Supremums and Infimums

Example

Consider $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$. Is A bounded above? Is A bounded below?

A is bounded above, for example 2 is an upper bound since $\frac{1}{n} \leq n \;\forall\, n \in \mathbb{N}$.

$\sup(A) = 1$ since 1 is an upper bound, and we can't go any smaller since $1 \in A$.

A is bounded below, for example -1 is a lower bound since $\frac{1}{n} > 0 \;\forall\, n \in \mathbb{N}$.

$\inf(A) = 0$ since we can get arbitrarily close to 0 (we will need the Archimedean Principle to prove this)

Definition : Least Upper Bound Property

Let S be an ordered field, then S has the least upper bound property if given any nonempty set, $A \subseteq S$ where A is bounded above, A has a least upper bound in S. That is, $\sup(A) \in S$ exists.

A set that has the least upper bound property is complete.

Example

$\mathbb{Q}$ is an ordered field, but show that it is not complete.

Consider $A = \{x \in \mathbb{Q} : x^2 < 2\}$. A is all the rational numbers between $-\sqrt{2}$ and $\sqrt{2}$. This is bounded above, but there is no rational least upper bound.

Note: There is a least upper bound in $\mathbb{R}$ ($\sqrt{2}$)

Why isn't there a greatest lower bound property?

Suppose A is a nonempty subset of a complete set S, and A is bounded below.

Let $b \in S$ be a lower bound for A

$$b \leq x \;\forall\, x \in A$$

Consider the subset $-A = \{-x : x \in A\}$. We can show that this subset is bounded above, so by completeness, $\sup(-A)$ exists. Then, we can show that $\inf(A) = -\sup(-A)$.

Theorem : Existence and Uniqueness of $\mathbb{R}$

There exists a unique complete ordered field, we call this field the real numbers.

We could relabel the elements, but the field would be isomorphic. The construction uses Dedekind cuts.

Proposition

If $\sup(A)$ or $\inf(A)$ of some $A \subseteq \mathbb{R}$ exists, it is unique.

Proof. Assume α, β are least upper bounds for A.

α is an upper bound and β is a least upper bound, so $\beta \leq \alpha$.

β is an upper bound and α is a least upper bound, so $\alpha \leq \beta$.

Thus, $\alpha = \beta$.

Therefore, if $\sup(A)$ exists, it is unique. Similarly, if $\inf(A)$ exists, it is unique. □

Example

Let $A = \{0.9, 0.99, 0.999, \ldots\}$. Find the supremum if it exists.

A has upper bounds, and $\mathbb{R}$ is complete, so $\sup(A)$ exists.

Claim: $\sup(A) = 1$

Idea: Any number even a little bit smaller than 1 can't be an upper bound.

$$1 - 0.1 = 0.9 \qquad \text{No, since } 0.99 \in A$$

$$1 - 0.0001 = 0.9999 \qquad \text{No, since } 0.99999 \in A$$

$$1 - 0.0000005 = 0.9999995 \qquad \text{No, since } 0.9999999 \in A$$

We want to formalize this: Use $\varepsilon > 0$ for "a little bit".

Theorem

Let $A \subseteq \mathbb{R}$, $\sup(A) = \alpha$ if and only if:

(i) α is an upper bound of A

(ii) Given any $\varepsilon > 0$, $\sup(A) - \varepsilon = \alpha - \varepsilon$ is not an upper bound of A. That is, there is some $x \in A$ such that $\alpha - \varepsilon < x$.

Likewise, $\inf(A) = \beta$ if and only if:

(i) β is a lower bound of A

(ii) Given any $\varepsilon > 0$, $\inf(A) + \varepsilon = \beta + \varepsilon$ is not a lower bound of A. That is, there is some $x \in A$ such that $x < \beta + \varepsilon$.