Countable Infinities

Facts about sizes that we can use:

- $|S| = |T|$ if and only if there is a bijection from S to T
- $|S| \leq |T|$ if and only if there is an injection from S to T
- $|S| \geq |T|$ if and only if there is a surjection from S to T

Example

Show that $\mathbb{N} = \{1, 2, 3, \dots\}$ is the same size as $A = \{2, 4, 6, \dots\}$

To show these are the same size, we need $f : \mathbb{N} \to A$, where f is a bijection. Let $f(n) = 2n$ (we can show this is a bijection)

Notice that A is a proper subset of $\mathbb{N}$.

Definition : Infinite Set

One way to define what it means to be an infinite set is that the set is the same as a proper subset of itself.

Example

Are $\mathbb{N}$ and $\mathbb{Z}$ the same size?

$\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} = \{0, 1, -1, 2, -2, \dots\}$.

$\mathbb{N}$	$\mathbb{Z}$
1	0
2	1
3	-1
4	2

Definition : Countable Sets

A countable set is either finite, or is equivalent to $\mathbb{N}$. If $|A| = |\mathbb{N}|$, then we say A is countably infinite, or listable. We can write $A = \{a_1, a_2, a_3, \dots\}$

Example

Are $\mathbb{N}$ and $\mathbb{Q}$ the same size?

Consider the positive rationals:

	1	2	3	4
1	1/1	2/1	3/1	4/1
2	1/2	2/2	3/2	4/2
3	1/3	2/3	3/3	4/3
4	1/4	2/4	3/4	4/4

We are looking for a map between $\mathbb{N}$ and the positive rationals. Consider drawing a line from the top right corner, that would hit every fraction (skipping the duplicates). For example:

$$\frac{1}{1} \to \frac{2}{1} \to \frac{1}{2} \to \frac{1}{3} \to \frac{3}{1} \to \frac{4}{1} \to \frac{3}{2} \to \frac{2}{3} \to \frac{1}{4} \to \dots$$

Then, we have the map:

$$1 \to \frac{1}{1}, 2 \to \frac{2}{1}, 3 \to \frac{1}{2}, 4 \to \frac{1}{3}, 5 \to \frac{3}{1}, \dots$$

If we going, this gives a way to map the positive integers to the positive rationals, but what about 0 and the negatives? Consider the integers, map the negative integers, to the negative rationals (the same way as above), map 0 to 0, and map the positive integers to the positive rational (shown above).

This shows $|\mathbb{Q}| = |\mathbb{Z}|$, but we know $|\mathbb{Z}| = |\mathbb{N}|$, so $|\mathbb{Q}| = |\mathbb{N}|$

Theorem

$|\mathbb{R}| > |\mathbb{N}|$

Proof. Suppose we can list the elements of $\mathbb{R}$. First prove that there are more reals in $(0, 1)$. Real numbers in $(0, 1)$ are of the form:

$$0.a_1a_2a_3a_4\dots \qquad \text{where each } a_i \in \{0, 1, 2, \dots, 8, 9\}$$

Consider:

$$a_1 = 0.a_{11}a_{12}a_{13}a_{14}\dots$$
$$a_2 = 0.a_{21}a_{22}a_{23}a_{24}\dots$$
$$a_3 = 0.a_{31}a_{32}a_{33}a_{34}\dots$$
$$\vdots$$

Cantor's Diagonal Proof: We want a new number that isn't listed.

$$b = 0.b_1b_2b_3\dots \qquad b_i = \begin{cases} 1 & \text{if } a_{i,i} \neq 1 \\ 2 & \text{if } a_{i,i} = 1 \end{cases}$$

This means that $b \neq a_i$ for all i. □

Sizes of infinities

- $|\mathbb{N}| = \aleph_0$ (aleph null)
- $|\mathbb{R}| = \mathfrak{c}$ (continuum)

There are different infinities!