1. (5 points) Transform the following propositional logic formula into an equivalent formula in Conjunctive Normal Form:

$$((A \lor \neg B) \land (\neg C \lor D)) \lor ((B \land C) \rightarrow (A \lor \neg D))$$

$$((A \lor \neg B) \land (\neg C \lor D)) \lor (\neg(B \land C) \lor (A \lor \neg D))$$

$$((A \lor \neg B) \land \neg C) \lor ((A \lor \neg B) \land D) \lor \neg B \lor \neg C \lor A \lor \neg D$$

$$\underbrace{\neg C \lor (\neg C \land (A \lor \neg B))}_{\neg C} \lor \underbrace{\neg D \lor (D \land (A \lor \neg B))} \lor \neg B \lor A$$

$$\underbrace{\underbrace{\neg D \lor D}_{T} \land (\neg D \lor A \lor \neg B)}_{\neg D \lor A \lor \neg B}$$

$$(\neg C \lor \neg D \lor A \lor \neg B)$$

2. (5 points) Consider the following statement:

"If the user is authenticated, then they have entered the correct password or they have used biometric authentication. If the user is not authenticated, then they do not have access to the system. If the user has access to the system, the firewall is active. Either the firewall is not active, or the system is secure. If the system is not secure, the user is not authenticated. The system is secure."

Formalise this statement and determine (with truth tables or otherwise) whether it is consistent. Use numbers to enumerate each statement, in doing so follow text order.
Remind: "either or" should always be considered as exclusive.
Remind: if something is not clear, write your interpretation on the right side

U: USER AUTHENTICATED
P: ENTERED CORRECT PASSWORD
B: USED BIOMETRIC AUTHENTICATION
A: access to THE SYSTEM
F: FIREWALL ACTIVE
S: SYSTEM SECURE

1) $U \rightarrow (P \lor B)$

2) $\neg U \rightarrow \neg A$

2) $\neg U \to \neg A$

3) $A \to F$

4) $(\neg F \vee S) \wedge \neg(\neg F \wedge S)$

5) $\neg S \to \neg U$

6) S

CONSISTENCY

6) $S = T$ CASE 1 S:T

5) $\bot \to \neg U$ ALWAYS TRUE CASE 1 S:T U:?

4) $\neg F$ XOR $\underbrace{S}_{T}$ SO $\neg F$ MUST BE $\bot$ SO F IS T

CASE 1 S:T F:T U:?

3) $A \to T$ SO A MUST BE T

CASE 1 S:T F:T A:T U:?

2) $\neg U \to \underbrace{\neg A}_{\bot}$ SO $\neg U$ MUST BE $\bot$ SO U IS T

CASE 1 S:T F:T A:T U:T

1) $\underbrace{U}_{T} \to (P \vee B)$

SOLUTION

CASE 1 S:T F:T A:T U:T P:T B:?

CASE 2 S:T F:T A:T U:T P:? B:T

3. (5 points) Say whether the following FOL formulas are valid or not, justifying your answer:

(a) $(\exists x\ x = x) \to (\forall y\ \exists z\ y = z)$
(b) $\forall x P(x) \wedge \forall x \neg P(x)$
(c) $(\exists x\ P(x)) \wedge (\exists x\ Q(x)) \to (\exists x\ (P(x) \wedge Q(x)))$

a) $\exists x \; x = x$ always true

$\forall y \exists z \; y = z$ always TRUE BECAUSE z CAN BE $z = y$

b) AN OBJECT IN THE DOMAIN CANNOT HAVE $P(x)$ AND $\neg P(x)$ AT THE SAME TIME SO IT IS INVALID

c) NOT VALID EXEMPLE

KNOWLEDGE BASE

$P(a) \leftarrow \exists x \, P(x) \quad x = a$

$Q(b) \leftarrow \exists x \, Q(x) \quad x = b$

but DOES NOT EXIST $\exists x \, (P(x) \wedge Q(x))$

4. (5 points) Given the following FOL formulas A and B:

$$A : \forall x \, (\forall y \, P(x, y)) \rightarrow Q(x)).$$

$$B : \forall x \, \exists y \, (P(x, y) \rightarrow Q(x))$$

(a) Express their meaning in natural language.

