New expression of Coupling Coefficient between Resonators Based on Overlap Integral of EM Field

Ikuo Awai Yangjun Zhang
School of Science and Technology, Ryukoku University
2-1 Yokotani, Seta-oecho, Otsu 520-2194, JAPAN

***Abstract*-The coupled mode theory is applied to analyze the coupling between resonators. According to the theory, the coupling coefficient is, in principle, described by adding the overlap integral of the electric field of the two resonators and the magnetic counterpart regardless of the configurations. But, depending on the configurations, such as two dielectric resonators, one dielectric resonator with degenerate modes, two metallic resonators or one metallic resonator with degenerate modes, simpler expressions are obtained that is useful for numerical calculations and physical interpretations. The present paper derives the general coupling equation based on the coupled mode theory and shows some numerical results for two dielectric resonators using a derived simple overlap integral expression.**

I. INTRODUCTION

The coupling coefficient between two resonators has long been calculated using two resonant frequencies split into different values due to the coupling which was originally the same each other, that is,

$$k = \frac{\omega_h^2 - \omega_l^2}{\omega_h^2 + \omega_l^2}, \tag{1}$$

where k is the coupling coefficient, ω_h the higher resonant frequency and ω_l the lower one [1]. Though this procedure is quite simple and versatile (applicable to any structures), it has a drawback that it cannot elucidate the nature of the coupling.

We have recently been trying to find a new expression for the coupling coefficient based on the coupled mode theory in order to investigate the physical meaning of the coupling [2]. It was, in addition, found out that the new method of calculation could even be faster than the conventional method for some configuration [2].

In this paper, the eigen functions for the coupling fields are taken complex (not real) as is common for ordinary manipulation of Maxwell equations, facilitating the succeeding integral calculation. We could derive overlap integral expressions for several typical coupling structures, e.g. two dielectric resonators, one dielectric resonator with two degenerate modes, two planar metal resonators and one metal resonator with two degenerate modes. We will present one of them, coupling of two dielectric resonators, and show its numerical result for evaluation of the validity of the theory.

II. DERIVATION OF COUPLING COEFFICIENT

The coupled mode theory expands the electromagnetic field in a coupled system of two resonators using the uncoupled mode of each resonator, that is,

$$\boldsymbol{E} = \sum_i^2 a_i \boldsymbol{E}_i, \tag{2.a}$$

$$\boldsymbol{H} = \sum_i^2 b_i \boldsymbol{H}_i, \tag{2.b}$$

where $\boldsymbol{E}$, $\boldsymbol{H}$ are the EM fields of the coupled system and $\boldsymbol{E}_i$, $\boldsymbol{H}_i$ are those of each uncoupled resonator. Though we follow the procedure in Ref.2, we have to be careful for the differences below:

1) The electromagnetic fields are complex.
2) The fields are not normalized.

Because of Condition 1), the field equations for the uncoupled eigen mode are given by

$$\nabla \times \boldsymbol{H}_i = j\omega_i \varepsilon_i \boldsymbol{E}_i, \tag{3.a}$$

$$\nabla \times \boldsymbol{E}_i = -j\omega_i \mu_i \boldsymbol{H}_i. \tag{3.b}$$

Condition 2) introduces the amplitude adjustment factor

$$\varsigma_i = \frac{1}{\int_v \varepsilon_i \left|\boldsymbol{E}_i\right|^2 dv}. \tag{4}$$

Substituting (2) into Maxwell equations and manipulating them, we finally obtain the following differential equations for the field amplitudes in (2),

$$\begin{aligned}
\frac{da_1}{dt} &= \frac{j}{|g|}\left[\left\{\omega_1\left(c_{11}g_{22} - c_{21}g_{12}\right) + j\sum_j^2\left(c_{1j}g_{22} - c_{2j}g_{12}\right)\varsigma_1 f_{j1}\right\}b_1\right.\\
&\left.+\left\{\omega_2\left(c_{12}g_{22} - c_{22}g_{12}\right) + j\sum_j^2\left(c_{1j}g_{22} - c_{2j}g_{i2}\right)\varsigma_2 f_{j2}\right\}b_2\right],\\
\frac{da_2}{dt} &= \frac{j}{|g|}\left[\left\{\omega_1\left(c_{21}g_{11} - c_{11}g_{21}\right) + j\sum_j^2\left(c_{2j}g_{11} - c_{1j}g_{21}\right)\varsigma_1 f_{j1}\right\}b_1\right.\\
&\left.+\left\{\left(g_{11}c_{21} - g_{21}c_{12}\right)k_2 + \sum_i\left(g_{11}c_{2i} - g_{21}c_{1i}\right)f_{i2}\right\}b_2\right],
\end{aligned} \tag{5}$$

