An Elementary Exposition of Grassmann's "Ausdehnungslehre," or Theory of Extension

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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

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[Continued from the June-July Number.]

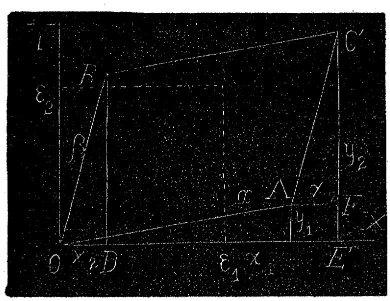
CHAPTER VII.

GEOMETRICAL MULTIPLICATION.

88. Basing our investigation on the fundamental law of combinatory multiplication (34), *let us seek the product of a non-posited point* (76) *and two vectors.* The vectors are thought of as denoting merely translation a given distance in a given direction (See 4—9). Let p denote the point and α and β the vectors.

Suppose

$$\alpha = x_1\varepsilon_1 + y_1\varepsilon_2$$
$$\beta = x_2\varepsilon_1 + y_2\varepsilon_2$$



where ε_1 and ε_2 are unit vectors at right angles to each other. Then by 45

$$[p\alpha\beta] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} [p\varepsilon_1\varepsilon_2].$$

Now the determinant $x_1y_2 - x_2y_1$ is the difference between two rectangles.

Let us seek the relation, if any exists, between this area and that of the parallelogram $AOCB$. We have

$$OACB = OBCE - OACE = BCED - OAFE$$
$$= \tfrac{1}{2}(y_1 + 2y_2)x_1 - \tfrac{1}{2}(x_1 + 2x_2)y_1 = x_1 y_2 - x_2 y_1 .$$

The equation $[p\alpha\beta] = (x_1 y_2 - x_2 y_1)[p\varepsilon_1\varepsilon_2]$ shows, therefore, that if $p\varepsilon_1\varepsilon_2$ is taken to denote the area of the unit square the two sides of which are ε_1 and ε_2, $p\alpha\beta$ denotes the area of the parallelogram whose adjacent sides are α and β.

Now to assume that $p\varepsilon_1\varepsilon_2$ is the area of the square is a perfectly natural assumption analogous to the theorem in geometry which says, The area of a rectangle equals the product of the base by the altitude. *Thus Grassman was led to define the product $\alpha\beta$ as the area generated while α moves (remaining, of course, constantly parallel to itself) over a distance determined by β.*

Attaching this meaning to $p\varepsilon_1\varepsilon_2$ we think first of the point p moving over a distance determined by ε_1 generating a line, and then this line moving over a distance determined by ε_2 generating the square.

In the next two articles we will drop the factor p and think of the first factor as a line rather than as mere translation.

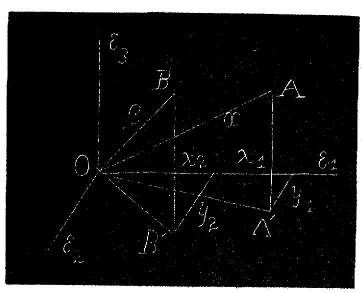
89. *Similarly let us seek the product of two vectors in space.* Let

$$\alpha = x_1\varepsilon_1 + y_1\varepsilon_2 + z_1\varepsilon_3 \text{ and } \beta = x_2\varepsilon_1 + y_2\varepsilon_2 + z_2\varepsilon_3$$

where ε_1, ε_2; ε_3 are three unit vectors each at right angles to the other two. Then by 47,

$$[\alpha\beta] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} [\varepsilon_1\ \varepsilon_2] + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} [\varepsilon_2\ \varepsilon_3] + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} [\varepsilon_3\varepsilon_1].$$

Here it is evident that the first determinant coefficient is the area of the projection of the parallelogram whose adjacent sides are α and β on the plane $\varepsilon_1\varepsilon_2$, and that the other coefficients are the areas of the corresponding projections on the other planes. The above equation expresses then, that the area of the parallelogram whose sides are α and β is equal to the sum of its projections on the three coördinate planes (75).



