# Limit Laws

Let $f, g$ be functions from $A \subseteq \mathbb{R}$ to $\mathbb{R}$, $c$ is a limit point of $A$, and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M$$

1. $\lim_{x \to c} k \cdot f(x) = k \cdot L$ for any $k \in \mathbb{R}$
2. $\lim_{x \to c} [f(x) + g(x)] = L + M$
3. $\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot M$
4. $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right]$ provided $M \neq 0$ and $g(x) \neq 0$ for $x \in A$
5. *To prove this, use the sequence limit laws*

**Theorem : Squeeze Theorem**

For $f, g, h : A \to \mathbb{R}$, $c$ is a limit point of $A$ and $f(x) \leq g(x) \leq h(x)$ for all $x \in A$, and $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} g(x) = L$

*Proof.* Use the sequence limit laws. □

# Continuity

**Definition : Continuity**

A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$ we have $|f(x) - f(c)| < \varepsilon$.

This means, for $x \in A$ with $x \in (c - \delta, c + \delta)$, we have $f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$. Note that we include $x = c$

If $f(x)$ is continuous at every point in its domain, we say it is continuous.

The following are equivalent (TFAE):

(i) $f$ is continuous at $x = c$

(ii) $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $\forall\, x \in A$ with $|x - c| < \delta$, we have $|f(x) - f(c)| < \varepsilon$

(iii) (This one uses $\varepsilon$-neighbourhoods from Chapter 5

(iv) For all sequences $(a_n)$ from $A$ which converge to $c$, we have $f(a_n) \to f(c)$

(v) $\lim_{x \to c} f(x) = f(c)$

Continuity Limit Laws

Let $f, g : A \to \mathbb{R}$, $c$ is a limit point of $A$, $f, g$ are both continuous at $c$

(i) $k \cdot f(x)$ is continuous at $c$ for any $k \in \mathbb{R}$

(ii) $f(x) + g(x)$ is continuous at $c$

(iii) $f(x)\, g(x)$ is continuous at $c$

(iv) $\frac{f(x)}{g(x)}$ is continuous at $c$, provided $g(x) \neq 0$ for all $x \in A$

**Example**

Use the definition to prove that $f(x) = x^2$ is continuous at $x = 3$ for $c = 3$, $f(c) = 9$

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Let $\varepsilon > 0$, $|f(x) - f(c)| = |x^2 - 9| = |(x+3)(x-3)| = |x+3| \cdot |x-3|$. We want to find $\delta > 0$ so that $|x+3| \cdot |x-3| < \varepsilon$ when $|x-3| < \delta$. We are staying close to 3, so we can find an upper bound for $|x+3|$. For $\delta \leq 1$, $|x-3| < \delta = 1 \Leftrightarrow x \in (2,4)$. This, an upper bound for $|x+3|$ for $x \in (2,4)$ is $|4+3| = 7$. So, assuming $\delta \leq 1$, $|x+3| \cdot |x-3| < 7 \cdot |x-3|$. Need $7 \cdot |x-3| < \varepsilon$ for $|x-3| < \delta$. Let $\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}$, so we have $|x+3| \leq 7$ and $7 \cdot |x-3| < 7\left(\frac{\varepsilon}{7}\right) = 7$ for $|x-3| < \delta$. Thus,

$$\begin{aligned} |x^2 - 9| = |(x+3)(x-3)| &\leq 7 \cdot |x-3| && \text{for } \delta \leq 1 \\ &< 7 \cdot \frac{\varepsilon}{7} && \text{for } \delta = \min\left\{1, \frac{\varepsilon}{7}\right\} \text{ and } |x-3| < \delta \end{aligned}$$

Compositions

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is continuous at 3, could show $g(x) = 2x + 5$ is continuous at 9, then $(g \circ f)(x) = 2x^2 + 5$ is continuous at 3.

Idea – When you compose continuous functions, you get a continuous function

**Proposition**

Let $A, B \subseteq \mathbb{R}$, $g : A \to B$ is continuous at $c$, $f : B \to \mathbb{R}$ is continuous at $g(c)$, then $(f \circ g)(x) : A \to \mathbb{R}$ is continuous at $c$

*Proof.* Use sequences. Consider an arbitrary sequence $(a_n)$ in $A$ with $a_n \to c$. $(g(a_n))$ is a sequence in $B$ with $g(a_n) \to g(c)$ because $g$ is continuous. $(f(g(a_n)))$ is a sequence in $\mathbb{R}$ with $f(g(a_n)) \to f(g(c))$ because $f$ is continuous. $\square$

Facts (Functions that are continuous on their domains)

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$
2. $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln(x)$
3. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ or $f(x) = \cos(x)$