Bifurcations

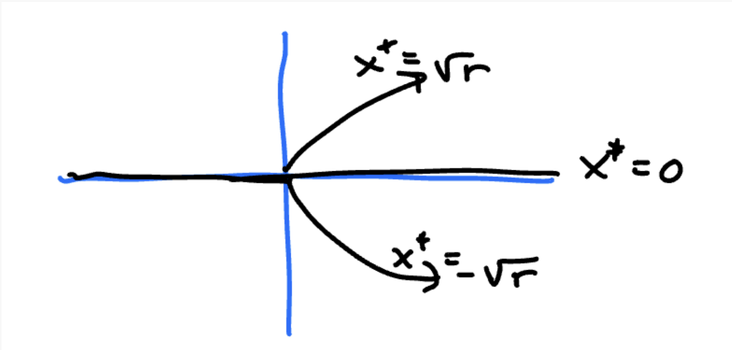
Example

Plot the bifurcation diagram for $\dot{x} = rx - x^3$.

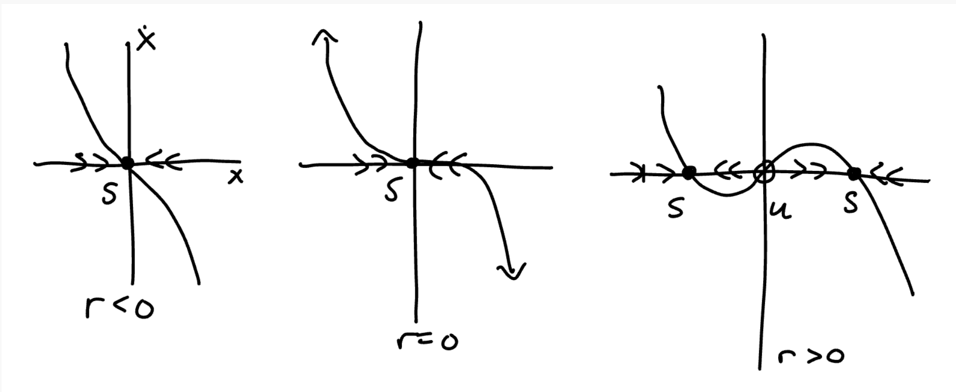
Solve for the equilibria:

$$rx - x^3 = 0$$
$$x\left(r - x^2\right) = 0$$

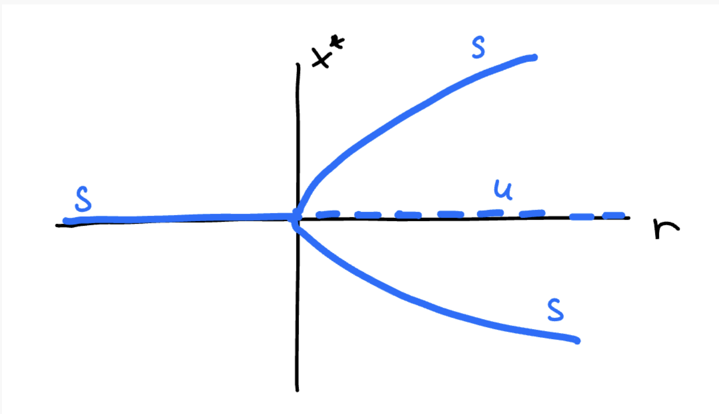
$$x^\star = 0, x^\star = \sqrt{r}, x^\star = \sqrt{-r}$$



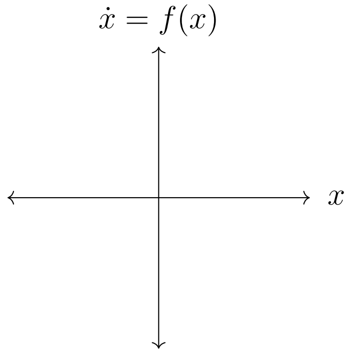
Stability: Notice a change at $r = 0$



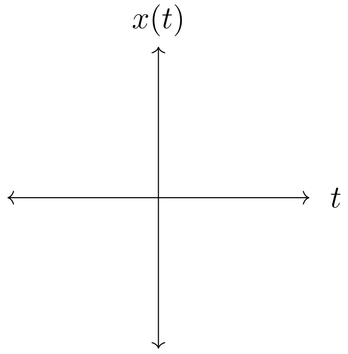
Finally, combine the curves with stability



