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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

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[Continued from December Number.]

CHAPTER V.

RELATIVE MULTIPLICATION.

56. Definition.—In the preceding chapter no reference was made to the space in which the factors multiplied were contained. Now, in ordinary multiplication of geometrical magnitudes, there is a limit beyond which one can not go. For instance, when one has multiplied the length, breadth, and thickness together, he can add no other dimension. This suggests the idea of taking any arbitrary number as n for the number of dimensions of the space considered. In any investigation, then, what we will call "the space considered" is that space of the original units which contains all the quantities involved. Multiplications made with reference to the space considered are called *Relative Multiplications.*

57. Definition.—If in a space of the nth order, the combinatory product of the original units $e_1 e_2 \; \ldots\ldots e_n$, is set equal to the scalar unity, and E is a unit of any order, (*i. e.* either one of the original units or a combinatory product of two or more of them), then the *complement of* E is $+E'$ or $-E'$, where E' is the combinatory product of all the units which do not appear in E. The complement of E is $+E'$ when $[EE']=+1$; and $-E'$ when $[EE']=-1$. Let the com-

plement of E be denoted by $|E$. This mark, the sign of the complement, in Grassmann is a vertical line somewhat longer that the caps and about as heavy as the vertical stroke in cap N. Then

$$|E=[EE']E'.$$

The end sought is to get $[E|E]=+1$. To show that this is attained, multiply the equation above through by E. Then

$$[E|E]=[EE'][EE']=+1,$$

since whether $[EE']=+1$ or -1, $[EE'][EE']=+1$.

The reason why we have this ambiguity of sign in the product $[EE']$ is because each original unit is allowed to have either the plus or minus sign.

In particular we have

$$|1=1, \text{ or } |\alpha=\alpha,$$

by multiplying the first equation through by α.

58. Definition —By the *Complement* of any quantity A will be understood that quantity $|A$ which is obtained by replacing each product of units in the derived expression for A by its complement. Expressing the same in a formula we have

$$|A\equiv|(\alpha E_1+\alpha_2 E_2+\ldots\ldots)=\alpha_1|E_1+\alpha_2|E_2+\ldots\ldots$$

where E_1, E_2, ... are products of units of any order.

59. *The complement of the complement of any quantity A is equal either to A, or to $-A$, according as $(-1)^{qr}$ is $+$ or $-$*, where q and r are the orders of the quantity and its complement.

The Proof depends on 41.

60. *If the order of a space n is odd. $||A=A$: if n is even. $||A=(-1)^q A$, where q is the order of A.*

Proof.—By 59, $||A=(-1)^{q(n-q)}A$. Then if n is odd, $q(n-q)$ is even, whether q be even or odd : but if n is even, then if q is even $q(n-q)$ is even, and if q is odd $q(n-q)$ is odd. The theorem as stated readily follows.

61. Definition.—If the sum of the orders of two units is less than or equal to the order n of the space considered, then by their *progressive product* is understood their outer product (52), with the provision, however, that the product of the n original units is unity. On the other hand if the sum of the orders of two units is greater than the order n of the space, then by their *regressive product* is understood that quantity whose complement is the progressive product of the complements of these units.

Thus, if the sum of the orders of E and $F>n$, we have that

$$|[EF]=[|E|F],$$

where $[e_1e_2\ldots\ldots e_n]=1$.

The regressive products can be made plainer by an example. Let 5 be the order of the space considered, $[e_1e_2\ldots\ldots e_5]=1$, and let the product of $E_1=[e_1e_2e_3e_4]$ and $E_2=[e_1e_2e_5]$ be required.

Changing the order of the factors of E_2, we write $E_2=[e_5e_1e_2]$ (37), there being two interchanges. Then

$$[E_1E_2]=[e_1e_2e_3e_4][e_5e_1e_2]=[e_1e_2e_3e_4e_5e_1e_2]\ (52)=[e_1e_2]\ \text{(Rem. Art. 13)}.$$

Thus the product of E_1 and E_2 is the product of their common factors e_1 and e_2. The question arises, how can the common factors be selected and their product formed out of the two given factors?

Since E_1 and E_2 together contain all five of the units and neiter $|E_1$ nor $|E_2$ contains e_1 or e_2, the product of $|E_1$ and $|E_2$ contains all the units except e_1 and e_2. Then $|[E_1E_2]$ contains the factors of $[|E_1|E_2]$. Hence the definition of a regressive product.

62. *If q and r are the orders of A and B and n that of the space considered, the order of the product $[AB]$ is equal to $q+r$ when $q+r<n$, but is equal to $q+r-n$ when $q+r\gtreqless n$. In the language of the Theory of Numbers if p is the order of the product*

$$p\equiv q+r\ \text{(Modulus } n).$$

The proof of this theorem follows the lines of the example of the preceding article.

