Mathematics Notes

March 8, 2025

Contents

1 Logic, Metric Spaces, and Set Theory

Why study analysis or mathematics in general? If you intend to reason and navigate the complexities of any system, circumstance, task, or structure, the patterns of reasoning covered in mathematics equips you with the skill of understanding and making inferences or deductions in and about complex systems. So we will study systems at an abstracted level so that our conclusions and hard work are applicable and will aid us in any vocation whether we really notice it or not. Before we begin the rigorous study of calculus, which is the system used to understand and gain insight to abstract dynamic magnitudes. To build this system, we need to first discuss what type of *connections* this systems structure allows.

The first *axiom* of the system is that a *mathematical statement* is either true or false. A mathematical statement is a relationship that is shown through a type of *expression(s)*. An expression is a sequence of mathematical symbols, concepts, and objects that produce some other mathematical object. One can make statements out of expressions by using *relations* such as $=$, $<$, $\geq$, $\in$, $\subset$ or by using *properties* such as "is prime", "is invertible", "is continuous". Then one can make a compound statement from other statements by using *logical connectives*. We show some of these below,

Conjunction: If X is a statement and Y is a statement then the statement "X and Y" is a true statement if X and Y are both true. Notice though that this only concerns truth, where the artist of the mathematics must bring the connotations that illustrate more information that just "X and Y". For example, "X and also Y", or "both X and Y", or even "X but Y". Notice that X but Y suggests that the statements X and Y are in contrast to each other, while X and Y suggests that they support each other. We can find such reinterpretations of every logical connective.

Disjunction: If X is a statement and Y is a statement then the statement "X or Y" is true if either X or Y is true, or both. The reason we include the "X and Y" part is because when we are talking about X or Y we want to be talking about *all of* X or Y, instead of talking about X and not Y or Y and not X. So talking about the *exclusive* "or" (the one that doesn't include "and") is basically talking about two statements.

Negation: The statement "X is not true" or "X is false" is called the *negation* of X and is true if and only if X is false and is false if and only if X is true. Negations convert "and" into "or" and vice versa. For instance, the negation of "Jane Doe has black hair and Jane Doe has blue eyes" is "Jane Doe doesn't have black hair or doesn't have blue eyes". Notice how important the "inclusive or" is here to interpret the meaning of this statement.

If and only if: If X is a statement and Y is a statement, we say that "X is true if and only if Y is true", whenever X is true, Y also has to be true, and whenever Y is true, X must too be true. This is sort of like a logical equivalence. So if we were trying to pin down some type of abstract causal structure of some system an if and only if statement tells me that X and Y will always cause each other.

Implication: If X is a statement and Y is a statement then if we want to know whether (using some abstract notion of "cause") X causes, implies, or leads to Y then we are trying to prove an *implication* which is given by "if X then Y" (the implication of X to Y). So for X to truly *imply* Y, we need that when X is true Y is also true, if X is false then whether Y is true or false doesn't matter. So the only way to disprove an implication is is by showing that when the hypothesis is true, the conclusion is false. One can also think of the statement "if X, then Y " as "Y is at least as true as X"—if X is true, then Y also has to be true, but if X is false, Y could be as false as X, but it could also be true.

Variables and Quantifiers: Notice when we talk about some abstract, general, X and Y, the truth of the statements involving them depends on the context of X and Y. More precisely, X and Y are *variables* since they are variables that are set to obey some properties but the actual value of them hasn't been specified yet. Then *quantifiers* allow us to talk about the different values of these variables. We can say that there exists X where, say, X implies Y is true, this is denoted $\exists$. Or we can say for all X (denoted $\forall$), X implies Y. **Equality:** Out of the different relations we have discussed, *equality* is the most obvious. We need to be able to express the relationship of equality. We will present the axioms of equality, called an *equivalence relation*

Definition 1.1 (Equivalence Relation)**.** *Given elements x, y, z in any set with the relation $=$ defined, we have*

1. *(Reflexivity): Given any object x, we have $x = x$.*
2. *(Symmetry): Given any two objects x and y of the same type, if $x = y$ then $y = x$*
3. *(Transitive): Given any three objects x, y, z of the same type, if $x = y$ and $y = z$, then $x = z$.*
4. *(Substitution): Given any two objects x and y of the same type, if $x = y$, then $f(x) = f(y)$ for all functions or operations f. Similarly, for any property $P(x)$ depending on x, if $x = y$, then $P(x)$ and $P(y)$ are equivalent statements.*

Definition 1.2. *A set is a well-defined collection of distinct objects, called elements or members considered as a single entity unified under the defining properties of the set. The membership of an element x in a set S is denoted by $x \in S$, while non-membership is written as $x \notin S$. A set containing no elements is called the empty set, denoted $\emptyset$.*

Proposition 1.1. *Let A, B, C be sets, and let X be a set containing A, B, C as subsets.*

1. *(Minimal element) We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$*
2. *(Maximal element) We have $A \cup X = X$ and $A \cap X = A$.*
3. *(Identity) We have $A \cup A = A$ and $A \cap A = A$*
4. *(Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$*
5. *(Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$*
6. *(Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*
7. *(Partition) We have $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$*
8. *(De Morgan Laws) We have $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$*

Definition 1.3. *An ordered set is a set S together with an ordering relation, denoted $<$, such that*

1. *(trichotomy) $\forall x, y \in S, \text{ exactly one of } x < y, x = y, \text{ or } y < x$ holds.*
2. *(transitivity) If $x, y, z \in S$ such that $x < y$ and $y < z \implies x < z$.*

Well ordering property of $\mathbb{N}$ *: Every nonempty subset of* $\mathbb{N}$ *has a least element.*

Definition 1.4. *We define the natural numbers* $\{1, 2, 3, 4, \ldots\}$ *to be a set* $\mathbb{N}$ *with the successor function* S *defined on it. The successor function* $S : \mathbb{N} \to \mathbb{N}$, *is defined by the following axioms,*

N1: $1 \in \mathbb{N}$

N2: *If* $n \in \mathbb{N}$ *then its successor* $n + 1 \in \mathbb{N}$

N3: 1 *is not the successor of any element in* $\mathbb{N}$

N4: *If* n *and* m *in* $\mathbb{N}$ *have the same successor, then* $n = m$.

N5: *A subset of* $\mathbb{N}$ *that contains* 1, *and contains* $n + 1$ *whenever it contains* n, *must be equivalent to* $\mathbb{N}$.

Theorem 1.1 (Principle of induction)**.** *Let* $P(n)$ *be a statement depending on a natural number* n. *Suppose that*

(i) (basis statement) $P(1)$ *is true.*

(ii) (induction step) If $P(n)$ *is true, then* $P(n + 1)$ *is true.*

Then $P(n)$ *is true for all* $n \in \mathbb{N}$.

Proof. Let S be the set of natural numbers n for which $P(n)$ is not true. Suppose for contradiction that S is nonempty. Then S has a least element by the well-ordering property. Call $m \in S$ the least element of S. We know $1 \notin S$ by hypothesis. So $m > 1$, and $m - 1$ is a natural number as well. Since m is the least element of S, we know that $P(m - 1)$ is true. But the induction step says that $P(m - 1 + 1) = P(m)$ is true, contradicting the statement that $m \in S$. Therefore, S is empty and $P(n)$ is true for all $n \in \mathbb{N}$.

□

Definition 1.5. *A set* F *is called a field if it has two operations defined on it, addition* $x + y$ *and multiplication* xy, *and if it satisfies the following axioms:*

(A1) If $x \in F$ *and* $y \in F$, *then* $x + y \in F$.

(A2) (commutativity of addition) $x + y = y + x$ *for all* $x, y \in F$.

(A3) (associativity of addition) $(x + y) + z = x + (y + z)$ *for all* $x, y, z \in F$.

(A4) There exists an element $0 \in F$ *such that* $0 + x = x$ *for all* $x \in F$.

(A5) For every element $x \in F$, *there exists an element* $-x \in F$ *such that* $x + (-x) = 0$.

(M1) If $x \in F$ *and* $y \in F$, *then* $xy \in F$.

