PAPER *Special Section on Wireless Technologies and Computational Electromagnetics*

New Expressions for Coupling Coefficient between Resonators

Ikuo AWAI[†a)], ***Fellow***

SUMMARY Coupling between resonators are analyzed theoretically on basis of the coupled mode theory. New and basic equations for the coupling coefficient are derived and compared with those of waveguides. They should be useful for understanding the physical background of coupling and designing a new coupling scheme.

key words: *resonator, coupling coefficient, coupled-mode theory, perturbation method, overlap integral, energy exchange, resonant frequency*

1. Introduction

The design of band pass filters (BPFs) requires the quantity “coupling coefficient between resonators” to determine the bandwidth. In the case of LC filter design, the coupling coefficient k is defined by the mutual inductance M or capacitance C_m between adjacent inductors or capacitors, respectively, coping with the design theory based on the LC circuits

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{C_m}{\sqrt{C_1 C_2}}, \tag{1}$$

where L_1, L_2 are the adjacent self inductances and C_1, C_2 are the capacitances.

For the higher frequency band such as the microwaves, however, circuits are not described by lumped elements, demanding an alternative expression for the coupling coefficient. In fact, it is usually given by the equation

$$k = \frac{\omega_h^2 - \omega_l^2}{\omega_h^2 + \omega_l^2} \tag{2}$$

where ω_h and ω_l are the splitted higher and lower frequencies respectively separating from the originally identical resonant frequencies due to the mutual coupling. The basis of this definition seems the extension from the LC circuit theory for the lower frequency band. One can easily obtain the relation (2) for coupled LC resonators [1] and they take it for granted that Eq. (2) is also true for the higher frequencies where the lumped circuit approximation is not adequate any more.

The purpose of the present article is to find the background for Eq. (2) through the concept of energy exchange between two systems. The coupled-mode theory has been successfully applied for the coupling of two propagating modes, clarifying that the coupling coefficient is directly connected with the energy exchange between two modes [2], [3]. The same theory should also be applicable to the resonators, because the electromagnetic field in the coupled resonator system could be expanded by the original eigen modes in the similar manner as the case of propagating modes. We will give a more essential and persuasive, we believe, definition and the relating equations based on this theory.

Manuscript received April 8, 2005.
Manuscript revised June 16, 2005.

†The author is with School of Science and Technology, Ryukoku University, Otsu-shi, 520-2194 Japan.
a) E-mail: awai@rins.ryukoku.ac.jp

DOI: 10.1093/ietele/e88–c.12.2295

2. Coupling between Two Guided Waves

The coupling between electromagnetic wave and electron beam was extensively studied for use of traveling wave tubes. We can find a comprehensive book written by Lousell [2] more than 40 years ago. But we would rather consult with a book on integrated optics written by Marcuse [3] 30 years ago. It covers the coupling between two light waveguides to make a directional coupler. According to him, multiple number of coupled lines are studied with the perturbation method. The Maxwell’s equations for a coupled system are

$$\begin{aligned} \nabla \times \boldsymbol{H} &= j\omega\varepsilon_0 n^2 \boldsymbol{E}, \\ \nabla \times \boldsymbol{E} &= -j\omega\mu_0 \boldsymbol{H}, \end{aligned} \tag{3}$$

where n is the refractive index varying place to place. The electric and magnetic field are divided into the transverse and axial part, respectively,

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_t + \boldsymbol{E}_z, \\ \boldsymbol{H} &= \boldsymbol{H}_t + \boldsymbol{H}_z. \end{aligned} \tag{4}$$

One substitutes Eq. (4) into (3) and takes out only transverse components from the equations

$$\begin{aligned} -\frac{1}{j\omega\mu_0}\nabla_t \times (\nabla_t \times \boldsymbol{E}_t) + \boldsymbol{a}_z \times \frac{\partial \boldsymbol{H}_t}{\partial z} &= j\omega\varepsilon_0 n^2 \boldsymbol{E}_t, \\ \frac{1}{j\omega\varepsilon_0}\nabla_t \times \left(\frac{1}{n^2}\nabla_t \times \boldsymbol{E}_t\right) + \boldsymbol{a}_z \times \frac{\partial \boldsymbol{E}_t}{\partial z} &= -j\omega\mu_0 \boldsymbol{H}_t. \end{aligned} \tag{5}$$

The solution of Eq. (5) is expanded by the eigen modes of each single waveguide

$$\begin{aligned} \boldsymbol{E}_t &= \sum_i a_i \boldsymbol{E}_{ti}, \\ \boldsymbol{H}_t &= \sum_i b_i \boldsymbol{H}_{ti}. \end{aligned} \tag{6}$$

