CS 577 # Assignment 1 – Discrete Review Spring 2023

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Related Readings: `http://pages.cs.wisc.edu/~hasti/cs240/readings/`

Name: Carson Miller Wisc id: 9081473028

Logic

1. Using a truth table, show the equivalence of the following statements.

(a) $P \vee (\neg P \wedge Q) \equiv P \vee Q$

Solution:

P	Q	$\neg P$	$\neg P \wedge Q$	$P \vee (\neg P \wedge Q)$	$P \vee Q$
F	F	T	F	F	F
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	F	T	T

(b) $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

Solution:

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$
F	F	T	T	T	F	T
F	T	T	F	T	F	T
T	F	F	T	T	F	T
T	T	F	F	F	T	F

(c) $\neg P \vee P \equiv \text{true}$

Solution:

P	$\neg P$	true	$\neg P \vee P$
F	T	T	T
T	F	T	T

(d) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Solution:

P	Q	R	$Q \wedge R$	$P \vee Q$	$P \vee R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
F	F	F	F	F	F	F	F
F	F	T	F	F	T	F	F
F	T	F	F	T	F	F	F
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	T	F	F	T	T	T	T
T	T	T	T	T	T	T	T

Sets

2. Based on the definitions of the sets A and B, calculate the following: $|A|$, $|B|$, $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.

   (a) $A = \{1, 2, 6, 10\}$ and $B = \{2, 4, 9, 10\}$

   Solution:
   $|A| = 4$ $A \cup B = \{1,2,4,6,9,10\}$
   $|B| = 4$ $A \cap B = \{2,10\}$

   $A \setminus B = \{1,6\}$ $B \setminus A = \{4,9\}$

   (b) $A = \{x \mid x \in \mathbb{N}\}$ and $B = \{x \in \mathbb{N} \mid x \text{ is even}\}$

   Solution:
   $A \cup B = \{x \mid x \in N\}$
   $|A| = \infty$
   $A \cap B = \{x \in N \mid x \text{ is even}\}$
   $|B| = \infty$

   $A \setminus B = \{x \in N \mid x \text{ is odd}\}$ $B \setminus A = \{\emptyset\}$

Relations and Functions

3. For each of the following relations, indicate if it is reflexive, antireflexive, symmetric, antisymmetric, or transitive.

   (a) $\{(x, y) : x \leq y\}$

   Solution: transitive, reflexive

   (b) $\{(x, y) : x > y\}$

   Solution: antireflexive, transitive

(c) $\{(x, y) : x < y\}$

Solution: antireflective, transitive

(d) $\{(x, y) : x = y\}$

Solution: reflexive, transitive, antisymmetric, symmetric

4. For each of the following functions (assume that they are all $f : \mathbb{Z} \to \mathbb{Z}$), indicate if it is surjective (onto), injective (one-to-one), or bijective.

(a) $f(x) = x$

Solution: bijective

(b) $f(x) = 2x - 3$

Solution: injective

(c) $f(x) = x^2$

Solution: None

5. Show that $h(x) = g(f(x))$ is a bijection if $g(x)$ and $f(x)$ are bijections.

Solution: First, I will show that $h(x)$ is surjective. Since g & f are both bijections, we may have $f(A) = B$ and $g(B) = C$. Then, we have

$$\begin{aligned} h(A) &= g(f(A)) \\ &= \{c \in C \mid g(f(a)) = c, \text{ for some } a \in A\} \\ &= \{c \in C \mid g(b) = c, \text{ for some } b \in f(A)\} \\ &= g(f(A)) \\ &= g(B) \\ &= C \end{aligned}$$

Next, we will show that h is injective. Suppose we have $h(a_1) = h(a_2)$. We will show that $a_1 = a_2$

$$\begin{aligned} h(a_1) &= h(a_2) \\ g(f(a_1)) &= g(f(a_2)) \\ f(a_1) &= f(a_2) \\ a_1 &= a_2 \checkmark \end{aligned}$$

Thus, h is injective & surjective, thus it is bijective.

Induction

6. Prove the following by induction.

(a) $\sum_{i=1}^{n} i = n(n+1)/2$

Solution: Base Case: $n=1$ $\quad \sum_{i=1}^{1} i = 1(1+1)/2$

$1 = 1(2)/2$

$1 = 1$

Inductive Step: If $\sum_{i=1}^{k} i = k(k+1)/2$, then $\sum_{i=1}^{k+1} i = (k+1)(k+2)/2$ ✓

$\sum_{i=1}^{k+1} i = (k+1)(k+2)/2$

$(k+1) + \sum_{i=1}^{k} i = \frac{k^2+3k+2}{2} \Rightarrow \sum_{i=1}^{k} i = \frac{k^2+3k+2-2k-2}{2} \Rightarrow \sum_{i=1}^{k} i = \frac{k^2+k}{2} \Rightarrow \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

(b) $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

Solution: Base Case: $n=1$, $\sum_{i=1}^{1} i^2 = 1(1+1)(2(1)+1)/6 \Rightarrow 1 = 1(2)(3)/6 \Rightarrow 1 = 1$

Inductive Step: If $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$, then $\sum_{i=1}^{k+1} i^2 = (k+1)(k+2)(2(k+1)+1)/6$.