(M2) (commutativity of multiplication) $xy = yx$ *for all* $x, y \in F$.

(M3) (associativity of multiplication) $(xy)z = x(yz)$ *for all* $x, y, z \in F$.

(M4) There exists an element $1 \in F$ *(with* $1 \neq 0$*) such that* $1x = x$ *for all* $x \in F$.

(M5) For every $x \in F$ *such that* $x \neq 0$, *there exists an element* $1/x \in F$ *such that* $x(1/x) = 1$.

(D) (distributive law) $x(y + z) = xy + xz$ *for all* $x, y, z \in F$.

Definition 1.6. *A field* F *is said to be an ordered field if* F *is also an ordered set such that*

(i) For $x, y, z \in F$, $x < y$ *implies* $x + z < y + z$.

(ii) For $x, y \in F$, $x > 0$ *and* $y > 0$ *implies* $xy > 0$.

If $x > 0$, *we say* x *is positive. If* $x < 0$, *we say* x *is negative. We also say* x *is nonnegative if* $x \geq 0$, *and* x *is nonpositive if* $x \leq 0$.

Proposition 1.2. *Let* F *be an ordered field and* $x, y, z, w \in F$. *Then*

(i) If $x > 0$, *then* $-x < 0$ *(and vice versa).*

(ii) *If* $x > 0$ *and* $y < z$*, then* $xy < xz$*.*

(iii) *If* $x < 0$ *and* $y < z$*, then* $xy > xz$*.*

(iv) *If* $x \neq 0$*, then* $x^2 > 0$*.*

(v) *If* $0 < x < y$*, then* $0 < 1/y < 1/x$*.*

(vi) *If* $0 < x < y$*, then* $x^2 < y^2$*.*

(vii) *If* $x \leq y$ *and* $z \leq w$*, then* $x + z \leq y + w$*.*

Note that (iv) implies, in particular, that $1 > 0$*.*

Proof. Let us prove (i). The inequality $x > 0$ implies by item (i) of the definition of ordered fields that $x + (-x) > 0 + (-x)$. Apply the algebraic properties of fields to obtain $0 > -x$. The "vice versa" follows by a similar calculation.

For (ii), note that $y < z$ implies $0 < z - y$ by item (i) of the definition of ordered fields. Apply item (ii) of the definition of ordered fields to obtain $0 < x(z - y)$. By algebraic properties, $0 < xz - xy$. Again, by item (i) of the definition, $xy < xz$.

Part (iii) is left as an exercise.

To prove part (iv), first suppose $x > 0$. By item (ii) of the definition of ordered fields, $x^2 > 0$ (use $y = x$). If $x < 0$, we use part (iii) of this proposition, where we plug in $y = x$ and $z = 0$.

To prove part (v), notice that $1/y$ cannot be equal to zero (why?). Suppose $1/y < 0$, then $-1/y > 0$ by (i). Apply part (ii) of the definition (as $x > 0$) to obtain $x(-1/y) > 0$ or $-1 > 0$, which contradicts $1 > 0$ by using part (i) again. Hence $1/y > 0$. Similarly, $1/x > 0$. Thus $(1/x)(1/y)x < (1/x)(1/y)y$.

By algebraic properties, $1/y < 1/x$.

Parts (vi) and (vii) are left as exercises.

□

Definition 1.7. *Let* $E \subset S$*, where* S *is an ordered set.*

(i) *If* $\exists b \in S$ such that $x \leq b$, $\forall x \in E \implies E$ *is bounded above and* b *is an upper bound of* E*.*

(ii) *If* $\exists b \in S$ such that $x \geq b$, $\forall x \in E \implies E$ *is bounded below and* b *is a lower bound of* E*.*

(iii) *If* $\exists b_0$ *an upper bound of* E *such that* $b_0 \leq b$, $\forall$ *upper bounds* b *of* E*, then* b_0 *is called the least upper bound or the supremum of* E*. We write:*

$$\sup E := b_0.$$

(iv) *If* $\exists b_0$ *a lower bound of* E *such that* $b_0 \geq b$, $\forall$ *lower bounds* b *of* E*, then* b_0 *is called the greatest lower bound or the infimum of* E*. We write*

$$\inf E := b_0.$$

When a set E *is both bounded above and bounded below, we say simply that* E *is bounded.*

Definition 1.8 (Least Upper Bound Property)**.** *An ordered set* S *has the least-upper-bound property if every nonempty subset* $E \subset S$ *that is bounded above has a least upper bound, that is,* $\sup E$ *exists in* S*.*

The least-upper-bound property is sometimes called the completeness property or the Dedekind completeness property.

Remark 1.1. *So since* A *is a subset of an ordered field that has the least upper bound property, which states that every set bounded above with the least upper bound property is bounde*

Proposition 1.3. *Let* F *be an ordered field with the least-upper-bound property. Let* $A \subset F$ *be a nonempty set that is bounded below. Then* $\inf A$ *exists.*

Proof. Let $B = \{-a \mid a \in A\}$. Then since A is bounded above with the least upper bound property, $\exists \sup A = b \in F$. Thus $\forall a \in A, a \leq b$ which implies $-b \leq -a$, which means that B is bounded below by $-b$. Now suppose $\exists M \in F$ such that

$$\forall -a \in B, \quad -b \leq M \leq -a \implies b \geq -M \geq a$$

Since this is contradicts $b = \sup A$. Therefore we have found that B is bounded below by $-b$ and $-b$ is greater than every other lower bound, so $\inf B$ exists.

$\square$

Exercise 1.1. *Let S be an ordered set, and let $B \subseteq S$ be a subset that is bounded above and below. Suppose that $A \subseteq B$ is a nonempty subset and that both $\inf A$ and $\sup A$ exist. Then we have the inequalities:*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Proof. Let $B \subset S$ be bounded above and below, and let $A \subset B$ be nonempty. By definition of greatest lower bound, every lower bound of B is also a lower bound of A (since $A \subset B$), and hence $\inf B \leq \inf A$. Also, every upper bound of B is an upper bound of A, so $\sup A \leq \sup B$. Furthermore, because A is nonempty, for any $x \in A$ we have $\inf A \leq x \leq \sup A$, which ensures $\inf A \leq \sup A$. Combining these gives

$$\inf B \quad \leq \quad \inf A \quad \leq \quad \sup A \quad \leq \quad \sup B,$$

as required. $\square$

Remark 1.2. *Notice that it seems like we are being imprecise about the infs and sups across subsets. We are actually using the definition, try to contradict and show that $\sup A > \sup B$.*

Proposition 1.4 (The Supremum is the least upper bound)**.** *Let $S \subset \mathbb{R}$ be nonempty, and $L \in \mathbb{R} \cup \{\infty, -\infty\}$. Then*

$$\sup S \leq L \iff s \leq L \quad \forall s \in S.$$

Proof. Suppose $\sup S \leq L$. Then by transitivity of ordering 1.3

$$s \leq \sup S \leq L \quad \forall s \in S$$

Which shows $s \leq L$.
Conversely, suppose for some $L \in \mathbb{R} \cup \{\infty, -\infty\}$ we have $s \leq L$, $\forall s \in S$. Since we can say that L is in the set of extended reals that bound the set S where $\sup S$ is the least element, so we have

$$s \leq \sup S \leq L \quad \forall s \in S.$$

$\square$

Exercise 1.2. *Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \leq y$ whenever $x \in A$ and $y \in B$. Assume A is bounded above, B is bounded below, and $\sup A \leq \inf B$. Then it follows that A is bounded below, B is bounded above, and moreover:*

$$\sup A \leq \inf B.$$

This inequality confirms that the upper bound of A does not exceed the lower bound of B, effectively placing A entirely below or at most touching B.

