dim | In generale:

$$X_n := \mathbb{E}[\,Y \mid \mathcal{F}_n\,] \quad , \quad n \in \mathbb{N}_0$$

è una martingala. Infatti:

$$\mathbb{E}[\,X_{n+1} \mid \mathcal{F}_n\,] = \mathbb{E}[\,\mathbb{E}[Y \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n\,]$$

$$\underset{\text{torre}}{\overset{\text{propr.}}{\longrightarrow}} = \mathbb{E}[\,Y \mid \mathcal{F}_n\,] = X_n \,.$$

- x la tesi del teorema

$$\left(Y_n \, \mathbb{E}^{Q}\left[\frac{Z}{Y_N} \mid \mathcal{F}_n \right] = X_n \, \mathbb{E}^{Q^x}\left[\frac{Z}{X_N} \mid \mathcal{F}_n \right] \right)$$

con $Z = X'_N \in \{ S^1_N, \ldots, S^d_N, B_N \}$, abbiamo

$$\cancel{Y_n} \underbrace{\mathbb{E}^{Q}\left[\frac{X'_N}{Y_N} \mid \mathcal{F}_n \right]}_{= \frac{X'_n}{\cancel{Y_n}}} = X_n \, \mathbb{E}^{Q^x}\left[\frac{X'_N}{X_N} \mid \mathcal{F}_n \right]$$

$$\Downarrow$$

$$\frac{X'_n}{X_n} = \mathbb{E}^{Q^x}\left[\frac{X'_N}{X_N} \mid \mathcal{F}_n \right]$$

$\forall \; n \leq N$ fissato

$$\Downarrow$$

X'_n / X_n è Q^x-marting. #

dim (teorema) Proviamo prima:

$$(*) \quad \mathbb{E}^{Q^X}[Z \mid \mathcal{F}_n] = \mathbb{E}^{Q}\left[Z \cdot \frac{Y_n}{Y_N}\left(\frac{X_n}{X_N}\right)^{-1} \mid \mathcal{F}_n\right]$$

$$\mathbb{E}^{Q^X}[Z \mid \mathcal{F}_n] \overset{\text{Verifica diretta}}{=} \frac{\mathbb{E}^{Q}[Z \cdot L \mid \mathcal{F}_n]}{\mathbb{E}^{Q}[L \mid \mathcal{F}_n]}$$

def. di L

$$= \frac{\mathbb{E}^{Q}\left[Z \cdot \frac{X_N}{Y_N} \mid \mathcal{F}_n\right]}{\underbrace{\mathbb{E}^{Q}\left[\frac{X_N}{Y_N} \mid \mathcal{F}_n\right]}_{\text{m } \mathcal{F}_n \text{, limitata } (\Omega \text{ finito})}}$$

Q E.M.M. risp. Y

$$= \mathbb{E}^{Q}\left[Z \cdot \frac{X_N}{Y_N} \cdot \left(\frac{X_n}{Y_n}\right)^{-1} \mid \mathcal{F}_n\right] \Rightarrow (*)$$

Ora: $Y_n \, \mathbb{E}^{Q}\left[\frac{Z}{Y_N} \mid \mathcal{F}_n\right]$

$$= \mathbb{E}^{Q}\left[Z \cdot \frac{Y_n}{Y_N} \mid \mathcal{F}_n\right]$$

$$\mathbb{E}^{Q}\left[\underbrace{Z \cdot \frac{Y_n}{Y_N} \cdot \frac{X_n}{X_N} \cdot \frac{X_N}{X_n}}_{\parallel} \,\middle|\, \mathcal{F}_n\right]$$

$$\parallel$$

$$\frac{Y_n}{Y_N}\left(\frac{X_n}{X_N}\right)^{-1} \cdot Z \cdot \frac{X_n}{X_N}$$

$$\parallel \quad (*)$$

$$\mathbb{E}^{Q^*}\left[Z \cdot \frac{X_n}{X_N} \,\middle|\, \mathcal{F}_n\right] = X_n \, \mathbb{E}^{Q^*}\left[\frac{Z}{X_N} \,\middle|\, \mathcal{F}_n\right]$$

