if

1. $B \in \mathcal{A}_\nu$
2. $A \subset B$
3. for each $E \in \mathcal{A}$, $\nu(A \cap E) = \nu(B \cap E)$

Exercise 5.3.0.26. Let X be a set, ν an outer measure on X, $\mathcal{A} \subset \mathcal{A}_\nu$ a σ-algebra on X, $B \in \mathcal{A}_\nu$ and $A \subset B$. Suppose that $\nu(A) = \nu(B)$ and $\nu(B) < \infty$. Then B is a $(\mathcal{A}, \nu)$-hull of A.

Proof. Let $E \in \mathcal{A}$. Since $A \subset B$ and $\nu(B) < \infty$, we have that $\nu(A) < \infty$. Since $\mathcal{A} \subset \mathcal{A}_\nu$ and $\nu(A), \nu(B) < \infty$, we have that $E \in \mathcal{A}_\nu$, $\nu(A \cap E) = \nu(A) - \nu(A \cap E^c)$ and $\nu(B \cap E) = \nu(B) - \nu(B \cap E^c)$. Since $A \subset B$, we have that $A \cap E \subset B \cap E$ and $A \cap E^c \subset B \cap E^c$. Therefore $\nu(A \cap E) \leq \nu(B \cap E)$ and $\nu(A \cap E^c) \leq \nu(B \cap E^c)$. Hence

$$\begin{aligned} \nu(A \cap E) &= \nu(A) - \nu(A \cap E^c) \\ &\geq \nu(A) - \nu(B \cap E^c) \\ &= \nu(B) - \nu(B \cap E^c) \\ &= \nu(B \cap E) \end{aligned}$$

So $\nu(A \cap E) = \nu(B \cap E)$. Since $E \in \mathcal{A}$ is arbitrary, B is a $(\mathcal{A}, \nu)$-hull of A. □

5.4 Subspace Measures

Definition 5.4.0.1. Let $(X, \mathcal{A})$ be a measurable space, μ a measure on $(X, \mathcal{A})$ and $E \in \mathcal{A}$.

- We define the **restriction of** μ **to** E, denoted $\mu|_E : \mathcal{A} \cap E \to [0, \infty]$, by
$$\mu|_E(A) := \mu(A)$$
- We define the **constriction of** μ **to** E **on** X, denoted $\mu_E : \mathcal{A} \to [0, \infty]$, by
$$\mu_E(A) := \mu(A \cap E)$$

reserve $\mu_{|E}$ to mean $\mu^E/\mu(E)$ if $0 < \mu(E) < \infty$ (i.e. the conditional of μ on E)

Exercise 5.4.0.2. Let $(X, \mathcal{A})$ be a measurable space, μ a measure on $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then

1. (a) $\mu|_E \in M_+(E, \mathcal{A} \cap E)$
   (b) $\mu_E \in M_+(X, \mathcal{A})$
2. Define $\iota : E \to X$ by $\iota(x) = x$. Then $\mu_E = \iota_* \mu|_E$

Proof. FINISH!!! □

Exercise 5.4.0.3. Let $(X, \mathcal{A})$ be a measurable space, μ a measure on $(X, \mathcal{A})$. Then

1. for each $(E_j)_{j \in \mathbb{N}} \subset \mathcal{A}$, $(E_j)_{j \in \mathbb{N}}$ is disjoint implies that
$$\mu_{\bigcup\limits_{j \in \mathbb{N}} E_j} = \sum_{j \in \mathbb{N}} \mu_{E_j},$$
2. for each $E, F \in \mathcal{A}$, if $E \subset F$, then $\mu_E \leq \mu_F$,
3. $\mu_X = \mu$,
4. $\mu_\varnothing = 0$.

Proof.

1. Suppose that $(E_j)_{j \in \mathbb{N}}$ is disjoint. Let $A \in \mathcal{A}$. Since $(E_j)_{j \in \mathbb{N}}$ is disjoint, we have that $(A \cap E_j)_{j \in \mathbb{N}}$ is disjoint. Therefore
$$\begin{aligned}
\mu_{\bigcup\limits_{j \in \mathbb{N}} E_j}(A) &= \mu\left[A \cap \Big(\bigcup_{j \in \mathbb{N}} E_j\Big)\right] \\
&= \mu\left[\bigcup_{j \in \mathbb{N}} (A \cap E_j)\right] \\
&= \sum_{j \in \mathbb{N}} \mu(A \cap E_j) \\
&= \sum_{j \in \mathbb{N}} \mu_{E_j}(A) \\
&= \left[\sum_{j \in \mathbb{N}} \mu_{E_j}\right](A).
\end{aligned}$$
Since $A \in \mathcal{A}$ is arbitrary, we have that for each $A \in \mathcal{A}$,
$$\mu_{\bigcup\limits_{j \in \mathbb{N}} E_j}(A) = \left[\sum_{j \in \mathbb{N}} \mu_{E_j}\right](A).$$
Thus
$$\mu_{\bigcup\limits_{j \in \mathbb{N}} E_j} = \left[\sum_{j \in \mathbb{N}} \mu_{E_j}\right].$$

2. Let $E, F \in \mathcal{A}$. Suppose that $E \subset F$. The previous part implies that

$$\begin{aligned}\mu_F &= \mu_{(F\cap E)\cup(F\cap E^c)} \\ &= \mu_{F\cap E} + \mu_{F\cap E^c} \\ &= \mu_E + \mu_{F\cap E^c} \\ &\geq \mu_E.\end{aligned}$$

3. Let $A \in \mathcal{A}$. Then

$$\begin{aligned}\mu_X(A) &= \mu(A \cap X) \\ &= \mu(A).\end{aligned}$$

   Since $A \in \mathcal{A}$ is arbitrary, we have that for each $A \in \mathcal{A}$, $\mu_X(A) = \mu(A)$. Hence $\mu_X = \mu$.

4. Let $A \in \mathcal{A}$. Then

$$\begin{aligned}\mu_\varnothing(A) &= \mu(A \cap \varnothing) \\ &= \mu(\varnothing) \\ &= 0.\end{aligned}$$

   Since $A \in \mathcal{A}$ is arbitrary, we have that for each $A \in \mathcal{A}$, $\mu_\varnothing(A) = 0$. Hence $\mu_\varnothing = 0$.

□

Definition 5.4.0.4. Let X be a set and ν an outer measure on X and $E \subset X$.

- We define the **restriction of ν to E**, denoted $\nu|_E : \mathcal{P}(E) \to [0, \infty]$, by

$$\nu|_E(A) := \nu(A)$$

- We define the **constriction of ν to E on X**, denoted $\nu_E : \mathcal{P}(X) \to [0, \infty]$, by

$$\nu_E(A) := \nu(A \cap E)$$

Exercise 5.4.0.5. Let X be a set, ν and outer measure on X and $E \subset X$.

1. (a) $\nu|_E$ is an outer measure on E
   (b) ν_E is an outer measure on X
2. Define $\iota : E \to X$ by $\iota(x) = x$. Then $\nu_E = \iota_* \nu|_E$. might need to put pushforward measure section after outer measure section instead of before, need to see if outer measure section uses pushforward, if not move it. Then can use pushforward here

Proof. FINISH!!! □

Exercise 5.4.0.6. Let X be a set, $E \in \mathcal{A}$ and $\mu \in M_+(X, \mathcal{A})$. Then $\mu|_E$ is σ-finite iff μ_E is σ-finite.

Proof. Define $\iota : E \to X$ by $\iota(x) = x$. Since $\mathcal{A} \cap E = \iota^* \mathcal{A}$, we have that ι is $(\mathcal{A} \cap E, \mathcal{A})$-measurable. Clearly ι is injective. Since $\mathcal{A} \cap E \subset \mathcal{A}$, we have that for each $A \in \mathcal{A} \cap E$,

$$\begin{aligned}\iota(A) &= A \\ &\in \mathcal{A}.\end{aligned}$$

Since $\mu_E = \iota_* \mu|_E$, Exercise 5.2.0.4 implies that $\mu|_E$ is σ-finite iff μ_E is σ-finite. □

Definition 5.4.0.7. Let X be a set, $E \in \mathcal{A}$ and $\mu \in M_+(X, \mathcal{A})$. Then E is said to be **σ-finite with respect to μ** if $\mu|_E$ is σ-finite

Exercise 5.4.0.8. Let $(X, \mathcal{A})$ be a measurable space, $\mu \in M_+(X, \mathcal{A})$ and $E, F \in \mathcal{A}$. Suppose that $E \subset F$. If F is σ-finite with respect to μ, then E is σ-finite with respect to μ.

Proof. Suppose that F is σ-finite with respect to μ. Then μ_F is σ-finite. Thus there exist $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ such that $X = \bigcup_{n\in\mathbb{N}} A_n$ and for each $n \in \mathbb{N}$, $\mu_F(A_n) < \infty$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned} \mu_E(A_n) &= \mu(A_n \cap E) \\ &\leq \mu(A_n \cap F) \\ &= \mu_F(A_n) \end{aligned}$$

□

Exercise 5.4.0.9. Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$. Then $\mu|_E^* = \mu^*|_E$.

Proof. Let $B \subset E$. Set

$$V(B) := \left\{ \sum_{j\in\mathbb{N}} \mu(F_j) : (F_j)_{j\in\mathbb{N}} \subset \mathcal{A} \text{ and } B \subset \bigcup_{j\in\mathbb{N}} F_j \right\}$$

and

$$V_E(B) := \left\{ \sum_{j\in\mathbb{N}} \mu|_E(F_j) : (F_j)_{j\in\mathbb{N}} \subset \mathcal{A} \cap E \text{ and } B \subset \bigcup_{j\in\mathbb{N}} F_j \right\}$$

Since $E \in \mathcal{A}$, we have that $\mathcal{A} \cap E \subset \mathcal{A}$. By definition, for each $F \in \mathcal{A} \cap E$, $\mu|_E(F) = \mu(F)$. Hence $V_E(B) \subset V(B)$ and

$$\begin{aligned} \mu^*|_E(B) &= \mu^*(B) \\ &= \inf V(B) \\ &\leq \inf V_E(B) \\ &= \mu|_E^*(B) \end{aligned}$$

- First, suppose that $\mu^*|_E(B) = \infty$. From before, we have that

  $$\begin{aligned} \infty &= \mu^*|_E(B) \\ &\leq \mu|_E^*(B) \end{aligned}$$

  so that

  $$\begin{aligned} \mu|_E^*(B) &= \infty \\ &= \mu^*|_E(B) \end{aligned}$$

  In particular, $\mu|_E^*(B) \leq \mu^*|_E(B)$.

- Now suppose that $\mu^*|_E(B) < \infty$. Then

  $$\begin{aligned} \mu^*(B) &= \mu^*|_E(B) \\ &< \infty \end{aligned}$$

  Let $\epsilon > 0$. Then there exists $(F_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $B \subset \bigcup_{j\in\mathbb{N}} F_j$ and $\sum_{j\in\mathbb{N}} \mu(F_j) < \mu^*(B) + \epsilon$. We observe that $(F_j \cap E)_{j\in\mathbb{N}} \subset \mathcal{A} \cap E$ and

  $$\begin{aligned} B &\subset \left[\bigcup_{j\in\mathbb{N}} F_j \right] \cap E \\ &= \bigcup_{j\in\mathbb{N}} (F_j \cap E) \end{aligned}$$

Hence

$$\begin{aligned}\mu|_E^*(B) &= \inf V_E(B) \\ &\leq \sum_{j\in\mathbb{N}} \mu|_E(F_j \cap E) \\ &= \sum_{j\in\mathbb{N}} \mu(F_j \cap E) \\ &\leq \sum_{j\in\mathbb{N}} \mu(F_j) \\ &< \mu^*(B) + \epsilon \\ &= \mu^*|_E(B) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\mu|_E^*(B) \leq \mu^*|_E(B)$.

Thus $\mu|_E^*(B) = \mu^*|_E(B)$. Since $B \subset E$ is arbitrary, $\mu|_E^* = \mu^*|_E$. $\square$

Exercise 5.4.0.10. Let $(X, \mathcal{A})$ be a measurable space, $\mu \in M_+(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then $(\mu^*)_E = (\mu_E)^*$.

Proof. Let $A \subset X$. Define $V(A), V_E(A)$ by

$$V(A \cap E) = \left\{ \sum_{j\in\mathbb{N}} \mu(F_j) : (F_j)_{j\in\mathbb{N}} \subset \mathcal{A} \text{ and } A \cap E \subset \bigcup_{j\in\mathbb{N}} F_j \right\}$$

and

$$V_E(A) := \left\{ \mu_E(F_j) : (F_j)_{j\in\mathbb{N}} \subset \mathcal{A} \text{ and } A \subset \bigcup_{j\in\mathbb{N}} F_j \right\}$$

Let $a \in V_E(A)$. Then there exists $(F_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $A \subset \bigcup\limits_{j\in\mathbb{N}} F_j$ and $a = \sum\limits_{j\in\mathbb{N}} \mu_E(F_j)$. Then $(F_j \cap E)_{j\in\mathbb{N}} \subset \mathcal{A}$ and

$$\begin{aligned}A \cap E &\subset \left(\bigcup_{j\in\mathbb{N}} F_j\right) \cap E \\ &= \bigcup_{j\in\mathbb{N}} (F_j \cap E)\end{aligned}$$

Thus

$$\begin{aligned}a &= \sum_{j\in\mathbb{N}} \mu_E(F_j) \\ &= \sum_{j\in\mathbb{N}} \mu(F_j \cap E) \\ &\in V(A \cap E)\end{aligned}$$

Since $a \in V_E(A)$ is arbitrary, we have that $V_E(A) \subset V(A \cap E)$. Therefore

$$\begin{aligned}(\mu^*)_E(A) &= \mu^*(A \cap E) \\ &= \inf V(A \cap E) \\ &\leq \inf V_E(A) \\ &= (\mu_E)^*(A)\end{aligned}$$

Exercise 5.3.0.17 implies that there exists $U \in \mathcal{A}$ such that $A \cap E \subset U$ and $\mu^*(A \cap E) = \mu(U)$. Then $A \cap E \subset U \cap E$. Since

$$\begin{aligned}\mu^*(A \cap E) &\leq \mu^*(U \cap E) \\ &= \mu(U \cap E) \\ &\leq \mu(U) \\ &= \mu^*(A \cap E)\end{aligned}$$

we have that $\mu^*(A \cap E) = \mu(U \cap E)$. Define $U' \in \mathcal{A}$ by $U' = (U \cap E) \cup E^c$. Then

$$\begin{aligned} A &= (A \cap E) \cup (A \cap E^c) \\ &\subset (U \cap E) \cup E^c \\ &= U' \end{aligned}$$

and

$$\begin{aligned} \mu_E(U') &= \mu_E(U' \cap E) + \mu_E(U' \cap E^c) \\ &= \mu_E(U \cap E) + \mu_E(E^c) \\ &= \mu(U \cap E) + \mu(\varnothing) \\ &= \mu(U \cap E) \\ &= \mu^*(A \cap E) \\ &= (\mu^*)_E(A) \end{aligned}$$

Therefore

$$\begin{aligned} (\mu_E)^*(A) &= \inf V_E(A) \\ &\leq \mu_E(U') \\ &= (\mu^*)_E(A) \end{aligned}$$

Since $A \subset X$ is arbitrary, we have that $(\mu_E)^* = (\mu^*)_E$. □

Exercise 5.4.0.11. Let X be a set, ν an outer measures on X, $E, F \subset X$ and $B \subset E$. If F ν-covers B, then $F \cap E$ $\nu|_E$-covers B.

Proof. Suppose that F ν-covers B. Since $B \subset E$, we have that $B \setminus (F \cap E) \subset E$ and therefore

$$\begin{aligned} \nu|_E[B \setminus (F \cap E)] &= \nu[B \setminus (F \cap E)] \\ &= \nu[B \cap (F \cap E)^c] \\ &= \nu[B \cap (F^c \cup E^c)] \\ &= \nu[(B \cap F^c) \cup (B \cap E^c)] \\ &= \nu[(B \cap F^c) \cup \varnothing] \\ &= \nu(B \cap F^c) \\ &= \nu(B \setminus F) \\ &= 0 \end{aligned}$$

So $F \cap E$ $\nu|_E$-covers B. □

Exercise 5.4.0.12. Let $(X, \mathcal{A})$ be a measurable space, ν, μ measures on $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then $\nu_\mu|_E = \nu|_{E_{\mu|_E}}$.

Proof. Let $B \subset E$. Set

$$V(B) = \{\nu(F) : F \in \mathcal{A} \text{ and } F \ \mu^*\text{-covers } B\}$$

and

$$V_E(B) = \{\nu|_E(F) : F \in \mathcal{A} \cap E \text{ and } F \ \mu|_E^*\text{-covers } B\}$$

Let $F \in \mathcal{A} \cap E$. Since $E \in \mathcal{A}$,

$$\begin{aligned} F &\in \mathcal{A} \cap E \\ &\subset \mathcal{A} \end{aligned}$$

Suppose that F $\mu|_E^*$-covers B. Since $B \subset E$, we have that $B \setminus F \subset E$. Exercise 5.4.0.9 implies that

$$\begin{aligned} \mu^*(B \setminus F) &= \mu^*|_E(B \setminus F) \\ &= \mu|_E^*(B \setminus F) \\ &= 0 \end{aligned}$$

and thus F μ^*-covers B. Since $F \in \mathcal{A} \cap E$ with F $\mu|_E^*$-covering B is arbitrary, $V_E(B) \subset V(B)$. Hence

$$\begin{aligned}\nu_\mu|_E(B) &= \nu_\mu(B)\\ &= \inf V(B)\\ &\leq \inf V_E(B)\\ &= \nu|_{E_{\mu|_E}}(B)\end{aligned}$$

- First, suppose that $\nu_\mu|_E(B) = \infty$. From before, we have that

$$\begin{aligned}\infty &= \nu_\mu|_E(B)\\ &\leq \nu|_{E_{\mu|_E}}(B)\end{aligned}$$

  Hence

$$\begin{aligned}\nu|_{E_{\mu|_E}}(B) &= \infty\\ &= \nu_\mu|_E(B)\end{aligned}$$

  In particular, $\nu|_{E_{\mu|_E}}(B) \leq \nu_\mu|_E(B)$.

- Now suppose that $\nu_\mu|_E(B) < \infty$. Then

$$\begin{aligned}\nu_\mu(B) &= \nu_\mu|_E(B)\\ &< \infty\end{aligned}$$

  Let $\epsilon > 0$. Choose $F \in \mathcal{A}$ such that F μ^*-covers B and $\nu(F) < \nu_\mu(B) + \epsilon$. Then $F \cap E \in \mathcal{A} \cap E$ and Exercise 5.4.0.11 implies that $F \cap E$ $\mu^*|_E$-covers B. Exercise 5.4.0.9 implies that $F \cap E$ $\mu|_E^*$-covers B. Hence

$$\begin{aligned}\nu|_{E_{\mu|_E}}(B) &= \inf V_E(B)\\ &\leq \nu|_E(F \cap E)\\ &= \nu(F \cap E)\\ &\leq \nu(F)\\ &< \nu_\mu(B) + \epsilon\\ &= \nu_\mu|_E(B) + \epsilon\end{aligned}$$

  Since $\epsilon > 0$ is arbitrary, $\nu|_{E_{\mu|_E}}(B) \leq \nu_\mu|_E(B)$.

Therefore, $\nu_\mu|_E(B) = \nu|_{E_{\mu|_E}}(B)$. Since $B \subset E$ is arbitrary, $\nu_\mu|_E = \nu|_{E_{\mu|_E}}$. $\square$

5.5 Product Measures

Definition 5.5.0.1. Let $(X,\mathcal{A},\mu),(Y,\mathcal{B},\nu)$ be σ-finite measurable spaces. Put $\mathcal{E}=\{A\times B : A\in\mathcal{A} \text{ and } B\in\mathcal{B}\}$. Then $\mathcal{E}$ is an elementary family and thus $\mathcal{M}_0=\{\bigcup_{i=1}^n M_i : (M_i)_{i=1}^n\subset\mathcal{E} \text{ are disjoint}\}$ is an algebra on $X\times Y$. We define $\pi_0:\mathcal{M}_0\to[0,\infty]$ by

$$\pi_0\left(\bigcup_{i=1}^n A_i\times B_i\right)=\sum_{i=1}^n \mu(A_i)\nu(B_i)$$

Then π_0 is a premeasure on $(X\times Y, M_0)$. Since $\mathcal{A}\otimes\mathcal{B}=\sigma(\mathcal{M}_0)$, we define the **product measure**, $\mu\otimes\nu$ on $(X\times Y,\mathcal{A}\otimes\mathcal{B})$, to be the unique extension of π_0 to $\mathcal{A}\otimes\mathcal{B}$. The existence of which is guaranteed by a theorem in the previous section. In particular,

$$\begin{aligned}\mu\otimes\nu(E)&=\inf\left\{\sum_{n\in\mathbb{N}}\pi_0(E_i) : (E_i)_{i\in\mathbb{N}}\subset\mathcal{M}_0 \text{ and } E\subset\bigcup_{i\in\mathbb{N}}E_i\right\}\\&=\inf\left\{\sum_{n\in\mathbb{N}}\mu(A_i)\nu(B_i) : (A_i\times B_i)_{i\in\mathbb{N}}\subset\mathcal{E} \text{ and } E\subset\bigcup_{i\in\mathbb{N}}A_i\times B_i\right\}\end{aligned}$$

5.6 Coproduct Measures

Chapter 6

The Lebesgue Integral

6.1 Integration of Nonnegative Functions

Definition 6.1.0.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in S^+(X, \mathcal{A})$. Suppose that the standard representation of f is $f = \sum_{k=1}^{n} a_k \chi_{E_k}$. We define the **integral of f with respect to μ**, denoted

$$\int_X f \, d\mu$$

by

$$\int_X f \, d\mu := \sum_{k=1}^{n} a_k \mu(E_k)$$

Exercise 6.1.0.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, $f, g \in S^+(X, \mathcal{A})$ and $\alpha \in [0, \infty)$. Then

$$\int_X f + \alpha g \, d\mu = \int_X f \, d\mu + \alpha \int_X g \, d\mu$$

Proof. FINISH!!! □

Definition 6.1.0.3. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in L^+(X, \mathcal{A})$. We define the **integral of f with respect to μ**, denoted

$$\int_X f \, d\mu$$

by

$$\int_X f \, d\mu := \sup \left\{ \int_X \phi \, d\mu : \phi \in S^+(X, \mathcal{A}) \text{ and } \phi \leq f \right\}.$$

Exercise 6.1.0.4. exercises about integrals of sums

Definition 6.1.0.5. Let $(X, \mathcal{A}, \mu)$ be a measure space, $f \in L^+(X, \mathcal{A})$ and $E \in \mathcal{A}$. We define the **integral of f over E with respect to μ**, denoted

$$\int_E f \, d\mu$$

by

$$\int_E f \, d\mu := \int_X f \chi_E \, d\mu.$$

Exercise 6.1.0.6. For each $f \in L^+(X, \mathcal{A})$,

$$\int_E f \, d\mu = \int f \, d\mu_E$$

Proof. FINISH!!! □

define and change notation from μ_f to $\mu^f(E) := \int_E f\,d\mu$.

Theorem 6.1.0.7. Monotone Convergence Theorem:
Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$. Then

$$\sup_{n\in\mathbb{N}} \int f_n = \int \sup_{n\in\mathbb{N}} f_n$$

.

Exercise 6.1.0.8. Let $(X, \mathcal{A})$ be a measurable space, $\mu_1, \mu_2 \subset M_+(X, \mathcal{A})$, $\lambda \in [0, \infty)$ and $f \in L^+(X, \mathcal{A})$. Define $\mu \in M_+(X, \mathcal{A})$ by $\mu := \mu_1 + \lambda\mu_2$. Then

$$\int_X f\,d\mu = \int_X f\,d\mu_1 + \lambda \int_X f\,d\mu_2.$$

Proof.

- Suppose that f is simple. Then there exist $(a_k)_{k=1}^n \subset [0, \infty)$ and $(E_k)_{k=1}^n \subset \mathcal{A}$ such that $(E_k)_{k=1}^n$ is disjoint and $f = \sum\limits_{k=1}^n a_k \chi_{E_k}$. Then

$$\begin{aligned}
\int_X f\,d\mu &= \sum_{k=1}^n a_k \mu(E_k) \\
&= \sum_{k=1}^n a_k \Big[\mu_1 + \lambda\mu_2\Big](E_k) \\
&= \sum_{k=1}^n a_k \Big[\mu_1(E_k) + \lambda\mu_2(E_k)\Big] \\
&= \sum_{k=1}^n a_k \mu_1(E_k) + \lambda \sum_{k=1}^n a_k \mu_2(E_k) \\
&= \int_X f\,d\mu_1 + \lambda \int_X f\,d\mu_2.
\end{aligned}$$

- Suppose that f is not simple. There exists $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that $\phi_n \to f$ pointwise and for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1}$. Then monotone convergence tells us that

$$\begin{aligned}
\int_X f\,d\mu &= \lim_{n\to\infty} \int_X \phi_n\,d\mu \\
&= \lim_{n\to\infty} \left[\int_X \phi_n\,d\mu_1 + \lambda \int_X \phi_n\,d\mu_2\right] \\
&= \lim_{n\to\infty} \int_X \phi_n\,d\mu_1 + \lambda \lim_{n\to\infty} \int_X \phi_n\,d\mu_2 \\
&= \int_X f\,d\mu_1 + \lambda \int_X f\,d\mu_2.
\end{aligned}$$

□

Exercise 6.1.0.9. Let $\mu_1, \mu_2 \in M_+(X, \mathcal{A})$. Suppose that $\mu_1 \leq \mu_2$. Then for each $f \in L^+$,

$$\int f d\mu_1 \leq \int f d\mu_2$$

Proof. First suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset [0,\infty)$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum\limits_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}\int f d\mu_1 &= \sum_{i=1}^n a_i \mu_1(E_i)\\ &\leq \sum_{i=1}^n a_i \mu_2(E_i)\\ &= \int f\, d\mu_2\end{aligned}$$

for general f,

$$\begin{aligned}\int f d\mu_1 &= \sup_{\substack{s\in S^+\\ s\leq f}} \int s\, d\mu_1\\ &\leq \sup_{\substack{s\in S^+\\ s\leq f}} \int s d\mu_2\\ &= \int f d\mu_2\end{aligned}$$

□

Theorem 6.1.0.10. Fatou's Lemma:
Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.$$

Theorem 6.1.0.11. Let $(f_n)_{n\in\mathbb{N}} \subset L^+$. Then

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n.$$

Exercise 6.1.0.12. Let $f \in L^+$ and suppose that $\int f < \infty$. Define $N, S \in \mathcal{B}(X)$ by

$$N = \{x \in X : f(x) = \infty\}, \quad S = \{x \in X : f(x) > 0\}.$$

Then $\mu(N) = 0$ and S is σ-finite.

Proof. Suppose that $\mu(N) > 0$. Define $f_n = n\chi_N \in L^+$. Then for each $n \in \mathbb{N}$, $f_n \leq f_{n+1} \leq f$ on N. So

$$\begin{aligned}\int f &\geq \int_N f\\ &= \lim_{n\to\infty} \int_N f_n\\ &= \lim_{n\to\infty} n\mu(N)\\ &= \infty, \text{ a contradiction.}\end{aligned}$$

Hence N is a null set. Now, put $S_n = \{x \in X : f(x) > 1/n\}$. Then $S = \bigcup\limits_{n\in\mathbb{N}} S_n$. Suppose that there exists some $n \in \mathbb{N}$ such that $\mu(S_n) = \infty$. Then

$$\begin{aligned}\int f &\geq \int_{S_n} f\\ &\geq \frac{1}{n}\mu(S_n)\\ &= \infty, \text{ a contradiction.}\end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(S_n) < \infty$ and S is σ-finite. □

Exercise 6.1.0.13. Let $f \in L^+$. Then $f = 0$ a.e. iff for each $E \in \mathcal{A}$, $\int_E f = 0$.

Proof. $f = 0$ a.e. implies that for each $E \in \mathcal{A}$, $\int_E f = 0$ is clear. Conversely, suppose that for each $E \in \mathcal{A}$, $\int_E f = 0$. For $n \in \mathbb{N}$ put $N_n = \{x \in X : f(x) > 1/n\}$ and define $N = \{x \in X : f(x) > 0\}$. So $N = \bigcup_{n\in\mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned} 0 &= \int_{N_n} f \\ &\geq \frac{1}{n}\mu(N_n) \\ &\geq 0. \end{aligned}$$

Hence for each $n \in \mathbb{N}$, $\mu(N_n) = 0$. Thus $\mu(N) = 0$ and $f = 0$ a.e. as required. □

Exercise 6.1.0.14. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(f_n)_{n\in\mathbb{N}} \subset L^+(X, \mathcal{A})$ and $f \in L^+(X, \mathcal{A})$. Suppose that $f_n \xrightarrow{\text{p.w.}} f$,

$$\lim_{n\to\infty} \int f_n = \int f \text{ and } \int f < \infty$$

Then for each $E \in \mathcal{A}$,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

This result may fail to be true if

$$\int f \, d\mu = \infty.$$

Proof. Let $E \in \mathcal{A}$. By Fatou's lemma,

$$\int_E f \, d\mu \leq \liminf_{n\to\infty} \int_E f_n.$$

Note that since

$$\int f < \infty,$$

we have that

$$\begin{aligned} \int_{E^c} f \, d\mu &\leq \int f \, d\mu \\ &< \infty. \end{aligned}$$

Thus we may write

$$\begin{aligned} \int_E f &= \int f - \int_{E^c} f \\ &\geq \int f - \liminf_{n\to\infty} \int_{E^c} f_n \\ &= \int f - \liminf_{n\to\infty} \left(\int f_n - \int_E f_n \right) \\ &= \int f - \int f + \limsup_{n\to\infty} \int_E f_n \\ &= \limsup_{n\to\infty} \int_E f_n. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} \int_E f_n \leq \int_E f \leq \liminf_{n\to\infty} \int_E f_n$$

and therefore

$$\lim_{n\to\infty}\int_E f_n = \int_E f.$$

If we drop the assumption that $\int f < \infty$, then the result would fail to be true for the functions $f = \infty\chi_{(0,1)}$ and $f_n = \infty\chi_{(0,1)} + n\chi_{(1,1+1/n)}$. Here $f_n \xrightarrow{\text{p.w.}} f$, $\lim\limits_{n\to\infty}\int f_n = \int f = \infty$ and $\lim\limits_{n\to\infty}\int_{(1,\infty)} f_n = 1$ while $\int_{(1,\infty)} f = 0$. $\square$

Exercise 6.1.0.15. Let X be a set and $f \in L^+(X, \mathcal{P}(X))$. Then

$$\int f\, d\# = \sup\left\{\sum_{x\in F} f(x) : F \subset X \text{ and } \#(F) < \infty\right\}$$

Proof. Define $A_1, A_2 \subset [0,\infty]$ by

$$A_1 = \left\{\int \phi\, d\# : \phi \in S^+(X,\mathcal{A}) \text{ and } \phi \leq f\right\}, \quad A_2 = \left\{\sum_{x\in F} f(x) : F \subset X \text{ and } \#(F) < \infty\right\}$$

- Let $y \in A_1$. Then there exists $\phi \in S^+(X,\mathcal{A})$ such that $\phi \leq f$ and

  $$y = \int \phi\, d\#$$

  Thus there exist $(E_j)_{j=1}^n \subset \mathcal{P}(X)$ and $(a_j)_{j=1}^n \subset [0,\infty)$ such that $\operatorname{Im}\phi = (a_j)_{j=1}^n$, $(E_j)_{j=1}^n$ is disjoint and $\phi = \sum\limits_{j=1}^n a_j\chi_{E_j}$.

  - Suppose that $y = \infty$. Then $\sup A_1 = \infty$. If for each $j \in \{1,\dots,n\}$, $a_j = 0$ or $\#(E_j) \neq \infty$, then

    $$\begin{aligned} y &= \int \phi\, d\# \\ &= \sum_{j=1}^n a_j \#(E_j) \\ &< \infty \end{aligned}$$

    which is a contradiction. Therefore, there exists $j_0 \in \{1,\dots,n\}$ such that $a_{j_0} > 0$ and $\#(E_{j_0}) = \infty$. Then there exists $(x_l)_{l\in\mathbb{N}} \subset E_{j_0}$ such that for each $k, l \in \mathbb{N}$, $k \neq l$ implies that $x_k \neq x_l$. For $k \in \mathbb{N}$, define $F_k \subset E_{j_0}$ by

    $$F_k = \bigcup_{l=1}^k \{x_l\}$$

    Then for each $k \in \mathbb{N}$, $\#(F_k) = k$ and

    $$\begin{aligned} a_{j_0}k &= \sum_{x\in F_k}\phi(x) \\ &\leq \sum_{x\in F_k} f(x) \\ &\in A_2 \end{aligned}$$

    which implies that

    $$\begin{aligned} \sup A_2 &\geq \sup_{k\in\mathbb{N}}\sum_{x\in F_k} f(x) \\ &\geq \sup_{k\in\mathbb{N}} a_{j_0}k \end{aligned}$$

    Since $k \in \mathbb{N}$ is arbitrary, $\sup A_2 = \infty$ and in particular, $y \leq \sup A_2$.

- Suppose that $y \neq \infty$. Then for each $j \in \{1, \ldots, n\}$, $a_j = 0$ or $\#(E_j) < \infty$. Define $J \subset \{1, \ldots, n\}$ and $F \subset X$ by

$$J = \{j \in \{1, \ldots, n\} : \#(E_j) < \infty\}, \quad F = \bigcup_{j \in J} E_j$$

We note that since J is finite, $\#(F) < \infty$ and for each $j \in J^c$, $a_j = 0$. Therefore

$$\begin{aligned} y &= \int \phi \, d\# \\ &= \sum_{j=1}^{n} a_j \#(E_j) \\ &= \sum_{j \in J} a_j \#(E_j) \\ &= \sum_{j \in J} \sum_{x \in E_j} \phi(x) \\ &= \sum_{x \in F} \phi(x) \\ &\leq \sum_{x \in F} f(x) \\ &\leq \sup A_2 \end{aligned}$$

Since $y \in A_1$ is arbitrary, we have that for each $y \in A_1$, $y \leq \sup A_2$. Hence $\sup A_1 \leq \sup A_2$.

- Conversely, let $y \in A_2$. Then there exists $F \subset X$ such that $\#(F) < \infty$ and $y = \sum_{x \in F} f(x)$. Define $\phi \in S^+(X, \mathcal{P}(X))$ by $\phi = f\chi_F$. Then $\phi \leq f$ and

$$\begin{aligned} y &= \int \phi \, d\# \\ &\in A_1 \end{aligned}$$

Since $y \in A_2$ is arbitrary, $A_2 \subset A_1$. Thus $\sup A_2 \leq \sup A_1$.

Since $\sup A_1 \leq \sup A_2$ and $\sup A_2 \leq \sup A_1$, we have that

$$\begin{aligned} \int f \, d\# &= \sup A_1 \\ &= \sup A_2. \end{aligned}$$

□

Exercise 6.1.0.16. Let X be a set and $f \in L^+(X, \mathcal{P}(X))$. If f is $\#$-integrable, then $\{x \in X : f(x) > 0\}$ is countable.

Proof. Suppose that f is integrable. For $n \in \mathbb{N}$, set $X_n = \{x \in X : f(x) > 1/n\}$ and define $X_+ = \{x \in X : f(x) > 0\}$. Then $X_+ = \bigcup_{n \in \mathbb{N}} X_n$. Since f is integrable, we have that for each $n \in \mathbb{N}$,

$$\begin{aligned} \infty &> \int f \, d\# \\ &\geq \int_{X_n} f \, d\# \\ &\geq \frac{1}{n} \#(X_n). \end{aligned}$$

Thus for each $n \in \mathbb{N}$, X_n is finite and X_+ is countable. □

Definition 6.1.0.17. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in L^+(X, \mathcal{A})$. We define $\mu^f : \mathcal{A} \to [0, \infty]$ by

$$\mu^f(E) := \int_E f \, d\mu.$$

Note 6.1.0.18. Exercise 6.1.0.6 implies that

$$\mu^f(E) = \int_X f \, d\mu_E.$$

Exercise 6.1.0.19. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in L^+(X, \mathcal{A})$. Then

1. μ^f is a measure on $(X, \mathcal{A})$
2. for each $g \in L^+(X, \mathcal{A})$,

$$\int g d\mu^f = \int g f d\mu$$

Proof.

1. Clearly $\mu^f(\varnothing) = 0$. Let $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$. Suppose that $(A_j)_{j \in \mathbb{N}}$ is disjoint.

   - Suppose that $f \in S^+(X, \mathcal{A})$. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0, \infty)$ such that $f = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}
\mu^f\bigg(\bigcup_{j \in \mathbb{N}} A_j\bigg) &= \int_{\bigcup_{j \in \mathbb{N}} A_j} f \\
&= \sum_{i=1}^n a_i \mu\bigg(E_i \cap \bigg(\bigcup_{j \in \mathbb{N}} A_j\bigg)\bigg) \\
&= \sum_{i=1}^n a_i \mu\bigg(\bigcup_{j \in \mathbb{N}} E_i \cap A_j\bigg) \\
&= \sum_{i=1}^n a_i \sum_{j \in \mathbb{N}} \mu(E_i \cap A_j) \\
&= \sum_{j \in \mathbb{N}} \sum_{i=1}^n a_i \mu(E_i \cap A_j) \\
&= \sum_{j \in \mathbb{N}} \int_{A_j} f \, d\mu \\
&= \sum_{j \in \mathbb{N}} \mu^f(A_j)
\end{aligned}$$

   - Suppose that $f \notin S^+(X, \mathcal{A})$. Then there exist $(\phi_n)_{n \in \mathbb{N}} \subset S^+(X, \mathcal{A})$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1}$ and $\phi_n \xrightarrow{\text{p.w.}} f$. Set $A = \bigcup_{j \in \mathbb{N}} A_j$ and for each $n \in \mathbb{N}$, define $\mu_n : \mathcal{A} \to [0, \infty]$ by

$$\mu_n(E) := \int_E \phi_n \, d\mu$$

   From above, we have that for each $n \in \mathbb{N}$, $\mu_n \in M_+(X, \mathcal{A})$. For $n \in \mathbb{N}$, we define $g_n \in L^+(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ by

$$g_n(j) = \int_{A_j} \phi_n \, d\mu$$

Then for each $n \in \mathbb{N}$, $g_n \leq g_{n+1}$. The monotone convergence theorem implies that

$$\begin{aligned}
\mu^f(A) &= \int_A f\, d\mu \\
&= \lim_{n\to\infty} \int_A \phi_n\, d\mu \quad \text{(by monotone convergence theorem)} \\
&= \lim_{n\to\infty} \mu_n(A) \\
&= \lim_{n\to\infty} \sum_{j\in\mathbb{N}} \mu_n(A_j) \\
&= \lim_{n\to\infty} \sum_{j\in\mathbb{N}} \int_{A_j} \phi_n\, d\mu \\
&= \lim_{n\to\infty} \sum_{j\in\mathbb{N}} g_n(j) \\
&= \lim_{n\to\infty} \int g_n\ d\# \\
&= \int \lim_{n\to\infty} g_n\ d\# \quad \text{(by monotone convergence theorem)} \\
&= \sum_{j\in\mathbb{N}} \lim_{n\to\infty} g_n(j) \\
&= \sum_{j\in\mathbb{N}} \lim_{n\to\infty} \int_{A_j} \phi_n\, d\mu \\
&= \sum_{j\in\mathbb{N}} \int_{A_j} f\, d\mu \\
&= \sum_{j\in\mathbb{N}} \mu^f(A_j).
\end{aligned}$$

Hence μ^f is a measure on $(X, \mathcal{A})$.

2. Let $g \in L^+$.

- Suppose that $g \in S^+(X, \mathcal{A})$. Then there exist $E_1, E_2, \cdots, E_n \in \mathcal{A}$ and $a_1, a_2, \cdots, a_n \in [0, \infty)$ such that $g = \sum_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}
\int g d\lambda &= \sum_{i=1}^n a_i \lambda(E_i) \\
&= \sum_{i=1}^n a_i \int_{E_i} f d\mu \\
&= \int \left(\sum_{i=1}^n a_i \chi_{E_i} \right) f d\mu \\
&= \int g f d\mu.
\end{aligned}$$

- Suppose that $g \notin S^+(X, \mathcal{A})$. Then there exist $(\psi_n)_{n\in\mathbb{N}} \subset S^+(X, \mathcal{A})$ such that for each $n \in \mathbb{N}$, $\psi_n \leq \psi_{n+1}$ and

$\psi_n \xrightarrow{\text{p.w.}} g$. Therefore for each $n \in \mathbb{N}$, $\psi_n f \leq \psi_{n+1} f \leq gf$ and $\psi_n f \xrightarrow{\text{p.w.}} gf$. Monotone convergence implies that

$$\begin{aligned}\int g\,d\lambda &= \lim_{n\to\infty} \int \psi_n\,d\lambda \\ &= \lim_{n\to\infty} \int \psi_n f\,d\mu \\ &= \int gf\,d\mu\end{aligned}$$

□

Exercise 6.1.0.20. Let $(f_n)_{n\in\mathbb{N}} \subset L^+$ and $f \in L^+$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$, $f_n \xrightarrow{\text{p.w.}} f$ and f_1 is integrable. Then

$$\lim_{n\to\infty} \int f_n = \int f$$

Proof. First we note that since $\int f_1 < \infty$, $f_1 < \infty$ a.e., for each $n \in \mathbb{N}$, $f_1 - f_n$ and $\int f_1 - \int f_n$ are well defined and $\int f_n \leq \int f_1 < \infty$. Also, for $n \in \mathbb{N}$, $f_1 - f_n \in L^+$. So we may write

$$\begin{aligned}\int (f_1 - f_n) &= \int (f_1 - f_n) + \int f_n - \int f_n \\ &= \int [(f_1 - f_n) + f_n] - \int f_n \\ &= \int f_1 - \int f_n\end{aligned}$$

Put $g_n = f + (f_1 - f_n)$. Then $g_n \in L^+$, for each $n \in \mathbb{N}$, $g_n \leq g_{n+1}$ and $g_n \xrightarrow{\text{p.w.}} f_1$. Monotone convergence tells us that

$$\begin{aligned}\int f_1 &= \lim_{n\to\infty} \int g_n \\ &= \lim_{n\to\infty} \left[\int f + (f_1 - f_n)\right] \\ &= \lim_{n\to\infty} \left[\int f + \int (f_1 - f_n)\right] \\ &= \lim_{n\to\infty} \left[\int f + \int f_1 - \int f_n\right]\end{aligned}$$

Since $\lim\limits_{n\to\infty} \int f$ and $\lim\limits_{n\to\infty} \int f_1$ exist, $\lim\limits_{n\to\infty} \int f_n = \int f$ as required. □

Exercise 6.1.0.21. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ a measurable space and $f : X \to Y$. Suppose that f is $(\mathcal{A}, \mathcal{B})$-measurable. Then for each $g \in L^+(Y, \mathcal{B})$ and $B \in \mathcal{B}$,

$$\int_{f^{-1}(B)} g \circ f\,d\mu = \int_B g\,df_*\mu$$

Proof. Let $g \in L^+(X, \mathcal{A})$ and $B \in \mathcal{B}$. Suppose that there exists $E \in \mathcal{B}$ such that $g = \chi_E$. Then $g \circ f = \chi_{f^{-1}(E)}$ and

$$\begin{aligned}\int_{f^{-1}(B)} g \circ f\,d\mu &= \int_{f^{-1}(B)} \chi_{f^{-1}(E)}\,d\mu \\ &= \mu(f^{-1}(E) \cap f^{-1}(B)) \\ &= \mu(f^{-1}(E \cap B)) \\ &= f_*\mu(E \cap B) \\ &= \int_B \chi_E\,df_*\mu \\ &= \int_B g\,df_*\mu\end{aligned}$$

Suppose that g is simple. Then there exist $(a_j)_{j=1}^n \subset [0,\infty)$ and $(E_j)_{j=1}^n \subset \mathcal{B}$ such that $g = \sum_{j=1}^n a_j\chi_{E_j}$. Then

$$\begin{aligned} g \circ f &= \left(\sum_{j=1}^n a_j\chi_{E_j}\right) \circ f \\ &= \sum_{j=1}^n a_j\chi_{E_j} \circ f \end{aligned}$$

and

$$\begin{aligned} \int_{f^{-1}(B)} g \circ f \, d\mu &= \int_{f^{-1}(B)} \sum_{j=1}^n a_j\chi_{E_j} \circ f \, d\mu \\ &= \sum_{j=1}^n a_j \int_{f^{-1}(B)} \chi_{E_j} \circ f \, d\mu \\ &= \sum_{j=1}^n a_j \int_B \chi_{E_j} \, df_*\mu \\ &= \int_B \sum_{j=1}^n a_j\chi_{E_j} \, df_*\mu \\ &= \int_B g \, df_*\mu \end{aligned}$$

Suppose that g is not simple. Then there exists $(\phi_n)_{n\in\mathbb{N}} \subset S^+(Y,\mathcal{B})$ such that $\phi_n \xrightarrow{\text{p.w.}} g$ and for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1}$. Then $\phi_n \circ f \xrightarrow{\text{p.w.}} g \circ f$ and for each $n \in \mathbb{N}$, $\phi_n \circ f \leq \phi_{n+1} \circ f$. The monotone convergence theorem implies that

$$\begin{aligned} \int_{f^{-1}(B)} g \circ f \, d\mu &= \lim_{n\to\infty} \int_{f^{-1}(B)} \phi_n \circ f \, d\mu \\ &= \lim_{n\to\infty} \int_B \phi_n \, df_*\mu \\ &= \int_B g \, df_*\mu \end{aligned}$$

$\square$

Exercise 6.1.0.22. Let $(X,\mathcal{A},\mu)$ be a measure space, $(Y,\mathcal{B})$ a measurable space and $f : X \to Y$. Suppose that f is $(\mathcal{A},\mathcal{B})$-measurable. Then for each $g \in L^+(Y,\mathcal{B})$,

$$f_*(\mu^{f^*g}) = (f_*\mu)^g.$$

DEFINE $f^*g := g \circ f$ in set theory section

Proof. Let $B \in \mathcal{B}$. Exercise 6.1.0.21 implies that

$$\begin{aligned} f_*(\mu^{f^*g})(B) &= \mu^{f^*g}(f^{-1}(B)) \\ &= \int_{f^{-1}(B)} g \circ f \, d\mu \\ &= \int_B g \, df_*\mu \\ &= (f_*\mu)^g(B). \end{aligned}$$

Since $B \in \mathcal{B}$ is arbitrary, we have that for each $B \in \mathcal{B}$, $f_*(\mu_{f^*g})(B) = (f_*\mu)_g(B)$. Thus $f_*(\mu^{f^*g}) = (f_*\mu)^g$. $\square$

Exercise 6.1.0.23. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ a measurable space and $f : X \to Y$. Suppose that f is $(\mathcal{A}, \mathcal{B})$-measurable. Let $g, h \in L^0(Y, \mathcal{B})$. Then $g \circ f = h \circ f$ μ-a.e. iff $g = h$ $f_*\mu$-a.e.

Proof.

- ($\Longrightarrow$):
  Suppose that $g \circ f = h \circ f$ μ-a.e. Then $|(g-h) \circ f| = 0$ μ-a.e. The previous exercise implies that

$$\begin{aligned}\int_Y |g-h| \, df_*\mu &= \int_X |g-h| \circ f \, d\mu \\ &= \int_X |(g-h) \circ f| \, d\mu \\ &= \int_X |g \circ f - h \circ f| \, d\mu \\ &= 0\end{aligned}$$

  Hence $|g-h| = 0$ $f_*\mu$-a.e. and $g = h$ $f_*\mu$-a.e.

- ($\Longleftarrow$):
  Suppose that $g = h$ $f_*\mu$-a.e. Then

$$\begin{aligned}\int_X |g \circ f - h \circ f| \, d\mu &= \int_X |(g-h) \circ f| \, d\mu \\ &= \int_X |g-h| \circ f \, d\mu \\ &= \int_Y |g-h| \, df_*\mu \\ &= 0\end{aligned}$$

  Hence $|g \circ f - h \circ f| = 0$ μ-a.e. and $g \circ f = h \circ f$ μ-a.e.

□

Note 6.1.0.24. The previous exercise says that in the category of measurable spaces where morphisms are measure preserving (under pushforward) measurable maps, then all morphisms are epimorphisms.

6.2 Integration of Complex Valued Functions

- In the previous section, define
$$\mu_f(A) := \int_A f \, d\mu$$
and in this section change notation to define
$$\mu^f(A) := \int_X f \, d\mu_A$$
and show that when f is $\pi^*\mathcal{B}$-measurable,
$$\pi_*(\mu^f) = (\pi_*\mu)^{\pi^* f}$$
show something similar for μ_f, However, this conflicts with borel measure on $\mathbb{R}$ notation μ_F
- Actually, get rid of μ_f, write μ^f. Turns out $\mu_f = \mu^f$ (see previous section). and hide proof of μ^f being a measure in Exercise **??**.

Definition 6.2.0.1. Let $f : X \to \mathbb{C}$ be measurable. Then f is said to be **integrable** if

$$\int |f| \, d\mu < \infty$$

need to define

$$\mathcal{L}^1(X, \mathcal{A}, \mu) := \left\{ f : X \to \mathbb{C} : f \text{ is measurable and } \int |f| < \infty \right\}$$

and $\sim_\mu$ on $\mathcal{L}^1(X, \mathcal{A}, \mu)$ by $f \sim_\mu g$ iff $f = g$ μ-a.e. Then define $L^1(X, \mathcal{A}, \mu) := \mathcal{L}(X, \mathcal{A}, \mu)/\sim_\mu$

Definition 6.2.0.2. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define

$$L^1(X, \mathcal{A}, \mu) = \left\{ f : X \to \mathbb{C} : f \text{ is measurable and } \int |f| < \infty \right\}$$

Exercise 6.2.0.3. Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff f^+ and f^- are integrable.

Proof. $f^+, f^- \leq |f| = f^+ + f^-$ □

Definition 6.2.0.4. Let $f : X \to \mathbb{R}$ be measurable. Then f is said to be **extended integrable** if

$$\int f^+ \, d\mu < \infty \text{ or } \int f^- \, d\mu < \infty$$

Exercise 6.2.0.5. Let $f : X \to \mathbb{R}$ be measurable. Then f is integrable iff $Re(f)$ and $Im(f)$ are integrable.

Proof. $|Re(f)|, |Im(f)| \leq |f| \leq |Re(f)| + |Im(f)|$ □

Exercise 6.2.0.6. Dominated Convergence Theorem:
Let $(f_n)_{n\in\mathbb{N}} \subset L^0$, $f \in L^0$ and $g \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$ and there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f \in L^1$ and

$$\int_X |f_n - f| \, d\mu \to 0$$

Hint: Fatou's lemma

Proof. Continuity imples that $|f| \leq g$ a.e. Since

$$\begin{aligned} |f_n - f| &\leq |f_n| + |f| \\ &\leq 2g \end{aligned}$$

Fatou's lemma implies that

$$\begin{aligned}\int 2g\,d\mu &= \int \liminf_{n\to\infty}(2g - |f_n - f|)\,d\mu \\ &\leq \liminf_{n\to\infty}\int 2g - |f_n - f|\,d\mu \\ &= \int 2g\,d\mu - \limsup_{n\to\infty}\int |f_n - f|\,d\mu\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\int |f_n - f|\,d\mu \leq 0$$

and thus

$$\int |f_n - f|\,d\mu \to 0$$

□

Exercise 6.2.0.7. Let μ_1, μ_2 be measures on $(X, \mathcal{A})$. Then

1. $L^1(\mu_1 + \mu_2) = L^1(\mu_1) \cap L^1(\mu_2)$
2. for each $f \in L^1(\mu_1 + \mu_2)$, we have that

$$\int f d(\mu_1 + \mu_2) = \int f\,d\mu_1 + \int f\,d\mu_2$$

Proof. 1. The firt part is clear since similar exercise from the section on nonnegative funtions tells us that

$$\int |f| d(\mu_1 + \mu_2) = \int |f|\,d\mu_1 + \int |f|\,d\mu_2$$

2. Suppose that f is simple. Then there exist $(a_n)_{i=1}^n \subset \mathbb{C}$ and $(E_i)_{i=1}^n \subset \mathcal{A}$ such that $f = \sum\limits_{i=1}^n a_i \chi_{E_i}$. Then

$$\begin{aligned}\int f d(\mu_1 + \mu_2) &= \sum_{i=1}^n a_i(\mu_1 + \mu_2)(E_i) \\ &= \sum_{i=1}^n a_i(\mu_1(E_i) + \mu_2(E_i)) \\ &= \sum_{i=1}^n a_i\mu_1(E_i) + a_i\mu_2(E_i) \\ &= \int f\,d\mu_1 + \int f\,d\mu_2\end{aligned}$$

Now for general f, choose $(\phi_n)_{n\in\mathbb{N}} \subset S$ such that $\phi_n \to f$ pointwise and for each $n \in \mathbb{N}$, $|\phi_n| \leq |\phi_{n+1}| \leq |f|$. Then dominated convergence tells us that

$$\begin{aligned}\int f d(\mu_1 + \mu_2) &= \lim_{n\to\infty}\int \phi_n d(\mu_1 + \mu_2) \\ &= \lim_{n\to\infty}\int \phi_n\,d\mu_1 + \lim_{n\to\infty}\int \phi_n\,d\mu_2 \\ &= \int f\,d\mu_1 + \int f\,d\mu_2\end{aligned}$$

□

Exercise 6.2.0.8. Let $(f_n)_{n\in\mathbb{N}} \subset L^1$. Suppose that

$$\sum_{n\in\mathbb{N}} \int |f_n| < \infty.$$

Then after redefinition on a set of measure zero, $\sum_{n\in\mathbb{N}} f_n \in L^1$ and

$$\int \sum_{n\in\mathbb{N}} f_n = \sum_{n\in\mathbb{N}} \int f_n$$

Proof. content... $\square$

Exercise 6.2.0.9. Let $f \in L^1$. Then for each $\epsilon > 0$, there exists $\phi \in L^1$ such that ϕ is simple and $\int |f - \phi| < \epsilon$. FINISH!!!

Proof. content... $\square$

Exercise 6.2.0.10. Generalized Fatou's Lemma: Let $(X, \mathcal{A}, \mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}} \subset L^0(X, \mathcal{A})$. Suppose that for each $n \in \mathbb{N}$, $f : X \to \mathbb{R}$, there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \geq -g$. Then

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

What is the analogue of Fatou's lemma for measurable, real valued functions that are appropriately bounded above?

Proof. First note that for each $n \in \mathbb{N}$, $\int f_n$ is well defined since $f_n^- \leq g \in L^1$. Since $g + f_n \geq 0$, we may use Fatou's lemma to write

$$\begin{aligned}
\int g \, d\mu + \int \liminf_{n\to\infty} f_n \, d\mu &= \int \liminf_{n\to\infty} (g + f_n) \, d\mu \\
&\leq \liminf_{n\to\infty} \int (g + f_n) \, d\mu \\
&= \int g \, d\mu + \liminf_{n\to\infty} \int f_n \, d\mu
\end{aligned}$$

Since $\int g < \infty$, $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$ as required. The analogue is as follows: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable real valued functions. Suppose that there exists $g \in L^1$ such that $g \geq 0$ and for each $n \in \mathbb{N}$, $f_n \leq g$. Then $\limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n$. To show this, just use the result from above with the sequence $(g_n)_{n\in\mathbb{N}}$ given by $g_n = -f_n$. $\square$

Exercise 6.2.0.11. Let $(f_n)_{n\in\mathbb{N}} \subset L^1(X, \mathcal{A}, \mu)$ and $f : X \to \mathbb{C}$. Suppose that $f_n \xrightarrow{\text{u}} f$. Then

1. if $\mu(X) < \infty$, then $f \in L^1(X, \mathcal{A}, \mu)$ and $\lim_{n\to\infty} \int f_n = \int f$
2. if $\mu(X) = \infty$, then the conclusion of (1) may fail (find an example on $\mathbb{R}$ with Lebesgue measure).

Proof. Choose $N \in \mathbb{N}$ such that for $n \geq N$ and $x \in X$, $|f(x) - f_n(x)| < 1$. Then $||f| - |f_N|| < 1$ and so $|f| < |f_N| + 1$. Thus $\int |f| \leq \int |f_N| + \mu(X) < \infty$ and $f \in L^1$. Similarly for $n \geq N$, $|f_n| < |f| + 1$. Dominated convergence then gives us that $\lim_{n\to\infty} \int f_n = \int f$ as required. To see the necessity that $\mu(X) < \infty$, consider $f \equiv 0$ and $f_n = (1/n)\chi_{(0,n)}$. Then $f_n \xrightarrow{\text{u}} f$, but $1 = \lim_{n\to\infty} \int f_n \neq \int f = 0$. $\square$

Exercise 6.2.0.12. Generalized Dominated Convergence Let $f_n, g_n, f, g \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$, $g_n \xrightarrow{\text{a.e.}} g$ and for each $n \in \mathbb{N}$, $|f_n| \leq g_n$. If

$$\int g_n \, d\mu \to \int g \, d\mu$$

then

$$\int f_n \, d\mu \to \int f \, d\mu$$

.

Proof. We simply use Fatou's lemma. Put $h_n = (g + g_n) - |f_n - f|$. Since for each $n \in \mathbb{N}$, $|f_n| \leq g_n$, we know that $|f| \leq g$. So $h_n \geq 0$ and $h_n \xrightarrow{\text{p.w.}} 2g$. Thus

$$\begin{aligned} 2\int g &= \int \liminf_{n\to\infty} h_n \\ &\leq \liminf_{n\to\infty} \left[\left(\int g + \int g_n\right) - \int |f_n - f|\right] \\ &= 2\int g + \liminf_{n\to\infty}\left(-\int |f_n - f|\right) \\ &= 2\int g - \limsup_{n\to\infty} \int |f_n - f| \end{aligned}$$

Hence $\limsup_{n\to\infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $\int f_n \to \int f$ as required. □

Exercise 6.2.0.13. Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$.

Proof. Suppose that $\int |f_n - f| \to 0$. Since

$$\begin{aligned} \left|\int |f_n| - \int |f|\right| &= \left|\int (|f_n| - |f|)\right| \\ &\leq \int ||f_n| - |f|| \\ &\leq \int |f_n - f|, \end{aligned}$$

we see that $\int |f_n| \to \int |f|$. Conversely, suppose that $\int |f_n| \to \int |f|$. Put $h_n = |f_n - f|$, $g_n = |f_n| + |f|$, $h \equiv 0$ and $g = 2f$. Then $h_n \xrightarrow{\text{a.e.}} h$, $g_n \xrightarrow{\text{a.e.}} g$ and for each $n \in \mathbb{N}$, $h_n \leq g_n$. Our assumption implies that $\int g_n \to \int g$. Thus the last exercise tells us that $\int h_n \to \int h$ as required.

□

Exercise 6.2.0.14. Let $(r_n)_{n\in\mathbb{N}}$ be an enummeration of the rationals. Define $f : \mathbb{R} \to [0, \infty)$ by

$$f(x) = \begin{cases} x^{-\frac{1}{2}} & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

and define $g : X \to [0, \infty]$ by

$$g(x) = \sum_{n\in\mathbb{N}} 2^{-n} f(x - r_n).$$

Then

1. $g \in L^1$ (perhaps after redefinition on a null set) and particularly $g < \infty$ a.e.

2. $g^2 < \infty$ a.e., but g^2 is not integrable on any subinterval of $\mathbb{R}$

3. Taking $g \in L^1$, g is unbounded on each subinterval of $\mathbb{R}$ and discontinuous everywhere and remains so after redefinition on a null set

Proof. For convenience, define $f_n : \mathbb{R} \to [0, \infty)$ by $f_n(x) = f(x - r_n)$ for $x \in \mathbb{R}$. To show (1) we note that for each $n \in \mathbb{N}$, $f_n \in L^1$ and

$$\begin{aligned} \int |2^{-n} f_n| &= 2^{-n} \int_0^1 x^{-1/2} dx \\ &= 2^{n-1} \end{aligned}$$

Hence

$$\sum_{n\in\mathbb{N}} \int |2^{-n} f_n| = 2 < \infty.$$

Therefore after redefinition on a null set, $g \in L^1$. In particular $\int |g| < \infty$ and so $|g|$ (and hence g) are finite almost everywhere. For (2), since $g < \infty$ a.e., so too is g^2. Let $a, b \in \mathbb{R}$ and suppose that $a < b$. Choose $N \in \mathbb{N}$ such that $r_N \in (a, b)$. Since all the terms in the sum are nonnegative, $g^2 \geq \sum_{n\in\mathbb{N}} 2^{-2n} f_n^2$ and so

$$\begin{aligned}
\int_{(a,b)} g^2 &\geq \int_{(a,b)} \sum_{n\in\mathbb{N}} 2^{-2n} f_n^2 \\
&= \sum_{n\in\mathbb{N}} 2^{-2n} \int_{(a,b)} f_n^2 \\
&\geq 2^{-2N} \int_{(a,b)} f_N^2 \\
&\geq 2^{-2N} \int_{r_N}^{b\wedge(r_N+1)} \frac{1}{x - r_N} dx \\
&= \infty
\end{aligned}$$

So g^2 is not integrable on any subinterval of $\mathbb{R}$. For (3), note that redefining g on a null set does not change the result of (2). Suppose that there is a finite subinterval $I \subset \mathbb{R}$ such that g is bounded on I. Hence there exists $M > 0$ such that for each $x \in I$, $g(x)^2 \leq M$. Then

$$\begin{aligned}
\int_I g^2 &\leq M^2 m(I) \\
&< \infty
\end{aligned}$$

which is a contradiction. So g is not bounded on any subinterval of $\mathbb{R}$. Now, suppose that there exists $x_0 \in \mathbb{R}$ such that g is continuous at x_0. Choose $\delta > 0$ such that for each $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then $|g(x) - g(x_0)| < 1$. The reverse triangle inequality tells us that for each $x \in (x_0 - \delta, x_0 + \delta)$, $|g(x)| < 1 + |g(x_0)|$. Hence g is bounded on $(x_0 - \delta, x_0 + \delta)$ which is a contradiction. So g is discontinuous everywhere. □

Exercise 6.2.0.15. Let $f \in L^1$.

1. If f is bounded, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$,

$$\mu(E) < \delta \text{ implies that } \int_E |f| < \epsilon.$$

2. For each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$,

$$\mu(E) < \delta \text{ implies that } \int_E |f| < \epsilon.$$

Proof.

1. Since f is bounded, there exists $M > 0$ such that $|f| \leq M$. Let $\epsilon > 0$. Choose $\delta = \epsilon/2M$. Let $E \in \mathcal{A}$. Suppose that $\mu(A) < \delta$. Then

$$\begin{aligned}
\int_E |f| &\leq M\mu(E) \\
&= M\frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

2. Suppose that f is unbounded. Let $\epsilon > 0$. Then there exists $\phi \in L^1$ such that ϕ is simple and
$$\int |f - \phi| < \epsilon/2.$$
Since ϕ is bounded, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then
$$\int_E |\phi| < \epsilon/2.$$
Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then
$$\begin{aligned}\int_E |f| &\leq \int_E |f - \phi| + \int_E |\phi| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon\end{aligned}$$

□

Exercise 6.2.0.16. Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Define $F : \mathbb{R} \to \mathbb{R}$ by
$$F(x) = \int_{(-\infty,x]} f\, dm$$
Then F is continuous.

Proof. Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Since $f \in L^1$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - x_0| < \delta$, then
$$\int_{(x \wedge x_0, x \vee x_0]} |f|\, dm < \epsilon.$$
Let $x \in \mathbb{R}$. Suppose that $|x - x_0| < \delta$. Then
$$\begin{aligned}|F(x) - F(x_0)| &= \left| \int_{(x \wedge x_0, x \vee x_0]} f\, dm \right| \\ &\leq \int_{(x \wedge x_0, x \vee x_0]} |f|\, dm \\ &< \epsilon\end{aligned}$$
So F is continuous.

□

Exercise 6.2.0.17. Let $x \in X$ and denote by δ_x the point mass measure at $x \in X$ on measurable space $(X, \mathcal{P}(X))$. Let $f : X \to \mathbb{C}$. Then
$$\int f d\delta_x = f(x)$$

Proof. First assume that f is simple. Then there exist $(a_j)_{j=1}^n \subset \mathbb{C}$ and $(E_j)_{j=1}^n \subset \mathcal{P}(X)$ such that $(E_j)_{j=1}^n$ is disjoint and $f = \sum_{i=1}^n a_i \chi_{E_i}$. Choose $j^* \in \{1, \dots, n\}$ such that $x \in E_{j^*}$. Thus
$$\begin{aligned}\int f d\delta_x &= \int \sum_{j=1}^n c_j \chi_{E_j} d\delta_x \\ &= \sum_{j=1}^n c_j \delta_x(E_j) \\ &= c_j \delta_x(E_{j^*}) \\ &= c_j \\ &= f(x)\end{aligned}$$

Now for $f \in L^+$, choose a sequence $(\phi_n)_{n\in\mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1}$ and $\phi_n \xrightarrow{\text{p.w}} f$. Then monotone convergence implies that

$$\begin{aligned}\int f d\delta_x &= \int \lim_{n\to\infty} \phi_n \delta_x \\ &= \lim_{n\to\infty} \int \phi_n \delta_x \\ &= \lim_{n\to\infty} \phi_n(x) \\ &= f(x)\end{aligned}$$

Now just extend to complex valued functions.

□

Exercise 6.2.0.18. Let X be a set and $f \in L^1(X, \mathcal{P}(X), \#)$. Then $\{x \in X : f(x) \neq 0\}$ is countable.

Proof. Since $\{x \in X : f(x) \neq 0\} = \{x \in X : |f|(x) > 0\}$ and $|f| \in L^1(X, \mathcal{P}(X))$, an exercise in the previous section implies that $\{x \in X : f(x) \neq 0\}$ is countable. □

Exercise 6.2.0.19. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f, g \in L^1(X, \mathcal{A}, \mu)$. Then $f \leq g$ μ-a.e. iff for each $E \in \mathcal{A}$,

$$\int_E f \leq \int_E g$$

Proof. Suppose $f \leq g$ a.e. Put $N = \{x \in X : f(x) > g(x)\} \subset N$. Then $\mu(N) = 0$ and $g - f \geq 0$ on N^c. So for each $E \in \mathcal{A}$,

$$\begin{aligned}\int_E g\, d\mu - \int_E f\, d\mu &= \int_E (g - f)\, d\mu \\ &= \int_{E\cap N^c} (g - f)\, d\mu \\ &\geq 0\end{aligned}$$

Conversely, suppose that for each $E \in \mathcal{A}$,

$$\int_E f\, d\mu \leq \int_E g\, d\mu$$

Put $N_n = \{x \in X : f(x) - g(x) > 1/n\}$ and $N = \{x \in X : f(x) > g(x)\}$. Then $N = \bigcup_{n\in\mathbb{N}} N_n$. Let $n \in \mathbb{N}$. Then our assumption tells us that

$$\begin{aligned}0 &\geq \int_{N_n} f - g \\ &\geq \frac{1}{n}\mu(N_n) \\ &\geq 0.\end{aligned}$$

So that $\mu(N_n) = 0$. Thus for each $n \in \mathbb{N}$, $\mu(N_n) = 0$ which implies $\mu(N) = 0$. Therefore $f \leq g$ a.e. as required. □

Exercise 6.2.0.20. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \times \mathbb{R} \to \mathbb{C}$. Suppose that for each $t \in \mathbb{R}$, $f(\cdot, t) \in L^1(\mu)$. Define $F : \mathbb{R} \to \mathbb{C}$ by

$$F(t) = \int_X f(x, t)\, d\mu(x)$$

1. Suppose that there exists $g \in L^1(\mu)$ such that for each $(x, t) \in X \times \mathbb{R}$, $|f(x, t)| \leq g(x)$. Let $t_0 \in \mathbb{R}$. If for each $x \in X$, $f(x, \cdot)$ is continuous at t_0, then F is continuous at t_0.
2. Suppose that $\partial f/\partial t$ exits and there exists $g \in L^1(\mu)$ such that for each $(x, t) \in X \times \mathbb{R}$, $|\partial f/\partial t(x, t)| \leq g(x)$. Then F is differentiable and for each $t \in \mathbb{R}$,
$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t)\, d\mu(x)$$

Proof.

1. Suppose that for each $x \in X$, $f(x, \cdot)$ is continuous at t_0. Let $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$. Suppose that $t_n \to t_0$. Then $f(\cdot, t_n) \xrightarrow{\text{p.w.}} f(\cdot, t_0)$. Since for each $n \in \mathbb{N}$, $|f(x, t_n)| \leq g(x)$, the dominated convergence theorem implies that $F(t_n) \to F(t_0)$.

