MATH254: Linear Algebra
Lecture 13

Moira MacNeil

February 4, 2025

Last Time

1. Finite dimensional spaces
2. Relationship between span and linear independence

Today

1. Linear transformations: properties and examples

Reminders:

- Midterm 1 is this Friday, February 7
- Assignment 3 is due Friday, February 14

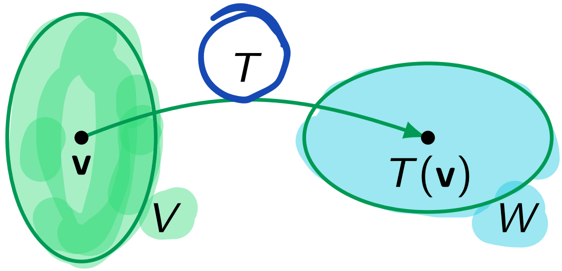
Linear Transformations

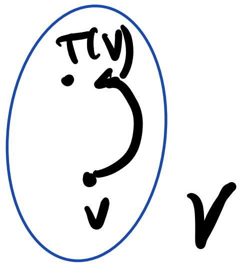
Linear Transformations of Vector Spaces

If V and W are two vector spaces, a function $T : V \to W$ is called a **linear transformation** if it satisfies the following axioms.

T1.	$T(\mathbf{v} + \mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1)$	for all $\mathbf{v}$ and $\mathbf{v}_1$ in V.
T2.	$T(r\mathbf{v}) = rT(\mathbf{v})$	for all $\mathbf{v}$ in V and r in $\mathbb{R}$.

A linear transformation $T : V \to V$ is called a **linear operator** on V.





Axioms of Linear Transformations

T1: $\underbrace{T(v+v_1)}_{V} = \underbrace{T(v)+T(v_1)}_{W}$

- Axiom T1 $\rightarrow$ T **preserves** vector addition. $T: V \to W$
- That is the result $T(\mathbf{v}+\mathbf{v}_1)$ of adding $\mathbf{v}$ and $\mathbf{v}_1$ then applying T is the same as applying T to get $T(\mathbf{v})$ and $T(\mathbf{v}_1)$ and then adding
- Axiom T2 means that T **preserves** scalar multiplication.
- Note: the additions in axiom T1 are both denoted by $+$, but the addition on the left forming $\mathbf{v}+\mathbf{v}_1$ is carried out in V, whereas the addition $T(\mathbf{v})+T(\mathbf{v}_1)$ is done in W
- The scalar multiplications $r\mathbf{v}$ and $rT(\mathbf{v})$ in axiom T2 refer to the spaces V and W, respectively.

Example 7.1.2

If V and W are vector spaces, the following are linear transformations:

Identity operator $V \to V$	$1_V : V \to V$	where $1_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V
Zero transformation $V \to W$	$0 : V \to W$	where $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in V
Scalar operator $V \to V$	$a : V \to V$	where $a(\mathbf{v}) = a\mathbf{v}$ for all $\mathbf{v}$ in V
		(Here a is any real number.)

The symbol 0 will be used to denote the zero transformation from V to W for *any* spaces V and W.

1_v satisfies axioms

let $v \in V$ and $u \in V$ and $r \in \mathbb{R}$

T1: $1_v(v+u) = v+u = 1_v(v) + 1_v(u)$

T2: $1_v(rv) = rv = r\,1_v(v)$

Example 7.1.3

sum elem on main diag.

Show that the transposition and trace are linear transformations. More precisely,

$$R : \mathbf{M}_{mn} \to \mathbf{M}_{nm} \qquad \text{where } R(A) = A^T \text{ for all } A \text{ in } \mathbf{M}_{mn}$$
$$S : \mathbf{M}_{mn} \to \mathbb{R} \qquad \text{where } S(A) = \text{tr } A \text{ for all } A \text{ in } \mathbf{M}_{nn}$$

are both linear transformations.

Transpose let $A, B \in M_{mn}$, $r \in \mathbb{R}$

T1: $R(A+B) = (A+B)^T = A^T + B^T = R(A) + R(B)$ ✓

T2: $R(rA) = (rA)^T = r(A^T) = r\,R(A)$ ✓

From properties of transpose

Trace

T1: $S(A+B) = tr(A+B) = tr(A) + tr(B) = S(A) + S(B)$ ✓

T2: $S(rA) = tr(rA) = r\,tr(A) = r\,S(A)$ ✓

Example 7.1.4

polynomials of deg at most n

If a is a scalar, define $E_a : \mathbf{P}_n \to \mathbb{R}$ by $E_a(p) = p(a)$ for each polynomial p in $\mathbf{P}_n$. Show that E_a is a linear transformation (called **evaluation** at a).

Let p, q be polynomials, $r \in \mathbb{R}$. Then

sum of $p+q$: $(p+q)(x) = p(x) + q(x)$

scalar mult. : $(rp)(x) = r\,p(x)$ for all x.

For all $p, q \in P_n$ and all $r \in \mathbb{R}$

T1: $E_a(p+q) = (p+q)(a) = p(a) + q(a) = E_a(p) + E_a(q)$

T2: $E_a(rp) = (rp)(a) = r\,p(a) = r\,E_a(p)$

$\therefore$ E_a is a linear transformation.

Theorem 7.1.5

Theorem

Let $T : V \to W$ be a linear transformation.

1. $T(\mathbf{0}) = \mathbf{0}$.
2. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.
3. $T(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k) = r_1T(\mathbf{v}_1) + r_2T(\mathbf{v}_2) + \cdots + r_kT(\mathbf{v}_k)$ for all $\mathbf{v}_i$ in V and all r_i in $\mathbb{R}$.

Proof.