Exercise 1.3. *If S and T are nonempty subsets of $\mathbb{R}$ and $T \subseteq S$, then $\sup T \leq \sup S$ and $\inf T \geq \inf S$. Note that the supremum and infimum could be finite or infinite.*

Proof. Suppose nonempty sets $T \subseteq S \subseteq \mathbb{R}$ exist. Then $\forall t \in T$, $\exists s_1, s_2 \in S$ such that $s_1 \leq t \leq s_2$. Then,

$$\inf S \leq s_1 \leq \inf T \leq t, \quad \forall t, \implies \inf T \geq \inf S.$$

$$t \leq \sup T \leq s_2 \leq \sup S, \quad \forall t, \implies \sup T \leq \sup S.$$

This states that every upper/lower bound of S is also an upper/lower bound of T so the maximum/minimum of such bounds must too satisfy the inequality. Which is exactly what we wanted to prove. Note that the inequalities above also hold if the sets are unbounded. We can see this by considering an example,

$$\text{If } \sup T = \infty \implies \sup S = \infty$$

but the converse does not hold, as T could just be a finite subset.

$\square$

Exercise 1.4. *Let A and B be two nonempty bounded sets of real numbers, and let $C = \{a + b : a \in A, b \in B\}$ and $D = \{ab : a \in A, b \in B\}$. Then*

1. $\sup C = \sup A + \sup B$ *and* $\inf C = \inf A + \inf B$.
2. $\sup D = (\sup A)(\sup B)$ *and* $\inf D = (\inf A)(\inf B)$.

Definition 1.9. *A function $f : A \to B$ is a subset f of $A \times B$ such that for each $x \in A$, there exists a unique $y \in B$ for which $(x, y) \in f$. We write $f(x) = y$. Sometimes the set f is called the graph of the function rather than the function itself.*

The set A is called the domain of f (and sometimes confusingly denoted $D(f)$). The set

$$R(f) := \{y \in B : \text{there exists an } x \in A \text{ such that } f(x) = y\}$$

is called the range of f. The set B is called the codomain of f.

Definition 1.10. *Consider a function $f : A \to B$. Define the image (or direct image) of a subset $C \subset A$ as*

$$f(C) := \{f(x) \in B : x \in C\}.$$

Define the inverse image of a subset $D \subset B$ as

$$f^{-1}(D) := \{x \in A : f(x) \in D\}.$$

In particular, $R(f) = f(A)$, the range is the direct image of the domain A.

Theorem 1.2. *Let $f : A \to B$ be a function. Then the inverse relation f^{-1} is a function from B to A if and only if f is bijective. Furthermore, if f is bijective, then f^-1 is also bijective.*

Proposition 1.5. *Consider $f : A \to B$. Let C, D be subsets of B. Then*

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D),$$

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D),$$

$$f^{-1}(C^c) = (f^{-1}(C))^c.$$

Read the last line of the proposition as $f^{-1}(B \setminus C) = A \setminus f^{-1}(C)$.

Proposition 1.6. *Consider $f : A \to B$. Let C, D be subsets of A. Then*

$$f(C \cup D) = f(C) \cup f(D),$$

$$f(C \cap D) \subseteq f(C) \cap f(D).$$

Definition 1.11. *Let $f : A \to B$ be a function. The function f is said to be injective or one-to-one if*

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2.$$

In other words, f is injective if for all $y \in B$, the set $f^{-1}(\{y\})$ is empty or consists of a single element. We call such an f an injection.

If $f(A) = B$, then we say f is surjective or onto. In other words, f is surjective if the range and the codomain of f are equal. We call such an f a surjection.

If f is both surjective and injective, then we say f is bijective or that f is a bijection.

Definition 1.12. *Let $f : A \to B$ and $g : B \to C$ be functions. Then we define the composition as $(g \circ f)(x) = g(f(x))$. So we first use f to map from A to B, then take the value of f in B and input into g and use it to map to C.*

Proposition 1.7. *If $f : A \to B$ and $g : B \to C$ are bijective functions, then $f \circ g$ is bijective.*

Definition 1.13. *Let A and B be sets. We say A and B have the same cardinality when there exists a bijection $f : A \to B$.*

We denote by $|A|$ the equivalence class of all sets with the same cardinality as A, and we simply call $|A|$ the cardinality of A.

Definition 1.14. *We write*

$$|A| \leq |B|$$

if there exists an injection from A to B.

We write $|A| = |B|$ if A and B have the same cardinality.

We write $|A| < |B|$ if $|A| \leq |B|$, but A and B do not have the same cardinality.
If $|A| \leq |\mathbb{N}$ then we say that A is countable. If $|A| = |\mathbb{R}|$ then we say that A is uncountable.

Theorem 1.3. *If there exists a bijective function between two sets A and B, then we have that the cardanalities, 1.13, are equivalenet.*

Exercise 1.5. *Let S be a nonempty collection of nonempty sets. A realation R is defined on S by A R B if there exists a bijective function from A to B. Then R is an equivalence relation 1.1.*

Proposition 1.8. *The set $\mathbb{Z}$ is countable*

Proposition 1.9. *Every infinite subset of a countable set is also countable*

Proposition 1.10. *If A and B are countable, then $A \times B$ is countable*

Theorem 1.4. *The set $\mathbb{Q}$ is countable*

Theorem 1.5. *The open interval $(0, 1)$ of real numbers is uncountable.*

Theorem 1.6. $|(0, 1)| = |\mathbb{R}|$

Theorem 1.7. $|\mathcal{P}(A)| = |2^A|$

Lemma 1.1. *Let $f : A \to B$ and $g : C \to D$ be one-to-one functions, where $A \cap C = \emptyset$, and where the function $h : A \cup C \to B \cup D$ is defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in C. \end{cases}$$

If $B \cap D = \emptyset$, then h is also a one-to-one function. Consequently, if f and g are bijective functions, then h is a bijective function.

Theorem 1.8. *Let A and B be nonempty sets such that $B \subseteq A$. If there exists an injective function from A to B, then there exists a bijective function from A to B.*

Theorem 1.9 (**Schröder-Bernstein Theorem**)**.** *If A and B are sets such that $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Theorem 1.10. $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

1.1 Metric Spaces

Definition 1.15. *Let X be a set, and let $d : X \times X \to \mathbb{R}$ be a function such that for all $x, y, z \in X$:*

1. $d(x, y) \geq 0$ *(nonnegativity)*
2. $d(x, y) = 0$ *if and only if* $x = y$ *(identity of indiscernibles)*
3. $d(x, y) = d(y, x)$ *(symmetry)*
4. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

The pair (X, d) is called a metric space. The function d is called the metric or the distance function. Sometimes we write just X as the metric space instead of (X, d) if the metric is clear from context.

Lemma 1.2. *(Cauchy-Schwarz inequality). Suppose $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Proposition 1.11. *Let (X, d) be a metric space and $Y \subset X$. Then the restriction $d|_{Y\times Y}$ is a metric on Y.*

Definition 1.16. *If (X, d) is a metric space, $Y \subset X$, and $d' := d|_{Y\times Y}$, then (Y, d') is said to be a subspace of (X, d).*

Definition 1.17. *Let (X, d) be a metric space. A subset $S \subset X$ is said to be bounded if there exists a $p \in X$ and a $B \in \mathbb{R}$ such that*

$$d(p, x) \leq B \quad \text{for all } x \in S.$$

We say (X, d) is bounded if X itself is a bounded subset.

Definition 1.18. *Let (X, d) be a metric space, $x \in X$, and $\delta > 0$. Define the open ball, or simply ball, of radius δ around x as*

$$B(x, \delta) := \{y \in X : d(x, y) < \delta\}.$$

Define the closed ball as

$$C(x, \delta) := \{y \in X : d(x, y) \leq \delta\}.$$

When dealing with different metric spaces, it is sometimes vital to emphasize which metric space the ball is in. We do this by writing $B_X(x, \delta) := B(x, \delta)$ or $C_X(x, \delta) := C(x, \delta)$.

