

What is a number?

As a reminder the rationals are: $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$. A nice property of \mathbb{Q} is that between any two rational numbers, there is another rational number. That is, if $x, y \in \mathbb{Q}$, $\exists z \in \mathbb{Q}$ with $x < z < y$. For example, $z = \frac{x+y}{2}$.

In general, \mathbb{Q} is closed under addition, subtraction, multiplication, and division (by a nonzero rational). That is, if $p, q \in \mathbb{Q}$, then:

- $p + q \in \mathbb{Q}$
- $p - q \in \mathbb{Q}$
- $p \times q \in \mathbb{Q}$
- $p/q \in \mathbb{Q}, q \neq 0$

Proof. (Closure under addition)

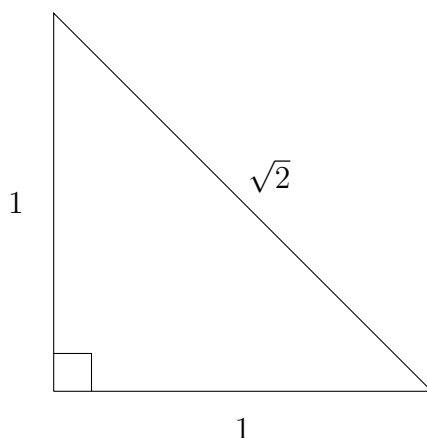
Assume $x, y \in \mathbb{Q}$, this means $\exists p_1, q_1, p_2, q_2 \in \mathbb{Z}$ with $q_1, q_2 \neq 0$ such that:

$$\begin{aligned} x &= \frac{p_1}{q_1}, & y &= \frac{p_2}{q_2} \\ x + y &= \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ &= \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \\ &= \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \in \mathbb{Z}, q \neq 0 \end{aligned}$$

□

\mathbb{N} is not closed under subtraction. For example, $5 - 7 = -2 \notin \mathbb{N}$.

There are some problems with \mathbb{Q} . We can have $p \in \mathbb{Q}$, but $\sqrt{p} \notin \mathbb{Q}$, for example, 2. $\sqrt{2}$ is irrational, although it is constructible.



There are holes in the rationals. You could have $p_i \in \mathbb{Q}$ for all $i \in \mathbb{N}$, but the sequence $\{p_i\}$ converges to some $p \notin \mathbb{Q}$.

\mathbb{Q} is not algebraically closed, we can have polynomials with rational coefficients whose roots are not rational. For example $x^2 - 2 = 0$.

Fields

Definition : Field

A field is a nonempty set \mathbb{F} along with 2 binary operations, addition (+) and multiplication (\cdot), satisfying the following axioms:

1. If $a, b \in \mathbb{F}$, then

$$\left. \begin{aligned} a + b &= b + a \\ a \cdot b &= b \cdot a \end{aligned} \right\} \text{Operations are commutative}$$

2. If $a, b, c \in \mathbb{F}$, then

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{Distributive property}$$

3. If $a, b, c \in \mathbb{F}$, then

$$\left. \begin{aligned} a + (b + c) &= (a + b) + c \\ a \cdot (b \cdot c) &= (a \cdot b) \cdot c \end{aligned} \right\} \text{Operations are associative}$$

4. There are elements $0, 1 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$:

$$\begin{aligned} a + 0 &= a & 0 \text{ is the additive identity} \\ a \cdot 1 &= a & 1 \text{ is the multiplicative identity} \end{aligned}$$

5. For each $a \in \mathbb{F}$ there is $-a \in \mathbb{F}$ such that:

$$a + (-a) = 0 \quad \text{Additive inverse}$$

For each $a \in \mathbb{F}$ with $a \neq 0$, there is $a^{-1} \in \mathbb{F}$ such that:

$$a \cdot a^{-1} = 1 \quad \text{Multiplicative inverse}$$

\mathbb{Q} is a field, but \mathbb{N} is not. (\mathbb{N} has no additive identity or inverses)

The identities in a field are unique.

Proof. (Proof for the additive identity)

Suppose 0_1 and 0_2 are additive identities. So, $0_1, 0_2 \in \mathbb{F}$, and for all $a \in \mathbb{F}$ we have that:

$$\begin{aligned} a + 0_1 &= a \\ a + 0_2 &= a \end{aligned}$$

Since $0_1 \in \mathbb{F}$, and 0_2 is an additive identity, we have:

$$\begin{aligned} 0_1 + 0_2 &= 0_1 \\ 0_1 + 0_2 &= 0_2 + 0_1 && \text{By commutativity} \\ &= 0_2 && \text{Since } 0_2 \in \mathbb{F} \text{ and } 0_1 \text{ is an additive identity} \end{aligned}$$

Thus, $0_1 = 0_2$, and the additive identity is unique. □

Ordered Fields

Definition : Ordered Field

An ordered field is a field \mathbb{F} along with:

6. There is a nonempty subset $P \subset \mathbb{F}$ called the positive elements such that:

(a) If $a, b \in P$ then $a + b \in P$ and $a \cdot b \in P$

(b) If $a \in \mathbb{F}$ and $a \neq 0$, then $a \in P$ or $-a \in P$ but not both. (This is called order)

0 is its own inverse, so $0 = -0$. For all $a \neq 0$, $a \neq -a$ so $0 \notin P$.