Lezione 33 MSC
Team Bisimulation for BPP nets

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How to generalize the idea of bisimulation on BPP nets?

- BPP nets: each transition has **singleton preset**
- Each token is behaviorally **independent** of any other token
- So, try to define a bisimulation-like relation relating not markings, but single places!
- Once a suitable notion of **bisimulation** is defined **on places**, “team” bisimulation is **extended to markings** by additive closure.

Additive closure (1)

Definition 19. (Additive closure) Given a BPP net $N = (S, A, T)$ and a *place relation* $R \subseteq S \times S$, we define a *marking relation* $R^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$, called the *additive closure* of R, as the least relation induced by the following axiom and rule.

$$\frac{}{(\theta, \theta) \in R^{\oplus}} \qquad \frac{(s_1, s_2) \in R \quad (m_1, m_2) \in R^{\oplus}}{(s_1 \oplus m_1, s_2 \oplus m_2) \in R^{\oplus}}$$

□

Proposition 9. For any BPP net $N = (S, A, T)$ and any place relation $R \subseteq S \times S$, if $(m_1, m_2) \in R^{\oplus}$, then $|m_1| = |m_2|$. □

An alternative way to define that two markings m_1 and m_2 are related by $R^{\oplus}$ is to state that m_1 can be represented as $s_1 \oplus s_2 \oplus \ldots \oplus s_k$, m_2 can be represented as $s'_1 \oplus s'_2 \oplus \ldots \oplus s'_k$ and $(s_i, s'_i) \in R$ for $i = 1, \ldots, k$.

Additive closure (2)

- If R is an equivalence relation, then $R^{\oplus}$ is an equivalence relation.
- $R^{\oplus}$ is additive and also subtractive when R is an equivalence relation

Proposition 11. Given a BPP net $N = (S, A, T)$ and a place relation R, the following hold:

- If $(m_1, m_2) \in R^{\oplus}$ and $(m_1', m_2') \in R^{\oplus}$, then $(m_1 \oplus m_1', m_2 \oplus m_2') \in R^{\oplus}$.
- If R is an equivalence relation, $(m_1 \oplus m_1', m_2 \oplus m_2') \in R^{\oplus}$ and $(m_1, m_2) \in R^{\oplus}$, then $(m_1', m_2') \in R^{\oplus}$. $\square$

Additive closure (3)

- Properties useful for proving properties of team bisimulations

Proposition 12. For any BPP net $N = (S, A, T)$ and any family of place relations $R_i \subseteq S \times S$, the following hold:

1. $\emptyset^{\oplus} = \{(\theta, \theta)\}$, i.e., the additive closure of the empty place relation is a singleton marking relation, relating the empty marking to itself.
2. $(\mathscr{I}_S)^{\oplus} = \mathscr{I}_M$, i.e., the additive closure of the identity relation on places $\mathscr{I}_S = \{(s,s) \mid s \in S\}$ is the identity relation on markings $\mathscr{I}_M = \{(m,m) \mid m \in \mathscr{M}(S)\}$.
3. $(R^{\oplus})^{-1} = (R^{-1})^{\oplus}$, i.e., the inverse of an additively closed relation R is the additive closure of its inverse R^{-1}.
4. $(R_1 \circ R_2)^{\oplus} = (R_1^{\oplus}) \circ (R_2^{\oplus})$, i.e., the additive closure of the composition of two place relations is the compositions of their additive closures.
5. $\bigcup_{i \in I}(R_i^{\oplus}) \subseteq (\bigcup_{i \in I} R_i)^{\oplus}$, i.e., the union of additively closed relations is included into the additive closure of their union. $\square$

Team Bisimulation

Definition 20. (Team bisimulation) Let $N = (S, A, T)$ be a BPP net. A *team bisimulation* is a place relation $R \subseteq S \times S$ such that if $(s_1, s_2) \in R$ then for all $\ell \in A$

- $\forall m_1$ such that $s_1 \xrightarrow{\ell} m_1$, $\exists m_2$ such that $s_2 \xrightarrow{\ell} m_2$ and $(m_1, m_2) \in R^{\oplus}$,
- $\forall m_2$ such that $s_2 \xrightarrow{\ell} m_2$, $\exists m_1$ such that $s_1 \xrightarrow{\ell} m_1$ and $(m_1, m_2) \in R^{\oplus}$.