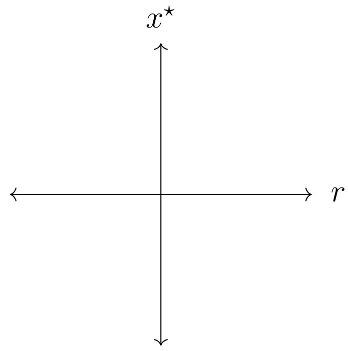
So far, we have lots of graphs. Keep an eye on the axes.



Phase Portrait



Trajectories



Bifurcation Diagram / Plot

Nondimensionalization

Think back to the logistic model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$$

Can we obtain an equivalent system with fewer equations? Yes, its all about units / dimension. We could measure population and time in many different ways.

Idea: Let's rescale P and t to convenient expressions, exploiting this freedom.

1. What are my independent and dependent variables?
   - t: independent
   - P: dependent
2. Redefine each variable as a unitless scaled version, let the scaling term contain units.

   $$t = \hat{T}\tau$$

   $$P = \hat{P}x$$

   Here, $t, \hat{T}$ have units of time, $P, \hat{P}$ have units of population, and τ, x are unit free.

   Idea: $(P, t) \to (x, \tau)$
3. Write the DE in terms of new scalings:

   $$\frac{dP}{dt} = \frac{dP}{d\tau} \cdot \frac{d\tau}{dt} \qquad t = \hat{T}\tau \Rightarrow \tau = \frac{1}{\hat{T}}t \Rightarrow \frac{d\tau}{dt} = \frac{1}{\hat{T}}$$

   $$\frac{dP}{dt} = \frac{dP}{d\tau} \cdot \frac{1}{\hat{T}}$$

   $$\frac{dP}{d\tau} = \frac{dP}{dx} \cdot \frac{dx}{d\tau} \qquad P = \hat{P}x \Rightarrow \frac{dP}{dx} = \hat{P}$$

   Plug it all in:

   $$\frac{dP}{dt} = \frac{\hat{P}}{\hat{T}}\frac{dx}{d\tau} \equiv \frac{\hat{P}}{\hat{T}}\dot{x}$$

Plug in the rest of the DE:

$$\frac{\hat{P}}{\hat{T}}\dot{x} = r\hat{P}x\left(1 - \frac{\hat{P}x}{k}\right)$$

The original DE has units of population / time, and so does the rescaled DE. This means k has units of population, and r has units of $\frac{1}{\text{time}}$.

4. Divide by parameters with dimension population / time.

   We know that rk has units of population / time.

$$\frac{\hat{T}}{rk\hat{T}}\dot{x} = \frac{r\hat{P}}{rk}x\left(1 - \frac{\hat{P}}{k}x\right)$$

   Note that the coefficients in front of each x, and the $\dot{x}$ are dimensionless groups. Now, choose $\hat{P}$ and $\hat{T}$ to keep it simple.

$$\begin{aligned}
\hat{P} = k &\Rightarrow \frac{k}{rk\hat{T}}\dot{x} = \frac{k}{k}x\left(1 - \frac{k}{k}x\right) \\
&\Rightarrow \frac{1}{r\hat{T}}\dot{x} = x\left(1 - x\right) \\
\hat{T} = \frac{1}{r} &\Rightarrow \frac{1}{r \cdot \frac{1}{r}}\dot{x} = x\left(1 - x\right) \\
&\Rightarrow \dot{x} = x\left(1 - x\right)
\end{aligned}$$

Now, x represents a fraction of carrying capacity, τ measure time scaled by $\frac{1}{r}$.

If $r = 0.4$ cats / year:

$$t = 1 \Rightarrow \tau = \frac{1}{\frac{1}{r}}t = 0.4\,(1)$$

Back to the original $(x, \tau) \to (P, t)$

$$t = \frac{1}{r}\tau, \quad P = kx$$