(b) Are the logically equivalent? Motivate the answer. (Hint: consider carefully the parenthesis and therefore the scope of quantifiers).

A: FOR EVERY X IF $P(X, Y)$ IS TRUE FOR ALL POSSIBLE Y THEN $Q(X)$ MUST BE TRUE

B FOR EVERY X EXIST A Y THAT MAKES $P(X, Y)$ TRUE THEN $Q(X)$ MUST BE TRUE

The Logical Proof:

1. **Start with A:** $\forall x((\forall y P(x, y)) \rightarrow Q(x))$
2. **Rewrite the implication:** Recall that $p \rightarrow q$ is equivalent to $\neg p \vee q$.

$$\forall x(\neg(\forall y P(x, y)) \vee Q(x))$$

3. **Apply De Morgan's for quantifiers:** $\neg \forall y$ becomes $\exists y \neg$.

$$\forall x((\exists y \neg P(x, y)) \vee Q(x))$$

REMEMBER

$\neg \forall_x F \longleftrightarrow \exists_x \neg F$

$\neg \exists_x A \longleftrightarrow \forall_x \neg A$

The Logical Proof:

1. **Start with A:** $\forall x((\forall y P(x,y)) \to Q(x))$
2. **Rewrite the implication:** Recall that $p \to q$ is equivalent to $\neg p \lor q$.

$$\forall x(\neg(\forall y P(x,y)) \lor Q(x))$$

3. **Apply De Morgan's for quantifiers:** $\neg\forall y$ becomes $\exists y \neg$.

$$\forall x((\exists y \neg P(x,y)) \lor Q(x))$$

4. **Move the existential quantifier out:** Since y does not appear in $Q(x)$, we can pull the $\exists y$ to the front of the parenthesis.

$$\forall x \exists y(\neg P(x,y) \lor Q(x))$$

5. **Convert back to implication:** Using the $p \to q$ rule again.

$$\forall x \exists y(P(x,y) \to Q(x))$$

6. **Result:** This is exactly **Formula B**.

REMEMBER

$\neg \forall_x F \longleftrightarrow \exists_x \neg F$

$\neg \exists_x A \longleftrightarrow \forall_x \neg A$

5. (6 points) Consider the following Prolog program:

```
q(a, X, Y) :- q(b, X, Y), r(Y).
q(b, X, f(Y)) :- p(X), r(Y), s(Y).
q(a, b, Z) :- r(Z), s(Z).
q(c, X, g(Y)) :- q(a, X, Y), p(Y).

p(c).
p(d).
p(e).

r(f(a)).
r(g(b)).
r(h(c)).

s(g(b)).
s(h(c)).
```

What is the result of evaluating the query:

```
q(a, c, W).
```

Provide all possible solutions for W, if any. Motivate the answer.

:- q(a, c, W)

MATCH q(a, X, Y)

:- q(b, X, Y), r(Y)

