An Elementary Exposition of Grassmann's "Ausdehnungslehre," or Theory of Extension

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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

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[Continued from the October Number.]

CHAPTER X.

146. Combinatory multiplication gives us a very neat solution of sets of simultaneous equations. Let n equations of the first degree containing n unknowns be given to find the value of the unknowns. Let the n equations be

$$\alpha_{11}x_1+\alpha_{21}x_2+\ldots\ldots+\alpha_{n1}x_n=\beta_1$$
$$\alpha_{12}x_1+\alpha_{22}x_2+\ldots\ldots+\alpha_{n2}x_n=\beta_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{1n}x_1+\alpha_{2n}x_2+\ldots\ldots+\alpha_{nn}x_n=\beta_n$$

We multiply the first of these equations through by e_1, the second through by e_2, and so on, and the last by e_n, (where $[e_1e_2\ldots\ldots e_n]=1$), and add the resulting equations. Then if

$$\alpha_{11}e_1+\alpha_{12}e_2+\ldots\ldots+\alpha_{1n}e_n=a_1,$$
$$\alpha_{21}e_1+\alpha_{22}e_2+\ldots\ldots+\alpha_{2n}e_n=a_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{n1}e_1+\alpha_{n2}e_2+\ldots\ldots+\alpha_{nn}e_n=a_n$$
$$\beta_1e_1+\beta_2e_2+\ldots\ldots\ldots+\beta_ne_n=b,$$

we have for the sum of the products referred to above, the equation

$$x_1a_1+x_2a_2+\ldots\ldots+x_na_n=b.$$

By 20 this one equation replaces or transforms the set originally given us. Now in order to find x_1, multiply the last equation by $[a_2a_3\ldots\ldots a_n]$. This gives (52, 43)

$$x_1[a_1a_2\ldots\ldots a_n]=[ba_2a_3\ldots\ldots a_n]$$
$$\text{or, } x=\frac{[ba_2a_3\ldots\ldots a_n]}{[a_1\,a_2\ldots\ldots a_n]}$$

Replacing the a's and b by their values, by 45 we have the usual expression for x_1 in terms of the coefficients.

CHAPTER XI.

APPLICATION TO TRIGONOMETRY.

147. Definition.—The angle between two quantities is that angle ($<\pi$) whose cosine equals the inner product of the two quantities divided by the product of their scalar coefficients. Thus

$$\cos\angle ab=[a\mid b]\div\alpha\beta$$

where a and b are two quantities and α and β are their numerical values (123).

Again, if a, b, $c\ldots.$ are quantities of the first order, α, β, $\gamma\ldots.$ are their respective numerical values. $\sin(a\ b\ c\ldots.)$ is that numerical quantity which equals

$$\frac{[a\ b\ c\ldots.]}{\alpha\beta\gamma\ \ldots}$$

and is not negative. Thus (123) $\sin^2(abc)=\dfrac{[a\ b\ c\ldots.]^2}{\alpha^2\beta^2\gamma^3\ldots.}$.

148. *If a and b are quantities of the first order*, $\sin(ab)=\sin\angle ab$.

Proof.— $\sin^2(ab)=\dfrac{[a\,b]^2}{\alpha^2\beta^2}=\dfrac{a^2b^2-[a\mid b]^2}{\alpha^2\beta^2}$ (144)

$$=\frac{\alpha^2\beta^2-[a\mid b]^2}{\alpha^2\beta^2}\ (123)=1-\frac{[a\mid b]^2}{\alpha^2\beta^2}\ (123)$$

$$=1-\cos^2\angle ab=\sin^2\angle ab\ (147).$$

Then if $\sin(ab)$ is never negative and $\angle ab<\pi$, $\sin(ab)=\sin\angle ab$.

149. *If* α, β, γ, δ *are the numerical values of* a, b, c, d, *by* 147 *and* 148,

$$[ab\mid cd]=\alpha\beta\gamma\delta\sin\angle ab\sin\angle cd\cos\angle(ab.ed).$$

150. *The normal projection of* A *on a quantity* B *of the same order is numerically equal to* $A\cos\angle AB$.

Proof.—If A' is the normal projection of A on B (134)

$$A'=\frac{[A\mid B]B}{\beta^2}=\frac{\alpha\beta\cos\angle AB.B}{\beta^2}\ (119,\ 144)$$

$$=\alpha\cos\angle AB\,.\,\frac{B}{\beta}=\text{(numerically) } A\cos\angle AB.$$

151. The two expressions $[a\mid b]$ and $[ab]$, where a and b are vectors, play a very important part in mathematics. They occur yoked together in quaternions and apart in the *Ausdehnungslehre*, typifying the two products, the inner and outer. Numerically, as we have just seen, $[a\mid b]$ is the *projection* of either vector on the other multiplied by the tensor of the other ; $[ab]$, on the other

hand, is the area of the parallelogram whose adjacent sides are a and b, or, when the tensor of one vector is unity, it is equal numerically to the *perpendicular* from the extremity of the other vector on the first, when they go out from the same origin, or, when both tensors are unity, it is equal numerically to the sine of the angle between the given vectors (148).

