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Gravitational Waves As a Means to Characterize Black Holes

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Abstract

An in-depth exploration of the mathematical and physical foundations, relativistic predictions, and current gravitational waves and their detectors. I will also present challenges with this method as well, as well as new technologies that are enhancing our understanding of black holes. We will consider various a restricted set

1. INTRODUCTION

The high - level notion of a gravity wave is simple. It is an oscillation in a gravitational field that transports energy as gravitational radiation. It is often compared to an electromagnetic wave propagating in an electromagnetic field. It results from the curvature of spacetime, which is caused by the presence of mass. If the motion of that mass is spherically *asymmetric*, the mass may induce gravitational waves. An observer experiencing a gravitational wave will find distortions in spacetime. Geodesic distances will change at the frequency of the wave.

2. HISTORY

They were initially proposed by Oliver Heaviside in 1893. He developed a rudimentary analogue of gravity waves. This is technically the first ideation of the concept. This comes from a note written by H. Poincare.

Eventually Einstein and Grossmann published “Entwurf,” or an ”Outine of a Generalized Relativity Theory and a Theory of Gravitation” in 1913. This work did not fully define the gravitational wave, and it was not treated as a legitimized phenomenon. At last, in 1915, Einstein published his work on General Relativity in its present form. Einstein eventually produced a gravity wave equation with three solutions (1917 - 1919). Arthur Eddington subsequently published on the mathematical types of waves, though but they only maintained the status of theory.

It is interesting to remark that Einstein publicly indicated on numerous occasions that he thought gravity waves did not exist. But eventually, Einstein and Rosen released the first publication on the rigorous cylindrical solutions in 1937.

Gravity waves received their first formal conference treatment in 1957 at Chapel Hill. After the session chaired by Herman Bondi, the search for gravity waves began officially.

3. MATHEMATICAL FOUNDATION

This paper will refer primarily to a thorough overview published by Bieri et al. (2017). Firstly, we give a brief presentation of the Einstein field equations (EFE). These equations relate spacetime curvature to mass distributions in spacetime. In full, the Einstein Field Equation (depending on your choice of Λ) takes the form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1}$$

The left hand side describes the curvature of spacetime by the metrics, and the right hand side corresponds to the stress, energy, and momentum of spacetime.

They transition to the notion of a spacetime manifold, which is a 4 dimensional, oriented, differentiable manifold M with a Lorentzian metric tensor g. The tensor is of a non-degenerate, quadratic form of index one. Each component of the manifold is described below:

1. A manifold is a topological space that is similar to Euclidean space on a local scale.

2. It has four dimensions, one temporal and the other three spatial.

3. Orientation implies that there is a distinction between right and left handed coordinate systems. This allows us to keep some useful mathematical relations.

4. The metric tensor g distinguishes the spacetime geometry. This Lorentzian metric accounts for the time dimension with a negative Eigenvalue.

The metric tensor g specified is:

$$g = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$$

According to Bieri et al. (2017) they say that any point q on the manifold M is defined, as well as the tangent space T. When they say that it varies smoothly, they mean that the components are differentiable (continuous).

These equations, along with the Geodesic Deviation (Jacobi) equations and many others, could produce a theoretical result of gravitational waves. But after reaching this point and searching for these solutions, physicists encountered an immediate problem. There are various stability issues and questions surrounding the dynamics of a gravitational field. The main issue became known as the *Cauchy problem*. The problem can be outlined as follows.

1. We have some initial data, along with a Riemannian manifold $\mathscr{H}$ with a metric $\bar{g}_{ij}$, and a symmetric 2 - tensor $\mathscr{K}_{ij}$ that satisfy some constraints and consistency elements.

2. We then have to solve for the spacetime (M, g) that satisfies the EFE which evolve from the data. Specifically the manifold $\mathscr{H}$ needs to be a space-like hypersurface in this spacetime solution M.

The Cauchy problem can be solved with modern analysis and geometric techniques, but for some cases, the equations can be solved numerically. Circa 1920, many elements of partial differential equation theory remained undiscovered. Hence, it took a long time before anyone could solve these equations for meaningful results.

