1 Linear Algebra

Definition 1.1. Let F be a field. A **vector space** over F is a set V equipped with two operations:

- **Vector addition**: A function $+ : V \times V \to V$ assigning to each pair $(v, w) \in V \times V$ a sum $v + w \in V$.
- **Scalar multiplication**: A function $\cdot : F \times V \to V$ assigning to each scalar $a \in F$ and vector $v \in V$ a product $av \in V$.

These operations satisfy the following axioms for all $u, v, w \in V$ and all $a, b \in F$:

1. **Axioms for Vector Addition:**
   (a) **Closure**: $v + w \in V$.
   (b) **Associativity**: $u + (v + w) = (u + v) + w$.
   (c) **Commutativity**: $v + w = w + v$.
   (d) **Existence of Additive Identity**: There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
   (e) **Existence of Additive Inverses**: For each $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.
2. **Axioms for Scalar Multiplication:**
   (a) **Closure**: $av \in V$ for all $a \in F$ and $v \in V$.
   (b) **Distributivity over Vector Addition**: $a(v + w) = av + aw$.
   (c) **Distributivity over Scalar Addition**: $(a + b)v = av + bv$.
   (d) **Associativity**: $(ab)v = a(bv)$.
   (e) **Multiplicative Identity**: There exists a scalar $1 \in F$ such that $1v = v$ for all $v \in V$.

Definition 1.2 (Subspace)**.** Let V be a vector space, and let W be a subset of V. We define W to be a *subspace* if W satisfies the following conditions:

1. If v, w are elements of W, their sum $v + w$ is also an element of W.
2. If v is an element of W and c is a scalar, then cv is an element of W.
3. The element O of V is also an element of W.

Then W itself is a vector space. Indeed, properties **VS1** through **VS8**, being satisfied for all elements of V, are satisfied *a fortiori* for the elements of W.

Definition 1.3 (Linear Combination)**.** Let V be an arbitrary vector space, and let $v_1, \ldots, v_n$ be elements of V. Let $x_1, \ldots, x_n$ be scalars. An expression of the form

$$x_1 v_1 + \cdots + x_n v_n$$

is called a *linear combination* of $v_1, \ldots, v_n$.

Definition 1.4 (Dot Product)**.** Let $V = K^n$. Let $A, B \in K^n$ with $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$. We define the *dot product* or *scalar product* as

$$A \cdot B = a_1 b_1 + \cdots + a_n b_n.$$

Remark 1.1. Geometrically we say that A and B are orthogonal MORE HERE

Definition 1.5 (Linear Independence)**.** Let $v_1, \ldots, v_n$ be vectors in a vector space. The set of vectors $\{v_1, \ldots, v_n\}$ is said to be *linearly independent* if the only solution to the equation

$$a_1 v_1 + \cdots + a_n v_n = O$$

is $a_1 = a_2 = \cdots = a_n = 0$. That is, the vectors are linearly independent if no nontrivial linear combination of them results in the zero vector.

Definition 1.6 (Basis)**.** Let V be a vector space. A set of vectors $\{v_1, \ldots, v_n\}$ in V is called a *basis* of V if:

1. The vectors $v_1, \ldots, v_n$ *generate* V, meaning that every vector in V can be written as a linear combination of $v_1, \ldots, v_n$.

2. The vectors $v_1, \ldots, v_n$ are *linearly independent*, meaning that the only solution to

$$a_1v_1 + \cdots + a_nv_n = O$$

   is $a_1 = a_2 = \cdots = a_n = 0$.

If these conditions are satisfied, we say that $\{v_1, \ldots, v_n\}$ *forms a basis* of V.

Theorem 1.1. *Let V be a vector space. Let $v_1, \ldots, v_n$ be linearly independent elements of V. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be scalars. Suppose that*

$$x_1v_1 + \cdots + x_nv_n = y_1v_1 + \cdots + y_nv_n.$$

Then $x_i = y_i$ for all $i = 1, \ldots, n$.

