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Intuitionistic Logic

An introduction to intuitionistic logic and its connections to modal logic, topology and theoretical computer science

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Contents

Part I

Preliminaries

Chapter 1
Overview

Intuitionistic logic arose from BrouwerâĂŹs effort to provide a new, constructive foundation for mathematics, and was developed by Heyting into a formally precise alternative to classical logic. In the second half of the 20th Century, this logic has come under attention again due to its strong ties to theoretical computer science.
This course is an introduction to intuitionistic logic. We will cover in depth the features of intuitionistic propositional logic and its connection with classical and modal logic, and introduce intuitionistic predicate logic. We will discuss the connections of intuitionistic logic to typed programming languages, introducing the proofs-as-programs correspondence discovered by Curry and Howard.
This is not a course on constructivism in philosophy of mathematics or a course on constructive mathematics.

1.1 How intuitionistic logic differs from classical logic

To start with, let us look at two notable differences between classical and intuitionistic logic:

Truth In classical logic, truth is understood as *correspondence*: a statement p is true if it corresponds to some external state of affairs. Brouwer thought that this picture is inaccurate with respect to mathematics. According to his philosophy of mathematics – called 'intuitionism' – the mathematician does not *describe* some external Platonic realm of mathematical objects but rather *constructs* these objects in their mind. Accordingly, in intuitionistic logic, truth is understood as *theoremhood*: a statement p is true if a proof for p has been constructed.
One prominent consequence of these different conceptions of truth is the following:

Excluded middle In classical logic, any statement p either obtains or does not obtain, because states of affairs are *complete*. As such, the law of exluded middle is valid in

	Classical logic	Intuitionistic logic
Truth	'truth is correspondence'	'truth is provability'
Excluded middle	$\vDash p \vee \neg p$	$\nvDash p \vee \neg p$

Table 1.1: *Two prominent differences between classical and intuitionistic logic*

classical logic. By contrast, if truth is understood as provability, then it is possible that there is some statement p that has, up to this point in the mathematicians mental life, neither been proved nor refuted. Consequently, in intuitionistic logic, the law of excluded middle is invalid.

1.2 Survey of topics to which intuitionistic logic is related

Let us briefly look at how intuitionistic logic relates to other topics in logic, philosophy, mathematics and theoretical computer science. This will give us a taste of just how natural and interesting the structure of intuitionistic logic really is—it shows up in a lot of different places! Topics that are discussed in detail in this course are marked with an asterix (*).

Philosophy

Constructivism In the philosophy of mathematics, constructivism involves the claim that one needs to 'find' or 'construct' a mathematical object in order to prove that it exists, or to show that it has a certain property. By contrast, in classical mathematics, one can prove the existence of a mathematical object indirectly by assuming its non-existence and then deriving a contradiction from that assumption, i.e. without 'finding' that object explicitly. Such a proof by contradiction is non-constructive and therefore deemed problematic by constructivists. To illustrate this point further, consider the statements $\exists x Px$ and $\neg\forall x \neg Px$. In classical logic, these statements are equivalent. For this reason, one can prove the former simply by proving the latter (and vice versa). But let us look more closely at these two statements and how one would prove them (directly). To prove $\exists x Px$, we would 'find' an object that has the property P. Such a proof would thus do *more* than just establish that $\exists x Px$—it would also give us an object that *witnesses* this statement. Such a proof would thus be *constructive*. By contrast, to prove $\neg\forall x \neg Px$, we would start out by assuming $\forall x \neg Px$, i.e. that there is no object

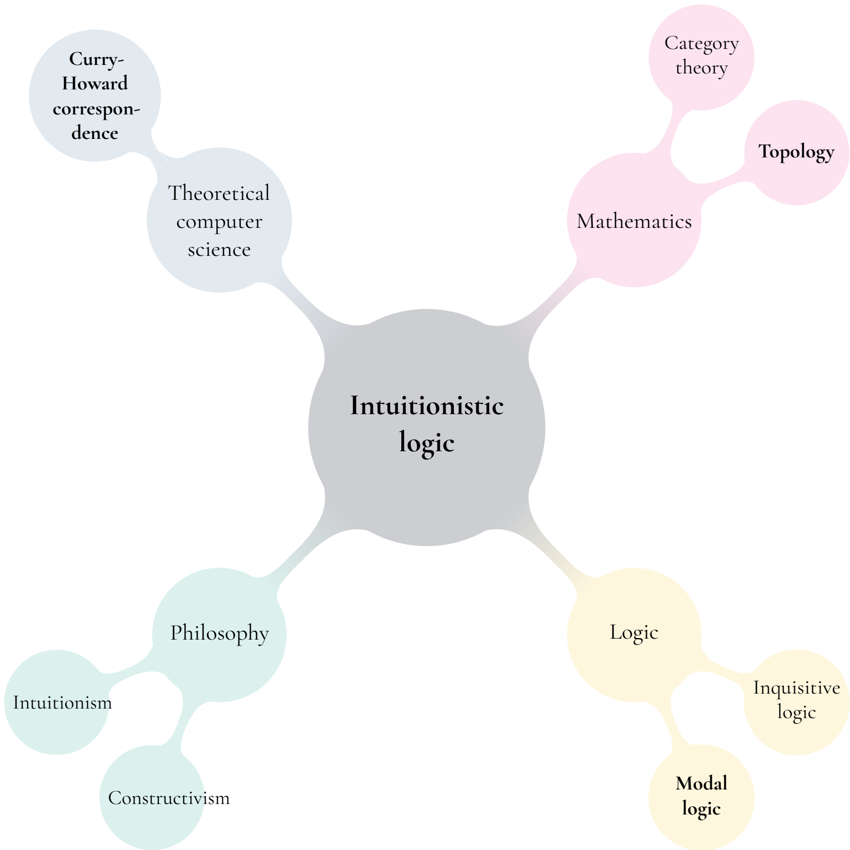


Figure 1.1: *How intuitionistic logic relates to topics in philosophy, mathematics and theoretical computer science. The topics that are discussed in detail in this course are marked printed in* ***bold****.*

that has property P, and then derive a contradiction to infer $\neg\forall x \neg Px$. Note that such a proof would *not* give us an object that witnesses the statement. As such, such a proof would be *non-constructive*. Thus, we can see that these statements are different with respect to their constructiveness. Classical logic glosses over this difference, but intuitionistic logic does not: intuitionistic logic makes this difference explicit at the level of formulas and is, in this sense, more nuanced. We can thus use intuitionistic logic as a tool to better *understand* such nuances in mathematics—it need not be intended to *replace* classical logic.

Intuitionism Intuitionism is a particular constructivist philosophy of mathematics according to which mathematics is the result of the constructive activity of some mind rather than the discovery of fundamental principles that exist in an objective reality. Intuitionism locates the foundations of mathematics in the individual mathematician's intuition, thereby turning mathematics into an intrinsically subjective activity. Note that constructivism is often identified with intuitionism, but intuitionism is only one type of constructivist program. As we will see in the history section below, Brouwer's intuitionism provided the philosophical motivation for intuitionistic logic.

Theoretical computer science

Curry-Howard correspondence (*) Constructive content is essentially algorithmic content. The Curry-Howard correspondence testifies to an exact match between intuitionistic logic on the one hand and type theories, which range from simple models of computation to essentially programming languages, on the other. In particular, it establishes that *propositions/formulas* correspond to *types*, *proofs* of formulas correspond to *objects/programs*, and *normalisation* of proof corresponds to *computation*:

Intuitionistic logic		**Type theories**
proposition/formula	$\leftrightarrow$	types
proof of a proposition	$\leftrightarrow$	objects/programs
normalisation of proofs	$\leftrightarrow$	computation

It turns out that the same mathematical structure is characterised by intuitionistic logic and by type theories; we can view it through the lens of logic or through the lens of computation. There exists a 1–1 correspondence: every theorem of intuitionistic logic corresponds exactly to a theorem on the computation side. Given this correspondence, a proof can be seen as an algorithm that, given a certain input, yields an output satisfying the property in question.

Logic

Modal logic (*) We can embed intuitionistic logic inside classical modal logic and thereby render the intuitionistic understanding of provability explicit in a classical setting. To do so, we will use a translation from the language of intuitionistic logic into the language of modal logic:

$$p \rightsquigarrow \Box p$$
$$\neg p \rightsquigarrow \Box \neg p$$
$$p \lor \neg p \rightsquigarrow \Box p \lor \Box \neg p$$

where '$\Box p$' can be read as 'it is provable that p'. Among other things, the classical perspective highlights why the law of excluded middle is not intuitionistically valid: the disjuncts are essentially modalised.

Inquisitive logic The basic idea behind inquisitive logic is to extend classical logic to deal with *questions* as well as declarative statements. For example, while classical logic allows us to formally state that q is a logical consequence of $p \vee q$ and $\neg p$:

$$p \vee q, \neg p \vDash q,$$

inquisitive logic allows us to formally state that if $p \leftrightarrow q$, then answering the question whether p also answers the question whether q:

$$p \leftrightarrow q, ?p \vDash ?q.$$

The logic of questions is very similar to, but not exactly the same as, intuitionistic logic.

Mathematics

Topology (*) Topology studies properties of space at a very abstract level. Formally, a topology is a certain family of sets. It turns out that the relation between intuitionistic logic and the notion of a topology on the one hand is the same relation as that between classical logic and the notion of a powerset on the other (note that a powerset is also a certain family of sets):

$$\text{Intuitionistic logic} : \text{topology} = \text{Classical logic} : \text{powerset}$$

The same algebraic structure is characterised.

Category theory Category theory allows us to reconstruct logic within the framework of categories. A part of category theory corresponds to intuitionistic logic.

1.3 History

Intuitionistic logic started out rather informally and was motivated by philosophical concerns, and, over time, it developed into a formal structure that could be described and studied by mathematical means.

Intuitionism (Brouwer 1907, 1920-30) The Dutch mathematician Brouwer finished his PhD dissertation in 1907 on a 'standard' mathematical topic (in it, Brouwer proved an important fixed-point theorem in topology). More than a decade later, he returned to constructive mathematics. According to his intuitionism, mathematical objects should be seen—contra Platonism—as mental constructions, and—contra formalism—propositions have content; it is not the case that they can be investigated with respect to their consistency only relative to axioms. Proofs should be seen as constructions that make statements true.

Program to reform mathematics (Brouwer 1920s) Brouwer thus embarked on a program to reform mathematics. Brouwer never explicitly articulated a system of intuitionistic logic. Brouwer thought that mathematics should be understood in terms of the mental life of a single mathematician and hence there was no significant role for communication or, by extension, for language on his picture. Therefore, there was also not much of a role for logic. Proofs are psychological objects that can be described in language, but they are not themselves linguistic in nature. Similarly, logic is not foundational—it is merely a post-hoc reconstruction of mathematics. However, his informal interpretation implicitly laid the groundwork for intuitionistic logic. In the end, Brouwer's program to reform mathematics fell on deaf ears; working mathematicians were not convinced by his philosophical concerns.

Intuitionistic logic (Kolmogorov, Heyting 1930s) Kolmogorov and Heyting articulated the so-called *Brouwer-Heyting-Kolmogorov interpretation* (BHK-interpretation for short) which is the intended semi-formal intuitionistic interpretation of the logical constants. They then formalised intuitionistic logic so that it became a mathematical object...

Mathematical investigations (Glivenko, Gödel, Beth, Kleene, Kripke, de Jongh, ... 1930s - 1960s) ...that could be studied by mathematical means.

Constructive analysis (Bishop 1960s) Bishop did analysis by constructive means. Importantly, in doing so, he was not trying to dictate what is or is not admissible in mathematics, but rather he wanted to investigate how much of analysis could be captured constructively and what constructive approximations are possible—his project, unlike Brouwer's, was not about *reforming* mathematics, but rather about *understanding* it better.

Constructive type theories (Martin-Löf, Cocquand & Huet 1980s) In set theory, a bunch of axioms describes a universe of sets and their properties. This universe can then be investigated for the further properties that sets have. By contrast, in constructive type theories, mathematical objects in the universe are *constructed* by formal rules rather than *described*.

Chapter 2
BHK Interpretation

The standard semi-formal interpretation of the logical operators in intuitionistic logic is the so-called *Brouwer-Heyting-Kolmogorov interpretation* (BHK-interpretation for short). This interpretation motivates the formal systems of intuitionistic logic that we will encounter in later chapters.

2.1 Propositional connectives

We use capitals $A, B, C, \ldots$ for arbitrary formulas. Our logical operators are $\wedge, \vee, \rightarrow$, $\perp$. We treat $\neg A$ as an abbreviation of $A \rightarrow \perp$.

Notation 1. We write '$\alpha : A$' for 'α is a proof of A'.

Definition 1 (BHK interpretation – propositional connectives)**.**

- *A proof of $A \wedge B$ is a pair $\langle \alpha, \beta \rangle$ where $\alpha : A$ and $\beta : B$.*
- *A proof of $A \vee B$ is either $\langle 0, \alpha \rangle$ with $\alpha : A$ or $\langle 1, \beta \rangle$ with $\beta : B$.*
- *A proof of $A \rightarrow B$ is a method σ such that if applied to $\alpha : A$ is guaranteed to return $\sigma(\alpha) : B$.*
- *There is no proof of $\perp$.*

Remark 1. Because $\neg A := A \rightarrow \perp$, a proof of $\neg A$ is a method τ such that if applied to $\alpha : A$ returns $\tau(\alpha) : \perp$

Definition 2 (Validity). *The propositional formula $\varphi(p_1, \ldots, p_n)$ with $p_1, \ldots, p_n$ atomic is* valid *iff for all mathematical statements $A_1, \ldots, A_n$ the sentence $A_1, \ldots, A_n$.*

Example 1. We will show that $p \wedge q \to p$ is valid. Take some arbitrary statements A and B. We need to construct a proof $\sigma : A \wedge B \to A$. That is, given $\gamma : A \wedge B$, we need to produce a method $\sigma(\gamma) : A$. By definition, $\gamma : A \wedge B$ means $\gamma = \langle \alpha, \beta \rangle$ with $\alpha : A$ and $\beta : B$. Define $\sigma(\gamma) := \alpha$.

2.2 Quantifiers

We add the logical operators $\exists, \forall$ to the propositional ones.

Definition 3 (BHK interpretation – quantifiers)**.**

- *A proof of $\exists(x)\varphi(x)$ consists of*
  1. *an object d in the intended domain, and*
  2. *a proof of $\varphi(d)$.*
- *A proof of $\forall(x)\varphi(x)$ is a method that, given an object d in the intended domain, produces a proof of $\varphi(d)$.*

Remark 2. A proof of $\forall x \exists x R(x, y)$ yields a computable Skolem function f such that $\forall R(x, f(x))$.

Remark 3. A proof of $\forall x(P(x) \vee \neg P(x))$ yields a decision procedure for P.

Part II

Intuitionistic Propositional Logic

Chapter 3
The Language $\mathcal{L}_{\mathcal{P}}$

Definition 4 ($\mathcal{L}_{\mathcal{P}}$)**.** *Let $\mathcal{P}$ be a set of atomic sentences $p, q, r, \ldots$ The language $\mathcal{L}_{\mathcal{P}}$ is defined by*

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$$

where $p \in \mathcal{P}$.

Remark 4. Note that it is not a matter of convenience that IPC uses those primitives, as it is in the *Classical Propositional Calculus* (CPC). The following

$$\varphi \wedge \psi \equiv \neg(\neg\varphi \vee \neg\psi)$$
$$\varphi \to \psi \equiv \neg\varphi \vee \psi$$

are not valid in IPC. None of the connectives are definable in terms of the others.

Remark 5 (Negation ($\neg$) is not primitive)**.**

$$\neg\varphi := \varphi \to \bot$$

Remark 6 (The biconditional ($\leftrightarrow$) is not primitive)**.**

$$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Remark 7 (Verum ($\top$) is not primitive)**.**

$$\top := \bot \to \bot$$

CHAPTER 4

Natural Deduction

4.1 Rules

The rules of the natural deduction system for IPC fall into two categories. The introduction rules tell us how to infer a formula with some main connective, while the elimination rules tell us what to infer from such a formula, i.e. how to eliminate the main connective.
Note how the following rules are derived from the BHK interpretation. For example, $(\wedge i)$ says that if you have a proof of φ and a proof of ψ, then you also have a proof of $\varphi \wedge \psi$.

connective	introduction rule(s)	elimination rule(s)
$\wedge$	$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\,(\wedge i)$	$\dfrac{\varphi \wedge \psi}{\varphi}\,(\wedge e) \qquad \dfrac{\varphi \wedge \psi}{\psi}\,(\wedge e)$
$\vee$	$\dfrac{\varphi}{\varphi \vee \psi}\,(\vee i) \qquad \dfrac{\psi}{\varphi \vee \psi}\,(\vee i)$	$\dfrac{\varphi \vee \psi \quad \begin{array}{c}[\varphi]\\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\chi}\,(\vee e)$
$\to$	$\dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \to \psi}\,(\to i)$	$\dfrac{\varphi \to \psi \quad \varphi}{\psi}\,(\to e)$
$\bot$	(no introduction rule)	$\dfrac{\bot}{\varphi}\,(\bot e)$
$\neg$	$\dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \bot\end{array}}{\neg\varphi}\,(\neg i)$	$\dfrac{\neg\varphi \quad \varphi}{\bot}\,(\neg e)$

Remark 8. Note that the rules for $\neg$ are simply derived from the rules for $\to$ using the definition $\neg\varphi := \varphi \to \bot$.

Remark 9. When applying $(\to i)$, you may discharge as many occurrences of φ as you would like, i.e. none, one, two, ..., or every occurrence of φ.

Remark 10. If we drop $(\bot e)$, we obtain minimal logic and $\bot$ becomes an atom that is not governed by any rules. As a consequence, a negation formula becomes just an ordinary implication formula which does not have any special status.

Remark 11. If we add one of the following equivalent rules to the system, we obtain classical logic:

$$\frac{\neg\neg\varphi}{\varphi}\,(dne)$$

$$\dfrac{\begin{array}{c}[\neg\varphi]\\ \vdots\\ \bot\end{array}}{\varphi}\ (con)$$

4.2 Proof-theoretic notions

Definition 5 (Derivability). *We say that* ψ *is* derivable *from* Φ *in* IPC, *and write '*$\Phi \vdash_{\mathsf{IPC}} \psi$*', if there is a proof of* ψ *whose set of undischarged assumptions is included in* Φ*.*

Definition 6 (Equivalence). *We say that* φ *and* ψ *are* equivalent *in* IPC, *and write '*$\varphi \equiv_{\mathsf{IPC}} \psi$*', if* $\phi \vdash_{\mathsf{IPC}} \psi$ *and* $\psi \vdash_{\mathsf{IPC}} \phi$*.*

Definition 7 (Theoremhood). *We say that* φ *is a* theorem *of* IPC, *and write '*$\vdash_{\mathsf{IPC}} \varphi$*', if* φ *is derivable from no assumptions in* IPC.

Definition 8 (IPC). $\mathsf{IPC} := \{\varphi \in \mathcal{L}_{\mathcal{P}} \mid\ \vdash_{\mathsf{IPC}} \varphi\}$

4.3 Tips

1. *Simplify the proof.* For example, transform a proof from A to $B \to C$ ($A \vdash B \to C$) into an easier proof from A, B to C ($A, B \vdash C$). This way you have more assumptions to utilise. Or transform a proof from $A \vee B$ to C ($A \vee B \vdash C$) into two smaller proofs from A to C ($A \vdash C$) and from B to C ($B \vdash C$). Once you have constructed these smaller proofs, they can usually be combined to form the larger, desired proof with relative ease.

2. *Reason from the bottom up.* Reason from the conclusion to the premises. That is, think about what you want to derive, find a rule that would allow you to derive this conclusion from the premises (or assumptions) available to you, and then think about which other 'ingredients' you need for this rule to work. Then try to derive those ingredients, and so on.

3. *Choose the right assumptions.* While guesswork at times, you can usually get a sense of which formulas might be used in the proof. For example, when proving $\bot$ you know that your proof will have to contain formulas that contradict your premises (or assumptions). Compile a list of your premises (or assumptions), as well as the formulas that contradict them. The more information is in front of you, the easier it is usually to see which rules to apply.

4. *Make the most of your premises (or assumptions).* Keep in mind that you can, and sometimes have to, use the same premise (or assumption) more than once. For example, when discharging temporary assumptions by means of the $(\neg i)$-rule, it is sometimes necessary to use the same premise (or assumption) multiple times.