But in 1952, Choque-Bruhat (1952) proved the existence and uniqueness theorem for the Einstein equations. Later, in 1969, she and R. Geroch demonstrated the global existence of a 'unique maximal future development' for each initial data set. Once mathematicians and physicists knew that local solutions existed, the task was to then solve the global Cauchy problem. Christodoulou and Klainerman did just that. They proved the following theorem:

Theorem 1. *(Christodoulou and Klainerman, 1993) Given strongly asymptotically flat initial data for the Einstein vacuum equations, which is sufficiently small, there exists a unique, causally geodesically complete and globally hyperbolic solution (M, g), which itself is globally asymptotically flat.*

This theorem from Christodoulou & Klainerman (1993) gave mathematicians and physicists an effective and fairly robust way of describing expected observational effects of gravitational wave propagation. Consider a black hole merger. Prior to the event, the total energy of the system is essentially the individual masses of the black holes. During the event, energy and momentum are radiated away in the form of gravitational waves. After the event, the remaining energy in the system is calculated.

Since gravitational waves travel on null hypersurfaces from very far away, we can think of waves arriving at the site of observation at null infinity. The definition of a *future null infinity* is given (Bieri et al. (2017)):

Definition 1. *$\mathscr{I}^+$ is the defined to be the endpoints of all future-directed null geodesics along with $r \to \infty$. It has the topology of $\mathbb{R} \times \mathbb{S}^2$ with the function u taking values in $\mathbb{R}$.*

The hypersurface of travel intersects $\mathscr{I}^+$ at infinity in a 2-sphere. A special kind of mass calculation, called a Trautman-Bondi mass $M(u)$ gives mass that remains in an isolated gravitational system at some later time.

$$M(u_2) = M(u_1) - C \int_{u_1}^{u_2} \int_{S^2} |\Xi|^2 d\mu_{\overset{\circ}{\gamma}} du \tag{2}$$

Equation 2 (Bieri et al. (2017)) gives the Bondi mass-loss formula. $|\Xi|^2$ is the norm of the shear tensor at $\mathscr{I}^+$ and $d\mu_{\overset{\circ}{\gamma}}$ is the canonical measure on S^2. Here is the critical point: The effects of gravitational waves on nearby geodesics is encoded in the Jacobi equation. Then the displacement equation in a gravitational field becomes:

$$\mathscr{F} = C \int_{-\infty}^{+\infty} |\Xi|^2 du \tag{3}$$

At this point, physicists linearize gravity and write the spacetime metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and then proceed to solve for wave polarizations. The eventual solutions take the forms (under the conditions of vanishing perturbations for large r and only two degrees of freedom):

$$h_+(t-z) = h^{TT}_{xx} = -h^{TT}_{yy} \tag{4}$$

and

$$h_\times(t-z) = h^{TT}_{xy} = h^{TT}_{yx} \tag{5}$$

These h essentially describe the oscillations of gravitational waves in orthogonal directions. Next, physicists make the post-Newtonian approximation (PN). This essentially assumes that the wave generating bodies do not travel appreciably close to the speed of light. But this assumption allows the extension of solutions into higher orders. Apparently, they are solved through Green function methods.

Computational numerical methods are required when the equations for high gravity, high dynamic systems. After assembling a system of equations, the main difficulty is described as follows:

"One then writes a computer program that implements this determination and runs the program. Sounds simple, right? So what could go wrong? Quite a lot, actually. It is best to think of the solution of the finite difference equation as something that is supposed to converge to a solution of the differential equation in the limit as the step size δ between the lattice points goes to zero. But it is entirely possible that the solution does not converge to anything at all in this limit. In particular, the coordinate invariance of general relativity allows one to express the Einstein field equations in many different forms, some of which are not strongly hyperbolic.

Computer simulations of these forms of the Einstein field equations generally do not converge" (From Bieri et al. (2017)).

