Illustration of interior point methods

Note the different notation used in these slides compared to the Lecture notes:

	Lecture notes	**These slides**
Optimization variables	x	w
Objective function	$f(x)$	$\Phi(w)$
Inequality constraint function	$g(x)$	$h(w)$
Linear constraint function	$h(x)$	$g(w)$

KKT conditions - Reminder

Consider the NLP problem:

$$\begin{aligned}\min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0\end{aligned}$$

KKT conditions with $\mathcal{L} = \Phi(\mathbf{w}) + \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{g}(\mathbf{w}) + \boldsymbol{\mu}^{\mathsf{T}} \mathbf{h}(\mathbf{w})$

$$\boxed{\begin{aligned}\text{Primal Feasibility:} \quad & \mathbf{g}(\mathbf{w}) = 0, \quad \mathbf{h}(\mathbf{w}) \leq 0, \\ \text{Dual Feasibility:} \quad & \nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0, \quad \boldsymbol{\mu} \geq 0, \\ \text{Complementarity Slackness:} \quad & \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0, \quad \forall\, i\end{aligned}}$$

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The difficulty of the KKT conditions is the non-smooth **Complementarity Slackness** conditions resulting form the inequality constraints.

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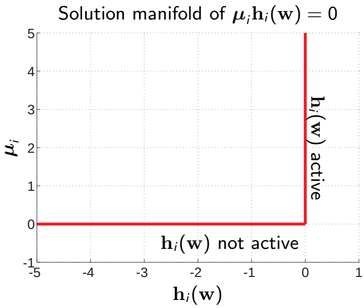

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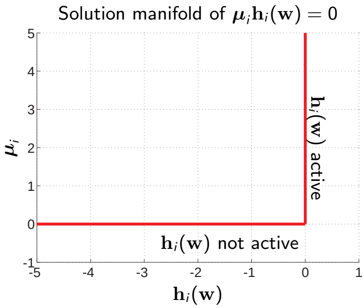

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The difficulty of the KKT conditions is the non-smooth **Complementarity Slackness** conditions resulting form the inequality constraints.

Key idea: get rid of the inequality constraints !!

Primal Interior-Point Methods

Log-barrier method: introduce the inequality constraints in the cost function

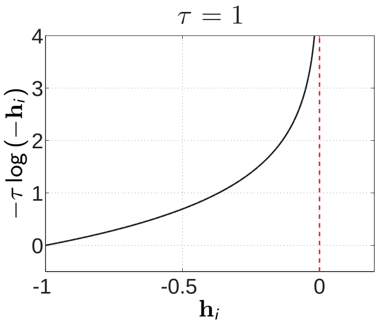
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w}) \qquad \text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

becomes

$$\min_{\mathbf{w}_\tau} \Phi_\tau\left(\mathbf{w}_\tau\right) = \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

Log-barrier approximates the characteristic function

$$\chi\left(\mathbf{h}_i\right) = \begin{cases} 0 & \text{if} \quad \mathbf{h}_i \leq 0 \\ \infty & \text{if} \quad \mathbf{h}_i > 0 \end{cases}$$



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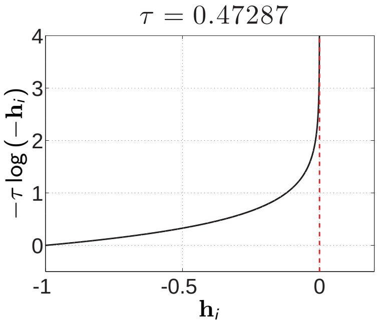

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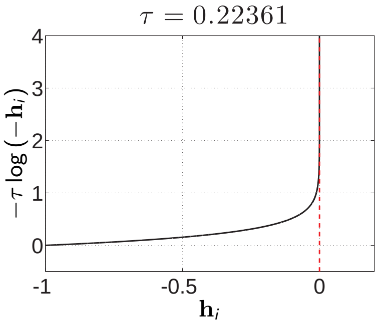

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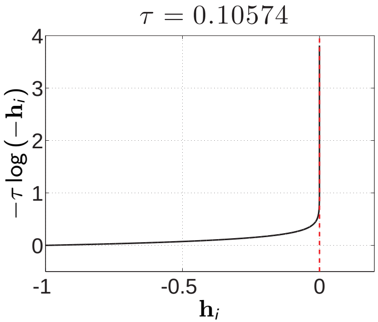

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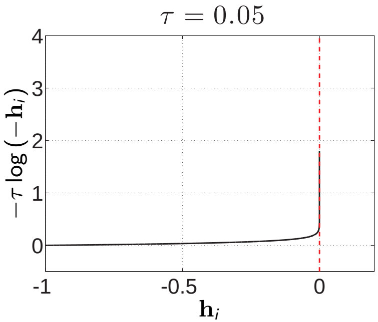

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Example:

$$\begin{array}{ll} \min\limits_{w} & \frac{1}{2}w^2 - 2w \\ \text{s.t.} & -1 \leq w \leq 1 \end{array}$$

I.e.

$$\Phi\left(w\right) = \frac{1}{2}w^2 - 2w$$

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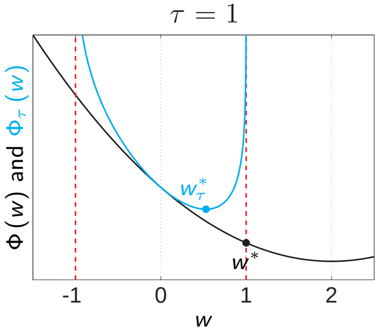

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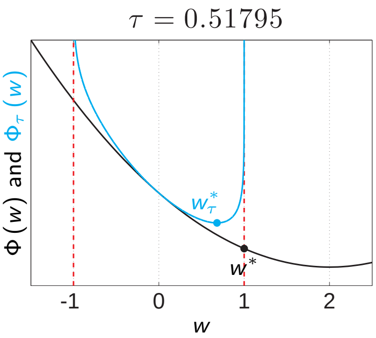

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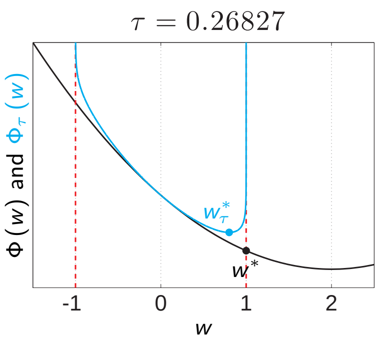

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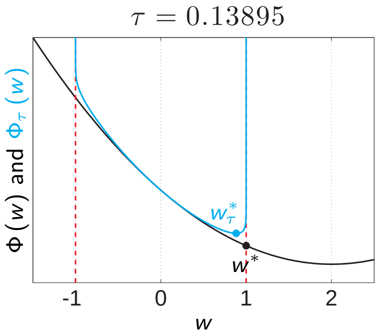

