1 Why Swap Moves Satisfy Detailed Balance

1.1 The Joint Distribution

Consider chains i and j with states x_i and x_j respectively. The joint distribution is:

$$\pi_{\text{joint}}(x_1, \ldots, x_N) = \prod_{k=1}^{N} \pi(x_k)^{\gamma_k} \tag{1}$$

Before swap, the joint probability is:

$$\pi_{\text{before}} = \pi(x_i)^{\gamma_i} \times \pi(x_j)^{\gamma_j} \times \prod_{k \neq i,j} \pi(x_k)^{\gamma_k} \tag{2}$$

After swap (exchanging x_i and x_j), the joint probability is:

$$\pi_{\text{after}} = \pi(x_j)^{\gamma_i} \times \pi(x_i)^{\gamma_j} \times \prod_{k \neq i,j} \pi(x_k)^{\gamma_k} \tag{3}$$

1.2 Detailed Balance Equation

For the swap move to preserve the joint distribution, we require:

$$\pi_{\text{before}} \times P(\text{swap } i \leftrightarrow j) = \pi_{\text{after}} \times P(\text{reverse swap } j \leftrightarrow i) \tag{4}$$

Explicitly:

$$\pi(x_i)^{\gamma_i}\pi(x_j)^{\gamma_j} \times \alpha_{\text{swap}}(i \leftrightarrow j) = \pi(x_j)^{\gamma_i}\pi(x_i)^{\gamma_j} \times \alpha_{\text{swap}}(j \leftrightarrow i) \tag{5}$$

1.3 The Metropolis-Hastings Acceptance Ratio

We choose the acceptance probability using the Metropolis-Hastings criterion:

$$\alpha_{\text{swap}} = \min\left(1, \frac{\pi_{\text{after}}}{\pi_{\text{before}}}\right) = \min\left(1, \frac{\pi(x_j)^{\gamma_i}\pi(x_i)^{\gamma_j}}{\pi(x_i)^{\gamma_i}\pi(x_j)^{\gamma_j}}\right) \tag{6}$$

This simplifies to:

$$\alpha_{\text{swap}} = \min\left(1, \left[\frac{\pi(x_j)}{\pi(x_i)}\right]^{\gamma_i - \gamma_j}\right) \tag{7}$$

1.4 Verification of Detailed Balance

Case 1: If $\pi_{\text{after}} > \pi_{\text{before}}$, then $\alpha_{\text{swap}} = 1$ and $\alpha_{\text{reverse}} = \frac{\pi_{\text{before}}}{\pi_{\text{after}}}$ Left side:

$$\pi_{\text{before}} \times 1 = \pi_{\text{before}} \tag{8}$$

Right side:

$$\pi_{\text{after}} \times \frac{\pi_{\text{before}}}{\pi_{\text{after}}} = \pi_{\text{before}} \quad \checkmark \tag{9}$$

Case 2: If $\pi_{\text{after}} < \pi_{\text{before}}$, then $\alpha_{\text{swap}} = \frac{\pi_{\text{after}}}{\pi_{\text{before}}}$ and $\alpha_{\text{reverse}} = 1$ Left side:

$$\pi_{\text{before}} \times \frac{\pi_{\text{after}}}{\pi_{\text{before}}} = \pi_{\text{after}} \tag{10}$$

Right side:

$$\pi_{\text{after}} \times 1 = \pi_{\text{after}} \quad \checkmark \tag{11}$$

1.5 Conclusion

In both cases, detailed balance holds. This ensures that the swap moves preserve the joint equilibrium distribution π_{joint}. Since the marginal distribution of chain N (where $\gamma_N = 1$) is:

$$\pi_{\text{marginal}}(x_N) \propto \pi(x_N)^{\gamma_N} = \pi(x_N) \tag{12}$$

chain N correctly samples from our target distribution $\pi(x)$.