2. Let $t_0 \in \mathbb{R}$. Choose $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $t_n \to t_0$ and for each $n \in \mathbb{N}$, $t_n < t_0$. For $n \in \mathbb{N}$, define $q_n : X \to \mathbb{R}$ by
$$q_n(x) = \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}$$
So $q_n(\cdot) \xrightarrow{\text{p.w.}} \partial f/\partial t(\cdot, t_0)$. The mean value theorem implies that for each $x \in X$ and $n \in \mathbb{N}$, there exists $c_{n,x} \in (t_n, t_0)$ such that $q_n(x) = \partial f/\partial t(x, c_{n,x})$. Therefore, for each $n \in \mathbb{N}$ and $x \in X$,
$$\begin{aligned} |q_n(x)| &= \left| \frac{\partial f}{\partial t}(x, c_{n,x}) \right| \\ &\leq g(x) \end{aligned}$$
The dominated convergence theorem then implies that $\partial f/\partial t(\cdot, t_0) \in L^1(\mu)$ and
$$\begin{aligned} \int \frac{\partial f}{\partial t}(x, t_0)\, d\mu(x) &= \lim_{n\to\infty} \int_X q_n \, d\mu \\ &= \lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} \\ &= F'(t_0^-) \end{aligned}$$
So that F is differentiable at t_0 from the left. Similarly, F is differentiable at t_0 from the right.

□

Exercise 6.2.0.21. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ a measurable space and $f : X \to Y$. Suppose that f is $(\mathcal{A}, \mathcal{B})$-measurable. Then for each $g \in L^0(Y, \mathcal{B})$ and $B \in \mathcal{B}$,

1. $g \circ f \in L^1(X, \mathcal{A})$ iff $g \in L^1(Y, \mathcal{B}, f_*\mu)$
2. if $g \circ f \in L^1(X, \mathcal{A}, \mu)$,
$$\int_{f^{-1}(B)} g \circ f \, d\mu = \int_B g \, df_*\mu$$

Proof. Let $g \in L^0(Y, \mathcal{B})$ and $B \in \mathcal{B}$.

1. Suppose that $g \circ f \in L^1(X, \mathcal{A}, \mu)$. Since $|g| \in L^+(X, \mathcal{A})$ and $|g \circ f| = |g| \circ f$, an exercise in the previous section implies that
$$\begin{aligned} \int_B |g| \, df_*\mu &= \int_{f^{-1}(B)} |g| \circ f \, d\mu \\ &= \int_{f^{-1}(B)} |g \circ f| \, d\mu \\ &< \infty \end{aligned}$$
Hence $g \in L^1(Y, \mathcal{B}, f_*\mu)$.
Conversely, suppose that $g \in L^1(Y, \mathcal{B}, f_*\mu)$. Since $|g \circ f| \in L^+(X, \mathcal{B})$, we have that
$$\begin{aligned} \int_{f^{-1}(B)} |g \circ f| \, d\mu &= \int_{f^{-1}(B)} |g| \circ f \, d\mu \\ &= \int_B |g| \, df_*\mu \\ &< \infty \end{aligned}$$
Hence $g \circ f \in L^1(X, \mathcal{A}, \mu)$.

2. Suppose that $g \circ f \in L^1(X, \mathcal{A}, \mu)$. Write $g = h_1^+ - h_1^- + i(h_2^+ - h_2^-)$. Since $h_1^+, h_1^-, h_2^+, h_2^- \in L^+(Y, \mathcal{B})$, an exercise in the previous section implies that

$$\begin{aligned}
\int_{f^{-1}(B)} g \circ f \, d\mu &= \int_{f^{-1}(B)} \left[h_1^+ - h_1^- + i(h_2^+ - h_2^-)\right] \circ f \, d\mu \\
&= \int_{f^{-1}(B)} h_1^+ \circ f \, d\mu - \int_{f^{-1}(B)} h_1^- \circ f \, d\mu \\
&+ i \int_{f^{-1}(B)} h_2^+ \circ f \, d\mu - i \int_{f^{-1}(B)} h_2^- \circ f \, d\mu \\
&= \int_B h_1^+ \, df_*\mu - \int_B h_1^- \, df_*\mu + i \int_B h_2^+ \, df_*\mu - i \int_B h_2^- \, df_*\mu \\
&= \int_B h_1^+ - h_1^- + i(h_2^+ - h_2^-) \, df_*\mu \\
&= \int_B g \, df_*\mu
\end{aligned}$$

□

Exercise 6.2.0.22. Change notation or define categories Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu) \in \mathrm{Obj}(\mathbf{Prob})$ and $f \in \mathrm{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B}))$. Then f is measure preserving iff for each $\phi \in L^1(Y, \mathcal{B}, \nu)$, $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and

$$\int_Y \phi \, d\nu = \int_X \phi \circ f \, d\mu$$

Proof.

- ($\Longrightarrow$):
  Suppose that f is measure preserving. $\phi \in L^1(Y, \mathcal{B}, \nu)$. Then the a basic result on the change of variables implies that $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and

$$\begin{aligned}
\int_Y \phi \, d\nu &= \int_Y \phi d \, f_* mu \\
&= \int_X \phi \, d\mu
\end{aligned}$$

- ($\Longleftarrow$):
  Suppose that for each $\phi \in L^1(Y, \mathcal{B}, \nu)$, $\phi \circ f \in L^1(X, \mathcal{A}, \mu)$ and

$$\int_Y \phi \, d\nu = \int_X \phi \circ f \, d\mu$$

  Let $B \in \mathcal{B}$. Since ν is a probability measure, $\chi_B \in L^1(Y, \mathcal{B}, \nu)$. Thus

$$\begin{aligned}
\nu(B) &= \int_Y \chi_B \, d\nu \\
&= \int_X \chi_B \circ f \, d\mu \\
&= \int_X \chi_{f^{-1}(B)} \, d\mu \\
&= \mu(f^{-1}(B)) \\
&= f_*\mu(B)
\end{aligned}$$

  Since $B \in \mathcal{B}$ is arbitrary, $f_*\mu = \nu$.

□

Definition 6.2.0.23. Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is said to be **uniformly integrable** if for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| < \epsilon$. (i.e. $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| = 0$).

Exercise 6.2.0.24. Suppose that μ is finite. Let $\mathcal{F} \subset L^1$. Then $\mathcal{F}$ is uniformly integrable iff

1. there exists $M > 0$ such that $\sup\limits_{f \in \mathcal{F}} \int |f| \leq M$
2. for each $\epsilon > 0$, there exists $\delta > 0$ such that for each $E \in \mathcal{A}$, if $\mu(E) < \delta$, then $\sup\limits_{f \in \mathcal{F}} \int_E |f| < \epsilon$.

Proof. ($\implies$): (1) Suppose that $\mathcal{F}$ is uniformly integrable. Then there exists $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| < 1$. Choose $M = \mu(X)K + 1$. Then for each $f \in \mathcal{F}$,

$$\begin{aligned} \int |f| &= \int_{\{|f|>K\}} |f| + \int_{\{|f|\leq K|\}} |f| \\ &\leq 1 + K\mu(X) \\ &= M \end{aligned}$$

(2) Let $\epsilon > 0$. Then choose $K \in \mathbb{N}$ such that $\sup\limits_{f \in \mathcal{F}} \int_{\{|f|>K\}} |f| < \epsilon/2$ and choose $\delta = \epsilon/2K$. Let $E \in \mathcal{A}$. Suppose that $\mu(E) < \delta$. Then for $f \in \mathcal{F}$,

$$\begin{aligned} \int_E |f| &= \int_{E \cap \{|f|>K\}} |f| + \int_{E \cap \{|f| \leq K\}} |f| \\ &\leq \epsilon/2 + K\delta \\ &= \epsilon \end{aligned}$$

($\Leftarrow$): Choose $M > 0$ as in (1). Suppose that there exists $\epsilon > 0$ such that for each $K \in \mathbb{N}$, there exists $f \in \mathcal{F}$ such that $\mu(\{|f| > K\}) \geq \epsilon$. Choose $K \in \mathbb{N}$ such that $K > M/\epsilon$. Then choose $f_K \in \mathcal{F}$ such that $\mu(\{|f_K| > K\}) \geq \epsilon$. Then

$$\begin{aligned} \int |f_K| &\geq \int_{\{|f_K|>K\}} |f| \\ &\geq K\mu(\{|f_K| > K\}) \\ &> \frac{M}{\epsilon} \cdot \epsilon \\ &= M, \end{aligned}$$

which is a contradiction. Hence for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $f \in \mathcal{F}$, $\mu(\{|f| > K\}) < \epsilon$. Since $\mu(\{|f| > k\})$ is a decreasing sequence in k, we have that $\lim\limits_{k \to \infty} \sup\limits_{f \in \mathcal{F}} \mu(\{|f| > k\}) = 0$. Now, let $\epsilon > 0$. Choose $\delta > 0$ as in (2). Choose $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$, $\mu(\{|f| > k\}) < \delta$. Then for each $k \in \mathbb{N}$, if $k \geq K$, then for each $f \in \mathcal{F}$,

$$\int_{\{|f|>k\}} |f| < \epsilon.$$

Thus

$$\lim_{k \to \infty} \sup_{f \in \mathcal{F}} \int_{\{|f|>k\}} |f| = 0$$

as required. □

Definition 6.2.0.25. Let $(X, \mathcal{A}, \mu)$ be a measure space. Define $\|\cdot\|_* : L^1(\mu) \to [0, \infty)$ by

$$\|f\|_* = \sup_{A \in \mathcal{A}} \left| \int_A f \, d\mu \right|$$

Exercise 6.2.0.26. Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $\| \cdot \|_*$ is a norm on $L^1(\mu)$ and there exists $C > 0$ such that $C\| \cdot \|_1 \leq \| \cdot \|_* \leq \| \cdot \|_1$.

6.3 Integration on Product Spaces

Note 6.3.0.1. Recall the definition of the sections of E and f from the section on product σ-algebras. It is often helpful to observe that $(\chi_E)_x = \chi_{E_x}$ and $(\chi_E)^y = \chi_{E^y}$.

Theorem 6.3.0.2. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then for each $E \in \mathcal{A} \otimes \mathcal{B}$, the maps $\phi : X \to [0, \infty]$ and $\psi : Y \to [0, \infty]$ defined by $\phi(x) = \nu(E_x)$ and $\psi(y) = \mu(E^y)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, respectively and

$$\mu \otimes \nu(E) = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y) d\nu(y)$$

Theorem 6.3.0.3. Fubini, Tonelli: Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.

1. (Tonelli) For each $f \in L^+(X \times Y)$, the functions $g : X \to [0, \infty]$, $h : Y \to [0, \infty]$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x)\, d\mu(x)$ are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable respectively and

$$\int_{X \times Y} f\, d\mu \otimes \nu = \int_X g\, d\mu = \int_Y h d\nu$$

2. (Fubini) For each $f \in L^1(X \times Y)$, $f_x \in L^1(\nu)$ for μ-a.e. $x \in X$ and $f^y \in L^1(\mu)$ for ν-a.e. $y \in Y$, respectively and (after redefinition of f on a null set) the functions $g : X \to \mathbb{C}$, $h : Y \to \mathbb{C}$ defined by $g(x) = \int_Y f_x(y) d\nu(y)$ and $h(y) = \int_X f^y(x)\, d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively. Furthermore

$$\int_{X \times Y} f\, d\mu \otimes \nu = \int_X g\, d\mu = \int_Y h d\nu$$

Note 6.3.0.4. We usually just write

$$\int \int f\, d\mu\, d\nu \text{ and } \int \int f\, d\nu\, d\mu$$

instead of

$$\int h d\nu$$

and

$$\int g\, d\mu$$

respectively. We have a similar result for complete product measure spaces. See

Exercise 6.3.0.5. Take $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1]), \mathcal{B} = \mathcal{P}([0, 1])$ and μ, ν to be Lebesgue measure and counting measure respectively. Define $D = \{(x, y) \in [0, 1]^2 : x = y\}$ Show that

$$\int \chi_D\, d\mu \otimes \nu, \int \int \chi_D\, d\mu d\nu \text{ and } \int \int \chi_D d\nu\, d\mu$$

are all different. (Hint: for the first integral, use the definition of $\mu \otimes \nu$)

Proof. Let $x, y \in [0, 1]$. Then $(\chi_D)_x = \chi_{D_x} = \chi_x$ and $(\chi_D)^y = \chi_{D^y} = \chi_y$. Thus

$$\begin{aligned} \int \int \chi_D\, d\mu d\nu &= \int \mu(\{y\}) d\nu \\ &= \int 0 d\nu \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}\int\int \chi_D \, d\mu d\nu &= \int \nu(\{x\}) \, d\mu \\ &= \int 1 \, d\mu \\ &= 1\end{aligned}$$

Now, Observe that $\int \chi_D \, d\mu \otimes \nu = \mu \otimes \nu(D)$. Recall from the section on product measures that $\mu \otimes \nu(D) = \inf\{\sum_{n \in \mathbb{N}} \mu(A_n)\nu(B_n) : (A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ and $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n\}$. Let $(A_n \times B_n)_{n \in \mathbb{N}} \subset \mathcal{E}$. Suppose that $D \subset \bigcup_{n \in \mathbb{N}} A_n \times B_n$. Then for each $x \in [0,1]$, $(x,x) \in \bigcup_{n \in \mathbb{N}} A_n \times B_n$. So for each $x \in [0,1]$, there exists $n \in \mathbb{N}$, such that $x \in A_n \cap B_n$. Thus $[0,1] \subset \bigcup_{n \in \mathbb{N}} A_n \cap B_n$. Since $1 = \mu([0,1]) \leq \sum_{n \in \mathbb{N}} \mu(A_n \cap B_n)$, we know that there exists $n \in \mathbb{N}$ such that $0 < \mu(A_n \cap B_n)$. Thus $\mu(A_n) > 0$ and $\mu(B_n) > 0$. Since $\mu(B_n) > 0$, B_n must be infinite and therefore $\nu(B_n) = \infty$. So $\sum_{n \in \mathbb{N}} \mu(A_n)\nu(B_n) = \infty$. □

Exercise 6.3.0.6. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $f : X \to [0, \infty) \in L^+$. Show that $G = \{(x,y) \in X \times [0,\infty) : f(x) \geq y\} \in \mathcal{A} \otimes \mathcal{B}([0,\infty))$ and $\mu \times m(G) = \int_X f \, d\mu$. The same is true if we replace "$\geq$" with "$>$". (Hint: to show that G is measurable, split up $(x,y) \mapsto f(x) - y)$ into the composition of measurable functions.

Proof. Define $\phi : X \times [0,\infty) \to [0,\infty)^2$ and $\psi : [0,\infty)^2 \to [0,\infty)$ by $\phi(x,y) = (f(x), y)$ and $\psi(z,y) = z - y$. Then $G = \{(x,y) \in X \times [0,\infty) : \psi \circ \phi(x,y) \geq 0\}$. Let $A, B \in \mathcal{B}([0,\infty))$. Then $\phi^{-1}(A \times B) = f^{-1}(A) \times B \in \mathcal{A} \times \mathcal{B}([0,\infty))$. Since $\mathcal{B}([0,\infty)^2) = \mathcal{B}([0,\infty)) \otimes \mathcal{B}([0,\infty)) = \sigma(\{A \times B : A, B \in \mathcal{B}([0,\infty))\})$, we have that ϕ is $\mathcal{A} \otimes \mathcal{B}([0,\infty))$-$\mathcal{B}([0,\infty)^2)$ measurable. Since ψ is continuous, we have that ψ is $\mathcal{B}([0,\infty)^2)$-$\mathcal{B}([0,\infty))$ measurable. This implies that $\psi \circ \phi$ is $\mathcal{A} \otimes \mathcal{B}([0,\infty))$-$\mathcal{B}([0,\infty))$ measurable. Thus $G = \psi \circ \phi^{-1}([0,\infty)) \in \mathcal{A} \otimes \mathcal{B}([0,\infty))$. Now for $x \in X$, $G_x = \{y \in [0,\infty) : f(x) \geq y\} = [0, f(x)]$. Thus

$$\begin{aligned}\mu \times m(G) &= \int \chi_G \, d\mu \times m \\ &= \int_X \int_{[0,\infty)} \chi_{G_x} \, dm \, d\mu(x) \\ &= \int_X f(x) \, d\mu(x)\end{aligned}$$

The same reasoning holds if we replace "$\geq$" with "$>$". □

Exercise 6.3.0.7. Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$. Define $h : X \times Y \to \mathbb{C}$ by $h(x,y) = f(x)g(y)$.

1. If f is $\mathcal{A}$-measurable and g is $\mathcal{B}$-measurable, then h is $\mathcal{A} \otimes \mathcal{B}$-measurable.
2. If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \otimes \nu)$ and

$$\int_{X \times Y} h \, d\mu \otimes \nu = \int_X f \, d\mu \int_Y g d\nu$$

Proof.

1. First suppose that f, g are simple. Then there exist $(A_i)_{i=1}^n \subset \mathcal{A}$, $(B_j)_{j=1}^m \subset \mathcal{B}$ and $(a_i)_{i=1}^n, (b_i)_{j=1}^m \subset \mathbb{C}$ such that $f = \sum_{i=1}^n a_i \chi_{A_i}$ and $g = \sum_{j=1}^m b_j \chi_{B_j}$. Then $h = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \chi_{A_i \times B_j}$. So h is $\mathcal{A} \otimes \mathcal{B}$-measurable. For general f, g, there exist $(f_n)_{n \in \mathbb{N}} \subset S(X, \mathcal{A})$ and $(g_n)_{n \in \mathbb{N}} \subset S(Y, \mathcal{B})$ such that $f_n \to f$ pointwise, $g_n \to g$ pointwise and for each $n \in \mathbb{N}$, $|f_n| \leq |f_{n+1}| \leq |f|$ and $|g_n| \leq |g_{n+1}| \leq |g|$. For $n \in \mathbb{N}$, define $h_n \in S(X \times Y, \mathcal{A} \otimes \mathcal{B})$ by $h_n = f_n g_n$. Then $h_n \to h$ pointwise and for each $n \in \mathbb{N}$, $|h_n| \leq |h_{n+1}| \leq |h|$. Thus h is $\mathcal{A} \otimes \mathcal{B}$-measurable.

2. First suppose f and g are simple as before. Then

$$\begin{aligned}\int_{X\times Y} |h|\, d\mu\otimes\nu &\le \sum_{i=1}^{n}\sum_{j=1}^{m} |a_i b_j|\mu(A_i)\nu(B_j)\\ &= \Big(\sum_{i=1}^{n}|a_i|\mu(A_i)\Big)\Big(\sum_{j=1}^{m}|b_j|\nu(B_j)\Big)\\ &= \int_X |f|\, d\mu \int_Y |g| d\nu\\ &< \infty\end{aligned}$$

So $h \in L^1(\mu\otimes\nu)$. Furthermore,

$$\begin{aligned}\int_{X\times Y} h\, d\mu\otimes\nu &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j\mu(A_i)\nu(B_j)\\ &= \Big(\sum_{i=1}^{n}a_i\mu(A_i)\Big)\Big(\sum_{j=1}^{m}b_j\nu(B_j)\Big)\\ &= \int_X f\, d\mu \int_Y g d\nu\end{aligned}$$

For general $f \in L^1(\mu), g \in L^1(\nu)$, take $(h_n)_{n\in\mathbb{N}}$ as before. Monotone convergence and the result above say that

$$\begin{aligned}\int_{X\times Y} |h|\, d\mu\times d\nu &= \lim_{n\to\infty}\int_{X\times Y} |h_n|\, d\mu\otimes\nu\\ &= \lim_{n\to\infty}\left(\int_X |f_n|\, d\mu \int_Y |g_n| d\nu\right)\\ &= \int_X |f|\, d\mu \int_Y |g| d\nu\\ &< \infty\end{aligned}$$

So $h \in L^1(\mu\otimes\nu)$. Dominated convergence and the result above then tell us that

$$\begin{aligned}\int_{X\times Y} h\, d\mu\times d\nu &= \lim_{n\to\infty}\int_{X\times Y} h_n\, d\mu\times d\nu\\ &= \lim_{n\to\infty}\left(\int_X f_n\, d\mu \int_Y g_n d\nu\right)\\ &= \int_X f\, d\mu \int_Y g d\nu\end{aligned}$$

$\square$

Note 6.3.0.8. In the above exercise part (2), we can replace L^1 with L^+ and get the same result by the same method.

Exercise 6.3.0.9. Let $f : \mathbb{R} \to [0,\infty) \in L^+$. Show that

$$\int_{\mathbb{R}} f\, dm = \int_{[0,\infty)} m(\{x\in\mathbb{R} : f(x) \ge t\})\, dm(t)$$

Proof. Note that

$$\int_{[0,\infty)} m(\{x \in \mathbb{R} : f(x) \geq t\}) = \int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x\in\mathbb{R}:f(x)\geq t\}}\, dm \right] dm(t)$$

Comparing this with Tonelli's theorem, we can put $\chi_{\{x\in\mathbb{R}:f(x)\geq t\}} = (\chi_E)^t = \chi_{E^t}$. Then $E = \{(x,t) \in \mathbb{R} \times [0,\infty) : f(x) \geq t\}$ and $E_x = \{t \in [0,\infty) : f(x) \geq t\} = [0, f(x)]$. Tonelli's theorem tells us that

$$\begin{aligned}\int_{[0,\infty)} \left[\int_{\mathbb{R}} \chi_{\{x\in\mathbb{R}:f(x)\geq t\}}(x)\, dm(x) \right] dm(t) &= \int_{\mathbb{R}} \left[\int_{[0,\infty)} \chi_{[0,f(x)]}(t)\, dm(t) \right] dm(x) \\ &= \int_{\mathbb{R}} f(x)\, dm(x)\end{aligned}$$

□

6.4 Modes of Convergence

Definition 6.4.0.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, (Y, d) a metric space, $(f_n)_{n\in\mathbb{N}} \subset L^0_Y(X, \mathcal{A}, \mu)$ and $f \in L^0_Y(X, \mathcal{A}, \mu)$. Then $(f_n)_{n\in\mathbb{N}}$ is said to **converge to f in measure**, denoted $f_n \xrightarrow{\mu} f$, if for each $\epsilon > 0$,

$$\mu(\{x \in X : d(f_n(x), f(x)) \geq \epsilon\}) \to 0 \quad \text{as } n \to \infty$$

Definition 6.4.0.2. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$. Then $(f_n)_{n\in\mathbb{N}}$ is said to be **Cauchy in measure** if for each $\epsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \quad \text{as } n, m \to \infty$$

i.e. for each $\epsilon, \delta > 0$, there exists $N \in \mathbb{N}$ such that for each $n, m \in \mathbb{N}$, $n, m \geq N$ implies that $\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) < \delta$.

Note 6.4.0.3. It is useful to observe that

$$\bigcup_{\epsilon>0} \limsup_{n\to\infty} \{x \in X : |f_n(x) - f(x)| \geq \epsilon\} = \{x \in X : f_n(x) \not\to f(x)\}$$

and

$$\bigcap_{\epsilon>0} \liminf_{n\to\infty} \{x \in X : |f_n(x) - f(x)| < \epsilon\} = \{x \in X : f_n(x) \to f(x)\}$$

Exercise 6.4.0.4. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. If $f_n \xrightarrow{\mu} f$, then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure.

Proof. Suppose that $f_n \xrightarrow{\mu} f$. For $\epsilon > 0$ and $n, m \in \mathbb{N}$, set

$$A_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$$

and

$$B_{n,m,\epsilon} = \{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}$$

Let $\epsilon > 0$, $n, m \in \mathbb{N}$ and $x \in A^c_{n,\frac{\epsilon}{2}} \cap A^c_{m,\frac{\epsilon}{2}}$. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

and $x \in B^c_{n,m,\epsilon}$. Therefore $A^c_{n,\frac{\epsilon}{2}} \cap A^c_{m,\frac{\epsilon}{2}} \subset B^c_{n,m,\epsilon}$. This implies that $B_{n,m,\epsilon} \subset A_{n,\frac{\epsilon}{2}} \cup A_{m,\frac{\epsilon}{2}}$. Let $\delta > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $\mu(A_{n,\frac{\epsilon}{2}}) < \delta/2$. Then for each $n, m \in \mathbb{N}$, $n, m \geq N$ implies that

$$\begin{aligned} \mu(B_{n,m,\epsilon}) &\leq \mu(A_{n,\frac{\epsilon}{2}}) + \mu(A_{m,\frac{\epsilon}{2}}) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta \end{aligned}$$

So for each $\epsilon > 0$,

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \quad \text{as } n, m \to \infty$$

and $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure. $\square$

Exercise 6.4.0.5. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$. Then $f = g$ a.e.

Proof. Set $B = \{x \in X : |f(x) - g(x)| \geq 0\}$ and for $n, k \in \mathbb{N}$, set

- $B_k = \{x \in X : |f(x) - g(x)| \geq \frac{1}{k}\}$
- $A_{f,n,k} = \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{k}\}$

- $A_{g,n,k} = \{x \in X : |f_n(x) - g(x)| \geq \frac{1}{k}\}$

As in the proof of Exercise 6.4.0.4, for each $n, k \in \mathbb{N}$

$$\mu(B_k) \leq \mu(A_{f,n,2k}) + \mu(A_{g,n,2k})$$

Let $\epsilon > 0$. Convergence in measure implies that for each $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N$ implies that $\mu(A_{f,n,2k}), \mu(A_{g,n,2k}) < \epsilon 2^{-(1+k)}$. Then

$$\begin{aligned}
\mu(B) &= \mu\bigg(\bigcup_{k\in\mathbb{N}} B_k\bigg) \\
&\leq \sum_{k\in\mathbb{N}} \mu(B_k) \\
&\leq \sum_{k\in\mathbb{N}} \mu(A_{f,N_k,2k}) + \sum_{k\in\mathbb{N}} \mu(A_{g,N_k,2k}) \\
&\leq \sum_{k\in\mathbb{N}} \epsilon 2^{-(1+k)} + \sum_{k\in\mathbb{N}} \epsilon 2^{-(1+k)} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\mu(B) = 0$ and $f = g$ a.e. $\square$

Exercise 6.4.0.6. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$. Suppose that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure.

1. There exists a subsequence $(f_{n_j})_{j\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that for each $j \in \mathbb{N}$,

$$\mu(\{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}) < 2^{-j}$$

2. For $j, k \in \mathbb{N}$ set

$$E_j = \{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}$$

and

$$F_k = \bigcup_{j\geq k} E_j$$

Then $(F_k)_{k\in\mathbb{N}}$ is decreasing and for each $k \in \mathbb{N}$, $\mu(F_k) \leq 2^{1-k}$ and for each $i, j, k \in \mathbb{N}$, $i \geq j \geq k$ implies that for each $x \in F_k^c$,

$$|f_{n_i}(x) - f_{n_j}(x)| \leq 2^{1-k}$$

So for each $k \in \mathbb{N}$, $(f_{n_j})_{j\in\mathbb{N}}$ is uniformly Cauchy on F_k^c and therefore $(f_{n_j})_{j\in\mathbb{N}}$ is pointwise Cauchy on F_k^c.
Hint: get a telescoping sum via the triangle inequality

3. Set

$$F = \bigcap_{k\in\mathbb{N}} F_k$$

Then $\mu(F) = 0$ and there exists $f \in L^0$ such that $f_{n_j} \xrightarrow{\text{a.e.}} f$.

4. Finally, $f_{n_j} \xrightarrow{\mu} f$, $f_n \xrightarrow{\mu} f$
Hint: consider showing $\{x \in X : |f_{n_k}(x) - f(x)| \geq \epsilon\} \subset F_k$ and use something similar to the proof of Exercise 6.4.0.4

Proof.

1. By definition, for each $j \in \mathbb{N}$, there exists $N_j \in \mathbb{N}$ such that for each $n, m \in \mathbb{N}$, $n, m \geq N_j$ implies that

$$\mu(\{x \in X : |f_n(x) - f_m(x)| \geq 2^{-j}\}) < 2^{-j}$$

Setting $n_1 = N_1$ and for $j \geq 2$, setting $n_j = \max(n_{j-1} + 1, N_j)$, we may obtain a subsequence (f_{n_j}) such that for each $j \in \mathbb{N}$,

$$\mu(\{x \in X : |f_{n_j}(x) - f_{n_{j+1}}(x)| \geq 2^{-j}\}) < 2^{-j}$$

2. Clearly $(F_k)_{k\in\mathbb{N}}$ is decreasing. Let $k \in \mathbb{N}$. Part (1) implies that

$$\begin{aligned}\mu(F_k) &\leq \sum_{j\geq k} 2^{-j} \\ &= 2^{1-k}\sum_{j\geq 1} 2^{-j} \\ &= 2^{1-k}\end{aligned}$$

Let $i, j \in \mathbb{N}$. Suppose that $i \geq j \geq k$. Let $x \in F_k^c$. Then

$$\begin{aligned}|f_{n_i}(x) - f_{n_j}(x)| &\leq \sum_{l=j}^{i-1} |f_{n_{l+1}}(x) - f_{n_l}(x)| \\ &< \sum_{l=j}^{i-1} 2^{-l} \\ &< \sum_{l\geq j} 2^{-l} \\ &= 2^{1-j} \\ &\leq 2^{1-k}\end{aligned}$$

Let $\epsilon > 0$. Choose $k' \in \mathbb{N}$ such that $k' \geq k$ and $2^{1-k'} < \epsilon$. Let $i, j \in \mathbb{N}$. Suppose that $i, j \geq k'$. Let $x \in F_k^c \subset F_{k'}^c$. Then

$$\begin{aligned}|f_{n_i}(x) - f_{n_j}(x)| &< 2^{1-k'} \\ &< \epsilon\end{aligned}$$

So $(f_{n_j})_{j\in\mathbb{N}}$ is uniformly Cauchy on F_k^c

3. Since $\mu(F_1) < \infty$, $(F_k)_{k\in\mathbb{N}}$ is decreasing and $F = \inf_{k\in\mathbb{N}} F_k$, we have that

$$\begin{aligned}\mu(F) &= \inf_{k\in\mathbb{N}} \mu(F_k) \\ &\leq \inf_{k\in\mathbb{N}} 2^{1-k} \\ &= 0\end{aligned}$$

Since for each $k \in \mathbb{N}$, $(f_{n_j})_{j\in\mathbb{N}}$ is pointwise Cauchy on F_k^c, $(f_{n_j})_{j\in\mathbb{N}}$ is pointwise Cauchy on F^c. Then $(f_{n_j}\chi_{F^c})_{j\in\mathbb{N}}$ is pointwise Cauchy.
Define $f : X \to \mathbb{C}$ pointwise by

$$f = \lim_{j\to\infty} f_{n_j}\chi_{F^c}$$

Then $f \in L^0$ since $(f_{n_j}\chi_{F^c})_{j\in\mathbb{N}} \subset L^0$ and $f_{n_j}\chi_{F^c} \xrightarrow{\text{p.w.}} f$. Since $\mu(F) = 0$ and $\{x \in X : f_{n_j}(x) \not\to f(x)\} \subset F$, we have that $f_{n_j} \xrightarrow{\text{a.e.}} f$.

4. For $n, m \in \mathbb{N}$ and $\epsilon > 0$, set

$$A_{n,\epsilon} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$$

and

$$B_{m,n,\epsilon} = \{x \in X : |f_m(x) - f_n(x)| \geq \epsilon\}$$

Let $\epsilon, \delta > 0$. Choose $k \in \mathbb{N}$ such that $2^{2-k} < \epsilon$ and $\mu(F_k) < \delta$. Let $x \in F_k^c$. Since $f_{n_j}(x) \to f(x)$, there exists $J \in \mathbb{N}$ such that $J \geq k$ and for each $j \in \mathbb{N}$, $j \geq J$ implies that $|f_{n_j}(x) - f(x)| < 2^{1-k}$. Let $l \in \mathbb{N}$. Suppose that $l \geq k$. Then part (2) implies that

$$\begin{aligned}|f_{n_l}(x) - f(x)| &\leq |f_{n_l}(x) - f_{n_J}(x)| + |f_{n_J}(x) - f(x)| \\ &\leq 2^{1-k} + 2^{1-k} \\ &\leq 2^{2-k} \\ &< \epsilon\end{aligned}$$

So $x \in A^c_{n_l,\epsilon}$. Hence $A_{n_l,\epsilon} \subset F_k$ and $\mu(A_{n_l,\epsilon}) < \delta$. So $f_{n_j} \xrightarrow{\mu} f$.
Let $\epsilon > 0$, $\delta > 0$. Since $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, there exists $J_1 \in \mathbb{N}$ such that for each $m,n \in \mathbb{N}$ $m,n \geq J_1$ implies that $\mu(B_{m,n,\frac{\epsilon}{2}}) < \frac{\delta}{2}$. Since $f_{n_j} \xrightarrow{\mu} f$, there exists J_2 such that for each $j \in \mathbb{N}$, $j \geq J_2$ implies that $\mu(A_{n_j,\frac{\epsilon}{2}}) < \frac{\delta}{2}$. Set $J = \max(J_1, J_2)$. Let $j \in \mathbb{N}$. Suppose that $j \geq J$. Since $n_j \geq j$, the proof of Exercise 6.4.0.4 implies that,

$$\begin{aligned}\mu(A_{j,\epsilon}) &\leq \mu(B_{j,n_j,\frac{\epsilon}{2}}) + \mu(A_{n_j,\frac{\epsilon}{2}}) \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta\end{aligned}$$

So that $f_n \xrightarrow{\mu} f$.

□

Exercise 6.4.0.7. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$.

1. If $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure, then there exists a $f_0 \in L^0$ and a subsequence $(f_{n_j})_{j\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_n \xrightarrow{\mu} f_0$ and $f_{n_j} \xrightarrow{\text{a.e.}} f_0$.
2. If $f_n \xrightarrow{\mu} f$, then there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_n \xrightarrow{\text{a.e.}} f$.

Proof.

1. Previous exercise.
2. Suppose that $f_n \xrightarrow{\mu} f$. Then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in measure. Part (1) implies that there exists a $f_0 \in L^0$ and a subsequence $(f_{n_j})_{j\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_n \xrightarrow{\mu} f_0$ and $f_{n_j} \xrightarrow{\text{a.e.}} f_0$. Since $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} f_0$, $f = f_0$ a.e. Hence $f_{n_j} \xrightarrow{\text{a.e.}} f$.

□

Exercise 6.4.0.8. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(f_n)_{n\subset\mathbb{N}} \subset L^0(X, \mathcal{A})$ and $f \in L^0(X, \mathcal{A})$. Suppose that $f_n \xrightarrow{\mu} f$.

1. If for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$ a.e., then $f_n \xrightarrow{\text{a.e.}} f$.
2. If for each $n \in \mathbb{N}$, $f_n \geq f_{n+1}$ a.e., then $f_n \xrightarrow{\text{a.e.}} f$.

Proof.

1. Suppose that for each $n \in \mathbb{N}$, $f_n \leq f_{n+1}$ a.e. Define $N_1 \in \mathcal{A}$ by

$$N_1 = \bigcap_{n\in\mathbb{N}} \{x \in X : f_n(x) \leq f_{n+1}(x)\}$$

By assumption, $\mu(N_1^c) = 0$. Since $f_n \xrightarrow{\mu} f$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$. Hence there exists $N_2 \in \mathcal{A}$ such that $\mu(N_2^c) = 0$ and $f_{n_k}\chi_{N_2} \xrightarrow{\text{p.w.}} f\chi_{N_2}$. Set $N = N_1 \cap N_2$. Then

$$\begin{aligned}\mu(N^c) &= \mu(N_1^c \cup N_2^c) \\ &\leq \mu(N_1^c) + \mu(N_2^c) \\ &= 0\end{aligned}$$

By construction, $f\chi_N = \sup_{k\in\mathbb{N}} f_{n_k}\chi_N$ which implies that for each $n \in \mathbb{N}$,

$$\begin{aligned} f_n\chi_N &\leq f_{n_n}\chi_N \\ &\leq f\chi_N \end{aligned}$$

Let $x \in N$ and $\epsilon > 0$. Choose $K \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geq K$ implies that $|f_{n_k}(x) - f(x)| < \epsilon$. Let $n \in \mathbb{N}$. Suppose that $n \geq n_K$. Then

$$\begin{aligned} |f_n(x) - f(x)| &= f(x) - f_n(x) \\ &\leq f(x) - f_{n_K}(x) \\ &= |f_{n_K}(x) - f(x)| \\ &< \epsilon \end{aligned}$$

Hence $f_n(x) \to f(x)$. Since $x \in N$ is arbitrary, $f_n\chi_N \xrightarrow{\text{p.w.}} f\chi_N$. Since $\mu(N^c) = 0$, $f_n \xrightarrow{\text{a.e.}} f$.

2. Similar to (1).

$\square$

Definition 6.4.0.9. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Then $(f_n)_{n\in\mathbb{N}}$ is said to **converge to f almost uniformly**, denoted $f_n \xrightarrow{\text{a.u.}} f$, if for each $\epsilon > 0$, there exists $N \in \mathcal{A}$ such that $\mu(N) < \epsilon$ and $f_n \xrightarrow{\text{u}} f$ on N^c.

Exercise 6.4.0.10. Egoroff's Theorem: Suppose that $\mu(X) < \infty$. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$. Then $f_n \xrightarrow{\text{a.u.}} f$.

Proof. For each $n, k \in \mathbb{N}$, define $E_{n,k} = \{x \in X : |f_n(x) - f(x)| \geq \frac{1}{k}\}$ and $F_{n,k} = \bigcup_{m\geq n} E_{m,k}$. Then $F_{n,k}$ is decreasing in n and

$$\bigcap_{n\in\mathbb{N}} F_{n,k} \subset \{x : f_n(x) \not\to f(x)\}$$

Thus $\mu(\bigcap_{n\in\mathbb{N}} F_{n,k}) = 0$. Since $\mu(X) < \infty$, $\inf_{n\in\mathbb{N}} \mu(F_{n,k}) = 0$. Let $\epsilon > 0$. We may choose a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\mu(F_{n_k,k}) \leq \frac{\epsilon}{2^k}$. Put $N = \bigcup_{k\in\mathbb{N}} F_{n_k,k}$. Then

$$\begin{aligned} \mu(N) &\leq \sum_{k\in\mathbb{N}} \mu(F_{n_k,k}) \\ &\leq \sum_{k\in\mathbb{N}} \frac{\epsilon}{2^k} \\ &= \epsilon \end{aligned}$$

Let $\delta > 0$. Choose $K \in \mathbb{N}$ such that $\frac{1}{K} < \delta$. Then for each $m \geq n_K$ and $x \in N^c = \bigcap_{k\in\mathbb{N}} \bigcap_{m\geq n_k} E^c_{m,k}$, $|f_m(x) - f(x)| < \frac{1}{K} < \delta$. So $f_n \xrightarrow{\text{u}} f$ on N^c. $\square$

Exercise 6.4.0.11. Let $(f_n)_{n\in\mathbb{N}} \subset L^1$ and $f \in L^1$. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.

Proof. Let $\epsilon > 0$. for $n \in \mathbb{N}$, define $E_{e,n} = \{x \in X : |f(x) - f_n(x)| \geq \epsilon\}$. Then for $n \in \mathbb{N}$,

$$\begin{aligned} \int |f - f_n| &\geq \int_{E_{\epsilon,n}} |f - f_n| \\ &\geq \epsilon\mu(E_{\epsilon,n}). \end{aligned}$$

So for each $n \in \mathbb{N}$, $\mu(E_{\epsilon,n}) \leq \epsilon^{-1} \int |f - f_n|$. Since $\int |f - f_n| \to 0$, we have that $\mu(E_{\epsilon,n}) \to 0$. Since $\epsilon > 0$ is arbitrary, $f_n \xrightarrow{\mu} f$ as required. $\square$

Exercise 6.4.0.12. Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose $\mu(X) < \infty$. Define $d : L^0 \times L^0 \to [0, \infty)$ by

$$d(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, d\mu$$

Then d is a metric on L^0 if we identify functions that are equal a.e. and convergence in this metric is equivalent to convergence in measure. Note that for each $f, g \in L^0$, $d(f, g) \leq \mu(X)$.

Proof. Let $f, g \in L^0$. Clearly $d(f, g) = d(g, f)$. If $f = g$ a.e. then clearly $d(f, g) = 0$. Conversely, if $d(f, g) = 0$, then $\frac{|f-g|}{1+|f-g|} = 0$ a.e and so $|f - g| = 0$ a.e. which implies $f = g$ a.e. It is not hard to show that $\phi : [0, \infty) \to [0, \infty)$ given by $\phi(x) = \frac{x}{1+x}$ satisfies $\phi(x + y) \leq \phi(x) + \phi(y)$. Thus satisfies the triangle inequality. Now, let $(f_n)_{n \in \mathbb{N}} \subset L^0$. Suppose that $f_n \overset{\mu}{\not\to} f$. Then there exists $\epsilon > 0, \delta > 0$ and a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that for each $k \in \mathbb{N}$, $\mu(E_{\epsilon, n_k}) = \mu(\{x \in X : |f_{n_k} - f| \geq \epsilon\}) \geq \delta$. It is not hard to show that ϕ from earlier is increasing. Thus for each $k \in \mathbb{N}$,

$$\begin{aligned}
d(f_{n_k}, f) &= \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\
&\geq \int_{E_{\epsilon, n_k}} \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} \\
&\geq \int_{E_{\epsilon, n_k}} \frac{\epsilon}{1 + \epsilon} \\
&\geq \frac{\epsilon \delta}{1 + \epsilon}
\end{aligned}$$

So $f_{n_k} \overset{d}{\not\to} f$. Hence $f_{n_k} \xrightarrow{d} f$ implies that $f_{n_k} \xrightarrow{\mu} f$. Conversely, suppose that $f_{n_k} \xrightarrow{\mu} f$. Let $\epsilon > 0$. Then $\delta = \frac{\epsilon}{1+\mu(X)} > 0$. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then $\mu(E_{\delta, n}) < \frac{\delta}{1+\delta}$. Let $n \in \mathbb{N}$. Suppose that $n \geq N$. Since ϕ is increasing and $\phi \leq 1$, we have that

$$\begin{aligned}
d(f_n, f) &= \int \frac{|f_n - f|}{1 + |f_n - f|} \\
&= \int_{E_{\delta, n}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E^c_{\delta, n}} \frac{|f_n - f|}{1 + |f_n - f|} \\
&\leq \mu(E_{\delta, n}) + \mu(X) \frac{\delta}{1 + \delta} \\
&< \frac{\delta}{1 + \delta}(1 + \mu(X)) \\
&\leq \delta(1 + \mu(X)) \\
&= \epsilon
\end{aligned}$$

□

Exercise 6.4.0.13. Let $(f_n)_{n \in \mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that for each $n \in \mathbb{N}$, $f_n \geq 0$ and $f_n \xrightarrow{\mu} f$. Then $f \geq 0$ a.e. and

$$\int f \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu$$

Proof. Since $f_n \xrightarrow{\mu} f$, there is a subsequence converging to f a.e. So clearly $f \geq 0$ a.e. Now, choose a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that $\int f_{n_k} \to \liminf_{n \to \infty} \int f_n$. Since $f_n \xrightarrow{\mu} f$ so does $(f_{n_k})_{k \in N}$. Therefore there exists a subsequence $(f_{n_{k_j}})_{k \in \mathbb{N}}$ of $(f_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Thus $f \geq 0$ a.e. and Fatou's lemma tells us that

$$\begin{aligned}
\int f &\leq \liminf_{j \in \mathbb{N}} \int f_{n_{k_j}} \\
&= \liminf_{n \to \infty} \int f_n.
\end{aligned}$$

□

Exercise 6.4.0.14. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $|f_n| \leq g$. Then $f_n \xrightarrow{\mu} f$ implies that $f \in L^1$ and $f_n \xrightarrow{L^1} f$.

Proof. Clearly $(f_n)_{n\in\mathbb{N}} \subset L^1$. Since $f_n \xrightarrow{\mu} f$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subset (f_n)_{n\in\mathbb{N}}$ such that $f_{n_k} \xrightarrow{\text{a.e.}} f$. This implies that $|f| \leq g$ a.e. and so $f \in L^1$. For $n \in \mathbb{N}$, put $h_n = 2g - |f_n - f|$. Then for each $n \in \mathbb{N}$, $h_n \geq 0$ and $h_n \xrightarrow{\mu} 2g$. By the previous exercise

$$\begin{aligned}\int 2g &\leq \liminf_{n\to\infty} \int (2g - |f_n - f|) \\ &= \int 2g - \limsup_{n\to\infty} \int |f_n - f|.\end{aligned}$$

So $\limsup\limits_{n\to\infty} \int |f_n - f| \leq 0$ which implies that $\int |f_n - f| \to 0$ and $f_n \xrightarrow{L^1} f$ as required. □

Exercise 6.4.0.15. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$.

1. If ϕ is continuous, and $f_n \xrightarrow{\text{a.e.}} f$ then $\phi \circ f_n \xrightarrow{\text{a.e.}} \phi \circ f$.
2. If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly or in measure, respectively.
3. Find a counter example to (2) if we drop the word "uniform".

Proof.

1. Clear
2. Suppose that ϕ is unifomly continuous.
   - uniformly:
     Suppose that $f_n \xrightarrow{\text{u}} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Now choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ if $n \geq n$ then for each $x \in X$, $|f_n(x) - f(x)| < \delta$. Let $n \in \mathbb{N}$, suppose $n \geq N$, Let $x \in X$. Then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Thus $\phi \circ f_n \xrightarrow{\text{u}} \phi \circ f$.
   - almost uniformly:
     Suppose that $f_n \xrightarrow{\text{a.u.}} f$. Let $\epsilon > 0$. Choose $N \in \mathcal{A}$ such $\mu(N) < \epsilon$ and $f_n \xrightarrow{\text{u}} f$ on N^c. Then from above, we know that $\phi \circ f_n \xrightarrow{\text{u}} \phi \circ f$ on N^c. Thus $\phi \circ f_n \xrightarrow{\text{a.u.}} \phi \circ f$.
   - in measure:
     Suppose that $f_n \xrightarrow{\mu} f$. Let $\epsilon > 0$. Choose $\delta > 0$ such that for each $z, w \in \mathbb{C}$, if $|z - w| < \delta$, then $|\phi(z) - \phi(w)| < \epsilon$. Observe that for $x \in X$, if $|f_n(x) - f(x)| < \delta$, then $|\phi(f_n(x)) - \phi(f(x))| < \epsilon$. Hence $E_{n,\epsilon} = \{x \in X : |\phi(f_n(x)) - \phi(f(x))| \geq \epsilon\} \subset F_{n,\delta} = \{x \in X : |f_n(x) - f(x)| \geq \delta\}$. By definition of convergence in measure, $\mu(F_{n,\delta}) \to 0$. Thus $\mu(E_{n,\epsilon}) \to 0$. Hence $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.
3.

□

Exercise 6.4.0.16. Let $(f_n)_{n\in\mathbb{N}} \subset L^0$ and $f \in L^0$. Suppose that $f_n \xrightarrow{\text{a.u}} f$. Then $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\text{a.e.}} f$.

Proof. (measure) Let $\epsilon > 0$, $\delta > 0$. Choose $M \in \mathcal{A}$ such that $\mu(M) < \delta$ and $f_n \xrightarrow{\text{u}} f$ on M^c. Choose $N \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N$, then for each $x \in M^c$, $|f_n(x) - f(x)| < \epsilon$. Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then $E_{\epsilon,n} \subset M$ and $\mu(E_{\epsilon,n}) < \delta$. Thus $\mu(E_{\epsilon,n}) \to 0$ and $f_n \xrightarrow{\mu} f$.

(a.e.) For each $n \in \mathbb{N}$, Choose $N_n \in \mathcal{A}$ such that $\mu(N_n) < 1/n$ and $f_n \xrightarrow{\text{u}} f$ on N_n^c. Observe that for $x \in X$, if $x \in \bigcup_{n\in\mathbb{N}} N_n^c$, then $f_n(x) \to f(x)$. Thus $N = \{x \in X : f_n(x) \not\to f(x)\} \subset \bigcap_{n\in\mathbb{N}} N_n$. Therefor $\mu(N) = 0$ and $f_n \xrightarrow{\text{a.e.}} f$. □

Exercise 6.4.0.17. Let $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset L^0$ and $f, g \in L^0$. Suppose that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Then

1. $f_n + g_n \xrightarrow{\mu} f + g$
2. if $\mu(X) < \infty$, then $f_n g_n \xrightarrow{\mu} fg$

Proof. 1. Let $\epsilon > 0$. For convenience, put $F_{n,\epsilon/2} = \{x \in X : |f_n(x) - f(x)| \geq \epsilon/2\}$, $G_{n,\epsilon/2} = \{x \in X : |g_n(x) - g(x)| \geq \epsilon/2\}$, and $(F+G)_{n,\epsilon} = \{x \in X : |f_n(x) + g_n(x) - (f(x) + g_n(x))| \geq \epsilon\}$ Observe that for $x \in X$, $|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$. Thus $(F+G)_{n,\epsilon} \subset F_{n,\epsilon/2} \cup G_{n,\epsilon/2}$. Since $\mu(F_{n,\epsilon/2} \cup G_{n,\epsilon/2}) \leq \mu(F_{n,\epsilon/2}) + \mu(G_{n,\epsilon/2}) \to 0$, we have that $\mu((F+G)_{n,\epsilon}) \to 0$. Hence $f_n + g_n \xrightarrow{\mu} f + g$.

2. Suppose that $\mu(X) < \infty$. Let $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(f_n g_n)_{n \in \mathbb{N}}$. Choose a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e}} f$ and $g_{n_{k_j}} \xrightarrow{\text{a.e}} g$. Then $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.e.}} fg$. Egoroff's theorem tells us that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\text{a.u.}} fg$, which implies that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$. Thus for each subsequence $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ of $(f_n g_n)_{n \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}} g_{n_{k_j}})_{j \in \mathbb{N}}$ of $(f_{n_k} g_{n_k})_{k \in \mathbb{N}}$ such that $f_{n_{k_j}} g_{n_{k_j}} \xrightarrow{\mu} fg$. Using the fact that this is equivalent to convergence in a metric defined in an earlier exercise, we have that $f_n g_n \xrightarrow{\mu} fg$.

$\square$

Exercise 6.4.0.18. Let $(f_n)_{n \in \mathbb{N}}, \subset L^0$ and $f \in L^0$. Suppose that $\mu(X) < \infty$. Then $f_n \xrightarrow{\mu} f_n$ iff for each subsequence $(f_{n_k})_{k \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$.

Proof. Suppose that $f_n \xrightarrow{\mu} f$. Let $(f_{n_k})_{k \in \mathbb{N}}$ be a subsequence. Then $f_{n_k} \xrightarrow{\mu} f$. By a previous theorem, there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Conversely, suppose that for each subsequence $(f_{n_k})_{k \in \mathbb{N}}$, there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Let $\epsilon > 0$. For $n \in \mathbb{N}$, define $E_n = \{x \in X : |f_n(x) - f(x)| \geq \epsilon\}$ and define $E = \{x \in X : f_n(x) \not\to f(x)\}$. Let $(f_{n_k})_{k \in \mathbb{N}}$ be a subsequence. Choose a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. Since $\left\{x \in X : \limsup_{j \to \infty} \chi_{E_{n_{k_j}}}(x) = 1\right\} = \limsup_{j \to \infty} E_{n_{k_j}} \subset E$ and $\mu(E) = 0$, we have that $\limsup_{j \to \infty} \chi_{E_{n_{k_j}}} = 0$ a.e. and $\chi_{E_{n_{k_j}}} \xrightarrow{\text{a.e}} 0$. Since $\mu(X) < \infty$, the dominated convergence theorem implies that

$$\mu(E_{n_{k_j}}) = \int \chi_{E_{n_{k_j}}} \, d\mu \to 0$$

So for each subsequence $(\mu(E_{n_k}))_{k \in \mathbb{N}}$, there exists a subsequence $(\mu(E_{n_{k_j}}))_{j \in \mathbb{N}}$ such that $\mu(E_{n_{k_j}}) \to 0$. Thus $\mu(E_n) \to 0$ and $f_n \xrightarrow{\mu} f$. $\square$

Exercise 6.4.0.19. Let $(f_n)_{n \in \mathbb{N}}, \subset L^0$, $f \in L^0$ and $\phi : \mathbb{C} \to \mathbb{C}$. Suppose that $\mu(X) < \infty$. If ϕ is continuous and $f_n \xrightarrow{\mu} f$, then $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$.

Proof. Suppose that ϕ is continuous and $f_n \xrightarrow{\mu} f$. Let $(\phi \circ f_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(\phi \circ f_n)_{n \in \mathbb{N}}$. Then $(f_{n_k})_{k \in \mathbb{N}}$ is a subsequence of $(f_n)_{n \in \mathbb{N}}$. Since $f_n \xrightarrow{\mu} f$, the previous exercise tells us that there exists a subsequence $(f_{n_{k_j}})_{j \in \mathbb{N}}$ such that $f_{n_{k_j}} \xrightarrow{\text{a.e.}} f$. A previous exercise implies that $\phi \circ f_{n_{k_j}} \xrightarrow{\text{a.e.}} \phi \circ f$. The previous exercise implies that $\phi \circ f_n \xrightarrow{\mu} \phi \circ f$. $\square$

Exercise 6.4.0.20. Let $(f_n)_{n \in \mathbb{N}} L^0$ and $f \in L^0$. Suppose that for each $\epsilon > 0$,

$$\sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) < \infty$$

Then $f_n \xrightarrow{\text{a.e.}} f$.

Proof. Let $\epsilon > 0$. By assumption we know that

$$\begin{aligned}
\int \left[\sum_{n \in \mathbb{N}} \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}}\right] d\mu &= \sum_{n \in \mathbb{N}} \int \chi_{\{x \in X : |f_n(x) - f(x)| > \epsilon\}} \, d\mu \\
&= \sum_{n \in \mathbb{N}} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \\
&< \infty
\end{aligned}$$

Thus we also know that $\sum_{n\in\mathbb{N}} \chi_{\{x\in X:|f_n(x)-f(x)|>\epsilon\}} < \infty$ a.e. Equivalently, we could say that for a.e. $x \in X$, $|\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\}| < \infty$. For $k \in \mathbb{N}$, define $N_k = \{x \in X : \sum_{n\in\mathbb{N}} \chi_{\{x\in X:|f_n(x)-f(x)|>1/k\}} = \infty\}$. Then for each $k \in \mathbb{N}$, $\mu(N_k) = 0$. Define $N = \bigcup_{k\in\mathbb{N}} N_k$. Then $\mu(N) = 0$. Let $x \in N^c$ and $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \epsilon$. Then $\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\} \subset \{n \in \mathbb{N} : f_n(x) - f(x) > 1/k\}$ which is finite because $x \in N_k^c$. Put $M = \max\{n \in \mathbb{N} : f_n(x) - f(x) > \epsilon\}$. Then for $m \geq M$, $|f_m(x) - f(x) \leq \epsilon|$. Thus $f_n(x) \to f(x)$. Hence $f_n \xrightarrow{\text{a.e.}} f$. $\square$

Chapter 7

The Radon-Nikodym Derivative

7.1 Mutually Singular and Absolutely Continuous Measures

Definition 7.1.0.1. Let $(X, \mathcal{A})$ be a measurable space and ν, μ measures on $(X, \mathcal{A})$. Then

- ν and μ are said to be **mutually singular**, denoted $\nu \perp \mu$, if there exists $A, B \in \mathcal{A}$ such that $A \cap B = \varnothing$, $A \cup B = X$, $\nu(A) = 0$ and $\mu(B) = 0$.
- ν is said to be **absolutely continuous with respect to** μ, denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.

Exercise 7.1.0.2. Let $(X, \mathcal{A})$ be a measurable space, ν, μ measures on $(X, \mathcal{A})$ and $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Suppose that

1. $X = \bigcup\limits_{n\in\mathbb{N}} E_n$
2. for each $n \in \mathbb{N}$
   (a) $E_n \subset E_{n+1}$
   (b) $\nu|_{E_n} \perp \mu|_{E_n}$

Then $\nu \perp \mu$.

Proof. Let $n \in \mathbb{N}$. Since $E_n \in \mathcal{A}$, $\mathcal{A} \cap E_n \subset \mathcal{A}$. Since $\nu|_{E_n} \perp \mu|_{E_n}$, there exist

$$\begin{aligned} A_n, B_n &\in \mathcal{A} \cap E_n \\ &\subset \mathcal{A} \end{aligned}$$

such that A_n is $\nu|_{E_n}$-null, B_n is $\mu|_{E_n}$-null, $A_n \cap B_n = \varnothing$ and $A_n \cup B_n = E_n$. Define $(A'_n)_{n\in\mathbb{N}}, (B'_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ by

$$A'_n = \begin{cases} A_n & n = 1 \\ A_n \setminus E_{n-1} & n \geq 2 \end{cases}$$

$$B'_n = \begin{cases} B_n & n = 1 \\ B_n \setminus E_{n-1} & n \geq 2 \end{cases}$$

Set $A' = \bigcup\limits_{n\in\mathbb{N}} A'_n$ and $B' = \bigcup\limits_{n\in\mathbb{N}} B'_n$. Let $n, j \in \mathbb{N}$.

- Suppose that $n < j$. Then $n \leq j - 1$. Since $E_n \subset E_{j-1}$, we have that $E_{j-1}^c \subset E_n^c$ and therefore

$$\begin{aligned} A'_n \cap B'_j &\subset E_n \cap (E_j \setminus E_{j-1}) \\ &= E_n \cap (E_j \cap E_{j-1}^c) \\ &\subset E_n \cap (E_j \cap E_n^c) \\ &= \varnothing \end{aligned}$$

  Hence $A'_n \cap B'_j = \varnothing$.

- Similarly, if $j < n$, then $A'_n \cap B'_j = \varnothing$.
- Suppose that $j = n$. Since $A'_n \subset A_n$ and $B'_n \subset B_n$, we have that

$$\begin{aligned} A'_n \cap B'_j &= A'_n \cap B'_n \\ &\subset A_n \cap B_n \\ &= \varnothing \end{aligned}$$

  Thus $A'_n \cap B'_j = \varnothing$.

Therefore

$$\begin{aligned} A' \cap B' &= \left[\bigcup_{n \in \mathbb{N}} A'_n \right] \cap \left[\bigcup_{j \in \mathbb{N}} B'_j \right] \\ &= \bigcup_{n \in \mathbb{N}} \left[A'_n \cap \left(\bigcup_{j \in \mathbb{N}} B'_j \right) \right] \\ &= \bigcup_{n \in \mathbb{N}} \left[\bigcup_{j \in \mathbb{N}} (A'_n \cap B'_j) \right] \\ &= \bigcup_{n \in \mathbb{N}} \left[\bigcup_{j \in \mathbb{N}} \varnothing \right] \\ &= \varnothing \end{aligned}$$

Let $x \in X$.

- Suppose that $x \in E_1$. Then

$$\begin{aligned} x &\in E_1 \\ &= A_1 \cap B_1 \\ &= A'_1 \cap B'_1 \\ &\subset \left[\bigcup_{n \in \mathbb{N}} A'_n \right] \cup \left[\bigcup_{n \in \mathbb{N}} B'_n \right] \\ &= A' \cup B' \end{aligned}$$

- Suppose that $x \notin E_1$.
  For the sake of contradiction, suppose that for each $n \in \mathbb{N}$, $n \geq 2$ and $x \in E_n$ implies that $x \in E_{n-1}$. Since $X = \bigcup_{n \in \mathbb{N}} E_n$, there exists $n \in \mathbb{N}$ such that $x \in E_n$. Since $x \notin E_1$, $n \geq 2$. By assumption, $x \in E_{n-1}$. By induction $x \in E_1$, which is a contradiction. Therefore there exists $N \in \mathbb{N}$ such that $N \geq 2$, $x \in E_N$ and $x \notin E_{N-1}$.
  Since $E_N = A_N \cup B_N$, $x \in A_N$ or $x \in B_N$. If $x \in A_N$, then

$$\begin{aligned} x &\in A_N \cap E^c_{N-1} \\ &= A_N \setminus E_{N-1} \\ &= A'_N \\ &\subset \bigcup_{n \in \mathbb{N}} A'_n \\ &= A' \\ &\subset A' \cup B' \end{aligned}$$

  If $x \in B_N$, then similarly, $x \in A' \cup B'$.

Since $x \in X$ is arbitrary, $X \subset A' \cup B'$. Hence $X = A' \cup B'$.
Let $n, j \in \mathbb{N}$.

- Suppose that $j \geq n$. Then $E_n \subset E_j$ so that $E_j^c \subset E_n^c$ and

$$\begin{aligned}
\nu(A_j' \cap E_n) &= \nu([A_j \setminus E_j] \cap E_n) \\
&= \nu([A_j \cap E_j^c] \cap E_n) \\
&\leq \nu([A_j \cap E_n^c] \cap E_n) \\
&= \nu(\varnothing) \\
&= 0
\end{aligned}$$

  Hence $\nu(A_j' \cap E_n) = 0$.

- Suppose that $j < n$. Since

$$\begin{aligned}
A_j &\subset E_j \\
&\subset E_n
\end{aligned}$$

  we have that

$$\begin{aligned}
\nu(A_j' \cap E_n) &= \nu([A_j \setminus E_j] \cap E_n) \\
&= \nu([A_j \cap E_j^c] \cap E_n) \\
&= \nu(A_j \cap E_j^c) \\
&\leq \nu(A_j) \\
&= \nu|_{E_j}(A_j) \\
&= 0
\end{aligned}$$

We note that

$$\begin{aligned}
A' &= A' \cap X \\
&= A' \cap \left[\bigcup_{n \in \mathbb{N}} E_n\right] \\
&= \bigcup_{n \in \mathbb{N}} (A' \cap E_n)
\end{aligned}$$

and for each $n \in \mathbb{N}$, $A' \cap E_n \subset A' \cap E_{n+1}$. Since

$$\begin{aligned}
A' \cap E_n &= \left[\bigcup_{j \in \mathbb{N}} A_j'\right] \cap E_n \\
&= \bigcup_{j \in \mathbb{N}} (A_j' \cap E_n)
\end{aligned}$$

we have that

$$\begin{aligned}
\nu(A') &= \sup_{n \in \mathbb{N}} \nu(A' \cap E_n) \\
&\leq \sup_{n \in \mathbb{N}} \left[\sum_{j \in \mathbb{N}} \nu_\mu^\perp(A_j' \cap E_n)\right] \\
&= 0
\end{aligned}$$

Similarly, $\mu(B') = 0$. Since $A' \cup B' = X$, $A' \cap B' = \varnothing$, A' is ν-null and B' is μ-null, $\nu \perp \mu$. $\square$

7.2 Signed Measures

Definition 7.2.0.1. Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$. Then ν is said to be a **signed measure** if

1. for each $E \in \mathcal{A}$, $\nu(E) < \infty$ or for each $E \in \mathcal{A}$, $\nu(E) > -\infty$.
2. $\nu(\varnothing) = 0$
3. for each $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ if $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is disjoint, then $\nu(\bigcup\limits_{n\in\mathbb{N}} E_n) = \sum\limits_{n\in\mathbb{N}} \nu(E_n)$ and if $|\sum\limits_{n\in\mathbb{N}} \nu(E_n)| < \infty$, then $\sum\limits_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely.

Exercise 7.2.0.2. Let $\nu : \mathcal{A} \to [0, \infty]$ be a signed measure and $(E_n)_{n\in\mathbb{N}}, (F_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. If $(E_n)_{n\in\mathbb{N}}$ is increasing, then $\nu(\bigcup\limits_{n\in\mathbb{N}} E_n) = \lim\limits_{n\to\infty} \nu(E_n)$. If $(F_n)_{n\in\mathbb{N}}$ is decreasing and $|\nu(E_1)| < \infty$, then $\nu(\bigcap\limits_{n\in\mathbb{N}} F_n) = \lim\limits_{n\to\infty} \nu(F_n)$.

Proof. Put $E_1' = E_1$, $F_1' = F_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $E_n' = E_n \setminus E_{n-1}$ and $F_n' = F_1 \setminus F_n$. Then $(E_n')_{n\in\mathbb{N}} \subset \mathcal{A}$ is disjoint. Thus

$$\begin{aligned}
\nu(\bigcup_{n\in\mathbb{N}} E_n) &= \nu(\bigcup_{n\in\mathbb{N}} E_n') \\
&= \sum_{n\in\mathbb{N}} \nu(E_n') \\
&= \lim_{n\to\infty} \sum_{n=1}^{n} \nu(E_n') \\
&= \lim_{n\to\infty} \nu(E_n)
\end{aligned}$$

Since $(F_n')_{n\in\mathbb{N}}$ is increasing, we now know that

$$\begin{aligned}
\nu(F_1) - \nu(\bigcap_{n\in\mathbb{N}} F_n) &= \nu(F_1 \setminus \bigcap_{n\in\mathbb{N}} F_n) \\
&= \nu(\bigcup_{n\in\mathbb{N}} F_n') \\
&= \lim_{n\to\infty} \nu(F_n') \\
&= \lim_{n\to\infty} \nu(F_1 \setminus F_n) \\
&= \nu(F_1) - \lim_{n\to\infty} \nu(F_n)
\end{aligned}$$

Since $|\nu(F_1)| < \infty$, we see that $\nu(\bigcap\limits_{n\in\mathbb{N}} F_n) = \lim\limits_{n\to\infty} \nu(F_n)$. □

Definition 7.2.0.3. Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to [-\infty, \infty]$ a signed measure and $E \in \mathcal{A}$. Then E is said to be ν**-positive**, ν**-negative** and ν**-null** if for each $F \in \mathcal{A}$, $F \subset E$ implies that $\nu(F) \geq 0$, $\nu(F) \leq 0$, $\nu(F) = 0$ respectively.

Exercise 7.2.0.4. Let $E \subset \mathcal{A}$. If E is positive, negative or null, then for each $F \in \mathcal{A}$, if $F \subset E$, then F is positive, negative or null respectively.

Proof. Clear □

Exercise 7.2.0.5. Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ be positive, negative or null. Then $\bigcup\limits_{n\in\mathbb{N}} E_n$ is positive, negative or null respectively.

Proof. Suppose that $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ is positive. Let $F \in \mathcal{A}$. Suppose that $F \subset \bigcup\limits_{n\in\mathbb{N}} E_n$. Put $P_1 = E_1$ and for $n \in \mathbb{N}$, $n \geq 2$, put $P_n = E_n \setminus (\bigcup_{j=1}^{n-1} E_j)$. So $\bigcup_{n\in\mathbb{N}} P_n = \bigcup_{n\in\mathbb{N}} E_n$ and $(P_n)_{n\in\mathbb{N}}$ is disjoint. Thus

$$\begin{aligned}\nu(F) &= \nu(F \cap \bigcup_{n\in\mathbb{N}} P_n)\\ &= \nu(\bigcup_{n\in\mathbb{N}} (F \cap P_n))\\ &= \sum_{n\in\mathbb{N}} \nu(F \cap P_n)\\ &\geq 0\end{aligned}$$

The process is the same if $(E_n)_{n\in\mathbb{N}}$ is negative and null. $\square$

Theorem 7.2.0.6. Hahn Decomposition:
Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist $P, N \in \mathcal{A}$ such that P is positive, N is negative, $X = N \cup P$ and $N \cap P = \varnothing$. Furthermore, these two sets are unique in the following sense: For any $P', N' \in \mathcal{A}$, if N, P satisfy the properties above, $P' \Delta P = N' \Delta N$ is ν-null.

Definition 7.2.0.7. Let ν be a signed measure on $(X, \mathcal{A})$ and $P, N \in \mathcal{A}$. Then P and N are said to form a **Hahn decomposition** of X with respect to ν if P, N satisfy the results in the above theorem.

Definition 7.2.0.8. Let μ, ν be signed measures on $(X, \mathcal{A})$. Then μ and ν are said to be **mutually singular** if there exist $E, F \in \mathcal{A}$ such that $X = E \cup F$, $E \cap F = \varnothing$ and E is μ-null and F is ν-null. We will denote this by $\mu \perp \nu$.

Theorem 7.2.0.9. Jordan Decomposition: Let ν be a signed measure on $(X, \mathcal{A})$. Then there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{A})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof. Choose a Hahn decomposition P, N of X with respect to ν. Define ν^+, ν^- by $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$. $\square$

Definition 7.2.0.10. Let ν be a signed measure on $(X, \mathcal{A})$. Then ν^+ and ν^- from the last theorem are called the **positive** and **negative variations** of ν respectively. We define the **total variation of** ν, denoted $|\nu| : \mathcal{A} \to [0, \infty]$ by

$$|\nu| = \nu^+ + \nu^-$$

Definition 7.2.0.11. Let ν be a signed measure on $(X, \mathcal{A})$. Then ν is said to be σ-finite if $|\nu|$ is σ-finite.

Exercise 7.2.0.12. Let ν be a signed measure and λ, μ positive measures on $(X, \mathcal{A})$. Suppose that $\nu = \lambda - \mu$. Then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. Choose a Hahn decomposition P, N of X with respect to ν. Let $E \in \mathcal{A}$. Then

$$\begin{aligned}\lambda(E \cap P) - \mu(E \cap P) &= \nu(E \cap P)\\ &= \nu^+(E \cap P)\end{aligned}$$

So $\lambda(E \cap P) \geq \nu^+(E \cap P)$ and therefore

$$\begin{aligned}\lambda(E) &= \lambda(E \cap P) + \lambda(E \cap N)\\ &\geq \nu^+(E \cap P) + \lambda(E \cap N)\\ &\geq \nu^+(E \cap P)\\ &= \nu^+(E)\end{aligned}$$

Similarly $\mu(E \cap N) \geq \nu^-(E \cap N)$ and $\mu(E) \geq \nu^-(E)$. $\square$

Exercise 7.2.0.13. Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1 + \nu_2$ is a signed measure. Then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Hint: use the last exercise)

Proof. Since

$$\begin{aligned}\nu_1 + \nu_2 &= (\nu_1^+ - \nu_1^-) + (\nu_2^+ - \nu_2^-) \\ &= (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)\end{aligned}$$

the previous exercise tells us that $\lambda = \nu_1^+ + \nu_2^+ \geq (\nu_1+\nu_2)^+$ and $\mu = \nu_1^- + \nu_2^- \geq (\nu_1+\nu_2)^-$. Therefore

$$\begin{aligned}|\nu_1 + \nu_2| &= (\nu_1+\nu_2)^+ + (\nu_1+\nu_2)^- \\ &\leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) \\ &= |\nu_1| + |\nu_2|\end{aligned}$$

□

Note 7.2.0.14. Recall that a previous exercise from the section on complex valued functions tells us that $L^1(|\nu|) = L^1(\nu^+) \cap L^1(\nu^-)$.

