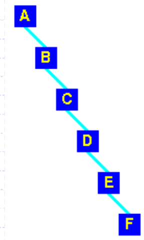
7. AVL Tree and 2-4 Tree

- AVL tree and 2-4 tree are balanced search trees.
- AVL tree, the definition, operations for searching, insertion, deletion, and re-balancing an AVL tree when it becomes unbalanced, the efficiency.
- 2-4 tree, the definition, operations of searching, insertion and deletion, the efficiency.

Problems of BST

- Binary search trees may be unbalanced.
- Insert this list of characters and form a tree

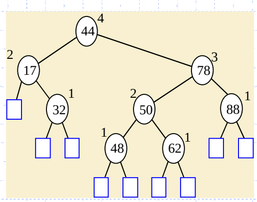
  A B C D E F
- An unbalanced BST may also be the result of repeated insertions and deletions.



BST degenerates to a linked list

AVL Tree (Adelson-Velskii and Landis)

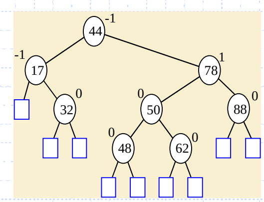
- **AVL trees are** **balanced.**
- An AVL tree is a ***binary search tree*** such that for every internal node v, the *heights of the children of v can differ by at most 1.*



An example of an AVL tree where the heights are shown next to the nodes:

Balance Factors in AVL Tree

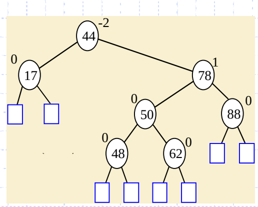
- In an AVL tree, every internal node is associated with a ***balance factor*** , which is calculated as the height of the left subtree minus the height of the right subtree.



An example of an AVL tree where each internal node is associated with a balance factor.

Balance Factors in AVL Tree

- In an AVL tree, if there is an internal node whose balance factor is less than -1 or greater than 1, the tree is said ***unbalanced***.



An example of an AVL tree which becomes unbalanced after a node is removed.

Height of an AVL Tree

- **Proposition**: The ***height*** of an AVL tree T storing n keys is O(log n).
- **Justification**: The easiest way to approach this problem is to find **n(h):** the *minimum number of internal nodes* of an AVL tree of height h.
- We see that n(1) = 1 and n(2) = 2
- For n ≥ 3, an AVL tree of height h contains the root node, one AVL subtree of height h-1 and the other AVL subtree of height at least h-2.
- i.e. n(h) = 1 + n(h-1) + n(h-2)

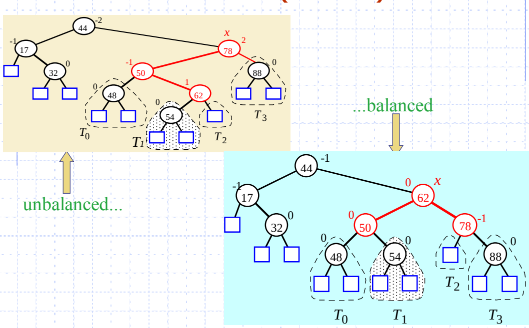
Height of an AVL Tree (cont)

- Knowing n(h-1) > n(h-2), we get n(h) > 2n(h-2)
  - **n(h) > 2n(h-2)**
  - **n(h) > 4n(h-4)** (n(h-2) > 2n(h-4))
  - **n(h) > 8n(h-6)** (n(h-4) > 2n(h-6))
  - **…**
  - **n(h) > 2^in(h-2i)**
  - **For any integer i such that h-2i ≥ 1**
  - Let h – 2i = 1, then i = (h – 1)/2
- Solving the base case we get: n(h) < $2^{(h-1)/2}$
- Taking logarithms: h < 2log n(h) + 1
- Thus the height of an AVL tree is O(log n)

Insertion

- A binary search tree T is said to have ***AVL property*** if for every node v, the height of v's children differ by at most one, or the balance factor is -1, 0, 1.
- Inserting a node into an AVL tree may change the balance factors of some of the nodes in T.
- If an insertion causes AVL tree T to become unbalanced, we travel up the tree from the newly created node until we find the first node x such that its balance factor is -2 or 2.
- Node x is the root of the subtree to be rebalanced.
- Now to rebalance...

