- $X \in m\mathcal{F}_N \to F(S) = F(S^1, \dots, S^N)$

$S : \Omega \to \mathbb{R}^N \qquad (d=1)$

$(\mathcal{F}_N^S = \sigma(S))$

- per $n = 1, \dots, N$ fissato

$$S^{(n)} := (S_k^{(n)})_{k=0,\dots,n}, \qquad S_k^{(n)} := S_k$$

$$S^{(n)} : \Omega \to \mathbb{R}^n, \qquad Y \in m\mathcal{F}_n \; (m\mathcal{F}_n^S)$$

$$\Downarrow$$

$$Y = F^{(n)}(S^{(n)})$$

$$\downarrow$$

$$\in$$

$$m\mathcal{B}$$

- ABUSO: $H_n = H_n(S^n)$ $\to$ funzione di pricing

$\downarrow$

prezzo risk-neutral
var. aleat.

- Sistema di replicazione:

$$H_n = V_n = \alpha_n S_n + \beta_n B_n$$

$$\Downarrow$$

$$(*)\begin{cases} \alpha_n \cdot \underset{\wedge}{u} \cdot S_{n-1} + \beta_n B_n = \overbrace{H_n(S^{(n-1)}, u S_{n-1})}^{H_n \text{ se } 1+\mu_n = u} \\ \alpha_n \cdot d \cdot S_{n-1} + \beta_n B_n = \underbrace{H_n(S^{(n-1)}, d S_{n-1})}_{H_n \text{ se } 1+\mu_n = d} \end{cases}$$

$\forall\, n = 1, \dots, N \qquad (H_N = F)$

- $(*)$ ha soluzione

$$\alpha_n = \frac{H_n(S^{(n-1)}, u \cdot S_{n-1}) - H_n(S^{(n-1)}, d \cdot S_{n-1})}{u S_{n-1} - d S_{n-1}}$$

$$\beta_n = \frac{u \cdot H_n(S^{(n-1)}, d S_{n-1}) - d \cdot H_n(S^{(n-1)}, u \cdot S_{n-1})}{(1+r)^n (u-d)}$$

$$H_{n-1} \underset{\uparrow}{=} V_{n-1} = \alpha_n S_{n-1} + \beta_n B_{n-1}$$

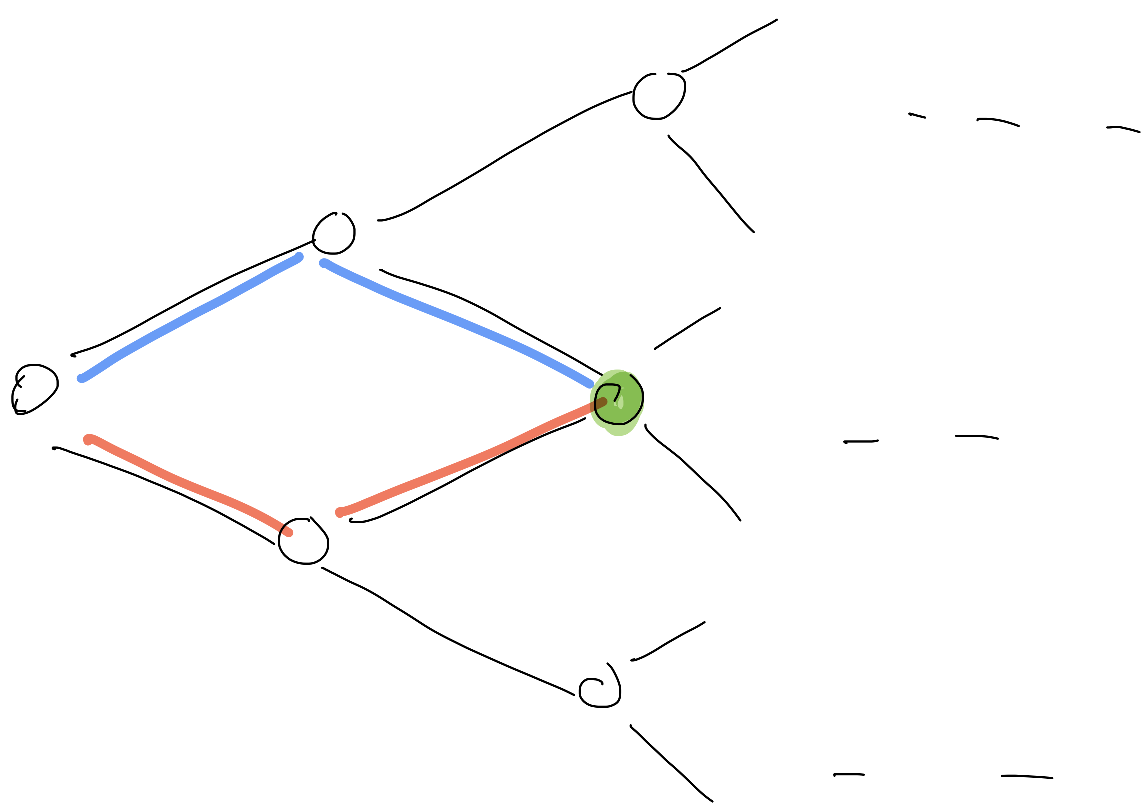
$\parallel$ auto-finanz.