Manipulating the complicated equations, one arrives at the simplified set of scalar differential equations for each mode amplitude

$$\frac{da_j}{dz} = -j\beta_j a_j + \sum_i K_{ji} a_i. \tag{7}$$

We further assume that only two modes exist in the system and obtain

$$\begin{aligned}\frac{da_1}{dz} &= -j\beta_1 a_1 + K_{12} a_2,\\ \frac{da_2}{dz} &= -j\beta_2 a_2 + K_{21} a_1,\end{aligned} \tag{8}$$

where K_{ji} is an overlap integral of the two modes and sometimes called coupling coefficient. The amplitude of each mode varies with propagation when a single mode is excited [3]. When $\beta_1 = \beta_2 = \beta_0$ and $K_{12} = -K_{21}^* = jK$ hold, the whole energy transfers alternately between two guides (See Appendix). The result is expressed as follows,

$$\begin{aligned}|a_1| &= |\cos Kz|,\\ |a_2| &= |\sin Kz|.\end{aligned} \tag{9}$$

Referring to Fig. 1, the period of energy interchange λ_m is given

$$K\lambda_m = \pi. \tag{10}$$

Thus larger K corresponds to faster energy interchange (shorter length of the waveguide needed for total interchange). The normalized K should be more appropriate as the coupling coefficient, since it shows the ratio of contribution to the variation of the wave intensity, referring Eq. (8). Hence, it is defined as

$$k = \frac{2K}{\beta_0} \tag{11}$$

and transformed into the following by use of Eq. (10) and , $\beta_0 = 2\pi/\lambda_0$,

$$k = \frac{\lambda_0}{\lambda_m}. \tag{12}$$

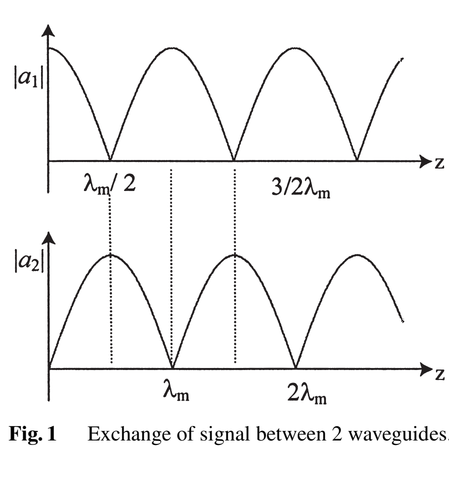

Fig. 1 Exchange of signal between 2 waveguides.

Equations (A· 3), and (11), on the other hand, are combined to give

$$k = \frac{2(\beta_h - \beta_l)}{\beta_h + \beta_l}, \tag{13}$$

since

$$\beta_0 \cong \frac{(\beta_h + \beta_l)}{2}. \tag{14}$$

Now we have three expressions for the coupling coefficient of guided waves. The second one, Eq. (12), looks most essential, because it is related with the energy, being given by the normalized rate of energy exchange. These expressions will be compared with those for resonator's coupling later.

3. Application of Coupled Mode Theory to Resonators

3.1 Mode Expansion

We expand the electromagnetic field of the coupled resonator system with each eigen modes of non-coupled system. As long as there is no internal loss nor external coupling, the field is expanded by the solenoidal functions [4], that is,

$$\begin{aligned}\boldsymbol{E} &= \sum_i a_i \boldsymbol{E}_i,\\ \boldsymbol{H} &= \sum_i b_i \boldsymbol{H}_i.\end{aligned} \tag{15}$$

where each eigen modes satisfy the relations

$$\nabla \times \boldsymbol{H}_i = k_i \varepsilon_s \boldsymbol{E}_i \tag{16a}$$

$$\nabla \times \boldsymbol{E}_i = k_i \mu_s \boldsymbol{H}_i \tag{16b}$$

Here, k_i is a positive real eigen value. The quantities ε_s and μ_s denote the permittivity and permeability of the non-coupled system respectively, which are allowed to be a function of the coordinates. The electromagnetic fields $\boldsymbol{E}_i$ and $\boldsymbol{H}_i$ are normalized as

$$\int \mu_s \boldsymbol{H}_n \cdot \boldsymbol{H}_m dv = \int \varepsilon_s \boldsymbol{E}_n \cdot \boldsymbol{E}_m dv = \begin{cases} 0 & (n \neq m) \\ 1 & (n = m) \end{cases}. \tag{17}$$

