Following from the Axioms Continued

Example

The chess clubs of two schools consist of 8 and 9 players respectively. Four members from each club are randomly chosen to participate in a contest between the two schools. The chosen players from one team are then randomly paired with those from the other team, and each pairing plays a game of chess. Suppose that Rebecca and her sister Elise are on the chess clubs at different schools. What is the probability that:
(a) Rebecca and Elise will be paired?
(b) Rebecca and Elise will be chosen to represent their schools but you will not play each other?
(c) either Rebecca or Elise will be chosen to represent her school?

Suppose Rebecca is from the school with 8 players, and Elise is from the school with 9 players.

(a) The probability of Rebecca being chosen is: $\binom{7}{3}/\binom{8}{4} = \frac{1}{2}$. The probability of Elise being chosen is: $\binom{8}{3}/\binom{9}{4} = \frac{4}{9}$. There are 4! ways of choosing the teams, and Elise and Rebecca are paired in 6 of those. Thus, the probability of Rebecca and Elise being paired is $\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{4}{9} = \frac{1}{18}$
(b) We know that the probability of them each being chosen to represent their team is $\frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9}$. The probability of both being paired given that they are chosen is $\frac{1}{4}$, so the probability of them not being paired given that they are chosen is $\frac{3}{4}$. Thus, The probability that they are chosen but not paired is $\frac{2}{9} \cdot \frac{3}{4} = \frac{1}{6}$
(c) It is easier to think the complement. The complement of either being chosen is that neither are chosen. In the school with 8 people, if Rebecca is not chosen, they are $\binom{7}{4}$ ways to choose the remaining players. In the school with 9 people, if Elise is not chosen, there are $\binom{8}{4}$ ways to choose the remaining players. The probability that neither is chosen is:

$$\frac{\binom{7}{4}}{\binom{8}{4}} \cdot \frac{\binom{8}{4}}{\binom{9}{4}} = \frac{5}{18}$$

Thus, the probability of either being chosen is $1 - \frac{5}{18} = \frac{13}{18}$

Conditional Probability

Definition : Conditional Probability

Let E and F be events in a sample space Ω. The conditional probability that E occurs, given that F has occurred is denote by $\Pr(E \mid F)$. If $\Pr(F) > 0$, then

$$\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$

Remark

$$\Pr(S^c \mid F) = 1 - \Pr(S \mid F)$$

Proof.

$$\begin{aligned} F &= F \cap \Omega \\ &= F \cap (S \cup S^c) \\ &= (F \cap S) \cup (F \cap S^c) \end{aligned}$$

$$\begin{aligned} \Pr(F) &= \Pr(F \cap S) \cup (F \cap S^c) \\ &= \Pr(F \cap S) + \Pr(F \cap S^c) - \Pr((F \cap S) \cap (F \cap S^c)) \\ &= \Pr(F \cap S) + \Pr(F \cap S^c) \\ &= \Pr(S \mid F) \cdot \Pr(F) + \Pr(S^c \mid F) \cdot \Pr(F) \\ 1 &= \Pr(S \mid F) + \Pr(S^c \mid F) \\ \Pr(S^c \mid F) &= 1 - \Pr(S \mid F) \end{aligned}$$

□

Example

A math teacher gave their class two tests. 25% of the class passed both tests and 42% of the class passed the first test.
(a) What percent of those who passed the first test also passed the second test?
(b) What is the probability that a student does not pass the second test given that they passed the first test?

(a) Let F be the event that a student passed the first test, and S be the event that a student passed the second test.

$$\Pr(S \mid F) = \frac{\Pr(F \cap S)}{\Pr(F)} = \frac{0.25}{0.42} = 0.5952$$

(b) Let S^c be the event that the student does not pass the second test.

$$\Pr(S^c \mid F) = 1 - (S \mid F) = 1 - 0.5952 = 0.4048$$

By rearranging the conditional probability rule, we get the (binary) multiplication rules:

$$\Pr(E \cap F) = \Pr(E \mid F) \times \Pr(F) \quad \text{or} \quad \Pr(E \cap F) = \Pr(F \mid E) \times \Pr(E)$$

Example

The probability that I go to the Tall and Small in the morning is 0.80. When I do go to the Tall and Small, there is a 0.75 probability that I order a flat white. What is the probability that I both go to Tall and Small and order a flat white?

Let T be the event that I go to the Tall and Small and let F be the event that I order a flat white.

$$\Pr(T \cap F) = \Pr(F \mid T) \times \Pr(T) = 0.75 \times 0.80 = 0.60$$

We can generalize to more than two events in the multiplication rule by chaining:

$$\begin{aligned}&\Pr(E_1 \cap E_2 \cap \cdots \cap E_n) \\ &= \Pr(E_1) \times \Pr(E_2 \mid E_1) \times \Pr(E_3 \mid E_1 \cap E_2) \times \cdots \times \Pr(E_n \mid E_1 \cap E_2 \cap \cdots \cap E_{n-1})\end{aligned}$$

Example

A recent college graduate is planning to take the first three actuarial examinations. They will take the first actuarial exam in June. If they pass that exam, then they will take the second exam in July, and if they also pass that one, then they will take the third exam in September. If they fail an exam, then they are not allowed to take any others. The probability that they pass the first exam is 0.9. If they pass the first exam, then the conditional probability that they pass the second one is 0.8. If they pass both the first and second exams, then the conditional probability that they pass the third exam is 0.7.
(a) What is the probability that they pass all three exams?
(b) Given that they did not pass all three exams, what is the conditional probability that they failed the second exam?

(a) Let E_1 be the event that they pass the first exam, E_2 be the event that they pass the second exam, and E_3 be the event that they pass the third exam.

$$\begin{aligned}\Pr(E_1 \cap E_2 \cap E_3) &= \Pr(E_1) \times \Pr(E_2 \mid E_1) \times \Pr(E_3 \mid E_1 \cap E_2) \\ &= 0.9 \times 0.8 \times 0.7 \\ &= 0.504\end{aligned}$$

(b) We are looking for $\Pr\left(E_2^c \mid (E_1 \cap E_2 \cap E_3)^c\right)$.

$$(E_1 \cap E_2 \cap E_3)^c = 1 - (E_1 \cap E_2 \cap E_3) = 1 - 0.504 = 0.496$$

$$\begin{aligned}\Pr\left(E_2^c \mid (E_1 \cap E_2 \cap E_3)^c\right) &= \frac{\Pr\left(E_2^c \cap (E_1 \cap E_2 \cap E_3)^c\right)}{\Pr\left((E_1 \cap E_2 \cap E_3)^c\right)} \\ &= \frac{\Pr(E_2^c \cap E_1)}{\Pr\left((E_1 \cap E_2 \cap E_3)^c\right)} \\ &= \frac{\Pr(E_2^c \mid E_1) \times \Pr(E_1)}{\Pr\left((E_1 \cap E_2 \cap E_3)^c\right)} \\ &= \frac{(1 - \Pr(E_2 \mid E_1)) \times \Pr(E_1)}{\Pr\left((E_1 \cap E_2 \cap E_3)^c\right)} \\ &= \frac{(1 - 0.8) \times 0.9}{0.496} \\ &\approx 0.3629\end{aligned}$$