Definition 1.19. *Let (X, d) be a metric space. A subset $V \subset X$ is open if for every $x \in V$, there exists a $\delta > 0$ such that $B(x, \delta) \subset V$. A subset $E \subset X$ is closed if the complement $E^c = X \setminus E$ is open. When the ambient space X is not clear from context, we say V is open in X and E is closed in X. If $x \in V$ and V is open, then we say V is an open neighborhood of x (or sometimes just neighborhood).*

Proposition 1.12. *Let (X, d) be a metric space.*

1. *$\emptyset$ and X are open.*
2. *If $V_1, V_2, \ldots, V_k$ are open subsets of X, then*
   $$\bigcap_{j=1}^{k} V_j$$
   is also open. That is, a finite intersection of open sets is open.
3. *If $\{V_\lambda\}_{\lambda\in I}$ is an arbitrary collection of open subsets of X, then*
   $$\bigcup_{\lambda\in I} V_\lambda$$
   is also open. That is, a union of open sets is open.

Proposition 1.13. *Let (X, d) be a metric space.*

1. *$\emptyset$ and X are closed.*
2. *If $\{E_\lambda\}_{\lambda\in I}$ is an arbitrary collection of closed subsets of X, then*
   $$\bigcap_{\lambda\in I} E_\lambda$$
   is also closed. That is, an intersection of closed sets is closed.

3. *If $E_1, E_2, \ldots, E_k$ are closed subsets of X, then*

$$\bigcup_{j=1}^{k} E_j$$

is also closed. That is, a finite union of closed sets is closed.

Proposition 1.14. *Let (X,d) be a metric space, $x \in X$, and $\delta > 0$. Then $B(x,\delta)$ is open and $C(x,\delta)$ is closed.*

Proposition 1.15. *Suppose (X,d) is a metric space, and $Y \subset X$. Then $U \subset Y$ is open in Y (in the subspace topology) if and only if there exists an open set $V \subset X$ (so open in X) such that $V \cap Y = U$.*

Proposition 1.16. *Suppose (X,d) is a metric space, $V \subset X$ is open, and $E \subset X$ is closed.*

1. *$U \subset V$ is open in the subspace topology if and only if U is open in X.*

2. *$F \subset E$ is closed in the subspace topology if and only if F is closed in X.*

Definition 1.20. *A nonempty metric space (X,d) is connected if the only subsets of X that are both open and closed (so-called clopen subsets) are $\emptyset$ and X itself. If a nonempty (X,d) is not connected, we say it is disconnected.*

When we apply the term connected to a nonempty subset $A \subset X$, we mean that A with the subspace topology is connected.

In other words, a nonempty X is connected if whenever we write $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and X_1 and X_2 are open, then either $X_1 = \emptyset$ or $X_2 = \emptyset$. So to show X is disconnected, we need to find nonempty disjoint open sets X_1 and X_2 whose union is X.

Proposition 1.17. *Let (X,d) be a metric space. A nonempty set $S \subset X$ is disconnected if and only if there exist open sets U_1 and U_2 in X such that $U_1 \cap U_2 \cap S = \emptyset$, $U_1 \cap S \neq \emptyset$, $U_2 \cap S \neq \emptyset$, and*

$$S = (U_1 \cap S) \cup (U_2 \cap S).$$

Proposition 1.18. *A nonempty set $S \subset \mathbb{R}$ is connected if and only if S is an interval or a single point.*

Definition 1.21. *Let (X,d) be a metric space and $A \subset X$. The closure of A is the set*

$$\bar{A} := \bigcap \{E \subset X : E \text{ is closed and } A \subset E\}.$$

That is, $\bar{A}$ is the intersection of all closed sets that contain A.

Proposition 1.19. *Let (X,d) be a metric space and $A \subset X$. The closure $\bar{A}$ is closed, and $A \subset \bar{A}$. Furthermore, if A is closed, then $\bar{A} = A$.*

Proposition 1.20. *Let (X,d) be a metric space and $A \subset X$. Then $x \in \bar{A}$ if and only if for every $\delta > 0$, $B(x,\delta) \cap A \neq \emptyset$.*

Definition 1.22. *Let (X,d) be a metric space and $A \subset X$. The interior of A is the set*

$$A^\circ := \{x \in A : \text{there exists a } \delta > 0 \text{ such that } B(x,\delta) \subset A\}.$$

The boundary of A is the set

$$\partial A := \bar{A} \setminus A^\circ.$$

Proposition 1.21. *Let (X,d) be a metric space and $A \subset X$. Then A° is open and ∂A is closed.*

Proposition 1.22. *Let (X,d) be a metric space and $A \subset X$. Then $x \in \partial A$ if and only if for every $\delta > 0$, $B(x,\delta) \cap A$ and $B(x,\delta) \cap A^c$ are both nonempty.*

Corollary 1.1. *Let (X,d) be a metric space and $A \subset X$. Then*

$$\partial A = \bar{A} \cap \overline{A^c}.$$

Proposition 1.23. *Let (X, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Then $\{x_n\}_{n=1}^{\infty}$ converges to $p \in X$ if and only if for every open neighborhood U of p, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n \in U$.*

Proof. *Suppose $\{x_n\}_{n=1}^{\infty}$ converges to p. Let U be an open neighborhood of p, then there exists an $\epsilon > 0$ such that $B(p, \epsilon) \subset U$. As the sequence converges, find an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $d(p, x_n) < \epsilon$, or in other words $x_n \in B(p, \epsilon) \subset U$.*

Conversely, given $\epsilon > 0$, let $U := B(p, \epsilon)$ be the neighborhood of p. Then there is an $M \in \mathbb{N}$ such that for $n \geq M$, we have $x_n \in U = B(p, \epsilon)$, or in other words, $d(p, x_n) < \epsilon$. □

A closed set contains the limits of its convergent sequences.

Proposition 1.24. *Let (X, d) be a metric space and $A \subset X$. Then $p \in \bar{A}$ if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements in A such that*

$$\lim_{n \to \infty} x_n = p.$$

Definition 1.23. *We say a metric space (X, d) is complete or Cauchy-complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to a $p \in X$.*

Proposition 1.25. *The space $\mathbb{R}^n$ with the standard metric is a complete metric space.*

Proposition 1.26. *The space of continuous functions $C([a, b], \mathbb{R})$ with the uniform norm as metric is a complete metric space.*

Definition 1.24. *Let (X, d) be a metric space and $K \subset X$. The set K is said to be compact if for every collection of open sets $\{U_\lambda\}_{\lambda \in I}$ such that*

$$K \subset \bigcup_{\lambda \in I} U_\lambda,$$

there exists a finite subset $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subset I$ such that

$$K \subset \bigcup_{j=1}^{m} U_{\lambda_j}.$$

A collection of open sets $\{U_\lambda\}_{\lambda \in I}$ as above is said to be an open cover of K. A way to say that K is compact is to say that every open cover of K has a finite subcover.

Proposition 1.27. *Let (X, d) be a metric space. If $K \subset X$ is compact, then K is closed and bounded.*

Lemma 1.3. *(Lebesgue covering lemma). Let (X, d) be a metric space and $K \subset X$. Suppose every sequence in K has a subsequence convergent in K. Given an open cover $\{U_\lambda\}_{\lambda \in I}$ of K, there exists a $\delta > 0$ such that for every $x \in K$, there exists a $\lambda \in I$ with $B(x, \delta) \subset U_\lambda$.*

Theorem 1.11. *Let (X, d) be a metric space. Then $K \subset X$ is compact if and only if every sequence in K has a subsequence converging to a point in K.*

Proposition 1.28. *Let (X, d) be a metric space and let $K \subset X$ be compact. If $E \subset K$ is a closed set, then E is compact.*

Theorem 1.12. *(Heine-Borel theorem). A closed bounded subset $K \subset \mathbb{R}^n$ is compact.*

So subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded, a condition that is much easier to check. Let us reiterate that the Heine-Borel theorem only holds for $\mathbb{R}^n$ and not for metric spaces in general. The theorem does not hold even for subspaces of $\mathbb{R}^n$, just in $\mathbb{R}^n$ itself. In general, compact implies closed and bounded, but not vice versa.

Definition 1.25. *Let (X, d_X) and (Y, d_Y) be metric spaces and $c \in X$. Then $f : X \to Y$ is continuous at c if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in X$ and $d_X(x, c) < \delta$, then $d_Y(f(x), f(c)) < \epsilon$.*

When $f : X \to Y$ is continuous at all $c \in X$, we simply say that f is a continuous function.

