

## What is a number?

As a reminder the rationals are:  $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0 \right\}$ . A nice property of  $\mathbb{Q}$  is that between any two rational numbers, there is another rational number. That is, if  $x, y \in \mathbb{Q}, \exists z \in \mathbb{Q}$  with  $x < z < y$ . For example,  $z = \frac{x+y}{2}$ .

In general,  $\mathbb{Q}$  is closed under addition, subtraction, multiplication, and division (by a nonzero rational). That is, if  $p, q \in \mathbb{Q}$ , then:

- $p + q \in \mathbb{Q}$
- $p - q \in \mathbb{Q}$
- $p \times q \in \mathbb{Q}$
- $p/q \in \mathbb{Q}, q \neq 0$

*Proof.* (Closure under addition)

Assume  $x, y \in \mathbb{Q}$ , this means  $\exists p_1, q_1, p_2, q_2 \in \mathbb{Z}$  with  $q_1, q_2 \neq 0$  such that:

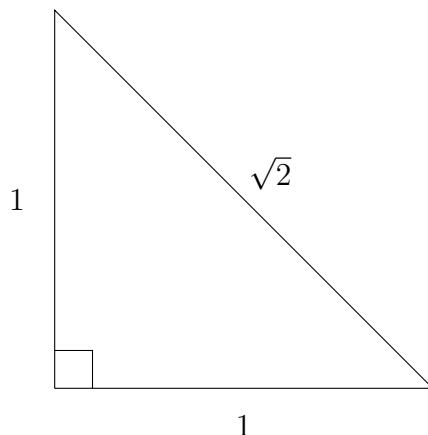
$$x = \frac{p_1}{q_1}, \quad y = \frac{p_2}{q_2}$$

$$\begin{aligned} x + y &= \frac{p_1}{q_1} + \frac{p_2}{q_2} \\ &= \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \\ &= \frac{p}{q} \in \mathbb{Q} \text{ with } p, q \in \mathbb{Z}, q \neq 0 \end{aligned}$$

□

$\mathbb{N}$  is not closed under subtraction. For example,  $5 - 7 = -2 \notin \mathbb{N}$ .

There are some problems with  $\mathbb{Q}$ . We can have  $p \in \mathbb{Q}$ , but  $\sqrt{p} \notin \mathbb{Q}$ , for example,  $2$ .  $\sqrt{2}$  is irrational, although it is constructible.



There are holes in the rationals. You could have  $p_i \in \mathbb{Q}$  for all  $i \in \mathbb{N}$ , but the sequence  $\{p_i\}$  converges to some  $p \notin \mathbb{Q}$ .

$\mathbb{Q}$  is not algebraically closed, we can have polynomials with rational coefficients whose roots are not rational. For example  $x^2 - 2 = 0$ .

## Fields

### Definition : Field

A field is a nonempty set  $\mathbb{F}$  along with 2 binary operations, addition (+) and multiplication ( $\cdot$ ), satisfying the following axioms:

1. If  $a, b \in \mathbb{F}$ , then

$$\left. \begin{array}{l} a + b = b + a \\ a \cdot b = b \cdot a \end{array} \right\} \text{Operations are commutative}$$

2. If  $a, b, c \in \mathbb{F}$ , then

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \text{Distributive property}$$

3. If  $a, b, c \in \mathbb{F}$ , then

$$\left. \begin{array}{l} a + (b + c) = (a + b) + c \\ a \cdot (b \cdot c) = (a \cdot b) \cdot c \end{array} \right\} \text{Operations are associative}$$

4. There are elements  $0, 1 \in \mathbb{F}$  such that for all  $a \in \mathbb{F}$ :

$$\begin{aligned} a + 0 &= a && 0 \text{ is the additive identity} \\ a \cdot 1 &= a && 1 \text{ is the multiplicative identity} \end{aligned}$$

5. For each  $a \in \mathbb{F}$  there is  $-a \in \mathbb{F}$  such that:

$$a + (-a) = 0 \quad \text{Additive inverse}$$

For each  $a \in \mathbb{F}$  with  $a \neq 0$ , there is  $a^{-1} \in \mathbb{F}$  such that:

$$a \cdot a^{-1} = 1 \quad \text{Multiplicative inverse}$$

$\mathbb{Q}$  is a field, but  $\mathbb{N}$  is not. ( $\mathbb{N}$  has no additive identity or inverses)

The identities in a field are unique.

*Proof.* (Proof for the additive identity)

Suppose  $0_1$  and  $0_2$  are additive identities. So,  $0_1, 0_2 \in \mathbb{F}$ , and for all  $a \in \mathbb{F}$  we have that:

$$\begin{aligned} a + 0_1 &= a \\ a + 0_2 &= a \end{aligned}$$

Since  $0_1 \in \mathbb{F}$ , and  $0_2$  is an additive identity, we have:

$$\begin{aligned} 0_1 + 0_2 &= 0_1 \\ 0_1 + 0_2 &= 0_2 + 0_1 && \text{By commutativity} \\ &= 0_2 && \text{Since } 0_2 \in \mathbb{F} \text{ and } 0_1 \text{ is an additive identity} \end{aligned}$$

Thus,  $0_1 = 0_2$ , and the additive identity is unique. □

## Ordered Fields

### Definition : Ordered Field

An ordered field is a field  $\mathbb{F}$  along with:

6. There is a nonempty subset  $P \subset \mathbb{F}$  called the positive elements such that:
    - (a) If  $a, b \in P$  then  $a + b \in P$  and  $a \cdot b \in P$
    - (b) If  $a \in \mathbb{F}$  and  $a \neq 0$ , then  $a \in P$  or  $-a \in P$  but not both. (This is called order)
- 0 is its own inverse, so  $0 = -0$ . For all  $a \neq 0$ ,  $a \neq -a$  so  $0 \notin P$ .