1 $T(\mathbf{0}) = T(0v) = 0\,T(v) = 0$ for all $v \in V$

2 $T(-v) = T([-1]v) = (-1)T(v) = -T(v)$ for $v \in V$

3 induction on k

Property (3)

- Property (3): If $T : V \to W$ is a linear transformation and $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ are known, then $T(\mathbf{v})$ can be computed for *every* vector $\mathbf{v}$ in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$
- If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V, then $T(\mathbf{v})$ is determined for all $\mathbf{v}$ in V by the choice of $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$
- Two linear transformations $T : V \to W$ and $S : V \to W$ are called **equal** $(T = S)$ if they have the same **action** $\to$ if $T(\mathbf{v}) = S(\mathbf{v})$ for all $\mathbf{v}$ in V

Example 7.1.6

Let $T : V \to W$ be a linear transformation. If $T(\mathbf{v} - 3\mathbf{v}_1) = \mathbf{w}$ and $T(2\mathbf{v} - \mathbf{v}_1) = \mathbf{w}_1$, find $T(\mathbf{v})$ and $T(\mathbf{v}_1)$ in terms of $\mathbf{w}$ and $\mathbf{w}_1$.

$$T(v-3v_1) \overset{③}{=} T(v) - 3T(v_1) = w$$

$$T(2v-v_1) \overset{③}{=} 2T(v) - T(v_1) = w_1$$

Solve for $T(v)$ and $T(v_1)$!

$$T(v) - 6T(v) = w - 3w_1$$

$$T(v) = -\tfrac{1}{5}(w - 3w_1)$$

$$-T(v_1) + 6T(v_1) = w_1 - 2w$$

$$T(v_1) = \tfrac{1}{5}(w_1 - 2w)$$

Theorem 7.1.7

Theorem

Let $T : V \to W$ and $S : V \to W$ be two linear transformations. Suppose that $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. If $\text{T}(\mathbf{v}_i) = S(\mathbf{v}_i)$ for each i, then $T = S$.

If we know what a linear transformation $T : V \to W$ does to each vector in a spanning set for V, then we know what T does to *every* vector in V.

Proof.

If $v \in V = \text{span}\{v_1, \cdots, v_n\}$, then let $v = a_1 v_1 + \cdots + a_n v_n$ where $a_i \in \mathbb{R}$. Since $T(v_i) = S(v_i)\ \forall i$ then

$$\begin{aligned} T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_n T(v_n) \end{aligned}$$

$$= a_1 S(v_1) + \cdots + a_n S(v_n)$$
$$= S(a_1 v_1 + \cdots + a_n v_n)$$
$$= S(v)$$

Since v arbitrary, $T = S$ $\square$

Example 7.1.8

Let $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $T : V \to W$ be a linear transformation. If $T(\mathbf{v}_1) = \cdots = T(\mathbf{v}_n) = \mathbf{0}$, show that $T = 0$, the zero transformation from V to W.

$0 : V \to W$

$0(v) = 0 \quad \forall\ v \in V$

so $\quad T(v_i) = 0 = 0(v_i) \qquad$ for all i

$\therefore \quad T = 0$

Theorem 7.1.9

Theorem

Let V and W be vector spaces and let $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis of V. Given any vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ in W (they need not be distinct), there exists a unique linear transformation $T : V \to W$ satisfying $T(\mathbf{b}_i) = \mathbf{w}_i$ for each $i = 1, 2, \ldots, n$. In fact, the action of T is as follows:

Given $\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n$ in V, v_i in $\mathbb{R}$, then

$$\underline{T(\mathbf{v}) = T(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{w}_1 + v_2\mathbf{w}_2 + \cdots + v_n\mathbf{w}_n}.$$

Linear Transformations and Basis

- This theorem shows that to define a linear transformation we can simply specify where the basis vectors go, and the rest of the action is dictated by the linearity
- Deciding whether two linear transformations are equal $\rightarrow$ determine whether they have the same effect on the basis vectors
- Given a basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of a vector space V, there is a different linear transformation $V \rightarrow W$ for every ordered selection $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ of vectors in W (not necessarily distinct)

Example 7.1.10

Find a linear transformation $T : \mathbf{P}_2 \to \mathbf{M}_{22}$ such that

$$T(1+x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T(x+x^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad T(1+x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\{1+x,\ x+x^2,\ 1+x^2\} \to$ basis of P_2 (check)

Every $p \in P_2$ can be written as lin. comb:

$p = a + bx + cx^2$

$$p(x) = \tfrac{1}{2}(a+b-c)(1+x) + \tfrac{1}{2}(-a+b+c)(x+x^2) + \tfrac{1}{2}(a-b+c)(1+x^2)$$

To be cont.

Recap

Today we saw:

- Properties of linear transformations

Next time: Two important subspaces associated with linear transformations

MATH254: Linear Algebra
Lecture 14

Moira MacNeil

February 5, 2025

Last Time

1. Properties of linear transformations

Today

1. Finish the example from yesterday

2. Two important subspaces associated with linear transformations

Reminders:

- Midterm 1 is this Friday, February 7
- Assignment 3 is due Friday, February 14

Recall: Theorem 7.1.9

Theorem

Let V and W be vector spaces and let $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ be a basis of V. Given any vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ in W (they need not be distinct), there exists a unique linear transformation $T : V \to W$ satisfying $T(\mathbf{b}_i) = \mathbf{w}_i$ for each $i = 1, 2, \dots, n$. In fact, the action of T is as follows:
Given $\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n$ in V, v_i in $\mathbb{R}$, then

$$T(\mathbf{v}) = T(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{w}_1 + v_2\mathbf{w}_2 + \cdots + v_n\mathbf{w}_n.$$

Example 7.1.10 (again)

polynomials of deg. ≤ 2 $\longrightarrow$ 2×2 matrices

Find a linear transformation $T : \mathbf{P}_2 \to \mathbf{M}_{22}$ such that

$$T(1+x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T(x+x^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad T(1+x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Set $B = \{1+x, x+x^2, 1+x^2\}$ is a basis of P_2

