Poisson Random Variables

Let $\lambda > 0$ be a fixed real number and let X be a random variable with possible values $0, 1, 2 \ldots$. Then, X is said to a Poisson random variable with parameter λ if its pmf is of the form

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, 3, \ldots \qquad \text{(and 0 otherwise)}$$

We write $X \sim \text{Pois}(\lambda)$ to indicate that X is Poisson with parameter λ. We have shown this is a valid pmf. Also, $E(X) = V(X) = \lambda$.

Example

Suppose that the number X of people that visit a professors student hours in a week is a Poisson random variable with $\lambda = 3$. What is the probability that at least four people visit the student hours next week?

We know $X \sim \text{Pois}(3)$.

$$\begin{aligned}
\Pr(X \geq 4) &= 1 - \Pr(X < 4) \\
&= 1 - (P(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3)) \\
&= 1 - \left(\frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!}\right) \\
&= 1 - \left(e^{-3} + 3e^{-3} + \frac{9}{2}e^{-3} + \frac{27}{6}e^{-3}\right) \\
&= 1 - 0.6474 \\
&= 0.3526
\end{aligned}$$

Poisson random variables arise as the results of the Poisson process. Suppose that we can assume the following about how events occur:

1. There exists a real $\lambda > 0$ such that for any short period of time h, the probability of exactly one event occurs if $\lambda h + o(h)$, where $o(h)$ is any function f with $\lim_{h \to 0} f(h)/h = 0$ (So, $f(h)$ approaches 0 faster than h does).

   Intuitively, the probability of one event occurring in an interval of length h is approximately equal to λh. For example, if the time units are weeks and $\lambda = 5$, we might think of this as "5 per week, on average", since $\Pr(X = 1) = \lambda h + o(h) \approx 5(1) = 5$. But it also means we can scale to smaller units of time. We could ask questions about days ($h = 1/7$ of a unit week) using $\Pr(1 \text{ in a day}) = \lambda h + o(h) \approx 5\left(\frac{1}{7}\right)$.

2. The probability that 2 or more events will occur in an interval of length h is equal to $o(h)$.

   Intuitively, the probability that 2 or more events occur is small when compared to h. I.e., the chance that 2 or more events occur decreases faster than you can narrow the interval.

3. The number of events occurring in an interval of length h is independent of the number of occur prior to this interval.

Under these three assumptions, the number of events occurring in an interval of length t is a Poisson random variable with parameter λt. This is a complicated definition, but it is natural in practice. In questions, we will be told that "X is Poisson" in this case. Know these assumptions however, as they like you split time intervals up like in Assumption 1.

Example

Suppose small aircraft arrive at a certain airport according to a Poisson process with a rate of $\lambda = 8$ per hour (so that the number of arrivals during a time period of t hours is a Poisson rv with parameter $\lambda = 8t$). The pmf for X is:

$$p(x) = \frac{e^{-8t}(8t)^x}{x!} \qquad x = 0, 1, 2, 3, \ldots \qquad \text{(and 0 otherwise)}$$

When $t = 1$, we have $p(x) = \frac{e^{-8}(8)^x}{x!}$. What is the probability that exactly 6 small aircraft arrive during a 1-hour period? What about at least 6 small aircrafts. What are the expected value and standard deviation of the number of small aircraft that arrive during 90-min period? What is the probability that 15 small aircraft arrive during a 2.5-hour period?

$$\Pr(X = 6) = \frac{e^{-8}8^6}{6!} = 0.1221$$

$$\begin{aligned}
\Pr(X \geq 6) &= 1 - \Pr(X \leq 5) \\
&= 1 - (\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) + \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)) \\
&= 1 - (0.000335 + 0.002684 + 0.010735 + 0.028625 + 0.057244 + 0.09161) \\
&= 1 - 0.1912 \\
&= 0.8088
\end{aligned}$$

For $t = 1.5$, $p(x) = \frac{e^{-12}12^x}{x!}$.