Two places s and s' are *team bisimilar* (or *team bisimulation equivalent*), denoted $s \sim s'$, if there exists a team bisimulation R such that $(s, s') \in R$. $\square$

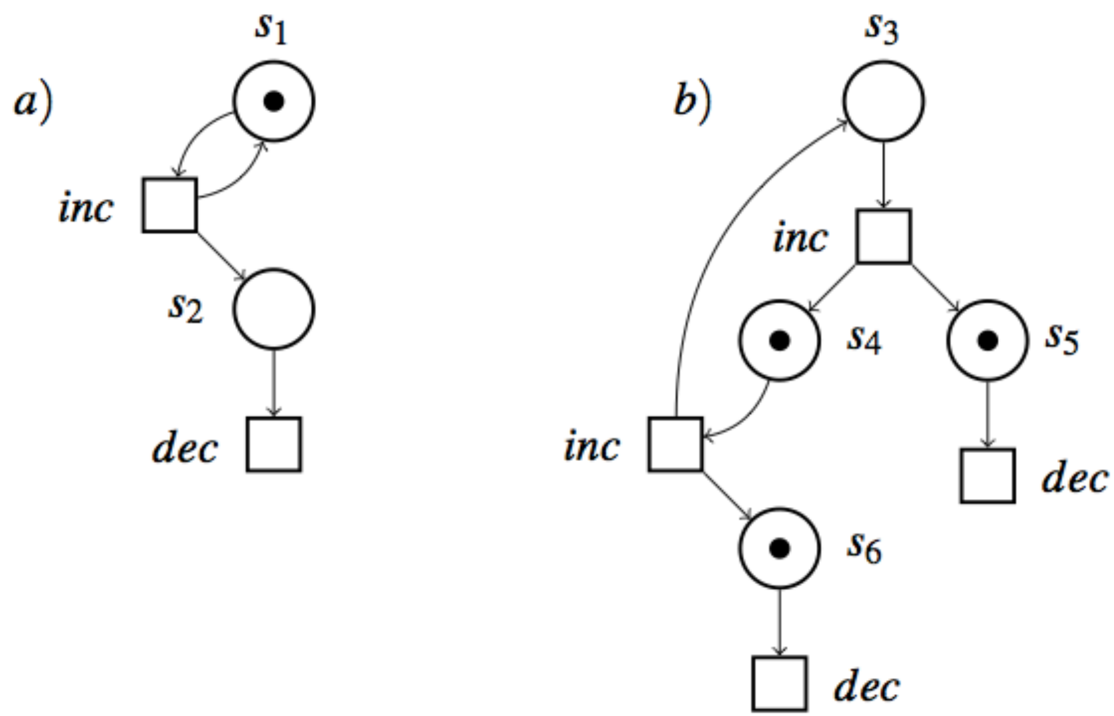
- Very similar to standard bisimulation on LTSs
- The reached markings are to be related by the additive closure of R
- Defined over an **unmarked** BPP net