If one takes the rotation of Eq. (16b) and substitute Eq. (16a) in it,

$$\nabla \times \frac{1}{\mu_s} \nabla \times \boldsymbol{E}_i = k_i^2 \varepsilon_s \boldsymbol{E}_i \tag{18}$$

holds. The similar equation is easily obtained for $\boldsymbol{H}_i$. According to the Maxwell equation, the rotation of the electric field is proportional to $\boldsymbol{H}$, and hence it should be expanded by $\boldsymbol{H}_i$ as,

$$\nabla \times \boldsymbol{E} = \sum_i \mu_s \boldsymbol{H}_i \int_V \nabla \times \boldsymbol{E} \cdot \boldsymbol{H}_i dv. \tag{19}$$

If we multiply Eq. (19) by $\boldsymbol{H}_i$ scalarly, integrate it in the total space V and refer to Eq. (17), we will find its validity. Using a well known vector relation and Eq. (16b), one obtains

$$\begin{aligned}\nabla \cdot \left(\boldsymbol{E} \times \frac{1}{\mu_s} \nabla \times \boldsymbol{E}_i\right) &= \nabla \times \boldsymbol{E} \cdot \frac{1}{\mu_s} \nabla \times \boldsymbol{E}_i - \boldsymbol{E} \cdot \nabla \times \frac{1}{\mu_s} \nabla \times \boldsymbol{E}_i \\ &= k_i \boldsymbol{H}_i \cdot \nabla \times \boldsymbol{E} - k_i^2 \varepsilon_s \boldsymbol{E} \cdot \boldsymbol{E}_i. \end{aligned} \tag{20}$$

Substitution of (20) into (19) gives

$$\begin{aligned}\nabla \times \boldsymbol{E} &= \sum_i \mu_s \boldsymbol{H}_i \left[k_i \int_V \varepsilon_s \boldsymbol{E} \cdot \boldsymbol{E}_i dv - \int_S \boldsymbol{n} \cdot (\boldsymbol{E} \times \boldsymbol{H}_i) dS\right] \\ &= \sum_i k_i a_i \mu_s \boldsymbol{H}_i, \end{aligned} \tag{21}$$

where it should be noted that $\boldsymbol{n}$ is a unit outward normal at the conductor surface. The surface integral in the equation above is transformed as

$$\int_S \boldsymbol{n} \cdot (\boldsymbol{E} \times \boldsymbol{H}_i) dS = \int_S \boldsymbol{H}_i \cdot (\boldsymbol{n} \times \boldsymbol{E}) dS, \tag{22}$$

and hence, it becomes 0 on the conductor surface S. It is also considered to be 0 for an open resonator like microstrip line, since $\boldsymbol{E}$ and $\boldsymbol{H}_i$ decreases according as $\exp(-\alpha r)$

One has the similar expression for the magnetic field

$$\nabla \times \boldsymbol{H} = \sum_i k_i b_i \varepsilon_s \boldsymbol{E}_i - \sum_i \boldsymbol{E}_i \int_S \boldsymbol{n} \cdot (\boldsymbol{H} \times \boldsymbol{E}_i) dS, \tag{23}$$

but the surface integral in this case does not necessarily 0 for S of the coupled system. Substituting Eqs. (15), (21) and (23) into the Maxwell equations for the coupled system

$$\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = 0, \ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{D}}{\partial t} = 0, \tag{24}$$

one obtains the expansions

$$\sum_i \varepsilon_s k_i b_i \boldsymbol{E}_i + \sum_i \boldsymbol{E}_i \int_S \boldsymbol{n} \cdot (\boldsymbol{E}_i \times \boldsymbol{H}) dS - \frac{d}{dt} \varepsilon \sum_i a_i \boldsymbol{E}_i = 0, \tag{25a}$$

$$\sum_i \mu_s k_i a_i \boldsymbol{H}_i + \frac{d}{dt} \mu \sum_i b_i \boldsymbol{H}_i = 0, \tag{25b}$$

where ε and μ denote permittivity and permeability of the coupled system, respectively. Expanding $\boldsymbol{H}$ in the surface integral of Eq. (25a) by use of Eq. (15), we have

$$\int_S \boldsymbol{n} \cdot (\boldsymbol{E}_i \times \boldsymbol{H}) dS = \sum_j \int_S b_j \boldsymbol{n} \cdot (\boldsymbol{E}_i \times \boldsymbol{H}_j) dS, \tag{26}$$

and then, we denote it as

$$\int_S \boldsymbol{n} \cdot (\boldsymbol{E}_i \times \boldsymbol{H}_j) dS = f_{ij}. \tag{27}$$

