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January 23, 2023

Lecture Notes
Advanced Quantum Mechanics

Transforming $\psi^\dagger(x)$

$$\psi'(x') = \left(1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi(x)$$

$$\psi'^\dagger(x') = \psi^\dagger(x)\left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu\dagger}\right)$$

$S^{\mu\nu}$: Hermitian for rotations ($S^{ij};\ i,j > 0$)
and anti-Hermitian for boosts (S^{0j} for $j > 0$)
Property: $S^{\mu\nu\dagger} \to \gamma^0 S^{\mu\nu}\gamma^0$ (Why?)

So,

$$\psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)\left(1 + \frac{i}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\gamma^0\right)\gamma^0$$

$$\bar{\psi}'(x') = \psi^\dagger(x)\left(\gamma^0 + \frac{i}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\right)$$

$$\bar{\psi}'(x') = \bar{\psi}(x)\left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

The product

$$\begin{aligned}
\bar{\psi}'(x')\psi'(x') &= \bar{\psi}(x)\left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \\
&\sim \bar{\psi}(x)\psi(x) \\
(\bar{\psi}\psi)'(x') &= (\bar{\psi}\psi)(x)
\end{aligned}$$

Therefore $\bar{\psi}\psi$ transforms like a scalar field.
Transformation of $\bar{\psi}(x)\gamma^\mu\psi(x)$ under Lorentz transform

Transformed: $\bar{\psi}'(x')\gamma^\mu\psi'(x')$ use the transformations if ... (I didn't understand this part)
The commutator for $S^{\mu\nu}$ and γ^μ pops up (figure it out)
Takes up a form,

$$\bar{\psi}(x')\gamma^\mu\psi(x') = \left(\delta^\mu_\sigma - \frac{i}{2}\omega_{\lambda\nu}(J^{\lambda\nu})^\mu_\sigma\right)\bar{\psi}(x)\gamma^\mu\psi(x)$$

same transformation is used for vector field

$\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)$ will transform like a scalar field.

$$\text{Covariant vectors: } x'_\mu = (\Lambda^{-1})^\nu_\mu x_\nu$$
$$\text{Contra: } x'^\mu = \Lambda^\mu_\nu x^\nu$$
$$x'_\mu x'^\mu = x_\mu x^\mu, \quad [\because (\Lambda^{-1})^\nu_\mu (\Lambda)^\mu_\lambda = \delta^\nu_\lambda]$$

$$\frac{\partial}{\partial x'^\mu} = \left((\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial x^\nu}\right)$$

$\hookrightarrow$ this transformation was missing

Constructing a Lagrangian

For a scalar field we used $\phi\phi^*$ (i.e. $\partial^\mu\phi^*\partial_\mu\phi$). For the fields under a Lorentz transformation

$$\begin{aligned}\mathcal{L} =& i\bar{\psi}\gamma^\mu\partial_\mu\psi - k\bar{\psi}\psi \\ &\hookrightarrow \bar{\psi}\gamma^0\frac{1}{c}\frac{d\psi}{dt} + \bar{\psi}\vec{\gamma}\cdot\nabla\psi\end{aligned}$$
$$\text{with } \vec{\gamma} = \begin{pmatrix}\gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}$$

$\gamma^0 \to$ Hermitian (time-like part)
$\vec{\gamma} \to$ anti-Hermitian (spatial part)

$$\begin{aligned}\gamma^0 &= \gamma_0 \\ \gamma_\mu &= -\gamma^\mu \\ \gamma_\mu &= g_{\mu\nu}\gamma^\nu\end{aligned}$$

the metric tensor transforms like the Dirac matrices

Equation of motion for the constructed lagrangian of ψ

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_l)}\right) = \frac{\partial\mathcal{L}}{\partial\phi_l}, \quad l = 1, ..N$$

We are treating ψ and $\bar{\psi}$ as independent variables
So, for the Lagrangian,

$$(i\gamma^\mu\partial_\mu - \kappa)\psi(x) = 0 \to \text{Dirac equation}$$

The equation for $\bar{\psi}$ turns out to be

$$\bar{\psi}(x)(i\gamma^\mu\partial_\mu + \kappa) = 0$$

$\hookrightarrow$ Derive this

(Just multiply Dirac equation by γ^0 and pull out a γ^0 after taking conjugate).
So for QED, there would be extra interaction terms in the Lagrangian

$$\mathcal{L} = \bar{\psi} i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Relevance of κ in the Dirac equation

$$(i\gamma^\mu\partial_\mu - \kappa)\psi(x) = 0 \qquad \times \big(i\gamma^\mu\partial_\mu + \kappa\big) \text{ on the left}$$

$$(i\gamma^\mu\partial_\mu + \kappa)(i\gamma^\mu\partial_\mu - \kappa)\psi(x) = 0$$

$$(- \underbrace{\gamma^\nu\gamma^\mu}_{\text{becomes}} \partial_\nu\partial_\mu - \kappa^2)\psi(x) = 0$$

$$\hookrightarrow \frac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu) = \frac{1}{2}\{\gamma^\nu, \gamma^\mu\} = g^{\mu\nu}\mathbb{1}$$

$$\Rightarrow (-g^{\mu\nu}\partial_\nu\partial_\mu - \kappa^2)\psi(x) = 0$$

$$(\partial^\mu\partial_\mu - \kappa^2)\psi(x) = 0$$

$$\hookrightarrow \text{ Klein-Gordon equation}$$

Therefore, the Dirac equation implies the Klein-Gordon equation.
$\mathcal{L}$