Example (1) – Semi-counters

$R = \{(s1,s3), (s1,s4),(s2,s5), (s2,s6)\}$

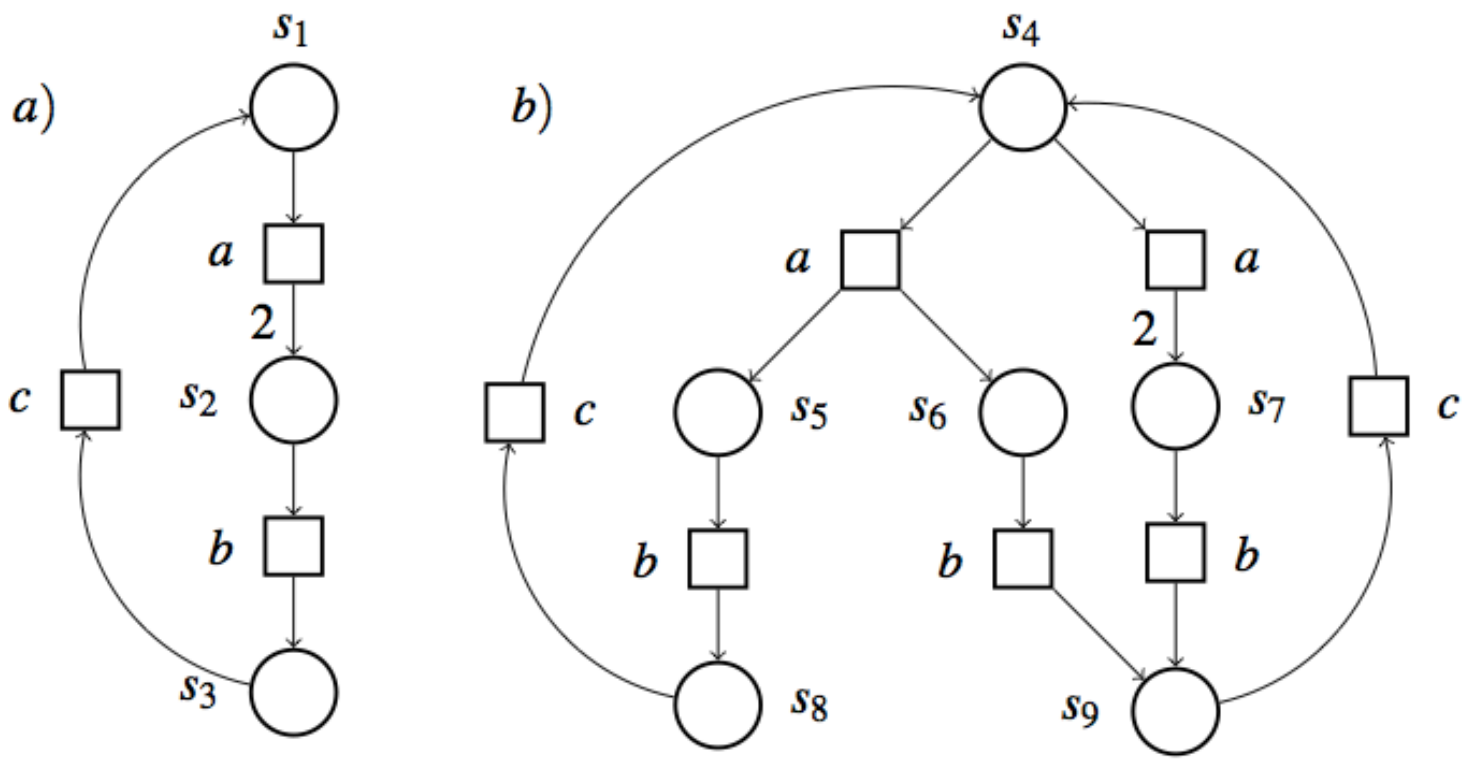
is a team bisimulation

Check this!



$R' = \{(s1 \oplus k \cdot s2, s3 \oplus k1 \cdot s5 \oplus k2 \cdot s6) \mid k = k1 + k2$ and $k, k1, k2 \geq 0\} \cup \{(s1 \oplus k \cdot s2, s4 \oplus k1 \cdot s5 \oplus k2 \cdot s6) \mid k = k1 + k2$ and $k, k1, k2 \geq 0\}$ is an interleaving bisimulation.

Example (2)



- R = {(s1,s4), (s2,s5), (s2,s6),(s2,s7),(s3,s8),(s3,s9)} is a team bisimulation.

Proposition 13. For any BPP net $N = (S, A, T)$, the following hold:

1. The identity relation $\mathscr{I}_S = \{(s,s) \mid s \in S\}$ is a team bisimulation;
2. the inverse relation $R^{-1} = \{(s', s) \mid (s, s') \in R\}$ of a team bisimulation R is a team bisimulation;
3. the relational composition $R_1 \circ R_2 = \{(s, s'') \mid \exists s'.(s, s') \in R_1 \wedge (s', s'') \in R_2\}$ of two team bisimulations R_1 and R_2 is a team bisimulation;
4. the union $\bigcup_{i \in I} R_i$ of team bisimulations R_i is a team bisimulation. □

Remember that $s \sim s'$ if there exists a team bisimulation containing the pair (s, s'). This means that $\sim$ is the union of all team bisimulations, i.e.,

$$\sim = \bigcup \{R \subseteq S \times S \mid R \text{ is a team bisimulation}\}.$$

By Proposition 13(4), $\sim$ is also a team bisimulation, hence the largest such relation.

Proposition 14. For any BPP net $N = (S, A, T)$, relation $\sim \subseteq S \times S$ is the largest team bisimulation relation. □

Proposition 15. For any BPP net $N = (S, A, T)$, relation $\sim \subseteq S \times S$ is an equivalence relation.

Complexity

Remark 6. (Complexity 1) It is well-known that the optimal algorithm for computing bisimulation equivalence over a finite-state LTS with n states and m transitions has $O(m \cdot \log n)$ time complexity [35]; this very same partition refinement algorithm can be easily adapted also for team bisimilarity over BPP nets: it is enough to consider the empty marking θ as an additional, special place which is team bisimilar to itself only (i.e., the initial partition is composed of two blocks: S and $\{\theta\}$), and to consider the little additional cost due to the fact that the reached markings are to be related by the additive closure of the current partition over places; this extra cost is related to the size of the post-set of the net transitions; if p is the size of the largest post-set of the net transitions (i.e., p is the least number such that $|t^\bullet| \leq p$, for all $t \in T$), then the time complexity is $O(m \cdot p^2 \cdot \log (n+1))$, where m is the number of the net transitions and n is the number of the net places. □

- The complexity of checking whether m1 and m2 are related by $R^\oplus$ is $O(k^2)$ if k is the size of the two markings

Team bisimilarity on markings

Starting from team bisimulation equivalence $\sim$, which has been computed over the places of an *unmarked* BPP net N, we can lift it over *the markings* of N in a distributed way: m_1 is team bisimulation equivalent to m_2 if these two markings are related by the additive closure of $\sim$, i.e., if $(m_1, m_2) \in \sim^{\oplus}$, usually denoted by $m_1 \sim^{\oplus} m_2$.

