Bijections

Consider the function $f(x) = \tan(x)$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. y could be any value of $\mathbb{R}$. We also have that $f(x)$ is a bijection. So, this function is a bijection between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to $\mathbb{R}$.

How would you show that $(0,1)$ and $\mathbb{R}$ are equivalent?

First, go from $(0,1)$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then go to $\mathbb{R}$. $g(0) = -\frac{\pi}{2}$, $g(1) = \frac{\pi}{2}$, so $g(x) = \pi x - \frac{\pi}{2}$. The composition of bijections is a bijection so, $f \circ g$ is:

$$h(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$$

is a bijection from $(0,1)$ to $\mathbb{R}$.

How would we show that $[0,1]$ and $(0,1)$ are the same size?

Most elements of $(0,1)$ are matched with themselves. We want a countable subset of $(0,1)$ and map 0 to the first element, 1 to the second, and a_1 to a_3 (shifting everything over).

$$A = \{a_1, a_2, a_3, \dots\}$$

Let $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{8}$, $a_n = \left(\frac{1}{n}\right)^n$. So, our function $f : [0,1] \to (0,1)$

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{4} & \text{if } x = 1 \\ \left(\frac{1}{2}\right)^{n+2} & \text{if } x = \left(\frac{1}{2}\right)^n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

How Many Infinities Are There?

Theorem

If A is a set and $\mathcal{P}(A)$ is the power set of A, then $|A| < |\mathcal{P}(A)|$

Proof. Assume $|A| \geq |\mathcal{P}(A)|$, so there is a surjection from A to $\mathcal{P}(A)$. We know that elements of $\mathcal{P}(A)$ are subsets of A. $f : A \to \mathcal{P}(A)$ so for every $T \in \mathcal{P}(A)$, $\exists t \in A$ s.t. $f(t) = T$. $T \in \mathcal{P}(A)$ means $T \subseteq A$. We will construct a subset $B \subseteq A$ that is not in the image.

$$B = \{a \in A : a \neq f(a)\}$$

Claim : There is no $b \in A$ where $f(b) = B$. If there is some $b \in A$ with $f(b) = B$, is $b \in B$?

- If $b \in B$, then $b \in f(b) \to$ Contradiction
- If $b \notin B$, then $b \in B \to$ Contradiction

Thus, there is no $b \in A$ with $f(b) = B$. f is not onto, this contradicts our assumption that f is onto. This means we cannot find a surjection from A to $\mathcal{P}(A)$. Hence, $|A| < |\mathcal{P}(A)|$. $\square$

In fact,

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

So, we can get infinitely many infinities by taking power sets over and over. We had that $|\mathbb{N}| = \aleph_0$, and now $|\mathcal{P}(\mathbb{N})| = \aleph_1$. But now, where does $|\mathbb{R}|$ fit?

Claim: $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$

Proof. We can represent every real in $[0, 1]$ uniquely in terms of binary numbers. A real in binary is of the form:

$$0.b_1 b_2 b_3 \dots \qquad \text{where } b_i = 0 \text{ or } 1 \text{ for all } i$$

If we convert this number to decimal we get:

$$b_1\left(\frac{1}{2}\right) + b_2\left(\frac{1}{2}\right)^2 + b^3\left(\frac{1}{2}\right)^3 + \cdots$$

For example, $0_{10} = 0.000\dots_2$ and $1_{10} = 0.111\dots_2$. Our goal is to find a bijection from $\mathcal{P}(\mathbb{N})$ to $[0, 1]$. We know that given $A \subseteq \mathbb{N}$, each natural number is either in A or not in A. So, $f : \mathcal{P}(\mathbb{N}) \to [0, 1]$ is:

$$f(A) = 0.a_1 a_2 a_3 \dots \text{ where } a_i = \begin{cases} 0 & \text{if } i \notin A \\ 1 & \text{if } i \in A \end{cases}$$

For example, $f(\emptyset) = 0.000\dots_2 = 0_{10}$, $f(\mathbb{N}) = 0.111\dots_2 = 1_{10}$, $f(\{2, 4, 6, \dots\}) = 0.010101\dots_2 = \frac{1}{3}_{10}$.

This function is a bijection, so $|\mathcal{P}(\mathbb{N})| = |[0, 1]| = |\mathbb{R}|$ $\square$