ne previo

Next: Series and Parallel Resonance **Up:** Chapter 3: AC Circuit **Previous:** AC Circuit Analysis II

Second Order RCL Circuit

Applying KVL to an RCL series circuit we get

$$v_L(t) + v_R(t) + v_C(t) = v(t)$$

where

$$v_L(t) = L\frac{d}{dt}i(t), \quad v_R(t) = Ri(t), \quad v_C(t) = \frac{Q(t)}{C}$$

Substituting

$$i(t) = \frac{dQ(t)}{dt} = C\frac{d}{dt}v_C(t)$$

into the equation above we get

$$LC\frac{d^2}{dt^2}v_C(t) + RC\frac{d}{dt}v_C(t) + v_C(t) = v(t)$$

or

$$\frac{d^2}{dt^2}v_C(t)+\frac{R}{L}\frac{d}{dt}v_C(t)+\frac{1}{LC}v_C(t) = \ddot{v}_C(t)+2\zeta\omega_n\dot{v}_C(t)+\omega_n^2 v_C(t) = \omega_n^2 v(t)$$

where

$$\omega_n = \frac{1}{LC}, \qquad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

are respectively the damping coefficient and natural frequency of the second order system. To find the homogeneous solution we substitute $v_C(t) = e^{st}$ into the homogeneous DE and get

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)e^{st} = 0$$

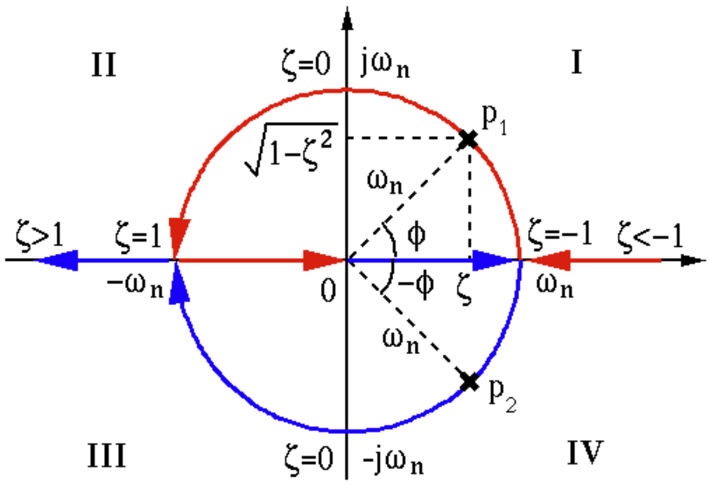
Solving the algebraic equation $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ we get the two roots

$$s_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n = (-\zeta \pm j\sqrt{1-\zeta^2})\omega_n = -\omega_n e^{\mp j\phi}$$

where

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

The two roots can be represented as the root locus on the complex plane, for a fixed value of ω_n.



The homogeneous solution takes the following form:

$$v_C(t) = Ae^{s_1 t} + Be^{s_2 t}$$

As in the RCL circuit $\zeta \geq 0$ and $\omega_n \geq 0$, we only need to consider the root locus in the II and III quadrants in the figure above. Specifically we consider the following two cases:

- $0 \leq \zeta < 1$. The system is underdampled.

$$s_{1,2} = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\omega_n = -\zeta\omega_n \pm j\omega_d$$

  where $\omega_d = \sqrt{1-\zeta^2}\omega_n$ is the damped nature frequency. Crrespondingly

$$e^{st} = e^{-\zeta\omega_n t} e^{\pm j\omega_d t} = e^{-\zeta\omega_n t}(\cos\omega_d t \pm \sin\omega_d t)$$

  is an exponentially decaying sinusoidal ocsillation. In particular, when $\zeta = 0$, $e^{st} = \cos\omega_n t \pm \sin\omega_n t$, the system is undamped.

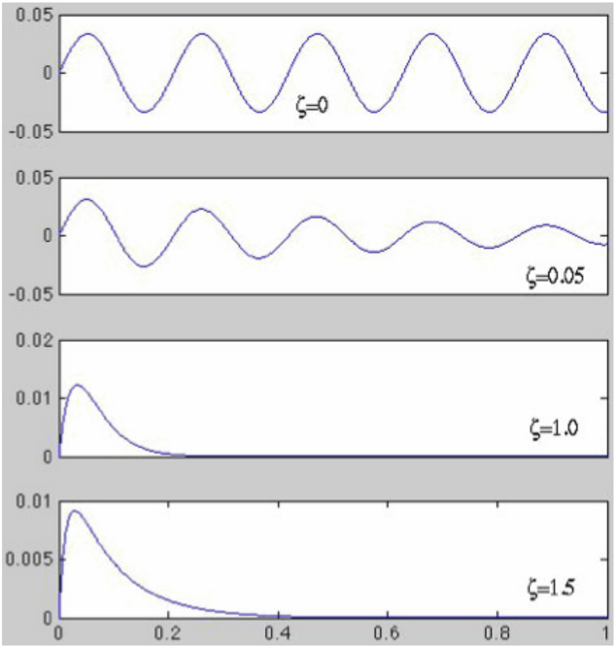
- $\zeta \geq 1$. The system is over-damped.

$$s_{1,2} = -(\zeta \pm \sqrt{\zeta^2 - 1})\omega_n < 0$$

  and the crresponding solution

$$e^{st} = e^{-(\zeta \pm \sqrt{\zeta^2-1})\omega_n t}$$

  decays exponentially. In particular, when $\zeta = 1$, the system is critically damped.