4.4 Examples

Example 2 ($\vdash_{\mathsf{IPC}} p \to \neg\neg p$)**.**
Recipe:

$$\begin{array}{l}
\vdash_{\mathsf{IPC}} p \to \neg\neg p \\
2 \Updownarrow (\to i)\text{, unary inference, discharge } p \text{ at this step} \\
p \vdash_{\mathsf{IPC}} \neg\neg p \\
1 \Updownarrow (\neg i)\text{, unary inference, discharge } \neg p \text{ at this step} \\
p, \neg p \vdash_{\mathsf{IPC}} \bot \ (\neg e)
\end{array}$$

Proof:

$$2\,\dfrac{1\,\dfrac{\dfrac{[p]^2 \qquad [\neg p]^1}{\bot}\,(\neg e)}{\neg\neg p}\,(\neg i)}{p \to \neg\neg p}\,(\to i)$$

Example 3 ($\vdash_{\mathsf{IPC}} (p \to q) \to (\neg q \to \neg p)$)**.**

Recipe:

$$
\begin{array}{c}
\vdash_{\mathsf{IPC}} (p \to q) \to (\neg q \to \neg p) \\
3 \Updownarrow (\to i), \text{ unary inference, discharge } p\to q \text{ at this step} \\
p \to q \vdash_{\mathsf{IPC}} \neg q \to \neg p \\
2 \Updownarrow (\to i), \text{ unary inference, discharge } \neg q \text{ at this step} \\
p \to q, \neg q \vdash_{\mathsf{IPC}} \neg p \\
1 \Updownarrow (\neg i), \text{ unary inference, discharge } p \text{ at this step} \\
p \to q, \neg q, p \vdash_{\mathsf{IPC}} \bot \\
ii \Updownarrow (\neg e), \text{ binary inference} \\
i\ p \to q, p \vdash_{\mathsf{IPC}} q\ (\to e) \quad \& \quad \neg q \vdash_{\mathsf{IPC}} \neg q
\end{array}
$$

Proof:

$$
{\scriptstyle 3}\,\dfrac{{\scriptstyle 2}\,\dfrac{{\scriptstyle 1}\,\dfrac{{\scriptstyle ii}\,\dfrac{{\scriptstyle i}\,\dfrac{[p \to q]^3 \qquad [p]^1}{q}\,(\to e) \qquad [\neg q]^2}{\bot}\,(\neg e)}{\neg p}\,(\neg i)}{\neg q \to \neg p}\,(\to i)}{(p \to q) \to (\neg q \to \neg p)}\,(\to i)
$$

Example 4 ($\vdash_{\mathsf{IPC}} \neg p \vee q \to (p \to q)$)**.**
Recipe:

$$
\begin{array}{ccc}
\multicolumn{3}{c}{\vdash_{\mathsf{IPC}} \neg p \vee q \to (p \to q)} \\
\multicolumn{3}{c}{4 \Updownarrow (\to i), \text{ unary inference, discharge } \neg p \vee q \text{ at this step}} \\
\multicolumn{3}{c}{\neg p \vee q \vdash_{\mathsf{IPC}} p \to q} \\
\multicolumn{3}{c}{3 \Updownarrow (\vee e), \text{ ternary inference, discharge } \neg p \text{ and } q \text{ at this step}} \\
\neg p \vdash_{\mathsf{IPC}} p \to q & \& & q \vdash_{\mathsf{IPC}} p \to q \\
1 \Updownarrow (\to i), \text{ u.f., d.} p & & 2 \Updownarrow (\to i), \text{ u.f., d.} q \\
\neg p, p \vdash_{\mathsf{IPC}} q & & q, p \vdash_{\mathsf{IPC}} q \\
\Updownarrow & & \\
ii\ \bot \vdash_{\mathsf{IPC}} q\ (ecq) & & \\
\& & & \\
i\ \neg p, p \vdash_{\mathsf{IPC}} \bot\ (\neg e) & &
\end{array}
$$

Proof:

$$
\begin{prooftree}
\AxiomC{$[\neg p \vee q]^4$}
\AxiomC{$[\neg p]^3$}
\AxiomC{$[p]^1$}
\LeftLabel{i}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{ii}
\RightLabel{(ecq)}
\UnaryInfC{q}
\LeftLabel{1}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$p \rightarrow q$}
\AxiomC{$[q]^{2,3}$}
\LeftLabel{2}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$p \rightarrow q$}
\LeftLabel{3}
\RightLabel{$(\vee e)$}
\TrinaryInfC{$p \rightarrow q$}
\LeftLabel{4}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$\neg p \vee q \rightarrow (p \rightarrow q)$}
\end{prooftree}
$$

Example 5 ($\vdash_{\mathsf{IPC}} \neg p \vee \neg q \rightarrow \neg(p \wedge q)$).

$$
\begin{prooftree}
\AxiomC{$[\neg p \vee \neg q]^4$}
\AxiomC{$[p \wedge q]^1$}
\RightLabel{$(\wedge e)$}
\UnaryInfC{p}
\AxiomC{$[\neg p]^3$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg(p \wedge q)$}
\AxiomC{$[p \wedge q]^2$}
\RightLabel{$(\wedge e)$}
\UnaryInfC{q}
\AxiomC{$[\neg q]^3$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg(p \wedge q)$}
\LeftLabel{3}
\RightLabel{$(\vee e)$}
\TrinaryInfC{$\neg(p \wedge q)$}
\LeftLabel{4}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$\neg p \vee \neg q \rightarrow \neg(p \wedge q)$}
\end{prooftree}
$$

Example 6 ($\vdash_{\mathsf{IPC}} (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow (p \vee q \rightarrow r)$).

$$
\begin{prooftree}
\AxiomC{$[p \vee q]^2$}
\AxiomC{$[(p \rightarrow r) \wedge (q \rightarrow r)]^3$}
\RightLabel{$(\wedge e)$}
\UnaryInfC{$p \rightarrow r$}
\AxiomC{$[p]^1$}
\RightLabel{$(\rightarrow e)$}
\BinaryInfC{r}
\AxiomC{$[(p \rightarrow r) \wedge (q \rightarrow r)]^3$}
\RightLabel{$(\wedge e)$}
\UnaryInfC{$q \rightarrow r$}
\AxiomC{$[q]^1$}
\RightLabel{$(\rightarrow e)$}
\BinaryInfC{r}
\LeftLabel{1}
\RightLabel{$(\vee e)$}
\TrinaryInfC{r}
\LeftLabel{2}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$p \vee q \rightarrow r$}
\LeftLabel{3}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$(p \rightarrow r) \wedge (q \rightarrow r) \rightarrow (p \vee q \rightarrow r)$}
\end{prooftree}
$$

Example 7 ($\vdash_{\mathsf{IPC}} (p \vee q \rightarrow r) \rightarrow (p \rightarrow r) \wedge (q \rightarrow r)$).

$$
\begin{prooftree}
\AxiomC{$[p]^1$}
\RightLabel{$(\vee i)$}
\UnaryInfC{$p \vee q$}
\AxiomC{$[p \vee q \rightarrow r]^3$}
\RightLabel{$(\rightarrow e)$}
\BinaryInfC{r}
\LeftLabel{1}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$p \rightarrow r$}
\AxiomC{$[q]^2$}
\RightLabel{$(\vee i)$}
\UnaryInfC{$p \vee q$}
\AxiomC{$[p \vee q \rightarrow r]^3$}
\RightLabel{$(\rightarrow e)$}
\BinaryInfC{r}
\LeftLabel{2}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$q \rightarrow r$}
\RightLabel{$(\wedge i)$}
\BinaryInfC{$(p \rightarrow r) \wedge (q \rightarrow r)$}
\LeftLabel{3}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$(p \vee q \rightarrow r) \rightarrow (p \rightarrow r) \wedge (q \rightarrow r)$}
\end{prooftree}
$$

Example 8 ($\vdash_{\mathsf{IPC}} \neg\neg(\varphi \wedge \psi) \equiv \neg\neg\varphi \wedge \neg\neg\psi$). Double negation commutes with conjunction.

$$
\begin{prooftree}
\AxiomC{$[\neg\varphi]^2$}
\AxiomC{$[\varphi \wedge \psi]^1$}
\RightLabel{$(\wedge e)$}
\UnaryInfC{φ}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg(\varphi \wedge \psi)$}
\AxiomC{$[\neg\neg(\varphi \wedge \psi)]^5$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\neg\varphi$}
\AxiomC{$[\neg\psi]^4$}
\AxiomC{$[\varphi \wedge \psi]^3$}
\RightLabel{$(\wedge e)$}
\UnaryInfC{ψ}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{3}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg(\varphi \wedge \psi)$}
\AxiomC{$[\neg\neg(\varphi \wedge \psi)]^5$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{4}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\neg\psi$}
\RightLabel{$(\wedge i)$}
\BinaryInfC{$\neg\neg\varphi \wedge \neg\neg\psi$}
\LeftLabel{5}
\RightLabel{$(\rightarrow i)$}
\UnaryInfC{$\neg\neg(\varphi \wedge \psi) \rightarrow \neg\neg\varphi \wedge \neg\neg\psi$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$[\neg(\varphi \wedge \psi)]^3$}
\AxiomC{$[\varphi]^1$}
\AxiomC{$[\psi]^2$}
\RightLabel{$(\wedge i)$}
\BinaryInfC{$\varphi \wedge \psi$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\varphi$}
\AxiomC{$[\neg\neg\varphi \wedge \neg\neg\psi]^4$}
\RightLabel{$(\wedge e)$}
\UnaryInfC{$\neg\neg\varphi$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\psi$}
\AxiomC{$[\neg\neg\varphi \wedge \neg\neg\psi]^4$}
\RightLabel{$(\wedge e)$}
\UnaryInfC{$\neg\neg\psi$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{3}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\neg(\varphi \wedge \psi)$}
\LeftLabel{4}
\RightLabel{$(\to i)$}
\UnaryInfC{$\neg\neg\varphi \wedge \neg\neg\psi \to \neg\neg(\varphi \wedge \psi)$}
\end{prooftree}
$$

Example 9 **($\vdash_{\mathsf{IPC}} \neg\neg(\varphi \to \psi) \equiv \neg\neg\varphi \to \neg\neg\psi$).** Double negation commutes with implication.

$$
\begin{prooftree}
\AxiomC{$[\varphi]^1$}
\AxiomC{$[\varphi \to \psi]^2$}
\RightLabel{$(\to e)$}
\BinaryInfC{ψ}
\AxiomC{$[\neg\psi]^3$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\varphi$}
\AxiomC{$[\neg\neg\varphi]^4$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg(\varphi \to \psi)$}
\AxiomC{$[\neg\neg(\varphi \to \psi)]^5$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{3}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\neg\psi$}
\LeftLabel{4}
\RightLabel{$(\to i)$}
\UnaryInfC{$\neg\neg\varphi \to \neg\neg\psi$}
\LeftLabel{5}
\RightLabel{$(\to i)$}
\UnaryInfC{$\neg\neg(\varphi \to \psi) \to (\neg\neg\varphi \to \neg\neg\psi)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$[\varphi]^1$}
\AxiomC{$[\neg\varphi]^2$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\RightLabel{(ecq)}
\UnaryInfC{ψ}
\LeftLabel{1}
\RightLabel{$(\to i)$}
\UnaryInfC{$\varphi \to \psi$}
\AxiomC{$[\neg(\varphi \to \psi)]^4$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\neg\varphi$}
\AxiomC{$[\neg\neg\varphi \to \neg\neg\psi]^5$}
\RightLabel{$(\to e)$}
\BinaryInfC{$\neg\neg\psi$}
\AxiomC{$[\psi]^3$}
\RightLabel{$(\to i)$}
\UnaryInfC{$\varphi \to \psi$}
\AxiomC{$[\neg(\varphi \to \psi)]^4$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{3}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\psi$}
\RightLabel{$(\neg e)$}
\BinaryInfC{$\bot$}
\LeftLabel{4}
\RightLabel{$(\neg i)$}
\UnaryInfC{$\neg\neg(\varphi \to \psi)$}
\LeftLabel{5}
\RightLabel{$(\to i)$}
\UnaryInfC{$(\neg\neg\varphi \to \neg\neg\psi) \to \neg\neg(\varphi \to \psi)$}
\end{prooftree}
$$

Note that you're allowed to discharge zero assumptions when using the $(\to i)$-rule.

The following two facts might be helpful in reconstructing and understanding the proof above:

$$\neg\varphi \vdash \varphi \to \psi \overset{\text{CP}}{\Longrightarrow} \neg(\varphi \to \psi) \vdash \neg\neg\varphi$$
$$\psi \vdash \varphi \to \psi \overset{\text{CP}}{\Longrightarrow} \neg(\varphi \to \psi) \to \neg\psi$$

Chapter 5
Intuitionistic Kripke Semantics

5.1 Informal picture

The basic idea behind Kripke semantics is that we evaluate formulas relative to 'points' that, intuitively, reflect states of affairs.

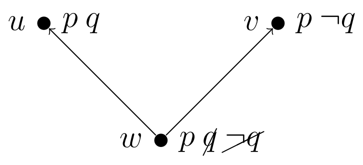
We have already remarked that in classical logic, truth is understood as correspondence. Consequently, we may understand a point as corresponding to an external state of affairs such as a possible world. Because such states of affairs are *complete*, points are complete: for any point, and any formula, either that formula is true at that point, or it is not.

By contrast, in intuitionistic logic, points are understood as stages in some construction. There are two important features of intuitionistic Kripke semantics that follow from this interpretation:

1. It is possible that at some point in the construction the mathematician has neither proved a formula nor its negation. Hence points in intuitionistic Kripke semantics are *partial*.

2. Successor points always 'inherit' proofs of formulas from their predecessor points. The idea is that, intuitively, once the mathematician has (correctly) proved a statement, that proof will not be forgotten at some later stage in the construction.

The model below depicts these two features. Note that at w, we have neither q nor $\neg q$. That is simply because at that stage in the construction, neither q nor $\neg q$ have been proved. Note that p holds at w and thus must also hold at all successor points.



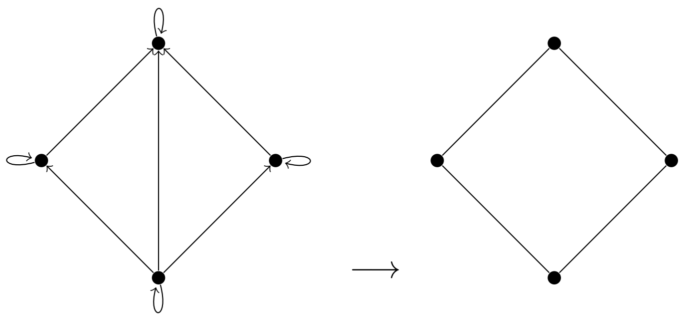
5.2 Frames

Definition 9 (i.K.f.). *An* intuitionistic Kripke frame *(short:* i.K.f.*) is a pair* $\mathfrak{F} = \{W, R\}$ *where* W *is a set and* R *is a partial order on* W*, i.e. a binary relation that satisfies:*

- ***reflexivity***: $\forall w \in W : \ wRw$
- ***transitivity***: $\forall w, v, u \in W : \ wRv \ \& \ vRu \Rightarrow wRu$
- ***anti-symmetry***: $\forall w, v \in W : \ wRv \ \& \ vRw \Rightarrow w = v$

Remark 12. We could drop anti-symmetry and still use Kripke frames to characterise intuitionistic logic; the anti-symmetry assumption is not needed to obtain a sound semantics. However, it will be convenient for us to work with that assumption.

Notation 2. To simplify our diagrams of i.K.f.s, we will indicate that wRv simply by locating w below v and drawing a line between them. Such a line is to be understood as an arrow whose tail is at w and whose tip is at v. We will omit reflexive and transitive arrows.



Definition 10 (Successors).

- $R[w] := \{v \in W \mid wRv\}$
- *If* $X \subseteq W$, $R[X] := \bigcup_{w \in X} R[w]$

5.3 Models

Definition 11 (i.K.m.)**.** *An* intuitionistic Kripke model *(short:* i.K.m.*) is a triple $M = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is an i.K.f. and $V : \mathcal{P} \to \wp(W)$ is a valuation function obeying:*

- ***persistency**: $w \in V(p)$ & $wRv \Rightarrow v \in V(p)$*

5.4 Forcing conditions

Definition 12 (Intuitionistic Kripke semantics)**.** *Let $M = \langle W, R, V \rangle$ be an i.K.m., $w \in W$ a point:*

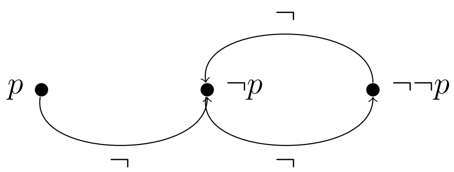
- $M, w \Vdash p \;\Leftrightarrow\; p \in V(p)$, *where $p \in \mathcal{P}$,*
- $M, w \nVdash \bot$,
- $M, w \Vdash \varphi \wedge \psi \;\Leftrightarrow\; M, w \Vdash \varphi$ *and* $M, w \Vdash \psi$,
- $M, w \Vdash \varphi \vee \psi \;\Leftrightarrow\; M, w \Vdash \varphi$ *or* $M, w \Vdash \psi$,
- $M, w \Vdash \varphi \to \psi \;\Leftrightarrow\; \forall v \in R[w] : \big[M, v \Vdash \varphi \;\Rightarrow M, v \Vdash \psi\big]$,
- $M, w \nVdash \neg\varphi \;\Leftrightarrow\; \forall v \in R[w] : \big[M, v \Vdash \varphi \;\Rightarrow M, v \Vdash \bot\big] \;\Leftrightarrow\; \forall v \in R[w] : M, v \nVdash \varphi$

Remark 13. The clauses for atomic formulas, $\bot$, conjunction and disjunction are the same as in classical logic. The clauses for implication and, derivatively, for negation are different.

Remark 14 (Negation)**.** Note that in CPC, negation is an operation that when applied twice to a formula yields a formula which is logically equivalent to the formula we started out with:

$$\begin{tikzcd} p \arrow[rr, bend right, "\neg"'] & & \neg p \arrow[ll, bend right, "\neg"'] \end{tikzcd}$$

By contrast, in IPC, negation behaves as follows:



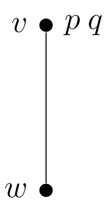
Two features are noteworthy: 1) $\neg$ behaves classically on negated formulas. 2) There exists a copy of the 'classical universe' inside intuitionistic logic. We will return to this feature later, when discussing the negative translation from classical logic into intuitionistic logic.

5.5 Consequence

Definition 13 (IPC-consequence). $\Phi \vDash_{\mathsf{IPC}} \psi \Leftrightarrow$ *for all i.K.m.s* M *and points* $w \in W$: *if* $M, w \Vdash \varphi$ *for all* $\varphi \in \Phi$, *then* $M, w \Vdash \psi$.

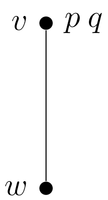
5.6 Examples

Example 10 ($\nvDash_{\mathsf{IPC}} p \vee \neg p$).



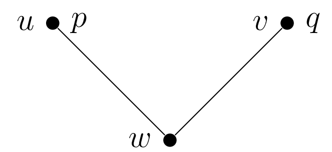
$w \nVdash p$
$w \nVdash \neg p$
$w \nVdash p \vee \neg p$

Example 11 ($\nvDash_{\mathsf{IPC}} (p \to q) \to \neg p \vee q$).



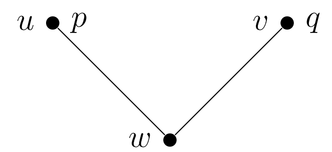
$w \Vdash p \to q$
$w \nVdash \neg p$
$w \nVdash q$
$w \nVdash \neg p \vee q$
$w \nVdash (p \to q) \to \neg p \vee q$

Example 12 ($\nvDash_{\mathsf{IPC}} \neg(p \wedge q) \to \neg p \vee \neg q$).



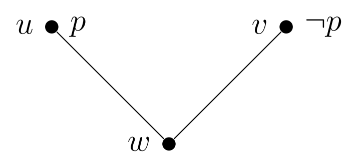
$w \Vdash \neg(p \wedge q)$
$w \nVdash \neg p$
$w \nVdash \neg q$
$w \nVdash \neg p \vee \neg q$
$w \nVdash \neg(p \wedge q) \to \neg p \vee \neg q$

Example 13 ($\nvDash_{\mathsf{IPC}}$ $(p \to q) \vee (q \to p)$).



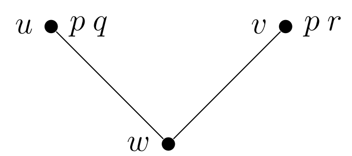
$w \nVdash p \to q$
$w \nVdash q \to p$
$w \nVdash (p \to q) \vee (q \to p)$

Example 14 (Weak excluded middle $\nvDash_{\mathsf{IPC}}$ $\neg p \vee \neg\neg p$).



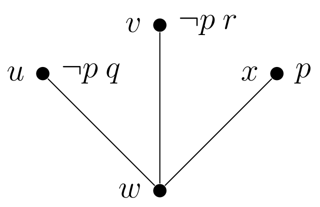
$w \nVdash \neg p \vee \neg\neg p$
$w \nVdash \neg p$
$w \nVdash \neg\neg p$

Example 15 ($\nvDash_{\mathsf{IPC}}$ $(p \to q \vee r) \to (p \to q) \vee (p \to r)$).



$w \nVdash (p \to q \vee r) \to (p \to q) \vee (p \to r)$
$w \Vdash p \to q \vee r$
$w \nVdash (p \to q) \vee (p \to r)$
$w \nVdash p \to q$
$w \nVdash p \to r$

Example 16 (Kreisel-Putnam axiom $\nvDash_{\mathsf{IPC}}$ $(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$). Note that the Kreisel-Putnam axiom is used to characterise the intermediate logic KP, which has the disjunction property.



$w \nVdash (\neg p \to q \vee r) \to$
$(\neg p \to q) \vee (\neg p \to r)$
$w \Vdash \neg p \to q \vee r$
$w \nVdash (\neg p \to q) \vee$
$(\neg p \to r)$
$w \nVdash \neg p \to q$
$w \nVdash \neg p \to r$

Notice that world x is necessary to avoid $w \Vdash \neg p$, which would falsify the antecedent. If, on the other hand, we allowed $w \Vdash \neg p$, then we would also have to have $w \Vdash q$ or $w \Vdash r$ in order to force the antecedent at w. However, by persistency, it would then follow that $u \Vdash q$ and $v \Vdash q$ or $u \Vdash r$ and $v \Vdash r$—thereby satisfying the consequent. We must thus ensure that $w \nVdash \neg p$ by adding a world x such that $v \Vdash p$.

5.7 Persistency

Proposition 1. *For all i.K.m.s* M*, points* w*, formulas* φ*:* $M, w \Vdash \varphi \,\&\, wRv \Rightarrow M, v \Vdash \varphi$

Proof. Exercise. ■

Chapter 6
Soundness and Completeness

6.1 Overview

In this section, we show that the proof-theoretic characterisation of IPC and the model-theoretic one coincide.

Theorem 1 (Soundness and completeness for IPC). $\varphi \vDash_{\mathsf{IPC}} \psi \;\Leftrightarrow\; \varphi \vdash_{\mathsf{IPC}} \psi$

Proof. "$\Rightarrow$" Canonical model construction
"$\Leftarrow$" Check soundness of each inference rule.
Example ($\to i$): Prove that $\varphi, \psi \vDash_{\mathsf{IPC}} \chi$ implies $\varphi \vDash_{\mathsf{IPC}} \psi \to \chi$.
Suppose $\varphi, \psi \vDash_{\mathsf{IPC}} \chi$. Take any M, w s.t. $M, w \Vdash \varphi$. Take any $v \in R[w]$ with $M, v \Vdash \psi$.
By persistency, also $M, v \Vdash \varphi$, and since $\varphi, \psi \vDash_{\mathsf{IPC}} \chi$, also $M, v \Vdash \chi$.
Since $v \in R[w]$ was arbitrary, this proves that $M, w \Vdash \psi \to \chi$. So, $\varphi \vDash_{\mathsf{IPC}} \psi \to \chi$.

6.2 Canonical model construction

Definition 14. *A set $\Gamma \subseteq \mathcal{L}_{\mathcal{P}}$ is an IPC-theory if $\forall \varphi : \; \Gamma \vdash_{\mathsf{IPC}} \varphi \;\Rightarrow\; \varphi \in \Gamma$.*

Definition 15. *Let Γ be an IPC-theory. We say that:*

- *Γ is consistent if $\bot \notin \Gamma$.*
- *Γ has the $\vee$-property if $\forall \varphi, \psi : \; \varphi \vee \psi \in \Gamma \;\Rightarrow\; \varphi \in \Gamma$ or $\psi \in \Gamma$.*

Definition 16 (Canonical model). *The canonical model for IPC is $M^c = \langle W^c, R^c, V^c \rangle$, where*

- *W^c is the set of consistent IPC-theories with the $\vee$-property,*
- $\Gamma R^c \Gamma' \ \Leftrightarrow\ \Gamma \subseteq \Gamma'$,
- $\forall p \in P : \ \Gamma \in V^c(p) \ \Leftrightarrow\ p \in \Gamma$.

