Extreme Value Theorem

Definition : Compact Set

A set is "compact" is the same as the set being closed and bounded.

Closed means it contains it limit points, for example $[a, b]$.

Theorem

Suppose $f : X \to \mathbb{R}$ is continuous. If $A \subseteq X$ is compact, then $f(A)$ is compact.

Example

Demonstrate the above theorem with $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, and $A = [1, 3]$.

We can see that A is compact. $f(A) = [1, 3]$, which is compact. The image is closed and bounded.

Theorem : Extreme Value Theorem (EVT)

A continuous function on a compact set attains it's sup and inf.

What does this mean? f is continuous, A is closed and bounded, $f(A)$ is closed and bounded. $f(A)$ has a sup (least upper bound) and inf (greatest lower bound). $f(A)$ is closed so the sup and inf are in $f(A)$, so, $f(A)$ has a max and a min.

Example

Demonstrate the EVT with $f : [-2, 1] \to \mathbb{R}$, $f(x) = x^2$, $A = [-2, 1]$.

$f(A) = [0, 4]$. $\sup(f(A)) = 4$, $\exists x$ s.t. $f(x) = 4$, $f(-2) = 4$. $\inf(f(A)) = 0$, $\exists x$ s.t. $f(x) = 0$, $f(0) = 0$.

Example

Show what happens when A is not closed for $A = (-2, 1]$ and $f(x) = x^2$.

$f(A) = [0, 4)$. $\sup(f(A)) = 4 \notin f(A)$. $f(A)$ does not have a max.

Example : S

ow what happens when A is not bounded for $A = [-2, \infty)$ and $f(x) = x^2$.

$f(A) = [0, \infty)$. $f(A)$ is no bounded above, so it doesn't have a supremum, and also doesn't have a max.

Example

Show what happens when f is not continuous for $A = [0, 4]$, $f : [0, 4]$, and

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, 2) \\ 2 & \text{if } x \in [2, 4] \end{cases}$$

$\lim_{x \to 2} f(x)$ does not exist, so the function is not continuous at $x = 0$. $f(A) = [0, 4)$. A was closed and bounded, but $f(A)$ is not since $4 \notin f(A)$. $f(A)$ has a sup, but not a max.

Intermediate Value Theorem

Lemma

If f is continuous and $f(c) > 0$, then $\exists\ \delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$

Proof. We need to pick $\varepsilon > 0$ so that $f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$ are all > 0. Try $\varepsilon = \frac{f(c)}{2}$. Since f is continuous, there exists δ such that $|f(x) - f(c)| < \varepsilon$ for $|x - c| < \delta$. Use this to show $f(x) > 0$. Likewise, if $f(c) < 0$, there is $\delta > 0$, such that $f(x) < 0$ for $x \in (c - \delta, c + \delta)$ $\hfill\square$

Proposition

If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have different signs, then there exists some $c \in (a, b)$ such that $f(c) = 0$

Proof. Idea – Assume $f(a) > 0$ and $f(b) < 0$. Let $A = \{t : f(x) > 0\ \forall\ x \in [a, t]\}$. We have $a \in A$, so A is non-empty. $f(b) < 0$, so A is bounded above by b. So, by completeness, $\sup(A)$ exists. Let $c = \sup(A)$. Then, we can show $f(c) = 0$ by the previous Lemma. $\hfill\square$

Theorem : Intermediate Value Theorem (IVT)

If f is continuous on $[a, b]$ and α is any number between $f(a)$ and $f(b)$. Then, there exists $c \in (a, b)$ such that $f(c) = \alpha$.

Proof. Use $g(x) = f(x) - \alpha$ and the previous proposition $\hfill\square$