63. *Similarly, for a larger number of factors,*

$$p\equiv q+r+s+t\ldots\ldots\ \text{(Modulus } n).$$

64. *The product of the complements of two quantities is the complement of the product of those quantities, that is to say,*

$$[|A\ |B]=|[AB].$$

Proof.—1. Suppose at first that the sum of the orders α and β of A and B is greater that n, that of the space under consideration. Let $A=\Sigma\alpha_rE_r$, $B=\Sigma\beta_sF_s$, where E_r and F_s are units. Then $|A=\Sigma\alpha_r|E_r$ and $|B=\Sigma\beta_s|F_s$ (58). Thus, we have,

$$[|A\ |B]\equiv[\Sigma\alpha_r|E_r.\Sigma\beta_s|F_s]=\Sigma\alpha_r\beta_s[|E_r|F_s\ \ldots\ldots\ldots\ldots(28)$$

$$=\Sigma\alpha_r\beta_s|[E_rF_s]\ \ldots\ldots(61)=|\Sigma\alpha_r\beta_s[E_rF_s]\ \ldots\ldots(58)$$

$$=|[\Sigma\alpha_rE_r.\Sigma\beta_sF_s]\ldots\ldots(28)\equiv|[AB].$$

2. Suppose $\alpha+\beta=n$. Let E and F be products of the original units. Two cases may be distinguished. First, when E and F contain a common factor e_1. It is plain that in this case both $[EF]$ and $[|E|F]$ contain common factors, so that they are each equal to zero (43). Second, when $[EF]=1$. Replacing $|E$ and $|F$ by their values from 57 and noting that $[FE][FE]=+1$, we get $[|E|F]=[EF]$. But as $[EF]=1$, $|[EF]=1$ (57). Thus the law holds for units. Then reasoning as in 1, above, it holds for any quantities.

3. Suppose $\alpha+\beta<n$. The proof of this case is based on 1. of this article by letting $A=|A'$, $B=|B'$, and writing

$$|[A'B']=[|A'|B']=[AB].$$

65. *The product of the complements of several quantities is the complement of the product of these quantities.*

Proved by mathematical induction from 64.

66. *The complement of a polynomial is the sum of the complements of its parts.*

Proof from 58.

67. *If E, F, G are units the sum of whose orders is n (the order of the space considered),*

$$[EF.EG]=[EFG]E.$$

Proof.—We distinguish two cases: either $[EFG]$ contains equal factors or it does not. If it does, then at least one of the units, say e_1, is missing in it. Now let $[EF]=|Q$; then by 57 Q must contain e_1; likewise, let $[EG]=|R$; then R contains e_1. Then $[QR]=0$ (43). Now we have

$$[EF.EG]=[|Q|R]=|[QR] \text{ (64)}=|0=0 \text{ (57)}.$$

But $[EFG]$ also equals zero since it contains equal factors (43). Thus in this case

$$[EF.EG]=[EFG]E.$$

If $[EFG]$ does not contain equal factors, then it contains all of the original units and no others. Then by 57 and 55,

$$|G=[GEF][EF], \quad |F=[FEG][EG].$$

Since $[GEF]$ and $[FEG]$ are equal to either $+1$ or -1, they can appear on either side of their equations. Hence we can write

$$[EF]=[GEF]|G, \quad [EG]=[FEG]|F. \quad \text{Whence}$$

$$[EF.EG]=[GEF][FEG][|G|F]=[GEF][FEG]|[GF \ldots\ldots \text{(64)}$$

$$=[GEF][FEG][GFE]E\ldots(57)=[GEF][EFG][GEF]E\ldots(41)$$

$$=[EFG]E,$$

since $[GEF][GEF]$, as in (57), equals $+1$.

68. *If A, B, C are simple quantities the sum of whose orders equals n, the order of the space of these quantities.*

$$[AB.AC]=[ABC]A.$$

The proof of this formula (based on (67) on account of its length, is omitted, as are also those of the next four formulas given below.

69.—71. *If A, B, C are simple quantities whose product is of the* 0*th order.*

69. $$[AB.AC]=[ABC]A.$$

70. $$[AB.BC]=[ABC]B.$$

71. $$[AC.BC]=[ABC]C.$$

72. *If A, B, C are simple quantities and the sum of the orders of A and C equals the order of the space considered and B is subordinate* (18) *to A, then*

$$[A.BC]=[AC]B \text{ and } [CB.A]=[CA]B.$$

Remark.—It seems proper to state here that the matter contained in Chapters II—V is taken direct from Grassmann's Ausdehnungslehre of 1862. What the writer has done has been to cut out everything which was not essential to the development of the main principles of the work. What to insert and what to omit constitutes the chief difficulty. In the following chapters (except Chapter VIII) we shall not follow Grassmann very closely.

MATHEMATICAL INDUCTION.

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There is such a general lack of presentation of the governing principles of mathematical induction in the text books which refer to the subject, and failure to give a working rule for its application, that it is thought a presentation of such a working rule specifically stated and not left to inference from examples merely, would be acceptable.

Mathematical Induction (not entering into the question of the appropriateness of the name) is a method of proof used where a primary operation and a