teo Se (S,B) è arbitrage-free allora,

(S,B) è completo $\iff$ $\exists!$ $\mathbb{Q}$ E.M.M.

- tra le ipotesi: $\mathcal{F}_N = \mathcal{F}$
- $\boxed{\Rightarrow}$: già vista

dim × ipotesi: $\exists$ $\mathbb{Q}$ E.M.M.

- B numeraire, $\mathbb{Q} > 0$ (non è restrittivo)
- $m := \# \Omega$

$$\underset{\text{v.a}}{Y} \longleftrightarrow \underbrace{(Y_1, \ldots, Y_m)}_{\in \mathbb{R}^m}$$

$$\mathcal{L} := \{ \underbrace{\tilde{V}_N^{(\alpha,\beta)}}_{\text{v.a}} : (\alpha, \beta) \in \mathcal{A} \} \subseteq \mathbb{R}^m$$

→ sotto-spazio lineare

(S,B) non è completo

$$\Downarrow$$

$$\mathcal{L} \neq \mathbb{R}^m \qquad (\mathcal{F}_N = \mathcal{F})$$

$$\Downarrow$$

$$\exists\, \xi \in \mathbb{R}^m \setminus \{0\} : \quad \mathbb{E}^{\mathbb{Q}}[\xi \cdot \tilde{V}_N^{(\alpha,\beta)}] = 0 \quad \forall (\alpha,\beta) \in \mathcal{A}$$

$$\sum_{j=1}^{m} \xi_j \cdot \left(\tilde{V}_N^{(\alpha,\beta)}\right)_j \cdot \underbrace{\mathbb{Q}(\{w_j\})}_{>0}$$

$$= \langle \xi, \tilde{V}_N^{(\alpha,\beta)} \rangle_{\mathbb{Q}}$$

<u>oss:</u> $(1, \dots, 1) \in \mathcal{L} \implies \mathbb{E}^{\mathbb{Q}}[\xi] = 0$

$\hookrightarrow \tilde{V}_N^{(\alpha,\beta)} \equiv 1$

- Fisso $\delta > 1$,

$$\mathbb{Q}_\delta(\{w_j\}) = \left(1 + \frac{\xi_j}{\delta \underbrace{\|\xi\|_\infty}_{\max_{1 \le j \le m} |\xi_j|}}\right) \cdot \underbrace{\mathbb{Q}(\{w_j\})}_{>0}$$

$\forall j = 1, \dots, m$

- $\mathbb{Q}_\delta \neq \mathbb{Q} \quad (\xi \neq 0)$

- $$1 + \underbrace{\frac{\xi_j}{\delta \|\xi\|_\infty}}_{>-1} > 0 \qquad \left(\delta > 1, \ \frac{\xi_j}{\|\xi\|_\infty} \in [-1, 1]\right)$$

$\implies \mathbb{Q}_\delta > 0 \implies \mathbb{Q}_\delta \sim \mathbb{Q}$

- $$Q_\delta(\Omega) = \sum_{j=1}^{m} Q_\delta(\{w_j\})$$

$$= \sum_{j=1}^{m} \left(1 + \frac{\xi_i}{\delta \|\xi\|_\infty}\right) Q(\{w_j\})$$

$$= Q(\Omega) + \frac{1}{\delta\|\xi\|_\infty} \underbrace{\sum_{j=1}^{m} \xi_j \, Q(\{w_j\})}_{\overset{\shortparallel}{\mathbb{E}^{Q}[\xi] = 0}}$$

$$= 1$$

- <u>Martingalità</u>: proviamo, $\forall i = 1, \dots, d$

$$\mathbb{E}^{Q_\delta}\Big[\underbrace{\sum_{n=1}^{N} \alpha_n (\tilde{S}^i_n - \tilde{S}^i_{n-1})}_{=: \, G^{(\alpha)} \text{ v.a.}}\Big] = 0$$

$\forall \alpha$ predicibile

$$\mathbb{E}^{Q_\delta}[G^{(\alpha)}] = \sum_{j=1}^{m} \left(1 + \frac{\xi_j}{\delta\|\xi\|_{+\infty}}\right) Q(\{w_j\}) \, G^{(\alpha)}_j$$

$$= \underbrace{\mathbb{E}^{Q}[G^{(\alpha)}]}_{\shortparallel} + \frac{1}{\delta\|\xi\|_{+\infty}} \underbrace{\sum_{j=1}^{m} \xi_j \cdot Q(\{w_j\}) \, G^{(\alpha)}_j}$$

$$\boxed{\begin{array}{c} \mathbb{Q} \text{ e E.M.M.} \\ \Updownarrow \\ \tilde{S}^i \text{ martingale sotto } \mathbb{Q} \\ \Updownarrow \\ \mathbb{E}[G^{(\alpha)}] = 0 \end{array}}$$

$$0 = \mathbb{E}^{\mathbb{Q}}[\xi \cdot \underbrace{G^{(\alpha)}}_{\cap\ \mathcal{L}}]$$

