# Lezione 7 MSC
# Equivalenze deboli – prima parte (di 2)

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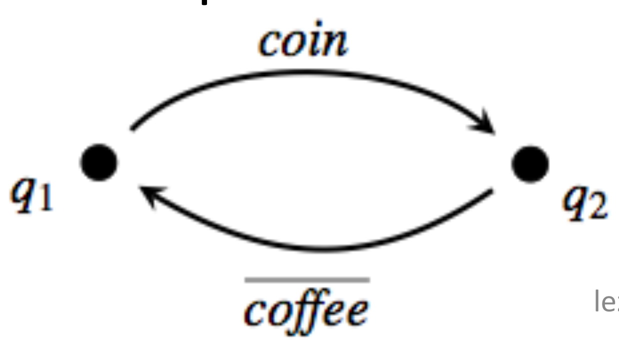
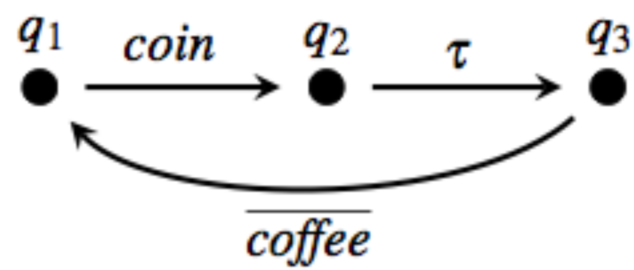
# Azione non osservabile

- Azione interna su cui non è possibile interagire, di solito rappresentata con tau $\tau$
- L'azione $\tau$ può essere il risultato di una sincronizzazione tra processi: comunicazione sincrona (hand-shake) punto-a-punto
- Tutte le nozioni di equivalenza viste possono essere adattate a (non) tener conto del tau
- I due processi sotto devono essere considerati equivalenti, in qualsiasi di queste varianti

