Continuous Random Variables

Example

Suppose that X is a continuous random variable with pdf:

$$f(x) = \begin{cases} C \cdot (1-x)^4 & x \in (0,1) \end{cases}$$

What is the conditional probability $\Pr(X > 0.4 \mid X > 0.1)$?

First, normalize $f(x)$.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} f(x)\ dx \\
&= \int_{-\infty}^{0} f(x)\ dx + \int_{0}^{1} f(x)\ dx + \int_{1}^{\infty} f(x)\ dx \\
&= 0 + \int_{0}^{1} C \cdot (1-x)^4\ dx + 0 \\
&= C \cdot \int_{0}^{1} (1-x)^4\ dx \qquad \text{let } u = 1-x,\ du = -dx \\
&= C \cdot \int_{1}^{0} u^4 (-1)\ du \\
&= C \int_{0}^{1} u^4\ du \\
&= C \cdot \left(\frac{1}{5} u^5 \Big|_{u=0}^{u=1} \right) \\
&= C \cdot \left(\frac{1}{5}(1)^5 - \frac{1}{5}(0)^5 \right) \\
&= \frac{C}{5} \\
C &= 5
\end{aligned}$$

Now,

$$\Pr(X > a) = \int_{a}^{1} f(x)\ dx = 5 \int_{a}^{1} (1-x)^5\ dx = 5 \left(-\frac{(1-x)^5}{5} \right) \Bigg|_{a}^{1} = (1-a)^5$$

$$\Pr(X > 0.4 \mid X > 0.1) = \frac{\Pr(X > 0.4 \cap X > 0.1)}{\Pr(X > 0.1)} = \frac{\Pr(X > 0.4)}{\Pr(X > 0.1)}$$

$$\Pr(X > 0.4 \mid X > 0.1) = \frac{(1-0.4)^5}{(1-0.1)^5} = \frac{0.6^5}{0.9^5} \approx 0.317$$

The Expected Value of a Continuous Random Variable

If X is a continuous random variable with pdf f, the expected value of X is

$$E\left(X\right) = \int_{\mathbb{R}} x \cdot f\left(x\right)\ dx$$

More generally, if $g : \mathbb{R} \to \mathbb{R}$ is any real-valued function, then

$$E\left(g\left(X\right)\right) = \int_{\mathbb{R}} g\left(x\right) \cdot f\left(x\right)\ dx$$

In particular, we can compute

$$E\left(X^2\right) = \int_{\mathbb{R}} x^2 \cdot f\left(x\right)\ dx$$

with which we can compute the variance of X:

$$V\left(X\right) = E\left(X^2\right) - \left(E\left(X\right)\right)^2$$

The Median of a Continuous Random Variable

Unlike the discrete case, the median of X is easily calculable. The median m is a real number satisfying

$$\int_{-\infty}^{m} f\left(x\right)\ dx = \frac{1}{2} = \int_{m}^{\infty} f\left(x\right)\ dx$$

In other words, it is equally likely that $X \geq m$ or $X \leq m$.

Example

Suppose that a continuous random variable X is uniform on the interval $[a, b]$. That is, the pdf f of X is constant on $[a, b]$ and 0 elsewhere:

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{elsewhere} \end{cases}$$

What are the expected value, variance, and median of X?

$$\begin{aligned} E(X) &= \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \int_a^b x\, dx = \frac{1}{b-a} \left(\frac{1}{2} x^2 \bigg|_a^b \right) \\ &= \frac{1}{b-a} \left(\frac{1}{2} b^2 - \frac{1}{2} a^2 \right) = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

$$\begin{aligned} E\left(X^2\right) &= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \int_a^b x^2\, dx = \frac{1}{b-a} \left(\frac{1}{3} x^3 \bigg|_a^b \right) \\ &= \frac{1}{b-a} \left(\frac{b^3}{3} - \frac{a^3}{3} \right) = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3} \end{aligned}$$

$$V(X) = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2} \right)^2 = \frac{(b-a)^3}{12}$$

