Series and Parallel Resonance

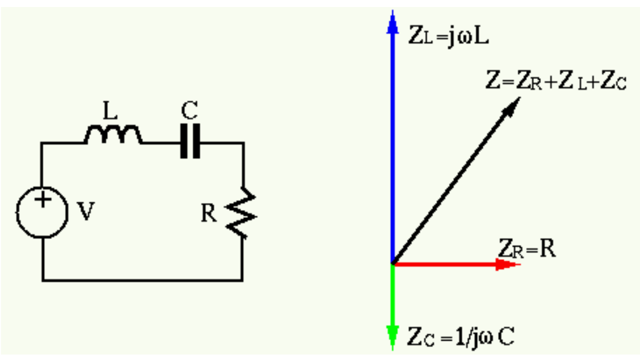
Series Resonance

Consider an RCL series circuit consists of a resistor R, an inductor L, and a capacitor C connected in series to a voltage source. The overall impedance of the three elements is

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right) = |Z| e^{j\angle Z}$$

where

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}, \quad \angle Z = tan^{-1}\frac{\omega L - 1/\omega C}{R} = \begin{cases} -90^\circ & \omega = 0 \\ 90^\circ & \omega \rightarrow \infty \end{cases}$$



In particular the **resonant frequency** is defined as

$$\omega_0 \triangleq \frac{1}{\sqrt{LC}}$$

When $\omega = \omega_0$, the impedances of the capacitor and the inductor have the same magnitude but opposite phase:

$$Z_L = j\omega_0 L = j\sqrt{\frac{L}{C}}, \qquad Z_C = \frac{1}{j\omega_0 C} = -j\sqrt{\frac{L}{C}}$$

and they add up to zero $Z_L + Z_C = 0$. Now the total impedance is minimized:

$$Z = Z_R + Z_C + Z_L = Z_R = R$$

and the current $\dot{I} = \dot{V}/Z = \dot{V}/R$ is maximized. The current $\dot{I}$ and voltage $\dot{V}$ are in phase. At the resonant frequency $\omega = \omega_0 = 1/\sqrt{LC}$, the ratio of the magnitude of the inductor/capacitor impedance and the resistance is defined as the **quality factor**:

$$Q \triangleq \frac{|Z_L|}{R} = \frac{|Z_C|}{R} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

When $\omega = \omega_0$, the voltages across each of the three components are:

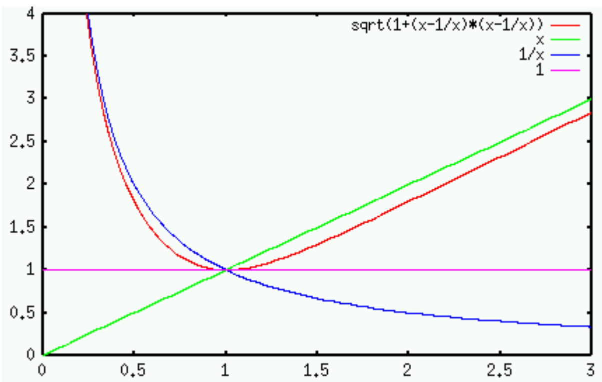
$$\dot{V}_R = \dot{I} Z_R = \dot{I} R = \dot{V}, \quad \text{i.e.} \quad \dot{I} = \frac{\dot{V}}{R}$$

$$\dot{V}_L = \dot{I} Z_L = \frac{\dot{V}}{R} j\omega_0 L = jQ\dot{V}$$

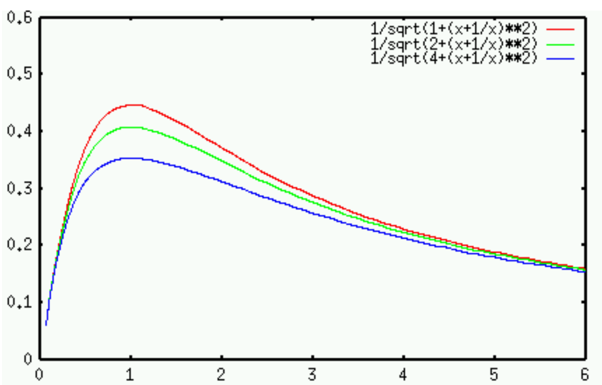
$$\dot{V}_C = \dot{I} Z_C = \frac{\dot{V}}{R} \frac{1}{j\omega_0 C} = -jQ\dot{V}$$

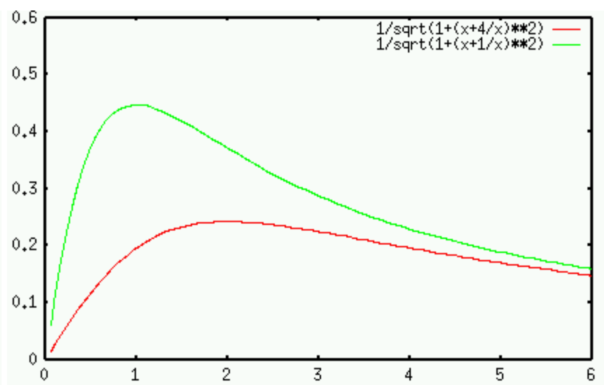
The magnitude of V_L and V_C are Q times larger than that of $\dot{V}_R$, which is equal to the source voltage $\dot{V}$. But as V_L and V_C are in opposite polarity (180° out of phase), they cancel each other.