Definition 7.2.0.15. Let ν be a signed measure on $(X, \mathcal{A})$. Then we define $L^1(\nu) = L^1(|\nu|)$. For $f \in L^1(\nu)$, we define

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-$$

Exercise 7.2.0.16. Let ν_1, ν_2 be signed measures on $(X, \mathcal{A})$. Suppose that $\nu_1+\nu_2$ is a signed measure. Then $L^1(\nu_1) \cap L^1(\nu_2) \subset L^1(\nu_1 + \nu_2)$

Proof. The previous exercise tells us that $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. Two previous exercises from the section on nonnegative functions imply that

$$\begin{aligned}\int |f| d|\nu_1 + \nu_2| &\leq \int |f| d(|\nu_1| + |\nu_2|) \\ &= \int |f| d|\nu_1| + \int |f| d|\nu_2|\end{aligned}$$

□

Exercise 7.2.0.17. Let ν, μ be signed measures on $(X, \mathcal{A})$ and $E \in \mathcal{A}$. Then

1. E is ν-null iff $|\nu|(E) = 0$
2. $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof.
1. Suppose that E is ν-null. Choose a Hahn decomposition P, N of X with respect to ν. Then $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = \nu(E \cap N) = 0$. Therefore $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Conversely, suppose that $|\nu|(E) = 0$. Then $\nu^+(E) = \nu^-(E) = 0$. Let $F \in \mathcal{A}$. Suppose that $F \subset E$. Then $\nu^+(F) = 0$ and $\nu^-(F) = 0$. Therefore $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. So E is ν-null.
2. Suppose that $\nu \perp \mu$. Then there exist $E, F \in \mathcal{A}$ such that $E \cup F = X$, $E \cap F = \varnothing$, E is μ-null and F is ν-null. By (1), F is $|\nu|$-null and thus $|\nu| \perp \mu$. If $|\nu| \perp \mu$, choose $E, F \in \mathcal{A}$ as before. Since F is $|\nu|$-null, we know that $\nu^+(F) + \nu^-(F) = |\nu|(F) = 0$. This implies that F is ν^+-null and F is ν^--null. So $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Finally assume that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. **FINISH!!!!**

□

Exercise 7.2.0.18. Let ν be a signed measure on $(X, \mathcal{A})$. Then

1. for $f \in L^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$
2. if ν is finite, then for each $E \in \mathcal{A}$,

$$|\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| : f \text{ is measurable and } |f| \leq 1 \right\}$$

Proof. 1. Let $f \in L^1(\nu)$. Then

$$\begin{aligned}\left|\int f d\nu\right| &= \left|\int f d\nu^+ - \int f d\nu^-\right| \\ &\leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- \\ &= \int |f| d(\nu^+ + \nu^-) \\ &= \int |f| d|\nu|\end{aligned}$$

2. Let $E \in \mathcal{A}$. Let $f : X \to \mathbb{R}$ be measurable and suppose that $|f| \leq 1$. Since ν is finite, so is $|\nu|$ and thus $f \in L^1(\nu)$. Then (1) tells us that

$$\begin{aligned}\left|\int_E f\, d\nu\right| &\leq \int_E |f|\, d|\nu| \\ &\leq |\nu|(E)\end{aligned}$$

Now, choose a Hahn decomposition P, N of X with respect to ν. Define $f = \chi_P - \chi_N$. Then $|f| \leq 1$, f is measurable and

$$\begin{aligned}\left|\int_E f d\nu\right| &= \left|\int_E f d\nu^+ - \int_E f d\nu^-\right| \\ &= |\nu^+(E \cap P) + \nu^-(E \cap N)| \\ &= \nu^+(E) + \nu^-(E) \\ &= |\nu|(E).\end{aligned}$$

□

Exercise 7.2.0.19. Let μ be a positive measure on $(X, \mathcal{A})$ and $f \in L^0(X, \mathcal{A})$ extended μ-integrable. Define ν on $(X, \mathcal{A})$ by

$$\nu(E) = \int_E f\, d\mu$$

Then

1. ν is a signed measure

2. for each $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E |f|\, d\mu$$

Proof. 1. Clearly $\nu(\varnothing) = 0$ and ν is finte by assumption. Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_n)_{n\in\mathbb{N}}$ is disjoint. Then

$$\begin{aligned}
\nu(\bigcup_{n\in\mathbb{N}} E_n) &= \int_{\bigcup_{n\in\mathbb{N}} E_n} f\,d\mu \\
&= \int_{\bigcup_{n\in\mathbb{N}} E_n} f^+\,d\mu - \int_{\bigcup_{n\in\mathbb{N}} E_n} f^-\,d\mu \\
&= \sum_{n\in\mathbb{N}} \int_{E_n} f^+\,d\mu - \sum_{n\in\mathbb{N}} \int_{E_n} f^-\,d\mu \\
&= \sum_{n\in\mathbb{N}} \left[\int_{E_n} f^+\,d\mu - \int_{E_n} f^-\,d\mu \right] \\
&= \sum_{n\in\mathbb{N}} \int_{E_n} f\,d\mu \\
&= \sum_{n\in\mathbb{N}} \nu(E_n)
\end{aligned}$$

If $|\nu(\bigcup_{n\in\mathbb{N}} E_n)| < \infty$, then $\int_{\bigcup_{n\in\mathbb{N}} E_n} f^+ d\mu < \infty$ and $\int_{\bigcup_{n\in\mathbb{N}} E_n} f^- d\mu < \infty$ because

$$\begin{aligned}
|\nu(\bigcup_{n\in\mathbb{N}} E_n)| &= \left| \int_{\bigcup_{n\in\mathbb{N}} E_n} f\,d\mu \right| \\
&= \left| \int_{\bigcup_{n\in\mathbb{N}} E_n} f^+\,d\mu - \int_{\bigcup_{n\in\mathbb{N}} E_n} f^-\,d\mu \right|
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |\nu(E_n)| &= \sum_{n\in\mathbb{N}} \left| \int_{E_n} f\,d\mu \right| \\
&= \sum_{n\in\mathbb{N}} \left| \int_{E_n} f^+\,d\mu - \int_{E_n} f^-\,d\mu \right| \\
&\leq \sum_{n\in\mathbb{N}} \int_{E_n} f^+\,d\mu + \sum_{n\in\mathbb{N}} \int_{E_n} f^-\,d\mu \\
&= \int_{\bigcup_{n\in\mathbb{N}} E_n} f^+\,d\mu + \int_{\bigcup_{n\in\mathbb{N}} E_n} f^-\,d\mu \\
&< \infty
\end{aligned}$$

So the sum $\sum_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely and ν is a signed measure.

2. Put $P = \{x \in X : f(x) \geq 0\}$ and $N = \{x \in X : f(x) < 0\}$. Then P, N form a Hahn decomposition of X with respect to ν. Thus for $E \in \mathcal{A}$,

$$\nu^+(E) = \int_{E\cap P} f\,d\mu = \int_E f^+\,d\mu$$

and

$$\nu^-(E) = \int_{E\cap N} f\,d\mu = \int_E f^-\,d\mu$$

So for $E \in \mathcal{A}$,

$$|\nu|(E) = \int_E f^+\,d\mu + \int_E f^-\,d\mu = \int_E |f|\,d\mu$$

□

Definition 7.2.0.20. Let $(X, \mathcal{A})$ be a measureable space, ν be a signed measure on $(X, \mathcal{A})$ and μ a measure on $(X, \mathcal{A})$. Then ν is said to be **absolutely continuous** with respect to μ, denoted $\nu \ll \mu$, if for each $E \in \mathcal{A}$, $\mu(E) = 0$ implies that $\nu(E) = 0$.

Note 7.2.0.21. If there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that for each $E \in \mathcal{A}$, $\nu(E) = \int_E f\, d\mu$, then we write $d\nu = f\, d\mu$.

Exercise 7.2.0.22. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measureable spaces, $f : X \to Y$ $\mathcal{A}$-$\mathcal{B}$ measurable, ν be a signed measure on $(X, \mathcal{A})$ and μ a measure on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$. Then $f_*\nu \ll f_*\mu$

Proof. Let $E \in \mathcal{B}$. Suppose that $f_*\mu(E) = 0$. By definition, $\mu(f^{-1}(E)) = 0$. Since $\nu \ll \mu$, $\nu(f^{-1}(E)) = 0$. Hence $f_*\nu(E) = 0$ and $f_*\nu \ll f_*\mu$. □

Theorem 7.2.0.23. Lebesgue Decomposition Theorem:
Let $(X, \mathcal{A})$ be a measureable space, ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists an extended μ-integrable $f \in L^0(X, \mathcal{A})$ such that $d\rho = f\, d\mu$ and f is unique μ-a.e.

Definition 7.2.0.24. The decomposition $\nu = \lambda + \rho$ is referred to as the **Lebesgue decomposition of ν with respect to μ**. In the case $\nu \ll \mu$, we have $\lambda = 0$ and $\rho = \nu$ and we define the **Radon-Nikodym derivative of ν with respect to μ**, denoted by $d\nu/d\mu$, to be $d\nu/d\mu = f$ where $d\nu = f\, d\mu$.

Theorem 7.2.0.25. Let $(X, \mathcal{A})$ be a measurable space, ν a σ-finite signed measure on $(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then

1. for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and
$$\int g d\nu = \int g \frac{d\nu}{d\mu}\, d\mu$$
2. $\nu \ll \lambda$ and
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Exercise 7.2.0.26. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures and μ a measure.

1. If for each $n \in \mathbb{N}$, $\nu_n \ll \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.
2. If for each $n \in \mathbb{N}$, $\nu_n \perp \mu$, then $\sum_{n\in\mathbb{N}} \nu_n \perp \mu$.

Proof.
1. Let $E \in \mathcal{A}$. Suppose that $\mu(E) = 0$. Then for each $n \in \mathbb{N}$, $\nu_i(E) = 0$ and thus $\sum_{n\in\mathbb{N}} \nu_n(E) = 0$. Hence $\sum_{n\in\mathbb{N}} \nu_n \ll \mu$.
2. For each $n \in \mathbb{N}$, there exist $N_i, M_i \in \mathcal{A}$ such that $N_i \cap M_i = \varnothing$, $N_i \cup M_i = X$ and $\nu_i(M_i) = \mu(N_i) = 0$. Put $N = \bigcup_{n\in\mathbb{N}} N_i$ and $M = N^c$. Note that for each $n \in \mathbb{N}$, $M \subset N_i^c = M_i$. So $\mu(N) \leq \sum_{n\in\mathbb{N}} \mu(N_i) = 0$ and $(\sum_{n\in\mathbb{N}} \nu_i)(M) \leq \sum_{n\in\mathbb{N}} \nu_i(M_i) = 0$. Thus $\sum_{n\in\mathbb{N}} \nu_i \perp \mu$.

□

Exercise 7.2.0.27. Choose $X = [0, 1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$. Let m be Lebesgue measure and μ the counting measure. Then

1. $m \ll \mu$ but for each $f \in L^+$, $dm \neq f\, d\mu$
2. There is no Lebesgue decomposition of μ with respect to m.

Proof.
1. Let $E \in \mathcal{A}$. If $\mu(E) = 0$, then $E = \varnothing$ and $m(E) = 0$. So $m \ll \mu$. Suppose for the sake of contradiction that there exists $f \in L^+$ such that $dm = f\, d\mu$. Then
$$\begin{aligned} 1 &= m(X) \\ &= \sum_{x\in X} f(x) \end{aligned}$$

Put $Z = \{x \in X : f(x) \neq 0\}$. Then Z is countable. So

$$\begin{aligned} 1 &= m(X \setminus Z) \\ &= \sum_{x \in X \setminus Z} f(x) \\ &= 0 \end{aligned}$$

This is a contradiction, so no such f exists.

2. Suppose for the sake of contradiction that there is a Lebesgue decomposition for μ with respect to m given by $\mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho \ll m$. We may assume λ and ρ are positive. Then for each $x \in X$, $m(\{x\}) = 0$ which implies that $\rho(\{x\}) = 0$. Let $E \subset X$, if E is countable, then $\lambda(E) = \mu(E)$. If E is uncountable, choose $F \subset E$ such that F is countable. Then

$$\begin{aligned} \lambda(E) &\geq \lambda(F) \\ &= \mu(F) \\ &= \infty \end{aligned}$$

So $\lambda = \mu$. This is a contradiction since $\mu \not\perp m$.

□

Exercise 7.2.0.28. Let $(X, \mathcal{A})$ be a measurable space and μ be a σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$. Then $d\nu/\, d\mu \geq 0$ μ-a.e.

Proof. Let $E \in \mathcal{A}$. Then

$$\begin{aligned} \int_E \frac{d\nu}{d\mu}\, d\mu &= \mu(E) \\ &\geq 0 \\ &= \int_E 0\, d\mu \end{aligned}$$

Since $E \in \mathcal{A}$ is arbitrary, Exercise 6.2.0.19 implies that $\frac{d\nu}{d\mu} \geq 0$ μ-a.e. fix this □

Exercise 7.2.0.29. Let $(X, \mathcal{A})$ be a measureable space, ν be a σ-finite signed measure on $(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$. Then $d\nu/\, d\mu > 0$ μ-a.e. iff for each $E \in \mathcal{A}$, $\mu(E) \neq 0$ implies that $\nu(E) > 0$.

Proof. Since ν is a measure, there exists $f \in L^+(X, \mathcal{A})$ such that $f = d\nu/d\mu$ μ-a.e. Suppose that there exists $E \in \mathcal{A}$ such that $\mu(E) > 0$ and $\nu(E) = 0$. Then

$$\begin{aligned} \int_E f\, d\mu &= \nu(E) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} \frac{d\nu}{d\mu}\chi_E &= f\chi_E \\ &= 0 \ \mu\text{-a.e.} \end{aligned}$$

Therefore $d\nu/\, d\mu \not> 0$ μ-a.e.
Conversely, suppose that $d\nu/\, d\mu \not> 0$ μ-a.e. Then there exists $E \in \mathcal{A}$ such that $\mu(E) > 0$ and $(d\nu/d\mu)\chi_E = 0$ μ-a.e. Therefore

$$\begin{aligned} \nu(E) &= \int_E \frac{d\nu}{d\mu}\chi_E\, d\mu \\ &= 0 \end{aligned}$$

fix this □

7.3 Complex Measures

Definition 7.3.0.1. Let $(X, \mathcal{A})$ be a measurable space and $\nu : \mathcal{A} \to \mathbb{C}$. Then ν is said to be a **complex measure** if

1. $\nu(\varnothing) = 0$
2. for each sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, if $(E_n)_{n\in\mathbb{N}}$ is disjoint, then $\nu\left(\bigcup\limits_{n\in\mathbb{N}} E_n\right) = \sum\limits_{n\in\mathbb{N}} \nu(E_n)$ and $\sum\limits_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely.

Definition 7.3.0.2. Let $(X, \mathcal{A})$ be a measurable space.

- Let μ a complex measure on $(X, \mathcal{A})$. Then $(X, \mathcal{A}, \mu)$ is called a **complex measure space**.
- We define
$$M(X, \mathcal{A}) := \{\mu : \mathcal{A} \to \mathbb{C} : \mu \text{ is a complex measure}\}.$$
When X is a topological space, we write $M(X)$ in place of $M(X, \mathcal{B}(X))$.

Exercise 7.3.0.3. Let $(X, \mathcal{A})$ be a measurable space. Then $M(X, \mathcal{A})$ is a vector space.

Proof. Let $\mu, \nu \in M(X, \mathcal{A})$ and $\lambda \in \mathbb{C}$. Then

1. Since $\mu, \nu \in M(X, \mathcal{A})$, $\mu(\varnothing) = 0$ and $\nu(\varnothing) = 0$. Hence
$$\begin{aligned}(\mu + \lambda\nu)(\varnothing) &= \mu(\varnothing) + \lambda\nu(\varnothing) \\ &= 0\end{aligned}$$
2. $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_n)_{n\in\mathbb{N}}$ is disjoint. Since $\mu, \nu \in M(X, \mathcal{A})$, we have that $\mu\left(\bigcup\limits_{n\in\mathbb{N}} E_n\right) = \sum\limits_{n\in\mathbb{N}} \mu(E_n)$, $\nu\left(\bigcup\limits_{n\in\mathbb{N}} E_n\right) = \sum\limits_{n\in\mathbb{N}} \nu(E_n)$ and $\sum\limits_{n\in\mathbb{N}} \mu(E_n)$, $\sum\limits_{n\in\mathbb{N}} \nu(E_n)$ converge absolutely. Therefore
$$\begin{aligned}[\mu + \lambda\nu]\left(\bigcup_{n\in\mathbb{N}} E_n\right) &= \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) + \lambda\nu\left(\bigcup_{n\in\mathbb{N}} E_n\right) \\ &= \sum_{n\in\mathbb{N}} \mu(E_n) + \lambda \sum_{n\in\mathbb{N}} \nu(E_n) \\ &= \sum_{n\in\mathbb{N}} \mu(E_n) + \sum_{n\in\mathbb{N}} \lambda\nu(E_n) \\ &= \sum_{n\in\mathbb{N}} [\mu(E_n) + \lambda\nu(E_n)] \\ &= \sum_{n\in\mathbb{N}} [\mu + \lambda\nu](E_n)\end{aligned}$$
and
$$\begin{aligned}\sum_{n\in\mathbb{N}} |(\mu + \lambda\nu)(E_n)| &= \sum_{n\in\mathbb{N}} |\mu(E_n) + \lambda\nu(E_n)| \\ &\leq \sum_{n\in\mathbb{N}} |\mu(E_n)| + |\lambda||\nu(E_n)| \\ &\leq \sum_{n\in\mathbb{N}} |\mu(E_n)| + |\lambda| \sum_{n\in\mathbb{N}} |\nu(E_n)| \\ &< \infty\end{aligned}$$
so that $\sum\limits_{n\in\mathbb{N}} |(\mu + \lambda\nu)(E_n)|$ converges absolutely.

Therefore $\mu + \lambda\nu \in M(X, \mathcal{A})$. Since $\mu, \nu \in M(X, \mathcal{A})$ and $\lambda \in \mathbb{C}$ are arbitrary, we have that $M(X, \mathcal{A})$ is a vector space. □

Exercise 7.3.0.4. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in M(X, \mathcal{A})$ with $\nu = \nu_1 + i\nu_2$. Then ν_1, ν_2 are signed measures.

Proof.

1. Let $E \in \mathcal{A}$. Since $\nu(E) \in \mathbb{C}$, $\nu_1(E), \nu_2(E) \in \mathbb{R}$. Since $E \in \mathcal{A}$ is arbitrary, we have that for each $E \in \mathcal{A}$, $\nu_1(E), \nu_2(E) < \infty$ or $\nu_1(E), \nu_2(E) > -\infty$.

2. Since $\nu(\varnothing) = 0$, $\nu_1(E) = 0$ and $\nu_2(E) = 0$.

3. Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. Suppose that $(E_n)_{n\in\mathbb{N}}$ is disjoint. Then

$$\begin{aligned}
\nu_1\bigg(\bigcup_{n\in\mathbb{N}} E_n\bigg) + i\nu_2\bigg(\bigcup_{n\in\mathbb{N}} E_n\bigg) &= \nu\bigg(\bigcup_{n\in\mathbb{N}} E_n\bigg) \\
&= \sum_{n\in\mathbb{N}} \nu(E_n) \\
&= \sum_{n\in\mathbb{N}} [\nu_1(E_n) + i\nu_2(E_n)] \\
&= \sum_{n\in\mathbb{N}} \nu_1(E_n) + i\sum_{n\in\mathbb{N}} \nu_2(E_n)
\end{aligned}$$

Therefore $\nu_1\bigg(\bigcup_{n\in\mathbb{N}} E_n\bigg) = \sum_{n\in\mathbb{N}} \nu_1(E_n)$ and $\nu_2\bigg(\bigcup_{n\in\mathbb{N}} E_n\bigg) = \sum_{n\in\mathbb{N}} \nu_2(E_n)$. Since $\nu \in M(X, \mathcal{A})$, we have that $\sum_{n\in\mathbb{N}} \nu(E_n)$ converges absolutely. Since $\|\cdot\|_1 : \mathbb{R}^2 \to [0, \infty)$ and $\|\cdot\|_2 : \mathbb{R}^2 \to [0, \infty)$ are equivalent, there exists $C > 0$ such that for each $x \in \mathbb{R}^2$, $\|x\|_1 \leq C\|x\|_2$. Therefore

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |\nu_1(E_n)|, \sum_{n\in\mathbb{N}} |\nu_2(E_n)| &\leq \sum_{n\in\mathbb{N}} |\nu_1(E_n)| + |\nu_2(E_n)| \\
&\leq \sum_{n\in\mathbb{N}} C|\nu_1(E_n) + i\nu_2(E_n)| \\
&= C\sum_{n\in\mathbb{N}} |\nu(E_n)| \\
&< \infty
\end{aligned}$$

Therefore ν_1 and ν_2 are signed measures. □

Exercise 7.3.0.5. Let $(X, \mathcal{A})$ be a measurable space and $\mu, \nu \in M(X, \mathcal{A})$. Set $\mathcal{L}_{\mu,\nu} = \{A \in \mathcal{A} : \mu(A) = \nu(A)\}$. If $X \in \mathcal{L}_{\mu,\nu}$, then $\mathcal{L}_{\mu,\nu}$ is a λ-system on X.

Proof. Suppose that $X \in \mathcal{L}_{\mu,\nu}$.

1. Since $X \in \mathcal{L}_{\mu,\nu}$, $\mathcal{L}_{\mu,\nu} \neq \varnothing$.

2. Let $A \in \mathcal{L}_{\mu,\nu}$. Then $\mu(A) = \nu(A)$. Thus

$$\begin{aligned}
\mu(A^c) &= \mu(X) - \mu(A) \\
&= \nu(X) - \nu(A) \\
&= \nu(A^c)
\end{aligned}$$

So $A^c \in \mathcal{L}_{\mu,\nu}$.

3. Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}_{\mu,\nu}$. So for each $n \in \mathbb{N}$, $\mu(A_n) = \nu(A_n)$. Suppose that $(A_n)_{n\in\mathbb{N}}$ is disjoint. Then

$$\begin{aligned}\mu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg) &= \sum_{n\in\mathbb{N}} \mu(A_n) \\ &= \sum_{n\in\mathbb{N}} \nu(A_n) \\ &= \nu\bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)\end{aligned}$$

Hence $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{L}_{\mu,\nu}$.

□

Exercise 7.3.0.6. Let $(X, \mathcal{A})$ be a measurable space, $\mu, \nu \in M(X, \mathcal{A})$ and $\mathcal{P} \subset \mathcal{A}$ a π-system on X and $X \in \mathcal{P}$. If for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Then for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$.

Proof. Suppose that for each $A \in \mathcal{P}$, $\mu(A) = \nu(A)$. Exercise 7.3.0.5 implies that $\mathcal{L}_{\mu,\nu}$ is a λ-system on X. By assumtion, $\mathcal{P} \subset \mathcal{L}_{\mu,\nu}$. Dynkin's theorem Exercise 3.9.0.6 implies that $\sigma(\mathcal{P}) \subset \mathcal{L}_{\mu,\nu}$. So for each $A \in \sigma(\mathcal{P})$, $\mu(A) = \nu(A)$. □

Exercise 7.3.0.7. Let $(X, \mathcal{T})$ be a topological space and $\mu, \nu \in M(X)$. If for each $A \in \mathcal{T}$, $\mu(A) = \nu(A)$, then $\mu = \nu$.

Proof. Since $\mathcal{T} \subset \mathcal{B}(X)$ is a π-system on X (cite previous exercise) and $X \in \mathcal{T}$, Exercise 7.3.0.6 implies that for each $A \in \sigma(\mathcal{T})$, $\mu(A) = \nu(A)$. Since $\sigma(\mathcal{T}) = \mathcal{B}(X)$, $\mu = \nu$. □

Exercise 7.3.0.8. Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subset \mathcal{T}$ a basis for $\mathcal{T}$ and $\mu, \nu \in M(X)$. Suppose that $\mathcal{B}$ is countable, $\mathcal{B}$ is a π-system on X and $X \in \mathcal{B}$. If for each $A \in \mathcal{B}$, $\mu(A) = \nu(A)$, then $\mu = \nu$.

Proof. Since $\mathcal{B}$ is countable Exercise 3.2.0.12 implies that $\mathcal{B}(X) = \sigma(\mathcal{B})$. Since $\mathcal{B}$ is a π-system, $X \in \mathcal{B}$, and for each $A \in \mathcal{B}$, $\mu(A) = \nu(A)$, Exercise 7.3.0.6 implies that $\mu = \nu$. □

Definition 7.3.0.9. Let $(X, \mathcal{A})$ be a measurable space, $\mu \in M(X, \mathcal{A})$ and $A \in \mathcal{A}$. Then μ is said to be **supported on** A if for each $E \in \mathcal{A}$, $\mu(E) = \mu(A \cap E)$.
need to define for signed measures and measures

Exercise 7.3.0.10. Let $(X, \mathcal{A})$ be a measurable space, $\mu \in M(X, \mathcal{A})$ and $f \in L^1(X, \mathcal{A}, \mu)$. Define $\nu : \mathcal{A} \to \mathbb{C}$ by

$$\nu(E) = \int_E f \, d\mu$$

Then $\nu \in M(X, \mathcal{A})$.

Proof. FINISH!!! □

Note 7.3.0.11. We use the same definitions for mutual orthogonality and absolute continuity when discussing complex measures instead of signed measures.

Definition 7.3.0.12. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in M(X, \mathcal{A})$ with $\nu = \nu_1 + i\nu_2$. We define $L^1(\nu) = L^1(\nu_1) \cap L^1(\nu_2)$. For $f \in L^1(\nu)$, we define

$$\int f d\nu = \int f d\nu_1 + i \int f d\nu_2$$

Theorem 7.3.0.13. Lebesgue-Radon-Nikodym Theorem:
Let $(X, \mathcal{A})$ be a measurable space, $\nu \in M(X, \mathcal{A})$ and μ a σ-finite measure on $(X, \mathcal{A})$. Then there exists unique $\lambda, \rho \in M(X, \mathcal{A})$ such that $\lambda \perp \mu$, $\rho \ll \mu$ and $\nu = \lambda + \rho$, and there exists $f \in L^1(\mu)$ such that $d\rho = f \, d\mu$ and f is unique μ-a.e.

Exercise 7.3.0.14. Let $\nu \in M(X, \mathcal{A})$ and μ, λ σ-finite measures on $(X, \mathcal{A})$. Suppose that $\nu \ll \mu$ and $\mu \ll \lambda$. Then

1. for each $g \in L^1(\nu)$, $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} \, d\mu$$

2. $\nu \ll \lambda$ and
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

Definition 7.3.0.15. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in M(X, \mathcal{A})$ with $\nu = \nu_1 + i\nu_2$. Define $\mu = |\nu_1| + |\nu_2|$. Then $\nu \ll \mu$ and thus there exists $f \in L^1(\mu)$ such that $d\nu = f\, d\mu$. We define the **total variation of** ν, denoted $|\nu| : \mathcal{A} \to [0, \infty)$, by
$$|\nu|(E) = \int_E |f|\, d\mu$$

Exercise 7.3.0.16. Let $(X, \mathcal{A})$ be a measurable space, $\nu \in M(X, \mathcal{A})$ and λ a σ-finite measure on $(X, \mathcal{A})$. Suppose that $\nu \ll \lambda$. Set $g = d\nu/d\lambda$. Then for each $E \in \mathcal{A}$,
$$|\nu|(E) = \int_E |g|\, d\lambda$$

Proof. Write $\nu = \nu_1 + i\nu_2$. Then $\nu_1, \nu_2 \ll \lambda$. Set $f_1 = d\nu_1/d\lambda$ and $f_2 = d\nu_2/d\lambda$. Then Exercise 7.2.0.19 implies that $d|\nu_1| = |f_1|\, d\lambda$ and $d|\nu_2| = |f_2| d\lambda$. Set $\mu = |\nu_1| + |\nu_2|$ and $f = d\nu/\, d\mu$ as in Definition 7.3.0.15. Then by construction,
$$\begin{aligned} d\mu &= d|\nu_1| + d|\nu_2| \\ &= |f_1|\, d\lambda + |f_2|\, d\lambda \\ &= (|f_1| + |f_2|)\, d\lambda \end{aligned}$$

So that $\mu \ll \lambda$ with $d\mu/d\lambda = |f_1| + |f_2|$. Then Exercise 7.3.0.14 implies that $\nu \ll \lambda$ with
$$\begin{aligned} \frac{d\nu}{d\lambda} &= \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \\ &= f(|f_1| + |f_2|) \\ &= g \end{aligned}$$

and for each $E \in \mathcal{A}$,
$$\begin{aligned} |\nu|(E) &= \int_E |f| d\,\mu \\ &= \int_E |f|(|f_1| + |f_2|)\, d\lambda \\ &= \int_E |g|\, d\lambda \end{aligned}$$

□

Exercise 7.3.0.17. Let $\nu \in M(X, \mathcal{A})$ and μ a σ-finite measures on $(X, \mathcal{A})$. If $\nu \ll \mu$, then $\{x \in X : d\nu/\, d\mu(x) = 0\}$ is ν-null.

Proof. Define $f = d\nu/\, d\mu$ and $E = \{x : f(x) = 0\}$. Let $A \in \mathcal{A}$ and suppose that $A \subset E$. Then
$$\begin{aligned} \nu(A) &= \int_A f\, d\mu \\ &= 0 \end{aligned}$$

□

Exercise 7.3.0.18. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in M(X, \mathcal{A})$ with $\nu = \nu_1 + i\nu_2$. Then $|\nu_1|, |\nu_2| \leq |\nu| \leq |\nu_1| + |\nu_2|$.

Proof. Let μ and $f = f_1 + if_2$ be as in the definition of $|\nu|$. Since for each $E \in \mathcal{A}$, we have
$$\begin{aligned} \nu(E) &= \int_E f\, d\mu \\ &= \int_E f_1\, d\mu + i \int_E f_2\, d\mu \end{aligned}$$

and

$$\nu(E) = \nu_1(E) + i\nu_2(E),$$

we know that $\nu_1 = f_1 \, d\mu$ and $\nu_2 = f_2 \, d\mu$. A previous exercise tells us that $d|\nu_1| = |f_1| \, d\mu$ and $d|\nu_2| = |f_2| \, d\mu$. Since $|f_1|, |f_2| \leq |f| \leq |f_1| + |f_2|$, we have that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \end{aligned}$$

□

Exercise 7.3.0.19. Let $(X, \mathcal{A})$ be a measurable space, $\nu \in M(X, \mathcal{A})$ and $c \in \mathbb{C}$. Then $|c\nu| = |c||\nu|$.

Proof. Define μ and f as before so that $d\nu = f \, d\mu$. Then $d(c\nu) = cf \, d\mu$. Hence

$$\begin{aligned} d|c\nu| &= |cf| \, d\mu \\ &= |c||f| \, d\mu \\ &= |c| d|\nu| \end{aligned}$$

So $|c\nu| = |c||\nu|$. □

Exercise 7.3.0.20. Define $\| \cdot \| : M(X, \mathcal{A}) \to [0, \infty)$ by

$$\|\mu\| = |\mu|(X)$$

Then $\| \cdot \|$ is a norm on $M(X, \mathcal{A})$.

Proof. Let $\mu, \nu \in M(X, \mathcal{A})$ and $\alpha \in \mathbb{C}$. The previous exercises tell us that $|\mu + \nu| \leq |\mu| + |\nu|$ and $|\alpha\mu| = |\alpha||\mu|$. So clearly $\|\mu + \nu\| \leq \|\mu\| + \|\nu\|$ and $\|c\mu\| = |c|\|\mu\|$. If $\|\mu\| = 0$, then X is μ-null and μ is the zero measure. □

Exercise 7.3.0.21. Let $(X, \mathcal{A})$ be a measurable space and $\nu \in M(X, \mathcal{A})$. Then

1. for each $E \in \mathcal{A}$, $|\nu(E)| \leq |\nu|(E)$.
2. $\nu \ll |\nu|$ and $\big|d\nu/d|\nu|\big| = 1$ $|\nu|$-a.e.
3. $L^1(\nu) = L^1(|\nu|)$ and for each $g \in L^1(\nu)$,
$$\left| \int g d\nu \right| \leq \int |g| d|\nu|$$

Proof. Let μ, $f \in L^1(\mu)$ be as in the definition of $|\nu|$.

1. Let $E \in \mathcal{A}$. Then
$$\begin{aligned} |\nu(E)| &= \left| \int_E f \, d\mu \right| \\ &\leq \int_E |f| \, d\mu \\ &= |\nu|(E) \end{aligned}$$
2. Let $E \in \mathcal{A}$ and suppose that $|\nu|(E) = 0$. The previous part implies $|\nu(E)| = 0$ and $\nu \ll |\nu|$. Put $g = d\nu/d|\nu|$. Then
$$\begin{aligned} f &= \frac{d\nu}{d\mu} \\ &= g|f| \quad \mu\text{-a.e.} \end{aligned}$$
Hence $|f| = |g||f|$ μ-a.e. Since $|\nu| \ll \mu$, $|f| = |g||f|$ $|\nu|$-a.e. A previous exercise tells us that $|f| \neq 0$ $|\nu|$-a.e. Thus $|g| = 1$ $|\nu|$-a.e.

3. Write $\nu = \nu_1 + i\nu_2$ and $f = f_1 + if_2$. First we observe that

$$\begin{aligned} L^1(\nu) &= L^1(\nu_1) \cap L^1(\nu_2) \\ &= L^1(|\nu_1|) \cap L^1(|\nu_2|) \\ &= L^1(|\nu_1| + |\nu_2|) \\ &= L^1(\mu) \end{aligned}$$

The previous exercise tells us that

$$\begin{aligned} |\nu_1|, |\nu_2| &\leq |\nu| \\ &\leq |\nu_1| + |\nu_2| \\ &= \mu \end{aligned}$$

Let $g \in L^1(\mu)$. Then

$$\begin{aligned} \int |g| d|\nu| &\leq \int |g| \, d\mu \\ &< \infty \end{aligned}$$

So $g \in L^1(|\nu|)$. Conversely, let $g \in L^1(|\nu|)$. Then

$$\begin{aligned} \int |g| d|\nu_1|, \int |g| d|\nu_2| &\leq \int |g| d|\nu| \\ &< \infty \end{aligned}$$

So

$$\begin{aligned} \int |g| \, d\mu &= \int |g| d|\nu_1| + \int |g| d|\nu_2| \\ &< \infty \end{aligned}$$

and $g \in L^1(\mu)$. Hence $L^1(\nu) = L^1(|\nu|)$. Now, let $g \in L^1(\nu) = L^1(|\nu|)$, then

$$\begin{aligned} \left| \int g d\nu \right| &= \left| \int gf \, d\mu \right| \\ &\leq \int |g||f| \, d\mu \\ &= \int |g| d|\nu| \end{aligned}$$

□

Exercise 7.3.0.22. Let $(X, \mathcal{A})$ be a measurable space, $\mu_1, \mu_2 \in M(X, \mathcal{A})$ and $\lambda \in \mathbb{C}$. Then for each $f \in L^1(\mu_1 + \lambda\mu_2)$,

$$\int f d(\mu_1 + \lambda\mu_2) = \int f \, d\mu_1 + \lambda \int f \, d\mu_2$$

Proof. Clear by an exercise in section 3.2. □

7.3.1 Pushforward and Radon-Nikodym Derivative:

Exercise 7.3.1.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $\mathcal{B} \subset \mathcal{A}$ a sub σ-algebra. Then $L^1(X, \mathcal{B}, \mu|_{\mathcal{B}}) \subset L^1(X, \mathcal{A}, \mu)$ and for each $f \in L^1(X, \mathcal{B}, \mu|_{\mathcal{B}})$ and $B \in \mathcal{B}$,

$$\int_B f \, d\mu|_{\mathcal{B}} = \int_B f \, d\mu$$

Proof.

- (need to show if $f \sim_{\mu|_\mathcal{B}} g$, then $f \sim_\mu g$, i.e. L^1 is set of equivalence classes, need to show this is respected for the inclusion)

- Set $\mu_\mathcal{B} = \mu|_\mathcal{B}$. Let $f \in L^1(X, \mathcal{B}, \mu_\mathcal{B})$ and $B \in \mathcal{B}$. Clearly f is $\mathcal{A}$-measurable. If f is simple, then there exist $(b_i)_{i=1}^n \subset [0, \infty)$ and $(B_i)_{i=1}^n \subset \mathcal{B}$ such that

$$f = \sum_{i=1}^n b_i \chi_{B_i}$$

such that for each $i \in \{1, \cdots, n\}$,

$$\begin{aligned} \infty &> \mu_\mathcal{B}(B_i) \\ &= \mu(B_i) \end{aligned}$$

So $f \in L^1(X, \mathcal{A}, \mu)$ and

$$\begin{aligned} \int_B f \, d\mu_\mathcal{B} &= \int_B \sum_{i=1}^n b_i \chi_{B_i} \, d\mu_\mathcal{B} \\ &= \sum_{i=1}^n b_i \mu_\mathcal{B}(B_i \cap B) \\ &= \sum_{i=1}^n b_i \mu(B_i \cap B) \\ &= \int_B \sum_{i=1}^n b_i \chi_{B_i} \, d\mu \\ &= \int_B f \, d\mu \end{aligned}$$

If $f \geq 0$, then there exist $(\phi_n)_{n \in \mathbb{N}} \subset S^+(X, \mathcal{B})$ such that for each $n \in \mathbb{N}$, $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \xrightarrow{\text{p.w.}} f$. The monotone convergence theorem implies that for each $B \in \mathcal{B}$,

$$\begin{aligned} \int_B f \, d\mu &= \lim_{n \to \infty} \int_B \phi_n \, d\mu \\ &= \lim_{n \to \infty} \int_B \phi_n \, d\mu_\mathcal{B} \\ &= \int_B f \, d\mu_\mathcal{B} \\ &< \infty \end{aligned}$$

So $f \in L^1(X, \mathcal{A}, \mu)$. Similarly, the statement also holds for general $f \in L^1(X, \mathcal{B}, \mu_B)$ by writing $f = g + ih$ and applying the above to g^+, g^-, h^+ and h^-.

$\square$

Note 7.3.1.2. Denote the L^1 norms on $L^1(X, \mathcal{A}, \mu)$ and $L^1(X, \mathcal{B}, \mu|_\mathcal{B})$ by N and $N_\mathcal{B}$ respectively. The previous exercise implies that $L^1(X, \mathcal{B}, \mu|_\mathcal{B})$ is a subspace of $L^1(X, \mathcal{A}, \mu)$ and $N|_{L^1(X, \mathcal{B}, \mu|_\mathcal{B})} = N_\mathcal{B}$.

Exercise 7.3.1.3. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces, $\mu, \nu \in M(X, \mathcal{A})$ and $f : X \to Y$. Suppose that f is $(\mathcal{A}, \mathcal{B})$-measurable. If $\nu \ll \mu$, then $f_*\nu \ll f_*\mu$ and

$$\frac{df_*\nu}{df_*\mu} \circ f = \frac{d\nu|_{f^*\mathcal{B}}}{d\mu|_{f^*\mathcal{B}}} \qquad \mu|_{f^*\mathcal{B}}\text{-a.e.}$$

Proof. Suppose that $\nu \ll \mu$. Let $E \in \mathcal{B}$. Suppose that $f_*\mu(E) = 0$. By definition, $\mu(f^{-1}(E)) = 0$. Since $\nu \ll \mu$, we have that

$$\begin{aligned} f_*\nu(E) &= \nu(f^{-1}(E)) \\ &= 0 \end{aligned}$$

Since $E \in \mathcal{B}$ is arbitrary, $f_*\nu \ll f_*\mu$.
Since f is $(\mathcal{A}, \mathcal{B})$-measurable, f is $(f^*\mathcal{B}, \mathcal{B})$-measurable. Since $d(f_*\nu)/d(f_*\mu)$ is $(\mathcal{B}, \mathcal{B}(\mathbb{C}))$-measurable and f is $(f^*\mathcal{B}, \mathcal{B})$-measurable, we have that $d(f_*\nu)/d(f_*\mu) \circ f$ is $(f^*\mathcal{B}, \mathcal{B}(\mathbb{C}))$-measurable. Set $\mu' = \mu|_{f^*\mathcal{B}}$ and $\nu' = \nu|_{f^*\mathcal{B}}$. Let $A \in f^*\mathcal{B}$. Then there exists $B \in \mathcal{B}$ such that $A = f^{-1}(B)$. Exercise 7.3.1.1 implies that

$$\begin{aligned} \int_A \frac{df_*\nu}{df_*\mu} \circ f \, d\mu' &= \int_{f^{-1}(B)} \frac{df_*\nu}{df_*\mu} \circ f \, d\mu \\ &= \int_B \frac{df_*\nu}{df_*\mu} \, df_*\mu \\ &= f_*\nu(B) \\ &= \nu(f^{-1}(B)) \\ &= \nu(A) \\ &= \nu'(A) \end{aligned}$$

Since $A \in f^*\mathcal{B}$ is arbitrary,

$$\frac{df_*\nu}{df_*\mu} \circ f = \frac{d\nu'}{d\mu'} \qquad \mu'\text{-a.e.}$$

□

Exercise 7.3.1.4. Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces, $\mu, \nu \in M(X, \mathcal{A})$ and $f : X \to Y$. Suppose that f is an isomorphism. If $\nu \ll \mu$, then $f_*\nu \ll f_*\mu$ and

$$\frac{df_*\nu}{df_*\mu} \circ f = \frac{d\nu}{d\mu} \qquad \mu\text{-a.e.}$$

Proof. Suppose that $\nu \ll \mu$. Exercise 7.3.1.3 implies that $f_*\nu \ll f_*\mu$ and

$$\frac{df_*\nu}{df_*\mu} \circ f = \frac{d\nu|_{f^*\mathcal{B}}}{d\mu|_{f^*\mathcal{B}}} \qquad \mu|_{f^*\mathcal{B}}\text{-a.e.}$$

Exercise **??** implies that $f^*\mathcal{B} = \mathcal{A}$ and therefore

$$\frac{df_*\nu}{df_*\mu} \circ f = \frac{d\nu}{d\mu} \qquad \mu\text{-a.e.}$$

Let $B \in \mathcal{B}$. Since $\mathcal{A} = f^*\mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$. Set $A = f^{-1}(B)$. Then

$$\begin{aligned} \int_B \frac{d\nu}{d\mu} \circ f^{-1} \, df_*\mu &= \int_{f^{-1}(B)} \frac{d\nu}{d\mu} \circ f^{-1} \circ f \, d\mu \\ &= \int_A \frac{d\nu}{d\mu} \, d\mu \\ &= \nu(A) \\ &= \nu(f^{-1}(B)) \\ &= f_*\nu(B) \end{aligned}$$

Since $B \in \mathcal{B}$ is arbitrary,

$$\frac{df_*\nu}{df_*\mu} = \frac{d\nu}{d\mu} \circ f^{-1} \qquad f_*\mu\text{-a.e.}$$

□

7.4 Differentiation on $\mathbb{R}^n$

Definition 7.4.0.1. Let $B \subset \mathbb{R}^n$. Then B is said to be a **ball** if there exists $x \in \mathbb{R}^n$ and $r > 0$ such that $B = B(x, r)$.

Definition 7.4.0.2. Let $f \in L^0(\mathbb{R}^n)$. Then f is said to be **locally integrable** (with respect to Lebesgue measure) if f is measurable and for each $K \subset \mathbb{R}$, K is compact implies $\int_K |f|\, dm < \infty$. We define $L^1_{\text{loc}}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ is locally integrable}\}$

Definition 7.4.0.3. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $r > 0$, $x \in \mathbb{R}^n$, we define the **average of f over** $B(x, r)$, denoted by $Af(x, r)$, to be

$$Af(x,r) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f\, dm$$

Definition 7.4.0.4. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We define its **Hardy-Littlewood maximal function**, denoted by Hf to be

$$Hf(x) = \sup_{r>0} A|f|(x,r) \quad x \in \mathbb{R}^n$$

Exercise 7.4.0.5. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Define

$$H^*f(x) = \sup\{\frac{1}{m(B)} \int_B |f|\, dm : B \text{ is a ball and } x \in B\} \quad (x \in \mathbb{R}^n)$$

Then $Hf \le H^*f \le 2^n Hf$.

Proof. Let $x \in \mathbb{R}^n$. Then

$$\left\{\frac{1}{m(B(x,r))} \int_{B(x,r)} |f|\, dm : r > 0\right\} \subset \left\{\frac{1}{m(B)} \int_B |f|\, dm : B \text{ is a ball and } x \in B\right\}$$

So $Hf(x) \le H^*f(x)$. Let B be a ball. Then there exists $y \in \mathbb{R}^n$, $R > 0$ such that $B = B(y, R)$ Suppose that $x \in B$. Then $B \subset B(x, 2R)$. Since $m(B(x, 2R)) = 2^n m(B(y, R))$, we have that

$$\begin{aligned}\frac{1}{m(B)} \int_B |f|\, dm &\le \frac{1}{m(B)} \int_{m(B(x,2R))} |f|\, dm \\ &= \frac{2^n}{m(B(x,2R))} \int_{m(B(x,2R))} |f|\, dm\end{aligned}$$

Thus $H^*f(x) \le 2^n Hf(x)$. □

Lemma 7.4.0.6. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $Af : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ is continuous.

Theorem 7.4.0.7. There exists $C > 0$ such that for each $f \in L^1(m)$ and $\alpha > 0$,

$$m(\{x \in \mathbb{R}^n : Hf(x) > \alpha\}) \le \frac{C}{a} \int |f|\, dm$$

Exercise 7.4.0.8. Let $f \in Ł^1(\mathbb{R}^n)$. Suppose that $||f||_1 > 0$. Then there exist $C, R > 0$ such that for each $x \in \mathbb{R}^n$, if $|x| > R$, then $Hf(x) \ge C|x|^{-n}$. Hence there exists $C' > 0$ such that for each $\alpha > 0$, $m(\{x \in X : Hf(x) > \alpha\}) > C'/\alpha$ when α is small.

Proof. Since $||f||_1 > 0$, there exists $R > 0$ such that $\int_{B(0,R)} |f|\, dm > 0$. Recall that there exists $K > 0$ such that for each $x \in R^n$ and $r > 0$, $m(B(x, r)) = Kr^n$. Choose

$$C = \frac{1}{K2^n} \int_{B(0,R)} |f|\, dm$$

. Let $x \in \mathbb{R}^n$. Suppose that $|x| > R$. Then $B(0,R) \subset B(x, 2|x|)$. Thus

$$\begin{aligned}
Hf(x) &\geq \frac{1}{m(B(x,2|x|))} \int_{B(x,2|x|)} |f| \, dm \\
&= \frac{1}{K2^n|x|^n} \int_{B(x,2|x|)} |f| \, dm \\
&\geq \frac{1}{K2^n|x|^n} \int_{B(0,R)} |f| \, dm \\
&= \frac{C}{|x^n|}
\end{aligned}$$

Let $a < \frac{C}{2R^n}$. Then $R^n < \frac{C}{2\alpha}$. Choose $C' = \frac{KC}{2}$. Let $A = \{x \in \mathbb{R}^n : R < |x| < (\frac{C}{\alpha})^{\frac{1}{n}}\}$. For $x \in A$,

$$\begin{aligned}
Hf(x) &\geq \frac{C}{|x|^n} \\
&> \alpha
\end{aligned}$$

Thus $A \subset m(\{x \in R^n : Hf(x) > \alpha\})$ and therefore

$$\begin{aligned}
m(\{x \in R^n : Hf(x) > \alpha\}) &\geq m(A) \\
&= m(B(0, (C/\alpha)^{1/n})) - m(B(0,R)) \\
&= K\left[\frac{C}{\alpha} - R^n\right] \\
&> K\left[\frac{C}{\alpha} - \frac{C}{2\alpha}\right] \\
&= \frac{KC}{2\alpha} \\
&= \frac{C'}{\alpha}
\end{aligned}$$

□

Theorem 7.4.0.9. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} Af(x,r) = f(x)$$

Equivalently, for a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} [f(y) - f(x)] \, dm(y)\right] = 0$$

Note 7.4.0.10. We can a stronger result of the same flavor.

Definition 7.4.0.11. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We define the **Lebesgue set of** f, denoted by L_f, to be

$$\begin{aligned}
L_f &= \{x \in \mathbb{R}^n : \lim_{r \to 0} A|f - f(x)|(x,r) = 0\} \\
&= \left\{x \in \mathbb{R}^n : \lim_{r \to 0} \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dm(y)\right] = 0\right\}
\end{aligned}$$

Exercise 7.4.0.12. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. If f is continuous at x, then $x \in L_f$.

Proof. Suppose that f is continuous at x. Let $\epsilon > 0$. By assumption, there exists $\delta > 0$ such that for each $y \in \mathbb{R}^n$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Let $r > 0$. Suppose that $r < \delta$. Then for each $y \in \mathbb{R}^n$, $y \in B(x,r)$ implies that $|f(x) - f(y)| < \epsilon$ and thus

$$\begin{aligned}
\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, dm(y) &\leq \frac{1}{m(B(x,r))} \epsilon m(B(x,r)) \\
&= \epsilon
\end{aligned}$$

Hence

$$\lim_{r\to 0}\left[\frac{1}{m(B(x,r))}\int_{B(x,r)}|f(y)-f(x)|\,dm(y)\right]=0$$

and $x \in L_f$. □

Theorem 7.4.0.13. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then $m((L_f)^c)=0$

Definition 7.4.0.14. Let $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then $(E_r)_{r>0}$ is said to **shrink nicely to** x if

1. for each $r > 0$, $E_r \subset B(x,r)$
2. there exists $\alpha > 0$ such that for each $r > 0$, $m(E_r) > \alpha m(B(x,r))$

Theorem 7.4.0.15. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $(E_r)_{r>0} \subset \mathcal{B}(\mathbb{R}^n)$. Then for each $x \in L_f$,

$$\lim_{r\to 0}\left[\frac{1}{m(E_r)}\int_{E_r}|f(y)-f(x)|\,dm(y)\right]=0$$

and

$$\lim_{r\to 0}\frac{1}{m(E_r)}\int_{E_r} f\,dm = f(x)$$

Definition 7.4.0.16. Let $\mu : \mathcal{B}(\mathbb{R}^n) \to [0,\infty]$ be a Borel measure. Then μ is said to be **regular** if

1. for each $K \subset \mathbb{R}^n$, if K is compact, then $\mu(K) < \infty$
2. for each $E \in \mathcal{B}(\mathbb{R}^n)$, $\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subset U\}$

Let ν be a signed or complex Borel measure on $\mathbb{R}^n$. Then ν is said to be regular if $|\nu|$ is regular.

Theorem 7.4.0.17. Let ν be a regular signed or complex measure on $\mathbb{R}^n$. Let $d\nu = d\lambda + f\,dm$ be the Lebesgue decomposition of ν with respect to m. Then for m-a.e. $x \in \mathbb{R}^n$ and $(E_r)_{r>0} \subset \mathcal{B}(R^n)$, if $(E_r)_{r>0}$ shrinks nicely to x, then

$$\lim_{r\to 0}\frac{\nu(E_r)}{m(E_r)} = f(x)$$

7.5 Functions of Bounded Variation

Definition 7.5.0.1. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Define $F_+ : \mathbb{R} \to \mathbb{R}$ and $F_- : \mathbb{R} \to \mathbb{R}$ by

$$F_+(x) = \lim_{t \to x^+} F(t) = \inf\{F(t) : t > x\}$$

and

$$F_-(x) = \lim_{t \to x^-} F(t) = \sup\{F(t) : t < x\}$$

respectively.

Exercise 7.5.0.2. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then

1. (a) $F \leq F_+$
   (b) F_+ is increasing
2. (a) $F_- \leq F$
   (b) F_- is increasing

Proof.

1. (a) Let $x \in \mathbb{R}$. Since F is increasing, for each $t > x$, $F(x) \leq F(t)$. Hence

$$\begin{aligned} F(x) &\leq \inf\{F(t) : t > x\} \\ &= F_+(x) \end{aligned}$$

   Since $x \in \mathbb{R}$ is arbitrary, $F \leq F_+$.

   (b) Let $x, y \in \mathbb{R}$. Suppose that $x \leq y$. Then $\{F(t) : t > y\} \subset \{F(t) : t > x\}$. Thus

$$\begin{aligned} F_+(x) &= \inf\{F(t) : t > x\} \\ &\leq \inf\{F(t) : t > y\} \\ &= F_+(y) \end{aligned}$$

2. Similar to (1).

□

Exercise 7.5.0.3. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and $x \in \mathbb{R}$. Then F is discontinuous at x iff $F_-(x) < F_+(x)$.

Proof. Since F is continuous at x iff $\lim_{t \to x^+} F(t) = F(x)$ and $\lim_{t \to x^-} F(t) = F(x)$, by definition, F is continuous at x iff $F_+(x) = F(x)$ and $F_-(x) = F(x)$. Then the previous exercise implies that F is discontinuous at x iff $F_+(x) > F(x)$ or $F_-(x) < F(x)$. Since $F_+(x) > F(x)$ implies that $F_-(x) < F_+(x)$ and $F_-(x) < F(x)$ implies that $F_-(x) < F_+(x)$, we have that F is discontinuous at x iff $F_-(x) < F_+(x)$. □

Exercise 7.5.0.4. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then for each $x \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in (x, x+\delta)$, $0 \leq F_+(y) - F(y) \leq \epsilon$.

Proof. For the sake of contradiction, suppose not. Then there exists $x \in \mathbb{R}$ and $\epsilon > 0$ such that for each $\delta > 0$, there exist $y \in (x, x+\delta)$ such that $F_+(y) - F(y) > \epsilon$. Then there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that for each $n \in \mathbb{N}$, $y_n \in (x, x + \frac{1}{n})$, $y_n > y_{n+1}$ and $F_+(y_n) - F(y_n) > \epsilon$. Choose $N \in \mathbb{N}$ such that $(N-1)\epsilon > F(y_1) - F(x)$. Note that for each $n \in \mathbb{N}$, $(y_n + y_{n+1})/2 < y_n$ which implies that

$$\begin{aligned} F_+(y_{n+1}) &\leq F((y_n + y_{n+1})/2) \\ &\leq F(y_n) \end{aligned}$$

Therefore

$$\begin{aligned}
F(y_1) - F(x) &= \sum_{j=1}^{N-1}\Big[F(y_j) - F_+(y_{j+1}) + F_+(y_{j+1}) - F(y_{j+1})\Big] + F(y_N) - F(x) \\
&= \sum_{j=1}^{N-1}\Big[F(y_j) - F_+(y_{j+1})\Big] + \sum_{j=1}^{N-1}\Big[F_+(y_{j+1}) - F(y_{j+1})\Big] + F(y_N) - F(x) \\
&\geq \sum_{j=1}^{N-1}\Big[F_+(y_{j+1}) - F(y_{j+1})\Big] \\
&\geq (N-1)\epsilon \\
&> F(y_1) - F(x)
\end{aligned}$$

This is a contradiction, so the claim holds. □

Exercise 7.5.0.5. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then F_+ is right continuous.

Proof. Let $x \in \mathbb{R}$. Let $\epsilon > 0$. By definition, there exists $\delta_1 > 0$ such that for each $y \in (x, x+\delta_1)$ $0 \leq F(y) - F_+(x) < \epsilon/2$. The previous exercise implies that there exists $\delta_2 > 0$ such that for each $y \in (x, x+\delta_2)$, $0 \leq F_+(y) - F(y) < \epsilon/2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Let $y \in (x, x+\delta)$.

$$\begin{aligned}
|F_+(x) - F_+(y)| &\leq |F_+(x) - F(y)| + |F(y) - F_+(y)| \\
&= (F(y) - F_+(x)) + (F_+(y) - F(y)) \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So $\lim\limits_{t \to x^+} F_+(t) = F_+(x)$ and F_+ is right continuous. □

Exercise 7.5.0.6. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing. Then

1. $\{x \in \mathbb{R} : F \text{ is not continuous at } x\}$ is countable
2. F and F_+ are differentiable a.e. and $F' = F'_+$ a.e.

Proof.

1.
2.

□

Definition 7.5.0.7. Let $F : \mathbb{R} \to \mathbb{C}$. Define $T_F : \mathbb{R} \to \mathbb{R}$ by

$$T_F(x) = \sup\left\{\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x\right\} \quad (x \in \mathbb{R})$$

T_F is called the **total variation function of** F.

Exercise 7.5.0.8. Let $F : \mathbb{R} \to \mathbb{C}$. Then T_F is increasing.

Proof. Let $x, y \in \mathbb{R}$. Suppose that $x < y_2$.
Define $A_x = \left\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x\right\}$ and
$A_y = \left\{\sum_{i=1}^n |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = y\right\}$. Let $z \in A_x$. Then there exists $(x_i)_{i=0}^n \subset \mathbb{R}$ such that $(x_i)_{i=0}^n$ is increasing,
$x_n = x$ and $z = \sum_{i=1}^n |F(x_i) - F(x_{i-1})|$. Then

$$\begin{aligned}
z &\le z + |F(y) - F(x)| \\
&= \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + |F(y) - F(x)| \\
&\in A_y
\end{aligned}$$

So $z \le \sup A_y = T_F(y)$ and thus $F_T(x) = \sup A_x \le T_F(y)$ □

Lemma 7.5.0.9. Let $F : \mathbb{R} \to \mathbb{R}$. Then $T_F + F$ and $T_F - F$ are increasing.

Exercise 7.5.0.10. For each $F : \mathbb{R} \to \mathbb{C}$, $T_{|F|} \le T_F$.

Proof. Let $F : \mathbb{R} \to \mathbb{C}$, $x \in R$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then by the reverse triangle inequality,

$$\sum_{i=1}^{n} \big||F(x_i)| - |F(x_{i-1})|\big| \le \sum_{i=1}^{n} \big|F(x_i) - |F(x_{i-1})\big|$$

Thus

$$\begin{aligned}
T_{|F|}(x) &= \sup\left\{ \sum_{i=1}^{n} \big||F(x_i)| - |F(x_{i-1})|\big| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&\le \sup\left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset \mathbb{R} \text{ is increasing and } x_n = x \right\} \\
&= T_F(x)
\end{aligned}$$

Hence $T_{|F|} \le T_F$ □

Definition 7.5.0.11. Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to have **bounded variation** if $\lim\limits_{x\to\infty} T_F(x) < \infty$. The **total variation of** F, denoted by $\mathrm{TV}(F)$, is defined to be $\mathrm{TV}(F) = \lim\limits_{x\to\infty} T_F(x)$. We define $\mathrm{BV} = \{F : \mathbb{R} \to \mathbb{C} : \mathrm{TV}(F) < \infty\}$.

Definition 7.5.0.12. Let $F : [a,b] \to \mathbb{C}$. Define $G_F : \mathbb{R} \to \mathbb{C}$ by $G_F = F(a)\chi_{(-\infty,a)} + F\chi_{[a,b]} + F(b)\chi_{(b,\infty)}$. Then F is said to have **bounded variation on** $[a,b]$ if $G_F \in \mathrm{BV}$. The **total variation of** F, denoted $\mathrm{TV}(F)$, is defined to be $\mathrm{TV}(F) = \mathrm{TV}(G_F)$. We define $\mathrm{BV}(a,b) = \{F : [a,b] \to \mathbb{C} : \mathrm{TV}(F) < \infty\}$.

Note 7.5.0.13. Equivalently, $\mathrm{TV}(F) = \sup\left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : (x_i)_{i=0}^n \subset [a,b] \text{ is increasing, } x_0 = a, \text{ and } x_n = b \right\}$ and $F \in \mathrm{BV}(a,b)$ iff $\mathrm{TV}(F) < \infty$. In general,

Exercise 7.5.0.14. Let $F \in \mathrm{BV}$. Then F is bounded.

Proof. If F is unbounded, then the supremum in the previous definition is clearly infinite. □

Exercise 7.5.0.15. Let $F : \mathbb{R} \to \mathbb{R}$. If F is bounded and increasing, then $F \in \mathrm{BV}$.

Proof. Suppose that F is bounded and increasing. Then $-\infty < \inf_{x\in\mathbb{R}} F(x) \le \sup_{x\in\mathbb{R}} F(x) < \infty$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then

$$\begin{aligned}
\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) \\
&= F(x) - F(x_0)
\end{aligned}$$

Thus

$$T_F(x) = F(x) - \inf_{x\in\mathbb{R}} F(x)$$

This implies that

$$\begin{aligned}\text{TV}(F) &= \sup_{x\in\mathbb{R}} F(x) - \inf_{x\in\mathbb{R}} F(x) \\ &< \infty\end{aligned}$$

Hence $F \in \text{BV}$. $\square$

Exercise 7.5.0.16. Let $F : \mathbb{R} \to \mathbb{C}$. If F is differentiable and F' is bounded on $[a, b]$, then, $F \in \text{BV}(a, b)$.

Proof. Suppose that F is differentiable and F' is bounded on $[a, b]$. Then there exists $M > 0$ such that for each $x \in [a, b]$, $|F(x)| \leq M$. Let $(x_i)_{i=1}^n \subset [a, b]$. Suppose that $(x_i)_{i=1}^n$ is strictly increasing, $x_0 = a$ and $x_n = b$. By the mean value theorem, for each $i = 1, 2, \cdots, n$, there exists $c_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$. Then

$$\begin{aligned}\sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n |F'(c_i)(x_i - x_{i-1})| \\ &\leq \sum_{i=1}^n M(x_i - x_{i-1}) \\ &= M(b-a)\end{aligned}$$

Hence $\text{TV}(F) \leq M(b - a)$. $\square$

Exercise 7.5.0.17. Define $F, G : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \begin{cases} x^2 \sin(x^{-1}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and

$$G(x) = \begin{cases} x^2 \sin(x^{-2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then F and G are differentiable, $F \in \text{BV}(-1, 1)$ and $G \notin \text{BV}(-1, 1)$.

Proof. On $\mathbb{R} \setminus \{0\}$,

$$\begin{aligned}F'(x) &= 2x\sin(x^{-1}) - \sin(x^{-1}) \\ &= \sin(x^{-1})(2x - 1)\end{aligned}$$

We see that F is also differentiable at $x = 0$ since

$$\begin{aligned}F'(0) &= \lim_{x\to 0} \frac{F(x) - F(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x^2 \sin(x^{-1})}{x} \\ &= \lim_{x\to 0} x\sin(x^{-1}) \\ &= 0\end{aligned}$$

Therefore for each $x \in [-1, 1]$, $|F'(x)| \leq 3$. Which by a previous exercise implies that $F \in \text{BV}(-1, 1)$.
On $\mathbb{R} \setminus \{0\}$,

$$\begin{aligned}G'(x) &= 2x\sin(x^{-2}) - \frac{2\sin(x^{-2})}{x} \\ &= \sin(x^{-2})(2x - \frac{2}{x})\end{aligned}$$

We see that G is also differentiable at $x = 0$ since

$$\begin{aligned}
G'(0) &= \lim_{x \to 0} \frac{G(x) - G(0)}{x - 0} \\
&= \lim_{x \to 0} \frac{x^2 \sin(x^{-2})}{x} \\
&= \lim_{x \to 0} x \sin(x^{-2}) \\
&= 0
\end{aligned}$$

For $n \in \mathbb{N}$, define $(x_i)_{i=0}^n \subset [-1, 1]$ by

$$x_i = \frac{-1}{\sqrt{\frac{\pi}{2} + i\pi}}$$

Then for each $n \in \mathbb{N}$, $(x_i)_{i=1}^n$ is strictly increasing and for each $i = 1, 2, \cdots, n$ we have that

$$\begin{aligned}
|G(x_i) - G(x_{i-1})| &= \frac{1}{\frac{\pi}{2} + i\pi} + \frac{1}{\frac{\pi}{2} + (i-1)\pi} \\
&= \frac{2}{\pi}\left[\frac{(2i-1) + (2i+1)}{(2i+1)(2i-1)}\right] \\
&= \frac{2}{\pi}\left[\frac{4i}{4i^2 - 1}\right] \\
&> \frac{2}{i\pi}
\end{aligned}$$

Hence for each $n \in \mathbb{N}$,

$$\begin{aligned} \mathrm{TV}(G, [-1,1]) &\geq \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})| \\ &> \frac{2}{\pi} \sum_{i=1}^{n} \frac{1}{i} \end{aligned}$$

Therefore $G \notin \mathrm{BV}([-1,1])$. □

Exercise 7.5.0.18. The following is stated for BV, but is also true for $\mathrm{BV}(a,b)$.

1. For each $F, G \in \mathrm{BV}$, $T_{F+G} \leq T_F + T_G$ and therefore BV is a vector space.
2. For each $F : \mathbb{R} \to \mathbb{C}$, $F \in \mathrm{BV}$ iff $Re(f) \in \mathrm{BV}$ and $Im(F) \in \mathrm{BV}$.
3. For each $F : \mathbb{R} \to \mathbb{R}$, $F \in \mathrm{BV}$ iff there exist functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$
4. For each $F \in \mathrm{BV}$ and $x \in \mathbb{R}$, $\lim_{t \to x^+} F(t)$ and $\lim_{t \to x^-} F(t)$ exist.
5. For each $F \in \mathrm{BV}$, $\{x \in R : F \text{ is not continuous at } x\}$ is countable.
6. For each $F \in \mathrm{BV}$, F and F_+ are differentiable a.e. and $F' = (F_+)'$ a.e.
7. For each $F \in \mathrm{BV}, c \in \mathbb{R}$, $F - c \in \mathrm{BV}$

Proof.

1. Let $F, G \in \mathrm{BV}$, $x \in \mathbb{R}$ and $\epsilon > 0$. Since $T_{F+G}(x) < \infty$, $T_{F+G}(x) - \epsilon < T_{F+G}(x)$. Thus there exists $(x_i)_{i=0}^n \subset \mathbb{R}$ such that $(x_i)_{i=0}^n$ is increasing, $x_n = x$ and $T_{F+G}(x) < \sum_{i=1}^n |(F+G)(x_i) - (F+G)(x_{i-1}))| + \epsilon$. Thuerefore

$$\begin{aligned} T_{F+G}(x) &< \sum_{i=1}^{n} |(F+G)(x_i) - (F+G)(x_{i-1}))| + \epsilon \\ &\leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| + \sum_{i=1}^{n} |G(x_i) - G(x_{i-1})| + \epsilon \\ &\leq T_F(x) + T_G(x) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $T_{F+G}(x) \leq T_F(x) + T_G(x)$. Therefore $\mathrm{TV}(F+G) \leq \mathrm{TV}(F) + \mathrm{TV}(G) < \infty$. Thus $F + G \in \mathrm{BV}$. It is straight forward to verify the other requirements needed to show that BV is a vector space.

2. Let $F : \mathbb{R} \to \mathbb{C}$. Write $F = F_1 + iF_2$ with $F_1, F_2 : \mathbb{R} \to \mathbb{R}$. Suppose that $F \in \mathrm{BV}$. Note that for each $x_1, x_2 \in \mathbb{R}$ and $j = 1, 2$, $|F_j(x_1) - F_j(x_2)| \leq |F(x_1) - F(x_2)|$. Let $x \in \mathbb{R}$ and $(x_i)_{i=0}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=0}^n$ is increasing and $x_n = x$. Then for $j = 1, 2$

$$\sum_{i=1}^{n} |F_j(x_i) - F_j(x_{i-1})| \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|$$

Thus for $j = 1, 2$ we have that $T_{F_j}(x) \leq T_F(x)$ which implies that $Re(f), Im(F) \in \mathrm{BV}$. Conversely, Suppose that $Re(f), Im(F) \in \mathrm{BV}$. Then $F = Re(f) + iIm(f) \in \mathrm{BV}$ by (1).

3. Suppose that $F \in \mathrm{BV}$. Choose $F_1 = \frac{1}{2}(T_F - F)$ and $F_2 = \frac{1}{2}(T_F + F)$. Then F_1, F_2 are bounded, increasing and $F = F_1 + F_2$. Conversely, if there exist $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ such that F_1, F_2 are bounded, increasing and $F = F_1 - F_2$, then $F_1, F_2 \in \mathrm{BV}$. By (1) $F \in \mathrm{BV}$.

4. This is clear by previous results and (3)

5. This is clear by previous results and (3)

6. This is clear by previous results and (3)

7. Clearly constant functions have zero total variation. The rest is implied by (1).

□

Lemma 7.5.0.19. Let $F \in BV$. Then $\lim_{x \to -\infty} T_F(x) = 0$ and if F is right continuous, then T_F is right continuous.

