Independence Continued

Example

Consider an urn containing four balls, numbered 110, 101, 011, and 000, from which one ball is drawn at random. For $k = 1, 2, 3$, let A_k be the event of drawing a ball with a 1 in the kth position.

$$\begin{aligned} A_1 &= \{110, 101\} \\ \Pr(A_1) &= \frac{2}{4} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} A_2 &= \{110, 011\} \\ \Pr(A_2) &= \frac{2}{4} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} A_3 &= \{101, 011\} \\ \Pr(A_3) &= \frac{2}{4} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} A_1 \cap A_2 &= \{110\} \\ \Pr(A_1 \cap A_2) &= \frac{1}{4} = \Pr(A_1) \cdot \Pr(A_2) \end{aligned}$$

$$\begin{aligned} A_1 \cap A_3 &= \{101\} \\ \Pr(A_1 \cap A_3) &= \frac{1}{4} = \Pr(A_1) \cdot \Pr(A_3) \end{aligned}$$

$$\begin{aligned} A_2 \cap A_3 &= \{011\} \\ \Pr(A_2 \cap A_3) &= \frac{1}{4} = \Pr(A_2) \cdot \Pr(A_3) \end{aligned}$$

$$\begin{aligned} A_1 \cap A_2 \cap A_3 &= \emptyset \\ \Pr(A_1 \cap A_2 \cap A_3) &= 0 \neq \Pr(A_1) \cdot \Pr(A_2) \cdot \Pr(A_3) \end{aligned}$$

This shows that events can be pairwise independent, but not 3-way independent.

More generally, a finite set $E_1, E_2, \ldots E_n$ of events is said to be independent if all 2-way, 3-way, $\ldots$, n-way product rules hold.

That is, for all indexing sets $I \subseteq \{1, 2, \ldots, n\}$ of order $2 \leq r \leq n$,

$$\Pr\left(\bigcap_{i \in I} E_i\right) = \prod_{i \in I} \Pr(E_i)$$

Definition : Random Variable

Let Ω be a sample space of some experiment. A random variable (rv) X is a function $X : \Omega \to \mathbb{R}$. That is, X is mapping of each outcome in $\omega \in \Omega$ to some real number $X(\omega) \in \mathbb{R}$. We define X by some numerical measure of interest that can about each $\omega \in \Omega$.

A random variable is discrete if its possible values either

(i) constitute a finite set, or

(ii) can be listed in an infinite sequence in which there is a first element, a second element, and so on (countably infinite)

Example

The number of pumps in use at both a six-pump and a four-pump station will be determined. The sample space Ω consists of all $2^{10} = 1024$ 10-tuples.

$$(A_1, A_2, A_3, A_4, A_5, A_6, B_1, B_2, B_3, B_4)$$

where $A_i = 1$ if pump i is in use at the first gas station, and 0 otherwise (similarly for B_j). What are the possible values for each of the following random variables?

(a) X is the total number of pumps in use

(b) X is the difference between the number of pumps in use at station 1 and station 2

(c) X is the maximum number of pumps in use at either station

(d) X is the number of stations having exactly two pumps in use

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(a) X is the total number of pumps in use

$$X = (A_1 + A_2 + A_3 + A_4 + A_5 + A_6) + (B_1 + B_2 + B_3 + B_4)$$

Minimum value $\Rightarrow X = 0$, Maximum value $\Rightarrow X = 10$.

$$X \in \{0, 1, \ldots, 9, 10\}$$

(b) X is the difference between the number of pumps in use each station

$$X = (A_1 + A_2 + A_3 + A_4 + A_5 + A_6) - (B_1 + B_2 + B_3 + B_4)$$

Minimum value $\Rightarrow X = -4$, Maximum value $\Rightarrow X = 6$.

$$X \in \{-4, -3, \ldots, 5, 6\}$$

(c) X is the maximum number of pumps in use at either station

$$X = \max\left((A_1 + A_2 + A_3 + A_4 + A_5 + A_6), (B_1 + B_2 + B_3 + B_4)\right)$$

Minimum value $\Rightarrow X = 0$, Maximum value $\Rightarrow X = 6$.

$$X \in \{0, 1, \ldots 6\}$$

(d) X is the numbers of stations having exactly two pumps in use.

Minimum value $\Rightarrow X = 0$, Maximum value $\Rightarrow X = 2$.

$$X \in \{0, 1, 2\}$$