Proposition 1.29. *Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f : X \to Y$ is continuous at $c \in X$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to c, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.*

Lemma 1.4. *Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a continuous function. If $K \subset X$ is a compact set, then $f(K)$ is a compact set.*

Theorem 1.13. *Let (X, d) be a nonempty compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then f is bounded and in fact f achieves an absolute minimum and an absolute maximum on X.*

Proof. As X is compact and f is continuous, $f(X) \subset \mathbb{R}$ is compact. Hence $f(X)$ is closed and bounded. In particular, $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$, because both the sup and the inf can be achieved by sequences in $f(X)$ and $f(X)$ is closed. Therefore, there is some $x \in X$ such that $f(x) = \sup f(X)$ and some $y \in X$ such that $f(y) = \inf f(X)$. □

Lemma 1.5. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $c \in X$ if and only if for every open neighborhood U of $f(c)$ in Y, the set $f^{-1}(U)$ contains an open neighborhood of c in X.*

Theorem 1.14. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if and only if for every open $U \subset Y$, $f^{-1}(U)$ is open in X.*

2 Linear Algebra

3 Analysis

Theorem 3.1 (Archimedean Property)**.** *If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that*

$$nx > y.$$

Proof. Notice that $nx > y \implies n > y/x$. So if this didn't hold, we would have that $\mathbb{N}$ is bounded above. Suppose by contradiction, we have

$$\exists t \in \mathbb{R}, \forall n \in \mathbb{N}, \quad n \leq t$$

Thus there must exist a least upper bound, call it $m \in \mathbb{R}$. Then

$$\exists n \text{ such that } m - 1 \leq n \leq m \leq t \implies m \leq n.$$

This contradicts that $\exists y, x$ so that $n \leq y/x \quad \forall n \in \mathbb{N}$. Hence, the Archimedean property holds. □

Theorem 3.2 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *If $x, y \in \mathbb{R}$ and $x < y$, then there exists an $r \in \mathbb{Q}$ such that*

$$x < r < y.$$

Proof. Let $r = \frac{m}{n}$ and $m, n \in \mathbb{Z}$ such that $n \neq 0$ and $gcd(m, n) = 1$. Then we want to show the existence of m and n such that for any x and y,

$$x < \frac{m}{n} < y \quad \implies \quad 0 < n(y - x).$$

Then by 3.1, we have that $\exists n \in \mathbb{N}$ such that

$$1 < n(y - x) \quad \text{or} \quad \frac{1}{n} < y - x \quad \text{or} \quad nx + 1 < ny.$$

So we have that the n *scaled difference* of y and x is greater than 1, this tells me I can fit an integer m between nx and ny. To pick this m, let $S = \{k \in \mathbb{Z} \mid k > nx\}$. By 3.1, we know S is nonempty, then by the Well Ordering Axiom, we have that there exists a least element, call it m. Then $m \in S$ so $nx < m$ or $x < m/n$. Now it remains to show that $m < ny$. Since m is the least element of S, we must have $m - 1 \notin S$. Thus

$$m - 1 < nx \quad \implies \quad m < nx + 1 < ny.$$

This gives us, $m/n < y$ which proves the statement.

□

3.1 Sequences

Definition 3.1 (Sequence)**.** *A sequence (of real numbers) is a function $x : \mathbb{N} \to \mathbb{R}$. Instead of $x(n)$, we usually denote the nth element in the sequence by x_n. To denote a sequence we write*

$$\{x_n\}_{n=1}^{\infty}$$

Definition 3.2 (Bounded Sequence)**.** *A sequence $\{x_n\}_{n=1}^{\infty}$ is bounded if there exists $M \in \mathbb{R}$ such that*

$$|x_n| \leq M \quad \textit{for all } n \in \mathbb{N}.$$

That is, the sequence x_n is bounded whenever the set $\{x_n \mid n \in \mathbb{N}\}$ is bounded.

Definition 3.3 (Monotone Sequence)**.** *A sequence $\{x_n\}_{n=1}^{\infty}$ is monotone increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{x_n\}_{n=1}^{\infty}$ is monotone decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. If a sequence is either monotone increasing or monotone decreasing, we can simply say the sequence is monotone.*

Definition 3.4 (Convergent Sequence)**.** *A sequence x_n is said to converge to a number $x \in \mathbb{R}$ if*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \textit{ such that } \forall n \geq N, |x_n - x| < \varepsilon.$$

Note that this is equivalently written $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$.

Remark 3.1. *The definition of a convergence sequence seems as though it does not lend itself easily to application, but a change in perspective of the definition allows you to see the usefulness. Think of it as, me and some other guy are both looking at x_n, he chooses $\varepsilon > 0$, this determines how precise our limit must be. So I then choose an $N \in \mathbb{N}$ such that x_n is always within ε of x for all n after the N which we specifically found given ε.*

Proposition 3.1. *A convergent sequence has a unique limit.*

Proof. Suppose x_n converges to both x and y. Then by definition 3.4, we have $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|x_n - x| < \varepsilon/2$, and for the same ε, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$, $|x_n - y| < \varepsilon/2$. Thus if we choose $N = max(N_1, N_2)$ we obtain,

$$|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $|y - x| < \varepsilon,\ \forall \varepsilon > 0$, is equivalent to $y = x$, this proves that if the limit exists, it is unique. □

Exercise 3.1. *Claim: The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is convergent and converges to* 0.
To apply the definition of convergence we would need to show that for any $\varepsilon > 0$, there exists some value $N \in \mathbb{N}$ such that x_n is bounded by ε for all n after that N. In other words, we would that $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$, we would have $|\frac{1}{n}| < \varepsilon \implies n > \frac{1}{\varepsilon}$. Notice this n exists by 3.1. This is how we find the N value that we use in our proof most of the time.

Exercise 3.2. *Let (s_n) be a sequence of non-negative real numbers and suppose $s = \lim_{n\to\infty}$. Then*

$$\lim_{n\to\infty} \sqrt{s_n} = \sqrt{\lim_{n\to\infty} s_n}$$

Proof. From the definition of convergence, we need to bound the magnitude of the difference of $\sqrt{s_n} - \sqrt{s}$. So we massage the expression that we are supposed to be concluding with to see if we find some bound.

$$\left|\sqrt{s_n} - \sqrt{s}\right| \implies \left|\frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}}\right| = \left|\frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}\right|$$

Since $\sqrt{s_n} \geq 0$, we have that $\left|\frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}\right| \leq \left|\frac{s_n - s}{\sqrt{s}}\right|$. This is the type of expression we want, we have that $s_n - s$ along with other elements, of which we can bound, are greater than the expression we are trying to bound by ε. So we choose $N \in \mathbb{N}$ such that

$$\mid s_n - s \mid < \sqrt{s}\varepsilon \implies \frac{|s_n - s|}{\sqrt{s}} < \varepsilon \implies \left|\frac{s_n - s}{\sqrt{s_n} + \sqrt{s}}\right| < \varepsilon \implies \left|\sqrt{s_n} - \sqrt{s}\right| < \varepsilon.$$

This proves the statement. □

Proposition 3.2. *Convergent sequences are bounded.*

Proof. Suppose $x_n \longrightarrow x$. Then there exists an $N \in \mathbb{N}$ such that $\forall n > N$ we have $|x_n - x| < 1$. Then for $n > N$,

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|.$$

Now consider the set

$$M = \{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x|\}.$$

Observe that M is finite. Then let

$$B = \max(\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |x|\}.$$

Then for all $n \in \mathbb{N}$,

$$|x_n| \leq B.$$

This satisfies definition 3.2. $\square$

Proposition 3.3 (Algebra of Limits)**.** *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be convergent sequences.*

1. $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$.
2. $\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$.
3. *If* $\lim_{n\to\infty} y_n \neq 0$ *and* $y_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$.