$$\frac{1}{1+r} \mathbb{E}^Q[H_n \mid \mathcal{F}_{n-1}] = \frac{1}{1+r} \mathbb{E}^Q[\overbrace{H_n(S^{(n)})}^{H_n(S^{(n-1)}, (1+\mu_n) S_{n-1})} \mid S^{(n-1)}]$$

$$\underset{\uparrow}{=} \frac{1}{1+r} \mathbb{E}^Q[H_n(\underset{(S_1, \dots, S_{n-1})}{\underset{\parallel}{s}}, (1+\mu_n) S_{n-1})] \Big|_{s = S^{(n-1)}}$$

lemma di freezing

$$= \frac{q\, M_n(S^{(n-1)}, u\, S_{n-1}) + (1-q)\, M_n(S^{(n-1)}, d \cdot S_{n-1})}{1+r}$$



oss. La complessità comput. è esponenz. in N. (2^N traiettorie fino a T) mentre S_N assume $N+1$ valori

esempio $X = F(S_N, A_N)$, dove

$A_n = \Psi(S^{(n)})$. Es:

$$\Psi_n = \begin{cases} \frac{1}{1+n} \sum_{k=0}^{n} S_k & \to \text{opz. asiatica aritm.} \\ \min_{k \le n} S_k & \to \text{look-back} \end{cases}$$

$\underline{es:}$ $F(s,a) = (s-a)^+ \longrightarrow$ Call floating-strike

$F(s,a) = (k-a)^+ \longrightarrow$ Put fixed-strike

$\underline{IDEA:}$ $(S, A) = (S_n, A_n)_n$ processo di Markov

$$A_n^u := \Psi_n(S^{(n-1)}, u \cdot S_{n-1})$$

$$A_n^d := \Psi_n(S^{(n-1)}, d \cdot S_{n-1})$$

OSS.

$$\mathbb{E}^Q[\underbrace{\varphi(S_n, A_n)}_{\varphi(S_n, \Psi_n(S^{(n)}))} \mid \mathcal{F}_{n-1}] =$$

$$\varphi(\underbrace{S_n}_{S_{n-1}(1+\mu_n)}, \underbrace{\Psi_n(S^{(n)})}_{\Psi_n(S^{(n-1)}, (1+\mu_n)S_{n-1})})$$

come sopra

$$= q \cdot \varphi(u \cdot S_{n-1}, \Psi_n(S^{(n-1)}, u \cdot S_{n-1}))$$

$$+ (1-q) \cdot \varphi(d \cdot S_{n-1}, \Psi_n(S^{(n-1)}, d \cdot S_{n-1}))$$

$$= q \cdot \varphi(u \cdot S_{n-1}, A_n^u) + (1-q)\, \varphi(d \cdot S_{n-1}, A_n^d)$$

funzioni di (A_{n-1}, S_{n-1})

$\overset{}{=} \mathbb{E}^{Q}\left[\varphi(S_n, A_n) \mid (S_{n-1}, A_{n-1}) \right]$

(Markov !)

<u>es:</u> asiatica $\rightarrow$ $A_n^u = \dfrac{A_{n-1} \cdot (n-1) + u \cdot S_{n-1}}{n}$

- Definiamo:
  - $S_{n,k} := S_0 \cdot u^k \cdot d^{n-k}$
  - $A_{n,k}(j)$, $j = 0, \dots, J(n,k)$

    valori possibili per A_n se $S_n = S_{n,k}$
  - $H_{n,k}(j) := H_n(S_{n,k}, A_{n,k}(j))$
- algoritmo binomiale:

$$H_{n-1}(S_{n-1}, A_{n-1}) = \frac{1}{r+1} \mathbb{E}^{Q}\left[H_n(S_n, A_n) \mid \mathcal{F}_{n-1} \right]$$

$$= \frac{1}{1+r} \left(q \, H_n(u S_{n-1}, A_n^u) + (1-q) \, H_n(d S_{n-1}, A_n^d) \right)$$

oss. preced.