Suppose there are scalars a, b, c such that

$(*)$ $a(1+x) + b(x+x^2) + c(1+x^2) = 0$

Collect our exponents of x to get system:

$a + c = 0$ Solve system:
$a + b = 0$ $a = -c$ $-2c = 0$
$b + c = 0$ $b = -c$ $c = 0$
$a = b = 0$

$\Rightarrow$ B is indept
dim $P_2 = 3$ and B has 3 vect.
$\Rightarrow$ B spans P_2
$\therefore$ B is a basis for P_2

Then every vector $p = a_0 + a_1 x + a_2 x^2$ in P_2 can be written as linear comb. of vectors in B.
System ⊛ but RHS coeff of p (not 0 vect).

$$p(x) = \tfrac{1}{2}(a+b-c)(1+x) + \tfrac{1}{2}(-a+b+c)(x+x^2) + \tfrac{1}{2}(a-b+c)(1+x^2)$$

Applying Thm

$$T(p(x)) = \tfrac{1}{2}(a+b-c)\,T(1+x) + \tfrac{1}{2}(-a+b+c)\,T(x+x^2) + \tfrac{1}{2}(a-b+c)\,T(1+x^2)$$

$$= \frac{1}{2}\left[(a+b-c)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + (-a+b+c)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + (a-b+c)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix}a+b-c & -a+b+c\\ -a+b+c & a-b+c\end{bmatrix}$$

Given $p(x) = a+bx+cx^2$

$T(p(x)) =$

Kernel and Image

Kernel and Image of a Linear Transformation

The **kernel** of T (denoted $\ker T$) and the **image** of T (denoted $\operatorname{im} T$ or $T(V)$) are defined by

$$\ker T = \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\}$$
$$\operatorname{im} T = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V)$$

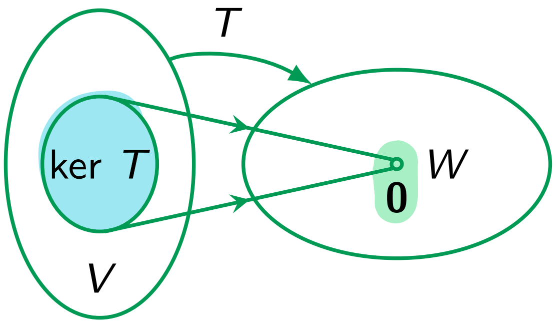
Nullspace and Range

$T: V \to W$

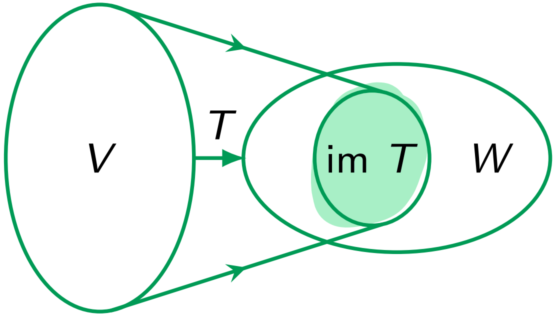
$\ker T = \{ v \text{ in } V \mid T(v) = 0 \}$ in V

$\operatorname{im} T = \{ T(v) \mid v \text{ in } V \}$ in W

- Kernel of T $\to$ called the **nullspace** of T because it consists of all vectors $\mathbf{v}$ in V satisfying the *condition* that $T(\mathbf{v}) = \mathbf{0}$



- Image of T $\to$ called the **range** of T and consists of all vectors $\mathbf{w}$ in W of the *form* $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V



Example 7.2.2

Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by the $m \times n$ matrix A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$. Then

$$\ker T_A = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} = \text{null } A \quad \text{and}$$
$$\text{im } T_A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\} = \text{im } A$$

Theorem 7.2.3

Theorem

Let $T : V \to W$ be a linear transformation.

1. ker T is a subspace of V.
2. im T is a subspace of W.

Proof. Subspace Test

① $T(0) = 0$ $\therefore$ both ker T and im T contain vector of V and W respectively.

ker T let v $v_1 \in$ ker T, then $T(v)=0=T(v_1)$

② $T(v+v_1) = T(v)+T(v_1) = 0+0 = 0$

$v + v_1 \in \ker T$

③ let $r \in \mathbb{R}$

$T(rv) = rT(v) = r0 = 0$ true for any $r \in \mathbb{R}$

$rv \in \ker T$

$\therefore$ $\ker T$ is subspace of V

im T let $w, w_1 \in \operatorname{im} T$ so

$w = T(v)$ $w_1 = T(v_1)$ $v, v_1 \in V$

② $w + w_1 = T(v) + T(v_1) = T(v + v_1)$

$w + w_1 \in \operatorname{im} T$

③ $rw = rT(v) = T(rv)$ $\forall r \in \mathbb{R}$

$rw \in \operatorname{im} T$

$\therefore$ $\operatorname{im} T$ is subspace of W

Nullity and Rank

Given a linear transformation $T : V \to W$.

- $\dim(\ker T)$ is called the **nullity** of T and denoted as $\text{nullity}(T)$
- $\dim(\text{im } T)$ is called the **rank** of T and denoted as $\text{rank}(T)$
- Recall: we defined the rank of a matrix A as dimension of col A, the column space of A

Example 7.2.4 $T_A : \mathbb{R}^n \to \mathbb{R}^m \quad T_A(x) = Ax$

Given an $m \times n$ matrix A, show that $\text{im } T_A = \text{col } A$, so $\text{rank } T_A = \text{rank } A$.

let $A = [c_1 \cdots c_n]$ (columns)

then $\text{im } T_A = \{Ax \mid x \in \mathbb{R}^n\}$

by def of matrix vect. multiplication $= \{x_1c_1 + x_2c_2 + \cdots + x_nc_n \mid x_i \in \mathbb{R}\}$

This is col space of A!

Example 7.2.5

Define a transformation $P : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ by $P(A) = A - A^T$ for all A in $\mathbf{M}_{nn}$. Show that P is linear and that:

(a) ker P consists of all symmetric matrices.

