

Eigenvalues

How do we compute λ_1 and λ_2 for

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. Subtract λ from the main diagonal entries

$$\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}$$

2. Calculate the determinant of the above matrix

$$\det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = (a - \lambda)(d - \lambda) - (b)(c)$$

3. Rearrange as a quadratic in λ

$$\lambda^2 + (a + d) \cdot \lambda + (ad - bc)$$

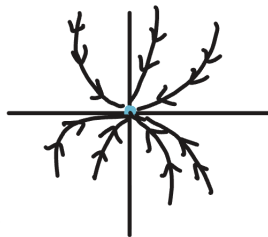
4. Solve this determinant equal to 0

$$\lambda^2 + (a + d) \cdot \lambda + (ad - bc) = 0$$

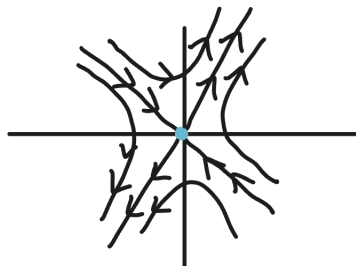
Either factor or use the quadratic formula to get λ_1 and λ_2

Classifying behaviour of $\dot{x} = ax + by$, $\dot{y} = cx + dy$ based on λ_1, λ_2

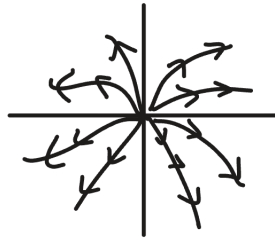
1. $\lambda_1 < 0, \lambda_2 < 0$, both $\lambda_1, \lambda_2 \in \mathbb{R}$. \Rightarrow Stable, called a sink or a node



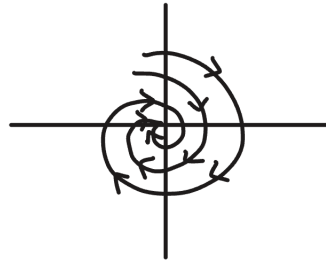
2. $\lambda_1 > 0, \lambda_2 < 0$, both $\lambda_1, \lambda_2 \in \mathbb{R}$. \Rightarrow Unstable, called a saddle



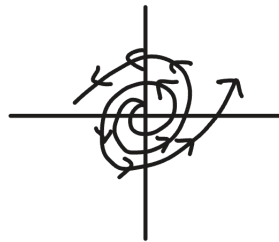
3. $\lambda_1 > 0, \lambda_2 > 0$, both $\lambda_1, \lambda_2 \in \mathbb{R}$. \Rightarrow Unstable, called a source



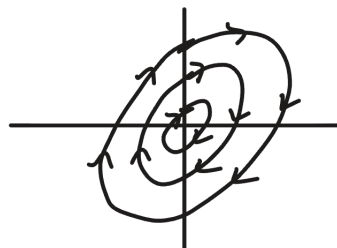
4. $\lambda_{1,2} = \alpha \pm \beta i, \alpha < 0. \Rightarrow$ Stable spiral



5. $\lambda_{1,2} = \alpha \pm \beta i, \alpha > 0. \Rightarrow$ Unstable spiral



6. $\lambda_{1,2} = \alpha \pm \beta i, \alpha = 0. \Rightarrow$ Neutrally stable



This is all linear theory. Most models we encounter are not linear, so what do we do? We can extend linear stability analysis to 2 dimensions via “the Jacobian”.

We will need a multivariate Taylor Series about the equilibrium (x^*, y^*) for a small perturbation (\hat{x}, \hat{y}) .

