Cauchy Criterion

Definition : Convergence

For every $\varepsilon > 0$, $\exists\ N \in \mathbb{R}$ such that $|a_n - a| < \varepsilon$ for $n > N$, that is, $a_n \in (a-\varepsilon, a+\varepsilon)$ for $n > N$.

For $n, m > N$, we have $a_n, a_m \in (a-\varepsilon, a+\varepsilon)$.

Definition : Cauchy Sequences

A sequence (a_n) is Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a_m| < \varepsilon$ for $n, m > N$.

Example

Show that the sequence $a_n = \frac{2}{n}$ is Cauchy.

$|a_n - a_m| = |\frac{2}{n} - \frac{2}{m}| \leq |\frac{2}{n}| + |\frac{2}{m}| = \frac{2}{n} + \frac{2}{m}$. Given $\varepsilon > 0$, we need $\frac{2}{n} + \frac{2}{m} < \varepsilon$. Try to get $\frac{2}{n} < \frac{\varepsilon}{2}$ and $\frac{2}{m} < \frac{\varepsilon}{2}$. We want $\frac{1}{N} < \frac{\varepsilon}{4}$. We know there exists $N \in \mathbb{R}$ to satisfy $\frac{1}{N} < \frac{\varepsilon}{4}$ since $\frac{\varepsilon}{4} > 0$. Then for $n, m > N$, $\frac{2}{n} + \frac{2}{m} < \frac{2}{N} + \frac{2}{N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus, the sequence is Cauchy.

Sequences of Partial Sums

Definition : Series

The sum of infinitely many numbers is called a series.

Consider the sum:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$$

The sequence is: $a_n = \left(\frac{1}{2}\right)^n$.

$$\begin{aligned}
S_1 &= a_1 = \frac{1}{2} \\
S_2 &= a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
S_3 &= a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
&\vdots \\
S_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \qquad \text{This is the } n^{\text{th}} \text{ partial sum}
\end{aligned}$$

Series Properties:

- A series converges to $L \in \mathbb{R}$, that is, $\sum_{k=1}^{\infty} a_k = L$ if the sequence of partial sums converges to L

- A series diverges if (S_n) diverges
- A series is bounded or monotone if (S_n) is bounded or monotone

Example

Show $\sum_{n=1}^{\infty} \frac{1}{2^k} = 1$.

$S_1 = \frac{1}{2}, S_2 = \frac{3}{4}, S_3 = \frac{7}{8}$, our claim is that $S_n = 1 - \left(\frac{1}{2}\right)^n$. We can use induction.

Base Case: $S_1 = 1 - \frac{1}{2} = \frac{1}{2}$.

Assume this is true for S_k: $S_k = 1 - \frac{1}{2^k}$.

Induction Step: Show $S_{k+1} = 1 = \frac{1}{2^{k+1}}$

$$\begin{aligned} S_{k+1} &= \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} \\ &= S_k + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \\ &= \frac{2^{k+1} - 2 + 1}{2^{k+1}} \\ &= \frac{2^{k+1} - 1}{2^{k+1}} \\ &= \frac{2^{k+1}}{2^{k+1}} - \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^{k+1}} \end{aligned}$$

So by induction, $S_n = 1 - \frac{1}{2^n}$. Now, take the limit of partial sums.

$$\lim_{n\to\infty} 1 - \frac{1}{2^n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0 = 1$$

Series Limit Laws:

- Assume $\sum_{k=1}^{\infty} a_k = \alpha$, $\sum_{k=1}^{\infty} b_k = \beta$, $C \in \mathbb{R}$

1. $\sum_{k=1}^{\infty} (a_k + b_k) = \alpha + \beta$
2. $\sum_{k=1}^{\infty} (a_k - b_k) = \alpha - \beta$
3. $\sum_{k=1}^{\infty} C \cdot a_k = c \cdot \alpha$
4. You cannot multiply and divide series in terms of their limits.

Example

Show that you cannot multiply series in terms of limits.

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1. $\left(\sum_{k=1}^{2} k\right) \cdot \left(\sum_{k=1}^{2} k^2\right) = (1+2) \cdot (1^2 + 2^2) = 3 \cdot 5 = 15$
2. $\sum_{k=1}^{2} k \cdot k^2 = \sum_{k=1}^{2} k^3 = 1^3 + 2^3 = 9$

These are not equal.

Series Convergence Tests

Proposition : k^{th} Term Test

If $a_k \not\to 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Contrapositive: If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$.

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges. Let $S_n = \sum_{k=1}^{n} a_k$. S_n converges, so it is Cauchy. Let $\varepsilon > 0$, $\exists\ N \in \mathbb{R}$ such that for $n, m > N$,

$$|S_n - S_m| < \varepsilon$$

$$S_n = a_1 + a_2 + \cdots + a_n$$
$$S_m = a_1 + a_2 + \cdots + a_m$$

Without loss of generality, assume $n \geq m$

$$S_n = a_1 + a_2 + \cdots + a_m + a_{m+1} + \cdots + a_n$$

$$S_n - S_m = a_{m+1} + \cdots + a_n = \sum_{k=m+1}^{n} a_k$$

We have $\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$ for $n, m > N$. Consider $m + 1 = n$, then $|\sum_{n}^{n} a_n| = |a_n| < \varepsilon$ for $n > N$. We can do this for any $\varepsilon > 0$. Thus, $a_n \to 0$. (Proved in chapter 1) $\square$

We will see series where $a_k \to 0$, but the series diverges. A classic example is the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$, we know $\frac{1}{k} \to 0$, but we will prove the series diverges.