Qualifying Exam Problems

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Misc

S06.3. The Bernoulli polynomials $\varphi_n(z)$ are defined by the expansion

$$\frac{e^{tz}-1}{e^t-1} = \sum_{n=1}^{\infty} \frac{\varphi_n(z)}{n!} t^{n-1}$$

Prove the following two statements

[(i)] $\varphi_n(z+1) - \varphi_n(z) = nz^{n-1}$

(ii) $\frac{\varphi_{n+1}(n+1)}{n+1} = 1 + 2^n + 3^n + \ldots + n^n$

S06.4. Let $f(z)$ be analytic and satisfy $|f(z)| \leq 100|z|^{-2}$ in the strip $\alpha_1 \leq \operatorname{Re} z \leq \alpha_2$. Prove the function

$$h(x) = \int_{-\infty}^{\infty} f(x+iy)dy$$

is a constant function of $x \in [\alpha_1, \alpha_2]$.

S06.7. Let f be analytic in the unit disc $D(0,1)$ and continuous on $\overline{D}(0,1)$. Assume that

$$|f(z)| = |e^z| \text{ for all } z \in \partial D(0,1) = \{z \in \mathbb{C} \colon |z| = 1\}$$

Find all such f.

F06.3. (Also F12.3) Let $P(z)$ be a polynomial in z. Assume that $P(z) \neq 0$ for $\operatorname{Re}(z) > 0$. Show that $P'(z) \neq 0$ for $\operatorname{Re}(z) > 0$.

F06.4. (Also F10.1) Let $z_1, \ldots, z_n$ be distinct complex numbers contained in the disk $D(0,R)$. Let f be analytic in the closed disk $\overline{D}(0,R)$. Let

$$Q(z) = (z - z_1) \ldots (z - z_n)$$

Prove that

$$P(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} f(\zeta) \frac{1 - \frac{Q(z)}{Q(\zeta)}}{\zeta - z} d\zeta$$

is a polynomial of degree $n-1$ having the same values as f at the points $z_1, \ldots, z_n$.

F07.5. Let $f : D(0,1) \to D(0,1)$ be holomorphic. Prove

$$\left|\frac{|f(z) - f(w)|}{1 - f(z)\overline{f(w)}}\right| \leq \left|\frac{z - w}{1 - z\overline{w}}\right|, \qquad z, w \in D(0,1)$$

F08.1. Compute the area of the image of the unit disc $D = \{z \mid |z| < 1\}$ under the map $f(z) = z + \frac{z^2}{2}$.

F11.6. Prove the Schwarz-Pick lemma: Let $f : D(0,1) \to D(0,1)$ be holomorphic. Then

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leq \left|\frac{z - a}{1 - \overline{a}z}\right|, \qquad a, z \in D(0,1)$$

F11.7. Let f be holomorphic in $D(0,1)$ and let

$$M(r, f) = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$$

Prove that $M(r, f)$ is an increasing convex function of r on $[0, 1)$.

F13.8. Suppose f is holomorphic on $\mathbb{C} \setminus \{0\}$ and suppose that $f\left(\frac{1}{z}\right) = f(z)$ for all $z \in \mathbb{C} \setminus \{0\}$. Suppose further that f is real on the unit circle. Show that f is real for all real $z \neq 0$.

F14.7. Let D be a simply connected domain in $\mathbb{C}$ and $z_0 \in D$. Let $\mathcal{F}$ be the set of all $f : D \to D(0,1)$ such that

(i) $f(z_0) = 0$

(ii) $f'(z_0) > 0$

(iii) f is one to one.

Then prove $\mathcal{F}$ is not empty set.

S15.1. Prove that for each $n \in \mathbb{N}$ every solution of the equation $(1 - iz)^n + z^n = 0$ must satisfy $\operatorname{Im} z = -\frac{1}{2}$.

S15.3. Expand in a series of powers each of the branches of $z(w)$ defined by the equation $w = 2z + z^2$ (for one branch $z(0) = 0$, for the other $z(0) = -2$.)

F16.2. Suppose f is analytic in the annulus $A = \{z \in \mathbb{C} \colon 1/2 < |z| < 2\}$, and there exists a sequence of polynomials p_n converging to f uniformly on the unit circle $|z| = 1$. Show that f can be extended to be an analytic function on the disc $D(0, 2)$.

S17.2. The Bernoulli polynomials $B_n(z)$ are defined by the expansion

$$t\frac{e^{tz} - 1}{e^t - 1} = \sum_{n=1}^{\infty} \frac{B_n(z)}{n!} t^n$$

Prove that $B_n(z + 1) - B_n(z) = nz^{n-1}$.

S17.3. Let $f(z)$ be analytic in $S = \{z = x + iy \colon -1 < x < 1\}$ and continuous on $\overline{S}$, the closure of S. Suppose that $f(z)$ are real when $\operatorname{Re} z = x = \pm 1$. Prove that $f(z)$ can be extended analytically to the whole plane and that the resulting entire function satisfies $f(z + 4) = f(z)$ for all $z \in \mathbb{C}$.

S18.2. Let f and g be analytic functions on the open set $U = D(1, 15) \setminus \{i\}$, i.e the open disc centered at 1 with radius 15 with the point i removed. Suppose $f'(z) = g'(z)$ for all $z \in U$. Prove that f and g differ by a constant, that is, there exists $a \in \mathbb{C}$ such that $f(z) - g(z) = a$ for all $z \in U$.

S18.5. Suppose $p(z)$ is a polynomial of degree $d \geq 2$ that has only simple zeros $r_1, r_2, \ldots, r_d$. Prove that $\frac{1}{p'(r_1)} + \frac{1}{p'(r_2)} + \ldots + \frac{1}{p'(r_d)} = 0$.

S18.7. Prove that the range of the function $f(z) = \sum_{n=1}^{2018} \cos^n(z)$ is the whole complex plane $\mathbb{C}$.

F19.7. Let $f : D(0, 1) \to D(0, 1)$ be a proper holomorphic map such that $f(z)$ is continuous on

$\overline{D(0,1)}$. Prove f is a rational function.

Integral

S06.5. Evaluate the integral:

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx$$

F06.2. Show that for $a > 0$,

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{\pi(1+a)}{4e^a}$$

F07.1. Prove the following Jordan's lemma. Let $f(z)$ be continuous in the region $D = \{z \in \mathbb{C} \colon |z| \geq R_0, \operatorname{Im} z \geq 0\}$ and $\lim_{z\to\infty} f(z) = 0$ uniformly on D. Then for any positive number a

$$\lim_{R\to\infty} \int_{\Gamma_R} e^{iaz} f(z) dz = 0$$

where Γ_R is the arc of the circle $\{z \in \mathbb{C} \colon |z| = R\}$, which lies in the semiplane $\operatorname{Im} z \geq 0$.

