Cantor Set

Start with the unit interval

$$C_0 = [0,1] = \{x \mid 0 \leq x \leq 1\}$$

Intuitive Idea: Remove the middle third and keep repeating this process.

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] = \left\{x \mid 0 \leq x \leq \frac{1}{3} \text{ or } \frac{2}{3} \leq x \leq 1\right\}$$

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

$$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right]$$

At each stage, the number of closed intervals doubles, but the lengths are scaled by $\frac{1}{3}$.

$$\mathbb{N} \text{ (the naturals) } = \{1, 2, 3, 4, \dots\}$$

For $n \in \mathbb{N}$, C_n has 2^n closed intervals, each of length $\left(\frac{1}{3}\right)^n$. Take the infinite intersection of these intervals, this is the Cantor Set.

$$\bigcap_{n=1}^{\infty} C_n = C$$

What elements are in C? Notice that the approximations are nested.

$$\dots C_3 \subseteq C_2 \subseteq C_1 \subseteq C_0$$

The endpoints of the closed intervals of all C_n will be in C.

$$C_0 = [0,1] \quad \{0,1\} \in C_0 \quad \{0,1\} \in C_n \text{ for all } n \in \mathbb{N} \text{ or } n = 0$$

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \text{ so } \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\} \in C_n \text{ for all } n \in \mathbb{N}, n \geq 1$$

We can keep going, C will have infinitely many elements. We will show that C has no length. This is why C has dimension between 0 and 1.