- Problema + generale:

$$U^{0,x} := \mathbb{E}\left[\mathcal{U}(X_N^{0,x,\eta}) + \underbrace{\cancel{\sum_{n=1}^{N} c(\eta, X_n^{0,x,\eta})}} \right]$$

es. (short-fall risk minimization)

$F(S)$ payoff non replicabile

- $X = (S, V)$

S: titolo rischioso; V: valore stragia autofin.

- $\mathcal{U}(s,v) := (F(s) - v)^+$

$$\inf_{(\alpha,\beta)\in\mathcal{A}} \mathbb{E}^{P}[\mathcal{U}(S_N, V_N)]$$

$F(S_N) - V_N >> 0 \longrightarrow$ MALE

$V_N - F(S_N) >> 0 \longrightarrow$ NON BENE

es. (ottimizzazione di portafoglio)

(S, B) mercato, $\mathcal{A}$ strategie ammissibili

- $X = V$
- $\mathcal{U}$: funzione di utilità

$$\sup_{(\alpha,\beta)\in A} \mathbb{E}^{\mathbb{P}}\big[\, \underbrace{U}_{\text{soggettiva}}(V_N) \,\big]$$

$$\left(\begin{array}{l} (\alpha,\beta) : \quad V_N^{(\alpha,\beta)} = X \quad \text{certamente} \\ \text{PROBLEMA DI REPLICAZ. : } \underline{\text{OGGETTIVO}} \end{array} \right)$$

<u>es:</u> $U(v) = v$

$$\Downarrow$$

$$\sup_{(\alpha,\beta)\in A} \mathbb{E}^{\mathbb{P}}[\, V_N \,] \qquad \underline{\text{risk-neutral}}$$

