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Quality Factor, Peak Frequency and Bandwidth

The physical meaning of the quality factor Q of an RCL series circuit is the ratio between the energy stored in the circuit (in C and L) and the energy dissipated (by R):

$$Q = 2\pi \frac{\mbox{maximum energy stored}}{\mbox{energy dissipated per cycle}}$$

The maximum energy stored in L is:

$$W_L = \int_0^T v(t)\; i(t) dt = \int_0^T i(t) L \frac{di(t)}{dt} dt = L \int_0^{I_p} i\; di = \frac{1}{2} L I_p^2 = L I_{rms}^2$$

where $I_p = \sqrt{2} I_{rms}$ is the peak current through L. The maximum energy stored in C is:

$$W_C = \int_0^T v(t)\; i(t) dt = \int_0^T v(t) C \frac{dv(t)}{dt} dt = C \int_0^{V_p} v\; dv = \frac{1}{2} C V_p^2 = C V_{rms}^2$$

where $V_p = \sqrt{2} V_{rms}$ is the peak voltage across C. We show that $W_L = W_C$ at resonant frequency $\omega = \omega_0 = 1/\sqrt{LC}$:

$$W_L = L I_{rms}^2 = L \frac{V_{rms}^2}{(\omega_0 L)^2} = L \frac{V_{rms}^2 LC}{L^2} = C V_{rms}^2 = W_C$$

where V_{rms} is the voltage across L which is the same as that across C when $\omega = \omega_0$. Here the energy $W_L = W_C$ is converted back and forth between magnetic energy in L and electrical energy in C.

The energy dissipated in R per cycle $T_0 = 2\pi/\omega_0 = 1/f_0$ is:

$$W_R = P_R T_0 = I_{rms}^2 R T_0$$

Following the definition of Q above, we have

$$Q = 2\pi \frac{W_L}{W_R} = 2\pi \frac{L I_{rms}^2}{I_{rms}^2 R T_0} = 2\pi f_0 \frac{L}{R} = \frac{\omega_0 L}{R}$$

which is indeed the same as the Q defined before.

Relationship between Q and ζ

If the voltage across R is treated as the output of the circuit, then the frequency response function (FRF) of this second order system can be expressed as (voltage divider):

$$H_R(\omega) = \frac{\mathbf{V}_R}{\mathbf{V}} = \frac{R}{R + j\omega L + 1/j\omega C} = \frac{j\omega R/L}{(j\omega)^2 + (j\omega) R/L + 1/LC} = \frac{j\omega 2\zeta\omega_0}{(j\omega)^2 + j\omega 2\zeta\omega_0 + \omega_0^2}$$

Here we multiplied both the numerator and the denominator by $j\omega/L$ so that the denominator matches the canonical form:

$$(j\omega)^2+\frac{j\omega R}{L}+\frac{1}{LC}=(j\omega)^2+j\omega 2\zeta\omega_0+\omega_0^2$$

where $\omega_0=1/\sqrt{LC}$ as shown above, and $R/L=2\zeta\omega_0$. Solving the second equation, we get

$$\zeta=\frac{R}{2L}\frac{1}{\omega_0}=\frac{R}{2L}\sqrt{LC}=\frac{R}{2}\sqrt{\frac{C}{L}}$$

Recalling that the quality factor is defined as

$$Q=\frac{1}{R}\sqrt{\frac{L}{C}}$$

therefore we have

$$\zeta=\frac{1}{2Q}$$

The quality factor Q can also be used to judge whether a second order system is under, critically or over damped:

$\zeta<1$	$Q>0.5$	under damped
$\zeta=1$	$Q=0.5$	critically damped
$\zeta>1$	$Q<0.5$	over damped

Peak Frequency and Bandwidth

The frequency response function above can be further expressed as:

$$H_R(\omega)=\frac{R}{R+j(\omega L-1/\omega C)}=\left[1+j\left(\frac{\omega L}{R}-\frac{1}{\omega CR}\right)\right]^{-1}$$