152. *If a, b, c, ... are normal to each other and k is any quantity numerically derived from them, we have*

$$\frac{k}{\varkappa}=\frac{a}{\alpha}\cos\angle ak+\frac{b}{\beta}\cos\angle bk+\ldots.$$

Proof.—Let $k=xa+yb+\ldots.$ Then to find x multiply each member by $|a$. There results, since $[b\,|\,a]=0$, etc., $[k\,|\,a]=x[a\,|\,a]$. Finding the value of $y\ldots.$ in the same way and substituting we get the equation as given above.

153. If a, b, $c\ldots.$ are normal to one another and k and l are two quantities numerically derivable from a, b, ...

$$\cos\angle kl=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

Proof.—From 147, we have

$$\cos\angle kl=\frac{[k\,|\,l]}{\varkappa\lambda}=\left[\frac{k}{\varkappa}\Big|\frac{l}{\lambda}\right]\equiv$$

$$[(\frac{a}{\alpha}\cos\angle ak+\frac{b}{\beta}\cos\angle bk+\ldots.)\,|\,(\frac{a}{\alpha}\cos\angle al+\frac{b}{\beta}\cos\angle bl+\ldots.)]\ (152)$$

$$=\frac{a^2}{\alpha^2}\cos\angle ak\cos\angle al+\frac{b^2}{\beta^2}\cos\angle bk\cos\angle bl+\ldots.$$

$\therefore\ \cos\angle kl=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$

154. If a, b, c, ... are normal to each other and k is numerically derivable from them, we have by putting $l=k$ in 153

$$1=\cos^2\angle ak+\cos^2\angle bk+\ldots.$$

155. If a, b, c, ... are normal to each other, and k and l two quantities numerically derivable from them are normal to each other, 153 gives

$$0=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

156. Writing in the formula of 144 a for p_1, b for p_2, c for q_1, d for q_2 gives

$$\sin\angle ab\sin\angle cd\cos\angle(ab.cd)=\cos\angle ac\cos\angle bd-\cos\angle bc\cos\angle ad.$$

In this formula if c and d are replaced respectively by a and c there results

$$\sin\angle ab \sin\angle ac \cos\angle(ab.ac)=\cos\angle bc-\cos\angle ba \cos\angle ac,$$

a familiar formula of spherical trigonometry.

157. The last formula of 145, by substituting a, b, c for p_1, p_2, p_3, and a, b, c for q_1, q_2, q_3 gives (147)

$$\sin^2(abc)=1-\cos^2\angle bc-\cos^2\angle ca-\cos^2\angle ab+2\cos\angle bc \cos\angle ca \cos\angle ab.$$

158. *The formula* $(a+b)^2=a^2+2[a\mid b]+b^2=\alpha^2+2\alpha\beta\cos\angle ab+b^2$ gives the familiar extension of the Pythagorean proposition.

159. Let a, b, c be plane segments whose sum is the fourth face of a tetraedron of which they are the other three (75). Then

$$(a+b+c)^2=a^2+b^2+c^2+2[b\mid c]+2[c\mid a]+2[a\mid b]$$
$$=\alpha^2+\beta^2+\gamma^2+2\beta\gamma\cos\angle bc+2\alpha\gamma\cos\angle ca+2\alpha\beta\cos\angle ab,$$

which is the extension of the preceding result to space.

In words:—The square of the base of any tetraedron is equal to the sum of the squares of the lateral faces dimished by twice the products of each pair of lateral faces times the cosine of the diedral angle between them.

CHAPTER XII.

APPLICATION TO ANALYTIC GEOMETRY.

160. Let p_1, p_2, p_3 represent three unit points, and suppose their product is unity (57). Then

$$[p_1p_2p_3]=[p_1(p_2p_3+p_3p_1+p_1p_2)]=1. \ (43)$$

But if p denote any other unit point in the plane of $[p_1p_2p_3]$, by 94 we may replace p_1 by p in this product, getting

$$[p(p_2p_3+p_3p_1+p_1p_2)]=[p\mid(p_1+p_2+p_3)]=1. \ (57,\ 58)$$

Let p_s denote a unit point which is the mean of the reference points. Then $p_1+p_2+p_3=3p_s$ (81). Substituting this value of $p_1+p_2+p_3$ in the equation above, we have

$$3[p\mid p_s]=1, \text{ or in solid space, } 4[p\mid p_s]=1.$$

161. *The equation* $p=xp_1+yp_2+zp_3$ *represents a straight line, provided* x, y, z *satisfy a linear equation, as* $ax+by+cz=0$.

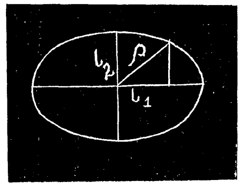
To see this let us eliminate z. Then

$$p=\frac{1}{c}\{x(cp_1-ap_3)+y(cp_2-bp_3)\}.$$

Thus, by 80, p lies on the right line through cp_1-ap_3 and cp_2-bp_3.

162. The equation $[p\,p_1p_2]=0$, in which p_1 and p_2 are constants and p is a variable is the equation of a straight line (94).