Definition 7.5.0.20. Define $\text{NBV} = \{F \in \text{BV} : F \text{ is right continuous and } \lim_{x\to-\infty} F(x) = 0\}$.

Theorem 7.5.0.21. Let $M(\mathbb{R})$ be the set of complex Borel measures on $\mathbb{R}$. For $F \in \text{NBV}$, define $\mu_F \in M(\mathbb{R})$ by $\mu_F((-\infty, x]) = F(x)$. Then $F \mapsto \mu_F$ defines a bijection $\text{NBV} \to M(\mathbb{R})$. In addition, $|\mu_F| = \mu_{T_F}$

Theorem 7.5.0.22. Let $F \in \text{NBV}$. Then $F' \in L^1(m)$, $\mu_F \perp m$ iff $F' = 0$ a.e. and $\mu_F \ll m$ iff for each $x \in \mathbb{R}$,

$$\int_{(-\infty,x]} F' \, dm = F(x)$$

Definition 7.5.0.23. Let $F : \mathbb{R} \to \mathbb{C}$. Then F is said to be **absolutely continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each disjoint $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.

Definition 7.5.0.24. Let $F : [a, b] \to \mathbb{C}$. Then F is said to be **absolutely continuous** if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each disjoint $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$, $\sum_{i=1}^n b_i - a_i < \delta$ implies that $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.

Exercise 7.5.0.25. Let $F : [a, b] \to \mathbb{C}$. If F is absolutely continuous, then $F \in \text{BV}$.

Proof. Suppose that F is absolutely continuous. Then for each $j \in \mathbb{N}$, there exists $\delta > 0$ such that for each disjoint $((a_i, b_i))_{i=1}^n \subset \mathcal{B}([a, b])$, $\sum\limits_{i=1}^n b_i - a_i < \delta$ implies that $\sum\limits_{i=1}^n |F(b_i) - F(a_i)| < 1$
Define Choose $n^* \in \mathbb{N}$ such that $(b - a)/n < \delta$ and define $(x_j^*)_{j=0}^{n^*} \subset [a, b]$ by

$$x_j^* = a + \frac{b-a}{n} j$$

Let $(x_j)_{j=1}^n \subset [a, b]$ be increasing. Consider the refinement

$$(x_j')_{j=0}^{n'} = (x_j)_{j=0}^n \cup (x_j^*)_{j=0}^{n^*}$$

For $j \in \{1, \dots, n\}$, set $k_0 = 0$ and $k_j = \max\{k : x_k' \in [x_{j-1}^*, x_j^*]\}$. Then for each $k \in \{k_{j-1} + 1, \dots, k_j\}$, $x_k' - x_{k-1}' < \delta$. Then

$$\begin{aligned}
\sum_{j=1}^{n'} |F(x_j') - F(x_{j-1}')| &= \sum_{j=1}^{n} \sum_{k=k_{j-1}+1}^{k_j} |F(x_k') - F(x_{k-1}')| \\
&< \sum_{j=1}^{n} 1 \\
&= n
\end{aligned}$$

So $\text{TV}(F) \leq n < \infty$ and $F \in \text{BV}$. □

Exercise 7.5.0.26. There exists $F : \mathbb{R} \to \mathbb{C}$ such that F is absolutely continuous and $F \notin \text{BV}$.

Proof. Define $F : \mathbb{R} \to \mathbb{C}$ by $F(x) = x$. □

Exercise 7.5.0.27. Let $F : \mathbb{R} \to \mathbb{C}$. Suppose that there exists $f \in L^1(m)$ such that for each $x \in \mathbb{R}$,

$$F(x) = \int_{(-\infty,x]} f \, dm$$

Then $F \in \text{NBV}$.

Proof. Let $x \in \mathbb{R}$ and $(x_i)_{i=1}^n \subset \mathbb{R}$. Suppose that $(x_i)_{i=1}^n$ is increasing and $x_n = x$. Then

$$\begin{aligned}\sum_{i=1}^n |F(x_i) - F(x_{i-1})| &= \sum_{i=1}^n \left| \int_{(x_{i-1},x_i]} f\,dm \right| \\ &\leq \sum_{i=1}^n \int_{(x_{i-1},x_i]} |f|\,dm \\ &= \int_{(x_0,x]} |f|\,dm \\ &< \int |f|\,dm\end{aligned}$$

Hence $T_F(x) \leq \int |f|\,dm$. Since $x \in \mathbb{R}$ is arbitrary, $\mathrm{TV}(F) \leq \int |f|\,dm$. Therefore $F \in \mathrm{BV}$. By the continuity from above and below for measures and the fact that $m(x) = 0$ for each $x \in \mathbb{R}$, F is continuous. By continuity from above for measures, $\lim\limits_{x \to -\infty} F(x) = 0$. So $F \in \mathrm{NBV}$. □

Lemma 7.5.0.28. Let $F \in \mathrm{NBV}$. Then F is absolutely continuous iff $\mu_F \ll m$.

Exercise 7.5.0.29. The Fundamental Theorem of Calculus:
Let $F : [a,b] \to \mathbb{C}$. The following are equivalent:

1. F is absolutely continuous on $[a,b]$.
2. there exists $f \in L^1([a,b],m)$ such that for each $x \in [a,b]$,
$$F(x) - F(a) = \int_{(a,x]} f\,dm$$
3. F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b],m)$ and for each $x \in [a,b]$,
$$F(x) - F(a) = \int_{(a,x]} F'\,dm$$

Proof. (1) $\implies$ (3)
Suppose that F is absolutely continuous on $[a,b]$. Then $F \in \mathrm{BV}[a,b]$. Extend F to $\mathbb{R}$ by setting $F(x) = F(a)$ for $x < a$ and $F(x) = F(b)$ for $x > b$. Then $G = F - F(a) \in \mathrm{NBV}$ and is absolutely continuus. The previous lemma implies that there exists $f \in L^1(m)$ such that $d\mu_G = f\,dm$. A previous theorem implies that for a.e. $x \in [a,b]$

$$\begin{aligned}F'(x) &= \lim_{r \to x} \frac{\mu_G((x,x+r])}{m((x,x+r])} \\ &= f(x)\end{aligned}$$

So F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b],m)$ and by construction, for each $x \in [a,b]$, we have that

$$\begin{aligned}F(x) - F(a) &= \mu_G((a,x]) \\ &= \int_{(a,x]} f\,dm \\ &= \int_{(a,x]} F'\,dm\end{aligned}$$

(3) $\implies$ (2)
Trivial.
(2) $\implies$ (1)
Suppose that there exists $f \in L^1([a,b],m)$ such that for each $x \in [a,b]$, $F(x) - F(a) = \int_{(a,x]} f\,dm$. Extend F as before and obtain G as before. Note that a previous exercise implies that $G \in \mathrm{NBV}$. Since $\mu_G \ll m$, the previous lemma implies that G is absolutely continuous. □

Exercise 7.5.0.30. Let $F : \mathbb{R} \to \mathbb{C}$. If F is absolutely continuous. Then F is differentiable a.e.

Proof. Let $n \in \mathbb{N}$. Since F is absolutely continuous on $\mathbb{R}$, F is absolutely continuous on $[-n, n]$. The FTC implies that F is differentiable a.e. on $[-n, n]$. Since $n \in \mathbb{N}$ is arbitrary, F is differentiable a.e on $\mathbb{R}$. □

Exercise 7.5.0.31. Let $F : \mathbb{R} \to \mathbb{C}$. Then F is Lipschitz continuous iff F is absolutely continuous and F' is bounded a.e.

Proof. Suppose that F is Lipschitz continuous. Then there exists $M > 0$ such that for each $x, y \in \mathbb{R}$, $|F(x)-F(y)| \leq M|x-y|$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{M}$. Let $((a_i, b_i))_{i=1}^n \subset \mathcal{B}(\mathbb{R})$, Suppose that $\sum_{i=1}^n b_i - a_i < \delta$. Then

$$\begin{aligned}\sum_{i=1}^n |F(b_i) - F(a_i)| &\leq \sum_{i=1}^n M(b_i - a_i) \\ &< M\delta \\ &= \epsilon\end{aligned}$$

Hence F is absolutely continuous. For each $x, y \in \mathbb{R}$, if $x \neq y$, then $\left|\frac{F(x)-F(y)}{x-y}\right| \leq M$. Hence for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$.
Conversely, suppose that F is absolutely continuous and F' is bounded a.e. Then there exits $M > 0$ such that for a.e. $x \in \mathbb{R}$, $|F'(x)| \leq M$. Let $x, y \in \mathbb{R}$. Suppose $x < y$. Then the FTC implies that

$$\begin{aligned}|F(y) - F(x)| &= \left|\int_{(x,y]} F'\,dm\right| \\ &\leq \int_{(x,y]} |F'|\,dm \\ &= M|y - x|\end{aligned}$$

and F is Lipschitz continuous. □

Exercise 7.5.0.32. Construct an increasing function $F : \mathbb{R} \to \mathbb{R}$ whose discontinuities is $\mathbb{Q}$.

Proof. Let $(q_n)_{n\in\mathbb{N}}$ be an ennumeration of $\mathbb{Q}$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F = \sum_{n\in\mathbb{N}} 2^{-n}\chi_{[q_n,\infty)}$$

Equivalently, if we define $S_x = \{n \in \mathbb{N} : q_n \leq x\}$, then we may write

$$F(x) = \sum_{n\in S_x} 2^{-n}$$

Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Then $S_x \subsetneq S_y$. So $F(x) < F(y)$ and therefore F is strictly increasing.
For each $x, y \in R$ with $x < y$, define $S_{x,y} = \{n \in \mathbb{N} : x < q_n \leq y\}$. Note that $\lim_{y\to x^+} \min(S_{x,y}) = \infty$ and if $y \in \mathbb{R} \setminus \mathbb{Q}$, then $\lim_{x\to y^-} \min(S_{x,y}) = \infty$.
Now, let $x \in \mathbb{R}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$. Choose $\delta > 0$ such that $\min(S_{x,x+\delta}) \geq N$. Let $y \in [x, \infty)$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned}|F(x) - F(y)| &= \sum_{n\in S_y} 2^{-n} - \sum_{n\in S_x} 2^{-n} \\ &= \sum_{n\in S_{x,y}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon\end{aligned}$$

Hence F is right continuous. Now let $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ as before and $\delta > 0$ such that $\min(S_{x-\delta,x}) \geq N$. Let $y \in (-\infty, x]$. Suppose that $|x - y| < \delta$. Then

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_x} 2^{-n} - \sum_{n \in S_y} 2^{-n} \\ &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\leq \sum_{n=N}^{\infty} 2^{-n} \\ &< \epsilon \end{aligned}$$

Hence F is left continuous on $\mathbb{R} \setminus \mathbb{Q}$.
Now, let $x \in \mathbb{Q}$. Then there exists $j \in \mathbb{N}$ such that $q_j = x$. Choose $\epsilon = 2^{-j}$. Let $\delta > 0$. Choose $y = x - \frac{\delta}{2}$. Then $|x - y| < \delta$ and

$$\begin{aligned} |F(x) - F(y)| &= \sum_{n \in S_{y,x}} 2^{-n} \\ &\geq 2^{-j} \\ &= \epsilon \end{aligned}$$

Hence F is discontinuous from the left at x. Since $x \in \mathbb{Q}$ is arbitrary, F is discontinuous from the left on $\mathbb{Q}$. □

Exercise 7.5.0.33. Let $(F_n)_{n \in \mathbb{N}} \in \text{NBV}$ be a sequence of nonnegative, increasing functions. If for each $x \in \mathbb{R}$, $F(x) = \sum_{n \in \mathbb{N}} F_n(x) < \infty$, then for a.e. $x \in \mathbb{R}$, F is differentiable at x and $F'(x) = \sum_{n \in \mathbb{N}} F_n'(x)$.

Proof. Define $\mu = \sum_{n \in \mathbb{N}} \mu_{F_n}$. Note that

$$\begin{aligned} \mu((-\infty, x]) &= \sum_{n \in \mathbb{N}} \mu_{F_n}((-\infty, x]) \\ &= \sum_{n \in \mathbb{N}} F_n(x) \\ &= F(x) \end{aligned}$$

Hence $F \in \text{NBV}$ and $\mu = \mu_F$. For each $n \in \mathbb{N}$, there exist $\lambda_n \in M(\mathbb{R})$ and $f \in L^1(\mathbb{R})$ such that $d\mu_{F_n} = d\lambda_n + f_n\, dm$ and $\lambda \perp m$. Since for each $n \in \mathbb{N}$, λ_n, f_n are nonnegative, we have that $d\mu_F = \sum_{n \in \mathbb{N}} d\lambda_n + (\sum_{n \in \mathbb{N}} f_n)\, dm$. By a previous theorem, for a.e. $x \in \mathbb{R}$,

$$\begin{aligned} F'(x) &= \lim_{r \to 0} \frac{\mu_F((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} f_n(x) \\ &= \sum_{n \in \mathbb{N}} \lim_{r \to 0} \frac{\mu_{F_n}((x, x+r])}{m((x, x+r])} \\ &= \sum_{n \in \mathbb{N}} F_n'(x) \end{aligned}$$

□

Exercise 7.5.0.34. Let $F : [0, 1] \to [0, 1]$ be the Cantor function. Extend F to $\mathbb{R}$ by setting $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. Let $([a_n, b_n])_{n \in \mathbb{N}}$ be an ennumeration of the closed subintervals of $[0, 1]$ with rational endpoints. For $n \in \mathbb{N}$, define $F_n : \mathbb{R} \to [0, 1]$ by $F_n(x) = F(\frac{x - a_n}{b_n - a_n})$. Define $G : \mathbb{R} \to \mathbb{R}$ by $G = \sum_{n \in \mathbb{N}} 2^{-n} F_n$. Then G is continuous, strictly increasing on $[0, 1]$ and $G' = 0$ a.e.

Proof. Since F is continuous on $\mathbb{R}$, we have that for each $n \in \mathbb{N}$, F_n is continuous on $\mathbb{R}$. We observe that for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $|2^{-n}F_n(x)| \leq 2^{-n}$. Thus the Weierstrass M-test implies that G converges uniformly on $\mathbb{R}$ and is therefore continuous. Since F is increasing, for each $n \in \mathbb{N}$, F_n is increasing. Let $x, y \in \mathbb{R}$. Suppose that $x < y$. Choose $j \in \mathbb{N}$ such that $x < a_j < y < b_j$. Then

$$\begin{aligned}
G(x) &= \sum_{n \in \mathbb{N}} 2^{-n} F_n(x) \\
&= \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(x) + 0 \\
&< \sum_{\substack{n \in \mathbb{N} \\ n \neq j}} 2^{-n} F_n(y) + 2^{-j} F_n(y) \\
&= \sum_{n \in \mathbb{N}} 2^{-n} F_n(y) \\
&= G(y)
\end{aligned}$$

So G is strictly increasing.
Now we observe that for each $n \in \mathbb{N}$, $F_n \in \text{NBV}$. The previous exercise implies that

$$G' = \sum 2^{-n} F_n' = 0 \text{ a.e.}$$

$\square$

7.6 Partial Information Projection

7.6.1 TO DO

- Look at compactifications, i.e. stone-cech,
- read conditional probabilities and conditional expectation by david simmons, try to define the weak* limit of measures on a topological space in terms of its compactification, show its support is on the fibers,

Note 7.6.1.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $\mathcal{B} \subset \mathcal{A}$ a sub σ-algebra. We recall Exercise 7.3.1.1 that Then $L^1(X, \mathcal{B}, \mu|_{\mathcal{B}}) \subset L^1(X, \mathcal{A}, \mu)$ and for each $f \in L^1(X, \mathcal{B}, \mu|_{\mathcal{B}})$ and $B \in \mathcal{B}$,

$$\int_B f \, d\mu|_{\mathcal{B}} = \int_B f \, d\mu$$

Exercise 7.6.1.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$ and $f \in L^1(X, \mathcal{A}, \mu)$. Define $\mu_{\mathcal{B}} : \mathcal{B} \to [0, \infty]$ and $\nu_f : \mathcal{B} \to [0, \infty)$ by $\mu_{\mathcal{B}} = \mu|_{\mathcal{B}}$ and

$$\nu_f(B) = \int_B f \, d\mu$$

Then $\nu_f \ll \mu_{\mathcal{B}}$.

Proof. Let $B \in \mathcal{B}$. Suppose that $\mu_{\mathcal{B}}(B) = 0$. By definition, $\mu(B) = 0$. So $\nu(B) = 0$ and $\nu \ll \mu_{\mathcal{B}}$. $\square$

Note 7.6.1.3. Since $\nu_f \ll \mu_{\mathcal{B}}$ and $\nu_f(X) < \infty$, if μ is σ-finite, then $d\nu_f/d\mu_{\mathcal{B}}$ exists and

$$\begin{aligned} d\nu_f/d\mu_{\mathcal{B}} &\in L^1(X, \mathcal{B}, \mu_{\mathcal{B}}) \\ &\subset L^1(X, \mathcal{A}, \mu) \end{aligned}$$

Definition 7.6.1.4. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$. We define the **projection from** $L^1(X, \mathcal{A}, \mu)$ **to** $L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$, denoted $P^\mu_{\mathcal{B}} : L^1(X, \mathcal{A}, \mu) \to L^1(X, \mathcal{B}, \mu_{\mathcal{B}})$ by

$$P^\mu_{\mathcal{B}} f = \frac{d\nu_f}{d\mu_{\mathcal{B}}}$$

Exercise 7.6.1.5. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$. Then

1. $P^\mu_{\mathcal{B}} \in L(L^1(X, \mathcal{A}, \mu))$ and $\|P^\mu_{\mathcal{B}}\| = 1$
2. $P^\mu_{\mathcal{B}}|_{L^1(X,\mathcal{B},\mu_{\mathcal{B}})} = \mathrm{id}_{L^1(X,\mathcal{B},\mu_{\mathcal{B}})}$
3. $P^\mu_{\mathcal{B}}$ is idempotent

Proof.

1. Let $f, g \in L^1(X, \mathcal{A}, \mu)$ and $\lambda \in \mathbb{C}$. For each $B \in \mathcal{B}$, we have that

$$\begin{aligned} \nu_{f+\lambda g}(B) &= \int_B f + \lambda g \, d\mu \\ &= \int_B f \, d\mu + \lambda \int_B g \, d\mu \\ &= \nu_f(B) + \lambda \nu_g(B) \\ &= (\nu_f + \lambda \nu_g)(B) \end{aligned}$$

Hence $\nu_{f+\lambda g} + \nu_f + \lambda \nu_g$. Thus

$$\begin{aligned} P^\mu_{\mathcal{B}}(f + \lambda g) &= \frac{d\nu_{f+\lambda g}}{d\mu_{\mathcal{B}}} \\ &= \frac{d\nu_f}{d\mu_{\mathcal{B}}} + \lambda \frac{d\nu_g}{d\mu_{\mathcal{B}}} \\ &= P^\mu_{\mathcal{B}} f + \lambda P^\mu_{\mathcal{B}} g \end{aligned}$$

So $P^\mu_\mathcal{B}$ is linear. Since $|P^\mu_\mathcal{B} f| \in L^1(X, \mathcal{B}, \mu_\mathcal{B})$, a previous exercise implies that

$$\begin{aligned}\|P^\mu_\mathcal{B} f\|_1 &= \int |P^\mu_\mathcal{B} f|\, d\mu \\ &= \int |P^\mu_\mathcal{B} f|\, d\mu_\mathcal{B} \\ &= |\nu_f|(X) \\ &= \int |f|\, d\mu \\ &= \|f\|_1\end{aligned}$$

Hence $\|P^\mu_\mathcal{B} f\|_1 = \|f\|_1$ and $P^\mu_\mathcal{B} \in L(L^1(X, \mathcal{A}, \mu))$.

2. Let $f \in L^1(X, \mathcal{B}, \mu_\mathcal{B})$. Then for each $B \in \mathcal{B}$,

$$\begin{aligned}\nu_f(B) &= \int_B f\, d\mu \\ &= \int_B f\, d\mu_\mathcal{B}\end{aligned}$$

Uniqueness of the Radon-Nikodym derivative implies that $P^\mu_\mathcal{B} f = f$. Since $f \in L^1(X, \mathcal{B}, \mu_\mathcal{B})$ is arbitrary, $P^\mu_\mathcal{B}|_{L^1(X,\mathcal{B},\mu_\mathcal{B})} = \mathrm{id}_{L^1(X,\mathcal{A},\mu_\mathcal{B})}$.

3. Let $f \in L^1(X, \mathcal{A}, \mu)$. Since $P^\mu_\mathcal{B} f \in L^1(X, \mathcal{B}, \mu_\mathcal{B})$ and $P^\mu_\mathcal{B}|_{L^1(X,\mathcal{B},\mu_\mathcal{B})} = \mathrm{id}_{L^1(X,\mathcal{A},\mu_\mathcal{B})}$, we have that

$$\begin{aligned}(P^\mu_\mathcal{B})^2 f &= P^\mu_\mathcal{B}(P^\mu_\mathcal{B} f) \\ &= \mathrm{id}_{L^1(X,\mathcal{B},\mu_\mathcal{B})}(P^\mu_\mathcal{B} f) \\ &= P^\mu_\mathcal{B} f\end{aligned}$$

Since $f \in L^1(X, \mathcal{B}, \mu_\mathcal{B})$ is arbitrary, $(P^\mu_\mathcal{B})^2 = P^\mu_\mathcal{B}$ and $P^\mu_\mathcal{B}$ is idempotent.

□

Exercise 7.6.1.6. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$, $f \in L^1(X, \mathcal{A}, \mu)$ and $g \in L^1(X, \mathcal{B}, \mu_\mathcal{B})$. Then $g = P^\mu_\mathcal{B} f$ iff for each $B \in \mathcal{B}$,

$$\int_B g\, d\mu = \int_B f\, d\mu$$

Proof. Suppose that $g = P^\mu_\mathcal{B} f$. Let $B \in \mathcal{B}$. Then

$$\begin{aligned}\int_B g\, d\mu &= \int_B g\, d\mu_\mathcal{B} \\ &= \nu_f(B) \\ &= \int_B f\, d\mu\end{aligned}$$

Since $B \in \mathcal{B}$ is arbitrary, for each $B \in \mathcal{B}$,

$$\int_B g\, d\mu = \int_B f\, d\mu$$

Conversely, suppose that for each $B \in \mathcal{B}$,

$$\int_B g\, d\mu = \int_B f\, d\mu$$

Then for each $B \in \mathcal{B}$,

$$\begin{aligned}\int_B g\, d\mu_{\mathcal{B}} &= \int_B g\, d\mu \\ &= \int_B f\, d\mu \\ &= \nu_f(B)\end{aligned}$$

By definition,

$$\begin{aligned}P^{\mu}_{\mathcal{B}} f &= \frac{d\nu_f}{d\mu_{\mathcal{B}}} \\ &= g\end{aligned}$$

□

Exercise 7.6.1.7. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$ and $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$. Suppose that $(A_j)_{j\in\mathbb{N}}$ is disjoint and $\mu\bigg(\bigcup\limits_{j\in\mathbb{N}} A_j\bigg) < \infty$. Then

1. $\chi_{\bigcup\limits_{j\in\mathbb{N}} A_j} \in L^1(X, \mathcal{A}, \mu)$
2. $P^{\mu}_{\mathcal{B}} \chi_{\bigcup\limits_{j\in\mathbb{N}} A_j} = \sum\limits_{j\in\mathbb{N}} P^{\mu}_{\mathcal{B}} \chi_{A_j}$

Proof.

1. Since $(A_j)_{j\in\mathbb{N}}$ is disjoint, we have that

$$\begin{aligned}\|\chi_{\bigcup\limits_{j\in\mathbb{N}} A_j}\|_1 &= \int \chi_{\bigcup\limits_{j\in\mathbb{N}} A_j}\, d\mu \\ &= \mu\bigg(\bigcup_{j\in\mathbb{N}} A_j\bigg) \\ &< \infty\end{aligned}$$

So $\chi_{\bigcup\limits_{j\in\mathbb{N}} A_j} \in L^1(X, \mathcal{A}, \mu)$.

2. Since $(A_j)_{j\in\mathbb{N}}$ is disjoint, we have that

$$\chi_{\bigcup\limits_{j\in\mathbb{N}} A_j} = \sum_{j\in\mathbb{N}} \chi_{A_j}$$

For each $n \in \mathbb{N}$, define $f_n = \sum\limits_{j=1}^{n} \chi_{A_j}$. Set $f = \chi_{\bigcup\limits_{j\in\mathbb{N}} A_j}$. Then for each $n \in \mathbb{N}$, $f_n \leq f$ and $f_n \xrightarrow{\text{p.w.}} f$. Since $f \in L^1(X, \mathcal{A}, \mu)$, the dominated convergence theorem implies that $f_n \xrightarrow{L^1(\mu)} f$. Since $P^{\mu}_{\mathcal{B}} \in L(L^1(X, \mathcal{A}, \mu))$,

$$\begin{aligned}\sum_{j=1}^{n} P^{\mu}_{\mathcal{B}} \chi_{A_j} &= P^{\mu}_{\mathcal{B}} \sum_{j=1}^{n} \chi_{A_j} \\ &= P_B f_n \\ &\xrightarrow{L^1(\mu)} P^{\mu}_{\mathcal{B}} f \\ &= P^{\mu}_{\mathcal{B}} \chi_{\bigcup\limits_{j\in\mathbb{N}} A_j}\end{aligned}$$

Hence $P^{\mu}_{\mathcal{B}} \chi_{\bigcup\limits_{j\in\mathbb{N}} A_j} = \sum\limits_{j\in\mathbb{N}} P^{\mu}_{\mathcal{B}} \chi_{A_j}$.

□

Exercise 7.6.1.8. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$ and $f \in L^1(X, \mathcal{A}, \mu)$. If $f \geq 0$, then $P^\mu_\mathcal{B} f \geq 0$ $\mu_\mathcal{B}$-a.e.

Proof. Suppose that $f \geq 0$. Then $\nu_f : \mathcal{B} \to [0, \infty)$ is a finite measure. For the sake of contradiction, suppose that Hence

$$\begin{aligned} P^\mu_\mathcal{B} f &= \frac{d\nu_f}{d\mu_\mathcal{B}} \\ &\geq 0 \ \mu_\mathcal{B}\text{-a.e.} \end{aligned}$$

cite exercise or fill in why □

delete rest of section

Exercise 7.6.1.9. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$ and $f : X \to \mathbb{R}$ $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable. For each $z \in \mathbb{R}$, define $h_z \in L^1(X, \mathcal{A}, \mu)$ by $h_z = \chi_{f^{-1}((-\infty, z])}$ and choose $f_z \in L_0(X, \mathcal{A})$ such that $f_z = P^\mu_\mathcal{B} h_z$ μ-a.e. Then there exists $M \in \mathcal{B}$ such that $\mu_\mathcal{B}(M^c) = 0$ and for each $x \in M$, $(f_q(x))_{q \in \mathbb{Q}}$ is increasing.

Proof. Let $q, r \in \mathbb{Q}$. Suppose that $q < r$. Then $\chi_{f^{-1}((-\infty, r])} - \chi_{f^{-1}((-\infty, q])} \geq 0$ and

$$\begin{aligned} f_r - f_q &= P^\mu_\mathcal{B} \chi_{f^{-1}((-\infty, r])} - P^\mu_\mathcal{B} \chi_{f^{-1}((-\infty, q])} \\ &= P^\mu_\mathcal{B} \bigg[\chi_{f^{-1}((-\infty, r])} - \chi_{f^{-1}((-\infty, q])} \bigg] \\ &\geq 0 \ \mu_\mathcal{B}\text{-a.e.} \end{aligned}$$

Hence $f_q \leq f_r$ $\mu_\mathcal{B}$-a.e. An exercise in the section on measures implies that $(f_q)_{q \in \mathbb{Q}}$ is increasing $\mu_\mathcal{B}$-a.e. and thus there exists $M \in \mathcal{B}$ such that $\mu_\mathcal{B}(M^c) = 0$ and for each $x \in M$, $(f_q(x))_{q \in \mathbb{Q}}$ is increasing. □

Exercise 7.6.1.10. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$ and $f : X \to \mathbb{R}$ $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable. Define $(h_z)_{z \in \mathbb{R}} \subset L^1(X, \mathcal{A}, \mu)$, $(f_z)_{z \in \mathbb{R}} \subset L^0(X, \mathcal{B})$ and $M \in \mathcal{B}$ as in the previous exercise. Choose $g \in \text{NBV}(\mathbb{R})$ such that $g : \mathbb{R} \to \mathbb{R}$, g is increasing and $\sup\limits_{z \in \mathbb{R}} g(z) = 1$. Define $G : \mathbb{R} \times X \to \mathbb{R}$ by

$$G(z, x) = \begin{cases} \inf\limits_{\substack{q \in \mathbb{Q} \\ q > z}} f_q(x) & x \in M \\ g(z) & x \in M^c \end{cases}$$

Then for each $x \in X$, $G(\cdot, x)$ is increasing and right continuous.

Proof. Let $x \in \mathbb{R}$. If $x \in M^c$, by defintion, $G(\cdot, x)$ is increasing and right continuous. Suppose that $x \in M$. Since $(f_q(x))_{q \in \mathbb{Q}}$ is increasing, slightly modifying the statement and proof of an exercise in the section on functions of bounded variation implies that $G(\cdot, x)$ is increasing and right continuous. □

Exercise 7.6.1.11. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$ and $f : X \to \mathbb{R}$ $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable. Define $(h_z)_{z \in \mathbb{R}} \subset L^1(X, \mathcal{B}, \mu)$, $(f_z)_{z \in \mathbb{R}} \subset L^0(X, \mathcal{B})$, $M \in \mathcal{B}$ and $G : \mathbb{R} \times X \to \mathbb{R}$ as in the previous exercise.

1. for each $z \in \mathbb{R}$, $G(z, \cdot) \in L^0(X, \mathcal{B})$ and $G(z, \cdot) = f_z$ $\mu_\mathcal{B}$-a.e.

2. $\sup\limits_{z \in \mathbb{R}} G(z, \cdot) = 1$ $\mu_\mathcal{B}$-a.e.

3. $\inf\limits_{z \in \mathbb{R}} G(z, \cdot) = 0$ $\mu_\mathcal{B}$-a.e.

Proof.

1. Let $z \in \mathbb{R}$. By definition,

$$G(z, \cdot) = \inf_{\substack{q \in \mathbb{Q} \\ q > z}} [f_q \chi_M](\cdot) + g(z)\chi_{M^c}(\cdot)$$

Since $(f_q \chi_M)_{q \in \mathbb{Q} \cap (z,\infty)} \subset L^0(X, \mathcal{B})$ and is point-wise bounded below, $\inf_{\substack{q \in \mathbb{Q} \\ q > z}} f_q \chi_M \in L^0(X, \mathcal{B})$. Hence $G(z, \cdot) \in L^0(X, \mathcal{B})$.

Choose $(q_n)_{n \in \mathbb{N}} \subset \mathbb{Q}$ such that for each $n \in \mathbb{N}$, $q_n \geq q_{n+1} > z$ and $q_n \to z$. Since for each $n \in \mathbb{N}$, $h_{q_n} - h_z = \chi_{f^{-1}((z,q_n])}$, $(z, q_{n+1}] \subset (z, q_n]$ and μ is finite, we have that

$$\begin{aligned}
\|h_{q_n} - h_z\|_1 &= \|\chi_{f^{-1}((z,q_n])}\|_1 \\
&= \mu(f^{-1}((z, q_n])) \\
&= f_*\mu((z, q_n]) \\
&\to f_*\mu(\varnothing) \\
&= 0
\end{aligned}$$

So that $h_{q_n} \xrightarrow{L^1(\mu)} h_z$. Therefore

$$\begin{aligned}
f_{q_n} &= P^{\mu}_{\mathcal{B}} h_{q_n} \\
&\xrightarrow{L^1(\mu_{\mathcal{B}})} P^{\mu}_{\mathcal{B}} h_z \\
&= f_z
\end{aligned}$$

This implies that $f_{q_n} \xrightarrow{\mu_{\mathcal{B}}} f_z$. Since $(f_{q_n})_{n \in \mathbb{N}}$ is decreasing $\mu_{\mathcal{B}}$-a.e., an exercise in the section on modes of convergence implies that $f_{q_n} \xrightarrow{\mu_{\mathcal{B}}\text{-a.e.}} f_z$. So there exists $N_1 \in \mathcal{B}$ such that $\mu_{\mathcal{B}}(N_1^c) = 0$ and $f_{q_n}\chi_{N_1} \xrightarrow{\text{p.w.}} f_z \chi_{N_1}$. Set $E = M \cap N_1$. Then

$$\begin{aligned}
\mu_{\mathcal{B}}(E^c) &= \mu_{\mathcal{B}}(M^c \cup N_1^c) \\
&\leq \mu_{\mathcal{B}}(M^c) + \mu_{\mathcal{B}}(N_1^c) \\
&= 0
\end{aligned}$$

and for each $x \in E$, $f_{q_n}(x) \to f_z(x)$ and $f_{q_n}(x) \to G(z, x)$. Hence $G(z, \cdot)\chi_E(\cdot) = f_z \chi_E(\cdot)$ which implies that $G(z, \cdot) = f_z$ $\mu_{\mathcal{B}}$-a.e.

2. Part (1) implies that for each $n \in \mathbb{N}$, there exists $E_n \in \mathcal{B}$ such that $E_n \subset M$, $\mu(E_n^c) = 0$ and $G(n, \cdot)\chi_{E_n}(\cdot) = f_n(\cdot)\chi_{E_n}(\cdot)$. Set $E = \bigcap_{n \in \mathbb{N}} E_n$. Since for each $n \in \mathbb{N}$, $\chi_X - h_n = \chi_{f^{-1}((n,\infty))}$, $(n+1, \infty) \subset (n, \infty)$ and μ is finite, we have that

$$\begin{aligned}
\|h_n - \chi_X\|_1 &= \mu(f^{-1}((n, \infty))) \\
&= f_*\mu((n, \infty)) \\
&\to f_*\mu(\varnothing) \\
&= 0
\end{aligned}$$

So that $h_n \xrightarrow{L^1(\mu)} \chi_X$. Therefore

$$\begin{aligned}
f_n &= P^{\mu}_{\mathcal{B}} h_n \\
&\xrightarrow{L^1(\mu_{\mathcal{B}})} P^{\mu}_{\mathcal{B}} \chi_X \\
&= \chi_X
\end{aligned}$$

This implies that $f_n \xrightarrow{\mu_{\mathcal{B}}} \chi_X$. Since $(f_n)_{n \in \mathbb{N}}$ is increasing $\mu_{\mathcal{B}}$-a.e., an exercise in the section on modes of convergence implies that $f_n \xrightarrow{\mu_{\mathcal{B}}\text{-a.e.}} \chi_X$. So there exists $N_2 \in \mathcal{B}$ such that $\mu_{\mathcal{B}}(N_2^c) = 0$ and $f_n \chi_{N_2} \xrightarrow{\text{p.w.}} \chi_{N_2}$. Set $M^+ = E \cap N_2$.

Then $M^+ \subset E \subset M$ and

$$\begin{aligned}
\mu_{\mathcal{B}}((M^+)^c) &= \mu_{\mathcal{B}}(E^c \cup N_2^c) \\
&\leq \mu_{\mathcal{B}}(E^c) + \mu_{\mathcal{B}}(N_2^c) \\
&= \mu_{\mathcal{B}}\bigg(\bigcup_{n \in \mathbb{N}} E_n^c\bigg) + \mu_{\mathcal{B}}(N_2^c) \\
&\leq \bigg[\sum_{n \in \mathbb{N}} \mu_{\mathcal{B}}(E_n^c)\bigg] + \mu_{\mathcal{B}}(N_2^c) \\
&= 0
\end{aligned}$$

Since $M^+ \subset M$, for each $x \in M^+$, $(f_n(x))_{n \in \mathbb{N}}$ is increasing. Hence for each $x \in M^+$,

$$\begin{aligned}
\sup_{z \in \mathbb{R}} G(z, x) &= \sup_{n \in \mathbb{N}} G(n, x) \\
&= \sup_{n \in \mathbb{N}} f_n(x) \\
&= 1
\end{aligned}$$

Thus $\sup\limits_{z \in \mathbb{R}} G(z, \cdot) = 1$ $\mu_{\mathcal{B}}$-a.e.

3. Part (2) implies that for each $n \in \mathbb{N}$, there exists $E_n \in \mathcal{B}$ such that $E_n \subset M$, $\mu(E_n^c) = 0$ and $G(n, \cdot)\chi_{E_n}(\cdot) = f_n(\cdot)\chi_{E_n}(\cdot)$. Set $E = \bigcap\limits_{n \in \mathbb{N}} E_n$. Since for each $n \in \mathbb{N}$, $h_{-n} = \chi_{f^{-1}((-\infty, -n])}$, $(-\infty, -(n+1)] \subset (-\infty, -n]$ and μ is finite, we have that

$$\begin{aligned}
\|h_{-n}\|_1 &= \mu(f^{-1}((-\infty, -n])) \\
&= f_*\mu((-\infty, -n]) \\
&\to \mu(\varnothing) \\
&= 0
\end{aligned}$$

So that $h_{-n} \xrightarrow{L^1(\mu)} 0$. Therefore

$$\begin{aligned}
f_{-n} &= P^{\mu}_{\mathcal{B}} h_{-n} \\
&\xrightarrow{L^1(\mu_{\mathcal{B}})} P^{\mu}_{\mathcal{B}} 0 \\
&= 0
\end{aligned}$$

This implies that $f_n \xrightarrow{\mu_{\mathcal{B}}} 0$. Since $(f_{-n})_{n \in \mathbb{N}}$ is decreasing $\mu_{\mathcal{B}}$-a.e., an exercise in the section on modes of convergence implies that $f_{-n} \xrightarrow{\mu_{\mathcal{B}}\text{-a.e.}} 0$. So there exists $N_3 \in \mathcal{B}$ such that $\mu_{\mathcal{B}}(N_3^c) = 0$ and $f_{-n}\chi_{N_3} \xrightarrow{\text{p.w.}} 0$. Set $M^- = E \cap N_3$. Then $M^- \subset E \subset M$ and

$$\begin{aligned}
\mu_{\mathcal{B}}((M^-)^c) &= \mu_{\mathcal{B}}(E^c \cup N_3^c) \\
&\leq \mu_{\mathcal{B}}(E^c) + \mu_{\mathcal{B}}(N_3^c) \\
&= \mu_{\mathcal{B}}\bigg(\bigcup_{n \in \mathbb{N}} E_n^c\bigg) + \mu_{\mathcal{B}}(N_3^c) \\
&\leq \bigg[\sum_{n \in \mathbb{N}} \mu_{\mathcal{B}}(E_n^c)\bigg] + \mu_{\mathcal{B}}(N_3^c) \\
&= 0
\end{aligned}$$

Since $M^- \subset M$, for each $x \in M^-$, $(f_{-n}(x))_{n \in \mathbb{N}}$ is decreasing. Hence for each $x \in M^-$,

$$\begin{aligned}
\inf_{z \in \mathbb{R}} G(z, x) &= \inf_{n \in \mathbb{N}} G(-n, x) \\
&= \inf_{n \in \mathbb{N}} f_{-n}(x) \\
&= 0
\end{aligned}$$

Thus $\inf_{z\in\mathbb{R}} G(z,\cdot) = 0$ $\mu_{\mathcal{B}}$-a.e.

$\square$

Exercise 7.6.1.12. Let $(X,\mathcal{A},\mu)$ be a finite measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$ and $f : X \to \mathbb{R}$ $(\mathcal{A},\mathcal{B}(\mathbb{R}))$-measurable. Then there exists $F : \mathbb{R}\times X \to [0,1]$ such that

1. for each $z\in\mathbb{R}$, $F(z,\cdot)\in L^0(X,\mathcal{B})$ and $F(z,\cdot) = P^{\mu}_{\mathcal{B}}\chi_{f^{-1}((-\infty,z])}$ μ_B-a.e.
2. for each $x\in X$, $F(\cdot,x)\in \text{NBV}(\mathbb{R})$, $F(\cdot,x)$, increasing and $\sup_{z\in\mathbb{R}} F(z,\cdot) = 1$.

Proof. Define $(h_z)_{z\in\mathbb{R}}\subset L^1(\mathbb{R},\mathcal{B}(\mathbb{R}),\mu)$, $(f_z)_{z\in\mathbb{R}}\subset L^0(\mathbb{R},\mathcal{B})$ as in the previous exercises. Choose $g\in\text{NBV}(\mathbb{R})$ such that $g:\mathbb{R}\to\mathbb{R}$, g is increasing and $\sup_{z\in\mathbb{R}} g(z) = 1$. Define $M, M^+, M^-\in\mathcal{B}$ and $G:\mathbb{R}\times X\to\mathbb{R}$ as in the previous exercises. Set $E = M\cap M^+\cap M^-$. Define $F:\mathbb{R}\times X\to\mathbb{R}$ by

$$F(z,x) = G(z,x)\chi_E(x) + g(z)\chi_{E^c}(x)$$

1. Let $z\in\mathbb{R}$. Then $F(z,\cdot) = G(z,\cdot)\chi_E(\cdot) + g(z)\chi_{E^c}(\cdot)$. Since $G(z,\cdot)\in L^0(X,\mathcal{B})$, $F(z,\cdot)\in L^0(X,\mathcal{B})$. Note that

$$\begin{aligned}\mu_{\mathcal{B}}(E^c) &= \mu_{\mathcal{B}}(M^c\cup(M^+)^c\cup(M^-)^c)\\ &\le \mu_{\mathcal{B}}(M^c) + \mu_{\mathcal{B}}((M^+)^c) + \mu_{\mathcal{B}}((M^-)^c)\\ &= 0\end{aligned}$$

Since $E\subset M$, by definition of G and F, we have that for each $x\in E$, $F(z,x) = G(z,x)$. Hence $\{x\in X : F(z,x)\neq G(z,x)\}\subset E^c$. Thus

$$\begin{aligned}F(z,\cdot) &= G(z,\cdot)\\ &= f_z \ \mu_{\mathcal{B}}\text{-a.e.}\end{aligned}$$

2. Let $x\in X$. Suppose that $x\in E$. The previous exercise implies that $G(\cdot,x)\in\text{NBV}(\mathbb{R})$, $G(\cdot,x)$ is increasing and $\sup_{z\in\mathbb{R}} G(z,x) = 1$. Since $F(\cdot,x) = G(\cdot,x)$, we have that $F(\cdot,x)\in\text{NBV}(\mathbb{R})$, $F(\cdot,x)$ is increasing and $\sup_{z\in\mathbb{R}} F(z,x) = 1$. If $x\in E^c$, then $F(\cdot,x) = g$. By definition of g, $F(\cdot,x)\in\text{NBV}(\mathbb{R})$, $F(\cdot,x)$, increasing and $\sup_{z\in\mathbb{R}} F(z,\cdot) = 1$.

$\square$

Definition 7.6.1.13. Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be measurable spaces and $\kappa : X\times\mathcal{B}\to[0,1]$. Then κ is said to be a **Markov kernel from** $(X,\mathcal{A})$ **to** $(Y,\mathcal{B})$ if

1. for each $x\in X$, $\kappa(x,\cdot)$ is a probability measure on $(Y,\mathcal{B})$
2. for each $B\in\mathcal{B}$, $\kappa(\cdot,B)$ is $\mathcal{A}$-measurable

Exercise 7.6.1.14. Let $(X,\mathcal{A},\mu)$ be a finite measure space, $\mathcal{B}$ a sub σ-algebra of $\mathcal{A}$ and $f : X\to\mathbb{R}$ $(\mathcal{A},\mathcal{B}(\mathbb{R}))$-measurable. Then there exists $\kappa : X\times\mathcal{B}(\mathbb{R})\to[0,1]$ such that

1. κ is a Markov kernel from $(X,\mathcal{B})$ to $(\mathbb{R},\mathcal{B}(\mathbb{R}))$.
2. For each $A\in\mathcal{B}(\mathbb{R})$, $\kappa(\cdot,A) = P^{\mu}_{\mathcal{B}}\chi_{f^{-1}(A)}$ $\mu_{\mathcal{B}}$-a.e.
3. For $\mu_{\mathcal{B}}$-a.e. $x\in X$, $\operatorname{supp}\kappa(x,\cdot) = f(x)$

Hint:

1. Consider $F:\mathbb{R}\times X\to[0,1]$ defined in the previous exercise and $\mu_x((a,b]) = F(b,x) - F(a,x)$.
2. Consider Dynkin's lemma with

$$\nu_B(A) = \int_B \kappa(x,A)\,d\mu_{\mathcal{B}}(x) \qquad\text{and}\qquad \lambda_B(A) = \mu(f^{-1}(A)\cap B)$$

Proof. Define $F : \mathbb{R} \times X \to [0,1]$ as in the previous exercise. For each $x \in X$, define $\mu_x : \mathcal{B}(\mathbb{R}) \to [0,1]$ to be the unique measure such that for each $a, b \in \mathbb{R}$, $a \leq b$ implies that $\mu_x((a,b]) = F(b,x) - F(a,x)$. Define $\kappa : X \times \mathcal{B}(\mathbb{R}) \to [0,1]$ by $\kappa(A,x) = \mu_x(A)$.

1.

(a) Let $x \in X$. By definition, $\kappa(x, \cdot) = \mu_x$ is a measure and

$$\begin{aligned}\kappa(x, \mathbb{R}) &= \sup_{n \in \mathbb{N}} \mu_x((-\infty, n]) \\ &= \sup_{n \in \mathbb{N}} F(n, x) \\ &= 1\end{aligned}$$

(b) Let $A \in \mathcal{B}(\mathbb{R})$. Recall that for each $x \in \mathbb{R}$,

$$\mu_x(A) = \inf\left\{\sum_{j \in \mathbb{N}} F(b_j, x) - F(a_j, x) : \text{ for each } j \in \mathbb{N},\, a_j, b_j \in \mathbb{R} \text{ and } A \subset \bigcup_{j \in \mathbb{N}} (a_j, b_j]\right\}$$

Therefore, for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$, there exist $(a^x_{n,j})_{j \in \mathbb{N}}$, $(b^x_{n,j})_{j \in \mathbb{N}} \subset \mathbb{R}$ such that $A \subset \bigcup\limits_{j \in \mathbb{N}} (a^x_{n,j}, b^x_{n,j}]$ and

$$\mu_x(A) \leq \sum_{j \in \mathbb{N}} F(b^x_{n,j}, x) - F(a^x_{n,j}, x) < \mu_x(A) + \frac{1}{n}$$

Define $(f_n)_{n \in \mathbb{N}} \subset L^0(X, \mathcal{B})$ by

$$f_n(x) = \sum_{j \in \mathbb{N}} F(b^x_{n,j}, x) - F(a^x_{n,j}, x)$$

Then $f_n \xrightarrow{\text{p.w.}} \kappa(\cdot, A)$ which implies that $\kappa(\cdot, A) \in L^0(X, \mathcal{B})$.

Hence κ is a markov kernel from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

2. Let $B \in \mathcal{B}$. Define $\nu_B, \lambda_B : \mathcal{B}(\mathbb{R}) \to [0, \infty)$ by

$$\nu_B(A) = \int_B \kappa(x, A)\, d\mu_{\mathcal{B}}(x)$$

and

$$\lambda_B(A) = \mu(f^{-1}(A) \cap B)$$

Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}\nu_B((a,b]) &= \int_B \kappa(x, (a,b])\, d\mu_{\mathcal{B}}(x) \\ &= \int_B F(b,x) - F(a,x)\, d\mu_{\mathcal{B}}(x) \\ &= \int_B P^{\mu}_{\mathcal{B}} \chi_{f^{-1}((-\infty,b])} - P^{\mu}_{\mathcal{B}} \chi_{f^{-1}((-\infty,a])}\, d\mu_{\mathcal{B}} \\ &= \int_B P^{\mu}_{\mathcal{B}} \chi_{f^{-1}((a,b])}\, d\mu_{\mathcal{B}} \\ &= \int_B \chi_{f^{-1}((a,b])}\, d\mu \\ &= \mu(f^{-1}((a,b]) \cap B) \\ &= \lambda_B((a,b])\end{aligned}$$

Define $\mathcal{P} \subset \mathcal{B}(\mathbb{R})$ by $\mathcal{P} = \{(a,b] : a, b \in \mathbb{R}\} \cup \{\varnothing, X\}$. A previous exercise in the sections on Dynkin's lemma implies that $\mathcal{P}$ is a π-system. Since $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$, an exercise in the section on complex measures implise that $\nu_B = \lambda_B$. Let $A \in \mathcal{B}(\mathbb{R})$. Then

$$\begin{aligned}\int_B \kappa(x, A)\, d\mu_{\mathcal{B}}(x) &= \nu_B(A) \\ &= \lambda_B(A) \\ &= \mu(f^{-1}(A) \cap B) \\ &= \int_B \chi_{f^{-1}(A)}\, d\mu \\ &= \int_B P^{\mu}_{\mathcal{B}} \chi_{f^{-1}(A)}\, d\mu \\ &= \int_B P^{\mu}_{\mathcal{B}} \chi_{f^{-1}(A)}\, d\mu_{\mathcal{B}}\end{aligned}$$

Since $B \in \mathcal{B}$ is arbitrary, $\kappa(\cdot, A) = P^{\mu}_{\mathcal{B}} \chi_{f^{-1}(A)}$ $\mu_{\mathcal{B}}$-a.e. Since $A \in \mathcal{B}(\mathbb{R})$ is arbitrary, we have that for each $A \in \mathcal{B}(\mathbb{R})$, $\kappa(\cdot, A) = P^{\mu}_{\mathcal{B}} \chi_{f^{-1}(A)}$ $\mu_{\mathcal{B}}$-a.e.

□

Exercise 7.6.1.15. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $(Y, \mathcal{B})$ a standard Borel space, $\mathcal{C}$ a sub σ-algebra of $\mathcal{A}$ and $f : X \to Y$ $(\mathcal{A}, \mathcal{B})$-measurable. Then there exists $\kappa : X \times \mathcal{B} \to [0, 1]$ such that

1. κ is a Markov kernel from $(X, \mathcal{C})$ to $(Y, \mathcal{B})$.
2. For each $B \in \mathcal{B}$, $\kappa(\cdot, B) = P^{\mu}_{\mathcal{C}} \chi_{f^{-1}(B)}$ $\mu_{\mathcal{C}}$-a.e.

Proof. **3 cases, $\mathcal{B}$ is finite, countably infinite and uncountable, in the latter case, can take** $E = \mathbb{R}$ Since $(Y, \mathcal{B})$ is a standard Borel space, there exists $E \in \mathcal{B}(\mathbb{R})$ and $\phi : (Y, \mathcal{B}) \to (E, \mathcal{B}(E))$ is an isomorphism. Let $\iota : (E, \mathcal{B}(E)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be the inclusion map. Then $\iota \circ \phi \circ f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The previous exercise implies that there exists $\kappa' : X \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that

1. κ' is a Markov kernel from $(X, \mathcal{C})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
2. For each $A \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}\kappa'(\cdot, A) &= P^{\mu}_{\mathcal{C}} \chi_{(\iota \circ \phi \circ f)^{-1}(A)} \\ &= P^{\mu}_{\mathcal{C}} \chi_{f^{-1}(\phi^{-1}(A \cap E))} \ \mu_{\mathcal{C}}\text{-a.e.}\end{aligned}$$

Define $\kappa : X \times \mathcal{B} \to [0, 1]$ by $\kappa(x, \cdot) = (\phi^{-1})_* \kappa'(x, \cdot)$.

1. (a) Let $x \in X$. Since $\kappa'(x, \cdot)$ is a probability measure on $(\mathbb{R}, \mathcal{B}(R))$, $(\phi^{-1})_* \kappa'(x, \cdot)$ is a probability measure on $(Y, \mathcal{B})$.
   (b) Let $B \in \mathcal{B}$. Then
2.

□

Exercise 7.6.1.16. Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ a measurable space, $f \in L^1(X, \mathcal{A}, \mu)$ and $g : X \to Y$. Suppose that for each $y \in Y$, $\{y\} \in \mathcal{B}$, g is surjective and g is $(\mathcal{A}, \mathcal{B})$-measurable. Then there exists a $\phi \in L^0(Y, \mathcal{B})$ such that $\phi \circ g = \mathcal{P}_{g^* \mathcal{B}} f$ μ-a.e. and ϕ is unique $g_* \mu$-a.e.
Hint: Doob-Dynkin lemma

Proof.

- **Existence:**
Since $P_{g^*\mathcal{B}}f \in L^1(X, g^*\mathcal{B}, \mu_{g^*\mathcal{B}})$ and $\mathcal{B}$, the Doob-Dynkin lemma implies that there exists a $\phi \in L^0(Y, \mathcal{B})$ such that $\phi \circ g = P_{g^*\mathcal{B}}f$.

- **Uniqueness:**
Suppose that there exists $\psi \in L^0(Y, \mathcal{B})$ such that $\psi \circ g = \mathcal{P}_{g^*\mathcal{B}}f$ μ-a.e. Then $\phi \circ g = \psi \circ g$ μ-a.e. An exercise in the section on integration of nonnegative functions implies that $\phi = \psi$ $g_*\mu$-a.e.

□

Exercise 7.6.1.17. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $(Y, \mathcal{B})$ a measurable space and $g : X \to Y$. Suppose that for each $y \in Y$, $\{y\} \in \mathcal{B}$, g is surjective and g is $(\mathcal{A}, \mathcal{B})$-measurable. Then there exists $\kappa : Y \times \mathcal{A} \to [0, \infty)$ such that κ is a transition kernel from $(Y, \mathcal{B})$ to $(X, \mathcal{A})$.
Hint: For $A \in \mathcal{A}$, define $\phi_A \in L^0(Y, \mathcal{B})$ to be the $g_*\mu$-a.e. unique $\phi \in L^0(Y, \mathcal{B})$ such that $\phi \circ g = P_{g^*\mathcal{B}}\chi_A$. Define $\kappa' : Y \times \mathcal{A} \to [0, \infty)$ by $\kappa'(y, A) = \phi_A(y)$. For each $A \in \mathcal{A}$, define $\kappa(\cdot, A)$ by redefining $\kappa'(\cdot, A)$ on a $g_*\mu$-null set.

Proof.

- Since $\chi_\varnothing = 0$, $P_{g^*\mathcal{B}}\chi_\varnothing = 0$ μ-a.e. Therefore

$$\begin{aligned} 0 \circ g &= 0 \\ &= P_{g^*\mathcal{B}}\chi_\varnothing \ \mu\text{-a.e.} \end{aligned}$$

Uniqueness of $\phi_\varnothing$ implies that $\phi_\varnothing = 0$ $g_*\mu$-a.e. Thus there exists $N_1 \in \mathcal{B}$ such that $g_*\mu(N_1) = 0$ and for each $y \in N_1^c$,

$$\begin{aligned} \kappa'(y, \varnothing) &= \phi_\varnothing(y) \\ &= 0 \end{aligned}$$

- Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$. Suppose that $(A_j)_{j\in\mathbb{N}}$ is disjoint. Since μ is finite, $\mu\left(\bigcup_{j\in\mathbb{N}} A_j\right) < \infty$. A previous exercise implies that

1. $\chi_{\bigcup_{j\in\mathbb{N}} A_j} \in L^1(X, \mathcal{A}, \mu)$
2. $P^\mu_\mathcal{B}\chi_{\bigcup_{j\in\mathbb{N}} A_j} = \sum_{j\in\mathbb{N}} P^\mu_\mathcal{B}\chi_{A_j}$

Therefore

$$\begin{aligned} \phi_{\bigcup_{j\in\mathbb{N}} A_j} \circ g &= P^\mu_\mathcal{B}\chi_{\bigcup_{j\in\mathbb{N}} A_j} \\ &= \sum_{j\in\mathbb{N}} P^\mu_\mathcal{B}\chi_{A_j} \\ &= \sum_{j\in\mathbb{N}} \phi_{A_j} \circ g \ \mu\text{-a.e.} \end{aligned}$$

Uniqueness of $\phi_{\bigcup_{j\in\mathbb{N}} A_j}$ implies that $\phi_{\bigcup_{j\in\mathbb{N}} A_j} = \sum_{j\in\mathbb{N}} \phi_{A_j}$ $g_*\mu$-a.e. $\phi_\varnothing$ implies that $\phi_\varnothing = 0$ $g_*\mu$-a.e. Thus there exists $N_2 \in \mathcal{B}$ such that $g_*\mu(N_2) = 0$ and for each $y \in N_2^c$,

$$\begin{aligned} \kappa'\left(y, \bigcup_{j\in\mathbb{N}} A_j\right) &= \phi_{\bigcup_{j\in\mathbb{N}} A_j}(y) \\ &= \sum_{j\in\mathbb{N}} \phi_{A_j}(y) \\ &= \sum_{j\in\mathbb{N}} \mu_y(A_j) \\ &= \sum_{j\in\mathbb{N}} \kappa'(y, A_j) \end{aligned}$$

Set $N = N_1 \cup N_2$. Then $g_*\mu(N) = 0$ and for each $y \in N^c$, $\kappa'(y,\cdot) : \mathcal{A} \to [0,\infty)$ is a measure on $(X, \mathcal{A})$. Choose $x \in X$. Define $\kappa : Y \times \mathcal{A} \to [0,\infty)$ by $\kappa(y, A) = \chi_N(y)\delta_x(A) + \chi_{N^c}(y)\kappa'(y, A)$.

1. Let $A \in \mathcal{A}$. Then

$$\begin{aligned}\kappa(\cdot, A) &= \chi_N(\cdot)\delta_x(A) + \chi_{N^c}(\cdot)\kappa'(\cdot, A)\\ &= \chi_N(\cdot)\delta_x(A) + \chi_{N^c}(\cdot)\phi_A(\cdot)\end{aligned}$$

   Hence for each $A \in \mathcal{A}$, $\kappa(\cdot, A)$ is $\mathcal{B}$-measurable.

2. Let $y \in Y$. Then

$$\kappa(y,\cdot) = \begin{cases}\delta_x(\cdot) & y \in N\\ \kappa'(y,\cdot) & y \in N^c\end{cases}$$

   Hence for each $y \in Y$, $\kappa(y, \cdot)$ is a measure on $(X, \mathcal{A})$.

Thus κ is a transition kernel from $(Y, \mathcal{B}, g_*\mu)$ to $(X, \mathcal{A})$. □

Definition 7.6.1.18. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $(Y, \mathcal{B})$ a measurable space and $g : X \to Y$. Suppose that for each $y \in Y$, $\{y\} \in \mathcal{B}$, g is surjective and g is $(\mathcal{A}, \mathcal{B})$-measurable. For $A \in \mathcal{A}$, define $\phi_A \in L^0(Y, \mathcal{B})$ to be the $g_*\mu$-a.e. unique $\phi \in L^0(Y, \mathcal{B})$ such that $\phi \circ g = P_{g^*\mathcal{B}}\chi_A$. For $y \in Y$, we define the **conditional of** μ **on** y, denoted $\mu_y : \mathcal{A} \to [0,\infty)$, by $\mu_y(A) = \phi_A(y)$.

Exercise 7.6.1.19. Disintegration of Measure:
Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $(Y, \mathcal{B})$ a measurable space and $g : X \to Y$. Suppose that for each $y \in Y$, $\{y\} \in \mathcal{B}$, g is surjective and g is $(\mathcal{A}, \mathcal{B})$-measurable. Then there exists a collection of measures $(\mu_y)_{y \in Y}$ such that

1. for each $A \in \mathcal{A}$,

$$\mu(A) = \int \mu_y(A)\, dg_*\mu(y)$$

2. for each $f \in L^1(X, \mathcal{A}, \mu)$,

$$\int f\, d\mu = \int \left[\int f\, d\mu_y(x)\right] dg_*\mu(y)$$

Chapter 8

L^p Spaces

8.1 Introduction

Definition 8.1.0.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in (0, \infty]$. Define $\|\cdot\|_p : L^0(X, \mathcal{A}, \mu) \to [0, \infty]$ by

$$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}} \qquad (p < \infty)$$

and

$$\|f\|_\infty = \inf\left\{\lambda > 0 : \mu\big(\{x \in X : \lambda < |f(x)|\}\big) = 0\right\}$$

We define

$$L^p(X, \mathcal{A}, \mu) = \{f \in L^0(X, \mathcal{A}, \mu) : \|f\|_p < \infty\}$$

Exercise 8.1.0.2. Let $(X, \mathcal{A}, \mu)$ be a measure space, $p \in (0, \infty]$ and $f, g \in L^p(X, \mathcal{A}, \mu)$. If $|f| \leq |g|$ μ-a.e., then $\|f\|_p \leq \|g\|_p$.

Proof. Suppose that $|f| \leq |g|$ μ-a.e. Then $|f|^p \leq |g|^p$ μ-a.e. This implies that

$$\int |f|^p \, d\mu \leq \int |g|^p \, d\mu$$

Hence $\|f\|_p \leq \|g\|_p$. □

Theorem 8.1.0.3. Hölder's Inequality: Let $(X, \mathcal{A}, \mu)$ be a measure space, $p, q \in [1, \infty)$ and $f, g \in L^0$. Suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Exercise 8.1.0.4. Minkowski Inequality: Let $(X, \mathcal{A}, \mu)$ be a measure space, $p \in [1, \infty)$ and $f, g \in L^p$. Then $f + g \in L^p$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. Define $\phi : \mathbb{R} \to [0, \infty)$ by $\phi(x) = |x|^p$. Then ϕ is convex because it is the composition of an increasing convex function with a convex function. By Jensen's inequality, we have that

$$\phi\left(\frac{1}{2}[f + g]\right) \leq \frac{1}{2}[\phi(f) + \phi(g)]$$

This implies that

$$\frac{1}{2^p}|f + g|^p \leq \frac{1}{2}\left(|f|^p + |g|^p\right)$$

Hence

$$\begin{aligned}\int |f+g|^p\,d\mu &\leq 2^{p-1}\int |f|^p + |g|^p\,d\mu\\ &= 2^{p-1}\bigg(\int |f|^p\,d\mu + \int |g|^p\,d\mu\bigg)\\ &= 2^{p-1}\bigg(\|f\|_p^p + \|g\|_p^p\bigg)\\ &< \infty\end{aligned}$$

So $f+g \in L^p$. Now, it is not hard to see that $|f+g|^p \leq \big(|f|+|g|\big)|f+g|^{p-1}$. Let q be the conjugate of p, so that $\frac{1}{p}+\frac{1}{q}=1$. Then $q(p-1)=p$. We use Hölder's inequality to show that

$$\begin{aligned}\|f+g\|_p^p &= \int |f+g|^p\,d\mu\\ &\leq \int |f||f+g|^{p-1}\,d\mu + \int |g||f+g|^{p-1}\,d\mu\\ &\leq \|f\|_p\bigg(\int |f+g|^{(p-1)q}\,d\mu\bigg)^{\frac{1}{q}} + \|g\|_p\bigg(\int |f+g|^{(p-1)q}\,d\mu\bigg)^{\frac{1}{q}}\\ &= \|f\|_p\bigg(\int |f+g|^p\,d\mu\bigg)^{\frac{1}{q}} + \|g\|_p\bigg(\int |f+g|^p\,d\mu\bigg)^{\frac{1}{q}}\\ &= (\|f\|_p + \|g\|_p)\bigg(\int |f+g|^p\,d\mu\bigg)^{\frac{1}{q}}\\ &= (\|f\|_p + \|g\|_p)\|f+g\|_p^{p/q}\end{aligned}$$

Since $\|f+g\|_p < \infty$, we see that

$$\begin{aligned}\|f\|_p + \|g\|_p &\geq \|f+g\|_p^{p-p/q}\\ &= \|f+g\|_p^{p(1-1/q)}\\ &= \|f+g\|_p^{p/p}\\ &= \|f+g\|_p\end{aligned}$$

□

Exercise 8.1.0.5. Let $(X,\mathcal{A},\mu)$ be a measure space, $p,q \in (0,\infty]$. Suppose that $\mu(X)<\infty$ and $p<q$. Then $L^q \subset L^p$. In particular, if $\mu(X)=1$, then for each $f \in L^q$, $\|f\|_p \leq \|f\|_q$.

Proof. Suppose that $q=\infty$. Let $f \in L^q$. Then

$$\begin{aligned}\|f\|_p &= \bigg(\int |f|^p\,d\mu\bigg)^{\frac{1}{p}}\\ &\leq \bigg(\int \|f\|_\infty^p\,d\mu\bigg)^{\frac{1}{p}}\\ &= \|f\|_\infty \mu(X)^{\frac{1}{p}}\end{aligned}$$

If $q < \infty$, then $\frac{q}{p} > 1$ and the conjugate of $\frac{q}{p}$ is $\frac{1}{1-p/q}$. By Hölder's inequality, we have that

$$\begin{aligned}\|f\|_p^p &= \|f^p\|_1 \\ &\leq \|f^p\|_{\frac{q}{p}} \|1\|_{\frac{1}{1-p/q}} \\ &= \left(\int |f|^{\frac{pq}{p}} \, d\mu\right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}} \\ &= \left(\int |f|^q \, d\mu\right)^{\frac{p}{q}} \mu(X)^{1-\frac{p}{q}} \\ &= \|f\|_q^p \mu(X)^{1-\frac{p}{q}}\end{aligned}$$

Hence

$$\begin{aligned}\|f\|_p &\leq \|f\|_q \mu(X)^{\frac{1}{p}-\frac{1}{q}} \\ &< \infty\end{aligned}$$

□

Exercise 8.1.0.6. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and $K \in L^0(X \times Y)$. Suppose that there exists $C > 0$ such that for μ-a.e $x \in X$,

$$\int_Y |K(x,y)| d\nu(y) < C$$

and for ν-a.e $y \in Y$,

$$\int_X |K(x,y)| \, d\mu(x) < C$$

Let $f \in L^p(\nu)$.

1. Then for μ-a.e. $x \in X$,

$$\int_Y K(x,y) f(y) d\nu(y)$$

exists.
Hint: Note that $|K(x,y)f(y)| = (|K(x,y)|^{1/q})(|K(x,y)|^{1/p}|f(y)|)$

2. Define $Tf \in L^0(X)$ by

$$Tf(x) = \int_Y K(x,y) f(y) d\nu(y)$$

Then $Tf \in L^p(\mu)$ and $\|Tf\|_p \leq C\|f\|_p$.

Proof. Let $p, q \in (0, \infty)$ be conjugate.

1. Define $h \in L^0(X \times Y)$ by $h(x,y) = K(x,y)f(y)$. By assumption, there exists $N \in \mathcal{A}$ such that $\mu(N) = 0$ and

$$\left\{x \in X : \int_Y |K(x,y)| d\nu(y) < C\right\} \subset N^c$$

Let $x \in N^c$. Then Holder's inequality implies that

$$\begin{aligned}\int_Y |h(x,y)| d\nu(y) &= \int_Y (|K(x,y)|^{1/q})(|K(x,y)|^{1/p}|f(y)|) d\nu(y) \\ &\leq \left(\int_Y |K(x,y)| d\nu(y)\right)^{1/q} \left(\int_Y |K(x,y)||f(y)|^p d\nu(y)\right)^{1/p} \\ &\leq C^{1/q} \left(\int_Y |K(x,y)||f(y)|^p d\nu(y)\right)^{1/p}\end{aligned}$$

Tonelli's theorem implies that the map

$$x \mapsto \int_Y |h(x,y)| d\nu(y)$$

is measurable and that

$$\begin{aligned}\int_X \left[\int_Y |h(x,y)|d\nu(y)\right]^p d\mu(x) &\le C^{p/q}\int_X \left[\int_Y |K(x,y)||f(y)|^p d\nu(y)\right] d\mu(x)\\ &= C^{p/q}\int_Y \left[\int_X |K(x,y)||f(y)|^p\, d\mu(x)\right] d\nu(y)\\ &= C^{p/q}\int_Y \left[\int_X |K(x,y)|\, d\mu(x)\right] |f(y)|^p d\nu(y)\\ &\le C^{1+p/q}\int_Y |f(y)|^p d\nu(y)\\ &= C^{1+p/q}\|f\|_p^p\end{aligned}$$

So for μ-a.e. $x \in X$,

$$\int_Y |h(x,y)|d\nu(y) < \infty$$

which implies that for μ-a.e. $x \in X$, $h(x,\cdot) \in L^1(\nu)$. Therefore, for μ-a.e. $x \in X$,

$$\int_Y h(x,y)d\nu(y)$$

exists. The case is similar when $p \in \{1, \infty\}$.

2. Let $x \in X$. Then

$$|Tf(x)| \le \int_Y |K(x,y)f(y)|d\nu(y)$$

which implies that

$$|Tf(x)|^p \le \left(\int_Y |K(x,y)f(y)|d\nu(y)\right)^p$$

By part (1),

$$\int_X |Tf|^p\, d\mu \le C^{1+p/q}\|f\|_p^p$$

So $Tf \in L^p(\mu)$ and $\|Tf\|_p \le C\|f\|_p$. The case is similar when $p \in \{1, \infty\}$.

□

8.2 l^p

Note 8.2.0.1. make definition for $\#_a \in M_+(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ by $\#_a(A) := \sum_{j \in A} a_j$ in section on counting measure, $\#$

Definition 8.2.0.2. Let $a \in \mathbb{C}^{\mathbb{N}}$. We define define $l^{p,a} := L^p(\#_a)$.

