# Spruce Budworm Continued

Last lecture, we took our spruce budworm model

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}$$

into a nondimensionalized form

$$\dot{x} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}, \quad \text{where } r = \frac{RA}{B}, k = \frac{K}{A}$$

Let's find the equilibria for this system

$$rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2} = 0$$

$$x\left(r\left(1 - \frac{x}{k}\right) - \frac{x}{1 + x^2}\right) = 0$$

One equilibrium point is $x^\star = 0$. But is this stable or unstable? We can do linear stability analysis.

$$\dot{x} = f(x) = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1 + x^2}$$

$$\begin{aligned}
f'(x) &= r\left(1 - \frac{x}{k}\right) + rx\left(-\frac{1}{k}\right) - \frac{2x\left(1 + x^2\right) - 2x\left(x^2\right)}{\left(1 + x^2\right)^2} \\
f'(0) &= r\left(1 - \frac{0}{k}\right) + r\left(0\right)\left(-\frac{1}{k}\right) - \frac{2\left(0\right)\left(1 + 0^2\right) - 2\left(0\right)\left(0^2\right)}{\left(1 + 0^2\right)^2} \\
&= r
\end{aligned}$$

Since $r = \frac{RA}{B}$, and $R > 0, A > 0, B > 0 \Rightarrow r > 0$. So $f'(0) > 0$, so $x^\star = 0$ is unstable.

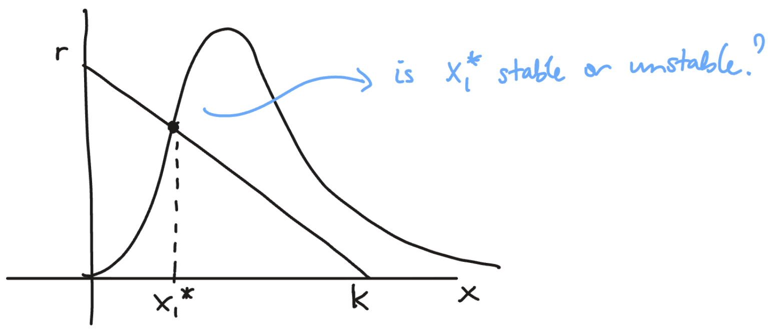
What about the other equilibria?

$$r\left(1 - \frac{x}{k}\right) - \frac{x}{1 + x^2} = 0$$

Or,

$$r\left(1 - \frac{x}{k}\right) = \frac{x}{1 + x^2}$$

So know we have a straight line, and a function free of parameters. Let's try plotting these. The intersections are equilibria.



An aside: Stability alternates stable / unstable between successive equilibria, so long as it doesn't reach a point that is half-stable.

So, $x_0^\star$ was unstable, meaning $x_1^\star$ is stable.

What happens to the equilibria as the parameters change? From the graph above, we can imagine the value, number and stability of equilibria might change.

What else contributes to the model? The growth of the forest!

- Think of how a bird searches for insects

$$A = A'S$$

  Here, $A$ is the search units of foliage, $A'$ s the budworm per unit foliage, and $S$ is the foliage (total tree branch / needle surface area). So as the forest grows, $S$ is increasing so $A$ increases.

- What about the carrying capacity?

$$K = K'S$$

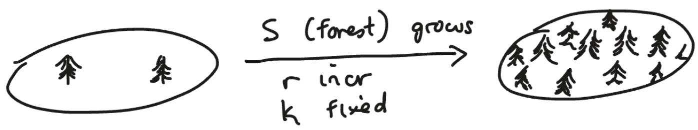
  Here, $K$ is the budworm carrying capacity. So as the forest grows, it can support more budworm.

So, as the forest grows, $S$ gets larger. What is the impact on $r$ and $k$?

$$r = \frac{RA'S}{B}, \quad k = \frac{K'S}{A'S} = \frac{K'}{A'}$$

As the forest grows, $S$ increases, so $r$ also increases. We can also see that $K$ is fixed as the forest grows. This gives us a nice bifurcation parameter ($S$).

Let's consider an initially barren forest




Next time, we will see the impact this has on our point of intersection.