The Lorentz Group Revisited

A Lorentz transformation is described by

$$\Lambda(\omega) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}$$

for an infinitesimal $\omega \ll 1$.

A Lorentz covariant four-vector thus satisfies

$$x'^{\rho} = \Lambda^{\rho}_{\sigma}x^{\sigma}$$

The generators of our infiniteismal Lorentz transformations

$$J^{\mu\nu} = -J^{\nu\mu}$$

(principal diagonal elements must be all 0) satisfy certain commutation relations

$[J^{\mu\nu}, J^{\rho\sigma}]$ are equal to some linear combination of[1] $J^{\alpha\beta}$ (i.e. are elements of the same vector space) such that they form an *algebra* [2] of the generators.

A spinor field $\psi_{\alpha}(x): \psi'_{\alpha}(x') = M(\Lambda)_{\alpha\beta}\psi_{\beta}(x)$.
Dirac's proposal for the anticommuting gamma matrices γ^{μ}, that satisfy

$$\{\gamma^{\mu}, \gamma_{\nu}\} = 2g^{\mu\nu}\mathbb{1}_{n\times n}$$

It is then posited that one can find generators that can be constructed from these matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}] = -S^{\nu\mu}$$

which also satisfy the commutation relations appropriate for the Lorentz group. That is to say $S^{\mu\nu}$ are a specific example of what $\mathcal{J}^{\mu\nu}$ can be (here, rotations for a spinor field).

Specific realization of Dirac's matrices

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$$

where $\sigma^{\mu} = (\mathbb{1}_{2\times 2}, \vec{\sigma})$ and $\{\sigma^i\}$ are the Pauli matrices
It is easy to verify that $(\gamma^0)^2 = \mathbb{1}_{4\times 4}$, and that the remaining three gamma matrices

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij}\mathbb{1}_{4\times 4}$$

where i, j are spatial indices. Furthermore, $\gamma^0, \gamma^j = 0$.
The different gamma matrices thus anticommute with each other, and the square of $\gamma^i = -1$.

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[1] The generators J^i, K^i (or if you prefer $J^{\mu\nu}$) serve as basis vectors and span the space of all possible transformations.

[2] An *algebra* is a vector space equipped with a bilinear product.

How unique are the gamma matrices?

They are unique up to a unitary transformation. Suppose for a second we find a different γ'^μ that are a unitary transformation of γ^μ, i.e.

$$\gamma'^m u = U^\dagger \gamma^\mu U, \quad U^\dagger U = \mathbb{1}$$

Then, the γ'^μ must also satisfy the same anticommutation relations. It is easy to explicitly compute $\{\gamma'^\mu, \gamma'^\nu\}$ and obtain, from the properties of γ^μ alone and show that it's equal to $2g^{\mu\nu}\mathbb{1}$.

Rotations

are a subgroup of the Lorentz group
The generators

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = -\frac{i}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & \underbrace{[\sigma^i, \sigma^j]}_{=2i\epsilon^{ijk}\sigma^k} \end{pmatrix}$$

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

are block diagonal matrices (with the same σ^k in the two blocks of the matrix).

For the moment, let's call the unit rotation matrices

$$\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = \Sigma^k$$

Now, for an infinitesimal rotation $|\omega| \ll 1$

$$M(\Lambda(\omega)) = \mathbb{1}_{4\times4} - \frac{i}{2}\omega_{ij}S^{ij}$$

if we are to rotate by an angle α in the xy-plane for example, $\omega_{12} = -\omega_{21} = \alpha$, then the $\frac{1}{2}\omega_{ij}\epsilon^{ijk} = \alpha\delta^{kz}$.
and the representation

$$M(\Lambda(\omega)) = \mathbb{1}_{4\times4} - \frac{i}{2}\alpha\Sigma^z$$

So, while rotation a $\psi_\alpha(x)$

$$\begin{pmatrix} \psi'_1(x') \\ \psi'_2(x') \\ \psi'_3(x') \\ \psi'_4(x') \end{pmatrix} = \begin{pmatrix} \mathbb{1} - \frac{i\alpha}{2}\sigma^3 & 0 \\ 0 & 1 - \frac{i\alpha}{2}\sigma^3 \end{pmatrix}\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

The two blocks transform without mixing (the upper two components, don't mix with the lower two).

The spin-1/2 electron

As a reminder of the problem, for an electron, the non-relativistic wavefunction

$$\Psi(x) = \begin{pmatrix} \Psi_\uparrow(x) \\ \Psi_\downarrow(x) \end{pmatrix}$$

the probability of measuring either the spin-up or spin-down component in a Stern-Gerlach apparatus should remain invariant under rotation of the coordinates. That is to say $\int d^3x |\Psi_{\uparrow,\downarrow}(x)|^2$ should remain unaffected

If we are to view the same electron in a rotated coordinate system

$$\Psi'(x') = \underbrace{D(R)}_{2\times 2} \Psi(x)$$

The infinitesimal rotation matrix

$$D(R) = -\mathbb{1}_{2\times 2}\, i\alpha\vec{n}\cdot\vec{s}/\hbar$$

where α is the angle of rotation, $\vec{n}$ is the axis of rotation unit vector, and $\vec{s}$ is the spin spector.

We can see how to turn this into a finite rotation $\to$ apply the rotation N times for an arbitrarily large N

$$D(R) = \lim_{N\to\infty} \left(\mathbb{1} - \frac{i\alpha}{N}\frac{\vec{n}\cdot\vec{s}}{\hbar}\right)^N = \exp(-i\alpha\vec{n}\cdot\vec{s}/\hbar)$$

As for a spin-1/2 particle s has eigenvalues $\pm\hbar/2$ and $\vec{s}/\hbar = \frac{1}{2}\vec{\sigma}$
the rotation

$$D(R) = \exp\left(\frac{-i\alpha\vec{n}\cdot\vec{\sigma}}{2}\right)$$

Boosts

Let's suppose we apply a boost along the x-axis of pseudorapidity η, $\omega_{01} = -\omega_{10} = \eta$ (where the velocity of the boost, $v = c\tanh\eta$)

$$M(\Lambda) = \mathbb{1}_{4\times 4} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} = 1 - i\omega_{01}\bar{J}^{01}$$

In Dirac's construction, the

$$\mathcal{J}^{0k} = S^{0k} = -\frac{i}{4}[\gamma^0, \gamma^k]$$

which we can evaluate to find are

$$-\frac{i}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}$$

stemming from the product of two matrices γ^0 and γ^k that themselves are zero-diagonal.

From this, we clearly see that the boost generators are anti-Hermitian $S^{ij} = (-S^{ij})^\dagger$, as opposed to the ones for rotations, which were Hermitian.

One can also verify that

$$(S^{\mu\nu})^\dagger = \gamma^0 S^{\mu\nu}\gamma^0, \quad \forall\mu,\nu$$

$\to$ Ultimately, our goal is to construct a Lagrangian: $\mathcal{L}(\psi, \psi^\dagger)$ which is a Lorentz scalar.