(b) im P consists of all skew-symmetric matrices.

P linear show satisfies axioms T1 and T2 (exercise)

(a) a matrix A is in ker P when $0 = P(A) = A - A^T$ this holds when $A = A^T \to$ i.e., A is symmetric.

(b) im P is all matrices $P(A)$ $A \in M_{nn}$

(skew sym $-A = A^T$)

$$P(A)^T = (A - A^T)^T = A^T - (A^T)^T$$
$$= A^T - A$$
$$= -(A - A^T)$$

Need to show all skew sym matrices are in im P

if S is skew sym then S lies in im P

$$P(1/2 S) = (1/2 S - 1/2 S^T) = 1/2(S - S^T)$$
$$= 1/2(S + S)$$
$$= S$$

Recap

Today we saw:

- Kernel and image
- One-to-one and onto transformations

Next time: Midterm 1

MATH254: Linear Algebra
Lecture 15

Moira MacNeil

February 11, 2025

Last Time

1. Midterm 1
2. Before that: kernel and image of transformations

Today

1. Review kernel and image
2. One-to-one and onto transformations
3. Review solutions for Midterm 1

Reminders:

- Assignment 3 is due this ~~Friday, February~~ 14

Tuesday after break
Feb 25

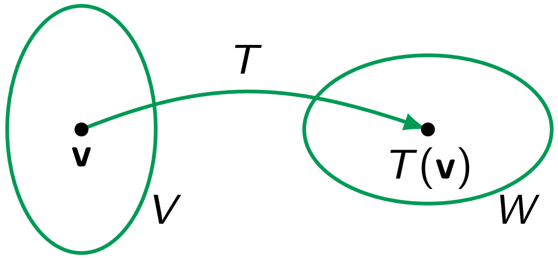
Recall: Linear Transformation

Linear Transformations of Vector Spaces

If V and W are two vector spaces, a function $T : V \to W$ is called a **linear transformation** if it satisfies the following axioms.

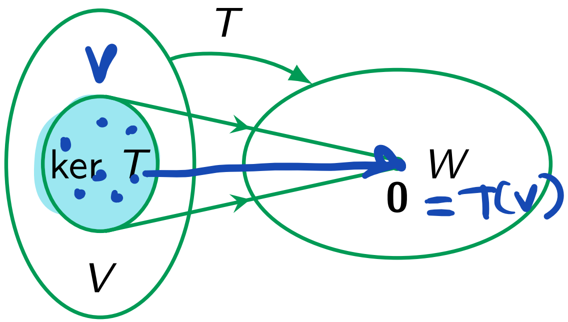
T1.	$T(\mathbf{v}+\mathbf{v}_1) = T(\mathbf{v}) + T(\mathbf{v}_1)$	for all $\mathbf{v}$ and $\mathbf{v}_1$ in V.
T2.	$T(\mathbf{rv}) = rT(\mathbf{v})$	for all $\mathbf{v}$ in V and r in $\mathbb{R}$.

A linear transformation $T : V \to V$ is called a **linear operator** on V.

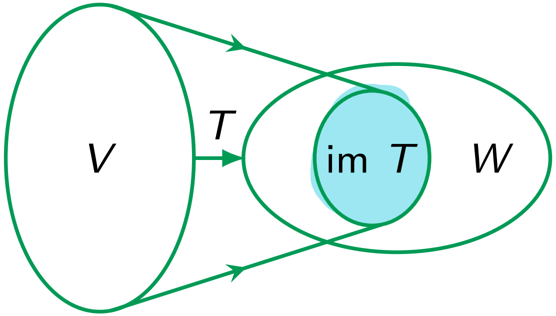


Recall: Kernel and Image of a Linear Transformation

- $\ker T = \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\}$
- Kernel of T $\rightarrow$ called the **nullspace** of T
- Consists of all vectors $\mathbf{v}$ in V satisfying the *condition* that $T(\mathbf{v}) = \mathbf{0}$
- $\ker T$ is a subspace of V



- $\operatorname{im} T = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V)$
- Image of T $\rightarrow$ called the **range** of T
- Consists of all vectors $\mathbf{w}$ in W of the *form* $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V
- $\operatorname{im} T$ is a subspace of W



Recall: Nullity and Rank

Given a linear transformation $T : V \to W$.

- $\dim(\ker T)$ is called the **nullity** of T and denoted as nullity(T)
- $\dim(\operatorname{im} T)$ is called the **rank** of T and denoted as rank(T)

Num vectors that are transformed to 0

↓ size of image space

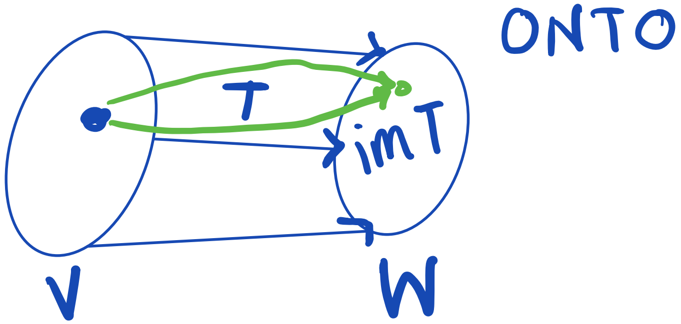
Num of vectors in W that are reached by T(v)

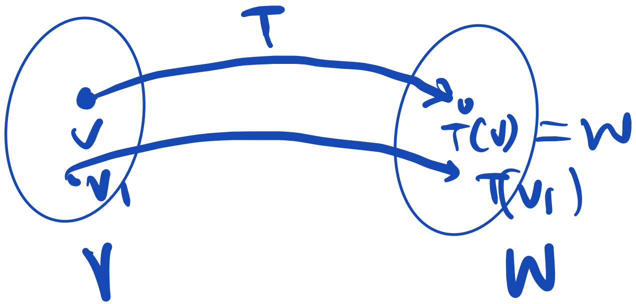
One-to-one and Onto Transformations

One-to-one and Onto Linear Transformations

Let $T : V \to W$ be a linear transformation.

