Limits

Limit Laws: Assume $a_n \to a$, $b_n \to b$, $a, b, c \in \mathbb{R}$

1. $(a_n + b_n) \to a + b$
2. $(a_n - b_n) \to a - b$
3. $(a_n \cdot b_n) \to a \cdot b$
4. $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$ provided $b_n \neq 0$ for all n, $b \neq 0$
5. $(c \cdot a_n) \to c \cdot a$

Proof. Proof of 3.

Scratch Work: Need $|a_n \cdot b_n - a \cdot b| < \varepsilon$ given that we can make $|a_n - a|$ and $|b_n - b|$ both really small ($< \varepsilon$ for any ε)

$$\begin{aligned} |a_n \cdot b_n - a \cdot b| &= |a_n \cdot b_n - a \cdot b_n + a \cdot b_n - a \cdot b| \\ &\leq |a_n \cdot b_n - a \cdot b_n| + |a \cdot b_n - a \cdot b| \\ &= |\,(a_n - a)\,(b_n)\,| + |\,(a)\,(b_n - b)\,| \\ &= |a_n - a||b_n| + |a||b_n - b| \end{aligned}$$

$b_n \to b$, so the sequence is bounded. $\exists\, C > 0$ such that $|b_n| \leq C$ for all n.

$$|a_n b_n - ab| \leq |a_n - a| \cdot C + |a||b_n - b|$$

$\exists\, N_1$ such that $|a_n - a| < \frac{\varepsilon}{2C}$ for $n > N_1$, then for $n > N_1$, $|a_n - a| \cdot C < \frac{\varepsilon}{2}$. We don't know that $a \neq 0$, we do know that $|a| \geq 0$, so $|a| + 1 \geq 0$. $\exists\, N_2$ such that $|b_n - b| < \frac{\varepsilon}{2(|a|+1)}$ for $n > N_2$, then for $n > N_2$, $|a||b_n - b| < |a|\frac{\varepsilon}{2(|a|+1)} < \frac{\varepsilon}{2}$.

Actual Proof: Let $\varepsilon > 0$, $b_n \to b$ so let $C > 0$ such that $|b_n| \leq C$ for all n. $\frac{\varepsilon}{2C} > 0$ so $\exists\, N_1$ such that $|a_n - a| < \frac{\varepsilon}{2C}$ for $n > N_1$. $\frac{\varepsilon}{2(|a|+1)} > 0$ so $\exists\, N_2$ such that $|b_n - b| < \frac{\varepsilon}{2(|a|+1)}$ for $n > N_2$. Let $N = \max(N_1, N_2)$ so both conditions hold for $n > N$.

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |a_n - a||b_n| + |a||b_n - b| \\ &\leq |a_n - a| \cdot C + |a||b_n - b| \\ &< \frac{\varepsilon}{2C} \cdot C + |a| + \frac{\varepsilon}{2\,(|a| + 1)} \qquad \text{for } n > N \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

$\square$

Theorem : Squeeze Theorem

Assume $a_n \leq x_n \leq b_n$ for all n and $a_n \to L$ and $b_n \to L$, then $x_n \to L$.

Proof. Let $\varepsilon > 0$, $\exists\, N_1$ such that $|a_n - L| < \varepsilon$ for $n > N_1$, and $\exists\, N_2$ such that $|b_n - L| < \varepsilon$ for $n > N_2$. Let $N = \max(N_1, N_2)$. So, $|a_n - L| < \varepsilon$ and $|b_n - L| < \varepsilon$ for $n > N$.

$$-\varepsilon < a_n - L < \varepsilon \quad \Rightarrow \quad L - \varepsilon < a_n < L + \varepsilon$$

$$-\varepsilon < b_n - L < \varepsilon \quad \Rightarrow \quad L - \varepsilon < b_n < L + \varepsilon$$

and, $a_n \leq x_n \leq b_n$. So,

$$L - \varepsilon < a_n \leq x_n \leq b_n < L + \varepsilon$$

Then, $L - \varepsilon < x_n < L + \varepsilon \Leftrightarrow |x_n - L| < \varepsilon$, for $n > N$. Therefore, $x_n \to L$. □

Example

Show that $x_n = \frac{\cos(n \cdot \pi)}{n}$ converges.

We know that $-1 \leq \cos(\theta) \leq 1$.

$$-1 \leq \cos(n \cdot \pi) \leq 1$$

$$-\frac{1}{n} \leq \frac{\cos(n \cdot \pi)}{n} \leq \frac{1}{n}$$

We know that $\frac{1}{n} \to 0$, and $(-1)\left(\frac{1}{n}\right) \to (-1)(0) = 0$ by Limit Law 5. Thus, $\frac{\cos(n \cdot \pi)}{n} \to 0$ by the Squeeze Theorem.