$$\text{Also} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = F(e,\ \tfrac{1}{2}\pi) \ldots\ldots \qquad (1).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\,d\theta = E(e,\ \tfrac{1}{2}\pi) \ldots\ldots \qquad (2).$$

Integrating both sides of (B) and (11_0) between the limits 2π and 0, we get with the aid of (1_0), (2_0), (1); (2), the following:

$$\pi A_0 = \int_0^{2\pi} \frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = 4F(e,\ \tfrac{1}{2}\pi).$$

$$\therefore\ A_0 = (4/\pi)F(e,\ \tfrac{1}{2}\pi) \ldots\ldots \qquad (12_0).$$

$$\pi A_1 = \int_0^{2\pi} \frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4e\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{1-e^2\sin^2\theta}} + \int_0^{2\pi} \cos\theta d\theta$$

$$= \frac{4}{e}\left[\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} - \int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\,d\theta\right] = \frac{4}{e}[F(e,\ \tfrac{1}{2}\pi) - E(e,\ \tfrac{1}{2}\pi)].$$

$$\therefore\ A_1 = (4/\pi e)[F(e,\ \tfrac{1}{2}\pi) - E(e,\ \tfrac{1}{2}\pi)].$$

[To be Continued.]

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION."

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[Continued from October Number.]

CHAPTER III.

MULTIPLICATION OF EXTENSIVE QUANTITIES. DIFFERENT KINDS OF MULTIPLICATION.

22. In the multiplication of extensive quantities expressed in terms of *units*, it is assumed that the distributive law holds, and that numerical coefficients may be treated as in elementary algebra (16).

Thus if $a = \Sigma\alpha_r e_r$ and $b = \Sigma\beta_s e_s$ are two extensive quantities in which α and β are numbers and the e's are extensive *units*, we may write

$$ab = [\Sigma\alpha_r e_r . \Sigma\beta_s e_s] = \Sigma\alpha_r\beta_s[e_r e_s],$$

that is to say, in the result each term of the multiplier is multiplied into every term of the multiplicand, and the partial products are added.

Notice that the law is assumed to hold only when the two factors are sums of *units*. In the theorems which follow it is shown that the same law applies when the factors are sums of *quantities*.

23. Before going on to the proofs of these theorems, we will illustrate the question involved by an example. (See Art. 9).

Let

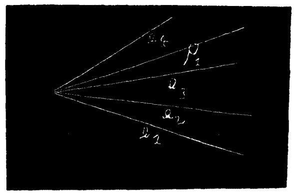
$$\rho_1 = r_{11}e_1 + r_{12}e_2 + r_{13}e_3 + \ldots\ldots$$

$$\rho_2 = r_{21}e_1 + r_{22}e_2 + r_{23}e_3 + \ldots\ldots$$

Also, let

$$a = \alpha_1\rho_1 + \alpha_2\rho_2 + \ldots\ldots$$

$$b = \beta_1\rho_1 + \beta_2\rho_2 + \ldots\ldots$$



It is to be shown, then, that if $\rho_1\rho_2 = \Sigma r_{1r}r_{2s}[e_r e_s]$, $ab = \Sigma\alpha_r\beta_s[\rho_r\rho_s]$.

The proof will be based on the definitions laid down and the theorems proved in the last chapter.

Remark.—It should be kept in mind in Articles 24—30 that α, β, ... denote numbers, the e's denote extensive *units* (11), and a, b, ... denote extensive *quantities* (12). Square brackets are used to indicate that the quantities inside are extensive quantities whose product is required.

24. *To show that* $[\Sigma\alpha_r e_r . b] = \Sigma\alpha_r[e_r b]$, *i. e.* to show that in multiplying $\Sigma\alpha_r e_r$ by b each term of $\Sigma\alpha_r e_r$ is multiplied by b.