Insertion (contd.)



Rotations

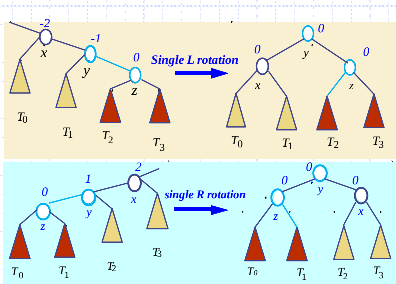
Rotations can be used to re-balance an AVL tree that becomes unbalanced after an insertion.

- There are four types of rotations: single left, single right, double right-left, double left-right.
- To re-balance an AVL tree, we travel up the tree from the newly inserted node until we find the first node x such that its balance factor is -2 or 2, then choose the type of rotation.

Rotations

Single Rotations:

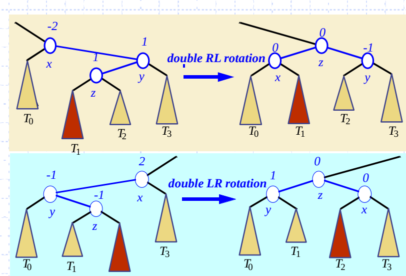
L: $(T_0\, x\, (T_1\, y\, z)) \Rightarrow ((T_0 x\, T_1) y\, z)$ R: $((z\, y\, T_2) x\, T_3) \Rightarrow (z\, y\, (T_2 x T_3\,))$



Rotations

Double rotations:

$RL: (T_0 x((T_1 z T_2) y T_3)) \Rightarrow ((T_0 x T_1) z (T_2 y T_3))$ $LR: ((T_0 y (T_1 z T_2)) x T_3) \Rightarrow ((T_0 x T_1) z (T_2 y T_3))$



Restructure Algorithm

function restructure(x):

Input: A node *x* of a binary search tree T that has both a child *y* and a grandchild *z*

Output: Tree T restructured involving nodes x, y, and z.

1: Let (*a, b, c*) be an inorder listing of the nodes *x, y, and z,* and let (T0, T1, T2, T3) be an inorder listing of the four subtrees of *x, y, and z.*

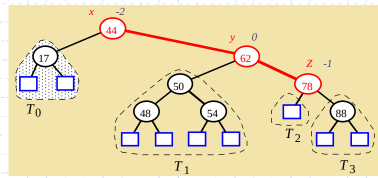
2. Replace the subtree rooted at x with a new subtree rooted at *b*

3. Let *a* be the left child of *b* and let T0, T1 be the left and right subtrees of *a*, respectively.

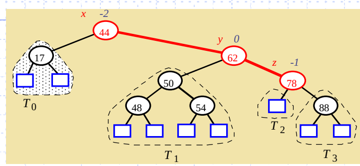
4. Let *c* be the right child of *b* and let T2, T3 be the left and right subtrees of *c*, respectively.

Cut/Link Restructure Algorithm

- Let's go into a little more detail on this algorithm...
- Any tree that needs to be balanced can be grouped into 7 parts: x, y, z, and the 4 trees anchored at the children of those nodes (T0-T3)



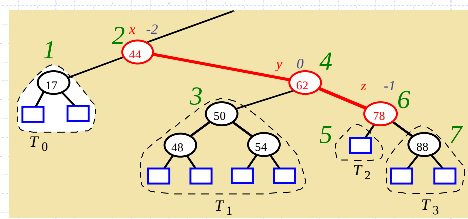
Cut/Link Restructure Algorithm



- Make a new tree which is balanced by putting the 7 parts from the old tree into the new tree so that the numbering is still correct when we do an in-order-traversal of the new tree.
- This works regardless of how the tree is originally unbalanced.
- Let's see how it works!

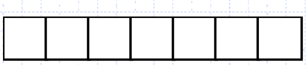
Cut/Link Restructure Algorithm

- Identify the x, y, z, where x has balance factor -2 or 2.
- Number the 7 parts by doing an in-order-traversal.