F07.2. Let $f(z)$ be holomorphic in the closed unti disc $\overline{D(0,1)}$. Prove

$$f(z) = \frac{1}{\pi} \int_{D(0,1)} \frac{f(w)}{(1 - z\overline{w})^2} dA(w), \ z \in D(0,1)$$

F07.4. Show that

$$F(z) = \int_0^1 \frac{e^{tz}}{1+t} dt$$

is holomorphic in $\mathbb{C}$.

F07.7. Let a be a real number, evaluate the following integral

$$\int_0^\infty \frac{\sin(ax)}{\sinh x} dx$$

F07.8. Let $f(z)$ be analytic on $\mathbb{C} \setminus \{1\}$ and have a simple pole at $z = 1$ with residue λ. Prove that for every $R > 0$,

$$\lim_{n\to+\infty} R^n \left| (-1)^n \frac{f^{(n)}(2)}{n!} - \lambda \right| = 0$$

S08.5. Evaluate the improper integral

$$\int_{-\infty}^{+\infty} \frac{x^2 \sin(\pi x)}{x^3 - 1} dx$$

S08.9. Show that there is no holomorphic function $f(z)$ on $\{z \in \mathbb{C} \mid 1 < |z| < 3\}$ satisfying

$$\left| \frac{f(z)^2}{z} - 1 \right| < 1$$

F08.5. Evaluate the integral

$$\int_0^{+\infty} \left(\frac{\sin x}{x} \right)^2 dx$$

S09.3. If $f(z)$ is continuous in the region $\operatorname{Re} z \geq \sigma$ (σ is a fixed real number) and $\lim_{z\to\infty} f(z) = 0$, then for any negative number t

$$\lim_{R\to\infty} \int_{\Gamma_R} e^{tz} f(z) dz = 0,$$

where Γ_R is the arc of the circle $|z| = R$, $\operatorname{Re} z \geq \sigma$.

S09.5. Let $0 < a < 1$ be any real number. Then

(a) Prove the following identity:

$$\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos\theta} d\theta = \frac{2\pi}{1 - a^2}$$

(b) Find the limit

$$\lim_{k\to+\infty} \int_{|z|=(k+\frac{1}{2})\pi} \frac{\pi}{z^2 \sin z} dz$$

F09.7. Find the integral

$$\int_0^\infty \frac{1-\cos(2x)}{x^2}dx$$

F10.5. Find the integral (where $a > b > 0$)

$$\int_0^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$$

F11.3. Evaluate

$$\int_0^\infty \frac{dx}{x^{1/3}(1+x)}$$

F12.1. Show that for $a > 0$,

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2}dx = \frac{\pi(1+a)}{4e^a}$$

S13.7. Evaluate the integral for $a > 0$

$$\int_{-\infty}^\infty \frac{\cos^3 x}{a^2+x^2}dx$$

F13.4. Evaluate the integral

$$\int_0^\infty \frac{1}{1+x^5}dx$$

S14.2. Complete the following two problems.

(a) Evaluate $\int_{|z|=1} \exp\left(\frac{1}{z^2}\right) dz$. (Here, $\exp(z) =: e^z$).

(b) Evaluate $\int_0^\infty \frac{x^2}{1+x^4}dx$.

F14.2. Evaluate the real integral

$$\int_0^\infty \frac{\ln x}{1+x^4} dx$$

S15.4. Evaluate the integral

$$\int_{-\infty}^\infty \frac{x \sin \pi x}{x^2+2x+5} dx$$

F15.3. Evaluate

$$\int_0^\infty \frac{\sin ax}{x(x^2+b^2)} dx, \ a, b > 0$$

S16.2. Show that for a positive integer $n \geq 1$

$$\int_0^\infty \frac{1}{x^{2n}+1} dx = \frac{\pi}{2n \sin \frac{\pi}{2n}}$$

F16.1. Evaluate the following integral

$$\int_0^\infty \frac{x}{1+x^5} dx$$

S17.1. Find the integral

$$\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}, \ a > 1$$

F17.6. Prove

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}$$

S18.6. Evaluate $\int_{-\infty}^\infty \frac{\sin x}{x+i} dx$.

F18.5. Evaluate

$$\int_\gamma \frac{1+z}{1-\cos(z)}dz, \text{ where}$$

(a) γ is the circle of radius 5 around 0, counterclockwise.

(b) γ is the circle of radius 7 around 0, counterclockwise.

F19.4. Prove that

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} = \frac{\pi^3}{8}$$

Sequence

S06.6. Prove or disprove that there is a sequence of analytic polynomials $\{p_n(z)\}_{n=1}^\infty$ so that $p_n(z) \to \overline{z}^4$ as $n \to \infty$ uniformly for $z \in \partial D(0,1) = \{z \in \mathbb{C}\colon |z| = 1\}$.

F09.8. Does there exist a sequence of holomorphic functions $\{f_n(z)\}_{n=1}^\infty$ on the unit disc $D(0,1)$ so that $f_n(z) \to 1/z$ uniformly on $\{z \in \mathbb{C}\colon |z| = 1/2\}$ as $n \to \infty$?

S14.9. Let $f_n : D(0,1) \to D(0,1)\backslash\{0\}$ be a sequence of holomorphic functions with $\sum_{n=1}^\infty |f_n(0)|^2 < \infty$. Prove that

$$\sum_{n=1}^{\infty} |f_n(z)|^3$$

converges uniformly on $\overline{D(0,1/5)}$.

S17.4. Let $f_n : D(0,1) \to D(0,1) \setminus \{0\}$ be analytic such that $\sum_{n=1}^\infty |f_n(0)| < \infty$.

(a) Prove that $\sum_{n=1}^\infty |f_n(z)|^3$ converges uniformly on $|z| \leq \frac{1}{2}$.

(b) Give an example of $\{f_n\}_{n=1}^\infty$ satisfying above conditions but $\sum_{n=1}^\infty |f_n(z)|^3$ diverges for any $|z| > 1/2$.

F17.8. Let $\{f_n\}_{n=1}^\infty$ be a sequence of holomorphic functions on the unit disk $D(0,1)$ such that

$$F(z) = \sum_{n=1}^{\infty} |f_n(z)|$$

defines a continuous function in $D(0,1)$ and $F(0) \geq F(z)$ on $D(0,1)$. Prove f_n are constant for all $n = 1, 2, 3, \ldots$

F19.5. Prove or disprove: there exist a sequence of holomorphic functions $\{f_n\}_{n=1}^\infty$ on $D(0,1)$ such that $f_n(z) \to |z|^2$ uniformly on a non-empty open subset of $D(0,1)$.