Proof.—Let $b = \Sigma\beta_s e_s$. Then

$$[\Sigma\alpha_r e_r . b] \equiv [\Sigma\alpha_r e_r . \Sigma\beta_s e_s] = \Sigma\alpha_r\beta_s[e_r e_s] \quad (22)$$

$$= \Sigma\alpha_1\beta_s[e_1 e_s] + \Sigma\alpha_2\beta_s[e_2 e_s] + \ldots\ldots \quad (14)$$

$$= \alpha_1\Sigma\beta_s[e_1 e_s] + \alpha_2\Sigma\beta_s[e_2 e_s] + \ldots\ldots \quad (15)$$

$$=\alpha_1[e_1.\Sigma\beta_s e_s]+\alpha_2[e_2.\Sigma\beta_s e_s]+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

$$=\alpha_1[e_1.b]+\alpha_2[e_2.b]+\ldots\ldots=\Sigma\alpha_r[e_r b].$$

25. *To show that* $[(a+b+\ldots\ldots)p]=[ap]+[bp]+\ldots\ldots$

$$[p(a+b+\ldots\ldots)]=[pa]+[pb]+\ldots\ldots$$

PROOF.—Let $a=\Sigma\alpha_r e_r$, $b=\Sigma\beta_r e_r\ldots\ldots$ Then

$$[(a+b+\ldots\ldots)p]=[(\Sigma\alpha_r e_r+\Sigma\beta_r e_r+\ldots\ldots)p]=[\Sigma(\alpha_r+\beta_r+\ldots\ldots)e_r.p]\ (14)$$

$$=\Sigma(\alpha_r+\beta_r+\ldots\ldots)[e_r p]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

$$=\Sigma[\alpha_r e_r.p]+\Sigma[\beta_r e_r.p]+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14,\ 24)$$

$$[ap]+[bp]+\ldots\ldots$$

26. *To show that* $[(\alpha a)b]=\alpha[ab]$, and $[b(\alpha a)]=\alpha[ba]$.

PROOF.—Let $a=\Sigma\alpha_r e_r$. Then

$$[(\alpha a)b]\equiv[(\alpha\Sigma\alpha_r e_r)b]=[\Sigma\alpha\alpha_r e_r)b]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

$$=\Sigma\alpha\alpha_r[e_r b]\ (24)=\alpha[\Sigma\alpha_r e_r.b]\ (16,\ 24)=\alpha[ab].$$

The other formula is obtained by making b the first factor in the above proof.

27. *To show that* $[(\alpha a+\beta b+\ldots\ldots)p]=\alpha[ap]+\beta[bp]+\ldots\ldots$

and $[p(\alpha a+\beta b+\ldots\ldots)]=\alpha[pa]+\beta[pb]+\ldots\ldots$

PROOF.—$[(\alpha a+\beta b+\ldots\ldots)p]=[(\alpha a)p]+[(\beta b)p]+\ \ldots\ldots\ldots\ldots\ldots(25)$

$$=\alpha[ap]+\beta[bp]+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

28. We are now in position to show that the distributive law holds when the two factors are sums of multiples of extensive *quantities* as well as when they are sums of multiples of units.

PROOF.—$[\Sigma\alpha_r a_r.\Sigma\beta_s b_s]=\Sigma\alpha_r[a_r.\Sigma\beta_s b_s]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$

$$=\Sigma\alpha_r(\Sigma\beta_s[a_r b_s])\ (27)=\Sigma\alpha_r\beta_s[a_r b_s]\ \ldots\ldots\ldots(16)$$

This theorem holds also for any number of factors, as can be shown by mathematical induction.

29. Let us denote a product containing a number of factors, a, b, $\ldots\ldots$

by $P_{a,\,b\ldots\ldots}$ In such a product suppose a factor p equals $qa+rb+\ldots\ldots$ where q, r, ... are numbers; to show that

$$P_{qa+rb+\ldots\ldots}=q.P_{a}+r.P_{b}+\ldots\ldots$$

PROOF.—However the product may be made up, we can always regard p as combined with another factor, then this product with other factors in turn. In each of the multiplications in which p enters as one factor Art. 27 applies.