Remark 15. M^c is an i.K.m.:

- $\subseteq$ is a partial order
- V^c is persistent, since $\Gamma \in V^c(p)$ and $\Gamma R^c \Gamma' \ \Rightarrow p \in \Gamma$ and $\Gamma \subseteq \Gamma'$, so $p \in \Gamma'$, so $\Gamma' \in V^c(p)$.

6.3 Lindenbaum-type lemma

Lemma 1 (Lindenbaum-type). $\varphi \nvdash_{\mathsf{IPC}} \psi \ \Rightarrow\ \exists \Gamma \in W^c : \ \varphi \subseteq \Gamma$ *but* $\psi \notin \Gamma$.

Proof. Suppose $\varphi \nvdash_{\mathsf{IPC}} \psi$. Enumerate $\mathcal{L}_{\mathcal{P}} = \{\varphi_0, \varphi_1, \ldots\}$.
Define Γ as follows:

- $\Gamma_0 := \varphi$
- $\Gamma_{n+1} := \begin{cases} \Gamma_n, & \text{if } \Gamma_n \cup \{\varphi_n\} \vdash_{\mathsf{IPC}} \psi \\ \Gamma_n \cup \{\varphi_n\}, & \text{if } \Gamma_n \cup \{\varphi_n\} \nvdash_{\mathsf{IPC}} \psi \end{cases}$
- $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$.

Obviously $\varphi \subseteq \Gamma$ and $\psi \notin \Gamma$. To show: $\Gamma \in W^c$, i.e. Γ is a consistent IPC-theory with $\vee$-property.

Remark 16. $\forall n : \ \Gamma_n \nvdash_{\mathsf{IPC}} \psi$. This follows from the definition of Γ_n by induction on n.

Remark 17. $\Gamma \nvdash_{\mathsf{IPC}} \psi$.
Suppose $\Gamma \vdash_{\mathsf{IPC}} \psi$. Then $\exists \gamma_1, \ldots, \gamma_n \in \Gamma$ s.t. $\gamma_1, \ldots, \gamma_n \vdash_{\mathsf{IPC}} \psi$.
Let Γ_m be s.t. $\gamma_1, \ldots, \gamma_n \in \Gamma_m$. Then $\Gamma_m \vdash_{\mathsf{IPC}} \psi$, which is a contradiction to the definition of Γ_m.

Claim 1. Γ is an IPC-theory.
Suppose $\Gamma \vdash_{\mathsf{IPC}} \chi$. Then, $\Gamma \cup \{\chi\} \nvdash_{\mathsf{IPC}} \psi$ (otherwise by transitivity $\Gamma \vdash_{\mathsf{IPC}} \psi$).
Let $\chi = \varphi_m$. Since $\Gamma_m \subseteq \Gamma$, $\Gamma_m \cup \{\varphi_m\} \nvdash_{\mathsf{IPC}} \psi$. Then, by definition $\varphi_m \in \Gamma_{m+1}$, therefore $\varphi_m = \chi \in \Gamma$.

Claim 2. Γ is consistent ($\bot \notin \Gamma$).
This is clear since $\Gamma \nvdash_{\mathsf{IPC}} \psi$. (A set of formulae is incosistent *iff* any formula can be derived from it.)

Claim 3. Γ has the $\vee$-property.
Proof: Suppose $\varphi_n, \varphi_m \notin \Gamma$. To show: $\varphi_n \vee \varphi_m \notin \Gamma$.
Since $\varphi_n \notin \Gamma$, also $\varphi_n \notin \Gamma_{n+1}$. This means that $\Gamma_n \cup \{\varphi_n\} \vdash_{\mathsf{IPC}} \psi$, so $\Gamma \cup \{\varphi_n\} \vdash_{\mathsf{IPC}} \psi$.
Similarly for φ_m.
By $(\vee\, e)$ we have $\Gamma \cup \{\varphi_n \vee \varphi_m\} \vdash_{\mathsf{IPC}} \psi$. But $\Gamma \nvdash_{\mathsf{IPC}} \psi$. So $\Gamma \neq \Gamma \cup \{\varphi_n \vee \varphi_m\}$, i.e. $\varphi_n \vee \varphi_m \notin \Gamma$.

■

6.4 Truth lemma

Lemma 2 (Truth Lemma). $M^c, \Gamma \Vdash \varphi \;\Leftrightarrow\; \varphi \in \Gamma$.

Proof. By induction on φ:

- $\varphi = p$: $M^c, \Gamma \Vdash p \;\Leftrightarrow\; \Gamma \in V^c(p) \;\Leftrightarrow\; p \in \Gamma$ (by definition of V^c).
- $\bot$, $\wedge$ are straightforward.
- $\varphi = \psi \vee \chi$: $M^c, \Gamma \Vdash \psi \vee \chi \Leftrightarrow M^c, \Gamma \Vdash \psi$ or $M^c, \Gamma \Vdash \chi \Leftrightarrow^{\mathrm{IH}} \psi \in \Gamma$ or $\chi \in \Gamma$

  $$\Leftrightarrow^{\vee\text{-property}} \psi \vee \chi \in \Gamma$$
- $\varphi = \psi \to \chi$:

"$\Rightarrow$": Suppose $\psi \to \chi \in \Gamma$. To show: $M^c, \Gamma \Vdash \psi \to \chi$.
Take $\Gamma' \supseteq \Gamma$ s.t. $M^c, \Gamma' \Vdash \psi$. By IH, $\psi \in \Gamma'$. Since $\Gamma' \supseteq \Gamma$, $\psi \to \chi \in \Gamma'$.
Since $\psi, \psi \to \chi \in \Gamma'$ and Γ' is an IPC-theory, $\chi \in \Gamma'$. By IH, $M^c, \Gamma' \Vdash \chi$.
Thus, $M^c, \Gamma \Vdash \psi \to \chi$.

"$\Leftarrow$": Suppose $\psi \to \chi \notin \Gamma$. To show: $M^c, \Gamma \nVdash \psi \to \chi$.
Since Γ is an IPC-theory, $\Gamma \nvdash_{\mathsf{IPC}} \psi \to \chi$. By $(\to i)$ we have $\Gamma \cup \{\psi\} \nvdash_{\mathsf{IPC}} \chi$.
By Lindenbaum-type Lemma $\exists \Gamma' \in W^c$ s.t. $\Gamma \cup \{\psi\} \subseteq \Gamma'$ but $\chi \notin \Gamma'$.
So we have:

$$\left.\begin{array}{lll} \Gamma \subseteq \Gamma' & \Leftrightarrow & \Gamma R^c \Gamma' \\ \psi \in \Gamma' & \Leftrightarrow^{\text{IH}} & M^c, \Gamma' \Vdash \psi \\ \chi \notin \Gamma' & \Leftrightarrow^{\text{IH}} & M^c, \Gamma' \nVdash \chi \end{array}\right\} \text{ So there is a successor of } \Gamma \text{ which}$$

forces ψ but not χ.

Thus $M^c, \Gamma \nVdash \psi \to \chi$. ■

6.5 Completeness

Proof of Completeness:
Suppose $\varphi \nvdash_{\mathsf{IPC}} \psi$. By Lindenbaum-type Lemma $\exists \Gamma \in W^c$ s.t. $\varphi \subseteq \Gamma$ but $\psi \notin \Gamma$.
By Truth Lemma $M^c, \Gamma \Vdash \varphi \ \ \forall \varphi \in \varphi$ but $M^c, \Gamma \nVdash \psi$.
Therefore $\varphi \nvDash_{\mathsf{IPC}} \psi$. ■

CHAPTER 7

Disjunction Property

7.1 Invariance under generated submodels

Proposition 2. *Let M be an i.K.m., w a point in M. Let M_w be the restriction of M to $R[w]$. Then:*

$$\forall\varphi : M, w \Vdash \varphi \;\Leftrightarrow\; M_w, w \Vdash \varphi.$$

Proof. Immediate by induction on φ. ∎

7.2 Rooted model property

Corollary 1 (Rooted model property)**.** If $\varphi \nVdash_{\mathsf{IPC}} \psi$, there is a model $M = \langle W, R, V\rangle$ and a point w with $R[w] = W$ s.t. $M, w \Vdash \varphi$ for all $\varphi \in \varphi$ but $M, w \nVdash \psi$. We say that w is the *root* of M (there is exactly one root because of antisymmetry of R).

The idea here is that every invalid entailment can be witnessed at the root of an i.K.m.

Proof of Corollary 1. If $\varphi \nVdash_{\mathsf{IPC}} \psi$, then $\exists N, w$ s.t. $N, w \Vdash \varphi$ for all $\varphi \in \varphi$ but $N, w \nVdash \psi$. Let $M := N_w$. Then w is the root of M and by invariance under generated submodels we have $M, w \Vdash \varphi$ for all $\varphi \in \varphi$ and $M, w \nVdash \psi$. ∎

7.3 Disjunction property

Theorem 2 (Disjunction property)**.** $\varphi \vee \psi \in \mathsf{IPC} \Leftrightarrow\; \varphi \in \mathsf{IPC}$ *or* $\psi \in \mathsf{IPC}$

Proof. "$\Leftarrow$": Obvious, since $\varphi \Vdash_{\mathsf{IPC}} \varphi \vee \psi$, $\psi \Vdash_{\mathsf{IPC}} \varphi \vee \psi$.
"$\Rightarrow$": By contraposition: Suppose $\varphi, \psi \notin$ IPC.
Then we have countermodels $M_1, v_1 \nVdash \varphi$ and $M_2, v_2 \nVdash \psi$. We may assume that v_i is the root of M_i for $i \in \{1, 2, \}$, and that W_1 and W_2 are disjoint. Define a new model $M = \langle W, R, V \rangle$ as follows:

- $W = W_1 \cup W_2 \cup \{w\}$ with $w \notin W_1 \cup W_2$,
- $R = R_1 \cup R_2 \cup \{< w, v > \mid v \in W\}$,
- $V(p) = V_1(p) \cup V_2(p)$ for all $p \in P$.

One can check that R is a partial order and V is persistent, so that M is indeed an i.K.m.
By invariance under generated submodels (since $M_{v_i} = M_i$ for $i \in \{1, 2\}$), we have $M, v_1 \nVdash \varphi$ and $M, v_2 \nVdash \psi$.
By persistency, it follows that $M, w \nVdash \varphi$ and $M, w \nVdash \psi$, so $M, w \nVdash \varphi \vee \psi$.
Thus, $\varphi \vee \psi \notin$ IPC. ■

Remark 18. This fails for classical logic. For instance, $p \notin \mathsf{CPC}$, $\neg p \notin \mathsf{CPC}$, but $p \vee \neg p \in \mathsf{CPC}$.

CHAPTER 8

Decidability

8.1 Finite model property

Fix a finite set of formulae φ which is subformula-closed: $\varphi \in \varphi$ and $\psi \in Sub(\varphi) \Rightarrow \psi \in \varphi$.[1]

Example 17. Consider $\varphi_0 = \{(p \to q) \vee (q \to p), p \to q, q \to p, p, q\}$.

Definition 17. *A φ-theory is a subset $\Gamma \subseteq \varphi$ s.t. $\forall \varphi \in \varphi : \ \Gamma \vdash_{\mathsf{IPC}} \varphi \ \Rightarrow \varphi \in \Gamma$.*

Example 18. $\Gamma_1 = \{p \to q\}$ is not a φ-theory, since $\Gamma_1 \vdash_{\mathsf{IPC}} (p \to q) \vee (q \to p)$, but $(p \to q) \vee (q \to p) \notin \Gamma_1$.
$\Gamma_2 = \{p \to q, (p \to q) \vee (q \to p)\}$ is a φ-theory: $\Gamma_2 \nvdash_{\mathsf{IPC}} q \to p$, $\Gamma_2 \nvdash_{\mathsf{IPC}} p$, $\Gamma_2 \nvdash_{\mathsf{IPC}} q$.

Exercise: Give models that force Γ_2 but not $p \to q$, p, q, respectively.

Definition 18. *The φ-canonical model is $M^c_\varphi = \langle W^c_\varphi, R^c_\varphi, V^c_\varphi \rangle$:*

- *W^c_φ is the set of consistent φ-theories with $\vee$-property,*
- $\Gamma R^c_\varphi \Gamma' \ \Leftrightarrow \ \Gamma \subseteq \Gamma'$,
- $\Gamma \in V^c_\varphi(p) \ \Leftrightarrow \ p \in \Gamma$.

Remark 19. M^c_φ is a finite model of size $\leq 2^{|\varphi|}$.

[1] $Sub(\varphi)$ is the set of all subformulae of a formulae φ.

Lemma 3 (Lindenbaum-type)**.** *Let $\Psi \cup \{\chi\} \subseteq \varphi$. If $\Psi \nvdash_{\mathsf{IPC}} \chi$, then $\exists \Gamma \in W^c_\varphi$ s.t. $\Psi \subseteq \Gamma$, $\chi \notin \Gamma$.*

Proof sketch. Let $\varphi = \{\varphi_0, \varphi_1, \ldots, \varphi_k\}$. We let:

- $\Gamma_0 := \Psi$,
- $\Gamma_{n+1} := \begin{cases} \Gamma_n & \text{if } \Gamma_n \cup \{\varphi_n\} \vdash_{\mathsf{IPC}} \chi, \\ \Gamma_n \cup \{\varphi_n\} & \text{if } \Gamma_n \cup \{\varphi_n\} \nvdash_{\mathsf{IPC}} \chi, \end{cases}$
- $\Gamma := \Gamma_{k+1}$.

Clearly $\Psi \subseteq \Gamma$ and $\chi \notin \Gamma$. To show: Γ is a consistent φ-theory with $\vee$-property, i.e. $\Gamma \in W^c_\varphi$.
The proof here is exactly analogous to the one for the previous Lindenbaum-type Lemma. ■

Lemma 4 (Truth Lemma)**.** *For all $\varphi \in \varphi$, $\Gamma \in W^c_\varphi$: $M^c_\varphi, \Gamma \Vdash \varphi \iff \varphi \in \Gamma$.*

Proof. By induction on φ, crucially using the fact that φ is subformula-closed. The proof is analogous to the one we saw for M^c. ■

Theorem 3 (IPC has the finite model property)**.** *If $\varphi \notin$ IPC then there is a finite model M and a world w in M s.t. $M, w \nVdash \varphi$.*

Proof. Suppose $\varphi \notin$ IPC. Let $\varphi := Sub(\varphi)$. Then φ is finite and subformula-closed. Thus, M^c_φ is finite. Since $\emptyset \cup \{\varphi\} \subseteq \varphi$ and $\emptyset \nvdash_{\mathsf{IPC}} \varphi$, by Lindenbaum-type Lemma there is a point $\Gamma \in W^c_\varphi$ with $\varphi \notin \Gamma$. Since $\varphi \in \varphi$, by Truth Lemma we have $M^c_\varphi, \Gamma \nVdash \varphi$. ■

Remark 20. The size of M^c_φ is $\leq 2^{|Sub(\varphi)|}$.

8.2 Decidability

Is IPC decidable, i.e. is there a procedure to decide, given a formula φ, whether $\varphi \in$ IPC? We know that CPC is decidable—for any formula φ, we can draw a truth-table to see whether $\varphi \in$ CPC. But for IPC, there is no analogue of truth-tables. However, we can give a constructive proof that IPC is, in fact, decidable.

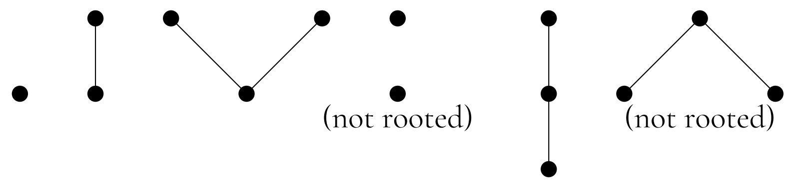
Theorem 4. IPC *is decidable.*

Proof. The following procedure decides whether $\varphi \in \mathsf{IPC}$. Given φ, proceed as follows:

1. Compute $n := |Sub(\varphi)|$.
2. Draw all rooted (for simplicity) i.K.m's of size $\leq 2^n$. Up to isomorphism, there are only finitely many.
3. For each of them, check whether φ holds at the root. This is possible in finite time since all the models are finite.
4. If all of the roots satisfy φ, then output "Yes", otherwise output "No".

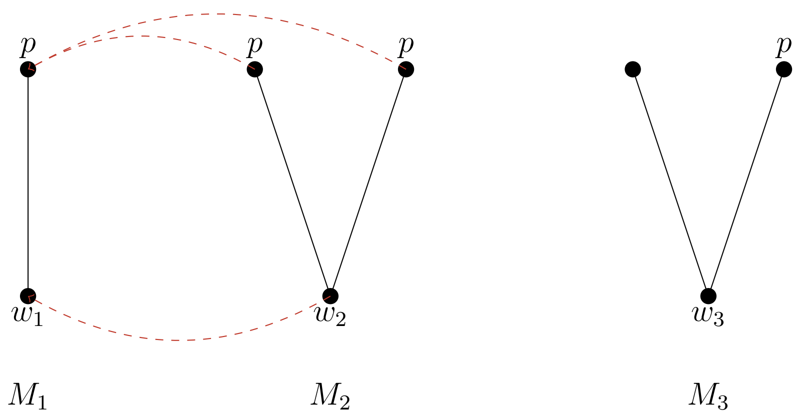
∎

Example 19 (All i.K.m's of size ≤ 3).



CHAPTER 9
Tree Model Property

9.1 p-morphisms



From the point of view of IPC, M_1, w_1 and M_2, w_2 encode essentially the same situation, while M_3, w_3 encodes a different situation. How can we make this precise?

Definition 19. *Let $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ be two i.K.m's. A map $f : W \to W'$ is a p-morphism in case:*

- *Atom:* $\forall p \in P : \ w \in V(p) \ \Leftrightarrow \ f(w) \in V'(p)$,
- *Forth:* $wRv \ \Rightarrow \ f(w)R'f(v)$,
- *Back:* $f(w)R'v' \ \Rightarrow \ \exists v \in R[w]$ *s.t.* $f(v) = v'$.

Proposition 3. *If $f : W \to W'$ is a p-morphism, then $\forall w \in W, \forall \varphi : M, w \Vdash \varphi \Leftrightarrow M', f(w) \Vdash \varphi$.*

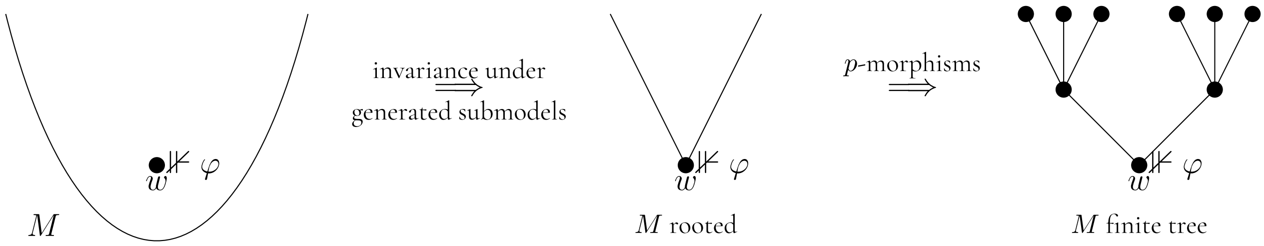
Proof. By induction on φ:

- $p \in P$: (Atom)-condition;
- $\vee, \wedge, \neg$ are straightforward;
- $\varphi = \psi \to \chi$:

"$\Leftarrow$": Suppose $M', f(w) \Vdash \psi \to \chi$. Take $v \in R[w]$ s.t. $M, v \Vdash \psi$. By (Forth), $f(v) \in R'[f(w)]$ and by IH, $M', f(v) \Vdash \psi$. Since $M', f(w) \Vdash \psi \to \chi$, it follows that $M', f(v) \Vdash \chi$, whence by IH on χ we have $M, v \Vdash \chi$. Thus, $M, w \Vdash \psi \to \chi$.

"$\Rightarrow$": Analogous, using the (Back)-condition. ■

Application of p-morphisms:
If $\varphi \notin \mathsf{IPC}$, then $\exists M$ finite, w a point in M s.t. $M, w \nVdash \varphi$.



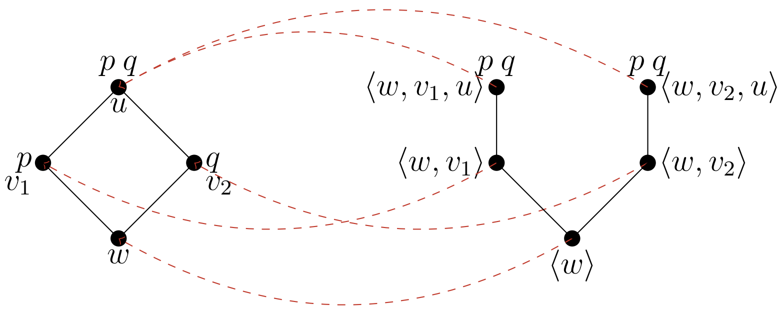
By invariance under generated submodels, we can restrict M to the rooted model M_w and obtain $M_w, w \nVdash \varphi$. Then, M_w is a p-morphic image of a finite tree M', i.e. a rooted model where the set of predecessors of any point is finite and linearly ordered. Trees are defined below:

9.2 Tree model property

Definition 20. *An i.K. frame $\mathcal{F} = \langle W, R \rangle$ is a tree if it is rooted and for all points*

$w \in W$, the set of w's predecessors $R^{-1}[w] = \{v \in W \mid vRw\}$ is finite and linearly ordered by R (i.e. for any two points in $R^{-1}[w]$, one of them sees the other).