Entire

S06.1. Prove or disprove that there exists an analytic function $f(z)$ in the unit disk $D(0,1)$ such that

$$f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}, \qquad \text{for all } n = 1, 2, 3 \ldots$$

S06.2. Complete the following problems:

(a) State the Liouville's Theorem.

(b) Prove the Liouville's Theorem by calculating the following integral:

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz$$

S06.8. Let $f(z)$ be an entire analytic function and satisfy

$$f(z+1) = f(z) \text{ and } |f(z)| \le e^{|z|}, z \in \mathbb{C}$$

Prove that $f(z)$ must be constant. Her $\mathbb{C}$ denotes the whole complex plane.

S08.2. Let $f(z)$ be an entire holomorphic function on $\mathbb{C}$ such that $|f(z)| \le |\cos z|$. Prove $f(z) = c\cos z$ for some constant c.

F08.2. Find all entire functions $f(z)$ that satisfy

$$f''\left(\frac{1}{n}\right) = 4f\left(\frac{1}{n}\right) \quad \text{for all } n \in \mathbb{N}$$

F08.3. Let $L \subset \mathbb{C}$ be the line $L = \{z = x + iy \mid x = y\}$. Assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that for any $z \in L$ we have $f(z) \in L$. Assume that $f(1) = 0$. Prove that $f(i) = 0$.

S09.2. Suppose that a function $f(z)$ is holomorphic in the unit disc $D(0,1)$ and has the property

$$f\left(\frac{1}{2n}\right) = f^{(4)}\left(\frac{1}{2n}\right) \quad \text{for all} \quad n \in \mathbb{N}$$

Prove that f can be extended to an entire function on $\mathbb{C}$. (Here $f^{(4)} = \frac{\partial^4 f}{\partial z^4}$).

S09.6. Let $f(z)$ be any non-constant entire function on $\mathbb{C}$. Use Liouville's Theorem to prove that the image of f (or $f(\mathbb{C})$) is dense in $\mathbb{C}$.

4. (blame spring 09 formatting, not me).

(a) State the Schwarz reflection principle for holomorphic function on the unit disk.

(b) Let $f(z)$ be holomorphic in the unit disc $D(0,1)$ and continuous on the closed disc $\overline{D(0,1)}$. Prove or disprove there exists such f so that $fe^{i\theta} = e^{-i\theta}$ for $0 < \theta < \pi/4$.

S09.9. Let f be an entire function on $\mathbb{C}$ with $|f(z)| = 1$ for $|z| = 1$ and $f'''(0) = 6$ (the third order derivative of f at $z = 0$). Find all such f.

F09.3. Let $L \subset \mathbb{C}$ be the line $L = \{z = x + iy \mid y = 2\}$. Assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that for any $z \in L$ we have $f(z) \in L$. Assume that $f(0) = i$. Find $f(4i)$.

F11.1. (Also F13.1) Describe all entire holomorphic functions f and g such that

(a) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2}$ for all positive integers n. Show your work.

(b) $g\left(\frac{1}{n}\right) = g\left(-\frac{1}{n}\right) = \frac{1}{n^3}$ for all positive integers n. Show your work.

F11.2. (Also F13.2) Let $f(z)$ be an entire holomorphic function such that

$$\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 0$$

Prove that f is a constant.

F12.6. Let $f(z)$ be an entire holomorphic function on $\mathbb{C}$ such that

$$|f(e^z)| \le |e^z|, \qquad z \in \mathbb{C}$$

Prove $f(z) \equiv az$ for some constant $|a| \le 1$.

S14.1. Complete the following two problems.

(a) Describe all entire holomorphic functions f with $|f(z)| \le |z|$ for all $z \in \mathbb{C}$.

(b) Describe all entire holomorphic functions f with $\lim_{z\to\infty} \frac{f(z)}{z} = 0$.

F14.1. Let f be an entire holomorphic function such that $f(z) \notin \mathbb{R}$ for all $z \in \mathbb{C}$, where $\mathbb{R}$ is the real line in the complex plane $\mathbb{C}$. Prove or disprove f is a constant.

F14.3. Let f be entire holomorphic such that $f(x + ix) \in \mathbb{R}$ for all $x \in \mathbb{R}$. If $f(2) = 1 - i$ then find $f(2i)$, where $i^2 = -1$.

F15.2. Let f be an entire function and suppose that there exists a bounded sequence $\{a_n\}$ of real numbers such that $f(a_n)$ is real for all $n \in \mathbb{N}$. Prove that $f(x)$ is real for all real x.

S16.4. Let f be an entire function. Prove the following two statements.

(a) If $|f(z)| \le M(1 + |z|^n)$ on $\mathbb{C}$ for some positive constant M then f is a polynomial of degree at most n.

(b) If $\lim_{|z|\to\infty} |f(z)| = \infty$ then f is a polynomial.

S16.5. Find all entire holomorphic functions f with justification such that

$$\operatorname{Im} f(z) = (y^2 - x^2),$$

where $\operatorname{Im} f$ denotes the imaginary part of f.

S17.8. Prove or disprove there is a non-constant entire function $f = u + iv$ satisfying $v(z) \neq u(z)^2$ when $u(z) \geq 0$.

F17.2. Prove or disprove there is a holomorphic function f on the unit disk $D(0,1)$ such that

$$f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{n}$$

for all positive integers n.

F17.3. Let f be an entire holomorphic function in $\mathbb{C}$ such that $f(x)$ and $f(ix)$ are real for $x \in (1,2)$. Prove there is an entire function g such that

$$f(z) = g(z^2), \quad z \in \mathbb{C}$$

F18.2. Let f and g be entire functions. Suppose that

(a) $g(z) \neq 0$ for all $z \in \mathbb{C}$.

(b) $|f(z)| \leq |z^7 g(z)|$ for all $z \in \mathbb{C}$.

Prove that there exists $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ such that $f(z) = \alpha z^7 g(z)$ for all $z \in \mathbb{C}$.

F19.6. Suppose that f is a non-constant entire function which satisfying

$$|f(z)| \geq 1 \quad \text{when} \quad |z| \geq 10$$

Prove that f is a polynomial.

Meromorphic

F06.1. Show that $\sum_{n=1}^{\infty} \frac{1}{z^2+n^2}$ is a meromorphic function on $\mathbb{C}$.

