AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

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[Continued from November Number.]

CHAPTER IV.

COMBINATORY MULTIPLICATION.

35. Definition.—A product containing only units of the same system as factors and such that if the last two factors (called simple factors) are interchanged the sign of the product is changed is called *a combinatory product.*

Thus if E (not equal to 0) is a product of units and e_1, e_2 are units, and

$$[Ee_1e_2]+[Ee_2e_1]=0,$$

the product $[Ee_1e_2]$ is a combinatory product.

36. *In the combinatory product $[Abc]$ in which A is any product of a series of factors and b and c are simple factors if b and c are interchanged, the sign of the product is changed.*

Proof. 1. Suppose b and c at first to be *units*. Since A is any series of factors and these factors are numerically derivable from the units, we may write, after removing the coefficients (by 28), $A=\Sigma\alpha_rE_r$, where E_r are products of the units. Substituting

$$[Abc]+[Acb]\equiv[\Sigma\alpha_rE_r.bc]+[\Sigma\alpha_rE_r.cb]=\Sigma\alpha_r[E_rbc]+\Sigma\alpha_r[E_rcb] \text{ (29)}$$

$$=\Sigma\alpha_r\{[E_rbc]+[E_rcb]\} \text{ (16, 11) } =0\ldots\ldots\ldots\ldots\ldots\text{(35)}.$$

2. Next supposing b and c to be not units but numerically derivable from them. Let $b=\Sigma\beta_re_r$, $c=\Sigma\gamma_re_r$. Then

$$[Abc]+[Acb]\equiv[A.\Sigma\beta_re_r.\Sigma\gamma_re_r]+[A.\Sigma\gamma_re_r.\Sigma\beta_re_r]$$

$$=\Sigma\beta_r\gamma_s[Ae_re_s]+\Sigma\gamma_s\beta_r[Ae_se_r]\ldots\ldots\ldots\ldots\ldots\text{(28)}$$

$$=\Sigma\beta_r\gamma_s\{[Ae_re_s]+[Ae_se_r]\} \text{ (16) } =0 \text{ (by 1 above)}.$$

37. *In a combinatory product one can interchange any two successive simple factors providing the sign of the product be changed, that is to say*

$$[AbcD]+[AcbD]=0,$$

where A and D are any factor series, and b and c are simple factors.

$$[AbcD]+[AcbD]=[\{Abc\}D]+[\{Acb\}D] \text{ (13, Rem.)}$$

$$=[([Abc]+[Acb])D] \text{ (25) } =0 \ldots\ldots\ldots\ldots \text{ (36).}$$

38. *In a combinatory product one can interchange any two simple factors by changing the sign of the product.*

Thus, $P_{a,b}=-P_{b,a}$.

PROOF.—Suppose n factors lie between a and b. Then n interchanges of adjacent factors will bring b into position next to a. After that $n+1$ interchanges of a with adjacent factors will but b in a's place and a in b's place. Thus there would be $2n+1$, or an odd number of changes of sign (37). Hence, ...

39. DEFINITION.—If each of two series of quantities contain a and b once and but once and a stands *before* b in both or *after* b in both, then these quantities in those series are said to be *similarly arranged;* otherwise they are said to be *oppositely arranged.*

40. *Two combinatory products, which contain the same simple factors but in different order, are equal to each other or opposite in value according as the number of oppositely arranged pairs of factors is even or odd.*

Thus, $Q=(-1)^rP$, where P and Q are the two products and r is the number of oppositely arranged pairs of factors.

PROOF.—If every pair of adjacent factors in Q were similarly arranged in P and Q, then, evidently, P and Q would be identical, and there would be no oppositely arranged pairs of factors in the two. If then there are oppositely arranged pairs of factors in P and Q, there must be at least one pair of factors adjacent in Q, which, as compared with the same in P, is oppositely arranged. Suppose after this pair of factors is interchanged in Q we call the result Q_1. Then $Q_1=-Q$. (37). Evidently P and Q_1 will have one less pair of oppositely arranged factor pairs than P and Q. Thus if r was the number at first, Q_1 and P will have $r-1$ such pairs. If r is not 1, there must be another such factor pair in Q_1 and P. Repeating the operation we get $Q_2=(-1)^2Q$. If therefore there were r oppositely arranged factor pairs at first,

$$Q=(-1)^rP.$$

41. *If B is a combinatory product containing r factors and C one containing s factors, then*

$$[ABC]=(-1)^{rs}[ACB].$$

PROOF.—Let $C=c_1c_2\ldots\ldots c_s$. Then since there will be a change of sign (37) each time c_1 interchanges with one of the r factors of B,

$$[ABc_1c_2\ldots\ldots c_s]=(-1)^r[Ac_1Bc_2c_3\ldots\ldots c_s]=(-1)^r(-1)^r[Ac_1c_2Bc_3\ldots\ldots c_s]$$

$$=(-1)^{sr}[Ac_1c_2c_3\ldots\ldots c_sB]=(-1)^{sr}[ACB].$$

42. *If A, B, C are products containing respectively q, r, s factors, then from the preceding article it is plain that*

$$[ABC]=(-1)^{rs+sq+qr}[CBA].$$

43. *If two simple factors of a combinatory product are equal, the product is zero.*