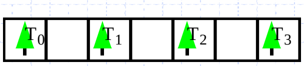
Cut/Link Restructure Algorithm

- Now create an Array, numbered 1 to 7 (the 0th element can be ignored with minimal waste of space)



1 2 3 4 5 6 7

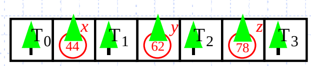
•Cut the 4 T trees and place them in their inorder rank in the array



1 2 3 4 5 6 7

Cut/Link Restructure Algorithm

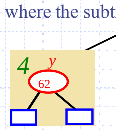
- Now cut x,y, and z in that order (child,parent,grandparent) and place them in their inorder rank in the array.



1 2 3 4 5 6 7

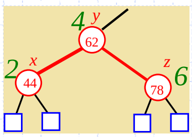
•Now we can re-link these subtrees to the main tree.

•Link in rank 4 (y) where the subtree's root formerly



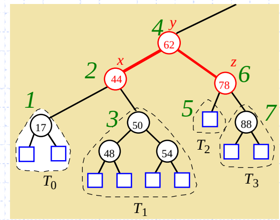
Cut/Link Restructure Algorithm

- Link in ranks 2 (x) and 6 (z) as 4's children.



Cut/Link Restructure Algorithm

- Finally, link in ranks 1,3,5, and 7 as the children of 2 and 6.



- Now you have a balanced tree!

Cut/Link Restructure algorithm

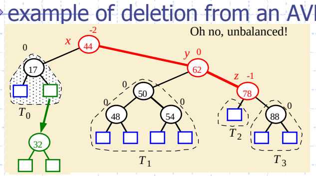
- This algorithm for restructuring has the same effect as using the four rotation cases discussed earlier.
- Advantages: no case analysis, more elegant
- Disadvantage: can be more code to write
- Same time complexity

Deletion

- We can easily see that performing a delete(w) can cause T to become unbalanced.
- Let *x* be the first unbalanced node encountered while traveling up the tree from w. Also, let y be the child of x with the larger height, and let z be the child of y with the larger height.
- We can perform operation restructure(x) to restore balance at the subtree rooted at x.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached

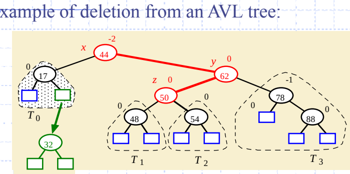
Deletion (contd.)

- example of deletion from an AVL tree:



Deletion (contd.)

- example of deletion from an AVL tree:



Oh no, unbalanced!

AVL Trees - Data Structures

- AVL trees can be implemented with a flag to indicate the balance state

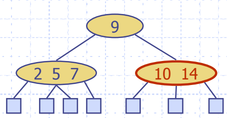
```
typedef enum {RightTooHeavy, RightHeavy,
  Balanced, LeftHeavy, LeftTooHeavy
          } BalanceFactor;

typedef struct node {
      BalanceFactor bf;
      void *item;
      struct node *left, *right;
} AVL_node;
```


Efficiency

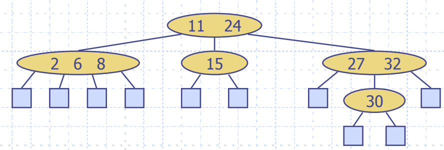
- The height of an AVL tree of *n* internal nodes is *O* (log *n*).
- The efficiency of searching an AVL tree is *O* (log *n*).
- The efficiency of a rotation or restructuring operation is *O* (1).
- The efficiency of insertion into an AVL tree is *O* (log *n*), including searching and rebalancing.
- The efficiency of deletion from an AVL tree is *O* (log *n*), including searching and rebalancing.

