- Modello trinomiale:

$$B_n = B_{n-1}(1+r)$$

$$S_n = S_{n-1}(1+\mu_n) \quad , \qquad n = 1, \dots, N$$

(VERSIONE INCOMPLETA)

$$1+\mu_n \sim p_1 \delta_u + p_2 \delta_m + (1-p_1-p_2)\delta_d$$

$$\Big(\mathbb{P}(\mu_n = u-1) = p_1 , \quad \mathbb{P}(\mu_n = m-1) = p_2$$
$$\mathbb{P}(\mu_n = d-1) = 1-p_1-p_2 \Big)$$

$(\mu_n)_n$ indipendenti

- Dato $X \in m\mathcal{F}_N$: come lo prezziamo?

$$H_n^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_n}{B_N} X \,\middle|\, \mathcal{F}_n \right]$$

$\downarrow$

prezzo risk-neutral

non introduce arbitraggi per alcuna scelta di $\mathbb{Q}$.

<u>Copertura</u>? short-fall risk minimization

$$\min_{(\alpha,\beta)\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\Big[\underbrace{\big(X - V_N^{(\alpha,\beta)}\big)^+}_{} \Big]$$

$$\hookrightarrow \begin{cases} + & \text{se sottoreplichiamo} \\ 0 & \text{se super-replichiamo} \end{cases}$$

es. $N=2, \quad S_0 = 1$

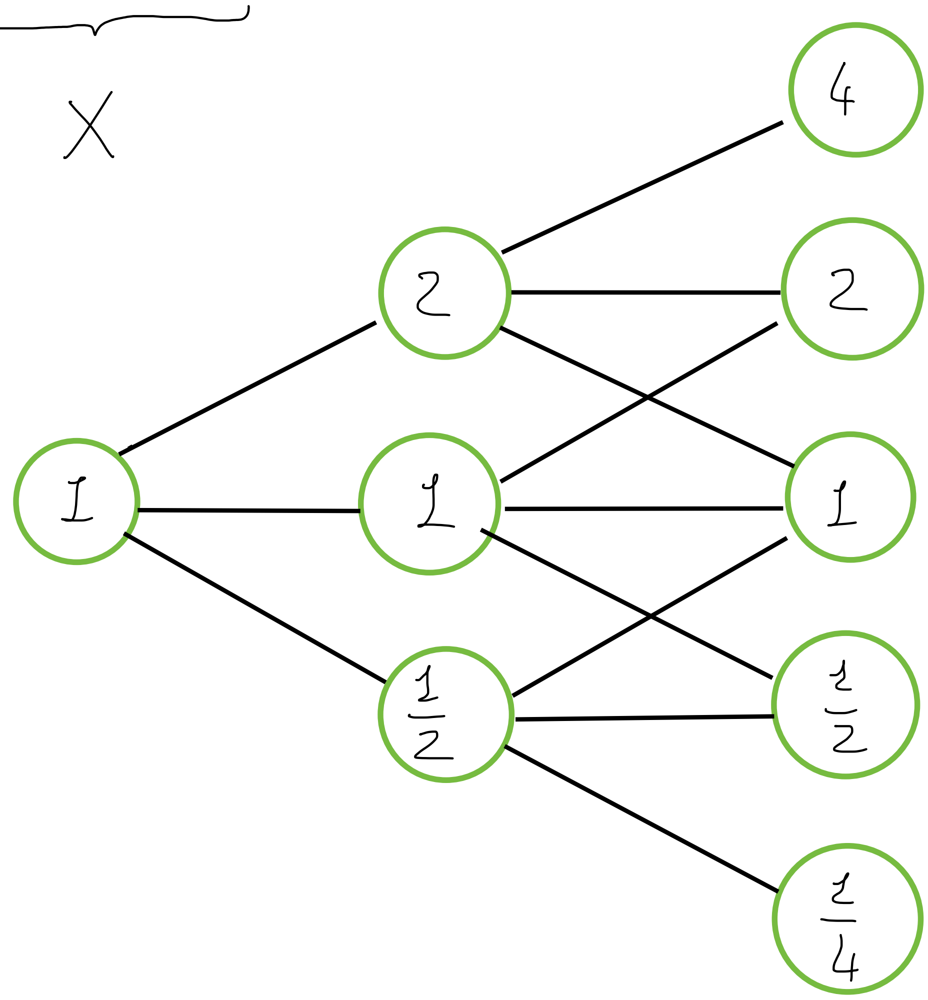
$$1+\mu_n = \begin{cases} 1/2 & \text{con } p_1 = 1/3 \\ 1 & \text{con } p_2 = 1/3 \\ 2 & \text{con } p_3 = 1/3 \end{cases} \qquad \text{(indipend.)}$$