Proposition 16. For any BPP net $N = (S, A, T)$, if $m_1 \sim^{\oplus} m_2$, then $|m_1| = |m_2|$.

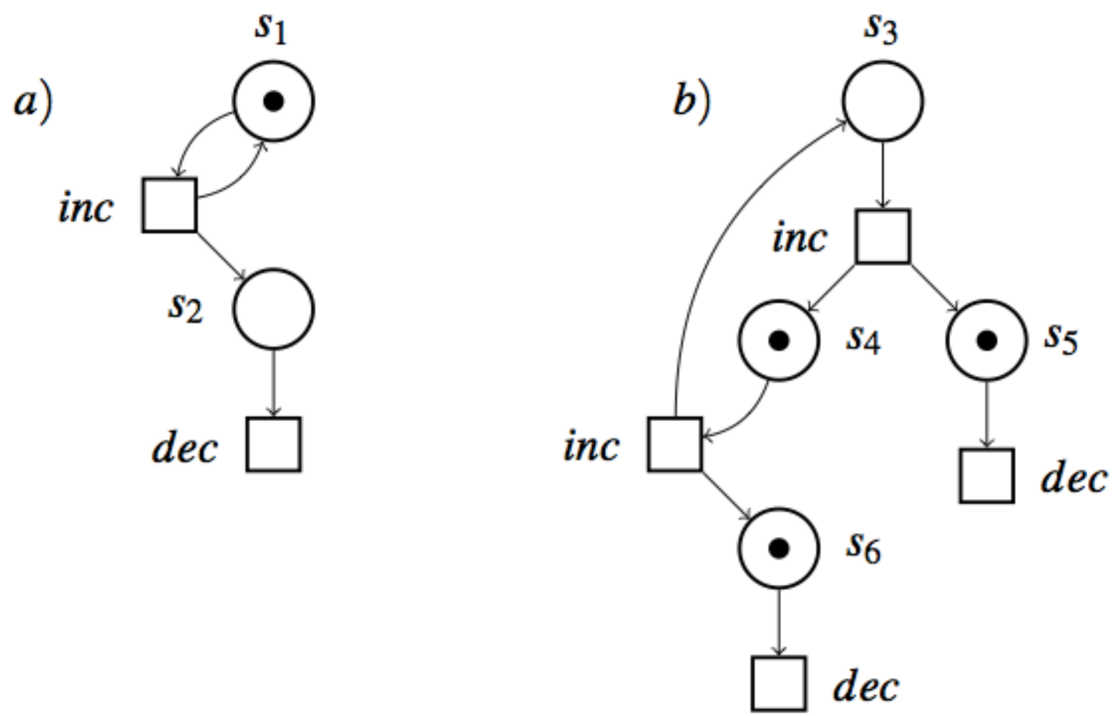
PROOF. By Proposition 9. $\square$

Proposition 17. For any BPP net $N = (S, A, T)$, relation $\sim^{\oplus} \subseteq \mathscr{M}(S) \times \mathscr{M}(S)$ is an equivalence relation.

Example – Semi-counters

$R = \{(s1,s3), (s1,s4), (s2,s5), (s2,s6)\}$

is a team bisimulation



Example 12. Continuing Example 8 about the semi-counters, the marking $s_1 \oplus 2 \cdot s_2$ is team bisimilar to the following markings of the net in (b): $s_3 \oplus 2 \cdot s_5$, or $s_3 \oplus s_5 \oplus s_6$, or $s_3 \oplus 2 \cdot s_6$, or $s_4 \oplus 2 \cdot s_5$, or $s_4 \oplus s_5 \oplus s_6$, or $s_4 \oplus 2 \cdot s_6$. $\square$

Relation with other equivalences

Theorem 2. Let $N = (S, A, T)$ be a BPP net. Two markings m_1 and m_2 are team bisimulation equivalent, $m_1 \sim^{\oplus} m_2$, if and only if $|m_1| = |m_2|$ and

