- **Modello binomiale:**

$d \leq 1+r \leq u \longrightarrow$ libero da arbitraggi

$$V_n = H_n = (1+r)^{-(N-n)} \mathbb{E}^{Q}[F(S_N) \mid \mathcal{F}_n]$$

$$= (1+r)^{-(N-n)} \mathbb{E}^{Q}[F(S_N) \mid S_n]$$

$$= (1+r)^{-(N-n)} \underbrace{\mathbb{E}^{Q}[F(S_N) \mid S_n = s]}_{\text{calcolare questa}}\Big|_{s=S_n}$$

$$V_{N-1} = H_{N-1} = (1+r)^{-1} \mathbb{E}^{Q}[F(S_N) \mid S_{N-1}]$$

$$\left(S_{n+1} = S_n (1 + \underbrace{\mu_n}_{\text{Sotto } Q\text{, indip. e identic. distr.}}), \quad n \in \mathbb{N}_0 \right.$$

$$\left. = f(S_n, \mu_n) \quad \text{con} \quad f(s,\mu) = s(1+\mu) \right)$$

$$= (1+r)^{-1} \mathbb{E}^{Q}[F(s(\underbrace{1+\mu_1}_{q\, s_u + (1-q)\, s_d}))]\Big|_{s=S_{N-1}}$$

$$= (1+r)^{-1} \left(F(\underbrace{S_{N-1}} \cdot u) \cdot q + F(\underbrace{S_{N-1}} \cdot d)(1-q) \right)$$

esercizio

Mostrare che

$$V_n = H_n = (1+r)^{-(N-n)} \mathbb{E}^Q \left[F(S_N) \mid \mathcal{F}_n \right]$$

$$= (1+r)^{-(N-n)} \sum_{k=0}^{N-n} \binom{N-n}{k} q^k (1-q)^{N-n-k} \cdot F(u^k d^{N-n-k} S_n)$$

notazione

$$H_n = V_n = H_n(S_n)$$

$H_n \to$ processo stoc.

$H_n(\cdot) \to$ funzione

$$\alpha_n = \alpha_n(S_n)$$

$$\beta_n = \beta_n(S_n)$$

- CASO $n = N-1$:

$$S_N = \begin{cases} u\, S_{N-1} & (\text{se } \mu_N = u-1) \\ d\, S_{N-1} & (\text{se } \mu_N = d-1) \end{cases}$$

$$V_N = \alpha_N \cdot S_N + \beta_N B_N = F(S_N)$$

$\Updownarrow$

$\downarrow$ derivato path- independ.

$$\begin{cases} \alpha_N \cdot u\, S_{N-1} + \beta_N \cdot B_N = F(u\, S_{N-1}) \\ \alpha_N \cdot d\, S_{N-1} + \beta_N \cdot B_N = F(d\, S_{N-1}) \end{cases}$$

$$\Updownarrow$$

"δ-hedging"

$$\begin{cases} \alpha_N = \dfrac{F(u\, S_{N-1}) - F(d\, S_{N-1})}{u\, S_{N-1} - d\, S_{N-1}} \\ \beta_N = \dfrac{u \cdot F(d\, S_{N-1}) - d\, F(u\, S_{N-1})}{(1+r)^N (u-d)} \end{cases}$$

$$\beta_N = \frac{1}{(1+r)^N}$$

$$\frac{F(d\, S_{N-1}) \cdot (u-d)\, S_{N-1} - d\, S_{N-1} \left(F(u\, S_{N-1}) - F(d\, S_{N-1})\right)}{(u-d)\, S_{N-1}}$$