Since $Q=\omega_0 L/R=1/\omega_0 CR$, we have

$$\frac{L}{R}=\frac{Q}{\omega_0},\qquad \text{and}\qquad \frac{1}{RC}=Q\omega_0$$

Substituting these into the equation above we get

$$H_R(\omega)=\left[1+jQ\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]^{-1}$$

At the resonant frequency $\omega=\omega_0$, $H=1$ reaches the maximum. When ω is either lower or higher than ω_0, H is smaller. The bandwidth is defined as

$$\triangle\omega\triangleq\omega_2-\omega_1$$

where $\omega_2>\omega_1$ are the two cut-off frequencies (or half-power frequency) at which $|H(\omega_{1,2})|=1/\sqrt{2}=0.707$ or $|H(\omega_{1,2})|^2=1/2$ (i.e., the power is halved):

$$H_R(\omega_{1,2})=\frac{1}{1\pm j},\qquad \text{i.e.}\quad Q\left(\frac{\omega_{1,2}}{\omega_0}-\frac{\omega_0}{\omega_{1,2}}\right)=\pm 1$$

Therefore the two cut-off frequencies should satisfy, respectively

$$Q\left(\frac{\omega_1}{\omega_0}-\frac{\omega_0}{\omega_1}\right)=-1,\quad \text{i.e.}\quad \omega_1^2+\frac{\omega_0}{Q}\omega_1-\omega_0^2=0$$

and

$$Q\left(\frac{\omega_2}{\omega_0}-\frac{\omega_0}{\omega_2}\right)=1,\quad \text{i.e.}\quad \omega_2^2-\frac{\omega_0}{Q}\omega_2-\omega_0^2=0$$

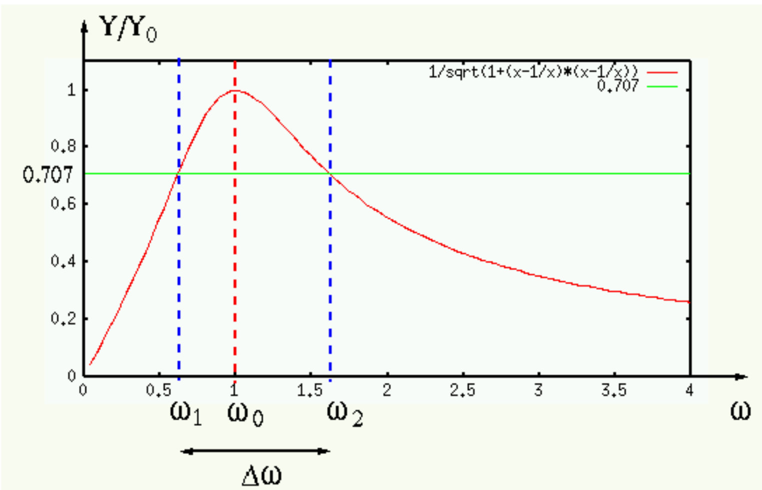
The positive solutions of these two equations are, respectively:

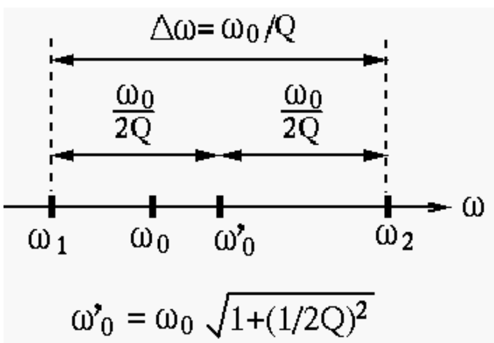
$$\omega_{1,2}=\omega_0\left[\sqrt{1+\left(\frac{1}{2Q}\right)^2}\mp\frac{1}{2Q}\right]=\omega_0(\sqrt{1+\zeta^2}\mp\zeta)$$

Note that in general a quadratic equation has two solutions. But here one of them is negative and therefore ignored (negative frequency has no physical meaning). The bandwidth can be found to be:

$$\triangle\omega=\omega_2-\omega_1=\frac{\omega_0}{Q}=2\zeta\omega_0,\quad \text{or}\quad \triangle f=f_2-f_1=\frac{f_0}{Q}=2\zeta f_0$$

i.e., the bandwidth is proportional to ζ or inversely proportional to Q. Also note that the middle point between ω_1 and ω_2 is $\omega_0'=(\omega_1+\omega_2)/2=\omega_0\sqrt{1+\zeta^2}>\omega_0$, i.e., $\omega_2-\omega_0>\omega_0-\omega_1$.