$\sum_{i=1}^{k+1} i^2 = (k+1)(k+2)(2(k+1)+1)/6$

$(k+1)^2 + \sum_{i=1}^{k} i^2 = (k^2+3k+2)(2k+3)/6$

$\sum_{i=1}^{k} i^2 = \frac{(2k^3+9k^2+13k+6) - 6k^2-12k-6}{6}$

$\sum_{i=1}^{k} i^2 = \frac{2k^3+3k^2+k}{6}$

$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ ✓

(c) $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$

$(k+1)(k+1)(k+1)$

$(k^2+2k+1)(k+1)$

$k^3+k^2+2k^2+2k+k+1$

k^3+3k^2+3k+1

Solution: Base Case: $n=1$, $\sum_{i=1}^{1} i^3 = 1^2(1+1)^2/4 \Rightarrow 1 = 1(4)/4 \Rightarrow 1 = 1$

Inductive Step: If $\sum_{i=1}^{k} i^3 = k^2(k+1)^2/4$, then $\sum_{i=1}^{k+1} i^3 = (k+1)^2(k+2)^2/4$

$\sum_{i=1}^{k+1} i^3 = (k+1)^2(k+2)^2/4$

$(k+1)^3 + \sum_{i=1}^{k} i^3 = (k^2+2k+1)(k^2+4k+4)/4$

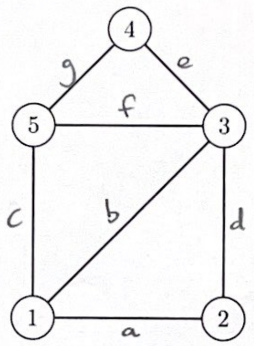
$\sum_{i=1}^{k} i^3 = \frac{k^4+6k^3+13k^2+12k+4-4k^3-12k^2-12k-4}{4}$

$\sum_{i=1}^{k} i^3 = \frac{k^4+2k^3+k^2}{4}$

$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ ✓

Graphs and Trees

7. Give the adjacency matrix, adjacency list, edge list, and incidence matrix for the following graph.



Solution: AM:

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 0 & 1 \\ 5 & 1 & 0 & 1 & 1 & 0 \end{array}$$

AL: $1 \to 2 \to 3 \to 5$; $2 \to 1 \to 3$; $3 \to 1 \to 2 \to 4 \to 5$; $4 \to 3 \to 5$; $5 \to 1 \to 3 \to 4$

EL: $\{(1,2), (1,3), (1,5), (2,3), (3,4), (3,5), (4,5)\}$, $\{1, 2, 3, 4, 5\}$

IM:

$$\begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}$$

8. How many edges are there is a complete graph of size n? Prove by induction.

Solution: Base Case: $n = 1$, A complete graph with one node (K_1) has zero edges.

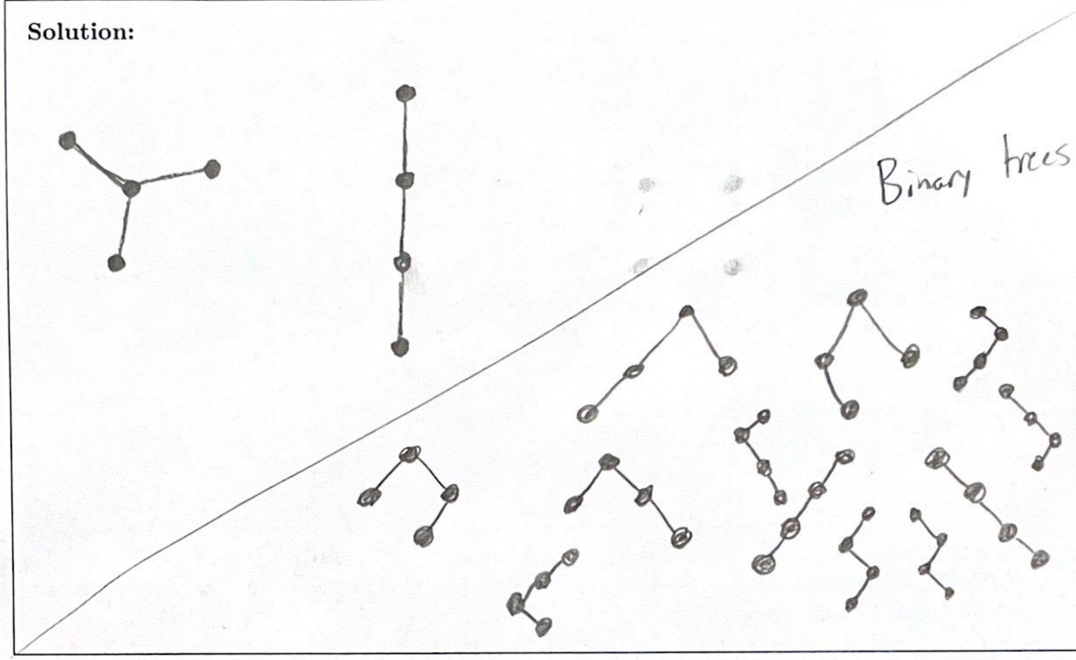
Inductive step: Suppose for $n \geq 2$ that K_n has $\frac{n(n-1)}{2}$ edges

