# Importance Sampling

# What is Importance Sampling?

**What?**

- Monte Carlo technique for estimating $\mathbb{E}_\pi[\phi(X)]$
- Sample from proposal $q$ instead of target $\pi$
- Reweight samples to correct bias

**Why?**

- Target $\pi$ difficult to sample from
- Focus sampling in important regions
- Works with unnormalized distributions
- All samples are used (unlike rejection)

**How?** The key identity:

$$\begin{aligned}\mathbb{E}_\pi[\phi(X)] &= \int \phi(x)\pi(x)dx \\ &= \int \phi(x)\frac{\pi(x)}{q(x)}q(x)dx \\ &= \mathbb{E}_q[\phi(X)w(X)]\end{aligned}$$

**Algorithm**

1. Sample $X_1, \ldots, X_n \sim q$
2. Compute $w(X_i) = \pi(X_i)/q(X_i)$
3. Estimate: $\hat{I} = \frac{1}{n}\sum_{i=1}^n \phi(X_i)w(X_i)$

# Key Properties & Unnormalized Distributions

**Properties of IS Estimator:**

- **Unbiased**: $\mathbb{E}_q[\hat{I}] = \mathbb{E}_\pi[\phi(X)]$
- **Consistent**: $\hat{I} \xrightarrow{n\to\infty} \mathbb{E}_\pi[\phi(X)]$ (LLN)
- **Variance**: $\text{Var}_q[\hat{I}] = \frac{1}{n}\text{Var}_q[\phi(X)w(X)]$

**Requirements:**

- $q(x) > 0$ whenever $\pi(x)\phi(x) \neq 0$
- $\mathbb{E}_q[|\phi(X)w(X)|] < \infty$

**Unnormalized Distributions:**

When $\pi(x) = \tilde{\pi}(x)/Z$ with unknown $Z$:

**Self-Normalized IS**

- Weights: $\tilde{w}(x) = \tilde{\pi}(x)/q(x)$
- Estimator:

$$\hat{I}_{SN} = \frac{\sum_{i=1}^n \phi(X_i)\tilde{w}(X_i)}{\sum_{i=1}^n \tilde{w}(X_i)}$$

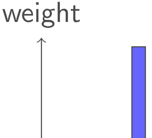
- Biased but **consistent**
- Bias: $O(1/n)$

# Why Weight Distribution Matters & ESS

**Weight Distribution Impact:**

- High weight variance $\Rightarrow$ poor estimates
- Few samples dominate the sum
- Ideal case: all weights equal $(q = \pi)$
- $\mathrm{Var}_q[w(X)]$ determines convergence

**Example Weight Degeneracy:**




**Effective Sample Size (ESS):**

**Definition**

$$\mathrm{ESS} = \frac{\left(\sum_{i=1}^{n} w_i\right)^2}{\sum_{i=1}^{n} w_i^2}$$

**Interpretation:**

- Number of "equivalent" samples from $\pi$
- Range: $1 \leq \mathrm{ESS} \leq n$
- $\mathrm{ESS} = n$ when all weights equal
- $\mathrm{ESS} = 1$ when one weight dominates

**Why ESS Matters:**

# Choosing Good Proposals & Dimensional Scaling

### Good Proposal Properties:

1. Heavier tails than $\pi$
2. Easy to sample from
3. Similar shape to $\pi|\phi|$
4. Covers support of $\pi$
5. Minimizes $\mathrm{Var}_q[\phi(X)w(X)]$

### Common Choices:

- Student-t for Gaussian targets
- Mixture distributions
- Previous MCMC output
- Laplace approximation

### Curse of Dimensionality:

**Gaussian Example**

For $\pi = \mathcal{N}(0, I_d)$, $q = \mathcal{N}(0, \sigma^2 I_d)$:

$$\mathrm{Var}_q[w(X)] = \left(\frac{\sigma^4}{2\sigma^2 - 1}\right)^{d/2} - 1$$

### Numerical Example:

| $d$ | $\sigma$ | $\mathrm{Var}_q[w(X)]$ |
|---|---|---|
| 10 | 1.2 | $\approx 5.6$ |
| 50 | 1.2 | $\approx 850$ |
| 100 | 1.2 | $\approx \mathbf{1.8 \times 10^4}$ |

# Importance Sampling vs. Rejection Sampling

| **Aspect** | **Importance Sampling** | **Rejection Sampling** |
|---|---|---|
| **Sample usage** | All samples (weighted) | Some samples rejected |
| **Efficiency** | Depends on weight variance | Depends on acceptance rate |
| **High dimensions** | Poor (variance explodes) | Very poor (accept rate $\to 0$) |
| **Proposal req.** | $q > 0$ where $\pi\phi \neq 0$ | Need $Mq \geq \pi$ everywhere |
| **Output** | Weighted samples | Exact samples from $\pi$ |
| **Normalizing const.** | Not required | Required (for bound $M$) |
| **Bias** | Unbiased (or consistent) | Unbiased (exact) |
| **Failure mode** | High variance | No samples produced |

## Key Insight

Both methods suffer from curse of dimensionality, but:

- IS degrades gracefully - still provides estimates (with high variance)

# Summary: Best Practices

**Strengths:**

- ✓ Handles complex distributions
- ✓ Unbiased estimates
- ✓ Works with unnormalized $\pi$
- ✓ All samples contribute
- ✓ Flexible proposal choice

**Limitations:**

- × Curse of dimensionality
- × Sensitive to proposal choice
- × Weight degeneracy issues
- × Can have infinite variance

**Best Practices:**

1. **Monitor ESS** regularly
2. **Check weight distribution** for outliers
3. **Use heavy-tailed proposals** (e.g., $t$-distribution)
4. **Consider adaptive IS** to improve $q$
5. **Multiple IS** for robustness
6. **Diagnostic plots**: weights, ESS over time

**Rule of Thumb**

If ESS $< n/10$, reconsider your proposal!