Proposition 4 (Unravelling of a model)**.** *Every finite rooted i.K.m. $M = \langle W, R, V \rangle$ is a p-morphic image of a finite tree $M' = \langle W', R', V' \rangle$, i.e. there exists a surjective p-morphism $f : W' \to W$.*



Proof sketch.
Define M' as follows, where w_0 is the root of the model M:

- $W' = \{< w_0, \ldots, w_n > \mid w_0 = w$ and w_{i+1} is an immediate successor of $w_i\}$,
- $\bar{v} R' \bar{u} \;\Leftrightarrow\; \bar{v}$ is an initial segment of $\bar{u}$,
- $< w_0, \ldots, w_n > \in V'(p) \;\Leftrightarrow\; w_n \in V(p)$,

where $\bar{v} = \{w_0, \ldots, w_i\}$, $\bar{u} = \{w_0, \ldots, w_j\}$ for $i, j \geq 0$, $w_i = v, w_j = u \in W$ are paths in the tree M'.
It's easy to check that M' is a finite tree with root $< w_0 >$ and that $f(< w_0, \ldots, w_n >) = w_n$ is a surjective p-morphism. ∎

Theorem 5 (Finite tree model property)**.** *If $\varphi \notin$ IPC, then φ can be falsified at the root of a finite tree.*

Proof. If $\varphi \notin$ IPC, there is a finite i.K.m M and a point w in M s.t. $M, w \nVdash \varphi$. Then, the generated submodel M_w is a finite rooted model with $M_w, w \nVdash \varphi$.
Let $(M_w)'$ be the finite tree obtained by unraveling the model M_w. Since formulae are invariant under p-morphisms and there is a surjective p-morphism f with $f(< w >) = w$, we have $(M_w)', < w > \nVdash \varphi$. So φ is falsified at the root of a finite tree. ∎

Chapter 10

Translations

$$\mathsf{CPC} \overset{\text{negative translation } (.)^n}{\hookrightarrow} \mathsf{IPC} \overset{\text{modal translation}(.)^\Box}{\hookrightarrow} \mathsf{S4}$$

Definition 21. *Let L, L' be two logics with corresponding languages $\mathcal{L}$, $\mathcal{L}'$. A translation from L to L' is a map*
$(.)^* : \mathcal{L} \to \mathcal{L}'$, *s.t.* $\forall \varphi_1, \dots, \varphi_n, \psi : \varphi_1, \dots, \varphi_n \vdash_L \psi \;\Leftrightarrow\; \varphi_1^*, \dots, \varphi_n^* \vdash_{L'} \psi^*$.

Lemma 5. *Let M be a finite i.K.m., w a point in M. Let $E[w] = \{e \in R[w] \mid e \text{ is an endpoint in } M\}$.*

$$M, w \Vdash \neg\neg\varphi \;\Leftrightarrow\; \forall e \in E[w] : M, e \Vdash \varphi$$

Proof. "⇒": Suppose $M, w \vDash \neg\neg\varphi$. Take $e \in E[w]$. By persistency, $M, e \Vdash \neg\neg\varphi$. Since endpoints behave classically, $M, e \Vdash \varphi$.
"⇐": Suppose $M, w \nVdash \neg\neg\varphi$. $\Rightarrow$ $\exists v \in R[w]$ s.t. $M, v \Vdash \neg\varphi$. Since M is finite, $E[v] \neq \emptyset$.
Let $e \in E[v]$. By persistency, $M, e \Vdash \neg\varphi$, so $M, e \nVdash \varphi$. By transitivity, $e \in E[w]$.
So $\neg\forall e \in E[w] : M, e \Vdash \varphi$. ∎

10.1 Glivenko's theorem

Theorem 6 (Glivenko)**.** $\varphi \in \mathsf{CPC} \;\Leftrightarrow\; \neg\neg\varphi \in \mathsf{IPC}$

Proof. "⇐": Suppose $\neg\neg\varphi \in \mathsf{IPC}$. $\Rightarrow \neg\neg\varphi \in \mathsf{CPC}$, i.e. $\varphi \in \mathsf{CPC}$ (since $\varphi \underset{\mathsf{CPC}}{\equiv} \neg\neg\varphi$).
"⇒": Suppose $\neg\neg\varphi \notin \mathsf{IPC}$. By there finite model property there is a finite i.K.m. M and a point w in M s.t. $M, w \nVdash \neg\neg\varphi$. By the Lemma $\exists e \in E[w] : M, e \nVdash \varphi$.

Let V_e be the prop. valuation associated with e. Then we know that $M, e \Vdash \varphi \Leftrightarrow V_e(\varphi) = 1$. Since $M, e \nVdash \varphi$, $V_e(\varphi) = 0$. So $\varphi \notin \mathsf{CPC}$. ■

10.2 Negative translation

Definition 22 (Negative translation of CPC to IPC)**.** *The negative translation of* CPC *to* IPC *is defined as follows:*

- $p^n = \neg\neg p$
- $\bot^n = \bot$
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$
- $(\varphi \to \psi)^n = \varphi^n \to \psi^n$
- $(\varphi \vee \psi)^n = \neg\neg(\varphi^n \vee \varphi^n)$

Remark 21. $\forall \varphi : \varphi \underset{\mathsf{CPC}}{\equiv} \varphi^n$

Lemma 6. $\forall \varphi : \varphi^n \underset{\mathsf{IPC}}{\equiv} \neg\neg\varphi^n$

Proof. By induction on φ. Recall from Exercise Session: (1) $\neg\neg(\varphi \wedge \psi) \underset{\mathsf{IPC}}{\equiv} \neg\neg\varphi \wedge \neg\neg\psi$ (2) $\neg\neg(\varphi \to \psi) \underset{\mathsf{IPC}}{\equiv} \neg\neg\varphi \to \neg\neg\psi$

- Atoms: $p^n = \neg\neg p \underset{\mathsf{IPC}}{\equiv} \neg\neg\neg\neg p = \neg\neg(\neg\neg p) = \neg\neg p^n$
- $\bot$: $\bot^n = \bot$
- $\wedge$: $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n \overset{\text{IH}}{\equiv} \neg\neg\varphi^n \wedge \neg\neg\psi^n \overset{(1)}{\equiv} \neg\neg(\varphi^n \wedge \psi^n) = \neg\neg(\varphi \wedge \psi)^n$
- $\to$: $(\varphi \to \psi)^n = \varphi^n \to \psi^n \overset{\text{IH}}{\equiv} \neg\neg\varphi^n \to \neg\neg\psi^n \overset{(2)}{\equiv} \neg\neg(\varphi^n \to \psi^n) = \neg\neg(\varphi \to \psi)^n$
- $\vee$: $(\varphi \vee \psi)^n = \neg\neg(\varphi^n \vee \psi^n) \underset{\mathsf{IPC}}{\equiv} \neg\neg\neg\neg(\varphi^n \vee \psi^n) = \neg\neg(\neg\neg(\varphi^n \vee \psi^n)) = \neg\neg(\varphi \vee \psi)^n$

■

Theorem 7 ($(.)^n$ is a translation)**.** $\forall \varphi_1, \ldots, \varphi_k, \psi$: $\varphi_1 \wedge \ldots \wedge \varphi_k \vdash_{\mathsf{CPC}} \psi \iff \varphi_1^n \wedge \ldots \wedge \varphi_k^n \vdash_{\mathsf{IPC}} \psi^n$

Proof.

$$
\begin{aligned}
\varphi_1, \ldots, \varphi_k \vdash_{\mathsf{CPC}} \psi &\overset{\chi \underset{\mathsf{CPC}}{\equiv} \chi^n}{\iff} \varphi_1^n, \ldots, \varphi_k^n \vdash_{\mathsf{CPC}} \psi^n \\
&\overset{\mathrm{DT}_{\mathsf{CPC}}}{\iff} \varphi_1^n, \ldots, \varphi_k^n \to \psi^n \in \mathsf{CPC} \\
&\overset{\text{Glivenko}}{\iff} \vdash_{\mathsf{IPC}} \neg\neg(\varphi_1^n, \ldots, \varphi_k^n \to \psi^n) \\
&\overset{(1),(2)}{\iff} \vdash_{\mathsf{IPC}} \neg\neg\varphi_1^n \wedge \ldots \wedge \neg\neg\varphi_k^n \to \neg\neg\psi^n \\
&\overset{\text{Lemma}}{\iff} \vdash_{\mathsf{IPC}} \varphi_1^n \wedge \ldots \wedge \varphi_k^n \to \psi^n \\
&\overset{\mathrm{DT}_{\mathsf{IPC}}}{\iff} \varphi_1^n \wedge \ldots \wedge \varphi_k^n \vdash_{\mathsf{IPC}} \psi^n
\end{aligned}
$$

■

10.3 Modal logic

Definition 23. *The modal language* $\mathcal{L}_\Box$ *is given by:*
$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi$.

Definition 24 (Modal logic $\mathsf{S4}$)**.** $\mathsf{S4}$ *is the modal logic defined by the following Hilbert-style proof system:*

Axioms:

- *All instances of* CPC*-tautologies*
- *K:* $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
- *T:* $\Box\varphi \to \varphi$
- *4:* $\Box\varphi \to \Box\Box\varphi$

Rules:

- *Modus ponens:*

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

– *Necessitation:*

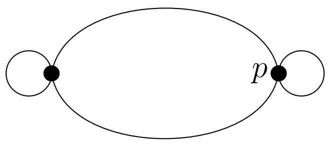
$$\frac{\varphi}{\Box \varphi}$$

Definition 25. $\varphi \in \mathsf{S4} \Leftrightarrow \varphi$ *is provable in above system;* $\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{S4}} \psi \Leftrightarrow \varphi_1 \wedge \ldots \wedge \varphi_n \to \psi \in \mathsf{S4}$.

Definition 26. *An* S4*-model is a structure* $M = \langle W, R, V \rangle$ *where*

- W *is a set*
- R *is a binary relation* $(R \subseteq W \times W)$ *that is reflexive and transitive*
- $V : P \to \wp(W)$

Remark 22. Any i.K.m. is an S4-model, but not conversely:



This is not an i.K.m. because persistency and antisymmetry are violated.

Definition 27 (S4 semantics)**.** *Let* M *be an* S4*-model and* $w \in W$.

- $M, w \vDash$ [1] $P \Leftrightarrow w \in V(p)$
- *connectives as in* CPC
- $M, w \vDash \Box \varphi \Leftrightarrow \forall v \in R[w] : M, v \vDash \varphi$

Definition 28.

$\varphi_1, \ldots, \varphi_k \vDash_{\mathsf{S4}} \psi \Leftrightarrow \forall$S4*-models* M $\forall w \in W : M, w \vDash \varphi_i$ *for all* $i \in$

[1] We use $\vDash$ for modal logic to distinguish it from $\Vdash_{\mathsf{IPC}}$.

$\{1, \ldots, k\} \to M, w \vDash \psi$

Theorem 8 (Soundness & Completeness). $\varphi_1, \ldots, \varphi_k \vDash_{\mathsf{S4}} \psi \iff \varphi_1, \ldots, \varphi_k \vdash_{\mathsf{S4}} \psi$

10.4 Modal translation

Definition 29 (Modal translation of IPC to S4). *The modal translation of* IPC *to* S4 *is defined as follows:*

- $p^\Box = \Box p$
- $\bot^\Box = \bot$
- $(\varphi \wedge \psi)^\Box = \varphi^\Box \wedge \psi^\Box$
- $(\varphi \vee \psi)^\Box = \varphi^\Box \vee \psi^\Box$
- $(\varphi \to \psi)^\Box = \Box(\varphi^\Box \to \psi^\Box)$
- $(\neg\varphi)^\Box = (\varphi \to \bot)^\Box = \Box(\varphi^\Box \to \bot^\Box) = \Box(\varphi^\Box \to \bot) = \Box\neg(\varphi^\Box)$

Example 20.

- $(p \to q)^\Box = \Box(p^\Box \to p^\Box) = \Box(\Box p \to \Box q)$
- $(\neg p)^\Box = \Box\neg(p^\Box) = \Box\neg\Box p$
- $(p \vee \neg p)^\Box = p^\Box \vee (\neg p)^\Box = \Box p \vee \Box\neg\Box p$

Lemma 7. *Let M be an i.K.m. Then M is also an* S4*-model, and it holds that* $\forall w \in W, \varphi \in \mathcal{L}_{\mathsf{IPC}} : M, w \Vdash_{\mathsf{IPC}} \varphi \Leftrightarrow M, w \vDash_{\mathsf{S4}} \varphi^\Box$.

Proof. By induction on φ.

- Atoms:
$$\begin{aligned} M, w \Vdash_{\mathsf{IPC}} p &\Leftrightarrow w \in V(p) \\ &\overset{\text{persistency}}{\Leftrightarrow} \forall v \in R[w] : v \in V(p) \\ &\overset{\text{Def. }\mathsf{S4}\text{ semantics}}{\Leftrightarrow} \forall v \in R[w] : M, v \vDash p \\ &\Leftrightarrow M, w \vDash_{\mathsf{S4}} \Box p \\ &\Leftrightarrow M, w \vDash_{\mathsf{S4}} p^{\Box} \end{aligned}$$

- $\bot, \wedge, \vee$ are straightforward; example:
$$\begin{aligned} M, w \Vdash_{\mathsf{IPC}} \varphi \wedge \psi &\Leftrightarrow M, w \Vdash_{\mathsf{IPC}} \varphi \text{ and } M, w \Vdash_{\mathsf{IPC}} \psi \\ &\overset{\text{IH}}{\Leftrightarrow} M, w \vDash_{\mathsf{S4}} \varphi^{\Box} \text{ and } M, w \vDash_{\mathsf{S4}} \psi^{\Box} \\ &\overset{\text{Def. S4 semantics}}{\Leftrightarrow} M, w \vDash_{\mathsf{S4}} \varphi^{\Box} \wedge \psi^{\Box} \\ &\overset{\text{Def.}}{\Leftrightarrow} M, w \vDash_{\mathsf{S4}} (\varphi \wedge \psi)^{\Box} \end{aligned}$$

- $\to$:
$$\begin{aligned} M, w \Vdash_{\mathsf{IPC}} \varphi \to \psi &\Leftrightarrow \forall v \in R[w] : M, v \Vdash \varphi \Rightarrow M, v \Vdash \psi \\ &\overset{\text{IH}}{\Leftrightarrow} \forall v \in R[w] : M, v \vDash_{\mathsf{S4}} \varphi^{\Box} \Rightarrow M, v \vDash_{\mathsf{S4}} \psi^{\Box} \\ &\Leftrightarrow \forall v \in R[w] : M, v \vDash_{\mathsf{S4}} \varphi^{\Box} \to \psi^{\Box} \\ &\Leftrightarrow M, w \vDash_{\mathsf{S4}} \Box(\varphi^{\Box} \to \psi^{\Box}) \\ &\Leftrightarrow M, w \vDash_{\mathsf{S4}} (\varphi \to \psi)^{\Box} \end{aligned}$$

■

Theorem 9 ($(.)^{\Box}$ is a translation)**.** $\varphi_1, \ldots, \varphi_k \vdash_{\mathsf{IPC}} \psi \;\Leftrightarrow\; \varphi_1^{\Box}, \ldots, \varphi_k^{\Box} \vdash_{\mathsf{S4}} \psi^{\Box}$

Proof. "$\Leftarrow$": Suppose $\varphi_1, \ldots, \varphi_k \nvdash_{\mathsf{IPC}} \psi$. Then $\exists$ i.K.m M, a point w in M s.t. $M, w \Vdash_{\mathsf{IPC}} \varphi_i$ for $i \in \{1, \ldots, k\}$ and $M, w \nVdash_{\mathsf{IPC}} \psi$.
By the previous Lemma, $M, w \vDash_{\mathsf{S4}} \varphi_i^{\Box}$ for $i \in \{1, \ldots, k\}$ and $M, w \nvDash_{\mathsf{S4}} \psi^{\Box}$.
Since M is an S4-model, $\varphi_1^{\Box}, \ldots, \varphi_k^{\Box} \nvdash_{\mathsf{S4}} \psi^{\Box}$.
"$\Rightarrow$":

1. Let $M = \langle W, R, V \rangle$ be a $\mathsf{S4}$ countermodel to $\varphi^{\Box}$.
2. Define an equivalence relation:
   $w \sim v :\Leftrightarrow wRv \;\&\; vRw$
   $\overline{w} := \{v \in W \mid w \sim v\}$
   Show that $\sim$ is an equivalence relation:

- reflexivity: yes, since M is an $\mathsf{S4}$ model
- symmetry: obvious
- transitivity: R is transitive, so $\sim$ is transitive

3. $W' := \{\overline{w} \mid w \in W\}$
   $\overline{w} R' \, \overline{v} :\Leftrightarrow wRv$
   well-defined since if $\overline{w} = \overline{w'}, \overline{v} = \overline{v'}$, then $wRw', w'Rw, vRv', v'Rv$
   $wRv \xRightarrow{\text{transitivity of } R} w'Rv'$

$$\begin{tikzcd} w \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & w' \arrow[d, leftrightarrow] \\ v \arrow[r, leftrightarrow] & v' \end{tikzcd}$$

4. $V'(p) = \{\overline{w} \in W' \mid R[w] \subseteq V(p)\}$
   M' is an i.K.m:
   - R' is reflexive: $\overline{w} R' \overline{w} \iff wRw$ since R is reflexive.
   - R' is transitive: $\overline{w} R' \overline{v} \,\&\, \overline{v} R' \overline{u} \iff wRv \,\&\, uRv \implies wRu \iff \overline{w} R' \overline{u}$
   - R' is antisymmetric: $\overline{w} R' \overline{v} \,\&\, \overline{v} R' \overline{w} \iff wRv \,\&\, vRv \iff \overline{w} = \overline{v}$
   - V' is persistent: $\overline{w} \in V'(p) \,\&\, \overline{w} R' \overline{v} \iff R[w] \subseteq V(p) \,\&\, wRv$
     $\xRightarrow{\text{transitivity of } R} R[v] \subseteq R[w] \subseteq V(p)$
     $\iff \overline{v} \in V(p)$

   $\forall \varphi : M, w \vDash \varphi^{\Box} \iff M', \overline{w} \Vdash \varphi$ By induction on φ

   <u>Atomic</u>:

$$\begin{aligned} M', \overline{w} \Vdash p \iff \overline{w} \in V'(p) &\iff R[w] \subseteq V(p) \\ &\iff \forall v \in R[w] : M', v \vDash p \\ &\iff M', w \vDash \underbrace{\Box p}_{p^{\Box}} \end{aligned}$$

   <u>$\wedge, \vee, \bot$</u>: Exercise.

$\underline{\rightarrow}$: $\varphi : \psi \rightarrow \chi$

$$\begin{aligned}
M', \overline{w} \Vdash \psi \rightarrow \chi &\iff \forall \overline{v} \in R'[\overline{w}] : (M', \overline{v}' \Vdash \psi \implies M', \overline{v} \Vdash \chi) \\
&\overset{I.H.}{\iff} \forall \overline{v} \in R'[\overline{w}] : (M', v \vDash \psi^{\Box} \implies M', v \vDash \chi^{\Box}) \\
&\iff \forall v \in R[w] : M', v \vDash \psi^{\Box} \rightarrow \chi^{\Box} \\
&\iff M', w \vDash \underbrace{\Box(\psi^{\Box} \rightarrow \chi^{\Box})}_{(\psi \rightarrow \chi)^{\Box}}
\end{aligned}$$

5. M' is a countermodel to φ modulo modal translation.

■

Chapter 11
Topological Semantics

11.1 Overview of different semantics for IPC

$$\text{Kripke} \underset{\text{special case}}{\subseteq} \text{Topological} \underset{\text{special case}}{\subseteq} \text{Algebraic}$$

11.2 Possible world semantics for CPC

Definition 30 (Possible world model). *A possible world model for* CPC *is a pair* $M = \langle W, V \rangle$*, where*

- W *is a set, and*
- $V : P \to \wp(W)$.

The *truth-set* $[\varphi]$ of the formula φ is understood as the set of points in W at which φ holds, i.e. $[\varphi] = \{w \in W \mid \varphi \text{ is true at } w\}$.

Definition 31 (Truth-set). *The truth-set of φ in M is given by:*

- $[p]_M = V(p)$
- $[\varphi \vee \psi]_M = [\varphi]_M \cup [\psi]_M$
- $[\bot]_M = \emptyset$
- $[\varphi \to \psi]_M = \overline{[\varphi]}_M \cup [\psi]_M$

- $[\varphi \wedge \psi]_M = [\varphi]_M \cap [\psi]_M$
- $[\neg\varphi]_M = [\varphi \to \bot]_M = \overline{[\varphi]}_M \cup \emptyset = \overline{[\varphi]}_M$

Proposition 5. $\varphi \in \mathsf{CPC} \Leftrightarrow$ *For all possible world models* $M = \langle W, V \rangle$*:* $[\varphi]_M = W$.

11.3 Topologies on sets

Definition 32 (Topology)**.** *Let* W *be a set and let* $\tau \subseteq \wp(W)$*. Then* τ *is called a* topology *if*

- $\emptyset, W \in \tau$
- $X_1, \ldots, X_n \in \tau \Rightarrow \bigcap_{i=1}^n X_i \in \tau$ *(closure under finite intersections)*
- $X_i \in \tau$ *for* $i \in I$ *an arbitrary index set* $\Rightarrow \bigcup_{i \in I} X_i \in \tau$ *(closure under arbitrary unions)*

Elements $X \in \tau$ *are called* open sets*. Complements of open sets are called* closed sets.

Some examples

Given any set W, the following are topologies:

Example 21 (Trivial topology)**.** $\tau = \{\emptyset, W\}$

The trivial topology on W is the *least* topology on W; it includes as few sets as possible.

Example 22 (Discrete topology)**.** $\tau = \wp(W)$

The discrete topology on W is the *greatest* topology on W; it includes as many sets as possible.

Euclidean topology on $\mathbb{R}^n$

Let's look at a special kind of topology, namely Euclidean topologies.

Example 23 (Euclidean topology on $\mathbb{R}^n$). $\tau = \{X \subseteq \mathbb{R}^n \mid \forall w \in X\ \exists \varepsilon > 0 : B(w, \varepsilon) \subseteq X\}$, where $B(w, \varepsilon) = \{x \in \mathbb{R}^n \mid d(w, x) < \varepsilon\}$ is the open ball with center w and radius ε and $d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$ is the Euclidean metric.

11.4 Topological spaces

Definition 33 (Topological space). *A* topological space *is a pair* $\langle W, \tau \rangle$ *where*

- W *is a set, and*
- τ *is a topology on* W.

11.5 Interior

Definition 34 (Interior). *Let* $\langle W, \tau \rangle$ *be a topological space. The interior of* $X \subseteq W$ *is*

$$X^{\text{int}} := \bigcup \{Y \in \tau | Y \subseteq X\}.$$

Remark 23.

- $X^{\text{int}} \in \tau$ (X^{int} is open)
- $X^{\text{int}} \subseteq X$
- $\forall Y \in \tau : Y \subseteq X \Rightarrow Y \subseteq X^{\text{int}}$ (X^{int} is the *greatest* open set included in X)
- $X \in \tau \Leftrightarrow X = X^{\text{int}}$

11.6 Topological models

Definition 35 (Topological model). *A topological model is a triple* $T = \langle W, \tau, V \rangle$ *where*

- $\langle W, \tau \rangle$ *is a topological space, and*

- $V : \mathcal{P} \to \tau$.

