Conditional Probability Continued

Note that $E = (E \cap F^c) \cup (E \cap F)$. Then,

$$\Pr(E) = \Pr(E \cap F) + \Pr(E \cap F^c) = \Pr(E \mid F)\Pr(F) + (E \mid F^c)\Pr(F^c)$$

That is, $\Pr(E)$ is a weighted average of the conditional probabilities of E when F has occurred, and E when F has not occurred.

Example

You have two urns, urn 1 contains 2 white and 4 red balls, whereas urn 2 contains 1 white and 1 red ball. A ball is randomly chosen from urn 1 and put into urn 1, and a ball is then randomly selected from urn 2. What is the probability that the ball selected from urn 2 is white?

Let W_1 be the event that the ball selected from urn 1 (and then put into urn 2) is white, and let W_2 be the event that the ball selected from urn 2 is white.

$$\begin{aligned}
W_2 &= W_2 \cap \Omega \\
&= W_2 \cap \left(W_1 \cup \overline{W_1}\right) \\
&= (W_2 \cap W_1) \cup \left(W_2 \cap \overline{W_1}\right) \\
\Pr(W_2) &= \Pr(W_2 \cap W_1) + \Pr\left(W_2 \cap \overline{W_1}\right) \\
&= \Pr(W_2 \mid W_1) \cdot \Pr(W_1) + \Pr\left(W_2 \mid \overline{W_1}\right) \cdot \Pr\left(\overline{W_1}\right) \\
&= \left(\frac{2}{3}\right) \cdot \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right) \\
&= \frac{4}{9}
\end{aligned}$$

We can generalize the formula from $F \cup F^c = \Omega$ to any (finite) set of events $\{F_1, F_2, \ldots, F_n\}$ which partition Ω.

Definition : Rule of Total Probability

If you have some finite set of events $\{F_1, F_2, \ldots, F_n\}$ which partition Ω. That is, $F_i \cap F_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=1}^{n} F_i = \Omega$, then

$$\Pr(E) = \sum_{i=1}^{n} \Pr(E \cap F_i) = \sum_{i=1}^{n} \Pr(E \mid F_i)\Pr(F_i)$$

Example

A family has j children with probability p_j, where $p_1 = 0.10$, $p_2 = 0.25$, $p_3 = 0.35$, and $p_4 = 0.30$. A child from this family is randomly chosen. What is the probability that this child is the eldest child in the family?

Let F_j be the event that the family has j children, and E be the event that the randomly chosen child is the eldest.

If a family has j children, then the probability of choosing the eldest child is $\frac{1}{j}$ since each child has an equal chance of being selected and there is only 1 eldest child.

$$\begin{aligned}
\Pr(E) &= \sum_{i=1}^{4} \Pr(E \mid F_j) \cdot \Pr(F_j) \\
&= \Pr(E \mid F_1) \cdot \Pr(F_1) + \Pr(E \mid F_2) \cdot \Pr(F_2) + \Pr(E \mid F_3) \cdot \Pr(F_3) + \\
&\quad \Pr(E \mid F_4) \cdot \Pr(F_4) \\
&= \frac{1}{1}p_1 + \frac{1}{2}p_2 + \frac{1}{3}p_3 + \frac{1}{4}p_4 \\
&= 0.10 + \frac{1}{2}(0.25) + \frac{1}{3}(0.35) + \frac{1}{4}(0.30) \\
&= 0.41\overline{66} = \frac{5}{12}
\end{aligned}$$

Theorem : Baye's Theorem

If $\{F_1, F_2, \ldots, F_n\}$ partitions Ω, then

$$\begin{aligned}
\Pr(F_i \mid E) &= \frac{\Pr(F_i \cap E)}{\Pr(E)} \\
&= \frac{\Pr(E \mid F_i) \cdot \Pr(F_i)}{\sum_{j=1}^{n} \Pr(E \mid F_j) \cdot \Pr(F_j)} \\
&= \frac{\Pr(E \mid F_i) \cdot \Pr(F_i)}{\Pr(E \mid F_1) \cdot \Pr(F_1) + \cdots + \Pr(E \mid F_n) \cdot \Pr(F_n)}
\end{aligned}$$

Example

A family has j children with probability p_j, where $p_1 = 0.10$, $p_2 = 0.25$, $p_3 = 0.35$, and $p_4 = 0.30$. A child from this family is randomly chosen. Given that this child is the eldest child in the family, find the conditional probability that the family has only 1 child and find the conditional probability that the family has 4 children.