Chapter 9

Borel Measures

9.1 Radon Measures

9.1.1 Introduction

Definition 9.1.1.1. Let X be a topological space, $\mu \in M_+(X)$.

- Let $E \in \mathcal{B}(X)$. We define
  - $V_O^\mu(E) = \{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$
  - $V_I^\mu(E) = \{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}$.

  Then μ is said to be

  1. **outer regular on** E if
  $$\mu(E) = \inf V_O^\mu(E)$$
  2. **inner regular on** E if
  $$\mu(E) = \sup V_I^\mu(E)$$
  3. **regular on** E if μ is inner regular on E and μ is outer regular on E

- Then μ is said to be

  1. **outer regular** if for each $E \in \mathcal{B}(X)$, μ is outer regular on E
  2. **inner regular** if for each $E \in \mathcal{B}(X)$, μ is inner regular on E
  3. **regular** if μ is inner regular and μ is outer regular

Exercise 9.1.1.2. Let X be a topological space and $\mu \in M_+(X)$. Suppose that μ is finite. If μ is inner regular, then μ is regular.

Proof. Suppose that μ is inner regular. Let $E \in \mathcal{B}(X)$. By definition, $\mu(E) \leq \inf V_O(E)$. Let $\epsilon > 0$. Since μ is finite, $\mu(E^c) - \epsilon < \mu(E^c)$. Inner regularity implies that there exists $K \subset E^c$ such that $\mu(K) > \mu(E^c) - \epsilon$. Define $U \in \mathcal{B}(X)$ by $U := K^c$. Then U is open and $E \subset U$. By construction,

$$\begin{aligned}
\mu(U) - \mu(E) &= [\mu(X) - \mu(U^c)] - [\mu(X) - \mu(E^c)] \\
&= \mu(E^c) - \mu(U^c) \\
&= \mu(E^c) - \mu(K) \\
&< \epsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
\inf V_O(E) &\leq \mu(U) \\
&< \mu(E) + \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\inf V_O(E) \leq \mu(E)$. Therefore $\mu(E) = \inf V_O(E)$ and μ is outer regular on E. Since $E \in \mathcal{B}(X)$ is arbitrary, we have that μ is outer regular. Hence μ is regular. $\square$

Definition 9.1.1.3. Let X be a topological space.

- Let $\mu \in M_+(X)$. Then μ is said to be **Radon** if for each $E \in \mathcal{B}(X)$,
  1. E is compact implies that $\mu(E) < \infty$
  2. μ is outer regular on E
  3. E is open implies that μ is inner regular on E
- We define
  - $\mathcal{M}_+(X) = \{\mu \in M_+(X) : \mu \text{ is Radon}\}$
  - $\mathcal{M}_1(X) = \{\mu \in \mathcal{M}_+(X) : \mu \text{ is a probability measure}\}$

Exercise 9.1.1.4. Let X be a topological space and $\mu, \nu \in \mathcal{M}_+(X)$ and $\lambda \geq 0$. Then $\mu + \lambda\nu \in \mathcal{M}_+(X)$.

Proof. Exercise 3.7.0.2 implies that $\mu + \lambda\nu \in M_+(X)$.

1. Let $K \in \mathcal{B}(X)$. Suppose that K is compact. Since $\mu, \nu \in \mathcal{M}_+(X)$, $\mu(K), \nu(K) < \infty$. Hence

$$\begin{aligned}(\mu + \lambda\nu)(K) &= \mu(K) + \lambda\nu(K) \\ &< \infty\end{aligned}$$

2. Let $E \in \mathcal{B}(X)$. By definition, $(\mu + \lambda\nu)(E) \leq \inf V_O^{\mu+\lambda\nu}(E)$.
   - Suppose that $(\mu + \lambda\nu)(E) = \infty$. Since $(\mu + \lambda\nu)(E) = \mu(E) + \lambda\nu(E)$, we have that $\mu(E) = \infty$ or $\nu(E) = \infty$.
     - Suppose that $\mu(E) = \infty$. Then

$$\begin{aligned}(\mu + \lambda\nu)(E) &= \mu(E) + \lambda\nu(E) \\ &\geq \mu(E) \\ &= \infty \\ &\geq \inf V_O^{\mu+\lambda\nu}(E)\end{aligned}$$

       Therefore $(\mu + \lambda\nu)(E) = \inf V_O^{\mu+\lambda\nu}(E)$.
     - Similarly, if $\nu(E) = \infty$, then $(\mu + \lambda\nu)(E) = \inf V_O^{\mu+\lambda\nu}(E)$.
   - Suppose that $(\mu + \lambda\nu)(E) < \infty$. Let $\epsilon > 0$. Then

$$\begin{aligned}\mu(E), \nu(E) &\leq \mu(E) + \lambda\nu(E) \\ &= (\mu + \lambda\nu)(E) \\ &< \infty\end{aligned}$$

   Since $\mu, \nu \in \mathcal{M}_+(X)$, there exist $U_\mu, U_\nu \in \mathcal{B}(X)$ such that U_μ, U_ν are open, $E \subset U_\mu$, $E \subset U_\nu$, $\mu(U_\mu) < \mu(E) + \epsilon/2$ and $\nu(U_\mu) < \nu(E) + \epsilon/(2[\lambda + 1])$. Set $U := U_\mu \cap U_\nu$. Then U is open, $E \subset U$ and

$$\begin{aligned}(\mu + \lambda\nu)(U) &= \mu(U) + \lambda\nu(U) \\ &\leq \mu(U_\mu) + \lambda\nu(U_\nu) \\ &< \left[\mu(E) + \frac{\epsilon}{2}\right] + \lambda\left[\nu(E) + \frac{\epsilon}{2(\lambda + 1)}\right] \\ &= \mu(E) + \frac{\epsilon}{2} + \lambda\nu(E) + \frac{\epsilon\lambda}{2(\lambda + 1)} \\ &< \mu(E) + \frac{\epsilon}{2} + \lambda\nu(E) + \frac{\epsilon}{2} \\ &= \mu(E) + \lambda\nu(E) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that

$$\begin{aligned}\inf V_O^{\mu+\lambda\nu}(E) &\leq (\mu+\lambda\nu)(U) \\ &\leq \mu(E) + \lambda\nu(E)\end{aligned}$$

Hence $(\mu+\lambda\nu)(E) = \inf V_O^{\mu+\lambda\nu}(E)$ and $\mu+\lambda\nu$ is outer regular on E. Since $E \in \mathcal{B}(X)$ is arbitrary, we have that for each $E \in \mathcal{B}(X)$, $\mu+\lambda\nu$ is outer regular on E.

3. Let $U \subset X$. Suppose that U is open. By definition, $V_I^{\mu+\lambda\nu}(U) \leq (\mu+\lambda\nu)(U)$. If $\lambda = 0$, then $\mu+\lambda\nu = \mu$ and since $\mu \in \mathcal{M}_+(X)$, we have that

$$\begin{aligned}\sup V_I^{\mu+\lambda\nu}(U) &= \sup V_I^{\mu}(U) \\ &= \mu(U) \\ &= (\mu+\lambda\nu)(U)\end{aligned}$$

Suppose that $\lambda \neq 0$.

- Suppose that $(\mu+\lambda\nu)(U) = \infty$. Since $(\mu+\lambda\nu)(U) = \mu(U) + \lambda\nu(U)$, we have that $\mu(U) = \infty$ or $\nu(U) = \infty$.
  - Suppose that $\mu(U) = \infty$. Let $M > 0$. Since $\mu \in \mathcal{M}_+(X)$, there exists $K \subset U$ such that K is compact and $\mu(K) > M$. Then

$$\begin{aligned}\sup V_I^{\mu+\lambda\nu}(U) &\geq (\mu+\lambda\nu)(U) \\ &\geq \mu(U) \\ &\geq \mu(K) \\ &> M\end{aligned}$$

    Since $M > 0$ is arbitrary, we have that

$$\begin{aligned}\sup V_I^{\mu+\lambda\nu}(U) &= \infty \\ &= (\mu+\lambda\nu)(U).\end{aligned}$$

  - Suppose that $\nu(U) = \infty$. Let $M > 0$. Since $\nu \in \mathcal{M}_+(X)$, there exists $K \subset U$ such that K is compact and $\nu(K) > M/\lambda$. Then

$$\begin{aligned}\sup V_I^{\mu+\lambda\nu}(U) &\geq (\mu+\lambda\nu)(U) \\ &\geq \lambda\nu(U) \\ &\geq \lambda\nu(K) \\ &> M\end{aligned}$$

    Since $M > 0$ is arbitrary, we have that

$$\begin{aligned}\sup V_I^{\mu+\lambda\nu}(U) &= \infty \\ &= (\mu+\lambda\nu)(U).\end{aligned}$$

- Suppose that $(\mu+\lambda\nu)(U) < \infty$. Let $\epsilon > 0$. Then $\mu(U) - \epsilon/2 < \mu(U)$ and $\nu(U) - \epsilon/(2\lambda) < \nu(U)$. Since $\mu, \nu \in \mathcal{M}_+(X)$, there exist $K_\mu, K_\nu \subset U$ such that K_μ, K_ν are compact and $\mu(K_\mu) > \mu(U) - \epsilon/2$ and $\nu(K_\nu) > \nu(U) - \epsilon/(2\lambda)$. Set $K := K_\mu \cup K_\nu$. Then K is compact, $K \subset U$ and

$$\begin{aligned}\sup V_I^{\mu+\lambda\nu}(U) &= (\mu+\lambda\nu)(K) \\ &= \mu(K) + \lambda\nu(K) \\ &\geq \mu(K_\mu) + \lambda\nu(K_\nu) \\ &> \left[\mu(U) - \frac{\epsilon}{2}\right] + \lambda\left[\nu(U) - \frac{\epsilon}{2\lambda}\right] \\ &= \left[\mu(U) - \frac{\epsilon}{2}\right] + \left[\lambda\nu(U) - \frac{\epsilon}{2}\right] \\ &= \mu(U) + \lambda\nu(U) + \epsilon \\ &= (\mu+\lambda\nu)(U) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\sup V_I^{\mu+\lambda\nu}(U) \geq (\mu+\lambda\nu)(U)$. Hence $\sup V_I^{\mu+\lambda\nu}(U) = (\mu+\lambda\nu)(U)$ and $\mu+\lambda\nu$ is inner regular on U. Since $U \subset X$ with U open is arbitrary, we have that for each $U \subset X$, if U is open, then $\mu+\lambda\nu$ is inner regular on U.

Hence $\mu+\lambda\nu$ is Radon. Thus $\mu+\lambda\nu \in \mathcal{M}_+(X)$. □

Definition 9.1.1.5. Let X be a topological space and $\mu \in \mathcal{M}_+(X)$. We define

- the **μ-null open sets**, denoted $\mathcal{N}_\mu \subset \mathcal{B}(X)$, by
$$\mathcal{N}_\mu := \{U \subset X : U \text{ is open and } \mu(U) = 0\}$$
- the **null set of** μ, denoted N_μ, by
$$N_\mu := \bigcup_{U \in \mathcal{N}_\mu} U$$
- the **support of** μ, denoted $\operatorname{supp}\mu$, by
$$\operatorname{supp}\mu := N_\mu^c$$

Exercise 9.1.1.6. Let X be a topological space and $\mu \in \mathcal{M}_+(X)$. Then

1. N_μ is open
2. $\operatorname{supp}\mu$ is closed
3. $\mu(N_\mu) = 0$

Hint: use inner regularity and compactness

Proof.

1. Since for each $U \in \mathcal{N}_\mu$, U is open and
$$N_\mu = \bigcup_{U \in \mathcal{N}_\mu} U$$
we have that N_μ is open.
2. Since N_μ is open and $\operatorname{supp}\mu = N_\mu^c$, we have that $\operatorname{supp}\mu$ is closed.
3. Let $K \subset N_\mu$. Suppose that K is compact. Since $\mathcal{N}_\mu$ is an open cover for K, there exist $U_1, \ldots, U_n \in \mathcal{N}_\mu$ such that
$$K \subset \bigcup_{j=1}^n U_j$$
This implies that
$$\begin{aligned}\mu(K) &\leq \mu\bigg(\bigcup_{j=1}^n U_j\bigg) \\ &\leq \sum_{j=1}^n \mu(U_j) \\ &= 0\end{aligned}$$
Inner regularity implies that
$$\begin{aligned}\mu(N_\mu) &= \sup\{\mu(K) : K \subset N_\mu \text{ and } K \text{ is compact}\} \\ &= 0\end{aligned}$$

□

Exercise 9.1.1.7. Let X be a topological space and $\mu \in \mathcal{M}_+(X)$. Let $x \in X$. Then $x \in \operatorname{supp} \mu$ iff for each $U \in \mathcal{N}(x)$, $\mu(U) > 0$.

Proof.

- **($\implies$):**
  Suppose that $x \in \operatorname{supp} \mu$. Let $U \in \mathcal{N}(x)$. For the sake of contradiction, suppose that $\mu(U) = 0$. Then $\mu(\operatorname{Int} U) = 0$ and thus $\operatorname{Int} U \in \mathcal{N}_\mu$. Therefore

$$\begin{aligned} x &\in \operatorname{Int} U \\ &\subset \bigcup_{V \in \mathcal{N}_\mu} V \\ &= N_\mu \\ &= (\operatorname{supp} \mu)^c \end{aligned}$$

  which is a contradiction. Hence $\mu(U) > 0$.

- **($\impliedby$):**
  Suppose that for each $U \in \mathcal{N}(x)$, $\mu(U) > 0$. For the sake of contradiction, suppose that $x \notin \operatorname{supp} \mu$. Then $x \in N_\mu$. Thus there exists $U \in \mathcal{N}_\mu$ such that $x \in U$. By definition, $U \in \mathcal{N}(x)$ and $\mu(U) = 0$. This is a contradiction. Hence $x \in \operatorname{supp} \mu$.

$\square$

Exercise 9.1.1.8. Let X be a topological space, $\mu \in \mathcal{M}_+(X)$ and $E \in \mathcal{B}(X)$. If $\mu(E) < \infty$, then for each $\epsilon > 0$,

1. there exists $U \in \mathcal{B}(X)$ such that U is open, $E \subset U$ and $\mu(U \setminus E) < \epsilon/4$
2. there exists $K \in \mathcal{B}(X)$ such that K is compact, $K \subset U$ and $\mu(U) - \epsilon/2 < \mu(K)$
3. there exists $V \in \mathcal{B}(X)$ such that V is open, $U \setminus E \subset V$ and $\mu(V) < \epsilon/2$

Proof. Suppose that $\mu(E) < \infty$. Let $\epsilon > 0$.

1. Outer regularity om E implies that there exists $U \in \mathcal{B}(X)$ such that U is open, $E \subset U$ and $\mu(U \setminus E) < \epsilon/4$.
2. Inner regularity on U implies that there exists $K \in \mathcal{B}(X)$ such that K is compact, $K \subset U$ and $\mu(U \setminus K) < \epsilon/2$. Therefore $\mu(U) - \epsilon/2 < \mu(K)$.
3. Outer regularity on $U \setminus E$ implies that there exists $V \in \mathcal{B}(X)$ such that V is open, $U \setminus E \subset V$ and

$$\begin{aligned} \mu(V) &< \mu(U \setminus E) + \epsilon/4 \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2} \end{aligned}$$

$\square$

Exercise 9.1.1.9. Let X be a topological space, $\mu \in \mathcal{M}_+(X)$ and $E \in \mathcal{B}(X)$. If $\mu(E) < \infty$, then μ is inner regular on E.
Hint: Define U, K_0 and V as in the previous exercise. Set $K = K_0 \setminus V$. Then K is compact, $K \subset E$ and $\mu(K) > \mu(E) - \epsilon$

Proof. Suppose that $\mu(E) < \infty$. Set $V(E) := \{\mu(K') : K' \subset E \text{ and } K' \text{ is compact}\}$. Clearly $\sup V(E) \leq \mu(E)$. Let $\epsilon > 0$. The previous exercise implies that there exist $U, K_0, V \in \mathcal{B}(X)$ such that

- U is open, $E \subset U$ and $\mu(U \setminus E) < \epsilon/4$
- K_0 is compact, $K_0 \subset U$ and $\mu(U) - \epsilon/2 < \mu(K_0)$
- V is open, $U \setminus E \subset V$ and $\mu(V) < \epsilon/2$

Set $K = K_0 \setminus V$. Then $K \subset K_0$ and since V is open, K is closed. Since K_0 is compact, K is compact. By construction,

$$\begin{aligned} U \cap E^c &= U \setminus E \\ &\subset V \end{aligned}$$

so that $V^c \subset U^c \cup E$. Since $K_0 \subset U$, we have that

$$\begin{aligned} K &= K_0 \setminus V \\ &= K_0 \cap V^c \\ &\subset K_0 \cap (U^c \cup E) \\ &= (K_0 \cap U^c) \cup (K_0 \cap E) \\ &= \varnothing \cup (K_0 \cap E) \\ &= K_0 \cap E \\ &\subset E \end{aligned}$$

The previous exercise implies that

$$\begin{aligned} \sup V(E) &\geq \mu(K) \\ &= \mu(K_0 \setminus V) \\ &= \mu(K_0 \cap V^c) \\ &= \mu(K_0) - \mu(K_0 \cap V) \\ &> \mu(U) - \frac{\epsilon}{2} - \mu(V) \\ &> \mu(E) - \frac{2\epsilon}{2} \\ &= \mu(E) - \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\sup V(E) \geq \mu(E)$. Thus $\mu(E) = \sup V(E)$. Thus μ is inner regular on E. □

Exercise 9.1.1.10. Let X be a topological space, $\mu \in \mathcal{M}_+(X)$ and $E \in \mathcal{B}(X)$. If E is σ-finite with respect to μ, then μ is inner regular on E.
Hint: use the previous exercise

Proof. Suppose that E is σ-finite with respect to μ. Then $\mu|_E$ is σ-finite.

- Suppose that $\mu|_E(E) < \infty$. Then

$$\begin{aligned} \mu(E) &= \mu|_E(E) \\ &< \infty \end{aligned}$$

  and the previous exercise implies that μ is inner regular on E.

- Suppose that $\mu|_E(E) = \infty$. Since $\mu|_E$ is σ-finite, there exists $(E_j)_{j\in\mathbb{N}} \subset \mathcal{B}(E)$ such that $E = \bigcup\limits_{j\in N} E_j$, for each $j \in \mathbb{N}$, $E_j \subset E_{j+1}$, $\mu|_E(E_j) < \infty$ and $\mu|_E(E_j) \to \infty$. Since $\mathcal{B}(E) \subset \mathcal{B}(X)$, we have that $(E_j)_{j\in\mathbb{N}} \subset \mathcal{B}(X)$ and for each $j \in \mathbb{N}$, $\mu|_E(E_j) = \mu(E_j)$. Let $N \in \mathbb{N}$. Choose $J \in \mathbb{N}$ such that $\mu(E_J) > N$. The above argument implies that there exists $K \in \mathcal{B}(X)$ such that K is compact, $K \subset E_J \subset E$ and $\mu(K) > N$. So

$$\begin{aligned} \mu(E) &= \infty \\ &= \sup_{\substack{K \subset E \\ K \text{ is compact}}} \mu(K) \end{aligned}$$

  and μ is inner regular on E.

□

Exercise 9.1.1.11. Let X be a topological space and $\mu \in \mathcal{M}_+(X)$. If μ is σ-finite, then μ is regular.

Proof. Clear by previous exercise. □

Exercise 9.1.1.12. Let X be a topological space and $\mu \in \mathcal{M}_+(X)$. If X is σ-compact, then μ is σ-finite and μ is regular.

Proof. If X is σ-compact, then μ is σ-finite. Exercise 9.1.1.11 implies that μ is regular. □

9.1.2 Radon Measures on Subspaces

Exercise 9.1.2.1. Let X be a topological space, $\mu \in \mathcal{M}_+(X)$ and $E \in \mathcal{B}(X)$. Then $\mu_E \in \mathcal{M}_+(E)$.

Proof.

1. Let $K \subset X$. Suppose that K is compact. Then

$$\begin{aligned}\mu_E(K) &= \mu(K \cap E)\\ &\leq \mu(K)\\ &< \infty\end{aligned}$$

2. Let $B \in \mathcal{B}(X)$. Set

$$V(B) := \{\mu(U) : U \subset X, U \text{ is open in } X \text{ and } B \subset U\}$$

and

$$V_E(B) := \{\mu_E(U) : U \subset X, U \text{ is open in } X \text{ and } B \subset U\}$$

Let $a \in V_E(B)$. Then there exists $U \subset X$ such that U is open in X, $B \subset U$ and $a = \mu_E(U)$. Since $B \subset U$,

$$\begin{aligned}\mu_E(B) &\leq \mu_E(U)\\ &= a\end{aligned}$$

Since $a \in V_E(B)$ is arbitrary, we have that $\mu_E(B)$ is a lower bound for $V_E(B)$. Therefore $\mu_E(B) \leq \inf V_E(B)$.

- First, suppose that $\mu_E(B) = \infty$. Then

$$\begin{aligned}\infty &= \mu_E(B)\\ &\leq \inf V_E(B)\end{aligned}$$

Hence

$$\begin{aligned}\mu_E(B) &= \infty\\ &= \inf V_E(B)\end{aligned}$$

In particular, $\inf V_E(B) \leq \mu_E(B)$.

- Now, suppose that $\mu_E(B) < \infty$. Then $\mu(B \cap E) < \infty$. Let $\epsilon > 0$. Since μ is outer regular, there exists $U_0 \subset X$ such that U_0 is open in X, $B \cap E \subset U_0$ and $\mu(U_0) < \mu(B \cap E) + \epsilon$.

$$\begin{aligned}\mu_E(B) &= \mu|_E(B)\\ &< \infty\end{aligned}$$

3.

□

Exercise 9.1.2.2. Let X be a topological space, $\mu \in \mathcal{M}_+(X)$ and $E \in \mathcal{B}(X)$. If μ is regular, then $\mu|_E$ is regular.

Proof. Suppose that μ is regular.

1. Let $K \subset E$. Suppose that K is compact in E. Then K is compact in X. Since μ is Radon, we have that

$$\begin{aligned}\mu|_E(K) &= \mu(K) \\ &< \infty\end{aligned}$$

2. Let $B \in \mathcal{B}(E)$. Set

$$V(B) := \{\mu(U) : U \subset X, U \text{ is open in } X \text{ and } B \subset U\}$$

and

$$V_E(B) := \{\mu|_E(U) : U \subset E, U \text{ is open in } E \text{ and } B \subset U\}$$

Since $\mathcal{B}(E) = \mathcal{B}(X) \cap E$ and $E \in \mathcal{B}(X)$, there exists $A \in \mathcal{B}(X)$ such that

$$\begin{aligned}B &= A \cap E \\ &\in \mathcal{B}(X) \cap E \\ &\subset \mathcal{B}(X)\end{aligned}$$

Clearly $\mu|_E(B) \leq \inf V_E(B)$.

- First, suppose that $\mu|_E(B) = \infty$. Then clearly

$$\begin{aligned}\mu|_E(B) &= \infty \\ &= \inf V_E(B)\end{aligned}$$

- Now, suppose that $\mu|_E(B) < \infty$. Then

$$\begin{aligned}\mu(B) &= \mu|_E(B) \\ &< \infty\end{aligned}$$

Let $\epsilon > 0$. Since μ is Radon, there exists $U_0 \subset X$ such that U_0 is open in X, $B \subset U_0$ and $\mu(U_0) < \mu(B) + \epsilon$. Set $U = U_0 \cap E$. Then U is open in E, $B \subset U$ and

$$\begin{aligned}\mu|_E(U) &= \mu(U) \\ &\leq \mu(U_0) \\ &< \mu(B) + \epsilon \\ &= \mu|_E(B) + \epsilon\end{aligned}$$

Therefore

$$\begin{aligned}\inf V_E(B) &\leq \mu|_E(U) \\ &< \mu|_E(B) + \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\inf V_E(B) \leq \mu|_E(B)$. Hence $\mu|_E(B) = \inf V_E(B)$.

Thus $\mu|_E$ is outer regular on B.

3. Let $B \in \mathcal{B}(E)$. Set

$$V_E(B) := \{\mu|_E(K) : K \subset B \text{ and } K \text{ is compact}\}$$

Clearly $\sup V_E(U) \leq \mu|_E(B)$. Since $E \in \mathcal{B}(X)$,

$$\begin{aligned}B &\in \mathcal{B}(E) \\ &\subset \mathcal{B}(X)\end{aligned}$$

- Suppose that $\mu(B) = \infty$. Let $M > 0$. Inner regularity of μ implies that there exists $K \subset B$ such that K is compact and $\mu(K) > M$. Then $K \in \mathcal{B}(E)$ and

$$\begin{aligned} \mu|_E(K) &= \mu(K) \\ &> M \end{aligned}$$

  So $\sup V_E(B) = \infty$ and therefore

$$\begin{aligned} \mu|_E(B) &= \infty \\ &= \sup V_E(B) \end{aligned}$$

- Suppose that $\mu(B) < \infty$. Let $\epsilon > 0$. Inner regularity of μ implies that there exists $K \subset B$ such that K is compact and $\mu(K) > \mu(B) - \epsilon$. Then $K \in \mathcal{B}(E)$ and

$$\begin{aligned} \sup V_E(B) &\geq \mu|_E(K) \\ &= \mu(K) \\ &> \mu(B) - \epsilon \\ &= \mu|_E(B) - \epsilon \end{aligned}$$

  Since $\epsilon > 0$ is arbitrary, we have that $\mu|_E(B) \leq \sup V_E(B)$. Hence $\mu|_E(B) = \sup V_E(B)$ and $\mu|_E$ is inner regular on B. Since $B \in \mathcal{B}(E)$ is arbitrary, $\mu|_E$ is inner regular.

So $\mu|_E$ is regular. □

Exercise 9.1.2.3. Show that μ_E is Radon if μ is Radon

9.1.3 Lebesgue Decomposition of Radon Measures

- maybe try to rework the section with μ_E instead of $\mu|_E$, might be able to drop the σ-finite assumption in various places.

Note 9.1.3.1. We recall ν_μ from the section on outer measures.
FINISH!!!

Exercise 9.1.3.2. Let X be a topological space and $\mu, \nu \in M_+(X)$. If ν is Radon, then for each $A \subset X$,

$$\nu_\mu(A) = \inf\{\nu(U) : \ U \subset X \text{ is open and } U \ \mu^*\text{-covers } A\}$$

Proof. Suppose that ν is Radon. For each $A \subset X$, set

$$V(A) := \{\nu(E) : \ E \in \mathcal{B}(X) \text{ and } E \ \mu^*\text{-covers } A\}$$

and

$$V'(A) := \{\nu(U) : \ U \subset X \text{ is open and } U \ \mu^*\text{-covers } A\}$$

Let $A \subset X$. Since $V'(A) \subset V(A)$, we have that

$$\begin{aligned} \nu_\mu(A) &= \inf V(A) \\ &\leq \inf V'(A) \end{aligned}$$

- Suppose that $\nu_\mu(A) = \infty$. Then

$$\begin{aligned} \infty &= \nu_\mu(A) \\ &\leq \inf V'(A) \\ &\leq \infty \end{aligned}$$

  Hence

$$\begin{aligned} \nu_\mu(A) &= \infty \\ &= \inf V'(A) \end{aligned}$$

- Suppose that $\nu_\mu(A) < \infty$. Let $\epsilon > 0$. Then there exists $E \in \mathcal{B}(X)$ such that $\mu^*(A \setminus E) = 0$ and $\nu(E) < \nu_\mu(A) + \epsilon/2$. Since ν is outer regular on E, there exists $U \subset X$ such that U is open, $E \subset U$ and $\nu(U) < \nu(E) + \epsilon/2$. Since $E \subset U$, we have that $U^c \subset E^c$ which implies that

$$\begin{aligned} \mu^*(A \setminus U) &= \mu^*(A \cap U^c) \\ &\leq \mu^*(A \cap E^c) \\ &= \mu^*(A \setminus E) \\ &= 0 \end{aligned}$$

Hence U μ^*-covers A and therefore

$$\begin{aligned} \inf V'(A) &\leq \nu(U) \\ &< \nu(E) + \epsilon/2 \\ &< \nu_\mu(A) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\inf V'(A) \leq \nu_\mu(A)$. Thus $\nu_\mu(A) = \inf V'(A)$.

□

Exercise 9.1.3.3. Let X be a topological space and $\mu, \nu \in M^+(X)$. Suppose that ν is a Radon. Then $\mathcal{B}(X) \subset \mathcal{A}_{\nu_\mu}$.
Hint: similar to the proof of (ii) of the Riesz representation theorem in Folland

Proof. Let $U \subset X$. Suppose that U is open. Let $A \subset X$. Suppose that $\nu_\mu(A) < \infty$. Let $\epsilon > 0$. The previous exercise implies that there exists $E \subset X$ such that E is open, E μ^*-covers A and $\nu(E) < \nu_\mu(A) + \epsilon/2$. Then $E \cap U$ is open. Since ν is inner regular on $E \cap U$, there exists $K_1 \subset E \cap U$ such that K_1 is compact and $\nu(K_1) > \nu(E \cap U) - \epsilon/4$. Similarly, since $E \cap K_1^c$ is open, there exists $K_2 \subset E \cap K_1^c$ such that K_2 is compact and $\nu(K_2) > \nu(E \cap K_1^c) - \epsilon/4$. We note that $K_1 \cap K_2 = \varnothing$ and $K_1 \cup K_2 \subset E$. Since $K_1 \subset E \cap U$, we have that

$$\begin{aligned} E^c \cup U^c &= (E \cap U)^c \\ &\subset K_1^c \end{aligned}$$

Hence

$$\begin{aligned} E \cap U^c &= \varnothing \cup (E \cap U^c) \\ &= (E \cap E^c) \cup (E \cap U^c) \\ &= E \cap (E^c \cup U^c) \\ &\subset E \cap K_1^c \end{aligned}$$

Exercise 5.3.0.20 implies that $\nu_\mu|_{\mathcal{B}(X)} \leq \nu$. Since E μ^*-covers A, Exercise 5.3.0.12 that E μ^*-covers $A \cap U$ and E μ^*-covers $A \cap U^c$. Exercise 5.3.0.24 implies that $\nu_\mu(A \cap U) = \nu_\mu[(A \cap U) \cap E]$ and $\nu_\mu(A \cap U^c) = \nu_\mu[(A \cap U^c) \cap E]$. Therefore

$$\begin{aligned} \nu_\mu(A) + \epsilon/2 &> \nu(E) \\ &\geq \nu(K_1 \cup K_2) \\ &= \nu(K_1) + \nu(K_2) \\ &\geq \nu(E \cap U) + \nu(E \cap K_1^c) - \epsilon/2 \\ &\geq \nu(E \cap U) + \nu(E \cap U^c) - \epsilon/2 \\ &\geq \nu_\mu(E \cap U) + \nu_\mu(E \cap U^c) - \epsilon/2 \\ &\geq \nu_\mu[(A \cap U) \cap E] + \nu_\mu[(A \cap U^c) \cap E] - \epsilon/2 \\ &= \nu_\mu(A \cap U) + \nu_\mu(A \cap U^c) - \epsilon/2 \end{aligned}$$

Hence $\nu_\mu(A) + \epsilon \geq \nu_\mu(A \cap U) + \nu_\mu(A \cap U^c)$. Since $\epsilon > 0$ is arbitrary, we have that

$$\nu_\mu(A) \geq \nu_\mu(A \cap U) + \nu_\mu(A \cap U^c)$$

Since $A \subset X$ with $\nu_\mu(A) < \infty$ is arbitrary, Exercise 5.3.0.4 implies that $U \in \mathcal{A}_{\nu_\mu}$. Since $U \subset X$ with U is open is arbitrary, $\mathcal{B}(X) \subset \mathcal{A}_{\nu_\mu}$. □

Exercise 9.1.3.4. Let X be a topological space, $\mu, \nu \in M_+(X)$. Suppose that ν is Radon. Then $\nu_\mu = (\nu_\mu)_{\operatorname{supp}\nu}$.

Proof. Let $A \subset X$.

- Since $A \cap \operatorname{supp}\nu \subset A$, we have that

$$\begin{aligned}\nu_\mu(A) &\geq \nu_\mu(A \cap \operatorname{supp}\nu) \\ &= (\nu_\mu)_{\operatorname{supp}\nu}(A)\end{aligned}$$

- Exercise 9.1.1.6 implies that $\operatorname{supp}\nu \in \mathcal{B}(X)$. Exercise 9.1.3.3 then implies that $\operatorname{supp}\nu \in \mathcal{A}_{\nu_\mu}$. Exercise 5.3.0.20 implies that $\nu_\mu|_{\mathcal{B}(X)} \leq \nu$. Therefore

$$\begin{aligned}\nu_\mu(A) &= \nu_\mu[A \cap \operatorname{supp}\nu] + \nu_\mu[A \cap (\operatorname{supp}\nu)^c] \\ &\leq \nu_\mu[A \cap \operatorname{supp}\nu] + \nu_\mu[(\operatorname{supp}\nu)^c] \\ &\leq \nu_\mu[A \cap \operatorname{supp}\nu] + \nu[(\operatorname{supp}\nu)^c] \\ &= \nu_\mu[A \cap \operatorname{supp}\nu] \\ &= (\nu_\mu)_{\operatorname{supp}\nu}(A)\end{aligned}$$

Thus $\nu_\mu(A) = (\nu_\mu)_{\operatorname{supp}\nu}(A)$. Since $A \subset X$ is arbitrary, we have that $\nu_\mu = (\nu_\mu)_{\operatorname{supp}\nu}$. □

Definition 9.1.3.5. Let X be a topological space and $\mu, \nu \in M_+(X)$. Suppose that ν is a Radon. We define $\nu_\mu^\perp \in M_+(X)$ by $\nu_\mu^\perp := \nu - \nu_\mu$.

Exercise 9.1.3.6. Let X be a topological space and $\mu, \nu \in M_+(X)$. Suppose that μ, ν are finite. If ν is Radon, then $\nu_\mu^\perp \perp \mu$.
Hint: consider lemma 3.7 in Folland

Proof. Suppose that ν is Radon. Since $\nu_\mu \leq \nu$, ν_μ is finite. For the sake of contradiction, suppose that $\nu_\mu^\perp \not\perp \mu$. Then Lemma 3.7 in Folland implies that there exists $\epsilon > 0$ and $E \in \mathcal{B}(X)$ such that $\mu(E) > 0$ and E is $(\nu_\mu^\perp - \epsilon\mu)$-positive.
Set $\delta = \epsilon\mu(E)/2$. Exercise 5.3.0.21 implies that there exists $F \in \mathcal{B}(X)$ such that $F \subset E$, F μ^*-covers E and $\nu(F) < \nu_\mu(E) + \delta$. Since $\mu^*|_{\mathcal{B}(X)} = \mu$, we have that

$$\begin{aligned}\mu(E) &= \mu(E \cap F) + \mu(E \cap F^c) \\ &= \mu(F) + \mu(E \setminus F) \\ &= \mu(F)\end{aligned}$$

Since F μ^*-covers E, Exercise 5.3.0.24 implies that

$$\begin{aligned}\nu_\mu(E) &= \nu_\mu(E \cap F) \\ &= \nu_\mu(F)\end{aligned}$$

Therefore

$$\begin{aligned}\epsilon\mu(E) &= \epsilon\mu(F) \\ &\leq \nu_\mu^\perp(F) \\ &= \nu(F) - \nu_\mu(F) \\ &< \nu_\mu(E) - \nu_\mu(F) + \delta \\ &= \nu_\mu(F) - \nu_\mu(F) + \delta \\ &= \delta \\ &= \epsilon\mu(E)\end{aligned}$$

which is a contradiction. Hence $\nu_\mu^\perp \perp \mu$. □

Exercise 9.1.3.7. Let X be a topological space and $\mu, \nu \in M_+(X)$. Suppose that μ, ν are σ-finite and ν is Radon. Then

1. $\nu_\mu \ll \mu$
2. $\nu_\mu^\perp \perp \mu$
3. $\nu = \nu_\mu^\perp + \nu_\mu$ is the Lebesgue decomposition of ν with respect to μ

Proof.

1. Let $E \in \mathcal{B}(X)$. Suppose that $\mu(E) = 0$. Then

$$\begin{aligned} \mu(E \setminus \varnothing) &= \mu(E \cap \varnothing^c) \\ &= \mu(E \cap X) \\ &= \mu(E) \\ &= 0 \end{aligned}$$

   Hence $\varnothing$ μ^*-covers E. Set

$$V(E) = \{\nu(F) : F \in \mathcal{B}(X) \text{ and } F \ \mu^*\text{-covers E}\}$$

   By definition of ν_μ, we have that

$$\begin{aligned} \nu_\mu(E) &= \inf V(E) \\ &\leq \nu(\varnothing) \\ &= 0 \end{aligned}$$

   Since $E \in \mathcal{B}(X)$ with $\mu(E) = 0$ is arbitrary, $\nu_\mu \ll \mu$.

2. Since μ, ν are σ-finite, Exercise 9.1.1.11 implies that μ, ν are regular and Exercise 5.1.0.14 implies that there exists $(E_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ such that

   (a) $X = \bigcup\limits_{n\in\mathbb{N}} E_n$

   (b) for each $n \in \mathbb{N}$,
      - $E_n \subset E_{n+1}$
      - $\nu(E_n), \mu(E_n) < \infty$

   Let $n \in \mathbb{N}$. Since $\nu(E_n), \mu(E_n) < \infty$, $\nu|_{E_n}$ and $\mu|_{E_n}$ are finite measures. Since μ, ν are regular, Exercise 9.1.2.2 implies that $\nu|_{E_n}$ is regular and therefore Radon. Exercise 5.4.0.12 implies that

$$\begin{aligned} \nu_\mu^\perp|_{E_n} &= (\nu - \nu_\mu)|_{E_n} \\ &= \nu|_{E_n} - \nu_\mu|_{E_n} \\ &= \nu|_{E_n} - \nu|_{E_n\,\mu|_{E_n}} \\ &= \nu|_{E_n}{}^{\perp}_{\mu|_{E_n}} \end{aligned}$$

   The previous exercise implies that

$$\begin{aligned} \nu_\mu^\perp|_{E_n} &= \nu|_{E_n}{}^{\perp}_{\mu|_{E_n}} \\ &\perp \mu|_{E_n} \end{aligned}$$

   Since for each $n \in \mathbb{N}$, E_n is closed, $(E_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$. Exercise 7.1.0.2 implies that $\nu_\mu^\perp \perp \mu$.

3. Clear by theorem and uniqueness.
   FINISH!!! and reference exercises

$\square$

Exercise 9.1.3.8. Let X be a topological space, $\mu, \nu \in M_+(X)$, $T \in \mathcal{A}_{\nu_\mu}$ and $S \subset T$. Suppose that ν is Radon. If T is a $(\mathcal{B}(X), \mu^*)$-hull of S, then T is a $(\mathcal{B}(X), \nu_\mu)$-hull of S.

Proof. Suppose that T is a $(\mathcal{B}(X), \mu^*)$-hull of S and T is a $(\mathcal{B}(X), \nu^*)$-hull of S. We note that since ν is radon $\mathcal{B}(X) \subset \mathcal{A}_{\nu_\mu}$. Let $F \in \mathcal{B}(X)$. Define

$$V(S,F) = \{G \in \mathcal{B}(X) : G \subset S \cap F \text{ and } G\ \mu^*\text{-covers } S \cap F\}$$

and

$$V(T,F) = \{G \in \mathcal{B}(X) : G \subset T \cap F \text{ and } G\ \mu^*\text{-covers } T \cap F\}$$

Let $G \in V(S,F)$. Then $G \in \mathcal{B}(X)$, $G \subset S \cap F$ and $\mu^*[(S \cap F) \setminus G] = 0$. Since $S \subset T$,

$$\begin{aligned} G &\subset S \cap F \\ &\subset T \cap F \end{aligned}$$

Since $G, F \in \mathcal{B}(X)$, we have that $F \cap G^c \in \mathcal{B}(X)$. By assumption, $\mu^*[S \cap (F \cap G^c)] = \mu^*[T \cap (F \cap G^c)]$. Therefore

$$\begin{aligned} \mu^*[(T \cap F) \setminus G] &= \mu^*[T \cap (F \cap G^c)] \\ &= \mu^*[S \cap (F \cap G^c)] \\ &= \mu^*[(S \cap F) \setminus G] \\ &= 0 \end{aligned}$$

and G μ^*-covers $T \cap F$. Hence $G \in V(T,F)$. Since $G \in V(S,F)$ is arbitrary, we have that $V(S,F) \subset V(T,F)$. Exercise 5.3.0.21 implies that

$$\begin{aligned} \nu_\mu(T \cap F) &= \inf_{G \in V(T,F)} \nu(G) \\ &\leq \inf_{G \in V(S,F)} \nu(G) \\ &= \nu_\mu(S \cap F) \end{aligned}$$

Since $S \cap F \subset T \cap F$, $\nu_\mu(S \cap F) \leq \nu_\mu(T \cap F)$. Therefore $\nu_\mu(S \cap F) = \nu_\mu(T \cap F)$. Since $F \in \mathcal{B}(X)$ is arbitrary, we have that T is a $(\mathcal{B}(X), \nu_\mu)$-hull of S. $\square$

Exercise 9.1.3.9. Let X be a topological space and $\mu, \nu \in M_+(X)$ and $A \in \mathcal{B}(X)$. Suppose that μ, ν are σ-finite, ν is Radon and $\nu(A) < \infty$. Let $E \in \mathcal{B}(X)$. Suppose that $E \subset A$, E μ^*-covers A and $\nu_\mu(A) = \nu(E)$. Then for each $S \subset A$, if $S \in \mathcal{B}(X)$, then

1. $\nu_\mu(S) \leq \nu(S \cap E)$
2. $\nu_\mu(A \setminus S) \leq \nu(E \setminus S)$
3. $\nu_\mu(S) - \nu(E \cap S) = \nu_\mu(E \setminus S) - \nu_\mu(A \setminus S)$
4. $\nu_\mu(S) = \nu(E \cap S)$

Proof. Let $S \subset A$. Suppose that $S \in \mathcal{B}(X)$.

1. Exercise 9.1.3.3 implies that $\mathcal{B}(X) \subset \mathcal{A}_{\nu_\mu}$. Thus $A, E, S \in \mathcal{A}_{\nu_\mu}$. Exercise 9.1.3.7 implies that $\nu_\mu|_{\mathcal{B}(X)} \ll \mu$. Exercise 5.3.0.12 implies that E μ^*-covers S. Therefore $\nu_\mu|_{\mathcal{B}(X)}(S \setminus E) = 0$. Hence

$$\begin{aligned} \nu_\mu(S) &= \nu_\mu(S \cap E) + \nu_\mu(S \setminus E) \\ &= \nu_\mu(S \cap E) \\ &\leq \nu(S \cap E) \end{aligned}$$

2. We note that

$$\begin{aligned}(A \setminus S) \setminus (E \setminus S) &= (A \cap S^c) \cap (E \cap S^c)^c \\ &= (A \cap S^c) \cap (E^c \cup S) \\ &= [(A \cap S^c) \cap E^c] \cup [(A \cap S^c) \cap S] \\ &= [(A \cap S^c) \cap E^c] \cup \varnothing \\ &= [(A \cap S^c) \cap E^c] \\ &\subset A \cap E^c \\ &= A \setminus E\end{aligned}$$

so that

$$\begin{aligned}\mu^*[(A \setminus S) \setminus (E \setminus S)] &\leq \mu^*(A \setminus E) \\ &= 0\end{aligned}$$

So $E \setminus S$ μ^*-covers $A \setminus S$. By definition, we then have that $\nu_\mu(A \setminus S) \leq \nu(E \setminus S)$.

3. We note that

$$\begin{aligned}\nu_\mu(S) + \nu_\mu(A \setminus S) &= \nu_\mu(A \cap S) + \nu_\mu(A \setminus S) \\ &= \nu_\mu(A) \\ &= \nu(E) \\ &= \nu(E \cap S) + \nu(E \setminus S)\end{aligned}$$

Since $\nu(A) < \infty$ and $\nu_\mu|_{\mathcal{B}(X)} \leq \nu$, we have that $\nu(E \cap S), \nu_\mu(A) < \infty$. Therefore $\nu_\mu(S) - \nu(E \cap S) = \nu_\mu(E \setminus S) - \nu_\mu(A \setminus S)$.

4. The previous parts imply that

$$\begin{aligned}\nu_\mu(S) - \nu(E \cap S) &= \nu_\mu(E \setminus S) - \nu_\mu(A \setminus S) \\ &\geq 0\end{aligned}$$

Hence $\nu_\mu(S) \geq \nu(E \cap S)$. Since $\nu_\mu(S) \leq \nu(E \cap S)$, we have that $\nu_\mu(S) = \nu(E \cap S)$.

□

Exercise 9.1.3.10. Let X be a topological space and $\mu, \nu \in M_+(X)$ and $A \in \mathcal{B}(X)$. Suppose that μ, ν are σ-finite, ν is Radon and $\mu(A), \nu(A) < \infty$. Let $E \in \mathcal{B}(X)$. Suppose that $E \subset A$, E μ^*-covers A and $\nu_\mu(A) = \nu(E)$. Then for each $S \subset A$, there exists $T \in \mathcal{B}(X)$ such that

1. $S \subset T \subset A$, $\mu^*(S) = \mu(T)$ and $\nu^*(S) = \nu(T)$
2. T is a $(\mathcal{B}(X), \mu^*)$-hull, a $(\mathcal{B}(X), \nu^*)$-hull and a $(\mathcal{B}(X), \nu_\mu)$-hull of S.

Proof. Let $S \subset A$.

1. Exercise 5.3.0.17 implies that there exist $T_1, T_2 \in \mathcal{B}(X)$ such that $S \subset T_1$, $S \subset T_2$, $\mu^*(S) = \mu(T_1)$ and $\nu^*(S) = \nu(T_2)$. Define $T \in \mathcal{B}(X)$ by $T = T_1 \cap T_2 \cap A$. Since $S \subset A$, we have that $S \subset T \subset A$. Therefore

$$\begin{aligned}\mu^*(S) &\leq \mu^*(T) \\ &= \mu(T)\end{aligned}$$

and

$$\begin{aligned}\nu^*(S) &\leq \nu^*(T) \\ &= \nu(T)\end{aligned}$$

Since $T \subset T_1$ and $T \subset T_2$, we have that

$$\begin{aligned}\mu(T) &= \mu^*(T)\\ &\leq \mu^*(T_1)\\ &= \mu(T_1)\\ &= \mu^*(S)\end{aligned}$$

and

$$\begin{aligned}\nu(T) &= \nu^*(T)\\ &\leq \nu^*(T_2)\\ &= \nu(T_2)\\ &= \nu^*(S)\end{aligned}$$

Hence $\mu^*(S) = \mu(T)$ and $\nu^*(S) = \nu(T)$.

2. Since $T \subset A$ and $\mu(A), \nu(A) < \infty$, we have that $\mu(T), \nu(T) < \infty$. Since $\mu^*(S) = \mu(T)$ and $\nu^*(S) = \nu^*(T)$, Exercise 5.3.0.26 implies that T is a $(\mathcal{B}(X), \mu^*)$-hull and a $(\mathcal{B}(X), \nu^*)$-hull of S. Exercise 9.1.3.8 implies that T is a $(\mathcal{B}(X), \nu_\mu)$-hull of S.

□

Exercise 9.1.3.11. Let X be a topological space and $\mu, \nu \in M_+(X)$ and $A \in \mathcal{B}(X)$. Suppose that μ, ν are σ-finite, ν is Radon and $\mu(A), \nu(A) < \infty$. Then there exists $E \in \mathcal{B}(X)$ such that

1. $E \subset A$, E μ^*-covers A, $\nu_\mu(A) = \nu(E)$
2. for each $S \subset A$, $\nu_\mu(S) = \nu^*(E \cap S)$
3. $\nu_\mu|_A = (\nu|_A)^*_E$

Proof.

1. Since $A \in \mathcal{B}(X)$, Exercise 5.3.0.22 implies that there exists $E \in \mathcal{B}(X)$ such that $E \subset A$, E μ^*-covers A and $\nu_\mu(A) = \nu(E)$.
2. Let $S \subset A$. Exercise 9.1.3.10 implies that there exists $T \in \mathcal{B}(X)$ such that T is a $(\mathcal{B}(X), \nu^*)$-hull and a $(\mathcal{B}(X), \nu_\mu)$-hull of S. Since $E \in \mathcal{B}(X)$, $E \subset A$, E μ^*-covers A, $\nu_\mu(A) = \nu(E)$, $T \subset A$ and $T \in \mathcal{B}(X)$, Exercise 9.1.3.9 implies that $\nu_\mu(T) = \nu(E \cap T)$. Since T is a $(\mathcal{B}(X), \nu_\mu)$-hull and a $(\mathcal{B}(X), \nu^*)$-hull of S and $S \subset T$, we have that

$$\begin{aligned}\nu_\mu(S) &= \nu_\mu(S \cap X)\\ &= \nu_\mu(T \cap X)\\ &= \nu_\mu(T)\\ &= \nu(E \cap T)\\ &= \nu^*(E \cap T)\\ &= \nu^*(E \cap S)\end{aligned}$$

3. Clear from the previous part.

□

Exercise 9.1.3.12. Let X be a topological space and $\mu, \nu \in M_+(X)$. Suppose that μ, ν are σ-finite and ν is Radon.

1. For each $A \in \mathcal{B}(X)$, if $\mu(A), \nu(A) < \infty$, then for each $S \subset A$, $\nu_\mu(S) = 0$ implies that there exists $F \in \mathcal{B}(A)$ such that $S \subset F$ and $\nu_\mu(F) = 0$.
2. for each $S \subset X$, $\nu_\mu(S) = 0$ implies that there exists $F \in \mathcal{B}(X)$ such that $S \subset F$ and $\nu_\mu(F) = 0$.

Proof.

1. Let $A \in \mathcal{B}(X)$. Suppose that $\mu(A), \nu(A) < \infty$. Define $\alpha, \beta \in M_+(X)$ by $\beta := \nu|_A$. Exercise 9.1.3.11 implies that there exists $E \in \mathcal{B}(X)$ such that $\nu_\mu|_A = (\beta^*)_E$. Let $S \subset A$. Suppose that $\nu_\mu(S) = 0$. Exercise 5.3.0.17 implies that there exists $F' \in \mathcal{B}(A)$ such that $E \cap S \subset F'$ and $\beta^*(E \cap S) = \beta(F')$. Then $E \cap S \subset E \cap F'$. Since

$$\begin{aligned} \beta^*(E \cap S) &\leq \beta^*(E \cap F') \\ &= \beta(E \cap F') \\ &\leq \beta(F') \\ &= \beta^*(E \cap S) \end{aligned}$$

we have that

$$\begin{aligned} \nu^*(E \cap S) &= \beta^*(E \cap S) \\ &= \beta(E \cap F') \\ &= \nu(E \cap F') \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= \nu_\mu(S) \\ &= \nu^*(S \cap E) \\ &= \nu(F' \cap E) \\ &= \nu_\mu(F' \cap E) \end{aligned}$$

Define $F \in \mathcal{B}(A)$ by $F := (F' \cap E) \cup E^c$. Then

$$\begin{aligned} S &= (S \cap E) \cup (S \cap E^c) \\ &\subset (F' \cap E) \cup E^c \\ &= F \end{aligned}$$

and

$$\begin{aligned} \nu_\mu(F) &= \nu_\mu(E \cap F) + \nu_\mu(E^c) \\ &= \nu_\mu(E \cap F) + \nu(E^c \cap E) \\ &= 0 \end{aligned}$$

2. Since μ, ν are σ-finite, Exercise 5.1.0.15 implies that there exists $(A_n)_{n \in \mathbb{N}} \subset \mathcal{B}(X)$ such that

   (a) $X = \bigcup\limits_{n \in \mathbb{N}} A_n$

   (b) for each $m, n \in \mathbb{N}$,

   - $m \neq n$ implies that $A_n \cap A_m = \varnothing$
   - $\nu(A_n), \mu(A_n) < \infty$

   Let $S \subset X$. Suppose that $\nu_\mu(S) = 0$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned} \nu_\mu(S \cap A_n) &\leq \nu_\mu(S) \\ &= 0 \end{aligned}$$

   Part (1) implies that for each $n \in \mathbb{N}$, there exists $F_n \in \mathcal{B}(A_n)$ such that $S \cap A_n \subset F_n$ and $\nu_\mu(F_n) = 0$.

Define $F \in \mathcal{B}(X)$ by $F := \bigcup_{n\in\mathbb{N}} F_n$. Then

$$\begin{aligned} S &= S \cap X \\ &= S \cap \left(\bigcup_{n\in\mathbb{N}} A_n \right) \\ &= \bigcup_{n\in\mathbb{N}} (S \cap A_n) \\ &\subset \bigcup_{n\in\mathbb{N}} F_n \\ &= F \end{aligned}$$

and

$$\begin{aligned} \nu_\mu(F) &= \nu_\mu\left(\bigcup_{n\in\mathbb{N}} F_n \right) \\ &\leq \sum_{n\in\mathbb{N}} \nu_\mu(F_n) \\ &= 0 \end{aligned}$$

□

9.1.4 Complex Radon Measures

Definition 9.1.4.1. Let X be a topological space.

- Let $\mu \in M(X)$. Then μ is said to be **Radon** if $|\mu|$ is Radon.
- We define $\mathcal{M}(X) = \{\mu \in M(X) : \mu \text{ is Radon}\}$

Exercise 9.1.4.2. Let X be a topological space and $\nu \in M(X)$. Then ν is Radon iff $\operatorname{Re}\nu$ is Radon and $\operatorname{Im}\nu$ is Radon.

Proof. FINISH!!! □

Exercise 9.1.4.3. Let X be a topological space. Then $\mathcal{M}(X)$ is a vector space.

Proof. Let $\mu, \nu \in \mathcal{M}(X)$ and $\lambda \in \mathbb{C}$. Exercise 9.1.4.2 implies that Exercise 9.1.1.4 implies that

$$\begin{aligned} \operatorname{Re}(\mu + \lambda\nu) &= \operatorname{Re}\mu + \operatorname{Re}(\lambda)\operatorname{Re}\nu \\ &\in M_+(X) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}(\mu + \lambda\nu) &= \operatorname{Im}\mu + \operatorname{Im}(\lambda)\operatorname{Im}\nu \\ &\in M_+(X) \end{aligned}$$

Exercise 9.1.4.2 implies that

$$\begin{aligned} \mu + \lambda\nu &= \operatorname{Re}(\mu + \lambda\nu) + i\operatorname{Im}(\mu + \lambda\nu) \\ &\in \mathcal{M}(X) \end{aligned}$$

FINISH!!! □

Exercise 9.1.4.4. Let X be a topological space, $\mu \in \mathcal{M}_+(X)$ and $f \in L^1(X, \mathcal{B}(X), \mu)$. Define $\nu \in M(X)$ by

$$\nu(E) = \int_E f \, d\mu.$$

Then

1. $|\nu|$ is inner regular,
   Hint: consider Exercise 6.1.0.12 and Exercise 6.2.0.15

2. $|\nu|$ is regular,

3. ν is Radon.

Proof.

1. Let $E \in \mathcal{B}(X)$. Define $V_I(E) = \{|\nu|(K') : K' \subset E \text{ and } K' \text{ is compact}\}$. By construction, $\sup V_I(E) \leq |\nu|(E)$. Let $\epsilon > 0$. Exercise 6.2.0.15 implies that there exists $\delta > 0$ such that for each $A \in \mathcal{B}(X)$, $\mu(A) < \delta$ implies that

$$\begin{aligned}|\nu|(A) &= \int_A |f|\, d\mu \\ &< \epsilon.\end{aligned}$$

   - Suppose that $\mu(E) < \infty$. Then $\mu(E) - \delta < \mu(E)$. Since μ is inner regular on Borel sets, there exists $K \subset E$ such that K is compact and $\mu(K) > \mu(E) - \delta$. Since $E = K \cup (E \setminus K)$, we have that

$$\mu(E) = \mu(K) + \mu(E \setminus K)$$

     Since $\mu(E) < \infty$, we have that

$$\begin{aligned}\mu(E \setminus K) &= \mu(E) - \mu(K) \\ &< \delta\end{aligned}$$

     Hence $|\nu|(E \setminus K) < \epsilon$ and therefore

$$\begin{aligned}|\nu|(K) &= |\nu|(E) - |\nu|(E \setminus K) \\ &> |\nu|(E) - \epsilon\end{aligned}$$

     So there exists $K \subset E$ such that K is compact, $K \subset E$ and $|\nu|(K) > |\nu|(E) - \epsilon$.
   - Suppose that $\mu(E) = \infty$. Define $A \in \mathcal{B}(X)$ by $A := \{x \in X : |f(x)| > 0\}$. Exercise 6.1.0.12 implies that A is σ-finite with respect to μ. Then Exercise 5.4.0.8 implies that $E \cap A$ is σ-finite with respect to μ. Exercise 9.1.1.10 implies that μ is inner regular on $E \cap A$. Since $E \cap A \in \mathcal{B}(A)$, there exists $K \subset E \cap A$ such that K is compact and $\mu(K) > \mu(E \cap A) - \delta$. Then $K \subset E$ and $\mu[(E \cap A) \setminus K] < \delta$. Therefore $|\nu|[(E \cap A) \setminus K] < \epsilon$. Since $|\nu|(E \cap A) < \infty$ and $|\nu|(E \cap A) = |\nu|(K) + |\nu|[(E \cap A) \setminus K]$, we have that

$$\begin{aligned}|\nu|(E \cap A) - |\nu|(K) &= |\nu|[(E \cap A) \setminus K] \\ &< \epsilon.\end{aligned}$$

     Hence

$$\begin{aligned}|\nu|(E) &= |\nu|(E \cap A) + |\nu|(E \cap A^c) \\ &= |\nu|(E \cap A) \\ &< |\nu|(K) + \epsilon\end{aligned}$$

     so that

$$\begin{aligned}\sup V_I(E) &\geq |\nu|(K) \\ &> |\nu|(E) - \epsilon\end{aligned}$$

   Since $\epsilon > 0$ is arbitrary, we have that $\sup V_I(E) \geq |\nu|(E)$. Therefore $\sup V_I(E) = |\nu|(E)$ and $|\nu|$ is inner regular on E. Since $E \subset X$ with E open is arbitrary, we have that $|\nu|$ is inner regular.

2. Since $\nu \in M(X)$, $|\nu|$ is finite. Since $|\nu|$ is inner regular, Exercise 9.1.1.2 implies that $|\nu|$ is regular.

3. Since $|\nu|$ is regular, $|\nu|$ is Radon. By definition, ν is Radon.

$\square$

(maybe make an exercise in the radon-nikodym derivative section establishing equivalence of ν being absolutely continuous wrt μ and having the form of integration over a set wrt μ, then establish the equivalence with the ϵ, δ instead of citing Exercise 6.2.0.15.)

Exercise 9.1.4.5. content...

Definition 9.1.4.6. Let X be a topological space, $\mu \in M_+(X)$ and $\nu \in \mathcal{M}(X)$.

- If $\operatorname{Im} \nu = 0$, we define $\nu_\mu \in M(X)$ by
$$\nu_\mu = \nu_\mu^+ - \nu_\mu^-$$
- We define $\nu_\mu \in M(X)$ by
$$\nu_\mu = [\operatorname{Re} \nu]_\mu + i[\operatorname{Im} \nu]_\mu$$

9.2 Differentiation of Radon Measures on Metric Spaces

9.2.1 Covering Lemmas

Add Besicovitch Covering Lemma

Note 9.2.1.1. We make use of some results about maps $\mathcal{P} : \mathbb{N} \to \text{Part}(X)$ which are decreasing. See the section on ultrametric spaces in the analysis notes for details.

Definition 9.2.1.2. Let (X, d) be a metric space. We define the set of closed balls in X, denoted $\bar{\mathcal{B}}_d$, by

$$\bar{\mathcal{B}}_d = \{\bar{B}_d(x, r) : x \in X \text{ and } r > 0\}$$

Let $A \subset X$.

- Let $\mathcal{V} \subset \bar{\mathcal{B}}_d$. Then $\mathcal{V}$ is said to be a d-**centered covering** of A if for each $x \in A$, there exists $r > 0$ such that $\bar{B}_d(x, r) \in \mathcal{V}$. We define
$$\mathcal{C}_d(A) = \{\mathcal{V} \subset \bar{\mathcal{B}}_d : \mathcal{V} \text{ is a } d\text{-centered covering of } A\}$$
- Let $\mathcal{V} \in \mathcal{C}_d(A)$ and $x \in A$. Then $\mathcal{V}$ is said to be d-**fine** at x if
$$\inf\{r > 0 : \bar{B}_d(x, r) \in \mathcal{V}\} = 0$$
- Let $\mathcal{V} \in \mathcal{C}_d(A)$. Then $\mathcal{V}$ is said to be d-fine on A if for each $x \in A$, $\mathcal{V}$ is d-fine at x.

Definition 9.2.1.3. Let (X, d) be a metric space and $\mu \in M_+(X)$. Then μ is said to be d-**Vitali** if for each $A \subset X$ and $\mathcal{V} \in \mathcal{C}_d(A)$, $\mathcal{V}$ is d-fine on A implies that there exists $\mathcal{F} \subset \mathcal{V}$ such that

1. $\mathcal{F}$ is countable
2. $\mathcal{F}$ is disjoint
3. $\mathcal{F}$ μ^*-covers A

Exercise 9.2.1.4. Let X be a set and $d_1, d_2 : X \times X \to [0, \infty)$ metrics on X and $\mu \in M_+(X)$. If $d_1 \sim_{\textbf{Top}} d_2$, then μ is d_1-Vitali iff μ is d_2-Vitali.

Proof. Suppose that μ is d_1-Vitali. Let $A \subset X$ and $\mathcal{V} \subset \mathcal{C}_{d_2}(A)$. Suppose that $\mathcal{V}$ is d_2-fine on A. Since μ is d_1-Vitali, there exists $\mathcal{F} \subset$ FINISH!!! $\square$

Note 9.2.1.5. We recall the characterization of ultrametrics in terms of ultrametric-equivalent $\mathcal{P} : (0, \infty) \to \text{Part}(X)$ outlined in the analysis notes.

Exercise 9.2.1.6. Let (X, d) be a separable ultrametric space and $\mu \in M_+(X)$. If $(X, \mathcal{T}_d)$ is compact, then μ is d-Vitali.
Hint:

- Since $(X, \mathcal{T}_d)$ is compact, an exercise in the analysis notes section on ultrametric spaces implies that $d(X \times X) \setminus \{0\}$ is discrete.
- For each $x \in A$, there exists a maximal $\bar{\pi}^d_{a_n}(x) \in \mathcal{V}$.

Proof. Suppose that X is compact. Define $R \subset [0, \infty)$ by $R := d(X \times X) \setminus \{0\}$. An exercise in the analysis notes section on ultrametric spaces implies that R is discrete.

- Suppose that R is finte. Write $R \cup \{0\} = (r_j)_{j=1}^n$ with $r_1 > \cdots > r_n = 0$. FINISH!!!
- Suppose that R is infinte. Write $R = (r_n)_{n\in\mathbb{N}}$ with $(r_n)_{n\in\mathbb{N}}$ strictly decreasing. An exercise in the analysis notes section on ultrametric spaces implies that $r_n \to 0$. Define $\mathcal{P} : \mathbb{N} \to \text{Part}(X)$ such that $\mathcal{P}_n := X/{\simeq_{r_n}}$. An exercise in the analysis notes section on ultrametric spaces implies that $\mathcal{P}$ separates points, collects points and is decreasing. Let $A \subset X$ and $\mathcal{V} \in \mathcal{C}(A)$. Suppose that $\mathcal{V}$ is d-fine on A. Let $a \in A$. Since $\mathcal{V} \in \mathcal{C}(A)$, there exists $r > 0$ such that $\bar{B}(a, r) \in \mathcal{V}$.

- Suppose that $r < r_1$. Since $(r_n)_{n\in\mathbb{N}}$ is strictly decreasing and $r_n \to 0$, there exists $n_0 \in \mathbb{N}$ such that $r_{n_0+1} \leq r < r_{n_0}$. Then

$$\begin{aligned}\pi^{\mathcal{P}}_{n_0+1}(a) &= \bar{B}(a, r_{n_0+1})\\ &= \bar{B}(a, r)\\ &\in \mathcal{V}\end{aligned}$$

- Suppose that $r \geq r_1$. Then

$$\begin{aligned}\pi^{\mathcal{P}}_{1}(a) &= \bar{B}(a, r_1)\\ &= \bar{B}(a, r)\\ &\in \mathcal{V}\end{aligned}$$

Since $a \in A$ is arbitrary, we have that for each $a \in A$, $\{n \in \mathbb{N} : \pi^{\mathcal{P}}_n(a) \in \mathcal{V}\} \neq \varnothing$. Since X is separable, A is separable (needs an exercise, not trivial). Thus there exists $(a_n)_{n\in\mathbb{N}} \subset A$ such that $(a_n)_{n\in\mathbb{N}}$ is dense in A. For each $k \in \mathbb{N}$, set $n_k := \min\{n \in \mathbb{N} : \pi^{\mathcal{P}}_n(a_k) \in \mathcal{V}\}$. Define $\mathcal{F} \subset \mathcal{V}$ by $\mathcal{F} = \{\pi^{\mathcal{P}}_{n_k}(a_k) : k \in \mathbb{N}\}$.

1. By construction $\mathcal{F}$ is countable.
2. Let $k_1, k_2 \in \mathbb{N}$. Suppose that $\pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) \cap \pi^{\mathcal{P}}_{n_{k_2}}(a_{k_2}) \neq \varnothing$. Then there exists $x \in X$ such that $x \in \pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) \cap \pi^{\mathcal{P}}_{n_{k_2}}(a_{k_2})$.
   - For the sake of contradiction, suppose that $n_{k_1} < n_{k_2}$. Since $\mathcal{P}$ is decreasing, an exercise in the section on ultrametric spaces in the analysis notes implies that $\pi^{\mathcal{P}}_{n_{k_2}}(a_{k_2}) \subset \pi^{\mathcal{P}}_{n_{k_1}}(a_{k_2})$. Then

     $$\begin{aligned}x &\in \pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) \cap \pi^{\mathcal{P}}_{n_{k_2}}(a_{k_2})\\ &\subset \pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) \cap \pi^{\mathcal{P}}_{n_{k_1}}(a_{k_2})\end{aligned}$$

     Since $\mathcal{P}_{n_{k_1}} \in \text{Part}(X)$ and $\pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) \cap \pi^{\mathcal{P}}_{n_{k_1}}(a_{k_2}) \neq \varnothing$, we have that

     $$\begin{aligned}\pi^{\mathcal{P}}_{n_{k_1}}(a_{k_2}) &= \pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1})\\ &\in \mathcal{V}\end{aligned}$$

     Therefore

     $$\begin{aligned}\min\{n \in \mathbb{N} : \pi^{\mathcal{P}}_n(a_{k_2}) \in \mathcal{V}\} &\leq n_{k_1}\\ &< n_{k_2}\\ &= \min\{n \in \mathbb{N} : \pi^{\mathcal{P}}_n(a_{k_2}) \in \mathcal{V}\}\end{aligned}$$

     which is a contradiction. Hence $n_{k_1} \geq n_{k_2}$.
   - Similarly, $n_{k_1} \leq n_{k_2}$.

   Thus $n_{k_1} = n_{k_2}$. Since $k_1, k_2 \in \mathbb{N}$ are arbitrary, we have that for each $k_1, k_2 \in \mathbb{N}$, $\pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) \cap \pi^{\mathcal{P}}_{n_{k_2}}(a_{k_2}) \neq \varnothing$ implies that $\pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) = \pi^{\mathcal{P}}_{n_{k_2}}(a_{k_2})$. Equivalently, for each $k_1, k_2 \in \mathbb{N}$, $\pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) \neq \pi^{\mathcal{P}}_{n_{k_2}}(a_{k_2})$ implies that $\pi^{\mathcal{P}}_{n_{k_1}}(a_{k_1}) \cap \pi^{\mathcal{P}}_{n_{k_2}}(a_{k_2}) = \varnothing$. Hence $\mathcal{F}$ is disjoint.
3. Let $a \in A$. Since $\mathcal{V} \in \mathcal{C}(A)$, there exists $r > 0$ such that $\bar{B}(a, r) \in \mathcal{V}$. Therefore there exists $n \in \mathbb{N}$ such that $\bar{B}(a, r) = \pi^{\mathcal{P}}_n(a)$. An exercise in the section on ultrametric spaces in the analysis notes implies that $\bar{B}(a, r)$ is open in X. Then $\bar{B}(a, r) \cap A$ is open in A. Since $(a_n)_{n\in\mathbb{N}}$ is dense in A, there exists $k \in \mathbb{N}$ such that

   $$\begin{aligned}a_k &\in \bar{B}(a, r) \cap A\\ &\subset \bar{B}(a, r)\\ &= \pi^{\mathcal{P}}_n(a)\end{aligned}$$

Since $\mathcal{P}_n \in \text{Part}(X)$,

$$\begin{aligned} a &\in \pi_n^{\mathcal{P}}(a) \\ &= \pi_n^{\mathcal{P}}(a_k) \end{aligned}$$

Since $\mathcal{P}$ is decreasing, we have that

$$\begin{aligned} a &\in \pi_n^{\mathcal{P}}(a_k) \\ &\subset \pi_{n_k}^{\mathcal{P}}(a_k) \\ &\in \mathcal{F} \end{aligned}$$

Since $a \in A$ is arbitrary, we have that $A \subset \bigcup_{S \in \mathcal{F}} S$. Therefore

$$\begin{aligned} \mu\bigg[A \setminus \bigg(\bigcup_{S \in \mathcal{F}} S\bigg)\bigg] &= \mu\bigg[A \cap \bigg(\bigcup_{S \in \mathcal{F}} S\bigg)^c\bigg] \\ &= \mu(\varnothing) \\ &= 0 \end{aligned}$$

Hence $\mathcal{F}$ μ^*-covers A.

