

The Archimedean Principle

We know that if $x, y \in \mathbb{Q}$, with $x < y$, then there exists $z \in \mathbb{Q}$ with $x < z < y$. For example, $\frac{x+y}{2}$ works. In fact, there are infinitely many other possible values for z . The size of the gap between x and y is $y - x$, and $y - x \in \mathbb{Q}$. If $r \in \mathbb{Q}$ and $0 < r < 1$, then $z = x + r(y - x)$ works. We say that \mathbb{Q} is dense in \mathbb{Q} .

Definition : Dense Sets

Suppose A and B are subsets of ordered fields. Then A is dense in B if for every $x, y \in B$, there exists $a \in A$ such that $x < a < y$.

Example

Show that the irrationals are dense in the rationals. That is, show that given $x, y \in \mathbb{Q}$ with $x < y$, there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$.

We know $\sqrt{2}$ is irrational, now consider $r = \frac{\sqrt{2}}{2}$ is between 0 and 1 (could prove $\frac{\sqrt{2}}{2}$ is also irrational). We have $y - x \in \mathbb{Q}$ and $\frac{\sqrt{2}}{2} \in \mathbb{R} \setminus \mathbb{Q}$, then $\frac{\sqrt{2}}{2}(y - x) \in \mathbb{R} \setminus \mathbb{Q}$ (could prove that if $a \in \mathbb{R} \setminus \mathbb{Q}$, and $b \in \mathbb{Q}$, then $ab \in \mathbb{R} \setminus \mathbb{Q}$, except for $b = 0$ since $ab = 0 \in \mathbb{Q}$). So,

$$z = x + \frac{\sqrt{2}}{2}(y - x) \in \mathbb{R} \setminus \mathbb{Q}$$

(Since the sum of a rational and an irrational is irrational)

Example

Show that \mathbb{Z} is not dense in \mathbb{Z} .

This is not true. For a counterexample, we need 2 integers without an integer between them. Consider $x = 1, y = 2$, x and y have no integer between them.

Example

Show that \mathbb{Z} is not dense in \mathbb{Q} .

This is not true. For a counterexample, we need 2 rationals without an integer between them. Consider $x = \frac{1}{2}, y = \frac{3}{4}$, x and y have no integer between them.

Example

Show that \mathbb{Q} is dense in \mathbb{Z} .

To do this, we need to show that between any 2 integers, there is a rational. Let $z = \frac{x+y}{2} \in \mathbb{Q}$. For any $x, y \in \mathbb{Z}$, $x < z < y$.

Definition : Maximal and Minimal Elements

Let $A \subseteq \mathbb{R}$, A has a maximal element if there exists $M \in A$ such that $x \leq M$ for all $x \in A$.

Likewise, A has a minimal element if there exists $m \in A$ such that $m \leq x$ for all $x \in A$.

Example

Consider the set $A = \{-3, 4, -7, 10, 11, -20\}$. What is $\max(A)$? What is $\min(A)$?

$$\max(A) = 11, \quad \min(A) = -20$$

Example

Consider the set $A = [0, 1) = \{x : 0 \leq x < 1\}$. What is $\max(A)$? What is $\min(A)$?

$$\min(A) = 0, \quad \text{There is no maximal element}$$

Sets can be bounded above, but not have a maximal element. Similarly for minimal elements.

Example : (Exercise 1.24)

Prove if A has a maximal element M , then $M = \sup(A)$. Likewise, if A has a minimal element m , then $m = \inf(A)$.

Suppose $M = \max(A)$. Then, $x \leq M$ for all $x \in A$, so M is an upper bound for A . $\sup(A)$ is the least upper bound, so $\sup(A) \leq M$. We also have $M \in A$, so $M \leq \sup(A)$ since $\sup(A)$ is an upper bound for A . Thus, $\sup(A) \leq M$ and $M \leq \sup(A)$ hence $M = \sup(A)$. Likewise, $m = \inf(A)$ if the minimal element exists.

Example : (Exercise 1.25)

Suppose that A is a nonempty set containing finitely many elements. Prove by induction that A has a maximal element. (Do induction on the number of elements in the set)

Base case: $|A| = 1$, $A = \{a_1\}$. Then, $\max(A) = a_1$ since $x \leq a_1$ for all $x \in A$.

Assume sets of size k all have maximal elements.

Let A be a set with $|A| = k + 1$.

$$\begin{aligned} A &= \{a_1, a_2, \dots, a_k, a_{k+1}\} \\ &= \{a_1, a_2, \dots, a_k\} \cup \{a_{k+1}\} \end{aligned}$$

Let $B = \{a_1, a_2, \dots, a_k\}$, B has a maximal element $\max(B)$ by the induction hypothesis. If $a_{k+1} < \max(B)$, then $\max(A) = \max(B)$ since $x \leq \max(B)$ for all $x \in A$. If $a_{k+1} > \max(B)$, then $\max(A) = a_{k+1}$ since $x \leq \max(A)$ for all $x \in A$. Thus sets of size $k + 1$ have a maximal element. Therefore, by induction, finite sets have maximal elements. Likewise, they would also have minimal elements.