F10.2. Show that $\sum_{n=1}^{\infty} \frac{1}{z^2+n^2}$ is a meromorphic function on $\mathbb{C}$.

S15.6. Let the function $f(z)$ be meromorphic in a neighbourhood of the unit disk $\{|z| \leq 1\}$ and suppose it has only one singular point z_0 on the circle $|z| = 1$ which is a simple pole. Show that $\frac{f^{(n)}(0)}{n!} = \frac{A}{z_0^n}(1 + \phi_n)$ where $\lim_{n\to\infty} \phi_n = 0$.

S16.1. Show that

$$\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}$$

defines a meromorphic function on $\mathbb{C}$.

F16.4. Let f be meromorphic in the complex plane $\mathbb{C}$ such that

$$|f(z)| = 1 \quad \text{on} \quad |z| = 1$$

Prove f is a rational function.

S17.7. Let f be meromorphic in $\mathbb{C}$ satisfying

$$|f(z)|^3 \leq |\tan z|, \qquad z \in \mathbb{C} \setminus P(f)$$

where $P(f)$ is the set of poles of f in $\mathbb{C}$. Prove $f(z) \equiv 0$.

Bound

F06.5. Let f be a function analytic in the unit disc $D(0,1)$ and $|f(z)-z| \leq 1$ on the unit circle $\partial D(0,1)$. Show that $\left|f'(\frac{1}{2})\right| \leq 7/3$.

S09.7. Let D be a bounded domain in $\mathbb{C}$ with $0 \in D$. If $f : D \to D$ is a holomorphic map so that $f(0) = 0$ and $f'(0) = 1$. Show that $f(z) = z$ on D.

F11.6. Prove the Schwarz-Pick lemma: Let $f : D(0,1) \to D(0,1)$ be holomorphic. Then

$$\left|\frac{f(z)-f(a)}{1-\overline{f(a)}f(z)}\right| \leq \left|\frac{z-a}{1-\bar{a}z}\right|, \qquad a, z \in D(0,1)$$

F12.5. Let f be analytic on the upper-half plane and satisfy $|f(z)| < 1$. Furthermore suppose $f(i) = 0$. Give an upper bound for $|f'(i)|$ and satisfy which functions realize this extrema.

S13.3. Suppose f is holomorphic on the upper half plane $\mathbb{H} = \{\operatorname{Im} z > 0\}$, $f(i) = 0$, and $|f(z)| \leq 1$ for all $z \in \mathbb{H}$. Prove that $|f(2i)| \leq \frac{1}{3}/$

S14.3. Let $f : D(0,1) \to D(0,1)$ be holomorphic with $f(0) = \frac{1}{3}$.

(a) Give a sharp upper bound estimate for $|f'(0)|$.

(b) Give an example of f such that $|f'(0)|$ achieves the upper bound you obtained in Part (a).

S15.5. Suppose $f : D(0,1) \to D(0,1)$ is a holomorphic mapping and $f(0) = \frac{1}{5}$. Give an upper bound for $|f'(0)|$ and characterize the functions for which the upper bound is an equality.

S16.8. Let $f : U \to U$ be holomorphic with U being the upper half plane. Prove that

$$|f'(i)| \leq |f(i)|$$

and provide an example that indicates the above inequality is an equality.

F16.3. Prove or disprove that there is a non-zero holomorphic function f in the complex plane $\mathbb{C}$ such that

$$|f(z)|^2 \leq |\cos z|$$

F16.7. Let f be analytic on the upper-half plane and satisfy $|f(z)| < 1$. Furthermore suppose $f(2+i) = 0$. Give an upper bound for $|f'(2+i)|$ and state which functions realize this extrema.

F18.8. Suppose $f : D(0,1) \to D(0,1)$ is a holomorphic mapping such that $f(0) = \frac{1}{5}$. Give an upper bound for $|f'(0)|$, and characterize the functions for which the upper bound is an equality.

S19.3. (a) State the Schwarz-Pick Lemma.

(b) Suppose $f : D(0,1) \to D(0,1)$ is a holomorphic mapping such that $f(0) = \frac{1}{6}$. Give an upper bound for $|f'(0)|$, and characterize the functions for which the upper bound is an equality.

S19.6. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ denote an entire function satisfying the estimate

$$|f(z)| \leq 10e^{|z|}, \qquad \text{for all } z \in \mathbb{C}$$

Prove that the coefficients a_j satisfy

$$|a_j| \leq 10(e/j)^j, \qquad \text{for all } j \in \mathbb{N}$$

Zeros

F06.6. Let real $a > 1$. Prove that $ze^{a-z} = 1$ has a single solution in the closed unit disk $\overline{D}(0,1)$ which is real and positive.

F06.8. Let $f : D(0,1) \to \mathbb{C}$ be a bounded analytic function. Let a_n be the non-zero zeros of f in D counting according to multiplicity. Prove

$$\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$$

F07.9. Suppose that $f(z)$ is an entire function such that

$$|f(z)| \leq Be^{A|z|}, \qquad z \in \mathbb{C}$$

for some positive numbers A, B. Let $\omega_1, \omega_2, \ldots$ be the zeros of f listed with appropriate multiplicity. Prove that

$$\sum_{n=1}^{\infty}(1 + |\omega_n|)^{-\alpha} < \infty$$

for all $\alpha > 1$.

F08.6. Suppose a function $f : D \to D$, where $D = \{|z| < 1\}$ is the unit disc, is holomorphic and $f(0) = \alpha \neq 0$. Show that f can't have a zero in the open disk $D(0, |\alpha|) = \{|z| < |\alpha|\}$.

F08.8. How many zeros does the function $f(z) = 14z^{100} - 5e^z$ have in the unit disc? What are the multiplicities of zeros?

S09.1. (a) State the Rouche's Theorem.

(b) Let $a > e$ be a real number. Prove that the equation

$$az^4e^{-z} = 1$$

has a single solution in $D(0,1)$, which is real and positive.

F09.6. Suppose that f is a polynomial such that all of its zeros are inside the unit disc. Prove that all zeros of f' are also inside of the unit disc.

F12.4. Determine the number of roots, counted with multiplicity, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $A(0, 1, 2) = \{z \in \mathbb{C} \colon 1 < |z| < 2\}$.

S13.5. Determine the number of roots, counted with multiplicity, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

inside the annulus $1 \leq |z| \leq 2$.

F13.6. How many solutions has the equation

$$z^4 + 3z^2 + z + 1 = 0$$

in the closed upper half unit disc?

