The Cantor Set

Recall that the Cantor set is the set of points obtained by repeatedly removing middle thirds from the interval $[0, 1]$. Another way to describe the Cantor Set is using Ternary (base 3). It is all the points in $[0, 1]$ that don't use the digit 1. It is possible for rational numbers to have 2 possible ternary representations. The numbers in the Cantor Set have a unique representation if we only use 0 and 2 however. So, the elements of the Cantor Set have a ternary representation

$$0.c_1c_2c_3\dots \qquad \text{where } c_i = 0 \text{ or } 1$$

We can map these to

$$0.b_1b_2b_3\dots \qquad \text{where } b_i = \begin{cases} 0 & \text{if } c_i = 0 \\ 1 & \text{if } c_i = 2 \end{cases}$$

All binary numbers give all the elements of $[0, 1]$. Thus, the Cantor Set is uncountable.

From a topology point of view, $[0, 1]$ and the Cantor Set are very different. $[0, 1]$ is dense in itself (if $x, y \in [0, 1]$, there exists $z \in [0, 1]$ such that $x < z < y$). We we find $x, y \in$ the Cantor Set with no element in the Cantor Set between them? The answer is yes, take $x = \frac{1}{3}$ and $y = \frac{2}{3}$, thus the Cantor Set is not dense in itself.

Sequences

Some examples of sequences are:

1. $1, 1, 2, 3, 5, 8, \dots$
   - Fibonacci Numbers
   - $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ (for $n \geq 3$)
2. $1, 3, 6, 10, 15, \dots$
   - Triangular Numbers
   - $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
3. $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots$
   - $a_n = \frac{1}{n}$
4. $1, 7, 19, 37, 61, \dots$
   - Hexagonal Numbers

Definition : Sequences

A sequence of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$. We usually write a_n instead of $a(n)$. The notation is:

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or} \qquad \{a_n\} \qquad \text{or} \qquad (a_1, a_2, \dots)$$

Sometimes, we have a formula, ex. $a_n = \frac{1}{n}$ or $a_n = n^2$. Sometimes the function is recursive, ex. $a_n = a_{n-1} + a_{n-2}$.

Bounded Sequences

Definition : Bounded Sequences

$\{a_n\}$ is bounded if the range $\{a_n : a \in \mathbb{N}\}$ is bounded. This means there exists $L, U \in \mathbb{R}$ such that

$$L \leq a_n \leq U \qquad \text{for all } n \in \mathbb{N}$$

Example

Is the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$ bounded?

Yes, this sequence is bounded since $0 \leq \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$.

Proposition

$\{a_n\}$ is bounded if and only if there exists a $C \in \mathbb{R}$ with $|a_n| \leq C$ for all $n \in \mathbb{N}$.

Proof. Suppose C exists, so $|a_n| \leq C$ for all $n \in \mathbb{N}$. This means:

$$-C \leq a_n \leq C \qquad \text{for all } n \in \mathbb{N}$$

Let $U = C$, $L = -C$. Then $\{a_n\}$ is bounded.

Suppose $\{a_n\}$ is bounded, this means there exists $L, U \in \mathbb{R}$ such that $L \leq a_n \leq U$ for all n. We don't know where 0 lies relative to L and U however.

$$\begin{aligned}
0 \leq L \leq U \quad &\Rightarrow \quad \text{let } C = U \\
L \leq 0 \leq U \quad &\Rightarrow \quad \text{let } C = \max\{|L|, U\} \\
L \leq U \leq 0 \quad &\Rightarrow \quad \text{let } C = |L|
\end{aligned}$$

Let $C = \max\{|L|, U\}$, this covers all cases. So $|a_n| \leq C$. $\square$