The RCL series circuit is a band-pass filter with the passing band centered around the resonant frequency $\omega_0 = 1/\sqrt{CL}$. The bandwidth is determined by the quality factor Q. The larger Q, the narrower the bandwidth. The impedance $Z = Z_R + R_L + Z_C$ as a function of ω is shown below:



and the admittances $Y = 1/Z$ for different Q (R) and C are shown below:





The bandpass effect can be intuitively explained. When ω is high, the inductor's impedance ωL is high, and when ω is low, the capacitor's impedance $1/\omega C$ is high. When $\omega = \omega_0$ the overall impedance is the smallest. If the input is a voltage source $v(t)$, the current through the circuit will reach a maximum value when $\omega = \omega_0$.

Example: In a series RLC circuit, $R = 5\Omega$, $L = 4\ mH$ and $C = 0.1\ \mu F$. The resonant frequency ω_0 can be found to be $\omega_0 = 1/\sqrt{LC} = 1/\sqrt{4 \times 10^{-3} \times 10^{-7}} = 5 \times 10^4$. The quality factor is

$$Q = \frac{\omega_0 L}{R} = \frac{(5 \times 10^4) \times (4 \times 10^{-3})}{5} = 40$$

or

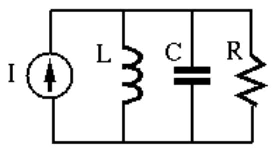
$$Q = \frac{1}{\omega_0 CR} = \frac{1}{(5\times 10^4)\times 10^{-7}\times 5} = 40$$

If the input voltage is $V_{rms} = 10V$ at the resonant frequency, the current is $I = V/R = 10/5 = 2A$, and the voltages across each of the elements are:

- $\dot{V}_R = R\dot{I} = V = 10V$
- $\dot{V}_L = j\omega L\dot{I} = j5\times 10^4 \times 4\times 10^{-3}\times 2 = j400V$
- $\dot{V}_C = \dot{I}/j\omega_0 C = -j2/(5\times 10^4 \times 0.1\times 10^{-6}) = -j400V$

Or, more conveniently, the amplitudes of $\dot{V}_L$ and $\dot{V}_C$ can be found by $|V_L| = |V_C| = QV = 40\times 10V = 400V$. Note that although input voltage is $10V$, the voltage across L and C (Q times the input) could be very high (but they are in opposite phase and therefore cancel each other).

Parallel Resonance: A GCL parallel circuit consists of a resistor $R = 1/G$, an inductor L and a capacitor connected in parallel to input voltage.



In this case, it is much easier to consider the conductance of the admittance $Y = 1/Z$ of each of the element. The overall admittance of the three elements in parallel is

$$Y = G + j\omega C + \frac{1}{j\omega L} = G + j(\omega C - \frac{1}{\omega L}) = |Y|e^{j\angle Y}$$

where

$$|Y| = \sqrt{G^2 + (\omega C - \frac{1}{\omega L})^2}, \quad \angle Y = tan^{-1}\frac{\omega C - 1/\omega L}{G}$$

In particular when ω is at the **resonant frequency**

$$\omega_0 \triangleq \frac{1}{\sqrt{LC}}$$

we have

$$Y_C + Y_L = j\omega C + \frac{1}{j\omega L} = j(\omega C - \frac{1}{\omega L}) = j(\sqrt{\frac{C}{L}} - \sqrt{\frac{C}{L}}) = 0$$

the effects of L and C cancel each other, and the complex admittance Y becomes real and its magnitude reaches the minimum

$$|Y| = \sqrt{G^2 + (\omega_0 C - \frac{1}{\omega_0 L})^2} = G, \quad \angle Y = \tan^{-1}\frac{0}{G} = 0$$

and the current $\dot{I}$ reaches a minimum value $\dot{I} = \dot{V}G$. In particular, if the resistor does not exist, i.e., $R = \infty$ and $G = 0$, then the admittance $Y = 0$ and $Z = \infty$.

The **Quality Factor** Q_p of a parallel resonance circuit is defined as the ratio of the magnitude of the inductor/capacitor susceptance and the conductance:

$$Q_p \triangleq \frac{\omega_0 C}{G} = \frac{1/\omega_0 L}{G} = \frac{1}{\omega_0 L G} = \frac{1}{G}\sqrt{\frac{C}{L}} = R\sqrt{\frac{C}{L}} = \frac{1}{Q_s}$$

Note that Q_p for a parallel RCL circuit is the reciprocal of Q_s for a series RCL circuit. The currents through each of the three components are:

- $$\dot{I}_R = \dot{V} G = \dot{I}$$
- $$\dot{I}_C = \dot{V} Y_C = \dot{V} j\omega_0 C|_{\omega_0=1/\sqrt{LC}} = j\frac{\dot{I}}{G}\sqrt{\frac{C}{L}} = jQ\dot{I}$$
- $$\dot{I}_L = \dot{V} Y_L = \dot{V}\frac{1}{j\omega_0 L}|_{\omega_0=1/\sqrt{LC}} = -j\frac{\dot{I}}{G}\sqrt{\frac{C}{L}} = -jQ\dot{I}$$

The magnitude of the current I_L through L and I_C through C are Q times larger than the current $\dot{V}_R$ through R which is the same as the current source $\dot{I}$). But as I_L and I_C are in opposite direction, they form a loop current through L and C with no effect to the rest of the circuit.

The parallel RCL circuit behaves like a bandstop filter which can be intuitively understood. When ω is high, the capacitor's impedance $1/\omega C$ is low, and when ω is low, the inductor's impedance ωL is low. When $\omega = \omega_0$ the overall impedance is the largest. However, if the input is a current source, the voltage across the elements $\dot{V} = \dot{I} Z = \dot{I}/Y$ will reach a maximum value when $\omega = \omega_0$.

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Next: Quality Factor, Peak Frequency **Up:** Chapter 3: AC Circuit **Previous:** AC Circuit Analysis II

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