Solutions are finicky, and computation hosts a myriad of other problems, as described by Bieri et al. (2017). But this concludes the mathematical overview of gravitational waves.

4. DETECTORS

This section will explore the various design features and functionalities of gravitational wave detectors. We begin with the largest and arguably most successful type of detector is the laser interferometer.

4.1. *Laser Interferometry*

Laser interferometry has proven to be the most successful method for detecting gravity waves. There have been two large scale projects to date, LIGO and Virgo.

The Laser Interferometer Gravitational Wave Observatory (LIGO) is a kind of detector that implements a Michelson interferometer with a Fabry-Perot resonant cavity. In essence, a laser is prepared and passed through a beam splitter. The beams then reach a photo-detector which measures the phase shift between the incident beams of light (Figure 1).

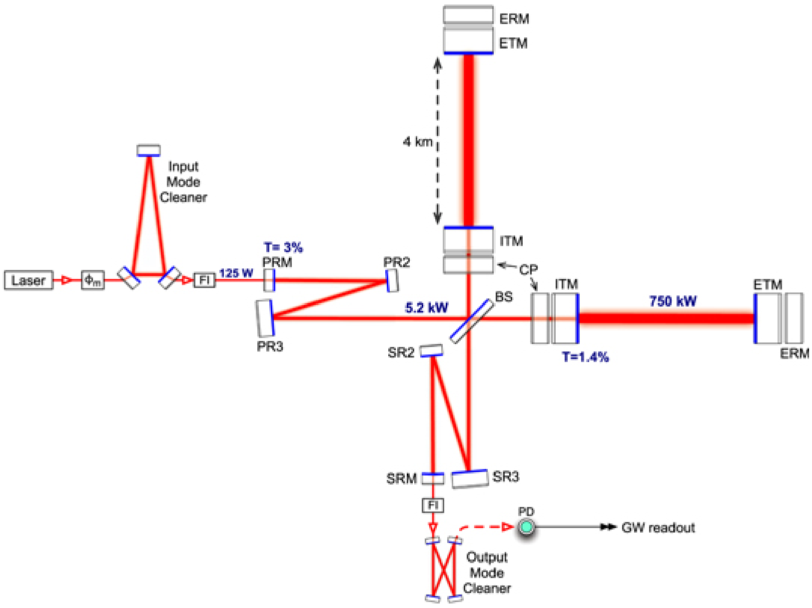

Figure 1: A high level instrument overview of a laser - interferometer gravitational wave detector.

From start to finish, the detector described in Aasi et al. (2015) functions as follows. A stable laser source emits light. The light is passed through a Faraday Isolator, which mitigates against back-reflections. The isolator polarizes the light using a crystal and a strong magnetic field. The desired polarized light transmits to the Input Mode Cleaner (IMC). Three mirrors resonate only for the spatial fundamental modes of the laser frequency, which reduces noise and cleans up the incoming laser profile. Power Recycling Mirrors between the mode cleaner and beam splitter. It reflects filtered light *back towards* the beam splitter in coherence with the main beam to boost the overall output signal. The beam then splits into two arms, each about 4 km in length. These arrive at test masses, which are very large cylindrical mirrors made of fused silica. The are suspended with highly complex isolation platforms to prevent seismic, thermal, and other vibrational noise. They act as free-falling

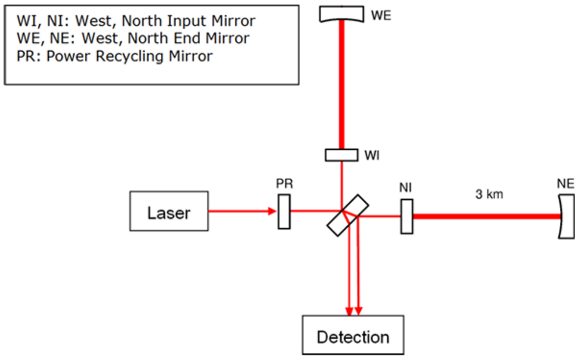

Figure 2: An overview of the French - Italian gravity wave detector, Virgo.

reference points in space. The gravitational wave will cause the proper distance between the input and end test masses to change, which is what the LIGO measures.