30. *To show that* $P_{qa,\,rb,\,sc,\ldots\ldots}=qrs\ldots\ldots P_{a,\,b,\,c\ldots\ldots}$

PROOF.—By 29, $P_{qa}=q.P_{a}$. Then

$$P_{qa,\,rb,\,sc\ldots\ldots}=q.P_{a,\,rb,\,sc\ldots\ldots}=qrs\ldots\ldots\ P_{a,\,b,\,c\ldots\ldots}$$

It evidently follows that $P_{qa,\,ra}=P_{ra,\,qa}$.

31. *Different Kinds of Multiplication.*—Different kinds of multiplication are obtained by laying down different laws for simplifying a distributed product.

To illustrate:—

(1) $(\alpha_1 e_1+\alpha_2 e_2)(\beta_1 e_1+\beta_2 e_2)=\alpha_1\beta_1 e_1^2+\alpha_1\beta_2 e_1 e_2+\alpha_2\beta_1 e_2 e_1+\alpha_2\beta_2 e_2^2$.

In ordinary algebra the result is simplified by supposing $e_1 e_2=e_2 e_1$. The result in this way becomes

$$\alpha_1\beta_1 e_1^2+(\alpha_1\beta_2+\alpha_2\beta_1)e_1 e_2+\alpha_2\beta_2 e_2^2,$$

or, say, $m_1 e_1^2+m_2 e_1 e_2++m_3 e_2^2$, where the m's are numerical coefficients.

(2) Similarly, writing three factors, we get,

$$(\alpha_1 e_1+\alpha_2 e_2)(\beta_1 e_1+\beta_2 e_2)(\gamma_1 e_1+\gamma_2 e_2)=m_1 e_1^3+m_2 e_1^2 e_2+m_3 e_1 e_2^2+m_4 e_2^3$$

by supposing, as in ordinary algebra, that

$$\begin{cases} e_1 e_1 e_2=e_1 e_2 e_1=e_2 e_1 e_1 \\ e_1 e_2 e_2=e_2 e_1 e_2=e_2 e_2 e_1 \end{cases}$$

Here the assumed law of simplification reduces a product which would otherwise have eight terms to one of four.

(3) The law of simplification in quaternions (which is a branch of mathematics using a certain kind of extensive quantities) may be seen by multiplying together two factors of three terms each.

Thus, $(\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3)(\beta_1 e_1+\beta_2 e_2+\beta_3 e_3)=$

$$-(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)+(\alpha_1\beta_2-\alpha_2\beta_1)e_3+(\alpha_2\beta_3-\alpha_3\beta_2)e_1+(\alpha_3\beta_1-\alpha_1\beta_3)e_2,$$

by supposing $e_1^2=e_2^2=e_3^2=-1$, $e_1 e_2=e_3$, $e_2 e_3=e_1$, and $e_3 e_1=e_2$.

32. Definition.—A multiplication is said to be *linear* when the same laws of simplication of the distributed product continue to hold when numerically derived quantities (10) replace the given units.

33. *To show that there are but four kinds of linear multiplication.*

Let

$$(a) \qquad \Sigma\alpha_{rs}[e_r e_s]=0$$

express a simplification law in the product $\Sigma\alpha_r\beta_s[e_r e_s]$.

Let, now, e_r be replaced by $\Sigma x_{r,u}\, e_u$ and e_s by $\Sigma x_{sv}e_v$. We thus get

$$\Sigma\alpha_{rs}[\Sigma x_{ru}e_u.\Sigma x_{sv}e_v]=0,$$

$$\text{whence, } \Sigma\alpha_{rs}\Sigma x_{ru}x_{sv}[e_u e_v]=0, \text{ (28)},$$

$$\text{or, } \Sigma\alpha_{rs}x_{ru}x_{sv}[e_u e_v]=0. \text{ (16)}$$

This equation is symmetrical in r and s and u and v and evidently will continue to hold true when these letters are interchanged. This gives

$$\Sigma\alpha_{sr}x_{sv}x_{ru}[e_v e_u]=0.$$

Adding the last two equations, we have

$$(b) \qquad \Sigma x_{ru}x_{sv}\{\alpha_{rs}[e_u e_v]+\alpha_{sr}[e_v e_u]\}=0.$$

Equation (*b*) may also be gotten by multiplying $\Sigma\alpha_{rs}x_{ru}x_{sv}[e_u e_v]$ by 2 and arranging the result with reference to equal coefficients, $(x_{rv}x_{sv})$. This derivation shows (*b*) to be a necessary, and not, as might appear, an arbitrary inference from the given equation.