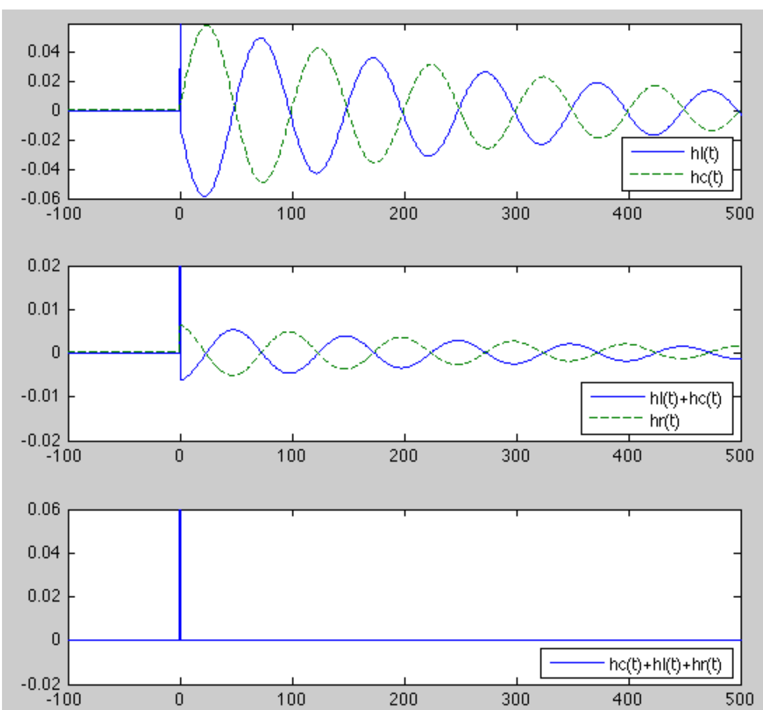


It can be shown that the impulse response of the system is

$$h_C(t)=\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t)u(t)$$

In particular, when $R=0$ and $\zeta=0$,

$$h_C(t)=\omega_n\sin(\omega_n t)u(t)$$



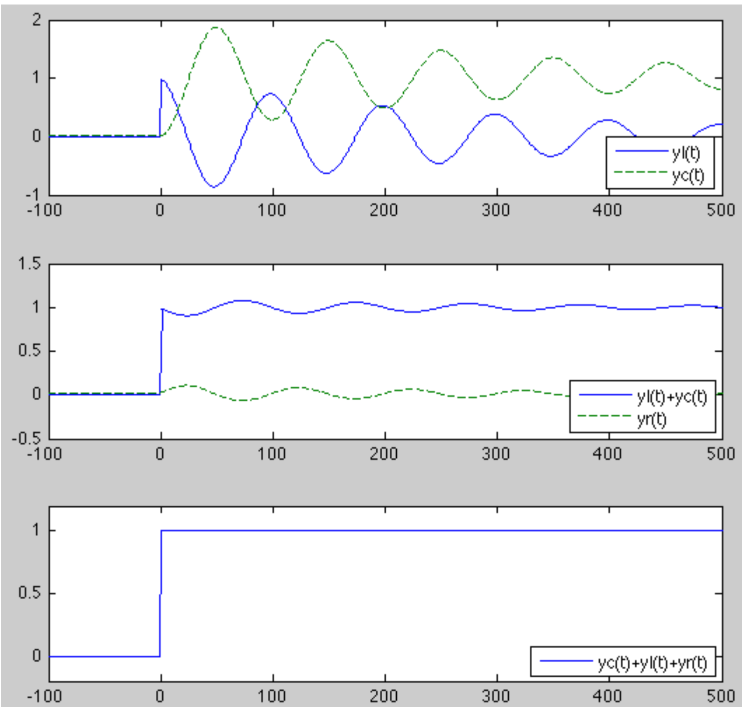
Impulse responses by R, L, and C of an RCL system. Top: impulse responses $h_L(t)$ (solid curve) and $h_C(t)$ (dashed curve); middle: their sum $h_L(t)+h_C(t)$ (solid curve) and impulse response $h_R(t)$ (dashed curve); bottom: the sum of all three: $\delta(t)=h_L(t)+h_R(t)+h_C(t)$.