- $\forall t_1$ such that $m_1[t_1\rangle m_1'$, $\exists t_2$ such that ${}^\bullet t_1 \sim {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \sim^{\oplus} t_2^\bullet$, $m_2[t_2\rangle m_2'$ and $m_1' \sim^{\oplus} m_2'$, and symmetrically,
- $\forall t_2$ such that $m_2[t_2\rangle m_2'$, $\exists t_1$ such that ${}^\bullet t_1 \sim {}^\bullet t_2$, $l(t_1) = l(t_2)$, $t_1^\bullet \sim^{\oplus} t_2^\bullet$, $m_1[t_1\rangle m_1'$ and $m_1' \sim^{\oplus} m_2'$. $\square$

Corollary 1. (Team bisimilarity is finer than interleaving bisimilarity) Let $N = (S, A, T)$ be a BPP net. If $m_1 \sim^{\oplus} m_2$, then $m_1 \sim_{int} m_2$. $\square$

Corollary 2. (Team bisimilarity and cn-bisimilarity coincide) Let $N = (S, A, T)$ be a BPP net. Then, $m_1 \sim_{cn} m_2$ if and only if $m_1 \sim^{\oplus} m_2$.

- What is cn-bisimilarity? Intuition?

Concrete truly-concurrent equivalence observing also the distributed state

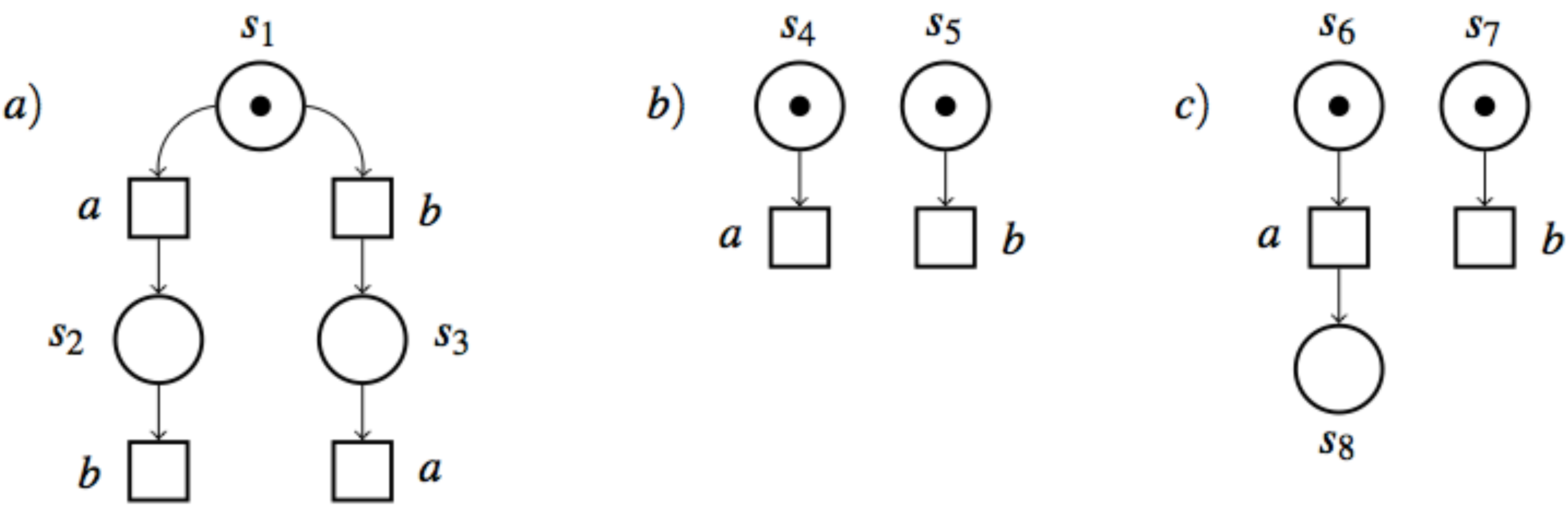


Fig. 4. Three net systems: $a.b.\mathbf{0} + b.a.\mathbf{0}$, $a.\mathbf{0} \mid b.\mathbf{0}$ and $a.C \mid b.\mathbf{0}$ (with $C \doteq \mathbf{0}$)

- s1 is not team bisimilar to s4⊕s5, and s4⊕s5 is not team bisimilar to s6⊕s7

Minimization

- Minimizing a BPP net w.r.t. team bisimilarity

Definition 11. (Reduced net) *Let $N = (S, A, T)$ be a BPP net and let $\sim$ be the team bisimulation equivalence relation over its places. The* reduced net *$N' = (S', A, T')$ is defined as follows:*