$\Downarrow$

$$H_{n-1,k}(j) = \frac{1}{1+r} \left(q \cdot H_n(u \cdot S_{n-1,k}, A_{n,k}^u(j)) \right.$$

$$+ (1-q)\, V_n\left(d \cdot S_{n-1,k},\, A^d_{n,k}(j)\right) \Big)$$

dove:

$$A^u_{n,k}(j) = A^u_n\left(S_{n-1,k},\, A_{n-1,k}(j)\right)$$

$$A^d_{n,k}(j) = A^d_n\left(S_{n-1,k},\, A_{n-1,k}(j)\right)$$

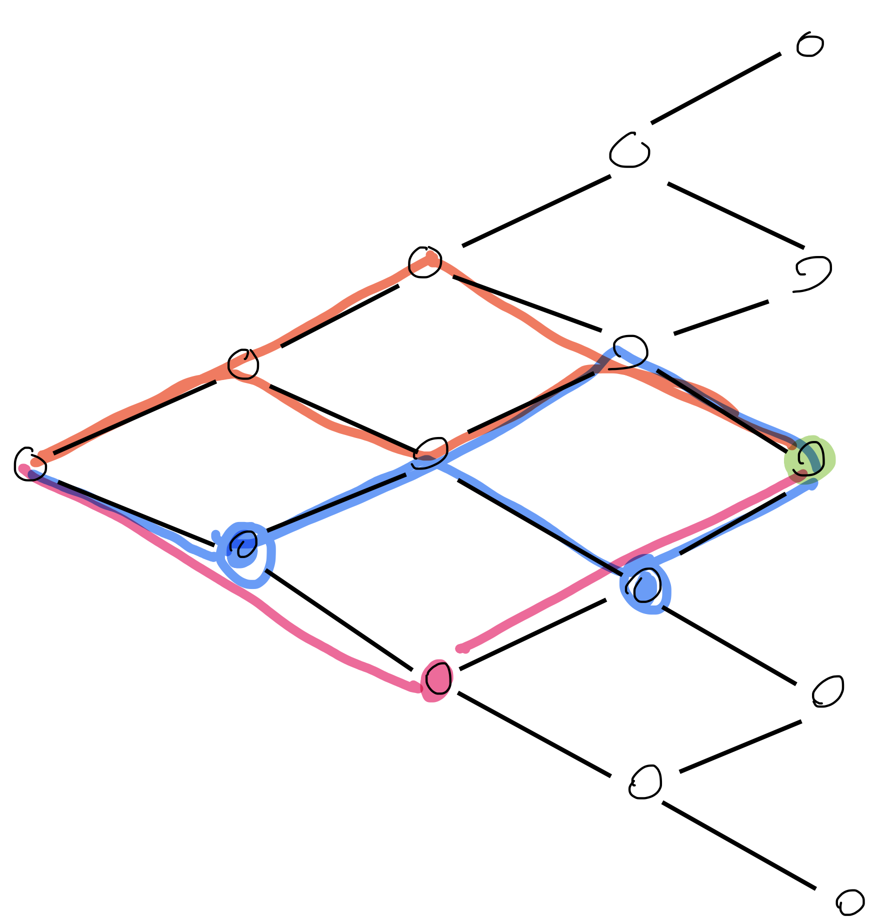
$$\forall\ n = 1, \ldots, N, \quad k = 0, \ldots, n-1, \quad j = 0, \ldots, J(n,k)$$

esempio $\quad A_n := \min_{k \le n} S_k \quad , \quad u \cdot d = 1$

$$\Downarrow$$

$S_0 = 1$

$$S_{n,k} = \begin{cases} d^{\,n-2k} & \text{se } 2k \le n \\ u^{\,2k-n} & \text{se } 2k \ge n \end{cases}$$



- $2k \le n$:

$$A_{n,k}(j) = d^{\,n-k-j}$$

$$j = 0, \ldots, k$$

- $2k \ge n$:

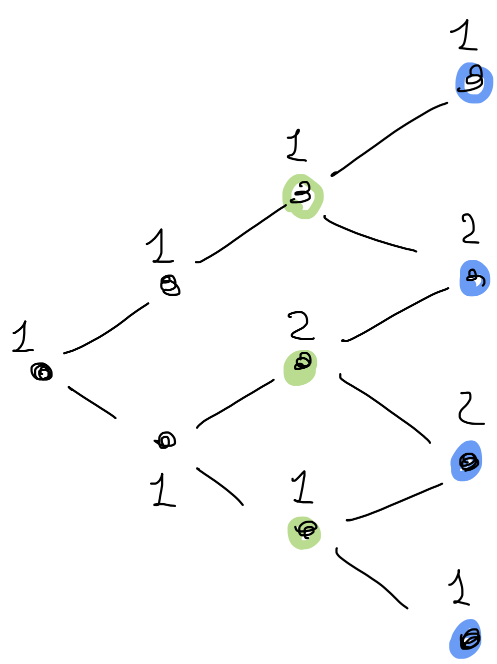
$$A_{n,k}(j) = u^{\,n-k-j}$$

$$j = 0, \ldots, n-k$$

es: $n=4,\ k=2 \longrightarrow S_{4,2} = 1$

$A_{4,2} \in \{1, d, d^2\}$

n	0	1	2	3	4	5	...
$\max_k J(n,k)$	1	1	2	2	3	3	

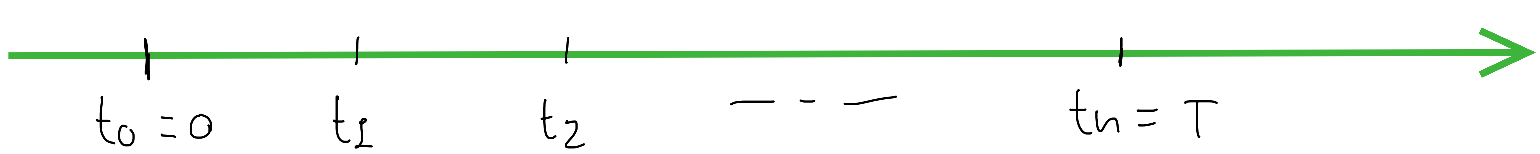


$$J(n,k) \sim \frac{n}{2}$$

$\Downarrow$

NO ESPONENZIALE

• Convergenza modello binomiale:



$$t_n - t_{n-1} = \frac{T}{N}$$

Problema: $H_n^{(N)} = H(S_n^{(N)}) \xrightarrow[N \to +\infty]{} ?$

- PARAMETRI: r_N, u_N, d_N

$$q_N = \frac{1 + r_N - d_N}{u_N - d_N}$$

$$\Big(\mathbb{Q}_N \left(1+\mu_n^{(N)} = u_N \right) = q_N \quad \forall\, n = 1, \dots, N ,$$

$$\left(\mu_n^{(N)}\right)_n \text{ sono i.i.d} \Big)$$

- Poniamo: $\delta_N := \dfrac{T}{N}$

$$r_N := e^{r \delta_n} - 1$$

$$\Updownarrow$$

$$(1+r_N) = e^{r\delta_n}$$

r → tasso osservato a capitaliz. comp.

$$u_N := e^{\sigma \sqrt{\delta_N} + \alpha \delta_N}$$

$$d_N := e^{-\sigma \sqrt{\delta_N} + \beta \delta_N}$$

σ volatilità di mercato stimata

α, β relativi alla media osservata dei ritorni

(PROCEDURA DI CALIBRAZIONE)

- Consideriamo:

$$H_0^{(N)} = (1+r_N)^{-N}\, \mathbb{E}^{Q_N}\left[F(S_N^{(N)}) \right]$$

$$= e^{-rT}\, \mathbb{E}^{Q_N}\Big[F\big(S_0 \underbrace{\prod_{n=1}^{N} (1+\mu_n^{(N)})}_{=\, e^{X^{(N)}}} \big) \Big]$$

$$X^{(N)} = \log \prod_{n=1}^{N} (1+\mu_n^{(N)})$$

$$= \sum_{n=1}^{N} \underbrace{\log(1+\mu_n^{(N)})}_{=:\, Y_n^{(N)}}$$

$$= e^{-rT}\, \mathbb{E}^{Q_N}\Big[\underbrace{F}_{(\in C_b)} \underbrace{\big(S_0\, e^{\overbrace{\sum_{n=1}^{N} Y_n^{(N)}}^{=X^{(N)}}} \big)}_{\text{funz. di } X^{(N)}} \Big] \xrightarrow[N\to+\infty]{} ?$$

