The Archimedean Principle

Note: $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$

Lemma

Let $x, y \in \mathbb{R}$, where $y - x > 1$. Then, there exists a $z \in \mathbb{Z}$ such that $x < z < y$.

Proof. If $x < 0 < y$, then let $z = 0$. So suppose $0 \leq x < y$, and $y - x > 1$. Let $A = \{n \in \mathbb{N}_0 : n \leq x\}$.

We have $0 \in A$, so A is nonempty. Also, the set A is finite (A contains non-negative integers less than or equal to x, so x is an upper bound. If A had infinitely many elements, there would be no upper bound).

$$A = \{0, 1, 2, \ldots, n\} \qquad n \leq x, \quad n + 1 > x$$

A is finite, so it contains a maximal element, let $M = \max(A)$. We know $M \leq x$ and $M + 1 > x$ since M is the maximal element of A.

$$M \leq x \to M + 1 \leq x + 1$$

$$y - x > 1 \to y > x + 1$$

$$\Rightarrow M + 1 < y$$

Now, let $z = M + 1$, then $z \in \mathbb{Z}$ with $x < z < y$.

If $x < y \leq 0$, and $y - x > 1$. Multiply by -1 to get $0 \leq -y < -x$, and $(-x) - (-y) > 1$. Apply the previous result to get z between $-y$ and $-x$, so $-z$ is an integer between x and y. $\square$

Theorem

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Assume $x, y \in \mathbb{R}$ with $x < y$. If $x < 0 < y$, let $z = 0$.

Assume $0 \leq x < y$, we know $y - x > 0$. By the Archimedean Principle, there is $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$. Multiply this by n to get:

$$1 < n\,(y - x) = ny - yx$$

$ny, nx \in \mathbb{R}$, with $ny - nx > 1$, so by the previous Lemma, there exists $m \in \mathbb{Z}$ such that $nx < m < ny$. Dividing by n gives:

$$x < \frac{m}{n} < y$$

We know $m \in \mathbb{Z}, n \in \mathbb{N}$ so $\frac{m}{n} \in \mathbb{Q}$.

Likewise, we can find a rational number between x and y when $x < y \leq 0$.

Therefore, $\mathbb{Q}$ is dense in $\mathbb{R}$. $\square$

Useful Definitions

Definition : Ceiling

The ceiling of x, denoted $\lceil x \rceil$ is the integer n such that

$$x \leq n < x + 1$$

Definition : Floor

The floor of x, denoted $\lfloor x \rfloor$ is the integer n such that

$$x - 1 < n \leq x$$

Definition : Closed Interval

A closed interval between a and b is the interval containing the endpoints a and b. Denoted by:

$$[a, b] = \{x : a \leq x \leq b\}$$

Definition : Open Interval

An open interval between a and b is the interval that doesn't contain the endpoints a and b. Denoted by:

$$(a, b) = \{x : a < x < b\}$$

Note: You can have half-open intervals.

Definition : Well Ordering Principle

Every nonempty subset of $\mathbb{N}$ has a smallest element.

Note: The Well Ordering Principle is equivalent to the Principle of Mathematical Induction

Cardinality

How do we compare the sizes of sets?

Example

Are the sets $A = \{1, 2, 3\}$, and $B = \{5, 7, 9\}$ the same size?

Yes, since $|A| = 3 = |B|$.

We want to say these are the same size. One way to understand "same size" is that elements can be matched to each other.

Definition

Two sets have the same size if and only if there is a bijection between them.

Some properties of bijections:

- If $f : A \to B$ is a bijection, then f has an inverse, $f^{-1} : B \to A$ defined by $f^{-1}(b) = a \Leftrightarrow f(a) = b$. That is, a bijection and it's inverse cancel each other out. f^{-1} is also a bijection.
- If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ is also a bijection.

Definition

We define a relation on sets by $A \sim B$ if there exists a bijection from A to B. This is an equivalence relation

Definition : Equivalence Relation

An equivalence relation is a binary relation that satisfies:

1. Reflexivity: $A \sim A$
   - Using identity, $i_A : A \to A$, $i_A(x) = x$
2. Symmetry: If $A \sim B$, then $B \sim A$
   - Since bijections are invertible
3. Transitivity: If $A \sim B$, and $B \sim C$, then $A \sim C$
   - By composition of bijections