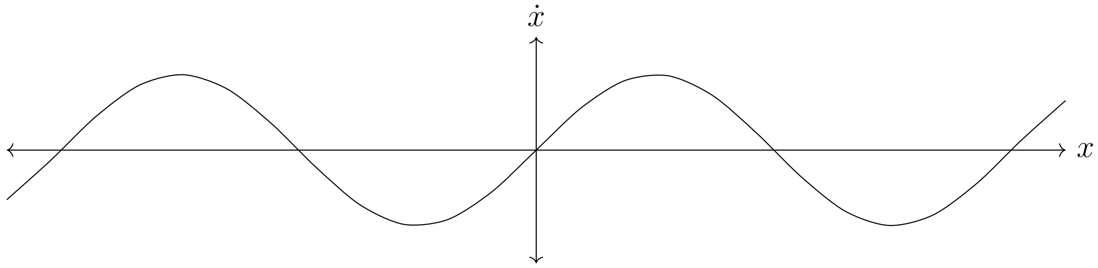
Phase Portraits & Equilibria

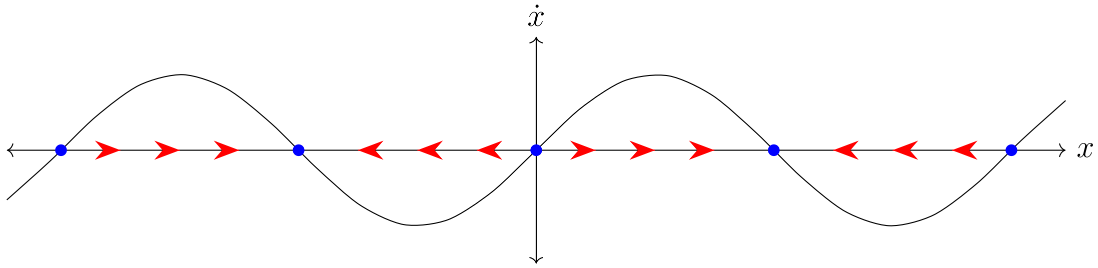
Let's return to $\dot{x} = \sin(x)$. Think of t as time, x as the position of a particle moving along the real line, and $\dot{x}$ as the particles velocity. Now, plot $\dot{x}$ vs x.



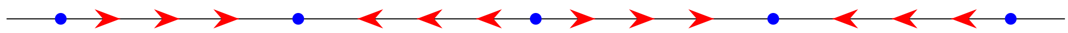
Guide: If $\dot{x}$ is positive: move forward, if $\dot{x}$ is negative, move backwards.

Idea: Draw arrows on the x axis that indicate the direction of motion of x.

The plot below is called a "phase portrait".



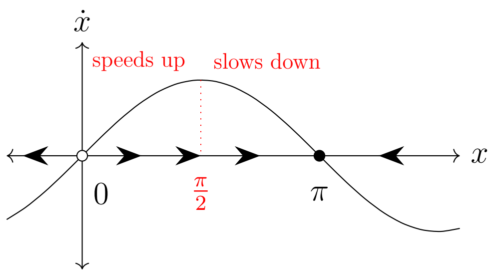
The horizontal axis part of the phase portrait (shown below) is called the vector field. The vector field characterizes the flow.



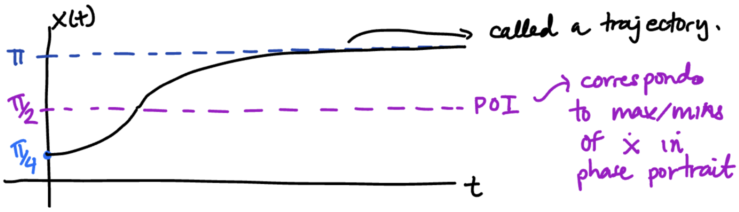
If $\dot{x} = 0$, (the x-axis intersections), then flow is 0. These points are called the equilibrium points / equilibria. Equilibria can be:

- Stable: Equilibria that have flow lines toward them, Notation: $\bullet$
- Unstable: Equilibria that have flow lines away from them, Notation: $\circ$

Let's focus on the simplest case: first order systems admit this approach easily. Next, we can plot curves, $x(t)$: trajectories. Take a subdomain of the plot above.

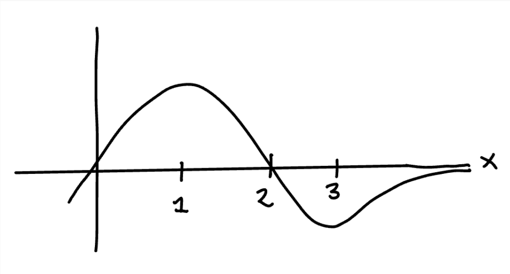


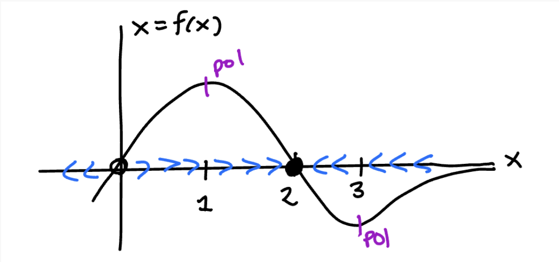
Try to plot $x(t)$ for $\dot{x} = \sin(x)$, $x_0 = x(0) = \frac{\pi}{4}$



Example

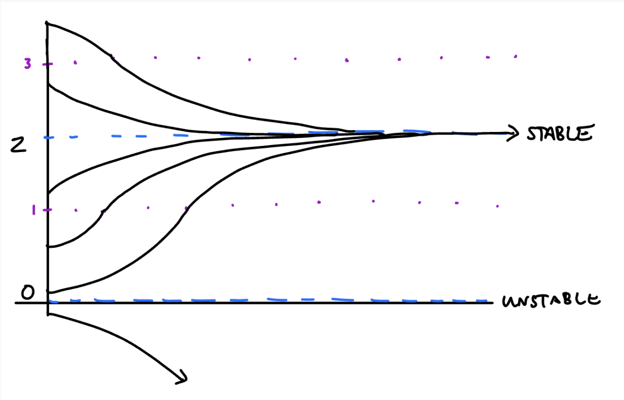
Given the follow plot of $\dot{x} = f(x)$, plot some trajectories.





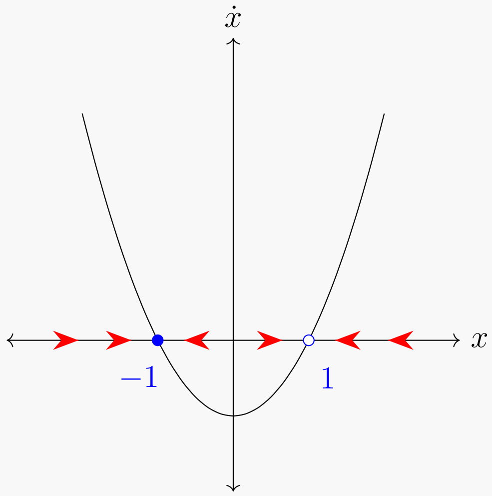
The equilibria are: $x = 0$ (unstable), and $x = 2$ (stable)

1. Plot dotted lines at equilibria
2. Indicate lines where POI occur
3. Draw enough trajectories to see all behaviours of solutions



Example

For $\dot{x} = x^2 - 1$, find equilibria, classify stability, draw phase portrait, and solution trajectories that show all types of solution behaviour.



The equilibria points are: $x = -1$ (stable), and $x = 1$ (unstable). There is a point of inflection at $x = 0$.