If $Q=1/2\zeta$ is much greater than 1 (typically $Q>10$, i.e., $\zeta<0.05$), we have $\sqrt{1+(1/2Q)^2}=\sqrt{1+\zeta^2}\approx 1$ and

$$\omega_1\approx\omega_0-\frac{\omega_0}{2Q}=\omega_0(1-\zeta),\qquad \omega_2\approx\omega_0+\frac{\omega_0}{2Q}=\omega_0(1+\zeta)$$

we therefore get these simple relations:

$$\omega_2-\omega_0=\omega_0-\omega_1=\frac{\omega_0}{2Q}=\omega_0\zeta$$

If we consider the voltage across each of the three components in the RCL series circuit as the output, then we have the following frequency response functions:

$$H_L(\omega)=\frac{\mathbf{V}_L}{\mathbf{V}}=\frac{j\omega L}{R+j\omega L+1/j\omega C}=\frac{(j\omega)^2}{(j\omega)^2+j\omega 2\zeta\omega_0+\omega_0^2}=\frac{(j\omega)^2}{(j\omega)^2+\triangle\omega j\omega+\omega_0^2}$$

$$H_R(\omega)=\frac{\mathbf{V}_R}{\mathbf{V}}=\frac{R}{R+j\omega L+1/j\omega C}=\frac{j\omega 2\zeta\omega_0}{(j\omega)^2+j\omega 2\zeta\omega_0+\omega_0^2}=\frac{j\omega 2\zeta\omega_0}{(j\omega)^2+\triangle\omega j\omega+\omega_0^2}$$

$$H_C(\omega)=\frac{\mathbf{V}_C}{\mathbf{V}}=\frac{1/j\omega C}{R+j\omega L+1/j\omega C}=\frac{\omega_0^2}{(j\omega)^2+j\omega 2\zeta\omega_0+\omega_0^2}=\frac{\omega_0^2}{(j\omega)^2+\triangle\omega j\omega+\omega_0^2}$$

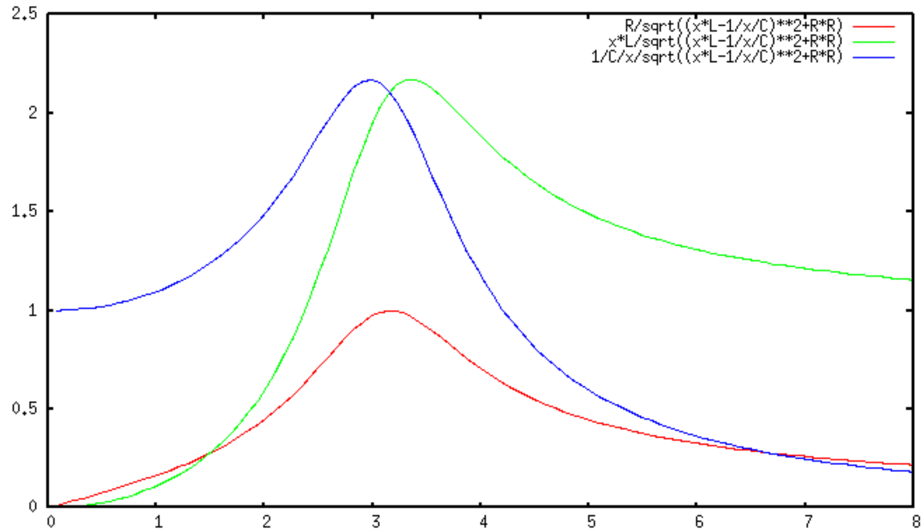
For a parallel RCL circuit with current input, due to the duality between current and voltage, parallel and series configuration, the same derivation of bandwidth can be carried out to obtain the same conclusions.