1. T is said to be **onto** if $\operatorname{im} T = W$.
2. T is said to be **one-to-one** if $T(\mathbf{v}) = T(\mathbf{v}_1)$ implies $\mathbf{v} = \mathbf{v}_1$.





One-to-one and Onto Linear Transformations

- A vector $\mathbf{w}$ in W is **hit** by T if $\mathbf{w} = T(\mathbf{v})$ for some $\mathbf{v}$ in V
- T is onto if every vector in W is hit at least once
- T is one-to-one if no element of W gets hit twice
- Onto transformations T are those for which im $T = W$ is as large a subspace of W as possible → entire vector space
- One-to-one transformations T are the ones with ker T as small a subspace of V as possible

Theorem 7.2.7

Theorem

If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Proof. $(\Rightarrow)$ T is 1-1 let $v \in \ker(T)$ then $T(v) = 0$, so $T(v) = T(0)$ since T is 1-1. $\therefore$ $\ker T = \{0\}$

$(\Leftarrow)$ Assume $\ker T = \{0\}$ let $T(v) = T(v_1)$ where $v, v_1 \in V$. $T(v - v_1) = T(v) - T(v_1) = 0$ so $v - v_1 \in \ker T$. Then $v - v_1 = 0 \Rightarrow v = v_1$

$\therefore$ T is one-to-one. $\square$

Example 7.2.8

The identity transformation $1_V : V \to V$ is both one-to-one and onto for any vector space V.

$$v \to 1_V(v) = v$$

Example 7.2.9

→ Satisfy axioms T1, T2

Consider the linear transformations

$$S : \mathbb{R}^3 \to \mathbb{R}^2 \quad \text{given by } S(x,y,z) = (x+y, x-y)$$
$$T : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{given by } T(x,y) = (x+y, x-y, x)$$

Show that T is one-to-one but not onto, whereas S is onto but not one-to-one.

T 1-1 means $\ker T = \{0\}$ zero vector in $\mathbb{R}^2$ is $(0,0)$

$\ker T = \{(x,y) \mid x+y = x-y = x = 0\} = \{(0,0)\}$

Not onto (counter example)

$(0,0,1)$ is not in im T because if

$(0,0,1) = (x+y, x-y, x) \Rightarrow$

$x+y=x-y=0$ and $x=1$ $\rightarrow$ impossible.

S is not 1-1 since $(0,0,1)$ is in $\ker S$

Every element (s,t) in $\mathbb{R}^2$ is in $\operatorname{im} S$

$(s,t)=(x+y,\ x-y)=S(x,y,z)$ for some x,y,z

$\rightarrow$ $x=\frac{1}{2}(s+t)$ $\quad y=\frac{1}{2}(s-t)$ $\quad z=0$

$\therefore$ S is onto.

Theorem 7.2.11

Theorem

Let A be an $m \times n$ matrix, and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation induced by A, that is $T_A(\mathbf{x}) = A\mathbf{x}$ for all columns $\mathbf{x}$ in $\mathbb{R}^n$.

1. T_A is onto if and only if rank $A = m$.
2. T_A is one-to-one if and only if rank $A = n$.

Recap

Today we saw:

- Kernel and image
- One-to-one and onto transformations

Next time: Dimension Theorem

MATH254: Linear Algebra
Lecture 16

Moira MacNeil

February 12, 2025

Last Time

1. One-to-one and onto transformations

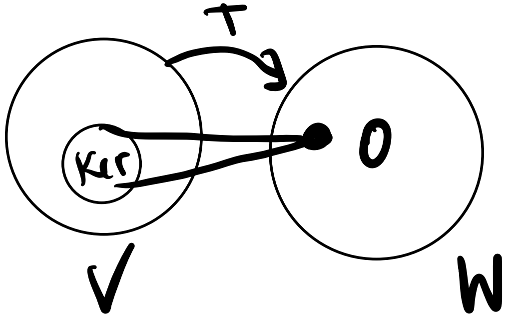
Today

1. Dimension Theorem
2. Isomorphisms

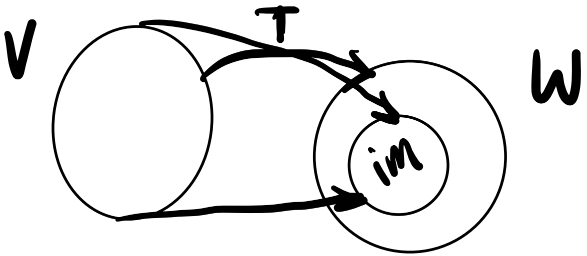
Reminders:

- Assignment 3 is now due Tuesday, February 25

Recall: Nullity and Rank



Given a linear transformation $T : V \to W$



- ker $T = \{\mathbf{v} \text{ in } V \mid T(\mathbf{v}) = \mathbf{0}\}$
- im $T = \{T(\mathbf{v}) \mid \mathbf{v} \text{ in } V\} = T(V)$
- dim(ker T) is called the **nullity** of T and denoted as nullity(T)
- dim(im T) is called the **rank** of T and denoted as rank(T)

One-to-one and Onto Transformations

One-to-one and Onto Linear Transformations

Let $T : V \to W$ be a linear transformation.

1. T is said to be **onto** if $\text{im } T = W$.
2. T is said to be **one-to-one** if $T(\mathbf{v}) = T(\mathbf{v}_1)$ implies $\mathbf{v} = \mathbf{v}_1$.

- T is onto if every vector in W is hit at least once
- T is one-to-one if no element of W gets hit twice

Recall: Theorem 7.2.7

Theorem

If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker T = \{\mathbf{0}\}$.