F14.5. Let D be a bounded domain in $\mathbb{C}$ with piecewise C^1 boundary. Let $f(z)$ be holomorphic in a bounded domain D and $f \in C(\overline{D})$ with all zeros $\{z_1, \ldots, z_n\} \subset D$ counting multiplicity. Let g be holomorphic in D and continuous on $\overline{D}$. Evaluate

$$\int_{\partial D} \frac{f'(z)}{f(z)} g(z) dz$$

F15.4. How many roots does the equation $z + e^{-z} = a$, $a \in \mathbb{R}$, $a > 1$, have in the right half plane?

F15.6. Consider a non-constant polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0,\ n \geq 1,\ a_0, a_n \neq 0$$

and set $B = \max_{0 \le j \le n-1} |a_j|$, $C = \max_{1 \le j \le n} |a_j|$. Prove that all roots of the polynomial P lie inside the annulus $r \le |z| \le R$, where

$$r = \frac{1}{1 + \frac{C}{|a_0|}}, R = 1 + \frac{B}{|a_n|}.$$

F17.5. For $a > 1$ Prove the equation $ze^{a-z} = 1$ has a unique solution in $|z| \le 1$, which is also real and positive.

S18.1. Determine the number of roots, counted with multiplicity, of the equation

$$2z^5 - 15z^2 + z + 2$$

inside the annulus $1 \le |z| \le 2$.

S18.5. Suppose $p(z)$ is a polynomial of degree $d \ge 2$ that has only simple zeros $r_1, r_2, \ldots, r_d$. Prove that $\frac{1}{p'(r_1)} + \frac{1}{p'(r_2)} + \ldots + \frac{1}{p'(r_d)} = 0$.

F18.4. Find the number of solutions with multiplicity of $e^z = 7z^9$ in the open unit disc around the origin.

F19.2. Let $f : D(0,1) \to D(0,1)$ be holomorphic such that $f(0) = 5^{-20}$. Give a sharp estimate for the number of zeros of f on $\overline{D(0, \frac{1}{5})}$.

Normal

F06.7. Let Ω be a bounded domain in $\mathbb{C}$, let $\{f_j\}_{j=1}^{\infty}$ be a sequence of analytic functions on Ω such that

$$\int_{\Omega} |f_j(z)|^2 dA(z) \leq 1$$

Prove that $\{f_j\}_{j=1}^{\infty}$ is a normal family in Ω.

S08.6. Prove that the product $\prod_{k=1}^{\infty}\left(\frac{z^n}{n!} + \exp\left(\frac{z}{2^n}\right)\right)$ converges uniformly on compact sets to an entire function.

S08.7. Let F be a family of holomorphic functions on the unit disc $D(0,1)$ such that each $f \in F$ satisfying

$$|f(0)|^2 + \int_{D(0,1)} |f'(z)|^2 dA(z) \leq 1$$

Prove that F is a normal family on $D(0,1)$.

F09.4. Let $\{f_\alpha\}_{\alpha\in A}$ be a family of holomorphic functions on the unit disc $D(0,1)$ such that

$$\text{for all } z \in D(0,1)\ \forall f \in \{f_\alpha\}_{\alpha\in A} \qquad \text{Im } f(z) \neq (\text{ Re } f(z))^2$$

Prove that $\{f_\alpha\}_{\alpha\in A}$ is a normal family (i.e every sequence in $\{f_\alpha\}_{\alpha\in A}$ has a subsequence that converges or tends to infinity uniformly on compact subsets of $D(0,1)$).

F10.3. Let $\mathcal{F}$ be a family of holomorphic functions on the unit disc so that for any $f \in \mathcal{F}$ one has

$$\int_{D} |f(z)|(1-|z|)^2 dA(z) \leq 1$$

Prove $\mathcal{F}$ is a normal family.

F12.2. Suppose f is analytic in an annulus $A(0,r,R) = \{z \in \mathbb{C} \colon r < |z| < R\}$, and there exists a sequence of polynomials p_n converging to f uniformly on compact sets of $A(0,r,R)$. Show that f is an analytic function on the disc $D(0,R)$.

S13.2. Let $\{f_j\}$ be a sequence of holomorphic functions from $D(0,1)$ to $D(0,1)\setminus\{0\}$. Prove that if $\sum_{j=1}^{\infty}|f_j(0)|$ converges, then $\sum_{j=1}^{\infty} f_j(z)^2$ converges absolutely and uniformly on compact sets in $D(0,1/3)$.

S13.6. (Also S15.8) Suppose f is analytic in an annulus $r<|z|<R$, and there exists a sequence of polynomials p_n converging to f uniformly on any compact subset of the annulus. Show that f is an analytic function on the disc $\{|z|<R\}$.

F15.7. TRUE or FALSE: The family $\mathcal{F}$ of functions holomorphic in a unit disc with power series $f(z)=\sum_{n=0}^{8`} a_n z^n$ that satisfy $|a_n|\le n^{2015}$ is normal.

S16.6. Prove or disprove there exists a family $\{f_n\}$ of holomorphic functions on $D(0,2)$ such that $f_n\to\overline{z}^3$ uniformly on the compact set $\{z\in\mathbb{C}\colon |z|=1 \text{ or } 1/2\}$ (two circles: $|z|=1$ and $|z|=1/2$).

F17.7. Let $\mathcal{F}$ be a family of holomorphic functions f on the unit disc $D(0,1)$ such that

$$|f(0)|^2+\int_{D(0,1)}|f'(z)|^2 dA\le 1$$

Prove that $\mathcal{F}$ is a normal family.

S18.8. Let us say that a holomorphic function $f:\mathbb{D}\to\mathbb{C}$ on the unit disc $\mathbb{D}$ is "good" if for some $n\in\{1,2,\ldots,2018\}$ the function f does not take values on the ray $\{te^{\frac{2\pi}{n}i}\mid t\ge 0\}$. Prove that the collection of all "good" functions is normal.

F18.3. Let $\{f_n\}$ be a uniformly bounded sequence of analytic functions on the open unit disc $D(0,1)$. Suppose $\lim_{n\to\infty} f_n(\frac{1}{k})$ exists for $k=1,2,\ldots$. Prove that there exists an analytic function f on $D(0,1)$ such that $f_n\to f$ uniformly on compact subsets of $D(0,1)$.

