Subsequences

Definition : Subsequences

Let (a_n) be a sequence. Let $n_1 < n_2 < n_3 < \dots$ be an increasing sequence of natural numbers. Then, $a_{n_1}, a_{n_2}, a_{n_3}, \dots$ is called a subsequence of (a_n) and is denoted a_{n_k}.

Some example subsequences are:

- Evens: $n_1 = 2, n_2 = 4, n_3 = 6, \dots$. Sequence: $a_{2n} = (a_2, a_4, a_6, \dots)$.
- Odds: $n_1 = 1, n_2 = 3, n_3 = 5, \dots$. Sequence: $a_{2n-1} = (a_1, a_3, a_5, \dots)$.
- $(n_k) = (5, 6, 11, 22, 101, \dots)$. Sequence: $(a_5, a_6, a_{11}, a_{22}, a_{101}, \dots)$.

Note: You cannot change the order, for example, $(a_1, a_3, a_{11}, a_4, a_2, \dots)$ is not a subsequence.

Proposition

A sequence converges to a if and only if every subsequence converges to a. (This is useful for showing a sequence diverges).

Corollary

If (a_n) has a pair of subsequences that converge to different limits, then (a_n) diverges.

Proposition

If (a_n) is a monotone sequence that has a convergent subsequence, then (a_n) converges to the same limit.

Proof. (a_n) is monotone increasing, we have a subsequence (a_{n_k}) that converges to a. We know that $\lim_{k\to\infty} a_{n_k} = a = \sup(\{a_{n_k} : k \in \mathbb{N}\})$.

The idea is to show that the original sequence (a_n) is bounded above, so you can use the monotone convergence theorem. $\square$

Example : Exercise 3.9

Consider the sequence (a_n) with $6 < a_n < 7$ for all n. Is it possible to have a subsequence that converges to 6 and another that converges to 7?

Idea: Evens go to 6, odds go to 7, we know that $\frac{1}{n} \to 0$. Evens: $a_{2n} = 6 + \frac{1}{2n}$, $a_{2n} \to 6$. Odds: $a_{2n-1} = 7 - \frac{1}{2n}$, $a_{2n-1} \to 7$.

Example : Exercise 3.20

Find a bounded sequence that does not convergence to $\frac{4}{9}$ but has a subsequence converging to $\frac{4}{9}$.

Try having the evens converge to $\frac{4}{9}$, $a_{2n} = \frac{4}{9}$. For odds, we just need them to be bounded and not converging to $\frac{4}{9}$, $a_{2n-1} = 0$.

Example : Exercise 3.28

Let $a_n = r^n$ for some $r \in \mathbb{R}$. Show $a_n \to 0$ if $r \in (-1, 1)$, $a_n \to 1$ if $r = 1$, and a_n diverges otherwise.

Case 1: $r = 1$, $a_n = (1)^n = 1$, constant sequence converges to 1.

Case 2: $r = -1$, $a_n = (-1)^n$, we have already shown this diverges.

Case 3: Suppose $|r| < 1$: Let $\varepsilon > 0$, need N such that for $n > N$ we have $|r^n - 0| < \varepsilon \Leftrightarrow |r|^n < \varepsilon$. If $r = 0$, we have a constant sequence that converges.

$$\begin{aligned} \ln\left(|r|^n\right) &< \ln(\varepsilon) \\ n \cdot \ln\left(|r|\right) &< \ln(\varepsilon) \\ n &> \frac{\ln(\varepsilon)}{\ln\left(|r|\right)} \end{aligned}$$

Let $N = \frac{\ln(\varepsilon)}{\ln(|r|)}$, so for $n > N$, we have $|r^n - 0| < \varepsilon$, so $r^n \to 0$.

Last case: $|r| > 1$, we can show $|r|^n$ are not bounded above. Thus, for $|r| > 1$, (r^n) is divergent.

Bolzano-Weierstrass Theorem

Lemma

Every sequence has a monotone subsequence.

Theorem : Bolzano-Weierstrass

Every bounded sequence has a convergent subsequence. *The idea is bounded and monotone gives convergence.*