Theorem 7.2.12

Dimension Theorem

Let $T : V \to W$ be any linear transformation and assume that ker T and im T are both finite dimensional. Then V is also finite dimensional and

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$$

In other words, $\dim V = \text{nullity}(T) + \text{rank}(T)$.

Theorem 7.2.13

Theorem

Let $T : V \to W$ be a linear transformation, and let $\{\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ be a basis of V such that $\{\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n\}$ is a basis of ker T. Then $\{T(\mathbf{e}_1), \ldots, T(\mathbf{e}_r)\}$ is a basis of im T, and hence $r = \text{rank } T$.

If either dim(ker T) or dim(im T) can be found, then the other is automatically known!

Example 7.2.15

If $T : V \to W$ is a linear transformation where V is finite dimensional, then

$$\dim(\ker T) \leq \dim V \quad \text{and} \quad \dim(\operatorname{im} T) \leq \dim V$$

Indeed, $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$ by Dimension Theorem. Of course, the first inequality also follows because $\ker T$ is a subspace of V.

Example 7.2.16 $P_n \rightarrow$ poly. of degree n

Let $D : \mathbf{P}_n \rightarrow \mathbf{P}_{n-1}$ be the differentiation map defined by $D\left[p(x)\right] = p'(x)$. Compute $\ker D$ and hence conclude that D is onto.

$p'(x) = 0$ then $p(x) = $ constant so we have $\dim(\ker D) = 1$

Since $\dim(P_n) = n+1$

By dimension thm

$$\dim(\operatorname{im} D) = \dim(P_n) - \dim(\ker D)$$
$$n+1 \quad - \quad 1$$
$$= n = \dim(P_{n-1})$$

$\therefore \operatorname{im} D = P_{n-1}$ so D is onto

Example 7.2.17

$p(x) = a_0 + a_1 x + \cdots + a_n x^n$
$q(x) = b_0 + b_1 x + \cdots + b_n x^n$

Given a in $\mathbb{R}$, the evaluation map $E_a : \mathbf{P}_n \to \mathbb{R}$ is given by $E_a[p(x)] = p(a)$. Show that E_a is linear and onto, and hence conclude that $\{(x-a), (x-a)^2, \ldots, (x-a)^n\}$ is a basis of $\ker E_a$, the subspace of all polynomials $p(x)$ for which $p(a) = 0$.

$\underline{E_a}$ Linear

T1: $T(v+v_1) = T(v) + T(v_1)$

$$E_a(p(x)+q(x)) = E_a\left[(a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n\right]$$
$$= (a_0+b_0) + (a_1+b_1)a + \cdots + (a_n+b_n)a^n$$
$$= (a_0 + a_1 a + \cdots + a_n a^n) + (b_0 + b_1 a + \cdots + b_n a^n)$$
$$= E_a(p(x)) + E_a(q(x))$$

T2: $T(rv) = rT(v)$ let $r \in \mathbb{R}$

$$E_a(r\,p(x)) = E_a(ra_0 + ra_1x + \cdots + ra_nx^n)$$
$$= ra_0 + ra_1a + \cdots + ra_na^n$$
$$= r(a_0 + a_1a + \cdots + a_na^n)$$
$$= r\,E_a(p(x))$$

<u>onto</u> $\dim(\text{im } E_a) = \dim(\mathbb{R}) = 1$

so $\dim(\ker E_a) = \underset{P_n}{n+1} - \underset{\mathbb{R}}{1} = n$ by dim thm

Each of the n polynomials

$(x-a), (x-a)^2, \ldots, (x-a)^n$ is in $\ker E_a$

since $p(a) = 0$ for any of them

Linearly independent since they have distinct degrees.

n vectors, $\dim(\ker(E_a)) = n$

$\therefore$ these polynomials are a basis for the $\ker(E_a)$.

Example 7.2.18

If A is any $m \times n$ matrix, show that rank $A =$ rank $A^TA =$ rank AA^T.

rank A = rank $A^T A$ let $B = A^T A$

$T_A : \mathbb{R}^n \to \mathbb{R}^m$
$x \mapsto Ax$

$T_B : \mathbb{R}^n \to \mathbb{R}^n$
$x \mapsto Bx$

im T_A = col A

im T_B = col B

rank T_A = rank A

rank T_B = rank B

rank A = rank T_A = dim(im T_A)
= n − dim(ker T_A)

$$\operatorname{rank} B = \operatorname{rank} T_B = \dim(\operatorname{im} T_B)$$
$$= n - \dim(\ker T_B)$$

it suffices to show $\ker(T_A) = \ker(T_B)$

$$Ax = 0 \Rightarrow Bx = A^T A x = A^T(0) = 0$$
$$\ker T_A \subseteq \ker T_B$$

$$Bx = 0 \Rightarrow A^T A x = 0$$
$$\|Ax\|^2 = (Ax)^T (Ax) = x^T \overbrace{A^T A}^{B} x$$
$$= x^T (0) = 0$$
$$\Rightarrow Ax = 0$$
$$\ker T_B \subseteq \ker T_A$$
$$\therefore \ker T_A = \ker T_B \Rightarrow \operatorname{rank} A = \operatorname{rank} B$$

When are two vectors spaces the "same"?

- Consider the spaces

$$\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\} \quad \text{and} \quad \mathbf{P}_1 = \{a + bx \mid a,b \in \mathbb{R}\}$$

- Compare the addition and scalar multiplication in these spaces:

$$\begin{array}{cc} (a,b) + (a_1,b_1) = (a+a_1, b+b_1) & (a+bx) + (a_1 + b_1x) = (a+a_1) + (b+b_1)x \\ r(a,b) = (ra, rb) & r(a+bx) = (ra) + (rb)x \end{array}$$

- These are the same vector space expressed in different notation: Change each (a,b) in $\mathbb{R}^2$ to $a+bx$, then $\mathbb{R}^2$ *becomes* $\mathbf{P}_1$
- The map $(a,b) \mapsto a+bx$ is a linear transformation $\mathbb{R}^2 \to \mathbf{P}_1$ that is both one-to-one and onto

Isomorphisms

Isomorphic Vector Spaces

A linear transformation $T : V \to W$ is called an **isomorphism** if it is both onto and one-to-one. The vector spaces V and W are said to be **isomorphic** if there exists an isomorphism $T : V \to W$, and we write $V \cong W$ when this is the case.