The equation $[pL]=0$, *where* p *is a point and* L *a line is the point equation of a straight line if* p *is variable and* L *is constant, and the line equation of the point* p *if* L *is variable and* p *is constant.*



163. *The Cartesian equations of the central conics, the ellipse and the hyperbola in the inner product notation* (151) *are*

$$\left(\frac{\rho\,|\,\iota_1}{a}\right)^2\pm\left(\frac{\rho\,|\,\iota_2}{b}\right)^2=1,$$

where ι_1 and ι_2 are unit vectors along the major and minor axes and ρ is the radius vector from the center to any point. Suppose we set

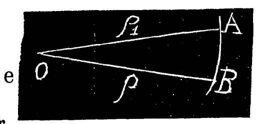
$$\frac{\rho\,|\,\iota_1}{a^2}\iota_1\pm\frac{\rho\,|\,\iota_2}{b^2}\iota_2\equiv\phi\rho.$$

Then the equation for the central conics reduces to

$$\rho\,|\,\phi\rho=1.$$

164. Differentiation.—Let ρ be a radius vector from an origin O to a curve AB. Then if ρ be made to approach indefinitely close to ρ_1, we have

$$\text{Limit}\frac{\rho-\rho_1}{AB}=\frac{d\rho}{ds}=a\ \textit{unit vector}$$



in the direction of the tangent at A. This is taken to be the meaning of $\frac{d\rho}{ds}$ no matter whether ρ be a *vector* from the origin O, or a *point* moving from B to A on the curve AB.

165. The function ϕ (163) possesses the property that $\rho\,|\,\phi\rho_1=\rho_1\,|\,\phi\rho$. Thus

$$\rho\,|\,\phi\rho_1\equiv\rho\,|\left(\frac{\rho_1\,|\,\iota_1}{a^2}\iota_1\pm\frac{\rho_1\,|\,\iota_2}{b^2}i_2\right)=\rho_1\,|\left(\frac{\rho\,|\,\iota_1}{a^2}\iota_1\pm\frac{\rho\,|\,\iota_2}{b^2}\iota_2\right).$$

166. Differentiating the equation $\rho\,|\,\phi\rho=1$ (163), we get

$$d\rho\,|\,\phi\rho+\rho\,|\,\phi d\rho=2d\rho\,|\,\phi\rho=0.\ (165)$$

Now if $d\rho$ is parallel to the tangent at the extremity of ρ, $\phi\rho$ is parallel to the normal (124).

If ρ_t and ρ_n be vectors to any point of the tangent and normal, respectively, and ρ_1 that to the point of contact, the equation of the tangent may be written $[(\rho_t-\rho_1)\,|\,\phi\rho_1]=0$, or $[\rho_t\,|\,\phi\rho_1]=1$, and that of the normal $[(\rho_n-\rho_1)\phi\rho_1]=0$.

The p's may also be thought of as representing points.

167. Let p_1, p_2, p_3 denote the vertices of a reference triangle whose sides are of unit length and p any point in their plane. Then $|p_1=p_2p_3$, $|p_2=p_3p_1$, $|p_3=p_1p_2$, and $p|p_1$, $p|p_2$, $p|p_3$ are proportional to the perpendiculars from p on the several sides of the triangle (94).

We shall consider only homogeneous equations. For, if any equation should not be homogeneous in p, all that is necessary to make it such is to introduce the factor $1=\Sigma p|p_s$ (160). Now the most general form of the equation of the second degree in trilinear coördinates is

$$a[p|p_1]^2+b[p|p_2]^2+c[p|p_3]^2+2d[p|p_2][p|p_3]$$
$$+2e[p|p_3][p|p_1]+2f[p|p_1][p|p_2]=0.$$

$$\text{Let } [(ap_1+fp_2+ep_3)p|p_1]+[(fp_1+bp_2+dp_3)p|p_2]$$
$$+[(ep_1+dp_2+cp_3)p|p_3]\equiv\phi p.$$

When this value of ϕp is substituted in the preceding equation it reduces to $p|\phi p=0$. Hence $p|\phi p=0$ is the equation for all quadric curves whether central or non-central. Had quadriplaner coördinates been employed and the corresponding expressions constructed, an equation would have resulted representing any and all quadric surfaces. The same method may be used in getting the equation of the quadric in n-dimensional space.

Remark.—The introduction of the ϕ function from Hamilton into the *Ausdehnungslehre* is due to Professor Hyde. (*Directional Calculus*, page 103). He shows that point analysis gives a means of changing the ordinary Cartesian equations into equations analogous to those of trilinear coördinates and then of generalizing the application of the equation $p|\phi p=0$ to include the case of quadrics, central and non-central.

[To be Concluded.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

132. Proposed by **WILLIAM SYMMONDS, A.M.**, Professor of Mathematics, Santa Rosa College, Sebastopol, Cal.

A road 60 feet wide crosses a square acre of land. The west line of the road passes through the southwest corner of the land, while the east line of the former passes through the northeast corner of the latter. What fraction of the land is included in the road?