Markov Chains

What is a Markov Chain?

Definition: A discrete-time process *Markov Chain* is a sequence of random variables $\{X_t\}_{t\geq 0}$ with the property that, given the present state, the future and past states are independent. Formally,

$$P(X_{t+1}|X_t, X_{t-1}, \ldots, X_0) = P(X_{t+1}|X_t).$$

The Markov Chain is **time-homogeneous** if the transition probabilities do not depend on time t:

$$\forall n \in N, \quad P(X_t = y|X_{t-1} = x) = P(X_{t+m} = y|X_{t+m-1} = x)$$

i.e. transition probabilities do not depend on t. The **key idea** is MCMC is to construct a Markov Chains such that x_t converges to a desired distribution π as $t \to \infty$ and

$$\frac{1}{n}\sum_{t=1}^{n}\phi(x_t) \to \mathbb{E}_{x\sim\pi}[\phi(x)] \quad \text{as } n \to \infty.$$

What kinds of conditions are required for this to hold?

Invariant / stationary distribution

A distribution π is called **invariant** (or **stationary**) for a Markov Chain with transition kernel P if

$$\pi(y) = \int \pi(x) P(x, y) dx.$$

Intuitively, if the chain starts with distribution π, it remains in distribution π at all future times.

Irreducible

A Markov Chain is called **irreducible** if it is possible to get to any state from any state. Formally, for any states x and y, there exists an integer $0 \leq n < \infty$ such that

$$P^n(x, y) > 0,$$

where $P^n(x, y)$ is the n-step transition probability from state x to state y.