# Rejection Sampling

# Introduction

**Context:** Transformation methods for sampling cannot be applied

**Goal:** Sample from target distribution with density $\pi(x)$ where no direct sampling is feasible.

**Basic idea:** Sample from instrumental proposal $q \neq \pi$; correct through rejection step to obtain a sample from $\pi$.

**Requirements:** Given two densities $\pi, q$ with $\pi(x) \leq Mq(x)$ for all $x$, and some $M$ where we note that $M$ is larger than 1 since both $\pi$ and $q$ integrate to 1.

# Intuition and Algorithm

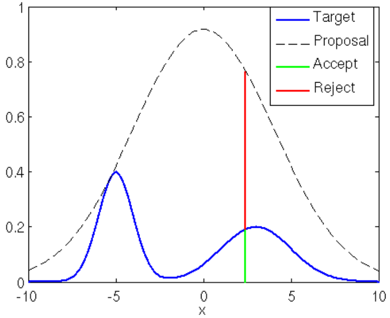
**Intuition:**

- Throw darts uniformly under $M \cdot q(x)$
- Keep only those under $\pi(x)$
- Kept points follow $\pi(x)$ exactly!

**The Algorithm:**

1. Sample $X \sim q(x)$
2. Sample $U \sim \text{Uniform}(0, 1)$
3. Accept if $U \leq \frac{\pi(X)}{M \cdot q(X)}$

**Why it works:** We're sampling uniformly from the area under $\pi(x)$!



# Mathematical Foundation

**Proposition**

*The distribution of accepted samples is exactly $\pi(x)$*

**Proof:** We need to show $P(X \in A | X \text{ accepted}) = \frac{P(X \in A, X \text{ accepted})}{P(X \text{ accepted})} = \pi(A)$.

$$\begin{aligned} P(X \in A, X \text{ accepted}) &= \int_X \int_0^1 \mathbb{I}_A(x) \cdot \mathbb{I}\left(u \leq \frac{\pi(x)}{Mq(x)}\right) q(x)\, du\, dx \\ &= \int_X \mathbb{I}_A(x) \cdot \frac{\pi(x)}{Mq(x)} \cdot q(x)\, dx \\ &= \int_X \mathbb{I}_A(x) \cdot \frac{\pi(x)}{M}\, dx = \frac{\pi(A)}{M} \end{aligned}$$

Similarly, $P(X \text{ accepted}) = \frac{1}{M}$. Therefore: $P(X \in A | X \text{ accepted}) = \frac{\pi(A)/M}{1/M} = \pi(A)$.

# Does this work for un-normalised distributions?

Often we only know $\pi$ and $q$ up to some normalising constants; i.e.

$$\tilde{\pi} = \frac{\pi}{Z_\pi} \quad \text{and} \quad \tilde{q} = \frac{q}{Z_q}$$

where $\pi$, $q$ are known but $Z_\pi$, $Z_q$ are unknown. We still need to be able to sample from $q(\cdot)$. If we can upper bound:

$$\frac{\tilde{\pi}(x)}{\tilde{q}(x)} \leq \widetilde{M},$$

then using $\tilde{\pi}$, $\tilde{q}$ and $\widetilde{M}$ in the algorithm is correct. Indeed we have

$$\frac{\tilde{\pi}(x)}{\tilde{q}(x)} \leq \widetilde{M} \iff \frac{\pi(x)}{q(x)} \leq \widetilde{M} \cdot \frac{Z_q}{Z_\pi} \stackrel{\text{def}}{=} M$$

**Waiting time:** Let $T$ denote the number of pairs $(X, U)$ that have to be generated until $X$ is accepted for the first time. $T$ is geometrically distributed with parameter $1/M$ and in particular $E(T) = M$.

This is why large $M$ is disastrous - it means you waste most of your computational effort on rejected samples!

**Dimensionality:** Assume $\pi$ and $q$ are Gaussian densities with same mean and $\sigma_q > \sigma_\pi$. Here $M = (\frac{\sigma_q}{\sigma_\pi})^d$ which grows with dimension. Say variance is 10% larger and $d = 100 : M = 1.1^{100} \approx 14,000$ (acceptance rate $\approx 0.007\%$)

**Extensions:** Squeezing techniques from exercises

Each trial (drawing $X$ from $q$ and $U$ from Uniform$[0, 1]$) is independent. Each trial has the same probability of success (acceptance). We stop at the first success. This is exactly the setup for a geometric distribution! **Why** $P($**accept**$) = 1/M$**?** From the previous proof, we showed: $P(X \text{ accepted}) = \frac{1}{M}$. So each trial succeeds (accepts) with probability $1/M$. **Why** $T \sim$ **Geometric**$(1/M)$**?** Think of it like coin flips: "Heads" = accept (probability $1/M$), "Tails" = reject (probability $1 - 1/M$). $T$ is the number of flips until the first heads. This is the definition of a geometric distribution! $P(T = k) = \left(1 - \frac{1}{M}\right)^{k-1} \cdot \frac{1}{M}$. We fail $(k-1)$ times, then succeed on the $k$-th try. **Why** $E(T) = M$**?** For a geometric distribution with success probability $p = 1/M$: $E(T) = \frac{1}{p} = \frac{1}{1/M} = M$.