Quiz 7

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1. Find the solution to the initial value problem.

$$4y'' + 3y' + 2y = u_2(t); y(0) = 0, y'(0) = 1$$

Answer:

$$\mathcal{L}\{4y'' + 3y' + 2y = u_2(t)\}$$

$$4(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y'(0)) + 2Y(s) = \frac{e^{-2s}}{s}$$

$$4s^2Y(s) - 4sy(0) - 4y'(0) + 3sY(s) - 3y'(0) + 2Y(s) = \frac{e^{-2s}}{s}$$

$$Y(s)(4s^2 + 3s + 2) - 4s(0) - 4(1) - 3(1) = \frac{e^{-2s}}{s}$$

$$Y(s)(4s^2 + 3s + 2) - 7 = \frac{e^{-2s}}{s}$$

$$Y(s) = e^{-2s}\frac{1}{4s^3 + 3s^2 + 2s} + \frac{7}{4s^2 + 3s + 2}$$

$$Y(s) = e^{-2s} \cdot \frac{1}{4s^3 + 3s^2 + 2s} + \frac{7}{4} \cdot \frac{1}{(s + 3/8)^2 + 23/64}$$

$$Y(s) = e^{-2s} \cdot \frac{1}{4s^3 + 3s^2 + 2s} + \frac{7}{4} \cdot \frac{\sqrt{64}}{\sqrt{23}} \cdot \frac{\sqrt{23}/\sqrt{64}}{(s + 3/8)^2 + (\sqrt{23}/\sqrt{64})^2}$$

The above satisfies the form $e^{at}\sin(bt)$. Conduct partial fraction decomposition for the first part.

$$\frac{A}{s} + \frac{Bs + C}{4s^2 + 3s + 2} = \frac{1}{4s^3 + 3s^2 + 2s}$$

$$A(4s^2 + 3s + 2) + s(Bs + C) = 1$$

$$4As^2 + 3As + 2A + Bs^2 + Cs = 1$$

$$\begin{Bmatrix} 4As^2 + Bs^2 = 0 \\ 3A + C = 0 \\ 2A = 1 \end{Bmatrix}$$

The system yields $A = 1/2$, $B = -2$, $C = -3/2$

$$e^{-2s} \cdot (\frac{1/2}{s} + \frac{-2s - 3/2}{4s^2 + 3s + 2}) + \frac{14}{\sqrt{23}} \cdot \frac{\sqrt{23}/\sqrt{64}}{(s + 3/8)^2 + (\sqrt{23}/\sqrt{64})^2}$$

$$e^{-2s} \cdot \frac{1}{2} \cdot (\frac{1}{s} + \frac{-4s - 3}{4s^2 + 3s + 2}) + \frac{14}{\sqrt{23}} \cdot \frac{\sqrt{23}/\sqrt{64}}{(s + 3/8)^2 + (\sqrt{23}/\sqrt{64})^2}$$

$$\frac{1}{2}e^{-2s}(\frac{1}{s} + \frac{-4s - 24/8}{4s^2 + 3s + 2}) + \frac{14}{\sqrt{23}} \cdot \frac{\sqrt{23}/\sqrt{64}}{(s + 3/8)^2 + (\sqrt{23}/\sqrt{64})^2}$$

$$\frac{1}{2}e^{-2s}(\frac{1}{s} + (-\frac{1}{4})\frac{s + 6/8}{4s^2 + 3s + 2}) + \frac{14}{\sqrt{23}} \cdot \frac{\sqrt{23}/\sqrt{64}}{(s + 3/8)^2 + (\sqrt{23}/\sqrt{64})^2}$$

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$$\frac{1}{2}e^{-2s}(\frac{1}{s} + (-\frac{1}{4})(\frac{s + 3/8}{(s + 3/8)^2 + (\sqrt{23}/\sqrt{64})^2} + \frac{3}{8}\frac{1}{(s + 3/8)^2 + (\sqrt{23}/\sqrt{64})})) \cdots$$

$$Y(s) = \mathcal{L}^{-1}\{(1)\}$$

$$\frac{1}{2}H(t-2)(1-(\frac{1}{4})(e^{-3t/8}\cos(\frac{t\sqrt{23}}{8}+\frac{3}{8}e^{-3t/8}\sin(\frac{t\sqrt{23}}{8}))))+\frac{14}{\sqrt{23}}e^{-3t/8}\sin(\frac{t\sqrt{23}}{8})$$

$$H(t-2)(\frac{1}{2} - \frac{1}{8}e^{-3t/8}\cos(\frac{t\sqrt{23}}{8}) - \frac{3}{32}e^{-3t/8}\sin(\frac{t\sqrt{23}}{8})) + \frac{14}{\sqrt{23}}e^{-3t/8}\sin(\frac{t\sqrt{23}}{8})$$

$$\boxed{H(t-2)(\frac{1}{2} - \frac{1}{8}e^{-3(t-2)/8}\cos(\frac{(t-2)\sqrt{23}}{8}) - \frac{3}{32}e^{-3(t-2)/8}\sin(\frac{(t-2)\sqrt{23}}{8})) + \frac{14}{\sqrt{23}}e^{-3t/8}\sin(\frac{t\sqrt{23}}{8})}$$

I hope I am not asked to do a problem of this length on the final. 2yx

2. Solve the given initial value problem and sketch the graph of the solution.

$$y'' + y = -\delta(t-\pi) + \delta(t-2\pi); y(0) = 0, y'(0) = 1$$

Answer:

$$\mathcal{L}\{y'' + y = -\delta(t-\pi) + \delta(t-2\pi)\}$$
$$s^2Y(s) - sy(0) - y'(0) + Y(s) = -e^{-\pi s} + e^{-2\pi s}$$
$$s^2Y(s) - s(0) - 1 + Y(s) = -e^{-\pi s} + e^{-2\pi s}$$
$$Y(s)(s^2+1) - 1 = -e^{-\pi s} + e^{-2\pi s}$$
$$Y(s) = -e^{-\pi s}\frac{1}{s^2+1} + e^{-2\pi s}\frac{1}{s^2+1} + \frac{1}{s^2+1}$$
$$\boxed{Y(s) = u_\pi \sin(t-\pi) + u_{2\pi}\sin(t-2\pi) + \sin(t)}$$