$$\frac{db_1}{dt} = \frac{j}{|h|}\left[\omega_1\left(d_{11}h_{22} - d_{21}h_{12}\right)a_1 + \omega_2\left(d_{12}h_{22} - d_{22}h_{12}\right)a_2\right],$$
$$\frac{db_2}{dt} = \frac{j}{|h|}\left[\omega_1\left(d_{21}h_{11} - d_{11}h_{21}\right)a_1 + \omega_2\left(d_{22}h_{11} - d_{12}h_{21}\right)a_2\right], \tag{6}$$

where

$$\begin{aligned} &|g| = g_{11}g_{22} - g_{12}g_{21}, |h| = h_{11}h_{22} - h_{12}h_{21}, \\ &g_{ij} = \int_V \varepsilon \boldsymbol{E}_i^* \cdot \boldsymbol{E}_j dv, c_{ij} = \int_V \varepsilon_j \boldsymbol{E}_i \cdot \boldsymbol{E}_j^* dv, \\ &h_{ij} = \int_V \mu \boldsymbol{H}_i \cdot \boldsymbol{H}_j^* dv, d_{ij} = \int_V \mu_j \boldsymbol{H}_i^* \cdot \boldsymbol{H}_j dv, \\ &f_{ij} = \int_S \boldsymbol{n} \cdot \left(\boldsymbol{E}_i^* \times \boldsymbol{H}_j\right) dS. \end{aligned} \tag{7}$$

If we put coefficients of b_1, b_2 in the right hand side of (5) to be $p_{11}, p_{12}, p_{21}, p_{22}$, we have

$$\begin{aligned} \frac{da_1}{dt} &= jp_{11}b_1 + jp_{12}b_2, \\ \frac{da_2}{dt} &= jp_{21}b_1 + jp_{22}b_2. \end{aligned} \tag{8}$$

In the similar manner, one obtains for (6)

$$\begin{aligned} \frac{db_1}{dt} &= jq_{11}a_1 + jq_{12}a_2, \\ \frac{db_2}{dt} &= jq_{21}a_1 + jq_{22}a_2. \end{aligned} \tag{9}$$

Differentiating (8) with t, and substituting (9) into their right hand side, one arrives at

$$\begin{aligned} &\frac{d^2a_1}{dt^2} + \left(p_{11}q_{11} + p_{12}q_{21}\right)a_1 + \left(p_{11}q_{21} + p_{12}q_{22}\right)a_2 = 0, \\ &\frac{d^2a_2}{dt^2} + \left(p_{21}q_{11} + p_{22}q_{21}\right)a_1 + \left(p_{21}q_{12} + p_{22}q_{22}\right)a_2 = 0. \end{aligned} \tag{10}$$

These are called coupled mode equations, which are solved in a well-known method. We will, for simplicity, assume the two resonators are the same and obtain the relations

$$\begin{aligned} &\omega_1 = \omega_2 = \omega_0 \\ &p_{11} = p_{22} = p, p_{12} = p_{21} = p' \\ &q_{11} = q_{22} = q, q_{12} = q_{21} = q' \end{aligned} \tag{11}$$

Then, (10) is simplified into

$$\begin{aligned} &\frac{d^2a_1}{dt^2} + \omega_r^2 a_1 + \omega_r^2 k a_2 = 0, \\ &\frac{d^2a_2}{dt^2} + \omega_r^2 a_2 + \omega_r^2 k a_1 = 0, \end{aligned} \tag{12}$$

where

$$\omega_r^2 = pq + p'q', k = \frac{pq' + p'q}{pq + p'q'}, \tag{13}$$

ω_r is the resonant frequency and k is the coupling coefficient. It is well known that a_1 and a_2 changes with time as shown in Fig.1.

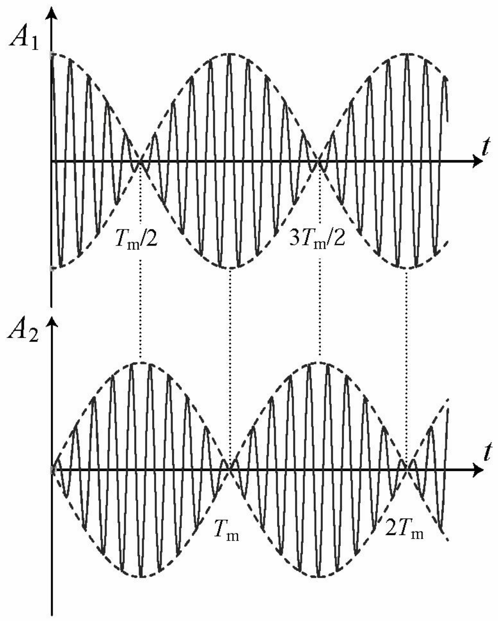


Fig.1 Energy exchange between two coupled resonators
(A_1 and A_2 express the amplitude of a_1 and a_2, respectively)

