Counting Principle

The simplest probabilities are computed by counting how many ways things happen:

$$\Pr(A) = \frac{\#\text{ of ways for } A \text{ to happen}}{\#\text{ of possibilities}}$$

Combinatorial analysis is the mathematical theory of counting.

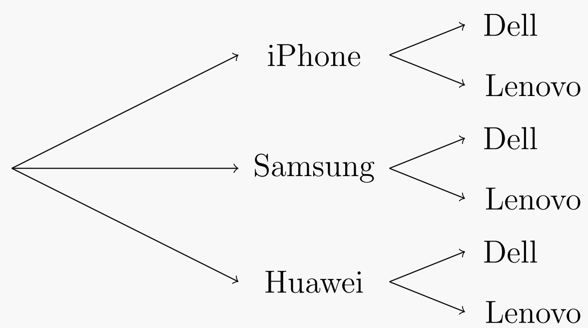
Counting Principle

The counting principle is the foundation of probability, combinatorics, and many real-life applications. It answers a very practical question: *If there are several choices to make, how many total possible outcomes are there?* For example,

- When creating a password, how many different codes can you form with letters and digits?
- When packing clothes for a trip, how many different outfits can you make from your shirts and pants?

Example

Suppose you start a new job at a company. They provide all their staff with a smartphone to be chosen from iPhone, Samsung, or Huawei; and a PC to be chosen from Dell or Lenovo. Write all possible options that you have in selecting the two devices.



The Multiplication Principle

Two step multiplication principle: Assume that a task can be broken up into two consecutive steps. If step 1 can be performed in m and ways, and for each of these, step 2 can be performed in n ways, then the task itself can be performed in $m \times n$ ways.

Example

Suppose you have 3 hat, hats A, B, and C, and 2 coats, coats 1 and 2 in your closet. Assuming that you feel comfortable with wearing any hat with any coat. How many different choices of hat / coat combinations do you have? List all combinations.

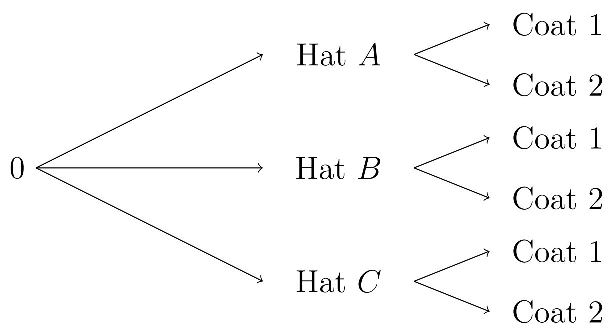
Since there are 3 choices for the first step, and 2 choices for the second step, by the multiplication principle, you have $3 \times 2 = 6$ different combinations.

- Hat A, Coat 1
- Hat A, Coat 2
- Hat B, Coat 1
- Hat B, Coat 2
- Hat C, Coat 1
- Hat C, Coat 2

We can get some insight into why the formula holds by representing all options on a tree diagram. We can break the decision making process into two steps here:

- Step 1: Choose a hat
- Step 2: Choose a coat

From the starting point 0, we can represent the three choices for step 1 by three branches whose endpoints are labeled by the choice names. From each of these endpoints we draw branches representing the options for step two with endpoints labeled appropriately. The result for the above example is shown below:



Each path on the tree diagram corresponds to a choice of hat and coat. Each of the three branches in step 1 is followed by two branches in step 2, giving us 3×2 distinct paths. If we had m hats and n coats, we would get $m \times n$ paths on our diagram. Of course, if the numbers m and n are large, it may be difficult to draw.

Example

Suppose that I have two sharpeners (Uni and Blackwing), one eraser, three pens (Fine, Extra Fine, and Medium), and two pencils (Staedtler, rOtring). How many ways can I pack my pencil case with one of each thing? If I randomly choose one of each thing, what is the probability that I pack my rOtring pencil?

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(1) By the multiplication principle, there are $2 * 1 * 3 * 2 = 12$ ways to pack your pencil case with one of each thing.

(2) By the multiplication principle, there are $2 * 1 * 3 * 1 = 6$ ways to pack your rOtring pencil. Therefore, the probability of randomly packing your rOtring pencil is:

$$\Pr(\text{rOtring}) = \frac{6}{12} = \frac{1}{2} = 0.5$$

The Generalized Basic Principle of Counting

If r experiments are to be performed such that the first experiment has n_1 possible outcomes. For each outcome of the first experiment, there are n_2 possible outcomes for the second experiment. For each combination of possible outcomes for the first two experiments, there are n_3 possible outcomes for the third experiment, etc. For each combination of possible outcomes for the first $r - 1$ experiments, there are n_r possible outcomes for the rth experiment, then there is a total of $n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_r$ possible outcomes of the r experiments.

Example

A Nova Scotian standard license plate has three letters and three numbers (e.g., CBR023). How many Nova Scotian standard license plates are possible? How many would be possible if no letters or numbers were allowed to be repeated?

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(1) By the multiplication principle, there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17576000$ possibilities for standard license plates.

(2) By the multiplication principle, there are $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11232000$ possibilities if no letters or numbers can be repeated.

Example

A function $f : A \mapsto B$ (A is the domain and B is the codomain) is an assignment rule of each $a \in A$ to exactly one $f(a) \in B$. In calculus, functions are usually infinite, real-valued, and defined by a rule $f(x)$. If the domain and codomain are finite, we can count how many functions $f : A \mapsto B$ exist. How many functions exist of the form $f : \{1, 2, 3, 4, 5, 6\} \mapsto \{1, 2, 3, 4\}$? Randomly choose such a function. What is the probability this function sends everything to an even number?

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(1) By the multiplication principle, there are $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^6 = 4096$ possible functions.

(2) By the multiplication principle, there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$ functions that only send to even numbers. Therefore, the probability of randomly selecting a function that only sends to even numbers is:

$$\Pr(\text{function only sending to evens}) = \frac{64}{4096} = \frac{1}{64} = 0.015625$$