Assignment 2
due Feb 3, 2014

Exercise 1 (50 points)**.**
Recall that a set S of logical operators is called a *functionally complete set* if every compound proposition is logically equivalent to a compound proposition that

- involves only logical operators from S, propositional variables, and parantheses
- and uses none of the constants `true` or `false`.

Consider the boolean operator NOR that has the following truth table.

p	q	p NOR q
T	T	F
T	F	F
F	T	F
F	F	T

Prove that NOR is functionally complete on its own, i.e., the set S that contains only the single logical operator NOR is a functionally complete set.

The following exercises in section 1.3 of the textbook can guide you: 42, 43, 45, 48, 49, 50.

Solution to Exercise 1.
Since the collection $\{\vee, \wedge, \neg\}$ is functionally complete it suffices to express these three operators in terms of NOR. This can be done as follows: $\neg p \equiv p$ NOR p, which can be readiliy checked with a truth table. $p \vee q \equiv \neg(p$ NOR $q)$, and we already know that we can replace every $\neg$ with a NOR-construction, so this implies that we can also replace every $\vee$ with a NOR-construction as well. We can get rid of $\wedge$ in our formulas using De Morgan as follows: $p \wedge q \equiv \neg(\neg p \vee \neg q)$ and we are done.

Exercise 2 (25 points)**.**
The situation is as in the previous exercise, but we use $\vee$ ("OR") instead of NOR. Prove that $\vee$ is not functionally complete on its own.

Solution to Exercise 2.
Since $\vee$ is associative, all formulas you can build using $\vee$ and parantheses are large disjunctions. Hence for every formula, at least half of the entries of its truth table have truth value T. In particular, $p \wedge q$ (which has only a single truth value T) cannot be formed using only $\vee$ and parantheses.

Exercise 3 (25 points).
The compound proposition

$$\neg((p \vee \neg s) \wedge (q \vee \neg r \vee \neg s) \wedge \neg(t \wedge s)) \wedge (p \vee \neg p)$$

can be thought of as a list of 33 characters from the following table.

$\neg$	$\vee$	$\wedge$	p	q	r	s	t	(	)

Write down a logically equivalent compound proposition using at most 14 characters of this table and prove the equivalence.

Operator precedence is important for this exercise. Note that by operator precedence we have $p \vee q \wedge r \equiv p \vee (q \wedge r)$, where the left hand side has 5 characters and the right hand side has 7.

Solution to Exercise 3.

$$\begin{aligned}
& \neg((p \vee \neg s) \wedge (q \vee \neg r \vee \neg s) \wedge \neg(t \wedge s)) \wedge (p \vee \neg p) \\
\overset{(p\vee\neg p\equiv T)}{\equiv}\; & \neg((p \vee \neg s) \wedge (q \vee \neg r \vee \neg s) \wedge \neg(t \wedge s)) \wedge T \\
\overset{(A\wedge T\equiv A)}{\equiv}\; & \neg((p \vee \neg s) \wedge (q \vee \neg r \vee \neg s) \wedge \neg(t \wedge s)) \\
\overset{\text{De Morgan}}{\equiv}\; & \neg(p \vee \neg s) \vee \neg(q \vee \neg r \vee \neg s) \vee (t \wedge s) \\
\overset{\text{De Morgan}}{\equiv}\; & (\neg p \wedge s) \vee (\neg q \wedge r \wedge s) \vee (t \wedge s) \\
\overset{\text{Distr.}}{\equiv}\; & (\neg p \vee (\neg q \wedge r) \vee t) \wedge s \\
\overset{\text{Op. precedence}}{\equiv}\; & (\neg p \vee \neg q \wedge r \vee t) \wedge s
\end{aligned}$$
