Variance

An important function of X is the "squared deviation" function $g(X) = (X - E(X))^2$. The expected value of g is called the variance $V(X)$ of X and represent the variation of X-values:

$$V(X) = E\left(X(X - E(X))^2\right) = E\left((X-\mu)^2\right)$$

We can compute variances using the shortcut formula $V(X) = E(X^2) - (E(X))^2$. (Note: $E(X) = \sum x^2 \cdot p(x)$)

The standard deviation of X is the square root of the variance: $SD(X) = \sigma(X) = \sqrt{V(X)}$

Example

Let X be a discrete random variable with mass function defined by $p(1) = 0.45$, $p(3) = 0.25$, and $p(5) = 0.30$. What is the variance of X?

$$\begin{aligned} E\left(X^2\right) &= 1^2 \cdot 0.45 + 3^2 \cdot 0.25 + 5^2 \cdot 0.30 \\ &= 0.45 + 9 \cdot 0.25 + 25 \cdot 0.30 \\ &= 0.45 + 2.25 + 7.50 \\ &= 10.20 \end{aligned}$$

$$\begin{aligned} (E(X))^2 &= (1 \cdot 0.45 + 3 \cdot 0.25 + 5 \cdot 0.30)^2 \\ &= (0.45 + 0.75 + 1.50)^2 \\ &= 2.70^2 \\ &= 7.29 \end{aligned}$$

$$\begin{aligned} V(X) &= E\left(X^2\right) - (E(X))^2 \\ &= 10.20 - 7.29 \\ &= 2.91 \end{aligned}$$

Example

Suppose that a coin is unfairly loaded to land on head with probability α and let X be the number of coin tosses required to observe a head. From previous lectures, we know that the pmf p of X is defined by:

$$p\left(x\right)=\begin{cases}\alpha\left(1-\alpha\right)^{x-1} & x=1,2,3,\dots\\ 0 & \text{otherwise}\end{cases}\qquad \text{with } E\left(X\right)=\frac{1}{\alpha}$$

What is the variance of X?

$$E\left(X\right)=\frac{1}{\alpha}\Rightarrow\left(E\left(X\right)\right)^2=\frac{1}{\alpha^2}$$

$$\begin{aligned}
E\left(X^2\right)&=\sum_{x=1}^{\infty}x^2\cdot p\left(x\right)\\
&=\sum_{x=1}^{\infty}x^2\cdot\alpha\cdot\left(1-\alpha\right)^{x-1}\qquad\text{let } q=1-\alpha\\
&=\sum_{x=1}^{\infty}x^2\cdot\alpha\cdot q^{x-1}\\
&=\alpha\sum_{x=1}^{\infty}x^2\cdot q^{x-1}\qquad\text{let } n=x-1\Rightarrow x=n+1\\
&=\alpha\sum_{n=0}^{\infty}\left(n+1\right)^2\cdot q^n\\
&=\alpha\sum_{n=0}^{\infty}\left(n^2+2n+1\right)\cdot q^n\\
&=\alpha\left(\sum_{n=0}^{\infty}n^2\cdot q^n+2\sum_{n=0}^{\infty}n\cdot q^n+\sum_{n=0}^{\infty}q^n\right)\\
&=\alpha\left(\frac{\left(1-\alpha\right)\left(2-\alpha\right)}{\alpha^3}+\frac{2\left(1-\alpha\right)}{\alpha^2}+\frac{1}{\alpha}\right)\\
&=\frac{\left(1-\alpha\right)\left(2-\alpha\right)}{\alpha^2}+\frac{2\left(1-\alpha\right)}{\alpha}+1
\end{aligned}$$

$$\begin{aligned}
V\left(X\right)&=E\left(X^2\right)-\left(E\left(X\right)\right)^2\\
&=\frac{\left(1-\alpha\right)\left(2-\alpha\right)}{\alpha^2}+\frac{2\left(1-\alpha\right)}{\alpha}+1-\frac{1}{\alpha^2}\\
&=\frac{1-\alpha}{\alpha^2}
\end{aligned}$$

If X is a discrete random variable and a and b are any constants,

$$V\left(aX+b\right)=a^2\cdot V\left(X\right)$$

Example

Let X be discrete random variable with mass function defined by $p(1) = 0.45$, $p(3) = 0.25$, and $p(5) = 0.30$. Let $g(X) = 3X + 2$ be a function of X. What is the variance of $g(X)$?

$$\begin{aligned} V(3X+2) &= 3^2 \cdot V(X) \\ &= 9 \cdot V(X) \\ &= 9 \cdot 2.91 \\ &= 26.19 \end{aligned}$$

Example

Suppose that two players $P1$ and $P2$ play a series of games that ends when one of them wins twice. Suppose that each game is played independently and $P1$ wins a game with probability α. Let X be the number of games played in the series. Compute $E(X)$ and $V(X)$. What value of α maximizes $E(X)$?

$$\Pr(X=2) = \alpha^2 + (1-\alpha)^2$$

$$\Pr(X=3) = 2\alpha^2(1-\alpha) + 2\alpha(1-\alpha)^2 = 2\alpha(1-\alpha)$$

$$\begin{aligned} E(X) &= 2 \cdot \left(\alpha^2 + (1-\alpha)^2\right) + 3 \cdot (2\alpha(1-\alpha)) \\ &= -2\alpha^2 + 2\alpha + 2 \end{aligned}$$

$$\begin{aligned} (E(X))^2 &= \left(-2\alpha^2 + 2\alpha + 2\right)^2 \\ &= 4\alpha^4 - 8\alpha^3 - 4\alpha^2 + 8\alpha + 4 \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) &= 2^2 \cdot \left(\alpha^2 + (1-\alpha)^2\right) + 3^2 \cdot (2\alpha(1-\alpha)) \\ &= 4 \cdot \left(\alpha^2 + (1-\alpha)^2\right) + 9 \cdot (2\alpha(1-\alpha)) \\ &= -10\alpha^2 + 10\alpha + 4 \end{aligned}$$

$$\begin{aligned} V(X) &= E\left(X^2\right) - (E(X))^2 \\ &= \left(-10\alpha^2 + 10\alpha + 4\right) - \left(4\alpha^4 - 8\alpha^3 - 4\alpha^2 + 8\alpha + 4\right) \\ &= -4\alpha^4 + 8\alpha^3 - 6\alpha^2 + 2\alpha \end{aligned}$$

In order to maximize $E(X)$, the vertex of $\alpha = -\frac{2}{-4} = \frac{1}{2}$. $a = -2 < 0$, so the graph is concave down, so the POI is a maximum. Therefore $\alpha = \frac{1}{2}$ maximizes $E(X)$.