3.2 Derivation of Coupled Mode Equation

To analyze the coupling of two resonators, we integrate Eq. (25a) after scalarly multiplying $\boldsymbol{E}_1$ to both sides of the equation

$$\begin{aligned} g_{11} \frac{da_1}{dt} + g_{12} \frac{da_2}{dt} &= \left(c_{11} k_1 + \sum_i c_{1i} f_{i1}\right) b_1 \\ &+ \left(c_{12} k_2 + \sum_i c_{1i} f_{i2}\right) b_2 \end{aligned} \tag{28}$$

where we put

$$\begin{aligned} g_{11} &= \int_V \varepsilon \boldsymbol{E}_1^2 dv, \ g_{22} = \int_V \varepsilon \boldsymbol{E}_2^2 dv, \\ g_{12} &= g_{21} = \int_V \varepsilon \boldsymbol{E}_1 \boldsymbol{E}_2 dv. \end{aligned} \tag{29}$$

$$\begin{aligned} c_{11} &= \int_V \varepsilon_s \boldsymbol{E}_1^2 dv, \ c_{22} = \int_V \varepsilon_s \boldsymbol{E}_2^2 dv, \\ c_{12} &= c_{21} = \int_V \varepsilon_s \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 dv \end{aligned} \tag{30}$$

Secondly, scalar multiplication of $\boldsymbol{E}_2$ to Eq. (25a) and integration gives

$$\begin{aligned} g_{21} \frac{da_1}{dt} + g_{22} \frac{da_2}{dt} &= \left(c_{21} k_1 + \sum_i c_{2i} f_{i1}\right) b_1 \\ &+ \left(c_{22} k_2 + \sum_i c_{2i} f_{i2}\right) b_2. \end{aligned} \tag{31}$$

Solving Eqs. (28) and (31) for da_i/dt, we get

$$\begin{aligned} \frac{da_1}{dt} &= \frac{1}{|g|} \left[\left\{(g_{22} c_{11} - g_{21} c_{21}) k_1 + \sum_i (g_{22} c_{1i} - g_{12} c_{2i}) f_{i1}\right\} b_1 \right. \\ &\left. + \left\{(g_{22} c_{12} - g_{12} c_{22}) k_2 + \sum_i (g_{22} c_{1i} - g_{12} c_{2i}) f_{i2}\right\} b_2\right] \\ \frac{da_2}{dt} &= \frac{1}{|g|} \left[\left\{(g_{11} c_{21} - g_{21} c_{11}) k_1 + \sum_i (g_{11} c_{2i} - g_{21} c_{1i}) f_{i1}\right\} b_1 \right. \\ &\left. + \left\{(g_{11} c_{21} - g_{21} c_{12}) k_2 + \sum_i (g_{11} c_{2i} - g_{21} c_{1i}) f_{i2}\right\} b_2\right] \end{aligned} \tag{32}$$

where

$$|g| = g_{11} g_{22} - g_{12} g_{21} \simeq g_{11} g_{22}. \tag{33}$$

Now, Eq. (32) contains three types of integrals, g_{ij}, c_{ij} and f_{ij}. But their properties become quite simple according to the resonator structure. For example, when the resonators are made of metal strips (strip line, microstrip, coplanar resonators and so on), g_{ij} is equal to c_{ij}. If the resonators are dielectric, on the other hand, f_{ij} becomes zero.

In order to simplify the expression, we will use p_{ij} to replace the coefficient of b_i in (32),

$$\frac{da_1}{dt} = p_{11}b_1 + p_{12}b_2,$$
$$\frac{da_2}{dt} = p_{21}b_1 + p_{22}b_2, \tag{34}$$

Operating the similar procedure to Eq. (25b), we find

$$\frac{db_1}{dt} = \frac{1}{|h|}\{(h_{12}d_{21} - h_{22}d_{11})k_1a_1 + (h_{12}d_{22} - h_{22}d_{12})k_2a_2\},$$
$$\frac{db_2}{dt} = \frac{1}{|h|}\{(h_{21}d_{11} - h_{11}d_{21})k_1a_1 + (h_{21}d_{21} - h_{11}d_{22})k_2a_2\}, \tag{35}$$

where

$$h_{11} = \int \mu \boldsymbol{H}_1^2 dv,\ h_{22} = \int \mu \boldsymbol{H}_2^2 dv,$$
$$h_{12} = h_{21} = \int \mu \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 dv,$$
$$|h| = h_{11}h_{22} - h_{12}h_{21} \cong h_{11}h_{22}.$$
$$d_{11} = \int_V \mu_s \boldsymbol{H}_1^2 dv,\ d_{22} = \int_V \mu_s \boldsymbol{H}_2^2 dv,$$
$$d_{12} = d_{21} = \int_V \mu_s \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 dv. \tag{36}$$