Lemma 3.1 (Squeeze lemma)**.** *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{x_n\}_{n=1}^{\infty}$ be sequences such that*

$$a_n \leq x_n \leq b_n \quad \text{for all } n \in \mathbb{N}.$$

Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Proof. Let $x := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Let $\varepsilon > 0$ be given. Find an M_1 such that for all $n \geq M_1$, we have $|a_n - x| < \varepsilon$, and an M_2 such that for all $n \geq M_2$, we have $|b_n - x| < \varepsilon$. Set $M := \max\{M_1, M_2\}$. Suppose $n \geq M$. In particular, $x - a_n < \varepsilon$, or $x - \varepsilon < a_n$. Similarly, $b_n < x + \varepsilon$. Putting everything together, we find

$$x - \varepsilon < a_n \leq x_n \leq b_n < x + \varepsilon.$$

In other words, $-\varepsilon < x_n - x < \varepsilon$ or $|x_n - x| < \varepsilon$. So $\{x_n\}_{n=1}^{\infty}$ converges to x. $\square$

We can also formally define divergent sequences even though we really already know from our definition of convergence.

Definition 3.5. *We say x_n diverges to infinity if*

$$\forall K \in \mathbb{R}, \exists M \in \mathbb{N}, \text{ such that } \exists n \geq M \text{ where} x_n > K.$$

This is written

$$\lim_{n\to\infty} x_n = \infty$$

Exercise 3.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n}.$$

Exercise 3.4. *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, then*

$$\lim_{n\to\infty} |x_n| = \left|\lim_{n\to\infty} x_n\right|.$$

Theorem 3.3 (Monotone Convergence Theorem)**.** *A monotone sequence* $\{x_n\}_{n=1}^{\infty}$ *is bounded if and only if it is convergent.*

Furthermore, if $\{x_n\}_{n=1}^{\infty}$ *is monotone increasing and bounded, then*

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

If $\{x_n\}_{n=1}^{\infty}$ *is monotone decreasing and bounded, then*

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

Exercise 3.5. *For any real number* a*,*

$$\lim_{n\to\infty} a^n = 0 \quad \text{for } |a| < 1.$$

Exercise 3.6. *Let* $n \in \mathbb{N}$ *and* $a > 0$*. Then,*

$$\lim_{n\to\infty} n^{1/n} = 1 \quad \text{and} \quad \lim_{n\to\infty} a^{1/n} = 1.$$

Exercise 3.7. *Let* $c > 0$*.*

1. *If* $c < 1$*, then*
$$\lim_{n\to\infty} c^n = 0.$$
2. *If* $c > 1$*, then* $\{c^n\}_{n=1}^{\infty}$ *is unbounded.*

Remark 3.2. *The idea of the proof in the next exercise uses the result of exercise 3.7. Notice if* $L < 1$*, then each term (since it's in absolute values) is less than the other by a ratio. But this only happens after we get to our limit, so its for all* n *after whatever* M *makes us convergent. But how exactly would I show that this sequence is a ratio (like a* $(1/c)^n$*) type)? This is where you are going to have to get weird. Break the sequence (mentally) into two parts, before* M *(meaning, before the terms are a ratio of each other) and after* M *(once the terms are a ratio of each other). So we could potentially express* x_n *using this.*

Exercise 3.8 (Ratio Test for Sequences)**.** *Let* $(x_n)_{n=1}^{\infty}$ *be a sequence such that* $x_n \neq 0$ $\forall n \in \mathbb{N}$ *and such that the limit*

$$L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists.

1. *If* $L < 1$*, then* $\lim\limits_{n\to\infty} x_n = 0$*.*
2. *If* $L > 1$*, then* $\{x_n\}_{n=1}^{\infty}$ *is unbounded.*

Proof. (1) Suppose $L < 1$. Since $\frac{|x_{n+1}|}{|x_n|} \geq 0$ for all n, we have that $L \geq 0$. Choose an $r \in \mathbb{R}$ such that $L < r < 1$. Since $r - L > 0$ we can treat $r - L$ like an ε such that, $\exists M \in \mathbb{N}$ such that $\forall n \geq M$, we have

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < r - L.$$

Therefore, for $n \geq M$,

$$\frac{|x_{n+1}|}{|x_n|} - L < r - L \quad \text{or} \quad \frac{|x_{n+1}|}{|x_n|} < r.$$

For $n > M$, use that each term is a multiple in $(0, 1)$ of the terms before it, so we write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} < |x_M| r r \cdots r = |x_M| r^{n-M} = (|x_M| r^{-M}) r^n.$$

The sequence $\{r^n\}_{n=1}^{\infty}$ converges to zero and hence $|x_M| r^{-M} r^n$ converges to zero. Since $\{x_n\}_{n=M+1}^{\infty}$ converges to zero, we have that $\{x_n\}_{n=1}^{\infty}$ converges to zero.

Now suppose $L > 1$. Pick r such that $1 < r < L$. As $L - r > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$,

$$\left| \frac{|x_{n+1}|}{|x_n|} - L \right| < L - r.$$

Therefore,

$$\frac{|x_{n+1}|}{|x_n|} > r.$$

Again, for $n > M$, write

$$|x_n| = |x_M| \frac{|x_{M+1}|}{|x_M|} \frac{|x_{M+2}|}{|x_{M+1}|} \cdots \frac{|x_n|}{|x_{n-1}|} > |x_M| rr \cdots r = |x_M| r^{n-M} = (|x_M| r^{-M}) r^n.$$

The sequence $\{r^n\}_{n=1}^{\infty}$ is unbounded (since $r > 1$), and so $\{x_n\}_{n=1}^{\infty}$ cannot be bounded. Consequently, $\{x_n\}_{n=1}^{\infty}$ cannot converge. □

Exercise 3.9. *For all $a \in \mathbb{R}$*

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$

Proof. Using 3.8, we have

$$\lim_{n\to\infty} \frac{a^{n+1} n!}{a^n (n+1)!} = \lim_{n\to\infty} \frac{a}{n+1} = 0$$

□

Exercise 3.10. *If $(x_n)_{n=1}^{\infty}$ is convergent and $k \in \mathbb{N}$ then*

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n \right)^k$$

Exercise 3.11. *If $(x_n)_{n=1}^{\infty}$ is a convergent sequence and $x_n \geq 0$ and $k \in \mathbb{N}$ then*

$$\lim_{n\to\infty} x_n^{1/k} = \left(\lim_{n\to\infty} x_n \right)^{1/k}$$

Definition 3.6. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers, that is, $n_i < n_{i+1}$ for all $i \in \mathbb{N}$ (in other words $n_1 < n_2 < n_3 < \cdots$). The sequence*

$$\{x_{n_i}\}_{i=1}^{\infty}$$

is called a subsequence of $\{x_n\}_{n=1}^{\infty}$.

Proposition 3.4. *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, then every subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ is also convergent, and*

$$\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_i}.$$

Definition 3.7. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Define the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ by*

$$a_n := \sup\{x_k : k \geq n\}, \quad b_n := \inf\{x_k : k \geq n\}.$$

Define, if the limits exist,

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} a_n, \quad \liminf x_n := \lim_{n\to\infty} b_n.$$

Theorem 3.4. *If $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n.$$

Similarly, there exists a (perhaps different) subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n.$$

Remark 3.3. *In the below proof, we are trying to find an x_{n_i} that converges to the same limit as the supremum. So we want the*

Proof. Define $a_n = \sup\{x_k : k \geq n\}$. Let $x := \limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$. We define the subsequence inductively. Let $n_1 = 1$, meaning $x_{n_1} = x_n$, and suppose $n_1, n_2, \ldots, n_{k-1}$ are defined for some $k \geq 2$. Since the subsequences index $(n_k)_{k=1}^{\infty}$ is strictly increasing, $n_k \geq n_{k-1}+1$, pick an $m \geq n_{k-1}+1$ such that

$$a_{n_k+1} - x_m < \frac{1}{k}.$$

Such an m exists as a_{n_k+1} is a supremum of the set $\{x_\ell : \ell \geq n_{k-1}+1\}$ and hence there are elements of the sequence arbitrarily close (or even possibly equal) to the supremum. Set $n_k = m$. The subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ is defined. Next, we must prove that it converges to x. For all $k \geq 2$, we have $a_{n_k+1} \geq a_{n_k}$ (why?) and $a_{n_k} \geq x_{n_k}$. Therefore, for every $k \geq 2$,

$$|a_{n_k} - x_{n_k}| = a_{n_k} - x_{n_k} \leq a_{n_k+1} - x_{n_k} < \frac{1}{k}.$$

