Functional Limits

Example

Prove that $\lim_{x\to 2} \frac{3x^2-12}{x-2} = 12$

Let $\varepsilon > 0$,

$$|f(x) - L| = \left|\frac{3x^2-12}{x-2} - 12\right| = \left|\frac{3x^2-12x+12}{x-2}\right| = \left|\frac{3(x-2)^2}{x-2}\right| = 3|x-2|$$

So, we have $|f(x) - L| = 3|x-2|$, and we want to find δ so that $0 < |x-2| < \delta$ implies $|f(x) - L| < \varepsilon$. Let $\delta = \frac{\varepsilon}{3}$, then $|f(x) - L| < 3|x-2| < 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ for $0 < |x-2| < \delta$. Therefore, $\lim_{x\to 2} \frac{3x^2-12}{x-2} = 12$

Properties of Limits

Proposition

$\lim_{x\to c} f(x)$ can converge to at most 1 value. That is, if the limit exists, then it is unique.

Proof. Suppose c is a limit point of A, and

$$\lim_{x\to c} f(x) = L_1 \quad \text{and} \quad \lim_{x\to c} f(x) = L_2$$

Let $\varepsilon > 0$, then $\frac{\varepsilon}{2} > 0$. There exists $\delta_1 > 0$ such that for $x \in A$ with $0 < |x-c| < \delta_1$, we have that $|f(x) - L_1| < \frac{\varepsilon}{2}$. Also, there exists $\delta_2 > 0$ such that for $x \in A$ with $0 < |x-c| < \delta_2$, we have that $|f(x) - L_2| < \frac{\varepsilon}{2}$. We want both conditions to hold, so let $\delta = \min\{\delta_1, \delta_2\}$.

$$\begin{aligned}
|L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| && \text{By the Triangle Inequality} \\
&\le |L_1 - f(x)| + |f(x) - L_2| \\
&= |f(x) - L_1| + |f(x) = L_2| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{for } x \in A \text{ with } 0 < |x-c| < \delta \\
&= \varepsilon
\end{aligned}$$

That is, $|L_1 - L_2| < \varepsilon$ for any $\varepsilon > 0$, thus $L_1 - L_2 = 0 \Rightarrow L_1 = L_2$ (This was a result from chapter 1). Therefore, if the limit exists, it is unique. □

Theorem

Let $A \subseteq \mathbb{R}$, c is a limit point of A, then $\lim_{x\to c} f(x) = L$ if and only if for every sequence (a_n) for which each $a_n \in A$ with $a_n \neq c$ and $a_n \to c$, we have $f(a_n) \to L$.

Proof. Assume $\lim_{x\to c} f(x) = L$. Let (a_n) be a sequence with $a_n \in A \setminus \{c\}$ where $a_n \to c$. We want to show that the sequence $(f(a_n))$ converges to L. Let $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x) - L| < \varepsilon$. For this $\delta > 0$, there exists

N such that $|a_n - c| < \delta$ for $n > N$ (because $a_n \to c$). Thus, for $n > N$, we have $f(x) \in (L - \varepsilon, L + \varepsilon) \Leftrightarrow |f(a_n) - L| < \varepsilon$. Therefore, $f(a_n) \to L$.

Assume that for every sequence with $a_n \to c$, we have $f(a_n) \to L$. We want to show $\lim_{x\to c} f(x) = L$. Suppose that $\lim_{x\to c} f(x) \neq L$, so there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$. In particular, let $\delta_n = \frac{1}{n}$ for $n \in \mathbb{N}$, there is a sequence (x_n) in $A \setminus \{c\}$ such that $0 < |x_n - c| < \delta_n$ for which $|f(x_n) - L| \geq \varepsilon$ for each $n \in \mathbb{N}$. Then, $x_n \to c$ because $|x_n - c| < \frac{1}{n}$ for all $n \in \mathbb{N}$, however $f(x_n)$ does not converge to L because $|f(x_n) - L| \geq \varepsilon$ for all n. This contradicts our assumption. The contradiction came from assuming the limit wasn't L. Thus, $\lim_{x\to c} f(x) = L$.

Therefore, $\lim_{x\to c} f(x) = L$ if and only if for every sequence $a_n \to c$ with $a_n \neq c$, we have $f(a_n) \to L$. $\square$

Example

Show that $\lim_{x\to 0} f(x)$ does not exist, where $f(x)$ is the Dirichlet function.

Idea - Find two sequences that both converge to 0, but have different limits when we apply f.

Let $a_n = \frac{1}{n}$, each $a_n \in \mathbb{Q}$, and $a_n \neq 0$ for all n, and $a_n \to 0$. $f(a_n) = 1$ for all n, so $(f(a_n)) \to 1$.

Let $b_n = \frac{\sqrt{2}}{n}$, each $b_n \notin \mathbb{Q}$, and $b_n \neq 0$ for all n, and $b_n \to 0$. $f(b_n) = 0$ for all n, so $(f(b_n)) \to 0$.

We have two sequences (a_n) and (b_n) that both converge to 0, but $(f(a_n))$ and $(f(b_n))$ have different limits. Thus, $\lim_{x\to 0} f(x)$ does not exist.