Definition 36. *If T is a topological model, then the truth-set of φ is given as follows:*

- $[p]_T = V(p)$
- $[\bot]_T = \emptyset$
- $[\varphi \vee \psi]_T = [\varphi]_T \cup [\psi]_T$
- $[\varphi \wedge \psi]_T = [\varphi]_T \cap [\psi]_T$
- $[\varphi \to \psi]_T = (\overline{[\varphi]}_T \cup [\psi]_T)^{int}$
- $[\neg\varphi]_T = [\varphi \to \bot]_T = (\overline{[\varphi]}_T \cup [\bot]_T)^{int} = (\overline{[\varphi]}_T \cup \emptyset)^{int} = \overline{[\varphi]}_T^{int}$

11.7 Consequence and validity

Definition 37 (Topological consequence). $\varphi_1, \ldots, \varphi_k \vDash_{top} \psi \Leftrightarrow$ *for every topological model T:*

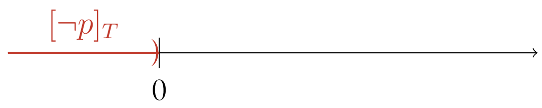
$$[\varphi_1 \wedge \ldots \wedge \varphi_k]_T \subseteq [\psi]_T.$$

Definition 38 (Topological validity). $\vDash_{top} \varphi \Leftrightarrow [\varphi]_T = W$ *for all topological models T.*

Example 24. Consider the topological model $T = \langle \mathbb{R}, \tau, V \rangle$ where τ is the Euclidean topology on $\mathbb{R}$ and $V(p) = \mathbb{R}^+$.

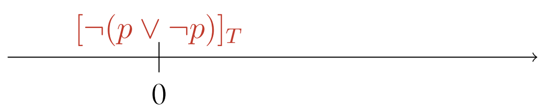
$$\begin{aligned} [p]_T &= V(p) \\ &= \mathbb{R}^+ \end{aligned}$$

$$
\begin{aligned}
[\neg p]_T &= \overline{[p]_T}^{\mathrm{int}} \\
&= (\mathbb{R}^- \cup \{0\})^{\mathrm{int}} \\
&= \mathbb{R}^-
\end{aligned}
$$

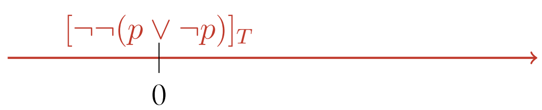


$$
\begin{aligned}
[p \vee \neg p]_T &= [p]_T \cup [\neg p]_T \\
&= \mathbb{R}^+ \cup \mathbb{R}^- \\
&= \mathbb{R} \setminus \{0\} \\
&\neq \mathbb{R} \quad \rightsquigarrow \quad p \vee \neg p \notin \mathsf{IPC}
\end{aligned}
$$

$$
\begin{aligned}
[\neg(p \vee \neg p)]_T &= \overline{[p \vee \neg p]_T}^{\mathrm{int}} \\
&= \{0\}^{\mathrm{int}} \\
&= \emptyset
\end{aligned}
$$



$$
\begin{aligned}
[\neg\neg(p \vee \neg p)]_T &= \overline{[\neg(p \vee \neg p)]_T}^{\mathrm{int}} \\
&= \overline{\emptyset}^{\mathrm{int}} \\
&= \mathbb{R}^{\mathrm{int}} \\
&= \mathbb{R} \quad \rightsquigarrow \quad \neg\neg(p \vee \neg p) \in \mathsf{IPC}
\end{aligned}
$$



11.8 Soundness and completeness

Theorem 10. *$\varphi \in \mathsf{IPC} \Leftrightarrow$ for every topological model $T : [\varphi]_T = w$.*

Soundness

Proof of Theorem 10 ($\Rightarrow$). Exercise. (Check all axioms in a Hilbert-style system for IPC, as well as modus ponens.) ■

Completeness

A canonical topological model construction is possible, but we will get completeness on the cheap by using the fact that intuitionistic Kripke semantics is a special case of topological semantics. Let $M = \langle W, R, V\rangle$ be an i.K.m. and let

$$\begin{aligned}\tau_R :&= \{X \subseteq W \mid X \text{ is an upward-closed set}\}\\ &= \{X \subseteq W \mid \forall w, v : w \in X \;\&\; v \in R[w] \Rightarrow v \in X\}\end{aligned}$$

Remark 24.

- τ_R is a topology on W.
- $T_M := \langle W, \tau_R, V\rangle$ is a topological model (by persistency $V(p) \in \tau_R$).
- $\forall w \in W : R[w]$ is the smallest/least open set which contains w.

Proposition 6. *For every i.K.m. M, point w, formula φ: $M, w \Vdash \varphi \;\Leftrightarrow\; w \in [\varphi]_{T_M}$.*

Proof. By induction on φ. Let's consider $\varphi = \psi \to \chi$.

"$\Rightarrow$": Suppose $M, w \Vdash \psi \to \chi$. Then $\forall v \in R[w] : M, v \nVdash \psi$ or $M, v \Vdash \chi$.
By IH: $\forall v \in R[w] : v \notin [\psi]_{T_M}$ or $v \in [\chi]_{T_M}$, i.e. $\forall v \in R[w] : v \in \overline{[\psi]}_{T_M} \cup [\chi]_{T_M}$, i.e. $R[w] \subseteq \overline{[\psi]}_{T_M} \cup [\chi]_{T_M}$. But $R[w] \in \tau$, so $R[w] \subseteq (\overline{[\psi]}_{T_M} \cup [\chi]_{T_M})^{\text{int}}$, so $w \in (\overline{[\psi]}_{T_M} \cup [\chi]_{T_M})^{\text{int}} \;\Leftrightarrow\; w \in [\psi \to \chi]_{T_M}$.

"$\Leftarrow$": Suppose $w \in [\psi \to \chi]_{T_M} = (\overline{[\psi]}_{T_M} \cup [\chi]_{T_M})^{\text{int}}$.
$\Rightarrow R[w] \subseteq (\overline{[\psi]}_{T_M} \cup [\chi]_{T_M})^{\text{int}}$ (since $(\overline{[\psi]}_{T_M} \cup [\chi]_{T_M})^{\text{int}}$ is open and $R[w]$ is the smallest open set containing w), and

$$\begin{aligned} R[w] \subseteq (\overline{[\psi]}_{T_M} \cup [\chi]_{T_M})^{\text{int}} &\Leftrightarrow \forall v \in R[w] : v \in (\overline{[\psi]}_{T_M} \cup [\chi]_{T_M})^{\text{int}} \subseteq \overline{[\psi]}_{T_M} \cup [\chi]_{T_M} \\ &\Leftrightarrow \forall v \in R[w] : v \in \overline{[\psi]}_{T_M} \text{ or } v \in [\chi]_{T_M} \\ &\overset{\text{IH}}{\Leftrightarrow} \forall v \in R[w] : M, v \nVdash \psi \text{ or } M, v \Vdash \chi \\ &\Leftrightarrow M, w \Vdash \psi \to \chi, \end{aligned}$$

so $M, w \Vdash \psi \to \chi$. ■

Proof of Theorem 10 ($\Leftarrow$). Suppose $\varphi \notin \mathsf{IPC}$. Then there exists an i.K.m. M and point w such that $M, w \nVdash \varphi$. By Proposition 6, T_M is a topological model with $w \notin [\varphi]_{T_M}$ and so $[\varphi]_{T_M} \neq W$. ■

C H A P T E R 12

Algebraic Semantics

$$\text{Kripke} \quad \underset{\text{special case}}{\subseteq} \quad \text{Topological} \quad \underset{\text{special case}}{\subseteq} \quad \text{Algebraic}$$

12.1 Heyting algebras

The general idea of algebraic semantics is to consider the propositions expressed by sentences as points in an algebra and to view connectives as expressing operations in this algebra. In the case of intuitionistic logic, the central notion is that of a *Heyting algebra*, which can be introduced in two equivalent ways, as partially-ordered sets, or as algebras, i.e. sets with a number of operations on them.

$$\begin{tikzcd}[column sep=small] & \text{Heyting algebras} \arrow[dl, dash] \arrow[dr, dash] & \\ \text{Posets } \langle A, \leq \rangle & & \text{Algebras } \langle A, \top, \bot, \wedge, \vee, \to \rangle \end{tikzcd}$$

Definition 39 (Heyting algebra)**.** *A Heyting algebra is a partially-ordered set* $\langle A, \leq \rangle$ *such that*

- *there is a top element* $\top_A$*, and a bottom element* $\bot_A$*.*

- $\forall a, b \in A$ *there is a meet, i.e. an element* $a \wedge b$ *s.t.*

$$\left.\begin{array}{l} -a \wedge b \leq a, a \wedge b \leq b \\ -\forall c: \ c \leq a \ \& \ c \leq b \ \Rightarrow \ c \leq a \wedge b \end{array}\right\} \forall c : c \leq a \ \& \ c \leq b \ \Leftrightarrow \ c \leq a \wedge b$$

- $\forall a, b \in A$ *there is a join, i.e. an element* $a \vee b$ *s.t.*

$$\left.\begin{array}{l} -a \vee b \geq a \ \& \ a \vee b \geq b \\ -\forall c : c \geq a \ \& \ c \geq b \ \Rightarrow \ c \geq a \vee b \end{array}\right\} \forall c : c \geq a \ \& \ c \geq b \ \Leftrightarrow \ c \geq a \vee b$$

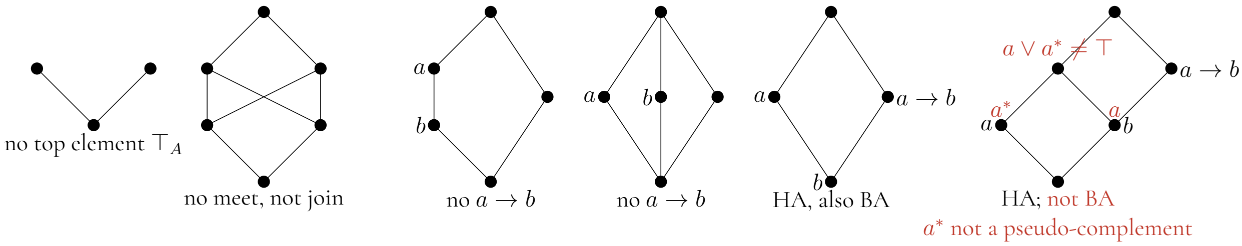
- $\forall a, b \in A$ *there is an implication of* a *and* b*, i.e. an element* $a \to b$ *s.t.*

$$\left.\begin{array}{l} -a \wedge (a \to b) \leq b \\ -\forall c : a \wedge c \leq b \ \Rightarrow \ c \leq (a \to b) \end{array}\right\} \forall c : a \wedge c \leq b \ \Leftrightarrow \ c \leq (a \to b)^1$$

[1] This is like a deduction theorem in algebraic form.

Remark 25. $a^* := a \to \bot_A$ is called the pseudo-complement of a.

Remark 26. If $\forall a \in A$ we have $a \vee a^* = \top_A$, then $\langle A, \leq \rangle$ is called a *Boolean algebra.*



12.2 Heyting algebra models

Definition 40. *An HA-model is* $\mathbb{A} = \langle A, \leq, V \rangle$*, where* $\langle A, \leq \rangle$ *is a Heyting algebra and* $V : P \to A$*. Given such a model, we define a map* $[.]_{\mathbb{A}} : \mathcal{L}_{\mathcal{P}} \to A$ *as follows:*

- $[p]_{\mathbb{A}} = V(p)$
- $[\bot]_{\mathbb{A}} = \bot_{\mathbb{A}}$
- $[\varphi \circ \psi]_{\mathbb{A}} = [\varphi]_{\mathbb{A}} \circ \ [\psi]_{\mathbb{A}}$ *for* $\circ \in \{\wedge, \vee, \to\}$

12.3 Consequence and validity

Definition 41 (Algebraic consequence). $\varphi_1, \ldots, \varphi_k \vDash_{alg} \psi \Leftrightarrow$ *for every HA-model* $\mathbb{A}$*:*

$$[\varphi_1]_{\mathbb{A}} \wedge \ldots \wedge [\varphi_k]_{\mathbb{A}} \leq [\psi]_{\mathbb{A}}.$$

Definition 42 (Algebraic validity). $\vDash_{alg} \varphi \Leftrightarrow [\varphi]_{\mathbb{A}} = \top$ *for all HA-models* $\mathbb{A}$.

12.4 Soundness and completeness

Theorem 11. $\varphi_1, \ldots, \varphi_k \vdash_{\mathsf{IPC}} \psi \Leftrightarrow \varphi_1, \ldots, \varphi_k \vDash_{alg} \psi$

Soundness

Proof of Theorem 11 (⇒). Check a Hilbert-style system for IPC.

1. Verify that all tautologies are mapped to the top element.
2. Check modus ponens: Verify that $[\varphi]_{\mathbb{A}} \wedge [\varphi \to \psi]_{\mathbb{A}} \leq [\psi]_{\mathbb{A}}$ for all HA-models $\mathbb{A}$.

■

Completeness

There are two ways to prove completeness:

1. By seeing that topological semantics is a special case of algebraic semantics.
2. By using a Lindenbaum-Tarski construction.

1. Topological semantics as a special case of algebraic semantics

Proposition 7. *Let* $\langle W, \tau \rangle$ *be a topological space. Then* $A_T = \langle \tau, \subseteq \rangle$ *is a Heyting algebra with*

- $\top = W$
- $\bot = \emptyset$
- $X \wedge Y = X \cap Y$
- $X \vee Y = X \cup Y$
- $X \to Y = (\overline{X} \cup Y)^{int}$

Proof. We show that $(\overline{X} \cup Y)^{\text{int}}$ is the implication of X and Y, i.e.:

(i) $X \cap (\overline{X} \cup Y)^{\text{int}} \subseteq Y$

(ii) $\forall Z \in \tau : X \cap Z \subseteq Y \;\Rightarrow\; Z \subseteq (X \to Y) = (\overline{X} \cup Y)^{\text{int}}$

Proof of (i): $X \cap (\overline{X} \cup Y)^{\text{int}} \subseteq X \cap (\overline{X} \cup Y) = (X \cap \overline{X}) \cup (X \cap Y) = X \cap Y \subseteq Y$

Proof of (ii): Suppose $Z \in \tau$ s.t. $X \cap Z \subseteq Y$. Then $Z \subseteq \overline{X} \cup Y$. Since $Z \in \tau$, $Z \subseteq (\overline{X} \cup Y)^{\text{int}}$. ■

Corollary 2. If $T = \langle W, \tau, V\rangle$ is a topological model, then $\mathbb{A}_T = \langle \tau, \subseteq, V\rangle$ is an HA-model and $\forall \varphi \in \mathcal{L}_\mathcal{P} : [\varphi]_T = [\varphi]_{\mathbb{A}_T}$.

Proof. Immediate, since the operations on $\mathbb{A}_T$ are exactly the ones used by topological semantics. ■

Proof of Theorem 11 (⇐). By contraposition. Suppose that $\varphi_1, \dots, \varphi_k \nvdash_{\mathsf{IPC}} \psi$. By the completeness of IPC for topological semantics, there exists a topological model $T = \langle W, \tau, V\rangle$ such that $[\varphi_1 \wedge \dots \wedge \varphi_k]_T \nsubseteq [\psi]_T$. By Corollary 2, there is an algebraic model $\mathbb{A}_T = \langle \tau, \subseteq, V\rangle$ such that $[\varphi_1 \wedge \dots \wedge \varphi_k]_{\mathbb{A}_T} \nsubseteq [\psi]_{\mathbb{A}_T}$. By Definition 40, $[\varphi_1]_{\mathbb{A}_T} \wedge \dots \wedge [\varphi_k]_{\mathbb{A}_T} \nsubseteq [\psi]_{\mathbb{A}_T}$. Therefore, by Definition 41, $\varphi_1, \dots, \varphi_k \nvDash_{\text{alg}} \psi$. ■

Remark 27. If $M = \langle W, R, V\rangle$ is an i.K.m., then $\mathbb{A}_M = \langle Up_R(W), \subseteq, V\rangle$ is a HA-model such that $M, w \Vdash \varphi \;\Leftrightarrow\; w \in [\varphi]_{\mathbb{A}_M}$.

In sum, we can say that the following relationships exist between Kripke semantics, topological semantics, and algebraic semantics:

$$\begin{tikzcd}
\text{Kripke} & \text{Topological} & \text{Algebraic} \\
\langle W, R, V\rangle \arrow[r] & \langle W, Up_R(W), V\rangle & \\
& \langle W, \tau, V\rangle \arrow[r] & \langle \tau, \subseteq, V\rangle
\end{tikzcd}$$

Definition 43. *The* IPC*-equivalence class of a formula* φ *is defined as*

$$\overline{\varphi}^{\mathsf{IPC}} := \{\psi \mid \varphi \underset{\mathsf{IPC}}{\equiv} \psi\}.$$

Definition 44 (Lindenbaum-Tarski algebra of IPC)**.** *The Lindenbaum-Tarski algebra of* IPC *is* $LT_{\mathsf{IPC}} = \langle A_{\mathsf{IPC}}, \leq_{\mathsf{IPC}} \rangle$ *where*

- $A_{\mathsf{IPC}} := \{\overline{\varphi}^{\mathsf{IPC}} \mid \varphi \in \mathcal{L}_{\mathcal{P}}\}$
- $\overline{\varphi}^{\mathsf{IPC}} \leq_{\mathsf{IPC}} \overline{\psi}^{\mathsf{IPC}} :\Leftrightarrow \varphi \vdash_{\mathsf{IPC}} \psi$

Note that $\leq_{\mathsf{IPC}}$ *is well-defined; the definition doesn't depend on the* φ *and* ψ *we use to represent the equivalence classes* $\overline{\varphi}^{\mathsf{IPC}}$ *and* $\overline{\psi}^{\mathsf{IPC}}$*. Moreover,* $\leq_{\mathsf{IPC}}$ *is a partial order because equivalence classes are disjoint or equal.*

Proposition 8. LT_{IPC} *is a Heyting algebra with*

- $\top_{LT_{\mathsf{IPC}}} = \overline{\top}^{\mathsf{IPC}}$
- $\bot_{LT_{\mathsf{IPC}}} = \overline{\bot}^{\mathsf{IPC}}$
- $\overline{\varphi}^{\mathsf{IPC}} \wedge \overline{\psi}^{\mathsf{IPC}} = \overline{\varphi \wedge \psi}^{\mathsf{IPC}}$
- $\overline{\varphi}^{\mathsf{IPC}} \vee \overline{\psi}^{\mathsf{IPC}} = \overline{\varphi \vee \psi}^{\mathsf{IPC}}$
- $\overline{\varphi}^{\mathsf{IPC}} \to \overline{\psi}^{\mathsf{IPC}} = \overline{\varphi \to \psi}^{\mathsf{IPC}}$

Proof. As an example, we will show $\overline{\varphi}^{\mathsf{IPC}} \wedge \overline{\psi}^{\mathsf{IPC}} = \overline{\varphi \wedge \psi}^{\mathsf{IPC}}$. Given the definition of a Heyting algebra, we need to show

(i) $\overline{\varphi \wedge \psi}^{\mathsf{IPC}} \leq \overline{\varphi}^{\mathsf{IPC}}$ and $\overline{\varphi \wedge \psi}^{\mathsf{IPC}} \leq \overline{\psi}^{\mathsf{IPC}}$;

(ii) $\forall \overline{\chi}^{\mathsf{IPC}} : \overline{\chi}^{\mathsf{IPC}} \leq \overline{\varphi}^{\mathsf{IPC}}$ and $\overline{\chi}^{\mathsf{IPC}} \leq \overline{\psi}^{\mathsf{IPC}} \;\Rightarrow\; \overline{\chi}^{\mathsf{IPC}} \leq \overline{\varphi \wedge \psi}^{\mathsf{IPC}}$.

Given the definition of LT_{IPC}, this means

- $\varphi \wedge \psi \vdash_{\mathsf{IPC}} \varphi$ and $\varphi \wedge \psi \vdash_{\mathsf{IPC}} \psi$;

- $\forall \chi : \chi \vdash_{\mathsf{IPC}} \varphi$ and $\chi \vdash_{\mathsf{IPC}} \psi \;\Rightarrow\; \chi \vdash_{\mathsf{IPC}} \varphi \wedge \psi$.

(i) holds by $(\wedge e)$, (ii) holds by $(\wedge i)$. ■

Definition 45 (Canonical algebraic model for IPC)**.** *The canonical algebraic model for* IPC *is*

$$\mathbb{A}_{\mathsf{IPC}} = \langle A_{\mathsf{IPC}}, \leq_{\mathsf{IPC}}, V_{\mathsf{IPC}} \rangle$$

where $V_{\mathsf{IPC}} : \mathcal{P} \to A_{\mathsf{IPC}}$ *such that* $V_{\mathsf{IPC}}(p) = \overline{p}^{\mathsf{IPC}}$.

Proposition 9. $\forall \varphi \in \mathcal{L}_{\mathcal{P}} : [\varphi]_{\mathbb{A}_{\mathsf{IPC}}} = \overline{\varphi}^{\mathsf{IPC}}$.

Proof. By induction on φ:

- $\varphi = p$: $[p]_{\mathbb{A}_{\mathsf{IPC}}} = V_{\mathsf{IPC}}(p) = \overline{p}^{\mathsf{IPC}}$
- $\varphi = \psi \wedge \chi$: $[\psi \wedge \chi]_{\mathbb{A}_{\mathsf{IPC}}} = [\psi]_{\mathbb{A}_{\mathsf{IPC}}} \wedge [\chi]_{\mathbb{A}_{\mathsf{IPC}}} \overset{\text{IH}}{=} \overline{\psi}^{\mathsf{IPC}} \wedge \overline{\chi}^{\mathsf{IPC}} \overset{\text{Prop. 8}}{=} \overline{\psi \wedge \chi}^{\mathsf{IPC}}$
- The other cases are analogous.

