An Elementary Exposition of Grassmann's "Ausdehnungslehre," or Theory of Extension

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of these conjugates would have to transform the other into itself for a substitution must transform every subgroup in which it occurs into itself.

While every subgroup whose order is half the order of the group is self-conjugate a subgroup of any lower order need not be self-conjugate. If a group contains a subgroup of one-third its order that is not selfconjugate it must contain three subgroups of this order which are transformed according to the symmetric group of order six by all the substitutions of the group.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS. Ph. D., Stevens Point, Wis.

[Concluded from November Number.]

APPLICATION TO MECHANICS.

168. A force is completely represented by a point vector. We will denote a force by $F\rho$, where F denotes the length and direction of a vector, indicating respectively the intensity and direction of the force, and ρ is the point of application. It is apparent here that two letters are needed to properly represent the complex concept of a force. (See 76).

158. In Chapter I, we saw that the sum of two vectors is the diagonal of a parallelogram whose adjacent sides are the two given vectors. Then the sum of two forces, or their resultant, is the diagonal of a parallelogram whose two adjacent sides represent the two given forces. Similarly, all the results obtained for vectors in that chapter hold equally well for forces. The condition for equilibrium of forces acting on a particle at the extremity of ρ is evidently $(\Sigma F)\rho=0$, or $\Sigma F=0$.

169. The formulas obtained in Chapter VI evidently hold true when the points are replaced by infinitesimal forces, as parallel forces acting on particles, and also when they are replaced by finite parallel forces. (See 80).

170. Let the resultant of the forces acting on a rigid body be denoted by R. Then if $\varepsilon=\rho-\rho_1$

$$R=\Sigma F\rho=\Sigma F\rho_1+\Sigma F(\rho-\rho_1)=(\Sigma F)\rho_1+(\Sigma F\varepsilon).$$

This result, called a "Wrench," contains two parts, a vector and a plane segment part. The vector ΣF represents the translation force, and the plane segment $\Sigma F\varepsilon$ gives the plane and magnitude of rotation. When the above result is interpreted geometrically, *i. e.* when ΣF is thought of as a line, and $\Sigma F\varepsilon$ as a plane segment, R is called a "Screw."

171. If R reduce to a single force, then by 34, $R^2=0$. We have then

$$R^2=(\Sigma F\rho_1+\Sigma F\varepsilon)^2=2(\Sigma F.\Sigma F\varepsilon)\rho_1=0.$$

This shows that ΣF and $\Sigma F\varepsilon$ must be parallel (112), *i. e.* the resultant force is parallel to the plane of the resultant couple. Evidently $\Sigma F.\Sigma F\varepsilon=0$ is satisfied also either by $\Sigma F=0$ whence R=a couple, or by $\Sigma F\varepsilon=0$, which makes of R a single force. If all of the forces lie in a single plane, the resultant can be reduced to either a single force or to a couple.

172. *For equilibrium we must have* (170), $\Sigma F=0$ *and* $\Sigma F\varepsilon=0$.

173. *A total resultant effect may be reduced to a single force and a couple whose place is perpendieular to the force by properly choosing the point of application.*

Proof.

$$R=\Sigma F\rho_1+\Sigma F(\rho-\rho_1)=\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1)+\Sigma F(\rho-\rho_1).$$

Let $\Sigma F(\rho-\rho_1)=|\varepsilon$. Then

$$R=\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1)+|\varepsilon.$$

Now the condition that $\Sigma F\rho_2$ shall be perpendicular to $|\varepsilon-\Sigma F(\rho_2-\rho_1)$ is

$$(|\varepsilon-\Sigma F(\rho_2-\rho_1)|\Sigma F\rho_2=|\varepsilon|\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1).|\Sigma F\rho_2=0.$$

$\therefore\ |\Sigma F\rho_2.\Sigma(\rho_2-\rho_1)F=-|\Sigma F\rho_2\varepsilon$, (64, 38), whence

$$\frac{|\Sigma F\rho_2.\Sigma(\rho_2-\rho_1)F}{(\Sigma F\rho_2)^2}=-\frac{|\Sigma F\rho_2\varepsilon}{(\Sigma F\rho_2)^2}.$$

Comparing the left member of this equation with the formula of 132, we see that the right member is the value of the orthogonal projection of $\rho_2-\rho_1$ on a plane perpendicular to $\Sigma F\rho_2$.

Multiplying $\Sigma F\rho_2$ by the members of the last equation and interchanging the factors of $\Sigma(\rho_2-\rho_1)F$, we get

$$\Sigma F(\rho_2-\rho_1)=\frac{\Sigma F\rho_2|\Sigma F\rho_2\varepsilon}{(\Sigma F\rho_2)^2}=\frac{|\Sigma F\rho_2.\Sigma F\rho_2|\varepsilon}{(\Sigma F\rho_2)^2}-|\varepsilon \text{ (144, last equation).}$$

Substituting the last value of $\Sigma F(\rho_2-\rho_1)$ in the first equation of this article, we have

$$R=\Sigma F\rho_2+\frac{\Sigma F\rho_2\Sigma F(\rho_2-\rho_1)}{(\Sigma F\rho_2)^2}.|\Sigma F\rho_2,$$

which gives the required reduction.