X = c Y = W

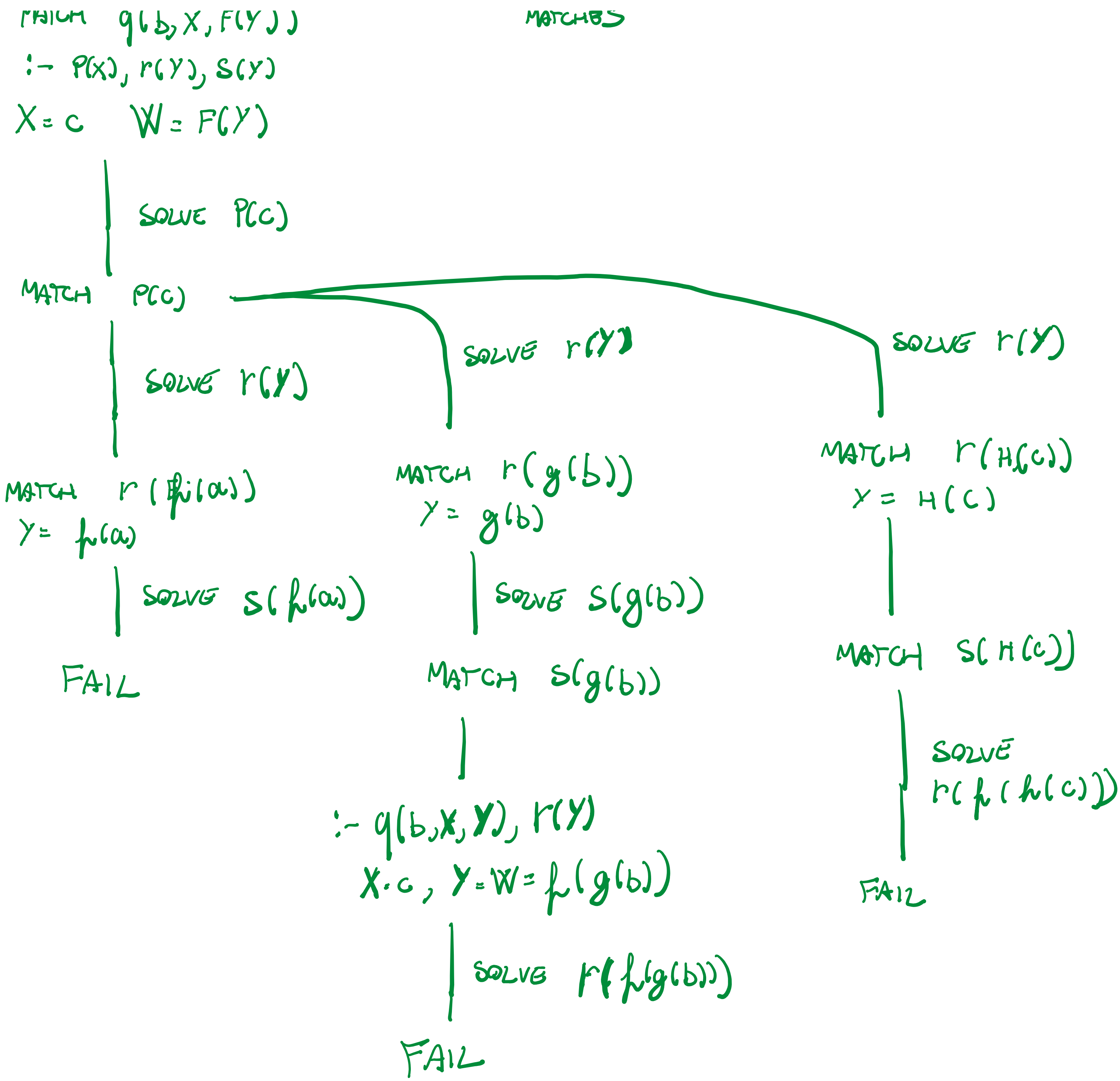
NO OTHER POSSIBLE MATCHES

SOLVE q(b, c, W)

NO OTHER POSSIBLE MATCHES

MATCH q(b, X, F(Y))

:- P(X), r(Y), S(Y)



6. (6 points) Four colleagues - Alice, Bob, Charlie, and Dana - share an €800 office rent but have specific financial constraints:

(a) Alice can only contribute €100, €200, or €300, but she must pay more than Charlie.
(b) Bob must pay exactly €50 more than Charlie and an even amount.
(c) Charlie refuses to pay more than €250, but his contribution must be a multiple of 50.
(d) Dana must pay at least €150, but no more than €300, and she must pay more than Bob.
(e) The total rent must sum to €800, and no one can pay more than €300.

```
VAR { 400, 200, 300 } : ALICE
VAR [0...300] : BOB
VAR [0...250] : CHARLIE
VAR [450..300] INT : DANA
VAR INT : TOTAL = ALICE + BOB + CHARLIE + DIANA
CONSTRAINT ALICE > CHARLIE
CONSTRAINT BOB = CHARLIE + 50
CONSTRAINT (BOB MOD 2) = 0
CONSTRAINT (CHARLIE MOD 50) = 0
CONSTRAINT DIANA > BOB
CONSTRAINT TOTAL = 800

SOLVE SATISFY
```