$$E(X) = \lambda t = 8(1.5) = 12$$

$$\sigma(X) = \sqrt{V(X)} = \sqrt{\lambda t} = \sqrt{8(1.5)} = \sqrt{12} = 3.464$$

For $t = 2.5$, $p(x) = \frac{e^{-20}20^x}{x!}$.

$$\Pr(X = 15) = \frac{e^{-20}20^{15}}{15!} = 0.0516488535$$

Poisson random variables also arise as limits of binomial distributions: Suppose that $X \sim \text{Bin}(n, p)$. As $n \to \infty$ and $p \to 0$ in such a way that $np \to \lambda > 0$, X approaches Poisson with $\lambda = np$. In practice, this means that a binomial distribution where n is large and p is small can be approximated by a Poisson distribution with $\lambda = np$. This is generally excellent if $n > 50$ and $np > 5$, but can be often be OK if n is smaller with $np < 5$. We say it is a good error if it roughly under 10% .

Example

Suppose that $X \sim \text{Bin}(75, 0.05)$. The exact value of $\Pr(X \leq 1)$ using the binomial pmf is

$$\Pr(X \leq 1) \binom{75}{0} (0.05)^0 (1-0.05)^{75-0} + \binom{75}{1} (0.05)^1 (1-0.05)^{75-1} = 0.1056$$

Find the Poisson approximation of $\Pr(X \leq 1)$. Is it good?

Since $n = 75 > 50$ and $np = (75)(0.05) = 3.75 < 5$, then we can approximate $X \sim \text{Bin}(75, 0.05)$ by $Y \sim \text{Pois}(3.75)$. $p(y) = \frac{e^{-3.75} 3.75^y}{y!}$.

$$\begin{aligned}
\Pr(Y \leq 1) &= \Pr(Y = 0) + \Pr(Y = 1) \\
&= \frac{e^{-3.75} 3.75^0}{0!} + \frac{e^{-3.75} 3.75^1}{1!} \\
&= 0.0235177459 + 0.088191547 \\
&= 0.1117092929
\end{aligned}$$

Is this approximation good?

$$\begin{aligned}
|\Pr(X \leq 1) - \Pr(Y \leq 1) &= |0.1056 - 0.111709| \\
&= |-0.06109| \\
&= 0.006109
\end{aligned}$$

Yes this is a good approximation since the error is small.

Computing the cdf F of a Poisson random variable X is to compute

$$F(x) = \Pr(X \leq x) = \sum_{i=0}^{x} \frac{e^{-\lambda} \lambda^i}{i!}$$

$F(x)$ has no closed form but can be computed relatively easily recursively using: If X is a Poisson random variable with parameter λ, then

$$p(x+1) = \frac{\lambda}{x+1} \cdot p(x)$$

Proof.

$$\frac{p(x+1)}{p(x)} = \frac{e^{-\lambda} \lambda^{x+1} / (x+1)!}{e^{-\lambda} \lambda^x / x!} = \frac{\lambda x!}{(x+1)!} = \frac{\lambda x!}{(x+1) \cdot x!} = \frac{\lambda}{x+1}$$

□

That is, we can use $p(x)$ to build the next $p(x+1)$ without directly using the Poisson pmf.

Example

Let X be a Poisson random variable with parameter $\lambda = 5$. Find the pmf p of X and use it to find the first couple values of the cdf of X.

We know $X \sim \text{Pois}(5)$, and $p(x) = \frac{e^{-5}5^x}{x!}$.

$$\begin{aligned}
p(0) &= \frac{e^{-5}5^0}{0!} = 0.006737947 \\
p(1) &= \frac{5}{1} \cdot p(0) = 0.033689735 \\
p(2) &= \frac{5}{2} \cdot p(1) = 0.0842243375 \\
p(3) &= \frac{5}{3} \cdot p(2) = 0.1403738958
\end{aligned}$$

So,

$$F(x) = \begin{cases} 0.006737947 & \text{if } x = 0 \\ 0.040427682 & \text{if } x = 1 \\ 0.124652020 & \text{if } x = 2 \\ 0.265025815 & \text{if } x = 3 \end{cases}$$