Let us show that $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x. Note that the subsequence need not be monotone. Let $\epsilon > 0$ be given. As $\{a_n\}_{n=1}^{\infty}$ converges to x, the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converges to x. Thus, there exists an $M_1 \in \mathbb{N}$ such that for all $k \geq M_1$, we have

$$|a_{n_k} - x| < \frac{\epsilon}{2}.$$

Find an $M_2 \in \mathbb{N}$ such that

$$\frac{1}{M_2} \leq \frac{\epsilon}{2}.$$

Take $M := \max\{M_1, M_2\}$. For all $k \geq M$,

$$|x - x_{n_k}| = |a_{n_k} - x_{n_k} + x - a_{n_k}| \leq |a_{n_k} - x_{n_k}| + |x - a_{n_k}| \leq \frac{1}{M_2} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Exercise 3.12. *Let $S \subset \mathbb{R}$ be a nonempty bounded set. Then there exist monotone sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $x_n, y_n \in S$ and*

$$\sup S = \lim_{n\to\infty} x_n \quad \textit{and} \quad \inf S = \lim_{n\to\infty} y_n.$$

Proposition 3.5. *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Then $\{x_n\}_{n=1}^{\infty}$ converges if and only if*

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Furthermore, if $\{x_n\}_{n=1}^{\infty}$ converges, then

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Exercise 3.13. *Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence and $(x_{n_k})_{k=1}^{\infty}$ is a subsequence. Then*

$$\liminf_{n\to\infty} x_n \leq \liminf_{k\to\infty} x_{n_k} \leq \limsup_{k\to\infty} x_{n_k} \leq \limsup_{n\to\infty} x_n$$

Exercise 3.14. *A bounded sequence $(x_n)_{n=1}^{\infty}$ converges to x $\iff$ every subsequence $(x_{n_k})_{k=1}^{\infty}$ converges to x.*

Definition 3.8 (Subsequential Limit)**.** *Let $(x_n)_{n=1}^{\infty}$ be a sequence. A subsequential limit is any extended real number that is the limit of some subsequence of $(x_n)_{n=1}^{\infty}$.*

Theorem 3.5 (Bolzano–Weierstrass)**.** *Suppose a sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is bounded. Then there exists a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$.*

Exercise 3.15. *Let (s_n) be any sequence of nonzero real numbers. Then we have*

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

Exercise 3.16. *If $\lim \left|\frac{s_{n+1}}{s_n}\right|$ exists and equals L then $\lim |s_n|^{1/n}$ exists and equals L.*

Definition 3.9 (Cauchy Sequence). *A sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$ and all $k \geq M$, we have*

$$|x_n - x_k| < \varepsilon.$$

Lemma 3.2. *If a sequence is Cauchy, then it is bounded.*

Theorem 3.6 (Convergent $\iff$ Cauchy). *A sequence of real numbers is Cauchy $\iff$ the sequence is convergent.*

3.2 Series

So we have built a good understanding of sequences, to make sense of what is about to come, consider the following example. Suppose you have an infinite number of people, each of them representing a number (like their age or something), if we give a calculator to the first person and tell them to put their age in then tell the next person to put their age in and tell the same person after them to do so. At any moment if we stop this process, say at person k, then the number on the calculator is the kth value of our sequence, where the sequence represents the sum of a sequence of numbers.

Definition 3.10 (Series). *Given a sequence $(x_n)_{n=1}^{\infty}$, we define*

$$\sum_{n=1}^{\infty} x_n$$

as a series. A series converges if the sequence $(s_k)_{k=1}^{\infty}$, called the partial sums, and defined by

$$s_k = \sum_{n=1}^{k} x_n = x_1 + x_2 + \cdots + x_k$$

converges. So a series converges if

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} \sum_{n=1}^{k} x_n.$$

Proposition 3.6 (Geometric Series). *Suppose $-1 < r < 1$. Then the geometric series $\sum_{n=0}^{\infty} r^n$ converges, and*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Exercise 3.17. *Let $\sum_{n=1}^{\infty} x_n$ be a series and let $M \in \mathbb{N}$. Then*

$$\sum_{n=1}^{\infty} x_n \text{ converges} \iff \sum_{n=M}^{\infty} x_n \text{ converges.}$$

Definition 3.11 (Cauchy Series). *A series $\sum_{n=1}^{\infty} x_n$ is said to be Cauchy if the sequence of the partial sums $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence.*
Note that a series is convergent if and only if it is Cauchy 3.6.

Exercise 3.18. *If a series $\sum_{n=1}^{\infty} x_n$ converges, then $\lim x_n = 0$.*

Proposition 3.7 (Linearity of Series). *Let $\alpha \in \mathbb{R}$ and $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series. Then*

1. *$\sum_{n=1}^{\infty} \alpha x_n$ is a convergent series and*

$$\sum_{n=1}^{\infty} \alpha x_n = \alpha \sum_{n=1}^{\infty} x_n.$$

2. *$\sum_{n=1}^{\infty} (x_n + y_n)$ is a convergent series and*

$$\sum_{n=1}^{\infty} (x_n + y_n) = \left(\sum_{n=1}^{\infty} x_n\right) + \left(\sum_{n=1}^{\infty} y_n\right).$$

Proof. For the first item, we simply write the kth partial sum

$$\sum_{n=1}^{k} \alpha x_n = \alpha \left(\sum_{n=1}^{k} x_n\right).$$

We look at the right-hand side and note that the constant multiple of a convergent sequence is convergent. Hence, we take the limit of both sides to obtain the result.

For the second item, we also look at the kth partial sum

$$\sum_{n=1}^{k} (x_n + y_n) = \left(\sum_{n=1}^{k} x_n\right) + \left(\sum_{n=1}^{k} y_n\right).$$

We look at the right-hand side and note that the sum of convergent sequences is convergent. Hence, we take the limit of both sides to obtain the proposition. ☐

Proposition 3.8. *If $x_n \geq 0$ for all n, then $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence of partial sums is bounded above.*

Definition 3.12 (Absolute Convergence)**.** *A series $\sum_{n=1}^{\infty} x_n$ converges absolutely if the series $\sum_{n=1}^{\infty} |x_n|$ converges. If a series converges, but does not converge absolutely, we say it converges conditionally*

Proposition 3.9. *If the series $\sum_{n=1}^{\infty} x_n$ converges absolutely, then it converges.*

Proposition 3.10 (Comparison Test)**.** *Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be series such that $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$.*

1. *If $\sum_{n=1}^{\infty} y_n$ converges, then so does $\sum_{n=1}^{\infty} x_n$.*
2. *If $\sum_{n=1}^{\infty} x_n$ diverges, then so does $\sum_{n=1}^{\infty} y_n$.*

Proposition 3.11 (P-Series)**.** *(p-series or the p-test). For $p \in \mathbb{R}$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Proposition 3.12 (Root Test)**.** *Let $\sum_{n=1}^{\infty} x_n$ be a series and let*

$$L = \limsup_{n\to\infty} |x_n|^{1/n}.$$

1. *If $L < 1$, then $\sum_{n=1}^{\infty} x_n$ converges absolutely.*
2. *If $L > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.*

Proposition 3.13 (Ratio Test)**.** *Let $\sum_{n=1}^{\infty} x_n$ be a series, $x_n \neq 0$ for all n, and such that*

1. *If $\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} x_n$ converges absolutely.*
2. *If $\liminf_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = L > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.*

Proposition 3.14 (Alternating Series Test)**.** *Let $\{x_n\}_{n=1}^{\infty}$ be a monotone decreasing sequence of positive real numbers such that $\lim_{n\to\infty} x_n = 0$. Then the alternating series*

$$\sum_{n=1}^{\infty} (-1)^n x_n$$

converges.