<u>RICHIAMO:</u> • $\mu_n \xrightarrow{d} \mu$ se (per def.)

$$\int \varphi(x)\, \mu_n(dx) \longrightarrow \int \varphi(x)\, \mu(dx)$$

$\forall \quad \varphi \in C_b$

- $X_n \xrightarrow{d} X \quad$ se

$$\mu_{X_n} \xrightarrow{d} \mu_X$$

$$\left(\Longleftrightarrow \mathbb{E}[\varphi(X_n)] \longrightarrow \mathbb{E}[\varphi(X)] \quad \forall \varphi \in C_b \right)$$

teorema

$$X^{(N)} \xrightarrow[N \longmapsto +\infty]{d} X \sim \mathcal{N}_{\left(r - \frac{\sigma^2}{2}\right)T,\, \sigma^2 T}$$

lemma

$$\underset{\left(\substack{\downarrow \\ \frac{1}{2}}\right)}{2 q_N - 1} = \frac{r - \frac{\sigma^2}{2} - \frac{\alpha+\beta}{2}}{\sigma} \sqrt{\delta_N} + o\left(\sqrt{\delta_N}\right)$$

per $N \longmapsto +\infty$

dim

$$q_n = \frac{1 + r_N - d_N}{u_N - d_N}$$

$$= \frac{e^{r\delta_N} - e^{-\sigma\sqrt{\delta_N} + \beta\delta_n}}{e^{\sigma\sqrt{\delta_N} + \alpha\delta_n} - e^{-\sigma\sqrt{\delta_N} + \beta\delta_n}}$$

$$\Downarrow$$

$$2q_N - 1 = \frac{2e^{r\delta_N} - e^{-\sigma\sqrt{\delta_N} + \beta\delta_N} - e^{\sigma\sqrt{\delta_N} + \alpha\delta_N}}{e^{\sigma\sqrt{\delta_N} + \alpha\delta_N} - e^{-\sigma\sqrt{\delta_N} + \beta}}$$

$$= \frac{2r\delta_N - (\alpha+\beta)\delta_N - \sigma^2\delta_N + o(\delta_N)}{2\sigma\sqrt{\delta_N}\,(1 + o(1))}$$

$$= \frac{\left(r - \frac{\alpha+\beta}{2} - \frac{\sigma^2}{2}\right)\delta_N + o(\delta_N)}{\sigma\sqrt{\delta_N}\,(1 + o(\cancel{1}))}$$

$$= \frac{r - \frac{\alpha+\beta}{2} - \frac{\sigma^2}{2}}{\sigma}\sqrt{\delta_N} + o\left(\sqrt{\delta_N}\right)$$

per $N \mapsto +\infty$

#

- osserviamo:

$$Q_N\left(Y_k^{(N)} = \sigma\sqrt{\delta_N} + \alpha\delta_N\right) = q_N$$

$$Q_N\left(Y_k^{(N)} = -\sigma\sqrt{\delta_N} + \beta\delta_N\right) = 1 - q_N$$

$\boxed{\text{lemma}}$

- $\mathbb{E}^{Q_N}\left[Y_1^{(N)}\right] = \left(r - \frac{\sigma^2}{2}\right)\delta_N + o(\delta_N)$

- $\mathbb{E}^{Q}\left[\left(Y_1^{(N)}\right)^2\right] = \sigma^2 \delta_N + o(\delta_N)$

- $\mathbb{E}^{Q}\left[\left|Y_1^{(N)}\right|^3\right] = o(\delta_N)$

per $N \mapsto +\infty$

- <u>Intuizione</u>:

$$\mathrm{var}(\cdot) \sim T\sigma^2 \cdot N$$

$$X^{(N)} = \frac{1}{N}\sum_{n=1}^{N} N \cdot Y_n^{(N)} \sim \mathcal{N}_{\left(r-\frac{\sigma^2}{2}\right)T,\ \sigma^2 T}$$

$$\mathbb{E}[\cdot] \sim \left(r - \frac{\sigma^2}{2}\right)T$$

teo contr. limite: Y_k i.i.d. con $\mathbb{E}[Y_k] = m$, $\mathrm{var}(Y_k) = \sigma^2$

$$\underbrace{\frac{1}{N}\sum_{k=1}^{N} Y_k}_{\downarrow\ \delta_m} \sim \mathcal{N}_{m,\ \frac{\sigma^2}{N}}$$

$= \mathrm{var}(Y_k)$

$m = \mathbb{E}[Y_k]$

(legge dei grandi numeri)