- α_N, β_N funzioni di S_{N-1}

$$V_{N-1} = \alpha_N S_{N-1} + \beta_N =$$

$$\shortparallel$$

$$H_{N-1} = \mathbb{E}^{Q}\left[F(S_N) \mid \mathcal{F}_{N-1} \right]$$

$$= (1+r)^{-1} \left(q\, F(u\, S_{N-1}) + (1-q)\, F(d\, S_{N-1}) \right)$$

dove $q = \dfrac{1+r-d}{u-d}$, $1-q = \dfrac{u-1-r}{u-d}$

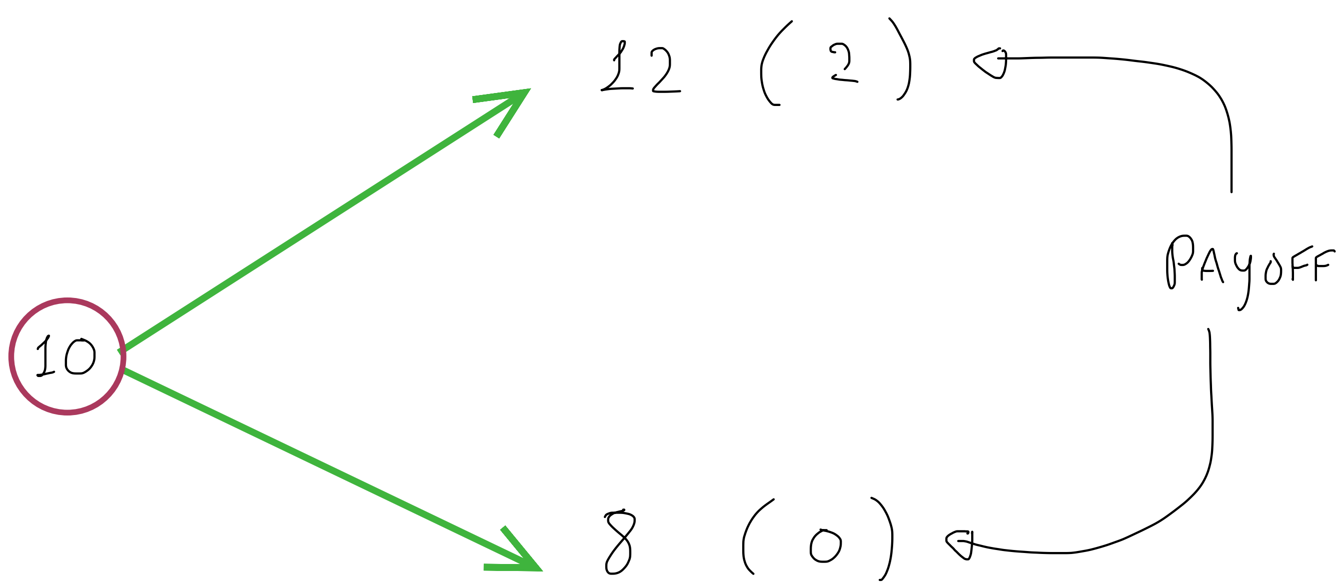
es.

$S_0 = 10\ \$$, $N = 1$

$u = 1{,}2$, $d = 0{,}8$

$r = 0{,}05$

$F(S_1) = (S_1 - K)^+$, $K = 10\ \$$



$$\begin{cases} 12\,\alpha + 1{,}05 \cdot \beta = 2 \\ 8\,\alpha + 1{,}05 \cdot \beta = 0 \end{cases}$$

$$\alpha = \frac{2-0}{12-8} = \frac{1}{2}$$

$$\beta = \frac{1{,}2 \cdot 0 - 0{,}8 \cdot 2}{1{,}05\,(1{,}2 - 0{,}8)} = -\frac{80}{21}$$

$$V_0 = \alpha \cdot 10 + \beta = \frac{25}{21} = H_0$$

- caso $n < N$:

$$V_n = \alpha_n S_n + \beta_n B_n = H_n = H_n(S_n)$$

$$S_n = \begin{cases} u\, S_{n-1} & (\text{se } \mu_n = u-1) \\ d\, S_{n-1} & (\text{se } \mu_n = d-1) \end{cases}$$

$$\begin{cases} \alpha_n \cdot u\, S_{n-1} + \beta_n B_n = H(u\, S_{n-1}) \\ \alpha_n \cdot d\, S_{n-1} + \beta_n B_n = H(d\, S_{n-1}) \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \alpha_n = \dfrac{H(u\, S_{n-1}) - H(d\, S_{n-1})}{u\, S_{n-1} - d\, S_{n-1}} \\[2ex] \beta_n = \dfrac{u\, H(d\, S_{n-1}) - d\, H(u\, S_{n-1})}{(1+r)^n\,(u-d)} \end{cases}$$

- α_n, β_n funzioni di S_{n-1}

$$V_{n-1} = \alpha_n S_{n-1} + \beta_n (S_{n-1}) =$$

$$\|$$

$$H_{n-1} = (1+r)^{-1} \mathbb{E}^{Q}[\underbrace{H_n}_{= H_n(S_n)} \mid S_{n-1}]$$

$$= (1+r)^{-1} \left(q \cdot H_n(u \cdot S_{n-1}) + (1-q) H_n(d \cdot S_{n-1}) \right)$$