$\#$

es. (Modello binomiale)

se $d \leq 1+r \leq u$, allora $\exists!$ $\mathbb{Q}$ misura martingala, data da

$$\begin{cases} \mathbb{Q}(\mu_n = u-1) =: q = \dfrac{1+r-d}{u-d} \\ (\mu_n)_n \text{ indipendenti} \end{cases}$$

inoltre $\mathbb{Q} \sim \mathbb{P}$ se

$$\begin{cases} d < 1+r < u & \text{se } p \in]0,1[\\ d = 1+r & \text{se } p = 0 \\ u = 1+r & \text{se } p = 1 \end{cases}$$

$\Longrightarrow$ $\mathfrak{X}(S,B)$ è arbitrage free allora è completo

(S.F.T.A.P)

$$H_n = \underbrace{(1+r)^{-(N-n)}}_{\frac{B_n}{B_N}} \mathbb{E}^{\mathbb{Q}}\left[F(S_N) \mid \mathcal{F}_n \right]$$

Probemi:

- calcolare H_n (v.a)
- trovare (α, β) replicante

- Se $n = 0$:

$$H_0 = (1+r)^{-N} \mathbb{E}^{\mathbb{Q}}\left[F(S_N) \right]$$

$$= (1+r)^{-N} \sum_{k=1}^{N} F(S_0 \cdot u^k d^{N-k}) \binom{N}{k} q^k (1-q)^{N-k}$$

- Se $n \neq 0$? Proprietà di Markov

$$\Downarrow$$

$$H_n = (1+r)^{-N} \underbrace{\mathbb{E}^{\mathbb{Q}}\left[F(S_N) \mid S_n \right]}_{\substack{\Downarrow \ (\text{Doob}) \\ \text{funz. di } S_n}}$$

- **Processi di Markov:**

def. Un processo X a tempo discreto

su $(\Omega, \mathcal{F}, (\mathcal{F}_n)_n, \mathbb{P})$ ha la proprietà di Markov se è adattato e

$$(M) \quad \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] = \mathbb{E}[\varphi(X_{n+1}) \mid X_n]$$

$\forall \ \varphi \in bm\mathcal{B}$ (limitata e Borel-mis.)

$\forall \ n \in \mathbb{N}_0$

! "=" per ogni coppia di versioni

Oss.
$$(M) \Rightarrow \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] = g_n^\varphi(X_n)$$

dove:
$$g_n^\varphi(x) := \int \varphi(y)\, \mu_{X_{n+1} \mid X_n = x}(dy)$$

$$= \mathbb{E}[\varphi(X_{n+1}) \mid X_n = x]$$

Anche:
$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] = \tilde{g}(X_n)$$

$$\Downarrow$$

$$\mathbb{E}[\varphi(X_{n+1}) \mid X_n] = \mathbb{E}[\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] \mid X_n]$$

$$= \mathbb{E}[\tilde{g}(X_n) \mid X_n]$$

(se $\tilde{g} \in m\mathcal{B}$) $$= \tilde{g}(X_n) = \mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n]$$

$$\Downarrow$$

$$(M)$$

es. $(X_n)_n$ v.a. indip., $\mathcal{F}_n = \mathcal{F}_n^X$

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n^X] \underset{\uparrow}{=} \mathbb{E}[\varphi(X_{n+1})] = C_n$$

($\varphi(X_{n+1})$ indip. da $\mathcal{F}_n^X$)

C_n: funz. di X_n (costante)

$$Y_n := \sum_{j=1}^{n} X_n \quad , \quad n \in \mathbb{N}$$

! $(Y_n)_n$ non sono indipend.

$$\mathbb{E}[\varphi(Y_{n+1}) \mid \mathcal{F}_n^X] = \mathbb{E}[\underbrace{\varphi(Y_n + X_{n+1})}_{\text{funz. mis. di } Y_n,\, X_{n+1}} \mid \mathcal{F}_n^X]$$

Y_n m $\mathcal{F}_n^X$, X_{n+1} indip. da $\mathcal{F}_n^X$

$$\underset{\text{(Lemma di freezing)}}{=} \mathbb{E}[\varphi(y + X_{n+1})]_{y=Y_n}$$

$$= g_n^{\varphi}(Y_n)$$

prop. (M) è equivalente a

(M') $\mathbb{E}[\mathbb{1}_{(X_{n+1}\in A)} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{1}_{(X_{n+1}\in A)} \mid X_n]$

$$\mathbb{P}(X_{n+1}\in A \mid \mathcal{F}_n) \qquad \mathbb{P}(X_{n+1}\in A \mid X_n)$$

