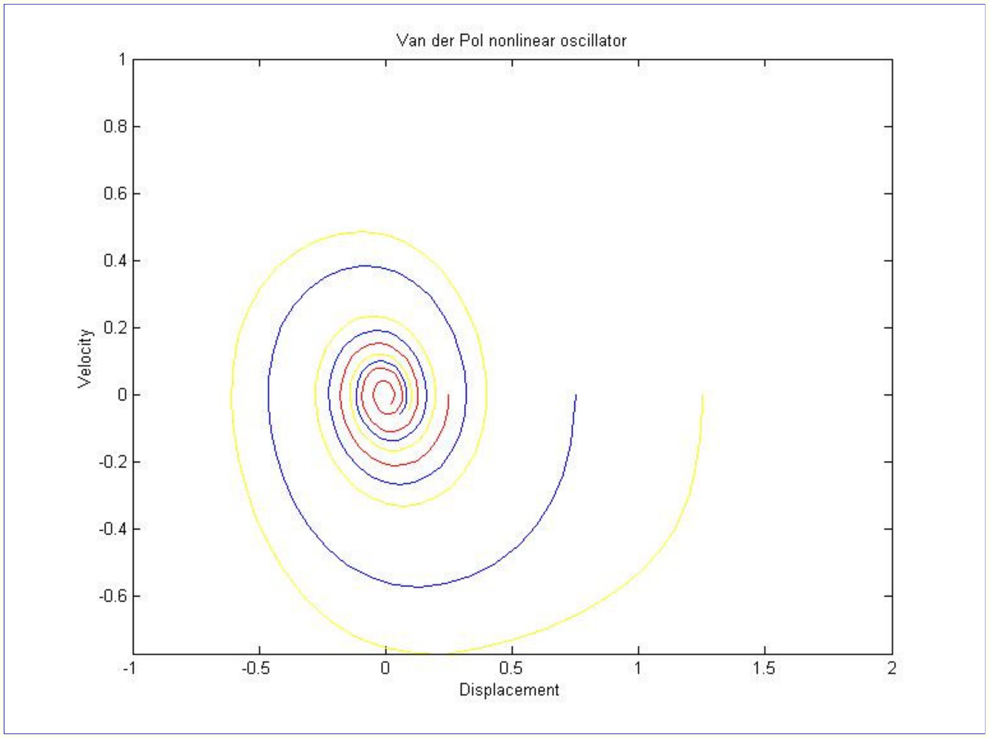
Van der Pol Nonlinear Oscillator

Historically, Balthazar van der Pol derived the equation for nonlinear oscillator when he studied self-sustained oscillations in electrical circuits containing vacuum tubes in 1920s. It was this work that prompted Lienard to study the nonlinear mechanics and limit cycles (asymptotically stable closed trajectories on a phase plane). The Van der Pol equation takes the form:

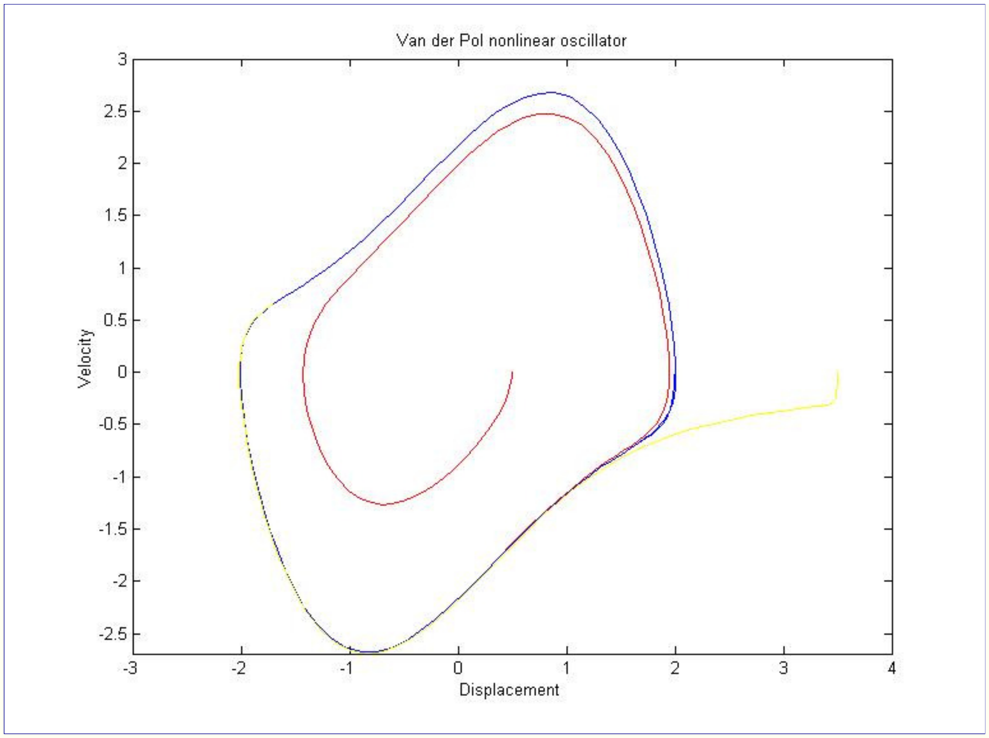
$$y'' + (\mu + y^2) y' + \omega^2 y = 0,$$