Now from the nature of the case the coefficients x_{ru}, x_{sv} must be capable of having any values, as would the x's in Articles 6—9. If we assume that the *products* $x_{ru}x_{sv}$ are arbitrary, then from the theory of equations we have

$$(c) \qquad \alpha_{rs}[e_u e_v]+\alpha_{sr}[e_v e_u]=0,$$

true for all values of r and s and u and v.*

If we put $u=v$ in (*c*) we get

$$(d) \qquad (\alpha_{rs}+\alpha_{sr})[e_u e_u]=0.$$

This equation is satisfied either by assuming (1) $\alpha_{rs}+\alpha_{sr}=0$, *i. e.*, $\alpha_{rs}=-\alpha_{sr}$; or, by assuming (2), $[e_u e_v]=0$.

***Grassmann's derivation of (*c*) does not assume that the products $x_{ru}x_{sv}$ are arbitrary. The writer gave another demonstration which does not assume this before the Mathematical Section of the American Association for the Advancement of Science, 1899 meeting. Though somewhat simpler than Grassmann's proof, it would add materially to the length of this article.**

(1) If $\alpha_{rs}=-\alpha_{sr}$, and we make this substitution in (c), there results

$$\alpha_{rs}\{[e_u e_v]-[e_v e_u]\}=0.$$

In this equation either $\alpha_{rs}=0$, or $[e_u e_v]=[e_v e_u]$. If $\alpha_{rs}=0$, all the coefficients reduce to zero, and equation (a) vanishes identically, which is contrary to hypothesis. If

$$(e) \qquad [e_u e_v]-[e_v e_u]=0, \text{ or } [e_u e_v]=[e_v e_u],$$

we have the law for a form of multiplication of extensive quantities which is analogous to ordinary multiplication in algebra. See Art. 31, (1).

(2) If we say $[e_u e_u]=0$, it is equivalent to making in equation (a) $\alpha_{rr}=1$, and all the other coefficients equal to 0. Making this substitution in (c), we get

$$(f) \qquad [e_u e_v]+[e_v e_u]=0,$$

which implies $[e_u e_u]=0$, as may be seen by making $u=v$ in (f).

We have seen that equations (e) and (f) are necessary conditions in order that a multiplication may be linear. That they are sufficient conditions may be seen as follows: If we start with

$$(a) \qquad [e_r e_s]\pm[e_s e_r]=0,$$

and substitute as above we get

$$(c) \qquad [e_r e_s]\pm[e_s e_r]=0.$$

Hence, by definition, (32) (e) and (f) give linear multiplications.

We have then four kinds of linear multiplication, viz:

1st. That in which there are no simplifying equations.

2nd. That in which all the coefficients in (a) are identically 0.

3d. That whose law of simplification is $[e_u e_v]=[e_v e_u]$.

4th. That whose law of simplification is $[e_u e_v]=-[e_v e_u]$.

As between (e) and (f), the latter gives the simpler species of multiplication. To see this let us take the distributed product in (1) Art. 31. Equation (e) reduces the product, as we saw in that Article to three terms. But taking (f) as the simplification law, we get a single term, viz., $(\alpha_1\beta_2-\alpha_2\beta_1)e_1e_2$.

34. The *Ausdehnungslehre* concerns itself very largely with the operation of multiplication, especially with what is called *combination* multiplication. This multiplication is based on the law described in the last Article, viz., $e_r e_s=-e_s e_r$, which also implies $e_r\, e_r=0$.

[To be Continued.]