<u>in generale:</u> U funzione di utilità se

1 - $U \in C^1$

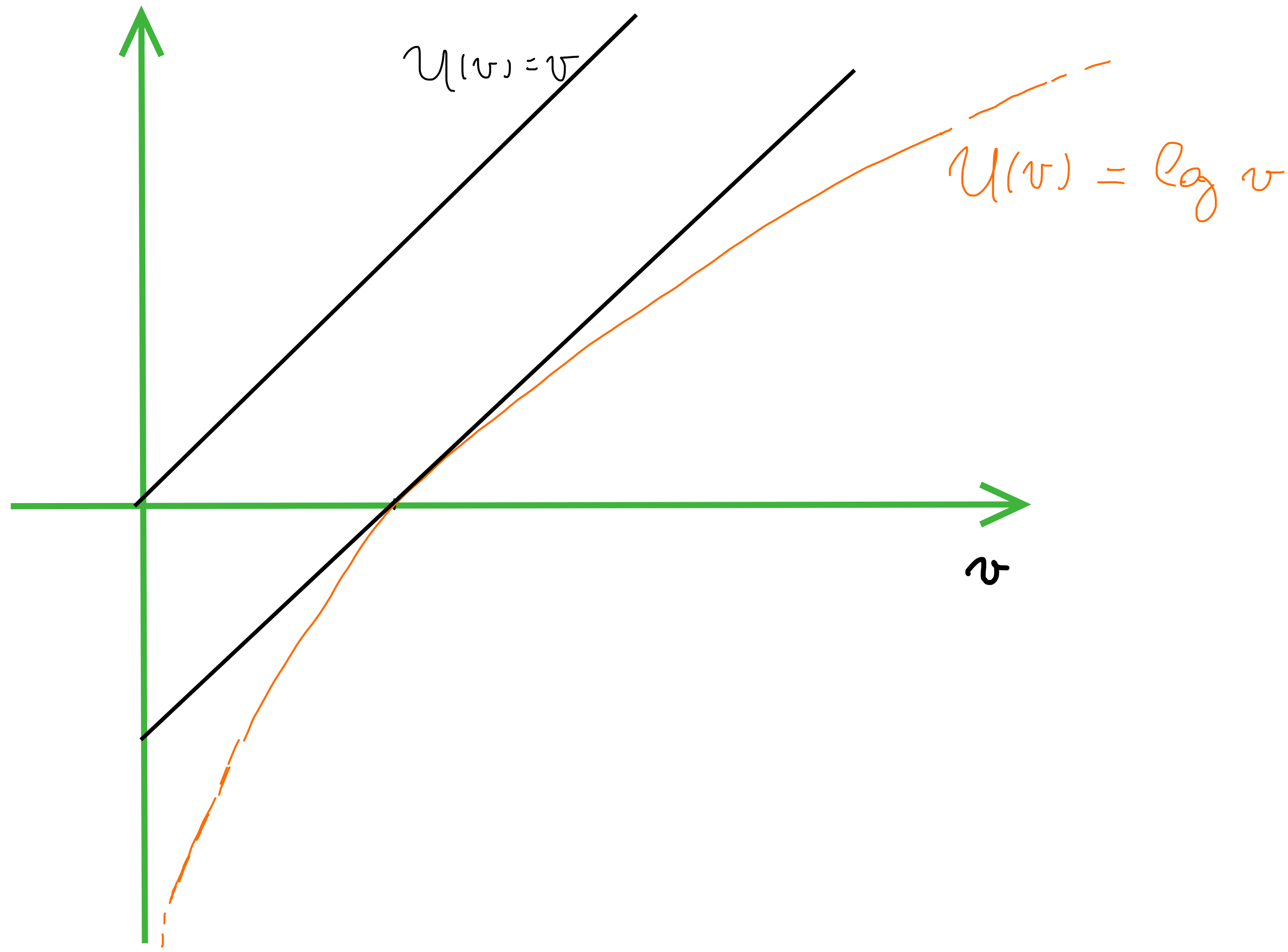
2 - $U \nearrow$

3 - U (strettam.) concava

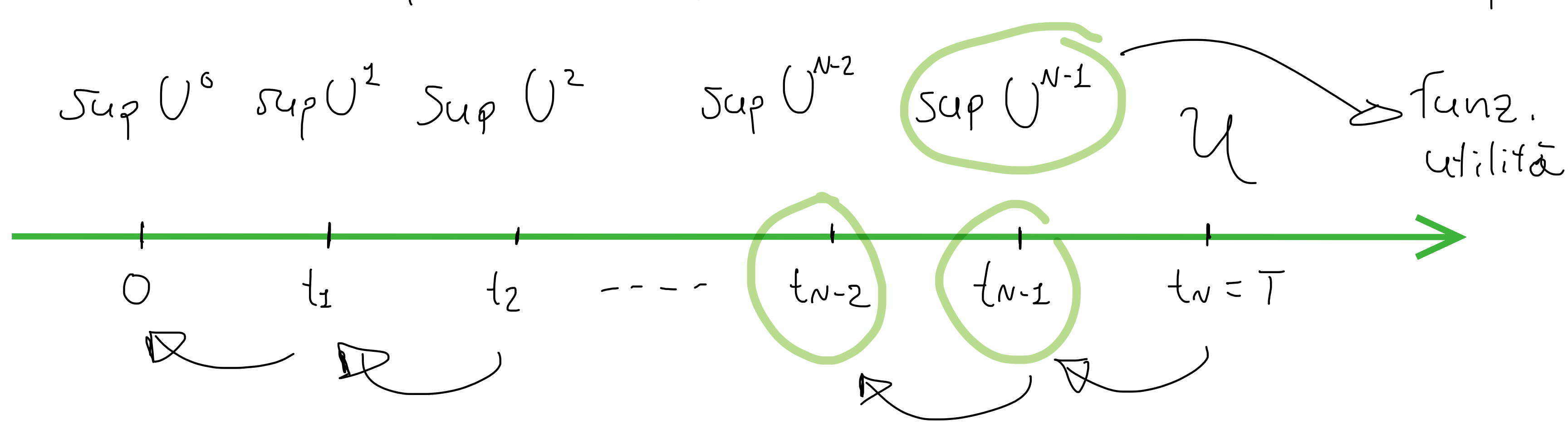
<u>es:</u> $U(v) = \log v \quad , \quad v > 0$

$U(v) = v^{\gamma}/\gamma \, , \quad v > 0 \, , \quad \gamma < 1 \, , \, \gamma \neq 0$

$$\mathcal{U}(v) = -e^{-v}, \qquad v \in \mathbb{R}$$



- <u>IDEA</u>: suddividere l'intervallo $[0, T]$ in sottoperiodi, e risolvere dei sottoproblemi



(PRINCIPIO DI OTTIMALITÀ DI BELLMAN)

