Warm Up

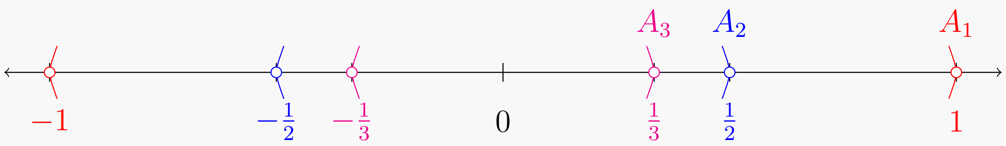
Example

Let $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$.

(1) Find $\bigcup_{n=1}^{\infty} A_n$

(2) Find $\bigcap_{n=1}^{\infty} A_n$

$$A_1 = (-1, 1) = \{x : -1 < x < 1\}$$
$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$$
$$A_3 = \left(-\frac{1}{3}, \frac{1}{3}\right)$$



$$\bigcup_{n=1}^{\infty} A_n = (-1, 1) \quad \text{since} \quad \bigcup_{n=1}^{\infty} A_n = \{x : \exists n \in \mathbb{N}, x \in A_n\}$$
$$\bigcap_{n=1}^{\infty} A_n = \{0\} \quad \text{since} \quad \bigcap_{n=1}^{\infty} A_n = \{x : \forall n \in \mathbb{N}, x \in A_n\}$$

Functions

Definition : Injective / One-to-one / 1-1

A function $f : A \to B$ is injective if $f(a) = f(b)$ forces $a = b$.

Remark : When is a function not injective?

A function is not injective when there exists a, b with $f(a) = f(b)$, and $a \neq b$.

Definition : Surjective / Onto

A function $f : A \to B$ is surjective if $f(A) = B$ (Note that $f(A)$ is the image, $f(A) = \{f(a) : a \in A\}$). That is, for all $b \in B$, there exists $a \in A$ with $f(a) = b$.

Remark : When is a function not surjective?

A function is not surjective when there exists $b \in B$ such that for all $a \in A$, $f(a) \neq b$.

Example

Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 1$ is injective and surjective.

Injective:

Suppose $f(a) = f(b)$, then

$$\begin{aligned} 3a + 1 &= 3b + 1 \\ 3a &= 3b \\ a &= b \end{aligned}$$

This is true for all $a, b \in \mathbb{R}$, thus f is injective.

Surjective:

Let $b \in \mathbb{R}$, now try to solve $f(x) = b$.

$$\begin{aligned} 3x + 1 &= b \\ 3x &= b - 1 \\ x &= \frac{b-1}{3} \end{aligned}$$

$x = \frac{b-1}{3} \in \mathbb{R}$ for all b, that is $f(x) = b$ will always have a solution. Thus, f is surjective.

Example

Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is neither injective nor surjective.

Injective:

We know $f(1) = f(-1)$, so f is not injective.

Surjective:

Let $b = -1$, then we have $x^2 = -1$, but this has no solution in $\mathbb{R}$, so f is not surjective.

Definition : Image of a subset

Suppose $f : X \to Y$, and let $A \subseteq X$. The image of the subset A is the set obtained by applying the function f to every element of A. This is denoted by $f(A)$.

$$f(A) = \{f(a) : a \in A\}$$

Definition : Preimage of a Subset

Suppose $f : X \to Y$, and let $B \subseteq Y$. The preimage of B is everything in X that gets mapped to B by f. This is denoted by $f^{-1}(B)$.

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

Example : Textbook Exercise 1.6

Suppose $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.

(a) Prove that $f(f^{-1}(B)) \subseteq B$

(b) Give an example where $f(f^{-1}(B)) \neq B$

(a) Let $b \in f(f^{-1}(B))$. So there exists $a \in f^{-1}(B)$ such that $f(a) = b$.

$a \in f^{-1}(B)$ so $f(a) \in B$ by the definition of the preimage. Thus, $b = f(a) \in B$.

$b \in f(f^{-1}(B))$ was arbitrary, so for all $b \in f(f^{-1}(B))$, we have $b \in B$. Thus, $f(f^{-1}(B)) \subseteq B$.

(b) Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, and $B = [-3, 4]$.

$$\begin{aligned} f^{-1}(B) &= \{x \in \mathbb{R} : f(x) \in [-3, 4]\} \\ &= [-2, 2] \end{aligned}$$

$$\begin{aligned} f(f^{-1}(B)) &= f([-2, 2]) \\ &= [0, 4] \end{aligned}$$

Here, $f(f^{-1}(B)) = [0, 4]$, and $[0, 4] \neq [-3, 4]$.

Bijection

A function that is both injective and surjective is a bijection. This means that every element in A gets mapped to a unique element in B, and everything in B has a solution in A. If we have $f : X \to Y$, and that f is a bijection, then we can define an inverse function $f^{-1} : B \to A$ where $f^{-1}(b) = a \Leftrightarrow f(a) = b$.