Replacement of the coefficients of b_i in Eq. (35) into q_{ij} gives the similar expression as Eq. (34)

$$\frac{db_1}{dt} = q_{11}a_1 + q_{12}a_2,$$
$$\frac{db_2}{dt} = q_{21}a_1 + q_{22}a_2. \tag{37}$$

Differentiating Eq. (34) with time and substituting Eq. (37), we can eliminate b_i,

$$\frac{d^2a_1}{dt^2} - (p_{11}q_{11} + p_{12}q_{21})a_1 - (p_{11}q_{12} + p_{12}q_{22})a_2 = 0,$$
$$\frac{d^2a_2}{dt^2} - (p_{21}q_{11} + p_{22}q_{21})a_1 - (p_{21}q_{12} + p_{22}q_{22})a_2 = 0. \tag{38}$$

It is common to have the same structure and the same resonant frequency for two coupled resonators when discussing the coupling, and hence, we assume that

$$k_1 = k_2 = k_0,$$
$$p_{11} = p_{22} = p,\ p_{12} = p_{21} = p',$$
$$q_{11} = q_{22} = -q,\ q_{12} = q_{21} = -q'. \tag{39}$$

Then, Eq. (38) reduces to

$$\frac{d^2a_1}{dt^2} + (pq + p'q')a_1 + (pq' + p'q)a_2 = 0,$$
$$\frac{d^2a_2}{dt^2} + (pq' + p'q)a_1 + (pq + p'q')a_2 = 0. \tag{40}$$

3.3 Coupling Characteristics

Equation (40) is expressed as

$$\frac{d^2a_1}{dt^2} + \omega_0^2 a_1 + \omega_0^2 \kappa a_2 = 0,$$
$$\frac{d^2a_2}{dt^2} + \omega_0^2 a_2 + \omega_0^2 \kappa a_1 = 0, \tag{41}$$

if we put

$$pq + p'q' = \omega_0^2,\ \frac{pq' + p'q}{pq + p'q'} = \kappa. \tag{42}$$

Assuming a_i is proportional to $\exp(j\omega t)$ and substituting it into Eq. (41), we have

$$(\omega^2 - \omega_0^2)a_1 - \omega_0^2\kappa a_2 = 0,$$
$$-\omega_0^2\kappa a_1 + (\omega^2 - \omega_0^2)a_2 = 0. \tag{43}$$

The condition for nontrivial solutions should be

$$\begin{vmatrix} \omega^2 - \omega_0^2 & -\omega_0^2\kappa \\ -\omega_0^2\kappa & \omega^2 - \omega_0^2 \end{vmatrix} = 0, \tag{44}$$

that is,

$$\omega^2 = \omega_0^2(1 \pm \kappa). \tag{45}$$

Representing the larger frequency by ω_h and the smaller one by ω_l, the amplitude a_i is given by the linear combination of two terms

$$a_1 = C_1 \exp(j\omega_h t) + C_1' \exp(j\omega_l t),$$
$$a_2 = C_2 \exp(j\omega_h t) + C_2' \exp(j\omega_l t). \tag{46}$$

where

$$\omega_h = \omega_0 \sqrt{1 \pm \kappa},\ \omega_l = \omega_0 \sqrt{1 \mp \kappa}. \tag{47}$$

according as κ is positive or negative, respectively.

Now, we consider the case where $a_1 = 1$, $a_2 = 0$ at $t = 0$, then

$$C_1 + C_1' = 1,$$
$$C_2 + C_2' = 0. \tag{48}$$

Adding the similar relation to Eq. (A· 5) for each eigen state, we obtain

$$C_1 = C_1' = -C_2 = C_2' = \frac{1}{2}. \tag{49}$$

The final expression for each resonator is

$$\begin{aligned} a_1 &= \frac{1}{2}\{\exp(j\omega_h t) + \exp(j\omega_l t)\} \\ &= \exp\left(j\frac{\omega_h + \omega_l}{2}t\right)\cos\left(\frac{\omega_h - \omega_l}{2}t\right), \\ a_2 &= \frac{1}{2}\{-\exp(j\omega_h t) + \exp(j\omega_l t)\} \\ &= -j\exp\left(j\frac{\omega_h + \omega_l}{2}t\right)\sin\left(\frac{\omega_h - \omega_l}{2}t\right). \end{aligned} \tag{50}$$

