Lecture 14
Polyhedral Convex Cones

Matthew T. Mason

Mechanics of Manipulation

Today's outline

What use is a cone? I: velocity twists.

- Reuleaux's analysis of unilateral constraint yields a set of nominally feasible motions.
  - What does the corresponding set of IC's look like, in the plane?
    A triangle of signed rotation centers.
  - What does the corresponding set of velocity twists look like, in three space?
    A batch of vertical, zero pitch screws hitting the horizontal plane in a triangle.
  - What does the corresponding set of velocity twists look like, in *velocity twist space* $(\omega_z, v_{0x}, v_{0y})$?
    A *polyhedral convex cone*.

What use is a cone? I: velocity twists.

- Reuleaux's analysis of unilateral constraint yields a set of nominally feasible motions.
  - What does the corresponding set of IC's look like, in the plane?
    A triangle of signed rotation centers.
  - What does the corresponding set of velocity twists look like, in three space?
    A batch of vertical, zero pitch screws hitting the horizontal plane in a triangle.
  - What does the corresponding set of velocity twists look like, in *velocity twist space* $(\omega_z, v_{0x}, v_{0y})$?
    A *polyhedral convex cone*.

What use is a cone? I: velocity twists.

- Reuleaux's analysis of unilateral constraint yields a set of nominally feasible motions.
  - What does the corresponding set of IC's look like, in the plane?
    A triangle of signed rotation centers.
  - What does the corresponding set of velocity twists look like, in three space?
    A batch of vertical, zero pitch screws hitting the horizontal plane in a triangle.
  - What does the corresponding set of velocity twists look like, in *velocity twist space* $(\omega_z, v_{0x}, v_{0y})$?
    A polyhedral convex cone.

What use is a cone? I: velocity twists.

- Reuleaux's analysis of unilateral constraint yields a set of nominally feasible motions.
  - What does the corresponding set of IC's look like, in the plane?
    A triangle of signed rotation centers.
  - What does the corresponding set of velocity twists look like, in three space?
    A batch of vertical, zero pitch screws hitting the horizontal plane in a triangle.
  - What does the corresponding set of velocity twists look like, in *velocity twist space* $(\omega_z, v_{0x}, v_{0y})$?
    A *polyhedral convex cone*.

What use is a cone? II: wrenches.

- What is the set of wrenches that can be applied by frictionless contacts on a rigid body?
  Polyhedral convex cones.
- What if we add friction?
  Polyhedral convex cones.

Cones, the agenda.

- Today: introduction to polyhedral convex cones;
- Next time: the *oriented plane*. (The plane of signed points, i.e. Reuleaux's plane. Convex polygons in the oriented plane are a way of representing cones.)
- Following: graphical methods, using cones and the oriented plane, to work with wrenches and velocity twists.
- Applications: grasping, manipulation.

Positive linear span

- For now, use n-dimensional vector space $\mathbf{R}^n$. Later, wrench space and velocity twist space.
- Let $\mathbf{v}$ be any non-zero vector in $\mathbf{R}^n$. Then the set of vectors
$$\{k\mathbf{v} \mid k \geq 0\}$$
describes a *ray*.
- Let $\mathbf{v}_1$, $\mathbf{v}_2$ be non-zero and non-parallel. Then the set of positively scaled sums
$$\{k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \mid k_1, k_2 \geq 0\}$$
is a planar cone—sector of a plane.
- We want to generalize to an arbitrary finite set of vectors . . .

Positive linear span

Definition

The positive linear span $\mathrm{pos}(\cdot)$ of a set of vectors $\{\mathbf{v}_i\}$ is

$$\mathrm{pos}(\{\mathbf{v}_i\}) = \{\sum k_i \mathbf{v}_i \mid k_i \geq 0\}$$

- (The positive linear span of the empty set is the origin.)

Relatives of positive linear span

Definition

The linear span $\mathrm{lin}(\cdot)$ is

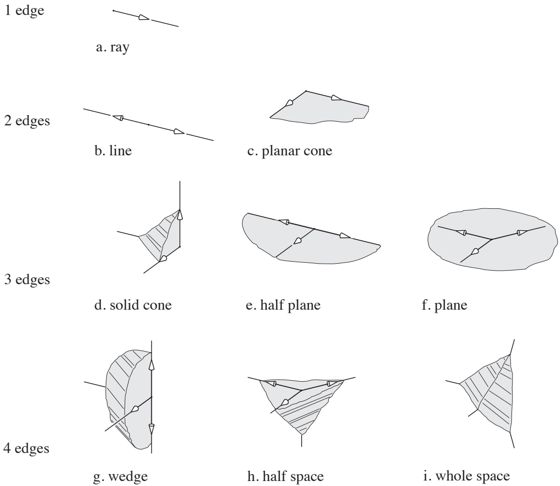
$$\mathrm{lin}(\{\mathbf{v}_i\}) = \{\sum k_i \mathbf{v}_i \mid k_i \in \mathbf{R}\}$$

Definition

The convex hull $\mathrm{conv}(\cdot)$ is

$$\mathrm{conv}(\{\mathbf{v}_i\}) = \{\sum k_i \mathbf{v}_i \mid k_i \geq 0, \sum k_i = 1\}$$

Varieties of cones in three space



- By the way, how many different ways are there of positively spanning a linear space? Start with Chandler Davis, Theory of positive linear dependence, American Journal of Mathematics, 76(4), Oct. 1954.