Theorem 1.2. *Let $\{v_1, \ldots, v_n\}$ be a set of generators of a vector space V. Let $\{v_1, \ldots, v_r\}$ be a maximal subset of linearly independent elements. Then $\{v_1, \ldots, v_r\}$ is a basis of V.*

Definition 1.7 (Dimension of a Vector Space)**.** Let V be a vector space having a basis consisting of n elements. We define n to be the *dimension* of V. If V consists only of the zero vector O, then V does not have a basis, and we define the dimension of V to be 0.

Theorem 1.3. *Let V be a vector space, and $\{v_1, \ldots, v_n\}$ a maximal set of linearly independent elements of V. Then $\{v_1, \ldots, v_n\}$ is a basis of V.*

Theorem 1.4. *Let V be a vector space of dimension n, and let $v_1, \ldots, v_n$ be linearly independent elements of V. Then $v_1, \ldots, v_n$ constitute a basis of V.*

Proof. According to Theorem 1.3, $\{v_1, \ldots, v_n\}$ is a maximal set of linearly independent elements of V. Hence it is a basis by Theorem 1.3. □

Corollary 1.1. *Let V be a vector space and let W be a subspace. If $\dim W = \dim V$, then $V = W$.*

Proof. A basis for W must also be a basis for V by Theorem 1.4. □

Corollary 1.2. *Let V be a vector space of dimension n. Let r be a positive integer with $r < n$, and let $v_1, \ldots, v_r$ be linearly independent elements of V. Then one can find elements $v_{r+1}, \ldots, v_n$ such that*

$$\{v_1, \ldots, v_n\}$$

is a basis of V.

Theorem 1.5. *Let V be a vector space having a basis consisting of n elements. Let W be a subspace which does not consist of O alone. Then W has a basis, and the dimension of W is $\leq n$.*

Proof. Let w_1 be a nonzero element of W. If $\{w_1\}$ is not a maximal set of linearly independent elements of W, we can find an element w_2 of W such that w_1, w_2 are linearly independent. Proceeding in this manner, one element at a time, there must be an integer $m \leq n$ such that we can find linearly independent elements $w_1, w_2, \ldots, w_m$, and such that

$$\{w_1, \ldots, w_m\}$$

is a maximal set of linearly independent elements of W (by Theorem 1.3, we cannot go on indefinitely finding linearly independent elements, and the number of such elements is at most n). If we now use Theorem 1.3, we conclude that $\{w_1, \ldots, w_m\}$ is a basis for W. □

Definition 1.8. Let V be a vector space over the field K. Let U, W be subspaces of V. We define the *sum* of U and W to be the subset of V consisting of all sums $u + w$ with $u \in U$ and $w \in W$. We denote this sum by $U + W$. It is a subspace of V. Indeed, if $u_1, u_2 \in U$ and $w_1, w_2 \in W$ then

$$(u_1 + w_1) + (u_2 + w_2) = u_1 + u_2 + w_1 + w_2 \in U + W.$$

If $c \in K$, then

$$c(u_1 + w_1) = cu_1 + cw_1 \in U + W.$$

Finally, $O + O \in W$. This proves that $U + W$ is a subspace.

We shall say that V is a *direct sum* of U and W if for every element v of V there exist *unique* elements $u \in U$ and $w \in W$ such that $v = u + w$.

Theorem 1.6. *Let V be a vector space over the field K, and let U, W be subspaces. If $U + W = V$, and if $U \cap W = \{O\}$, then V is the direct sum of U and W.*

Theorem 1.7. *Let V be a finite-dimensional vector space over the field K. Let W be a subspace. Then there exists a subspace U such that V is the direct sum of W and U.*

Theorem 1.8. *If V is a finite-dimensional vector space over K, and is the direct sum of subspaces U, W, then*

$$\dim V = \dim U + \dim W.$$