Consider K_{n+1}, we will prove that is has $\frac{(n+1)(n)}{2}$ edges. If we add a node to K_n, we must connect it to all other nodes to remain complete. This result in the addition of n edges.

$$\frac{n(n-1)}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2+n}{2} = \frac{(n+1)n}{2} \checkmark$$

9. Draw all possible (unlabelled) trees with 4 nodes.

Solution:



10. Show by induction that, for all trees, $|E| = |V| - 1$.

Solution: Base Case: $|V| = 1$, For any tree with 1 node, there are no edges

$|E| = |V| - 1$

$0 = 1 - 1$

$0 = 0$ ✓

Inductive Step: Assume that for a k-node tree, $|E_k| = k - 1$.
We will prove that for a $(k+1)$-node tree, $|E_{k+1}| = (k+1) - 1$

$|E_k| = k - 1$

$|E_k| + 1 = (k+1) - 1$

If we remove a leaf node from the $(k+1)$-node tree, we are also only removing 1 edge. We can only remove leaf nodes because trees must be connected, acyclic graphs.

Counting

11. How many 3 digit pin codes are there?

Solution: Assuming 0-9, $10^3 = 1000$

12. What is the expression for the sum of the ith line (indexing starts at 1) of the following:

i	s	
1	1	1
2	5	2 3
3	15	4 5 6
4	34	7 8 9 10
5	65	11 12 13 14 15
		⋮

Solution:

1st term $\sum_{j=1}^{i-1} j = \frac{i(i-1)}{2} + 1$

last term $\sum_{j=1}^{i} j = \frac{i(i+1)}{2}$

$$\frac{i}{2}\left(\frac{i(i-1)}{2} + 1 + \frac{i(i+1)}{2}\right)$$

$$\frac{i}{2}\left(\frac{i^2 - i + 2 + i^2 + i}{2}\right)$$

$$\frac{2i^3 + 2i}{4} = \frac{i^3 + i}{2}$$

13. A standard deck of 52 cards has 4 suits, and each suit has card number 1 (ace) to 10, a jack, a queen, and a king. A standard poker hand has 5 cards. For the following, how many ways can the described hand be drawn from a standard deck.

(a) A royal flush: all 5 cards have the same suit and are 10, jack, queen, king, ace.

Solution: 4

(b) A straight flush: all 5 cards have the same suit and are in sequence, but not a royal flush.

Solution: 36

(c) A flush: all 5 cards have the same suit, but not a royal or straight flush.

Solution: 5,108

(d) Only one pair (2 of the 5 cards have the same number/rank, while the remaining 3 cards all have different numbers/ranks):

Solution: 1,098,240

Proofs

14. Show that $2x$ is even for all $x \in \mathbb{N}$.

(a) By direct proof.

Solution: Assume that all $x \in N$

$N \in Z$ $\quad y = 2k$, y is an even number given that $k \in Z$. Since $x \in N$, $x \in Z$. We can substitute and get $y = 2x$. y is an even number for any x.

(b) By contradiction.

Solution: Assume $2x$ is not even for all $x \in N$. Then, $2x$ must be odd for all $x \in N$.

$2x = 2k+1$, where $k \in Z$

$x = k + \frac{1}{2}$

if $k \in Z$, then $(x - k) \in Z$. $x - k = \frac{1}{2}$, but $1/2$ is not $\in Z$.

Thus, $2x$ must be even for all $x \in N$

15. For all $x, y \in \mathbb{R}$, show that $|x + y| \leq |x| + |y|$. (Hint: use proof by cases.)

Solution:

Case 1: Both x & y are positive $\quad |x+y| \leq |x|+|y|$

$x + y \leq x + y$ ✓

Case 2: Both x & y are negative $\quad |x+y| \leq |x|+|y|$

$-x-y \leq -x + -y$ ✓

Case 3: One of them is negative (Eg. x)

$|x+y| \leq |x|+|y|$

$-x-y \leq -x+y$

$\leq y-x$

Program Correctness (and Invariants)

16. For the following algorithms, describe the loop invariant(s) and prove that they are sound and complete.

(a)

Algorithm 1: findMin

Input: a: A non-empty array of integers (indexed starting at 1)
Output: The smallest element in the array

```
1 begin
2     min ← ∞
3     for i ← 1 to len(a) do
4         if a[i] < min then
5             min ← a[i]
6         end
7     end
8     return min
9 end
```

Solution:

Partial Correctness: This algorithm compares every element in a to min, finding the smallest value through brute force.