CHAPTER XIV.

APPLICATION TO LOGIC.

174. The law of the inner product is $e_r e_s = 0$. For in 117 if F is different from E, $[E|F]$ contains equal factors and is therefore zero, (43). This law, which is the opposite of that of the outer product (34), is made the basis of the study of spaces. Now a space in the *Ausdehnungslehre* corresponds to a *concept* or notion in logic. Hence the former science can be applied in the latter.

175. Definition.—The *Combined Space*, or the *sum* of two spaces, is the totality of quantities which belong to one or other of them (17).

176. Definition.—The *Common Space*, or the *product* of two spaces, is the totality of quantities which are common to both, (see 104). The product of two spaces which have no quantity of the first order common is zero (174).

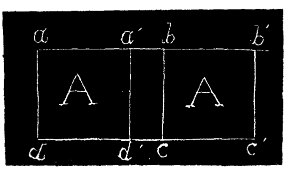
Thus, using $°(L_1 L_2)$ and $°(L_1 L_3)$ to denote the two spaces which contain these lines, we see that L_1 is the common space.

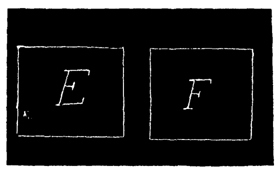
177. *The sum of the orders m and n of two spaces equals the sum of the orders p and q of their common and combined spaces.*

Evidently $m+n$ duplicates the number expressing the order of the common space, and $p+q$ does the same.

178. *All the laws of space analysis continue to hold true in logic when the word "space" is replaced by the logical term "concept."*

Proof.—It is evident from the definitions 175, 176 that $°(e+e)=°e$, and $°(e.e)=°e$, and also that $°e_r+°e_s$ is greater or less than 0 and $°(e_r).°(e_s)=0$, when r is greater or less than s. But these are likewise the basic formulas of logic. (See H. and R. Grassmanns' *Formenlehre*, B. II., *Die Begriffslehre, oder Logik*,





1872, page 43. See also Encyclopedia Britannica article 'Logic,' section 35, paragragh 6.)

179. Two spaces can contain equal simple quantities only when they overlap. Thus if $A=abcd$ and $A'=a'b'c'd'$, these spaces have $a'bcd'$ in common, and quantities in $a'bcd'$ are in both A and A'. But E and F have no simple quantities in common. Since they are of the same size and lie in the same plane E and F in *geometry* would be equal (94); but in the theory of space or logic they are altogether different, having not one point, element, or bit of surface in common. It is highly important to note this difference if the reader is to avoid misconception.

180. *The associative and commutative laws for addition and multiplication hold for space analysis.*

Proof.—It is evident from the definition 175 that $a+b=b+a$. Similarly from the definition in 176 it follows that $ab=ba$, and $abc=a(bc)$.

181. *Every sum*, $n.^\circ A$, *or product*, $(^\circ A)^n$, *formed from the same space*, $^\circ A$, *equals this space.* This follows from proof in 178.

182. *One can add to any space a product of two spaces, one of whose factors is the given space, without altering its value.* Thus

$$^\circ A = {^\circ A} + {^\circ A}{^\circ B}.$$

One can multiply any space by a sum of two spaces, one of whose parts is the given space, without altering its value. Thus,

$$^\circ A = {^\circ A}({^\circ A} + {^\circ B}).$$

Proof.—The product $^\circ A ^\circ B$ is that part of $^\circ A$ which is common to both factors, 176. But adding a part of $^\circ A$ to $^\circ A$ gives $^\circ A$ by 181. In the other case we see that what is common to $^\circ A$ and $^\circ A + {^\circ B}$ is $^\circ A$.

183. *If* $^\circ A + {^\circ B} = {^\circ B}$, $^\circ A . {^\circ B} = {^\circ A}$.

Proof.—$^\circ A . {^\circ B} = {^\circ A}({^\circ A} + {^\circ B})$ (Hyp.)

$= {^\circ A}$ (82)

If $^\circ A . {^\circ B} = {^\circ A}$, $^\circ A + {^\circ B} = {^\circ B}$.

Proof.—$^\circ A + {^\circ B} = {^\circ A}{^\circ B} + B^\circ$ (Hyp.)