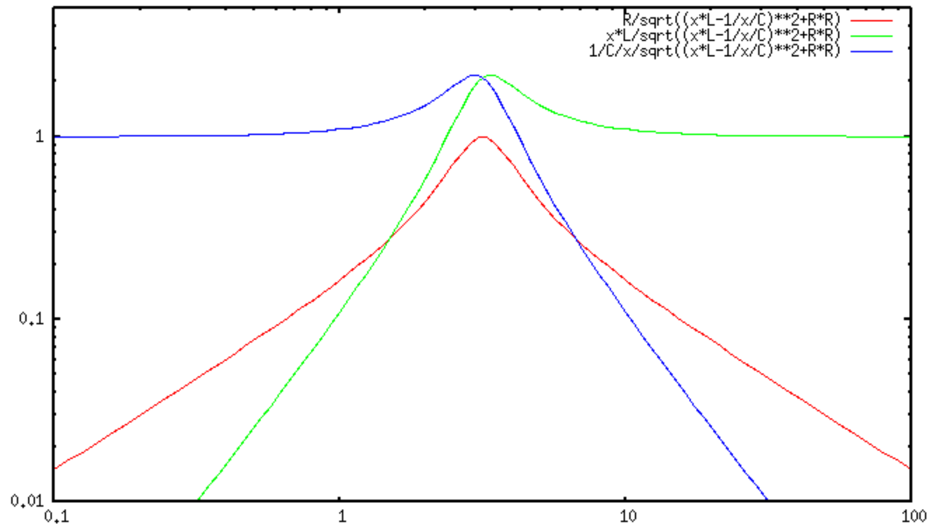
Summary:

- The resonant frequency of both series and parallel RCL circuits is completely determined by L and C: $\omega_0=1/\sqrt{LC}$, independent of the resistance R in the circuit.
- At the resonant frequency $\omega=\omega_0$, the impedance $Z=1/Y$ of a series RCL circuit is real and reaches minimum, and the current through the three components reaches maximum; the admittance $Y=1/Z$ of a parallel RCL circuit is real and reaches minimum and the voltage across the three components reaches maximum.
- In series RCL with voltage input and parallel RCL with current input, the quality factor Q is proportional to the ratio between L and C:

$$Q_s=\frac{1}{R}\sqrt{\frac{L}{C}},\quad Q_p=R\sqrt{\frac{C}{L}}=\frac{1}{Q_s}$$

- In series RCL, Q_s is inversely proportional R (the larger R, the smaller Q_s, the more energy lost and the wider bandwidth), while in parallel RCL, Q_p is proportional to R (the larger R, the larger Q_p, the less energy lost and the narrower bandwidth).
- At resonant frequency, the impedance of a series RCL circuit reaches minimum, consequently the current $\mathbf{I}$ reaches maximum and so does the voltage across the resistor $\mathbf{V_R}=\mathbf{I}R$. However, the voltage across the inductor $\mathbf{V_L}=\mathbf{I}Z_L$ reaches maximum at a frequency slightly higher than the resonant frequency as $Z_L=j\omega L$ is proportional to ω, and the voltage across the capacitor $\mathbf{V_C}=\mathbf{I}Z_C$ reaches maximum at a frequency slightly lower than the resonant frequency as $Z_C=1/j\omega C$ is inversely proportional to ω, as shown in the linear and log-scale plots below.





See this website for more detailed discussions of second-order systems.

Example 1:

A series RCL circuit composed of an inductor $L=80\mu H$ and $R=8\Omega$ and a capacitor C is connected to a voltage source. Find the value of C for this circuit to resonate at $f=400\;kHz$, also find the bandwidth.

$$\omega_0=\sqrt{\frac{1}{LC}},\quad C=\frac{1}{\omega_0^2 L}=\frac{1}{(2\pi 400\times 10^3)^2\times 80\times 10^{-6}}=20nF$$

The quality factor is

$$Q=\frac{\omega_0 L}{R}=\frac{2\pi 400\times 10^3\times 80\times 10^{-6}}{8}=25.13$$

