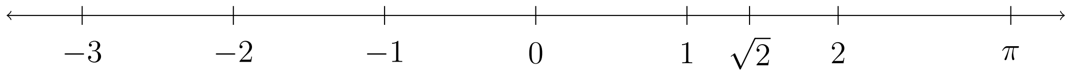
The Reals

Real Analysis is all about functions of the real numbers. What do we know about the real numbers?



There are no gaps, it is a continuum. Between any two real numbers, there is another real number. There is an order, given two real numbers, x and y, we can compare them:

$$x < y, \ x = y, \quad \text{or } x > y$$

We can ask many questions about the real numbers such as: What are the properties? What operations can we do? What are some important subsets? How many reals are there?

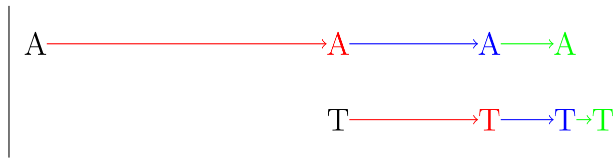
$$\mathbb{N} = \{1, 2, 3, \dots\} \ \text{(the naturals)}$$

$$\mathbb{R} \ \text{(the reals)}$$

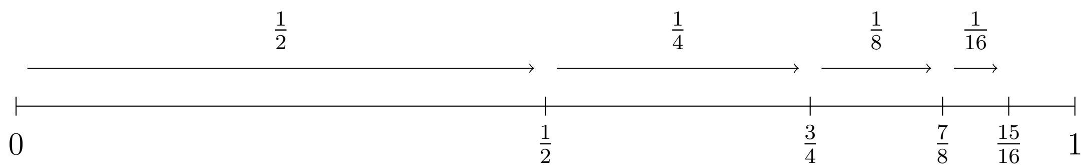
Both $\mathbb{N}$ and $\mathbb{R}$ are infinite, but we will see that they are not equivalent sets. This means that they do not have the same size, thus there are different infinities.

Zeno's Paradoxes

Achilles and the Tortoise: Imagine Achilles and a tortoise are having a race. Achilles is fast, so he gives the tortoise a head start. Once the race starts, Achilles runs to where the tortoise was but in this time the tortoise has moved forward. The tortoise is still ahead. Even though Achilles is faster, the tortoise is always ahead in the race.



Zeno claimed that motion is not possible because he claimed that you can only take finitely many steps. However, you can add infinitely many numbers and have their sum be finite.



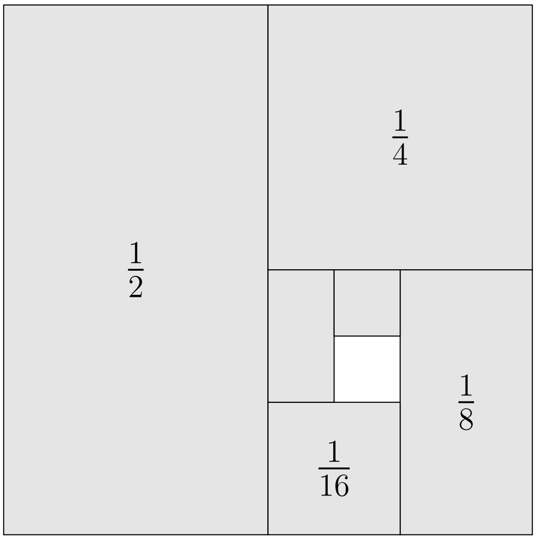
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

At a starting point, go halfway to the destination, then keep going halfway. Do we ever get there? Yes, but it will take infinitely many steps.

Geometric Series: Start with some constant a, then keep multiplying by some ratio r. Add all of the terms together.

$$a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \dots$$

In the above example, we have $a = \frac{1}{2}$ and $r = \frac{1}{2}$. When $|r| < 1$, this converges to $\frac{a}{1-r}$ (this will be proven later). Here, $\frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$



Basic Set Theory

Definition : Set

A set is an unordered collection of distinct objects which are called elements. A set must be well defined.

Definition : Well Defined

A set is well defined if when given an object, you can determine if the object is in the set or not.

Example

$A = \{1, 2, 3\}$, universe is $\mathbb{N}$. 2 is an element of A, $2 \in A$. 4 is not an element of A, $4 \notin A$. This is a set.

Example

B is the collection of books that are good. This is not a set since "good" is subjective.

Example

C is the collection of books written by Bertrand Russell. This is a set since given a book, we can see if it is written by Bertrand Russell.