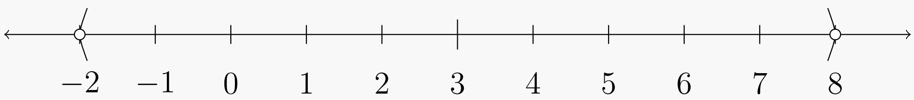
Absolute Values

Example

What values of x satisfy the following equation?

$$|x-3| < 5$$

We are looking for values where the difference between x and 3 is strictly less than 5.

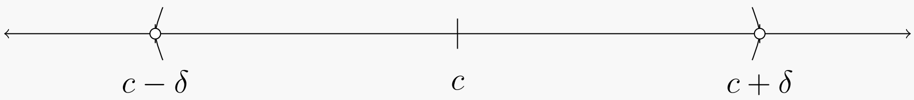


The equation holds for $x \in (-2, 8)$, or $\{x : -2 < x < 8\}$.

Example

What values of x satisfy the following equation?

$$|x-c| < \delta, \quad \text{for some } \delta > 0$$



The equation holds for $x \in (c-\delta, c+\delta)$, or $\{x : c-\delta < c < c+\delta\}$.

Standard Properties of Absolute Values you can use without proof:

- $|a| \geq 0$ for all a, and $|a| = 0$ if and only if $a = 0$
- $|a| = |-a|$
- $-|a| \leq a \leq |a|$
  - $a = |a|$ if $a \geq 0$
  - $a = -|a|$ if $a < 0$
- $|a \cdot b| = |a| \cdot |b|$
- $\frac{1}{|a|} = \left|\frac{1}{a}\right|$ for $a \neq 0$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for $b \neq 0$
- $|a| \leq b$ if and only if $-b \leq a \leq b$

Proposition

If $\mathbb{F}$ is an ordered field and $a, b \in \mathbb{F}$, then $|a| \leq b$ if and only if $-b \leq a \leq b$.

Proof. Suppose $|a| \leq b$. We know $-|a| \leq a \leq |a|$. We have that $a \leq |a|$ and $|a| \leq b$ so $a \leq b$. Now consider $|a| \leq b$, and multiply by -1.

$$-|a| \geq -b$$

We have $-b \leq -|a|$ and $-|a| \leq a$, so $-b \leq a$. Putting both together we have:

$$-b \leq a \leq b$$

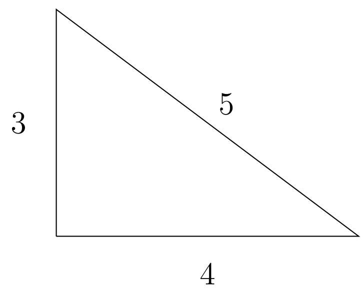
Now suppose $-b \leq a \leq b$.

Case 1: $a \geq 0$, then $|a| = a$, so $a \leq b$ gives $|a| \leq b$.

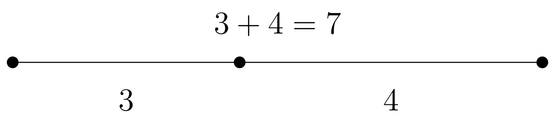
Case 2: $a < 0$, then $|a| = -a$, so $a = -|a|$. $-b \leq a$ by assumption, so $-b \leq -|a|$. Multiplying by -1 gives $b \geq |a|$.

In either case, we have $|a| \leq b$. Therefore, $|a| \leq b$ if and only if $-b \leq a \leq b$. $\square$

Consider the right 3-4-5 triangle:



If one side has length 3 and one has length 4, what could the length of the third side be? Consider the degenerate triangle (all 3 vertices are on the same line / collinear) shown below:



7 is an upper bound for the length of the third side (the lower bound would be 1).

Theorem : The Triangle Inequality

If $x, y \in \mathbb{F}$, then
$$|x + y| \leq |x| + |y|$$

Proof. We know $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$.

$$-|x| - |y| \leq x + y \leq |x| + |y|$$

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$
$$|x + y| \leq |x| + |y|$$

$\square$