$$\mathbb{P}(\mu_n = -1/2) = \mathbb{P}(\mu_n = 0)$$

$$= \mathbb{P}(\mu_n = 1) = \frac{1}{3}$$

$r = 0$

$$\underbrace{F(S_2)}_{X} = (S_2 - 1)^+$$



Risolvere: $$\min_{(\alpha,\beta)\in A_{\geq 0}} \mathbb{E}^{\mathbb{P}}\Big[\underbrace{\big((S_2-1)^+ - V_2^{(\alpha,\beta)}\big)^+}_{U(V_2^{(\alpha,\beta)},\, S_2)}\Big]$$

$$\mathcal{U}(v,s) = \left((s-1)^+ - v\right)^+$$

$$\mathcal{A}_{\geq 0} := \left\{ (\alpha,\beta) \in \mathcal{A} : V_n^{(\alpha,\beta)} \geq 0 \;, n = 0,1,2 \right\}$$

- condizione sufficiente per $(\alpha,\beta) \in \mathcal{A}_{\geq 0}$:

$$\begin{cases} V_0 \geq 0 \\ -\dfrac{V_{n-1}}{S_{n-1}} \leq \alpha_n \leq \dfrac{2V_{n-1}}{S_{n-1}} \end{cases} \qquad (*)$$

$$\underset{\substack{\uparrow \\ \text{autofinanziam.}}}{V_n =} V_{n-1} + \alpha_n S_{n-1} \mu_n$$

$$= V_{n-1} + \begin{cases} -\dfrac{\alpha_n S_{n-1}}{2} & \text{se } \mu_n = -\frac{1}{2} \\ 0 & \text{se } \mu_n = 0 \\ \alpha_n \cdot S_{n-1} & \text{se } \mu_n = 1 \end{cases}$$

$$\geq 0$$

- algoritmo di programm. dinamica:

$$W_2(v,s) = \mathcal{U}(v,s) = \left((s-1)^+ - v\right)^+$$

$$W_{n-1}(v,s) = \min_{\alpha \in [-\frac{v}{s}, \frac{2v}{s}]} \mathbb{E}^{\mathbb{P}}\Big[W_n \underbrace{\left(v + s\alpha\mu_n, s(1+\mu_n) \right)}_{\substack{\| \\ G_n((v,s),\mu_n;\alpha)}} \Big]$$

per $n = 1, 2$

- $n=2$:

$$W_1(v,2) = \min_{\alpha \in [-\frac{v}{2}, v]} \mathbb{E}^{\mathbb{P}}\left[U\left(v + 2\alpha\mu_2,\ 2(1+\mu_2)\right)\right]$$

$$= \min_{\alpha \in [-\frac{v}{2}, v]} \underbrace{\mathbb{E}^{\mathbb{P}}\left[\left(\left(2(1+\mu_2)-1\right)^+ - (v + 2\alpha\mu_2) \right)^+ \right]}$$

$$\frac{1}{3}\left[\cancel{\left(0 - \underbrace{(v-\alpha)}_{\geq 0}\right)^+} + (1-v)^+ \right.$$