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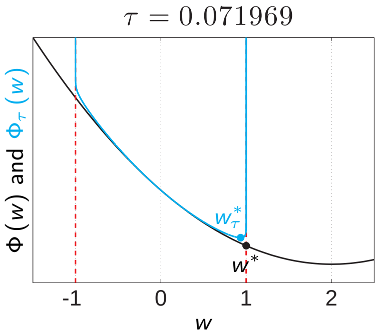

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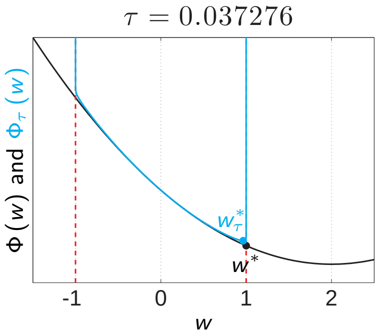

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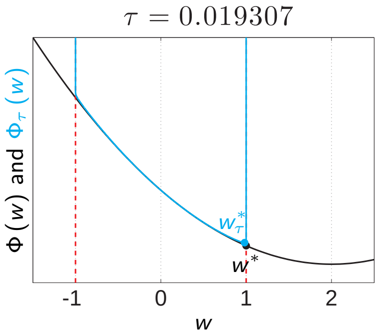

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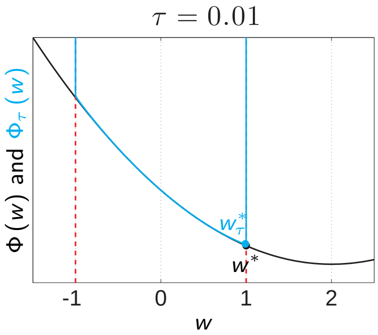

How accurate is the log-barrier approximation ?

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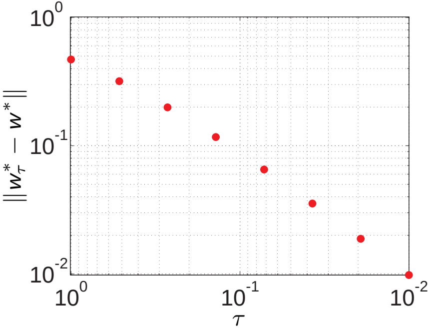

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If $\mathbf{w}^*$ is LICQ & SOSC:

$$\boxed{\|\mathbf{w}_\tau^* - \mathbf{w}^*\| = O(\tau)}$$

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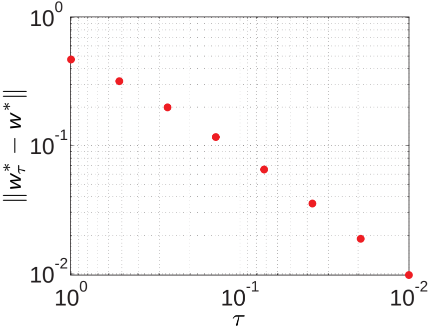

Newton on the Primal Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

Barrier formulation:

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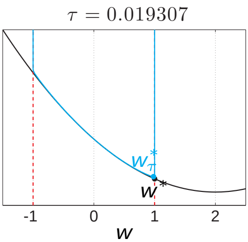

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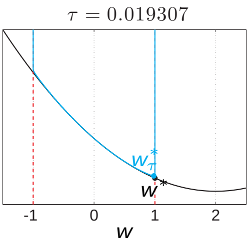

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$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \\ \boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) &= 0 \end{aligned}$$

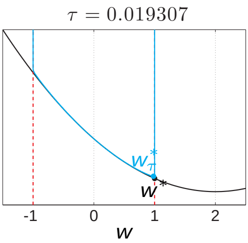
Barrier formulation:

$$\min_{\mathbf{w}} \; \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \; \Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$

KKT conditions*:

$$\nabla\Phi_\tau(\mathbf{w}) = \nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-1}\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$



Newton on the Primal Interior-Point method

Problem:

$$\begin{aligned}\min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0\end{aligned}$$

KKT conditions:

$$\begin{aligned}\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \\ \boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) &= 0\end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}} \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \Phi(\mathbf{w}) - \tau\sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$

KKT conditions*:

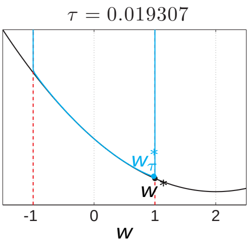
$$\nabla\Phi_\tau(\mathbf{w}) = \nabla\Phi(\mathbf{w}) - \tau\sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-1}\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Newton direction for the Primal Interior-Point KKTs:

$$\underbrace{\left(\nabla^2\Phi(\mathbf{w}) + \tau\sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-2}\nabla\mathbf{h}_i\nabla\mathbf{h}_i^\top\right)}_{=\nabla^2\Phi_\tau(\mathbf{w})}\Delta\mathbf{w} + \nabla\Phi_\tau(\mathbf{w}) = 0$$

for $\mathbf{h}$ affine



Newton on the Primal Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$

Barrier formulation:

$$\min_{\mathbf{w}} \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$

KKT conditions*:

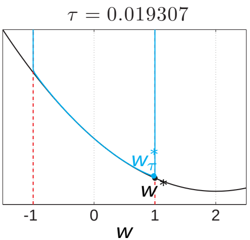
$$\nabla\Phi_\tau(\mathbf{w}) = \nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-1} \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Newton direction for the Primal Interior-Point KKTs:

$$\left(\nabla^2\Phi(\mathbf{w}) + \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-2} \nabla\mathbf{h}_i \nabla\mathbf{h}_i^\top\right) \Delta\mathbf{w} + \Phi_\tau(\mathbf{w}) = 0$$

for $\mathbf{h}$ affine



Newton on the Primal Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$

Barrier formulation:

$$\min_{\mathbf{w}} \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$

KKT conditions*:

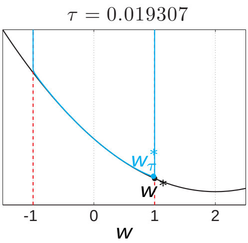
$$\nabla\Phi_\tau(\mathbf{w}) = \nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-1} \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Newton direction for the Primal Interior-Point KKTs:

$$\left(\nabla^2\Phi(\mathbf{w}) + \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-2} \nabla\mathbf{h}_i \nabla\mathbf{h}_i^\top\right) \Delta\mathbf{w} + \Phi_\tau(\mathbf{w}) = 0$$

for $\mathbf{h}$ affine



The term $\mathbf{h}_i^{-2}(\mathbf{w})$ reflects the "very strong curvature" of the problem when $\mathbf{h}_i$ tends to 0, which hinders the convergence

Primal-Dual Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Primal-Dual Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w}) \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Primal-Dual Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w})\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w})\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Define $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}(\mathbf{w})$

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w})\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Define $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}(\mathbf{w})$, then the Primal-Dual KKT conditions† read as:

$$\nabla\Phi(\mathbf{w}) + \sum_{i=1}^{m_i} \boldsymbol{\nu}_i \nabla\mathbf{h}_i(\mathbf{w}) = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) = -\tau$$

†valid for $\mathbf{h}_i(\mathbf{w}) < 0$, $\boldsymbol{\nu}_i > 0$

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau\sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau\sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w})\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Define $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}$, then the Primal-Dual Interior-Point KKT conditions read as:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w})\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Define $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}$, then the Primal-Dual Interior-Point KKT conditions read as:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

- Primal-Dual IP conditions yield the same solution as the Barrier problem
- Observe the similitude with the original KKT conditions !!