(2,4) Trees



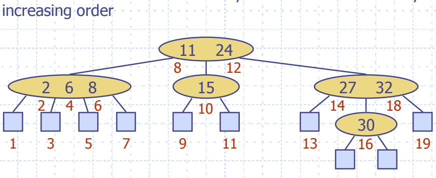
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\, v_2 \ldots v_d$ storing keys $k_1\, k_2 \ldots k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i $(i = 2, \ldots, d-1)$
    - keys in the subtree of v_d are greater than k_{d-1}



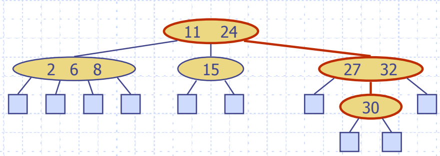
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order



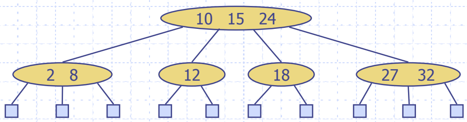
Multi-Way Searching

- Similar to search in a binary search tree
- At each internal node with children $v_1\, v_2 \ldots v_d$ and keys $k_1\, k_2 \ldots k_{d-1}$
  - $k = k_i$ $(i = 1, \ldots, d-1)$: the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ $(i = 2, \ldots, d-1)$: we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



(2,4) Tree

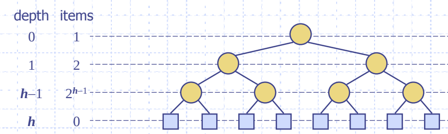
- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



Height of a (2,4) Tree

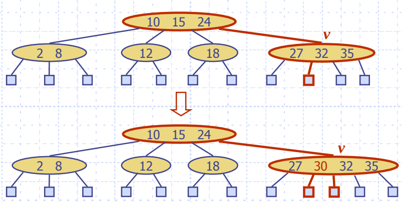
- Theorem: A (2,4) tree storing n items has height $O(\log n)$

  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \dots, h-1$ and no items at depth h, we have
    $n \geq 1 + 2 + 4 + \dots + 2^{h-1} = 2^h - 1$
  - Thus, $h \leq \log(n + 1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



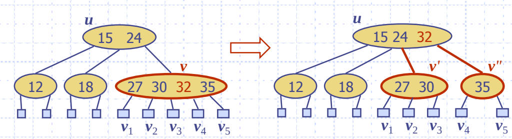
Insertion

- We insert a new item (k, o) at the parent v of the leaf reached by searching for k
  - We preserve the depth property but
  - We may cause an overflow (i.e., node v may become a 5-node)
- Example: inserting key 30 causes an overflow



Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \dots v_5$ be the children of v and $k_1 \dots k_4$ be the keys of v
  - node v is replaced by nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u



Analysis of Insertion

function *insertItem*(k, o)

1. We search for key k to locate the insertion node v

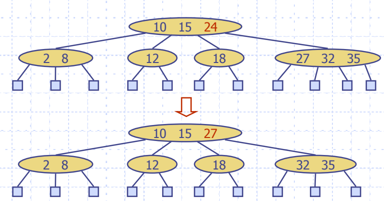
2. We add the new item (k, o) at node v

3. **while** (*overflow*(v)){
   if (*isRoot*(v))
   create a new empty root above v;
   $v \leftarrow$ *split*(v) // return parent of v;
   }

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Step 2 takes $O(1)$ time
  - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits
- Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

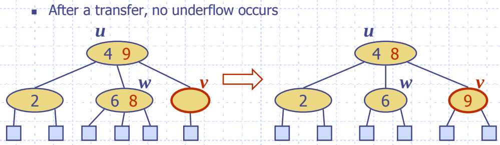
Deletion

- We reduce deletion of an item to the case where the item is at the node with leaf children
- Otherwise, we replace the item with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter item
- Example: to delete key 24, we replace it with 27 (inorder successor)



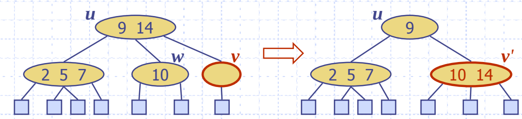
Underflow and Transfer

- Deleting an item from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases
- Case 1: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - After a transfer, no underflow occurs



Underflow and Fusion

- Case 2: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an item from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u



Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the item
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time