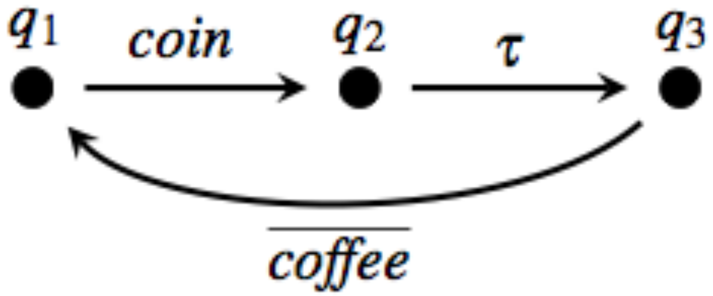


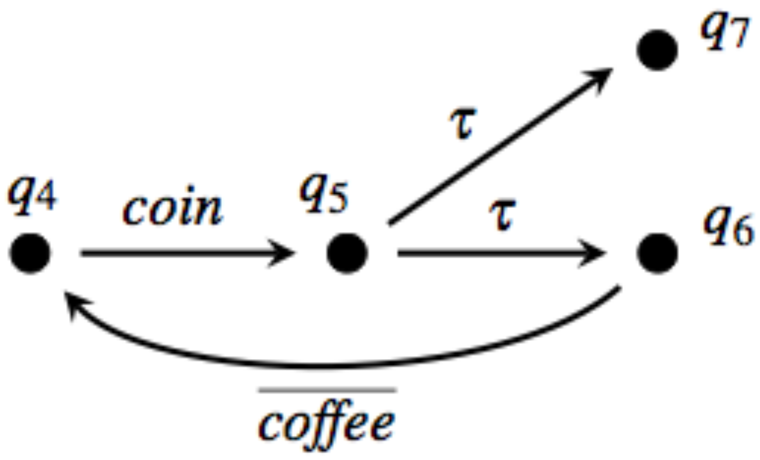



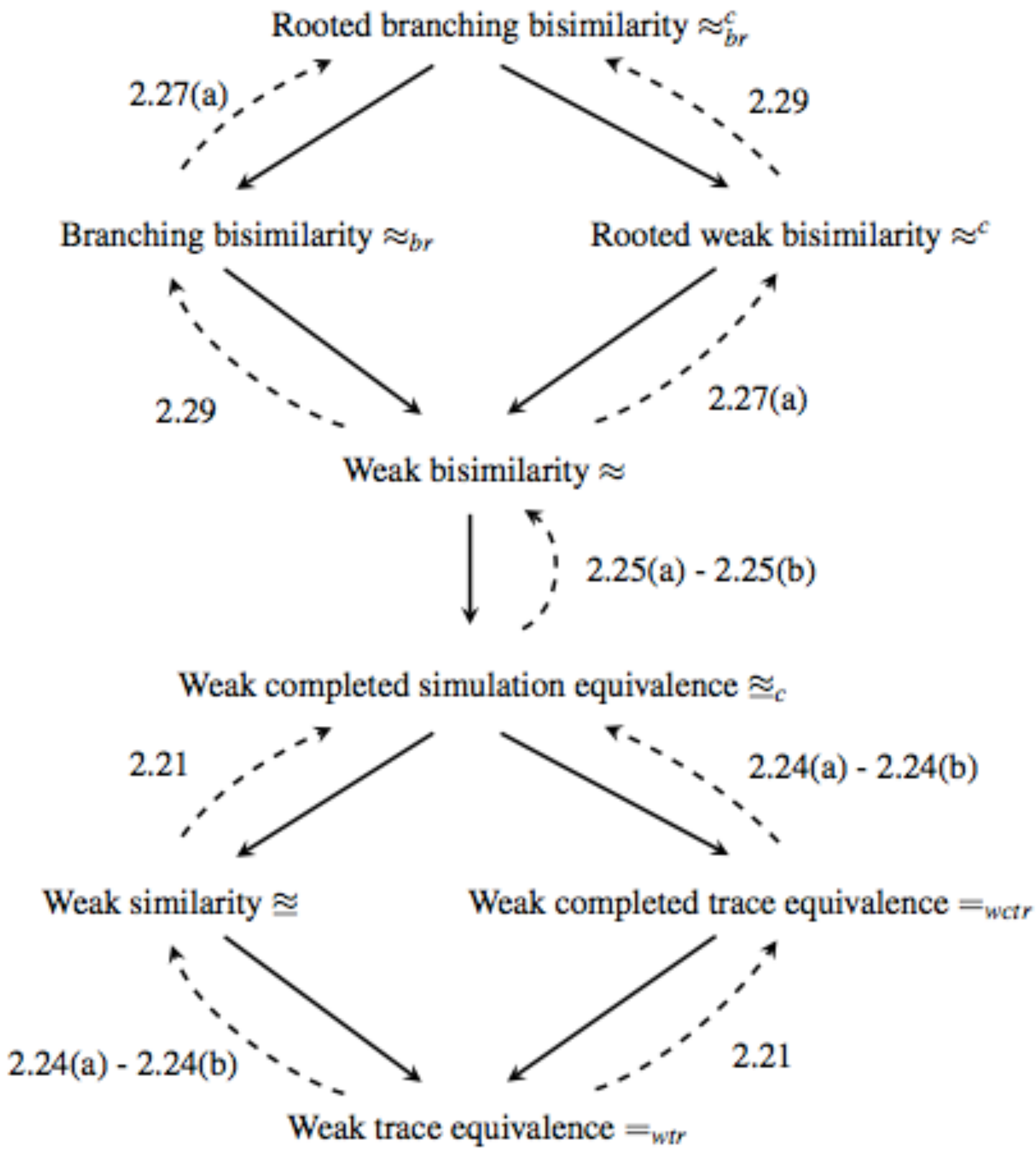
# A volte il tau ha effetti osservabili

Il tau può escludere azioni possibili, cioè operare delle scelte! Quindi non è totalmente non osservabile ….

N.B. $q_7$ è un deadlock




Rooted branching bisimilarity $\approx_{br}^{c}$
2.27(a)
2.29
Branching bisimilarity $\approx_{br}$
Rooted weak bisimilarity $\approx^{c}$
2.29
2.27(a)
Weak bisimilarity $\approx$
2.25(a) - 2.25(b)
Weak completed simulation equivalence $\eqsim_{c}$
2.21
2.24(a) - 2.24(b)
Weak similarity $\eqsim$
Weak completed trace equivalence $=_{wctr}$
2.24(a) - 2.24(b)
2.21
Weak trace equivalence $=_{wtr}$

# Chiusura riflessiva e transitiva weak

**Definition 2.16.** For any LTS $TS = (Q, A \cup \{\tau\}, \rightarrow)$, where $\tau \notin A$, define relation $\Longrightarrow \subseteq Q \times A^* \times Q$ as the *weak* reflexive and transitive closure of $\rightarrow$ (*cf.* Definition 2.4), i.e., as the least relation induced by the following axiom and rules, where $\varepsilon$ is the empty trace:

$$\frac{q_1 \xrightarrow{\alpha} q_2}{q_1 \xRightarrow{\alpha} q_2} \qquad \frac{q_1 \xrightarrow{\tau} q_2}{q_1 \xRightarrow{\varepsilon} q_2} \qquad \frac{}{q \xRightarrow{\varepsilon} q} \qquad \frac{q_1 \xRightarrow{\sigma_1} q_2 \quad q_2 \xRightarrow{\sigma_2} q_3}{q_1 \xRightarrow{\sigma_1 \sigma_2} q_3}$$

$\square$

Note that a path $q_1 \xrightarrow{\tau} q_2 \xrightarrow{\tau} \dots q_n \xrightarrow{\tau} q_{n+1}$ (with $n \geq 0$) yields that $q_1 \xRightarrow{\varepsilon} q_{n+1}$. Moreover, it can be proved that $q \xRightarrow{\alpha} q'$ if and only if there exist two states $q_1$ and $q_2$ such that $q \xRightarrow{\varepsilon} q_1 \xrightarrow{\alpha} q_2 \xRightarrow{\varepsilon} q'$. Finally, if $\sigma = \alpha_1 \alpha_2 \dots \alpha_n$, then $q_1 \xRightarrow{\sigma} q_{n+1}$ if and only if there exist $q_2, \dots, q_n$ such that $q_1 \xRightarrow{\alpha_1} q_2 \xRightarrow{\alpha_2} \dots q_n \xRightarrow{\alpha_n} q_{n+1}$.

# Weak traces

**Definition 2.17. (Weak trace equivalence)** Let $(Q, A \cup \{\tau\}, \rightarrow)$ be an LTS, where $\tau \notin A$. A *weak trace* of $q \in Q$ is a sequence $\sigma \in A^*$ such that $q \stackrel{\sigma}{\Longrightarrow} q'$ for some $q'$. Hence, the set $WTr(q)$ of weak traces of $q$ is

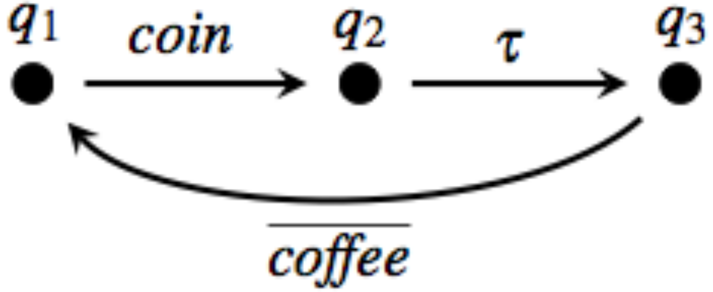
$$WTr(q) = \{\sigma \in A^* \mid \exists q' \in Q.\, q \stackrel{\sigma}{\Longrightarrow} q'\}.$$