The real part of the intensity is given

$$A_1 = \mathrm{Re}[a_1] = \cos\left(\frac{\omega_h + \omega_l}{2}t\right)\cos\left(\frac{\omega_h - \omega_l}{2}t\right),$$
$$A_2 = \mathrm{Re}[a_2] = \sin\left(\frac{\omega_h + \omega_l}{2}t\right)\sin\left(\frac{\omega_h - \omega_l}{2}t\right). \tag{51}$$

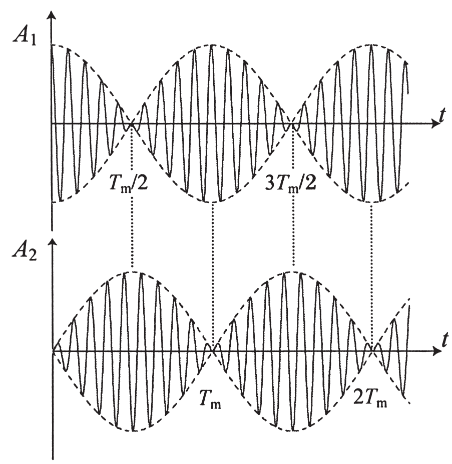

Fig. 2 Exchange of signal between 2 resonators.

Looking at the time variation of A_1 and A_2, we find the fast oscillation with $(\omega_h + \omega_l)/2$ is modulated by the lower frequency $(\omega_h - \omega_l)/2$ and A_1, A_2 are 90 degrees out of phase each other. In other words, the electromagnetic energy is going back and forth between the resonators with the frequency $(\omega_h - \omega_l)/2$.

The period of energy exchange is found by

$$\frac{\omega_h - \omega_l}{2} T_m = \pi.$$

$$\therefore T_m = \frac{2}{\omega_h - \omega_l} = \frac{2\pi}{\omega_0 \kappa}. \tag{52}$$

Normalizing it with the period of self oscillation and taking the reciprocal, we have

$$\frac{T_0}{T_m} = \kappa = \frac{2(\omega_h - \omega_l)}{\omega_h + \omega_l}, \tag{53}$$

which corresponds well with the relations (12) and (13) for the coupling between two guided waves. Equation (51) is described in Fig. 2.

4. Comparison of Guided Wave and Resonator

We have shown that coupled wave guides exchange energy between each wave guides and the same thing happens for coupled resonators, too. Since energy is the most essential physical quantity, it looks most natural to define its exchange rate as the coupling coefficient. But normalization by the period or the wave length would be required, because the higher the frequency of signal, the period of exchange should become shorter. Our proposal for the definition of coupling coefficient is, thus

$$\begin{array}{ll} \textit{coupling coefficient} \\ \textit{of guided waves} & = \dfrac{1 \textit{ wavelength}}{\textit{length of energy exchange}} \\ \textit{coupling coefficient} \\ \textit{of resonators} & = \dfrac{1 \textit{ period}}{\textit{time of energy exchange}} \end{array}$$

They are summarized in Table 1 together with other useful expressions obtained so far. The correspondence of waveguides and resonators is clearly demonstrated for each of three expressions. The difference in the overlap integral expressions is due to that of normalization. In the resonator's case, κ is normalized from the beginning as shown in Eq. (41).

Table 1 Definition of coupling coefficient and its calculation method.

	Definition	Spectrum space	Overlap integral
Coupling coefficient between guides	$\dfrac{\lambda_0}{\lambda_m}$	$\dfrac{2(\beta_h - \beta_l)}{\beta_h + \beta_l}$	$\dfrac{2K}{\beta_0}$
Coupling coefficient between resonators	$\dfrac{T_0}{T_m}$	$\dfrac{2(\omega_h - \omega_l)}{\omega_h + \omega_l}$	κ

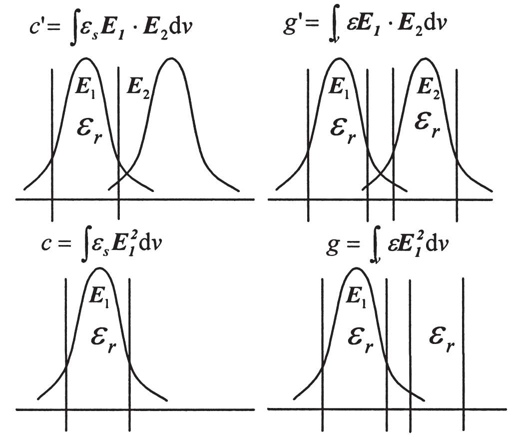

Fig. 3 How to calculate overlap integral.