$$\left. + \underbrace{(3 - v - 2\alpha)^+} \right]$$

$$\min_{\alpha \in [-\frac{v}{2}, v]} (\cdot) = (3-3v)^+$$

per $\alpha = v$

$$= \min_{\alpha \in [-\frac{v}{2}, v]} \frac{1}{3}(1-v)^+ + \frac{1}{3}(3-v-2\alpha)^+$$

$$= \frac{1}{3}(1-v)^+ + (1-v)^+ = \frac{4}{3}(1-v)^+$$

$$\underset{\alpha \in [-\frac{v}{2}, v]}{\arg\min} = v \quad \Rightarrow \quad \alpha_2(v, 2) = v$$

$$W_1(v, 1) = \min_{\alpha \in [-v, 2v]} \mathbb{E}^{\mathbb{P}}\left[\underset{\substack{\shortparallel \\ U}}{W_2}\left(v + \alpha \mu_2, \overset{1+}{\mu_2} \right) \right]$$

$$= \min_{\alpha \in [-v, 2v]} \underbrace{\mathbb{E}^{\mathbb{P}}\left[\left(\mu_2^+ - (v + \alpha \mu_2) \right)^+ \right]}_{\shortparallel}$$

$$\vdots$$

$$= \frac{1}{3}(1 - 3v)^+ \quad \text{con } \alpha = 2v$$

$$\Rightarrow \quad \alpha_2(v, 1) = 2v$$

$$W_1\left(v, \tfrac{1}{2}\right) = \min_{\alpha \in [-2v, 4v]} \mathbb{E}^{\mathbb{P}}\left[\;----\; \right]$$

$$\vdots$$

$$= 0 \quad \forall \alpha \in [-2v, 4v]$$

$$\Rightarrow \quad \alpha_2(v, 1/2) = \, ? \quad \text{qualsiasi}$$

$$W_0(v,1) = \min_{\alpha \in [-v, 2v]} \underbrace{\mathbb{E}^{\mathbb{P}}\left[W_1(v+\alpha\mu_1, 1+\mu_1) \right]}_{\shortparallel}$$

$$\frac{1}{3}\Big[\cancel{W_1\left(v-\frac{\alpha}{2}, \frac{1}{2}\right)} + W_1(v, 1) + W_1(v+\alpha, 2) \Big]$$

$$= \frac{5}{9}(1-3v)^+ \quad \text{con min in } \alpha = 2v$$

$$\Rightarrow \alpha_1(v,1) = 2v$$

<u>oss</u>: $\underbrace{v \geq \frac{1}{3}}$ $\Rightarrow$ $\underbrace{W_0(v,1)}_{} = 0$ $\nearrow$ $V_2^\alpha \geq X$ $\mathbb{P}$-q.c.

$$= \min_{\alpha} \mathbb{E}^{\mathbb{P}}[(X - V_2^\alpha)^+]$$

$\hookrightarrow$ candidato per il prezzo di X in 0

- **Procediamo in avanti**:

- V_0 fissato (≥ 0)
- $\alpha_1 = \alpha_1(v,1) = 2v$

$$- \; V_1 = V_0 + \alpha_1 \overset{\overset{1}{\shortparallel}}{S_0} \cdot \mu_1$$

$$= V_0 + \begin{cases} -V_0 & \text{se } \mu_1 = -\frac{1}{2} \\ 0 & \text{se } \mu_1 = 0 \\ 2V_0 & \text{se } \mu_1 = 1 \end{cases}$$

