Basic Set Theory

Given a set S, an object is in S, or not in S.

$$x \in S \qquad x \text{ is an element of } S$$

$$x \notin S \qquad x \text{ is not an element of } S$$

There are different ways to describe sets, and the elements within them.

1. List the elements, ex. $\{1, 2, 3\}$
2. Describe in words, ex. The set of reals
3. Set Builder Notation, ex. $\{x : x \text{ is an even natural number}\}$

The empty set, denoted $\emptyset = \{\,\}$ is the set that has no elements in it. The universe, denoted $\mathcal{U}$ is where the elements are coming from (often $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{R}$).

We say that A is a subset of B if every element of A is in B, denoted by $A \subseteq B$, or $A \subset B$. A is a proper subset of B if $A \subseteq B$, and there exists some element $x \in B$ such that $x \notin A$. This is denoted by $A \subsetneq B$.

The cardinality of a set A is the number of distinct elements in A, denoted by $|A|$.

There are a number of operations we can do on sets:

- Intersection ($A \cap B$)
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
- Union ($A \cup B$)
$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
- Difference ($A \setminus B$ or $A - B$)
$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$
- Complement (A^c or $\overline{A}$)
$$A^c = U \setminus A = \{x : x \notin A\}$$
- Cartesian Product ($A \times B$)
$$A \times B = \{(a, b) : a \in A, b \in B\}$$
- Power Set ($\mathcal{P}(A)$)
  - The power set of A is the set of all possible subsets of A. The power set of A contains $2^{|A|}$ elements.
- *A binary operation is commutative if order doesn't matter. The intersection and union are commutative ($A \cup B = B \cup A$, and $A \cap B = B \cap A$), but the difference is not. (In general, $A \setminus B \neq B \setminus A$)*

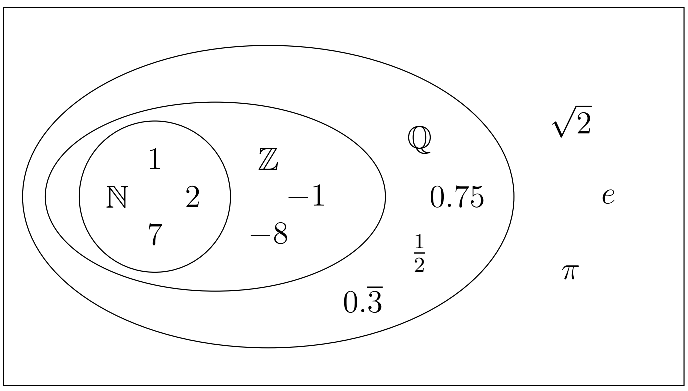
Example

Let $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$, and $B = \{3, 4\}$. Find:
(a) $A \cap B$
(b) $A \cup B$
(c) $A \setminus B$
(d) $B \setminus A$
(e) A^c
(f) $A \times B$
(g) $\mathcal{P}(A)$
(h) $|A|$

(a) $A \cap B = \{3\}$
(b) $A \cup B = \{1, 3, 4, 5\}$
(c) $A \setminus B = \{1, 5\}$
(d) $B \setminus A = \{4\}$
(e) $A^c = \{2, 4\}$
(f) $A \times B = \{(1, 3), (3, 3), (5, 3), (1, 4), (3, 4), (5, 4)\}$
(g) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{3, 5\}, \{1, 5\}, \{1, 3, 5\}\}$
(h) $|A| = 3$

Number Sets

- The Naturals
$$\mathbb{N} = \{1, 2, 3, \dots\}$$
- The Integers
$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$
- The Rationals
$$\mathbb{Q} = \left\{\frac{p}{q}, \ p, q \in \mathbb{Z}, \text{ and } q \neq 0\right\}$$
- The Reals
  - Numbers that can be written as a decimal expansion
- The Irrationals
$$\mathbb{R} \setminus \mathbb{Q}$$



Universe is $\mathbb{R}$

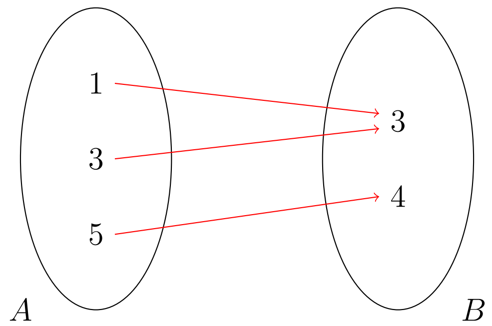
Quantifiers

- Existential Quantifier, $\exists$, "there exists"
- Universal Quantifier, $\forall$, "for every"

$$A \subseteq B \Leftrightarrow \forall x \in A, x \in B$$
$$A \subsetneq B \Leftrightarrow \forall x \in A, x \in B \text{ and } \exists y \in B \text{ s.t. } y \notin A$$

Functions

Given a pair of sets A and B, suppose each element $a \in A$ is associated with an element $b \in B$, denoted by $f(a)$.



$$f(1) = 3, \qquad f(3) = 3, \qquad f(5) = 4$$