Interpretation of the Primal-Dual Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Primal-Dual IP KKT conditions

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} &= 0 \\ \boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau &= 0 \\ \mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} &> 0 \end{aligned}$$

Interpretation of the Primal-Dual Interior-Point method

Problem:

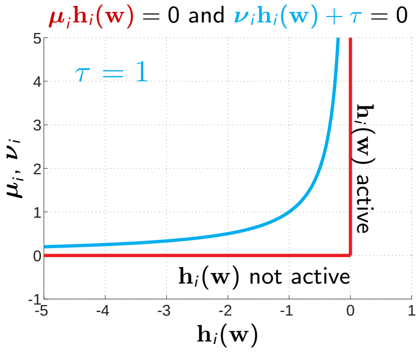
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

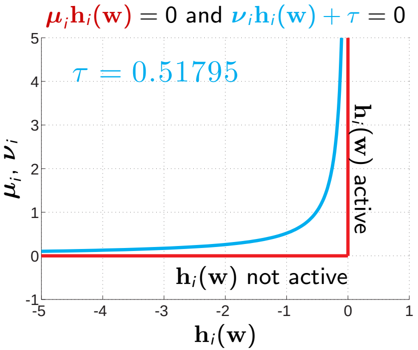
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

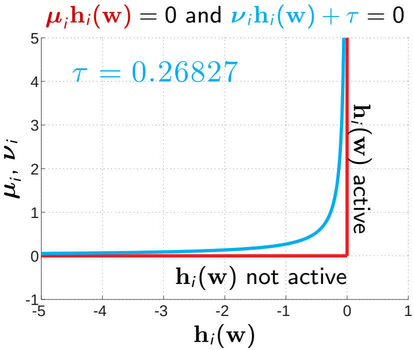
$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Primal-Dual IP KKT conditions

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} &= 0 \\ \boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau &= 0 \\ \mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} &> 0 \end{aligned}$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

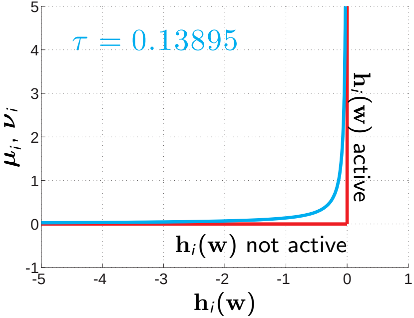
KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$ and $\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$



Interpretation of the Primal-Dual Interior-Point method

Problem:

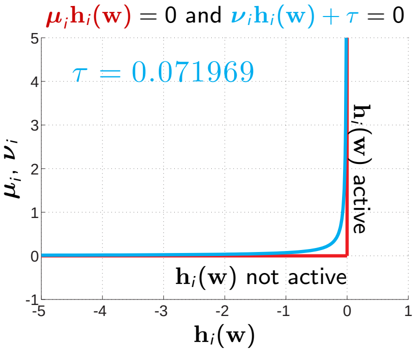
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

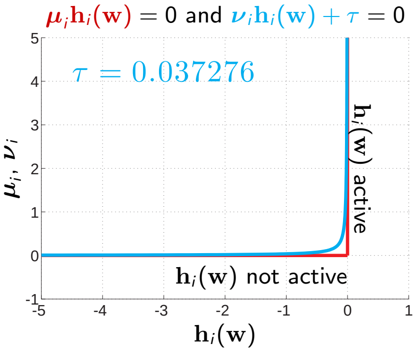
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

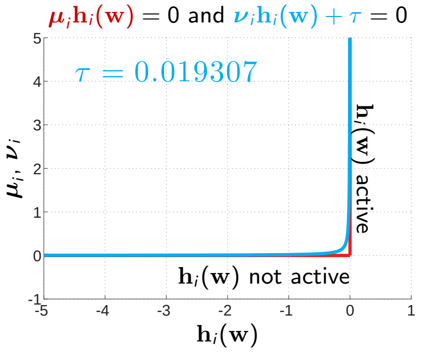
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

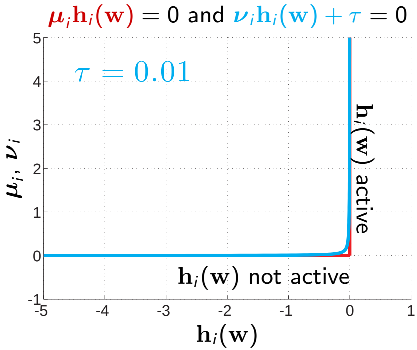
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



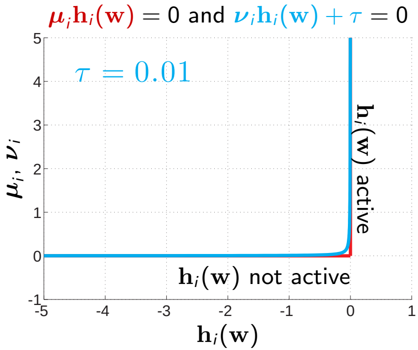
Interpretation of the Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$



Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

- Primal-Dual IP method solves KKT conditions with smoothed complementarity slackness

Interpretation of the Primal-Dual Interior-Point method

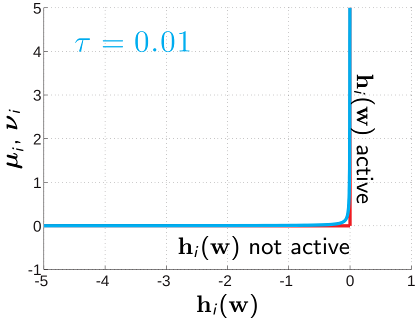
Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$ and $\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$



Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

- Primal-Dual IP method solves KKT conditions with smoothed complementarity slackness
- IP approximation

$$\|\boldsymbol{\mu}^* - \boldsymbol{\nu}^*\| = \mathcal{O}(\tau)$$
$$\|\mathbf{w}^* - \mathbf{w}^*_\tau\| = \mathcal{O}(\tau)$$

$\mathbf{w}^*_\tau$, $\boldsymbol{\nu}^*$ and $\mathbf{w}^*$, $\boldsymbol{\mu}^*$ are equivocated

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \tag{1a}$$

$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0 \tag{1b}$$

$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \tag{1c}$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Slack formulation of the Primal-Dual Interior-Point conditions

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- need to backtrack every Newton step $\Delta\mathbf{w}$ to ensure that

  $$\mathbf{h}(\mathbf{w} + t\Delta\mathbf{w}) < 0$$

  which is expensive if evaluating $\mathbf{h}$ is expensive !!

