Arithmetic Functions

Definition : Arithmetic Functions

An arithmetic function is a function whose domain is the set of positive integers.

The function $d(n)$, $\sigma(n)$ and $\phi(n)$ are all arithmetic functions. The Möbius Inversion Formula can be used to obtain nontrivial identities among arithmetic functions from trivial identities.

Theorem : (9.5)

Let f be an arithmetic function for $n \in \mathbb{Z}$ with $n > 0$. Then, consider the following arithmetic function.

$$F(n) = \sum_{d|n} f(d)$$

If f is multiplicative, then F is multiplicative.

Proof. Let m and n be relatively prime positive integers. Then, we have that

$$F(mn) = \sum_{d|mn} f(d)$$

Since $(m, n) = 1$, each divisor d of mn can be written uniquely as $d_1 d_2$, where $d_1, d_2 > 0$, $d_1 \mid m$, $d_2 \mid n$, and $(d_1, d_2) = 1$. Each product $d_1 d_2$ corresponds to a divisor d of mn, so we have that

$$\begin{aligned} F(mn) &= \sum_{d_1|m, d_2|n} f(d_1 d_2) \\ &= \sum_{d_1|m, d_2|n} f(d_1) f(d_2) \\ &= \sum_{d_1} f(d_1) \sum_{d_1} f(d_2) \\ &= F(m) F(n) \end{aligned}$$

□

Theorem : Gauss' Theorem

Let $n \in \mathbb{Z}$ with $n > 0$. Then

$$\sum_{d|n} \phi(d) = n$$

Proof. By Theorem 9.2 and Theorem 9.5, the following arithmetic function is multiplicative.

$$F(d) = \sum_{d|n} \phi(d)$$

Therefore, by Theorem 7.5, the arithmetic function F is completely determined by its values at powers of prime numbers. If p is a prime number and $a \in \mathbb{Z}$ with $a > 0$, then

$$\begin{aligned} F(p^a) &= \sum_{d \mid n} \phi(d) \\ &= \phi(1) + \phi(p) + \phi\left(p^2\right) + \cdots + \phi(p^a) \\ &= 1 + (p-1) + \left(p^2 - p\right) + \cdots + \left(p^a - p^{a_1}\right) \\ &= p^a \end{aligned}$$

Therefore, if the prime decomposition of n is $n = p_1^{e_1} \dots p_r^{e_r}$, then by Theorem 7.5, we have that

$$\begin{aligned} F(n) &= F\left(p_1^{e_1}\right) F\left(p_2^{e_2}\right) \dots F\left(p_r^{e_r}\right) \\ &= p_1^{e_1} p_2^{e_2} \dots p_r^{e_r} \\ &= n \end{aligned}$$

□

Example

Verify that Gauss' Theorem holds for $n = 12$.

The divisors of 12 are 1, 2, 3, 4, 6, and 12. For each divisor, we evaluate Euler's totient function.

$$\begin{aligned} &\phi(1) = 1, \quad \phi(2) = 1, \quad \phi(3) = 2, \\ &\phi(4) = 2, \quad \phi(6) = 2, \quad \phi(12) = 4 \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \sum_{d \mid 14} \phi(d) &= \phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(6) + \phi(12) \\ &= 1 + 1 + 2 + 2 + 2 + 4 \\ &= 12 \end{aligned}$$

The Möbius Function

Definition : The Möbius μ Function

If $n \in \mathbb{Z}$ with $n > 0$, then the Möbius μ-function, denoted $\mu(n)$, is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2 \mid n \text{ with } p \text{ prime} \\ (-1)^r & \text{if } n = p_1 p_2 \dots p_r \text{ with } p_1, \dots, p_r \text{ distinct primes} \end{cases}$$

Consider the first few values of the Möbius μ-function.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0

Theorem : (9.6)

The Möbius μ-function is multiplicative.

Proof. Let m and n be relatively prime positive integers. If $m = 1$, then by definition of μ, we have that $\mu(m) = 1$. Thus,

$$\begin{aligned}\mu(mn) &= \mu(1n) \\ &= 1 \times \mu(n) \\ &= \mu(m)\,\mu(n)\end{aligned}$$

If m is divisible by the square of a prime number, then mn is divisible by the square of a prime number. Therefore, by the definition of μ, we would have that $\mu(m) = 0$ and $\mu(mn) = 0$.

$$\begin{aligned}\mu(mn) &= 0 \\ &= 0 \times \mu(n) \\ &= \mu(m)\,\mu(n)\end{aligned}$$

Assume that $m = p_1 \dots p_r$ and that $n = q_1 \dots q_t$, where all the prime numbers are distinct. Then, by the definition of μ, we have that

$$\begin{aligned}\mu(mn) &= \mu(p_1 \dots p_r q_1 \dots q_t) \\ &= (-1)^{r+t} \\ &= (-1)^r (-1)^t \\ &= \mu(m)\,\mu(n)\end{aligned}$$

□

Corollary : (9.7)

Let $n \in \mathbb{Z}$ with $n > 0$. Then

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. By Theorem 9.5 and Theorem 9.6, the following arithmetic function is multiplicative.

$$F(d) = \sum_{d|n} \mu(d)$$

Therefore, by Theorem 7.5, the arithmetic function F is completely determined by its values at powers of prime powers. If p is a prime number and $a \in \mathbb{Z}$ with $a > 0$, then

$$\begin{aligned}F(p^a) &= \sum_{d|n} \mu(d) \\ &= \mu(1) + \mu(p) + \mu\left(p^2\right) + \cdots + \mu(p^a) \\ &= 1 - 1 + 0 + \cdots + 0 \\ &= 0\end{aligned}$$

Therefore, if the prime decomposition of n is $n = p_1^{e_1} \dots p_r^{e_r}$, then by Theorem 7.5, we have that

$$\begin{aligned}F(n) &= F(p_1^{e_1})\,F(p_2^{e_2}) \dots F(p_r^{e_r}) \\ &= 0 \cdot 0 \cdot \ldots \cdot 0 \\ &= 0\end{aligned}$$

If $n = 1$, then $F(n) = \mu(1) = 1$. $\square$

Example

Verify that Corollary 9.7 holds for $n = 12$.

The divisors of 12 are 1, 2, 3, 4, 6, and 12. For each divisor, we evaluate the Möbius μ-function

$$\begin{aligned} &\mu(1) = 1, \quad \mu(2) = -1, \quad \mu(3) = -1, \\ &\mu(4) = 0, \quad \mu(6) = 1, \quad \mu(12) = 0 \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \sum_{d \mid 14} \mu(d) &= \mu(1) + \mu(2) + \mu(3) + \mu(4) + \mu(6) + \mu(12) \\ &= 1 - 1 - 1 + 0 + 1 + 0 \\ &= 0 \end{aligned}$$