Series

F07.3. (Also F09.1, S16.3) Let α, β, and γ be positive real numbers. Then find the radius of convergence for the series

$$\sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\gamma(\gamma+1)\ldots(\gamma+n-1)} z^n$$

F08.4. Find the largest disk centered at 1 in which the Taylor series for

$$\frac{1}{1+z^2} = \sum a_k (z-1)^k$$

will converge. (Hint: You do not actually have to find the coefficients a_k nor the full series to answer this question).

F10.8. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence $R = 64$. Determine the region of convergence of the Laurent series

$$\sum_{n=-\infty}^{-1} a_{2|n|} z^{3n} + \sum_{n=0}^{\infty} a_{3n} z^{2n}$$

F12.8. Find the largest set in $\mathbb{C}$ where the Laurent series

$$\sum_{j=-\infty}^{\infty} 2^j z^{j^3}$$

converges.

S13.1. Find the largest set D where the power series

$$\sum_{n=1}^{\infty} \frac{n}{2^n} z^{n^2}$$

S14.4. Prove that there is an N such that if $n \geq N$ then

$$\sum_{k=0}^{n} (k+1)z^k \neq 0, \qquad z \in D(0, 3/4)$$

F14.4. Let $h(x)$ be a twice differentiable function on $[-1, 1]$ such that $h(0) = h'(0) = 0$ and $h''(0) \neq 0$. Prove

$$\sum_{n=1}^{\infty} h\left(\frac{1}{n}\right) z^n$$

defines a holomorphic function on $D(0, 1)$ which is continuous on $\overline{D(0, 1)}$.

F15.1. Prove that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges uniformly in the unit disc $|z| \leq 1$. Does the series obtained by term-by-term differentiation converge uniformly in the unit disc? Explain your answer.

F15.7. TRUE or FALSE: The family $\mathcal{F}$ of functions holomorphic in a unit disc with power series $f(z) = \sum_{n=0}^{8`} a_n z^n$ that satisfy $|a_n| \leq n^{2015}$ is normal.

F16.5. Find the radius of convergence for

$$\sin\left(\frac{2}{(z - 2i + 2)(z - 3 + i)}\right) = \sum_{n=0}^{\infty} a_n z^n$$

S19.5. Assume that the power series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

defines a holomorphic function for $|z| < 1$ and assume that

$$|a_1| > \sum_{j=2}^{\infty} j|a_j|$$

Show that the function $f(z)$ is one-to-one in the open unit disk $D(0, 1)$.

F19.3. Given a series

$$\sum_{n=1}^{\infty} \left(\frac{2019+i}{2019-i}\right)^{n^2} \cdot \left(\frac{z-2019}{z+2019}\right)^{n}$$

(i) Find all complex numbers z such that the series converges absolutely;

(ii) Find all complex numbers z such that the series convergence.

F19.8. (a) Prove

$$f(z) = \sum_{k=-\infty}^{\infty} \frac{1}{(k+z)^2} - \frac{p^2}{\sin^2(\pi z)}$$

is entire holomorphic.

(b) Prove

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+z)^2} = \frac{\pi^2}{\sin^2(\pi z)}, \ z \notin \mathbb{Z}$$

Automorphism

F07.6. Let $\Omega \neq \mathbb{C}$ be a simply connected domain in $\mathbb{C}$. Let $f : \Omega \to \Omega$ be a holomorphic mapping which fixes two distinct points in Ω (i.e there are $p, q \in \Omega$ so that $f(p) = p$ and $f(q) = q$). Show that $f(z) \equiv z$ on Ω.

S08.4. Denote by D the unit disk, $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Does there exist a holomorphic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$?

F09.5. Let $f(z) : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C} \setminus \{0, 1\}$ be holomorphic. Prove that f must be constant.

F13.3. Describe explicitly the automorphism group $\mathrm{Aut}(\mathbb{C} \setminus \{0, 1\})$.

S14.7. Let D be a simply connected domain in $\mathbb{C}$ and $z_0 \in D$. If $\phi_1, \phi_2 \in \mathrm{Aut}(D)$ such that

$$\phi_1(z_0) = \phi_2(z_0) \text{ and } \phi_1'(z_0) = \phi_2'(z_0)$$

then $\phi_1 \equiv \phi_2$. (Hint: Try $D = D(0, 1)$ and $D = \mathbb{C}$ first.)

Conformal

S08.1. Find explicitly a conformal mapping of the domain

$$\{z \in \mathbb{C} \mid |z| > 1,\ \text{Re } z > 0, \text{Im } z > 0\}$$

to the unit disc.

S09.4. (a) State the Riemann mapping theorem.

(b) Find explicitly a conformal mapping of the domain

$$\{z \in \mathbb{C} \mid |z| < 1,\ \text{Re } z > 0,\ \text{Im } z > 0\}$$

to the unit disc.

5. (this question is weird, ok). Does there exist a conformal automorphism φ of the unit disc such that $\varphi(1/2) = 0$ and $\varphi(0) = \frac{i}{3}$?

1. Find the integral $\int_0^\infty \frac{x \cos(ax)}{\sinh x} dx$.

3. Prove that if $|a| \neq R$, then

$$\int_{|z|=R} \frac{|dz|}{|z-a||z+a|} \leq \frac{2\pi R}{|R^2 - |a|^2|}$$

F10.4. Find an explicit conformal transformation of an open set $U = \{|z| > 1\} \setminus [1, +\infty)$ to the unit disc.

F10.6. Let U be an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$ and $z_0 \in U$. Write $f = u + iv$, i.e u, v are the real and imaginary parts of f. We say that f is complex differentiable at z_0 if $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

(i) Prove that if f is complex differentiable at z_0, then u, v satisfy the Cauchy-Riemann equations.

(ii) Prove that if f is complex differentiable and $f'(z) \neq 0$ in U then f is an orientation preserving conformal map, i.e. for any two differentiable curves α, β in U with $\alpha(0) = \beta(0)$ the angle from $\alpha'(0)$ to $\beta'(0)$ is equal to the angle from $(f \circ \alpha)'(0)$ to $(f \circ \beta)'(0)$.

S13.8. Find explicitly a conformal mapping of the domain

$$U = \{|z| < 1, z \notin [1, 2, 1)\} = \mathbb{D} \setminus [1/2, 1)$$

to the unit disc $\mathbb{D} = \{|z| < 1\}$.

S14.6. Let $D = \{z \in \mathbb{C} \colon 1 < |z + 1| \text{ and } |z + 2| < 2\}$. Construct a conformal holomorphic map which maps D onto the unit disc $D(0, 1)$.

F14.6. Let $D = \{z \in \mathbb{C} \colon |z| < 1, \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$. Construct a conformal holomorphic map which maps D onto the unit disc $D(0, 1)$.