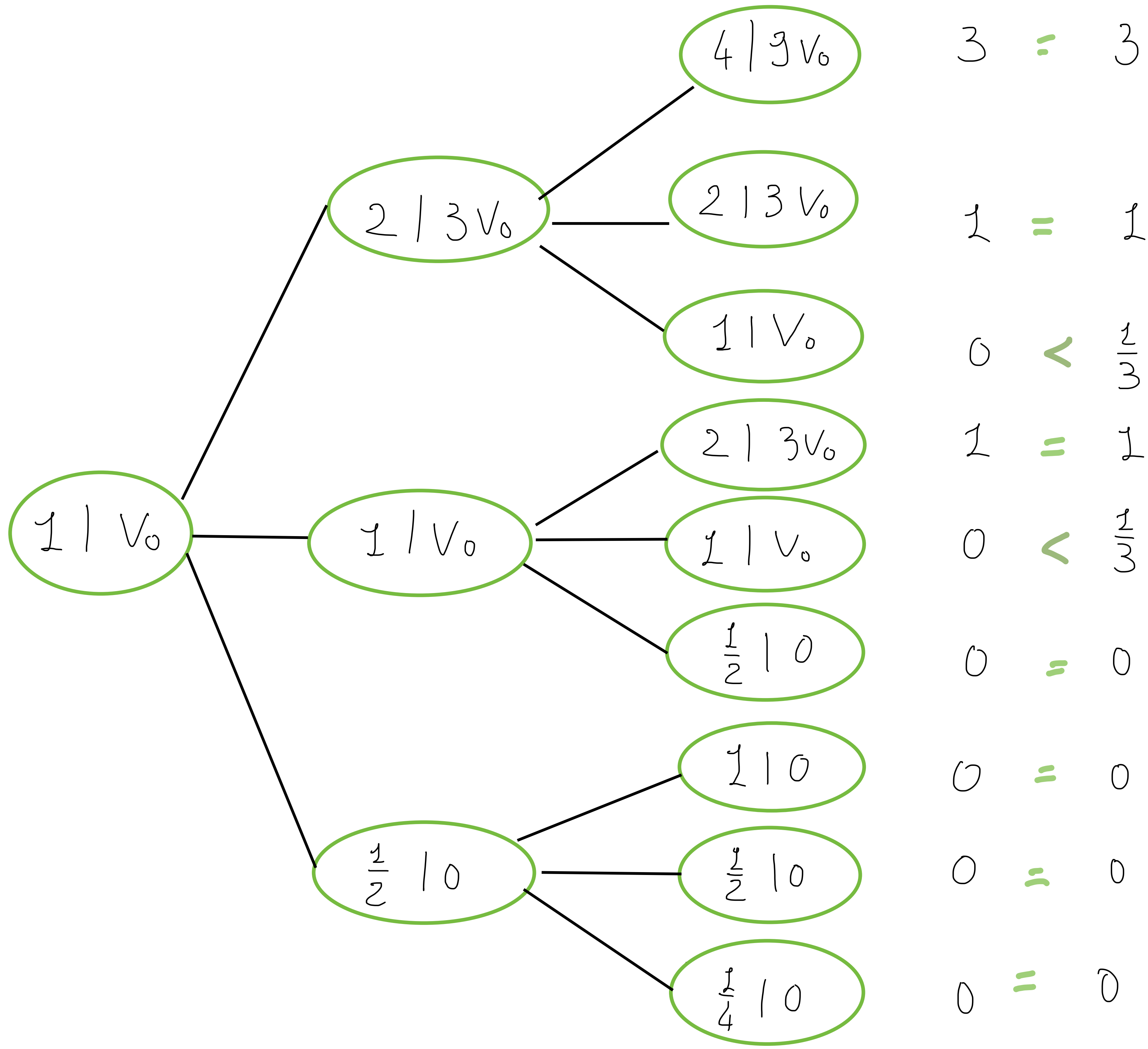
$$- \; \alpha_2 = \alpha_2(V, S)$$

$$= \begin{cases} 0 & \text{se } S_1 = \frac{1}{2} \quad (\text{se } \mu_1 = -\frac{1}{2}) \\ 2V_0 & \text{se } S_1 = 1 \quad (\text{se } \mu_1 = 0) \\ V_0 & \text{se } S_1 = 2 \quad (\text{se } \mu_1 = 1) \end{cases}$$

