

Why can you not divide by the additive identity?

The additive identity is 0, such that $a + 0 = a$ for all $a \in \mathbb{F}$. Also, note that “divide” means to multiply by the multiplicative inverse (In a field, every non-zero element has a multiplicative inverse a^{-1} with $a \cdot a^{-1} = 1$). So, why doesn’t 0 have a multiplicative inverse?

If $a, b, c \in \mathbb{F}$ and,

$$a + c = b + c, \quad \text{then } b = c \quad (\text{cancellation law})$$

Proof. Let $a, b, c \in \mathbb{F}$ with $a + c = b + c$.

By axiom 5, c has an additive inverse, $-c$ where $c + (-c) = 0$.

$$\begin{aligned} (a + c) &= (b + c) \\ (a + c) + (-c) &= (b + c) + (-c) \\ a + (c + (-c)) &= b + (c + (-c)) && \text{by axiom 3 (associativity)} \\ a + 0 &= b + 0 && \text{by axiom 5 (additive inverse)} \\ a &= b && \text{by axiom 4 (identity)} \end{aligned}$$

□

Now, $0 \cdot a = 0$ for all $a \in \mathbb{F}$

Proof. We know $0 = 0 + 0$ by axiom 4.

$$\begin{aligned} 0 &= 0 + 0 \\ a \cdot 0 &= a \cdot (0 + 0) \\ a \cdot 0 &= a \cdot 0 + a \cdot 0 && \text{by axiom 2 (distributive property)} \\ a \cdot 0 + 0 &= a \cdot 0 + a \cdot 0 && \text{by axiom 4} \\ 0 + a \cdot 0 &= a \cdot 0 + a \cdot 0 && \text{by axiom 1 (commutativity)} \\ 0 &= a \cdot 0 && \text{by cancellation law} \end{aligned}$$

Thus, $a \cdot 0 = 0$ for any a .

□

For an element b to have a multiplicative inverse, we need $b \cdot b^{-1} = 1$

$$\begin{aligned} 0 \cdot a &= a \cdot 0 && \text{by axiom 1} \\ &= 0 \end{aligned}$$

So, there is no element $a \in \mathbb{F}$ with $0 \cdot a = 1$. Thus, 0 does not have a multiplicative inverse.

Ordered Fields

Recall from lecture 5:

Definition : Ordered Field

An ordered field is a field \mathbb{F} along with:

6. There is a nonempty subset $P \subset \mathbb{F}$ called the positive elements such that:

- (a) If $a, b \in P$ then $a + b \in P$ and $a \cdot b \in P$
- (b) If $a \in \mathbb{F}$ and $a \neq 0$, then $a \in P$ or $-a \in P$ but not both. (This is called order)

0 is its own inverse, so $0 = -0$. For all $a \neq 0$, $a \neq -a$ so $0 \notin P$.

Claim: $1 \in P$

Proof. If $1 \notin P$, then $-1 \in P$. Suppose $a \in P$, then $(a) \cdot (-1) = -a \in P$ by axiom 6. However, we know $(a) \cdot (1) = 1$ since 1 is the multiplicative identity. So, both $a, -a \in P$, contradicting axiom 6. Thus, $1 \in P$. \square

Definition

Let \mathbb{F} be an ordered field with $a, b \in \mathbb{F}$. We say “ $a < b$ ” (a is less than b) if $b - a \in P$, and “ $a \leq b$ ” (a is less than or equal to b) if either $a = b$ or $a < b$.

Note: $c > 0$ means $c \in P$, $c < 0$ means $-c \in P$.

Some Nice Properties – Let $a, b, c \in \mathbb{F}$

- (a) If $a < b$, then $a + c < b + c$
- (b) If $a < b$ and $b < c$, then $a < c$
- (c) If $a < b$, then $ac < bc$ for $c > 0$ and $ac > bc$ if $c < 0$
- (d) If $a \neq 0$, then $a^2 > 0$

Proof. Proof of (a)

Assume $a < b$. Then, $b - a \in P$, that is $b - a > 0$. We want to prove $a + c < b + c$. Consider:

$$\begin{aligned}
 (b + c) - (a + c) &= b + c - a - c && \text{(associativity)} \\
 &= b - a + c - c && \text{(commutativity)} \\
 &= b - a + 0 && \text{(additive inverse)} \\
 &= b - a > 0 && \text{(identity)}
 \end{aligned}$$

\square

Definition : Absolute Value

If \mathbb{F} is an ordered field, define the absolute value function $| \cdot | : \mathbb{F} \rightarrow \mathbb{F}$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Properties of absolute values will be important for limits.

$$\lim_{x \rightarrow a} f(x) = b$$

In calculus, we say we can make the values of $f(x)$ get as close to b as we like by taking x close to a .

In Real Analysis, we say: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |f(x) - b| < \varepsilon$ when $|x - a| < \delta$.