The bandwidth is

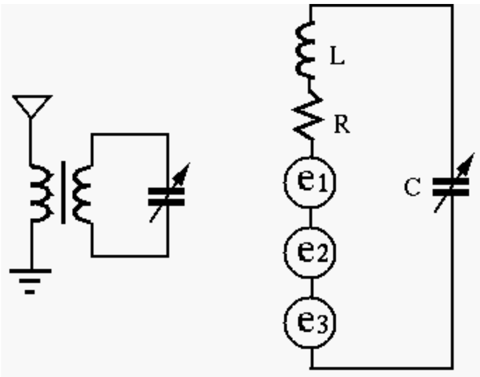
$$\triangle f=\frac{f_0}{Q}=\frac{400\times 10^3}{25.13}=15.9\;kHz$$

or

$$\triangle\omega = \frac{\omega_0}{Q} = \frac{R}{L} = 10^5$$

Example 2:

Resonant circuit is widely used in radio and TV receivers to select a desired station from many stations available. The circuit are shown in the figure below. Assume $L = 0.3mH$, $R = 16\Omega$, and C is a variable capacitor, which can be adjusted to match the resonant frequency of the circuit to the frequency of the desired station. Assume the frequency of the desired station is $\underline{640\ kHz}$, find the value of C. If the induced voltage in the circuit is $e = 2\mu V$ (rms), find the current (rms) in the resonant circuit, and the output voltage (rms) across the capacitor.



Solution: At the desired resonant frequency $f = 640\ kHz$, the reactance of the inductor is

$$X_L = \omega L = 2\pi f L = 2 \times 3.14 \times 640 \times 10^3 \times 0.3 \times 10^{-3} = 1206\Omega$$

and the quality factor Q of this circuit is

$$Q = \frac{\omega L}{R} = \frac{X_L}{R} = \frac{1206}{16} = 75$$

The bandwidth is:

$$\triangle f = \frac{f_0}{Q} = \frac{640}{75} \approx 8.53\ kHz$$

The resonant frequency can be expressed as:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 640\ kHz$$

Solving this we get

$$C = \frac{1}{(2\pi f_0)^2 L} = 206 \times 10^{-12} F = 206\ pF$$

The current in circuit is

$$I_{rms} = e/R = 2 \times 10^{-6}/16 = 0.125\ \mu A$$

The output voltage across C is

$$V_C = V_L = I_{rms} X_L = 0.125 \times 10^{-6} \times 1206 = 151\ \mu V$$

Radio/TV Broadcasting and Frequency Allocation

In either radio or TV broadcasting, the audio or video signal is used to modulate the amplitude, frequency or phase of the *carrier frequency*, which is transmitted through the air. In amplitude modulation (AM) radio broadcasting, if the highest frequency component contained in the audio signal is $s(t)=\cos(\omega_s t)$, and the carrier is $c(t)=\cos(\omega_c t)$, where the carrier frequency is much higher than the signal frequency, $\omega_c >> \omega_s$, then the signal transmitted is the carrier signal $c(t)$ with its amplitude modulated by the signal $s(t)$:

$$c(t)\,s(t)=\cos(\omega_c t)\cos(\omega_s t)=\frac{1}{2}[\cos(\omega_c+\omega_s)+\cos(\omega_c-\omega_s)]$$

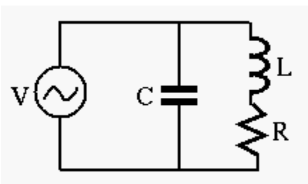
i.e., a certain bandwidth of $\Delta\omega=2\omega_s$ around the carrier (or central) frequency ω_c is needed to transmitting all signal frequencies up to ω_s. Consequently, the Q value of the tuner of the receiver needs to be very carefully chosen. It needs to be high enough for good selectivity between different radio stations, but it cannot be too high in order to have a bandwidth wide enough to contain all frequency components in the signal.