3.3 Continuity

Remark 3.4. *Now we will generalize the results up to now so we can apply it to mappings between sets.*

Definition 3.13 (Cluster Point)**.** *A number $x \in \mathbb{R}$ is called a cluster point of a set $S \subset \mathbb{R}$ if for every $\epsilon > 0$, the set*

$$(x - \epsilon, x + \epsilon) \cap (S \setminus \{x\})$$

is nonempty.

Equivalently, x is a cluster point of S if for every $\epsilon > 0$, there exists some $y \in S$ such that $y \neq x$ and $|x - y| < \epsilon$.

A cluster point of S need not belong to S.

Proposition 3.15. *Let $S \subset \mathbb{R}$. Then $x \in \mathbb{R}$ is a cluster point of S if and only if there exists a convergent sequence of numbers $\{x_n\}_{n=1}^{\infty}$ such that $x_n \neq x$ and $x_n \in S$ for all n, and $\lim_{n\to\infty} x_n = x$.*

Definition 3.14. *Let $f : S \to \mathbb{R}$ be a function and c a cluster point of $S \subset \mathbb{R}$. Suppose there exists an $L \in \mathbb{R}$ and for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in S \setminus \{c\}$ and $|x - c| < \delta$, we have*

$$|f(x) - L| < \epsilon.$$

We then say $f(x)$ converges to L as x goes to c, and we write

$$f(x) \to L \quad \text{as} \quad x \to c.$$

We say L is a limit of $f(x)$ as x goes to c, and if L is unique (it is), we write

$$\lim_{x\to c} f(x) := L.$$

If no such L exists, then we say that the limit does not exist or that f diverges at c.

Proposition 3.16. *Let c be a cluster point of $S \subset \mathbb{R}$ and let $f : S \to \mathbb{R}$ be a function such that $f(x)$ converges as x goes to c. Then the limit of $f(x)$ as x goes to c is unique.*

Lemma 3.3. *Let $S \subset \mathbb{R}$, let c be a cluster point of S, let $f : S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then $f(x) \to L$ as $x \to c$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S \setminus \{c\}$ for all n, and such that $\lim_{n\to\infty} x_n = c$, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.*

Proof. Suppose $f(x) \to L$ as $x \to c$, and $\{x_n\}_{n=1}^{\infty}$ is a sequence such that $x_n \in S \setminus \{c\}$ and $\lim_{n\to\infty} x_n = c$. We wish to show that $\{f(x_n)\}_{n=1}^{\infty}$ converges to L. Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - L| < \epsilon$. As $\{x_n\}_{n=1}^{\infty}$ converges to c, find an M such that for $n \geq M$, we have that $|x_n - c| < \delta$. Therefore, for $n \geq M$,

$$|f(x_n) - L| < \epsilon.$$

Thus $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

For the other direction, we use proof by contrapositive. Suppose it is not true that $f(x) \to L$ as $x \to c$. The negation of the definition is that there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists an $x \in S \setminus \{c\}$, where $|x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.

Let us use $1/n$ for δ in the statement above to construct a sequence $\{x_n\}_{n=1}^{\infty}$. We have that there exists an $\epsilon > 0$ such that for every n, there exists a point $x_n \in S \setminus \{c\}$, where $|x_n - c| < 1/n$ and $|f(x_n) - L| \geq \epsilon$. The sequence $\{x_n\}_{n=1}^{\infty}$ just constructed converges to c, but the sequence $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L. And we are done. □

Proposition 3.17. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist, and*

$$f(x) \leq g(x) \quad \text{for all } x \in S \setminus \{c\}.$$

Then

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

Proposition 3.18. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $h : S \to \mathbb{R}$ are functions such that*

$$f(x) \leq g(x) \leq h(x) \quad \textit{for all } x \in S \setminus \{c\}.$$

Suppose the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x).$$

Then the limit of $g(x)$ as x goes to c exists and

$$\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x).$$

Proposition 3.19. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist. Then*

1. $\lim_{x \to c}(f(x) + g(x)) = (\lim_{x \to c} f(x)) + (\lim_{x \to c} g(x))$.
2. $\lim_{x \to c}(f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$.
3. $\lim_{x \to c}(f(x)g(x)) = (\lim_{x \to c} f(x)) (\lim_{x \to c} g(x))$.
4. *If $\lim_{x \to c} g(x) \neq 0$ and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}.$$

Proposition 3.20. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Then*

$$\lim_{x \to c} |f(x)| = \left| \lim_{x \to c} f(x) \right|.$$

Definition 3.15. *Let $f : S \to \mathbb{R}$ be a function and $A \subset S$. Define the function $f|_A : A \to \mathbb{R}$ by*

$$f|_A(x) := f(x) \quad \textit{for } x \in A.$$

We call $f|_A$ the restriction of f to A.

Proposition 3.21. *Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, and let $f : S \to \mathbb{R}$ be a function. Suppose $A \subset S$ is such that there is some $\alpha > 0$ such that*

$$(A \setminus \{c\}) \cap (c - \alpha, c + \alpha) = (S \setminus \{c\}) \cap (c - \alpha, c + \alpha).$$

1. *The point c is a cluster point of A if and only if c is a cluster point of S.*
2. *Supposing c is a cluster point of S, then $f(x) \to L$ as $x \to c$ if and only if $f|_A(x) \to L$ as $x \to c$.*

Proposition 3.22. *Let $S \subset \mathbb{R}$ be such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, let $f : S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then c is a cluster point of S and*

$$\lim_{x \to c} f(x) = L \quad \textit{if and only if} \quad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

Definition 3.16. *Suppose $S \subset \mathbb{R}$ and $c \in S$. We say $f : S \to \mathbb{R}$ is continuous at c if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.*

When $f : S \to \mathbb{R}$ is continuous at all $c \in S$, then we simply say f is a continuous function.

Proposition 3.23. *Consider a function $f : S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}$ and let $c \in S$. Then:*

1. *If c is not a cluster point of S, then f is continuous at c.*
2. *If c is a cluster point of S, then f is continuous at c if and only if the limit of $f(x)$ as $x \to c$ exists and*

$$\lim_{x \to c} f(x) = f(c).$$

3. *The function f is continuous at c if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \in S$ and $\lim_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.*

Proposition 3.24. *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial. That is,*

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0,$$

for some constants $a_0, a_1, \ldots, a_d$. Then f is continuous.

Proposition 3.25. *Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be functions continuous at $c \in S$.*

1. *The function $h : S \to \mathbb{R}$ defined by $h(x) := f(x) + g(x)$ is continuous at c.*
2. *The function $h : S \to \mathbb{R}$ defined by $h(x) := f(x) - g(x)$ is continuous at c.*
3. *The function $h : S \to \mathbb{R}$ defined by $h(x) := f(x)g(x)$ is continuous at c.*
4. *If $g(x) \neq 0$ for all $x \in S$, the function $h : S \to \mathbb{R}$ given by $h(x) := \frac{f(x)}{g(x)}$ is continuous at c.*

Proposition 3.26. *Let $A, B \subset \mathbb{R}$ and $f : B \to \mathbb{R}$ and $g : A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g : A \to \mathbb{R}$ is continuous at c.*

Proposition 3.27. *Let $f : S \to \mathbb{R}$ be a function and $c \in S$. Suppose there exists a sequence $\{x_n\}_{n=1}^{\infty}$, where $x_n \in S$ for all n, and $\lim_{n\to\infty} x_n = c$ such that $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(c)$. Then f is discontinuous at c.*

Lemma 3.4. *A continuous function $f : [a, b] \to \mathbb{R}$ is bounded.*

Theorem 3.7 (Minimum-maximum theorem / Extreme value theorem)**.** *A continuous function $f : [a, b] \to \mathbb{R}$ achieves both an absolute minimum and an absolute maximum on $[a, b]$.*

Lemma 3.5. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) < 0$ and $f(b) > 0$. Then there exists a number $c \in (a, b)$ such that $f(c) = 0$.*

Theorem 3.8 (Bolzano's Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $y \in \mathbb{R}$ is such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then there exists a $c \in (a, b)$ such that $f(c) = y$.*