Divergent Sequences

Example

Consider $a_n = (-1)^n \Rightarrow (-1, 1, -1, 1, \dots)$. Show this sequence does not converge to 1.

To show it is not convergent to 1, we need $\varepsilon > 0$ such that for all $N \in \mathbb{R}$ there is some $n > N$ such that $|a_n - 1| \geq \varepsilon$. Consider $\varepsilon = 1$, $|a_n - 1| < 1 \Leftrightarrow a_n \in (0, 2)$. Given any N, there is some $n \in \mathbb{N}$ with n odd and $n > N$ so $a_n = -1 \notin (0, 1)$

Example

Consider $a_n = (-1)^n \Rightarrow (-1, 1, -1, 1, \dots)$. Show this sequence diverges.

To show the sequence diverges, we need to show it can't converge to any $a \in \mathbb{R}$. Let $a \in \mathbb{R}$, let $\varepsilon = \frac{1}{2}$, assume $a_n \to a$, so $\exists\, N \in \mathbb{R}$ such that $|a_n - a| < \frac{1}{2}$ for $n > N$. For $n > N$, there are two cases:

- if n is even, $a_n = (-1)^n = 1$

$$\begin{aligned} |1 - a| < \frac{1}{2} &\Leftrightarrow -\frac{1}{2} < 1 - a < \frac{1}{2} \\ &\Leftrightarrow \frac{1}{2} < a < \frac{3}{2} \end{aligned}$$

- if n is odd, $a_n = (-1)^n = -1$

$$\begin{aligned} |-1 - a| < \frac{1}{2} &\Leftrightarrow -\frac{1}{2} < -1 - a < \frac{1}{2} \\ &\Leftrightarrow -\frac{3}{2} < a < -\frac{1}{2} \end{aligned}$$

This gives two conditions on a

$$\frac{1}{2} < a < \frac{3}{2} \qquad \text{and} \qquad -\frac{3}{2} < a < -\frac{1}{2}$$

a cannot satisfy both. The contradiction came from assuming $a_n \to a$. a was arbitrary, so the sequence diverges.

Example : Exercise 1.13

(a) Prove if $a < b + \varepsilon$ for every $\varepsilon > 0$, then $a \leq b$.

(b) Prove if $|a - b| < \varepsilon$ for all $\varepsilon > 0$, then $a = b$.

(a) Assume $a < b + \varepsilon$ for every $\varepsilon > 0$ and $a > b$. If $a > b$, then $a - b > 0$, let $\varepsilon = a - b$. $b + \varepsilon = b + a - b = a$. Thus, $a < b + \varepsilon = a$, this is not possible. Thus, $a \leq b$.

(b) Let $\varepsilon > 0$, $|a - b| < \varepsilon$.

$$\begin{aligned} &\Leftrightarrow -\varepsilon < a - b < \varepsilon \\ &\Leftrightarrow b - \varepsilon < a < b + \varepsilon \end{aligned}$$

$$a < b + \varepsilon \Rightarrow a \leq b \qquad \forall\, \varepsilon > 0$$
$$b - \varepsilon < a \Rightarrow b < a + \varepsilon \Rightarrow b \leq a \qquad \forall\, \varepsilon > 0$$

Thus, $a = b$.

Proposition

If a sequence converges, the limit is unique.

Proof. Suppose $a_n \to a$ and $a_n \to b$. Let $\varepsilon > 0$, then $\frac{\varepsilon}{2} > 0$. $\exists\, N_1 \in \mathbb{R}$ such that $|a_n - a| < \frac{\varepsilon}{2}$ for $n > N_1$. $\exists\, N_2 \in \mathbb{R}$ such that $|a_n - b| < \frac{\varepsilon}{2}$ for $n > N_2$. To have both conditions hold, let $N = \max(N_1, N_2)$. Let $n > N$

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |a - a_n| + |a_n - b| && \text{by the Triangle Inequality} \\ &= |a_n - a| + |a_n - b| && \text{since } |x - y| = |y - x|\ \forall\, x, y \in \mathbb{R} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

For every $\varepsilon > 0$, $|a - b| < \varepsilon$, thus $a = b$. $\square$

Recall: A sequence (a_n) is bounded if $\exists C \in \mathbb{R}$ such that $|a_n| \leq C$ for all n.

Proposition

If a sequence is convergent, it is bounded.

Proof. Suppose $a_n \to a$. Let $\varepsilon = 1$, $\exists\, N \in \mathbb{R}$ such that $\forall\, n > N$ we have $|a_n - a| < 1$

$$\Leftrightarrow a_n \in (a - 1, a + 1)$$

Let $m = \lfloor N \rfloor$, $m \leq N < m + 1$. If $m < 1$, then every term of the sequence is in $(a - 1, a + 1)$, we have $a - 1 \leq a_n \leq a + 1$ for all n, so the sequence is bounded.

Suppose $m \in \mathbb{N}$, consider $A = \{|a_1|, |a_2|, \dots, |a_m|, |a-1|, |a+1|\}$. Let $C = \max(A)$ since A is finite. For $1 \leq n \leq m$, $|a_n| \leq \max(A)$. For $N < m+1 < n$, $a_n \in (a-1, a+1)$ so $|a_n| \leq \max(\{|a-1|, |a+1|\}) \leq \max(A)$. Thus, $|a_n| < \max(A)$ for all n. Thus, the sequence is bounded. $\square$

The above proof also proves that if a sequence is not bounded, then it is not convergent.

Can a sequence be bounded but not convergent? Yes, consider $a_n = (-1)^n$. $|a_n| \leq 1$ for all n, so it is bounded, but it is not convergent (shown above).