#

Proposiz. Sia Q una E.M.M. rispetto a $Y > 0$ numerario, sia $(\alpha, \beta) \in \mathcal{A}$. Allora:

$$\tilde{V}_n^{(\alpha,\beta)} := \frac{V_n^{(\alpha,\beta)}}{Y_n}, \qquad n = 0, \dots, N$$

è una Q-martingala

Oss. $\tilde{V}_0^{(\alpha,\beta)} \underbrace{\left(= V_0^{(\alpha,\beta)}\right)}_{\uparrow \text{ se } Y = B} = \mathbb{E}^{Q}\left[\tilde{V}_n^{(\alpha,\beta)}\right]$

dim (CASO $Y = B$)

$$\tilde{V}_n^{(\alpha,\beta)} = \tilde{V}_0^{(\alpha,\beta)} + \sum_{k=1}^{n} \underbrace{\alpha_k}_{m\mathcal{F}_{k-1}} \cdot (\underbrace{\tilde{S}_k}_{\mathbb{Q}\text{-mart.}} - \tilde{S}_{k-1})$$

(teo trasf. S per α) $\Rightarrow$ $\tilde{V}^{(\alpha,\beta)}$ è una $\mathbb{Q}$-marting.

esempio (Modello binomiale)

Assumiamo $\exists$ E.M.M. $\mathbb{Q}$ risp. a B.

$$\tilde{S}_{n-1} = \mathbb{E}^{\mathbb{Q}}[\tilde{S}_n \mid \mathcal{F}_{n-1}]$$

$$\Updownarrow \quad (B_n = (1+r)^n)$$

$$(1+r)\underbrace{\cancel{S_{n-1}}}_{>0} = \mathbb{E}^{\mathbb{Q}}[\underbrace{S_n}_{= \cancel{S_{n-1}}(1+\mu_n)} \mid \mathcal{F}_{n-1}]$$

$$\Updownarrow$$

$$(1+r) = \mathbb{E}^{\mathbb{Q}}[\underbrace{1+\mu_n}_{= u \cdot \mathbb{1}_{\{\mu_n = u-1\}} + d \cdot \mathbb{1}_{\{\mu_n = d-1\}}} \mid \mathcal{F}_{n-1}]$$

! NON SAPPIAMO ANCORA SE μ_n sono indip. sotto $\mathbb{Q}$

$$= u \cdot \mathbb{Q}(\mu_n = u - 1 \mid \mathcal{F}_{n-1}) + d \cdot \underbrace{\mathbb{Q}(\mu_n = d-1 \mid \mathcal{F}_{n-1})}_{= 1 - \mathbb{Q}(\mu_n = u-1 \mid \mathcal{F}_{n-1})}$$

$$\boxed{\mathbb{Q}(F \mid \mathcal{G}) := \mathbb{E}[\mathbb{1}_F \mid \mathcal{G}]}$$

$$= \mathbb{Q}(\mu_n = u - 1 \mid \mathcal{F}_{n-1})(u-d) + d$$

$$\Downarrow$$

$$q := \mathbb{Q}(\mu_n = u-1 \mid \mathcal{F}_{n-1}) = \frac{1+r-d}{u-d}$$

costante

$$\Downarrow$$

$(\mathcal{F}_{n-1} = \mathcal{F}^{\mu}_{n-1})$

$(\mu_n)_n$ sono v.a. indipend. $\Leftarrow$ μ_n indip. da $\mathcal{F}_{n-1}$

$$\Downarrow$$

$$q = \mathbb{Q}(\mu_n = u - 1)$$

$$\Downarrow$$

sotto $\mathbb{Q}$, μ_n i.i.d, con

$$1+\mu_n \sim q\cdot\delta_u + (1-q)\cdot\delta_d$$

$$\Downarrow$$

unica $\mathbb{Q}$ su $\Omega = \{u, d\}^N$.