<u>es</u>:

$$\min_{(\alpha,\beta)\in\mathcal{A}} \operatorname{var}(V_N) \qquad (\text{minimizz. l'aleatorietà})$$

soggetto al vincolo $\mathbb{E}[V_N] = \mu$
(Markowitz)

teorema Ponendo:

$$W_N(x) := \mathcal{U}(x)$$

$$W_n(x) := \sup_{\eta_n, \dots, \eta_{N-1}} U^{n,x}(\eta_n, \dots, \eta_{N-1})$$

$$n = 0, \dots, N-1$$

"funzione valore"

$\boxed{\text{EQ. DI BELLMAN}}$

vale:

$$W_{n-1}(x) = \sup_{\xi} \mathbb{E}\left[W_n\left(\underbrace{G_n(x, \mu_n; \xi)}_{\parallel} \right) \right]$$

$$= \sup_{\xi} \mathbb{E}\left[W_n\left(X_n^{n-1, x, \xi} \right) \right]$$

- operativamente:

- $W_{N-1}(x) = \sup_{\xi} \mathbb{E}\left[\mathcal{U}\left(X_N^{N-1, x, \xi} \right) \right]$

- $W_{N-2}(x) = \sup_{\xi} \mathbb{E}\left[W_{N-1}\left(X_{N-1}^{N-2, x, \xi} \right) \right]$

$\vdots$

$$- W_0(x) = \sup_{\xi} \mathbb{E}\left[W_1\left(X_1^{0,x,\xi}\right)\right]$$

corollario Siano $\bar{\eta}_0, \dots, \bar{\eta}_{N-1}$ funzioni di controllo tali che

$$\bar{\eta}_n(x) \in \arg\max_{\xi} \mathbb{E}\left[W_n\left(G_n(x, \mu_n; \xi)\right)\right]$$

allora

$$(*)$$

$$U^{n,x}(\bar{\eta}_n, \dots, \bar{\eta}_{N-1}) = \underbrace{\max_{\eta_n, \dots, \eta_{N-1}} U^{n,x}(\eta_n, \dots, \eta_{N-1})}_{= W_n(x)}$$

lemma Se X, Y sono v.a indip su $(\Omega, \mathcal{F}, \mathbb{P})$ e $g = g(x,y) \in m\mathcal{B}$, allora:

$$\mathbb{E}[g(X,Y)] = \mathbb{E}\left[\mathbb{E}[g(x,Y)]\big|_{x=X}\right]$$

dim proprietà torre + lemma di Freezing dim (corollario)

$$U^{N-1,x}(\bar{\eta}_{N-1}) = \mathbb{E}\Big[\mathcal{U}\big(\underbrace{X_N^{N-1,x,\bar{\eta}}}_{=\,G_N(x,\mu_N;\,\bar{\eta}_{N-1}(x))}\big)\Big]$$

x ipotesi $\to$ $$= \max_{\xi} \underbrace{\mathbb{E}\big[\mathcal{U}(G_N(x,\mu_N;\xi))\big]}_{=\,W_N}$$

eq. di Bellman $\to$ $= W_{N-1}(x)$

- Assumiamo (*) vera per $n \leq N-1$,

$$W_{n-1}(x) \underset{\text{EQ. DI BELLMAN}}{=} \max_{\xi} \mathbb{E}\Big[W_n\big(X_n^{n-1,x,\xi}\big)\Big]$$

(x def di $\bar{\eta}$) $\to$ $$= \mathbb{E}\Big[W_n\big(X_n^{n-1,x,\bar{\eta}_{n-1}(x)}\big)\Big]$$

(x ipotesi induttiva) $$= \mathbb{E}\Big[U^{n,X_n^{n-1,x,\bar{\eta}_{n-1}(x)}}(\bar{\eta}_n,\dots,\bar{\eta}_{N-1})\Big]$$

$\downarrow$ def. di $U^{n,y}$

$$= \mathbb{E}\Big[\mathbb{E}\big[\mathcal{U}(X_N^{n,y,\bar{\eta}})\big]\Big|_{y=\underbrace{X_n^{n-1,x,\bar{\eta}_{n-1}}}_{=\,G_n(x,\mu_n;\,\bar{\eta}_{n-1}(x))}}\Big]$$

$$\| \atop g(y, \mu_{n+1}, \dots, \mu_N)$$

$$\begin{pmatrix}\text{lemma}\\ \text{precedente}\end{pmatrix} = \mathbb{E}\left[\, \mathcal{U}(X_N^{n-1,x,\bar\eta})\,\right] = U^{n-1,x}(\bar\eta)$$

$$\left(X_N^{n,y,\bar\eta}\Big|_{y = X_n^{n-1,x,\bar\eta}} = X_N^{n-1,x,\bar\eta} \right) \qquad \#$$