- $S' = \{[s] \mid s \in S\}$, *where* $[s] = \{s' \in S \mid s \sim s'\}$;
- $T' = \{([s], a, [m]) \mid (s, a, m) \in T\}$,

where $[m]$ is defined as follows: $[\theta] = \theta$ and $[m_1 \oplus m_2] = [m_1] \oplus [m_2]$. If the net N has initial marking $m_0 = k_1 \cdot s_1 \oplus \ldots \oplus k_n \cdot s_n$, then N' as initial marking $[m_0] = k_1 \cdot [s_1] \oplus \ldots \oplus k_n \cdot [s_n]$. □

Theorem 4. *Let $N = (S, A, T)$ be a BPP net and let $N' = (S', A, T')$ be its reduced net w.r.t. $\sim$. For any $m \in \mathscr{M}(S)$, we have that $m \sim^{\oplus} [m]$.*

Minimization (2) - Example

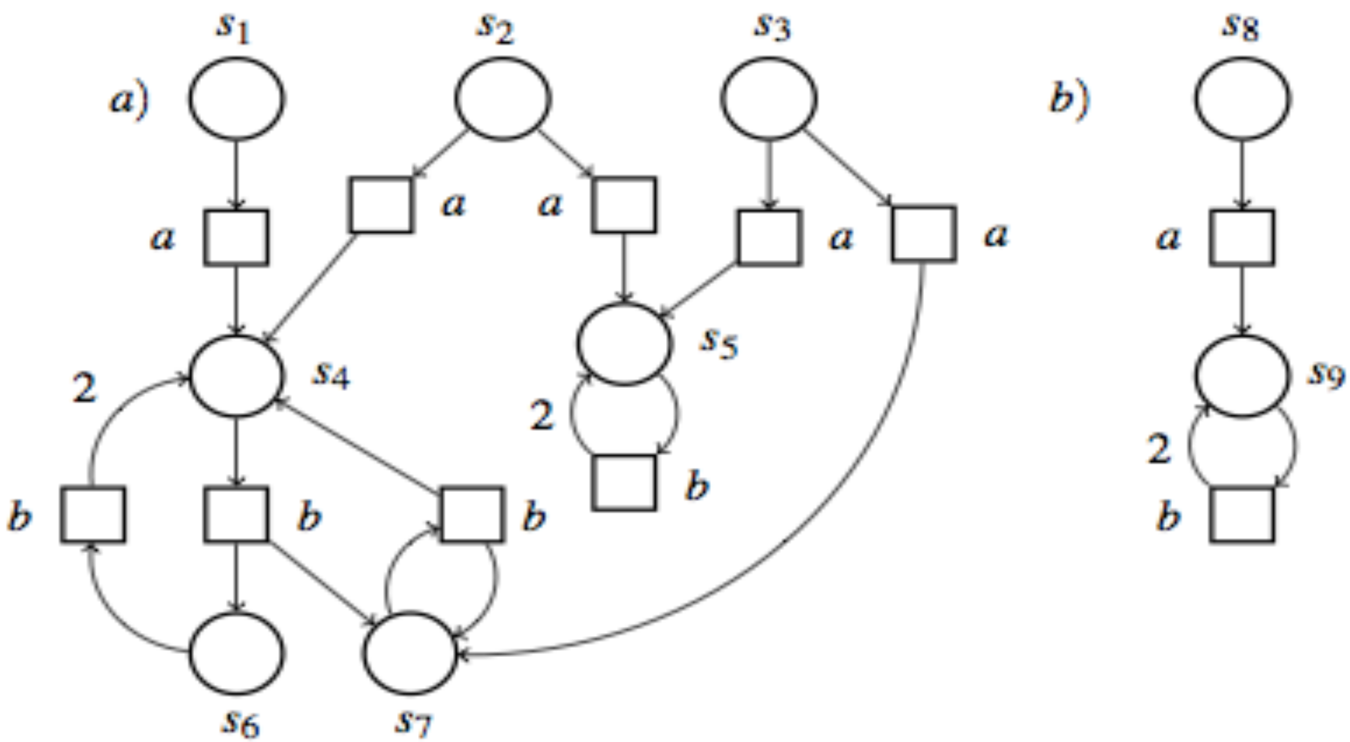


Fig. 4. A BPP net in (a) and its reduced net in (b)

- For the net in (a), the equivalence classes of ~ are {*s*1,*s*2,*s*3} and {*s*4,*s*5,*s*6,*s*7}. Hence, the reduced net is isomorphic to the net in (b).

