Aayush Arya

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Lecture Notes
Advanced Quantum Mechanics

For constructing a field theory of fermions, we need to find a Lagrangian density $\mathcal{L}$ that is Lorentz invariant, using some combination of $\psi(x)$ and its conjugate transpose.

Transforming $\psi^\dagger(x)$ works in an analogous fashion to $\psi(x)$, by taking the conjugate transpose of the whole expression

$$\psi'(x') = \left(1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi(x)$$

and thereby getting

$$\psi'^\dagger(x') = \psi^\dagger(x)\left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu\dagger}\right)$$

A couple of things to remember about the generators:

$$\begin{aligned} S^{\mu\nu} &: \text{Hermitian for rotations } (S^{ij};\ i,j>0) \\ &\text{and anti-Hermitian for boosts } (S^{0j} \text{ for } j>0) \\ \text{also, the property } & S^{\mu\nu\dagger} \to \gamma^0 S^{\mu\nu}\gamma^0 \text{ holds} \end{aligned}$$

This can be understood as follows: γ^0 is equivalent to the parity operator P.
— Therefore $\gamma^0 S^{\mu\nu}\gamma^0 = P^{-1}S^{\mu\nu}P$ (since γ^0 is unitary, $(\gamma^0)^{-1} = \gamma^0$). This transformation will either cause a sign flip as we obtain $S^{\mu\nu\dagger}$ (for the anti-Hermitian boost matrices) or remain the same (for rotation matrices).

We define $\bar{\psi} \equiv \psi^\dagger\gamma^0$, such that multiplying both sides by γ^0

$$\psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)\left(1 + \frac{i}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\gamma^0\right)\gamma^0$$

since $(\gamma^0)^2 = 1$,

$$\bar{\psi}'(x') = \psi^\dagger(x)\left(\gamma^0 + \frac{i}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\right)$$

we can then pull out a γ^0 from the right, which will act on the $\psi^\dagger$

$$\boxed{\bar{\psi}'(x') = \bar{\psi}(x)\left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)}$$

The product

$$\begin{aligned} \bar{\psi}'(x')\psi'(x') &= \bar{\psi}(x)\left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi(x) \\ &\sim \bar{\psi}(x)\psi(x),\ \text{if we ignore } \mathcal{O}(\omega_{\mu\nu}^2) \text{ terms} \\ \Rightarrow (\bar{\psi}\psi)'(x') &= (\bar{\psi}\psi)(x) \end{aligned}$$

Therefore $\bar{\psi}\psi$ transforms like a scalar field.
We may ask, how does $\bar{\psi}(x)\gamma^\mu\psi(x)$ transform?

As it turns out, the transformed $\bar{\psi}'(x')\gamma^\mu\psi'(x')$ takes up the form[1]

$$\bar{\psi}(x')\gamma^\mu\psi(x') = \left(\delta^\mu_\sigma - \frac{i}{2}\omega_{\lambda\nu}(J^{\lambda\nu})^\mu_\sigma\right)\bar{\psi}(x)\gamma^\mu\psi(x)$$

same transformation is used for vector field ($\overset{?}{\leftarrow}$)

However, $\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)$ will transform like a scalar field.
What went missing in $\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x)$? Let's remind ourselves

$$\text{Covariant vectors: } x'_\mu = (\Lambda^{-1})^\nu_\mu x_\nu$$
$$\text{Contra: } x'^\mu = \Lambda^\mu_\nu x^\nu$$
$$x'_\mu x'^\mu = x_\mu x^\mu, \quad [\because (\Lambda^{-1})^\nu_\mu(\Lambda)^\mu_\lambda = \delta^\nu_\lambda]$$

The way ∂_μ changes is

$$\frac{\partial}{\partial x'^\mu} = \left((\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial x^\nu}\right)$$

$\hookrightarrow$ this transformation was missing

Constructing a Lagrangian

For a scalar field we used ϕ, ϕ^* and $\partial^\mu\phi^*\partial_\mu\phi$. For the spinor field, the following $\mathcal{L}$ seems to recover the Dirac equation as its equation of motion.

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - k\bar{\psi}\psi$$

where

$$\bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}\gamma^0\frac{1}{c}\frac{d\psi}{dt} + \bar{\psi}\vec{\gamma}\cdot\nabla\psi$$

and our $\vec{\gamma} = \begin{pmatrix}\gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}$

It's worth noting that since γ^0 is Hermitian (the time-like part), and $\vec{\gamma}$ is anti-Hermitian (the spatial part), raising and lowering indices has the effect

$$\gamma^0 = \gamma_0$$
$$\gamma_i = -\gamma^i$$
$$\gamma_\mu = g_{\mu\nu}\gamma^\nu$$

and so apparently, the metric tensor can take care of the signs.

Now, we derive the equation of motion for the Lagrangian we constructed from ψ

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_l)}\right) = \frac{\partial\mathcal{L}}{\partial\phi_l}, \quad l = 1, ..N$$

We are treating ψ and $\bar{\psi}$ as independent variables

Computing these quantities yields

[1] I didn't understand this part. Rohan scribbled that the commutator for $S^{\mu\nu}$ and γ^μ pops up (figure it out).

$$\boxed{(i\gamma^\mu \partial_\mu - \kappa)\psi(x) = 0}$$

which is the celebrated Dirac equation.

There's an analogous equation for $\bar{\psi}$. If we take the conjugate transpose of the above equation, then multiply by γ^0 on the right,

$$(-i\partial_\mu \psi^\dagger \gamma^{\mu\dagger} - \psi^\dagger(x)\kappa)\gamma^0 = 0$$

insert the identity $\gamma^0\gamma^0 = 1$ in between $\partial_\mu \psi^\dagger$ and $\gamma^{\mu\dagger}$, and make use of $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$

$$(i\,\underbrace{\partial_\mu \psi^\dagger \gamma^0}_{\partial_\mu \bar{\psi}}\,\underbrace{\gamma^0\gamma^{\mu\dagger}\gamma^0}_{\gamma^\mu} + \kappa\psi^\dagger(x)\gamma^0) = 0$$

after cleaning this up, we arrive at

$$\boxed{\bar{\psi}(x)(i\gamma^\mu \partial_\mu + \kappa) = 0}$$

$\Rightarrow$ In QED, there would be extra interaction terms in the Lagrangian

$$\mathcal{L} = \bar{\psi} i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Relevance of the mass term in the Dirac equation

Let's take

$$(i\gamma^\mu \partial_\mu - \kappa)\psi(x) = 0 \qquad \times \left(i\gamma^\mu \partial_\mu + \kappa\right) \text{ on the left}$$

such that

$$(i\gamma^\mu \partial_\mu + \kappa)(i\gamma^\mu \partial_\mu - \kappa)\psi(x) = 0$$

and doing the multiplication gives

$$(-\underline{\gamma^\nu\gamma^\mu}\partial_\nu \partial_\mu - \kappa^2)\psi(x) = 0$$

the underlined $\gamma^\nu\gamma^\mu$, due to the anticommutator relation

$$\frac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu) = \frac{1}{2}\{\gamma^\nu, \gamma^\mu\} = g^{\mu\nu}\mathbb{1}$$

gives

$$\begin{aligned} \Rightarrow (-g^{\mu\nu}\partial_\nu \partial_\mu - \kappa^2)\psi(x) &= 0 \\ (\partial^\mu \partial_\mu - \kappa^2)\psi(x) &= 0 \end{aligned}$$

which is the Klein-Gordon equation.

Therefore, the Dirac equation implies the Klein-Gordon equation.