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Slack reformulation: define $-\mathbf{s} = \mathbf{h}(\mathbf{w})$

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$$\mathbf{h}(\mathbf{w}) + \mathbf{s} = 0$$

$$-\boldsymbol{\mu}_i s_i + \tau = 0$$

$$-\mathbf{s} < 0, \quad \boldsymbol{\mu} > 0$$

Slack formulation of the Primal-Dual Interior-Point conditions

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$$\boldsymbol{\mu}_i s_i - \tau = 0$$
$$\mathbf{s} > 0, \quad \boldsymbol{\mu} > 0$$

Newton on the slack formulation

- initialize with $\mathbf{s}$, $\boldsymbol{\mu} > 0$ and $\boldsymbol{\mu}_i s_i = \tau$

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Newton on the slack formulation

- initialize with $\mathbf{s}$, $\boldsymbol{\mu} > 0$ and $\boldsymbol{\mu}_i s_i = \tau$
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Newton on the slack formulation

- initialize with $\mathbf{s}$, $\boldsymbol{\mu} > 0$ and $\boldsymbol{\mu}_i s_i = \tau$
- $\mathbf{h}(\mathbf{w}) > 0$ does not matter...
- trivial backtracking enforces:
  $$\mathbf{s} + t\Delta\mathbf{s} > 0$$
  $$\boldsymbol{\mu} + t\Delta\boldsymbol{\mu} > 0$$
  for $t \in]0,\, 1]$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{g}(\mathbf{w})\boldsymbol{\lambda} + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{g}(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i s_i &= 0 \\ \mathbf{s} > 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

Newton on the Primal-Dual Interior-Point KKT conditions

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PD-IP KKT conditions

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{g}(\mathbf{w})\boldsymbol{\lambda} + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{g}(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i s_i - \tau &= 0 \\ \mathbf{s} > 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

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PD-IP KKT conditions

$$\begin{aligned} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 0 \\ \mathbf{g}(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i s_i - \tau &= 0 \\ \mathbf{s} > 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

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Solve the conditions

$$\begin{bmatrix} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

with $\mathbf{s} > 0, \quad \boldsymbol{\mu} > 0$

Newton on the Primal-Dual Interior-Point KKT conditions

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Newton direction $\mathbf{d}$ given by $\nabla \mathbf{r}_\tau^\top(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\, \mathbf{d} + \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$

Newton on the Primal-Dual Interior-Point KKT conditions

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$$\underbrace{\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix}}_{=\nabla \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})} \underbrace{\begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix}}_{=\mathbf{d}} = -\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})$$

with $H = \nabla^2 \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu})$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0$$

Solve the conditions

$$\begin{bmatrix} \nabla\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$$

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Newton direction $\mathbf{d}$ given by $\nabla\mathbf{r}_\tau^\top(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})\,\mathbf{d} + \mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$

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with $H = \nabla^2\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu})$

Observe the specific structure of the matrix $\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})$!!

Solving an NLP using the Primal-Dual Interior-Point method

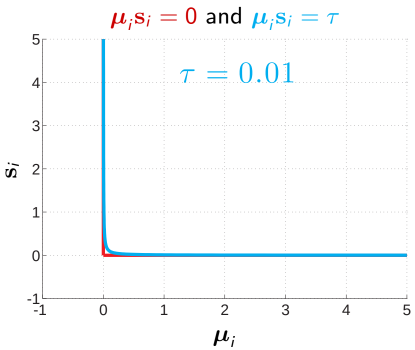
Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=\left[\begin{array}{c}\nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right)\\ \mathbf{g}\left(\mathbf{w}\right)\\ \mathbf{h}(\mathbf{w})+\mathbf{s}\\ \boldsymbol{\mu}_i s_i-\tau\end{array}\right]=0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top\mathbf{d}+\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=0$$



Solving an NLP using the Primal-Dual Interior-Point method

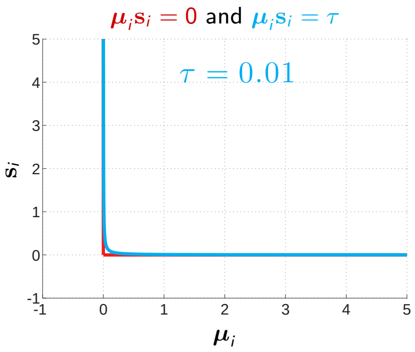
Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$



We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$ *for a very small* τ.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

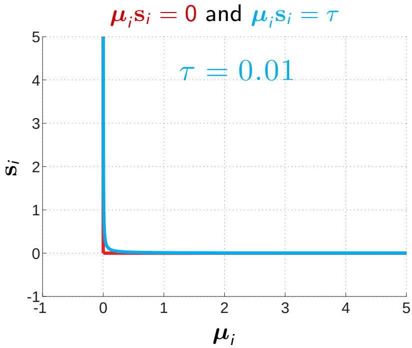
$$\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = \begin{bmatrix} \nabla\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})^\top \mathbf{d} + \mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$$

Reminder: Newton convergence depends on the Lipschitz constant of $\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})$, i.e. Newton does not "like" strong nonlinearities



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Solving an NLP using the Primal-Dual Interior-Point method

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$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

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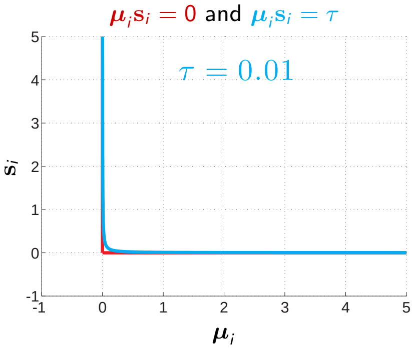

We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=0$ for a very small τ. But we do not want to get through the "corner" in $\boldsymbol{\mu}_i\mathbf{s}_i=\tau$ when τ is small.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