Focusing on the resonator coupling, the spectrum space expression is actually the same as Eq. (2), thus providing the physical background for the use of Eq. (2) even in the higher frequency band. The coupled mode theory starting from Maxwell equations has made a connecting bridge between the conventional expression Eq. (2) and our expression T_0/T_m.

The coupling coefficient κ in Eq. (42) is simplified if we specify the coupled system. Assume the resonators are encapsulated in a metal shield and their distance is changed to control the coupling. This configuration allows f_{ij} in Eq. (27) to be zero and κ is given by

$$\kappa = \left(\frac{c'}{c} - \frac{g'}{g}\right) + \left(\frac{d'}{d} - \frac{h'}{h}\right) = \kappa_e + \kappa_m. \tag{54}$$

where c', c, g' and g are given in Fig. 3, and κ_e, κ_m denote the electric and magnetic part of coupling coefficient, respectively.

The quantity in the first parenthesis is calculated referring to Eqs. (29), (30), and Fig. 3 to obtain

$$\kappa_e = \frac{\int_{res} \varepsilon_0(\varepsilon_r - 1)\boldsymbol{E}_1 \cdot \boldsymbol{E}_2 dv}{\int_V \varepsilon \boldsymbol{E}_1^2 dv}, \tag{55}$$

where "res" means that the integration is carried out in one resonator only, and ε_r denotes the specific permittivity of the single resonator. If the materials of the resonators are non-magnetic, the second parenthesis in Eq. (54) becomes zero, applying the similar consideration as Fig. 3,

$$\kappa_m = 0. \tag{56}$$

Thus, the total coupling coefficient is given by Eq. (55) in this case. The integral expression in Eq. (55) corresponds quite well to Eq. (3.2-27) of Ref.3 which was derived for the coupling of two optical waveguides.

5. Example

We will show the validity of the present theory giving a simple example. The structure shown in Fig. 4 is easily analyzed and the field distribution is derived [5]. Though the integral calculus is rather tedious, we have obtained the final result of Eq. (55) for the $\mathrm{TE}_{10\delta}$ mode as

$$\kappa_e = \frac{2\alpha^2 e^{-\alpha D}}{k_0^2 \left\{ 1 + \varepsilon_r \alpha \dfrac{\sin(\beta d) + \beta d}{2\beta \cos^2(\beta d/2)} \right\}}, \tag{57}$$

where k_0, α and β are the free space wave number, attenuation constant outside of the dielectrics and the propagation

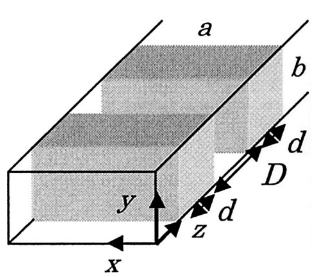

Fig. 4 Dielectric resonators in a rectangular waveguide.

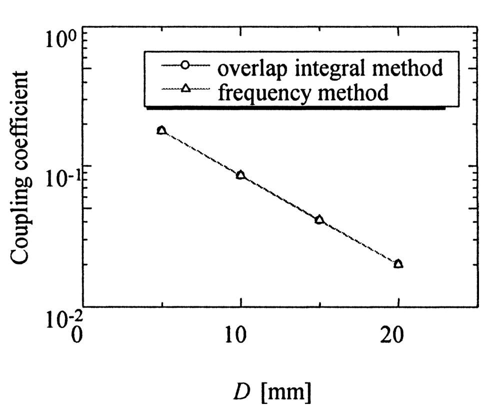

Fig. 5 Coupling coefficient for resonators in Fig. 4. (a = 20.0, b = 10.0, d = 10.0 mm, ε_r = 16.4)

constant in the dielectrics, respectively. The coupling coefficient is depicted in Fig. 5 as a function of the resonator distance together with the result obtained by Eq. (2). The perfect coincidence indicates the validity of the present theory and the conventional theory at the same time.

6. Conclusion

The coupled mode theory has given a new definition of coupling coefficient between resonators which looks more essential since it is based on the energy concept. It gives not only the physical background for the widely accepted but groundless definition Eq. (2), but also new means for the evaluation of the coupling coefficient. In fact, the time domain calculation of coupling would be effective in the FDTD algorithm [6], since it does not need the Fourier transform to use Eq. (2).