F15.8. Set $U_1 = \{1 < |z| < 2\}$ and $U_2 = \{0 < |z| < 1\}$.

(a) Show that homeomorphism $f : U_1 \to U_2$ given $f(re^{i\theta}) = (r - 1)e^{i\theta}$ is not a conformal mapping.

(b) Does there exist a conformal mapping $g : U_1 \to U_2$?

S16.7. Construct a conformal map ϕ which maps D_1 onto D_2, where

$$D_1 = \{z = x + iy \in D(0, 1) \colon y > x\}; \text{ and } D_2 = \{z \in \mathbb{C} \colon |z| > 1\}$$

S17.6. Find a conformal map which maps U_1 onto U_2, where

$$U_1 = \{z = x + iy \in \mathbb{C} \colon y > 0\} \setminus \{z = iy \colon 1 \le y \le 2\} \text{ and } U_2 = D(0, 1) \setminus \{0\}$$

S18.3. Let $L \subset \mathbb{C}$ be the ray $\{t + it \mid t \ge 1\}$, and $U = \{\operatorname{Re} z > 0, \operatorname{Im} z > 0\}$. Find an explicit conformal mapping of $U \setminus L$ to the unit disc.

F18.6. Find a surjective holomorphic map φ from the open unit disc $D = D(0, 1)$ to the punctured disc $D^* = D \setminus \{0\}$, with $\varphi'(z) \neq 0$ for any $z \in D$.

S19.7. Construct a conformal map ϕ which maps D_1 onto D_2, where

$$D_1 = \{z \in \mathbb{C}\colon 0 < \operatorname{Re} z < 2\} \text{ and } D_2 = \{z \in \mathbb{C}\colon |z| > 1\}/$$

Holomorphic

S08.3. (Also F11.4) Show that there is a holomorphic function defined in the set

$$\Omega = \{z \in \mathbb{C} \mid |z| > 4\}$$

whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

Is there a holomorphic function on Ω whose derivative is

$$\frac{z^2}{(z-1)(z-2)(z-3)}?$$

S08.8. Let F be holomorphic on the upper half plane U and continuous on $U \cup [0,1]$. Assume that

$$f(x) = x^2 - x + 1,\ x \in (0,1)$$

Find all such functions f.

F09.2. Prove or disprove there is a holomorphic function $f(z)$ on the unit disc $D(0,1)$ so that

$$\{z \in D(0,1)\colon f^{(k)}(z) = 0 \text{ for some non-negative integer } k\ \} = (-1,1)$$

where $f^{(k)}$ is the k'th derivative of f.

F11.5. Let $f : [0,1] \to \mathbb{C}$ be a continuous function. Define the function $F : \mathbb{C} \setminus [0,1] \to \mathbb{C}$ by

$$F(z) = \int_0^1 \frac{f(t)}{t-z}dt, \qquad z \in \mathbb{C} \setminus [0,1].$$

Prove that F is holomorphic on $\mathbb{C} \setminus [0,1]$.

F12.2. Suppose f is analytic in an annulus $A(0,r,R) = \{z \in \mathbb{C}\colon r < |z| < R\}$, and there exists a sequence of polynomials p_n converging to f uniformly on compact sets of $A(0,r,R)$. Show that f is an analytic function on the disc $D(0,R)$.

S13.4. (Also S18.4) Suppose $f(z) = u(x, y) + iv(y)$ is a holomorphic function. Show that there exists $a \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ such that $f(z) = az + \lambda$.

F16.6. Prove or disprove there is a holomorphic function f on $\mathbb{C} \setminus D(0, 3)$ such that

$$f'(z) = \frac{z^2 + 1}{z(z-1)(z-2)}$$

S17.5. Let f be holomorphic in $D = \{z \in \mathbb{C} \colon 2 < |z| < \infty\}$ satisfying

$$\int_{|z|=3} f(z)dz = 0$$

Prove that there is a holomorphic function F in D such that $F'(z) = f(z)$ on D.

S19.4. Let

$$f(z) = \frac{z^3}{(z^2 + 1)e^{1/z}}$$

(a) Find and classify all the singularities of $f(z)$ in the extended complex plane.

(b) Evaluate

$$\oint_{\Gamma} f(z)dz,$$

where $\Gamma = \{z \in \mathbb{C} \mid |\operatorname{Re} z| + |\operatorname{Im} z| = 3\}$.

(c) Does there exist a holomorphic function F on $|z| > 3$ such that $F'(z) = f(z)$ on $|z| > 3$? (Justify your answer.)

F19.1. Complete the following parts.

(a) Find all points where the function $f(z) = \overline{z}^2 + 3\overline{z}$ is analytic.

(b) Let D be a domain in $\mathbb{C}$ and let f be C^1 function on D such that

$$\int_{\partial D(z_0, r)} f(z)dz = 0,$$

for any $z_0 \in D$ and $0 < r < \delta(z_0) = \operatorname{dist}(z_0, \delta D)$. Prove f is holomorphic in D.

Product

S08.6. Prove that the product $\prod_{k=1}^{\infty}\left(\frac{z^n}{n!}+\exp\left(\frac{z}{2^n}\right)\right)$ converges uniformly on compact sets to an entire function.

Harmonic

F08.7. Let u be a harmonic function on $\mathbb{R}^2$ that does not take zero value (i.e $u(x) \neq 0 \; \forall x \in \mathbb{R}^2$). Show that u is constant.

S09.8. Let $f(z)$ be holomorphic on a domain in the complex plane. If $|f(z)|^2$ is harmonic in D. What can you conclude on f? (Show your work.)

F10.7. (i) State the Mean Value Theorem for analytic functions and use the Cauchy integral formula to prove it.

(ii) Prove that if $f = u + iv$ is an analytic function from an open subset U of $\mathbb{C}$ then the real and imaginary parts u and v of f are harmonic, i.e, $\Delta u = \Delta v = 0$.

(iii) Let U be an open subset of $\mathbb{R}^2$, and $u : U \to \mathbb{R}$ a harmonic function. Prove that if there is $p_0 \in U$ such that $u(p_0) = \inf_{x \in U} u(x)$, then u is a constant.

F12.9. Let u be a real-valued harmonic function in $\mathbb{C} \setminus \{0\}$. Show that then

$$u(z) = c \log |z| + \mathrm{Re}(f(z))$$

for some real constant c and a holomorphic function f on $\mathbb{C} \setminus \{0\}$.

F13.5. Prove that the function

$$u(x, y) = y \cos y \sinh x + x \sin y \cosh x$$

is harmonic in $\mathbb{R}^2$ and find its harmonic conjugate.

