Geometric Series

Consider the following, with $a, r \in \mathbb{R}$

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$$

We have seen that r^n:

- Converges to 0 for $r \in (-1, 1)$
- Converges to 1 for $r = 1$
- Diverges otherwise

Proposition

$$\sum_{k=0}^{\infty} ar^k = \begin{cases} \frac{a}{1-r} & \text{if } r \in (-1,1) \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Proof. We know that if the k^{th} term doesn't go to 0, then the series diverges. So the series diverges for $|r| \geq 1$.

Consider $|r| < 1$

$$(1-r)\left(1 + r + r^2 + \dots + r^n\right) = 1 - r^{n+1}$$

So,

$$a + ar + ar^2 + \dots + ar^n = a\left(1 + r + r^2 + \dots + r^n\right) = a\left(\frac{1 - r^{n+1}}{1-r}\right)$$

$$\begin{aligned} \lim_{n\to\infty} a\left(\frac{1-r^{n+1}}{1-r}\right) &= \frac{a}{1-r} \cdot \lim_{n\to\infty}\left(1 - r^{n+1}\right) \\ &= \frac{a}{1-r} \cdot (1-0) \quad \text{since } |r| < 1 \\ &= \frac{a}{1-r} \end{aligned}$$

□

Proposition

If $a_k \geq 0$ for all k, then $\sum a_k$ either converges, or diverges to $+\infty$.

Proof. The idea is to look at sequences of partial sums

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_k \leq S_{k+1} \qquad \text{Since } a_{k+1} \geq 0 \text{ and } S_{k+1} = S_k + a_{k+1}$$

So, the sequence is monotone increasing, and we can apply the monotone convergence theorem. □

Proposition : Comparison Test

Assume $0 \leq a_k \leq b_k$ for all k.

1. If $\sum b_k$ converges, then so does $\sum a_k$
2. If $\sum a_k$ diverges, then so does $\sum b_k$

Proof. Let (S_n) be the sequence of partial sums for $\sum a_k$, and let (t_n) be the sequence of partial sums for $\sum b_k$.

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$t_n = b_1 + b_2 + \cdots + b_n$$

$S_n \leq t_n$ for all n since $a_k \leq b_k$ for all k.

1. Suppose $\sum b_k$ converges, thus (t_n) converges, so it is bounded. Then, S_n is also bounded since $0 \leq S_n < t_n$ for all n. There is $M \in \mathbb{R}$ such that $t_n \leq M$ for all n (bounded). For this M, we have $0 \leq S_n \leq t_n \leq M$ so $S_n \leq M$. Thus, (S_n) is bounded and mono increasing since all terms are non-negative. So, this converges by the MCT. Therefore, $\sum a_k$ converges.
2. Idea: (S_n) must be unbounded, so use this to prove (t_n) is also unbounded.

$\square$

Harmonic Series

We know that $\frac{1}{n} \to 0$ as $n \to \infty$. The Harmonic Series is:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Proposition

The Harmonic Series diverges.

Proof. Look at the n^{th} partial sums

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \frac{1}{2} \\
S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right) \\
S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
&= 1 + 3\left(\frac{1}{2}\right)
\end{aligned}$$

You can show that $S_{2^n} \geq 1 + n\left(\frac{1}{2}\right)$. The subsequence (S_{2^n}) diverges because the sequence $b_n = 1 + n\left(\frac{1}{2}\right)$ diverges:

$$\lim_{n\to\infty} 1 + n\left(\frac{1}{2}\right) = +\infty$$

We know $0 \leq b_n \leq S_{2^n}$ for all n, and since (b_n) diverges to $+\infty$, so S_{2^n} also diverges by comparison. Thus, (S_n) must diverge because if it converged, then every subsequence would also converge.

Therefore, $\sum \frac{1}{n}$ diverges. □

So, for the harmonic series, k^{th} term goes to 0, but the series diverges.