The AM radio frequency range is from 535 to 1605 kHz with 10 kHz frequency spacing or bandwidth, i.e., the highest signal frequency allowed is about 5 kHz, while the upper limit of the audible frequencies is 20 kHz. In the case of FM radio, the frequency range is from 87.8 to 108 MHz with 0.2 MHz=200 kHz frequency spacing, corresponding to a much wider bandwidth that makes high fidelity and stereo broadcasting possible. The TV broadcasting is also in MHz frequency range with a much wider spacing of 6 MHz, needed to carry video as well as audio signals.

The Federal Communications Commission (FCC) has very specific frequency allocation regulations, see the FCC frequency allocations and the frequency allocation chart.

Example 3:

In reality, all inductors have a non-zero resistance, therefore a parallel resonance circuit should be modeled as shown in the figure.



The admittance is:

$$Y(\omega)=\frac{1}{R+j\omega L}+j\omega C=\frac{R-j\omega L+j\omega C(R^2+\omega^2L^2)}{R^2+\omega^2L^2}=\frac{1}{R^2+\omega^2L^2}[R-j(\omega L-\omega C(R^2+\omega^2L^2))]$$

As frequency ω appears in the real part $Re[Y(\omega)]$ as well as in the imaginary part $Im[Y(\omega)]$, the resonant frequency that minimizes $|Y(\omega)|$ has to be found by

$$\frac{d}{d\omega}\,|Y(\omega)|=0$$

However, when the quality factor $Q=\omega_0 L/R$ associated with the non-ideal inductor is large enough ($Q>20$), all previous discussed relations for ideal inductors still hold approximately, and the resonant frequency ω_0 can still be found approximately by the previous approach by letting $Im[Y(\omega)]=0$:

$$\omega_0 L=\omega_0 C(R^2+\omega_0^2L^2),\qquad\Longrightarrow\qquad \omega_0=\sqrt{\frac{1}{LC}-(\frac{R}{L})^2}$$

For ω_0 to be real, we must have

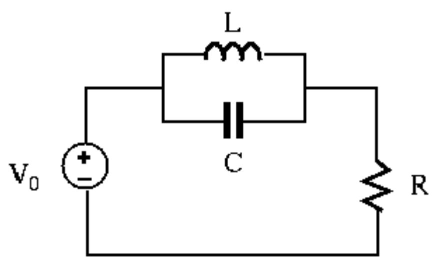
$$\frac{1}{LC}>(\frac{R}{L})^2,\qquad \text{i,e,}\qquad R<\sqrt{\frac{L}{C}}$$

Typically we have $R\ll\sqrt{L/C}$, and the resonant frequency is

$$\omega_0=\sqrt{\frac{1}{LC}-(\frac{R}{L})^2}\approx\frac{1}{\sqrt{LC}}$$

Note: For the same reason, when considering the transfer function of a series RCL circuit when the output is the voltage across either C or L, the peak frequency ω_p is not exactly the same as the resonant frequency ω_0, which only minimizes the denominator, but the numerator is still a function of ω. Only when the output is the voltage across R (i.e., the numerator is R, no longer a function of ω), will the resonant frequency ω_0 be the same as the peak frequency.

Example: Consider the output voltage $V_{out}=V_R$ across the resistor R of the circuit shown below.



$$V_R=V_0\frac{R}{R+Z_C||Z_L}$$

where

$$Z_C||Z_L=\frac{Z_LZ_C}{Z_L+Z_C}=\frac{j\omega L/j\omega C}{j\omega L+1/j\omega C}=\frac{j\omega L}{1-\omega^2LC}=\frac{j\omega L}{1-(\omega/\omega_0)^2}$$

where $\omega_0=1/\sqrt{LC}$.

- When $\omega\rightarrow 0$, $Z_C||Z_L\approx Z_L\rightarrow 0$, $V_{out}\approx V_0$
- When $\omega\longrightarrow\infty$ $Z_C||Z_L\approx Z_C\rightarrow 0$, $V_{out}\approx V_0$
- When $\omega\rightarrow\omega_0$, $Z_C||Z_L\rightarrow\pm j\infty$, $V_{out}\approx 0$.

This is a band-stop or band-block filter which attenuates frequencies around $\omega_0=1/\sqrt{LC}$.

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