Viscously Damped Free Vibration

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Viscous damping force is expressed by the equation

$F_d = c\dot{x}$ (14)

where c is a constant of proportionality.

Symbolically. it is designated by a dashpot, as shown in Fig. 3. From the free body diagram, the equation of motion is .seen to be

$m\ddot{x} + c\dot{x} + kx = F(t)$ (15)

The solution of this equation has two parts. If $F(t) = 0$, we have the homogeneous differential equation whose solution corresponds physically to that of ***free-damped vibration***. With $F(t)$ ¹ 0, we obtain the particular solution that is due to the excitation irrespective of the homogeneous solution. We will first examine the homogeneous equation that will give us some understanding of the role of damping.

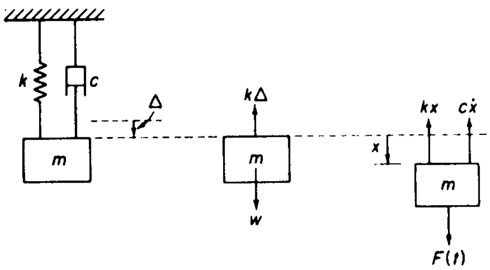

Figure 3 Viscously Damped Free Vibration

With the homogeneous equation :

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (16)$$

the traditional approach is to assume a solution of the form :

$$x = e^{st} \quad (17)$$

where s is a constant. Upon substitution into the differential equation, we obtain :

$$\left(ms^2 + cs + k\right)e^{st} = 0$$

which is satisfied for all values of t when

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0 \quad (18)$$

Equation (18), which is known as the ***characteristic equation***, has two roots :

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad (19)$$

Hence, the general solution is given by the equation:

$$x = Ae^{s_1 t} + Be^{s_2 t} \quad (20)$$

where A and B are constants to be evaluated from the initial conditions $x(0)$ and $\dot{x}(0)$.

Equation (19) substituted into (20) gives :

$$x = e^{-(c/2m)t}\left(Ae^{\left(\sqrt{(c/2m)^2 - k/m}\right)t} + Be^{-\left(\sqrt{(c/2m)^2 - k/m}\right)t}\right) \quad (21)$$

The first term, $e^{-(c/2m)t}$, is simply an exponentially decaying function of time. The behavior of the terms in the parentheses, however, depends on whether the numerical value within the radical is positive, zero, or negative.

When the damping term *(c/2m)2* is larger than *k/m*, the exponents in the previous equation are real numbers and no oscillations are possible. We refer to this case as ***overdamped***.

When the damping term *(c/2m)2* is less than *k/m*, the exponent becomes an imaginary number, $\pm i\sqrt{k/m - (c/2m)^2}\,t$. Because

$$e^{\pm i\left(\sqrt{k/m - (c/2m)^2}\right)t} = \cos\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t \pm i\sin\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\,t$$

the terms of Eq. (21) within the parentheses are oscillatory. We refer to this case as ***underdamped***.

In the limiting case between the oscillatory and non oscillatory motion $(c/2m)^2 = k/m$, and the radical is zero. The damping corresponding to this case is called ***critical damping, cc***.

$$c_c = 2m\sqrt{\frac{k}{m}} = 2m\,\omega_n = 2\sqrt{km} \quad (22)$$

Any damping can then be expressed in terms of the critical damping by a non dimensional number z , called the ***damping ratio***:

$$\zeta = \frac{c}{c_c} \quad (23)$$

and

$$\frac{c}{2m} = \zeta\left(\frac{c_c}{2m}\right) = \zeta\omega_n \quad (24)$$

(i) Oscillatory Motion (z < 1.0) Underdamped Case :

$$x = e^{-\zeta\omega_n t}\left(Ae^{i\left(\sqrt{1-\zeta^2}\right)\omega_n t} + Be^{-i\left(\sqrt{1-\zeta^2}\right)\omega_n t}\right) \quad (25)$$

The frequency of damped oscillation is equal to :

$$\omega_d = \frac{2\pi}{\tau_d} = \omega_n\sqrt{1-\zeta^2} \quad (26)$$

Figure 4 shows the general nature of the oscillatory motion.

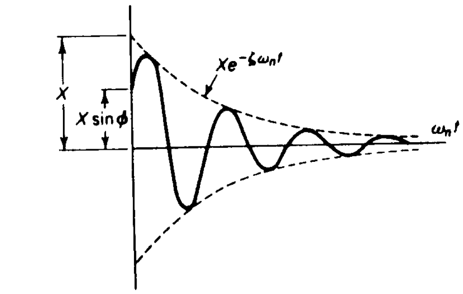

Figure 4 Damped Oscillation z < 1

(ii) Non oscillatory Motion (z > 1.0) Overdamped Case :

$$x = Ae^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + Be^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \quad (27)$$

The motion is an exponentially decreasing function of time as shown in Fig. 5.

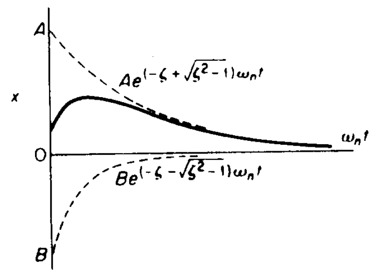

Figure 5 Aperiodic Motion z > 1

(iii) Critically Damped Motion (z = 1.0) :

$$x = (A+Bt)e^{-\omega_n t} \quad (28)$$

Figure 6 shows three types of response with initial displacement x(0).

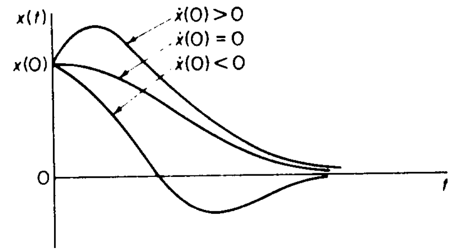

Figure 6 Critically Damped Motion z = 1