Functional Limits

Definition : Limit Point

Given a set $A \subseteq \mathbb{R}$, x is a limit point of A if there is a sequence of points in $A - \{x\}$ such that $x_n \to x$.

Example

Show that $x = 0$ is a limit point of $A = (0, 2)$.

First, we know $0 \notin A$. Consider the sequence $x_n = \frac{1}{n} \in A - \{0\} = A$.

Note: Any point in A is a limit of A when A is an interval.

Definition : Epsilon Neighbourhood

The ε-neighbourhood of x is $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$.

Proposition : Exercise 5.7

x is a limit point of A if and only if every ε-neighbourhood of x intersects A in some point other than x.

Example

Show that $x = 0$ is not a limit point of $A = \mathbb{Z}$.

Consider $\varepsilon = \frac{1}{2}$. The ε-neighbourhood of $x = 0$ is $\left(-\frac{1}{2}, \frac{1}{2}\right)$. This does not contain any integers other than 0. Thus, 0 is not a limit point of A, in fact, no elements of A are limit points.

If we have a function $y = f(x)$, and $c \in \mathbb{R}$ and we want to take $\lim_{x\to c} f(x)$, we think of values of x getting and closer and closer to c, but $x \neq c$.

Idea – c is a limit point of A if we can get closer and closer to c while staying in A

Definition : Functional Limit

Let $f : A \to \mathbb{R}$ and c is a limit point of A, then $\lim_{x\to c} f(x) = L$ if for all $\varepsilon > 0$, $\exists\, \delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

Example

Find $\lim_{x\to 1} f(x)$ using the definition for $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f(x) = \frac{x^2+x-2}{x-1}$.

From Calculus 1,

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1} (x+2) = 1+2 = 3$$

So, we will use $L = 3$. Let $\varepsilon > 0$

$$\begin{aligned} |f(x) - L| &= \left| \frac{x^2+x-2}{x-1} - 3 \right| = \left| \frac{x^2+x-2-3(x-1)}{x-1} \right| = \left| \frac{x^2+x-2-3x+3}{x-1} \right| \\ &= \left| \frac{x^2-2x+1}{x-1} \right| = \left| \frac{(x-1)^2}{x-1} \right| = |x-1| \end{aligned}$$

We need δ so that for $x \in A$ with $0 < |x-c| < \delta$ we have $|f(x) - L| < \varepsilon$. $c = 1$, so we need δ o that $0 < |x-1| < \delta$ guarantees $|x-1| < \varepsilon$. Let $\delta = \varepsilon$, then $|f(x) - 3| < \varepsilon$ when $x \in A$ and $0 < |x-1| < \delta = \varepsilon$.

Note: We are able to cancel $(x-1)$ since we know $x \neq 1$

Definition : Negation of Functional Limit

There exists $\varepsilon > 0$, such that for all $\delta > 0$ there is some $x \in A$ with $0 < |x-c| < \delta$ and $|f(x) - L| \geq \varepsilon$.

Example

Let $f(x) = \begin{cases} x^2 & \text{if } x \geq 2 \\ x & \text{if } x < 2 \end{cases}$. Show that $\lim_{x\to 2} f(x) \neq 2$.

Let $\varepsilon = 1$, s owe want $|f(x) - 2| \geq 1$, that is, $f(x) \notin (1,3)$. Let $\delta > 0$ be arbitrary. We want $x \in (2-\delta, \delta) \cup (\delta, 2+\delta)$ with $f(x) \notin (1,3)$. For $\delta > 0$, $2 + \frac{\delta}{2} \in (2, 2+\delta)$.

$$f\left(2 + \frac{\delta}{2}\right) = \left(2 + \frac{\delta}{2}\right)^2 = 4 + 2\delta + \frac{\delta^2}{4} \geq 4 \qquad \text{since } \delta > 0$$

Thus, $f\left(2 + \frac{\delta}{2}\right) \notin (1,3)$. Therefore, $\lim_{x\to 2} f(x) \neq 2$.

For the example above, if we wanted to show $\lim_{x\to 2} f(x) \neq 4$, we would take x on the left side. For any $L \in \mathbb{R}$, you could show $\lim_{x\to 2} f(x) \neq L$ by taking $\varepsilon = \frac{1}{2}$, there will be values of x on either on the right or left of $x = 2$ where $f(x)$ is outside of $(L-\varepsilon, L+\varepsilon)$.