

Convergence

Proposition : p -Series

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$.

Proof.

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k^p} &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \dots \\ &< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p} \right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} \right) + \dots \\ &= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \dots \\ &= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^2)^{p-1}} + \dots \\ &= 1 + \left(\frac{1}{2} \right)^{p-1} + \left(\left(\frac{1}{2} \right)^{p-1} \right)^2 + \dots \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{2^{p-1}} \right)^k\end{aligned}$$

Which is a geometric series with $r = \left(\frac{1}{2}\right)^{p-1} \in (-1, 1)$ since $p > 1$.

This converges, so by comparison, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges. \square

The cases for a p -series are:

- $p = 1$: Harmonic: $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ which we have proven diverges
- $p < 1$: $\frac{1}{k^p} > \frac{1}{k}$, so we can use comparison
- $p > 1$: Ex. $p = 2$: $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$ converges.

Absolute Convergence

Proposition : Alternating Series Test

Assume (a_k) is monotonically decreasing with $a_k \rightarrow 0$, then

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

converges.

Note: a_k is decreasing with a limit of 0, which implies each $a_k \geq 0$.

Example

Show that the Alternating Harmonic Series converges. What does it converge to?

$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$, $a_k = \frac{1}{k}$ is monotonically decreasing, and $\frac{1}{k} \rightarrow 0$.

To find what it converges to, use the Taylor series for $\ln(1+x)$. For $x \in (-1, 1)$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

For $x = 1$: $\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$, so the Alternating Harmonic Series converges to $\ln(2)$. (Although $x \notin (-1, 1)$.)