III. Dielectric Resonators

We will take a dielectric resonator as an example. Then, the relations $\mu = \mu_1 = \mu_2 = \mu_0$ in (7) gives

$$h_{ij} = d_{ij}. \tag{14}$$

Now, (6) reduces to

$$\begin{aligned} \frac{db_1}{dt} &= j\omega_1 a_1, \\ \frac{db_2}{dt} &= j\omega_2 a_2. \end{aligned} \tag{15}$$

Since dielectric resonators shown in Fig.2 are placed in a metallic cavity and it usually is not changed before and after coupling of resonators

$$f_{ij} = 0 \tag{16}$$

holds in (5). Thus, the relations

$$\begin{aligned} \frac{da_1}{dt} &= \frac{j}{g_{11}g_{22}}\left[\omega_1 c_{11} g_{22} b_1 + \omega_2 \left(c_{12}g_{22} - c_{22}g_{12}\right) b_2\right], \\ \frac{da_2}{dt} &= \frac{j}{g_{11}g_{22}}\left[\omega_1 \left(c_{21}g_{11} - c_{11}g_{21}\right) b_1 + \omega_1 c_{22} g_{11} b_2\right] \end{aligned} \tag{17}$$

are obtained, where the fractions less than the 2^{nd} order are discarded.

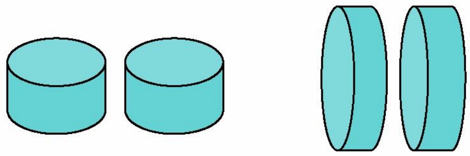


Fig.2 Simplest coupling structures for two dielectric resonators

If the two resonators are identical as shown in Fig.2, we can put

$$\begin{aligned} &\omega_1 = \omega_2 = \omega_0, \\ &c_{11} = c_{22} = c, c_{12} = c_{21} = c', \\ &g_{11} = g_{22} = g, g_{12} = g_{21} = g'. \end{aligned} \tag{18}$$

Then, we finally obtain the relations

$$\omega_r^2 = \omega_0^2 \frac{c}{g}, k = \frac{c'}{c} - \frac{g'}{g}, \tag{19}$$

where

$$\begin{aligned} &g' = \int_V \varepsilon \boldsymbol{E}_1^* \cdot \boldsymbol{E}_2 dv, c' = \int_V \varepsilon_1 \boldsymbol{E}_1^* \cdot \boldsymbol{E}_2 dv, \\ &g = \int_V \varepsilon \left|\boldsymbol{E}_1\right|^2 dv, c = \int_V \varepsilon_1 \left|\boldsymbol{E}_1\right|^2 dv. \end{aligned} \tag{20}$$

The concept of integration in (20) is explained in Fig.3 for one dimensional system of coupled dielectric resonators. If the resonators are not placed too close, the denominators c and g in (19) are taken the same and we can rewrite the expression for the coupling coefficient,

$$k = \frac{\int_{res} \varepsilon_0 \left(\varepsilon_r - 1\right) \boldsymbol{E}_1^* \cdot \boldsymbol{E}_2 \mathrm{d}v}{\int_V \varepsilon \left|\boldsymbol{E}_1\right|^2 \mathrm{d}v}, \tag{21}$$

where "*res*" indicates that the integration is carried out inside one resonator only. The numerator of (21) can be put

$$\varepsilon_0 (\varepsilon_r - 1) \boldsymbol{E}_1 \cdot \boldsymbol{E}_2^* = \boldsymbol{P}_1 \cdot \boldsymbol{E}_2^*, \tag{22}$$

which means the polarization in one resonator exited by the other resonator determines the coupling coefficient.

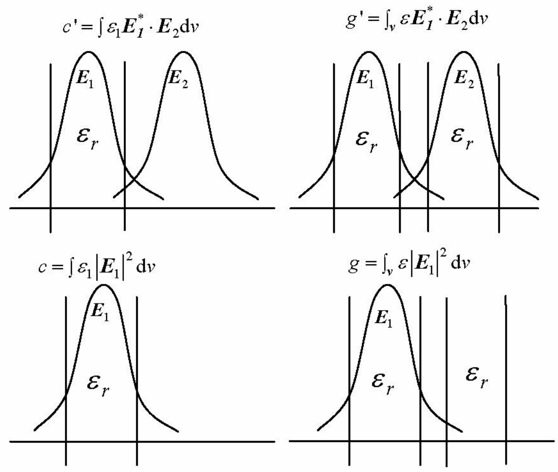


Fig.3 Calculation method of overlap integrals

IV. Numerical Examples

Two numerical examples for the dielectric resonator coupling shown in Fig.4 will be shown. Both results in Fig.5 are the coupling coefficient for the lowest TE mode versus the resonator spacing. The results compare well with those by the standard "frequency method" mentioned in INTRODUCTION. Calculations for the higher modes are under way.

