Permutations

A permutation is an ordered rearrangement of objects. For example, there are 6 permutations of the objects, A, B, and C: ABC, ACB, BAC, BCA, CAB, CBA.

If n distinct objects are to be arranged in order, there are:

- n choices for the first object,
- $n-1$ choices for the second object,
- $n-2$ choices for the third object,
- ...
- 2 choices for the second-last object, and
- 1 choice left for the last object

By the generalized principle of counting, there are

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

different permutations of the n objects. We read $n!$ as "n factorial" and define $0! = 1$.

Example

Twenty workers are to be assigned to twenty different jobs, one to each job. How many work assignments are possible?

The first job can be assigned to any of the 20 workers. The second job can be assigned to any of the remaining 19 workers, and so on. Each assignment has 1 fewer worker to choose from. So there are

$$20! = 20 \cdot 19 \cdot 18 \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = 2.432902 \cdot 10^{18}$$

possible work assignments

Example

In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if
(a) the books can be in any order?
(b) the mathematics books must be together and the novels must be together?
(c) the novels must be together, but the other books can be arranged in any order?

(a) If the books can be arranged in any order, there are $6! = 720$ ways
(b) If the mathematics books must be together and the novels must be together, there would be $3!\cdot 2!\cdot 3! = 72$ ways. The math "block" has 2! permutations, the novels "block" has 3! permutations, and the "blocks" themselves have 3! permutations.
(c) If the novels must be together, there would be $4! \cdot 3!$ ways. The novels "block" has 3! permutations, and the "blocks" have 4! permutations.

Remark

If some objects are indistinguishable from each other, the number of permutations is reduced.

In general, there are

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, etc.

Example

When all letters are used, how many letter arrangements can be made from the letters in the word ABBA?

There are $\frac{4!}{2! \cdot 2!} = 6$ arrangements. They are, AABB, ABBA, BBAA, ABBA, BAAB, ABAB, and BABA.

Combinations

A combination is an unordered selection of r objects from a list of n objects. In general, there are n choose r possible combinations of r objects chosen from n objects.

$$\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

Example

A student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if:
(a) both books are on the same subject?
(b) both books are to be on a different subject?

(a) This means both are either math, science or economics.

$$\binom{6}{2} + \binom{7}{2} + \binom{4}{2} = 42$$

(b) This means either 1 math, 1 science, or 1 science, 1 economics, or 1 math, 1 economics.

$$\binom{17}{2} - \left[\binom{6}{2} + \binom{7}{2} + \binom{4}{2}\right] = 136 - 42 = 94$$

This could also be done as follows:

$$6 \cdot 7 + 7 \cdot 4 + 6 \cdot 4 = 42 + 28 + 24 = 94$$

Example

A committee of 7 consists of 2 Republicans, 2 Democrats, and 3 Independents. If this committee is chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents, how many committees are possible?

There would be

$$\binom{5}{2} \cdot \binom{6}{2} \cdot \binom{4}{3} = \frac{5!}{3! \cdot 2!} \cdot \frac{6!}{4! \cdot 2!} \cdot \frac{4!}{1! \cdot 3!} = 10 \cdot 15 \cdot 4 = 600$$

possible committees.

Example

1. Eight birds are to be placed in eight different cages. How many arrangements are possible if each bird is placed in a separate cage?
2. Decide whether each of the following problems involves a permutation or a combination and then work out the answer.
   (a) How many 4 digit numbers can be made from the digits 2, 3, 5, 6, 7 and 9 if no repetition of digits is allowed.
   (b) A student has to answer 8 out of 10 questions in an exam. How many different choices does she have?
   (c) How many different car number plates can be made if each plate contains 3 different letters followed by 3 distinct digits?
   (d) How many ways are there of playing a game of lotto requiring you to select 6 correct numbers out of 44?
3. For the following questions, there are 5 men and 3 women.
   (a) How many committees of 4 can be chosen?
   (b) How many of these would be all men?
   (c) How many would consist of 2 men and 2 women?

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(1) There are $8! = 40320$ arrangements.
(2) (a) Permutations, $\frac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$
(2) (b) Combinations, $\binom{10}{8} = \frac{10!}{2! \cdot 8!} = 45$
(2) (c) Permutations, $\frac{26! \cdot 10!}{23! \cdot 7!} = 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11232000$
(2) (d) Combinations, $\binom{44}{6} = \frac{44!}{38! \cdot 6!} = 7059052$
(3) (a) There would be $\binom{8}{4} = \frac{8!}{4! \cdot 4!} = 70$ committees
(3) (b) There would be $\binom{5}{4} = \frac{5!}{1! \cdot 4!} = 5$ committees.
(3) (c) There would be $\binom{5}{2} \cdot \binom{3}{2} = \frac{5!}{3! \cdot 2!} \cdot \frac{3!}{1! \cdot 2!} = 10 \cdot 3 = 30$

Binomial Theorem

The binomial coefficient $\binom{n}{r}$ represents the number of distinct subsets of r that can be chosen from a set of n. The Binomial Theorem uses binomial coefficients to expand polynomials with two terms:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example

Expand $(a+b)^5$

$$\begin{aligned}(a+b)^5 &= \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

Example

Simplify

$$\sum_{k=0}^{n} \binom{n}{k} 2^k 5^{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^k 5^{n-k} = (2+5)^n = 7^n$$

Example

How many subsets does an n-element set have?

An n-element subset has 2^n subsets.