- Algoritmo binomiale: $(X = F(S_N))$

notazione

$$S_{n,k} := S_0 u^k d^{n-k}$$

$$H_{n,k} := H_n(S_{n,k}) \quad , \quad n = 0, \dots N \quad k = 0, \dots, n$$

valore di S_n se k "up" e $n-k$ "down" fino a t_n

$$\alpha_{n,k} := \alpha_n(S_{n-1,k})$$

$$\beta_{n,k} := \beta_n(S_{n-1,k})$$

per $n = 1, \dots, N$

$k = 0, \dots, n-1$

- $H_{N,k} = H_N(S_{N,k}) = F(S_{N,k}) \quad , \quad k = 0, \dots, N$

$$- \; H_{n-1,k} = H_{n-1}(S_{n-1,k}) = \frac{q \cdot H_n(u S_{n-1,k}) + (1-q) \cdot H_n(d \cdot S_{n-1,k})}{1+r}$$

$$= \frac{q \cdot H_n(S_{n,k+1}) + (1-q) \cdot H_n(S_{n,k})}{1+r}$$

$$= \frac{q \cdot H_{n,k+1} + (1-q) \; H_{n,k}}{1+r}$$

$\forall \; n = 1, \ldots, N$ (ITERAZIONE ALL'INDIETRO)

$$- \; \alpha_{n,k} = \alpha_n(S_{n-1,k}) = \frac{H_n(u S_{n-1,k}) - H_n(d S_{n-1,k})}{u S_{n-1,k} - d S_{n-1,k}}$$

$$= \frac{H_n(S_{n,k+1}) - H_n(S_{n,k})}{S_{n,k+1} - S_{n,k}}$$

$$= \frac{H_{n,k+1} - H_{n,k}}{S_{n,k+1} - S_{n,k}}$$

$$- \; \beta_{n,k} = \frac{u \; H_{n,k} - d \; H_{n,k+1}}{(1+r)^n \; (u-d)}$$

es. Opzione Put :

$$S_0 = 1 , \quad K = \frac{5}{2}$$

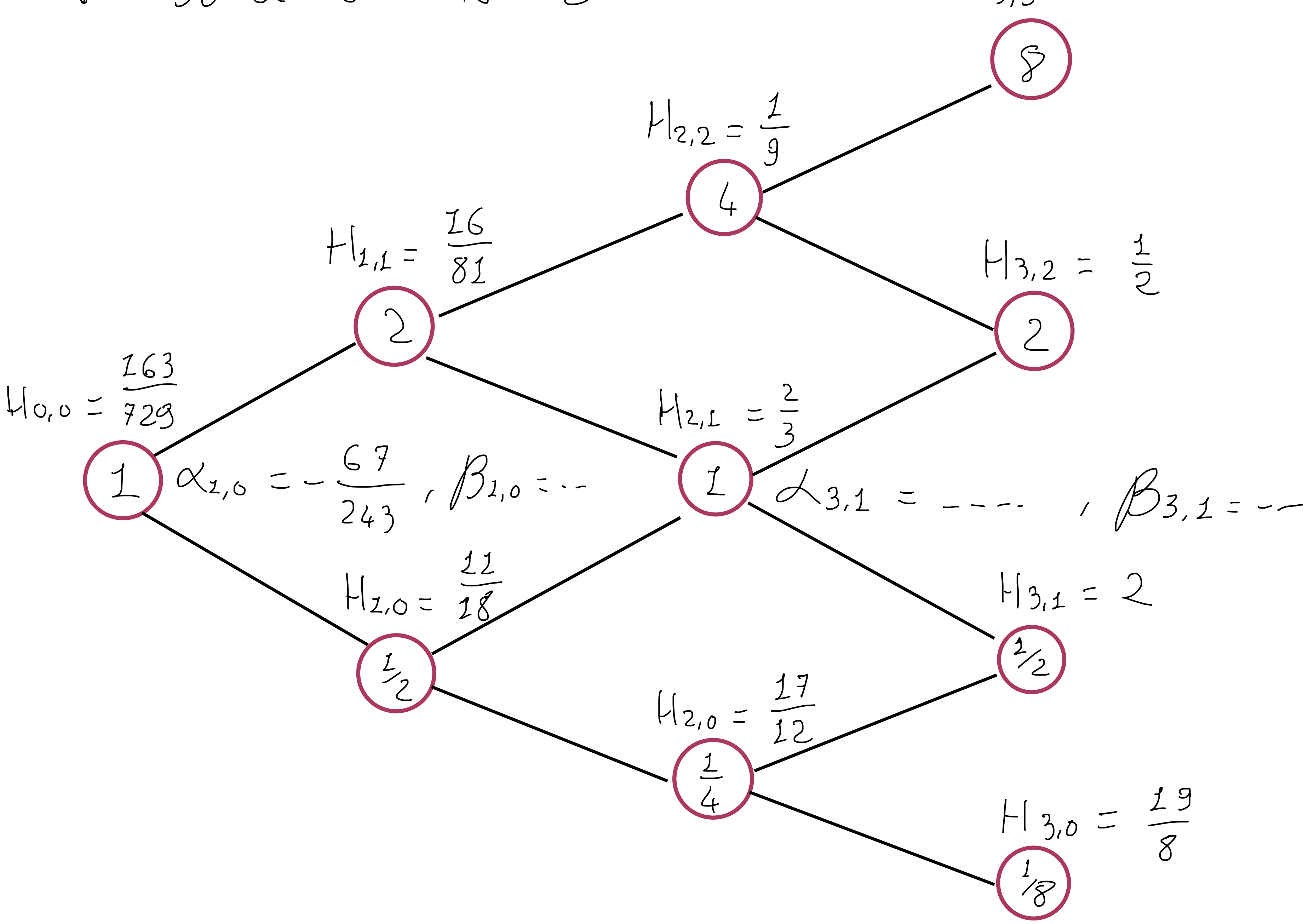
$$F(s) = \left(\frac{5}{2} - s \right)^+$$