The step response is

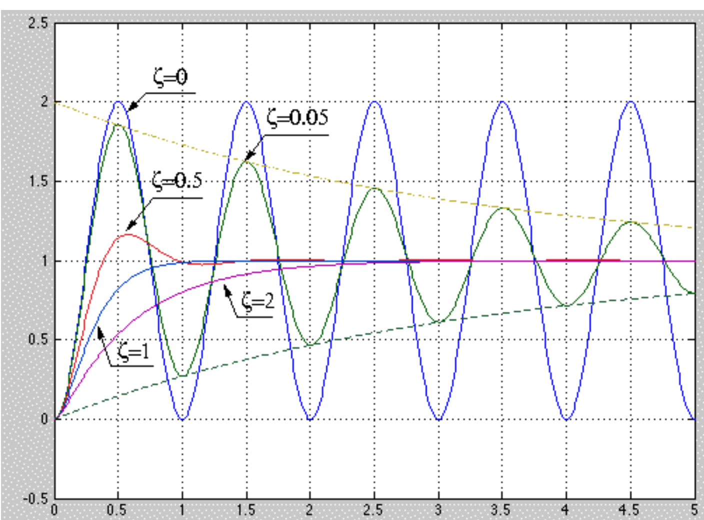
$$v_C(t)=\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi)\right]u(t)$$

In particular, when $R=0$, $\zeta=0$ and $\phi=\pi/2$, then the general form of the step response becomes

$$y(t)=\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi)\right]u(t)=[1-\sin(\omega_n t+\pi/2)]u(t)=[1-\cos(\omega_n t)]u(t)$$



Step responses by R, L as well as C of an RCL system. Top: step responses $v_L(t)$ (solid curve) and $v_C(t)$ (dashed curve); middle: their sum $v_L(t)+v_C(t)$ (solid curve) and step responses $y_R(t)$ (dashed curve); bottom: the sum of all three: $v_C(t)+v_L(t)+v_R(t)=u(t)$.



Step response of second-order system for different ζ. Step responses corresponding to five different values of ζ: 0, 0.05, 0.5, 1, and 2. The envelope of the step response for $\zeta=0.05$ is also plotted to show the exponential decay of the sinusoid.

Example It is often desirable for a second order system to reach a steady state value, e.g., $v=2$, without overshoot. This can be achieved by applying first a step input $u(t)$ and then another one $u(t-T/2)$, where $T=2\pi/\omega_n$. The response of the first step input is $[1-\cos(\omega_n t)]u(t)$ and that to the second is

$$[1-\cos(\omega_n(t-T/2))]u(t-T/2)=[1-\cos(\omega_n t-\pi)]u(t-T/2)=[1+\cos(\omega_n t)]u(t-T/2)$$

As the sum of the two individual responses the total response for $t>T/2$ is 2, i.e.,

$$v_C(t)=\left\{\begin{array}{ll}[1-\cos\omega_n t]u(t) & 0<t<T/2=\pi/\omega_n\\ 2 & t>T/2=\pi/\omega_n\end{array}\right.$$

Example Consider the response $y(t)=v_C(t)$ of the RCL system when $R=0$.

- Find the step response $y(t)$ of the RLC circuit (with zero initial conditions $v_C(0)=\dot{v}_C(0)=0$).

  The general solution is the sum of the homogeneous solution $A\sin\omega_n t+B\cos\omega_n t$ and the particular solution $V=1$:

$$y(t)=1+A\sin\omega_n t+B\cos\omega_n t$$

  Based on the zero initial conditions

$$y(0)=1+B=0,\qquad \dot{y}(0)=A\omega_n=0$$

  we get $A=0$ and $B=-1$ and

$$y(t)=1-\cos\omega_n t$$

- Find the system's response to a square impulse

$$x(t)=\left\{\begin{array}{ll}1 & 0\le t<t_0\\ 0 & \text{else}\end{array}\right.$$

  As $x(t)=u(t)-u(t-t_0)$, the response is

$$y(t)=[[1-\cos(\omega_n t)]u(t)-[1-\cos(\omega_n(t-t_0))]u(t-t_0)]$$

  i.e.,

$$y(t)=[-\cos(\omega_n t)+\cos(\omega_n(t-t_0))],\qquad t>t_0$$

- Find the impulse response $h(t)$. The impulse input can be written as

$$\delta(t)=\lim_{t_0\rightarrow 0}\left\{\begin{array}{ll}1/t_0 & 0\le t<t_0\\ 0 & \text{else}\end{array}\right.$$

  When $t_0\rightarrow 0$, we have first order approximations $\cos(\omega_n t_0)\approx 1$, $\sin(\omega_n t_0)\approx\omega_n t_0$, and get

$$\cos(\omega_n(t-t_0))=\cos(\omega_n t)\ \cos(\omega_n t_0)+\sin(\omega_n t)\ \sin(\omega_n t_0)\approx\cos(\omega_n t)+\sin(\omega_n t)\,\omega_n t_0$$

  Substituting this into the equation above we get the impulse response

$$y(t) = \omega_n \sin(\omega_n t)$$

- When $t_0 = T/2 = \pi/\omega_n$, we have

$$\cos(\omega_n(t-t_0)) = \cos(\omega_n t - \pi) = -\cos(\omega_n t)$$

and the response for $t > T/2$ is

$$y(t) = -2\cos(\omega_n t)$$

---

ne previo

Next: Series and Parallel Resonance **Up:** Chapter 3: AC Circuit **Previous:** AC Circuit Analysis II

Ruye Wang 2011-12-28