Figure 1 is from the LIGO collaboration. The advanced LIGO system employs a technique called signal recycling. Two signal recycling mirrors (SRM) at the output lowers the cavity finesse to access a broad range of frequencies. In interferometry, finesse determines the sharpness of the peaks produced:

$$f \equiv \frac{2\pi}{\Phi_{FWHM}} \tag{6}$$

This equation comes from Peatross & Ware (2015), where Φ is the phase shift between the incoming signals. These mirrors can also be tuned to a narrow-band mode of operation. This technique of modifying finesse to allow for different modes of operation is what sets the advanced LIGO detector apart.

Virgo is another Michelson Interferometer in essence, but has a slightly lower sensitivity than LIGO. Structurally, it is very similar to LIGO (see Figure 2, but with different recycling specifications, arm lengths, and collimators. The length of each arm is about 3 km in length.

5. CURRENT RESULTS

We now review the success and results of different gravity wave searches. In Table 1, an overview of some of the initial data surveys from two gravity wave detector facilities are listed. A surplus of measurements and data are available. We note here that dealing with data artifacts has been a unique challenge– according to Riles (2013), "...separate detection thresholds are frequently used in gravitational wave searches. Why? One technical reason is that the computational cost of pursuing cadidate outliers can be reduced with negligible loss in efficiency by applying low individual thresholds as an initial step... Nominal false alarm probabilities for a combined detection statistic can skyrocket if even one detector misbehaves."

Data Run	Period	Days
LIGO S1	August - September 2022	17
LIGO S2	February - April 2003	59
LIGO S3	OCtober 2003 - January 2004	70
LIGO S4	February - March 2005	30
LIGO S5	November 2005 - September 2007	696
LIGO S6	July 2009 - October 2010	470
Virgo VSR1	July 2009 - January 2010	136
Virgo VSR2	July 2009 - January 2010	187
Virgo VSR3	August - October 2010	71
Virgo VSR4	June - September 2011	110

Table 1: Adapted from Riles (2013), shows period and durations of surveys taken by LIGO and Virgo.

Up through 2013, all sky searches for gravity waves had been unsuccessful, though significant improvements had been made to detector sensitivity. The first breakthrough in detection came in 2016 (B. et al. (2016)), a momentous discovery and confirmation of physical theory. Since then, other detections have been made, but few as remarkable as the **GW 190521** incident. In May 2019, both LIGO and Virgo detected an extremely large gravity wave induced by a supermassive black hole merger. One of the black holes was $85M_{\odot}$ and the other $66M_{\odot}$ (Jia (2024)). If these measurements are accurate, it would be the largest incident to date. Even more interesting is the observation of GW190521 made by the Zwicky Transient Facility (ZTF). Typically, these events are non-emissive. But the ZTF detected a flash of light during the event. This indicates that the merger event occurred near another supermassive black hole accretion disk or is the result of some other kind of process. If it occurred near another SMBH, the accretion rate of would have spiked and we would have seen an increase in luminous output. To justify, Jia (2024) uses the familiar formulae:

$$L_{BHL} = \eta \dot{M}_{BHL} c^2 \tag{7}$$

$$\dot{M}_{BHL} = \frac{4\pi G^2 M_{MBH}^2 \rho}{v_{rel}^3} \tag{8}$$

6. CONCLUSION

Gravity waves, if nothing else, are an amazing phenomenon. Their detection is one of the hallmark discoveries of the century. The initial proposal which developed into a solved and provable physical phenomenon was incredible enough. But the successful development of large scale detectors over decades is even more astounding. And since the advent of LIGO and Virgo, numerous gravity wave events have been confirmed, observed, and measured. In studying them, gravity waves have been critical in confirming and building our understanding of fundamental physics. The future is bright, as we continue to probe the deepest secrets and mysteries of spacetime itself.

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