PROPR. FLUSSO

dim (teo. eq. di Bellman) (bozza)

$n = 1, \dots, N$,

$$W_{n-1}(x) = \sup_{\eta_{n-1},\dots,\eta_{N-1}} \underbrace{U^{n-1,x}(\eta_{n-1},\dots,\eta_{N-1})}_{\| \atop \mathbb{E}\left[\mathcal{U}\left(\underbrace{X_N^{n-1,x,\eta}}_{\| \atop X_N^{n, X_n^{n-1,x,\eta}, \eta}}\right)\right]}$$

flusso

$$\text{lemma} \rightarrow = \sup_{\eta_{n-1},\dots,\eta_{N-1}} \mathbb{E}\left[\underbrace{\mathbb{E}\left[\mathcal{U}(X_N^{n,y,\eta})\right]}_{\| \atop U^{n,y}(\eta)} \Big|_{y = X_n^{n-1,x,\eta}} \right]$$

$$= \sup_{\eta_{n-1},\dots,\eta_{N-1}} \mathbb{E}\left[U_N^{n, X_n^{n-1,x,\eta}, \eta} \right]$$

$$! \; = \; \sup_{\eta_{n-1}} \mathbb{E}\Bigg[\underbrace{\sup_{\eta_n,\dots,\eta_{N-1}} U_N^{n, X_n^{n-1,x,\eta}, \eta}}_{= \, W_n\big(\underbrace{X_n^{n-1,x,\eta}}_{= \, G_n(x, \mu_n ; \eta_{n-1}(x))}\big)} \Bigg]$$

$$= \sup_{\xi} \mathbb{E}\left[W_n\left(X_n^{n-1,x,\xi}\right) \right] \qquad \#$$

• Modello trinomiale

• B titolo non rischioso:

$$B_0 = 1 ; \qquad B_n = (1+r) B_{n-1}, \quad n = 1, \dots, N$$

- $(h_n)_{n=1,\dots,N}$ v.a. i.i.d t.c.

$$h_n \sim p_1 \cdot \delta_1 + p_2 \cdot \delta_2 + \underset{(1-p_1-p_2)}{\underset{\parallel}{p_3}} \cdot \delta_3$$

$[p_1, p_2 \in [0,1]]$

$$\left(\mathbb{P}(h_n = 1) = p_1 ,\ \mathbb{P}(h_n = 2) = p_2 ,\ \mathbb{P}(h_n = 3) = p_3 \right)$$

- 2 versioni: 1 o 2 titoli rischiosi

$$S_n^i = S_{n-1}^i \left(1 + \mu^i(h_n)\right), \quad n = 1, \dots, N$$
$$i = 1, 2$$