Termination: The for loop on line 3 will stop executing after len(a) iterations. Then, the program returns.

(b)

Algorithm 2: InsertionSort

Input: a: A non-empty array of integers (indexed starting at 1)
Output: a sorted from largest to smallest

```
begin
    for i ← 2 to len(a) do
        val ← a[i]
        for j ← 1 to i − 1 do
            if val > a[j] then
                shift a[j..i − 1] to a[j + 1..i]
                a[j] ← val
                break
            end
        end
    end
    return a
end
```

Solution:

Partial Correctness: This algorithm starts with looking at the first two elements, sorts them, then adds in the next element one-by-one, placing it in the correct index & shifting any other already-sorted elements to accomodate.

Termination: The inner for loop iterates until $j = i-1$, and i reaches a maximum of $len(a)$ in the outer for loop. Thus, once $i = len(a)$ and $j = len(a)-1$, the loops will both terminate & the program will return.

Recurrences

17. Solve the following recurrences.

(a) $c_0 = 1; c_n = c_{n-1} + 4$

Solution:

$C_1 = C_0 + 4 = 5$

$C_2 = C_1 + 4 = 9$

$C_3 = C_2 + 9 = 13$

$O(n)$

As n increases, C increases linearly.

(b) $d_0 = 4; d_n = 3 \cdot d_{n-1}$

Solution:

$d_0 = 4$

$d_1 = 12$

$d_2 = 48$

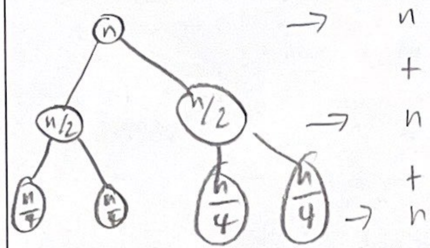
$d_3 = 144$

$O(3^n)$

As n increases, d multiplies by 3 with each increment

(c) $T(1) = 1; T(n) = 2T(n/2) + n$ (An upper bound is sufficient.)

Solution:



tree height = log n

$O(n \log n)$

(d) $f(1) = 1; f(n) = \sum_{1}^{n-1} (i \cdot f(i))$
(Hint: compute $f(n+1) - f(n)$ for $n > 1$)

Solution: For $n > 1$, $f(n+1) - f(n)$.

$f(n) = f(n-1) + (n-1)f(n-1)$ $= \sum_{1}^{n} i \cdot f(i) - \sum_{1}^{n-1} i \cdot f(i)$

$= f(n-1)(1 + n - 1)$

$= f(n-1)(n)$ $= n f(n)$

Incremental work done with each +1 increase in n.

$f(2) = 1(2) = 2$

$f(3) = 2(3) = 6$

$f(4) = 24$

$f(5) = 120$

$f(6) = 720$

$f(7) = 5040$

$O(n!)$

Coding Question

Most assignments will have a coding question. You can code in C, C++, C#, Java, Python, or Rust. You will submit a Makefile and a source code file.

Makefile: In the Makefile, there needs to be a build command and a run command. Below is a sample Makefile for a C++ program. You will find this Makefile in assignment details. Download the sample Makefile and edit it for your chosen programming language and code.

```
#Build commands to copy:
#Replace g++ -o HelloWorld HelloWord.cpp below with the appropriate command.
#Java:
#        javac source_file.java
#Python:
#        echo "Nothing to compile."
#C#:
#        mcs -out:exec_name source_file.cs
#C:
#        gcc -o exec_name source_file.c
#C++:
#        g++ -o exec_name source_file.cpp
#Rust:
#        rustc source_file.rs

build:
        g++ -o HelloWorld HelloWord.cpp

#Run commands to copy:
#Replace ./HelloWorld below with the appropriate command.
#Java:
#        java source_file
#Python 3:
#        python3 source_file.py
#C#:
#        mono exec_name
#C/C++:
#        ./exec_name
#Rust:
#        ./source_file

run:
        ./HelloWorld
```

HelloWorld Program Details The input will start with a positive integer, giving the number of instances that follow. For each instance, there will be a string. For each string s, the program should output Hello, s! on its own line.

A sample input is the following:

```
3
World
Marc
Owen
```

The output for the sample input should be the following:

```
Hello, World!
Hello, Marc!
Hello, Owen!
```