■

Now we can prove the completeness of IPC for algebraic semantics directly:

Proof of Theorem 11 ($\Leftarrow$*).*

$$\begin{aligned}
\varphi_1, \dots, \varphi_k \vDash_{\text{alg}} \psi &\overset{\text{Def. 41, } \mathbb{A}_{\mathsf{IPC}} \text{ is an HA-model}}{\Rightarrow} [\varphi_1]_{\mathbb{A}_{\mathsf{IPC}}} \wedge \dots \wedge [\varphi_k]_{\mathbb{A}_{\mathsf{IPC}}} \leq_{\mathsf{IPC}} [\psi]_{\mathbb{A}_{\mathsf{IPC}}} \\
&\overset{\text{Prop. 9}}{\Leftrightarrow} \overline{\varphi_1}^{\mathsf{IPC}} \wedge \dots \wedge \overline{\varphi_k}^{\mathsf{IPC}} \leq_{\mathsf{IPC}} \overline{\psi}^{\mathsf{IPC}} \\
&\overset{\text{Prop. 8}}{\Leftrightarrow} \overline{\varphi_1 \wedge \dots \wedge \varphi_k}^{\mathsf{IPC}} \leq_{\mathsf{IPC}} \overline{\psi}^{\mathsf{IPC}} \\
&\overset{\text{Def. 44}}{\Leftrightarrow} \varphi_1 \wedge \dots \wedge \varphi_k \vdash_{\mathsf{IPC}} \varphi \\
&\Leftrightarrow \varphi_1, \dots, \varphi_k \vdash_{\mathsf{IPC}} \varphi
\end{aligned}$$

■

Chapter 13

Lindenbaum-Tarski Algebras & Local Tabularity

13.1 The set $\mathcal{L}_{p_1,\dots,p_n}/L$ of L-equivalence classes over $\mathcal{L}_{p_1,\dots,p_n}$

$\mathcal{L}_{p_1,\dots,p_n}$ is the set of formulas built up from the propositional variables $p_1,\dots,p_n$ as specified by Definition 4:

Definition 46. $\mathcal{L}_{p_1,\dots,p_n} := \mathcal{L}_{\mathcal{P}}$ *where* $\mathcal{P} = \{p_1,\dots,p_n\}$

If L is a logic, then $\mathcal{L}_{p_1,\dots,p_n}/L$ is the set of all L-equivalence classes of formulas in $\mathcal{L}_{p_1,\dots,p_n}$:

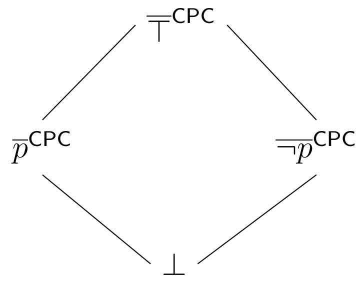
Definition 47. $\mathcal{L}_{p_1,\dots,p_n}/L := \{\overline{\varphi}^L \mid \varphi \in \mathcal{L}_{p_1,\dots,p_n}\}$

Example 25.

$$\begin{aligned}\mathcal{L}_p/\mathsf{CPC} &= \{\overline{\varphi}^{\mathsf{CPC}} \mid \varphi \in \mathcal{L}_p\}\\ &= \{\overline{\top}^{\mathsf{CPC}}, \overline{\bot}^{\mathsf{CPC}}, \overline{p}^{\mathsf{CPC}}, \overline{\neg p}^{\mathsf{CPC}}\}\end{aligned}$$

13.2 Lindenbaum-Tarski algebra for $\mathcal{L}_p/\mathsf{CPC}$

The Lindenbaum-Tarski algebra LT^p_{CPC} of CPC in propositional letter p is:



Proposition 10. *$\mathcal{L}_{p_1,\ldots,p_n}/\mathsf{CPC}$ has 2^{2^n} elements.*

Proof. An equivalence class $\overline{\varphi}^{\mathsf{CPC}}$ is determined by a set of valuations—those that make φ true. Conversely, each set of valuations determines an equivalence class. Thus, the elements of $\mathcal{L}_{p_1,\ldots,p_n}/\mathsf{CPC}$ are in 1-1 correspondence with the sets of valuations for $p_1, \ldots, p_n$. There are 2^n valuations for $p_1 \ldots p_n$ and so 2^{2^n} such sets. ■

13.3 Local tabularity

Definition 48. *A propositional logic L is locally tabular if $\forall n \in \mathbb{N}, \mathcal{L}_{p_1,\ldots,p_n}/L$ is finite.*

Intuitively, if L is locally tabular then with finitely many atoms we can express only finitely-many non-equivalent meanings.

Example 26. CPC is locally tabular.

Theorem 12. *IPC is not locally tabular.*

That is, in IPC even with finitely many atoms—in fact, even a single atom—we can express infinitely many non-equivalent meanings. To show this, we consider a special frame called the *Rieger-Nishimura ladder*.

13.4 Rieger-Nishimura ladder

We want to show that $\mathcal{L}_{\mathcal{P}}/_{\mathsf{IPC}}$ is infinite. Given that equivalent formulas must be forced at the same points in intuitionistic Kripke semantics, we can do so by constructing an i.K.m. with an infinite set of points such that no two points force the same formulas. The basis for our model will be provided by the following frame, known as the *Rieger-Nishimura ladder*:

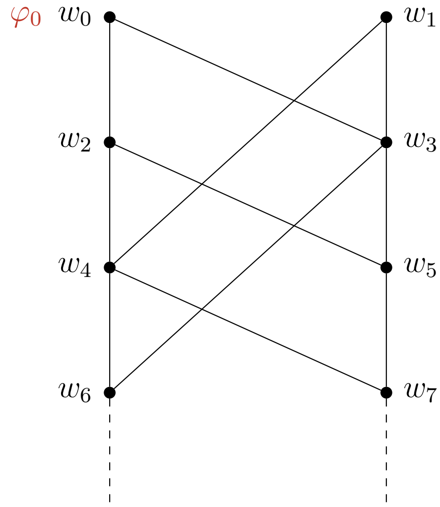


Figure 13.1: *The Rieger-Nishimura ladder*

13.5 The model RN

Next, we turn this frame into a model, call it RN, such that each point w_i is associated with a formula φ_i such that φ_i is forced *only* at w_i (and, by persistency, at its successors). We define the φ_is inductively as follows:

$$
\begin{aligned}
\varphi_0 &:= p \\
\varphi_1 &:= \neg p \\
\varphi_2 &:= \neg\neg p \\
\varphi_3 &:= \neg\neg p \to p \\
\varphi_{n+4} &:= \varphi_{n+3} \to \varphi_n \vee \varphi_{n+1}
\end{aligned}
$$

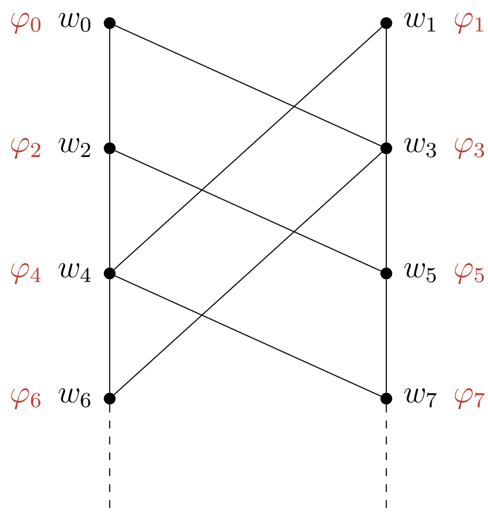


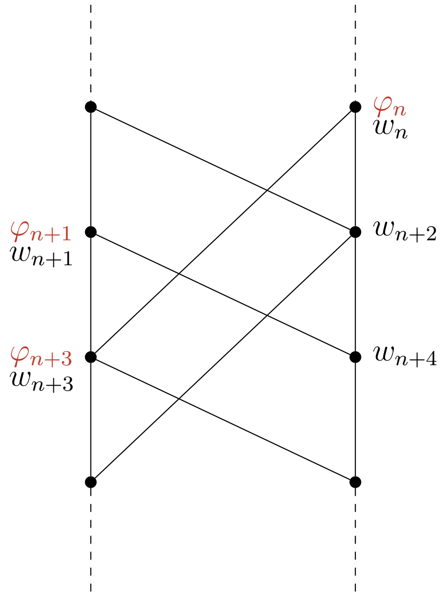
Figure 13.2: *The model RN, based on the Rieger-Nishimura ladder*

Next, we have to check that defining the φ_is in this way does indeed work, i.e. that φ_i is forced *only* at w_i and its successors.

Proposition 11. $\forall n \in \mathbb{N} : RN, w_i \Vdash \varphi_n \iff w_i \in R[w_n]$

Proof. By induction on n.

- Base case: $n = 0, 1, 2, 3$: Check directly.
- Inductive step: Suppose the claim holds for $k < n + 4$. We show it holds for $k = n + 4$ as well. Consider the case where n is odd (the case for even is similar).



(⇐) Suppose $v \notin R[w_{n+4}]$. Then vRw_{n+3}. By IH, $w_{n+3} \Vdash \varphi_{n+3}$ but $w_{n+3} \nVdash \varphi \vee \varphi_{n+1}$. Therefore $v \nVdash \varphi_{n+3} \to \varphi_n \vee \varphi_{n+1}$, i.e. $v \nVdash \varphi_{n+4}$.

(⇒) Conversely, suppose $v \nVdash \varphi_{n+4}$. Then vRu for some $u \Vdash \varphi_{n+3}$, $u \nVdash \varphi_{n1} \vee \varphi_{n+1}$. The only such u is w_3. But if vRw_{n+3} it follows that $v \notin R[w_{n+4}]$.

■

Corollary 3. For $n \neq m$, $\varphi_n \not\equiv_{\mathsf{IPC}} \varphi_m$

Remark 28. For all $n \in \mathbb{N}$, $\varphi_n \in \mathcal{L}_{\mathcal{P}}$, since only the latter p occurs in φ_n.

Corollary 4. $\mathcal{L}_{\mathcal{P}}/\mathsf{IPC}$ is infinite.

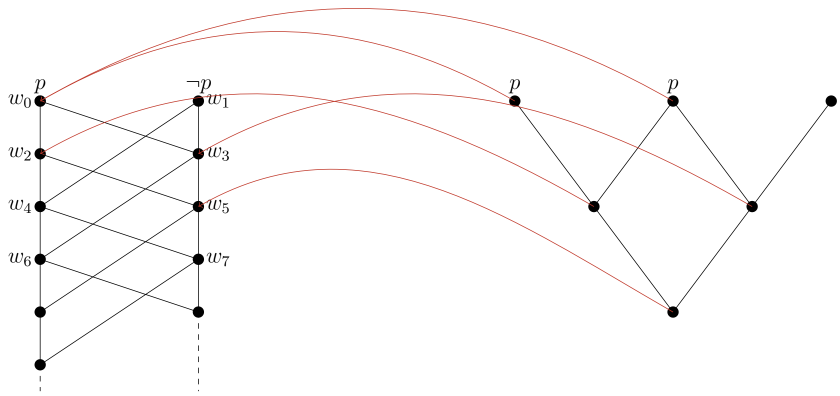
Corollary 5. IPC is not locally finite.

A natural next question is: what does the Lindenbaum-Tarski agebra of the one-letter fragment of IPC, LT^p_{IPC}, look like? To answer this question we can once more use the Rieger-Nishimura ladder, showing that it is 'universal' in the sense that it makes all possible distinctions between formulas.

13.6 RN is a 1-universal model

Proposition 12 (Universality of the Rieger-Nishimura ladder). *Let M be any finite rooted i.K.m. for $\{p\}$. Then there exists a unique p-morphism $f : M \to RN$.*

Proof. By induction on the depth of M, i.e. the maximum length of a chain in M. We'll just look at an example. Proceed inductively from endpoints to the root:



This generalises. Abstract away from this example for the general proof (in each instance, there will be only one $w_i \in RN$ to map a point to). ■

Corollary 6. If $\varphi \in \mathcal{L}_p$ and $\varphi \notin \mathsf{IPC}$, then φ can be refuted in RN, i.e. there exists a point $w \in RN$ such that $RN, w \nVdash \varphi$.

Let us denote by $|\varphi|_{RN}$ the set of points in RN where φ is forced, i.e.

$$|\varphi|_{RN} = \{w \mid RN, w \Vdash \varphi\}$$

Corollary 7. $\forall \varphi, \psi \in \mathcal{L}_p : \varphi \equiv_{\mathsf{IPC}} \psi \iff |\varphi|_{RN} = |\psi|_{RN}$.

Proof.

($\Rightarrow$) Obvious.

($\Leftarrow$) Suppose $\varphi \not\equiv_{\mathsf{IPC}} \psi$. Then $\varphi \to \psi \notin \mathsf{IPC}$ or $\psi \to \varphi \notin \mathsf{IPC}$. Wlog, suppose the former. By Corollary 6, there exists a point $w \in RN$ s.t. $RN, w \nVdash \varphi \to \psi$. Hence there exists a point $v \in R[w]$ s.t. $RN, v \Vdash \varphi$ and $RN, v \nVdash \psi$. Hence $v \in |\varphi|_{RN}$ but $v \notin |\psi|_{RN}$, i.e. $|\varphi|_{RN} \neq |\psi|_{RN}$.

■

13.7 Lindenbaum-Tarski algebra for $\mathcal{L}_p/\mathsf{IPC}$ (Rieger-Nishimura lattice)

Proposition 13. *LT^p_{IPC} is isomorphic to $\langle Up(RN), \subseteq \rangle$ via* $\overline{\varphi}^{\mathsf{IPC}} \overset{f}{\mapsto} |\varphi|_{RN}$.

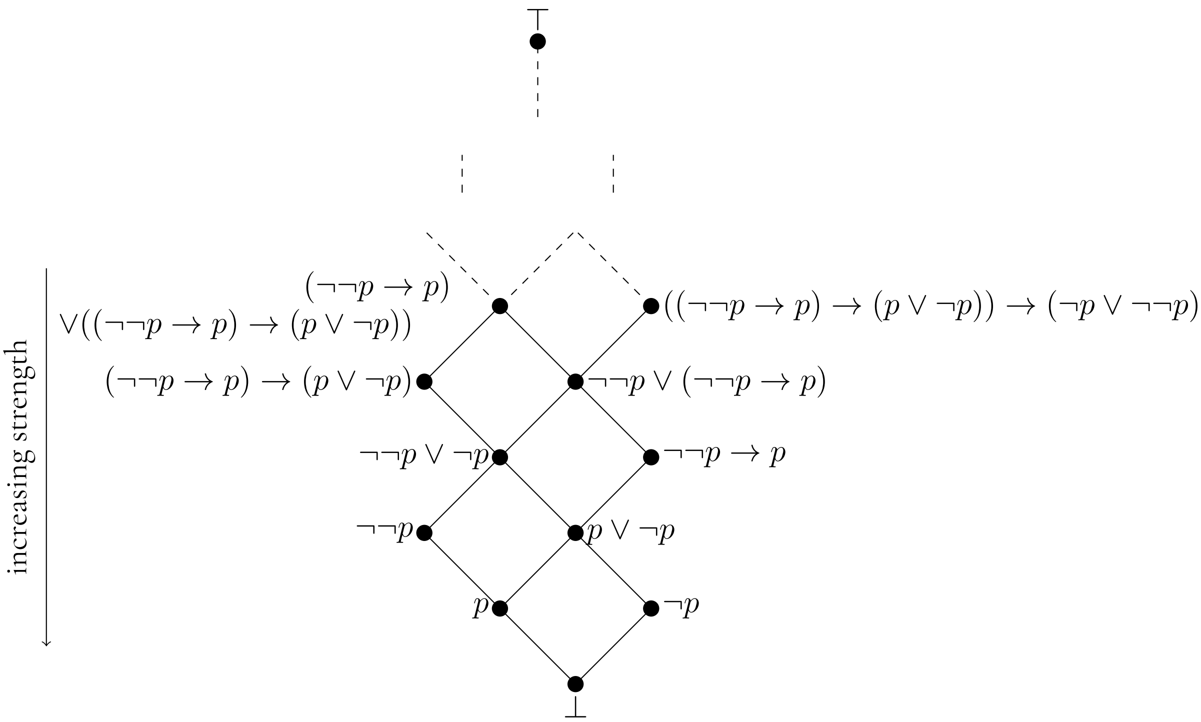
Proof.

- f is well-defined.
- f is order-preserving: $\overline{\varphi}^{\mathsf{IPC}} \leq \overline{\psi}^{\mathsf{IPC}} \Leftrightarrow \varphi \vdash_{\mathsf{IPC}} \psi \overset{\text{Soundness (for } RN)}{\Rightarrow} |\varphi|_{RN} \subseteq |\psi|_{RN}$
- f is injective: $\overline{\varphi}^{\mathsf{IPC}} \neq \overline{\psi}^{\mathsf{IPC}} \Leftrightarrow \varphi \underset{\mathsf{IPC}}{\not\equiv} \psi \Leftrightarrow |\varphi|_{RN} \neq |\psi|_{RN}$
- f is surjective: Take any upset $U \in Up(RN)$. There are four cases:
  - $U = \emptyset$: $\Rightarrow U = f(\overline{\bot}^{\mathsf{IPC}}) = |\bot|_{RN}$;
  - $U = W$: $\Rightarrow f(\overline{\top}^{\mathsf{IPC}}) = |\top|_{RN}$;
  - $U = R[w_n]$ for some n: $U = f(\overline{\varphi_n}^{\mathsf{IPC}}) = |\varphi_n|_{RN}$;
  - $U = R[w_n] \cup R[w_{n+1}]$ for some n: $\Rightarrow U = f(\overline{\varphi_n \vee \varphi_{n+1}}) = |\varphi_n \vee \varphi_{n+1}|_{RN}$.

  In each case $U = |\varphi|_{RN}$ for some φ.

■

By looking at the upsets of RN we can see that the algebra LT^p_{IPC} looks as follows:



From this picture we immediately obtain some interesting insights about the one-letter fragment of IPC. For instance:

Corollary 8. There is no consistent formula $\varphi \in \mathcal{L}_p$ s.t. φ properly entails p. (The same holds for $\neg p$.)

Corollary 9. For any $\varphi \in \mathcal{L}_p$, $\varphi \not\equiv_{\mathsf{IPC}} \top$, there are only finitely many nonequivalent formulas in $\mathcal{L}_p$ which entail φ. (But there are infinitely many nonequivalent formulas in $\mathcal{L}_p$ which are entailed by φ.)

Part III

Intermediate Logics

CHAPTER 14

Frame validity

14.1 Frame validity

Definition 49 (Frame validity). *We say that φ is valid in $\mathcal{F} = \langle W, R \rangle$, notation $\mathcal{F} \Vdash \varphi$, in case for all valuations V on $\mathcal{F}$ and all $w \in W$:*

$$\langle \mathcal{F}, V \rangle, w \Vdash \varphi.$$

The class of frames defined by φ is $\mathcal{F}(\varphi) = \{\mathcal{F} \mid \mathcal{F} \Vdash \varphi\}$.

Remark 29.

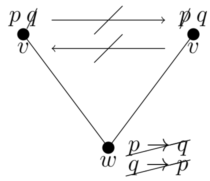
- $\varphi \in \mathsf{IPC} \;\Rightarrow\; \mathcal{F}(\varphi) =$ all i.K.frames
- $\varphi \notin \mathsf{CPC} \;\Rightarrow\; \mathcal{F}(\varphi) = \emptyset$

14.2 Dummett's axiom

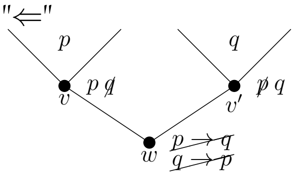
Example 27. $(p \to q) \vee (q \to p)$ (Dummett's axiom)

$$\mathcal{F} \nVdash (p \to q) \vee (q \to p) \;\Leftrightarrow\; \exists w \exists v, v' \in R[w] : v \not R v', v' \not R v$$

"⇒"



"⇐"



$V(p) = R[v]$, $V(q) = R[v']$;

$v \notin R[v']$, so $v \nVdash q$,

but $v \in R[v]$, so $v \Vdash p$.

Since wRv, $w \nVdash p \to q$.

Therefore

$$\mathcal{F} \Vdash (p \to q) \vee (q \to p) \;\Leftrightarrow\; \forall w \forall v, v' \in R[w] : vRv' \text{ or } v'Rv,$$

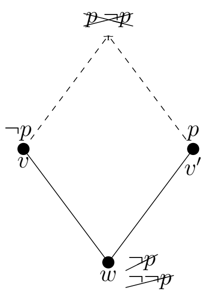
$$\text{i.e. } \mathcal{F} \Vdash (p \to q) \vee (q \to p) \;\Leftrightarrow\; \mathcal{F} \text{ is locally linear.}^1$$

14.3 Weak excluded middle

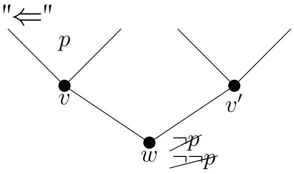
Example 28. $\neg p \vee \neg\neg p$ (weak excluded middle)

$$\mathcal{F} \nVdash \neg p \vee \neg\neg p \;\Leftrightarrow\; \exists w \exists v, v' \in R[w] : R[v] \cap R[v'] = \emptyset$$

"⇒"



"⇐"



$V(p) = R[v]$;

$v \in R[v] \Rightarrow v \Vdash p$;

$R[v'] \cap V(p) = \emptyset$

$\Rightarrow v' \Vdash \neg p$,

so $w \nVdash \neg p \vee \neg\neg p$.

Therefore

$$\mathcal{F} \Vdash \neg p \vee \neg\neg p \;\Leftrightarrow\; \forall w \forall v, v' \in R[w] : R[v] \cap R[v'] \neq \emptyset,$$

$$\text{i.e. } \mathcal{F} \Vdash \neg p \vee \neg\neg p \;\Leftrightarrow\; \mathcal{F} \text{ is locally confluent.}^2$$

---

[1] $\mathcal{F}$ is linearly ordered (or totally ordered) iff $\forall w, v \in W : wRv$ or vRw. If $R[w]$ is linearly ordered for all $w \in W$, then $\mathcal{F}$ is locally linear.

[2] $\mathcal{F}$ is confluent iff $\forall w, v \in W : R[w] \cap R[v] \neq \emptyset$. If $R[w]$ is confluent for all $w \in W$, then $\mathcal{F}$ is locally confluent.

CHAPTER 15

Intermediate Logics

15.1 Overview

$$\mathsf{IPC} \subseteq \text{ intermediate logics } \subseteq \mathsf{CPC}$$

15.2 Substitution

Definition 50. *A substitution is a map* $(.)^* : P \to \mathcal{L}_{\mathcal{P}}$*. It extends naturally to* $(.)^* : \mathcal{L}_{\mathcal{P}} \to \mathcal{L}_{\mathcal{P}}$*.*

Example 29. Suppose $p^* := r \vee s,\ q^* := \neg s$. Then $(p \to p \wedge \neg q)^* = (r \vee s) \to (r \vee s) \wedge \neg(\neg s)$.

If $\varphi = p \vee \neg p$, then

$(p \to q) \vee \neg(p \to q)$,

$\neg p \vee \neg\neg p$,

$q \vee \neg q$

are called *substitution instances* of φ.

