Advanced Quantum Mechanics	WiSe 2022/23
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Theoretical Physics 5	Instructor: Harvey Meyer

Recap

$$(\gamma^\mu k_\mu - \kappa)u(k) = \begin{pmatrix} 0 & k\cdot\sigma \\ k\cdot\bar{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{k\cdot\sigma} \\ \sqrt{k\cdot\bar{\sigma}} \end{pmatrix} = \begin{pmatrix} (k\cdot\sigma)\sqrt{k\cdot\bar{\sigma}} \\ (k\cdot\bar{\sigma})\sqrt{k\cdot\sigma} \end{pmatrix}$$

$$(k\cdot\sigma)k\cdot\bar{\sigma} = k^2 \mathbb{1}_{2\times 2}$$

Indeed: $k_\mu\sigma^\mu k_\nu\bar{\sigma}^\nu = k_\mu k_\nu \frac{1}{2}\{\sigma^\mu, \bar{\sigma}^\nu\}$

$$= k_\mu k_\nu \frac{1}{2}(\sigma^\mu\bar{\sigma}^\mu + \sigma^\nu\bar{\sigma}^\nu)$$

$$= \begin{cases} \mathbb{1} & \mu=\nu=0 \\ -\delta^{ij} & \mu=i, \nu=j \\ 0 & \mu=0, v=j \\ 0 & \mu=i, \nu=0 \end{cases}$$

$$= g^{\mu\nu}\mathbb{1}_{2\times 2}$$

Another ansatz:

$$\psi(x) = v(k)e^{+ikx}$$

for the antiparticle case, but also with $k^0 = +\sqrt{k^2+\kappa^2}$.

That is to say, k^0 is still kept positive (positive energy, negative frequency).

Putting this back into the Dirac equation

$$(-\gamma^\mu k_\mu - \kappa)v(k) = 0$$

$$\implies (\gamma^\mu k_\mu + \kappa)\ v(k) = 0$$

At $\vec{k}=0$:

$$(\gamma^0 + 1)\ v(\vec{k}=0) = 0$$

the eigenvector of γ^0 with digenvalue $_1$

$$v(\vec{k}=0) = \begin{pmatrix} \eta \\ -\eta \end{pmatrix}, \eta \in \mathbb{C}^2$$

Apply boost to obtain $v(k)$ for a general k (not at rest).

$$v(k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\ \eta \\ -\sqrt{k\cdot\bar{\sigma}}\ \eta \end{pmatrix}$$

The full set of solutions of Dirac equation

$$\psi(x) = \begin{pmatrix} \sqrt{k\cdot\sigma} \\ \sqrt{k\cdot\bar{\sigma}} \end{pmatrix} e^{-ik\cdot x}$$

and

$$\psi(x) = \begin{pmatrix} \sqrt{k\cdot\sigma}\ \eta \\ -\sqrt{k\cdot\sigma}\ \eta \end{pmatrix} e^{ik\cdot x}$$

Basis of solutions $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
and $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Orthogonality and completenesss of the solutions

$$\bar{u}^r(k)u^s(k) = 2\kappa\xi^{r\dagger}\cdot\xi^s = 2\kappa\delta^{rs}; \quad \bar{u} \equiv u^\dagger\cdot\gamma^0$$

The same also holds for $\bar{v}^r(k)v^s(k) = -2\kappa\delta^{rs}$
Additionally

$$\bar{u}^r(k)v^s(k) = 0,\ \forall r,s\in\{1,2\}$$

and same goes the other way
Another set of relations

$$u^{r\dagger}(k)\ u^s(k) = 2\frac{\omega_k}{c}\delta^{rs} = 2k^0\delta^{rs}$$

$$v^{r\dagger}(k)\ v^s(k) = 2k^0\delta^{rs}$$

Note: $u^{r\dagger}v^s(k) \neq 0$ but $u^r(\vec{k})^\dagger\ v^s(\vec{k}) = 0$

"Spin sum rules": $\gamma^\mu k_\mu \equiv \not{k}$ (Feynman notation)

$$\sum_{s=1,2} u^s(k)\bar{u}^s(k) = \not{k} + \kappa$$

note that this $\not{k}+\kappa$ is a projector (i.e., $P^2 = P$)

$$\not{p}\cdot\not{k} = (p\cdot k)\mathbb{1}_{4\times4}$$

$$\left(\frac{\not{k}+\kappa}{2\kappa}\right) = \frac{1}{4\kappa^2}(\underbrace{k^2}_{\kappa^2} + 2\kappa k + \kappa^2) = \frac{1}{2\kappa}(k+\not{k})$$

$$\boxed{\sum_{r=1,2} v^r(k)\ \bar{v}^r(k) = \not{k} - \kappa}$$

This is analogous to

$$\sum_{\lambda=1,2} e^i_\lambda(k)e^j_\lambda(k) = \delta^{ij}\frac{k^i\cdot j}{\vec{k}^2}$$

Behavior of the solutions for $k^3 \to$infty

We've already discussed the special case when our $\vec{k} = 0$.

For $k^3 \to \infty$, we can forget $k^1 = k^2 = 0$

$$u(k)^{s=1} = \begin{pmatrix} \sqrt{k^0 - k^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{k^0 + k^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \xrightarrow{k^3 \to \infty} = \sqrt{2|k^3|} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u^{s=2}(k) = \begin{pmatrix} \sqrt{k^0 + k^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{k^0 - k^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \xrightarrow{k^3 \to +\infty} = \sqrt{2|k^3|} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

This also corresponds to the limit $\kappa \to 0$

In this limit, only two components survive.

$\Rightarrow$ Why did Dirac choose a 4 dimensional representation of the Lorentz group? This looks like a waste of 2 degrees of freedom.

It's because we wanted to describe a particle of a finite mass (e.g. electron). If we wanted to describe a massless fermion of spin$-1/2$ then we could make do with just the upper two components in the block-diagonal

$$S^{\mu\nu} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

and that would be a perfectly fine representation of the Lorentz group.

The mass term in the $\mathcal{L}$, that is $-\kappa\bar{\psi}\psi$ couples the upper two and the lower two components. Otherwise we wouldn't need that many degrees of freedom.

$\Rightarrow$ To describe waves (read: particles) with a dispersion relation $\kappa \neq 0$, it is necessary to use four-component spinors.

For $\kappa = 0$, a two-component spinor suffices.

$$\textbf{Define} : \gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$$
$$= i\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$
$$= \text{Lorentz scalar}$$

Actually, this is a pseudo-scalar. Under T and P reversal, this picks up a $-$ sign. But for the purpose of a *proper orthochronous* Lorentz group, this is sufficient.

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix}$$

$$\frac{\mathbb{1} + \gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} : \text{Projector onto the 2 lower components}$$

$$\frac{\mathbb{1} - \gamma^5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} : \text{Projector onto the upper two components}$$