Assignment 3
due Feb 10, 2014

Exercise 1 (30 points).
We define the proposition p as follows.

$$p \quad := \quad \forall x > 0\ \exists y\ (x + y = 0 \land \forall z(x + z = 0 \to y = z)),$$

where the domain for all variables is the set of all integers. Write down the negation $\neg p$ such that no negation symbol $\neg$ appears. To do so, you can make use of the symbol $\neq$.

Solution to Exercise 1.

$$\begin{aligned}
\neg p &= \neg \forall x > 0 \exists y(x + y = 0 \land \forall z(x + z = 0 \to y = z)) \\
&= \exists x > 0 \neg \exists y(x + y = 0 \land \forall z(x + z = 0 \to y = z)) \\
&= \exists x > 0 \forall y \neg (x + y = 0 \land \forall z(x + z = 0 \to y = z)) \\
&= \exists x > 0 \forall y(x + y \neq 0 \lor \neg \forall z(x + z = 0 \to y = z)) \\
&= \exists x > 0 \forall y(x + y \neq 0 \lor \exists z \neg (x + z = 0 \to y = z)) \\
&= \exists x > 0 \forall y(x + y \neq 0 \lor \exists z(x + z = 0 \land \neg y = z)) \\
&= \exists x > 0 \forall y(x + y \neq 0 \lor \exists z(x + z = 0 \land y \neq z))
\end{aligned}$$

Exercise 2 (35 points).
Prove or disprove the truth of p from Exercise 1.

Solution to Exercise 2.
Proof: Let x be arbitrary. Choose y to be $-x$. If follows $x + y = x + (-x) = 0$.

It remains to show that $\forall z(x + z = 0 \to y = z)$. Let z be arbitrary. If $x + z \neq 0$, then the implication is true, so let $x + z = 0$. We have to show that $y = z$, i.e., that $-x = z$. But this follows from the equation $x + z = 0$ by substraction x on both sides.

Exercise 3 (35 points).
Translate the following mathematical statement into the language of predicate calculus using the predicate $P(x)$ for the statement "x is a prime number".

At least one of two distinct prime numbers is always ≥ 3.

Solution to Exercise 3.
First we rephrase the statement as

For all pairs of distinct prime numbers, at least one of them is ≥ 3.

and then translate

$$\forall x, y : (P(x) \land P(y) \land x \neq y \to x \geq 3 \lor y \geq 3).$$