where ω is the frequency of a linear oscillator and μ is the damping constant.

When $\mu > 0$, the only equilibrium point at $y = 0$ is asymptotically stable. It is either a stable spiral point for an underdamped oscillator or a stable node for an overdamped oscillator. The phase portrait is simple and shows global stability of the zero equilibrium:



When $\mu < 0$, the Van der Pol nonlinear oscillator represents a physical system, where for small oscillations, energy is fed into the system, whereas for large oscillations, energy is taken from the system. For such systems, large oscillations are damped, while small oscillations are magnified due to the negative damping. Indeed, the equilibrium point $y = 0$ is an unstable spiral point for $-2 |\omega| < \mu < 0$ and an unstable node for $\mu < -2 |\omega|$. It is clear that the damping is negative for $y^2 < |\mu|$. On the other hand, the damping is positive for $y^2 > |\mu|$, large oscillations are damped. As a balance between negative damping for small y and positive damping for large y, we can expect that the Van der Pol nonlinear oscillator approaches asymptotically to a periodic behavior. The periodic behavior is described by a special closed trajectory on a phase plane, called a limit cycle. The limit cycle is shown on the phase plane of the Van der Pol nonlinear oscillator for $\mu = -1$ and $\omega^2 = 1$:



The limit cycle is shown by the blue curve. It is a unique closed orbit that corresponds to the periodic oscillations of the nonlinear system with special initial data lying on the orbit. All other trajectories starting from the inside and from the outside of the limit cycle spiral around the origin and approach the limit cycle. As a result, the limit cycle is globally asympotically stable in nonlinear dynamics of the Van der Pol oscillator. The non-periodic oscillations become asymptotically periodic over large periods of time.

The size of the limit cycle depends on the damping parameter μ. The limit cycle emerges from the center equilibruim point $y = 0$ at $\mu = 0$. The value $\mu = 0$ remarks the famous bifurcation, called the Hopf bifurcation. When the parameter μ transits from positive to negative values, the zero equilibrium becomes unstable, while the non-zero periodic trajectory (the limit cycle) becomes asymptotically stable. The size of the limit cycle grows with the larger values of μ.

You are suggested to check the conclusions above by running numerical simulations of the Van der Pol oscillator software demo. It is recommended to attend the following questions in your simulations:

1. Set $\omega^2 = 1$ and consider the case of positive damping $\mu = 1$. Specify three trajectories with zero initial velocity $y' = 0$ and different initial displacement: $y = 0.5,1,1.5$. What type of dynamics do you see? Are trajectories finite or infinite? Is the zero critical point globally stable?
2. For the same parameters, specify three trajectories with zero initial displacement $y = 0$ and different initial velocities $y' = 1,2,3$. Do you have any changes in the phase portrait?
3. Consider now the case of negative damping $\mu = -1$ and repeat simulations in (1) and (2). Continue your simulations for sufficiently long time in order to observe convergence to the limit cycle. Is the limit cycle globally stable? Are the solutions for the three given trajectories periodic or asymptotically periodic? Control the flow of the three trajectories to sketch the whole phase portrait of the Van der Pol nonlinear oscillator.
4. Fix the flow of trajectories as in (1) and change the negative damping constant: $\mu = -0.1,-0.5,-1,-1.5$ for $\omega^2 = 1$. Do you observe any changes in the size of the limit cycle? Any changes in the type of asymptotic dynamics of the nonlinear oscillator?
5. Fix the flow of trajectories as in (1) and change the frequency constant: $\omega^2 = 1,2,3$ for $\mu = -1$. Do you observe any changes in the period of oscillations? Any changes in the size of the limit cycle? Any changes in the type of asymptotic dynamics of the nonlinear oscillator?

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