Convergence and Divergence

Proposition

Suppose $S \subseteq \mathbb{R}$ that is bounded above, then there exists a sequence (a_n), $a_n \in S$ for all n and $\lim_{n\to\infty} = \sup(S)$

Proof. Let $\alpha = \sup(S)$. For each $n \in \mathbb{N}$, $\alpha - \frac{1}{n} < \alpha$ so it is not an upper bound. For each n, there exists $a_n \in S$ such that $a_n > \alpha - \frac{1}{n}$.

$$a_1 > \alpha - 1, \qquad a_2 > \alpha - \frac{1}{2}, \qquad a_3 > \alpha - \frac{1}{3}$$

This sequence is well-defined, we know that $\frac{1}{n} \to 0$. We have $\alpha - \frac{1}{n} < a_n \leq \alpha$ for all n.

Let $c_n = \alpha$, we have $c_n \to \alpha$ (limit of a constant sequence). Let $b_n = \alpha - \frac{1}{n}$, we have $b_n \to \alpha - 0 = \alpha$ (by limit laws).

Thus, $a_n \to \alpha$ by the Squeeze Theorem.

Similarly, if S is bounded below, there exists a sequence in S that converges to $\inf(S)$. $\square$

Example

Demonstrate the above proposition with $S = (0, 1)$.

We know that $\sup(S) = 1$, each $a_n \in (0, 1)$ and $\lim_{n\to\infty} a_n = 1$. We could start with:

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{3}{4}, \quad , a_3 = \frac{7}{8}, \quad \dots, \quad a_n = 1 - \left(\frac{1}{2}\right)^n$$

We could show that this is increasing and bounded, so it converges to $\sup(S)$.

Example

Is it possible to have a divergent sequence with a convergent subsequence? If so, do all convergent subsequences have the same limit?

Let $a_n = (-1)^n = (-1, 1, -1, 1, \dots)$. We have proven that this diverges. What if we take a subset of the even naturals?

$$a_{2n} = (-1)^{2n} = (1, 1, 1, \dots)$$

What if we take a subset of the odd naturals?

$$a_{2n-1} = (-1)^{2n-1} = (-1, -1, -1, \dots)$$

Clearly, $a_{2n} \to 1$ and $a_{2n-1} \to -1$. Thus, this divergent sequence has two convergent subsequences with different limits.