Thus μ is d-Vitali

□

9.2.2 Differentiation of Radon Measures

Definition 9.2.2.1. Let (X, d) be a metric space and $\mu \in \mathcal{M}_+(X)$. Then

- μ is said to be **d-finite** if for each $x \in X$ and $r > 0$, $\mu(\bar{B}(x,r)) < \infty$.
- μ is said to be **locally d-finite** if for each $x \in X$, there exists $r > 0$ such that $\mu(\bar{B}(x,r)) < \infty$.

We define $\mathcal{M}_+(X,d)$, $\mathcal{M}_+^{\text{loc}}(X,d) \subset \mathcal{M}_+(X)$ by

- $\mathcal{M}_+(X,d) := \{\mu \in \mathcal{M}_+(X) : \mu \text{ is } d\text{-finite}\}$
- $\mathcal{M}_+^{\text{loc}}(X,d) := \{\mu \in \mathcal{M}_+(X) : \mu \text{ is } d\text{-locally finite}\}$

Exercise 9.2.2.2. Let (X, d) be a metric space and $\mu \in \mathcal{M}_+(X)$. Then $\mu \in \mathcal{M}_+^{\text{loc}}(X,d)$ iff for each $x \in \operatorname{supp}\mu$, there exists $r > 0$ such that $\mu(\bar{B}(x,r)) < \infty$.

Proof.

- $(\implies)$:
  By definition, if $\mu \in \mathcal{M}_+^{\text{loc}}(X,d)$, then for each $x \in \operatorname{supp}\mu$, there exists $r > 0$ such that $\mu(\bar{B}(x,r)) < \infty$.
- $(\impliedby)$:
  Suppose that for each $x \in \operatorname{supp}\mu$, there exists $r > 0$ such that $\mu(\bar{B}(x,r)) < \infty$. Let $x \in X$.
  - Suppose that $x \in \operatorname{supp}\mu$. By assumption there exists $r > 0$ such that $\mu(\bar{B}(x,r)) < \infty$.
  - Suppose that $x \in (\operatorname{supp}\mu)^c$. Since $\mu \in \mathcal{M}_+(X)$, Definition 9.1.1.5 implies that there exists $U \subset X$ such that $x \in U$, U is open and $\mu(U) = 0$. Since U is open, there exists $s > 0$ such that $B(x,s) \subset U$. Set $r := s/2$. Then $\bar{B}(x,r) \subset U$ and therefore

$$\begin{aligned} \mu(\bar{B}(x,r)) &\le \mu(U) \\ &= 0 \\ &< \infty \end{aligned}$$

Since $x \in X$ is arbitrary, we have that for each $x \in X$, there exists $r > 0$ such that $\mu(\bar{B}(x,r)) < \infty$.

$\square$

Exercise 9.2.2.3. Let (X,d) be a metric space and $\mu \in \mathcal{M}_+(X,d)$. Then μ is σ-finite.

Proof. Since $X \neq \varnothing$, there exists $x \in X$. Define $E_n := \bar{B}(x,n)$. Then $X = \bigcup_{n \in \mathbb{N}} E_n$ and since $\mu \in \mathcal{M}_+(X,d)$, we have that for each $n \in \mathbb{N}$, $\mu(E_n) < \infty$. Hence μ is σ-finite. $\square$

Definition 9.2.2.4. Let (X,d) be a metric space and $\mu, \nu \in \mathcal{M}_+^{\text{loc}}(X,d)$.

- We define the **upper derivate of** ν **by** μ, denoted $\underline{D}_{\nu,\mu} : X \to [0,\infty]$, by

$$\underline{D}_\mu \nu(x) := \begin{cases} \liminf\limits_{r \to 0^+} \frac{\nu(\bar{B}(x,r))}{\mu(\bar{B}(x,r))}, & x \in \operatorname{supp} \mu \\ \infty, & x \in (\operatorname{supp} \mu)^c \end{cases}$$

- We define the **lower derivate of** ν **by** μ, denoted $\overline{D}_{\nu,\mu} : X \to [0,\infty]$, by

$$\overline{D}_\mu \nu(x) := \begin{cases} \limsup\limits_{r \to 0^+} \frac{\nu(\bar{B}(x,r))}{\mu(\bar{B}(x,r))}, & x \in \operatorname{supp} \mu \\ \infty, & x \in (\operatorname{supp} \mu)^c \end{cases}$$

Exercise 9.2.2.5. Let (X,d) be a metric space, $\alpha, \beta \in \mathcal{M}_+^{\text{loc}}(X,d)$ and $c \in (0,\infty)$. Then

1. for each $x \in \operatorname{supp} \alpha \cap \operatorname{supp} \beta$, $\underline{D}_\alpha \beta(x) < c$ iff $\overline{D}_\beta \alpha(x) > c^{-1}$
2. for each $x \in \operatorname{supp} \beta$, $\underline{D}_\alpha \beta(x) < c$ iff $\overline{D}_\beta \alpha(x) > c^{-1}$

Proof.

1. Let $x \in \operatorname{supp} \alpha \cap \operatorname{supp} \beta$. Since $\alpha, \beta \in \mathcal{M}_+^{\text{loc}}(X,d)$, there exists $r_0 > 0$ such that for each $s \in (0, r_0]$, $\alpha(\bar{B}(x,s)), \beta(\bar{B}(x,s)) < \infty$. Since $x \in \operatorname{supp} \alpha \cap \operatorname{supp} \beta$, Exercise 9.1.1.7 implies that for each $r > 0$, $\alpha(\bar{B}(x,r)), \beta(\bar{B}(x,r)) > 0$. Hence $(\alpha(\bar{B}(x,s)))_{s \leq r_0}$, $(\beta(\bar{B}(x,s)))_{s \leq r_0} \subset (0,\infty)$. Since $x \in \operatorname{supp} \alpha \cap \operatorname{supp} \beta$, the definition of $\underline{D}_\alpha \beta(x)$ and $\overline{D}_\beta \alpha(x)$ implies that
$$\overline{D}_\beta \alpha(x) = \limsup_{r \to 0^+} \frac{\alpha(\bar{B}(x,r))}{\beta(\bar{B}(x,r))}$$
and
$$\underline{D}_\alpha \beta(x) = \liminf_{r \to 0^+} \frac{\beta(\bar{B}(x,s))}{\alpha(\bar{B}(x,s))}$$
Since the map $(0,\infty) \to (0,\infty)$ given by $x \mapsto x^{-1}$ is order reversing, we have that
$$\begin{aligned} \overline{D}_\beta \alpha(x) > c^{-1} &\iff \limsup_{r \to 0^+} \frac{\alpha(\bar{B}(x,s))}{\beta(\bar{B}(x,s))} > c^{-1} \\ &\iff \left[\limsup_{r \to 0^+} \frac{\alpha(\bar{B}(x,s))}{\beta(\bar{B}(x,s))} \right]^{-1} < c \\ &\iff \liminf_{r \to 0^+} \left[\frac{\alpha(\bar{B}(x,s))}{\beta(\bar{B}(x,s))} \right]^{-1} < c \\ &\iff \liminf_{r \to 0^+} \frac{\beta(\bar{B}(x,s))}{\alpha(\bar{B}(x,s))} < c \\ &\iff \underline{D}_\alpha \beta(x) < c \end{aligned}$$

2. Let $x \in \operatorname{supp} \beta$.

- ($\implies$) :
  Suppose that $\underline{D}_\alpha\beta(x) < c$. Since $c < \infty$, we have that $\underline{D}_\alpha\beta(x) < \infty$. By definition of $\underline{D}_\alpha\beta(x)$, $x \in \operatorname{supp}\alpha$. Since $x \in \operatorname{supp}\alpha \cap \operatorname{supp}\beta$, part (1) implies that $\underline{D}_\alpha\beta(x) < c$ iff $\overline{D}_\beta\alpha(x) > c^{-1}$.
- ($\impliedby$) :
  Suppose that $\overline{D}_\beta\alpha(x) > c^{-1}$. Since $x \in \operatorname{supp}\beta$, Exercise 9.1.1.7 implies that for each $s > 0$, $\beta(\bar{B}(x,s)) > 0$ and by definition of $\overline{D}_\beta\alpha(x)$, we have that

$$\begin{aligned}\inf_{r>0}\left[\sup_{s\in(0,r]}\frac{\alpha(\bar{B}(x,s))}{\beta(\bar{B}(x,s))}\right] &= \limsup_{r\to 0^+}\frac{\alpha(\bar{B}(x,r))}{\beta(\bar{B}(x,r))}\\ &= \overline{D}_\beta\alpha(x)\\ &> c^{-1}\end{aligned}$$

  Let $r > 0$. Then there exists $s \in (0,r]$ such that $\alpha(\bar{B}(x,s)) > c^{-1}\beta(x,s)$. Therefore

$$\begin{aligned}\alpha(\bar{B}(x,r)) &\geq \alpha(\bar{B}(x,s))\\ &> c^{-1}\beta(x,s)\\ &> 0\end{aligned}$$

  Since $r > 0$ is arbitrary, Exercise 9.1.1.7 implies that $x \in \operatorname{supp}\alpha$. Then $x \in \operatorname{supp}\alpha \cap \operatorname{supp}\beta$ and part (1) implies that $\underline{D}_\alpha\beta(x) < c$ iff $\overline{D}_\beta\alpha(x) > c^{-1}$.

$\square$

Exercise 9.2.2.6. Let (X,d) be a metric space and $\alpha,\beta \in \mathcal{M}_+^{\mathrm{loc}}(X,d)$ and $\mu \in \mathcal{M}_+(X)$. Suppose that μ is d-Vitali. Let $c \in (0,\infty)$ and $A \subset X$.

1. If $A \subset \{x \in X : \underline{D}_\beta\alpha(x) < c\}$, then $\alpha_\mu(A) \leq c\beta_\mu(A)$.
2. If $A \subset \{x \in X : \overline{D}_\beta\alpha(x) > c\}$, then $\alpha_\mu(A) \geq c\beta_\mu(A)$.

Proof.

1. Suppose that $A \subset \{x \in X : \underline{D}_\beta\alpha(x) < c\}$.
   - If $\beta_\mu(A) = \infty$, then

$$\begin{aligned}\alpha_\mu(A) &\leq \infty\\ &= c\beta_\mu(A)\end{aligned}$$

   - Suppose that $\beta_\mu(A) < \infty$. Let $\epsilon > 0$. Since $\beta_\mu(A) < \infty$, $\beta_\mu(A) < \beta_\mu(A) + \epsilon/c$. Exercise 9.1.3.2 implies that there exists $U \subset X$ such that U is open, $\mu^*(A \setminus U) = 0$ and $\beta(U) < \beta_\mu(A) + \epsilon/c$. Define

$$\mathcal{V}_U := \{\bar{B}(x,r) \subset U : x \in A \cap U, r > 0 \text{ and } \alpha(\bar{B}(x,r)) < c\beta(\bar{B}(x,r))\}$$

     Let $x \in A \cap U$. Since $\underline{D}_\beta\alpha(x) < c$ and $c < \infty$, we have that $x \in \operatorname{supp}\beta$. Exercise 9.1.1.7 then implies that for each $r > 0$, $\beta(\bar{B}(x,r)) > 0$. Since

$$\begin{aligned}\sup_{r>0}\inf_{s\in(0,r]}\frac{\alpha(\bar{B}(x,s))}{\beta(\bar{B}(x,s))} &= \liminf_{r\to 0^+}\frac{\alpha(\bar{B}(x,s))}{\beta(\bar{B}(x,s))}\\ &= \underline{D}_\beta\alpha(x)\\ &< c,\end{aligned}$$

     we have that for each $r > 0$, there exists $s \in (0,r]$ such that $\alpha(\bar{B}(x,s)) < c\beta(\bar{B}(x,s))$. Since U is open, there exists $r_0 > 0$ such that $\bar{B}(x,r_0) \subset U$. Let $\delta > 0$. Choose $s_0 \in (0, r_0 \wedge \delta]$ such that $\alpha(\bar{B}(x,s_0)) < c\beta(\bar{B}(x,s_0))$. Then $\bar{B}(x,s_0) \in \mathcal{V}_U$ and

$$\begin{aligned}\inf\{r > 0 : \bar{B}(x,r) \in \mathcal{V}_U\} &\leq s_0\\ &\leq r_0 \wedge \delta\\ &\leq \delta\end{aligned}$$

Since $\delta > 0$ is arbitrary, we have that $\inf\{r > 0 : \bar{B}(x,r) \in \mathcal{V}_U\} = 0$. Since $x \in A \cap U$ is arbitrary, we have that $\mathcal{V}_U \in \mathcal{C}(A \cap U)$ and $\mathcal{V}_U$ is fine on $A \cap U$. Since μ is d-Vitali, there exists $\mathcal{F} \subset \mathcal{V}_U$ such that $\mathcal{F}$ is countable, $\mathcal{F}$ is disjoint and $\mathcal{F}$ μ^*-covers $A \cap U$. Since U μ^*-covers A, Exercise 5.3.0.24 implies that $\alpha_\mu(A) = \alpha_\mu(A \cap U)$. Since $\bigcup\limits_{B \in \mathcal{F}} B$ μ^*-covers $A \cap U$, Exercise 5.3.0.24 implies that $\alpha_\mu(A \cap U) = \alpha_\mu\Big((A \cap U) \cap \Big[\bigcup_{B \in \mathcal{F}} B\Big]\Big)$. Then

$$\begin{aligned}
\alpha_\mu(A) &= \alpha_\mu(A \cap U) \\
&= \alpha_\mu\bigg((A \cap U) \cap \bigg[\bigcup_{B \in \mathcal{F}} B\bigg]\bigg) \\
&= \alpha_\mu\bigg(A \cap \bigg[\bigcup_{B \in \mathcal{F}} B\bigg]\bigg) \\
&\leq \alpha_\mu\bigg(\bigcup_{B \in \mathcal{F}} B\bigg) \\
&\leq \alpha\bigg(\bigcup_{B \in \mathcal{F}} B\bigg) \\
&= \sum_{B \in \mathcal{F}} \alpha(B) \\
&\leq c \sum_{B \in \mathcal{F}} \beta(B) \\
&= c\beta\bigg(\bigcup_{B \in \mathcal{F}} B\bigg) \\
&\leq c\beta(U) \\
&\leq c\bigg(\beta_\mu(A) + \frac{\epsilon}{c}\bigg) \\
&= c\beta_\mu(A) + \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\alpha_\mu(A) \leq c\beta_\mu(A)$.

2. Suppose that $A \subset \{x \in X : \overline{D}_\beta \alpha(x) > c\}$. Exercise 9.2.2.5 implies that for each $x \in \operatorname{supp}\beta$, $\underline{D}_\alpha \beta(x) < c^{-1}$ iff $\overline{D}_\beta \alpha(x) > c$. Hence $A \cap \operatorname{supp}\beta \subset \{x \in X : \underline{D}_\alpha \beta(x) < c^{-1}\}$. Exercise 9.1.3.4 implies that $\beta_\mu = (\beta_\mu)_{\operatorname{supp}\beta}$. Part (1) then implies that

$$\begin{aligned}
\beta_\mu(A) &= (\beta_\mu)_{\operatorname{supp}\beta}(A) \\
&= \beta_\mu(A \cap \operatorname{supp}\beta) \\
&\leq c^{-1}\alpha_\mu(A \cap \operatorname{supp}\beta) \\
&\leq c^{-1}\alpha_\mu(A)
\end{aligned}$$

So $\alpha_\mu(A) \geq c\beta_\mu(A)$.

$\square$

Exercise 9.2.2.7. Let (X,d) be a metric space and $\alpha, \beta \in \mathcal{M}_+^{\text{loc}}(X,d)$ and $\mu \in \mathcal{M}_+(X)$. Suppose that μ is d-Vitali. Let $c \in (0,\infty)$ and $A \subset X$.

1. If $A \subset \{x \in X : \underline{D}_\beta \alpha(x) \leq c\}$, then $\alpha_\mu(A) \leq c\beta_\mu(A)$.

2. If $\beta_\mu(A) < \infty$ and $A \subset \{x \in X : \overline{D}_\beta \alpha(x) \geq c\}$, then $\alpha_\mu(A) \geq c\beta_\mu(A)$.

Proof.

1. Suppose that $A \subset \{x \in X : \underline{D}_\beta \alpha(x) \leq c\}$.

- If $\beta_\mu(A) = \infty$, then as before, $\alpha_\mu(A) \leq c\beta_\mu(A)$.
- Suppose that $\beta_\mu(A) < \infty$. Let $\epsilon > 0$. Set $\epsilon' = \epsilon/(\beta_\mu(A)+1)$. Then $\epsilon' > 0$ and therefore $A \subset \{x \in X : \underline{D}_\beta \alpha(x) < c + \epsilon'\}$. Part (1) then implies that

$$\begin{aligned} \alpha_\mu(A) &\leq (c+\epsilon')\beta_\mu(A) \\ &= c\beta_\mu(A) + \epsilon'\beta_\mu(A) \\ &= c\beta_\mu(A) + \epsilon\frac{\epsilon'\beta_\mu(A)}{\beta_\mu(A)+1} \\ &< c\beta_\mu(A) + \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\alpha_\mu(A) \leq c\beta_\mu(A)$.

2. Suppose that $\beta_\mu(A) < \infty$ and $A \subset \{x \in X : \overline{D}_\beta \alpha(x) \geq c\}$. Let $\epsilon > 0$. Set $\epsilon' = \epsilon/(\beta_\mu(A)+1)$. Then $\epsilon' > 0$ and therefore $A \subset \{x \in X : \overline{D}_\beta \alpha(x) > c - \epsilon'\}$. Part (3) then implies that

$$\begin{aligned} \alpha_\mu(A) &\geq (c-\epsilon')\beta_\mu(A) \\ &= c\beta_\mu(A) - \epsilon'\beta_\mu(A) \\ &= c\beta_\mu(A) - \epsilon\frac{\epsilon'\beta_\mu(A)}{\beta_\mu(A)+1} \\ &\geq c\beta_\mu(A) - \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\alpha_\mu(A) \geq c\beta_\mu(A)$.

$\square$

Exercise 9.2.2.8. Let (X,d) be a metric space and $\mu, \nu \in \mathcal{M}_+^{\mathrm{loc}}(X,d)$. Suppose that μ, ν are σ-finite. If μ is d-Vitali, then

1. $0 \leq \overline{D}_\mu \nu < \infty$ μ-a.e.
Hint: consider $I := \{x \in X : \overline{D}_\mu \nu(x) = \infty\}$, Exercise 9.2.2.6 and Exercise 9.1.3.12
2. $\underline{D}_\mu \nu = \overline{D}_\mu \nu$ μ-a.e.
Hint: for $a, b \in (0,\infty)$, consider $R_{a,b} := \{x \in X : \underline{D}_\mu(x) < a < b < \overline{D}_\mu \nu(x)\}$, Exercise 9.2.2.6 and Exercise 9.1.3.12

Proof. Suppose that μ is d-Vitali.

1. Define $I \subset X$ by $I := \{x \in X : \overline{D}_\mu \nu(x) = \infty\}$. Since ν is σ-finite, there exists $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X)$ such that
   - $X = \bigcup_{n\in\mathbb{N}} A_n$
   - for each $n \in \mathbb{N}$, $\nu(A_n) < \infty$.

   Let $n \in \mathbb{N}$ and $\epsilon > 0$. Since $\nu_\mu|_{\mathcal{B}(X)} \leq \nu$, $\nu_\mu(A_n) < \infty$. Define $c \in (0,\infty)$ by $c := (\nu_\mu(I \cap A_n) + 1)/\epsilon$. Since $I \cap A_n \subset \{x \in X : \overline{D}_\mu \nu(x) > c\}$, Exercise 9.2.2.6 implies that

$$\begin{aligned} \mu_\mu(I \cap A_n) &\leq c^{-1}\nu_\mu(I \cap A_n) \\ &= \frac{\epsilon\nu_\mu(I \cap A_n)}{\nu_\mu(I \cap A_n) + 1} \\ &\leq \epsilon \end{aligned}$$

   Since $\epsilon > 0$ is arbitrary, $\mu_\mu(I \cap A_n) = 0$. Since $n \in \mathbb{N}$ is arbitrary, we have that for each $n \in \mathbb{N}$, $\mu_\mu(I \cap A_n) = 0$. Therefore

$$\begin{aligned} \mu_\mu(I) &= \mu_\mu\bigg[I \cap \bigg(\bigcup_{n\in\mathbb{N}} A_n\bigg)\bigg] \\ &= \mu_\mu\bigg[\bigcup_{n\in\mathbb{N}} (I \cap A_n)\bigg] \\ &\leq \sum_{n\in\mathbb{N}} \mu_\mu(I \cap A_n) \\ &= 0 \end{aligned}$$

Since $\mu \ll \mu$, Exercise 9.1.3.7 implies that $\mu_\mu|_{\mathcal{B}(X)} = \mu$. Exercise 9.1.3.12 implies that there exists $N \in \mathcal{B}(X)$ such that $I \subset N$ and $\mu_\mu(\mu)(N) = 0$. Thus

$$\begin{aligned}\mu(N) &= \mu_\mu(N) \\ &= 0\end{aligned}$$

Since $I = \{x \in X : \overline{D}_\mu \nu(x) = \infty\}$ and there exists $N \in \mathcal{B}(X)$ such that $I \subset N$ and $\mu(N) = 0$, we have that $\overline{D}_\mu \nu < \infty$ μ-a.e.

2. For each $a, b \in (0, \infty)$, define $R_{a,b} \subset X$ by $R_{a,b} := \{x \in X : \underline{D}_\mu(x) < a < b < \overline{D}_\mu \nu(x)\}$. Let $a, b \in (0, \infty)$. Suppose that $a < b$. Then $R_{a,b} \subset \{x \in X : \underline{D}_\mu \nu(x) < a\} \cap \{x \in X : \overline{D}_\mu \nu(x) > b\}$. Exercise 9.2.2.6 implies that $\nu_\mu(R_{a,b}) \leq a\mu_\mu(R_{a,b})$ and $b\mu_\mu(R_{a,b}) \leq \nu_\mu(R_{a,b})$. Therefore

$$\begin{aligned}b\mu_\mu(R_{a,b}) &\leq \nu_\mu(R_{a,b}) \\ &\leq a\nu_\mu(R_{a,b})\end{aligned}$$

Hence $\mu_\mu(R_{a,b}) \leq ab^{-1}\mu_\mu(R_{a,b})$ and $(1 - ab^{-1})\mu_\mu(R_{a,b}) \leq 0$. Since $a < b$, $1 - ab^{-1} > 0$ and thus $\mu_\mu(R_{a,b}) \leq 0$. Therefore $\mu_\mu(R_{a,b}) = 0$. Since $a, b \in (0, \infty)$ with $a < b$ are arbitrary, we have that for each $a, b \in (0, \infty)$, $a < b$ implies that $\mu_\mu(R_{a,b}) = 0$.

Define $R \subset X$ by $R := \{x \in X : \underline{D}_\mu(x) < \overline{D}_\mu \nu(x)\}$. Let $x \in R$. Then there exist $a, b \in (0, \infty) \cap \mathbb{Q}$ such that $a < b$ and $\underline{D}_\mu(x) < a < b < \overline{D}_\mu \nu(x)$. Thus

$$\begin{aligned}x &\in R_{a,b} \\ &\subset \bigcup_{\substack{a,b \in \mathbb{Q} \\ 0<a<b}} R_{a,b}\end{aligned}$$

Since $x \in R$ is arbitrary, we have that

$$R \subset \bigcup_{\substack{a,b \in \mathbb{Q} \\ 0<a<b}} R_{a,b}$$

Therefore

$$\begin{aligned}\mu_\mu(R) &\leq \mu_\mu\bigg(\bigcup_{\substack{a,b \in \mathbb{Q} \\ 0<a<b}} R_{a,b}\bigg) \\ &\leq \sum_{\substack{a,b \in \mathbb{Q} \\ 0<a<b}} \mu_\mu(R_{a,b}) \\ &= 0\end{aligned}$$

Exercise 9.1.3.12 implies that there exists $N \in \mathcal{B}(X)$ such that $R \subset N$ and

$$\begin{aligned}\mu(N) &= \mu_\mu(N) \\ &= 0\end{aligned}$$

Since $R = \{x \in X : \underline{D}_\mu(x) < \overline{D}_\mu \nu(x)\}$ and there exists $N \in \mathcal{B}(X)$ such that $R \subset N$ and $\mu(N) = 0$, we have that $\underline{D}_\mu = \overline{D}_\mu \nu$ μ-a.e.

□

Exercise 9.2.2.9. Let X be a metric space and $\mu \in \mathcal{M}_+(X, d)$. Let $r > 0$. Define $f : X \to [0, \infty)$ by $f(x) := \mu(\bar{B}(x, r))$. Then f is upper semi-continuous.

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset X$, $x \in X$ and $\epsilon > 0$. Let $k \in \mathbb{N}$. Suppose that $x_n \to x$. An exercise in the introduction section on metric spaces in the analysis notes implies that there exists $N_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N_0$ implies that $\bar{B}(x_n, r) \subset \bar{B}(x, r + k^{-1})$. Let $n \in \mathbb{N}$. Suppose that $n \geq N_0$. Then $\mu(\bar{B}(x_n, r)) \leq \mu(\bar{B}(x, r + k^{-1}))$. Since $n \in \mathbb{N}$ with $n \geq N_0$ is arbitrary, we have that $\sup_{n \geq N_0} \mu(\bar{B}(x_n, r)) \leq \mu(\bar{B}(x, r + k^{-1}))$. Therefore

$$\begin{aligned}\limsup_{n\to\infty} f(x_n) &= \limsup_{n\to\infty} \mu(\bar{B}(x_n, r)) \\ &= \inf_{N\in\mathbb{N}} \sup_{n\geq N} \mu(\bar{B}(x_n, r)) \\ &\leq \sup_{n\geq N_0} \mu(\bar{B}(x_n, r)) \\ &\leq \mu(\bar{B}(x, r + k^{-1}))\end{aligned}$$

Since $k \in \mathbb{N}$ is arbitrary, we have that

$$\limsup_{n\to\infty} f(x_n) \leq \inf_{k\in\mathbb{N}} \mu(\bar{B}(x, r + k^{-1}))$$

Since $\mu \in \mathcal{M}_+(X, d)$, $\mu(\bar{B}(x, r + 1)) < \infty$. Since $(\bar{B}(x, r + k^{-1}))_{k\in\mathbb{N}}$ is decreasing and $\inf_{k\in\mathbb{N}} \bar{B}(x, r + k^{-1}) = \bar{B}(x, r)$, we have that

$$\begin{aligned}\limsup_{n\to\infty} f(x_n) &\leq \inf_{k\in\mathbb{N}} \mu(\bar{B}(x, r + k^{-1})) \\ &= \mu(\bar{B}(x, r)) \\ &= f(x)\end{aligned}$$

Since $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$ with $x_n \to x$ are arbitrary, we have that f is upper semicontinuous. □

Definition 9.2.2.10. Let (X, d) be a metric space, $\mu, \nu \in \mathcal{M}_+^{\text{loc}}(X, d)$ and $h \in \mathcal{L}^0(X, \mathcal{B}(X))$. Then h is said to be a **derivate of** ν **by** μ if $h = \underline{D}_\mu \nu$ μ-a.e. and $h = \overline{D}_\mu \nu$ μ-a.e.

Exercise 9.2.2.11. Let (X, d) be a metric space, $\mu, \nu \in \mathcal{M}_+^{\text{loc}}(X, d)$ and $h \in \mathcal{L}^0(X, \mathcal{B}(X))$. If h is a derivate of ν by μ, then h is unique μ-a.e.

Proof. Suppose that h is a derivate of ν by μ. Then $h = \underline{D}_\mu \nu$ μ-a.e. and $h = \overline{D}_\mu \nu$ μ-a.e. Let $h_0 \in L(X, B(X))$. Suppose that h_0 are derivates of ν by μ. Then $h_0 = \underline{D}_\mu \nu$ μ-a.e. and $h_0 = \overline{D}_\mu \nu$ μ-a.e. Therefore

$$\begin{aligned}h &= \overline{D}_\mu \nu \; \mu\text{-a.e.} \\ &= h_0 \; \mu\text{-a.e.}\end{aligned}$$

(maybe just make an exercise about equality of function a.e. being an equivalence relation and reference) □

Exercise 9.2.2.12. Let (X, d) be a metric space and $\mu, \nu \in \mathcal{M}_+(X, d)$. If μ is d-Vitali, then there exists $h \in \mathcal{L}^0(X, \mathcal{B}(X))$ such that h is a derivate of ν by μ.

Proof. Suppose that μ is d-Vitali. Since $\mu, \nu \in \mathcal{M}_+(X, d)$, we have that $\mu, \nu \in \mathcal{M}_+^{\text{loc}}(X, d)$ and Exercise 9.2.2.3 implies that μ, ν are σ-finite. Exercise 9.2.2.8 implies that $\overline{D}_\mu \nu < \infty$ and $\underline{D}_\mu \nu = \overline{D}_\mu \nu$ μ-a.e. Thus there exists $N_1, N_2 \in \mathcal{B}(X)$ such that $\mu(N_1), \mu(N_2) = 0$, $\{\overline{D}_\mu \nu(x) = \infty\} \subset N_1$ and $\{x \in X : \underline{D}_\mu \nu < \overline{D}_\mu \nu\} \subset N_2$. Define $N \in \mathcal{B}(X)$ by $N = N_1 \cup N_2 \cup (\operatorname{supp} \mu)^c$. Then

$$\begin{aligned}\mu(N) &\leq \mu(N_1) + \mu(N_2) + \mu[(\operatorname{supp} \mu)^c] \\ &= 0.\end{aligned}$$

Define $h : X \to [0, \infty)$ by $h = \overline{D}_\mu \nu \chi_{N^c}$. Definition 9.2.2.1 implies that for each $x \in X$ and $n \in \mathbb{N}$, $\mu(\bar{B}(x, 1/n)), \nu(\bar{B}(x, 1/n)) < \infty$. For each $n \in \mathbb{N}$, define $f_n, g_n : X \to [0, \infty)$ by $f_n(x) := \nu(\bar{B}(x, 1/n))$ and $g_n(x) := \mu(\bar{B}(x, 1/n))$. Exercise 9.2.2.9 implies that for each $n \in \mathbb{N}$, f_n and g_n are upper semicontinuous. Exercise 3.3.0.24 then implies that $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset \mathcal{L}^0(X, \mathcal{B}(X))$.

Exercise 9.1.1.7 implies that for each $x \in \operatorname{supp} \mu$ and $n \in \mathbb{N}$, $\mu(\bar{B}(x, 1/n)) > 0$ and therefore $g_n|_{N^c} > 0$. For each $n \in \mathbb{N}$, define $h_n : X \to [0, \infty)$ by

$$h_n(x) = \begin{cases} f_n(x)/g_n(x), & x \in N^c \\ 0, & x \in N \end{cases}$$

Since $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subset \mathcal{L}^0(X, \mathcal{B}(X))$ and $N \in \mathcal{B}(X)$, we have that $(h_n)_{n\in\mathbb{N}} \subset \mathcal{L}^0(X, \mathcal{B}(X))$ (make this an exercise in the earlier chapters). Then $h_n \xrightarrow{\text{p.w.}} h$. Exercise 3.3.0.25 implies that $h \in \mathcal{L}^0(X, \mathcal{B}(X))$. Since $\{x \in X : h(x) \neq \overline{D}_\mu \nu(x)\} \cup \{x \in X : h(x) \neq \underline{D}_\mu \nu(x)\} \subset N$, we have that $h = \underline{D}_\mu \nu$ μ-a.e. and $h = \overline{D}_\mu \nu$ μ-a.e. Thus h is a derivate of ν by μ. □

Definition 9.2.2.13. Let (X, d) be a metric space, $\mu, \nu \in \mathcal{M}_+^{\text{loc}}(X, d)$ and $h \in \mathcal{L}^0(X, \mathcal{B}(X))$. Suppose that h is a derivate of ν by μ. We define $D_\mu \nu \in L^0(X, \mathcal{B}(X), \mu)$ by

$$D_\mu \nu := \pi_{L^0}(h).$$

define $\pi_{L^0} : \mathcal{L}^0(X, \mathcal{A}) \to L^0(X, \mathcal{A})$ somewhere in measurable function section

Exercise 9.2.2.14. Let (X, d) be a metric space and $\mu, \nu \in \mathcal{M}_+(X, d)$. Suppose that μ is d-Vitali.

1. We have that
$$D_\mu \nu = \frac{d\nu_\mu}{d\mu}$$

2. for μ-a.e. $x \in X$,
$$\frac{d\nu_\mu}{d\mu}(x) = \lim_{r\to 0^+} \frac{\nu(\bar{B}(x, r))}{\mu(\bar{B}(x, r))}$$

Proof.

1. Define $Z \in \mathcal{B}(X)$ by $Z = \{x \in X : D_\mu \nu(x) = 0\}$. Let $\epsilon > 0$. Set $c := \epsilon/(\mu(Z) + 1)$. Then $c > 0$ and $Z \subset \{x \in X : D_\mu \nu(x) \leq c\}$. Exercise 9.2.2.7 implies that
$$\begin{aligned} \nu_\mu(Z) &\leq c\mu_\mu(Z) \\ &= c\mu(Z) \\ &= \epsilon \frac{\mu(Z)}{\mu(Z) + 1} \\ &< \epsilon \end{aligned}$$
Since $\epsilon > 0$ is arbitrary, we have that $\nu_\mu(Z) = 0$. By construction,
$$\begin{aligned} \nu_\mu(Z) &= 0 \\ &= \int_Z D_\mu \nu \, d\mu \end{aligned}$$
Let $E \in \mathcal{B}(X)$ and $t > 1$. For each $m \in \mathbb{Z}$, define $E_m \in \mathcal{B}(X)$ by $E_m := E \cap \{x \in X : t^m \leq D_\mu \nu(x) < t^{m+1}\}$. We note that $E \cap Z^c = \bigcup_{m\in\mathbb{Z}} E_m$ and $(E_m)_{m\in\mathbb{Z}}$ is disjoint. Exercise 9.2.2.7 implies that
$$\begin{aligned} t^m \mu(E_m) &= t^m \mu_\mu(E_m) \\ &\leq \nu_\mu(E_m) \\ &\leq t^{m+1} \mu_\mu(E_m) \\ &= t^{m+1} \mu(E_m) \end{aligned}$$

Therefore

$$\begin{aligned}
\nu_\mu(E) &= \nu_\mu(E \cap Z^c) + \nu_\mu(E \cap Z) \\
&= \nu_\mu(E \cap Z^c) \\
&= \nu_\mu\left(\bigcup_{m \in \mathbb{Z}} E_m\right) \\
&= \sum_{m \in \mathbb{Z}} \nu_\mu(E_m) \\
&\leq \sum_{m \in \mathbb{Z}} t^{m+1} \mu(E_m) \\
&= t \sum_{m \in \mathbb{Z}} t^m \mu(E_m) \\
&= t \sum_{m \in \mathbb{Z}} \int_{E_m} t^m \, d\mu \\
&\leq t \sum_{m \in \mathbb{Z}} \int_{E_m} D_\mu \nu \, d\mu \\
&= t \int_{E \cap Z^c} D_\mu \nu \, d\mu \\
&\leq t \int_E D_\mu \nu \, d\mu
\end{aligned}$$

Similarly, since

$$\begin{aligned}
\int_E D_\mu \nu \, d\mu &= \int_{E \cap Z^c} D_\mu \nu \, d\mu + \int_{E \cap Z} D_\mu \nu \, d\mu \\
&= \int_{E \cap Z^c} D_\mu \nu \, d\mu,
\end{aligned}$$

we have that

$$\begin{aligned}
\nu_\mu(E) &= \sum_{m \in \mathbb{Z}} \nu_\mu(E_m) \\
&\geq \sum_{m \in \mathbb{Z}} t^m \mu(E_m) \\
&= t^{-1} \sum_{m \in \mathbb{Z}} t^{m+1} \mu(E_m) \\
&= t^{-1} \sum_{m \in \mathbb{Z}} \int_{E_m} t^{m+1} \, d\mu \\
&\geq t^{-1} \sum_{m \in \mathbb{Z}} \int_{E_m} D_\mu \nu \, d\mu \\
&= t^{-1} \int_{E \cap Z^c} D_\mu \nu \, d\mu \\
&= t^{-1} \int_E D_\mu \nu \, d\mu
\end{aligned}$$

Thus

$$\begin{aligned}
t^{-1} \int_E D_\mu \nu \, d\mu &\leq \nu_\mu(E) \\
&\leq t \int_E D_\mu \nu \, d\mu
\end{aligned}$$

Since $t > 1$ is arbitrary, we have that

$$\nu_\mu(E) = \int_E D_\mu \nu \, d\mu.$$

Since $E \in \mathcal{B}(X)$ is arbitrary, we have that $D_\mu \nu = d\nu_\mu / d\mu$.

2. By Definition 9.2.2.4,

□

9.2.3 Differentiation of Complex Radon Measures

show that if $\nu \in \mathcal{M}(X)$ is Radon, then its real and imaginary parts are Radon and if ν is finite signed, then its positive and negative parts are Radon.

Definition 9.2.3.1. Let (X, d) be a metric space and $\nu \in \mathcal{M}(X)$.

- If $\operatorname{Im} \nu = 0$, we define $D_\mu \nu = D_\mu \nu^+ - D_\mu \nu^-$.
- We define $D_\mu \nu = D_\mu(\operatorname{Re} \nu) + i D_\mu(\operatorname{Im} \nu)$.

Exercise 9.2.3.2. Let (X, d) be a metric space, $\mu \in \mathcal{M}_+(X, d)$ and $\nu \in \mathcal{M}(X)$. Suppose that μ is d-Vitali.

1. We have that
$$D_\mu \nu = \frac{d\nu_\mu}{d\mu}$$
2. for μ-a.e. $x \in X$,
$$\frac{d\nu_\mu}{d\mu}(x) = \lim_{r \to 0^+} \frac{\nu(\bar{B}(x, r))}{\mu(\bar{B}(x, r))}$$

Proof.

1. By definition,
$$\begin{aligned}
D_\mu \nu &= D_\mu(\operatorname{Re} \nu) + i D_\mu(\operatorname{Im} \nu) \\
&= [D_\mu(\operatorname{Re} \nu)^+ - D_\mu(\operatorname{Re} \nu)^-] + i[D_\mu(\operatorname{Im} \nu)^+ - D_\mu(\operatorname{Im} \nu)^-] \\
&= \left[\frac{d(\operatorname{Re} \nu)^+_\mu}{d\mu} - \frac{d(\operatorname{Re} \nu)^-_\mu}{d\mu}\right] + i\left[\frac{d(\operatorname{Im} \nu)^+_\mu}{d\mu} - \frac{d(\operatorname{Im} \nu)^-_\mu}{d\mu}\right] \\
&= \frac{d\nu_\mu}{d\mu}
\end{aligned}$$
2. Exercise 9.2.2.12 implies that for μ-a.e. $x \in X$,
$$\begin{aligned}
D_\mu \nu(x) &= [D_\mu(\operatorname{Re} \nu)^+ - D_\mu(\operatorname{Re} \nu)^-] + i[D_\mu(\operatorname{Im} \nu)^+ - D_\mu(\operatorname{Im} \nu)^-] \\
&= \lim_{r \to 0^+} \frac{(\operatorname{Re} \nu)^+(\bar{B}(x, r))}{\mu(\bar{B}(x, r))} + \lim_{r \to 0^+} \frac{(\operatorname{Re} \nu)^-(\bar{B}(x, r))}{\mu(\bar{B}(x, r))} \\
&\qquad + \lim_{r \to 0^+} \frac{(\operatorname{Im} \nu)^+(\bar{B}(x, r))}{\mu(\bar{B}(x, r))} + \lim_{r \to 0^+} \frac{(\operatorname{Im} \nu)^-(\bar{B}(x, r))}{\mu(\bar{B}(x, r))} \\
&= \lim_{r \to 0^+} \frac{\nu(\bar{B}(x, r))}{\mu(\bar{B}(x, r))}
\end{aligned}$$

□

Exercise 9.2.3.3. Let (X, d) be a metric space, $\mu \in \mathcal{M}_+(X, d)$ and $\nu \in \mathcal{M}(X)$. If $\nu \ll \mu$, Then $\nu_\mu = \nu$.

Proof. FINISH!!! □

Definition 9.2.3.4. Let (X, d) be a metric space, $\mu \in \mathcal{M}_+(X, d)$ and $f \in L^1(X, \mathcal{B}(X), \mu)$. For each $r > 0$, we define the **r-average of f**, denoted $A_r f : X \to \mathbb{C}$ by

$$A_r f(x) = \frac{1}{\mu(\bar{B}(x,r))} \int_{\bar{B}(x,r)} f \, d\mu$$

Exercise 9.2.3.5. Lebesgue Differentiation Theorem:
Let (X, d) be a metric space and $\mu \in \mathcal{M}_+(X, d)$ and $f \in L^1(X, \mathcal{B}(X), \mu)$. Suppose that μ is d-Vitali. Then for μ-a.e. $x \in X$,

$$\lim_{r \to 0^+} A_r f(x) = f(x).$$

Proof. Define $\nu \in M(X)$ by

$$\nu(E) = \int_E f \, d\mu$$

Since $\mu \in \mathcal{M}_+(X)$ and $f \in L^1(X, \mathcal{A}, \mu)$, Exercise 9.1.4.4 implies that $\nu \in \mathcal{M}(X)$. Since $\nu \ll \mu$, Exercise 9.2.3.3 implies that $\nu_\mu = \nu$. Exercise 9.2.3.2 implies that for μ-a.e. $x \in X$,

$$\begin{aligned} f(x) &= \frac{d\nu}{d\mu}(x) \\ &= \frac{d\nu_\mu}{d\mu}(x) \\ &= D_\mu \nu(x) \\ &= \lim_{r \to 0^+} \frac{\nu(\bar{B}(x,r))}{\mu(\bar{B}(x,r))} \\ &= \lim_{r \to 0^+} \frac{1}{\mu(\bar{B}(x,r))} \int_{\bar{B}(x,r)} f \, d\mu \\ &= \lim_{r \to 0^+} A_r f(x) \end{aligned}$$

□

9.3 Radon Measures on LCH Spaces

Exercise 9.3.0.1. Let X be a LCH space and $\mu \in \mathcal{M}_+(X)$. Then

- for each $p \in [1, \infty]$, $C_c(X) \subset L^p(\mu)$
- for each $p \in [1, \infty)$, $C_c(X)$ is dense in $L^p(\mu)$.

Proof.

- Let $p \in [1, \infty]$ and $f \in C_c(X)$. Then $|f|^p \in C_c(X)$ and

$$\begin{aligned} \|f\|_p &= \int |f|^p \, d\mu \\ &\leq \|f\|_\infty^p \mu(\text{supp}(f)) \\ &< \infty \end{aligned}$$

- Let $p \in [1, \infty)$. Let $E \in \mathcal{B}(X)$. Suppose that $\mu(E) < \infty$. Then is $\mu|_E$ is σ-finite. A previous exercise implies that μ is inner regular on E. Let $\epsilon > 0$. Since μ is inner regular on E, a previous exercise (that might need doing) implies that there exists $U, K \in \mathcal{B}(X)$ such that, U is open, K is compact, $K \subset E \subset U$ and $\mu(U \setminus K) < \epsilon$. Urysohn's lemma implies that there exists $f \in C_c(X)$ such that $f|_K = 1$ and $\text{supp}\, f \subset U$. Then
FINISH!!!

$\square$

Definition 9.3.0.2. Let X be a topological space and $I : C_c(X) \to \mathbb{C}$ a linear functional. Then I is said to be **positive** if for each $f \in C_c(X, \mathbb{R})$, $f \geq 0$ implies that $I(f) \geq 0$.

Exercise 9.3.0.3. Let X be a topological space, $I : C_c(X) \to \mathbb{C}$ a positive linear functional and $f, g \in C_c(X, \mathbb{R})$. If $f \geq g$, then $I(f) \geq I(g)$.

Proof. Suppose that $f \geq g$. Then $f - g \geq 0$. So

$$\begin{aligned} I(f) - I(g) &= I(f - g) \\ &\geq 0 \end{aligned}$$

$\square$

Exercise 9.3.0.4. Let X be a LCH space, $I : C_c(X) \to \mathbb{C}$ a positive linear functional. Then for each $K \subset X$, K is compact implies that there exists $C_K \geq 0$ such that for each $f \in C_c(X)$, if $\text{supp}(f) \subset K$, then $I(f) \leq C_K \|f\|_\infty$.
Hint: Urysohn's lemma

Proof. Let $K \subset X$. Suppose that K is compact. Then Urysohn's lemma implies that there exists $\phi \in C_c(X)$ such that $0 \leq \phi \leq 1$ and $\phi|_K = 1$. Then $I(\phi) \geq 0$. Choose $C_K = I(\phi)$. Let $f \in C_c(X)$. Suppose that $\text{supp}(f) \subset K$. Then

$$\begin{aligned} f, -f &\leq |f| \\ &\leq \|f\|_\infty \phi \end{aligned}$$

The previous exercise implies that $I(f), -I(f) \leq \|f\|_\infty I(\phi)$. So

$$\begin{aligned} |I(f)| &\leq \|f\|_\infty I(\phi) \\ &\leq C_K \|f\|_\infty \end{aligned}$$

$\square$

Note 9.3.0.5. Let X be a LCH space, $U \subset X$ open and $f \in C_c(X)$. We write $f \prec U$ to mean $0 \leq f \leq 1$ and $\text{supp}(f) \subset U$.

Exercise 9.3.0.6. Let X be a LCH space, $I : C_c(C) \to \mathbb{C}$ a positive linear functional and $\mu : \mathcal{B}(X) \to [0, \infty]$ a Radon measure. Suppose that for each $f \in C_c(X)$,

$$I(f) = \int f \, d\mu$$

Then

1. for each $U \subset X$, U is open implies that

$$\mu(U) = \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\}$$

2. μ is the unique Radon measure such that for each $f \in C_c(X)$,

$$I(f) = \int f d\, \mu$$

Proof.

1. Let $U \subset X$. Suppose that U is open. For $f \in C_c(X)$, if $f \prec U$, then

$$\begin{aligned} I(f) &= \int f \, d\mu \\ &\leq \mu(U) \end{aligned}$$

   Let $K \subset U$. Suppose that K is compact. Then Urysohn's lemma implies that there exists $f \in C_c(X)$ such that $f \prec U$ and $f|_K = 1$. Then

$$\begin{aligned} \mu(K) &\leq \int f \, d\mu \\ &= I(f) \end{aligned}$$

   Inner regularity implies that

$$\begin{aligned} \mu(U) &= \sup\{\mu(K) : K \subset X \text{ and } K \text{ is compact} \\ &\leq \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\} \\ &\leq \mu(U) \end{aligned}$$

2. Let $\nu : \mathcal{B}(X) \to [0, \infty]$ be a Radon measure. Suppose that for each $f \in C_c(X)$,

$$I(f) = \int f d\nu$$

   Part (1) implies that for each $U \subset X$, if U is open, then

$$\begin{aligned} \nu(U) &= \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\} \\ &= \mu(U) \end{aligned}$$

   Outer regularity implies that for each $E \in \mathcal{B}(X)$,

$$\begin{aligned} \nu(E) &= \inf\{\nu(U) : E \subset U \text{ and } U \text{ is open}\} \\ &= \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\} \\ &= \mu(E) \end{aligned}$$

   So $\nu = \mu$ and μ is unique.

$\square$

Theorem 9.3.0.7. Representation Theorem I:
Let X be a LCH space and $I : C_c(C) \to \mathbb{C}$ a positive linear functional. Then there exists a unique Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$ such that for each $f \in C_c(X)$,

$$I(f) = \int f \, d\mu$$

In addition,

1. for each $U \subset X$, U is open implies that

$$\mu(U) = \sup\{I(f) : f \in C_c(X) \text{ and } f \prec U\}$$

2. for each $K \subset X$, K is compact implies that

$$\mu(U) = \inf\{I(f) : f \in C_c(X) \text{ and } f \geq \chi_K\}$$

Exercise 9.3.0.8. Let X be a topological space.
and $\|\cdot\| : \mathcal{M}(X) \to [0, \infty)$ given by $\|\mu\| = |\mu|(X)$ is a norm on $\mathcal{M}(X)$. FINISH!!! or FIX!!!, should be an exercise in the section on complex measures, but here might show that radon complex measures form a closed subspace

Definition 9.3.0.9. Let X be a topological space. For $\mu \in \mathcal{M}(X)$, define $I_\mu : C_0(X) \to \mathbb{C}$ by

$$I_\mu(f) = \int f \, d\mu$$

.

Exercise 9.3.0.10. Let X be a topological space. For each $\mu \in \mathcal{M}(X)$, $I_\mu \in C_0(X)^*$.

Proof. Let $\mu \in \mathcal{M}(X)$ and $f \in C_0(X)$. An exercise in section (4.3) implies that

$$\begin{aligned} |I_\mu(f)| &= \left| \int f \, d\mu \right| \\ &\leq \int |f| d|\mu| \\ &\leq \|\mu\| \|f\|_\infty \end{aligned}$$

So I_μ is bounded and $I_\mu \in C_0(X)^*$. □

Theorem 9.3.0.11. Let $I \in C_0(X, \mathbb{R})^*$, then there exist positive linear functionals $I^+, I^- \in C_0(X, \mathbb{R})^*$ such that $I = I^+ - I^-$

Exercise 9.3.0.12. Let X be a LCH space. Then the map $\phi : \mathcal{M}(X) \to C_0(X)^*$ given by $\phi(\mu) = I_\mu$ is a linear surjection.

Proof. An exercise in section (4.3) implies that ϕ is linear. Let $I \in C_0(X)^*$. Then there exists positive linear functionals $I^\pm$, $J^\pm \in C_0(X)^*$ such that $I = I^+ - I^- + i(J^+ - J^-)$. The first representation theorem implies that there exist Radon measures $\mu^\pm$, $\nu^\pm$ such that $I^\pm = I_{\mu^\pm}$ and $J^\pm = I_{\mu^\pm}$. Set $\mu = \mu^+ - \mu^- + i(\nu^+ - \nu^-)$. Then $I = \phi(\mu)$ □

Theorem 9.3.0.13. Representation Theorem II:
Let X be a LCH space. Then the map $\phi : \mathcal{M}(X) \to C_0(X)^*$ given by $\phi(\mu) = I_\mu$ is an isometric linear isomorphism.

Definition 9.3.0.14. Let X be a LCH space, $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Then μ_n is said to **converge to μ in weak-***, denoted $\mu_n \xrightarrow{w^*} \mu$, if $I_{\mu_n} \xrightarrow{w^*} I_\mu$, i.e. for each $f \in C_0(X)$,

$$\int f \, d\mu_n \to \int f \, d\mu$$

Exercise 9.3.0.15. Let X be a compact Hausdorff topological space and Y a LCH space and $f : X \to Y$. If $f : X \to Y$ is continuous, then $f_* : \mathcal{M}(X) \to \mathcal{M}(Y)$ is continuous.

Proof. Suppose that f is continuous. Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Suppose that $\mu_n \xrightarrow{w^*} \mu$. Let $\phi \in C_0(Y)$. Since X is compact, $C_0(X) = C(X)$. Therefore $\phi \circ f \in C_0(X)$ and

$$\begin{aligned}\int_Y \phi \, d(f_*\mu_n) &= \int \phi \circ f \, d\mu_n \\ &\to \int \phi \circ f \, d\mu \\ &= \int_Y \phi \, d(f_*\mu)\end{aligned}$$

Since $\phi \in C_0(Y)$ is arbitrary, $f_*\mu_n \xrightarrow{w^*} f_*\mu$. Since $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$ are arbitrary, we have that $f_* : \mathcal{M}(X) \to \mathcal{M}(Y)$ is continuous. □

Exercise 9.3.0.16. Let X be a LCH space and $\mu \in \mathcal{M}(X)$. Then $C_c(X)$ is dense in $L^p(X)$.

Proof. Let $E \in \mathcal{B}(X)$. Suppose that $\mu(E) < \infty$. □

Exercise 9.3.0.17. Let X be a LCH space. Then

1. $\mathcal{M}_1(X)$ is weak-* closed
2. $\mathcal{M}_1(X)$ is norm closed

Proof. Let $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_1(X)$ and $\mu \in \mathcal{M}(X)$.

1. Suppose that $\mu_n \xrightarrow{w^*} \mu$.
2. Since $\mathcal{M}_1(X)$ is weak-* closed, an exercise in the analysis notes in the duality section of the Banach spaces chapter implies that $\mathcal{M}_1(X)$ is norm closed.

□

9.4 Borel Measures on Metric Spaces

Note 9.4.0.1. Let X be a metric space and $A \subset X$. For $\epsilon > 0$, we write $A_\epsilon = \{x \in X : d(x, A) < \epsilon\}$ and recall that A_ϵ is open.

Exercise 9.4.0.2. Let X be a metric space, $\mu : \mathcal{B}(X) \to [0, \infty)$ be a finite measure and $E \in \mathcal{B}(X)$. Then $\mu(E) = \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$ iff $\mu(E^c) = \sup\{\mu(C) : C \subset E^c \text{ and } C \text{ is closed}\}$
move to previous section

Proof. Suppose that $\mu(E) = \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}$. Let $\epsilon > 0$. Then there exists $U \subset X$ such that $E \subset U$, U is open and $\mu(U) < \mu(E) + \epsilon$. Choose $C = U^c$. Then $C \subset E^c$, C is closed and

$$\begin{aligned}
\mu(E^c) - \epsilon &= \mu(E^c \cap C) + \mu(E^c \cap C^c) - \epsilon \\
&= \mu(C) + \mu(E^c \cap U) - \epsilon \\
&= \mu(C) + [\mu(U) - \mu(E)] - \epsilon \\
&< \mu(C) + \epsilon - \epsilon \\
&= \mu(C)
\end{aligned}$$

So for each $\epsilon > 0$, there exists $C \subset E^c$ such that C is closed and $\mu(C) < \mu(E^c) - \epsilon$. is arbitrary, $\mu(E^c) = \sup\{\mu(C) : C \subset E^c \text{ and } E \text{ is closed}\}$.
The converse is similar. □

Exercise 9.4.0.3. Let X be a metric space and $\mu : \mathcal{B}(X) \to [0, \infty)$ be a finite measure. Then for each $C \subset X$, if C is closed, then μ is outer regular on C.
Hint: For $\epsilon > 0$, consider $C_\epsilon = \{x \in X : d(x, C) < \epsilon\}$.

Proof. Let $n \in \mathbb{N}$. Set $V_n = C_{1/n}$. Then V_n is open and $C \subset V_n$. Since C is closed, $C = \bigcap_{n \in \mathbb{N}} V_n$. Since for each $n \in \mathbb{N}$, $V_{n+1} \subset V_n$ and μ is finite, we have that $\mu(C) = \inf\limits_{n \in \mathbb{N}} \mu(V_n)$. So for each $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\mu(V_n) < \mu(C) + \epsilon$. Hence $\mu(C) = \inf\{\mu(U) : C \subset U \text{ and } U \text{ is open}\}$ and μ is outer regular on C. □

Exercise 9.4.0.4. Let X be a metric space and $\mu : \mathcal{B}(X) \to [0, \infty)$ be a finite measure. Set

$$\mathcal{A} = \left\{ E \in \mathcal{B}(X) : \mu \text{ is outer regular on } E \text{ and } E^c \right\}$$

Then $\mathcal{A}$ is a σ-algebra on X.

Proof.

1. Clearly, $\varnothing \in \mathcal{A}$.

2. Let $E \in \mathcal{A}$. Since $(E^c)^c = E$, by definition, $E^c \in \mathcal{A}$.

3. Let $(E_n)_{n \in \mathbb{N}} \subset \mathcal{A}$. Set $E = \bigcup\limits_{n \in \mathbb{N}} E_n$. Let $\epsilon > 0$.

   - For each $n \in \mathbb{N}$, there exists $U_n \subset X$ such that U_n is open, $E_n \subset U_n$ and $\mu(U_n) < \mu(E_n) + \epsilon 2^{-n-1}$. Set $U = \bigcup\limits_{n \in \mathbb{N}} U_n$.

Then U is open, $E \subset U$ and

$$\begin{aligned} U \setminus E &= \bigg(\bigcup_{n \in \mathbb{N}} U_n \bigg) \cap E^c \\ &= \bigg(\bigcup_{n \in \mathbb{N}} U_n \cap E^c \bigg) \\ &= \bigg(\bigcup_{n \in \mathbb{N}} U_n \cap \bigg[\bigcap_{j \in \mathbb{N}} E_j^c \bigg] \bigg) \\ &= \bigg(\bigcup_{n \in \mathbb{N}} \bigg[\bigcap_{j \in \mathbb{N}} (U_n \cap E_j^c) \bigg] \bigg) \\ &\subset \bigcup_{n \in \mathbb{N}} (U_n \cap E_n^c) \\ &= \bigcup_{n \in \mathbb{N}} (U_n \setminus E_n) \end{aligned}$$

Therefore

$$\begin{aligned} \mu(U) - \mu(E) &= \mu(U \setminus E) \\ &\leq \mu\bigg(\bigcup_{n \in \mathbb{N}} [U_n \setminus E_n] \bigg) \\ &\leq \sum_{n \in \mathbb{N}} \mu(U_n \setminus E_n) \\ &= \sum_{n \in \mathbb{N}} [\mu(U_n) - \mu(E_n)] \\ &\leq \sum_{n \in \mathbb{N}} \epsilon 2^{-n-1} \\ &= \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

So for each $\epsilon > 0$, there exists $U \subset X$ such that U is open, $\bigcup_{n \in \mathbb{N}} E_n \subset U$ and $\mu(U) < \mu\Big(\bigcup_{n \in \mathbb{N}} E_n \Big) + \epsilon$. Therefore

$$\mu\bigg(\bigcup_{n \in \mathbb{N}} E_n \bigg) = \inf \bigg\{ \mu(U) : \bigcup_{n \in \mathbb{N}} E_n \subset U \text{ and } U \text{ is open} \bigg\}$$

and μ is outer regular on $\bigcup\limits_{n \in \mathbb{N}} E_n$.

- A previous exercise implies that for each $n \in \mathbb{N}$, there exists $C_n \subset E_n$ such that C_n is closed and $\mu(C_n) > \mu(E_n) - 2^{-n-1}\epsilon$. Since

$$\mu\bigg(\bigcup_{n \in \mathbb{N}} C_n \bigg) = \sup_{K \in \mathbb{N}} \mu\bigg(\bigcup_{n=1}^{K} C_n \bigg)$$

there exists $K \in \mathbb{N}$ such that $\mu\bigg(\bigcup\limits_{n=1}^{K} C_n \bigg) > \mu\bigg(\bigcup\limits_{n \in \mathbb{N}} C_n \bigg) - \epsilon/2$. Set $C = \bigcup\limits_{n=1}^{K} C_n$. Then C is closed, $C \subset E$ and similar to the previous part, we have that

$$\begin{aligned}\mu(E) - \mu(C) &< \mu(E) - \mu\bigg(\bigcup_{n\in\mathbb{N}} C_n\bigg) + \frac{\epsilon}{2} \\ &= \mu\bigg(E \setminus \bigcup_{n\in\mathbb{N}} C_n\bigg) + \frac{\epsilon}{2} \\ &= \mu\bigg(\bigcup_{n\in\mathbb{N}} \bigg[\bigcap_{j\in\mathbb{N}} (E_n \cap C_j^c)\bigg]\bigg) + \frac{\epsilon}{2} \\ &\leq \mu\bigg(\bigcup_{n\in\mathbb{N}} (E_n \cap C_n^c)\bigg) + \frac{\epsilon}{2} \\ &\leq \bigg[\sum_{n\in\mathbb{N}} \mu(E_n \cap C_n^c)\bigg] + \frac{\epsilon}{2} \\ &= \bigg[\sum_{n\in\mathbb{N}} \mu(E_n) - \mu(C_n)\bigg] + \frac{\epsilon}{2} \\ &\leq \bigg[\sum_{n\in\mathbb{N}} 2^{-n-1}\epsilon\bigg] + \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

So for each $\epsilon > 0$, there exists $C \subset X$ such that C is closed, $C \subset \bigcup_{n\in\mathbb{N}} E_n$ and $\mu(C) > \mu\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) - \epsilon$. Therefore

$$\mu\bigg(\bigcup_{n\in\mathbb{N}} E_n\bigg) = \sup\bigg\{\mu(C) : C \subset \bigcup_{n\in\mathbb{N}} E_n \text{ and } C \text{ is closed}\bigg\}$$

which implies that

$$\mu\bigg(\bigg[\bigcup_{n\in\mathbb{N}} E_n\bigg]^c\bigg) = \inf\bigg\{\mu(U) : \bigg[\bigcup_{n\in\mathbb{N}} E_n\bigg]^c \subset U \text{ and } U \text{ is open}\bigg\}$$

and μ is outer regular on $\bigg(\bigcup\limits_{n\in\mathbb{N}} E_n\bigg)^c$.

Hence $\bigcup\limits_{n\in\mathbb{N}} E_n \in \mathcal{A}$.

Therefore $\mathcal{A}$ is a σ-algebra on X. □

Exercise 9.4.0.5. Let X be a metric space and $\mu : \mathcal{B}(X) \to [0, \infty)$ be a finite measure. Then μ is outer regular.

Proof. Set $\mathcal{T} := \{U \subset X : X \text{ is open}\}$ and define $\mathcal{A}$ as in the previous exercise. The previous exercises imply that $\mathcal{T} \subset \mathcal{A}$. Since $\mathcal{B}(X) = \sigma(\mathcal{T})$, we have that $\mathcal{B}(X) \subset \mathcal{A}$. Therefore $\mathcal{B}(X) = \mathcal{A}$ and μ is outer regular. □

Exercise 9.4.0.6. Let X be a metric space and $\mu : \mathcal{B}(X) \to [0, \infty)$ a finite measure. If μ is inner regular on X, then μ is inner regular.

Proof. Suppose that is inner regular on X. Let $E \in \mathcal{B}(X)$ and $\epsilon > 0$. Then there exists $K_0 \subset X$ such that K_0 is compact and $\mu(K_0) > \mu(X) - \epsilon/2$. The previous exercise implies that there exists $C \subset E$ such that C is closed and $\mu(C) > \mu(E) - \epsilon/2$.

Set $K = K_0 \cap C$. Then $K \subset E$, K is compact and

$$\begin{aligned}\mu(E) &< \mu(C) + \frac{\epsilon}{2} \\ &= [\mu(C \cap K_0) + \mu(C \cap K_0^c)] + \frac{\epsilon}{2} \\ &\leq \mu(C \cap K_0) + \mu(X \cap K_0^c) + \frac{\epsilon}{2} \\ &= \mu(K) + [\mu(X) - \mu(K_0)] + \frac{\epsilon}{2} \\ &< \mu(K) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \mu(K) + \epsilon\end{aligned}$$

So for each $\epsilon > 0$, there exists $K \subset E$ such that K is compact and $\mu(K) > \mu(E) - \epsilon$. Hence $\mu(E) = \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\}$ and μ is inner regular on E. Since $E \in \mathcal{B}(X)$ is arbitrary, μ is inner regular. □

Exercise 9.4.0.7. Let X be a Polish space and $\mu : \mathcal{B}(X) \to [0, \infty)$ a finite measure. Then μ is inner regular.
Hint: If $(x_n)_{n\in\mathbb{N}}$ is a countable dense of X, consider $K \subset X$ of the form

$$K = \bigcap_{m\in\mathbb{N}} \bigcup_{n=1}^{n_m} \operatorname{cl} B(x_n, 1/m)$$

Proof. Let $\epsilon > 0$. Since X is separable, there exists a a countable dense subset $(x_n)_{n\in\mathbb{N}}$ of X. Let $m \in \mathbb{N}$. Then $X = \bigcup\limits_{n\in\mathbb{N}} \operatorname{cl} B(x_n, 1/m)$. This implies that there exists $n_m \in \mathbb{N}$ such that

$$\mu\bigg(\bigcup_{n=1}^{n_m} \operatorname{cl} B(x_n, 1/m)\bigg) > \mu(X) - 2^{-m-1}\epsilon$$

Set

$$K = \bigcap_{m\in\mathbb{N}} \bigcup_{n=1}^{n_m} \operatorname{cl} B(x_n, 1/m)$$

Then K is closed. Let $\delta > 0$. Choose $m_\delta \in \mathbb{N}$ such that $1/m_\delta < \delta$. Then

$$\begin{aligned}K &= \bigcap_{m\in\mathbb{N}} \bigcup_{n=1}^{n_m} \operatorname{cl} B(x_n, 1/m) \\ &\subset \bigcup_{n=1}^{n_{m_\delta}} \operatorname{cl} B(x_n, 1/m_\delta) \\ &\subset \bigcup_{n=1}^{n_{m_\delta}} B(x_n, \delta)\end{aligned}$$

Hence K is totally bounded. Since X is complete, K is compact. Finally, we have that

$$\begin{aligned}
\mu(X) - \mu(K) &= \mu(K^c) \\
&= \mu\bigg(\bigcup_{m\in\mathbb{N}} \bigg[\bigcup_{n=1}^{n_m} \operatorname{cl} B(x_n, 1/m)\bigg]^c\bigg) \\
&\leq \sum_{m\in\mathbb{N}} \mu\bigg(\bigg[\bigcup_{n=1}^{n_m} \operatorname{cl} B(x_n, 1/m)\bigg]^c\bigg) \\
&= \sum_{m\in\mathbb{N}} \bigg[\mu(X) - \mu\bigg(\bigcup_{n=1}^{n_m} \operatorname{cl} B(x_n, 1/m)\bigg)\bigg] \\
&\leq \sum_{m\in\mathbb{N}} 2^{-m-1}\epsilon \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

So for each $\epsilon > 0$, there exists $K \subset X$ such that K is compact and $\mu(K) > \mu(X) - \epsilon$. Thus

$$\mu(X) = \sup\{\mu(K) : K \subset X \text{ and } K \text{ is compact}\}$$

and μ is inner regular on X. The previous exercise implies that μ is inner regular. □

Exercise 9.4.0.8. Ulam's Theorem:
Let X be a Polish space and $\mu : \mathcal{B}(X) \to [0, \infty)$ a finite measure. Then μ is regular and Radon.

Proof. Clear by preceeding exercises (list exercises specifically). □

Note 9.4.0.9. Recall definition of ν_μ. We will mean the restriction of ν_μ to $\mathcal{B}(X)$.

Exercise 9.4.0.10. Suppose that ν is Radon. Show that $\nu_\mu(E) = \nu(E \cap \operatorname{supp} \mu)$ for each $E \in \mathcal{B}(X)$

Exercise 9.4.0.11. Suppose that μ is Radon. Then $\nu_\mu : \mathcal{B}(X) \to [0, \infty]$ is Radon. Show that $\nu_\mu(E) = \nu(E \cap \operatorname{supp} \mu)$ for each $E \in \mathcal{B}(X)$.

9.4.1 Weak* Convergence

Definition 9.4.1.1. Let X be a topological space. For $f \in C_b(X)$, define $\lambda_f : \mathcal{M}(X) \to \mathbb{C}$ by

$$\lambda_f(\mu) = \int f \, d\mu$$

.

Exercise 9.4.1.2. Let X be a topological space. For each $f \in C_b(X)$, $\lambda_f \in \mathcal{M}(X)^*$.

Proof. Let $f \in C_b(X)$ and $\mu \in \mathcal{M}(X)$. Then

$$\begin{aligned} |\lambda_f(\mu)| &= \left| \int f \, d\mu \right| \\ &\leq \int |f| \, d|\mu| \\ &\leq \|f\|_u \|\mu\| \end{aligned}$$

Exercise 7.3.0.22 implies that λ_f is linear. So $\lambda_f \in \mathcal{M}(X)^*$. □

Definition 9.4.1.3. Let X be a topological space. We define the **weak topology on** $\mathcal{M}(X)$ to be the weak topology generated by $\{\lambda_f \in \mathcal{M}(X)^* : f \in C_b(X)\}$.