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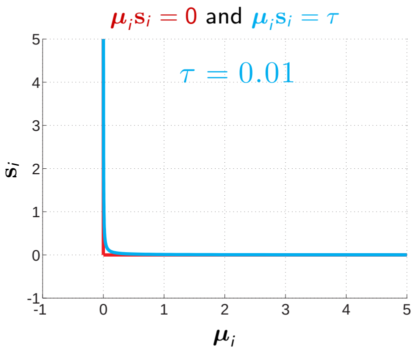

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Key idea: get the Newton iteration to "choose its side" at large τ, then reduce it.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

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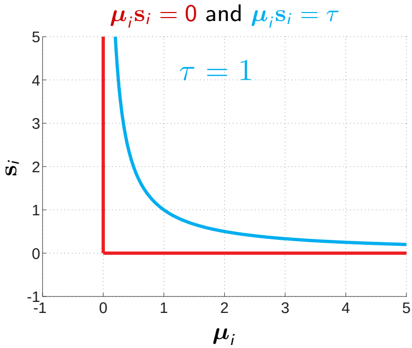

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Solving an NLP using the Primal-Dual Interior-Point method

Solve:

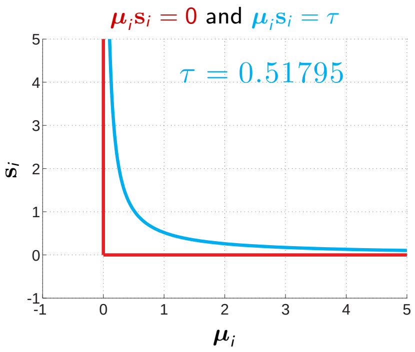
$$\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = \begin{bmatrix} \nabla\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})^\top \mathbf{d} + \mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$$

Reminder: Newton convergence depends on the Lipschitz constant of $\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})$, i.e. Newton does not "like" strong nonlinearities



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Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

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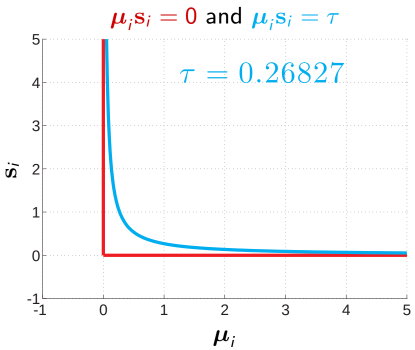
Reminder: Newton convergence depends on the Lipschitz constant of $\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)$, i.e. Newton does not "like" strong nonlinearities



We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$ for a very small τ. But we do not want to get through the "corner" in $\boldsymbol{\mu}_i\mathbf{s}_i = \tau$ when τ is small.

Key idea: get the Newton iteration to "choose its side" at large τ, then reduce it.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

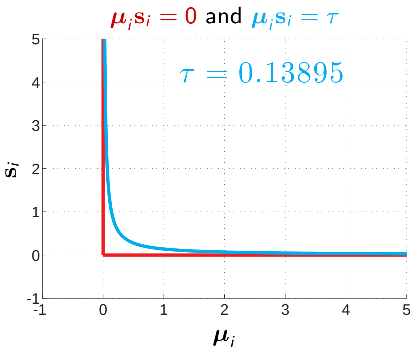
$$\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = \begin{bmatrix} \nabla\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})^\top \mathbf{d} + \mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$$

Reminder: Newton convergence depends on the Lipschitz constant of $\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})$, i.e. Newton does not "like" strong nonlinearities



We want to solve $\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$ *for a very small* τ. *But we do not want to get through the "corner" in* $\boldsymbol{\mu}_i\mathbf{s}_i = \tau$ *when* τ *is small.*

Key idea: get the Newton iteration to "choose its side" at large τ, then reduce it.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

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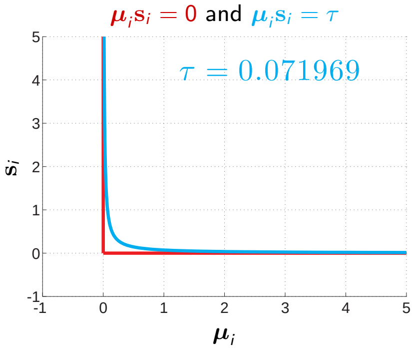

We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$ for a very small τ. But we do not want to get through the "corner" in $\boldsymbol{\mu}_i\mathbf{s}_i = \tau$ when τ is small.

Key idea: get the Newton iteration to "choose its side" at large τ, then reduce it.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=\begin{bmatrix}\nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right)\\ \mathbf{g}\left(\mathbf{w}\right)\\ \mathbf{h}(\mathbf{w})+\mathbf{s}\\ \boldsymbol{\mu}_i s_i-\tau\end{bmatrix}=0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top\mathbf{d}+\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=0$$

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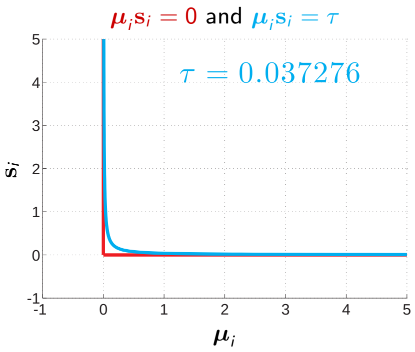

We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=0$ for a very small τ. But we do not want to get through the "corner" in $\boldsymbol{\mu}_i\mathbf{s}_i=\tau$ when τ is small.

Key idea: get the Newton iteration to "choose its side" at large τ, then reduce it.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

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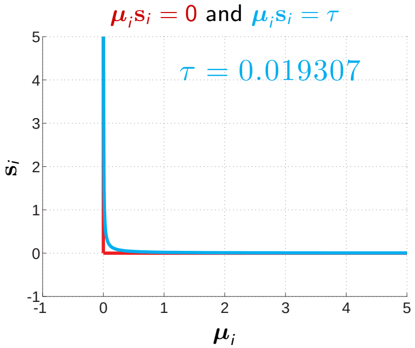

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Key idea: get the Newton iteration to "choose its side" at large τ, then reduce it.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

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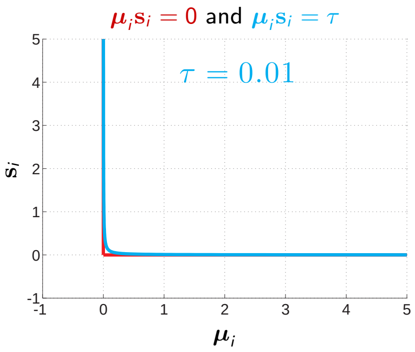

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Key idea: get the Newton iteration to "choose its side" at large τ, then reduce it.

Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
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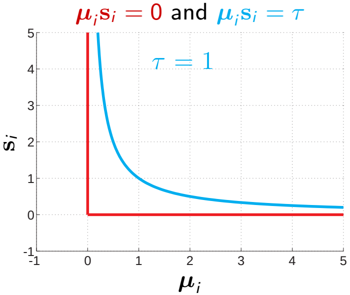
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

Algorithm: PD-IP solver

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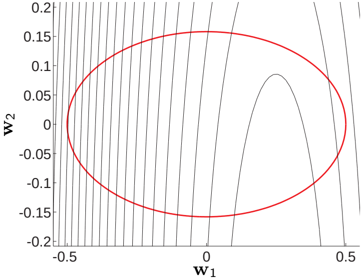
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w} - \mathbf{w}_0)^\top Q\,(\mathbf{w} - \mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

Algorithm: PD-IP solver

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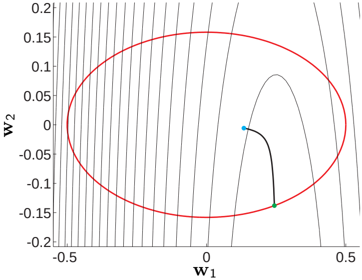


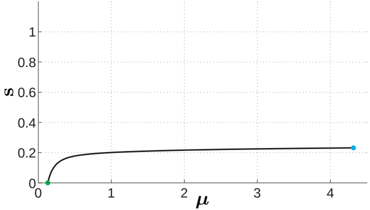
Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

Algorithm: PD-IP solver

Set $\tau,\ \boldsymbol{\mu},\ \mathbf{s} \leftarrow 1$
while $\tau > \text{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0) \\ \text{s.t.} \quad & \mathbf{w}^\top S\mathbf{w} \leq 1 \end{aligned}$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

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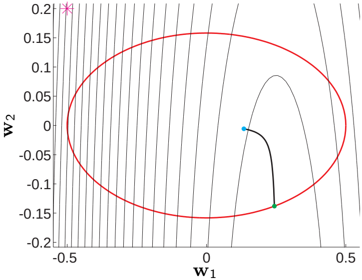
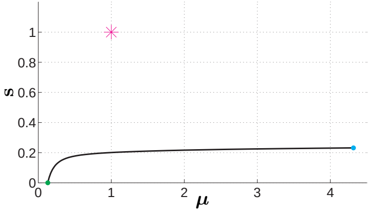


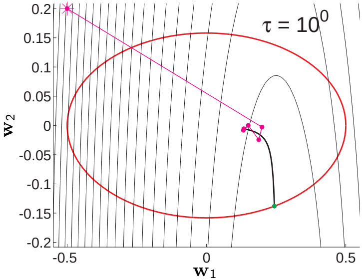


Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

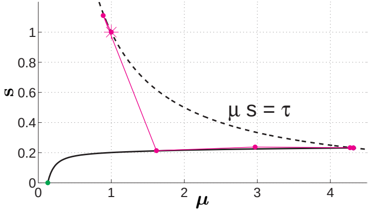
Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w} - \mathbf{w}_0\right)^\top Q\left(\mathbf{w} - \mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$



Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\, 0[$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

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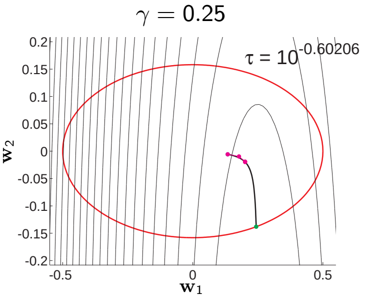
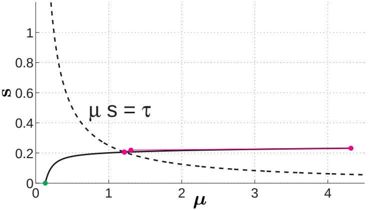


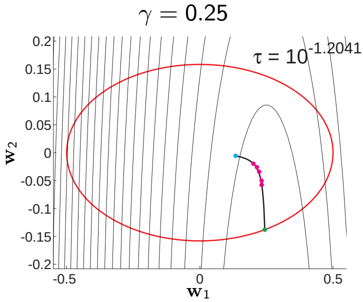


Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



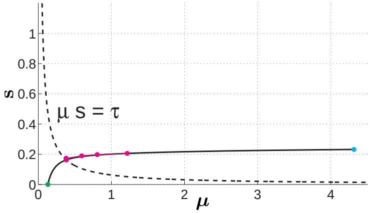
Example

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0) \\ \text{s.t.} \quad & \mathbf{w}^\top S\mathbf{w} \leq 1 \end{aligned}$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1, 0[$

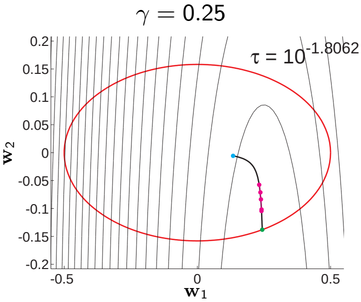


Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



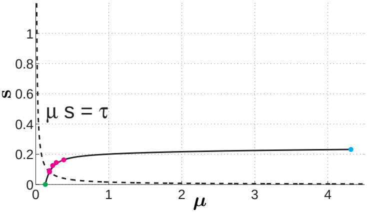
Example

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0) \\ \text{s.t.} \quad & \mathbf{w}^\top S \mathbf{w} \leq 1 \end{aligned}$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\, 0[$

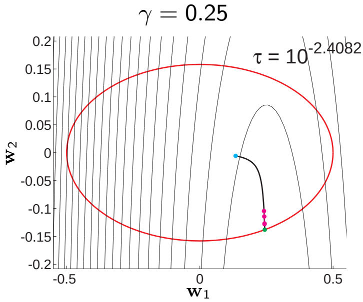


Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



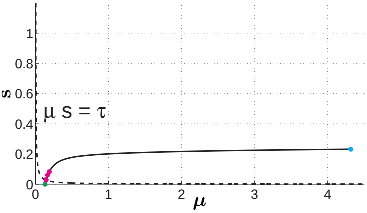
Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w} - \mathbf{w}_0)^\top Q\,(\mathbf{w} - \mathbf{w}_0)$$
$$\mathrm{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$

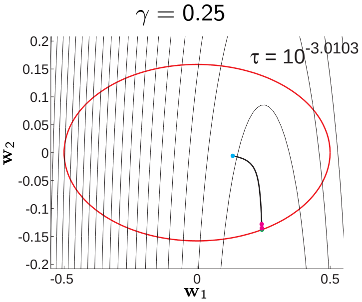


Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
    Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