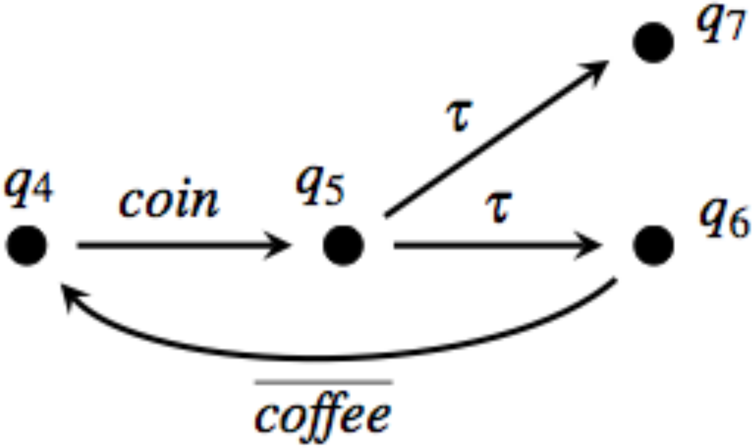
Two states $q_1, q_2 \in Q$ are *weak trace equivalent* if $WTr(q_1) = WTr(q_2)$, and this is denoted $q_1 =_{wtr} q_2$. This definition can be adapted to rooted LTSs: the set $WTr(TS)$ of weak traces of the rooted LTS $TS = (Q, A \cup \{\tau\}, \rightarrow, q_0)$ is $WTr(q_0)$. Two rooted LTSs, $TS_1$ and $TS_2$, are *weak trace equivalent* if $WTr(TS_1) = WTr(TS_2)$. $\square$

- Esercizio: definire il weak trace preorder.
- Dimostrare che weak trace equiv è meno fine di (strong) trace equiv. (dove le label sono su (A U {tau})* !)
- Esercizio: osservare che le vending machine del lucido 2 sono weak trace equivalent, ma non strong trace equivalent (mentre quelle del lucido 3 sono anche strong trace equivalent.)

# Weak trace equiv. vs deadlock

- Si può notare come anche queste due siano weak trace equivalent, quindi, non sorprendentemente, weak trace equiv. è insensibile al deadlock.



# Completed weak traces

**Definition 2.18. (Weak completed traces)** Given an LTS $TS = (Q, A \cup \{\tau\}, \to)$, where $\tau \notin A$, and a state $q \in Q$, the set of the *weak completed traces* of $q$ is

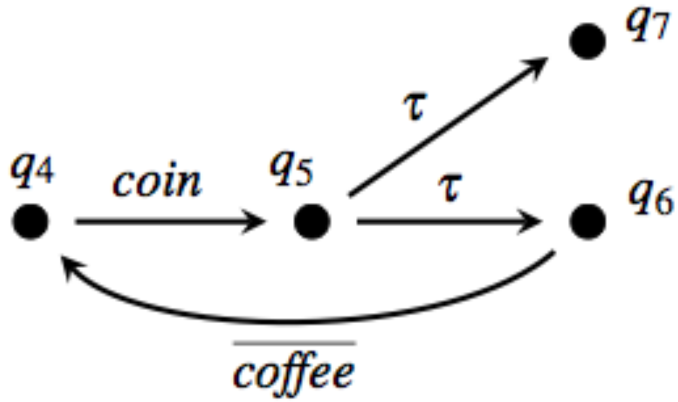
$$WCTr(q) = \{\sigma \in A^* \mid \exists q' \in Q.\, q \stackrel{\sigma}{\Longrightarrow} q' \wedge q' \stackrel{\alpha}{\not\Longrightarrow} \text{ for all observable } \alpha \in A\}.$$

Note that state $q'$ above need not to be a deadlock state, as it may still perform silent, $\tau$-labeled transitions. Two states $q_1, q_2 \in Q$ are *weakly completed trace equivalent* if $WTr(q_1) = WTr(q_2)$ and $WCTr(q_1) = WCTr(q_2)$, denoted as $q_1 =_{wctr} q_2$.

This definition can be lifted to rooted lts's as follows. The set $WCTr(TS)$ of weak completed traces of the rooted lts $TS = (Q, A, \to, q_0)$ is $WCTr(q_0)$. Two rooted lts's $TS_1$ and $TS_2$ are weakly completed trace equivalent if $WTr(TS_1) = WTr(TS_2)$ and $WCTr(TS_1) = WCTr(TS_2)$. □

Ovviamente, completed weak trace equivalence non rispetta il branching time: ad esempio, le solite due vending machine sono completed weak trace equivalent, visto che sono completed (strong) trace equivalent.

# Possiamo eliminare i tau?




**Exercise 2.48. (Tau-free lts)** An lts is $\tau$-free if it is labeled only on observable actions in $A$. Show that there is a $\tau$-free lts weakly (completed) trace equivalent to the lts in Figure 2.21(b). □

The exercise above can be generalized to show that, given an lts with $\tau$ transitions, we can always build an associated $\tau$-free lts which is weak (completed) trace equivalent.

- Analogia con le mosse epsilon in teoria degli automi: NFA equivalenti a DFA (ma qui non voglio rendere l'lts deterministico, ma solo far sparire le transizioni tau!)
- Vedi tau-abstracted LTS sul libro (Exercise 2.49)

# Lts finite-state e linguaggi regolari (1)

- Sia L un linguaggio regolare. Allora esiste un DFA $M = (Q, A, \delta, F, q_0)$ tale che $L = L[M]$.
- Dato M, possiamo costruire un rooted lts $TS = (Q \cup F', A, \rightarrow, q_0)$ dove $F' = \{q' \mid q \text{ in } F\}$ è un insieme di copie degli stati finali e la relazione di transizione $\rightarrow$ estende $\delta$ aggiungendo a $\delta$ transizioni del tipo $(q_1, a, q_2')$ per ogni transizione in $\delta$ del tipo $\delta(q_1, a) = q_2$ con $q_2$ in F.
- Allora le tracce complete di TS, $CTr(TS) = L[M] = L$.
- Nel caso $\varepsilon$ in L, allora bisogna aggiungere una transizione $(q_0, tau, q_0')$ e vale che $WCTr(TS) = L = L[M]$.