$\forall\, n\in\mathbb{N}_0$, $\forall\, A\in\mathcal{B}$

dim Sia $(\varphi_m)_m$ famiglia di funzioni semplici e limitate t.c. $\varphi_m \xrightarrow[m\to+\infty]{} \varphi \in b m\mathcal{B}$

puntualmente ($\exists$ sempre)

$\left(\varphi_m = \sum_j c_j^m \mathbb{1}_{A_j^m}\right)$

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] = \lim_{m\to+\infty} \mathbb{E}[\varphi_m(X_{n+1}) \mid \mathcal{F}_n]$$

teo. Lebesgue condizionato

$$= \lim_{m\to+\infty} \sum_j c_j^m \underbrace{\mathbb{P}(X_{n+1}\in A_j^m \mid \mathcal{F}_n)}_{(M')}$$

$$= \qquad\qquad \mathbb{P}(X_{n+1}\in A_j^m \mid X_n)$$

$$=$$

$$= \lim_{m \to +\infty} \mathbb{E}[\varphi_m(X_{n+1}) \mid X_n]$$

$$= \mathbb{E}[\varphi(X_{n+1}) \mid X_n] \qquad \#$$

prop.

$$(M) \iff \mathbb{P}(X_k \in A \mid \mathcal{F}_n) = \mathbb{P}(X_k \in A \mid X_n)$$

$$\forall\, k > n, \quad \forall A \in \mathcal{B} \qquad (M'')$$

dim

$\boxed{\Rightarrow}$: assumiamo (M'') vera per k, n e proviamo che è vera per $k+1, n$

$$\mathbb{P}(X_{k+1} \in A \mid \mathcal{F}_n) = \mathbb{P}(\underbrace{\mathbb{P}(X_{k+1} \in A \mid \mathcal{F}_k)}_{\underbrace{\mathbb{P}(X_{k+1} \in A \mid X_k)}_{\text{funz. } \varphi \text{ di } X_k}} \mid \mathcal{F}_n)$$

Ipotesi induttiva + dim. preced.

$$= \mathbb{P}(\underbrace{\mathbb{P}(X_{k+1} \in A \mid X_k)}_{} \mid X_n)$$

$$(M) \qquad (\sigma(X_n) \subseteq \sigma(X_k))$$

$$= \mathbb{P}(X_{k+1} \in A \mid \mathcal{F}_k)$$

$$= \mathbb{P}(X_{n+1} \in A \mid X_n) \quad \#$$

Oss. $(M)_{(\mathcal{F}_n)_n} \Rightarrow (M)_{(\tilde{\mathcal{F}}_n)_n}$ se $\tilde{\mathcal{F}}_n \subset \mathcal{F}_n \ \forall n$
e X è adatt. a $(\tilde{\mathcal{F}}_n)_n$

$$\mathbb{E}[\varphi(X_{n+1}) \mid \tilde{\mathcal{F}}_n] =$$

$$= \mathbb{E}\Big[\underbrace{\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n]}_{\substack{\shortparallel \\ \underbrace{\mathbb{E}[\varphi(X_{n+1}) \mid X_n]}_{m\,\tilde{\mathcal{F}}_n}}} \mid \tilde{\mathcal{F}}_n \Big]$$

$\boxed{\text{se } X \text{ rimane adattato rispetto a } \tilde{\mathcal{F}}_n}$

$$= \mathbb{E}[\varphi(X_{n+1}) \mid X_n]$$

- In particolare:

$$(M)_{(\mathcal{F}_n)_n} \Rightarrow (M)_{(\mathcal{F}^X_n)_n}$$

$$\not\Leftarrow$$

$\hookrightarrow$ es: $\mathcal{F}_n = \mathcal{F} \ \forall n$

def Un processo stoc. a tempo discreto a valori in uno spazio $E \subset \mathbb{R}^d$

discreto si dice un processo discreto

notazione Denotiamo con $i, j, k, h, \ldots$

oppure $i_0, i_1, \ldots$, $j_1, j_2, \ldots$

gli elementi generici di E

def. Un processo discreto si dice una catena di Markov discreta se vale (M) rispetto alla filtraz. naturale.

proposizione Un processo discreto X è una catena di Markov discreta se e solo se:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$

$$\| \qquad (*)$$

$$\mathbb{P}(X_{n+1} = j \mid X_n = i_n)$$

$\forall\, n \in \mathbb{N}_0$, $\forall\, i_0, i_1, \ldots, i_n, j \in E$:

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) > 0$$

- $(*)$ è la definizione classica di catena di Markov discreta (Bremaud - "Markov chains ...")

def.) Una catena di Markov si dice OMOGENEA se il termine destro in (*) non dipende da n

es: $$\mathbb{P}(X_3 = 4 \mid X_2 = 1, X_1 = 0, X_0 = 5)$$

$$\|$$

$$\mathbb{P}(X_3 = 4 \mid X_2 = 1)$$

$$\|$$

$$\mathbb{P}(X_{10} = 4 \mid X_9 = 1)$$