15.3 Intermediate logics

Definition 51. *An* ***intermediate logic (IL)*** *is a set of formulae $L \subseteq \mathcal{L}_{\mathcal{P}}$ s.t.:*

1. $\mathsf{IPC} \subseteq L \subseteq \mathsf{CPC}$
2. *L is closed under modus ponens: $\varphi, \varphi \to \psi \in L \Rightarrow \psi \in L$.*
3. *L is closed under uniform substitution: $\varphi \in L \Rightarrow \varphi^* \in L$ for all substitutions $(.)^*$.*

Remark 30. Condition 2 is equivalent to
2'. closure under $\vdash_{\mathsf{IPC}}$: $L \vdash_{\mathsf{IPC}} \varphi \Rightarrow \varphi \in L$.
We can use a Hilbert-style system for IPC to see this equivalence.

15.4 Characterising intermediate logics

As we'll see in the next two chapters, there are two ways to characterize ILs:

- syntactic: add axioms to IPC.
- semantic: restrict available i.K. frames.

CHAPTER 16

Syntactic Characterisations

16.1 Adding axioms

Definition 52. *Let* $\Gamma \subseteq \mathcal{L}_{\mathcal{P}}$. *The logic axiomatized by* Γ, *denoted* $\mathsf{IPC} \oplus \Gamma$, *is the closure of* $\mathsf{IPC} \cup \Gamma$ *under modus ponens and uniform substitution.*

Remark 31. If $\Gamma \subseteq \mathsf{CPC}$, then $\mathsf{IPC} \oplus \Gamma$ is an IL, the smalles IL containing Γ.

Example 30. $\mathsf{IPC} \oplus \{p \vee \neg p\} = \mathsf{CPC}$; $\mathsf{IPC} \oplus \{\neg\neg p \to p\} = \mathsf{CPC}$

16.2 Some salient intermediate logics

- LC $= \mathsf{IPC} \oplus \{(p \to q) \vee (q \to p)\}$ (GÃűdel-Dummet)
- KC $= \mathsf{IPC} \oplus \{\neg p \vee \neg\neg p\}$ (logic of weak excluded middle)
- KP $= \mathsf{IPC} \oplus \{(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to s)\}$ (Kreisel-Putnam)
- S $= \mathsf{IPC} \oplus \{((\neg\neg p \to p) \to p \vee \neg p) \to \neg p \vee \neg\neg p\}$ (Scott)
  (The other direction is valid intuitionistically.)

Proposition 14. *KC* $\subseteq$ *LC*

Proof. If we can prove that $\neg p \vee \neg\neg p \in$ LC the claim follows since KC is the smallest IL containing $\neg p \vee \neg\neg p$. To do so, we show the following two equivalences * and **

$$\dfrac{\dfrac{\dfrac{p \to \neg p \qquad [p]}{\neg p} \qquad [p]}{\perp}}{\neg p}$$

$$\dfrac{\dfrac{\dfrac{\neg p \to p \qquad [\neg p]}{p} \qquad [\neg p]}{\perp}}{\neg\neg p}$$

$$\dfrac{\dfrac{\dfrac{\neg\neg p \qquad \neg p}{\perp}}{p}}{\neg p \to p}$$

(one direction of * is trivial):

* $\neg p \equiv_{\mathsf{IPC}} p \to \neg p$:   ** $\neg\neg p \equiv_{\mathsf{IPC}} \neg p \to p$:

Hence, $\neg p \vee \neg\neg p \equiv_{\mathsf{IPC}} (p \to \neg p) \vee (\neg p \to p) \ := \eta$.

η is a substitution instance of $(p \to q) \vee (q \to p)$. Since the latter is in LC, so is the former, i.e. $\eta \in$ LC. Since $\eta \equiv_{\mathsf{IPC}} \neg p \vee \neg\neg p$ and LC is closed under $\vdash_{\mathsf{IPC}}$, we conclude that $\neg p \vee \neg\neg p \in$ LC. ■

CHAPTER 17

Semantic Characterisations

17.1 Restricting frame class

Definition 53. *Let $C \neq \emptyset$ be a class of i.K.frames. The logic induced by these frames is*

$$Log(C) = \{\varphi \in \mathcal{L}_{\mathcal{P}} \mid \forall \mathcal{F} \in C : \mathcal{F} \Vdash \varphi\}.$$

Proposition 15. *If $C \neq \emptyset$, then $Log(C)$ is an IL.*

Proof. $\varphi \in \mathsf{IPC} \Leftrightarrow C =$ the set of all i.K. frames, i.e. for any $C \neq \emptyset$ $\mathsf{IPC} \subseteq Log(C)$.
We check closure under uniform substitution:
By contraposition, suppose $\varphi^* \notin Log(C)$ for some substitution instance φ^* of φ.
Then there exist a frame $\mathcal{F} \in C$, a valuation V, and a world w s.t. $\mathcal{F}, V, w \nVdash \varphi^*$.
We define a new valuation V' s.t. $V'(p) = \{v \text{ in } \mathcal{F} \mid \mathcal{F}, V, v \Vdash p^*\}$.
By induction on ψ we prove $\forall\psi$: $\mathcal{F}, V', u \Vdash \psi \;\Leftrightarrow\; \mathcal{F}, V, u \Vdash \psi^*$ for any u in $\mathcal{F}$.
So in particular $\mathcal{F}, V', w \nVdash \varphi$, i.e. $\mathcal{F} \nVdash \varphi$. Since $\mathcal{F} \in C$, we have $\varphi \notin Log(C)$. ■

17.2 Medvedev's logic

Definition 54 (Medvedev's logic)**.** *Medvedev's logic is defined as $ML := Log(C)$ where*

$$C = \left\{ \langle \wp(X) \setminus \{\emptyset\}, \supseteq \rangle \;\middle|\; X \text{ is finite} \right\}.$$

It is an open problem whether ML is axiomatizable by a decidable set $\Gamma \subseteq \mathcal{L}_{\mathcal{P}}$, i.e. whether
ML = $\mathsf{IPC} \oplus \Gamma$ for Γ decidable.

Theorem 13 (Maksimova). *$ML \neq \mathsf{IPC} \oplus \Gamma$ for any finite set Γ.*

Definition 55. *Let L be an IL.*

$$\varphi \vdash_L \psi \;\Leftrightarrow\; \varphi,\ L \vdash_{\mathsf{IPC}} \psi,$$

(where "L" stands for all the validities of L taken as axioms).

Definition 56. *Let C be a class of i.K.frames.*
$\varphi \Vdash_C \psi \;\Leftrightarrow\; \forall \mathcal{F} \in C\ \forall$ *valuations* $V, w \in \mathcal{F}$: *if* $\forall \varphi \in \varphi : \mathcal{F}, V, w \Vdash \varphi$, *then* $\mathcal{F}, V, w \Vdash \psi$.

Chapter 18
Soundness and Completeness

18.1 Overview

Definition 57. *Let L be an IL, C a class of i.K.frames.*

- *L is sound for C:* $\varphi \vdash_L \psi \;\Rightarrow\; \varphi \Vdash_C \psi$
- *L is strongly complete for* C*:* $\varphi \Vdash_C \psi \;\Rightarrow\; \varphi \vdash_L \psi$
- *L is (weakly) complete for* C*:* $\Vdash_C \psi \;\Rightarrow \vdash_L \psi$ *(i.e.* $Log(C) \subseteq L$*)*

18.2 Completeness

Definition 58. $\Gamma \subseteq \mathcal{L}_{\mathcal{P}}$ *is an L-theory for an IL L if* $\forall \varphi \in \mathcal{L}_{\mathcal{P}} : \; \Gamma \vdash_L \varphi \;\Rightarrow\; \varphi \in \Gamma$.

Definition 59. *The canonical model* M^c_L *for L is defined in the same way as* M^c_{IPC}*, but:*

- W^c_L := *the set of consistent L-theories with* $\vee$*-property.*

Lemma 8 (Lindenbaum-type)**.** *If* $\varphi \nvdash_L \psi$*, then* $\exists \Gamma \in W^c_L$ *s.t.* $\varphi \subseteq \Gamma$ *but* $\psi \notin \Gamma$.

Lemma 9 (Truth Lemma). $M^c_L, \Gamma \Vdash \varphi \iff \varphi \in \Gamma$.

The proofs for these two lemmas are exactly like those for IPC.

Proposition 16. *If* $\varphi \nvdash_L \psi$, *then* $\exists \Gamma \in W^c_L$ *s.t.* $M^c_L, \Gamma \Vdash \varphi \, \forall \varphi \in \varphi$ *but* $M^c_L, \Gamma \nVdash \psi$.

Theorem 14. *Let* $\mathcal{F}^c_L$ *be the canonical frame, i.e.* $\mathcal{F}^c_L := \langle W^c_L, \subseteq \rangle$. *If* $\mathcal{F}^c_L \in C$ *for a class of i.K.frames* C, *then* L *is strongly complete for* C.

18.3 KC

Theorem 15. *KC is sound and strongly complete for* $C_{\mathrm{KC}} = \{\mathcal{F} \mid \mathcal{F}$ *is locally confluent*$\}$.[a]

[a] Remember that a frame $\mathcal{F}$ is locally confluent *iff* $\forall w$ in $\mathcal{F}, \forall v, v' \in R[w] : R[v] \cap R[v'] \neq \emptyset$.

Proof. KC = IPC $\oplus \{\neg p \vee \neg\neg p\}$
Soundness: We saw that $\neg p \vee \neg\neg p$ is valid on the frames in C_{KC}, so $\neg p \vee \neg\neg p \in Log(C_{\mathrm{KC}})$.
Hence, KC $\subseteq Log(C_{\mathrm{KC}})$.
Completeness: We show that $\mathcal{F}^c_{\mathrm{KC}}$ is locally confluent.
Take $\Gamma \in W^c_{\mathrm{KC}}$, $\Gamma', \Gamma'' \in R[\Gamma]$. To show: $R[\Gamma'] \cap R[\Gamma''] \neq \emptyset$.
Suppose $\Gamma' \cup \Gamma'' \nvdash_{\mathrm{KC}} \bot$ (consistent).
By Lindenbaum-type Lemma $\exists \Gamma''' \in W^c_{\mathrm{KC}}$ s.t. $\Gamma''' \supseteq \Gamma' \cup \Gamma''$. So $\Gamma' \subseteq \Gamma'''$ and $\Gamma'' \subseteq \Gamma'''$, i.e.
$\Gamma''' \in R[\Gamma']$ and $\Gamma''' \in R[\Gamma'']$, hence $\Gamma''' \in R[\Gamma'] \cap R[\Gamma''] \neq \emptyset$.
What is left to show now is $\Gamma' \cup \Gamma'' \nvdash_{\mathrm{KC}} \bot$. Suppose otherwise.
Then (wlog) $\exists \alpha_1, \ldots, \alpha_n \in \Gamma''$ s.t. $\Gamma' \cup \{\alpha_1, \ldots, \alpha_n\} \vdash_{\mathrm{KC}} \bot$. Let $\alpha := \alpha_1 \wedge \ldots \wedge \alpha_n$.
(1) $\Gamma' \cup \{\alpha\} \vdash_{\mathrm{KC}} \bot$, so $\Gamma' \vdash_{\mathrm{KC}} \neg\alpha$. But Γ' is a KC-theory, so $\neg\alpha \in \Gamma'$.
(2) $\Gamma'' \vdash_{\mathrm{KC}} \alpha$. Since Γ'' is a KC-theory, $\alpha \in \Gamma''$.
Since $\neg\alpha \vee \neg\neg\alpha \in$ KC and Γ is a KC-theory, $\neg\alpha \vee \neg\neg\alpha \in \Gamma$. Since Γ has $\vee$-property, either (i) $\neg\alpha \in \Gamma$ or (ii) $\neg\neg\alpha \in \Gamma$.
Suppose (i). Since $\Gamma \subseteq \Gamma''$, $\neg\alpha \in \Gamma''$, contrary to the consistency of Γ''.
Suppose (ii). Since $\Gamma \subseteq \Gamma'$, $\neg\neg\alpha \in \Gamma'$, contrary to the consistency of Γ'.
Therefore $\Gamma' \cup \Gamma'' \nvdash_{\mathrm{KC}} \bot$. ∎

Part IV

Intuitionistic First-Order Logic

Chapter 19
The language $\mathcal{L}_{\mathcal{P}}^{\mathcal{Q}}$

Definition 60 ($\mathcal{L}_{\mathcal{P}}$)**.** *Let $\mathcal{P}$ be a set of predicate symbols, each with a corresponding arity $n \in \mathbb{N}$. The language $\mathcal{L}_{\mathcal{P}}^{\mathcal{Q}}$ is defined by*

$$\varphi := P(x_1, \ldots, x_n) \mid x = y \mid \varphi \to \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \bot \mid \forall x\varphi \mid \exists x\varphi$$

where P is an n-ary predicate and $x_1, \ldots, x_n, y$ are variables.

Remark 32. As in IPC, we define $\neg\varphi := \varphi \to \bot$.

Chapter 20
Natural Deduction System

We have the IPC rules for the connectives, plus the following rules for the quantifiers and for equality. In what follows, we assume that the variable y is free for x in the formula φ.

$$\frac{\varphi[\frac{y}{x}]}{\exists x\varphi}\ (\exists i) \qquad \frac{\exists x\varphi \qquad \begin{array}{c}\varphi[\frac{y}{x}]\\ \vdots \\ \psi\end{array}}{\psi}\ (\exists e)^{\star}$$

$$\frac{\varphi[\frac{y}{x}]}{\forall x\varphi}\ (\forall i)^{\dagger} \qquad \frac{\forall x\varphi}{\varphi[\frac{y}{x}]}\ (\forall e)$$

$$\frac{}{y = y}\ (=i) \qquad \frac{y = y' \qquad \varphi[\frac{y}{x}]}{\varphi[\frac{y'}{x}]}\ (=e)$$

* *Provided that y does not occur free in ψ or any undischarged assumption besides $\varphi[\frac{y}{x}]$.*
† *Provided that y does not occur free in any undischarged assumptions.*

> **Definition 61** (Derivability). *Write '$\varphi \vdash_{\mathsf{IQC}} \psi$' if ψ is derivable from assumptions in φ in this system.*

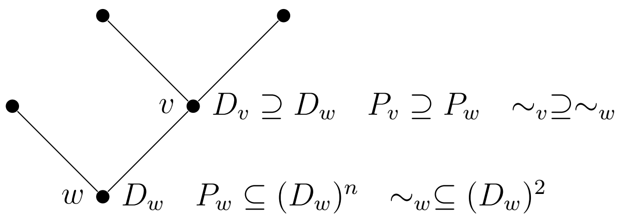
Example 31 ($\forall x \neg P(x) \equiv_{\mathsf{IQC}} \neg\exists x P(x)$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x \neg P(x)}{\neg P(y)}\,(\forall e) \qquad [P(y)]}{\bot}\,(\neg e) \qquad [\exists x P(x)]}{\bot}\,(\exists e)}{\neg \exists x P(x)}\,(\neg i)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[P(y)]}{\exists x P(x)}\,(\exists i) \qquad \neg \exists x P(x)}{\bot}\,(\neg e)}{\neg P(y)}\,(\neg i)}{\forall x \neg P(x)}\,(\forall i)$$

Remark 33. Compare $\forall x \neg P(x) \equiv_{\mathsf{IQC}} \neg \exists x P(x)$ with $\neg(p \vee q) \equiv_{\mathsf{IPC}} \neg p \wedge \neg q$.

Chapter 21
Kripke Semantics



- D_w is the domain at w, which is the set of objects that have been constructed at w.
- P_w is the extension of P at w, which is the set of n-tuples which have been proved to stand in the P-relation at w.
- $\sim_w$ is identity at w, which is the set of pairs of objects which have been proved to be identical at w. Notice that $d \not\sim_w d'$ does not mean that d and d' are distinct; they may later be proved to be identical.
- At a successor, all the old objects will still be present; more objects might have been constructed at this point.
- At a successor, all tuples which have already been proved to stand in a relation still stand in that relation, plus possibly more.

Definition 62 (IQC-model)**.** *An* IQC-model *is a tuple*

$$\mathfrak{M} = \langle W, R, \{D_w \mid w \in W\}, \{P_w \mid w \in W, P \in \mathcal{P}\}, \{\sim_w \mid w \in W\}\rangle$$

where

- $\langle W, R \rangle$ *is a partial order (i.e. reflexive, anti-symmetric and transitive),*
- $P_w \subseteq (D_w)^n$ *if* P *is an* n*-ary predicate symbol,*
- $D_w \neq \emptyset$,
- $\sim_w \subset D_w \times D_w$ *is a* congruence, *i.e. an equivalence relation (i.e. reflexive, symmetric and transitive) such that*

  $$d_1 \sim_w d_1', \ldots, d_n \sim_w d_n' \Rightarrow [\langle d_1, \ldots, d_n \rangle \in P_w \Leftrightarrow \langle d_1', \ldots, d_n' \rangle \in P_w].$$

If wRw'*, then we require that*

- $D_w \subseteq D_w'$,
- $P_w \subseteq P_w'$,
- $\sim_w \subseteq \sim_w'$.

Definition 63 (Assignment). *An* assignment *into* D_w *is a map*

$$g : Var \to D_w.$$

Remark 34. Assignments can be 'inherited': If wRw', then an assignment into D_w is also into $D_w' \supseteq D_w$.

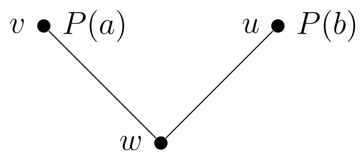
Definition 64 (Intuitionistic Kripke semantics for IQC).

- $\mathfrak{M}, w \Vdash_g P(x_1, \ldots, x_n) \Leftrightarrow \langle g(x_1), \ldots, g(x_n) \rangle \in P_w$
- $\mathfrak{M}, w \Vdash_g x = y \Leftrightarrow g(x) \sim_w g(y)$
- $\mathfrak{M}, w \nVdash_g \bot$
- $\mathfrak{M}, w \Vdash_g \varphi \vee \psi \Leftrightarrow \mathfrak{M}, w \Vdash_g \varphi$ *or* $\mathfrak{M}, w \Vdash_g \psi$
- $\mathfrak{M}, w \Vdash_g \varphi \wedge \psi \Leftrightarrow \mathfrak{M}, w \Vdash_g \varphi$ *and* $\mathfrak{M}, w \Vdash_g \psi$
- $\mathfrak{M}, w \Vdash_g \varphi \to \psi \Leftrightarrow$ *for all* $w' \in R[w]$: *if* $\mathfrak{M}, w' \Vdash_g \varphi$ *then* $\mathfrak{M}, w' \Vdash_g \psi$
- $\mathfrak{M}, w \Vdash_g \exists x \varphi \Leftrightarrow$ *for some* $d \in D_w$: $\mathfrak{M}, w \Vdash_{g[x \mapsto d]} \varphi$

- $\mathfrak{M}, w \Vdash_g \forall x\varphi \;\Leftrightarrow\;$ *for all* $w' \in R[w]$, *for all* $d \in D_w : \mathfrak{M}, w' \Vdash_{g[x\mapsto d]} \varphi$

where g *is an assignment into* D_w.

Example 32 ($\neg\forall x P(x) \nvDash_{\mathsf{IQC}} \exists x \neg P(x)$).



$$D_w = D_v = D_u = \{a, b\}$$
$$P_w = \emptyset$$
$$P_v = \{a\}$$
$$P_u = \{b\}$$

$$\text{there is no } w' \text{ such that } w' \Vdash \forall x P(x) \;\Rightarrow\; w \Vdash \neg\forall x P(x)$$

$$\left.\begin{array}{l} w \nVdash_{[x\mapsto a]} \neg P(x) \\ w \nVdash_{[x\mapsto b]} \neg P(x) \end{array}\right\} \;\Rightarrow\; w \nVdash \exists x \neg P(x)$$

Proposition 17 (Basic features of intuitionistic Kripke semantics for IQC).

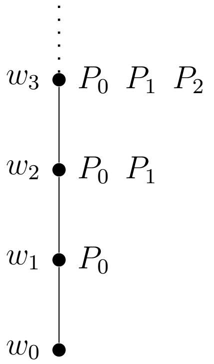
- ***Persistency***: $\mathfrak{M}, w \Vdash_g \varphi \;\&\; wRw' \;\Rightarrow\; \mathfrak{M}, w' \Vdash_g \varphi$
- ***Endpoints are complete***: *If* e *is an endpoint, then* $\mathfrak{M}, e \Vdash_g \varphi$ *or* $\mathfrak{M}, e \Vdash_g \neg\varphi$.
- ***Endpoints behave classically***: *If* $\varphi \in \mathsf{CQC}$ *and* e *is an endpoint, then* $\mathfrak{M}, e \Vdash_g \varphi$.
- ***Invariance under generated submodels***: $\mathfrak{M}, w \Vdash_g \varphi \;\Leftrightarrow\; \mathfrak{M}_w, w \Vdash_g \varphi$

Proof. Exercise. ■

Proposition 18. *Glivenko's theorem fails in* IQC:

$$\varphi \in \mathsf{CQC} \;\not\Rightarrow\; \neg\neg\varphi \in \mathsf{IQC}$$

Proof. $\forall x(P(x) \vee \neg P(x)) \in \mathsf{CQC}$, but $\neg\neg\forall x(P(x) \vee \neg P(x)) \notin \mathsf{IQC}$.



$D_{w_i} = \mathbb{N}$ for all i

For every $n \in \mathbb{N}$:
$w_n \nVdash_{[x \mapsto n]} P(x) \vee \neg P(x)$
$w_n \nVdash \forall x(P(x) \vee \neg P(x))$

$w_0 \Vdash \neg \forall x(P(x) \vee \neg P(x))$

$w_0 \nVdash \neg\neg \forall x(P(x) \vee \neg P(x))$

∎

Corollary 10. φ is a contradiction in CQC $\not\Leftarrow$ φ is a contradiction in IQC (Unlike in IPC.)

Proposition 19. *φ is a contradiction in* CPC $\Leftrightarrow$ *φ is a contradiction in* IPC

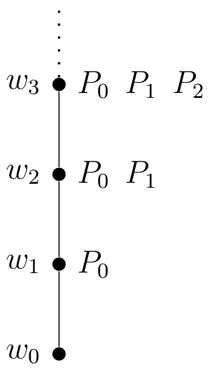
Proof.

$$
\begin{aligned}
\varphi \equiv_{\mathsf{CPC}} \bot &\Leftrightarrow \neg\varphi \in \mathsf{CPC} \\
&\xLeftrightarrow{\text{Glivenko}} \neg\neg\neg\varphi \in \mathsf{IPC} \\
&\xLeftrightarrow{\neg\neg\neg = \neg} \neg\varphi \in \mathsf{IPC} \\
&\Leftrightarrow \varphi \equiv_{\mathsf{IPC}} \bot
\end{aligned}
$$

∎

Remark 35. The reason underlying the failure of Glivenko's theorem is that $\neg\neg$ does not commute with $\forall$. (Contrast this with $\neg\neg(\varphi \wedge \psi) \equiv \neg\neg\varphi \wedge \neg\neg\psi$.)