Spanning all of $\mathbf{R}^n$

Theorem

A set of vectors $\{\mathbf{v}_i\}$ positively spans the entire space $\mathbf{R}^n$ if and only if the origin is in the interior of the convex hull:

$$\text{pos}(\{\mathbf{v}_i\}) = \mathbf{R}^n \leftrightarrow \mathbf{0} \in \text{int}(\text{conv}(\{\mathbf{v}_i\}))$$

(Review meaning of “interior”.)

Theorem

It takes at least $n+1$ vectors to positively span $\mathbf{R}^n$.

Representing cones

- Two ways to represent cones: *edge representation* and *face representation*.
- Edge representation uses positive linear span. Given a set of edges $\{\mathbf{e}_i\}$, the cone is given by $\text{pos}(\{\mathbf{e}_i\})$.

Face representation of cones

- First represent *planar half-space* by inward pointing normal vector **n**.

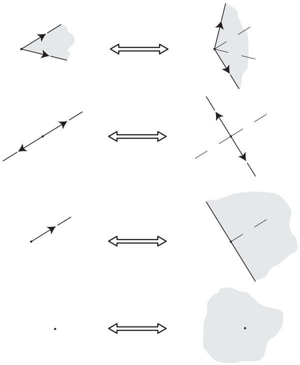
$$\text{half}(\mathbf{n}) = \{\mathbf{v} \mid \mathbf{n} \cdot \mathbf{v} \geq 0\}$$

  (Here we use dot product, but when working with twists and wrenches we will use reciprocal product.)

- Consider a cone with face normals $\{\mathbf{n}_i\}$. Then the cone is the intersection of the half-spaces:

$$\cap\{\text{half}(\mathbf{n}_i)\}$$

Supplementary cone; polar

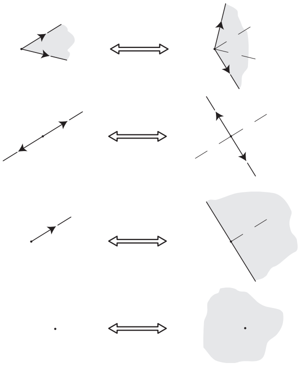


Definition

Given a polyhedral convex cone V, the supplementary cone supp(V) (also known as the polar) comprises the vectors that make non-negative dot products with *all* the vectors in V:

$$\{u \in \mathbf{R}^n \mid u \cdot v \geq 0 \; \forall v \in V\}$$

Supplementary cone; polar; representation



- The supplementary cone's *edges* are the original cone's *face normals*, and vice versa. So if

$$V = \text{pos}(\{\mathbf{e}_i\}) = \cap\{\text{half}(\mathbf{n}_i)\}$$

then

$$\text{supp}(V) = \text{pos}(\{\mathbf{n}_i\}) = \cap\{\text{half}(\mathbf{e}_i)\}$$

Frictionless contact

Characterize contact by *set of possible wrenches*.

Assume uniquely determined contact normal.

Assume frictionless contact can give arbitrary magnitude force along inward-pointing normal.

Then a frictionless contact gives a ray in wrench space, pos($\mathbf{w}$), where $\mathbf{w} = (\mathbf{c}, \mathbf{c}_0)$ is the contact screw.

Two contacts

Given two frictionless contacts $\mathbf{w}_1$ and $\mathbf{w}_2$, total wrench is the sum of possible positive scalings of $\mathbf{w}_1$ and $\mathbf{w}_2$:

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2; k_1, k_2 \geq 0$$

i.e. the positive linear span $\text{pos}(\{\mathbf{w}_1, \mathbf{w}_2\})$.

Generalizing:

Theorem

If a set of frictionless contacts on a rigid body is described by the contact normals $\mathbf{w}_i = (\mathbf{c}_i, \mathbf{c}_{0i})$ then the set of all possible wrenches is given by the positive linear span $\text{pos}(\{\mathbf{w}_i\})$.

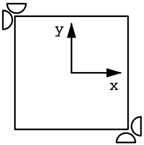
Force closure

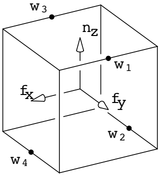
Definition

Force closure means that the set of possible wrenches exhausts all of wrench space.

It follows from a previous theorem that a frictionless force closure requires at least 7 contacts. Or, since planar wrench space is only three-dimensional, frictionless force closure in the plane requires at least 4 contacts.

Example wrench cone





Construct unit magnitude force at each contact.

Write screw coords of wrenches.

Take positive linear span.

Exhausts wrench space?

Cones in velocity twist space

Cannot use finite displacement twists. They are not vectors.

Velocity twists are vectors!

Let $\{\mathbf{w}_i\}$ be a set of contact normals.

Let $W = \text{pos}(\{\mathbf{w}_i\})$ be set of possible wrenches.

First order analysis: velocity twists T must be *reciprocal or repelling* to contact wrenches: $T = \text{supp}(W)$.

“Reciprocal or repelling” $\equiv$ supp($\cdot$)

- The PCC of possible wrenches, and the PCC of feasible velocity twists, are supplementary cones!
- This raises some interesting questions. Most specifically, we have a keen two-dimensional way of representing the feasible velocity twists: Reuleaux’s method. Can we do the same thing for wrenches? More later!

Conclusion

- You can represent contact constraints as PCC's in wrench space.
- You can represent feasible velocities as PCC's in velocity twist space.
- The twist cone will be supplementary to the wrench cone.
- So ... what about those defective cases, where Reuleaux gives false positives? Example: triangle with a *force focus*.