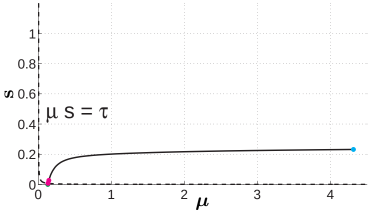
Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S \mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\, 0[$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w} - \mathbf{w}_0)^\top Q (\mathbf{w} - \mathbf{w}_0)$$
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Central path: solution manifold of

$$\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

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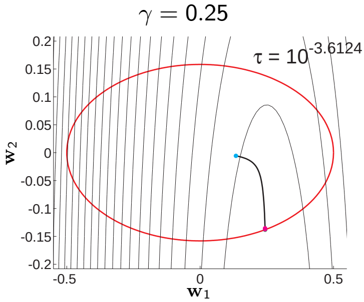
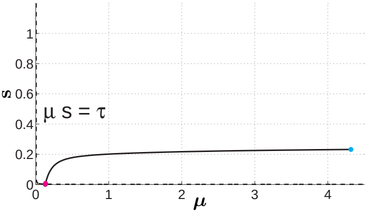




Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau >$ tol **or** $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})$
  if $\|\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\|_\mathrm{X} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

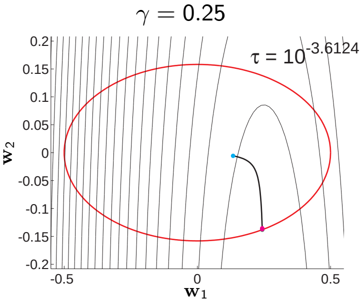
Example

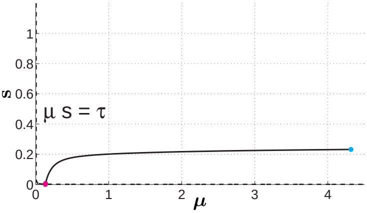
$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w} - \mathbf{w}_0)^\top Q(\mathbf{w} - \mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

for $\tau \in [1, 0[$





Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_\mathrm{X} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

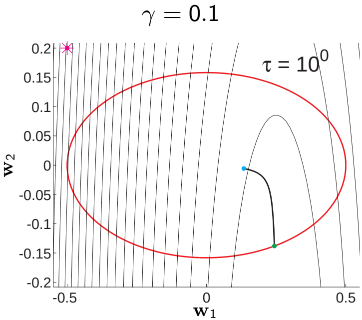
Example

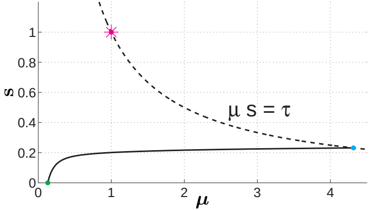
$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1, 0[$





Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

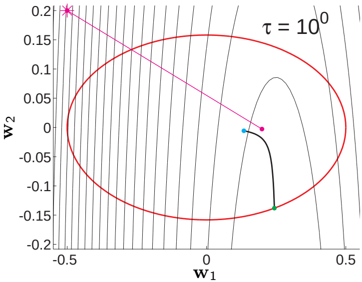
$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

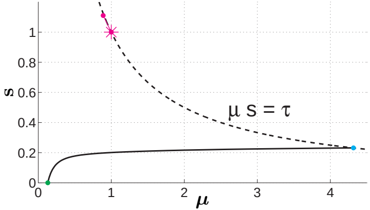
Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$

$\gamma = 0.1$





Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

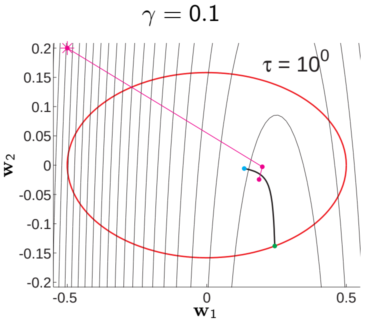
Example

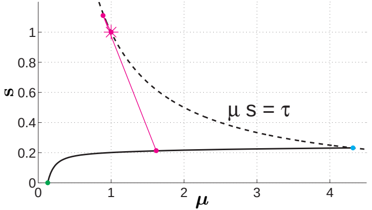
$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0) \\ \text{s.t.} \quad & \mathbf{w}^\top S\mathbf{w} \leq 1 \end{aligned}$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$





Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

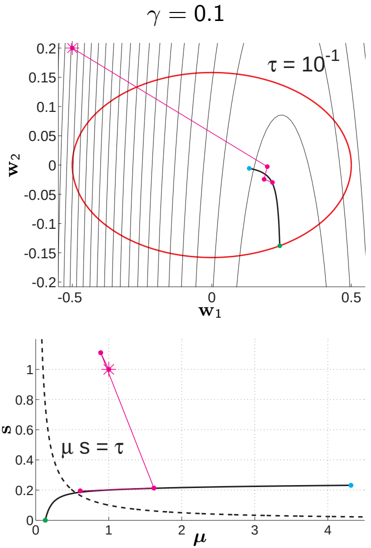
Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w} - \mathbf{w}_0)^\top Q\,(\mathbf{w} - \mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S \mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\, 0[$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau >$ tol **or** $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w}-\mathbf{w}_0\right)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

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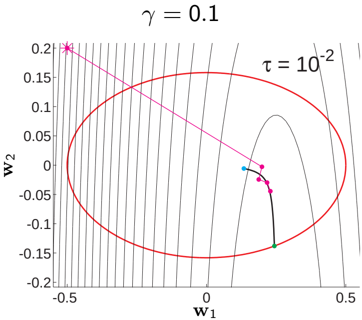
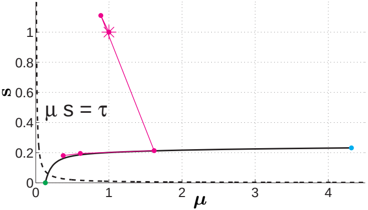




Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

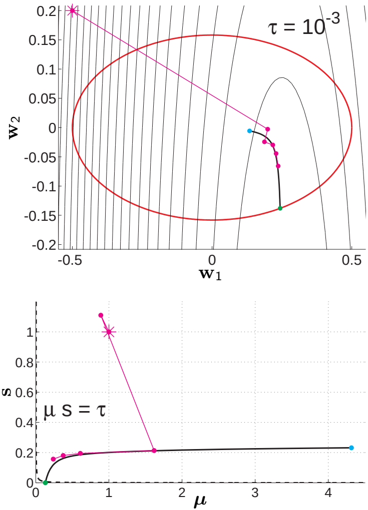
$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w}-\mathbf{w}_0\right)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S \mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$

$\gamma = 0.1$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$

while $\tau > \text{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \text{tol}$ **do**

  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$

  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_X \leq 1$ **then**

    Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

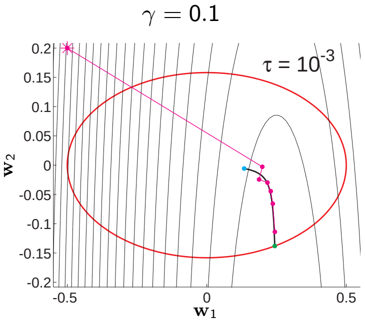
Example

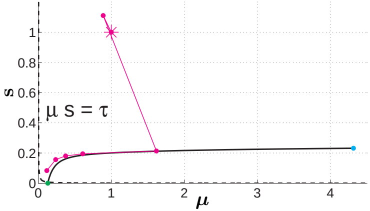
$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right) \\ \text{s.t.} \quad & \mathbf{w}^\top S\mathbf{w} \leq 1 \end{aligned}$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1, 0[$





Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \text{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \text{tol}$ **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

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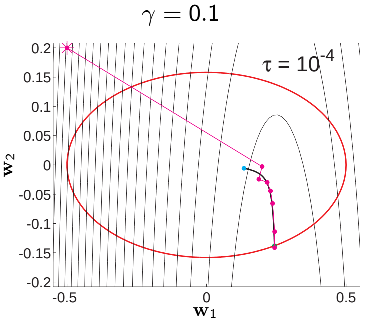
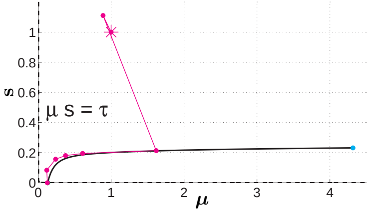




The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

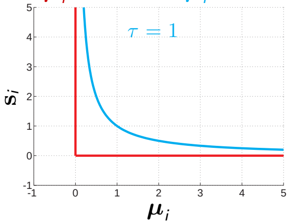
Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$

$\boldsymbol{\mu}_i \mathbf{s}_i = 0$ and $\boldsymbol{\mu}_i \mathbf{s}_i = \tau$



The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

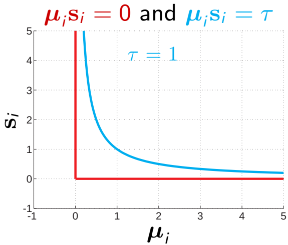
Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

  Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$



The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

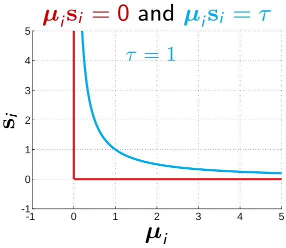
while $\tau > \text{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \text{tol}$ **do**

Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

Compute the Newton direction given by

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$



The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

Compute the Newton direction given by

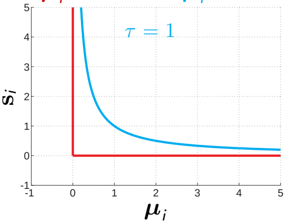
$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

Compute a step-size $t_{\max} \leq 1$ ensuring:

$$\mathbf{s} + t_{\max}\Delta\mathbf{s} \geq \epsilon\mathbf{s}, \quad \boldsymbol{\mu} + t_{\max}\Delta\boldsymbol{\mu} \geq \epsilon\boldsymbol{\mu}$$

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$

$\boldsymbol{\mu}_i\mathbf{s}_i = 0$ and $\boldsymbol{\mu}_i\mathbf{s}_i = \tau$



The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

Compute the Newton direction given by

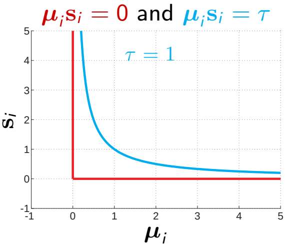
$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

Compute a step-size $t_{\max} \leq 1$ ensuring:

$$\mathbf{s} + t_{\max}\Delta\mathbf{s} \geq \epsilon\mathbf{s}, \quad \boldsymbol{\mu} + t_{\max}\Delta\boldsymbol{\mu} \geq \epsilon\boldsymbol{\mu}$$

Backtrack $t \in]0,\ t_{\max}]$ to ensure progress

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$



The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

Compute the Newton direction given by

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

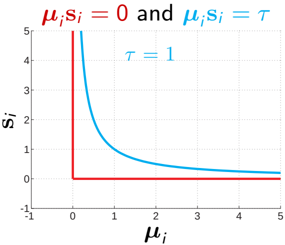
Compute a step-size $t_{\max} \leq 1$ ensuring:

$$\mathbf{s} + t_{\max}\Delta\mathbf{s} \geq \epsilon\mathbf{s}, \quad \boldsymbol{\mu} + t_{\max}\Delta\boldsymbol{\mu} \geq \epsilon\boldsymbol{\mu}$$

Backtrack $t \in \,]0, t_{\max}]$ to ensure progress

Take Newton step: $\mathbf{w} \leftarrow \mathbf{w} + t\Delta\mathbf{w}$, ...

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$



The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

Compute the Newton direction given by

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

Compute a step-size $t_{\max} \leq 1$ ensuring:

$$\mathbf{s} + t_{\max}\Delta\mathbf{s} \geq \epsilon\mathbf{s}, \quad \boldsymbol{\mu} + t_{\max}\Delta\boldsymbol{\mu} \geq \epsilon\boldsymbol{\mu}$$

Backtrack $t \in \,]0, t_{\max}]$ to ensure progress

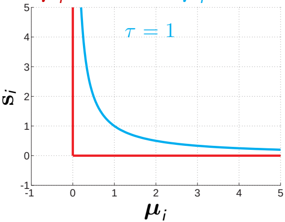
Take Newton step: $\mathbf{w} \leftarrow \mathbf{w} + t\Delta\mathbf{w}$, ...

if $\|\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\|_X \leq 1$ **then**

Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$

$\boldsymbol{\mu}_i\mathbf{s}_i = 0$ and $\boldsymbol{\mu}_i\mathbf{s}_i = \tau$



The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \mathrm{tol}$ **or** $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

Compute the Newton direction given by

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

Compute a step-size $t_{\max} \leq 1$ ensuring:

$$\mathbf{s} + t_{\max}\Delta\mathbf{s} \geq \epsilon\mathbf{s}, \quad \boldsymbol{\mu} + t_{\max}\Delta\boldsymbol{\mu} \geq \epsilon\boldsymbol{\mu}$$

Backtrack $t \in]0, t_{\max}]$ to ensure progress

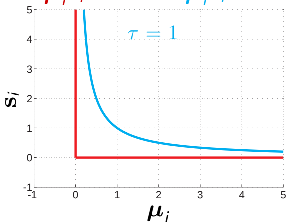
Take Newton step: $\mathbf{w} \leftarrow \mathbf{w} + t\Delta\mathbf{w}$, ...

if $\|\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\|_{\mathrm{X}} \leq 1$ **then**

Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$

$\boldsymbol{\mu}_i\mathbf{s}_i = 0$ and $\boldsymbol{\mu}_i\mathbf{s}_i = \tau$



Some subtleties:

- Measuring progress
- Choice of $\|.\|_{\mathrm{X}}$
- Mehrotra predictor