Gibbs Sampling

Gibbs Sampling Algorithm

Algorithm 1 2D Gibbs Sampler with Systematic Scan

Require: Target conditionals $\pi(x|y)$ and $\pi(y|x)$, number of iterations N, initial values $(x^{(0)}, y^{(0)})$

Ensure: Sequence of samples $\{(x^{(t)}, y^{(t)})\}_{t=1}^{N}$

```
Initialize t ← 0
Set starting values (x^(0), y^(0))
for t = 1 to N do
    Sample x^(t) ~ π(x|y^(t−1))
    Sample y^(t) ~ π(y|x^(t))
    Store (x^(t), y^(t))
end for
return {(x^(t), y^(t))}_{t=1}^N
```

Transition Kernel of Gibbs Sampler

The transition kernel for a 2D systematic Gibbs sampler from state (x, y) to state (x', y') is:

$$K((x, y), (x', y')) = \pi(x'|y) \cdot \pi(y'|x')$$

This represents the composition of two steps:

1. Sample $x' \sim \pi(x|y)$ (transition from (x, y) to (x', y))
2. Sample $y' \sim \pi(y|x')$ (transition from (x', y) to (x', y'))

The kernel is the product of these conditional probabilities since the updates are performed sequentially within each iteration.

To show that π is invariant under K, we need to verify:

$$\pi(x', y') = \int\int \pi(x, y)\, K((x, y), (x', y'))\, dx\, dy \tag{1}$$

Substituting the kernel:

$$\int\int \pi(x, y)\, K((x, y), (x', y'))\, dx\, dy = \int\int \pi(x, y)\, \pi(x'|y)\, \pi(y'|x')\, dx\, dy \tag{2}$$

Using the fact that $\pi(x, y) = \pi(y)\pi(x|y)$:

$$= \int\int \pi(y)\, \pi(x|y)\, \pi(x'|y)\, \pi(y'|x')\, dx\, dy \tag{3}$$

$$= \int \pi(y)\, \pi(x'|y)\, \pi(y'|x') \left[\int \pi(x|y)\, dx\right] dy \tag{4}$$

$$= \int \pi(y)\, \pi(x'|y)\, \pi(y'|x')\, dy \tag{5}$$

Since $\pi(y)\pi(x'|y) = \pi(x', y) = \pi(x')\pi(y|x')$:

$$= \int \pi(x')\, \pi(y|x')\, \pi(y'|x')\, dy \tag{6}$$