$$- \; V_2 = V_1 + \text{-----}$$

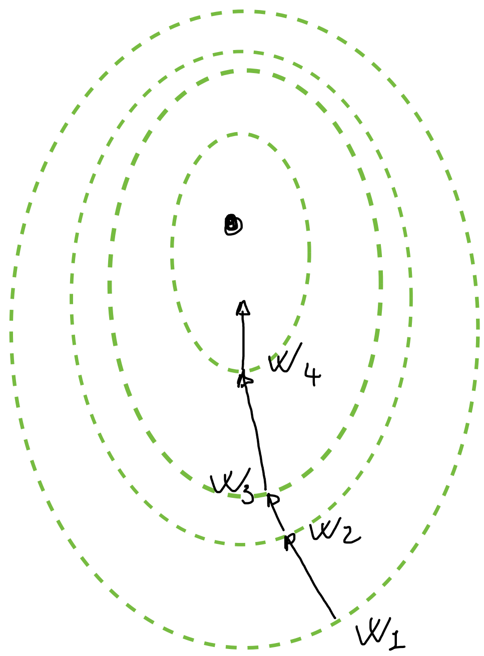
V_2

Payoff $\left(V_0 = \frac{1}{3}\right)$



- Discesa stocastica del gradiente:

$$\begin{cases} w_1 = 0 \\ w_{n+1} = w_n - \underbrace{\eta}_{\text{fissato}} \cdot \nabla \underbrace{f}_{\text{convessa}}(w_n), \quad n = 1, \dots, N-1 \end{cases}$$



- 2 possibilità:
  - $W_{out} := W_N$
  - $W_{out} := \frac{1}{N} \sum_{n=1}^{N} w_n$

    ↳ nostra scelta

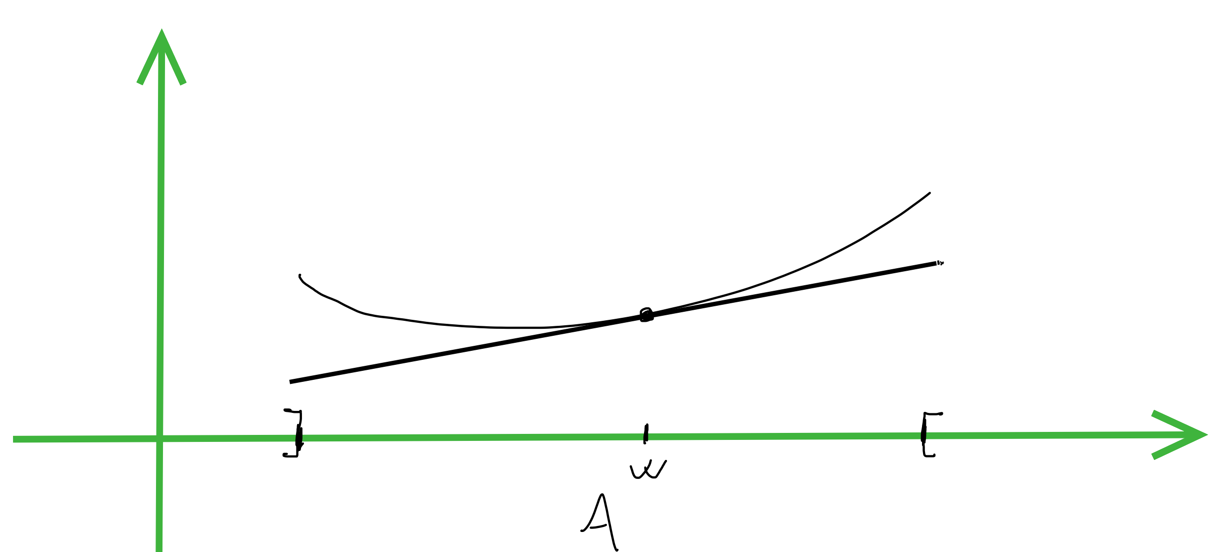
lemma Sia $A \subset \mathbb{R}^d$ aperto e convesso, e sia $f: A \to \mathbb{R}$ funzione. Allora

