# Convergence Tests

### Proposition

If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

*Proof.* Look at the partial sums. Let $S_n = a_1 + a_2 + \cdots + a_n$. The goal is to show this converges, so we will prove it is Cauchy. Let $\varepsilon > 0$

$$|S_n - S_m| = |\,(a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_m)\,|$$

Without loss of generality, assume $n \geq m$

$$|S_n - S_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|$$

We know that $\sum_{k=1}^{\infty} |a_k|$ converges. Let, $t_n = |a_1| + |a_2| + \cdots + |a_n|$. $(t_n)$ converges, so it is Cauchy. For $\varepsilon > 0$, $\exists\ N$ such that $|t_n - t_m| < \varepsilon$ when $n, m > N$. Without loss of generality, assume $n \geq m$.

$$|t_n - t_m| = ||a_{m+1}| + |a_{m+2}| + \cdots + |a_n|| = |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|$$

$$|S_n - S_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| = |t_n - t_m| < \varepsilon$$

So, $|S_n - S_m| < \varepsilon$ for $n, m > N$. $(S_n)$ is Cauchy, so it converges.

Therefore, if $\sum_{k=1}^{\infty} |a_k|$ converges, so does $\sum_{k=1}^{\infty} a_k$. $\square$

### Definition : Absolute Convergence

If $\sum_{k=1}^{\infty} |a_k|$ converges, we say $\sum_{k=1}^{\infty} a_k$ converges absolutely.

If $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k|$ does not, we say $\sum_{k=1}^{\infty} a_k$ converges conditionally.

A classic example of conditional convergence is the alternating harmonic series.

### Example : Exercise 4.1 (a)

Does $a_k = (-1)^k \cdot \frac{1}{\sqrt{k}}$ converge absolutely, converge conditionally, or diverge?

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$|a_k| = \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$, which is a $p$-series with $p = \frac{1}{2} < 1$. So, $\sum_{k=1}^{\infty} |a_k|$ diverges.

Alternating series test: each $\frac{1}{\sqrt{k}} > 0$ and $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$. So, $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{1}{\sqrt{k}}$ converges.

Thus, $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}}$ is conditionally convergent.

### Example : Exercise 4.1 (b)

Does $a_k = (-1)^k \cdot \frac{k}{k+7}$ converge absolutely, converge conditionally, or diverge?

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$\lim_{k\to\infty} (-1)^k \cdot \frac{k}{k+7} \neq 0$ (limit doesn't exist because of the oscillating terms). So, $\sum_{k=1}^{\infty} a_k$ diverges.

**Example : Exercise 4.1 (c)**

Does $a_k = \frac{1}{\ln(4)^k}$ converge absolutely, converge conditionally, or diverge?

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$a_k = \frac{1}{\ln(4)^k} = \left(\frac{1}{\ln(4)}\right)^k = (0.721)^k$. $\sum_{k=1}^{\infty}(0.721)^k$ is a geometric series with $r = 0.721 \in (-1,1)$. $a_k$ converges absolutely since $|(0.721)^k| = (0.721)^k$.

**Example : Exercise 4.1 (d)**

Does $a_k = \frac{1}{k!}$ converge absolutely, converge conditionally, or diverge? If it converges, what does it converge to?

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$\sum_{k=1}^{\infty} a_k = \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots$. Intuitively, the denominators are getting very large, so maybe it converges. Try to compare it to a $p$-series that we know converges. $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$. We could use induction to prove that for $k \geq 4$, $k^2 \leq k!$.

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k!} &= 1 + \frac{1}{2} + \frac{1}{6} + \sum_{k=4}^{\infty} \frac{1}{k!} \\ &\leq 1 + \frac{1}{2} + \frac{1}{6} + \sum_{k=4}^{\infty} \frac{1}{k^2}\end{aligned}$$

We know $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\sum_{k=4}^{\infty} \frac{1}{k^2}$ also converges (we can leave off finitely many terms), and $0 \leq \frac{1}{k!} \leq \frac{1}{k^2}$ for $k \geq 4$, so we can use to comparison test to say that $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges. This also converges absolutely since $\frac{1}{k!} = \left|\frac{1}{k!}\right|$ for $k \geq 0$.

To find what it converges to, consider the Taylor series for $e^x$

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ e^1 &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \\ e - 1 &= 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \end{aligned}$$

So, $\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1$.

**Example**

Express $67.\overline{67}$ as a geometric series to show that it is rational.

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$$\begin{aligned}
67.\overline{67} &= 67 + 0.67 + 0.0067 + \dots \\
&= 67\left[1 + \frac{1}{100} + \frac{1}{10000} + \dots\right] \\
&= 67\left[1 + \frac{1}{100} + \frac{1}{100^2} + \dots\right] \\
&= 67\left[\sum_{k=0}^{\infty}\left(\frac{1}{100}\right)^k\right] \qquad r = \frac{1}{100} \in (-1,1) \text{ so it converges} \\
&= 67\left(\frac{1}{1-\frac{1}{100}}\right) = 67\left(\frac{100}{99}\right) = \frac{6700}{99}
\end{aligned}$$

Therefore, $67.\overline{67}$ is rational.