90. *To find the product of three vectors, α, β, and γ.*

If ε_1, ε_2, ε_3 are three mutually perpendicular vectors, and

$$\alpha = x_1\varepsilon_1 + y_1\varepsilon_2 + z_1\varepsilon_3,\ \beta = x_2\varepsilon_1 + y_2\varepsilon_2 + z_2\varepsilon_3,\ \gamma = x_3\varepsilon_1 + y_3\varepsilon_2 + z_3\varepsilon_3,$$

$$[\alpha\beta\gamma] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\varepsilon_1\varepsilon_2\varepsilon_3] \tag{45}$$

Hence if $[\varepsilon_1\varepsilon_2\varepsilon_3]$ is the volume of a cube each side of which is a linear unit, $[\alpha\beta\gamma]$ devotes the volume of the parallelopiped whose adjacent edges are α, β, γ, since, as is well known in analytic geometry, the determinant expresses the number of units of volume in the parallelopiped. In the *Ausdehnungslehre* attention must be paid to the *order* of the factors, *i. e.* to the order of generation. Thus (37) $[\alpha\beta]=-[\beta\alpha]$, and $[\alpha\beta\gamma]=-[\alpha\gamma\beta]$.

It is apparent from the preceding articles that the *Ausdehnungslehre* is especially well adapted for the investigation of propositions concerning the areas and volumes of rectilinear figures.

91. *To find the product of a posited point and vector.*

Let $p\rho$ and ε denote the point and vector. Following the areal interpretation given above this product should be a *line* whose length is ε and whose other dimension is the infinitesimal p, the whole fixed in position by the radius vector ρ.

Now $p(\rho+x\varepsilon)$ denotes any point on the line through $p\rho$. Then since

$$[p(\rho+x\varepsilon)\varepsilon]=[p\rho\varepsilon]+x[p\varepsilon\varepsilon]=[p\rho\varepsilon] \text{ (34)},$$

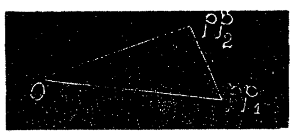
we see that the product of a posited point and a vector determine a line segment, *but this line segment may have any position on the vector through the given point.*

92. *To find the product of two posited points.*

Let $p\rho_1$ and $p\rho_2$ be two unit points. Then

$$[p\rho_1 . p\rho_2]=[p\rho_1(p\rho_2-p\rho_1)], \text{ since } [p\rho_1 . p\rho_1]=0. \text{ (34)}$$

But $[p\rho_1(p\rho_2-p\rho_1)]$ is the product of a point and a vector (78) which, by 91, is the line from the extremity of ρ_1 to that of ρ_2. Thus, *The product of two posited points is the line joining the first to the second.*



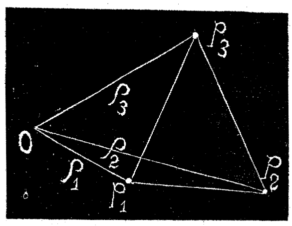
93. We have illustrated in the last article a principle which will be found to hold generally in the *Ausdehnungslehre*, viz., *That the product of posited quantities which have no common figure is some multiple of the connecting figure.*

94. *To find the product of three posited points.*

Let us use p_1, p_2, p_3 to denote the three unit points instead of $p\rho_1$, $p\rho_2$, $p\rho_3$ as heretofore. It will be understood when p is used to denote the posited point $p\rho$, that it stands for the complex quantity described in 76.

$$[p_1p_2p_3]=[p_1p_2 . p_3] \text{ (Rem. 13) } =[p_1p_2(p_3-p_1)], \text{ since } [p_1p_2p_1]=0 \text{ by 43.}$$

But $[p_1p_2]$ is the line from p_1 to p_2 (92), and by 88 the product of a line $[p_1p_2]$ and a vector p_3-p_1 (78) equals the parallelogram whose adjacent sides are $[p_1p_2]$ and $[p_1p_3]$



Thus the product of three given posited points is twice the area of the triangle whose vertices are the three given points. (93)

Let $p_1+xp_2+yp_3$ denote any point in the plane of $[p_1p_2p_3]$.

