Convergent Sequences

Definition : Convergent Sequence

(a_n) converges to $a \in \mathbb{R}$ if $\forall\, \varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $|a_n - a| < \varepsilon$ when $n > N$. a is called the limit of a_n.

$$\begin{aligned} |a_n - a| < \varepsilon &\Leftrightarrow -\varepsilon < a_n - a < \varepsilon \\ &\Leftrightarrow a - \varepsilon < a_n < a + \varepsilon \\ &\Leftrightarrow a \in (a - \varepsilon, a + \varepsilon) \end{aligned}$$

That is, for some $N \in \mathbb{R}$ and all $n > N$, we are far enough in the sequence to stay close within the open interval $(a - \varepsilon, a + \varepsilon)$,

Notation: (a_n) converges to a can be written as:

- $a_n \to a$ as $n \to \infty$
- $a_n \to a$
- $\lim_{n\to\infty} a_n = a$

Example

Show $\lim_{n\to\infty} \frac{1}{n} = 0$.

Let $\varepsilon > 0$,

$$|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n}$$

By the Archimedean Principle, $\exists N \in \mathbb{R}$ such that $\frac{1}{N} < \varepsilon$. For this N, if $n > N$, we have $\frac{1}{n} < \frac{1}{N} < \varepsilon$. Thus, $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example

Consider $a_n = \frac{3n}{n+1}$, Show $\lim_{n\to\infty} \frac{3n}{n+1} = 3$.

$$|a_n - a| = \left|\frac{3n}{n+1} - 3\right| = \left|\frac{3n - 3n - 3}{n+1}\right| = \frac{3}{n+1}$$

Let $\varepsilon > 0$, we want $\frac{3}{n+1} < \varepsilon$.

$$\frac{3}{n+1} < \varepsilon \Leftrightarrow \frac{3}{\varepsilon} < n + 1 \Leftrightarrow \frac{3}{\varepsilon} - 1 < n$$

Let $N = \frac{3}{\varepsilon} - 1$, then $|a_n - a| = \left|\frac{3n}{n+1} - 3\right| = \frac{3}{n+1} < \varepsilon$ for $n > N = \frac{3}{\varepsilon} - 1$. Thus, $\lim_{n\to\infty} \frac{3n}{n+1} = 3$

ε-Neighbourhoods

The ε of a point is everything within a distance of ε. In $\mathbb{R}$, we use the distance between a_n and a given by $|a_n - a|$. So, the ε-neighbourhood is $(a - \varepsilon, a + \varepsilon)$.

This generalizes based on the space we are in and which distance we use.

In $\mathbb{R}^2$, $a_n = (x_n, y_n)$, $a = (x, y)$. The Euclidean distance is $\sqrt{(x_n - x)^2 + (y_n - y)^2}$. The ε-neighbourhood is:

$$\sqrt{(x_n - x)^2 + (y_n - y)^2} < \varepsilon$$
$$(x_n - x)^2 + (y_n - y)^2 < \varepsilon^2$$

This is the interior of a circle centered at (x, y) with radius $r = \varepsilon$.

Divergent Sequences

Definition : Divergent Sequence

If a sequence doesn't converge, then it diverges. There are 3 forms of this:

1. (a_n) diverges to $+\infty$

   $\lim_{n\to\infty} a_n = +\infty$ if for all $M \in \mathbb{R}$ there is some $N \in \mathbb{R}$ such that $a_n > M$ for $n > N$. (Eventually the sequence stays bigger than M).

2. (a_n) diverges to $-\infty$

   $\lim_{n\to\infty} a_n = -\infty$ if for all $M \in \mathbb{R}$ there is some $N \in \mathbb{R}$ such that $a_n < M$ for $n > N$. (Eventually sequence stays smaller than M).

3. Limit doesn't exist

   We need to negate the definition: a_n does not converge to a if there exists $\varepsilon > 0$ such that for all $N \in \mathbb{R}$.

   $$|a_n - a| \geq \varepsilon \qquad \text{for some } n > N$$

Example

Consider $a_n = n^2$, show this sequence diverges.

Let $M \in \mathbb{R}$, if $M \leq 0$, let $N = 0$. Then for all $n > 0, n^2 > 0$. Assume $M > 0$, need $N \in \mathbb{R}$ such that $a_n > M$ for $n > N$, $a_n > M \Leftrightarrow n^2 > M \Leftrightarrow n > \sqrt{M}$. Let $N = \sqrt{M}$. Then, for $n > N$, we have $n^2 > N^2 = M$. Thus, $lim_{n\to\infty} n^2 = \infty$.