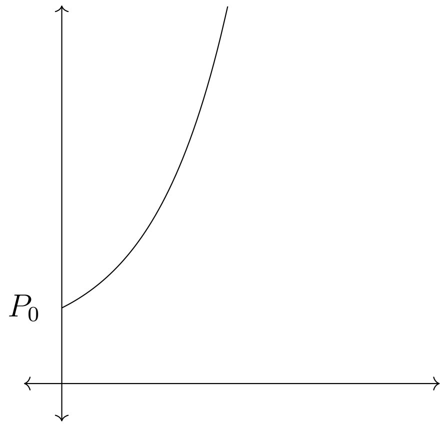
Population Growth Models

The law of natural growth,

$$\frac{dP}{dt} = rP \Rightarrow P = P_0 e^{rt}$$

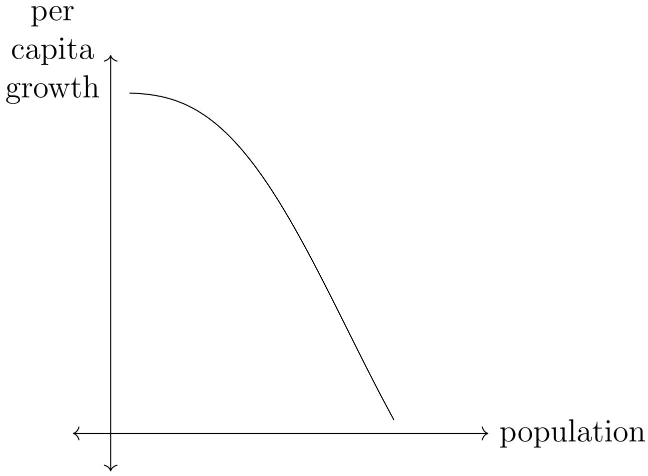
is good conceptually, but not very useful when growth is limited by the environment. Generally, it is not realistic.



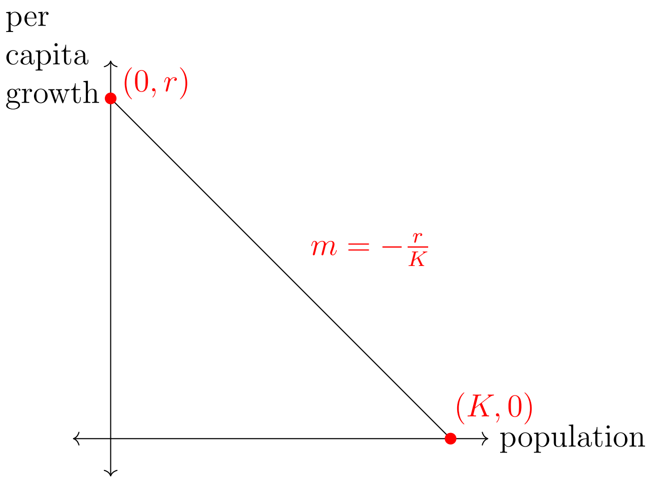
We want a better model, with environmental limitations. First, lets clarify growth and per capita growth

- Growth
  - $\frac{dP}{dt}$
  - Units of $\frac{\text{population}}{\text{time}}$
- Per Capita Growth
  - AKA percentage growth
  - $\left(\frac{dP}{dt}\right)/P$
  - Units of $\frac{\frac{\text{population}}{\text{time}}}{\text{population}} = \frac{1}{\text{time}}$

Back to the population model, the idea is that per capita growth decreases as population increases.



Let's use the simplest representation of a decreasing relationship, a linear function.



Now, using the base of $y = mx + b$, we can plug in our values.

$$\left(\frac{dP}{dt}\right)/P = -\frac{r}{K}(P) + r$$

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

This is the logical model for growth, where

- r : Competition Free Growth Rate
- K : Carrying Capacity
  - This is the population level the environment can sustainably support, $\frac{dP}{dt} = 0$.
- P : Population
- Note: $r, k > 0$

Let's study this with our toolkit.

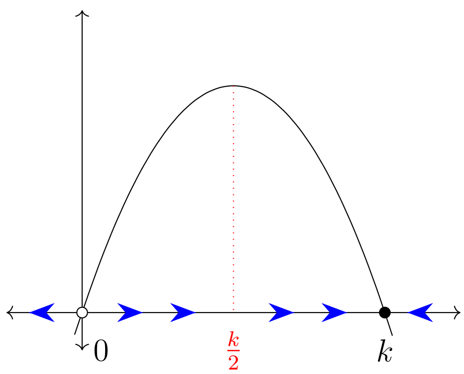
$$\dot{P} = rP\left(1 - \frac{P}{K}\right)$$

(1) Equilibrium Solutions

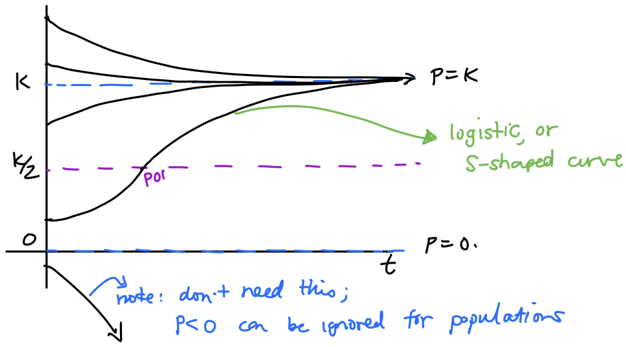
$$0 = rP\left(1 - \frac{P}{K}\right)$$

$$P^* = 0, P^* = K$$

(2) Phase Portrait:



(3) Trajectories ($P(t)$)

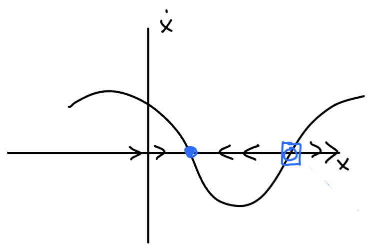


The last example was able to be plotted, but what about something like $\frac{ax+b}{1+be^{xc}}, a, b, c \in \mathbb{R}$? We can't plot the phase portrait unless we know what a, b, c are.

Let's develop an additional technique to determine the stability of equilibria in $\dot{x} = f(x)$

Linear Stability Analysis

Consider the function:



The tangent line to the curve at stable equilibria have a negative slope, and they have a positive slope and unstable equilibria. The question is: Do small perturbations from an equilibrium point shrink (stable) or grow (unstable)?

The idea is to linearize $f(x)$ in a neighbourhood of the equilibria points. Let x^s be an equilibrium point, so $f(x^s) = 0$. Define $\eta(t) = x(t) - x^s$ (small perturbation). We want to know if $\eta(t)$ grows or shrinks, and we can do this with Taylor Polynomials.