Continuity

What does it mean for a function to be continuous?

- We can draw the function without lifting our pen or pencil

- The function has no holes or jumps

- $f(x)$ is continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$

- $f(x)$ is continuous at $x = c$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$

$$|x - c| < \delta \Leftrightarrow x \in (c - \delta, c + \delta)$$
$$|f(x) - f(c)| < \varepsilon \Leftrightarrow f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$$

Definition : Continuity at $x = c$

A function $f(x)$ is continuous at $x = c$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$

Weird Examples of Continuity

Example

Can a function be continuous at one point, but not at another?

Yes, consider any function with a jump at some point. For example,

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x + 4 & \text{if } x \geq 0 \end{cases},$$

is continuous everywhere except $x = 0$

Could a function be continuous nowhere? We know that an open interval of the reals (a, b) has infinitely many elements that can be rational or irrational. We also know that between any two rationals is an irrational, and between any two irrationals is a rational.

Definition : Negation of Continuity at $x = c$

A function $f(x)$ is not continuous at $x = c$ if there exists $\varepsilon > 0$ such that $|x - c| < \delta$ for some x, but $|f(x) - f(c)| \geq \varepsilon$

Definition : Dirichlet Function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Example

Show that the Dirichlet function is not continuous at $x = 0$

We just need $\varepsilon \leq 1$, so let $\varepsilon = \frac{1}{2}$. Consider $x \in (c - \delta, c + \delta) = (-\delta, \delta)$. This open interval contains irrationals, so let $x \in (-\delta, \delta)$ with $x \notin \mathbb{Q}$, then, $f(x) = 0$

$$|f(x) - f(c)| = |0 - 1| = 1 \geq \frac{1}{2} = \varepsilon$$

Therefore, the Dirichlet function is not continuous at $x = 0$.

What about sequences? Consider $f : \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$, and a sequence (c_n) that converges to c. This gives a new sequence $(f(c_n))$. Does this converge to $f(c)$? Is f of the limit of of a sequence equal to the limit of the sequence where we apply f first?

Example

Show that for $f(x) = x^2$ and $c_n = 2 + \frac{1}{n}$, $f(\lim_{n \to \infty} c_n) = \lim_{n \to \infty} f(c_n)$.

We know that $c_n \to 2$, so $c = 2$, and $f(c) = f(2) = 2^2 = 4$. Now, $f(c_n) = f\left(2 + \frac{1}{n}\right) = \left(2 + \frac{1}{n}\right)^2 = 4 + \frac{4}{n} + \frac{1}{n^2}$. We could show that $f(c_n) \to 4$. Thus, $f(\lim_{n \to \infty} c_n) = \lim_{n \to \infty} f(c_n)$.

Example

Show that for $f(x)$, where f is the Dirichlet function, and $c_n = 2 + \frac{1}{n}$ that $f(\lim_{n \to \infty} c_n) = \lim_{n \to \infty} f(c_n)$.

We know that $c_n \to 2$, and also that $c_n = 2 + \frac{1}{n} \in \mathbb{Q}$ for each n. So, $f(c) = f(2) = 1$. Also, $f(c_n) = 1$ for each n. Thus, $f(\lim_{n \to \infty} c_n) = \lim_{n \to \infty} f(c_n)$.

Example

Show that for $f(x)$, where f is the Dirichlet function, and $c_n = 2 + \frac{\sqrt{2}}{n}$ that $f(\lim_{n \to \infty} c_n) \neq \lim_{n \to \infty} f(c_n)$.

We know that $c_n \to 2$, and also that $c_n \notin \mathbb{Q}$ for all n. So, $f(c_n) = 0$, $\lim_{n \to \infty} f(c_n) = \lim_{n \to \infty} 0 = 0 \neq 1 = f(\lim_{n \to \infty} c_n) = f(2)$.

Can a function be continuous at only 1 point?

Definition : Modified Dirichlet Function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

For $x = 0$, $f(0) = 0 = \lim_{x \to 0} f(x)$. For any $x \neq 0$, $(x - \delta, x + \delta)$ will have both rationals and irrationals, so there will be a problem.