Now $[p_1p_2p_3]$=twice area of triangle whose vertices are p_1, p_2, p_3; but we have

$$[(p_1+xp_2+yp_3)(p_2-p_1)(p_3-p_1)]=[p_1p_2p_3] \quad (22, 43).$$

This shows that *the value of the product remains the same whatever be the position of the triangle $[p_1p_2p_3]$ in the plane of these points.*

95. *To find the product of four posited points.*

Let p_1, p_2, p_3, p_4, represent four unit points. Then

$$[p_1p_2p_3p_4]=[p_1(p_2-p_1)(p_3-p_1)(p_4-p_1)] \quad (43)$$
$$=6\times\text{tetraedron whose vertices are } p_1,\ p_2,\ p_3,\ p_4, \text{ by } 90.$$

Let $p_1+xp_2+yp_3+zp_4$ be any point whatever. Then

$$[(p_1+xp_2+yp_3+zp_4)(p_2-p_1)(p_3-p_1)(p_4-p_1)]=[p_1p_2p_3p_4]. \quad (22, 43)$$

Hence the product of four points in solid space is the same no matter where located.

96. We have used the terms "line" and "line segment" to denote a quantity whose length and the line in which it must lie are given but not its position in that line. Similarly we will use the terms "plane" and "plane segment" (See 74) to denote the corresponding areal quantity described in 94. Grassmann's terms for them are respectively "*Linientheil*" and "*Flächentheil.*" For the quantity described in 95 he uses the term "*Körpertheil.*"

97. *To find the sum or difference of two lines or two planes.*

Let $[p_1p_2]$, $[p_1p_3]$ be the lines, $[p_1p_2p_3]$, $[p_1p_2p_4]$, the planes, and let $p_3+p_4=2p_s$, and $p_3-p_4=\varepsilon$. Then

$$[p_1p_2]\pm[p_1p_3]=[p_1(p_2\pm p_3)]=2[p_1p_s], \text{ or } [p_1\varepsilon], \text{ a line.} \quad (82, 91)$$

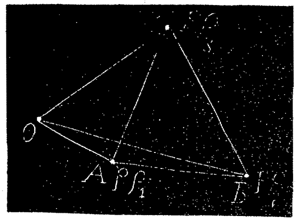
$$[p_1p_2p_3]\pm[p_1p_2p_4]=[p_1p_2(p_3\pm p_4)]=2[p_1p_2p_s], \text{ or } [p_1p_2\varepsilon], \text{ a plane.} \quad (94)$$

98. *To find the sum of the sides of a triangle.*

Let $p\rho_1$, $p\rho_2$, $p\rho_3$ represent the three vertices of a triangle. Then

$$[p\rho_1\cdot p\rho_2]+[p\rho_2\cdot p\rho_3]+[p\rho_3\cdot p\rho_1]=[(p\rho_2-p\rho_1)(p\rho_3-p\rho_1)], \text{ since } [p\rho_1p\rho_2]=0.$$

Thus the sum of the three sides of a triangle equals the product of the vectors $p\rho_2 - p\rho_1$ and $p\rho_3 - p\rho_1$. This product differs from the expression for the area of the triangle (94) *by the absence of the first factor* $p\rho_1$. An interpretation of the expression given above for the sum of the sides which makes it equal to the area of the triangle may be had by thinking of $p\rho$ as a *generative* product of p and ρ. Using the period to denote generative multiplication, we have



$$p.\rho_1.p\rho_2 = OAB\ ;\ p.\rho_2.p\rho_3 = OBC\ ;\ p.\rho_3.p\rho_1 = -OCA.$$

$$\text{Thus, } OAB + OBC - OCA = p.\rho_1.p.\rho_2 + p.\rho_3.p\rho_2 + p.\rho_2.p\rho_3 = ABC.$$

Remark.—By Grassmann's formulas the sum of the sides of a triangle equals its area, for he treats a point as that which has position only, and considers that the product of two vectors alone equals the area described in 88. The use of the definition of a point given in 76 has the effect of making some of the theorems of this chapter depart in certain respects from Grassmann's. The writer thinks however that regarding a line as generated by a point in motion agrees well with Grassmann's conception of "generative" multiplication.

99. *If* $p_1, p_2, p_3, p_4, p_5, p_6$ *are six points,* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ *are four vectors and* x *is any scalar, by the preceding articles we have the following conditions:*

(1) $p_1 = xp_2$ is the condition that points p_1 and p_2 coincide.

(2) $[p_1p_2] = x[p_3p_4]$, is the condition that the (unlimited) lines $[p_1p_2]$ and $[p_3p_4]$ coincide.

(3) $[p_1p_2p_3] = x[p_4p_5p_6]$ is the condition that the (unbounded) planes $[p_1p_2p_3]$ and $[p_4p_5p_6]$ coincide.