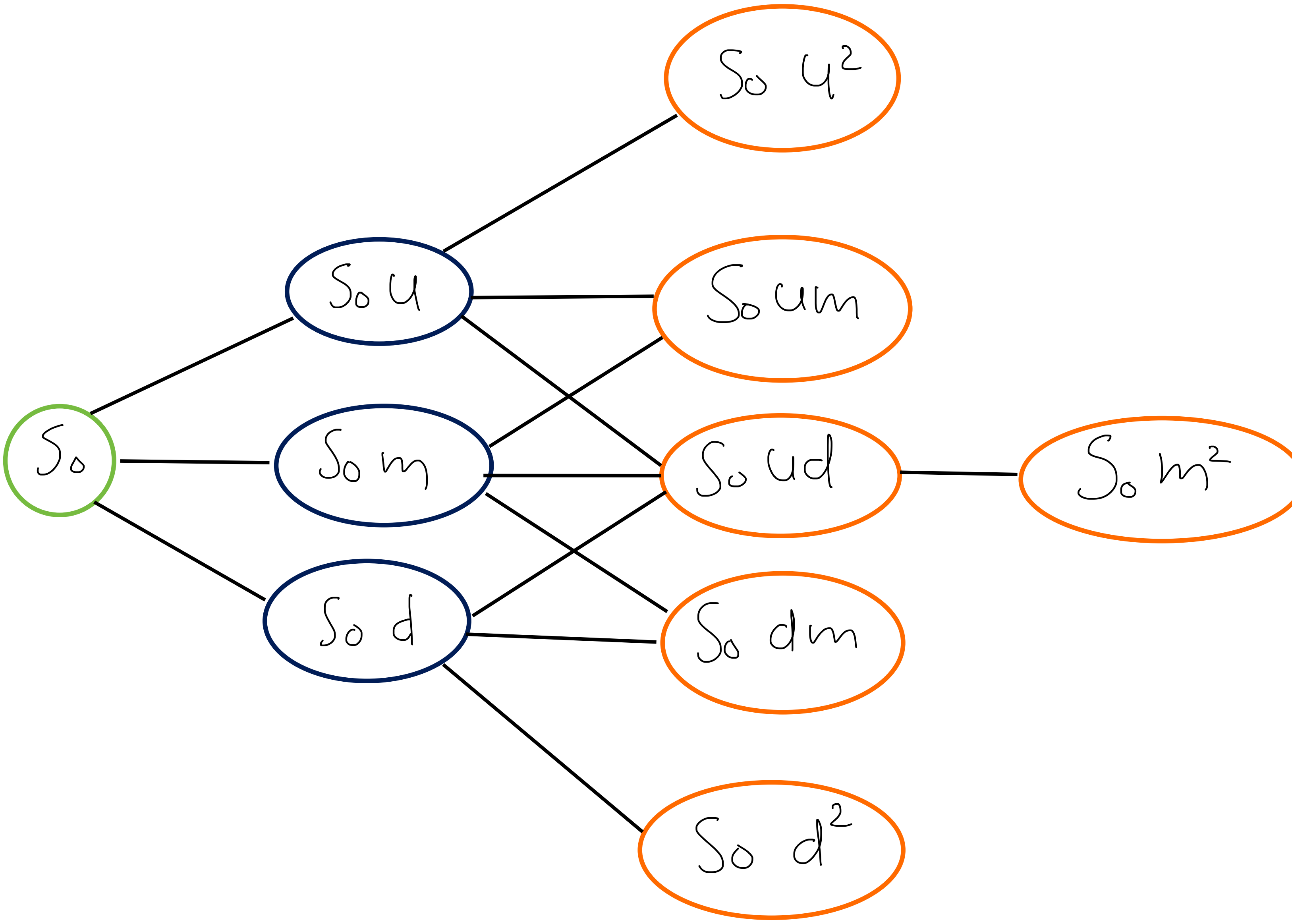
$$1 + \mu^i(h) = \begin{cases} u_i & \text{se } h = 1 \\ m_i & \text{se } h = 2 \\ d_i & \text{se } h = 3 \end{cases}, \quad d_i < m_i < u_i$$

- Costruzione canonica:

$$\Omega = \{1, 2, 3\}^N, \quad \mathcal{F} = \mathcal{P}(\Omega), \ \dots$$

- 2 versioni: 1 titolo rischioso $\rightarrow$ incompleto

  2 " rischiosi $\rightarrow$ completo



• CASO 1 titolo rischioso:

$$\cancel{S_{n-1}} = \frac{1}{r+1} \mathbb{E}^{\mathbb{Q}}\left[\cancel{S_{n-1}} (1 + \mu(h_n)) \mid \mathcal{F}_{n-1} \right]$$

$$= \frac{1}{r+1} \mathbb{E}^{\mathbb{Q}}\left[u \cdot \mathbb{1}_{\{h_n = 1\}} + m \, \mathbb{1}_{\{h_n = 2\}} + d \, \mathbb{1}_{\{h_n = 3\}} \mid \mathcal{F}_{n-1} \right]$$

$$\iff \quad r + 1 = u \cdot \mathbb{Q}(h_n = 1 \mid \mathcal{F}_{n-1}) =: q_1$$

$$+ \quad m \cdot \mathbb{Q}(h_n = 2 \mid \mathcal{F}_{n-1}) =: q_2$$

$$+ \quad d \cdot \mathbb{Q}(h_n = 3 \mid \mathcal{F}_{n-1}) =: q_3 = 1 - q_1 - q_2$$

$$\Longleftrightarrow \quad r + 1 = u \underbrace{q_1}_{v.a} + m \underbrace{q_2}_{v.a} + d(1 - q_1 - q_2)$$

NO UNICITÀ $\Longrightarrow$ mercato incompleto

$\boxed{\text{N.B.}}$ q_1, q_2 rimangono v.a.

$\Downarrow$

$\mu(h_n)$ non sono indip. sotto $\mathbb{Q}$

$\Downarrow$ (in generale)

S non è un processo di Markov.

DOMANDA: come valutiamo un derivato X?