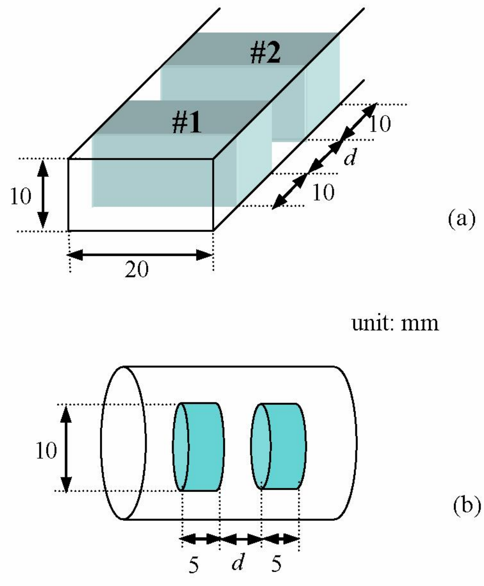
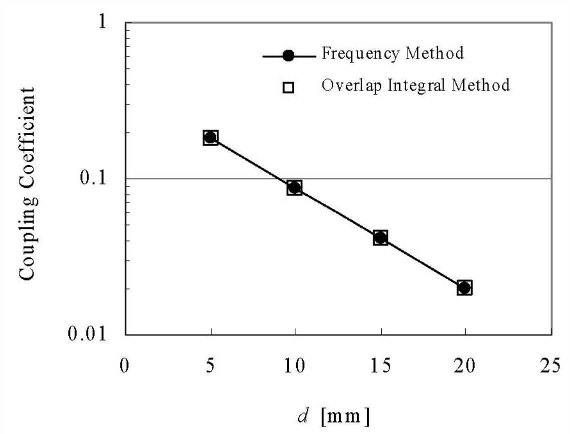
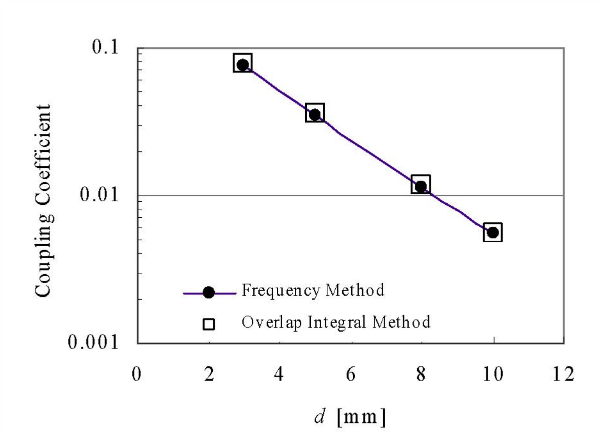


Fig.4 Two basic structures for coupling between dielectric resonators
(a) Resonators inserted into rectangular waveguide, ε_r=16
(b) Disk resonators in a circular cylindrical case, ε_r=30



(a)



(b)

Fig.5 Numerical results for coupling coefficient
(a) $TE_{10\delta}$ mode of resonators inserted into rectangular waveguide
(b) $TE_{01\delta}$ mode of disk resonators in a circular cylindrical case

V. CONCLUSION

A new expression for coupling coefficient of two resonators has been found out based on the coupled mode theory. A convenient integral expression is derived for dielectric resonators which also gives the meaning of the coupling of dielectric resonators. Some numerical examples have validated the new method. Further study is being carried out, including higher modes of dielectric resonators, planar metallic resonators, and degenerate resonators.

ACKNOWLEDGEMENT

The authors are thankful to Shintaro Iwamura for helping numerical calculation, and Prof. Hiroshi Kubo and Prof. Atsushi Sanada for fruitful discussions, all at Yamaguchi University. This work was partly supported by a Grant-in-Aid for Developmental Scientific Research (No.17560324) from Japan Society for the Promotion of Science.

REFERENCES

[1]. J. S. Hong, M. J. Lancaster, *Microstrip filters for rf/microwave applications*, pp.235-272, John Wiley & Sons, Inc., New York (2001).

[2]. I. Awai, "New Expressions for Coupling Coefficient between Resonators", *IEICE Trans. Electron.*, E88C, No.12, Dec. 2005 (Conditionally accepted).