Corollaries from the Triangle Inequality

Corollary : Reverse Triangle Inequality

$$||x| - |y|| \leq |x - y|$$

Proof. We know that $|a + b| \leq |a| + |b|$.

Let $a = x - y$, $b = y$, $\Rightarrow a + b = x$

$$\begin{aligned} |x| &\leq |x - y| + |y| \\ |x| - |y| &\leq |x - y| \end{aligned}$$

Now, $|c + d| \leq |c| + |d|$

Let $c = y - x$, $d = x$, $\Rightarrow c + d = y$

$$\begin{aligned} |y| &\leq |y - x| + |x| \\ |y| - |x| &\leq |y - x| \\ -(|x| - |y|) &\leq |x - y| \\ |x| - |y| &\geq -|x - y| \end{aligned}$$

So, we have

$$-|x - y| \leq |x| - |y| \leq |x - y|$$

Therefore,

$$||x| - |y|| \leq |x - y|$$

□

Corollary

$$|x - y| \leq |x| + |y|$$

Proof. We know that $|a + b| \leq |a| + |b|$.

Let $a = x$, $b = -y$

$$\begin{aligned} |x - y| &\leq |x| + |-y| \\ &\leq |x| + |y| \end{aligned}$$

Therefore, we have that $|x - y| \leq |x| + |y|$ □

Corollary

$$|x + y| \geq ||x| - |y||$$

Proof. From the reverse triangle inequality, we know that $||a| - |b|| \leq |a - b|$.

Let $a = x$, $b = -y$

$$\begin{aligned} ||x| - |-y|| &\leq |x + y| \\ ||x| - |y|| &\leq |x + y| \end{aligned}$$

Therefore, we have that

$$|x + y| \geq ||x| - |y||$$

$\square$

Distance and Metrics

Definition : Distance Function / Metric

A function $d : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is called a distance function, or metric, if it satisfies:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$

In $\mathbb{R}$, $d(x, y) = |x - y|$ is the standard Euclidean metric. In $\mathbb{R}^2$, let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. Then the distance between P and Q is given by: $d = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$.

Definition : Metric Spaces

A metric space is a space that has a metric. This allows us to talk about limits and continuity in terms of the metric.

Bounds

Let $A = \{x \in \mathbb{Q} : x^2 < 2\}$, that is, A contains all rationals numbers between $-\sqrt{2}$ and $\sqrt{2}$. There are upper bounds for A, for example, A is bounded above by 2 since for every $a \in A$, $a < 2$. 1.5 is another upper bound for A. Is there a least upper bound? Yes, but it is $\sqrt{2}$, and $\sqrt{2} \in \mathbb{R}$ and $\sqrt{2} \notin \mathbb{Q}$.

Definition

Let S be an ordered field, and let $A \subseteq S$ be nonempty.

(i) A is bounded above if $\exists b \in S$ such that $\forall x \in A$, $x \leq b$. b is called an upper bound of A.

(ii) The least upper bound of A (if it exists) is some $b_0 \in S$ such that b_0 is an upper bound of A, and if b is any other upper bound of A, then $b_0 \leq b$. b_0 is called the supremum of A, denoted by $\sup(A)$.

(iii) A is bounded below if $\exists c \in S$ such that $\forall x \in A$, $c \leq x$. c is called a lower bound of A.

(iv) The greatest upper bound of A (if it exists) is some $c_0 \in S$ such that c_0 is a lower bound of A, and if c is any other lower bound of A, then $c_0 \geq c$. c_0 is called the infimum of A, denoted by $\inf(A)$.

(v) If a set is both bounded above and bounded below, the set is called bounded.

Example

Find the supremum and infimum for the following sets (if they exist):

1. $A = \mathbb{N} = \{1, 2, 3, \dots\}$
2. $A = \mathbb{Q}$

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1. This set has no upper bound, but it does have lower bounds.

$$\inf(A) = 1$$

Since $\forall x \in A,\ 1 \leq x$, and if c is any other lower bound, $c \leq 1$ since $1 \in A$.

2. This set is not bounded above or below.

3. This set has both an upper bound, and a lower bound.

$$\sup(A) = 1, \quad \inf(A) = 0$$