Another advantage comes from Eq. (54), which indicates the coupling results from the excitation of the electric and/or magnetic polarization. It gives the coupling coefficient by the integration of the EM fields.

There could be more ways of definition and calculation method of coupling, corresponding to the variety of physical pictures of coupling scheme. We are still trying to find some more in order to clarify the physics of coupling.

Acknowledgement

The author appreciates the kindness of Prof. A. Sanada and Prof. H. Kubo at School of Engineering, Yamaguchi University for valuable discussion. This work was partly supported by a Grant-in-Aid for Developmental Scientific Research (No.15560297) from Japan Society for the Promotion of Science.

References

[1] J.S. Hong and M.J. Lancaster, Microstrip filters for rf/microwave applications, pp.235–272, John Wiley & Sons, New York, 2001.
[2] W.H. Louisell, Coupled mode parametric electronics, pp.1–34, John Wiley & Sons, New York, 1960.
[3] D. Marcuse, Theory of dielectric optical waveguide, pp.95–106, Academic Press, New York and London, 1974.
[4] K. Kurokawa, Introduction to microwave circuit, pp.132–145, Maruzen, Tokyo, 1963.
[5] A. Munir, N. Hamanaga, H. Kubo, and I. Awai, "Artificial dielectric resonator with novel anisotropic permittivity and its TE10 mode waveguide filter application," IEICE Trans. Electron., vol.E88-C, no.1, pp.40–46, Jan. 2005.
[6] I. Awai and Y. Oda, "FDTD calculation of coupling coefficient between two resonators," 1998 IEEE MTT-S Digest, pp.833–836, June 1998.

Appendix

We will derive $a_1(z)$ and $a_2(z)$ assuming that only guide 1 is excited at the end. If we postulate a solution $a_1(z)$, $a_2(z) \sim e^{-j\beta z}$, Eq. (8) gives

$$j(\beta-\beta_0)a_1 + jKa_2 = 0,$$
$$-jKa_1 + j(\beta-\beta_0)a_2 = 0. \quad \text{(A· 1)}$$

In order that a nontrivial solution exists, the determinant of the coefficient matrix is to be zero, that is,

$$\beta = \beta_0 \pm K. \quad \text{(A· 2)}$$

If we put

$$\beta_h = \beta_0 + K, \quad \beta_l = \beta_0 - K, \quad \text{(A· 3)}$$

both $a_1(z)$ and $a_2(z)$ are expressed as follows,

$$a_1 = A_{11}e^{-j\beta_h z} + A_{12}e^{-j\beta_l z},$$
$$a_2 = A_{21}e^{-j\beta_h z} + A_{22}e^{-j\beta_l z}. \quad \text{(A· 4)}$$

where the following relations hold between A_{ij} considering that β_i and A_{ij} satisfy Eq. (A· 1).

$$j(\beta_h-\beta_0)A_{11} + jKA_{21} = 0$$
$$j(\beta_l-\beta_0)A_{12} + jKA_{22} = 0 \quad \text{(A· 5)}$$

On the other hand, the initial condition stated above insists

$$A_{11} + A_{12} = 1,$$
$$A_{21} + A_{22} = 0. \quad \text{(A· 6)}$$

Solving Eqs. (A· 5) and (A· 6),

$$A_{11} = A_{12} = -A_{21} = A_{22} = \frac{1}{2}. \quad \text{(A· 7)}$$

If one substitutes Eq. (A· 7) into (A· 4), one obtains

$$|a_1| = |\cos Kz|,$$
$$|a_2| = |\sin Kz|. \quad \text{(A· 8)}$$

Ikuo Awai received the B.S. degree in 1963, M.S. degree in 1965, and Ph.D. in 1978, all from Kyoto University, Kyoto, Japan. In 1968, he joined Department of Electronics, Kyoto University, Kyoto, Japan, as a research associate, where he was engaged in microwave magnetic waves and integrated optics. From 1984 to 1990, he worked for Uniden Corporation, Tokyo, Japan, developing microwave communication equipments. He joined Yamaguchi University as a professor in 1990, where he has studied magnetostatic wave devices, dielectric wave-guide components, superconducting devices and artificial dielectric resonators for microwave application. In 2004, he started to work for Ryukoku University as a professor, being mainly engaged in microwave filters and meta-materials. He is a member of MTT Society of IEEE.