Aayush Arya

January 9, 2023

Lecture Notes
Advanced Quantum Mechanics

# Quantization of the (real) scalar field

$$\mathcal{L}[\phi, \partial_\mu \phi] = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\kappa^2 \phi^2 - V(\phi)$$

The classical physics that emerges from this Lagrangian is that we have wave solutions of the field that oscillate in the time. And the fields obey the equation of motion

$$\Box \phi + \kappa^2 \phi + V'(\phi) = 0$$

where $\Box = \partial^2$. For $V = 0$ the solutions look like $e^{i(\vec{k}\cdot\vec{x} - \omega_k t)}$ with the dispersion relation $\omega_k = c\sqrt{\vec{k}^2 + \kappa^2}$

We reached a certain form for the Hamiltonian of lattice. We had this pair of degrees of freedom for $\phi_{\vec{x}}$ and the canonically conjugate field (momenta) $\pi_{\vec{x}'}$ with the commutator relations $[\phi_{\vec{x}}, \pi_{\vec{x}'}] = i\delta^3(\vec{x} - \vec{x}')$

after returning from the lattice to the continuum[1], we found

$$H = \int d^3x \left( \frac{c^2}{2}\pi(\vec{x})^2) + \frac{1}{2}(\vec{\nabla})^2 + \frac{\kappa^2}{2}\phi_{\vec{x}}^2 + V(\phi_{\vec{x}}) \right)$$

with conjugate momentum $\pi(\vec{x}) = \frac{1}{c^2}\dot{\phi}$

## Direct derivation of the Hamiltonian

is to introduce the

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

and the

$$H = \int d^3x \left( \pi(\vec{x})\dot{\phi}(\vec{x}) - \mathcal{L}[\phi, \partial_\mu \phi] \right)$$

$\rightarrow$ Now, what does this quantum field theory describe?

[1]In our notation, the fields with the position vector in the subscript, e.g. $\phi_{\vec{x}}$ shall be used to represent the lattice fields

## Expansion of the field in plane waves

$$\phi_{\vec{x}} = \sum_{\vec{x}} g_{\vec{k}} \left( a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + a^\dagger e^{-i\vec{k}\cdot\vec{x}} \right)$$

with $g_k \in \mathbb{R}$ being just a normalization factor for the operators

$\rightarrow$ The sum is a discrete one; we're still working inside the box $L^3$. And also, this is in the Schrodinger picture ($\vec{k}$ is the 3-component one; we haven't introduced the time dependence yet).

And the conjugate momentum

$$\pi(\vec{x}) = \frac{-i}{c^2} \sum_{\vec{k}} \omega_k \left( a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} - a^\dagger e^{-i\vec{k}\cdot\vec{x}} \right)$$

If we want to have commutator relations for then we need to have separate expressions for the creation/annihilation operators. The trick is a Fourier transform.

$$\int d^3x e^{-i\vec{k}'\cdot\vec{x}} \phi(x) = L^3 g_{k'} \left( a_{\vec{k}'} + a^\dagger_{\vec{k}'} \right)$$

from the other (conjugate field), we have

$$\int d^3x e^{-\vec{k}'\cdot\vec{x}} \pi(\vec{x}) = -\frac{i}{c^2} g_{k'} \omega_{k'} \left( a_{k'} - a^\dagger_{k'} \right)$$

from a linear combination of these two expressions, we can isolate $a$ and $a^\dagger$

$$a_k = \frac{1}{2L^3 g_k} \int d^3x e^{-i\vec{k}\cdot\vec{x}} \left( \phi(\vec{x}) + \frac{ic^2}{\omega_k} \pi(\vec{x}) \right)$$

and $a^\dagger$ is obviously just the complex conjugate of this.

The commutator

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \frac{1}{(2L^3)^2 g_k g_{k'}} \int d^3x \int d^3y e^{i(\vec{k}'\cdot\vec{x}' - \vec{k}\cdot\vec{x})} \left[ \phi_{\vec{x}'} - \frac{ic^2}{\omega_{k'}} \pi(\vec{x'}), \phi_{\vec{x}} + \frac{ic^2}{\omega_k} \pi(\vec{x}) \right]$$

the commutator inside is $c^2\hbar\ \delta^3(\vec{x} - \vec{x}') \left( \frac{1}{\omega_{k'}} + \frac{1}{\omega_k} \right)$. The delta function will take away one of the integrals.

And then we'll have $\int d^3x\ e^{i(\vec{k}-\vec{k}')\vec{x}}$ giving us another delta function $L^3\delta(\vec{k} - \vec{k}')$

by choosing an appropriate normalization $g_k$, then this simplifies to

$$\boxed{[a_k, a^\dagger_{k'}] = \delta(\vec{k} - \vec{k}')}$$

and it can also be shown that $[a_k, a_{k'}] = [a^\dagger_k, a^\dagger_{k'}] = 0$

## Hamiltonian in terms of the $a_k$ and $a^\dagger_k$

$\rightarrow$ at this point I stopped noting down some stuff because I already know it and the algebra is boring. The way $\pi(x)^2$ term introduced a common $\omega_k^2$, the gradient term brings $k^2$ outside

we finally get

$$H = \sum_{\vec{k}} g_k^2 L^3 / c^2 \omega_k^2 \left( a_k a^\dagger_k + a^\dagger_k a_k \right)$$

writing $a_k a^\dagger_k = a^\dagger_k a_k + 1$ gives us

$$H = \sum_{\vec{k}} \hbar\omega_k (a^\dagger_k a_k + \frac{1}{2})$$

Note that what we have contains the number operator $n_{\vec{k}} \left| n \right\rangle = n \left| n \right\rangle$

Energy of one quantum of mode $\vec{k}$

$$\mathcal{E}_k = \hbar\omega_k = \hbar c\sqrt{\vec{k}^2 + \kappa^2} = c\sqrt{\vec{p}^2 + (\hbar k)^2}$$

and by comparing with the relativistic dispersion relation $E = c\sqrt{\vec{p}^2 + m^2c^2}$, we get the mass of the particle

$$m = \frac{\hbar k}{c}$$

when $\phi_{\vec{x}}$ acts on the vacuum $\left|0\right\rangle$ then it creates a linear superposition of one-particle states $\left|\delta^k_{k'}\right\rangle$

$$\phi_{\vec{x}} \left|0\right\rangle = \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \left|\delta^k_{k'}\right\rangle$$

When there's a non-linear potential, say $V(\phi(\vec{x})) = \phi^4$ there's the possibility of inelastic scattering.

..