# Lts finite-state e linguaggi regolari (2)

- Sia TS = (Q, A U {tau}, $\rightarrow$, $q_0$) un rooted lts. Allora possiamo costruire un NFA M = (Q, A, $\delta$, F, $q_0$) tale che WCTr(TS) = L[M].
- F = {q in Q | q è un deadlock}
- $q_2 \in \delta(q_1, a)$ sse $q_1 -a\rightarrow q_2$
- $q_2 \in \delta(q_1, \varepsilon)$ sse $q_1 -tau\rightarrow q_2$

In conclusione, tutti (per il lucido precedente) e soli (per questo lucido) i linguaggi regolari possono essere espressi da finite-state lts per mezzo delle weak completed traces.

# Weak simulation: Definizione

**Definition 2.19.** For any lts $TS = (Q, A \cup \{\tau\}, \rightarrow)$, where $\tau \notin A$, a *weak simulation* is a relation $R \subseteq Q \times Q$ such that if $(q_1, q_2) \in R$ then for all $\alpha \in A$

- $\forall q_1'$ such that $q_1 \xrightarrow{\alpha} q_1'$, $\exists q_2'$ such that $q_2 \stackrel{\alpha}{\Longrightarrow} q_2'$ and $(q_1', q_2') \in R$
- $\forall q_1'$ such that $q_1 \xrightarrow{\tau} q_1'$, $\exists q_2'$ such that $q_2 \stackrel{\varepsilon}{\Longrightarrow} q_2'$ and $(q_1', q_2') \in R$

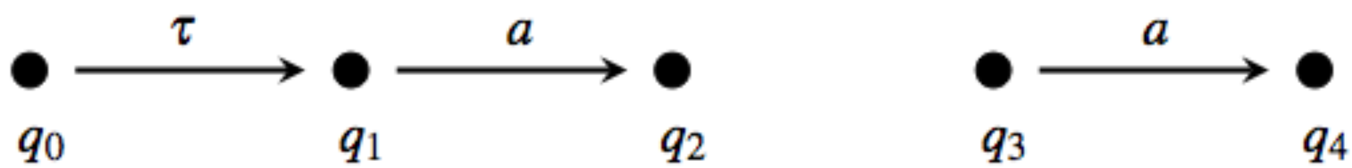
State $q$ is *weakly simulated* by $q'$, denoted $q \lessapprox q'$, if there exists a weak simulation $R$ such that $(q, q') \in R$. Two states $q$ and $q'$ are *weakly simulation equivalent*, denoted $q \approxeq q'$, if $q \lessapprox q'$ and $q' \lessapprox q$. $\square$

In other words, the weak simulation preorder $\lessapprox$ is the union of all the weak simulations:

$$\lessapprox = \bigcup \{R \subseteq Q \times Q \mid R \text{ is a weak simulation}\}.$$

Esistono ancora rooted lts massimi e minimi per il preordine fatti di un solo stato? SI

# Weak Simulation: Esempio




- Nota che ad una transizione q0 –tau→q1, q3 può anche rispondere stando fermo: q3 =ε⇒ q3.
- Se così non fosse, allora non potremmo dire che il primo è simulato debolmente dal secondo, come ci aspetteremmo che sia. S1 = {(q0,q3),(q1,q3),(q2,q4)}.
- A rovescio, una possibile weak simulation è S2 = {(q3,q0),(q4,q2)}. Nota che alla transizione q3 –a→q4, q0 risponde con q0 =a⇒q2.

# Strong vs weak simulation

- La definizione di (strong) simulation tratta l'etichetta tau come se fosse osservabile.
- Ovviamente se q è simulato strong da q', allora q è pure simulato weak da q': una strong simulation è anche una weak simulation (ma il rovescio non è vero).

**Exercise 2.51. (Strong simulation vs weak simulation)** Following Definition 2.12 for any LTS $TS = (Q, A \cup \{\tau\}, \rightarrow)$, a (strong) simulation is a relation $R \subseteq (Q \times Q)$ such that if $(q_1, q_2) \in R$ then for all $\mu \in A \cup \{\tau\}$

- $\forall q_1'$ such that $q_1 \xrightarrow{\mu} q_1'$, $\exists q_2'$ such that $q_2 \xrightarrow{\mu} q_2'$ and $(q_1', q_2') \in R$,

and we denote by $q \lesssim q'$ that there exists a strong simulation $R$ containing the pair $(q, q')$. Prove that $q \lesssim q'$ implies $q \lessapprox q'$ by showing that a strong simulation is also a weak simulation. Show that the inverse implication does not hold by providing a suitable counterexample. (*Hint:* Consider the LTSs in Figure 2.23.) $\square$

# Esempio

- Costruire LTS per

  F = tau.G   G = b.H+a.I    H = tau.G    I = b.I

- Verificare che Id = = {(F,F), (G,G), (H,H), (I,I)} è una strong simulation, e quindi anche una weak simul.
- Verificare che R = {(F,G), (G,G), (H,H), (I,I)} è una weak sim, ma non è strong.
- Verificare che R' = {(F,F),(G,G), (H,H),(I,I), (F,G), (G,F), (G,H), (H,G), (F,H), (H,F), (I,G), (I, F), (I, H)} è la più grande weak simulation.
- Verificare che A = b.A + a.B   B = b.B   è tale che F è weakly simile ad A, ma A ed F non sono strong simili.