Definition 9.4.1.4. Let X be a topological space and $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Then $(\mu_n)_{n\in\mathbb{N}}$ is said to **converge weakly** to μ, denoted $\mu_n \xrightarrow{w} \mu$, if $(\mu_n)_{n\in\mathbb{N}}$ converges to μ in the weak topology, i.e. for each $f \in C_b(X)$,

$$\int f \, d\mu_n \to \int f \, d\mu$$

Exercise 9.4.1.5. Portmanteau Theorem: Let X be a topological space and $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$. Suppose that for each $n \in \mathbb{N}$, $\mu_n(X) = \mu(X)$. Then the following are equivalent:

1. $\mu_n \xrightarrow{w} \mu$
2. for each $A \in \mathcal{B}(X)$, A is open implies that $\mu(A) \leq \liminf\limits_{n\to\infty} \mu_n(A)$
3. for each $A \in \mathcal{B}(X)$, A is closed implies that $\mu(A) \geq \limsup\limits_{n\to\infty} \mu_n(A)$
4. for each $A \in \mathcal{B}(X)$, $\mu(\partial A) = 0$ implies that $\mu_n(A) \to \mu(A)$

Proof.

- (2) $\iff$ (3):
  Suppose (2). Let $A \in \mathcal{B}(X)$. Suppose that A is closed. Then A^c is open. By assumption, $\mu(A^c) \leq \liminf\limits_{n\to\infty} \mu_n(A^c)$. Hence

$$\begin{aligned} \mu(A) &= \mu(X) - \mu(A^c) \\ &\geq \mu(X) - \liminf_{n\to\infty} \mu_n(A^c) \\ &= \mu(X) + \limsup_{n\to\infty} [-\mu_n(A^c)] \\ &= \limsup_{n\to\infty} \bigg[\mu(X) - \mu_n(A^c)\bigg] \\ &= \limsup_{n\to\infty} \bigg[\mu_n(X) - \mu_n(A^c)\bigg] \\ &= \limsup_{n\to\infty} \mu_n(A) \end{aligned}$$

  So (3) holds. Similarly, (3) implies (2).

- (2) $\iff$ (4):
  Suppose (2). From above, (3) holds. Let $A \in \mathcal{B}(X)$. Then

$$\begin{aligned}\mu(A^\circ) &\leq \liminf_{n\to\infty} \mu_n(A^\circ)\\ &\leq \liminf_{n\to\infty} \mu_n(A)\\ &\leq \limsup_{n\to\infty} \mu_n(A)\\ &\leq \limsup_{n\to\infty} \mu_n(\overline{A})\\ &\leq \mu(\overline{A})\end{aligned}$$

  Suppose that $\mu(\partial A) = 0$. Then

$$\begin{aligned}\mu(A^\circ) &\leq \mu(A)\\ &\leq \mu(\overline{A})\\ &= \mu(A^\circ) + \mu(\partial A)\\ &= \mu(A^\circ)\end{aligned}$$

  which implies that $\mu_n(A) \to \mu(A)$. Conversely, suppose (4).

-

$\square$

9.5 Disintegration of Radon Measures

Exercise ?? Exercise 9.2.2.14

Definition 9.5.0.1. Define disintegration here or in section 6.6.1.19? Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{B})$ a measurable space, $\pi : X \to Y$ and $(\mu_y)_{y \in Y} \subset M_+(X, \mathcal{A})$. Suppose that π is $(\mathcal{A}, \mathcal{B})$-measurable. Then $(\mu_{|y})_{y \in Y}$ is said to be a **disintegration of μ with respect to π** if

1. for each $A \in \mathcal{A}$, $\mu_{|\cdot}(A) \in \text{Ł}^+(Y, \mathcal{B})$,
2. for each $A \in \mathcal{A}$,
$$\mu(A) = \int \mu_{|y}(A) \, d\pi_*\mu(y)$$
3. for each $f \in L^1(X, \mathcal{A}, \mu)$,
$$\int f \, d\mu = \int \left[\int f \, d\mu_{|y}(x) \right] d\pi_*\mu(y)$$

Exercise 9.5.0.2. Disintegration of Measure:
Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $(Y, \mathcal{B})$ a measurable space and $g : X \to Y$. Suppose that for each $y \in Y$, $\{y\} \in \mathcal{B}$, g is surjective and g is $(\mathcal{A}, \mathcal{B})$-measurable. Then there exists a collection of measures $(\mu_y)_{y \in Y}$ such that

1. for each $A \in \mathcal{A}$,
$$\mu(A) = \int \mu_y(A) \, dg_*\mu(y)$$
2. for each $f \in L^1(X, \mathcal{A}, \mu)$,
$$\int f \, d\mu = \int \left[\int f \, d\mu_y(x) \right] dg_*\mu(y)$$

Exercise 9.5.0.3. Let X be a Polish space. Then there exists

Exercise 9.5.0.4. Let X be a locally compact Polish space nonarchimedean.

9.6 Projective Systems of Complex Measures

need to move this section or include sections on other structures like products, subspaces, etc

Definition 9.6.0.1. Let $(J, \leq)$ be a directed set $((X_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$ a projective system of topological spaces and $(\mu_j)_{j\in J} \in \prod_{j\in J} M(X_j)$. Set $(X, (\pi_j)_{j\in J}) := \varprojlim_{j\in J}((X_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$. Suppose that $(X, (\pi_j)_{j\in J})$ is perfect. For each $j \in J$, we define the **quotient measure induced by** π_j, denoted $\mu_j^Q \in M(X_j^Q)$, by $\mu_j^Q := (\bar{\pi}_j^{-1})_*\mu_j$.

Exercise 9.6.0.2. Let $(J, \leq)$ be a directed set $((X_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$ a projective system of topological spaces and $(\mu_j)_{j\in J} \in \prod_{j\in J} M(X_j)$. Set $(X, (\pi_j)_{j\in J}) := \varprojlim_{j\in J}((X_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$. Suppose that $(X, (\pi_j)_{j\in J})$ is perfect and $((X_j, \mathcal{B}(X_j), \mu_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$ is a projective system of complex measure spaces. Then $((X_j^Q, \mathcal{B}(X_j^Q), \mu_j^Q), (\pi_{j,k}^Q)_{(j,k)\in\leq})$ is a projective system of complex measure spaces.

Proof. Let $(j,k) \in \leq$. Since $((X_j, \mathcal{B}(X_j), \mu_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$ is a projective system of complex measure spaces, $(\pi_{j,k})_*\mu_k = \mu_j$. Therefore

$$\begin{aligned}
(\pi_{j,k}^Q)_*\mu_k^Q &= (\bar{\pi}_j^{-1} \circ \pi_{j,k} \circ \bar{\pi}_k)_*[(\bar{\pi}_k^{-1})_*\mu_k] \\
&= (\bar{\pi}_j^{-1} \circ \pi_{j,k} \circ \bar{\pi}_k \circ \bar{\pi}_k^{-1})_*\mu_k \\
&= (\bar{\pi}_j^{-1} \circ \pi_{j,k})_*\mu_k \\
&= (\bar{\pi}_j^{-1})_*[(\pi_{j,k})_*\mu_k] \\
&= (\bar{\pi}_j^{-1})_*\mu_j \\
&= \mu_j^Q.
\end{aligned}$$

Since $(j,k) \in \leq$ is arbitrary, we have that for each $(j,k) \in \leq$, $(\pi_{j,k}^Q)_*\mu_k^Q = \mu_j^Q$. Hence $((X_j^Q, \mathcal{B}(X_j^Q, \mu_j^Q), (\pi_{j,k}^Q)_{(j,k)\in\leq})$ is a projective system of complex measure spaces. □

Exercise 9.6.0.3. Let $(J, \leq)$ be a countable directed set $((X_{j\in J}), (\pi_{j,k})_{(j,k)\in\leq})$ be a projective system of topological spaces and $(\mu_j)_{j\in J} \in \prod_{j\in J} \mathcal{M}(X_j)$. Suppose that $((X_j, \mathcal{B}(X_j), \mu_j)_{j\in J}, (\pi_{j,k})_{(j,k)\in\leq})$ is a projective system of complex measure spaces and for each $j \in J$, X_j is a compact Hausdorff space. Set $(X, (\pi_j)_{j\in J}) := \varprojlim_{j\in J}((X_j)_{j\in\mathbb{N}}, (\pi_{j,k})_{(j,k)\in\leq})$.

1. For each $j \in J$, define $C_j \subset C(X)$ by

$$C_j := \{f \in C(X) : \text{ for each } x, y \in X,\ \pi_j(x) = \pi_j(y) \text{ implies that } f(x) = f(y)\}$$

   Define $C \subset C(X)$ by $C := \bigcup_{j\in J} C_j$. Then C is a subalgebra of $C(X)$ and C is dense in $C(X)$.

2. For each $j \in J$, define $\sim_j \subset X \times X$ by $x \sim_j y$ iff $\pi_j(x) = \pi_j(y)$. For each $j \in J$, denote the projection of X onto $X/{\sim_j}$ by $\pi_j^Q : X \to X/{\sim_j}$. Then for each $j \in J$,

   (a) $C_j = \{f \in C(X) : f \text{ is } \sim_j\text{-invariant}\}$

   (b) there exists a unique $\bar{\pi} : X/{\sim_j} \to X_j$ such that $\bar{\pi}_j \circ \pi_j^Q = \pi_j$ and $\bar{\pi}_j$ is a homeomorphism,

   (c) for each $f \in C_j$, there exists a unique $\bar{f} : X/{\sim_j} \to \mathbb{C}$ such that $\bar{f} \circ \bar{\pi}_j^{-1} \circ \pi_j = f$

   for each $j \in J$, $C_j = \{f \in C(X) : f \text{ is } \sim_j\text{-invariant}\}$. there exists $\phi : X/{\sim_j} \to X_j$ such that ϕ homeomorphic to X_j and and $\phi_j \in C(X/{\sim_j})^*$ by $\phi_j(\bar{f}) = \int$. Then

3.

Proof.

1. Let $f_1, f_2 \in C$ and $\lambda \in \mathbb{C}$. Then there exist $j_1, j_2 \in J$ such that $f_1 \in C_{j_1}$ and $f_2 \in C_{j_2}$. Since J is directed, there exists $j_0 \in J$ such that $j_0 \geq j_1, j_2$. Let $x, y \in X$. Suppose that $\pi_{j_0}(x) = \pi_{j_0}(y)$. Then

$$\begin{aligned} \pi_{j_1}(x) &= \pi_{j_1,j_0} \circ \pi_{j_0}(x) \\ &= \pi_{j_1,j_0} \circ \pi_{j_0}(y) \\ &= \pi_{j_1}(y) \end{aligned}$$

and similarly, $\pi_{j_2}(x) = \pi_{j_2}(y)$. Therefore $f_1(x) = f_1(y)$ and $f_2(x) = f_2(y)$. Hence

$$\begin{aligned} (f_1 + \lambda f_2)(x) &= f_1(x) + \lambda f_2(x) \\ &= f_1(y) + \lambda f_2(y) \\ &= (f_1 + \lambda f_2)(y) \end{aligned}$$

and

$$\begin{aligned} (f_1 \cdot f_2)(x) &= f_1(x) f_2(x) \\ &= f_1(y) f_2(y) \\ &= (f_1 \cdot f_2)(y). \end{aligned}$$

Since $x, y \in X$ with $\pi_{j_0}(x) = \pi_{j_0}(y)$ are arbitrary, we have that for each $x, y \in X$, $\pi_{j_0}(x) = \pi_{j_0}(y)$ implies that $(f_1 + \lambda f_2)(x) = (f_1 + \lambda f_2)(y)$ and $(f_1 \cdot f_2)(x) = (f_1 \cdot f_2)(y)$. Thus

$$\begin{aligned} f_1 + \lambda f_2 &\in C_{j_0} \\ &\subset C \end{aligned}$$

and

$$\begin{aligned} f_1 \cdot f_2 &\in C_{j_0} \\ &\subset C \end{aligned}$$

Since $f_1, f_2 \in C$ and $\lambda \in \mathbb{C}$ are arbitrary, for each $f_1, f_2 \in C$ and $\lambda \in \mathbb{C}$, $f_1 + \lambda f_2 \in C$ and $f_1 \cdot f_2 \in C$. Hence C is a subalgebra of $C(X)$.

2.

$\square$

DELETE:Exercise 3.8.0.2

idea: use this consistency theorem to obtain guassian measure μ on $\mathbb{C}^{\mathbb{N}}$ with the product sigma algebra where $(\pi_F)_*\mu$ is a gaussian measure on $\mathbb{C}^{|F|}$ for finite subsets $F \subset \mathbb{N}$ with a prescribed mean and covariance function, then try to generalize theorem 2.4 in the book on gaussian measures in hilbert spaces, for rkhs.

Exercise 9.6.0.4. Kolmogorov Extension Theorem:
Let $(J, \leq)$ be a directed set and $((X_j, \mathcal{T}_j, \mu_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$ a **TopRadMsr**$_+$-projective system. Set $X := \varprojlim_{j \in J} X_j$ and $\mathcal{T} := \varprojlim_{j \in J} T_j$. Suppose that

- J is countable
- for each $j \in J$, X_j is a Polish space and π_j is surjective
- $\sup_{j \in J} \mu_j(X_j) < \infty$

Define $\mathcal{A}_0 \subset \mathcal{P}(X)$ by $\mathcal{A}_0 := \bigcup_{j \in J} \pi_j^* \mathcal{B}(X_j, \mathcal{T}_j)$ and define $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ by $\mu_0(\pi_j^{-1}(E)) := \mu_j(E)$ $(j \in J, E \in \mathcal{B}(X_j, \mathcal{T}_j))$. Then

1. $\mathcal{A}_0$ is an algebra on X and $\sigma(\mathcal{A}_0) = \mathcal{B}(X, \mathcal{T})$
2. μ_0 is well-defined
3. $\mu_0(X) = \sup\limits_{j \in J} \mu_j(X_j)$
4. μ_0 is finitely-additive.
5. Let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$. Then there exists $(j_n)_{n\in\mathbb{N}} \in J$ and $(E_n)_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} \mathcal{B}(X_{j_n})$ such that
    - for each $n \in \mathbb{N}$, $B_n = \pi_{j_n}^{-1}(E_n)$
    - for each $j \in J$, there exists $n \in \mathbb{N}$ such that $j \leq j_n$
6. Suppose that $(B_n)_{n\in\mathbb{N}}$ is decreasing and $\inf\limits_{n\in\mathbb{N}} B_n = \varnothing$. Set $\delta := \inf\limits_{n\in\mathbb{N}} \mu_0(B_n)$. Suppose that $\delta > 0$. Then
    (a) for each $n \in \mathbb{N}$, there exists $K_n \subset E_n$ such that K_n is compact and $\mu_0(E_n \setminus K_n) < \delta 2^{-n}$
    (b) there exists $(D_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$ such that
        - for each $n \in \mathbb{N}$ and $m \in [n]$, $\pi_{j_m}(D_n) \subset K_m$
        - $\bigcap\limits_{n\in\mathbb{N}} D_n = \varnothing$
        - $\mu_0(B_n \setminus D_n) < \delta$
    (c) for each $n \in \mathbb{N}$, $\mu_0(D_n) > 0$
    (d) there exists $(x_n)_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} D_n$ and $(y_{0,m})_{m\in\mathbb{N}} \in \prod\limits_{m\in\mathbb{N}} K_m$ such that for each $m \in \mathbb{N}$, $\lim\limits_{n\to\infty} \pi_{j_m}(x_n) = y_{0,m}$
    Hint: sequential compactness
    (e) there exists $y_0 \in X$ such that for each $m \in \mathbb{N}$, $\pi_{j_m}(y_0) = y_{0,m}$ and $y_0 \in \bigcap\limits_{m\in\mathbb{N}} B_m$
    Hint: diagonal argument
7. μ_0 is a premeasure on $(X, \mathcal{A}_0)$
   Hint: Exercise 5.3.0.8
8. there exists a unique measure $\mu \in \mathcal{M}_+(X)$ such that $((X,\mu),(\pi_j)_{j\in J})$ is a **TopRadMsr**$_+$-projective limit of $((X_j,\mu_j)_{j\in J}$, $(\pi_{j,k})_{(j,}$

Proof.

1. Exercise 3.8.0.4 implies that $\mathcal{A}_0$ is an algebra on X. By definition, $\sigma(\mathcal{A}_0) = \varprojlim\limits_{j\in J} \mathcal{B}(X_j, \mathcal{T}_j)$. Exercise 3.8.0.3 implies that

$$\begin{aligned} \sigma(\mathcal{A}_0) &= \varprojlim_{j\in J} \mathcal{B}(X_j, \mathcal{T}_j) \\ &= \mathcal{B}\left(\varprojlim_{j\in J} X_j, \varprojlim_{j\in J} \mathcal{T}_j\right) \end{aligned}$$

2. Let $A_1, A_2 \in \mathcal{A}_0$. Then there exist $j_1, j_2 \in J$ and $E_1 \in \mathcal{B}(X_{j_1}), E_2 \in \mathcal{B}(X_{j_2})$ such that $A_1 = \pi_{j_2}^{-1}(E_1)$ and $A_2 = \pi_{j_2}^{-1}(E_2)$. Suppose that $A_1 = A_2$. Since J is a directed set, there exists $j_0 \in J$ such that $j_1, j_2 \leq j_0$. Therefore

$$\begin{aligned} \pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1)) &= (\pi_{j_1,j_0} \circ \pi_{j_0})^{-1}(E_1) \\ &= \pi_{j_1}^{-1}(E_1) \\ &= A_1 \\ &= A_2 \\ &= \pi_{j_2}^{-1}(E_2) \\ &= (\pi_{j_2,j_0} \circ \pi_{j_0})^{-1}(E_2) \\ &= \pi_{j_0}^{-1}(\pi_{j_2,j_0}^{-1}(E_2)). \end{aligned}$$

Since $(X, (\pi_j)_{j\in J})$ is perfect, π_{j_0} is surjective. Hence

$$\begin{aligned}\pi_{j_1,j_0}^{-1}(E_1) &= \pi_{j_0}(\pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1)))\\ &= \pi_{j_0}(\pi_{j_0}^{-1}(\pi_{j_2,j_0}^{-1}(E_2)))\\ &= \pi_{j_2,j_0}^{-1}(E_2).\end{aligned}$$

Hence

$$\begin{aligned}\mu_0(A_1) &= \mu_0(\pi_{j_1}^{-1}(E_1))\\ &= \mu_0((\pi_{j_1,j_0}\circ\pi_{j_0})^{-1}(E_1))\\ &= \mu_0(\pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1)))\\ &= \mu_0(\pi_{j_0}^{-1}(\pi_{j_2,j_0}^{-1}(E_2)))\\ &= \mu_0((\pi_{j_2,j_0}\circ\pi_{j_0})^{-1}(E_2))\\ &= \mu_0(\pi_{j_2}^{-1}(E_2))\\ &= \mu_0(A_2).\end{aligned}$$

Thus μ_0 is well-defined.

3. Set $S := \sup_{j\in J}\mu_j(X_j)$. Since J is a directed set, $J \neq \varnothing$. Thus there exists $j_0 \in J$. Then

$$\begin{aligned}\mu_0(X) &= \mu_0(\pi_j^{-1}(X_j))\\ &= \mu_{j_0}(X_{j_0})\\ &\leq S.\end{aligned}$$

Let $\epsilon > 0$. Since $S < \infty$, $S - \epsilon < S$. Hence there exists $j_0 \in J$ such that $\mu_{j_0}(X_{j_0}) > S - \epsilon$. Then

$$\begin{aligned}\mu_0(X) &= \mu_0(\pi_j^{-1}(X_j))\\ &= \mu_{j_0}(X_{j_0})\\ &\geq S - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have that $\mu_0(X) \geq S$. Hence $\mu_0(X) = S$.

4. Let $A_1, A_2 \in \mathcal{A}_0$. Then there exist $j_1, j_2 \in J$ and $E_1 \in \mathcal{B}(X_{j_1})$ and $E_2 \in \mathcal{B}(X_{j_2})$ such that $A_1 = \pi_{j_1}^{-1}(E_1)$ and $A_2 = \pi_{j_2}^{-1}(E_2)$. Since J is directed, there exists $j_0 \in J$ such that $j_1, j_2 \leq j_0$. Suppose that $A_1 \cap A_2 = \varnothing$. Then

$$\begin{aligned}\varnothing &= A_1 \cap A_2\\ &= \pi_{j_1}^{-1}(E_1) \cap \pi_{j_2}^{-1}(E_2)\\ &= (\pi_{j_1,j_0}\circ\pi_{j_0})^{-1}(E_1) \cap (\pi_{j_2,j_0}\circ\pi_{j_0})^{-1}(E_2)\\ &= \pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1)) \cap \pi_{j_0}^{-1}(\pi_{j_2,j_0}^{-1}(E_2))\\ &= \pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1) \cap \pi_{j_2,j_0}^{-1}(E_2)).\end{aligned}$$

Since $(X, (\pi_j)_{j\in J})$ is perfect, π_{j_0} is surjective. Therefore

$$\begin{aligned}\varnothing &= \pi_{j_0}(\varnothing)\\ &= \pi_{j_0}(\pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1) \cap \pi_{j_2,j_0}^{-1}(E_2)))\\ &= \pi_{j_1,j_0}^{-1}(E_1) \cap \pi_{j_2,j_0}^{-1}(E_2).\end{aligned}$$

We note that

$$\begin{aligned}A_1 \cup A_2 &= \pi_{j_1}^{-1}(E_1) \cup \pi_{j_2}^{-1}(E_2)\\ &= (\pi_{j_1,j_0}\circ\pi_{j_0})^{-1}(E_1) \cup (\pi_{j_2,j_0}\circ\pi_{j_0})^{-1}(E_2)\\ &= \pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1)) \cup \pi_{j_0}^{-1}(\pi_{j_2,j_0}^{-1}(E_2))\\ &= \pi_{j_0}^{-1}(\pi_{j_1,j_0}^{-1}(E_1) \cup \pi_{j_2,j_0}^{-1}(E_2)).\end{aligned}$$

Since μ_{j_0} is a complex measure and $\pi^{-1}_{j_1,j_0}(E_1) \cap \pi^{-1}_{j_2,j_0}(E_2) = \varnothing$, we have that

$$\begin{aligned}
\mu_0(A_1 \cup A_2) &= \mu_0(\pi^{-1}_{j_0}(\pi^{-1}_{j_1,j_0}(E_1) \cup \pi^{-1}_{j_2,j_0}(E_2))) \\
&= \mu_{j_0}(\pi^{-1}_{j_1,j_0}(E_1) \cup \pi^{-1}_{j_2,j_0}(E_2)) \\
&= \mu_{j_0}(\pi^{-1}_{j_1,j_0}(E_1)) + \mu_{j_0}(\pi^{-1}_{j_2,j_0}(E_2)) \\
&= \mu_0(\pi^{-1}_{j_0}(\pi^{-1}_{j_1,j_0}(E_1))) + \mu_0(\pi^{-1}_{j_0}(\pi^{-1}_{j_2,j_0}(E_2))) \\
&= \mu_0((\pi_{j_1,j_0} \circ \pi_{j_0})^{-1}(E_1)) + \mu_0((\pi_{j_2,j_0} \circ \pi_{j_0})^{-1}(E_2)) \\
&= \mu_0(\pi^{-1}_{j_1}(E_1)) + \mu_0(\pi^{-1}_{j_2}(E_2)) \\
&= \mu_0(A_1) + \mu_0(A_2).
\end{aligned}$$

Thus μ_0 is finitely-additive. need exercise showing that finitely additive functions are increasing wrt inclusion

5. Since J is countable (assume), we may write $J = (j^1_n)_{n \in \mathbb{N}}$. By definition of $\mathcal{A}_0$, for each $n \in \mathbb{N}$, there exists $j^2_n \in J$ and $E^2_n \in \mathcal{B}(X_{j^2_n})$ such that $B_n = \pi^{-1}_{j^2_n}(E^2_n)$. The axiom of countable choice implies that there exists $(j^2_n)_{n \in \mathbb{N}} \subset J$ and $(E^2_n)_{n \in \mathbb{N}} \in \prod\limits_{n \in \mathbb{N}} \mathcal{B}(X_{j^2_n})$ such that for each $n \in \mathbb{N}$, $B_n = \pi^{-1}_{j^2_n}(E^2_n)$.

   Since J is directed, for each $n \in \mathbb{N}$, there exists $j_n \in J$ such that $j^1_n, j^2_n \leq j_n$. The axiom of countable choice implies that there exists $(j_n)_{n \in \mathbb{N}} \subset J$ such that for each $n \in \mathbb{N}$, $j^1_n, j^2_n \leq j_n$. For each $n \in \mathbb{N}$, define $E_n := \pi^{-1}_{j^2_n, j_n}(E^2_n)$.

   - Then for each $n \in \mathbb{N}$,

   $$\begin{aligned}
   B_n &= \pi^{-1}_{j^2_n}(E^2_n) \\
   &= (\pi_{j^2_n, j_n} \circ \pi_{j_n}) - 1(E^2_n) \\
   &= \pi^{-1}_{j_n}(\pi^{-1}_{j^2_n, j_n}(E^2_n)) \\
   &= \pi^{-1}_{j_n}(E_n).
   \end{aligned}$$

   - Let $j \in J$. Then there exists $n \in \mathbb{N}$ such that $j = j^1_n$. By construction,

   $$\begin{aligned}
   j &= j^1_n \\
   &\leq j_n.
   \end{aligned}$$

   Since $j \in J$ is arbitrary, we have that for each $j \in J$, there exists $n \in \mathbb{N}$ such that $j \leq j_n$.

6. (a) Let $n \in \mathbb{N}$. Since μ_{j_n} is Radon and $\mu_{j_n}(E_n) < \infty$, there exists $K_n \subset E_n$ such that K_n is compact and $\mu_{j_n}(E_n \setminus K_n) < \delta 2^{-n}$.

   (b) Define $(C_n)_{n \in \mathbb{N}} \subset \mathcal{A}_0$ and $(D_n)_{n \in \mathbb{N}} \subset \mathcal{A}_0$ by $C_n := \pi^{-1}_{j_n}(K_n)$ and $D_n := \bigcap_{m=1}^n C_m$.

   - Let $n \in \mathbb{N}$ and $m \in [n]$. Since π_{j_m} is surjective (assume),

   $$\begin{aligned}
   \pi_{j_m}(D_n) &\subset \pi_{j_m}(C_m) \\
   &= \pi_{j_m}(\pi^{-1}_{j_m}(K_m)) \\
   &= K_m.
   \end{aligned}$$

   - For each $n \in \mathbb{N}$,

   $$\begin{aligned}
   D_n &\subset C_n \\
   &\subset B_n.
   \end{aligned}$$

   and therefore

   $$\begin{aligned}
   \bigcap_{n \in \mathbb{N}} D_n &\subset \bigcap_{n \in \mathbb{N}} B_n \\
   &= \varnothing.
   \end{aligned}$$

- By construction, for each $m \in \mathbb{N}$,

$$\begin{aligned}\mu_0(B_m \setminus C_m) &= \mu_0\left[\pi_{j_m}^{-1}(E_m) \cap (\pi_{j_m}^{-1}(K_m)^c)\right] \\ &= \mu_0\left[\pi_{j_m}^{-1}(E_m) \cap \pi_{j_m}^{-1}(K_m^c)\right] \\ &= \mu_0\left[\pi_{j_m}^{-1}(E_m \cap K_m^c)\right] \\ &= \mu_0\left[\pi_{j_m}^{-1}(E_m \setminus K_m)\right] \\ &= \mu_{j_m}(E_m \setminus K_m) \\ &< \frac{\delta}{2^m}.\end{aligned}$$

Since μ_0 is finitely additive, Exercise **??** (ref ex here) implies that μ_0 is finitely-subadditive. Since $(B_n)_{n\in\mathbb{N}}$ is decreasing, we have that for each $n \in \mathbb{N}$,

$$\begin{aligned}\mu_0(B_n \setminus D_n) &= \mu_0\left[B_n \cap (\bigcap_{m=1}^{n} C_m)^c\right] \\ &= \mu_0\left[B_n \cap (\bigcup_{m=1}^{n} C_m^c)\right] \\ &= \mu_0\left[\bigcup_{m=1}^{n} (B_n \cap C_m^c)\right] \\ &\leq \mu_0\left[\bigcup_{m=1}^{n} (B_m \cap C_m^c)\right] \\ &\leq \sum_{m=1}^{n} \mu_0(B_m \cap C_m^c) \\ &= \sum_{m=1}^{n} \mu_0(B_m \setminus C_m) \\ &< \sum_{m=1}^{n} \frac{\delta}{2^m} \\ &< \delta.\end{aligned}$$

(c) Let $n \in \mathbb{N}$. Since

$$\begin{aligned}\mu_0(B_n) &= \mu_{j_n}(E_n) \\ &< \mu_{j_n}(X_n) \\ &< \infty,\end{aligned}$$

Exercise **??** (ref exercise about finitely additive set functions) implies

$$\begin{aligned}\mu_0(B_n \setminus D_n) &= \mu_0(B_n) - \mu_0(D_n) \\ &\geq \delta - \mu_0(D_n).\end{aligned}$$

Thus

$$\begin{aligned}\mu_0(D_n) &\geq \delta - \mu_0(B_n \setminus D_n) \\ &> \delta - \delta \\ &= 0.\end{aligned}$$

(d) Let $n \in \mathbb{N}$. Since $\mu_0(D_n) > 0$, $D_n \neq \varnothing$. Since $n \in \mathbb{N}$ is arbitrary, we have that for each $n \in \mathbb{N}$, $D_n \neq \varnothing$. The axiom of countable choice implies that there exists $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} D_n$. For each $m, n \in \mathbb{N}$, set

$$x_{n,m} := \begin{cases} \pi_{j_m}(x_n), & n \geq m \\ \pi_{j_m}(x_m), & n < m \end{cases}$$

Then for each $m, n \in \mathbb{N}$, $x_{n,m} \in K_m$. Let $m \in \mathbb{N}$. Since X_{j_m} is Hausdorff, K_m is Hausdorff and thus K_m is first-countable. Since K_m is compact, an exercise in the analysis notes section on sequential compactness implies that K_m is sequentially compact. Since $m \in \mathbb{N}$ is arbitrary, we have that for each $m \in \mathbb{N}$, K_m is sequentially compact. Since $(x_{n,1})_{n\in\mathbb{N}} \subset K_1$ and K_1 is sequentially compact, there exist $\phi_1 : \mathbb{N} \to \mathbb{N}$ and $y_{0,1} \in K_1$ such that ϕ_1 is strictly increasing and $x_{\phi_1(n),1} \to y_{0,1}$. Since $(x_{\phi_1(n),2})_{n\in\mathbb{N}} \subset K_2$ and K_2 is sequentially compact, there exists $\phi_2 : \mathbb{N} \to \mathbb{N}$ and $y_{0,2} \in K_2$ such that ϕ_2 is strictly increasing and $x_{\phi_1(\phi_2(n)),2} \to y_{0,2}$. Continuing inductively, for each $m \in \mathbb{N}$, there exist $\phi_m : \mathbb{N} \to \mathbb{N}$ and $y_{0,m} \in K_m$ such that ϕ_m is strictly increasing and $x_{\phi_1\circ\cdots\circ\phi_m(n),m} \to y_{0,m}$. For each $m \in \mathbb{N}$, define $\psi_m : \mathbb{N} \to \mathbb{N}$ by $\psi_m := \phi_1 \circ \cdots \circ \phi_m$. Then for each $m \in \mathbb{N}$, ψ_m is strictly increasing and $x_{\psi_m(n),m} \to y_{0,m}$. For each $m, n \in \mathbb{N}$, define $y_{n,m} \in K_m$ by $y_{n,m} := x_{\psi_n(n),m}$. Let $m \in \mathbb{N}$. Since $(y_{n,m})_{n\in\mathbb{N}}$ is a subsequence of $(x_{\psi_m(n),m})_{n\in\mathbb{N}}$ and $x_{\psi_m(n),m} \xrightarrow{n} y_{0,m}$, we have that $y_{n,m} \xrightarrow{n} y_{0,m}$. Since $m \in \mathbb{N}$ is arbitrary, we have that for each $m \in \mathbb{N}$, $y_{n,m} \xrightarrow{n} y_{0,m}$.

(e) For each $j \in J$, set $A_j := \{m \in \mathbb{N} : (j, j_m) \in \leq\}$. Define $y_0 \in \prod_{j\in J} X_j$ by

$$\mathrm{proj}_j(y_0) := \pi_{j,j_m}(y_{0,m}), \qquad m \in A_j$$

- Let $j \in J$. A previous part implies that there exists $m \in \mathbb{N}$ such that $j \leq j_m$. Then $m \in A_j$ and $A_j \neq \varnothing$. Since $j \in J$ is arbitrary, we have that for each $j \in J$, $A_j \neq \varnothing$.
- Let $j \in J$ and $m_1, m_2 \in A_j$. Then

$$\begin{aligned}
\pi_{j,j_{m_1}}(y_{0,m_1}) &= \pi_{j,j_{m_1}}(\lim_{n\to\infty} y_{n,m_1}) \\
&= \lim_{n\to\infty} \pi_{j,j_{m_1}}(y_{n,m_1}) \\
&= \lim_{n\to\infty} \pi_{j,j_{m_1}}(x_{\psi_n(n),m_1}) \\
&= \lim_{n\to\infty} \pi_{j,j_{m_1}}(\pi_{j_{m_1}}(x_{\psi_n(n)})) \\
&= \lim_{n\to\infty} \pi_j(x_{\psi_n(n)}) \\
&= \lim_{n\to\infty} \pi_{j,j_{m_2}}(\pi_{j_{m_2}}(x_{\psi_n(n)})) \\
&= \lim_{n\to\infty} \pi_{j,j_{m_2}}(x_{\psi_n(n),m_2}) \\
&= \pi_{j,j_{m_2}}(\lim_{n\to\infty} y_{n,m_2}) \\
&= \pi_{j,j_{m_2}}(y_{0,m_2})
\end{aligned}$$

Therefore y_0 is well-defined. Let $(j,k) \in \leq$. A previous part implies that there exists $n \in \mathbb{N}$ such that $k \leq j_n$. Then $n \in A_k$, $n \in A_j$ and

$$\begin{aligned}
\pi_{j,k} \circ \mathrm{proj}_k(y_0) &= \pi_{j,k} \circ \pi_{k,j_n}(y_0, n) \\
&= \pi_{j,j_n}(y_0, n) \\
&= \mathrm{proj}_j(y_0).
\end{aligned}$$

Since $(j,k) \in \leq$ is arbitrary, we have that for each $(j,k) \in \leq$, $\pi_{j,k} \circ \mathrm{proj}_k(y_0) = \mathrm{proj}_j(y_0)$. Thus $y_0 \in X$. By construction, for each $m \in \mathbb{N}$,

$$\begin{aligned}
\pi_{j_m}(y_0) &= \mathrm{proj}_{j_m}|_X(y_0) \\
&= \pi_{j_m,j_m}(y_{0,m}) \\
&= \mathrm{id}_{X_m}(y_{0,m}) \\
&= y_{0,m} \\
&\in K_m.
\end{aligned}$$

Thus for each $m \in \mathbb{N}$,

$$\begin{aligned} y_0 &\in \pi_{j_m}^{-1}(K_m) \\ &= C_m. \end{aligned}$$

Hence

$$\begin{aligned} y_0 &\in \bigcap_{m \in \mathbb{N}} C_m \\ &\subset \bigcap_{m \in \mathbb{N}} B_m. \end{aligned}$$

7. Let $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}_0$. Suppose that $(B_n)_{n \in \mathbb{N}}$ is decreasing and $\inf_{n \in \mathbb{N}} B_n = \varnothing$. Set $\delta := \inf_{n \in \mathbb{N}} \mu_0(B_n)$. For the sake of contradiction, suppose that $\delta > 0$. The previous parts imply that $\bigcap_{m \in \mathbb{N}} B_m \neq \varnothing$. This is a contradiction. Hence $\delta = 0$. Since $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}_0$ such that $(B_n)_{n \in \mathbb{N}}$ is decreasing and $\inf_{n \in \mathbb{N}} B_n = \varnothing$ is arbitrary, we have that for each $(B_n)_{n \in \mathbb{N}} \subset \mathcal{A}_0$, $(B_n)_{n \in \mathbb{N}}$ is decreasing and $\inf_{n \in \mathbb{N}} B_n = \varnothing$ implies that $\inf_{n \in \mathbb{N}} \mu_0(B_n) = 0$. Exercise 5.3.0.8 then implies that μ_0 is a premeasure on $(X, \mathcal{A}_0)$.

8. Set $\mathcal{C} := \mathbf{TopRadMsr}_+$. Since $\sigma(\mathcal{A}_0) = \mathcal{B}(X, \mathcal{T})$ and μ_0 is finite, Exercise 5.3.0.9 implies that there exists a unique $\mu \in M_+(X)$ such that $\mu|_{\mathcal{A}_0} = \mu_0$. Since for each $j \in J$, X_j is a Polish space, an exercise in the analysis notes section on Polish spaces implies that X is a Polish space. Since

$$\begin{aligned} \mu(X) &= \mu_0(X) \\ &= \sup_{j \in J} \mu_j(X) \\ &< \infty, \end{aligned}$$

Exercise 3.7.0.2 (Ulam's theorem) exercise on radon measures implies that $\mu \in \mathcal{M}_+(X)$.

- Let $j \in J$ and $E \in \mathcal{B}(X_j, \mathcal{T}_j)$. By construction,

$$\begin{aligned} (\pi_j)_* \mu(E) &= \mu(\pi_j^{-1}(E)) \\ &= \mu|_{\mathcal{A}_0}(\pi_j^{-1}(E)) \\ &= \mu_0(\pi_j^{-1}(E)) \\ &= \mu_j(E). \end{aligned}$$

  Since $E \in \mathcal{B}(X_j, \mathcal{T}_j)$ is arbitrary, we have that $(\pi_j)_* \mu = \mu_j$. Since $j \in J$ is arbitrary, we have that for each $j \in J$, $(\pi_j)_* \mu = \mu_j$. Hence $(\pi_j)_{j \in J} \in \prod_{j \in J} \mathrm{Hom}_{\mathcal{C}}((X, \mu), (X_j, \mu_j))$. Since $(X, (\pi_j)_{j \in J})$ is a **Top**-projective limit of $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$, for each $(j,k) \in \leq$,

$$\begin{aligned} \pi_{j,k} \circ_{\mathcal{C}} \pi_k &= \pi_{j,k} \circ \pi_k \\ &= \pi_j. \end{aligned}$$

- **Universal Property:**
  Let $(Y, \nu) \in \mathrm{Obj}(\mathcal{C})$ and $(\tau_j)_{j \in J} \in \prod_{j \in J} \mathrm{Hom}_{\mathcal{C}}((Y, \nu), (X_j, \mu_j))$. Suppose that for each $(j,k) \in \leq$, $\pi_{j,k} \circ_{\mathcal{C}} \tau_k = \tau_j$.

  - **Existence:**
    Since $Y \in \mathrm{Obj}(\mathbf{Top})$, $(\tau_j)_{j \in J} \in \prod_{j \in J} \mathrm{Hom}_{\mathbf{Top}}(Y, X_j)$ and $(X, (\pi_j)_{j \in J})$ is a **Top**-projective limit of $((X_j)_{j \in J}, (\pi_{j,k})_{(j,k) \in \leq})$, there exists a unique $\phi \in \mathrm{Hom}_{\mathbf{Top}}(Y, X)$ such that for each $j \in J$, $\pi_j \circ \phi = \tau_j$. Then for each $j \in J$,

$$\begin{aligned} (\pi_j)_*[\phi_* \nu] &= (\pi_j \circ \phi)_* \nu \\ &= (\tau_j)_* \nu \\ &= \mu_j. \end{aligned}$$

Let $A \in \mathcal{A}_0$. Then there exists $j \in J$ and $E \in \mathcal{B}(X_j, \mathcal{T}_j)$ such that $A = \pi_j^{-1}(E)$. Then

$$\begin{aligned}
\phi_*\nu(A) &= \nu(\phi^{-1}(A)) \\
&= \nu(\phi^{-1}(\pi_j^{-1}(E)) \\
&= \nu([\pi_j \circ \phi]^{-1}(E)) \\
&= \nu(\tau_j^{-1}(E)) \\
&= (\tau_j)_*\nu(E) \\
&= (\mu_j)(E) \\
&= (\pi_j)_*\mu(E) \\
&= \mu(\pi_j^{-1}(E)) \\
&= \mu(A).
\end{aligned}$$

Since $A \in \mathcal{A}_0$ is arbitrary, we have that

$$\begin{aligned}
\phi_*\nu|_{\mathcal{A}_0} &= \mu|_{\mathcal{A}_0} \\
&= \mu_0.
\end{aligned}$$

By uniqueness of μ satisfying $\mu|_{\mathcal{A}_0} = \mu_0$, we have that $\phi_*\nu = \mu$. Hence $\phi \in \mathrm{Hom}_{\mathcal{C}}((Y,\nu),(X,\mu))$.

– **Uniqueness:**
  Clear by uniqueness of ϕ in **Top**. (add details)

Hence $((X,\mathcal{T},\mu),(\pi_j)_{j\in J})$ is a $\mathcal{C}$-projective limit of $((X_j,\mathcal{T}_j,\mu_j)_{j\in J},(\pi_{j,k})_{(j,k)\in\leq})$.

□

Exercise 9.6.0.5. Here make an exercise about $\sup_{j\in J}\mu_j(X_j) < \infty$ iff there exists a complex measure $\mu \in \mathcal{M}(X)$ such that it gives a $\textbf{TopRadMsr}_{\mathbb{C}}$ projective limit. Then relate/comment/explore how this is related to the feynman path integral nonexistence, which is why we do a wick rotation.

Proof. FINISH!!! □

Note 9.6.0.6. Recall here the definitions from the subsection on products of measurable spaces

Exercise 9.6.0.7. Let $(X_j,\mu_j)_{j\in\mathbb{N}} \subset \mathrm{Obj}(\textbf{TopRadMsr}_1)$. Suppose that for each $j \in \mathbb{N}$, X_j is a Polish space. Let $\mu \in \mathcal{M}_1(\prod_{j\in\mathbb{N}} X_j)$. Suppose that for each $(A_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} \mathcal{B}(X_j)$,

$$\mu\bigg(\prod_{j\in\mathbb{N}} A_j\bigg) = \prod_{j\in\mathbb{N}} \mu_j(A_j).$$

Then for each $j \in \mathbb{N}$, $\pi_j^*\mu = \mu_j$

Proof. Let $j \in \mathbb{N}$ and $A \in \mathcal{B}_j$. Then

$$\begin{aligned}
(\pi_j)_*\mu(A) &= \mu(\pi_j^{-1}(A)) \\
&= \mu\bigg(\bigg[\prod_{k=1}^{j-1} X_k\bigg] \times A \times \bigg[\prod_{k\geq j+1} X_k\bigg]\bigg) \\
&= \bigg(\prod_{k=1}^{j-1}\mu_k(X_k)\bigg)\mu_j(A)\bigg(\prod_{k\geq j-1}\mu_k(X_k)\bigg) \\
&= \mu_j(A).
\end{aligned}$$

Since $A \in \mathcal{B}_j$ is arbitrary, we have that $(\pi_j)_*\mu = \mu_j$. □

Exercise 9.6.0.8. Let $(X_j, \mu_j)_{j\in\mathbb{N}} \subset \mathrm{Obj}(\mathbf{TopRadMsr}_1)$. Suppose that for each $j \in \mathbb{N}$, X_j is a Polish space. Then there exists a unique $\mu \in \mathcal{M}_1(\prod_{j\in\mathbb{N}} X_j)$ such that for each $(A_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} \mathcal{B}(X_j)$,

$$\mu\bigg(\prod_{j\in\mathbb{N}} A_j\bigg) = \prod_{j\in\mathbb{N}} \mu_j(A_j),$$

Proof.

- **Existence:**
  For each $n \in \mathbb{N}$, define $Y_n := \prod_{j=1}^{n} X_j$. Set $Y := \prod_{j\in\mathbb{N}} X_j$ and for each $n, k \in \mathbb{N}$, if $n \leq k$, define $\tau_{n,k} : Y_k \to Y_n$ by $\tau_{n,k}((x_j)_{j=1}^k) := (x_j)_{j=1}^n$. For each $n \in \mathbb{N}$, define $\tau_n : Y \to Y_n$ by $\tau_n((x_j)_{j\in\mathbb{N}}) := (x_j)_{j=1}^n$. An exercise implies that $(Y, (\tau_n)_{n\in\mathbb{N}})$ is a **Top**-projective limit of $((Y_n)_{n\in\mathbb{N}}, (\tau_{n,k})_{(n\leq k)})$. For each $n \in \mathbb{N}$, define $\nu_n \in \mathcal{M}_+(Y_n)$ by $\nu_n := \bigotimes_{j\in[n]} \mu_j$. need exercise showing products of Radon measures is Radon, see theorem 7.20 in Folland, I dont think we need to assume that the spaces are LCH, so maybe just cite ulams theorem since any partial product ν_n is finite on a product of Polish spaces which is Polish. Then $(Y_n, \nu_n)_{n\in\mathbb{N}} \subset \mathrm{Obj}(\mathbf{TopRadMsr}_1)$. Let $n, k \in \mathbb{N}$. Suppose that $n \leq k$. Let $(A_j)_{j\in[k]} \in \prod_{j=1}^{n} \mathcal{B}(X_k)$. Then

$$\begin{aligned}
\pi_{n,k}^* \nu_k\bigg[\prod_{j=1}^{n} A_j\bigg] &= \nu_k\bigg[\pi_{n,k}^{-1}\bigg(\prod_{j=1}^{n} A_j\bigg)\bigg] \\
&= \nu_k\bigg[\prod_{j=1}^{n} A_j \times \prod_{j=n+1}^{k} X_j\bigg] \\
&= \bigg(\prod_{j=1}^{n} \mu_j(A_j)\bigg)\bigg(\prod_{j=n+1}^{k} \mu_j(X_j)\bigg) \\
&= \prod_{j=1}^{n} \mu_j(A_j) \\
&= \nu_n\bigg[\prod_{j=1}^{n} A_j\bigg].
\end{aligned}$$

  Exercise 3.5.0.5 (or maybe other exercise) and need exercise about uniqueness of product measure in section on product measures implies that $\pi_{n,k}^* \nu_k = \nu_n$. Thus $((Y_n, \nu_n)_{n\in\mathbb{N}}, (\tau_{n,k})_{n\leq k})$ is a $\mathbf{TopRadMsr}_+$-projective system. Exercise **??** make exercise about this implies that $(Y, (\tau_n)_{n\in\mathbb{N}}) = \varprojlim_{n\in\mathbb{N}} ((Y_n)_{n\in\mathbb{N}}, (\tau_{n,k})_{n\leq k})$ in **Top**. We observe that

  - the analysis notes section on products of polish spaces implies that for each $n \in \mathbb{N}$, Y_n is Polish
  - by construction, for each $n \in \mathbb{N}$ τ_n is surjective,
  - since for each $n \in \mathbb{N}$, $\mu_j(X_j) = 1$, $\sup_{n\in\mathbb{N}} \nu_n(Y_n) = 1$.

  Exercise 9.6.0.4 implies that there exists a unique $\mu \in \mathcal{M}_+(Y)$ such that $((Y, \mu), (\tau_n)_{n\in\mathbb{N}})$ is a $\mathbf{TopRadMsr}_+$-projective limit of $((Y_n, \nu_n)_{n\in\mathbb{N}}, (\tau_{n,k})_{n\leq k})$. Let $(A_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} \mathcal{B}(X_j)$. For $n \in \mathbb{N}$, define $B_n \in \mathcal{B}(Y)$ by $B_n := \tau_n^{-1}\bigg(\prod_{j=1}^{n} A_j\bigg)$.

Then for each $n \in \mathbb{N}$,

$$\begin{aligned}
B_{n+1} &= \tau_{n+1}^{-1}\bigg(\prod_{j=1}^{n+1} A_j\bigg) \\
&= \prod_{j=1}^{n+1} A_j \times \prod_{j \geq n+2} X_j \\
&\subset \prod_{j=1}^{n} A_j \times \prod_{j \geq n+1} X_j \\
&= \tau_n^{-1}\bigg(\prod_{j=1}^{n} A_j\bigg) \\
&= B_n.
\end{aligned}$$

Thus $(B_n)_{n\in\mathbb{N}}$ is decreasing. By construction,

$$\begin{aligned}
\mu(B_1) &= \mu\Big[\tau_1^{-1}(A_1)\Big] \\
&= (\tau_1)_*\mu(A_1) \\
&= \nu_1(A_1) \\
&= \mu_1(A_1) \\
&< \infty.
\end{aligned}$$

Since $\prod\limits_{j\in\mathbb{N}} A_j = \bigcap\limits_{n\in\mathbb{N}} B_n$, Exercise 5.1.0.8 implies that

$$\begin{aligned}
\mu\bigg[\prod_{j\in\mathbb{N}} A_j\bigg] &= \mu\bigg[\bigcap_{n\in\mathbb{N}} B_n\bigg] \\
&= \lim_{n\to\infty} \mu(B_n) \\
&= \lim_{n\to\infty} \mu\bigg[\tau_n^{-1}\bigg(\prod_{j=1}^{n} A_j\bigg)\bigg] \\
&= \lim_{n\to\infty} (\tau_n)_*\mu\bigg[\prod_{j=1}^{n} A_j\bigg] \\
&= \lim_{n\to\infty} \nu_n\bigg[\prod_{j=1}^{n} A_j\bigg] \\
&= \lim_{n\to\infty} \prod_{j=1}^{n} \mu_j(A_j) \\
&= \prod_{j\in\mathbb{N}} \mu_j(A_j).
\end{aligned}$$

maybe need some results on infinite products of (positive) real numbers (especially in $[0,1]$) like if order matters, etc so that the last equality is true

- **Uniqueness:**
  Suppose that there exists $\nu \in \mathcal{M}_+(\prod\limits_{j\in\mathbb{N}} X_j)$ such that for each $(A_j)_{j\in\mathbb{N}} \in \prod\limits_{j\in\mathbb{N}} \mathcal{B}(X_j)$,

$$\nu\bigg(\prod_{j\in\mathbb{N}} A_j\bigg) = \prod_{j\in\mathbb{N}} \mu_j(A_j).$$

Set $\mathcal{T}_X := \bigotimes_{j\in\mathbb{N}} \mathcal{T}_{X_j}$. Since for each $j \in \mathbb{N}$, X_j is a Polish space, we have that for each $j \in \mathbb{N}$, $(X_j, \mathcal{T}_{X_j})$ is second-countable. Therefore, for each $j \in \mathbb{N}$, there exists $\mathcal{B}_j \subset \mathcal{T}_{X_j}$ such that $\mathcal{B}_j$ is a basis for $\mathcal{T}_j$ and $\mathcal{B}_j$ is countable. Define $\mathcal{B} \subset \mathcal{T}_X$ by

$$\mathcal{B} = \Bigg\{ \prod_{j\in\mathbb{N}} U_j : \text{there exists } J \subset \mathbb{N} \text{ such that } \#J < \infty, \\ \text{for each } j \in J,\ U_j \in \mathcal{B}_j \text{ and for each } j \in J^c,\ U_j = X_j \Bigg\}.$$

An exercise in the analysis notes section on product topology implies that $\mathcal{B}$ is a basis for $\mathcal{T}_X$ and $\mathcal{B}$ is countable. need to show $\mathcal{B}$ is a π-system on X and $X \in \mathcal{B}$ (put this in an exercise in product measurable space section).

Let $U \in \mathcal{B}$. Then there exists $N \in \mathbb{N}$ and $(U_j)_{j=1}^N \in \prod_{j=1}^N \mathcal{B}_j$ such that $U = \left[\prod_{j=1}^N U_j\right] \times \prod_{j\geq N+1} X_j$. Therefore

$$\begin{aligned}
\nu(U) &= \nu\left(\left[\prod_{j=1}^N U_j\right] \times \left[\prod_{j\geq N+1} X_j\right]\right) \\
&= \left[\prod_{j=1}^N \mu_j(U_j)\right]\left[\prod_{j\geq N+1} \mu_j(X_j)\right] \\
&= \mu\left(\left[\prod_{j=1}^N U_j\right] \times \left[\prod_{j\geq N+1} X_j\right]\right) \\
&= \mu(U).
\end{aligned}$$

Since $U \in \mathcal{B}$ is arbitrary, we have that for each $U \in \mathcal{B}$, $\nu(U) = \mu(U)$.

Since $\mathcal{B}$ is countable, $\mathcal{B}$ is a π-system on X, $X \in \mathcal{B}$ and for each $U \in \mathcal{B}$ $\nu(U) = \mu(U)$, Exercise 7.3.0.8 implies that $\mu = \nu$. FINISH!!!, see red above

□

Definition 9.6.0.9. Let $(X_j, \mu_j)_{j\in\mathbb{N}} \subset \mathrm{Obj}(\mathbf{TopRadMsr}_1)$. Suppose that for each $j \in \mathbb{N}$, X_j is a Polish space. We define the **product measure of** $(\mu_j)_{j\in\mathbb{N}}$, denoted $\bigotimes_{j\in\mathbb{N}} \mu_j \in \mathcal{M}_1(\prod_{j\in\mathbb{N}} X_j)$, to be the unique $\mu \in \mathcal{M}_1(\prod_{j\in\mathbb{N}} X_j)$ such that for each $(A_j)_{j\in\mathbb{N}} \in \prod_{j\in\mathbb{N}} \mathcal{B}(X_j)$,

$$\mu\left(\prod_{j\in\mathbb{N}} A_j\right) = \prod_{j\in\mathbb{N}} \mu_j(A_j).$$

Chapter 10

Haar Measure

10.1 Introduction

Note 10.1.0.1. This section assumes familiarity with topological groups. See section 8.1 of [2] for details.

Definition 10.1.0.2. Let G be a group and $g \in G$. Define $l_g : G \to G$ and $r_g : G \to G$ by $l_g(x) = gx$ and $r_g(x) = xg^{-1}$.

Definition 10.1.0.3. Let G be a topological group, $y \in G$ and $f \in L^0$. Define $L_y, R_y : L^0(G) \to L^0(G)$ by $L_y f = f \circ l_y^{-1}$ and $R_y f = f \circ r_y^{-1}$, that is, $L_y f(x) = f(y^{-1}x)$ and $R_y f(x) = f(xy)$.

Definition 10.1.0.4. Let G be a topological group and μ a Radon measure on G. Then μ is said to be a **left Haar measure on** G if

1. μ is nonzero
2. for each $U \in \mathcal{B}(G)$ and $g \in G$, $\mu(gU) = \mu(U)$.

Similarly, μ is said to be a **right Haar measure on** G if

1. μ is nonzero
2. for each $U \in \mathcal{B}(G)$ and $g \in G$, $\mu(Ug) = \mu(U)$.

Exercise 10.1.0.5. Let G be a topological group, μ a Radon measure on G. Then μ is a left Haar measure on G iff $\iota_*\mu$ is a right Haar measure on G.

Proof. Suppose that μ is a left Haar measure on G. Let $U \in \mathcal{B}(G)$ and $g \in G$. Then

$$\begin{aligned}
\iota_*\mu(Ug) &= \mu(\iota^{-1}(Ug)) \\
&= \mu(g^{-1}U^{-1}) \\
&= \mu(U^{-1}) \\
&= \mu(\iota^{-1}(U)) \\
&= \iota_*\mu(U)
\end{aligned}$$

So $\iota_*\mu$ is a right Haar measure on G. The converse is similar. □

Exercise 10.1.0.6. Let G be a topological group, and μ a left Haar measure on G. Then for each $g \in G$, $r_{g*}\mu$ is a left Haar measure on G.

Proof. Let $g \in G$ and $U \in \mathcal{B}(G)$. Observe that $r_{g*}\mu(U) = \mu(Ug)$. So for each $h \in G$,

$$\begin{aligned}
r_{g*}\mu(hU) &= \mu(hUg) \\
&= \mu(Ug) \\
&= r_{g*}\mu(U)
\end{aligned}$$

□

Exercise 10.1.0.7. Let G be a topological group, μ a left Haar measure on G and ν a right Haar measure on G. Then for each $f \in L^1 \cup L^+$ and $y \in G$,

$$\int L_y f \, d\mu = \int f \, d\mu \tag{10.1}$$

$$\int R_y f d\nu = \int f d\nu \tag{10.2}$$

Proof.

1. Let $y \in G$ and $E \in \mathcal{B}(G)$. Put $f = \chi_E$. Then

$$\begin{aligned}\int L_y f \, d\mu &= \int L_y \chi_E \, d\mu \\ &= \int \chi_{yE} \, d\mu \\ &= \mu(yE) \\ &= \mu(E) \\ &= \int \chi_E \, d\mu \\ &= \int f \, d\mu\end{aligned}$$

By linearity of L_y, for $f \in S^+$ we have that,

$$\int L_y f \, d\mu = \int f \, d\mu$$

For $f \in L^+$, choose $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$ $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$. Then for each $n \in \mathbb{N}$ $L_y \phi_n \leq L_y \phi_{n+1} \leq L_y f$ and $L_y \phi \to L_y f$. So MCT implies that

$$\begin{aligned}\int L_y f \, d\mu &= \lim_{n \to \infty} \int L_y \phi_n \, d\mu \\ &= \lim_{n \to \infty} \int \phi_n \, d\mu \\ &= \int f \, d\mu\end{aligned}$$

Let $f \in L^1$. If f is real valued, write $f = f^+ - f^-$. Then $L_y f = L_y f^+ - L_y f^-$ and

$$\begin{aligned}\int L_y f \, d\mu &= \int L_y f^+ \, d\mu - \int L_y f^- \, d\mu \\ &= \int f^+ \, d\mu - \int f^- \, d\mu \\ &= \int f \, d\mu\end{aligned}$$

If f is complex valued, write $f = g + ih$ with $g, h \in L^1$ real valued. Then

$$\begin{aligned}\int L_y f \, d\mu &= \int L_y g \, d\mu + i \int L_y h \, d\mu \\ &= \int g \, d\mu + i \int h \, d\mu \\ &= \int f \, d\mu\end{aligned}$$

2. Similar

□

Exercise 10.1.0.8. Let G be a topological group and μ a left Haar measure on G. Then for each $U \subset G$, if U is open and $U \neq \varnothing$, then $\mu(U) > 0$

Proof. Let $U \subset G$. Suppose that U is open and $U \neq \varnothing$. Suppose that $\mu(U) = 0$. Since μ is nonzero, inner regularity implies that there exists $K \subset G$ such that K is compact and $\mu(K) > 0$. Then $\{xU : x \in K\}$ is an open cover of K. Then there exist $x_1, \cdots, x_n \in K$ such that $K \subset \bigcap\limits_{k=1}^{n} x_k U$. Then

$$\begin{aligned} \mu(K) &\leq \sum_{k=1}^{n} \mu(x_k U) && (10.3) \\ &= \sum_{k=1}^{n} \mu(U) && (10.4) \\ &= 0 && (10.5) \end{aligned}$$

This is a contradiction. So $\mu(U) > 0$. □

Exercise 10.1.0.9. Let G be a locally compact group and μ a left Haar measure on G. Then there exists $S \in \mathcal{B}(G)$ such that S is symmetric, $e \in S$ and $\mu(E) > 0$

Proof. Since G is locally compact, there exists a compact neighborhood K of e. Then $\mu(K) > 0$. Put $S = KK^{-1} \in \mathcal{B}(G)$. Then S is symmetric. Since $e \in K$, $K \subset S$ and $0 < \mu(K) \leq \mu(S)$. □

Exercise 10.1.0.10. Let G be a locally compact group and μ a left Haar measure on G. Then

1. $\mu(\{e\}) > 0$ iff there exists $\lambda > 0$ such that $\mu = \lambda\#$.
2. μ is finite iff G is compact

Proof.

1. If there exists $\lambda > 0$ such that $\mu = \lambda\#$, then $\mu(\{e\}) > 0$ Conversely, suppose that $\mu(\{e\}) > 0$. Define $\lambda = \mu(\{e\}) > 0$. Let $B \in \mathcal{B}(G)$. If B is finite, then

$$\begin{aligned} \mu(B) &= \sum_{x \in B} \mu(\{x\}) \\ &= \sum_{x \in B} \mu(x\{e\}) \\ &= \sum_{x \in B} \mu(\{e\}) \\ &= \sum_{x \in B} \lambda \\ &= \lambda\#(\{e\}) \end{aligned}$$

   If B is infinite, then we may choose a countable subset and the same reasoning as above tells us that

$$\mu(B) = \infty = \lambda\#(B)$$

2. If G is compact, then μ is finite since μ is Radon. Conversely, suppose that μ is finite. Then **FINISH**

□

Theorem 10.1.0.11. Let G be a locally compact group. Then there exists a left Haar measure on G.

Theorem 10.1.0.12. Let G be a locally compact group and μ_1, μ_2 left Haar measures on G. Then there exists $\lambda > 0$ such that $\mu_1 = \lambda \mu_2$.

Definition 10.1.0.13. Let G be a locally compact group and μ a left Haar measure on G. A previous exercise tells us that for each $g \in G$, $r_{g*}\mu$ is a left Haar measure on G. The previous result tells us that for each $g \in G$ there exists $\lambda_g > 0$ such that $r_{g*}\mu = \lambda_g \mu$. Define $\Delta : G \to (0, \infty)$ by $\Delta(g) = \lambda_g$. We call Δ the **modular function of** G.

Exercise 10.1.0.14. Let G be a locally compact group and μ a left Haar measure on G. Then

1. Δ is a homomorphism
2. for each $f \in L^1 \cup L^+$,
$$\int R_{y^{-1}} f \, d\mu = \Delta(y) \int f \, d\mu$$

Proof.

1. Recall that for each $g \in G$, $\Delta(g)\mu(U) = r_{g*}\mu(U) = \mu(Ug)$. Let $g, h \in G$ and $U \in \mathcal{B}(G)$. Then $\Delta(gh)\mu(U) = \mu(Ugh) = \Delta(h)\mu(Ug) = \Delta(g)\Delta(h)\mu(U)$. So $\Delta(gh) = \Delta(g)\Delta(h)$.
2. Let $y \in G$ and $U \in \mathcal{B}(G)$. Put $f = \chi_U$ Then
$$\begin{aligned}
\int R_{y^{-1}} f \, d\mu &= \int R_{y^{-1}} \chi_U \, d\mu \\
&= \int \chi_{Uy} \, d\mu \\
&= \mu(Uy) \\
&= \mu(r_y^{-1}(U)) \\
&= r_{y*}\mu(U) \\
&= \Delta(y)\mu(U) \\
&= \Delta(y) \int \chi_U \, d\mu \\
&= \Delta(y) \int f \, d\mu
\end{aligned}$$
By linearity of $R_{y^{-1}}$, for $f \in S^+$,
$$\int R_{y^{-1}} f \, d\mu = \Delta(y) \int f \, d\mu$$
For $f \in L^+$, choose $(\phi_n)_{n \in \mathbb{N}} \subset S^+$ such that for each $n \in \mathbb{N}$ $\phi_n \leq \phi_{n+1} \leq f$ and $\phi_n \to f$. Then for each $n \in \mathbb{N}$ $R_{y^{-1}}\phi_n \leq R_{y^{-1}}\phi_{n+1} \leq R_{y^{-1}} f$ and $R_{y^{-1}}\phi \to R_{y^{-1}} f$. So the monotone convergence theorem implies that
$$\begin{aligned}
\int R_{y^{-1}} f \, d\mu &= \lim_{n \to \infty} \int R_{y^{-1}} \phi_n \, d\mu \\
&= \lim_{n \to \infty} \Delta(y) \int \phi_n \, d\mu \\
&= \Delta(y) \int f \, d\mu
\end{aligned}$$
Let $f \in L^1$. If f is real valued, write $f = f^+ - f^-$. Then $R_{y^{-1}} f = R_{y^{-1}} f^+ - R_{y^{-1}} f^-$ and
$$\begin{aligned}
\int R_{y^{-1}} f \, d\mu &= \int R_{y^{-1}} f^+ \, d\mu - \int R_{y^{-1}} f^- \, d\mu \\
&= \Delta(y) \int f^+ \, d\mu - \Delta(y) \int f^- \, d\mu \\
&= \Delta(y) \int f \, d\mu
\end{aligned}$$

If f is complex valued, write $f = g + ih$ with $g, h \in L^1$ real valued. Then

$$\begin{aligned}\int R_{y^{-1}} f \, d\mu &= \int R_{y^{-1}} g \, d\mu + i \int R_{y^{-1}} h \, d\mu \\ &= \Delta(y) \int g \, d\mu + i\Delta(y) \int h \, d\mu \\ &= \Delta(y) \int f \, d\mu\end{aligned}$$

□

Definition 10.1.0.15. Let G be a locally compact group. Then G is said to be **unimodular** if $\ker \Delta = G$.

Exercise 10.1.0.16. Let G be a locally compact group. Then the following are quivalent:

1. G is unimodular
2. there exists a left Haar measure μ on G such that μ is a right Haar measure on G.
3. for each nonzero Radon measure μ on G, μ is a left Haar measure on G iff μ is a right Haar measure on G.

Proof.

- (1) $\implies$ (2):
  Since G is a locally compact group, there exists a left Haar measure μ on G. Let $g \in G$ and $U \in \mathcal{B}(G)$. Then
  $$\mu(Ug) = \Delta(g)\mu(U) = \mu(U)$$
  Since G is unimodular, $\Delta(g) = 1$. Then μ is a right Haar measure on G.
- (2) $\implies$ (3):
  By assumption, there exists a left Haar measure μ' on G such that μ' is a right Haar measure on G. Let μ be a nonzero Radon measure on G. If μ is a left Haar measure on G, then there exists $\lambda > 0$ such that $\mu = \lambda\mu'$ and therefore μ is a right Haar measure. The same reasoning implies that if μ is a right Haar measure on G, then μ is a left Haar measure on G.
- (3) $\implies$ (1):
  Since G is locally compact, there exists a left *Haar* measure μ on G. By assumption, μ is a right Haar measure on G. By inner regularity there exists $K \in \mathcal{B}(G)$ such that $\mu(K) > 0$. Let $g \in G$. Then
  $$\Delta(g)\mu(K) = \mu(Kg) = \mu(K)$$
  So $\Delta(g) = 1$.

□

Note 10.1.0.17. If G is a locally compact abelian group, then G is unimodular.

Exercise 10.1.0.18. Let G be a locally compact group and μ a left Haar measure on G. If G is unimodular then $\iota_*\mu = \mu$.

Proof. Suppose that G is unimodular. A previous exercise tells us that $\iota_*\mu$ is a right Haar measure on G. The unimodularity of G implies that $\iota_*\mu$ a left Haar measure on G. Then there exists $\lambda > 0$ such that $\iota_*\mu = \lambda\mu$. Since G is locally compact, there exists $S \in \mathcal{B}(G)$ such that S is symmetric and $\mu(S) > 0$. Then

$$\begin{aligned}\mu(S) &= \mu(S^{-1}) \\ &= \iota_*\mu(S) \\ &= \lambda\mu(S)\end{aligned}$$

So $\lambda = 1$ and $\iota_*\mu = \mu$.

it is also (Since G is locally compact, there exists $S \in \mathcal{B}(G)$ such that S is symmetric and $\mu(S) > 0$. Then

$$\mu(S) = \mu(S^{-1}) = \iota_*\mu(S)$$

Since $\iota_*\mu$ is a right Haar measure on G and G is unimodular, $\iota_*\mu(S)$ is also a left Haar measure on G. Then there exists $\lambda > 0$ such that $\mu(S) = \lambda\iota_*\mu(S)$. □

Exercise 10.1.0.19. Let $(X, \mathcal{A}, \lambda)$ be a probability space, G a locally compact group and μ a left Haar measure on G. Suppose that G is unimodular and $f_*\lambda \ll \mu$. Then

1. for each $f \in \mathrm{Hom}_{\mathbf{Meas}}[(X, \mathcal{A}), (G, \mathcal{B}(G))]$, $(f^{-1})_*\lambda \ll \mu$ and
$$\frac{df_*^{\odot -1}\lambda}{d\mu} = \frac{df_*\lambda}{d\mu} \circ \iota_\mu \quad \mu\text{-a.e.}$$
2. for each $f, g \in \mathrm{Hom}_{\mathbf{Meas}}[(X, \mathcal{A}), (G, \mathcal{B}(G))]$, $(f^{-1})_*\lambda \ll \mu$ and
$$\frac{d(f \odot g)_*\lambda}{d\mu} = \frac{df_*\lambda}{d\mu} * \frac{dg_*\lambda}{d\mu} \quad \mu\text{-a.e.}$$

Proof.

1. Let $f \in \mathrm{Hom}_{\mathbf{Meas}}[(X, \mathcal{A}), (G, \mathcal{B}(G))]$. The previous exercise implies that $(\iota_\mu)_*\mu = \mu$. SInce we have that ι_μ is an isomorphism, Exercise 7.3.1.4 implies that $(\iota_\mu)_* f_*\lambda \ll (\iota_\mu)_*\mu$ and therefore
$$\begin{aligned}\frac{d(f^{\odot -1})_*\lambda}{d\mu} &= \frac{d(\iota_\mu \circ f)_*\lambda}{d(\iota_\mu)_*\mu} \\ &= \frac{df_*\lambda}{d\mu} \circ \iota_\mu^{-1} \\ &= \frac{df_*\lambda}{d\mu} \circ \iota_\mu\end{aligned}$$

$$\frac{d[f_*^{-1}\lambda]}{d\mu} = \frac{f_*\lambda}{d\mu} \circ \iota_\mu \quad \mu\text{-a.e.}$$
2. for each $f, g \in \mathrm{Hom}_{\mathbf{Meas}}[(X, \mathcal{A}), (G, \mathcal{B}(G))]$, $(f^{-1})_*\lambda \ll \mu$ and
$$\frac{d[(f \odot g)_*\lambda]}{d\mu} = \frac{df_*\lambda}{d\mu} * \frac{dg_*\lambda}{d\mu} \quad \mu\text{-a.e.}$$

$\square$

10.2 Fundamental Examples

Note 10.2.0.1. The Haar measure on $(\mathbb{R}^n, +)$ is m.

