Rejection Sampling

why

Basic idea: Sample from instrumental proposal $q \neq \pi$; correct through rejection step to obtain a sample from π.
Given two densities π, q with $\pi(x) \leq Mq(x)$ for all x, and some $M > 0$, we can generate a sample from π by

What is Rejection Sampling?

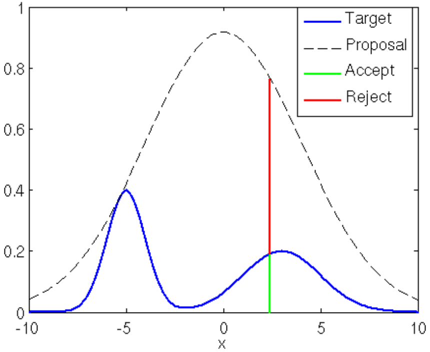
Intuition:

- Throw darts uniformly under $M \cdot q(x)$
- Keep only those under $\pi(x)$
- Kept points follow $\pi(x)$ exactly!

The Algorithm:

1. Sample $X \sim q(x)$
2. Sample $U \sim \text{Uniform}(0, 1)$
3. Accept if $U \leq \frac{\pi(X)}{M \cdot q(X)}$

Why it works: We're sampling uniformly from the area under $\pi(x)$!



Mathematical Foundation

Proposition

The distribution of accepted samples is exactly $\pi(x)$

Proof: We need to show $P(X \in A | X \text{ accepted}) = \pi(A)$. From the definition of conditional probability: $P(X \in A | X \text{ accepted}) = \frac{P(X \in A, X \text{ accepted})}{P(X \text{ accepted})}$.

$$
\begin{aligned}
P(X \in A, X \text{ accepted}) &= \int_X \int_0^1 \mathbb{I}_A(x) \cdot \mathbb{I}\left(u \leq \frac{\pi(x)}{Mq(x)}\right) q(x)\, du\, dx \\
&= \int_X \mathbb{I}_A(x) \cdot \frac{\pi(x)}{Mq(x)} \cdot q(x)\, dx \\
&= \int_X \mathbb{I}_A(x) \cdot \frac{\pi(x)}{M}\, dx = \frac{\pi(A)}{M}
\end{aligned}
$$

Similarly, $P(X \text{ accepted}) = \frac{1}{M}$. Therefore: $P(X \in A | X \text{ accepted}) = \frac{\pi(A)/M}{1/M} = \pi(A)$.

Does this work for un-normalised distributions?

Often we only know π and q up to some normalising constants; i.e.

$$\pi = \frac{\pi}{Z_\pi} \quad \text{and} \quad q = \frac{q}{Z_q}$$

where $\tilde{\pi}$, $\tilde{q}$ are known but Z_π, Z_q are unknown. You still need to be able to sample from $q(\cdot)$. If you can upper bound:

$$\frac{\tilde{\pi}(x)}{\tilde{q}(x)} \leq \tilde{M},$$

then using $\tilde{\pi}$, $\tilde{q}$ and $\tilde{M}$ in the algorithm is correct.
Indeed we have

$$\frac{\tilde{\pi}(x)}{\tilde{q}(x)} \leq \tilde{M} \iff \frac{\pi(x)}{q(x)} \leq \tilde{M} \cdot \frac{Z_q}{Z_\pi} \stackrel{\text{def}}{=} M$$

- forudsætninger og konstant M
- waiting time to accepted sample is geometric with mean M
- M storre end 1
- find M kan være svært
- kan være ineffektiv hvis M er stor
- dimensionalitet
- squeezing