Supremums and Infimums

Last time we saw the following theorem, now we will prove it.

Theorem

Let $A \subseteq \mathbb{R}$, $\sup(A) = \alpha$ if and only if:

(i) α is an upper bound of A

(ii) Given any $\varepsilon > 0$, $\sup(A) - \varepsilon = \alpha - \varepsilon$ is not an upper bound of A. That is, there is some $x \in A$ such that $\alpha - \varepsilon < x$.

Likewise, $\inf(A) = \beta$ if and only if:

(i) β is a lower bound of A

(ii) Given any $\varepsilon > 0$, $\inf(A) + \varepsilon = \beta + \varepsilon$ is not a lower bound of A. That is, there is some $x \in A$ such that $x < \beta + \varepsilon$.

Proof. This is the proof for the supremum. A similar argument can be used for the infimum.

First, assume $\sup(A) = \alpha$, thus α is an upper bound by definition. Let $\varepsilon > 0$, thus $\alpha - \epsilon < \alpha$, so $\alpha - \varepsilon$ cannot be an upper bound for A, otherwise it would contradict that α is the least upper bound. Then, there exists some $x \in A$ with $x > \alpha - \varepsilon$ (or else $\alpha - \varepsilon$ would be an upper bound.

Now, assume (i) and (ii) hold for α. Then α is an upper bound by (i), so we need to show that α is the least upper bound. Let $\beta < \alpha$, we want to show that β can't be an upper bound for A. Consider $\varepsilon = \alpha - \beta$, then $\varepsilon > 0$ since $\alpha > \beta$. By assumption (ii), for this ε, there exists $x \in A$ such that $\alpha - \varepsilon$.

$$\alpha - \varepsilon = \alpha - (\alpha - \beta) = \beta$$

So $x > \beta$, and β can't be a lower bound.

Therefore, $\sup(A) = \alpha$ if and only if (i) and (ii) hold. $\square$

Example

Let $A = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$. Prove that $\sup(A) = 0$.

(i) 0 is an upper bound since $x < 0 \; \forall \; x \in A$ by construction

(ii) Let $\varepsilon > 0$, we want to show $\exists \; x \in A$ with $x > \sup(A) - \varepsilon = 0 - \varepsilon = -\varepsilon$. We want $x \in A$ with $x > -\varepsilon$, so we need $-\varepsilon < x < 0$. Let $x = \frac{-\varepsilon}{2}$, then x satisfies $-\varepsilon < x < 0$ and condition (ii) holds.

Therefore, $\sup(A) = 0$

The Archimedean Principle

Example

Let $a = 3, b = 58$. Find $n \in \mathbb{N}$ such that $na > b$.

$$\begin{aligned} na &> b \\ n\,(3) &> 58 \\ n &> \frac{58}{3} \\ n &\in \{20, 21, \dots\} \end{aligned}$$

Lemma : (Archimedean Principle

If $a, b \in \mathbb{R}$ with $a > 0$ then there exists a rational number n such that

$$na > b \Leftrightarrow n > \frac{b}{a}$$

Proof. We know $a > 0$, we need $na > b \Leftrightarrow n > \frac{b}{a}$. If $b \leq 0$, then $n = 1$ works since $\frac{b}{a} \leq 0$. So we can assume $b > 0$. Then, let $x = \frac{b}{a}$. Suppose there is no natural number larger than x.

Thus, x is an upper bound for $\mathbb{N}$, $\mathbb{N}$ is bounded above, so by completeness of $\mathbb{R}$, $\mathbb{N}$ has a least upper bound in $\mathbb{R}$. We will call this $\alpha = \sup(\mathbb{N})$. Let $\varepsilon = 1 > 0$. $\alpha - 1$ is not an upper bound for $\mathbb{N}$. So $\exists\, m \in \mathbb{N}$ such that $\alpha - 1 < m$, otherwise, $\alpha - 1$ would be an upper bound. We have $m > \alpha - 1 \Rightarrow m + 1 > \alpha$ (We know that $m + 1 \in \mathbb{N}$ since $m \in \mathbb{N}$). This contradicts that α is an upper bound for $\mathbb{N}$.

The contradiction came from assuming there is no natural number than $\frac{b}{a}$. Thus, there exists some $n \in \mathbb{N}$ with $n > \frac{b}{a}$, or $na > b$. $\square$

This is often used for showing that for any $\varepsilon > 0$, $\exists\, n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$.

Example

Consider $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Prove that $\inf(A) = 0$.

(i) 0 is a lower bound since $0 \leq \frac{1}{n}\ \forall\, n \in \mathbb{N}$

(ii) By the Archimedean Principle, $\exists\, n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Thus, $0 + \varepsilon$ is not a lower bound for A.

Therefore, $\inf(A) = 0$.