Convergence and Divergence

Example

Suppose $a_n = a$ for some $a \in \mathbb{R}$ for all n. Prove that a_n converges to a.

Idea: The limit of a constant sequence is the constant. Let $\varepsilon > 0$, we need to find $N \in \mathbb{R}$ such that for $n > N$, we have $|a_n - a| < \varepsilon$. For any $n \in \mathbb{N}$, $a_n = a$, so

$$|a_n - a| = |a - a| = 0 < \varepsilon \qquad \text{for any } \varepsilon > 0$$

So, $N = 0$ works to prove a_n converges to a.

Example : Exercise 3.7 (a)

Suppose we have a sequence (a_n) where each $a_n \in \mathbb{Z}$. Is it possible that $a_n \to 3.5$?

No, find $\varepsilon > 0$ such that $(3.5 - \varepsilon, 3.5 + \varepsilon)$ doesn't contain any integers. Consider $\varepsilon = \frac{1}{2}$, $(3, 4)$ doesn't contain any integers. Thus, for all $n \in \mathbb{N}$, $a_n \notin (3, 4)$. That is, for any $N \in \mathbb{R}$, there is some $n > N$ such that $a_n \notin (3, 4)$.

$$|a_n - 3.5| \geq \varepsilon = \frac{1}{2} \qquad \text{for some } n > N$$

Thus, a_n does not converge to 3.5

Example : Exercise 3.7 (b)

Suppose we have a sequence (a_n) where each $a_n \in \mathbb{Z}$. If $a_n \neq a_M$ for all $n \neq m$, prove that the sequence doesn't converge.

Suppose $a_n \to a$ for some $a \in \mathbb{R}$. Consider $\left(a - \frac{1}{2}, a + \frac{1}{2}\right)$. There is at most one integer in this interval (there may be none).

If there are no integers in $\left(a - \frac{1}{2}, a + \frac{1}{2}\right)$, then $a_n \notin \left(a - \frac{1}{2}, a + \frac{1}{2}\right)$ for all n, so it is not possible for $a_n \to a$ (like in part (a))

Suppose $z \in \left(a - \frac{1}{2}, a + \frac{1}{2}\right)$ with $z \in \mathbb{Z}$.

If $\exists\, m \in \mathbb{N}$ with $a_m = z$, then for all $n \neq m$, we have $a_n \notin \left(a - \frac{1}{2}, a + \frac{1}{2}\right)$. Thus, for any N, $\exists\, n \in \mathbb{N}$ with $a_n \notin \left(a - \frac{1}{2}, a + \frac{1}{2}\right)$.

If $a_n \neq z$ for any n, then again, $a_n \notin \left(a - \frac{1}{2}, a + \frac{1}{2}\right)$ and a_n can't converge to a.

Example : Exercise 3.7 (c)

Suppose we have a sequence (a_n) where each $a_n \in \mathbb{Z}$. If (a_n) converges, what can we say?

So a_n can converge to an integer. Eventually, for some $N \in \mathbb{R}$ we have $a_n = z$ for all $n > N$. The sequence does not have to be constant, but eventually it stays the same.

Monotone Convergence Theorem

Definition : Monotone Increasing and Monotone Decreasing

A sequence (a_n) is monotone increasing if

$$a_n \leq a_{n+1} \qquad \text{for all } n$$

Likewise, a sequence (a_n) is monotone decreasing if

$$a_n \geq a_{n+1} \qquad \text{for all } n$$

Definition : Monotone

If a sequence (a_n) is either monotone increasing or monotone decreasing, we say that it is monotone.

Example

Is $a_n = \frac{1}{n}$ monotone increasing, monotone decreasing, both, or neither?

This is the sequence: $\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$. This is monotone decreasing since

$$\frac{1}{n} \geq \frac{1}{n+1} \qquad \text{for all } n \in \mathbb{N}$$

Example

Is $a_n = n^2$ monotone increasing, monotone decreasing, both, or neither?

This is the sequence: $(1, 4, 9, \dots)$. This is monotone increasing since

$$n^2 \leq (n+1)^2 \qquad \text{for all } n \in \mathbb{N}$$

Example

Is $a_n = (-1)^2$ monotone increasing, monotone decreasing, both, or neither?

This is the sequence: $(-1, 1, -1, 1, \dots)$. This is neither monotone increasing, nor monotone decreasing.

Example

Is $a_n = (1)^2$ monotone increasing, monotone decreasing, both, or neither?

This is the sequence: $(1, 1, 1, \dots)$. This is both monotone increasing and monotone decreasing.

Theorem : Monotone Convergence Theorem (MCT)

Suppose (a_n) is monotone. Then (a_n) converges if and only if it is bounded. Moreover, if (a_n) is increasing, then (a_n) either diverges to $+\infty$ or $\lim_{n\to\infty} a_n = \sup(\{a_n : n \in \mathbb{N}\})$. If (a_n) is decreasing, then either it diverges to $-\infty$ or $\lim_{n\to\infty} a_n = \inf(\{a_n : n \in \mathbb{N}\})$.

Proof. Assume (a_n) is monotone. If a_n is convergent, then it is bounded. (We have already proven that any convergent sequence is bounded).

Suppose that it is not bounded. Then, it is not convergent. Assume (a_n) is increasing. Let $M > 0$, M is not an upper bound for the sequence. There exists $N \in \mathbb{N}$ with $a_N > M$. Thus, for $n \geq N$ we have $a_n \geq a_N > M$. So, $a_n > M$ for $n > N$. Thus, $a_n \to +\infty$.

Likewise, if (a_n) is decreasing and unbounded, $a_n \to -\infty$.

Consider the set $\{a_n : n \in \mathbb{N}\}$ in the case that the sequence is monotone increasing and bounded above. We have a subset of $\mathbb{R}$ that is bounded above, so by completeness, it has a least upper bound. Let $\alpha = \sup(\{a_n : n \in \mathbb{N}\})$. Our claim is that $\lim_{n\to\infty} a_n = \alpha$.

Let $\varepsilon > 0$, $\alpha - \varepsilon$ is not an upper bound. There exists $N \in \mathbb{N}$ such that $a_N > \alpha - \varepsilon$ so for all $n \geq N$, $a_n \geq a_N \geq \alpha - \varepsilon$. Thus, $\alpha - \varepsilon < a_n \leq \alpha < \alpha + \varepsilon$ for $n \geq N$. Thus, $|a_n - \alpha| < \varepsilon$ for $n \geq N$, so $\lim_{n\to\infty} a_n = \alpha$. Note: We have $a_n \leq \alpha$ for all n since α is an upper bound.

Similarly if (a_n) is monotone decreasing and bounded below, then $\lim_{n\to\infty} \inf(\{a_n : n \in \mathbb{N}\})$. $\square$