(4) $\varepsilon_1 = x\varepsilon_2$ is the condition that the vectors ε_1 and ε_2 are parallel.

(5) $[\varepsilon_1\varepsilon_2] = x[\varepsilon_3\varepsilon_4]$ is the condition that the planes of $[\varepsilon_1\varepsilon_2]$ and $[\varepsilon_3\varepsilon_4]$ are parallel.

100. A point is regarded as a space of the first order; an unlimited line as a space of the second order ; an unbounded plane as a space of the third order ; and solid space as a space of the fourth order. See 17 and 85. Since a vector may be regarded as a point at infinity, a vector also may be regarded as a quantity of the first order. See Chapter I.

101. Relative Products. A *Planimetric* product is a relative product whose factors are in a plane, or space of the third order. A Stereometric product is one whose factors are in a space of the fourth order (100).

102. *To find the planimetric product of two line segments* p_1p_2 and p_1p_3. We have

$$[p_1p_2.p_1p_3] = [p_1p_2p_3]p_1.\ (67)$$

Here $[p_1p_2p_3]$ is a scalar (101, 61). Thus the product is the point of intersection multiplied by a scalar.

103. *To find the planimetric product of two parallel line segments* $[p_1p_2]$ *and* $[p_3p_4]$.

We have, by hypothesis, $p_3-p_4=x(p_1-p_2)$ (99, (4)). Then

$$[p_1p_2.p_3p_4]=[(p_1-p_2)p_2.(p_3-p_4)p_4]\ (49)=x[(p_1-p_2)p_2.(p_1-p_2)p_4]\ (\text{Hyp.})$$

$$=x[(p_1-p_2)p_2p_4](p_1-p_2)\ (67)=[(p_3-p_4)p_2p_4](p_1-p_2)\ (\text{Hyp.})$$

$$=[p_3p_2p_4](p_1-p_2)\ (49)=[p_2p_3p_4](p_2-p_1).\ (38)$$

Hence the product is the point at infinity (the vector p_2-p_1) which is the intersection of the two lines multiplied by the scalar $[p_2p_3p_4]$.

104. The last two articles illustrate a principle of general application in the *Ausdehnungslehre*, viz., *That the relative product of posited quantities which have a common figure is that common figure multiplied by a scalar.* See 93.

105. *To find the planimetric product of two lines and a posited point.*

Let $[p_1p_2]$ and $[p_1p_3]$ be the lines and p the point. Then

$$[p_1p_2.p_1p_3.p]=[(p_1p_2p_3)p_1.p]\ (13,\ \text{Rem.},\ 67)=[p_1p_2p_3][p_1p]$$

since $[p_1p_2p_3]$ is a scalar. (101, 61)

106. *To find the planimetric product of the three line segments* $[p_1p_2]$, $[p_1p_3]$, $[p_4p_5]$.

We have $[p_1p_2.p_1p_3.p_4p_5]=[p_1p_2p_3][p_1.p_4p_5]\ (67)=[p_1p_2p_3][p_1p_4p_5]$ (55).

Corollary. *If the three line segments are the sides of a triangle we may write* $[p_2p_3]$ *instead of* $[p_4p_5]$. *Then the product is* $[p_1p_2p_3]^2$.

107. The following general principle is illustrated in the two preceding articles: *If at any time the product of factors combined in regular order from the left gives rise to a scalar or to a scalar times an extensive quantity, this scalar is to be regarded as a simple numerical factor, and the extensive quantity part of the product, if there is such, is to be combined with the remaining extensive factors, and so on.* Such products are described as *mixed, i. e.* as both progressive and regressive. (61)

108. *The stereometric products of a line and a point and of two lines are commutative; but that of a point and a plane is non-commutative.*

Let L_1 denote the line $[p_1p_2]$, L_2 the line $[p_3p_4]$, and P the plane $[p_2p_3p_4]$.

$$[L_1p_3]\equiv[p_1p_2.p_3]=[p_1p_2p_3]=[p_3.p_1p_2]\ (40,\ 55)\equiv[p_3L_1]\ ;$$

$$[L_1L_2]\equiv[p_1p_2.p_3p_4]=[p_1p_2p_3p_4]\ (55)=[p_3p_4.p_1p_2]\ (40,\ 55)\equiv[L_1L_2]\ ;$$

$$[Pp_4]\equiv[p_1p_2p_3p_4]=-[p_4.p_1p_2p_3]\ (40,\ 55)\equiv-[p_4P].$$

109. *To find the stereometric product of two non-incident plane segments* $[p_1p_2p_3]$ and $[p_1p_2p_4]$.

