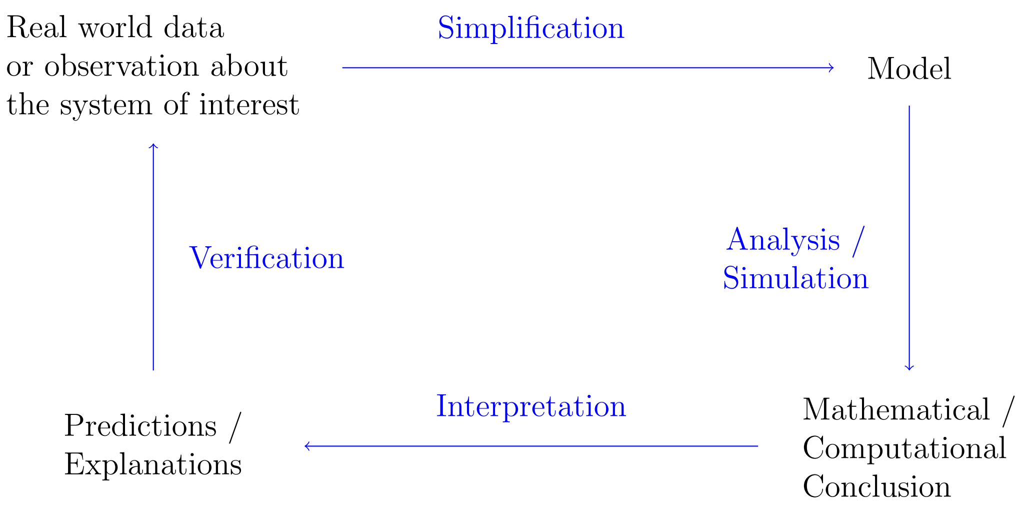
Modeling Intro Continued

When studying dynamical systems there are two main types:

- Discrete
  - Time is discretized into time steps, $t_0 \to t_1 \to t_2 \to \ldots$
  - Typically use difference equations
- Continuous
  - Time is continuous, no breaks
  - Typically use differential equations
  - We often simulate these with software, this leads to time becoming discretized with a small Δt jump. This puts us into the discrete category.

Modeling Process



We begin with the simplest model, and iterate through complexity until we get to a point where it fulfills our needs.

Math 106/126/107/127 Review

A derivative, $\frac{dy}{dx}$, is the instantaneous rate of change of y with respect to x. We will use different letters, $\frac{dx}{dt}$ is the rate of change of x with respect to t (time) $\Rightarrow x(t)$.

Example

Solve the following differential equation using separation of variables: $\frac{dx}{dt} = x(t)$.

$$\begin{aligned} \frac{dx}{dt} &= x(t) \\ \frac{1}{x}\,dx &= dt \\ \int \frac{1}{x}\,dx &= \int 1\,dt \\ \ln|x| &= t + \hat{c} \\ x &= e^{t+\hat{c}} \\ x &= Ce^t, \quad C = e^{\hat{c}} \end{aligned}$$

This is a solution to the differential equation.

If $\frac{dx}{dt} = kx$, k constant $\Rightarrow x(t) = Ce^{kt}$, C, k constant. In practice, these equations often have an initial condition like $x(0) = 1$, this specifies C. The above example is a statement that describes a population with no resource limitation that grows proportional to the size of the population. For example,

$$\frac{dP}{dt} = kP$$

- $\frac{dP}{dt}$ is the rate of change of the population
- $= k$ means "is proportional to"
- P is the size of the population
- The solution is $P = P_0e^{kt}$, where P_0 is the initial population
- This population model is known as the "law of natural growth"

We can explicitly solve some models, like the one above, but this is not the focus of this course. Here, we will qualitatively understand how solutions behave.

Notation

Derivative: $\frac{dx}{dt}$, $x'(t)$, $\dot{x}$

2nd Derivative: $\frac{d^2x}{dt^2}$, $x''(t)$, $\ddot{x}$

Let's focus on the simplest case of differential equations, first order systems. These are equations of an unknown function and it's first derivative only. The general form is $\dot{x} = f(x)$.

- These do not explicitly involve t, they are autonomous

- There are two types:
  - Linear
    - Simple dynamics
    - Easy to solve
  - Nonlinear
    - E.g., $\dot{x} = x^2$, $\dot{x} = \sin(x)$
    - More interesting dynamics
    - Harder to solve explicitly

If we try to solve $\dot{x} = \sin(x)$, $x(0) = x_0$, we find the solution to be:

$$t = \ln\left|\frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)}\right|$$

- This gives us limited value
- Tells us little about the behavior of the system
- What happens as $t \to \infty$? It is unclear
- This does not suit our purpose