Exercise 10.2.0.2. The Haar measure on $(\mathbb{R}^\times, \cdot)$ is

$$d\mu(x) = \frac{1}{|x|}\, dm(x)$$

Proof. Let $0 < a < b$ and $c > 0$. Then

$$\begin{aligned}
\mu(c(a,b)) &= \mu((ca, cb)) \\
&= \int_{(ca,cb)} \frac{1}{|x|}\, dm(x) \\
&= \int_{(ca,cb)} \frac{1}{x}\, dm(x) \\
&= \bigg[\log|x|\bigg]_{ca}^{cb} \\
&= \log(cb) - \log(ca) \\
&= \log b - \log a \\
&= \bigg[\log|x|\bigg]_{a}^{b} \\
&= \int_{(a,b)} \frac{1}{x}\, dm(x) \\
&= \mu((a,b))
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mu(-c(a,b)) &= \mu((-cb, -ca)) \\
&= \int_{(-cb,-ca)} \frac{1}{|x|}\, dm(x) \\
&= -\int_{(-cb,-ca)} \frac{1}{x}\, dm(x) \\
&= -\bigg[\log|x|\bigg]_{-cb}^{-ca} \\
&= \log(cb) - \log(ca) \\
&= \log b - \log a \\
&= \bigg[\log|x|\bigg]_{a}^{b} \\
&= \int_{(a,b)} \frac{1}{x}\, dm(x) \\
&= \mu((a,b))
\end{aligned}$$

□

Exercise 10.2.0.3. Define $f : [0, 1) \to \mathbb{T}$ by $f(x) = e^{i2\pi x}$. Let m be Lebesgue measure on $[0, 1)$, then the Haar measure on $\mathbb{T}$ is $f_* m$.

Proof. Note that f is a bijection and the topology on $\mathbb{T}$ is generated by sets of the form $f((a, b))$ where $a, b \in [0, 1)$ and $a < b$. Let $a, b \in [0, 1)$ and suppose that $a < b$. Put $A = f((a, b))$. Let $z \in \mathbb{T}$. Then there exists $\theta \in [0, 1)$ such that $z = f(\theta)$. If

$1 \notin zA$, then $f^{-1}(zA) = (\theta + a, \theta + b)$. If $1 \in zA$, then $f^{-1}(zA) = (\theta + a, 1) \cup [0, \theta + b - 1)$. Suppose that $1 \notin zA$. Then

$$= f_* m(zA) \qquad\qquad = m(f^{-1}(zA))$$
$$\begin{aligned}
&= m((\theta + a, \theta + b)) \\
&= b - a \\
&= m((a, b)) \\
&= m(f^{-1}(A)) \\
&= f_* m(A)
\end{aligned}$$

Similarly if $1 \in zA$, $f_* m(zA) = f_* m(A)$. $\square$

Exercise 10.2.0.4. Let p be a prime. Define $|\cdot|_p : \mathbb{Q} \to [0, \infty)]$ by

$$\begin{cases} |\frac{a}{b} p^n|_p = p^{-n}, \text{ if } \gcd(a,p) = \gcd(b,p) = 1 \\ |0|_p = 0 \end{cases}$$

Then $|\cdot|_p$ is an absolute value on $\mathbb{Q}$. Define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to the metric induced by $|\cdot|_p$. Define $\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1\}$. It is well known that $\mathbb{Q}_p$ is a locally compact field and $\mathbb{Z}_p$ is compact. Define $P = \{0, 1, \cdots, p-1\}$. It is known that the topology is generated by

$$\{x + p^n \mathbb{Z}_p : \text{ for } n \in \mathbb{Z}, x \in \mathbb{Q}_p\}$$

Another useful fact is that

$$\mathbb{Q}_p = \{\sum_{j=-n}^{\infty} a_j p^j : a_j \in P, n \in \mathbb{N}_0\}$$

and

$$\mathbb{Z}_p = \{\sum_{j=0}^{\infty} a_j p^j : a_j \in P\}$$

Let μ be the Haar measure on $\mathbb{Q}_p$. Then μ is completely determined by the value $\mu(\mathbb{Z}_p)$

Proof. We observe that for $n \in \mathbb{Z}$, we may write $p^n \mathbb{Z}_p$ as the following disjoint union:

$$p^n \mathbb{Z}_p = \bigcup_{j \in P} jp^n + p^{n+1} \mathbb{Z}^p$$

Thus $\mu(p^n \mathbb{Z}^p) = p\mu(p^{n+1}\mathbb{Z}_p)$. If we set $\mu(\mathbb{Z}_p) = 1$, we obtain that $\mu(\mathbb{Z}_p) = p^n \mu(p^n \mathbb{Z}_p)$, which implies that

$$\mu(p^n \mathbb{Z}_p) = \frac{1}{p^n} \mu(\mathbb{Z}_p)$$

. $\square$

Exercise 10.2.0.5. Let ν be the Haar measure on $\mathbb{Q}_p$. Then the Haar measure on $\mathbb{Q}_p^\times$ is $d\mu = \frac{1}{|x|_p} d\nu$.

Proof. Let $x, y \in P^\times$ and $\alpha = xp^{n-1} + p^n \mathbb{Z}_p$. Then

$$\alpha(yp^{k-1} + p^k \mathbb{Z}_p) = p^{(n-1)+(k-1)}(xy + p^{n+k}\mathbb{Z}_p)$$

$\square$

10.3 Action on Measures

Exercise 10.3.0.1. Let G be a locally compact group, μ a left Haar measure on G and $\nu \in \mathcal{M}(G)$. If $\nu \ll \mu$, then $l_{g*}\nu \ll \mu$.

Proof. Suppose that $\nu \ll \mu$. Let $A \in \mathcal{B}(G)$. Then

$$\begin{aligned}
\mu(A) = 0 &\implies \mu(g^{-1}A) = 0 \\
&\implies \nu(g^{-1}A) = 0 \\
&\implies \nu(l_{g^{-1}}(A)) = 0 \\
&\implies \nu(l_g^{-1}(A)) = 0 \\
&\implies l_{g*}\nu(A) = 0
\end{aligned}$$

So $l_{g*}\nu \ll \mu$. □

Definition 10.3.0.2. Let G be a locally compact group and μ a left Haar measure on G. Define $\mathcal{M}_\mu \subset \mathcal{M}(G)$ by

$$\mathcal{M}_\mu = \{\nu \in \mathcal{M}(G) : \nu \ll \mu\}$$

We define an action $\phi : G \times \mathcal{M}_\mu \to \mathcal{M}_\mu$ by

$$g \cdot \nu = l_{g*}\nu$$

Exercise 10.3.0.3. Let G be a locally compact group, μ a σ-finite left Haar measure on G, $\nu \in \mathcal{M}_\mu$ and $g \in G$. Then

$$\frac{d(g \cdot \nu)}{d\mu} = L_g \frac{d\nu}{d\mu}$$

Proof. Set $f = d\nu/d\mu$. Let $A \in \mathcal{B}(X)$. Then

$$\begin{aligned}
\int_A L_g f \, d\mu &= \int_A f \circ l_g^{-1} \, d\mu \\
&= \int_A f \circ l_g^{-1} \, d\mu \\
&= \int_{l_g^{-1}(A)} f \, d(l_g^{-1}{}_*\mu) \\
&= \int_{l_g^{-1}(A)} f \, d(l_{g^{-1}*}\mu) \\
&= \int_{l_g^{-1}(A)} f \, d\mu \\
&= \nu(l_g^{-1}(A)) \\
&= l_{g*}\nu(A) \\
&= g \cdot \nu(A)
\end{aligned}$$

Since A is arbitrary, uniqueness implies that

$$\frac{d(g \cdot \nu)}{d\mu} = L_g \frac{d\nu}{d\mu}$$

□

Exercise 10.3.0.4. Let G be a locally compact group, μ a σ-finite left Haar measure on G, $\nu \in \mathcal{M}_\mu$ and $g \in G$. Then $\|g \cdot \nu\| = \|\nu\|$.

Proof. Exercise 7.3.0.16 implies that

$$\begin{aligned}
\|g \cdot \nu\| &= \int \left| \frac{d(g \cdot \nu)}{d\mu} \right| d\mu \\
&= \int \left| L_g \frac{d\nu}{d\mu} \right| d\mu \\
&= \int L_g \left| \frac{d\nu}{d\mu} \right| d\mu \\
&= \int \left| \frac{d\nu}{d\mu} \right| d\mu \\
&= \|\nu\|
\end{aligned}$$

□

10.4 Measures Invariant under Group Actions

Definition 10.4.0.1. Let G be a group, X a set, $\phi : G \times X \to X$ a group action and $g \in G$. Define $l_g : X \to G$ by $l_g(x) = g \cdot x$.

Definition 10.4.0.2. Let G be a topological group, X a set, $\phi : G \times X \to X$ a group action and $g \in G$. Define $L_g : L^0(G) \to L^0(G)$ by

$$L_g f = f \circ l_g^{-1}$$

i.e. $L_g f(x) = f(g^{-1} \cdot x)$

Definition 10.4.0.3. Let G be a group, $(X, \mathcal{A}, \mu)$ a measure space, $\phi : G \times X \to X$ a group action and $\zeta : G \to (0, \infty)$. Then μ is said to be **relatively ϕ-invariant with multiplier** ζ if for each $g \in G$ and $U \in \mathcal{A}$ $\mu(g^{-1} \cdot U) = \zeta(g)\mu(U)$. If for each $g \in G$, $\zeta(g) = e$, then μ is said to be **ϕ-invariant**.

Exercise 10.4.0.4. Let G be a locally compact group and $\mu : \mathcal{B}(G) \to [0, \infty]$ a left Haar measure. Define the actions $\phi, \psi : G \times G \to G$ by $\phi(g, x) = gx$ and $\psi(g, x) = xg^{-1}$. Then μ is ϕ-invariant and μ is relatively ψ-invariant with multiplier Δ.

Proof. Clear. □

Exercise 10.4.0.5. Let G be a group, $(X, \mathcal{A}, \mu)$ a semifinite measure space, $\phi : G \times X \to X$ a group action and $\zeta : G \to (0, \infty)$. Suppose that $\mu \neq 0$. If μ is relatively ϕ-invariant with multiplier ζ, then

1. ζ is a homomorphism
2. for each $g \in G$, $f \in L^1(\mu) \cup L^+$,
$$\int L_g f \, d\mu = \zeta(g) \int f \, d\mu$$

Proof.

1. Let $g, h \in G$. Choose $U \in \mathcal{A}$ such that $\mu(U) \in (0, \infty)$. Then
$$\begin{aligned} \zeta(gh)\mu(U) &= \mu(gh \cdot U) \\ &= \mu(g \cdot (h \cdot U)) \\ &= \zeta(g)\mu(h \cdot U) \\ &= \zeta(g)\zeta(h)\mu(U) \end{aligned}$$
Then $\zeta(gh) = \zeta(g)\zeta(h)$. Since $g, h \in G$ are arbitary, ζ is a homomorphism.

2. Let $g \in G$ and $U \in \mathcal{A}$. Set $f = \chi_U$. Then
$$\begin{aligned} \int L_g f \, d\mu &= \int \chi_{gU} \, d\mu \\ &= \mu(gU) \\ &= \zeta(g)\mu(U) \\ &= \zeta(g) \int f \, d\mu \end{aligned}$$
Linearity of L_g implies that for each $f \in S^+$,
$$\int L_g f \, d\mu = \zeta(g) \int f \, d\mu$$

Let $f \in L^+$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset S^+$ such that $f_n \xrightarrow{\text{p.w.}} f$ and for each $N \in \mathbb{N}$, $f_n \leq f_{n+1}$. Hence $L_g f_n \xrightarrow{\text{p.w.}} L_g f$ and for each $N \in \mathbb{N}$, $L_g f_n \leq L_g f_{n+1}$. The monotone convergence theorem then implies that

$$\begin{aligned}\int L_g f \, d\mu &= \lim_{n\to\infty} \int L_g f_n \, d\mu \\ &= \lim_{n\to\infty} \zeta(g) \int f_n \, d\mu \\ &= \zeta(g) \lim_{n\to\infty} \int f_n \, d\mu \\ &= \zeta(g) \int f \, d\mu\end{aligned}$$

Let $f \in L^1(\mu)$. If $f : X \to \mathbb{R}$, then $f = f^+ - f^-$ and

$$\begin{aligned}\int L_g f \, d\mu &= \int L_g(f^+ - f^-) \, d\mu \\ &= \int L_g f^+ \, d\mu - \int L_g f^- \, d\mu \\ &= \zeta(g) \int f^+ \, d\mu - \zeta(g) \int f^- \, d\mu \\ &= \zeta(g) \int f^+ - f^- \, d\mu \\ &= \zeta(g) \int f \, d\mu\end{aligned}$$

If $f : X \to \mathbb{C}$, then there exist $a, b : X \to \mathbb{R}$ such that $f = a + ib$. Then

$$\begin{aligned}\int L_g f \, d\mu &= \int L_g(a + ib) \, d\mu \\ &= \int L_g a \, d\mu + i \int L_g b \, d\mu \\ &= \zeta(g) \int a \, d\mu + i\zeta(g) \int b \, d\mu \\ &= \zeta(g) \int a + ib \, d\mu \\ &= \zeta(g) \int f \, d\mu\end{aligned}$$

□

Definition 10.4.0.6. Let X be a set, G a group, $\phi : G \times X \to X$ a group action, $f : X \to \mathbb{C}$ and $x \in X$. We define $f^x : G \to \mathbb{C}$ by

$$f^x(g) = f(g^{-1} \cdot x)$$

Exercise 10.4.0.7. Let X be a LCH space, G a locally compact group $\phi : G \times X \to X$ a proper group action and $f \in C_c(X)$. Then for each $x \in X$, $f^x \in C_c(G)$.

Proof. □

Exercise 10.4.0.8. Let X be a LCH space, G a locally compact group with left Haar measure μ, $\phi : G \times X \to X$ a group action and $f \in C_c(X)$. Define $f^* : X \to \mathbb{C}$ by

$$f^*(x) = \int f(g^{-1} \cdot x) \, d\mu(g)$$

Chapter 11

Hausdorff Measure

11.1 Introduction

Definition 11.1.0.1. Let X be a metric space and $\mu^* : \mathcal{P}(X) \to [0,\infty]$ an outer measure on X. Then μ^* is said to be a **metric outer measure on** X if for each $A, B \subset X$, $d(A,B) > 0$ implies that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

Exercise 11.1.0.2. Let X be a metric space and $\mu^* : \mathcal{P}(X) \to [0,\infty]$ a metric outer measure on X. Then for each $A \in \mathcal{B}(X)$, A is μ^*-outer measurable.

Proof. $\square$

Definition 11.1.0.3. Let X be a metric space, $E \subset X$ and $\delta > 0$. Define $\mathcal{A}_{E,\delta} \subset \mathcal{P}(X)^{\mathbb{N}}$ by

$$\mathcal{A}_{E,\delta} = \inf \left\{ (A_j)_{j\in\mathbb{N}} \subset \mathcal{P}(X) : E \subset \bigcup_{j\in\mathbb{N}} A_j \text{ and for each } j \in \mathbb{N},\ \operatorname{diam}(A_j) < \delta \right\}$$

Exercise 11.1.0.4. Let X be a metric space, $E \subset X$ and $\delta_1, \delta_2 > 0$. If $\delta_1 \le \delta_2$, then $\mathcal{A}_{E,\delta_1} \subset \mathcal{A}_{E,\delta_2}$.

Proof. Clear. $\square$

Definition 11.1.0.5. Let X be a metric space, $d \ge 0$ and $\delta > 0$. Define $H_{d,\delta} : \mathcal{P}(X) \to [0,\infty]$ by

$$H_{d,\delta}(E) = \inf \left\{ \sum_{j\in\mathbb{N}} \operatorname{diam}(A_j)^d : (A_j)_{j\in\mathbb{N}} \in \mathcal{A}_{E,\delta} \right\}$$

Exercise 11.1.0.6. Let X be a metric space, $d \ge 0$ and $\delta_1, \delta_2 > 0$. If $\delta_1 \le \delta_2$, then $H_{d,\delta_2} \le H_{d,\delta_1}$.

Proof. Clear. $\square$

Definition 11.1.0.7. Let X be a metric space and $d \ge 0$. We define the **d-dimensional Hausdorff outer measure**, denoted $H_d : \mathcal{P}(X) \to [0,\infty]$, by

$$\begin{aligned} H_d(E) &= \sup_{\delta > 0} H_{d,\delta}(E) \\ &= \lim_{\delta \to 0^+} H_{d,\delta}(E) \end{aligned}$$

Exercise 11.1.0.8. Let X be a metric space and $d \ge 0$. Then $H_d : \mathcal{P}(X) \to [0,\infty]$ is an outer measure on X.

Proof. $\square$

Exercise 11.1.0.9. Let X be a metric space and $d \ge 0$. Then $H_d : \mathcal{P}(X) \to [0,\infty]$ is a metric outer measure on X.

Proof. $\square$

11.2 Hausdorff Measure on Smooth Manifolds

11.3 Induced Measures on Isometric Orbit Spaces

Note 11.3.0.1. This section assumes familiarity with induced metrics on orbit spaces of metric spaces under isometric group actions. See section 9.1 of [2] for details.

Note 11.3.0.2.

Definition 11.3.0.3. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $(X/G, \bar{d})$ is a metric space. Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a measure on X. We define $\bar{\mu} : \mathcal{B}(X/G) \to [0, \infty]$ by $\bar{\mu} = \pi_* \mu$.

Note 11.3.0.4. If $\mu \ll H_p^X$, where X has Hausdorff dimension p, I want to be able to define $\bar{\mu}$ in terms of $H_q^{X/G}$ where X/G has Hausdorff dimension q. I was unable to do this. It might be possible with some manifold theory, for instance $O(2)$ acting on $\mathbb{R}^2$.

Definition 11.3.0.5. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $(X/G, \bar{d})$ is a metric space. Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a measure on X. Then μ is said to be G-invariant if for each $g \in G$, $U \in \mathcal{B}(X)$,

$$\mu(g \cdot U) = \mu(U)$$

Exercise 11.3.0.6. Let X be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Then for each $p \geq 0$, H_p is G-invariant.

Proof. Clear. $\square$

Exercise 11.3.0.7. Let X be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a measure on X. Suppose that $\mu \ll H_p$. Then μ is G-invariant iff $d\mu/dH_p$ is G-invariant.

Proof. Suppose that μ is G-invariant. Let $g \in G$ and $U \in \mathcal{B}(X)$. Then

$$\begin{aligned}
\int_U L_g \frac{d\mu}{dH_p}(x)\, dH_p(x) &= \int_U \frac{d\mu}{dH_p} \circ l_g^{-1}(x)\, dH_p(x) \\
&= \int_{l_g^{-1}(U)} \frac{d\mu}{dH_p}(x)\, d(l_g^{-1})_* H_p(x) \\
&= \int_{g^{-1} \cdot U} \frac{d\mu}{dH_p}(x)\, dH_p(x) \\
&= \mu(g^{-1} \cdot U) \\
&= \mu(U)
\end{aligned}$$

So that

$$L_g \frac{d\mu}{dH_p} = \frac{d\mu}{dH_p}$$

The Converse is similar. $\square$

Exercise 11.3.0.8. Let (X, d) be a metric space, G a group, and $\phi : G \times X \to X$ an isometric group action. Suppose that $\bar{d}$ is a metric. Let $\mu : \mathcal{B}(X) \to [0, \infty]$ be a measure on X. Suppose that μ is G-invariant, $\mu \ll H_p^X$ and $d\mu/dH_p^X$ is continuous. Then $\bar{\mu} \ll \bar{H}_p^X$, $d\bar{\mu}/d\bar{H}_p^X$ is G-invariant, $d\bar{\mu}/d\bar{H}_p^X$ is continuous and

$$\frac{d\bar{\mu}}{d\bar{H}_p^X} = \overline{\frac{d\mu}{dH_p^X}}$$

Proof. A previous exercise implies that $\bar{\mu} \ll \bar{H}_p^X$. Set $f = d\mu/dH_p^X$. Since μ is G-invariant, f is G-invariant. Since f is continuous, an exercise in section 9.2 of [2] implies that $\bar{f}$ is continuous and $f = \bar{f} \circ \pi$. Let $E \in \mathcal{B}(X/G)$. Then

$$\begin{aligned} \int_E \bar{f} d\bar{H}_p^X &= \int_{\pi^{-1}(E)} \bar{f} \circ \pi dH_p^X \\ &= \int_{\pi^{-1}(E)} f dH_p^X \\ &= \mu(\pi^{-1}(E)) \\ &= \bar{\mu}(E) \end{aligned}$$

Therefore, by definition, we have that

$$\frac{d\bar{\mu}}{d\bar{H}_p^X} = \bar{f} = \overline{\frac{d\mu}{dH_p^X}}$$

$\square$

Chapter 12

Measure and Integration on Frechet Spaces

12.1 Borel Measures on Frechet Spaces

Definition 12.1.0.1. Let X be a topological vector space. We define the **cylindrical σ-algebra on** X, denoted $\mathcal{E}(X)$, by

$$\mathcal{E}(X) := \sigma_X(X^*)$$

Exercise 12.1.0.2. Let $(X, \mathcal{A})$ be a measurable space, Y a normed vector space and $f : X \to Y$. Then f is ($\mathcal{A}$-$\mathcal{E}(Y)$) measurable iff for each $\phi \in X^*$, $\phi \circ f$ is ($\mathcal{A}$-$\mathcal{B}(\mathbb{C})$) measurable.

Proof. Immediate by exercise about initial σ-algebra. □

Exercise 12.1.0.3. Let X be a normed vector space. Then $\mathcal{E}(X) \subset \mathcal{B}(X)$.

Proof. Let $\phi \in X^*$. Since ϕ is continuous, ϕ is $\mathcal{B}(X)$-measurable. Hence for each $E \in \mathcal{B}_\mathbb{C}$, $\phi^{-1}(E) \in \mathcal{B}(X)$. Thus $\{\phi^{-1}(E) : E \in \mathcal{B}(\mathbb{C}) \text{ and } \phi \in X^*\} \subset \mathcal{B}(X)$. This implies that

$$\begin{aligned}\mathcal{E}(X) &= \sigma_X(X^*) \\ &= \sigma(\{\phi^{-1}(E) : E \in \mathcal{B}(\mathbb{C}) \text{ and } \phi \in X^*\}) \\ &\subset \mathcal{B}(X)\end{aligned}$$

□

Exercise 12.1.0.4. Mourier's Theorem:
Let X be a normed vector space. If X is separable, then $\mathcal{E}(X) = \mathcal{B}(X)$.
Hint: Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a dense subset. An exercise in the section on duality implies that there exist $(\phi_n)_{n\in\mathbb{N}} \subset X^*$ such that for each $n \in \mathbb{N}$, $\|\phi_n\| = 1$ and $\phi_n(x_n) = \|x_n\|$ and for each $x \in X$, $\|x\| = \sup\limits_{n\in\mathbb{N}} |\phi_n(x)|$. Then $\operatorname{cl} B(0,1) \in \mathcal{E}(X)$.
probability distributions on Banach spaces - Vakhania, pg 17

Proof. Suppose that X is separable. Then there exists $(x_n)_{n\in\mathbb{N}} \subset X$ such that $(x_n)_{n\in\mathbb{N}}$ is dense in X. An exercise from the section on duality in [2] implies that there exists $(\phi_n)_{n\in\mathbb{N}} \subset X^*$ such that for each $n \in \mathbb{N}$, $\|\phi_n\| = 1$ and $\phi_n(x_n) = \|x_n\|$. A previous exercise implies that for each $x \in X$, $\|x\| = \sup\limits_{n\in\mathbb{N}} |\phi_n(x)|$. Let $x \in X$ and $r > 0$. Then $r^{-1}\|x - y\| = \sup\limits_{n\in\mathbb{N}} |r^{-1}\phi_n(x - y)|$

and

$$\begin{aligned}
\operatorname{cl} B(x,r) &= \{y \in X : \|x-y\| \leq r\} \\
&= \{y \in X : r^{-1}\|x-y\| \leq 1\} \\
&= \bigcap_{n\in\mathbb{N}} \{y \in X : |r^{-1}\phi_n(x-y)| \leq 1\} \\
&= \bigcap_{n\in\mathbb{N}} \{y \in X : |\phi_n(x-y)| \leq r\} \\
&= \bigcap_{n\in\mathbb{N}} \{y \in X : |\phi_n(x)-\phi_n(y)| \leq r\} \\
&= \bigcap_{n\in\mathbb{N}} \phi_n^{-1}(\operatorname{cl} B_{\mathbb{C}}(\phi_n(x), r)) \\
&\in \mathcal{E}(X)
\end{aligned}$$

Let $A \subset X$. Suppose that A is open. Since X is separable, there exist $(a_n)_{n\in\mathbb{N}} \subset A$ and $(r_n)_{n\in\mathbb{N}} \subset (0,\infty)$ such that

$$\begin{aligned}
A &= \bigcup_{n\in\mathbb{N}} \operatorname{cl} B(a_n, r_n) \\
&\in \mathcal{E}(X)
\end{aligned}$$

Therefore, $\mathcal{B}(X) \subset \mathcal{E}(X)$.
The previous exercise implies that $\mathcal{E}(X) \subset \mathcal{B}(X)$. So $\mathcal{E}(X) = \mathcal{B}(X)$. □

Exercise 12.1.0.5. Let X be a separable normed vector space and $\mu, \nu \in M(X)$. Then $\mu = \nu$ iff for each $\phi \in X^*$, $\phi_*\mu = \phi_*\nu$.

Proof. If $\mu = \nu$, then clearly for each $\phi \in X^*$, $\phi_*\mu = \phi_*\nu$.
Conversely, suppose that for each $\phi \in X^*$, $\phi_*\mu = \phi_*\nu$. Let $E \in \mathcal{B}(\mathbb{C})$ and $\phi \in X^*$. Then

$$\begin{aligned}
\mu(\phi^{-1}(E)) &= \phi_*\mu(E) \\
&= \phi_*\nu(E) \\
&= \nu(\phi^{-1}(E))
\end{aligned}$$

Set $\mathcal{P} = \{\phi^{-1}(E) : \phi \in X^* \text{ and } E \in \mathcal{B}(\mathbb{C})\}$. Then $\mathcal{P}$ is a π-system. Since

$$\begin{aligned}
\sigma(\mathcal{P}) &= \mathcal{E}(X) \\
&= \mathcal{B}(X)
\end{aligned}$$

An exercise from the section on complex measures that uses Dynkin's lemma implies that $\mu = \nu$. □

Definition 12.1.0.6. Let X be a real normed vector space and $\mu \in M(X)$. We define the **Fourier transform of** μ, denoted $\widehat{\mu} : X^* \to \mathbb{C}$, by

$$\widehat{\mu}(\phi) = \int_X e^{-i\phi(x)}\, d\mu(x)$$

Exercise 12.1.0.7. Let X be a real normed vector space and $\mu \in M(X)$. Then $\widehat{\mu} : X^* \to \mathbb{C}$ is bounded.

Proof. Let $\phi \in X^*$.

$$\begin{aligned}
|\widehat{\mu}(\phi)| &= \left| \int_X e^{-i\phi(x)}\, d\mu(x) \right| \\
&\leq \int_X |e^{-i\phi(x)}|\, d|\mu|(x) \\
&= |\mu|(X)
\end{aligned}$$

So $\widehat{\mu}$ is bounded. □

Exercise 12.1.0.8. Let X be a real normed vector space and $\mu \in M(X)$. Then $\widehat{\mu} \in C_b(X^*)$.

Proof. Let $(\phi_n)_{n\in\mathbb{N}} \subset X^*$ and $\phi \in X^*$. Suppose that $\phi_n \to \phi$. Then $e^{-i\phi_n} \xrightarrow{\text{p.w.}} e^{-i\phi}$ and for each $n \in N$,

$$\begin{aligned}|e^{-i\phi_n}| &= 1 \\ &\in L^1(|\mu|)\end{aligned}$$

The dominated convergence theorem implies that

$$\begin{aligned}|\widehat{\mu}(\phi_n) - \widehat{\mu}(\phi)| &= \left| \int_X e^{-i\phi_n(x)}\, d\mu(x) - \int_X e^{-i\phi(x)}\, d\mu(x) \right| \\ &= \left| \int_X e^{-i\phi_n(x)} - e^{-i\phi(x)}\, d\mu(x) \right| \\ &\leq \int_X |e^{-i\phi_n(x)} - e^{-i\phi(x)}|\, d|\mu|(x) \\ &\to 0\end{aligned}$$

So $\widehat{\mu} : X^* \to \mathbb{C}$ is continuous (in the norm topology). Hence $\widehat{\mu} \in C_b(X^*)$. □

Definition 12.1.0.9. Let X be a real normed vector space. We define $\mathcal{F} : M(X) \to C_b(X^*)$ by

$$\mathcal{F}(\mu) = \widehat{\mu}$$

Exercise 12.1.0.10. Let X be a real normed vector space. Then $\mathcal{F} : M(X) \to C_b(X^*)$ is linear.

Proof. Let $\mu, \nu \in M(X)$ and $\phi \in X^*$. Then

$$\begin{aligned}\mathcal{F}[\mu + \nu](\phi) &= \int_X e^{-i\phi(x)}\, d[\mu + \nu](x) \\ &= \int_X e^{-i\phi(x)}\, d\mu(x) + \int_X e^{-i\phi(x)}\, d\nu(x) \\ &= \mathcal{F}[\mu](\phi) + \mathcal{F}[\nu](\phi)\end{aligned}$$

Since $\phi \in X^*$ is arbitrary, $\mathcal{F}(\mu + \nu) = \mathcal{F}(\mu) + \mathcal{F}(\nu)$ and $\mathcal{F}$ is linear. □

Exercise 12.1.0.11. Let X be a real normed vector space. If X is separable, then $\mathcal{F}$ is injective.

Proof. Suppose that X is separable. Let $\mu \in M(X)$. Suppose that $\mu \in \ker \mathcal{F}$. Then $\widehat{\mu} = 0$ and for each $\phi \in X^*$,

$$\begin{aligned}0 &= \widehat{\mu}(\phi) \\ &= \int_X e^{-i\phi(x)}\, d\mu(x) \\ &= \int_{\mathbb{R}} e^{-ix}\, d[\phi_* \mu](x)\end{aligned}$$

FINISH!!! □

Exercise 12.1.0.12. Let X be a real normed vector space. Then $\mathcal{F} \in L(M(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$.

Proof. For $\mu \in M(X)$ and $\phi \in X^*$, we have that

$$\begin{aligned}|\mathcal{F}[\mu](\phi)| &= \left| \int_X e^{-i\phi(x)}\, d\mu(x) \right| \\ &\leq \int_X |e^{-i\phi(x)}|\, d|\mu|(x) \\ &= |\mu|(X) \\ &= \|\mu\|\end{aligned}$$

Hence

$$\begin{aligned}\|\mathcal{F}(\mu)\| &= \sup_{\phi \in X^*} |\mathcal{F}[\mu](\phi)| \\ &\leq \|\mu\|\end{aligned}$$

which implies that $\mathcal{F} \in L(M(X), C_b(X^*))$ and $\|\mathcal{F}\| \leq 1$. $\square$

12.2 Weak integration on Frechet Spaces

Definition 12.2.0.1. Let $(X, \mathcal{A})$ be a measurable space, Y a topological vector space and $f : X \to Y$. Then f is said to be **weakly measurable** if for each $\phi \in Y^*$, $\phi \circ f \in L^0(X, \mathcal{A})$.

Exercise 12.2.0.2. Let $(X, \mathcal{A})$ be a measurable space, Y a topological vector space and $f : X \to Y$. Then f is weakly measurable iff f is $(\mathcal{A}, \mathcal{B}(Y))$-measurable.

Proof. Suppose that f is weakly measurable. Then for each $\phi \in Y^*$, $\phi \circ f \in L^0(X, \mathcal{A})$. Let $E \in \mathcal{B}(Y)$. □

12.3 The Bochner Integral

Definition 12.3.0.1. Let $(X, \mathcal{A})$ be a measurable space, Y a Banach space and $f : X \to Y$. Then f is said to be **strongly measurable** if

1. f is $(\mathcal{A}$-$\mathcal{B}(Y))$ measurable
2. $f(X)$ is separable

We define $L^0_Y(X, \mathcal{A}) = \{f : X \to Y : f \text{ is strongly measurable}\}$

Exercise 12.3.0.2. Let $(X, \mathcal{A})$ be a measurable space, Y a Banach space and $f : X \to Y$. Then f is strongly measurable iff

1. f is $(\mathcal{A}$-$\mathcal{E}(Y))$ measurable
2. $f(X)$ is separable

Proof. □

Exercise 12.3.0.3. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. Then $L^0_Y(X, \mathcal{A})$ is a vector space.

Proof. Let $f, g \in L^0_Y(X, \mathcal{A})$ and $\lambda \in \mathbb{C}$. By definition, f and g are measurable. Since $f + \lambda g$ is a composition of measurable maps, $f + \lambda g$ is measurable. Therefore $f + \lambda g \in L^0_Y(X, \mathcal{A})$. Clearly constant maps are measurable and hence $0 \in L^0_Y(X, \mathcal{A})$. So $L^0_Y(X, \mathcal{A})$ is a vector space. □

Definition 12.3.0.4. Let $(X, \mathcal{A})$ be a measurable space, Y a Banach space and $\phi : X \to Y$. Then ϕ is said to be **simple** if

1. ϕ is $(\mathcal{A}, \mathcal{B}(X))$-measurable,
2. $\phi(X)$ is finite.

We define
$$S_Y(X, \mathcal{A}) = \{f \in L^0_Y(X, \mathcal{A}) : f \text{ is simple}\}$$

Exercise 12.3.0.5. Let $(X, \mathcal{A})$ be a measurable space, Y a Banach space. Then $S_Y(X, \mathcal{A})$ is a subspace of $L^0_Y(X, \mathcal{A})$.

Proof. FINISH!!! □

Definition 12.3.0.6. Let $(X, \mathcal{A})$ be a measurable space and $\phi \in S_Y(X, \mathcal{A})$. Set $(y_j)_{j=1}^n := \phi(X)$ and for each $j \in [n]$, set $E_j := \phi^{-1}(y_j)$. We define the **standard representation of** ϕ to be the sum

$$\phi = \sum_{j=1}^n \chi_{E_j} y_j$$

Exercise 12.3.0.7. Let $(X, \mathcal{A})$ be a measurable space, Y a Banach space. Let $\phi, \psi \in S_Y(X, \mathcal{A})$ and $\lambda \in \mathbb{C}$. Suppose that the standard representations of ϕ and ψ are

$$\phi = \sum_{j=1}^n \chi_{A_j} a_j$$

and

$$\psi = \sum_{j=1}^m \chi_{B_k} b_k$$

respectively.

1. Then $(E_j)_{j=1}^n$ are disjoint and $\bigcup_{j=1}^n E_j = X$.

2. Set

$$L := \{(j,k) \in \mathbb{N}^2 : j \in [n], k \in [m], \text{ and } A_j \cap B_k \neq \varnothing\}$$

Then the standard representation of $\phi + \psi$ is

$$\phi + \psi = \sum_{(j,k)\in L} \chi_{A_j \cap B_k}(a_j + b_k)$$

Clean UP, look at definition of standard rep in measruability section 3.3.028

Proof. 1.

2.
FINISH!!! □

Definition 12.3.0.8. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space and $p \in [1, \infty]$. Define $\|\cdot\|_p : L^0_Y(X, \mathcal{A}, \mu) \to [0, \infty]$ by

$$\|f\|_p = \left(\int \|f\|^p d\mu\right)^{\frac{1}{p}} \qquad (p < \infty)$$

and

$$\|f\|_\infty = \inf\left\{\lambda > 0 : \mu\big(\{x \in X : \lambda < \|f(x)\|\}\big) = 0\right\}$$

We define

$$L^p_Y(X, \mathcal{A}, \mu) = \{f \in L^0_Y(X, \mathcal{A}, \mu) : \|f\|_p < \infty\}$$

Exercise 12.3.0.9. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space and $p \in [1, \infty]$. Then $L^p_Y(X, \mathcal{A}, \mu)$ is a subspace of $L^0_Y(X, \mathcal{A}, \mu)$.

Proof. Let $f, g \in L^p_Y(X, \mathcal{A}, \mu)$ and $\lambda \in \mathbb{C}$. Then $\|f\|_p, \|g\|_p < \infty$.

1. Clearly $\|\lambda f\|_p = |\lambda|\|f\|_p < \infty$. So $\lambda f \in L^p_Y$.
2. Let $\|\cdot\|'_p : L^0(X, \mathcal{A}, \mu) \to [0, \infty]$ denote the usual L^p norm. Since $\|f + g\| \leq \|f\| + \|g\|$, we have that

$$\begin{aligned}
\|f + g\|_p &= \|\|f + g\|\|'_p \\
&\leq \|\|f\| + \|g\|\|'_p \\
&\leq \|\|f\|\|'_p + \|\|g\|\|'_p \\
&= \|f\|_p + \|g\|_p \\
&< \infty
\end{aligned}$$

So $f + g \in L^p_Y$.

Hence L^p_Y is a subspace. □

Exercise 12.3.0.10. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space and $p \in [1, \infty]$. Then

1. $\|\cdot\|_p$ is a seminorm on $L^p_Y(X, \mathcal{A}, \mu)$
2. if we identify functions that are equal μ-a.e., then $\|\cdot\|_p$ is a norm on $L^p_Y(X, \mathcal{A}, \mu)$

Proof. Let $f, g \in L^p_Y X, \mathcal{A}, \mu)$ and $\lambda \in \mathbb{C}$.

1. The previous exercise implies that, $\|\lambda f\|_p = |\lambda|\|f\|_p$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. So $\|\cdot\|_p$ is a seminorm on L^p_Y.
2. If $f = 0$ μ-a.e., then $\|f\| = 0$ μ-a.e. Hence

$$\begin{aligned}
\|f\|_p &= \|\|f\|\|'_p \\
&= 0
\end{aligned}$$

So if we identify functions that are equal μ-a.e., $\|\cdot\|_p$ becomes a norm on L^p_Y.

□

Note 12.3.0.11. So for $(f_n)_{n\in\mathbb{N}} \subset L^p_Y$ and $f \in L^p_Y$,

$$f_n \xrightarrow{L^p_Y} f \text{ iff } \int \|f_n - f\|^p \to 0$$

Definition 12.3.0.12. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space and $\phi : X \to Y$. Then ϕ is said to be **simple** if ϕ is measurable, $\phi(X)$ is finite and for each $y \in \phi(X) \setminus \{0\}$, $\mu(\phi^{-1}(y)) < \infty$. If ϕ is simple then the **standard representation of** ϕ is defined to be the sum

$$\phi = \sum_{j=1}^{n} \chi_{E_j} y_j$$

where $(y_j)_{j=1}^n = \phi(X)$ and for each $j \in \{1, \cdots, n\}$, $E_j = \phi^{-1}(y_j)$. We define

$$S_Y(X, \mathcal{A}, \mu) = \{f \in L^0_Y(X, \mathcal{A}) : f \text{ is simple}\}$$

Note 12.3.0.13. If $\phi = \sum\limits_{j=1}^{n} \chi_{E_j} y_j$ is in the standard representation, then $(E_j)_{j=1}^n$ are disjoint and $\bigcup\limits_{j=1}^{n} E_j = X$.

Exercise 12.3.0.14. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. Then $S_Y \subset L^1_Y$.

Proof. Let $\phi \in S_Y$. Write $\phi = \sum\limits_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. Then $\|\phi\| = \sum\limits_{j=1}^{n} \|y_j\| \chi_{E_j}$. By definition, for each $j \in \{1, \cdots, n\}$, $y_j \neq 0$ implies that $\mu(E_j) < \infty$. Then

$$\begin{aligned}\int \|\phi\| d\mu &= \sum_{j=1}^{n} \|y_j\| \mu(E_j) \\ &< \infty\end{aligned}$$

So $\phi \in L^1_Y$. □

Exercise 12.3.0.15. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. Then $S_Y(X, \mathcal{A}, \mu)$ is a subspace of $L^0_Y(X, \mathcal{A})$

Proof. Clear. □

Note 12.3.0.16. For the remainder of this section, we will use the shorthand notation L^0_Y, L^p_Y and S_Y unless the context underlying measure space $(X, \mathcal{A}, \mu)$ is unclear.

Definition 12.3.0.17. Let $(X, \mathcal{A}, \mu)$ be a measure space and Y a Banach space. Let $\phi \in S_Y$. Write $\phi = \sum\limits_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. With the convention that $\infty \cdot 0_Y = 0_Y$, we define

$$\int \phi d\mu = \sum_{j=1}^{n} \mu(E_j) y_j$$

For $A \in \mathcal{A}$, define

$$\int_A \phi d\mu = \int \chi_A \phi d\mu$$

Exercise 12.3.0.18. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $\phi \in S_Y$ and $A \in \mathcal{A}$. Write $\phi = \sum\limits_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. Then

$$\int_A \phi d\mu = \sum_{j=1}^{n} \mu(A \cap E_j) y_j$$

Proof. Note that $\chi_A \phi = \sum\limits_{j=1}^{n} \chi_{A \cap E_j} y_j$. □

Exercise 12.3.0.19. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $\phi, \psi \in S_Y$ and $\lambda \in \mathbb{C}$. Then

$$\int \phi + \lambda \psi d\mu = \int \phi d\mu + \lambda \int \psi d\mu$$

Proof. If $\lambda = 0$, then the result clearly holds. Suppose that $\lambda \neq 0$. Write $\phi = \sum_{j=1}^{n} \chi_{A_j} a_j$ and $\psi = \sum_{j=k}^{m} \chi_{B_k} b_k$ in the standard representation. Put

$$L = \{(j,k) \in \mathbb{N}^2 : j \leq n, k \leq m, \text{ and } A_j \cap B_k \neq \varnothing\}$$

Then the standard representation of $\phi + \lambda\psi$ is given by $\phi + \lambda\psi = \sum_{(j,k)\in L} \chi_{A_j \cap B_k}(a_j + \lambda b_k)$. So

$$\begin{aligned}
\int \phi + \lambda \psi d\mu &= \int \sum_{(j,k)\in L} \chi_{A_j \cap B_k}(a_j + \lambda b_k) d\mu \\
&= \sum_{(j,k)\in L} \mu(A_j \cap B_k)(a_j + \lambda b_k) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} \mu(A_j \cap B_k)(a_j + \lambda b_k) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} \mu(A_j \cap B_k) a_j + \lambda \sum_{j=1}^{n} \sum_{k=1}^{m} \mu(A_j \cap B_k) b_k \\
&= \sum_{j=1}^{n} \mu(A_j) a_j + \lambda \sum_{k=1}^{m} \mu(B_k) b_k \\
&= \int \phi d\mu + \lambda \int \psi d\mu
\end{aligned}$$

□

Exercise 12.3.0.20. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $\phi \in S_Y$. Then

$$\left\| \int \phi d\mu \right\| \leq \int \|\phi\| d\mu$$

Proof. Write $\phi = \sum_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. Note that $\|\phi\| = \sum_{j=1}^{n} \chi_{E_j} \|y_j\|$. Then

$$\begin{aligned}
\left\| \int \phi d\mu \right\| &= \left\| \int \sum_{j=1}^{n} \chi_{E_j} y_j d\mu \right\| \\
&= \left\| \sum_{j=1}^{n} \mu(E_j) y_j \right\| \\
&\leq \sum_{j=1}^{n} \mu(E_j) \|y_j\| \\
&= \int \sum_{j=1}^{n} \|y_j\| \chi_{E_j} d\mu \\
&= \int \|\phi\| d\mu
\end{aligned}$$

□

Exercise 12.3.0.21. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $f \in L^1_Y$ and $(\phi_n)_{n\in\mathbb{N}} \subset S_Y$. If $\phi_n \xrightarrow{L^1_Y} f$, then

$$\lim_{n\to\infty} \int \phi_n d\mu$$

exists.

Proof. Suppose that $\phi \xrightarrow{L^1_Y} f$. Then by definition,

$$\int \|\phi_n - f\| d\mu \to 0$$

Let $m, n \in \mathbb{N}$. Then

$$\begin{aligned}\left\| \int \phi_m d\mu - \int \phi_n d\mu \right\| &= \left\| \int \phi_m - \phi_n d\mu \right\| \\ &\leq \int \|\phi_m - \phi_n\| d\mu \\ &\leq \int \|\phi_m - f\| d\mu + \int \|f - \phi_n\| d\mu\end{aligned}$$

Hence $(\int \phi_n d\mu)_{n\in\mathbb{N}} \subset Y$ is Cauchy and $\lim\limits_{n\to\infty} \int \phi_n d\mu$ exists. □

Exercise 12.3.0.22. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $f \in L^1_Y$ and $(\phi_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}} \subset S_Y$. If $\phi_n \xrightarrow{L^1_Y} f$ and $\psi_n \xrightarrow{L^1_Y} f$, then

$$\lim_{n\to\infty} \int \phi_n d\mu = \lim_{n\to\infty} \int \psi_n d\mu$$

Proof. Suppose that $\phi_n \xrightarrow{L^1_Y} f$ and $\psi_n \xrightarrow{L^1_Y} f$. Let $\epsilon > 0$. By defintion, there exist $N_1 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $n \geq N_1$ implies that $\int \|\phi_n - f\| d\mu < \frac{\epsilon}{6}$ and $\int \|\psi_n - f\| d\mu < \frac{\epsilon}{6}$. Similarly to the previous exercise we have that for each $n \in \mathbb{N}$, $n \geq N_1$ implies that

$$\begin{aligned}\left\| \int \phi_n d\mu - \int \psi_n d\mu \right\| &= \left\| \int \phi_n - \psi_n d\mu \right\| \\ &\leq \int \|\phi_n - \psi_n\| d\mu \\ &\leq \int \|\phi_n - f\| d\mu + \int \|f - \psi_n\| d\mu \\ &< \frac{\epsilon}{6} + < \frac{\epsilon}{6} \\ &= \frac{\epsilon}{3}\end{aligned}$$

Put $I_\phi = \lim\limits_{n\to\infty} \int \phi_n d\mu$ and $I_\psi = \lim\limits_{n\to\infty} \int \psi_n d\mu$. Then there exists $N_2 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $n \geq N_2$, then

$$\left\| \int \phi_n d\mu - I_\phi \right\| < \frac{\epsilon}{3}$$

and

$$\left\| \int \psi_n d\mu - I_\psi \right\| < \frac{\epsilon}{3}$$

Choose $N = \max(N_1, N_2)$. Then for each $n \in \mathbb{N}$, $n \geq N$ implies that

$$\begin{aligned}\|I_\phi - I_\psi\| &\leq \left\| I_\phi - \int \phi_n d\mu \right\| + \left\| \int \phi_n d\mu - \int \psi_n d\mu \right\| + \left\| \int \psi_n d\mu - I_\psi \right\| \\ &=< \frac{\epsilon}{3} + < \frac{\epsilon}{3} + < \frac{\epsilon}{3} \\ &= \epsilon\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $I_\phi = I_\psi$. □

Exercise 12.3.0.23. Let Y be a Banach space and $(y_n)_{n\in\mathbb{N}} \subset Y$ a countable dense subset. For $\epsilon > 0$ and $n \in \mathbb{N}$, define $B_n^\epsilon \in \mathcal{B}(Y)$ by

$$B_n^\epsilon = \{y \in Y : \|y - y_n\| < \epsilon \|y_n\|\}$$

Then for each $\epsilon \geq 0$,

1. $$Y \setminus \{0\} \subset \bigcup_{n\in\mathbb{N}} B_n^\epsilon$$

2. if $\epsilon \leq 1$,
$$Y \setminus \{0\} = \bigcup_{n\in\mathbb{N}} B_n^\epsilon$$

Proof. Let $\epsilon \geq 0$.

1. For the sake of contradiction, suppose that $Y \setminus \{0\} \not\subset \bigcup_{n\in\mathbb{N}} B_n^\epsilon$. Then there exists $y \in Y$ such that $y \neq 0$ and for each $n \in \mathbb{N}$, $\|y - y_n\| \geq \epsilon \|y_n\|$. Since $(y_n)_{n\in\mathbb{N}}$ is dense in Y, there exists a subsequence $(y_{n_j})_{j\in\mathbb{N}} \subset (y_n)_{n\in\mathbb{N}}$ such that for each $j \in \mathbb{N}$, $\|y_{n_j} - y\| < 1/j$. Then for each $j \in \mathbb{N}$,
$$\begin{aligned} \|y_{n_j}\| &\leq \epsilon^{-1}\|y - y_{n_j}\| \\ &< \epsilon^{-1} 1/j \end{aligned}$$
So that $y_{n_j} \to y$ and $y_{n_j} \to 0$. Since $y \neq 0$, this is a contradiction and thus
$$Y \setminus \{0\} \subset \bigcup_{n\in\mathbb{N}} B_n^\epsilon$$

2. Suppose that $\epsilon \leq 1$. For the sake of contradiction, suppose that $0 \in \bigcup_{n\in\mathbb{N}} B_n^\epsilon$. Then there exists $n \in \mathbb{N}$ such that $0 \in B_n^\epsilon$. By definition,
$$\begin{aligned} \|y_n\| &= \|0 - y_n\| \\ &< \epsilon \|y_n\| \\ &\leq \|y_n\| \end{aligned}$$
Which is a contradiction. So $0 \notin \bigcup_{n\in\mathbb{N}} B_n^\epsilon$. Hence $\{0\} \subset \left(\bigcup_{n\in\mathbb{N}} B_n^\epsilon\right)^c$ and $\bigcup_{n\in\mathbb{N}} B_n^\epsilon \subset \{0\}^c$. Hence $\bigcup_{n\in\mathbb{N}} B_n^\epsilon \subset Y \setminus \{0\}$ and $Y \setminus \{0\} = \bigcup_{n\in\mathbb{N}} B_n^\epsilon$.

□

Exercise 12.3.0.24. Let $(X, \mathcal{A})$ be a measurable space, Y a separable Banach space and $f \in L^0_Y(X, \mathcal{A})$. Let $(y_n)_{n\in\mathbb{N}} \subset Y$ be a countable dense subset. For $j \in \mathbb{N}$, define $(A_n^j)_{n\in\mathbb{N}} \subset \mathcal{B}(Y)$ and $(E_n^j)_{n\in\mathbb{N}} \subset \mathcal{A}$ by

- $A_1^j = B_1^{1/j}$
- $A_n^j = B_n^{1/j} \setminus \left(\bigcup_{k=1}^{n-1} B_k^{1/j}\right)$
- $E_n^j = f^{-1}(A_n^j)$

Let $j \in \mathbb{N}$. Then

1. $(A_n^j)_{n\in\mathbb{N}}$ is disjoint and
$$\bigcup_{n\in\mathbb{N}} A_n^j = Y \setminus \{0\}$$

2. $(E_n^j)_{n\in\mathbb{N}}$ is disjoint and
$$\bigcup_{n\in\mathbb{N}} E_n^j = X \setminus f^{-1}(\{0\})$$
3. if $j \geq 2$, then for each $n \in \mathbb{N}$ and $x \in E_n^j$,
$$\|y_n\| < \frac{j}{j-1}\|f(x)\|$$

Hint: reverse triangle inequality

Proof.

1. Clear by previous exercice
2. Clear
3. Suppose that $j \geq 2$. Let $n \in \mathbb{N}$ and $x \in E_n^j$. Then $f(x) \in A_n^j \subset B_n^{1/j}$. Hence
$$\begin{aligned}\|y_n\| - \|f(x)\| &\leq \Big|\|y_n\| - \|f(x)\|\Big| \\ &\leq \|y_n - f(x)\| \\ &< \frac{1}{j}\|y_n\|\end{aligned}$$
Thus $(1-1/j)\|y_n\| < \|f(x)\|$. Since $j-1>0$, we have that
$$\|y_n\| < \frac{j}{j-1}\|f(x)\|$$

□

Exercise 12.3.0.25. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a separable Banach space and $f \in L^1_Y(X, \mathcal{A}, \mu)$. Let $(y_n)_{n\in\mathbb{N}} \subset Y$ be a countable dense subset. For $j \in \mathbb{N}$, define $(E_n^j)_{n\in\mathbb{N}} \subset \mathcal{A}$ as in the previous exercise and $(\psi_j)_{j\in\mathbb{N}} \subset L^0_Y(X, \mathcal{A})$ by
$$\psi_j = \sum_{n\in\mathbb{N}} \chi_{E_n^j} y_n$$
Then for each $j \in \mathbb{N}$, $j \geq 2$ implies that

1. $\psi_j \in L^1(X, \mathcal{A}, \mu)$
2. $\|\psi_j - f\| < \frac{1}{j-1}\|f\|_1$

Proof. Let $j \in \mathbb{N}$. Suppose that $j \geq 2$. Then

1.
$$\begin{aligned}\|\psi_j\|_1 &= \int \|\psi_j\|\, d\mu \\ &= \int \sum_{n\in\mathbb{N}} \|y_n\|\chi_{E_n^j}\, d\mu \\ &= \sum_{n\in\mathbb{N}} \int_{E_n^j} \|y_n\|\, d\mu \\ &\leq \frac{j}{j-1} \sum_{n\in\mathbb{N}} \int_{E_n^j} \|f\|\, d\mu \\ &= \frac{j}{j-1} \int_{\bigcup\limits_{n\in\mathbb{N}} E_n^j} \|f\|\, d\mu \\ &= \frac{j}{j-1} \int \|f\|\, d\mu \\ &= \frac{j}{j-1}\|f\|_1\end{aligned}$$

So $\psi_j \in L^1_Y(X, \mathcal{A}, \mu)$.

2. Similarly, we have that

$$\begin{aligned}
\|\psi_j - f\|_1 &= \int \|\psi_j - f\| \, d\mu \\
&= \int_{f^{-1}(\{0\})} \|\psi_j - f\| \, d\mu + \sum_{n\in\mathbb{N}} \int_{E_n^j} \|\psi_j - f\| \, d\mu \\
&= \sum_{n\in\mathbb{N}} \int_{E_n^j} \|y_n - f\| \, d\mu \\
&\leq \sum_{n\in\mathbb{N}} \int_{E_n^j} \frac{1}{j-1} \|y_n\| \, d\mu \\
&\leq \sum_{n\in\mathbb{N}} \int_{E_n^j} \frac{1}{j-1} \|f\| \, d\mu \\
&= \frac{1}{j-1} \int \|f\| \, d\mu \\
&= \frac{1}{j-1} \|f\|_1
\end{aligned}$$

So $\|\psi_j - f\| < \frac{1}{j-1}\|f\|_1$.

$\square$

Exercise 12.3.0.26. such that $\phi_n \xrightarrow{\text{a.e.}} f$ and $\phi_n \xrightarrow{L^1_Y} f$.
Hint: Choose a countable dense subset $(y_n)_{n\in\mathbb{N}} \subset f(X)$ and define

Definition 12.3.0.27. Bochner Integral:
Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a separable Banach space and $f : X \to Y$. Then f is said to be **Bochner** integrable if $f \in L^1_Y$. If f is Bochner integrable, then there exists $(\phi_n)_{n\in\mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{\text{a.e.}} f$ and $\phi_n \xrightarrow{L^1_Y} f$ and the **Bochner integral of** f with respect to μ, denoted

$$\int f d\mu$$

is defined to be

$$\int f d\mu = \lim_{n\to\infty} \int \phi_n d\mu$$

Exercise 12.3.0.28. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a separable Banach space, $f, g \in L^1_Y$ and $\lambda \in \mathbb{C}$. Then

$$\int f + \lambda g d\mu = \int f d\mu + \lambda \int g d\mu$$

Proof. Choose $(\phi_n)_{n\in\mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{L^1_Y} f$ and $(\psi_n)_{n\in\mathbb{N}} \subset S_Y$ such that $\psi_n \xrightarrow{L^1_Y} g$. Since addition and scalar multiplication are continuous, $\phi_n + \lambda\psi_n \xrightarrow{L^1_Y} f + \lambda g$. By definition, we have that

$$\int \phi_n + \lambda\psi_n d\mu \to \int f + \lambda g d\mu$$

$$\int \phi_n d\mu \to \int f d\mu$$

and

$$\int \psi_n d\mu \to \int g d\mu$$

Hence

$$\begin{aligned}\int f+\lambda g d\mu &= \lim_{n\to\infty}\int \phi_n+\lambda\psi_n d\mu\\ &= \lim_{n\to\infty}\int \phi_n d\mu+\lambda\lim_{n\to\infty}\int \psi_n d\mu\\ &= \int f d\mu+\lambda\int g d\mu\end{aligned}$$

□

Exercise 12.3.0.29. Let $(X,\mathcal{A},\mu)$ be a measure space and Y a separable Banach space. Define $I: L^1_Y \to Y$ by

$$If = \int f d\mu$$

Then $I \in L(L^1_Y, Y)$ and $\|I\| \le 1$.

Proof. Let $f \in L^1_Y$. Choose $(\phi_n)_{n\in\mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{L^1_Y} f$. Then

$$\begin{aligned}\left|\int \|\phi_n\|d\mu - \int \|f\|d\mu\right| &= \left|\int \|\phi_n\|-\|f\|d\mu\right|\\ &\le \int |\|\phi_n\|-\|f\||d\mu\\ &\le \int \|\phi_n - f\|d\mu\\ &\to 0\end{aligned}$$

So

$$\int \|\phi_n\|d\mu \to \int \|f\|d\mu$$

By continuity of $\|\cdot\| : Y \to [0,\infty)$,

$$\begin{aligned}\|If\| &= \left\|\int f d\mu\right\|\\ &= \left\|\lim_{n\to\infty}\int \phi_n d\mu\right\|\\ &= \lim_{n\to\infty}\left\|\int \phi_n d\mu\right\|\\ &\le \lim_{n\to\infty}\int \|\phi_n\|d\mu\\ &= \int \|f\|d\mu\\ &= \|f\|_1\end{aligned}$$

□

Exercise 12.3.0.30. Let Y be a separable Banach space and $f : [a,b] \to Y$ continuous. Then f is Banach-integrable.

Proof. Continuity implies that $f \in L^\infty_Y$ and

$$\begin{aligned}\int \|f\|dm &\le \|f\|_\infty(b-a)\\ &< \infty\end{aligned}$$

so that $f \in L^1_Y$ and f is Bochner integrable. □

Exercise 12.3.0.31. Dominated Convergence Theorem:
Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a separable Banach space, $(f_n)_{n\in\mathbb{N}} \subset L^1_Y$ and $f \in L^0_Y$. Suppose that $f_n \xrightarrow{\text{a.e.}} f$ and there exists $g \in L^1$ such that for each $n \in \mathbb{N}$, $\|f_n\| \leq g$. Then $f \in L^1_Y$ and $f_n \xrightarrow{L^1} f$.

Proof. Since $f_n \xrightarrow{\text{a.e.}} f$, $\|f\| \leq g$ a.e. and $f \in L^1_Y$. Also,

$$\begin{aligned}\|f_n - f\| &\leq \|f_n\| + \|f\| \\ &\leq 2g \text{ a.e.}\end{aligned}$$

Hence $2g - \|f_n - f\| \geq 0$ a.e. Fatou's lemma implies that

$$\begin{aligned}\int 2g\, d\mu &= \int \liminf_{n\to\infty}(2g - \|f_n - f\|)\, d\mu \\ &\leq \liminf_{n\to\infty}\left[\int 2g - \|f_n - f\|\, d\mu\right] \\ &= \int 2g\, d\mu - \limsup_{n\to\infty}\int \|f_n - f\|\, d\mu\end{aligned}$$

Hence

$$0 \leq \limsup_{n\to\infty}\int \|f_n - f\|\, d\mu \leq 0$$

and $f_n \xrightarrow{L^1_Y} f$. □

Exercise 12.3.0.32. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y, Z separable Banach spaces and $f \in L^1_Y$ and $T \in L(Y, Z)$. Then $T \circ f \in L^1_Z$ and

$$\int T \circ f d\mu = T\left(\int f d\mu\right)$$

Note 12.3.0.33. The statement remains true if T is continuous and conjugate-linear.

Proof. Suppose that $f \in S_Y$. Write $f = \sum\limits_{j=1}^{n} \chi_{E_j} y_j$ in the standard representation. Then $T \circ f = \sum\limits_{j=1}^{n} \chi_{E_j} T(y_j)$ and

$$\begin{aligned}\int T \circ f d\mu &= \sum_{j=1}^{n} \mu(E_j) T(y_j) \\ &= T\left(\sum_{j=1}^{n} \mu(E_j) y_j\right) \\ &= T\left(\int f d\mu\right)\end{aligned}$$

For $f \in L^1_Y$, choose $(\phi_n)_{n\in\mathbb{N}} \subset S_Y$ such that $\phi_n \xrightarrow{\text{a.e.}} f$ and $\phi_n \xrightarrow{L^1_Y} f$. Then

$$\begin{aligned}\|T \circ \phi_n - T \circ f\| &= \|T \circ (\phi_n - f)\| \\ &\leq \|T\|\|\phi_n - f\|\end{aligned}$$

So $T \circ \phi_n \xrightarrow{\text{a.e.}} T \circ f$ and $T \circ \phi_n \xrightarrow{L^1_Z} T \circ f$. Thus

$$\begin{aligned}\int T \circ f d\mu &= \lim_{n\to\infty}\int T \circ \phi_n d\mu \\ &= \lim_{n\to\infty} T\left(\int \phi_n d\mu\right) \\ &= T\left(\lim_{n\to\infty}\int \phi_n d\mu\right) \\ &= T\left(\int f d\mu\right)\end{aligned}$$

□

Note 12.3.0.34. Recall that for a function $f : X \times Y \to Z$, $x \in X$ and $y \in Y$, the functions $f_x : Y \to Z$ and $f^y : X \to Z$ are defined by $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$.

Exercise 12.3.0.35. Let $(X, \mathcal{A}, \mu)$ be a measure space, Y a Banach space, $A \subset Y$ open and $f : X \times A \to Z$. Suppose that for each $y \in A$, $f^y \in L^1(\mu)$. Define $F : Y \to \mathbb{C}$ by

$$F(y) = \int_X f^y \, d\mu$$

1. Suppose that there exists $g \in L^1(\mu)$ such that for each $(x, y) \in X \times A$, $\|f(x, y)\| \leq g(x)$. Let $y_0 \in A$. If for each $x \in X$, f_x is continuous at y_0, then F is continuous at y_0.
2. Suppose that for each $x \in X$, $f_x : A \to Z$ is Gateaux differentiable and there exists $g \in L^1(\mu)$ such that for each $(x, y) \in X \times A$,$h \in Y$, $|df_x(y)(h)| \leq g(x)$. Then F is Gateaux differentiable and for each $y \in A$, $h \in Y$,

$$dF(y)(h) = \int_X df_x(y)(h) \, d\mu(x)$$

Proof.

1. Suppose that for each $x \in X$, f_x is continuous at y_0. Let $(y_n) \subset A$. Suppose that $y_n \to y_0$. Continuity implies that $f^{y_n} \xrightarrow{\text{p.w.}} f^{y_0}$. Since for each $n \in \mathbb{N}$, $|f^{y_n}| \leq g$, the dominated convergence theorem implies that $F(y_n) \to F(y_0)$.
2. Let $y_0 \in \mathbb{R}$. Choose $(y_n)_{n \in \mathbb{N}}$ such that $y_n \to y_0$ and for each $n \in \mathbb{N}$, $y_n \neq y_0$. For $n \in \mathbb{N}$, define $q_n : X \to \mathbb{R}$ by

$$q_n(x) = \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}$$

So $h_n(\cdot) \xrightarrow{\text{p.w.}} \partial f / \partial t(\cdot, t_0)$. The mean value theorem implies that for each $x \in X$ and $n \in \mathbb{N}$, there exists $c_{n,x} \in (t_n, t_0)$ such that $h_n(x) = \partial f / \partial t(x, c_{n,x})$. Then for each $n \in \mathbb{N}$, $|h_n| \leq g$. The dominated convergence theorem then implies that $\partial f / \partial t(\cdot, t_0) \in L^1(\mu)$ and

$$\begin{aligned} \int \frac{\partial f}{\partial t}(x, t_0) d\mu(x) &= \lim_{n \to \infty} \int_X h_n d\mu \\ &= \lim_{n \to \infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} \\ &= F'(t_0^-) \end{aligned}$$

So that F is differentiable at t_0 from the left. Similarly, F is differentiable at t_0 from the right.

FINISH!!! $\square$

Chapter 13

Banach Space Valued Measures

Chapter 14

TODO

- Add background for banach space valued measures like riesz representation theorem and radon-nikodym derivatives to be be able to talk about condition expectation of banach space valued random variables
- Discuss disintegration of measures independently of probability by discussing the projection of $L^1(X, \mathcal{A})$ onto $L^1(X, \mathcal{B})$ for $\mathcal{B} \subset \mathcal{A}$ and the Doob-Dynkin Lemma. Use this to define the disitegration measure. Also do this for disintegration of vector measures.
- Talk about homology when conditioning measures on a value in relation to the entropy of that distribution (maybe make a new set of notes about entropy and put it there)
- Consider the category $\mathcal{C}$ of measurable spaces with measurable singletons. Fix an object $(X, \mathcal{A}) \in \mathcal{C}$. Consider the coslice category of $\mathcal{C}$ under $(X, \mathcal{A})$. Introduce an equivalence relation on objects in the coslice category by $f : X \to (Y, \mathcal{F}) \sim g : X \to (Z, \mathcal{G})$ iff $f^*\mathcal{F} = g^*\mathcal{G}$. Introduce a partial order on the quotient by $f : X \to (Y, \mathcal{F}) \leq g : X \to (Z, \mathcal{G})$ iff $f^*\mathcal{F} \subset g^*\mathcal{G}$. Describe the Doob-Dynkin Lemma in this context, i.e. that $f \leq g$ implies that there is exactly one morphism from g to f in the coslice category.
- Replace the notation "Imf" with h where $f = g + ih$ so that Imf can refer to **image of** f.
- Define $L^0(X, \mathcal{A}, \mu)$ somewhere as the measurable maps $L(X, \mathcal{A})$ modulo null sets of μ.
- Talk abour sober measurable spaces

14.1 Applications to Hilbert Spaces

Exercise 14.1.0.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, H a separable Hilbert space, $f \in L^1_H$ and $a \in H$. Then

$$\int \langle f(x), a \rangle d\mu(x) = \left\langle \int f(x) d\mu(x), a \right\rangle$$

Proof. Define $T \in L^*(H, \mathbb{C})$ by $T(x) = \langle x, a \rangle$ and apply a previous exercise. $\square$

Appendix A

Summation

Definition A.0.0.1. Let $f : X \to [0, \infty)$, Then we define

$$\sum_{x \in X} f(x) := \sup_{\substack{F \subset X \\ F \text{ finite}}} \sum_{x \in F} f(x)$$

This definition coincides with the usual notion of summation when X is countable. For $f : X \to \mathbb{C}$, we can write $f = g + ih$ where $g, h : X \to \mathbb{R}$. If

$$\sum_{x \in X} |f(x)| < \infty,$$

then the same is true for g^+, g^-, h^+, h^-. In this case, we may define

$$\sum_{x \in X} f(x)$$

in the obvious way.

The following note justifies the notation $\sum_{x \in X} f(x)$ where $f : X \to \mathbb{C}$.

Note A.0.0.2. Let $f : X \to \mathbb{C}$ and $\alpha : X \to X$ a bijection. If $\sum_{x \in X} |f(x)| < \infty$, then $\sum_{x \in X} f(\alpha(x)) = \sum_{x \in X} f(x)$.

Appendix B

Categories

Definition B.0.0.1.

- **Meas**:
  - Obj(**Meas**) $:= \{(X, \mathcal{A}) : (X, \mathcal{A})$ is a measurable space$\}$
  - $\mathrm{Hom}_{\mathbf{Meas}}((X, \mathcal{A}), (Y, \mathcal{B})) := \{f : X \to Y : f$ is $(\mathcal{A}, \mathcal{B})$-measurable$\}$.
- $\mathbf{TopRadMsr}_+$:
  - Obj(**Meas**) $:= \{(X, \mu) : X \in \mathrm{Obj}(\mathbf{Top})$ and $\mu \in \mathcal{M}_+(X)\}$
  - $\mathrm{Hom}_{\mathbf{Meas}}((X, \mu), (Y, \nu)) := \{f \in \mathrm{Hom}_{\mathbf{Top}}(X, Y) : f_*\mu = \nu\}$.
- $\mathbf{TopRadMsr}_1$:
  - Obj(**Meas**) $:= \{(X, \mu) : X \in \mathrm{Obj}(\mathbf{Top})$ and $\mu \in \mathcal{M}_1(X)\}$
  - $\mathrm{Hom}_{\mathbf{Meas}}((X, \mu), (Y, \nu)) := \{f \in \mathrm{Hom}_{\mathbf{Top}}(X, Y) : f_*\mu = \nu\}$.

Bibliography

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[2] Introduction to Analysis

[3] Introduction to Fourier Analysis

[4] Introduction to Measure and Integration