Following from the Axioms

Definition : The Complement Rule

Since E and E^c are complementary, $E \cup E^c = \Omega$, and thus, from Axioms 2 and 3, follow the complement rule:

$$1 = \Pr(\Omega) = \Pr(E \cup E^c) = \Pr(E) + \Pr(E^c)$$

or,

$$\Pr(E^c) = 1 - \Pr(E) \quad \text{and} \quad \Pr(E) = 1 - \Pr(E^c)$$

Example

The U.S. Census of 2010 gives the proportions / fractions of households in the United States with certain numbers of habitants:

Size of Household	1	2	3	4	5	6	7 or more
Probability	0.267	0.336	0.158	0.137	0.063	0.024	0.015

The most common household size is 2 people. What is the probability that a randomly chosen household is not this size?

Let E: "the household size is 2", then E^c: "the household size is not 2".

$$\Pr(E) = 0.336$$

$$\Pr(E^c) = 1 - \Pr(E) = 1 - 0.336 = 0.664$$

Definition : The Addition Rule

If E and F are events, then $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$.

Example

A standard deck of playing cards has 52 cards:

- Thirteen ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
- And for each rank, four suits: Heart, Diamond, Clubs, Spade

Randomly sample a card. What is the probability of choosing a king or a club?

Let K = Choose King = $\{K\heartsuit, K\diamondsuit, K\clubsuit, K\spadesuit\}$

Let C = Choose Club = $\{2\clubsuit, 3\clubsuit, 4\clubsuit, 5\clubsuit, 6\clubsuit, 7\clubsuit, 8\clubsuit, 9\clubsuit, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit\}$

The probability of choosing a king or club is:

$$\Pr(K \cup C) = \Pr(K) + \Pr(C) - \Pr(K \cap C) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} \approx 0.308$$

Example

A retail establishment accepts either American Express or VISA credit cards. A total of 24 percent of its customers carry American Express, 61 percent carry a VISA, and 11 percent carry both. What is the probability that a randomly chosen customer will carry a credit card that this establishment accepts?

Let $A =$ Carry American Express and $V =$ Carry VISA, then:

$$\begin{aligned}\Pr(\text{Accepts}) &= \Pr(\text{Carry American Express or VISA}) \\ &= \Pr(A \cup V) = \Pr(A) + \Pr(V) - \Pr(A \cap V) \\ &= 0.24 + 0.61 - 0.11 \\ &= 0.74\end{aligned}$$

Definition : The Generalized Addition Rule

If $E_1, E_2, \dots E_n$ is a finite sequence of events, then:

$$\begin{aligned}\Pr(E_1 \cup E_2 \cup \dots \cup E_n) = &\sum_{i=1}^{n} \Pr(E_i) - \sum_{i<j}^{n} \Pr(E_i \cap E_j) + \dots \\ &+ (-1)^{r+1} \sum_{i_1 < i_2 < \dots < i_r} \Pr(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_r}) + \dots \\ &+ (-1)^{n+1} \Pr(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_n})\end{aligned}$$

That is, add the singles subtract the doubles, add the triples, etc.

For example, with three events:

$$\begin{aligned}\Pr(E \cup F \cup G) = &\Pr(E) + \Pr(F) + \Pr(G) - \Pr(E \cap F) \\ &- \Pr(E \cap G) - \Pr(F \cap G) + \Pr(E \cap F \cap G)\end{aligned}$$

Example

A marketing firm estimates that 31.2% of their customers are professionals, 47.0% are married, 52.5% are college graduates, 4.2% are professional college graduates, 14.7% are married college graduates, 8.6% are married professionals, and 2.5% are married professional college graduates. How can we tell this firm's estimates are wrong?

Let P: "The customer is professional", M: "The customer is married", and G: "The customer is a college graduate".

$$\begin{aligned}\Pr(P \cup M \cup G) &= 0.312 + 0.47 + 0.525 - 0.086 - 0.147 - 0.042 + 0.025 \\ &= 1.057\end{aligned}$$

Since $\Pr(P \cup M \cup G) > 1$, so this is wrong.

In many cases, we naturally assume that all outcomes in Ω are equally likely. In the case where $\Omega = \{a_1, a_2, \ldots, a_n\}$ is finite, we may assume that:

$$\Pr(\{a_1\}) = \Pr(\{a_2\}) = \cdots = \Pr(\{a_n\})$$

which means, using Axioms 2 and 3, that

$$\Pr(\{a_i\}) = \frac{1}{n} \quad \text{for } i = 1, 2, \ldots, n$$

Then, by Axiom 3, for all events $E \subseteq \Omega$,

$$\Pr(E) = \frac{\text{Number of outcomes in } E}{\text{Number of outcomes in } \Omega}$$

Example

A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than the first one?

The set of outcomes for this experiment is:

$$\Omega = \{(1,1), (1,2), (1,3), \ldots, (1,6), (2,1), (2,2), \ldots, (6,6)\}$$

with $|\Omega| = 36$ and the set of outcomes in the event that the second die lands on a higher value than does the first is:

$$\begin{aligned} E = \{\, & (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6), \\ & (2,3),\ (2,4),\ (2,5),\ (2,6), \\ & (3,4),\ (3,5),\ (3,6), \\ & (4,5),\ (4,6), \\ & (5,6)\,\} \end{aligned}$$

with $|E| = 15$. Therefore, the probability that the second die lands on a higher value than the first is:

$$\Pr(E) = \frac{15}{36} = \frac{5}{12}$$

Example

An urn contains 5 red, 6 blue, and 8 green balls. If a set of three balls is randomly selected, what is the probability that each of the balls will be:
(a) of the same color?
(b) of different colors

Note that the number of outcomes in the sample space is:

$$|\Omega| = \binom{19}{3} = \frac{19!}{3!16!} = 969$$

(a) The number of ways you can select three of the same color balls is:

$$\binom{5}{3} + \binom{6}{3} + \binom{8}{3} = 86$$

So, the probability of picking three of the same color balls is:

$$\Pr\left(\text{3 of the same color}\right) = \frac{86}{969} \approx 0.08875$$

(b) The number of ways you can select three different color balls is:

$$\binom{5}{1} \cdot \binom{6}{1} \cdot \binom{8}{1} = 240$$

So, the probability of choosing three different color balls is:

$$\Pr\left(\text{3 different colors}\right) = \frac{240}{969} = \frac{80}{323} \approx 0.24768$$