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Lecture Notes
Advanced Quantum Mechanics

The Dirac equation

$$(i\gamma^\mu \partial_\mu - \kappa)\psi(x) = 0$$

- derives from the Lagrangian density $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu)\psi$
- implies the Klein-Gordon equation, $(\Box + \kappa^2)\psi = 0$

k^μ and p^μ are two four vectors

$$p_\mu \gamma^\mu k_\nu \gamma^\nu = \frac{1}{2}(p_\mu k_\nu \gamma^\mu \gamma^\nu + p_\nu k_\mu \gamma^\nu \gamma^\mu)$$

we can replace $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu + 2g^{\mu\nu}\mathbb{1}_{4\times 4}$

which then takes the neat expression

$$\boxed{(p \cdot \gamma)(k \cdot \gamma) = (p \cdot k)\mathbb{1}_{4\times 4}}$$

Hamiltonian for the Dirac field: $\bar{\psi} \equiv \psi^+ \gamma^0$

$$\frac{\partial \mathcal{L}}{\partial \psi_\alpha} = \frac{1}{c}\psi^+_\alpha = \Pi_\alpha, \quad \underbrace{(\;\dots\;)}_{\bar{\psi}} \begin{pmatrix} \cdot\cdot & \cdot & \cdot & \cdot \\ \vdots & & & \vdots \\ \cdot & & & \cdot \\ \cdot & & & \cdot \end{pmatrix} \underbrace{\begin{pmatrix} \vdots \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}}_{\psi}$$

$$\begin{aligned} \mathcal{H} &= \int d^3\vec{x} \left(\frac{i}{c}\psi^\dagger_\alpha \dot{\psi}_\alpha - \mathcal{L} \right) \\ &= \int d^3\vec{x} \left(\cancel{\frac{i}{c}\psi^+ \dot{\psi}} - (\cancel{\bar{\psi} i\gamma^0 \partial_0 \psi} + \bar{\psi} i\gamma^k \partial_k \psi - \kappa\bar{\psi}\psi) \right) \\ \mathcal{H} &= \int d^3\vec{x} \; \bar{\psi}(-i\vec{\gamma} \cdot \vec{\nabla} + \kappa)\psi, \quad \vec{\gamma} = \begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix} = -\vec{\gamma}^+ \end{aligned}$$

Solving the Dirac equation in vacuum

First, note that with the *ansatz*

$$\psi(x) = u(k)e^{-ik\cdot x}, \quad k\cdot x = \vec{k}\cdot\vec{x} - \underbrace{k^0 ct}_{\omega_k t}$$

$$\omega_k = \pm c\sqrt{\vec{k}^2 + \kappa^2}$$

which is to say

$$\boxed{k^0 = \pm\sqrt{\vec{k}^2 + \kappa^2}}$$

The case $k^0 = +\sqrt{\vec{k}^2 + \kappa^2}$

$$(i\gamma^\mu \partial_\mu - \kappa)u(k)e^{-ikx} = 0$$

Given that $\partial_\mu e^{-ikx} = -ik_\mu e^{-ikx}$

$$\Rightarrow (\gamma^\mu k_\mu - \kappa)u(k)\cancel{e^{-ikx}} = 0$$

rearranging this

$$(k\cdot\gamma)u(k) = \kappa u(k)$$

That is to say, $u(k)$ is an eigenvector of $k\cdot\gamma$ with eigenvalue κ

Strategy I: Problem for $k = 0$

i.e. $k^0 = \kappa$

Apply a boost to the obtain the solution for general $\vec{k}$. So here $k\cdot\gamma^\mu = k^0\gamma^0$

$$\kappa\gamma^0 u(k) = \kappa u(k)$$
$$\gamma^0 u(k) = u(k)$$

In our choice of Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \sigma^\mu = \begin{pmatrix} \mathbb{1}_{1\times 1} \\ \vec{\sigma} \end{pmatrix} \text{ and } \bar{\sigma}^\mu = \begin{pmatrix} \mathbb{1}_{1\times 1} \\ -\vec{\sigma} \end{pmatrix}$$

$$\gamma^0 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

Eigenvectors with eigenvalue +1: $(k\cdot\gamma)U = kU$

$$\begin{pmatrix} a \\ a \end{pmatrix} \quad \forall a \in \mathbb{C}^2$$

e.g. $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, a = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} a \\ -a \end{pmatrix}$ have eigenvalue -1, $\forall a \in \mathbb{C}^2$

