Introduction to Quantum Computing

Exercise Sheet 3

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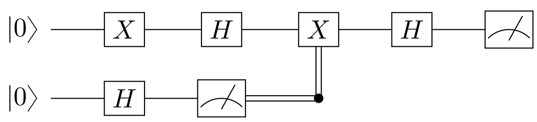
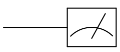
Exercise 1. One day Alice shows up carrying a closed box and proposes Bob a game. Alice explains the rules to Bob as follows: the box contains a penny, which may show heads (H) or tails (T); although it is possible to flip the penny inside the box from the outside, the box is closed and its content is not visible; Bob has to flip another penny and, depending on whether it shows H or T, he will flip or not flip the penny inside the closed box. After that Alice will have a last chance to flip the penny in the box, again without being able to see it. Finally, the box will be opened and the penny inside revealed. Alice wins if it shows heads (H), while Bob wins if it shows tails (T).

What Alice has not told Bob is that the closed box is a quantum box, and that Alice has prepared the penny in the box in a superposition of the states H and T. Taking $|0\rangle$ as a representation of the state T and $|1\rangle$ as a representation of the state H, the whole game may be represented by the circuit below: the first wire represents the penny inside the box, which is prepared by Alice in a superposition state via the X and Hadamard gates, is then conditionally flipped by Bob, then again flipped by Alice via a second Hadamard gate, and finally measured. The second wire represents the penny that is flipped by Bob, conditioning the value of the penny in the box.



[The symbol  *stands for measurement.]*

What are the percentages at which Alice and Bob may win this game? Motivate your answer by describing the behavior of the circuit above.

Exercise 2. Given the definition of the Quantum Fourier Transform as

$$\mathrm{QFT}_{2^n}(|x\rangle) = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i \frac{x}{2^n} y} |y\rangle,$$

write explicitly the matrix of QFT_{2^n} and $\mathrm{QFT}_{2^n}^{-1}$ for the cases $n = 1, n = 2, n = 3$, and explain how you did it.

Exercise 3. Let U be the 1-qubit unitary operator

$$U = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and consider the controlled-U 2-qubit operator $(c - U)$.

Show that $(c-U)$ has eigenvalues $\{i, -i\}$ and find two eigenvectors $|\psi_1\rangle, |\psi_2\rangle$ with corresponding eigenvalues $i, -i$, respectively. Then, construct the appropriate eigenvalue estimation circuit and show that, over the input states $|\psi_1\rangle, |\psi_2\rangle$, it correctly predicts the associated eigenvalues.