Example 33 ($\forall x \neg\neg P(x) \nvDash_{\mathsf{IQC}} \neg\neg \forall x P(x)$).



$\forall i, n : w_i \nVdash_{[x \mapsto n]} \neg P(x)$
$\forall i, n : w_i \Vdash_{[x \mapsto n]} \neg\neg P(x)$
Thus, $w_0 \Vdash \forall x \neg\neg P(x)$

$\forall i : w_i \nVdash \forall x P(x)$
Thus, $w_0 \Vdash \neg \forall x P(x)$
Thus, $w_0 \nVdash \neg\neg \forall x P(x)$

Proposition 20. $\forall x \neg\neg\varphi \to \neg\neg\forall x\varphi$ *is valid on* finite *frames.*

Proof. By contraposition. Let $\mathfrak{M}$ be a model with a finite frame. Suppose $\mathfrak{M}, w \nVdash \neg\neg\forall x\varphi$. Then for some endpoint $e \in R[w] : \mathfrak{M}, e \Vdash \neg\forall x\varphi$. Therefore $\mathfrak{M}, e \nVdash \forall x\varphi$ and thus $\mathfrak{M}, e \nVdash_{[x\mapsto d]} \varphi$ for some $d \in D_e$. Since endpoints are complete, $\mathfrak{M}, e \Vdash_{[x\mapsto d]} \neg\varphi$. So $\mathfrak{M}, e \nVdash_{[x\mapsto d]} \neg\neg\varphi$. Therefore $\mathfrak{M}, w \nVdash \forall x\neg\neg\varphi$. ∎

Corollary 11. IQC does not have the finite model property.

Proof. $\forall x\neg\neg\varphi \to \neg\neg\forall x\varphi$ is invalid in IQC, but valid in all models based on finite frames. Hence $\forall x\neg\neg\varphi \to \neg\neg\forall x\varphi$ has no finite countermodel. ∎

Chapter 22

Negative Translation

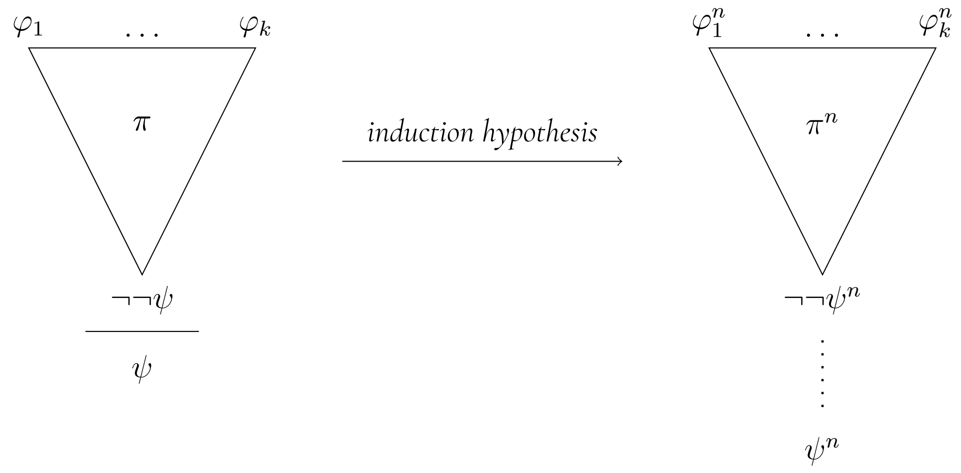
Definition 65 (Negative translation of CQC into IQC).

- $\varphi^n = \neg\neg\varphi$ *if* φ *is atomic*
- $\bot^n = \bot$
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$
- $(\varphi \to \psi)^n = \varphi^n \to \psi^n$
- $(\varphi \vee \psi)^n = \neg(\neg\varphi^n \wedge \neg\psi^n)$
- $(\forall x\varphi)^n = \forall x\varphi^n$
- $(\exists x\varphi)^n = \neg\forall x\neg\varphi^n$

Theorem 16. $\varphi_1, \ldots, \varphi_k \vdash_{\mathsf{CQC}} \psi \;\Leftrightarrow\; \varphi_1^n, \ldots, \varphi_k^n \vdash_{\mathsf{IQC}} \psi^n$

Proof. (Sketch.)

1. Show that $\forall\chi : \chi^n \equiv_{\mathsf{IQC}} \neg\neg\chi^n$ by induction on χ.
2. Using the previous point, show that a proof π of $\varphi_1, \ldots, \varphi_k \vdash_{\mathsf{CQC}} \psi$ can be translated into a proof π^n of $\varphi_1^n, \ldots, \varphi_k^n \vdash_{\mathsf{IQC}} \psi^n$ by induction on π. (Key step: Simulate the $\neg\neg$ rule by providing a proof of $\neg\neg\psi^n \vdash_{\mathsf{IQC}} \psi^n$, which exists by the first point.)



■

Remark 36. So, we can still view CQC as a fragment of IQC. However, this fragment no longer coincides with the set of formulas equivalent to a negation. Moreover, the translation *must* be defined inductively; we cannot simply add $\neg\neg$ in front of a formula.

Corollary 12. IQC is undecidable.

Proof. If IQC were decidable, then CQC would be decidable. Given φ:

1. compute φ^n,
2. decide whether $\varphi^n \in \mathsf{IQC}$,
3. return answer.

But CQC is undecidable (Church's theorem). ■

Part V

Theoretical Computer Science

CHAPTER 23
Simply typed lambda calculus

The simply typed λ-calculus is a system developed in the '30s by Alonso Chruch and Haskell Curry as a restricted version of a more general model of computation - Chruch's untyped λ-calculus.
This is a very simple model of computation. We have terms, denoting programs, that look like this:

$\lambda x.x$ the program that, given an input x, returns x

$\lambda x \lambda y.xy$ the program that, given x and y, returns the result of applying x to y.

Such terms can be applied to each other freely. Computation happens by substitution:

$$(\lambda x.\lambda y.xy)(\lambda x.x) \to \lambda y.(\lambda x.x)y \to \lambda y.y. \tag{23.1}$$

We can code the natural numbers and all computable function $f : \mathbb{N}^k \mapsto \mathbb{N}$, i. e., the untyped λ-calculus is **Turing-complete**.
This makes it a very expressive model, but it also means that it lacks certain desirable properties. In particular, we cannot be sure that a computation will eventually terminate and yield an output.

Example 34. $\Omega = \lambda x.xx$

Consider Ω applied to itself:

$\Omega\Omega = (\lambda x.xx)\Omega \to \Omega\Omega$
This means that when we try to compute the value of Ω on itself we end up in a loop and never get a value.

In the typed λ-calculus, terms are associated with types which determine which terms they can take in input and what kind of output they return. Terms in typed λ-calculus look like this:

$\lambda x_{\alpha\to\beta}.\lambda y_\alpha.xy$ the program that, given an input of type $\alpha \to \beta$ and an input of type α, returns the result of applying the first to the second.

Such a term cannot be applied to any term, but only to a term of the type it expects ($\alpha \to \beta$.
The typing system guarantees that computations always terminate. But this also means, by general results of computability theory, that not all computable functions can be expressed in this system.

> **Definition 66** (Types). *Given a set A of atomic types, the set τ_A of types over A is given by:*
>
> $$\tau_A := \alpha \,|\, \tau \to \tau \textit{ where } \alpha \in A$$

We call this system the symply typed λ-calculus because $\to$ is the only type constructor, more complex λ-calculi have a richer repertoire of constructors.

Example 35. $A = \alpha, \beta$
The following are types: α, $\alpha \to \alpha$, $\alpha \to (\beta \to \alpha)$

Each type comes with an infinite stock of variables: $\mathrm{Var}_\tau = x^0_\tau, x^1_\tau, x^2_\tau, \ldots$

> **Definition 67** (Terms). *The set of terms of a type is given by the following inductive rules:*
>
> *(Variables) if $x \in Var_\tau$ then $x : \tau$*
>
> *(Abstraction) if $x \in Var_\sigma$, and $M : \tau$ then $(\lambda x.M) : \sigma \to \tau$*
>
> *(Application) if $M : \sigma \to \tau, N : \sigma$, then $(MN) : \tau$*

We take an operator λx to bind the variable x in the term to which it is applied. The set FV(M) of variables free in M is defined as usual. If FV(M)=$\emptyset$ we say that M is a closed term. If $M : \tau$ for some closed M we say τ is inhabited.

Example 36. For every type τ, the type $\tau \to \tau$ is inhabited. Since given $x \in \mathrm{Var}_\tau$ we have $\lambda x.x : \tau \to \tau$

Example 37. For every τ, σ the type $\tau \to (\sigma \to \tau)$ is inhabited, since given $x \in \mathrm{Var}_\tau$, $y \in \mathrm{Var}_\tau$: $\lambda x.\lambda y.x : \tau \to (\sigma \to \tau)$

We regard two terms M and M' as the same if they can be converted into each other by a renaming of bound variables. Thus, eg., if $x, y \in \text{Var}_\tau$, then $\lambda.xx = \lambda y.y$

Definition 68 (Computation). *Let M, N be terms, x a variable of the same type as N. Then $M[^N/_x]$ is the term obtained by replacing each free occurence of x in M by N, taking care that no $y \in FV(N)$ ends up bound in the process.*

$$(\lambda x.M)N \to_\beta M[^N/_x]$$

Definition 69 (β-reduction).

- *$M \to_\beta N$ if N is obtained from M by replacing a redex with its reduct.*

  Notice that in general $M \to_\beta N$ for several N, since M may contain multiples redexes.

- *$M \twoheadrightarrow_\beta$ N if there is a chain $M \to_\beta \ldots \to_\beta N$ (possibly of length 0, so we have $M \twoheadrightarrow_\beta M$)*

- *M is in normal form if $\forall N : M \not\twoheadrightarrow_\beta N$*

Example 38. Let $z, z', x, y \in \text{Var}_\alpha$.

$$((\lambda x.\lambda y.x)z)z' \to_\beta (\lambda y.z)z' \to_\beta z$$

We think of the β-reduction process as the computation of a term; and of the normal form reached in this way as the output.
The following are key results of the theory of the simply typed λ-calculus:

Theorem 17 (Church-Rossier). *If $M \twoheadrightarrow_\beta M'$ and $M \twoheadrightarrow_\beta M''$ then for some term N, $M \twoheadrightarrow_\beta N$ and $M'' \twoheadrightarrow_\beta N$*

This expresses a confluence property of β-reduction process:

Theorem 18 (Weak normalization). *For any M there is always N in normal form such that $M \twoheadrightarrow_\beta N$*

This says that given a term, there is always a way to get to its value (in finitely many steps). That is, computations can always terminate f if we reduce in a suitable way. As a corollary of weak normalization and Church-Rossier we have that the "value" that we get from a computation of a term is uniquely determined.

Corollary 13 (Uniqueness of normal form)**.** If $M \twoheadrightarrow_\beta N$ and $M \twoheadrightarrow_\beta N'$ and if N and N' are in normal form, then $N = N'$.
Denote the unique N in n. f. such that $M \twoheadrightarrow_\beta N$ as nf(M). Think of this as the value of the term M.

Finally, the following result says that any chain of β-reduction from a term will eventually terminate, leading to its value.

Theorem 19 (Strong normalization)**.** *There are no infinite chains* $M \rightarrow_\beta M' \rightarrow_\beta \ldots$

Example 39. Let's look at how natural numbers and a computable function like product can be encoded in typed λ-calculus.

$\mathbb{N} = (\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha)$ for all $\alpha \in A$

$\overline{n} = \lambda f_{\alpha\rightarrow\alpha}.\lambda x_\alpha.f(\ldots(fx))$

Prod: $\mathbb{N} \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$

Prod = $\lambda x_{\mathbb{N}}.\lambda y_{\mathbb{N}}.\lambda f_{\alpha\rightarrow\alpha}.x(yF)$

Let us try to compute Prod(1)(2)

$$\begin{aligned}
(\text{Prod } \bar{1})\bar{2} \quad &= ((\lambda x.\lambda y.\lambda f.x(yf))\bar{1})\bar{2} \\
&\rightarrow_\beta (\lambda y.\lambda f.\bar{1}(yf))\bar{2} \\
&\rightarrow_\beta (\lambda f.\bar{1}(\bar{2}f)) \\
&\twoheadrightarrow_\beta \lambda f.\bar{1}(\lambda x.f(fx)) \\
&\twoheadrightarrow_\beta \lambda f.\lambda x.f(fx)(\lambda x.f(fx)) = \bar{2}
\end{aligned}$$

$$\begin{aligned}
\bar{2}f \quad &= (\lambda f.\lambda x.f(fx))f \\
&= (\lambda g.\lambda x.g(gx))f && \text{rename bound variable for convenience} \\
&\rightarrow_\beta \lambda x.f(fx)
\end{aligned}$$

$$
\begin{aligned}
\bar{1}(\lambda x.f(fx)) &= (\lambda f.\lambda x.fx)(\lambda x.f(fx)) \\
&= (\lambda g.\lambda y.gx)(\lambda x.f(fx)) && \text{rename bound variables for convenience} \\
&\to_\beta \lambda y((\lambda x.f(fx))y) \\
&\to_\beta \lambda y.f(fy) \\
&= \lambda x.f(fx) && \text{rename bound variables for convenience}
\end{aligned}
$$

CHAPTER 24
Curry-Howard correspondence

In this chapter we look at how the structure of intuitionistic logic matches that of typed λ-calculus. Since we've focused on the simply typed λ-calculus, where the only type constructor is $\to$, we should also focus on the $\to$-fragment of IPC, i.e. the fragment with $\to$ as unique connective. Call this fragment $\mathsf{IPC}[\to]$. However, the results we will look at extend to the full language of IPC and even IQC, provided the λ-calculus is enriched accordingly.
In order to fully appreciate the correspondence, we must make a minor change to our notion of natural deduction proof.

Labelled Natural Deduction proofs

Just like usual ND proofs, but:

- Every assumption comes with an index $k \in \mathbb{N}$
- Every application of $\to_i$ also comes with an index
- When we apply $\to_{(k)}$ to conclude $\varphi \to \psi$, we dicharge all and only assumptions of φ with index k above the rule.

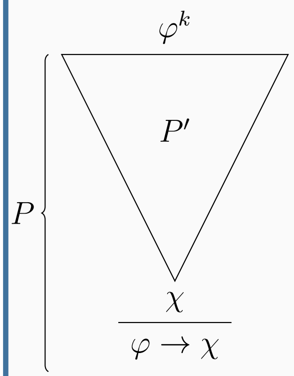
$$\dfrac{\dfrac{\dfrac{[\varphi] \qquad [\varphi \to \psi]}{\psi}\,(\to\text{-e})}{(\varphi \to \psi) \to \psi}\,(\to\text{-i})}{\varphi \to ((\varphi \to \psi) \to \psi)}\,(\to\text{-i})$$

Definition 70. *To each proof P in $\mathsf{IPC}[\to]$ with conclusion ψ we associate a term M_p : ψ*

We define the term by induction on P:

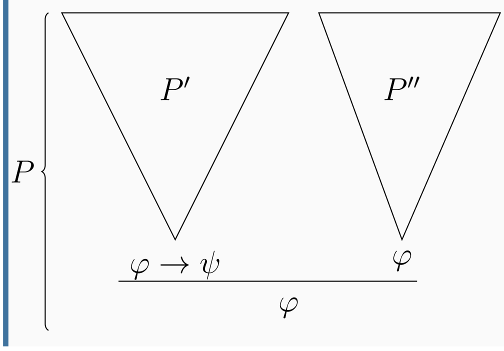
(Case 1) *P consists only of undischarged assumptions:*
$P\varphi^k \iff M_p := x^k_\psi : \psi$

(Case 2) *Last rule in P is ($\to$-i)*



$$\longrightarrow \quad M_p = \lambda x^k_\varphi . M'_p : \varphi \to \chi$$

(Case 3) *Last rule in P is ($\to$-e)*



$$\longrightarrow \quad M_p = (M_{p'} : \varphi \to \psi M_{p''} : \varphi)\psi$$

Example 40. Consider the formula

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)).$$

We will

1. construct a natural deduction proof in $\mathsf{IPC}[\to]$ of this formula, and then
2. convert this proof into a λ-term according to the Curry-Howard Correspondence.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi^1] \qquad [\varphi \to \psi^2]}{\varphi}\,(\to\text{-e}) \qquad [\varphi \to \chi^3]}{\chi}\,(\to\text{-e})}{\varphi \to \chi}\,(\to\text{-i}_1)}{(\psi \to \chi) \to (\varphi \to \chi)}\,(\to\text{-i}_3)}{(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))}\,(\to\text{-i}_2) \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^1_\varphi \qquad x^2_{\varphi\to\psi}}{x^2x^1} \qquad x^3_{\varphi\to\chi}}{x^3(x^2x^1)}}{\lambda x^1.x^3(x^2x^1)}}{\lambda x^3\lambda x^1.x^3(x^2x^1))}}{\lambda x^2\lambda x^3\lambda x^1.x^3(x^2x^1)}$$

$$\lambda x^2\lambda x^3\lambda x^1.x^3(x^2x^1) \xrightarrow{\text{renaming variables}} \lambda g_{\psi\to\chi}.\lambda f_{\varphi\to\psi}.\lambda x_\varphi.g(fx)$$

Replace each term by its type and you get back the original proof. You can thus reconstruct the proof given a λ-term: a λ-term is a linearized record of the three-like structure of a proof.

Example 41. Consider the formula

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r)).$$

We will

1. construct a natural deduction proof in $\mathsf{IPC}[\to]$ of this formula, and then
2. convert this proof into a λ-term according to the Curry-Howard Correspondence.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p \to (q \to r)^1] \qquad [p^2]}{q \to r}\,(\to\text{-e}) \qquad \dfrac{[p \to q^3] \qquad [p^2]}{q}\,(\to\text{-e})}{r}\,(\to\text{-e})}{p \to r}\,(\to\text{-i}_2)}{(p \to q) \to (p \to r)}\,(\to\text{-i}_3)}{(p \to (q \to r)) \to ((p \to q) \to (p \to r))}\,(\to\text{-i}_1) \qquad \longrightarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^1_{p\to(q\to r)} \qquad x^2_p}{x^1x^2} \qquad \dfrac{x^3_{p\to q} \qquad x^2_p}{x^3x^2}}{(x^1x^2)(x^3x^2)}}{\lambda x^2.(x^1x^2)(x^3x^2)}}{\lambda x^3.\lambda x^2.(x^1x^2)(x^3x^2)}}{\lambda x^1.\lambda x^3.\lambda x^2.(x^1x^2)(x^3x^2)}$$

$$\lambda x^1.\lambda x^3.\lambda x^2.(x^1x^2)(x^3x^2) \xrightarrow{\text{renaming variables}} \lambda f_{p\to(q\to r)}.\lambda g_{p\to q}.\lambda x_p.(fx)(gx)$$

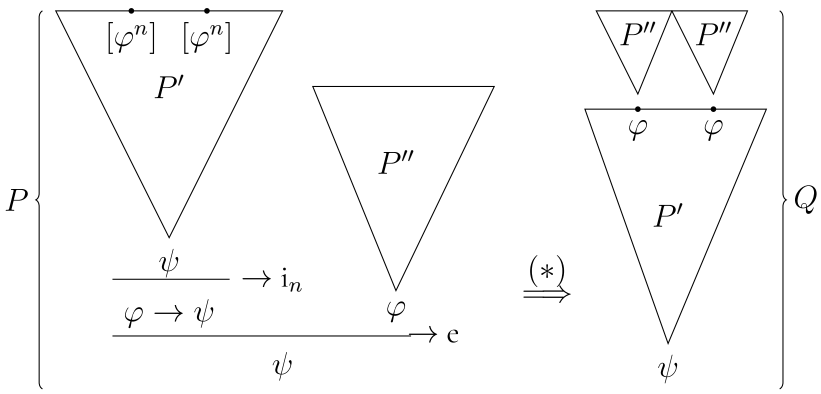
Proposition 21. *The map $P \mapsto M_P$ is a 1-to-1 correspondence between proofs of ψ (with undischarged assumptions) and terms of type ψ.*
Moreover $FV(M_P) = \{x^n_\varphi \mid \varphi^n$ is an undischarged assumption in $P\}$.
In particular: P is a proof without undischarged assumptions iff M_P is a closed term.

So we have the following perfect match:

Logic (IPC[$\to$])	**Computation (simply typed λ-calculus**
formula	type
proof	term
assumption	variable
discharged/undischarged	bound/free
proof without undischarged assumptions	closed term
theorem	inhabited type

One important piece is still missing: What is the logic counterpart of computation? More precisely, what are the notions that, on the logic side of the correspondence, are the counterpart of β-reduction ($\to_\beta$), β-reducibility ($\twoheadrightarrow_\beta$), and normal form? These are proof-theoretic notions which we have not discussed in the course, but which play an important role in proof theory: *normalization* of a natural deduction proof and *normal form* of a proof. In the context of IPC[$\to$] these can be explained very simply.

Proof normalization in IPC [$\to$]



Definition 71. *Let P and P' be proofs of φ. Then:*

- $P \to_N P'$ *if P' is obtained from P by replacing a sub-proof of P in accordance to $(*)$.*
- $P \twoheadrightarrow_N P'$ *if there is a finite chain $P \to_N \ldots \to_N P'$*
- P *is in NF if $P \not\twoheadrightarrow_N P'$ for any P'*

$$M_P := (\lambda x^k_\varphi . M_P)M'_P \xrightarrow{\textit{basic } \beta\textit{-reduction step}} M'_P\,[^{M_p}/x^k_p] = \underbrace{M_p}_{\textit{this can be checked on induction on } P'}$$

Remark 37. $P \to_N P' \iff M_P \to_\beta M'_P$

$P \twoheadrightarrow_N P' \iff M_P \twoheadrightarrow_\beta M'_P$

P in NF $\iff M_P$ in NF

Corollary 14 (Church-Rossier)**.** If $P \twoheadrightarrow_N Q$ and $P \twoheadrightarrow_N Q'$ then $\exists Q''$ such that $Q' \twoheadrightarrow_N Q''$ and $Q \twoheadrightarrow_N Q''$

Corollary 15 (Weak normalization)**.** For any P, $P \twoheadrightarrow_N Q$ for some Q in NF (different proofs have the same NF)

Corollary 16 (Strong normalization)**.** There are no infinite chains $P \to_N P' \to_N P'' \to_N \dots$

So the table can now be completed as follows:

IPC[$\to$]	**Simply typed λ-calculus**
formula	type
proof	term
assumption	variable
discharged/undischarged	bound/free
proof without undischarged assumptions	closed term
theorem	inhabited type
normalization	β-reduction
proof in normal form	term in normal form