Linear Stability Analysis

The question is, do small perturbations from equilibrium grow (unstable) or shrink (stable)?

Let x^s be an equilibrium point, so $f(x^s) = 0$, for $\dot{x} = f(x)$. Define $\eta(t) = x(t) - x^s$ as a small perturbation. So,

$$x = \eta + x^s, \quad \text{and } \dot{x} = \dot{\eta} + \dot{x}^s = \dot{\eta}$$

Plug into DE:

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{\eta} &= f(\eta + x^s) \end{aligned}$$

Now, do a Taylor Expansion in $\eta + x^s$ about $\eta + x^s = x^s$.

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \dots \\ f(\eta + x^s) &= f(x^s) + f'(x^s)(\eta + x^s - x^s) + O(\eta^2) \\ &= f'(x^s)\eta + O(\eta^2) \end{aligned}$$

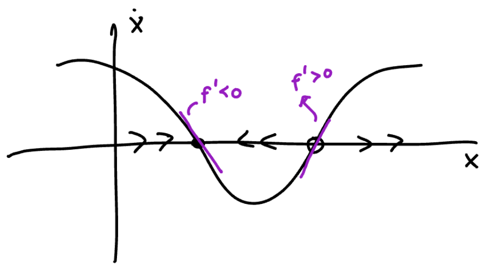
$f(x^s) = 0$, since x^s is the equilibrium. $f'(x^s)\eta$ is a linearized version of f which we will plug into the DE. $O(\eta^2)$ means the higher order terms, which we will ignore.

$$\dot{\eta} = f'(x^s)\eta, \quad \text{for perturbation } \eta$$

The solution of this DE is:

$$\eta(t) = Ce^{f'(x^s)t}$$

So, η grows if $f'(x^s) > 0$, and shrinks if $f'(x^s) < 0$.



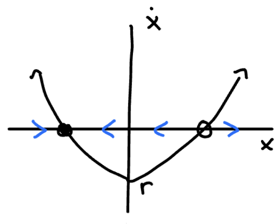
Check equilibrium stability by:

1. Get equilibrium: $x^\star$
2. Compute $f'(x)$ (derivative of RHS of DE)
3. Plug in $x^\star$ to get $f(x^\star)$
4. Check if positive (unstable) or negative (stable)

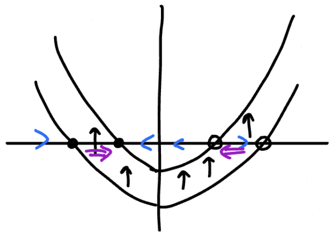
Bifurcation Analysis

Consider $\dot{x} = r + x^2$, where $r \in \mathbb{R}$. Call r a parameter.

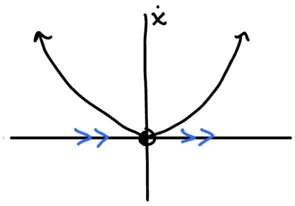
Start by imagining that $r < 0$:



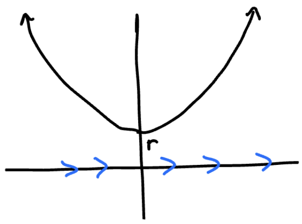
Start increasing r:



As $r \to 0^-$, equilibria $\to 0$, then when $r = 0$ (we call this point half-stable, but consider it unstable)



Now continue, $r > 0$



By changing the parameter r, equilibria can move and even disappear. When changing a parameter leads to a qualitative difference (change in the number or equilibria, or a change in the stability of equilibria) in the system, we call that a bifurcation.

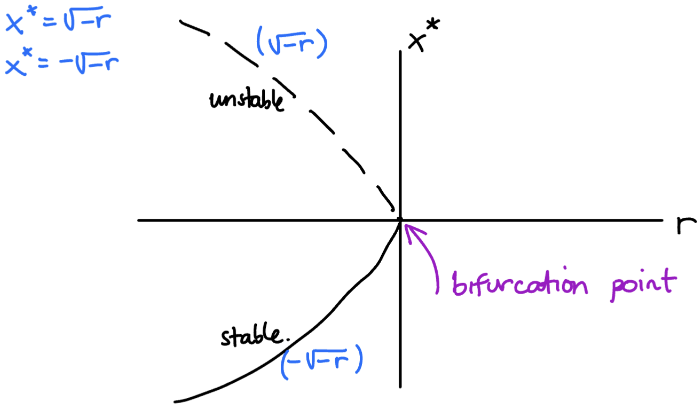
We can summarize response of the system to changes in a parameter with a bifurcation diagram.

Our example: $\dot{x} = r + x^2$

1. Get equilibria: solve $\dot{x} = 0$

$$\begin{aligned} r + x^2 &= 0 \\ x^2 &= -r \\ x^\star &= \pm\sqrt{-r} \end{aligned}$$

2. Decide, using phase portraits, for $x^\star$ expressions, when do changes occur?
    - $r < 0$: 2 equilibria (stable, then unstable)
    - $r = 0$: 1 equilibrium (unstable)
    - $r > 0$: 0 equilibria
3. Plot $x^\star(r)$ curves, indicating stability (solid line for stable, dotted line for unstable)



This is known as a bifurcation diagram