We have $[p_1p_2p_3 \cdot p_1p_2p_4]=[p_1p_2p_3p_4][p_1p_2]$ (67). See 104.

110. *To find the stereometric product of three plane segments* $[p_1p_2p_3]$, $[p_1p_2p_4]$, $[p_1p_3p_4]$ *which intersect in* p_1.

$$[p_1p_2p_3 \cdot p_1p_2p_4 \cdot p_1p_3p_4]=[p_1p_2p_3p_4][p_1p_2 \cdot p_1p_3p_4] \text{ (67)}=[p_1p_2p_3p_4]^2p_1 \text{ (67)}.$$

111. *To find the stereometric product of a plane segment* $[p_1p_2p_3]$ *and line segment* $[p_1p_4]$ *which do not lie in the same plane. Is the product commutative?*

We have $[p_1p_2p_3 \cdot p_1p_4]=[p_1p_2p_3p_4]p_1$ (67) (See 104).

Also, $[p_1p_4 \cdot p_1p_2p_3]=[p_1p_4p_2p_3]p_1$ (67)$=[p_1p_2p_3p_4]p_1$ (40).

112. *The stereometric product of two line segments* $[p_1p_2]$ *and* $[p_3p_4]$ *equals zero when and only when they lie in the same plane* (55, 95) ; *the stereometric product of two quantities of the first, second, or third orders, but not both at the same time of the second orders, equals zero when and only when the quantities are incident, i. e. when one falls in the space of the other ; as two coincident points, two plane segments if the planes coincide, a point and a line- or a plane-segment if the point lies in the line or plane, a line segment and a plane segment if the line lies in the plane.*

113. *Algebraic Curves and Surfaces. The equation of a variable point* p *which lies in the same straight line as* $[p_1p_2]$ *is* $[p\ p_1p_2]=0$. (112)

114. *The equation of a straight line* L *which passes through the intersection of* L_1 *and* L_2 *and lies in their plane is* $[L_1L_2L]=0$ *where* $[L_1L_2L]$ *is a planimetric product* (See 106).

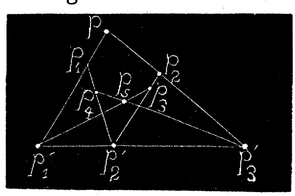
115. *If* $P_{n,p}$ *is a planimetric product of order zero which contains the variable point* p, n *times and besides only constant points and lines as factors, then* $P_{n,p}=0$ *is the point equation of an algebraic curve of the nth order, that is to say, the point* p *moves in an algebraic curve of the nth order.*

Proof. Let p_1, p_2, p_3 be any three points in the plane. Then

$$p=x_1p_1+x_2p_2+x_3p_3$$

may be any point in the plane.. Substituting this value of p in $P_{n,p}=0$ there results a homogeneous equation of the nth degree in x_1, x_2, x_3 whose terms are all of the form $Ax^a{}_1x^b{}_2x^c{}_3$ where $a+b+c=n$. A is the product of constant lines and points, and since this product is by hypothesis of the zeroth order, A is a constant. Regarding x_1, x_2, x_3 as trilinear coördinates, we see that $P_{n,p}=0$ now becomes an ordinary cartesian equation for a curve of the nth order.

116. As an example we give a proof of Pascal's Hexagram Theorem.



Let $p_1 \ldots . p_5$ be five given points and p a variable point which moves so as to leave p'_2 on the line $[p'_1p'_3]$, p'_1, p'_2, p'_3 being defined by the following equations : $p'_1=[pp_1 \cdot p_3p_5]$; $p'_2=[p_2p_3 \cdot p_1p_4]$; $p'_3=[pp_2 \cdot p_4p_5]$. $[(pp_1 \cdot p_3p_5)(p_2p_3 \cdot p_1p_4)(pp_2 \cdot p_4p_5)]=0$ is the equation of a conic passing through the five given points. For it is of the second degree in p and is satisfied by putting p equal to any one of the five given points. By changing points into lines in the above we have Brianchon's Theorem.

[To be Continued.]