$\Rightarrow$ solution for $\vec{k} = 0$,

$$u(\vec{0}) = \sqrt{\kappa}\begin{pmatrix} \zeta \\ \zeta \end{pmatrix}, \quad \zeta \in \mathbb{C}^2$$

Choose the normalization $\zeta^+ \cdot \zeta = 1$

Boost that brings $\begin{pmatrix} k \\ \vec{\sigma} \end{pmatrix}$ into the vector $\begin{pmatrix} k^0 = \sqrt{\vec{k}^2 + \kappa^2} \\ \vec{k} \end{pmatrix}$ is

$$\Lambda^\mu_\nu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}$$

$$k^\mu = \begin{pmatrix} k^0 \\ 0 \\ 0 \\ k^3 \end{pmatrix} = (\Lambda^\mu_\nu) \begin{pmatrix} \kappa \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \kappa\cosh\eta \\ 0 \\ 0 \\ \kappa\sinh\eta \end{pmatrix}$$

So this boost turns our rest frame wavevector with $k^0 = \kappa$ into a more general one

$$\Rightarrow k^0 + 3 = \kappa e^\eta \qquad (\because\ \cosh x + \sinh x = e^x)$$

and

$$k^0 - k^3 = \kappa e^{-\eta}$$

So now e have to transform our spinor u in a way that ... We know how it transforms, it's not with the matrix Λ^μ_ν, but with the spinor representations (using the matrices $\S^{\mu\nu}$).

Transformation of the spinor

$$\psi'(x') = \exp(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu})\psi(x)$$

Here, we pick $\omega_{03} = -\omega^{30} = \eta$, all other $\omega_{\mu\nu}$ vanish.

$$\Rightarrow \exp\left(-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) = \exp(-i\eta\delta^{03}) = \exp\left(-\frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right)$$

which is

$$= -\frac{i}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$$

Note that exponentiating such a complicated 4×4 matrix here reduces to exponentiating a 2×2 matrix because in $\exp\lambda = \sum_k^\infty \frac{\lambda^n}{n!}$ the different matrices don't talk to each other [1]

$$\text{the exponential becomes } = \begin{pmatrix} \frac{\mathbb{1}-\sigma^3}{2}e^{i\eta} + \frac{\mathbb{1}+\sigma^3}{2}e^{-i\eta} & 0 \\ 0 & \frac{\mathbb{1}-\sigma^3}{2}e^{i\eta} + \frac{\mathbb{1}+\sigma^3}{2}e^{-i\eta} \end{pmatrix}$$

So,

$$\exp\left(-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) = \frac{1}{\sqrt{k}}\left[\begin{matrix} \left(\frac{\mathbb{1}-\sigma^3}{2}\sqrt{k^0+k^3} - \frac{\mathbb{1}+\sigma^3}{2}\sqrt{k^0-k^3}\right) & 0 \\ 0 & \left(\sqrt{k^0+k^3}\frac{\mathbb{1}-\sigma^3}{2} - \sqrt{k^0-k^3}\left(\frac{\mathbb{1}+\sigma^3}{2}\right)\right) \end{matrix}\right]$$

[1] The even powers give $\mathbb{1}$ and the odd powers give σ^3

if $A^2 = \lambda\mathbb{1}$,

$$e^A = \left(\sum_{\text{even}} \lambda^n\right)\mathbb{1} + \left(\sum_{\text{odd}} \frac{\lambda^{(n-1)}}{n!}A\right) = \cosh\lambda\mathbb{1} + \frac{1}{\lambda}\sinh\lambda\cdot A$$

if $A = \sqrt{\lambda}\mathbb{1}e^{\sqrt{\lambda}}$...

What does the equation really say? These matrices (e.g. σ^3) are projectors. Which means if you square the matrix, you get the same matrix.

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{1+\sigma^3}{2}, \text{ projects onto subspace with eigenvalue } +1$$
$$\frac{1-\sigma^3}{2}, \text{ projects onto subspace with eigenvalue } -1$$

General k:

$$u(k) = \sqrt{k} \begin{pmatrix} \sqrt{k^0 \mathbb{1} - \vec{k}\vec{\sigma}} \\ \sqrt{k^0 \mathbb{1} + \vec{k}\vec{\sigma}} \end{pmatrix} = \begin{pmatrix} \sqrt{k \cdot \sigma} \\ \sqrt{k \cdot \bar{\sigma}} \end{pmatrix}$$