In partic.:

$$\mathbb{Q}\left(S_n = S_0\cdot u^k d^{n-k}\right) = \binom{n}{k} q^k (1-q)^{n-k}$$

- $$\mathbb{Q}\sim\mathbb{P} \iff \begin{cases} q\in\,]0,1[& (d<1+r<u) & \text{se } p\in\,]0,1[\\ q=1 & (1+r=u) & \text{se } p=1 \\ q=0 & (1+r=d) & \text{se } p=0 \end{cases} \quad (*)$$

- Viceversa, ponendo $\mathbb{Q}$ data da:
  - $\mathbb{Q}(\mu_n = u-1) := q := \dfrac{1+r-d}{u-d}$
  - $\mu_1, \ldots, \mu_N$ indip.

$$\Downarrow \text{ (ripercorrendo step prec. all'indietro)}$$

$$\tilde{S}_{n-1} = \mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_n \mid \mathcal{F}_{n-1}\right].$$

oss. $\exists$ misura martingala $\mathbb{Q}$ s.se

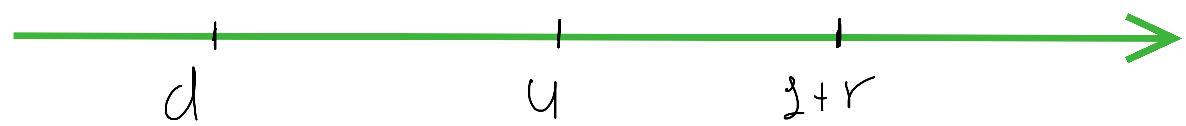
$$q\in[0,1] \iff d\le 1+r\le u \quad (\triangle)$$

Se $(\triangle)$ violata, es. $1+r>u$

$$\Downarrow$$

$B_n > S_n$ certam.

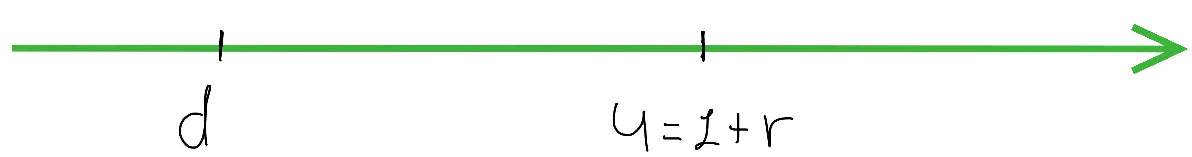
arbitraggio: compro B vendo S.



oss. Se (Δ) vale, ma $q = 1$ mentre $p \in]0,1[$

$\Downarrow$ (S mal prezzato)

$u = 1 + r$



- Arbitraggio: • in $t_0 = 0$: vendo 1 \$ di S / compro 1 \$ di B $\Big]\ V_0 = 0$

$\Rightarrow \forall\, n,\quad B_n \geq S_n$ e $\mathbb{P}(\underbrace{B_n > S_n}_{\{\mu_k = 1 - d \text{ per almeno un } k = 1, \ldots, n\}}) > 0$!

$\uparrow$ certamente

• stessa cosa se $q = 0$

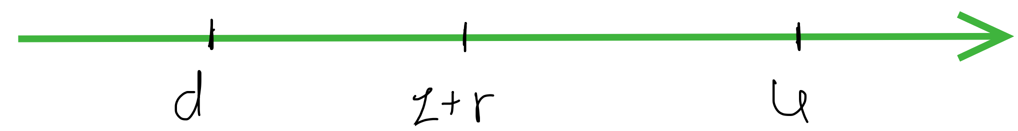
oss. Se (Δ) vale, con $q \in]0,1[$

$\Downarrow$

$$d < 1+r < u$$

ma $p = 1$ (GIOCO TRUCCATO)

$\Downarrow$



$$S_n = u^n \quad \mathbb{P}\text{-q.c.}$$

$$\vee$$

$$(1+r)^n = B_n$$

- Arbitraggio: - in $t_0 = 0$: vendo 1 \$ di B
  compro 1 \$ di S

$$\Rightarrow \mathbb{P}(\underset{\substack{\parallel \\ S_n - B_n}}{V_n} > 0) = 1$$

esercizio

$N = 1$, $d = 2$, $\Omega = \{\omega_1, \omega_2\}$

$\mathcal{F} = \mathcal{P}(\Omega)$

$$S_1^1 = \begin{cases} S^{1,+} & \text{se } \{\omega_1\} \\ S^{1,-} & \text{se } \{\omega_2\} \end{cases}$$

$$S_1^2 = \begin{cases} S^{2,+} & \text{se } \{\omega_1\} \\ S^{2,-} & \text{se } \{\omega_2\} \end{cases}, \quad r_1 = 0$$

