Assignment 1
due Jan 27, 2014

- Learn and understand what we covered in class.

- To check if you understood the material, do some exercises from the textbook that have solutions provided in the back of the book.

- Come to the office hours to ask questions. Also ask your fellow students for help and discuss the material.

On this assignment sheet we are only interested in nonnegative functions, so we use the following simplified version of the big-O notation. We define that $x^2 + 16 \in O(x^2)$ if there exists a pair or nonnegative real numbers (C, k) such that

- for all real numbers $x \geq k$ we have that $x^2 + 16 \leq C \cdot x^2$.

Such a pair is called a *witness pair to* $x^2 + 16 \in O(x^2)$. On this assignment sheet, witness pairs are always restricted to have nonnegative C and nonnegative k.

Exercise 1 (10 points).
Prove that $x^2 + 16 \in O(x^2)$.

Solution to Exercise 1.
For example, we can just use Theorem 1 of Section 3.2. But it is also possible to provide a witness pair. This is possible due to Proposition A from the lecture. □

Exercise 2 (20 points).
Prove that there exist two different witness pairs to $x^2 + 16 \in O(x^2)$.

Solution to Exercise 2.
Take the witness pair from Exercise 1 and increase C by 1 to obtain a second witness pair. □

Exercise 3 (35 points).
Two pairs (C, k) and (D, ℓ) of nonnegative real numbers are called *incomparable*, if one of the following two conditions holds:

- $C > D$ and $k < \ell$
  or
- $C < D$ and $k > \ell$.

Prove that there exist two incomparable witness pairs to $x^2 + 16 \in O(x^2)$.

Solution to Exercise 3.
(5,2) is a witness pair, as is checked as follows: For $x > 2$ we have

$$x^2 + 16 < x^2 + 4x^2 = 5x^2.$$

(17,1) is a witness pair, as is checked as follows: For $x > 1$ we have

$$x^2 + 16 < x^2 + 16x^2 = 17x^2.$$

The witness pairs are incomparable.   □

Exercise 4 (35 points)**.**
Draw a graphical representation of the set of all witness pairs (C, k) to the relationship $x^2 + 16 \in O(x^2)$. One axis should be the C-axis and one axis should be the k-axis. Explain your approach.

Solution to Exercise 4.
Let (C, k) be a witness pair to $x^2 + 16 \in O(x^2)$. This means that

- for all $x > k$ we have $x^2 + 16 \le Cx^2$.

Note that if $C \le 1$, this cannot be true for any nonnegative k, so we conclude $C > 1$.

Every witness pair must satisfy $k^2 + 16 \le C \cdot k^2$. Moreover, assuming $C > 1$, if the pair (C, k) satisfies $k^2 + 16 \le C \cdot k^2$, then it also satisfies $x^2 + 16 \le C \cdot x^2$ for $x > k$ and hence is a witness pair. These calculations show that witness pairs are exactly those (C, k) with $C > 1$ that satisfy $k^2 + 16 \le C \cdot k^2$. To draw this set we first draw the set described by $k^2 + 16 = C \cdot k^2$. We calculate

$$k^2 + 16 = C \cdot k^2 \Leftrightarrow 16 = (C - 1) \cdot k^2 \Leftrightarrow \frac{16}{C - 1} = k^2 \Leftrightarrow k = \sqrt{\frac{16}{C - 1}} = 4\frac{1}{\sqrt{C - 1}}.$$

Now we can easily plot the function (pointing to the right is C and pointing upwards is k):
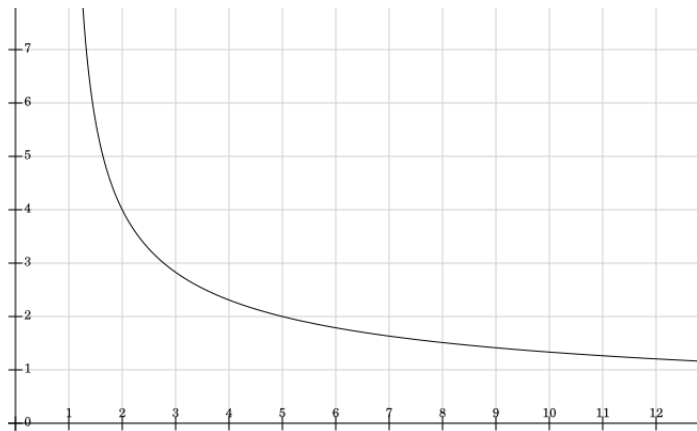
The witness pairs are all points in the upper right area.