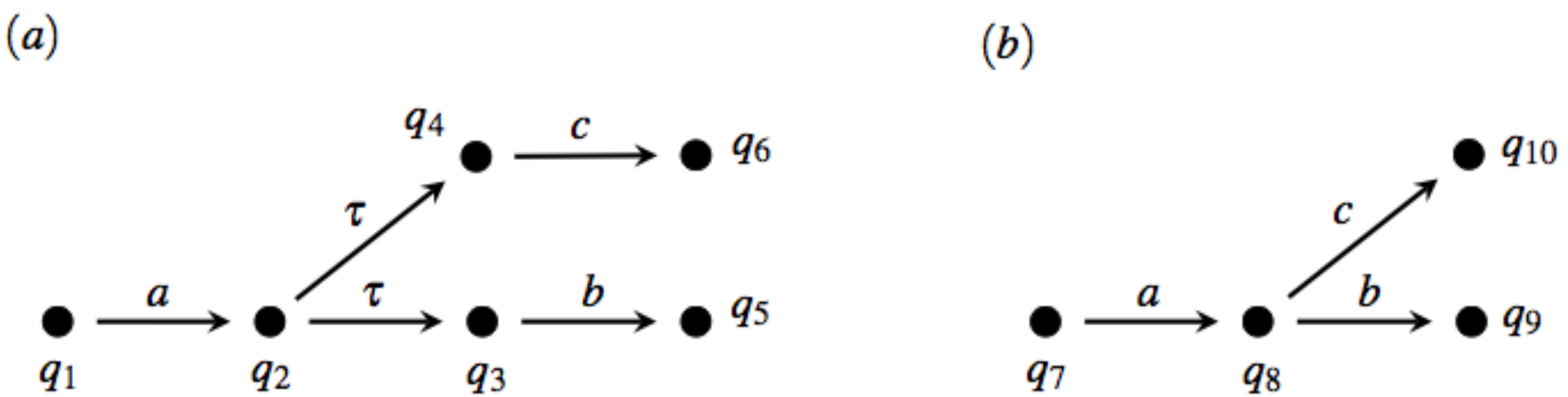
# Proprietà

**Exercise 2.53.** For any LTS $TS = (Q, A \cup \{\tau\}, \rightarrow)$, where $\tau \notin A$, given a weak simulation $R \subseteq Q \times Q$, prove that if $(q_1, q_2) \in R$ and $q_1 \stackrel{\delta}{\Longrightarrow} q_1'$, then there exists $q_2'$ such that $q_2 \stackrel{\delta}{\Longrightarrow} q_2'$ with $(q_1', q_2') \in R$, for $\delta \in A \cup \{\varepsilon\}$. (*Hint:* By induction on the proof of $q_1 \stackrel{\delta}{\Longrightarrow} q_1'$, according to Definition 2.16.) $\square$

**Exercise 2.54.** Prove that, for any LTS $(Q, A \cup \{\tau\}, \rightarrow)$, relation $\lessapprox \subseteq Q \times Q$ is a preorder, while relation $\approxeq \subseteq Q \times Q$ is an equivalence relation. (*Hint:* Follow the same steps of Proposition 2.2 and Proposition 2.4. In proving that the relational composition of weak simulations is a weak simulation, you need also the result of Exercise 2.53.) $\square$

# Esercizio (1)

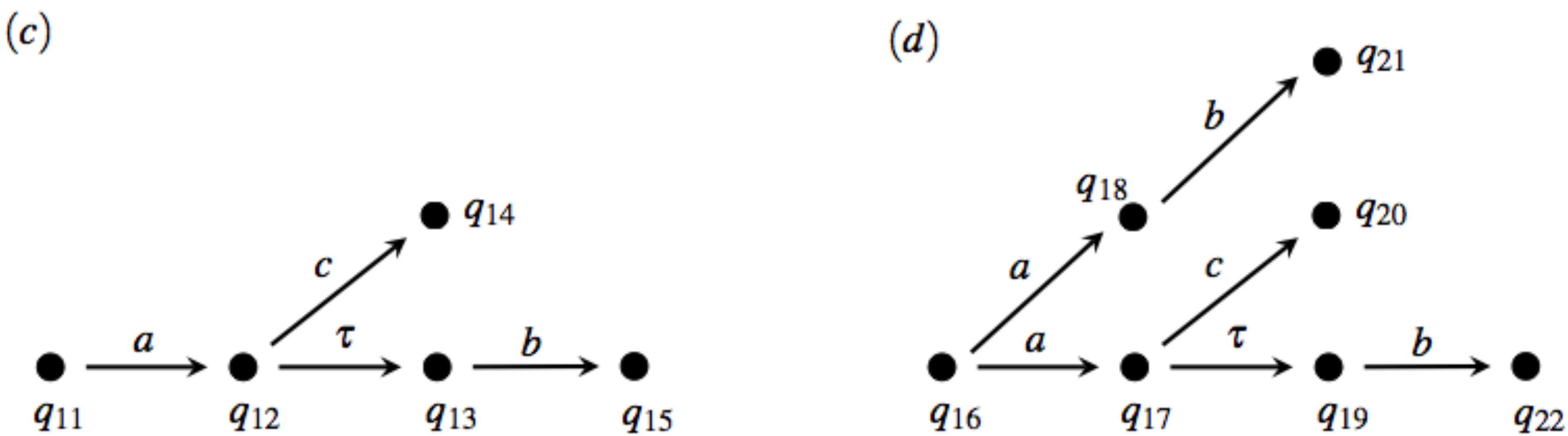
- Prova che R = {(q1,q7),(q2,q8),(q3,q8),(q4,q8),(q5,q9),(q6,q10)} è una weak simulation
- Prova anche che esiste una weak simulation che contiene la coppia (q7,q1).




- Ovviamente, così come (strong) simulation equivalence non rispetta correttamente il branching time, lo stesso dicasi per weak simulation equivalence.

# Esercizio (2)

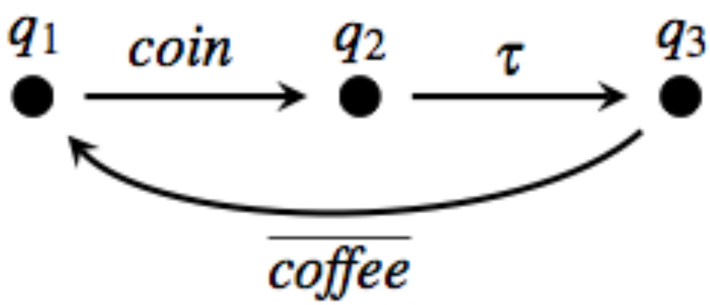
- Mostra che q11 e q16 sono weak simulation equivalent