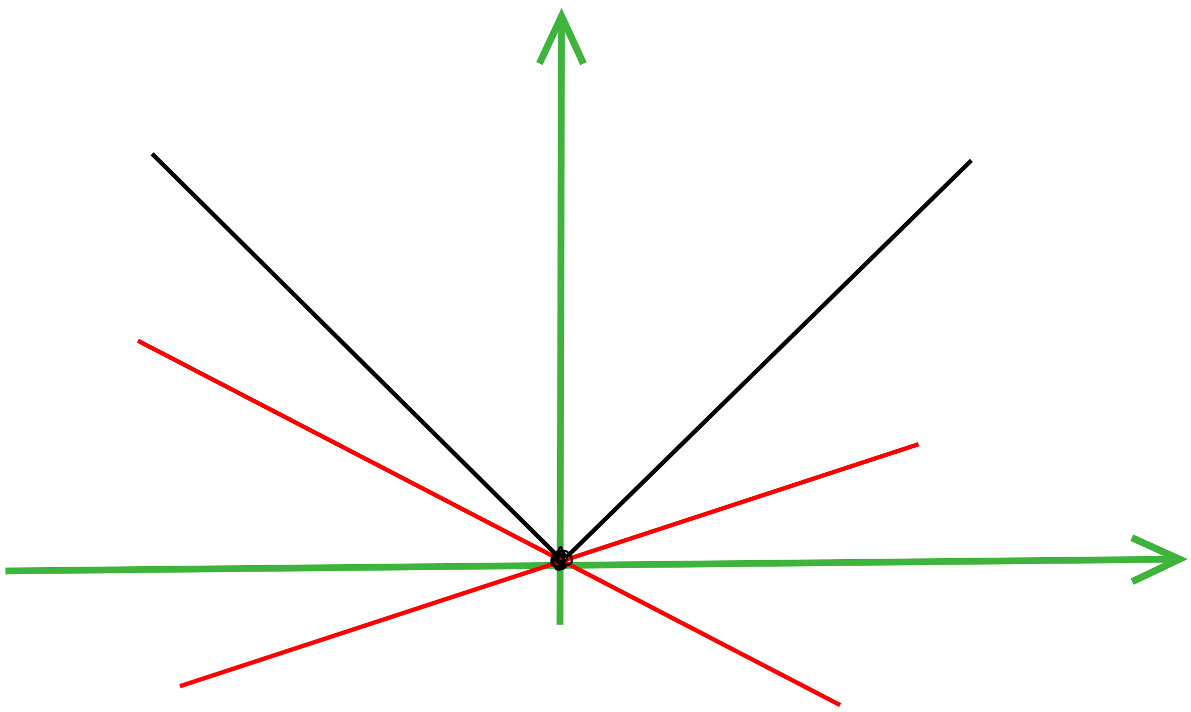
f è convessa $\iff$

$\forall \; w \in A \;\; \exists \; v = v(w) \in \mathbb{R}^d$:

$$f(u) \geq f(w) + \langle u-w, v \rangle \quad \forall \; u \in A$$

Inoltre, se f è differenziale, $v = v(w)$ è unico e $v(w) = \nabla f(w)$.





dim classica.

f è differenziabile — ASSURDO

$$\frac{f(u)-f(w)}{u-w} \geq v > f'(w)$$

$$\Rightarrow \liminf_{u \to w} \frac{f(u)-f(w)}{u-w} > f'(w)$$

notazione Dato $A \subset \mathbb{R}^d$, $\rho > 0$. Diciamo che f è ρ-Lipschitziana se

$$\| f(x) - f(y) \| \leq \rho \| x - y \| , \quad x, y \in A$$

teorema Sia A aperto e convesso. Sia

$f : A \to \mathbb{R}$ convessa, differenz e ρ-Lipschitziana.