S14.5. Let $f_1, \dots, f_n$ be holomorphic in a domain D in $\mathbb{C}$ and $p \in (0, \infty)$. Prove

(a) $\sum_{j=1}^{n} |f_j(z)|^p$ is subharmonic in D.

(b) If there is a $z_0 \in D$ such that $\sum_{j=1}^{n} |f_j(z_0)|^p \geq \sum_{j=1}^{n} |f_j(z)|^p$ for all $z \in D$, then f_j is constant for $j = 1, 2, \dots, n$.

F14.8. Let $u(z)$ be harmonic in $D =: D(0,1) \setminus \{0\}$ such that

$$\lim_{z\to 0} \frac{u(z)}{\log|z|} = 0$$

Prove that u can be extended to be harmonic in $D(0,1)$.

S15.7. TRUE or FALSE: There exists a bounded harmonic function on the upper half plane $\mathbb{H}$ that cannot be extended to any larger domain. Explain your answer.

F15.5. Find a real valued function $u(z)$ that is continuous in the closed disc $\overline{D(0,R)}$ (that is, closed disc centered at 0 of radius $R > 0$) and harmonic in $D(0,R)$, and satisfies

$$u(Re^{i\theta}) = \frac{1}{2}(1 + \cos^3\theta),\ \theta \in [0, 2\pi)$$

F16.8. Let u be a real-valued harmonic function in $\overline{D(0,1)} \setminus \{0\}$ such that

$$\lim_{z\to 0} \frac{u(z)}{\log z} = 0.$$

Show that there is a harmonic function U on $D(0,1)$ such that $u(z) = U(z)$ for all $z \in D(0,1) \setminus \{0\}$.

F17.1. Let u be a real-valued continuous function on $\mathbb{C}$ such that $e^{u(z)}$ is harmonic in $\mathbb{C}$. Then u is a constant.

S19.2. Let $u : \mathbb{C} \to \mathbb{R}$ be a nonconstant real harmonic function. Show that there exists a sequence of points $\{z_n\} \in \mathbb{C}$ such that $\lim_{n\to\infty} u(z_n) = -\infty$.

S19.8. Let u be harmonic in $D(0,1) \setminus \{0\}$ satisfying

$$\lim_{z\to 0} \frac{u(z)}{\ln|z|} = 0$$

Prove that u is harmonic on $D(0,1)$.

Singularity

F11.8. Let f be meromorphic in $D(0,1) \setminus \{0\}$ such that

$$\int_{D(0,1)\setminus\{0\}} |f(z)|^3 dA(z) \leq 1$$

Prove $z = 0$ is a removable singularity of f.

F12.7. Let f be holomorphic in $D(0,1) \setminus \{0\} = \{z \in \mathbb{C}\colon 0 < |z| < 1\}$. If

$$\int_D |f(z)|^3 dA(z) = \int_D |f(x+iy)|^3 dxdy < \infty$$

then $z = 0$ is removable singularity of f.

F13.7. Assume that $z = 0$ is an essential singularity of a holomorphic function f. Show that it is also an essential singularity of f^2, f^3, and, in fact, of f^n for every $n \in \mathbb{N}$.

S14.8. Let $f(z)$ be holomorphic in $D =: D(0,1) \setminus \{0\}$ such that

$$\int_D |f(z)| dA(z) < \infty$$

Prove that $z = 0$ is either removable or a simple pole.

S15.2. Classify all the singularities and find the associated residues for

$$f(z) = \frac{e^{-\frac{1}{z}}}{(z-1)(z+1)^2}$$

S15.6. Let the function $f(z)$ be meromorphic in a neighbourhood of the unit disk $\{|z| \leq 1\}$ and suppose it has only one singular point z_0 on the circle $|z| = 1$ which is a simple pole. Show that $\frac{f^{(n)}(0)}{n!} = \frac{A}{z_0^n}(1 + \phi_n)$ where $\lim_{n\to\infty} \phi_n = 0$.

F18.1. Let f be an analytic function on $D(z_0, r) \setminus \{z_0\}$, where $r > 0$, such that $f(z) \neq 0$ for all $z \in D(z_0, r) \setminus \{z_0\}$. Consider the analytic function $g(z) = \frac{1}{f(z)}$ for $z \in D(z_0, r) \setminus \{z_0\}$. Prove that f has an essential singularity at z_0 if and only if g has an essential singularity at z_0.

S19.1. Let $f(z)$ be analytic in the region $0 < |z| < 1$, which satisfies $\operatorname{Re} f(z) < 2$. Show that $z = 0$ is a removable singularity of f.

S19.4. Let

$$f(z) = \frac{z^3}{(z^2+1)e^{1/z}}$$

(a) Find and classify all the singularities of $f(z)$ in the extended complex plane.

(b) Evaluate

$$\oint_\Gamma f(z)dz,$$

where $\Gamma = \{z \in \mathbb{C} \mid |\operatorname{Re} z| + |\operatorname{Im} z| = 3\}$.

(c) Does there exist a holomorphic function F on $|z| > 3$ such that $F'(z) = f(z)$ on $|z| > 3$? (Justify your answer.)

Subharmonic

S14.5. Let $f_1, \ldots, f_n$ be holomorphic in a domain D in $\mathbb{C}$ and $p \in (0, \infty)$. Prove

(a) $\sum_{j=1}^n |f_j(z)|^p$ is subharmonic in D.

(b) If there is a $z_0 \in D$ such that $\sum_{j=1}^n |f_j(z_0)|^p \geq \sum_{j=1}^n |f_j(z)|^p$ for all $z \in D$, then f_j is constant for $j = 1, 2, \ldots, n$.

Cauchy

F17.4. Let $z_1, \ldots, z_n \in D(0, R)$ and

$$Q(z) = (z - z_1) \ldots (z - z_n)$$

Let f be a holomorphic function on $\overline{D(0, R)}$. PRove

$$P(z) = \frac{1}{2\pi i} \int_{|w|=R} f(w) \frac{Q(w) - Q(z)}{(w - z)Q(w)} dw$$

is a polynomial of degree $n - 1$ such that $f(z_j) = P(z_j)$ for $1 \leq j \leq n$.

F18.7. Let $f : \{z \mid |z| > 0\} \to \mathbb{C}$ be analytic. Furthermore suppose that $\lim_{z \to \infty} f(z) = 0$. Show that for $|z| > 1$, one has

$$\frac{1}{2\pi i} \int_{\nu=1} \frac{f(\nu)}{\nu - z} = -f(z)$$