Median:

$$\begin{aligned} \frac{1}{2} &= \int_a^m \frac{1}{b-a}\, dx = \frac{x}{b-a} \bigg|_a^m = \frac{m-a}{b-a} \\ b - a &= 2m - 2a \\ m &= \frac{b+a}{2} \end{aligned}$$

Note: For a uniform distribution,

$$F(x) = \frac{x-a}{b-a} \qquad \text{for } x \in [a, b]$$

Exponential Random Variables

A random variable X is called an exponential random variable if, for some $\lambda > 0$, its pdf is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

Example

Prove that the pdf for an exponential random variable, is in fact a valid pdf.

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\ dx &= \int_{0}^{\infty} \lambda e^{-\lambda x}\ dx \\
&= \lambda \int_{0}^{\infty} e^{-\lambda x}\ dx \quad \text{let } u = -\lambda x, dx = -\frac{1}{\lambda} \\
&= -\int_{0}^{-\infty} e^{u}\ du \\
&= \int_{-\infty}^{0} e^{u}\ du \\
&= e^{u}\Big|_{-\infty}^{0} \\
&= e^{0} - e^{-\infty} \\
&= 1 - 0 \\
&= 1
\end{aligned}
$$

Also, since $\lambda e^{-\lambda x} > 0$ for all $x \in [0, \infty)$, f is a valid pdf.

Example

Find the cdf of an exponential random variable.

We are looking for a function such that

$$
\begin{aligned}
F(x) &= \Pr(X \leq x) \\
&= \int_{-\infty}^{x} f(t)\ dt \\
&= \int_{0}^{x} f(t)\ dt \\
&= \lambda \int_{0}^{x} e^{-\lambda t}\ dt \\
&= \lambda \left(\frac{1}{-\lambda} e^{-\lambda t} \Big|_{0}^{x} \right) \\
&= -\left(e^{-\lambda x} - 1\right) \\
&= 1 - e^{-\lambda x}
\end{aligned}
$$

So,

$$
F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}
$$

Example

What is the expected value of an exponential random variable?

$$\begin{aligned}
E\left(X\right) &= \int_{-\infty}^{\infty} x \cdot f\left(x\right)\ dx \\
&= \int_{0}^{\infty} x \cdot f\left(x\right)\ dx \\
&= \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\ dx \\
&= \lambda \int_{0}^{\infty} x \cdot e^{-\lambda x}\ dx \quad \text{let } u = x, v' = e^{-\lambda x} \\
&= \lambda \cdot \left[\left(-\frac{x}{\lambda} e^{-\lambda x} \right) \Big|_{0}^{\infty} + \frac{1}{\lambda} \int_{0}^{\infty} e^{-\lambda x}\ dx \right] \\
&= \lambda \cdot \left[0 + \frac{1}{\lambda} \left(-\frac{1}{\lambda} e^{-\lambda x} \Big|_{0}^{\infty} \right) \right] \\
&= \lambda \cdot \left(\frac{1}{\lambda^2} \right) \\
&= \frac{1}{\lambda}
\end{aligned}$$

Note: The Gamma Integral Identity

$$\int_{0}^{\infty} x^n e^{-\lambda x} = \frac{n!}{\lambda^{n+1}} \quad \text{for } n = 0, 1, 2, \ldots$$

Example

What is the variance of an exponential random variable?

$$\begin{aligned}
E\left(X^2\right) &= \int_{0}^{\infty} x^2 \lambda e^{-\lambda x}\ dx \\
&= \lambda \int_{0}^{\infty} x^2 e^{-\lambda x}\ dx \\
&= \lambda \cdot \frac{2}{\lambda^3} \quad \text{by the Gamma Integral Identity} \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

So,

$$V\left(X\right) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{1}{\lambda^2}$$