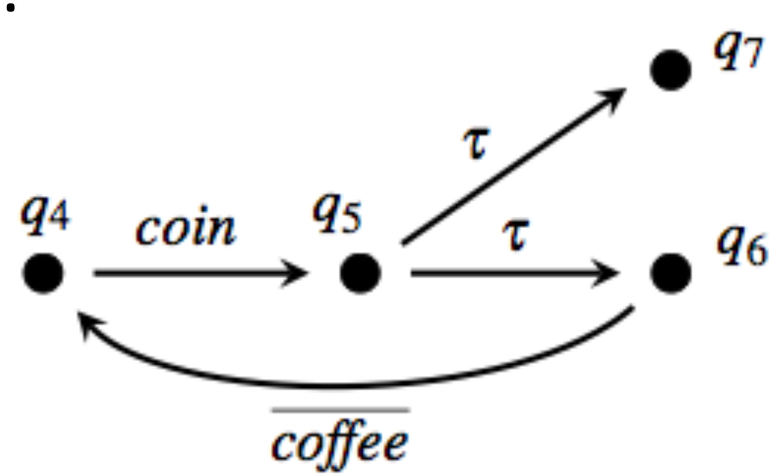



- Questi sono simulation equivalent ai due del lucido precedente!
- È vero che, dato un lts con mosse tau, è sempre possibile trovare un lts tau-free che sia simulation equivalent con esso? SI

# Esercizio (3): weak sim. vs deadlock

- Dimostra che i due sono weak simulation equivalent, verificando che S1 = {(q1,q4),(q2,q5),(q3,q6)} e S2 = {(q4,q1),(q5,q2),(q6,q3),(q7,q3)} sono entrambe relazioni di weak simulation.



# Weak completed simulation

**Exercise 2.60. (Weak completed simulation)** A *weak completed simulation* $R$ is a weak simulation such that for all $(q_1, q_2) \in R$ if $q_1 \nrightarrow$ then $q_2 \stackrel{\alpha}{\nRightarrow}$ for all observable $\alpha$ (but $q_2$ may still perform silent transitions). State $q_1$ is weakly completed simulated by $q_2$, denoted $q_1 \lessapprox_c q_2$, if there exists a weak completed simulation $R$ such that $(q_1, q_2) \in R$. States $q_1$ and $q_2$ are *weakly completed simulation equivalent*, $q_1 \approxeq_c q_2$, if $q_1 \lessapprox_c q_2$ and $q_2 \lessapprox_c q_1$.

Show that the two weakly simulation equivalent vending machines of Figure 2.21 are actually not weakly completed simulation equivalent. □

- La figura 2.21 è quella del lucido precedente.
- Ovviamente, come nel caso strong, anche la weak completed simulation non è in grado di percepire correttamente "the timing of choice", per cui bisogna passare alla weak bisimulation.

# Weak bisimulation

**Definition 2.20. (Weak bisimulation)** For any lts $TS = (Q, A \cup \{\tau\}, \rightarrow)$, where $\tau \notin A$, a *weak bisimulation* is a relation $R \subseteq (Q \times Q)$ such that both $R$ and its inverse $R^{-1}$ are weak simulations. More expicitly, a weak bisimulation is a relation $R$ such that if $(q_1, q_2) \in R$ then for all $\alpha \in A$

- $\forall q_1'$ such that $q_1 \xrightarrow{\alpha} q_1'$, $\exists q_2'$ such that $q_2 \stackrel{\alpha}{\Longrightarrow} q_2'$ and $(q_1', q_2') \in R$
- $\forall q_1'$ such that $q_1 \xrightarrow{\tau} q_1'$, $\exists q_2'$ such that $q_2 \stackrel{\varepsilon}{\Longrightarrow} q_2'$ and $(q_1', q_2') \in R$

and symmetrically

- $\forall q_2'$ such that $q_2 \xrightarrow{\alpha} q_2'$, $\exists q_1'$ such that $q_1 \stackrel{\alpha}{\Longrightarrow} q_1'$ and $(q_1', q_2') \in R$
- $\forall q_2'$ such that $q_2 \xrightarrow{\tau} q_2'$, $\exists q_1'$ such that $q_1 \stackrel{\varepsilon}{\Longrightarrow} q_1'$ and $(q_1', q_2') \in R$

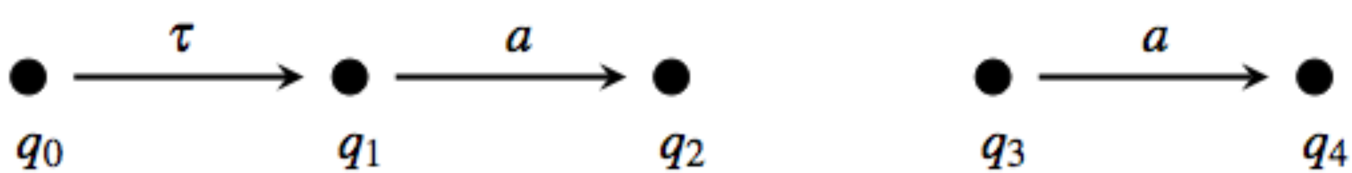
States $q$ and $q'$ are *weakly bisimilar* (or *weak bisimulation equivalent*), denoted with $q \approx q'$, if there exists a weak bisimulation $R$ such that $(q, q') \in R$. □

In other words, weak bisimulation equivalence is the union of all weak bisimulations:

$$\approx = \bigcup \{R \subseteq Q \times Q \mid R \text{ is a weak bisimulation}\}.$$

