Qualifying Exam Problems

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misc

F08.7. Let u be a harmonic function on $\mathbb{R}^2$ that does not take zero value (i.e $u(x) \neq 0 \; \forall x \in \mathbb{R}^2$). Show that u is constant.

S09.8. Let $f(z)$ be holomorphic on a domain in the complex plane. If $|f(z)|^2$ is harmonic in D. What can you conclude on f? (Show your work.)

F10.7. (i) State the Mean Value Theorem for analytic functions and use the Cauchy integral formula to prove it.

(ii) Prove that if $f = u + iv$ is an analytic function from an open subset U of $\mathbb{C}$ then the real and imaginary parts u and v of f are harmonic, i.e, $\Delta u = \Delta v = 0$.

(iii) Let U be an open subset of $\mathbb{R}^2$, and $u : U \to \mathbb{R}$ a harmonic function. Prove that if there is $p_0 \in U$ such that $u(p_0) = \inf_{x \in U} u(x)$, then u is a constant.

F12.9. Let u be a real-valued harmonic function in $\mathbb{C} \setminus \{0\}$. Show that then

$$u(z) = c \log |z| + \mathrm{Re}(f(z))$$

for some real constant c and a holomorphic function f on $\mathbb{C} \setminus \{0\}$.

F13.5. Prove that the function

$$u(x, y) = y \cos y \sinh x + x \sin y \cosh x$$

is harmonic in $\mathbb{R}^2$ and find its harmonic conjugate.

S14.5. Let $f_1, \ldots, f_n$ be holomorphic in a domain D in $\mathbb{C}$ and $p \in (0, \infty)$. Prove

(a) $\sum_{j=1}^n |f_j(z)|^p$ is subharmonic in D.

(b) If there is a $z_0 \in D$ such that $\sum_{j=1}^n |f_j(z_0)|^p \geq \sum_{j=1}^n |f_j(z)|^p$ for all $z \in D$, then f_j is constant for $j = 1, 2, \ldots, n$.

F14.8. Let $u(z)$ be harmonic in $D =: D(0, 1) \setminus \{0\}$ such that

$$\lim_{z \to 0} \frac{u(z)}{\log |z|} = 0$$

Prove that u can be extended to be harmonic in $D(0, 1)$.

S15.7. TRUE or FALSE: There exists a bounded harmonic function on the upper half plane $\mathbb{H}$ that cannot be extended to any larger domain. Explain your answer.

F15.5. Find a real valued function $u(z)$ that is continuous in the closed disc $\overline{D(0,R)}$ (that is, closed disc centered at 0 of radius $R > 0$) and harmonic in $D(0,R)$, and satisfies

$$u(Re^{i\theta}) = \frac{1}{2}(1 + \cos^3\theta),\ \ \theta \in [0, 2\pi)$$

F16.8. Let u be a real-valued harmonic function in $\overline{D(0,1)} \setminus \{0\}$ such that

$$\lim_{z\to 0} \frac{u(z)}{\log z} = 0.$$

Show that there is a harmonic function U on $D(0,1)$ such that $u(z) = U(z)$ for all $z \in D(0,1) \setminus \{0\}$.

F17.1. Let u be a real-valued continuous function on $\mathbb{C}$ such that $e^{u(z)}$ is harmonic in $\mathbb{C}$. Then u is a constant.

S19.2. Let $u : \mathbb{C} \to \mathbb{R}$ be a nonconstant real harmonic function. Show that there exists a sequence of points $\{z_n\} \in \mathbb{C}$ such that $\lim_{n\to\infty} u(z_n) = -\infty$.

S19.8. Let u be harmonic in $D(0,1) \setminus \{0\}$ satisfying

$$\lim_{z\to 0} \frac{u(z)}{\ln|z|} = 0$$

Prove that u is harmonic on $D(0,1)$.