Sia poi $w^* \in A$: $\|w^*\| \leq R$.

Ponendo $\eta := \frac{R}{\rho\sqrt{N}}$, e assumendo che

$w_n \in A \quad \forall \; n = 1, \dots, N$, vale:

$$f(\underbrace{\bar{w}}_{\frac{1}{N}\sum_{n=1}^{N} w_n}) - f(w^*) \leq \frac{R\rho}{\sqrt{N}}$$

corollario Sotto le ipotesi del teorema precedente,

$$w^* \in \arg\min_{\|w\| \leq R} f(w)$$

$$\Downarrow$$

$$\Big| f(\bar{w}) - f(w^*) \Big| \leq \frac{R\rho}{\sqrt{N}}$$

dipende da N

In particolare:

$$N \geq \frac{R^2 \rho^2}{\varepsilon^2} \Rightarrow |f(\bar{w}) - f(w^*)| \leq \varepsilon$$

oss.

$$f(\bar{w}) \xrightarrow[N \to +\infty]{} f(w^*)$$

$$\text{non } \bar{w} \xrightarrow[N \to +\infty]{} w^*$$

es.

$$f(x,y) = x^2 \,, \qquad w_1 = (1,1)$$

$$w^* = (0,0)$$

$$\bar{w}_N \to (0,1) \neq w^*$$

lemma

Sia $v_1, \ldots, v_N \in \mathbb{R}^d$ sequenza arbitraria.

Sia $w^* \in \mathbb{R}^d$ vettore arbitrario.

Ponendo:

$$\begin{cases} w_1 = 0 \\ w_{n+1} = w_n - \eta v_n \end{cases} \qquad \eta > 0$$

vale:

$$\sum_{n=1}^{N} \langle v_n, w_n - w^* \rangle \leq \frac{\|w^*\|^2}{2\eta} + \frac{\eta}{2} \sum_{n=1}^{N} \|v_n\|^2 \qquad (*)$$

- In particolare:

se $\eta = \dfrac{R}{\rho\sqrt{N}}$ con $R, \rho > 0$ tali che

$$\boxed{\|v_n\| \le \rho \quad , \quad \|w^*\| \le R}$$

allora:

$$\boxed{\frac{1}{N}\sum_{n=1}^{N} \langle v_n, w_n - w^* \rangle \le \frac{R\rho}{\sqrt{N}}}$$

$$\wedge\!\mid \qquad //$$

$$\frac{R^2}{2\eta N} + \frac{\eta}{2N} N \rho^2$$

dim (lemma)

$$\langle w_n - w^*, v_n \rangle = \frac{1}{\eta} \langle w_n - w^*, \eta v_n \rangle$$

$$= \frac{1}{2\eta}\left(- \| \underbrace{w_n - w^* - \eta v_n} \|^2 + \| w_n - w^* \|^2 + \eta^2 \|v_n\|^2 \right)$$

$$= \frac{1}{2\eta}\left(- \| w_{n+1} - w^* \|^2 + \| w_n - w^* \|^2 \right)$$

$$+ \frac{\eta}{2} \| v_n \|^2$$

$$\sum_{n=1}^{N} \langle w_n - w^*, v_n \rangle = \frac{1}{2\eta} \left(\| w_1 - w^* \|^2 - \| w_{N+1} - w^* \| \right)$$

$$+ \frac{\eta}{2} \sum_{n=1}^{N} \| v_n \|^2$$

$$\leq \frac{1}{2\eta} \| \underset{\substack{\shortparallel \\ 0}}{\cancel{w_1}} - w^* \|^2 + \frac{\eta}{2} \sum_{n=1}^{N} \| v_n \|^2 \qquad \#$$