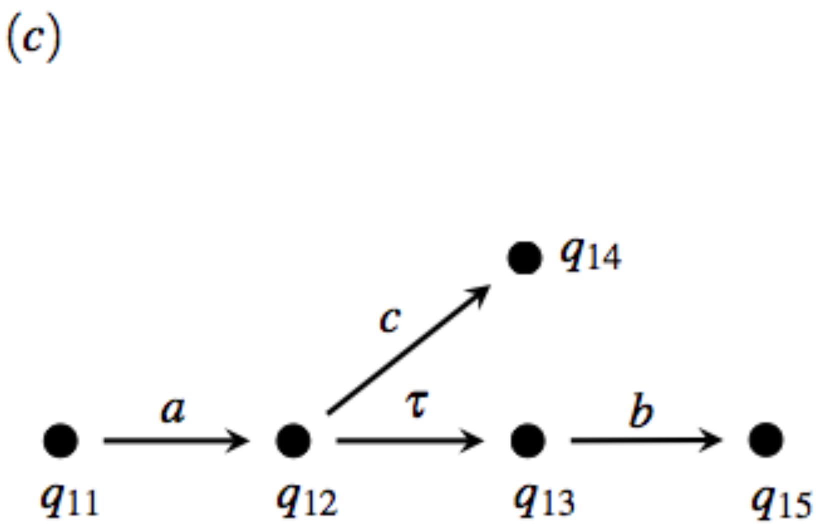
# Esempio (1)

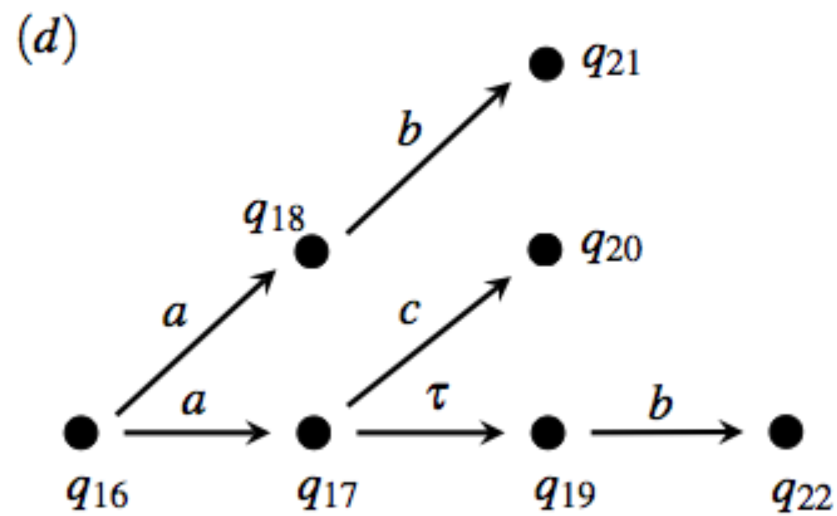
- S = {(q0,q3),(q1,q3),(q2,q4)} è una weak bisimulation.
- Infatti , S è una weak simulation, ed anche $S^{-1}$ = {(q3,q0),(q3,q1),(q4,q2)} lo è. Perciò q0 ≈ q3.



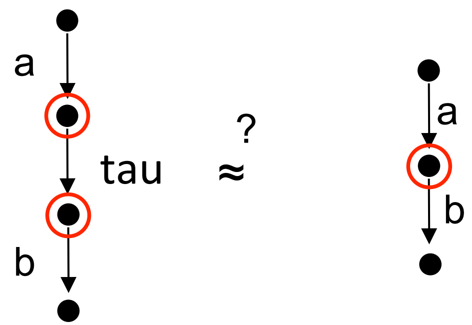
# Esempio (2)

- Verifica che R = {(q11,q16),(q12,q17),(q13,q19),(q13,q18),(q14,q20),(q15,q21),(q15,q22)} è una weak bisimulation.
- Osserva che quando $q_{16}$ –a→ $q_{18}$, allora $q_{11}$ =a⇒ $q_{13}$ e gli stati raggiunti possono solo fare b.

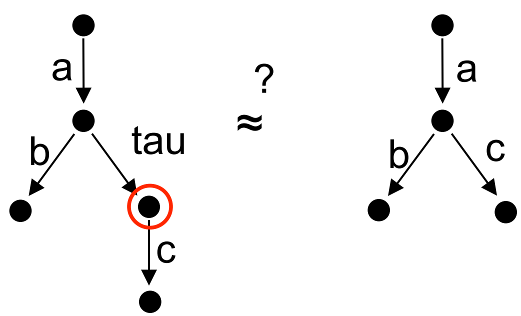


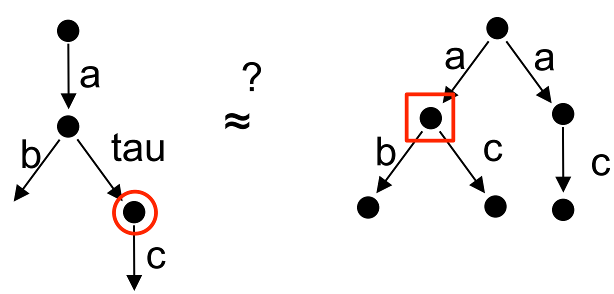
# ESEMPI



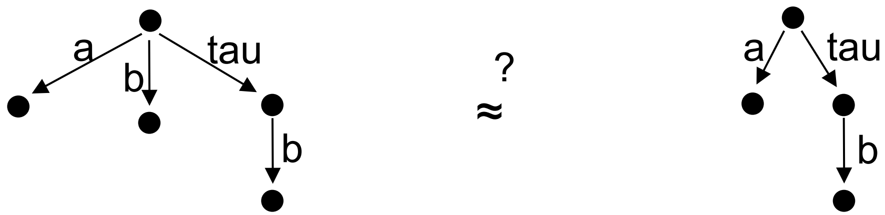

Sì, perché gli stati ○ sono equivalenti



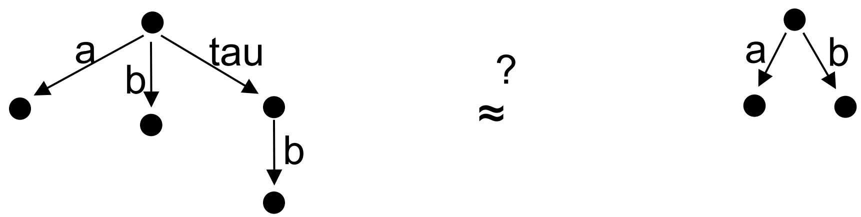

No, perché ○ non ha equivalenti nel 2°.



