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Lecture Notes
Advanced Quantum Mechanics

A word on the zero-point energy

In the Hamiltonian

$$H = \sum_{\vec{k}} \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right)$$

The constant $\hbar\omega_k \frac{1}{2}$ term is the zero point energy for each mode $\vec{k}$. There are an infinite number of modes $\vec{k}$, and the value of $|k|$ is apparently not bounded from above (leading to arbitrarily large values of $\hbar\omega_k$) and there are infinite such terms. This causes H to diverge.

Mathematically this can be dodged by inserting a constant 'vacuum' energy density term Ω_0 that exactly cancels the zero-point energy. One place where using an arbitrary energy density Ω_0 is not allowed is *gravity*. In general relativity, this constant term has real physical implications. There are dark energy models that attribute the expansion of the universe to a negative energy density.

Another way to eliminate such an infinity is to take the sum $\sum_k^\lambda$, where λ dictates the **ultraviolet cutoff**.

Infinite volume limit ($Ł \to \infty$)

$$\phi(\vec{x}) = \sqrt{\frac{hc^2}{L^3}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}} \right)$$

The way to take the continuum limit is

$$\frac{1}{L^3} \sum_k f(\vec{k}) \to \int \frac{d^3\vec{k}}{(2\pi)^3} f(\vec{k})$$

We shall rescale the creation and annihilation operators that have the appropriate normalization.

$$a(\vec{k}) := L^{3/2} a_{\vec{k}}$$

which leads to a commutation relation

$$[a(\vec{k}), a(\vec{k}')^\dagger] = L^3 \delta_{kk'} \to (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

when we go from discrete to continous variables, the Kronecker delta becomes a Dirac delta. Now, making this substitution

$$\phi(\vec{x}) = \sqrt{\hbar c^2} \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega_k}} \left(a(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + a(\vec{k})^\dagger e^{-i\vec{k}\cdot\vec{x}} \right)$$

and so $L \to \infty$ we arrive at a nice form of the Hamiltonian

$$H_0 = \frac{1}{L^3}\sum_{\vec{k}} \hbar\omega_k \left(a(\vec{k})^\dagger a(\vec{k}) + \frac{1}{2} \right) \to \int \frac{d^3\vec{k}}{(2\pi)^3} \hbar\omega_k (a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2})$$

with H_0 we mean that this is the non-interacting part. The total Hamiltonian

$$H = H_0 + \Delta H, \quad \Delta H = \int d^3\vec{x}\ V(\phi)$$

Momentum operator

$$\vec{P} = -\frac{1}{c^2}\int d^3\vec{x}\dot{\phi}\vec{\nabla}\phi$$

It's also convenient to replace $\dot{\phi}$ with $\pi(\vec{x})$

$$\vec{P} = -\int d^3x \pi(\vec{x})\vec{\nabla}\phi(\vec{x}) = \sum_{\vec{k}} \hbar k a^\dagger_k a_k = \frac{1}{L^3}\sum_{\vec{k}} a(\vec{k})^\dagger a(\vec{k}) \to \int \frac{d^3k}{(2\pi)^3}\hbar k a^\dagger_k a_k$$

Extension to a complex scalar field $\phi(x) \in \mathbb{C}$

Remember that the Lagrangian density has to be real ($\mathcal{L} \in \mathbb{R}$).

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^\dagger - \kappa^2\phi^\dagger\phi + \Omega_0 + (V(\phi^\dagger\phi))$$

The mode expansion for such a complex field[1]

$$\phi(\vec{x}) = \sqrt{\hbar c^3}\int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a(\vec{k})e^{i\vec{k}\cdot\vec{x}} + b(\vec{k})^\dagger e^{-i\vec{k}\cdot\vec{x}} \right)$$

and the conjugate momentum would be defined normally as

$$\pi(\vec{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{1}{c}\partial^0\phi^\dagger = \frac{1}{c^2}\dot{\phi}^\dagger$$

So this theory describes to particle species.
$a(\vec{k})^\dagger |0\rangle$ is a one-particle state for particles of type 'a'. and $b(\vec{k})^\dagger |0\rangle$ for those of type 'b'.
Here, due to the form of the κ^2 term in $\mathcal{L}$, the mass of both the particles is $m = \hbar\kappa/c$

Additional conserved quantity

Because of the U(1) symmetry[2], there exists a conserved current J^μ such that $\partial_\mu J^\mu = 0$. The time-like component of this current corresponds to a charge.

$$Q = \frac{i}{\hbar}\int d^3\vec{x}\left(\phi^\dagger\pi^\dagger - \pi\phi\right) = \int \frac{d^3k}{(2\pi)^3}\left(a(\vec{k})^\dagger a(\vec{k}) - b(\vec{k})^\dagger b(\vec{k}) \right)$$

Type 'a' and type 'b' particles have equal and opposite charge.

[1] The reason for the choice of the expansion in terms of a_k and $b^\dagger_k$ and not just one of them is that otherwise the time-dependence would be more sophisticated, not first-order in the derivative.

[2] $\phi(x) \to e^{i\alpha}\phi$ in turn implies $\phi^\dagger \to e^{-i\alpha}\phi^\dagger$ which for a constant α leaves the $\mathcal{L}$ unchanged.

If we wanted to interpret this conserved charge as the electromagnetic charge, we would add a coupling term to the $\mathcal{L}_{em}$ equal to $-eA_\mu j^\mu$.

Note that we need to impose extra conditions and another term to introduce a full *gauge invariance* in the theory[3].

We do not yet have a theory that can describe the electron[4] because it has a spin-1/2.

A relativistic quantum field theory of the electron

So far we've had

- scalar field $\phi'(x') = \phi(x)$, where
$$x'^\mu = A^\mu_\nu x^\nu + b^\mu$$
- a vector field
$$A^\mu : A'^\mu(x') = \Lambda^\mu_\nu A^\nu(x)$$
which led to the photon with polarization states $\lambda = \pm$
the angular momentum operator J_3 still had eigenvalues $\pm\hbar$ ($J_3\ a^\dagger_k\,|0\rangle = \pm\hbar a^\dagger_k\,|0\rangle$), but an e$^-$ shall have only half-spin.

While trying to resolve this, we shall come to the Dirac theory.

[3] In the Hamiltonian for the em field that we had before, the $\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$ part has a term quadratic in $\vec{A}$ which is needed for H to be gauge invariant

[4] The spin-statistics theorem says that in a relativistic theory one cannot have bosons with spin-$\frac{1}{2}$ and fermions with integer-spin.