$= {^\circ A}{^\circ B} + {^\circ B^2}$ (181)

$= {^\circ B}({^\circ A} + {^\circ B}$ (Dist. Law)

$= {^\circ B}$ (182)

184. *Unity added to any space gives unity, and any space multiplied by unity gives the same space; nought added to any space gives the same space, and any space multiplied by nought gives nought.*

Proof.—The last three of these results are self-evident. From the first we have

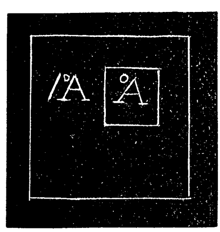
$$1 = 1(1 + 1 \times {^\circ A}) \qquad (182)$$

$$= 1 + {^\circ A} \qquad \text{(Dist. law and 2. of this Article.)}$$

185. *If* $^\circ A + {^\circ C} = {^\circ B} + {^\circ C}$ *and also* $^\circ A . {^\circ C} = {^\circ B} . {^\circ C}$, $^\circ A = {^\circ B}$.

Proof.—$^\circ A = {^\circ A}({^\circ A} + {^\circ C})$ (182) $= {^\circ A}({^\circ B} + {^\circ C})$ (hyp.) $= {^\circ A}.{^\circ B} + {^\circ A}{^\circ C}$ (Dist. law) $= {^\circ A}{^\circ B} + {^\circ B}{^\circ C}$(hyp.) $= {'B}({^\circ A} + {^\circ C}) = {^\circ B}({^\circ B} + {^\circ C})$(hyp.) $= {^\circ B}$ (182).

Definition.—The Non-Space or Complementary Space of $^\circ A$ is that space $|{^\circ A}$ (read non-A) which contains all the units not found in $^\circ A$, and none of the units which are found in $^\circ A$.



187. *The sum of a space* $^\circ A$ *and* $|{^\circ A}$ *is unity: the product of a space* $^\circ A$ *and* $|A$ *is zero.*

Proof.—The truth of the second part of the theorem follows directly from 176. For the other we write, by 184,

$$^\circ A + |{^\circ A} = ({^\circ A} + |{^\circ A})|.$$

But by 176 the product of two spaces cantains all the quantities which are common to both. Then all the quantities of the left member $^\circ A + |{^\circ A}$ are contained in $|$. But $^\circ A$ and $|{^\circ A}$ contain all the units there are. Then $|$ contains all possible units.

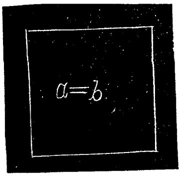
188. *All non-spaces of the same spaces are equal.*

PROOF.—Suppose $|{}^\circ A$ and $|{}^\circ A_1$ to be two non-spaces of ${}^\circ A$. Then ${}^\circ A+|{}^\circ A=1={}^\circ A+|{}^\circ A_1$; whence by 185. remembering that ${}^\circ A.|{}^\circ A=0$ and ${}^\circ A.|{}^\circ A_1=0$, $|{}^\circ A=|{}^\circ A_1$.

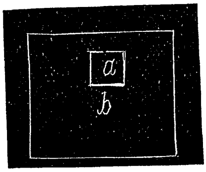
189. *The non-space of the non-space of a given space is the given space.*

PROOF.—We have $|{}^\circ A+||{}^\circ A=1=|{}^\circ A+{}^\circ A$, and $||{}^\circ A.|{}^\circ A=0={}^\circ A\,|{}^\circ A$(187). Whence, by 185, $||{}^\circ A={}^\circ A$.

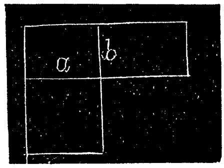
190. We give below four diagrams with descriptive names.



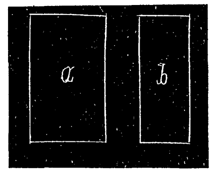
Identical Spaces.



Incident Space.



Cutting Spaces.



Disjunctive Spaces.

191. *Each of two spaces is incident to their sum. The product of two spaces is incident to each of them. Unity is the highest space, to which all other spaces are incident Nought is the lowest space, being included in all other spaces.* Compare the use of unity here with its use in Chapter V.

The theorems already given will suffice to show that the language and subject matter of the *Ausdehnungslehre* can be utilized in the study of logic. The material for this chapter is taken from Robert Grassmann's *Ausdehnungslehre* (Slettin, 1891). Robert Grassmann aided his brother Hermann in the preparation of the *Ausdehnungslehre* of 1862 and the *Formenlehre* of 1872, and was always deeply interested in the subject.

The article in the Britannica already referred to and Professor Stokes' article in the January (1900) number of THE AMERICAN MATHEMATICAL MONTHLY show that there is considerable diversity of opinion regarding the philosophy underlying the application of mathematics to logic.

INTEGRATION OF ELLIPTIC INTEGRALS.

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[Concluded from November Number.]

$$\int\frac{d\theta}{\cos^6\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{1-e^2}\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^6\theta}$$

$$-\frac{e^2}{1-e^2}\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-19e^2+15e^4)}{15(1-e^2)^2}F(e,\ \theta)$$