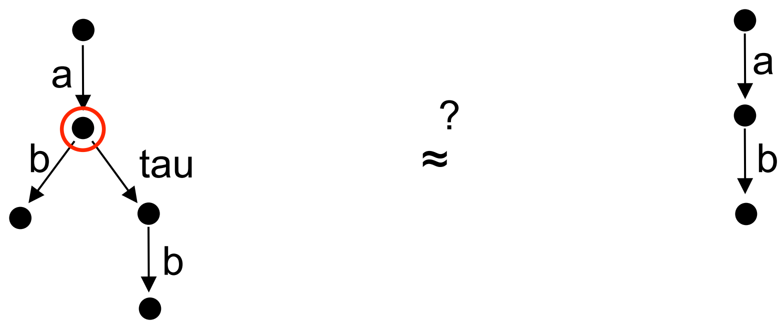

No, perché □ non ha equivalenti nel 2°.



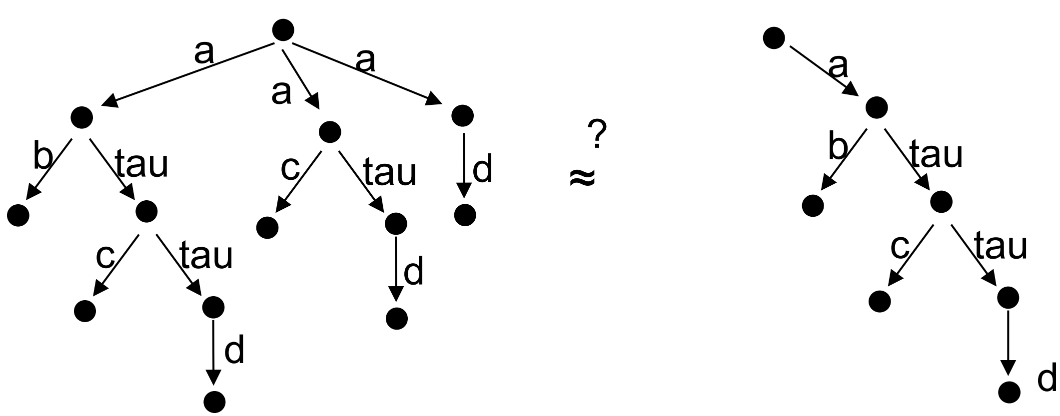

Sì.




No.




Sì, perché
non ha altre alternative
(cf. caso precedente)




Sì.

# Strong vs weak bisimulation

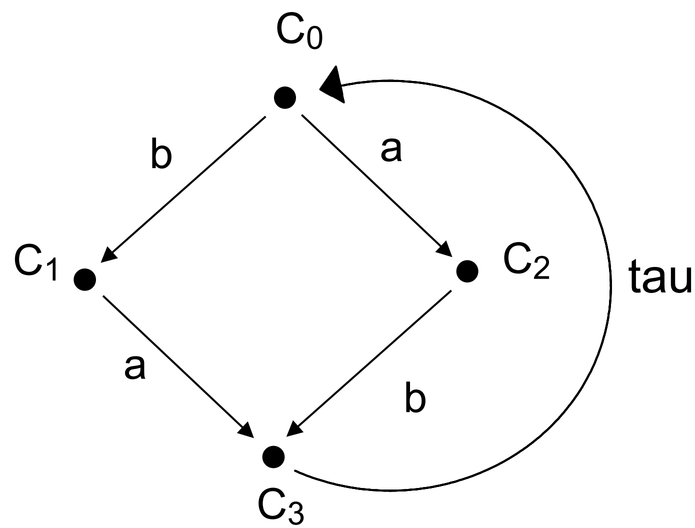
- La definizione di (strong) bisimulation tratta l'etichetta tau come se fosse osservabile.
- Ovviamente se q è bisimile strong a q', allora q è pure bisimile weak a q': una strong bisimulation è anche una weak bisimulation (ma il rovescio non è vero).

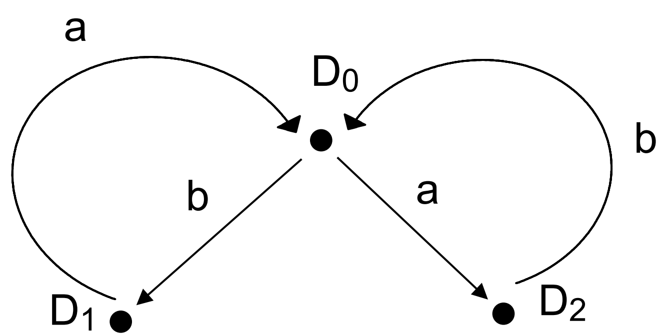
**Exercise 2.52. (Strong vs weak bisimulation)** Following Definition 2.13, for any lts $TS = (Q, A \cup \{\tau\}, \rightarrow)$, a (strong) bisimulation is a relation $R \subseteq Q \times Q$ such that if $(q_1, q_2) \in R$ then for all $\mu \in A \cup \{\tau\}$

- $\forall q_1'$ such that $q_1 \xrightarrow{\mu} q_1'$, $\exists q_2'$ such that $q_2 \xrightarrow{\mu} q_2'$ and $(q_1', q_2') \in R$
- $\forall q_2'$ such that $q_2 \xrightarrow{\mu} q_2'$, $\exists q_1'$ such that $q_1 \xrightarrow{\mu} q_1'$ and $(q_1', q_2') \in R$.

Two states $q$ and $q'$ are bisimilar, denoted $q \sim q'$, if there exists a strong bisimulation $R$ such that $(q, q') \in R$.

Prove that $q \sim q'$ implies $q \approx q'$ by showing that a strong bisimulation is also a weak bisimulation. Show that the reverse implication does not hold by providing a suitable counterexample. □




$S$ = { ($C_0$, $D_0$), ($C_1$, $D_1$), ($C_2$, $D_2$), ($C_3$, $D_0$) }
è una bisimulazione debole.
Non esiste una bisimulazione forte tra i due.