Let F_j be the event that the family has j children, and E be the event that the randomly chosen child is the eldest. From Baye's theorem we have:

$$\Pr(F_1 \mid E) = \frac{\Pr(E \mid F_1) \cdot \Pr(F_1)}{\Pr(E)} = \frac{1 \cdot 0.10}{\frac{5}{12}} = 0.24$$

$$\Pr(F_4 \mid E) = \frac{\Pr(E \mid F_4) \cdot \Pr(F_4)}{\Pr(E)} = \frac{\left(\frac{1}{4}\right)(0.30)}{\frac{5}{12}} = 0.18$$

Independence

Definition

Let E and F be events in a sample space Ω. E and F are independent if knowledge of one does not give knowledge of the other. That is,

$$\Pr(E \mid F) = \Pr(E) \quad \text{or,} \quad \Pr(F \mid E) = \Pr(F)$$

Remark

Two events E and F are independent if and only if

$$\Pr(E \cap F) = \Pr(E) \times \Pr(F)$$

Remark

Three events, E, F, and G are independent if and only if all of following hold

(i) $\Pr(E \cap F \cap G) = \Pr(E) \times \Pr(F) \times \Pr(G)$

(ii) $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$

(iii) $\Pr(E \cap G) = \Pr(E) \times \Pr(G)$

(iv) $\Pr(F \cap G) = \Pr(F) \times \Pr(G)$

Lemma

If a family of events are independent, then so are their complements. That is, if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

then

$$\Pr\left(\overline{A} \cap \overline{B}\right) = \Pr\left(\overline{A}\right) \cdot \Pr\left(\overline{B}\right)$$

Proof. Suppose $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$. We need to show $\Pr\left(\overline{A} \cap \overline{B}\right) = \Pr\left(\overline{A}\right) \cdot \Pr\left(\overline{B}\right)$.

$$\begin{aligned}
\Pr\left(\overline{A} \cap \overline{B}\right) &= 1 - \Pr(A \cup B) \\
&= 1 - \Pr(A) - \Pr(B) + \Pr(A) \cdot \Pr(B) \\
&= (1 - \Pr(A)) \cdot (1 - \Pr(B)) \\
&= \Pr\left(\overline{A}\right) \cdot \Pr\left(\overline{B}\right)
\end{aligned}$$

□

Example

A system with 4 parallel components functions when at least one of the components functions. Let A_i denote the event that the component i functions, and let p_i denote the probability that component i functions. Let $p_1 = 0.90$, $p_2 = 0.95$, $p_3 = 0.45$ and $p_4 = 0.14$. If the components operate independently, what is the probability that the system functions?

Let $\overline{A_i}$ be the event that the components fails

$$\begin{aligned}
\Pr\left(\overline{A_1}\right) &= 1 - \Pr(A_1) = 1 - 0.90 = 0.10 \\
\Pr\left(\overline{A_2}\right) &= 1 - \Pr(A_2) = 1 - 0.95 = 0.05 \\
\Pr\left(\overline{A_3}\right) &= 1 - \Pr(A_3) = 1 - 0.45 = 0.55 \\
\Pr\left(\overline{A_4}\right) &= 1 - \Pr(A_4) = 1 - 0.14 = 0.86
\end{aligned}$$

Now, the probability that the system fails is

$$\begin{aligned}
\Pr(\text{system fails}) &= \Pr\left(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}\right) \\
&= \Pr\left(\overline{A_1}\right) \cdot \Pr\left(\overline{A_2}\right) \cdot \Pr\left(\overline{A_3}\right) \cdot \Pr\left(\overline{A_4}\right) \\
&= (0.10) \cdot (0.05) \cdot (0.55) \cdot (0.86) \\
&= 0.002365
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Pr(\text{system functions}) &= 1 - \Pr(\text{system fails}) \\
&= 1 - 0.002365 \\
&= 0.997635
\end{aligned}$$

We need all the possible 2- and 3- way terms for three events to be independent. The example below shows that the 3-way condition can be held even when the pairwise conditions do not.

Example

Suppose that three coins are tossed and has the sample space

$$\Omega = \begin{Bmatrix} (H,H,H),\ (H,H,T),\ (H,T,H),\ (H,T,T), \\ (T,H,H),\ (T,H,T),\ (T,T,H),\ (T,T,T) \end{Bmatrix}$$

Let A be the event that the first coin is a head.

$$\Pr(A) = \frac{1}{2}$$

Let B be the event that the first two coins are heads, or the last two are tails.

$$\Pr(B) = \frac{1}{2}$$

Let C be the event that the last two coins are not the same.

$$\Pr(C) = \frac{1}{2}$$

$$\Pr(A \cap B \cap C) = \frac{1}{8} = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

$$\Pr(A \cap B) = \frac{3}{8} \neq \Pr(A) \cdot \Pr(B)$$