Positive Divisors

Definition

Let n be a positive integer. Then, $d(n)$ is the number of positive divisors of n, including 1 and n. Also, $\sigma(n)$ is the sum of the positive divisors of n. That is,

$$d(n) = \sum_{d|n} 1 \quad \text{and} \quad \sigma(n) = \sum_{d|n} d$$

(Note $\sum_{d|n}$ means the sum over the positive divisors of n)

Notice that when p is prime, $d\left(p^n\right) = n + 1$, since the positive divisors of p^n are $1, p, p^2, \ldots, p^n$.

Theorem : (7.1)

If $p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ is the prime-power decomposition of n, then we have that

$$d(n) = d\left(p_1^{e_1}\right) \times d\left(p_2^{e_2}\right) \times \cdots \times d\left(p_k^{e_k}\right)$$

Proof. Consider the set

$$D = \left\{p_1^{f_1} p_2^{f_2} \ldots p_k^{f_k} : 0 \leq f_i \leq e_i\right\}$$

Notice that D is exactly the set of divisors of n. If $d \mid n$, then $d \in D$ (left as an exercise to show). By the unique factorization theorem, $d(n) = |D|$.

$$\begin{aligned} d(n) &= |D| \\ &= (e_1 + 1) \times (e_2 + 1) \times \cdots \times (e_r + 1) \\ &= d(p_1^{e_1}) \times d(p_2^{e_2}) \times \cdots \times d(p_r^{e_r}) \end{aligned}$$

□

Example

Calculate $d(540)$ and $d(6300)$.

$$\begin{aligned} 540 &= 2^2 \cdot 3^3 \cdot 5 \\ d(540) &= d(2^2) \cdot d(3^3) \cdot d(5) \\ &= 3 \cdot 4 \cdot 2 \\ d(540) &= 24 \end{aligned}$$

$$\begin{aligned} 6300 &= 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \\ d(6300) &= d(2^2) \cdot d(3^2) \cdot d(5^2) \cdot d(7) \\ &= 3 \cdot 3 \cdot 3 \cdot 2 \\ &= 54 \end{aligned}$$

Now, notice that $\sigma(p^n) = 1 + p + \cdots + p^n$ for all primes p. This is because the only factors of p^n are $1, p, \ldots, p^n$.

Lemma : (7.1)

If p and q are different primes, then

$$\sigma\left(p^e q^f\right) = \sigma\left(p^e\right) \cdot \sigma\left(q^f\right)$$

Proof. The divisors of $p^e q^f$ are given by

$$\begin{array}{ccccc} 1, & p, & p^2, & \ldots, & p^e \\ q, & pq, & p^2 q, & \ldots, & p^e q \\ & & \vdots & & \\ q^f, & pq^f, & p^2 q^f, & \ldots, & p^e q^f \end{array}$$

If we add across each row, we get that

$$\begin{aligned} \sigma\left(p^e q^f\right) &= \left(1 + p + \cdots + p^e\right) + \cdots + q^f\left(1 + p + \cdots + p^e\right) \\ &= \left(1 + p + \cdots + p^e\right)\left(1 + q + \cdots + q^f\right) \\ &= \sigma\left(p^e\right)\sigma\left(q^f\right) \end{aligned}$$

□

Theorem : (7.2)

If $p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ is a prime-power decomposition of n, then

$$\sigma(n) = \sigma\left(p_1^{e_1}\right)\sigma\left(p_2^{e_2}\right)\ldots\sigma\left(p_k^{e_k}\right)$$

Proof. By Lemma 7.1, the theorem is true for $k = 2$. To prove by induction, suppose that the theorem is true for $k = r - 1$. We will show that this implies the theorem is true for $k = r$. Let

$$n = p_1^{e_1} p_2^{e_2} \ldots p_{r-1}^{e_{r-1}} p_r^{e_r} = N p_r^{e_r}, \qquad \text{let } N = p_1^{e_1} p_2^{e_2} \ldots p_{r-1}^{e_{r-1}}$$

Let $1, d_1, \ldots, d_t$ denote all the divisors of N, since $(N, p_r) = 1$, all the divisors of n are given by

$$\begin{array}{ccccc} 1, & d_1, & d_2, & \ldots, & d_t \\ p_r, & d_1 p_r, & d_2 p_r, & \ldots, & d_t p_r \\ & & \vdots & & \\ p_r^{e_r}, & d_1 p_r^{e_r}, & d_2 p_r^{e_r}, & \ldots, & d_t p_r^{e_r} \end{array}$$

Summing across the rows, we get that

$$\sigma(n) = \left(1 + d_1 + \cdots + d_t\right)\left(1 + p_r + \cdots + p_r^{e_r}\right) = \sigma(N)\,\sigma\left(p_r^{e_r}\right)$$

From the induction hypothesis, we get that

$$\sigma(n) = \sigma\left(p_1^{e_1}\right)\sigma\left(p_2^{e_2}\right)\ldots\sigma\left(p_{r-1}^{e_{r-1}}\right)\sigma\left(p_r^{e_r}\right)$$

□

Example

Calculate $\sigma(540)$.

$$\begin{aligned} 540 &= 2^2 \cdot 3^3 \cdot 5 \\ \sigma(540) &= \sigma\left(2^2\right) \cdot \sigma\left(3^3\right) \cdot \sigma(5) \\ &= (1+2+4) \cdot (1+3+9+27) \cdot (1+5) \\ &= 7 \cdot 40 \cdot 6 \\ &= 1680 \end{aligned}$$

Definition : Multiplicative Functions

A function f, defined for the positive integers, is said to be multiplicative if and only if

$$(m, n) = 1 \quad \text{implies} \quad f(mn) = f(m)\, f(n)$$