

## Bernoulli Random Variables

For a fixed  $\alpha \in (0, 1)$ , a Bernoulli random variable is a random variable with only two possible values, and pmf:

$$p(x) = \begin{cases} \alpha & x = 1 \\ 1 - \alpha & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

## Binomial Random Variables

Suppose that  $n$  independent trials, each of which results in a success with probability  $\alpha$  or in a failure with probability  $1 - \alpha$ , are performed. If  $X$  is the number of successes observed in the  $n$  trials, then  $X$  is said to be a binomial random variable with parameters  $(n, \alpha)$ . The pmf of a binomial random variable  $X$  with parameters  $(n, \alpha)$  is:

$$p(x) = \begin{cases} \binom{n}{x} \alpha^x (1 - \alpha)^{n-x} & x = 0, 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

We write  $X \sim \text{Bin}(n, \alpha)$  to denote that  $X$  is a binomial random variable with parameters  $(n, \alpha)$ .

Note: A binomial random variable is repeated, independent Bernoulli random variables.

**Example**

Suppose that a 70% head weighted coin is tossed 5 times. Let  $X$  be the number of heads observed in the 5 tosses. What is the mass function of  $X$ ? What is  $E(X)$ ?

The possible values of  $X$  are  $0, 1, 2, 3, 4, 5$ , and thus,  $p(x) = 0$  for  $x \notin \{0, 1, 2, 3, 4, 5\}$ . Computing the possible values of  $X$ , we get the mass function:

$$\begin{aligned} p(0) &= \binom{5}{0} 0.7^0 0.3^5 = 0.00243 \\ p(1) &= \binom{5}{1} 0.7^1 0.3^4 = 0.02835 \\ p(2) &= \binom{5}{2} 0.7^2 0.3^3 = 0.1323 \\ p(3) &= \binom{5}{3} 0.7^3 0.3^2 = 0.3087 \\ p(4) &= \binom{5}{4} 0.7^4 0.3^1 = 0.36015 \\ p(5) &= \binom{5}{5} 0.7^5 0.3^0 = 0.16807 \end{aligned}$$

The expected value of  $X$  is:

$$\begin{aligned} E(X) &= \sum_{x=0}^5 x \cdot p(x) \\ &= 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) + 5 \cdot p(5) \\ &= 0 \cdot 0.00243 + 1 \cdot 0.02835 + 2 \cdot 0.1323 + 3 \cdot 0.3087 + 4 \cdot 0.36015 + 5 \cdot 0.16807 \\ &= 3.5 \end{aligned}$$

If  $X \sim \text{Bin}(n, \alpha)$ , then the expected value  $E(X^k)$  of any power of  $X$  is reasonable to compute. The expected value  $E(X)$  and the variance  $V(X) = E(X^2) - (E(X))^2$  are then easy to compute.

Let's compute  $E(X)$  and  $V(X)$  for  $X \sim \text{Bin}(n, \alpha)$ . We know that  $X = X_1 + X_2 + \dots + X_n$ .  $X$  is the sum of Bernoulli Random Variable  $X_i$  with  $X_i \sim \text{Bernoulli}(\alpha)$ .

$$\begin{aligned} E(X) &= E\left(\sum_{i=1}^n X_i\right) \\ &= \sum_{i=1}^n E(X_i) \end{aligned}$$

Since  $X_i \sim \text{Bernoulli}(\alpha)$ , then  $E(X_i) = \alpha$ , so

$$E(X) = \sum_{i=1}^n p = n \cdot p$$

For the variance  $V(X)$ :

$$\begin{aligned}
 V(X) &= V\left(\sum_{i=1}^n X_i\right) \\
 &= \sum_{i=1}^n V(X_i) \quad \text{Since } V(X_i) = \alpha \cdot (1 - \alpha) \\
 &= n \cdot \alpha \cdot (1 - \alpha)
 \end{aligned}$$

### Example

A professor gives his statistics students a multiple-choice test with 5 questions and 4 answers per question. For each question on the test, there is only answer which is considered to be correct. One student writing the test did not study at all and decides to select their answers completely at random. What is the probability that the student answers exactly two questions correct? What is the probability that the student scores 80% or higher? What score should the student expect to get on the test?