Team Modal Logic: TML

$$F \ ::= \ nn \mid vv \mid tt \mid ff \mid F \wedge F \mid F \vee F \mid \neg F \mid \langle a \rangle F \mid [a]F \mid F \otimes F$$

$$[\![nn]\!] = S \qquad [\![vv]\!] = \{\theta\} \qquad [\![tt]\!] = \mathscr{M}(S) \qquad [\![ff]\!] = \emptyset$$

$$[\![F_1 \wedge F_2]\!] = [\![F_1]\!] \cap [\![F_2]\!] \qquad [\![F_1 \vee F_2]\!] = [\![F_1]\!] \cup [\![F_2]\!] \qquad [\![\neg F]\!] = \mathscr{M}(S) \setminus [\![F]\!]$$

$$[\![\langle a \rangle F]\!] = \{s \in S \mid \exists m. s \xrightarrow{a} m \text{ and } m \in [\![F]\!]\}$$

$$[\![[a]F]\!] = \{s \in S \mid \forall m (s \xrightarrow{a} m \text{ implies } m \in [\![F]\!])\}$$

$$[\![F_1 \otimes F_2]\!] = [\![F_1]\!] \otimes [\![F_2]\!]$$

where $M_1 \otimes M_2 = \{m_1 \oplus m_2 \mid m_1 \in M_1, m_2 \in M_2\}$

TML characterizes Team Bisimilarity on BPP nets

The semantics of a formula F is the set of markings that satisfy it; hence, the semantic function is parametrized with respect to some given BPP net $N = (S, A, T)$. Let $[\![-]\!] : \mathscr{F}_A \to \wp(\mathscr{M}(S))$ be the denotational semantics function, defined in Table 1.

Definition 12. (TML satisfaction relation) *We say that a marking m* satisfies *a formula F, written $m \models F$, if $m \in [\![F]\!]$.* □

Theorem 5. (Coherence) *Let $N = (S, A, T)$ be a BPP net. It holds that $m_1 \sim^{\oplus} m_2$ if and only if $\{F_1 \in \mathscr{F}_A \mid m_1 \models F_1\} = \{F_2 \in \mathscr{F}_A \mid m_2 \models F_2\}$.*

- Extension of Hennessy-Milner Theorem (HML characterizes bisimilarity on LTSs)

Algebraic Properties and Axiomatization

- Since the BPP process algebra represents all (and only) the BPP nets, we can extend the definition of team bisimilarity to BPP terms

Definition 14. *Two BPP processes p and q are team bisimilar, denoted $p \sim^{\oplus} q$, if, by considering the (union of the) nets $[\![p]\!]_\emptyset$ and $[\![q]\!]_\emptyset$, we have that $dec(p) \sim^{\oplus} dec(q)$.* □

- Hence, we can now study team bisimilarity on the BPP calculus! Hence, showing its algebraic properties, checking that it is a congruence for all the operators of the calculus and, finally, axiomatizing it!

BPP: syntax

$$
\begin{array}{ll}
s ::= \mathbf{0} \mid \mu.p \mid s+s & \textit{guarded processes} \\
q ::= s \mid C & \textit{sequential processes} \\
p ::= q \mid p|p & \textit{parallel processes}
\end{array}
$$

- A constant C is equipped with a defining equation C = r, **with r in syntactic category s**.
- Constants are guarded by construction.
- **Finite calculus**: finitely many constants and finitely many actions can be used.
- **Asynchronous parallel composition** (no communication is allowed).

Congruence

- On sequential process term, team bisimilarity $\sim^{\oplus}$ is simply $\sim$

Proposition 15. *For each $p, q \in \mathscr{P}_{BPP}^{grd}$, if $p \sim q$ (or $p = q = \mathbf{0}$), then*
$$p + r \sim q + r \quad \textit{for all } r \in \mathscr{P}_{BPP}^{grd}.$$

Proposition 16. *For each $p, q \in \mathscr{P}_{BPP}$, if $p \sim^{\oplus} q$, then $\mu.p \sim \mu.q$ for all $\mu \in Act$.*

Proposition 17. *For every $p, q, r \in \mathscr{P}_{BPP}$, if $p \sim^{\oplus} q$, then $p \mid r \sim^{\oplus} q \mid r$.*

- Recursion congruence

Theorem 8. *Let p and q be two open guarded BPP terms, with one variable x at mo*
Let $A \doteq p\{A/x\}$, $B \doteq q\{B/x\}$ and $p \sim q$. Then $A \sim B$.