$$r = 1/2 , \quad u = 2 , \quad d = 1/2$$

$$q = \frac{1+r-d}{u-d} = \frac{3/2 - 1/2}{3/2} = \frac{2}{3}$$

$$1-q = 1/3$$

- Fissiamo $N = 3$



- $H_{3,k} = F(S_{3,k}) = \left(\frac{5}{2} - S_{3,k}\right)^+$

- $H_{n-1,k} = \frac{2}{3}\left(\frac{2}{3} H_{n,k+1} + \frac{1}{3} H_{n,k}\right)$

$$\downarrow$$

- $H_{2,2} = \frac{2}{3}\left(\frac{2}{3} H_{3,3} + \frac{1}{3} H_{3,2}\right)$

$$= \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{9}$$

- $H_{2,1} = \frac{2}{3}\left(\frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 2\right) = \frac{2}{3}$

- $H_{2,0} = \frac{2}{3}\left(\frac{2}{3} \cdot 2 + \frac{1}{3} \cdot \frac{19}{8}\right) = \frac{17}{12}$

- $H_{1,1} = \frac{2}{3}\left(\frac{2}{3} \cdot \frac{1}{9} + \frac{1}{3} \cdot \frac{2}{3}\right) = \frac{16}{81}$

- $H_{1,0} = \frac{2}{3}\left(\frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{17}{12}\right) = \frac{11}{18}$

- $H_{0,0} = \frac{2}{3}\left(\frac{2}{3} \cdot \frac{16}{81} + \frac{1}{3} \cdot \frac{11}{18}\right) = \frac{163}{729}$

$$- \alpha_{1,0} = \frac{H_{1,1} - H_{1,0}}{S_{1,1} - S_{1,0}}$$

$$= \frac{16/81 - 11/18}{2 - 1/2} = -\frac{67}{243}$$

$$- \beta_{1,0} = \frac{u \cdot H_{1,0} - d\, H_{1,1}}{3/2\,(2 - 1/2)}$$

$$= \frac{2 \cdot 11/18 - 1/2 \cdot 16/81}{3/2\,(2 - 1/2)} = \frac{364}{729}$$

- <u>CASO</u> <u>PATH-DEPENDENT</u>: $X \in m\mathcal{F}_N \; (= m\mathcal{F}_N^S)$

$$\not\Downarrow$$

$$F(S_N)$$

$$S : \Omega \longrightarrow \mathbb{R}^N$$

$$\Downarrow$$

$$= \mathcal{F}_N^S = \sigma(S)$$

$$\overbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}^{N \text{ volte}}$$

$$\|$$

$$\{ S^{-1}(H),\; H \in \mathcal{B}_N \}$$

$$\|$$

$$\sigma\left(\left\{S^{-1}(A_1 \times \cdots \times A_N),\ A_1, \ldots, A_N \in \mathcal{B}\right\}\right)$$

$$\|$$

$$= \quad \sigma\left(\left\{S_n^{-1}(A),\ A \in \mathcal{B},\ n = 1, \ldots, N\right\}\right)$$

$$\Downarrow \left(X \in m\mathfrak{F}_N^S\right)$$

$$X = F(S) = F(S_1, \ldots, S_N)$$