Example 7.3.2

The identity transformation $1_V : V \to V$ is an isomorphism for any vector space V.

$$v \mapsto v$$

Example 7.3.3

If $T : \mathbf{M}_{mn} \to \mathbf{M}_{nm}$ is defined by $T(A) = A^T$ for all A in $\mathbf{M}_{mn}$, then T is an isomorphism. Hence $\mathbf{M}_{mn} \cong \mathbf{M}_{nm}$.

m x n zero matrix

Verify: 1-1 $\ker(T) = \{0\}$

$A^T = 0 \to A = 0$

Verify: onto dimension thm

notice that $\dim(M_{mn}) = mn$

$= \dim(M_{nm})$

Example 7.3.4

Isomorphic spaces can "look" quite different. For example, $\mathbf{M}_{22} \cong \mathbf{P}_3$ because the map $T : \mathbf{M}_{22} \to \mathbf{P}_3$ given by $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + bx + cx^2 + dx^3$ is an isomorphism.

verify linear transformation

verify 1-1 and onto

Theorem 7.3.5

Theorem

If V and W are finite dimensional spaces, the following conditions are equivalent for a linear transformation $T : V \to W$.

1. T is an isomorphism.
2. If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is any basis of V, then $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)\}$ is a basis of W.
3. There exists a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of V such that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)\}$ is a basis of W.

Isomorphisms are the linear transformations that preserve bases.

Recap

Today we saw:

- Dimension Theorem
- Isomorphisms

Next time: More on isomorphisms, composition

MATH254: Linear Algebra
Lecture 17

Moira MacNeil

February 14, 2025

Last Time

1. Dimension Theorem
2. Isomorphisms

Today

1. Isomorphisms
2. Composition of linear transformations

Reminders:

- Assignment 3 is now due Tuesday, February 25

Recall: Isomorphisms

Isomorphic Vector Spaces

A linear transformation $T : V \to W$ is called an **isomorphism** if it is both onto and one-to-one. The vector spaces V and W are said to be **isomorphic** if there exists an isomorphism $T : V \to W$, and we write $V \cong W$ when this is the case.

- An isomorphism $T : V \to W$ induces a pairing $\mathbf{v} \leftrightarrow T(\mathbf{v})$ between vectors $\mathbf{v}$ in V and vectors $T(\mathbf{v})$ in W that preserves vector addition and scalar multiplication
- The spaces V and W are identical except for notation, that is all vector space properties of either space are completely determined by those of the other

Theorem 7.3.5

Theorem

If V and W are finite dimensional spaces, the following conditions are equivalent for a linear transformation $T : V \to W$.

1. T is an isomorphism.
2. If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is any basis of V, then $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)\}$ is a basis of W.
3. There exists a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of V such that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)\}$ is a basis of W.

Isomorphisms are the linear transformations that preserve bases!

Theorem 7.3.6

Theorem

If V and W are finite dimensional vector spaces, then $V \cong W$ if and only if $\dim V = \dim W$.

Corollary 7.3.7

Corollary

Let U, V, and W denote vector spaces. Then:

1. $V \cong V$ for every vector space V.
2. If $V \cong W$ then $W \cong V$.
3. If $U \cong V$ and $V \cong W$, then $U \cong W$.

The relation $\cong$ is called an equivalence relation on the class of finite dimensional vector spaces.

Corollary 7.3.8

Corollary

If V is a vector space and $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$.

Coordinate Isomorphism

- If V is a vector space of dimension n, there are important explicit isomorphisms $V \to \mathbb{R}^n$
- Fix a basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of V and write the standard basis of $\mathbb{R}^n$ $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$
- There is a unique linear transformation $C_B : V \to \mathbb{R}^n$ given by

$$C_B(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

  where each v_i is in $\mathbb{R}$
- Moreover, $C_B(\mathbf{b}_i) = \mathbf{e}_i$ for each i so C_B is an isomorphism, called the **coordinate isomorphism** corresponding to the basis B

Example 7.3.9

Let V denote the space of all 2×2 symmetric matrices. Find an isomorphism $T : \mathbf{P}_2 \to V$ such that $T(1) = I$, where I is the 2×2 identity matrix.

$\{1, x, x^2\}$ is a basis of P_2

Want a basis of V containing I

The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is indept. in V (check)

it has dim 3 $\therefore$ a basis since $\dim V = 3$ (previous Eg)

Define Transformation $T: P_2 \to V$

$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $T(x^2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$p(x) = a + bx + cx^2$

$$M = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(a + bx + cx^2) = T(a) + T(bx) + T(cx^2)$$

$$= a\,T(1) + b\,T(x) + c\,T(x^2)$$

$$= \begin{bmatrix} a & b \\ b & a+c \end{bmatrix}$$

Theorem 7.3.10

Theorem

If V and W have the same dimension n, a linear transformation $T : V \to W$ is an isomorphism if it is either one-to-one or onto.

Proof. Dimension Thm

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T) = n$$

so $\dim(\ker T) = 0 \iff \dim(\operatorname{im} T) = n$

thus T is one-to-one <u>iff</u> T is onto.

