Expectation

Definition : Expected Value

The expected value $E(X)$ (or μ_X or μ) of a discrete random variable X with pmf p is a weighted average of all possible X-values, weighted by the probability that they occur.

$$E(X) = \mu_X = \sum_{x:p(x)>0} x \cdot p(x)$$

Example

Suppose that you play a game in which three dice are rolled. Based on the results, you either lose or win money:

Result	111, 222, or 333	444, or 555	666	All Different	Other
Prize	+$5	+$20	+$100	-$5	$0

What is the expected value of this game?

First, find the probability of each event:

$$\begin{aligned}
\Pr(111, 222, 333) &= \frac{3}{216} = \frac{1}{72} \\
\Pr(444, 555) &= \frac{2}{216} = \frac{1}{108} \\
\Pr(666) &= \frac{1}{216} \\
\Pr(\text{all different}) &= \frac{6 \cdot 5 \cdot 4}{216} = \frac{120}{216} = \frac{5}{9} \\
\Pr(\text{other}) &= 1 - \left(\frac{1}{72} + \frac{1}{108} + \frac{1}{216} + \frac{5}{9}\right) = \frac{126}{216} = \frac{5}{12}
\end{aligned}$$

So, we have:

$$\begin{aligned}
E(X) &= 5 \cdot \frac{3}{216} + 20 \cdot \frac{2}{216} + 100 \cdot \frac{1}{216} + (-5) \cdot \frac{120}{216} + 0 \cdot \frac{126}{216} \\
&= \frac{15}{216} + \frac{40}{216} + \frac{100}{216} - \frac{600}{216} + 0 \\
&= -\frac{445}{216}
\end{aligned}$$

Example

Suppose that a coin is unfairly loaded to land on heads with a probability of α and let X be the number of coin tosses required to observe a head. From a previous lecture, we know that the pmf p of X is defined as:

$$p\left(x\right) = \alpha\left(1-\alpha\right)^{x-1} \quad \text{for } x = 1, 2, 3, \ldots$$

What is the expected value of X? How does it vary with α?

First, let's find the expected value of X:

$$\begin{aligned}
E\left(X\right) &= \sum_{x=1}^{\infty} x \cdot p\left(x\right) \\
&= \sum_{x=1}^{\infty} x \cdot \alpha\left(1-\alpha\right)^{x-1} \\
&= \alpha \sum_{x=1}^{\infty} x \cdot \left(1-\alpha\right)^{x-1}, \quad \text{let } q = 1-\alpha \\
&= \alpha \sum_{x=1}^{\infty} x \cdot q^{x-1}, \quad \text{let } n = x-1 \\
&= \alpha \sum_{n=0}^{\infty} \left(n+1\right) \cdot q^{n}, \quad \text{let } n = x-1 \\
&= \alpha \left[\sum_{n=0}^{\infty} n \cdot q^{n} + \sum_{n=0}^{\infty} q^{n}\right] \\
&= \alpha \left[\frac{q}{\left(1-q\right)^2} + \frac{1}{1-q}\right] \\
&= \frac{\alpha}{\left(1-1+\alpha\right)^2} \qquad = \frac{1}{\alpha}
\end{aligned}$$

Now, as α increases, $E\left(X\right)$ decreases, and as α decreases, $E\left(X\right)$ increases.

We can also find the expected value of a function $g\left(X\right)$ of X. In general, suppose that X is a discrete random variable that takes on possible values x_i, $i \geq 1$, with probabilities $p\left(x_i\right)$. Then, for any real-valued function g,

$$E\left(g\left(X\right)\right) = \sum_{i} g\left(x_i\right) p\left(x_i\right)$$

Example

Suppose that squares of side length X are randomly generated, and that X has pmf p defined by $p(1) = 0.4$, $p(2) = 0.25$, $p(3) = 0.2$, $p(4) = 0.1$, and $p(5) = 0.05$. What is the expected area of this square?

Suppose $g(X) = X^2$.

$$\begin{aligned} E\left(X^2\right) &= 1^2(0.4) + 2^2(0.25) + 3^2(0.2) + 4^2(0.1) + 5^2(0.05) \\ &= 0.4 + 4(0.25) + 9(0.2) + 16(0.1) + 25(0.05) \\ &= 0.40 + 1.00 + 1.80 + 1.60 + 1.25 \\ &= 6.05 \end{aligned}$$

A useful identity: If X is a discrete random variable, and a and b are any constants,

$$E(aX + b) = aE(X) + b$$

Example

Let X be a discrete random variable with a probability mass function defined by $p(1) = 0.45$, $p(3) = 0.25$, and $p(5) = 0.30$. Let $g(X) = 3X + 2$ be a function of X. What is the expected value of $g(X)$?

$g(X) = 3X + 2$, $E(g(X)) = E(3X + 2) = 3E(X) + 2$.

$$\begin{aligned} E(X) &= 1 \cdot 0.45 + 3 \cdot 0.25 + 5 \cdot 0.30 \\ &= 0.45 + 0.75 + 1.50 \\ &= 2.7 \end{aligned}$$

So, $E(3X + 2) = 3(2.7) + 2 = 10.1$