Introduction to Quantum Computing

Exercise Sheet 0 (not to be handed over)

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Exercise 1. Express the following complex numbers in standard algebraic form $a + bi$ (suggestion: to simplify a number of the form $\frac{a+bi}{c+di}$, multiply it by $1 = \frac{c-di}{c-di}$). Then compute their absolute value:

1. $1 + i - \frac{i}{1-5i}$;
2. $(2 + i)(2 - i)(1 + \sqrt{3}i)$;
3. $\left(\frac{1+i}{1-i} - 1\right)^2$.

Exercise 2. Express the following complex numbers in trigonometric and exponential forms:

1. $z = 1 + i$;
2. $z = \frac{1}{3+3i}$.

Express the complex number $z = e^{-i\frac{\pi}{4}} + e^{-i\frac{\pi}{3}}$ in algebraic form.

Exercise 3. Compute the three 3rd roots of unity and check that they satisfy $z^3 = 1$.

Exercise 4. Show that for an arbitrary vector $|\psi\rangle \in \mathbb{C}^2$ and for any real number θ,

$$\|e^{i\theta}|\psi\rangle\| = \||\psi\rangle\|.$$

Exercise 5. Express the vectors of $\mathbb{C}^8$ corresponding to the following expressions:

$$|010\rangle \qquad |110\rangle \qquad |011\rangle.$$

Exercise 6. Consider the following two vectors

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle - 2i|1\rangle) \qquad\qquad |\phi\rangle = i|0\rangle + \frac{\pi}{6}|1\rangle.$$

Compute:

- the complex number $\langle \psi \mid \phi \rangle$;
- the vector $|\psi\rangle \otimes |\phi\rangle \in \mathbb{C}^4$;
- the matrix $|\psi\rangle\langle\phi| \in \mathbb{C}^4$.

Exercise 7. Express the vector

$$|\psi\rangle = \frac{3}{5}|0\rangle + 2i|1\rangle$$

as a linear combination of the vectors of the Hadamard basis $|+\rangle, |-\rangle$.

Exercise 8. Consider the following vector:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

Show that there exist no complex numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$|\psi\rangle = (\alpha_1|0\rangle + \alpha_2|1\rangle) \otimes (\beta_1|0\rangle + \beta_2|1\rangle).$$

Exercise 9. Decide if the following vectors can be decomposed via tensors, or if they are entangled.

$$\frac{1}{\sqrt{2}}(|01\rangle + |00\rangle) \qquad \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad \frac{1}{\sqrt{3}}(|011\rangle + |000\rangle + |101\rangle).$$

Exercise 10. Consider the following matrices

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Are they unitary? Hermitian? Projections?

For each of the two matrices, find an orthonormal basis composed of eigenvectors, as well as the associated eigenvalues.