Per trovare $\mathbb{Q}$ devo risolvere

$$\begin{cases} S_0^1 = \mathbb{E}^{\mathbb{Q}}[S_T^1] = q \cdot S^{1,+} + (1-q) S^{1,-} \\ S_0^2 = \mathbb{E}^{\mathbb{Q}}[S_T^2] = q \cdot S^{2,+} + (1-q) S^{2,-} \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} q = \dfrac{S_0^1 - S^{1,-}}{S^{1,+} - S^{1,-}} \\ q = \dfrac{S_0^2 - S^{2,-}}{S^{2,+} - S^{2,-}} \end{cases}$$

$\nexists$ sol. in generale

$\Downarrow$ 1° F.F.T.A.P

$\exists$ arbitraggi, quali?

Proposiz. Se (S, B) mercato discreto è libero da arbitraggi ($\exists$ $\mathbb{Q}$ E.M.M)

$$V_N^{(\alpha,\beta)} \leq V_N^{(\alpha',\beta')}$$

$\mathbb{P}$-q.c. ($\mathbb{Q}$-q.c.)

$$\Downarrow$$

(* *) $\quad V_n^{(\alpha,\beta)} \leq V_n^{(\alpha',\beta')}$, $n=0,\dots,N-1$

$\mathbb{P}$-q.c. ($\mathbb{Q}$ q.c.)

oss. $V_N^{(\alpha,\beta)} = V_N^{(\alpha',\beta')} \Rightarrow V_n^{(\alpha,\beta)} = V_n^{(\alpha',\beta')}$, $n=0,\dots,N-1$

dim x F.F.T.A.P $\exists$ $\mathbb{Q}$ EMM (risp a Y num.)

$\Rightarrow \tilde{V}^{(\alpha,\beta)}, \tilde{V}^{(\alpha',\beta')}$ sono $\mathbb{Q}$-martingale

$$\Rightarrow \tilde{V}_n^{(\alpha,\beta)} = \mathbb{E}^{\mathbb{Q}}\left[\tilde{V}_N^{(\alpha,\beta)} \mid \mathcal{F}_n\right]$$

$$\mathbb{P}\sim\mathbb{Q} \; ! \quad \leq \mathbb{E}^{\mathbb{Q}}\left[\tilde{V}_N^{(\alpha',\beta')} \mid \mathcal{F}_n\right] = \tilde{V}_n^{(\alpha',\beta')}$$

Ma $\tilde{V}_n = V_n / B_n$ #

dim (alternativa) se (* *) non vale

$\Downarrow$

$\exists$ arbitraggio (quale?) .

ASSURDO x F.F.T.A.P.

dim (F.F.T.A.P)

I parte : $\exists$ $\mathbb{Q}$ E.M.M. $\Rightarrow \nexists$ arbitraggi

x ASSURDO , $\exists$ $(\alpha,\beta) \in \mathcal{A}$ arbitraggio ,

$$(\star)\quad \begin{cases} V_{\bar{n}} \geq 0 & \mathbb{P}\text{-q.c.} \\ \mathbb{P}(V_{\bar{n}} > 0) > 0 \end{cases}$$

per un $\bar{n} \in \{1, \ldots, N\}$

$\mathbb{P} \sim \mathbb{Q} \implies (\star)$ soddisfatte anche sotto $\mathbb{Q}$

x ipotesi

$\implies \mathbb{E}^{\mathbb{Q}}[V_{\bar{n}}] > 0 \implies \mathbb{E}^{\mathbb{Q}}[\tilde{V}_{\bar{n}}] > 0$

Però, $\mathbb{Q}$ misura mart.,

$0 \overset{}{=} V_0 = \mathbb{E}[\tilde{V}_{\bar{n}}] > 0$ $\underline{\text{ASSURDO}}$ #

prop. (i) di arbitraggio

$$\boxed{\int_\Omega \underset{\substack{\vee\!\mid \\ 0}}{X}\, d\mathbb{Q} = 0 \implies X = 0 \ \mathbb{Q}\text{-q.c.}}$$