PROOF.—$P_{a,\,b}+P_{b,\,a}=0$ ....(38).

Then, if $b=a$, $P_{a,\,a}=0$.

44. *A combinatory product equals zero if a numerical relation exists between its simple factors.*

Let $a_1=\alpha_2 a_2+\alpha_3 a_3+\ \ldots\ldots \alpha_m a_m$. Then

$$[a_1 a_2 \ldots\ldots a_m]\equiv[(\alpha_2 a_2+\alpha_3 a_3+\ldots\ldots \alpha_m a_m)a_2 a_3 \ldots\ldots a_m]$$

$$=\alpha_2[a_2 a_2 a_3\ \ldots\ldots a_m]+\alpha_3[a_3 a_2 a_3 \ldots\ldots a_m]+\ldots\ldots(29)$$

$$=0+0+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43).$$

45. *The combinatory product of n simple factors which are numerically derivable from the n quantities a_1, a_2, $\ldots\ldots a_n$ equals the determinant formed from the numerical coefficients in their values times* $[a_1 a_2 \ldots\ \ldots a_n]$. Thus

$$[(\alpha_{11}a_1+\ldots\ldots\ \alpha_{n1}a_n)(\alpha_{12}a_1+\ldots\ldots \alpha_{n2}a_n)\ldots\ldots(\alpha_{1n}a_1+\ldots\ldots \alpha_{nn}a_n)]$$

$$=\begin{vmatrix} \alpha_{11} & \alpha_{12}\ldots\ldots\alpha_{1n} \\ \alpha_{21} & \alpha_{22}\ldots\ldots\alpha_{2n} \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots \\ \alpha_{n1} & \ldots\ldots\ldots\ldots\alpha_{nn} \end{vmatrix}[a_1 a_2 a_3 \ldots\ldots a_n].$$

For, the product on the left side equals by (28),

$$\Sigma\alpha_{s1}\alpha_{t2}\ldots\ldots\alpha_{wn}[a_s a_t \ldots\ldots a_w],$$

where the n subscripts s, t, ...w assume in turn all the values from 1 to n. Now those terms of the distributed product in which two or more of the a's are equal disappear by (43). There remains then only those terms which contain all the a's used each once. We have now

$$[a_s a_t\ \ldots\ldots a_w]=(-1)^r[a_1 a_2 a_3 \ldots\ldots a_n]\ldots\ldots\ldots\ldots(40),$$

where r denotes the number of pairs of factors in the left member oppositely arranged as compared with $[a_1 a_2 a_3 \ldots\ldots a_n]$. But this is the law which determines the sign of any term of a determinant. See *e. g. Salmon's Higher Algebra*, § 8.

46. Definition.—By the *multiplicative combinations* of a series of quantities are meant those products which are their combinations without repetition. The simple factors are called the elements of the combination.

47. *Every combinatory product of m factors which are numerically expressed in terms of the n independent quantities a_1, a_2, ... a_n is numerically expressible in terms of the multiplicative combinations of the mth class of $a_1 \ldots\ldots a_n$, and each of these combinations has for its coefficient the determinant formed out of the m^2 numerical coefficients belonging to its m elements. Thus,*

$$[\Sigma \alpha_a a_a . \Sigma \beta_b b_b \ldots\ldots] = \Sigma \begin{vmatrix} \alpha_r & \cdots & \cdots \\ \vdots & \beta_s & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \ddots & \vdots \end{vmatrix} [a_r a_s \ldots\ldots]$$

where $r < s < \ldots\ldots$

Proof.—Evident from that of Art. 45.