We know that  $\alpha = 0.25$ ,  $n = 5$ ,  $P(X = x) = \binom{5}{x} (0.25)^x (0.75)^{5-x}$ . So,  $X \sim \text{Bin}(5, 0.25)$ .

$$\begin{aligned}
 P(X = 2) &= \binom{5}{2} (0.25)^2 (0.75)^3 \\
 &= (10) (0.0625) (0.421875) \\
 &= 0.263671875
 \end{aligned}$$

Score higher than 80% means 4 or 5 correct.

$$\begin{aligned}
 P(X \geq 4) &= P(X = 4) + P(X = 5) \\
 &= 0.0146484375 + 0.0009765625 \\
 &= 0.015625
 \end{aligned}$$

The expected value is:

$$\begin{aligned}
 E(X) &= n \cdot \alpha \\
 &= 5 \cdot 0.25 \\
 &= 1.25
 \end{aligned}$$

**Example**

Suppose that 10 cars in a fleet are inspected independently of each other and that each car has a 5% chance of failing the inspection. If more than one car fails then the company is issued a warning. What is the probability that a warning is issued?

We know  $\alpha = 0.05$ ,  $n = 10$ ,  $X \sim \text{Bin}(10, 0.05)$ ,  $P(X = x) = \binom{10}{x} (0.05)^x (0.95)^{10-x}$  for  $x = 0, 1, \dots, 10$ . A warning is issued if more than 1 car fails the inspection, the probability of this is  $P(X > 1)$ .

$$\begin{aligned}
 P(X > 1) &= 1 - P(X \leq 1) \\
 &= 1 - (P(X = 1) + P(X = 0)) \\
 &= 1 - \left[ \binom{10}{0} (0.95)^{10} + \binom{10}{1} (0.05) (0.95)^9 \right] \\
 &= 1 - (0.3151242049 + 0.5987369392) \\
 &= 1 - 0.9137616441 \\
 &= 0.086138559
 \end{aligned}$$

Computing the cdf  $F$  of a binomial random variable  $X \sim \text{Bin}(n, \alpha)$  is to compute

$$F(x) = \Pr(X \leq x) = \sum_{i=0}^x \binom{n}{i} \alpha^i \cdot (1 - \alpha)^{n-i}$$

$F(x)$  has no closed form but can be computed relatively easily recursively using: If  $X$  is a binomial random variable with parameters  $(n, \alpha)$ , then

$$p(x+1) = \frac{\alpha}{1 - \alpha} \cdot \frac{n - x}{x + 1} \cdot p(x)$$

That is, we can use  $p(x)$  to build the next  $p(x+1)$  without directly using the binomial pmf.

**Example**

Suppose that  $X \sim \text{Bin}(5, 0.3)$ . Find the pmf  $p$  of  $X$  and use it to find the cdf of  $X$ .

We know  $P(X = x) = \binom{5}{x} (0.3)^x (0.7)^{5-x}$  for  $x = 0, 1, \dots, 5$ .

$$\begin{aligned} p(0) &= \binom{5}{0} (0.3)^0 (0.7)^{5-0} = 0.16807 \\ p(1) &= \frac{0.3}{1-0.3} \cdot \frac{5-0}{0+1} \cdot p(0) = 0.36015 \\ p(2) &= \frac{0.3}{1-0.3} \cdot \frac{5-1}{1+1} \cdot p(1) = 0.3087 \\ p(3) &= \frac{0.3}{1-0.3} \cdot \frac{5-2}{2+1} \cdot p(2) = 0.1323 \\ p(4) &= \frac{0.3}{1-0.3} \cdot \frac{5-3}{3+1} \cdot p(3) = 0.02835 \\ p(5) &= \frac{0.3}{1-0.3} \cdot \frac{5-4}{4+1} \cdot p(4) = 0.00243 \end{aligned}$$

$$F(x) = \begin{cases} 0.16807 & \text{if } x = 0 \\ 0.52822 & \text{if } x = 1 \\ 0.83592 & \text{if } x = 2 \\ 0.96922 & \text{if } x = 3 \\ 0.99757 & \text{if } x = 4 \\ 1.00000 & \text{if } x = 5 \end{cases}$$