Composition

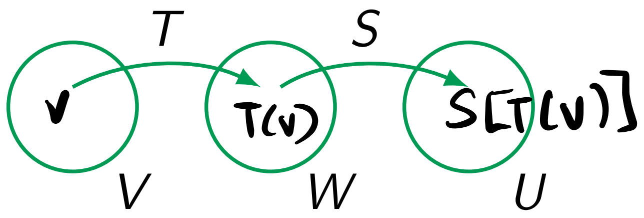
$T: V \to W \qquad S: W \to U$

Composition of Linear Transformations

Given linear transformations $V \xrightarrow{T} W \xrightarrow{S} U$, the **composite** $ST : V \to U$ of T and S is defined by

$$ST(\mathbf{v}) = S[T(\mathbf{v})] \quad \text{for all } \mathbf{v} \text{ in } V$$

The operation of forming the new function ST is called **composition** (sometimes denoted $S \circ T$).



Not all linear transformations can be composed

- If $T : V \to W$ and $S : W \to U$ are linear transformations then $ST : V \to U$ is defined, but TS *cannot* be formed unless $U = V$
- Even if ST and TS can both be formed, they may not be equal
- For example, if $S : \underbrace{\mathbb{R}^m}_{V} \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \underbrace{\mathbb{R}^m}_{U}$ are induced by matrices A and B respectively, then ST and TS can both be formed (they are induced by AB and BA respectively), but the matrix products AB and BA may not be equal (they may not even be the same *size*)

$T: \mathbb{R}^n \to \mathbb{R}^m \qquad S: \mathbb{R}^m \to \mathbb{R}^n$

$ST: x \mapsto ABx \qquad TS: x \mapsto BAx$

Example 7.3.12

Define: $S : \mathbf{M}_{22} \to \mathbf{M}_{22}$ and $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ by $S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ and $T(A) = A^T$ for $A \in \mathbf{M}_{22}$. Describe the action of ST and TS, and show that $ST \neq TS$.

$$S\left[T\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right] = S\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

$$TS\begin{bmatrix} a & b \\ c & d \end{bmatrix} = T\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

in general $ST \neq TS$

Theorem 7.3.13

$$1_V : \begin{array}{c} V \to V \\ v \mapsto v \end{array}$$

Theorem

Let $V \xrightarrow{T} W \xrightarrow{S} U \xrightarrow{R} Z$ be linear transformations.

1. The composite ST is again a linear transformation.
2. $T1_V = T$ and $1_W T = T$.
3. $(RS)T = R(ST)$.

Theorem 7.3.14

Theorem

Let V and W be finite dimensional vector spaces. The following conditions are equivalent for a linear transformation $T : V \to W$.

(a) T is an isomorphism.

(b) There exists a linear transformation $S : W \to V$ such that $ST = 1_V$ and $TS = 1_W$.

Moreover, in this case S is also an isomorphism and is uniquely determined by T:

$$\text{If } \mathbf{w} \text{ in } W \text{ is written as } \mathbf{w} = T(\mathbf{v}), \text{ then } S(\mathbf{w}) = \mathbf{v}.$$

Inverse of a Linear Transformation

- Given an isomorphism $T : V \to W$, the unique isomorphism $S : W \to V$ satisfying condition (b) of the previous Theorem is called the **inverse of T** and is denoted by T^{-1}.
- Hence $T : V \to W$ and $T^{-1} : W \to V$ are related by the **fundamental identities**:

$$T^{-1}\left[T(\mathbf{v})\right] = \mathbf{v} \text{ for all } \mathbf{v} \text{ in } V \quad \text{and} \quad T\left[T^{-1}(\mathbf{w})\right] = \mathbf{w} \text{ for all } \mathbf{w} \text{ in } W$$

- Each of T and T^{-1} reverses the action of the other.

Example 7.3.15

Define $T : \mathbf{P}_1 \to \mathbf{P}_1$ by $T(a+bx) = (a-b)+ax$. Show that T has an inverse, and find the action of T^{-1}.

First step is to check T is linear. Preserve addition and scalar multiplication.

$T(1+0x) = 1+x$ $\quad$ $T(0+1x) = -1$

$B = \{1, x\}$ is a basis of P_1 so T takes B to $D = \{1+x, -1\}$ T is isomorphism.

T^{-1} takes D back to B

$T^{-1}(1+x) = 1$ $\quad$ $T^{-1}(-1) = x$

$$a + bx = b(1+x) + (b-a)(-1) \quad \text{so}$$

$$T^{-1}(a+bx) = b\,T^{-1}(1+x) + (b-a)\,T^{-1}(-1)$$

$$= b + (b-a)\,x$$

Example 7.3.16

If $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis of a vector space V, the coordinate transformation $C_B : V \to \mathbb{R}^n$ is an isomorphism defined by

$$C_B(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n) = (v_1, v_2, \ldots, v_n)^T$$

The way to reverse the action of C_B is clear: $C_B^{-1} : \mathbb{R}^n \to V$ is given by

$$C_B^{-1}(v_1, v_2, \ldots, v_n) = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n \quad \text{for all } v_i \text{ in } V$$

Example 7.3.17

Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = (z, x, y)$. Show that $T^3 = 1_{\mathbb{R}^3}$, and hence find T^{-1}.

$$T^2(x,y,z) = T[T(x,y,z)] = T(z,x,y)$$
$$= (y,z,x)$$

$$T^3(x,y,z) = T(y,z,x) = (x,y,z)$$

$$\therefore T^3 = 1_{\mathbb{R}^3} = T(T^2)$$

Thus $T^{-1} = T^2$

Example 7.3.18

Define $T : \mathbf{P}_n \to \mathbb{R}^{n+1}$ by $T(p) = (p(0), p(1), \ldots, p(n))$ for all p in $\mathbf{P}_n$. Show that T^{-1} exists.

$T(p) = 0$ $\quad p(k) = 0$ for $k = 0, \ldots, n$

so p has $n+1$ distinct roots

because p is deg at most n, $n+1$ distinct roots $\Rightarrow$ $p = 0$ $\therefore$ T is 1-1

$\dim P_n = n+1 = \dim \mathbb{R}^{n+1}$ so T is also onto

$\therefore$ T is an iso. and T^{-1} exists.

Recap

Today we saw:

- Isomorphisms
- Composition

Next time: A new chapter - Orthogonality