48. *If $a_1 \ldots$ a_n are independent, then their multiplicative combinations of any particular class are independent.*

Proof.—Let $\alpha A + \beta B + \ldots\ldots = 0$, in which A, B, ... are the multiplicative combinations of any one class formed out of $a_1 \ldots\ldots a_n$, and α, β ...... are numbers. Let ns multiply the equation through by A', the product of all the factors not found in A. Then B, C, ... would each contain one or more of the elements of A', and the products $[BA']$, $[CA']$, ... would each equal zero (43). Then we have $\alpha[AA'] = 0$. Now, $[AA']$ is not equal to zero. Hence $\alpha = 0$. In the same way we can prove that β, γ, ... must each equal zero. Hence there can be no such equation as $\alpha A + \beta B + \ldots\ldots = 0$, which expresses a dependence between A, B, ... Thus, A. B, ... are independent.

49. *A combinatory product remains constant when to any simple factor an arbitrary multiple of another is added.*

Proof.—$P_{a,\, b+qa} = P_{a,\, b} + qP_{a,\, a}$ (29) $= P_{a,\, b}$ (43).

50. Definition.—If from a series of quantities a second is derived by adding to any quantity a multiple of an adjacent quantity, then the first series is said to be changed into the second by a *simple linear alteration*. If the operation is repeated it is called a *multiple linear alteration*.

From what we saw in 49, it appears that the value of a quantity is not affected by linear alteration.

51. Definition.—The multiplicative combinations of the original units of the mth class is called a *unit* of the mth order, and a quantity numerically derived from such units is called a *quantity* of the mth order.

The space derived from the simple factors of a quantity (17) is called the space of this quantity. A quantity is subordinate to another if its space is.

52. Definition.—The outer product of two *units* of a higher order is obtained by merely uniting their simple factors into a combinatory product.

Thus, $[(e_1 e_2 \ldots\ldots e_m)(e_{m+1} \ldots\ldots e_n)] = [e_1 e_2 \ldots\ldots e_n]$.

53. *In order to multiply two simple quantities,* $[ab\ldots\ldots]$ and $[cd\ldots\ldots]$, *it is sufficient to unite their simple factors taken in order into a single combinatory product* $[ab\ldots\ldots cd\ldots\ldots]$.

Proof.—Let $e_1\ldots\ldots e_n$ be the original units, and let $a=\Sigma\alpha_a e_a$, $b=\Sigma\beta_b e_b$, $c=\Sigma\gamma_c e_c$, $d=\Sigma\delta_d e_d$. Then

$$[(ab\ldots\ldots)(cd\ldots\ldots)] \equiv [(\Sigma\alpha_a e_a.\Sigma\beta_b e_b\ldots\ldots)(\Sigma\gamma_c e_c.\Sigma\delta_d e_d\ldots\ldots)]$$

$$=[\Sigma\{\alpha_a\beta_b\ldots\ldots[e_a e_b]\}\Sigma\{\gamma_c\delta_d\ldots\ldots[e_c e_d]\}]\ldots\ldots\ldots\ldots(28)$$

$$=\Sigma\{\alpha_a\beta_b\ldots\ldots\gamma_c\delta_d\ldots\ldots[(e_a e_b\ldots\ldots)(e_c e_d\ldots\ldots)]\}\ldots\ldots(28)$$

$$=\Sigma\{\alpha_a\beta_b\ldots\ldots\gamma_c\delta_d\ldots\ldots[e_a e_b\ldots\ldots e_c e_d\ldots\ldots]\}\ldots\ldots\ldots(52)$$

$$=[\Sigma\alpha_a e_a.\Sigma\beta_b e_b\ldots\ldots\Sigma\gamma_c e_c.\Sigma\delta_d e_d\ldots\ldots]\ldots\ldots\ldots\ldots(28)$$

$$\equiv[ab\ldots\ldots cd\ldots\ldots].$$

54. Corollary to 53.—If a simple quantity A is subordinate to B (18), then B may be written $B=[AC]$, where C is a simple factor.

55. *To show that* $[A(BC)]=[ABC]$, *i. e. to show that the associative law holds.*

Proof.—1. When A, B, and C are the products of simple factors, the truth of this case follows readily from 53.

2. When A, B, and C are sums of simple quantities, $A=\Sigma A_a$, $B=\Sigma B_b$, $C=\Sigma C_c$.

$$[(A(BC)]\equiv[\Sigma A_a.(\Sigma B_b\Sigma C_c)]=\Sigma[A_a(B_b C_c)]\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

$$=\Sigma[A_a B_b C_c] \text{ (By 1, above) } =[\Sigma A_a.\Sigma B_b.\Sigma C_c] \text{ (28) } =[ABC].$$

[To be Continued.]

A SOLUTION OF THE OBLIQUE TRIANGLE GIVEN TWO SIDES AND THE INCLUDED ANGLE.

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Take the data to be b, c, A.

$$p(\cot A+\cot B)=c.$$

Divide by $p=b\sin A$ and transpose $\cot A$.