Algebraic Properties

Proposition 18. **(Laws of the choice operator)** *For each $p, q, r \in \mathscr{P}_{BPP}^{grd}$, the following hold:*

$$\begin{array}{rlll} p+(q+r) & \sim (p+q)+r & & \textit{(associativity)} \\ p+q & \sim q+p & & \textit{(commutativity)} \\ p+\mathbf{0} & \sim p & \textit{if } p \neq \mathbf{0} & \textit{(identity)} \\ p+p & \sim p & \textit{if } p \neq \mathbf{0} & \textit{(idempotency)} \end{array}$$

- 0 is **not** team bisimilar to 0+0

Proposition 20. **(Laws of the parallel operator)** *For each $p, q, r \in \mathscr{P}_{BPP}$, the following hold:*

$$\begin{array}{rcll} p\,|\,(q\,|\,r) & \sim^{\oplus} & (p\,|\,q)\,|\,r & \textit{(associativity)} \\ p\,|\,q & \sim^{\oplus} & q\,|\,p & \textit{(commutativity)} \\ p\,|\,\mathbf{0} & \sim^{\oplus} & p & \textit{(identity)} \end{array}$$

- 0 **is** team bisimilar to 0|0

Algebraic Properties (2)

Proposition 19. **(Laws of the constant)** *For each $p \in \mathscr{P}^{grd}_{BPP}$, and each $C \in \mathscr{C}$, the following hold:*

if $C \doteq \mathbf{0}$, then	$C \sim \mathbf{0}+\mathbf{0}$	*(stuck)*
if $C \doteq p$ and $p \neq \mathbf{0}$, then	$C \sim p$	*(unfolding)*
if $C \doteq p\{C/x\}$ and $q \sim p\{q/x\}$ then	$C \sim q$	*(folding)*

where, in the third law, p is actually open on x (while q is closed).

- For the folding property, if C =° a.C and q ~ a.q, then we can conclude that C ~ q

Axiomatization

- Open BPP (with variables), flattened with only one syntactic category, but we require that each ground instantiation of an axiom must respect the syntactic definition of (closed) BPP
- This means that we can write the axiom $x + (y + z) = (x + y) + z$, but it is invalid to instantiate it to $C+(a.0+b.0|0) = (C+a.0)+(b.0|0)$ because these are not legal BPP processes (the constant C and the parallel process $b.0 \mid 0$ cannot be used as summands).

The set of axioms are outlined in Table 4. We call E the set of axioms $\{$**A1, A2, A3, A4, R1, R2, R3, P1, P2, P3**$\}$. By the notation $E \vdash p = q$ we mean that there exists an equational deduction proof of the equality $p = q$, by using the axioms in E. Besides the usual equational deduction rules of reflexivity, symmetry, transitivity, substitutivity and instantiation (see, e.g., [28]), in order to deal with constants we need also the following recursion congruence rule:

$$\frac{p = q \;\wedge\; A \doteq p\{A/x\} \;\wedge\; B \doteq q\{B/x\}}{A = B}$$

Axiomatization (2)

A1	Associativity	$x+(y+z)=(x+y)+z$	
A2	Commutativity	$x+y=y+x$	
A3	Identity	$x+\mathbf{0}=x$	if $x\neq\mathbf{0}$
A4	Idempotence	$x+x=x$	if $x\neq\mathbf{0}$
R1	Stuck	if $C\doteq\mathbf{0}$, then $C=\mathbf{0}+\mathbf{0}$	
R2	Unfolding	if $C\doteq p\ \wedge\ p\neq\mathbf{0}$, then $C=p$	
R3	Folding	if $C\doteq p\{C/x\}\ \wedge\ q=p\{q/x\}$, then $C=q$	
P1	Associativity	$x\,\vert\,(y\,\vert\,z)=(x\,\vert\,y)\,\vert\,z$	
P2	Commutativity	$x\,\vert\,y=y\,\vert\,x$	
P3	Identity	$x\,\vert\,\mathbf{0}=x$	

Table 4. Axioms for team bisimulation equivalence

Axiomatization (3)

- Axiomatization of a truly-concurrent behavioral semantics for a process algebra with **recursive behavior**!

Theorem 9. **(Soundness)** *For every* $p, q \in \mathscr{P}_{BPP}$, *if* $E \vdash p = q$, *then* $p \sim^{\oplus} q$.

Theorem 10. **(Completeness)** *For every* $p, q \in \mathscr{P}_{BPP}$, *if* $p \sim^{\oplus} q$, *then* $E \vdash p = q$.

Conclusion

- Team bisimulation is a natural extension of bisimulation on LTS to BPP nets
- Intuitively correct, as it corresponds to the causal semantics of Petri nets, observing also the size of the distributed state
- Very efficient verification techniques
- Modal logic characterization
- Axiomatization