Advanced Quantum Mechanics	Aayush Arya January 18, 2023	WiSe 2022/23
Theoretical Physics 5		Instructor: Harvey Meyer

The Lorentz Group Revisited

A Lorentz transformation is described by

$$\Lambda(\omega) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}$$

for an infinitesimal $\omega \ll 1$.

A Lorentz covariant four-vector thus satisfies

$$x'^{\rho} = \Lambda^{\rho}_{\sigma} x^{\sigma}$$

The generators of our infiniteismal Lorentz transformations

$$J^{\mu\nu} = -J^{\mu\nu}$$

(principal diagonal elements must be all 0) satisfy certain commutation relations $[J^{\mu\nu}, J^{\rho\sigma}]$ are equal to some linear combination of $J^{\alpha\beta}$ (i.e. are elements of the same vector space) such that they form an *algebra* [1] of the generators.

A spinor field $\psi_\alpha(x):\ \psi'_\alpha(x') = M(\Lambda)_{\alpha\beta}\psi_\beta(x)$.

Dirac's proposal for the anticommuting gamma matrices γ^μ, that satisfy

$$\{\gamma^\mu, \gamma_\nu\} = 2g^{\mu\nu}\mathbb{1}_{n\times n}$$

It is then posited that one can find generators that can be constructed from these matrices

$$\S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = -S^{\nu\mu}$$

which also satisfy the commutation relations appropriate for the Lorentz group. That is to say $S^{\mu\nu}$ are a specific example of what $\mathcal{J}^{\mu\nu}$ can be.

Specific realization of Dirac's matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma^\mu} & 0 \end{pmatrix}$$

where $\sigma^\mu = (\mathbb{1}_{2\times 2}, \vec{\sigma})$ and $\{\sigma^i\}$ are the Pauli matrices

It is easy to verify that $(\gamma^0)^2 = \mathbb{1}_{4\times 4}$, and that the remaining three gamma matrices

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij}\mathbb{1}_{4\times 4}$$

where $i,\ j$ are spatial indices. Furthermore, $\gamma^0, \gamma^j = 0$.

The different gamma matrices thus anticommute with each other, and the square of $\gamma^i = -1$.

[1] An *algebra* is a vector space equipped with a bilinear product.

How unique are the gamma matrices?

They are unique up to a unitary transformation. Suppose for a second we find a different γ'^{μ} that are a unitary transformation of γ^{μ}, i.e.

$$\gamma'^{m}u = U^{\dagger}\gamma^{\mu}U, \quad U^{\dagger}U = \mathbb{1}$$

Then, the γ'^{μ} must also satisfy the same anticommutation relations. It is easy to explicitly compute $\{\gamma'^{\mu}, \gamma'^{\nu}\}$ and obtain, from the properties of γ^{μ} alone and show that it's equal to $2g^{\mu\nu}\mathbb{1}$.

Rotations

are a subgroup of the Lorentz group

The generators

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = -\frac{i}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & \underbrace{[\sigma^i, \sigma^j]}_{=2i\epsilon^{ijk}\sigma^k} \end{pmatrix}$$

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

are block diagonal matrices (with the same σ^k in the two blocks of the matrix).

For the moment, let's call the unit rotation matrices

$$\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = \Sigma^k$$

Now, for an infinitesimal rotation $|\omega| \ll 1$

$$M(\Lambda(\omega)) = \mathbb{1}_{4\times 4} - \frac{i}{2}\omega_{ij}S^{ij}$$

if we are to rotate by an angle α in the xy-plane for example, $\omega_{12} = -\omega 21 = \alpha$, then the $\frac{1}{2}\omega_{ij}\epsilon^{ijk} = \alpha\delta^{kz}$.

and the representation

$$M(\Lambda(\omega)) = \mathbb{1}_{4\times 4} - \frac{i}{2}\alpha\Sigma^z$$

So, while rotation a $\psi_{\alpha}(x)$

$$\begin{pmatrix} \psi'_1(x') \\ \psi'_2(x') \\ \psi'_3(x') \\ \psi'_4(x') \end{pmatrix} = \begin{pmatrix} \mathbb{1} - \frac{i\alpha}{2}\sigma^3 & 0 \\ 0 & 1 - \frac{i\alpha}{2}\sigma^3 \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

The two blocks transform without mixing (the upper two components, don't mix with the lower two).

The spin-1/2 electron

As a reminder of the problem, for an electron, the non-relativistic wavefunction

$$\Psi(x) = \begin{pmatrix} \Psi_\uparrow(x) \\ \Psi_\downarrow(x) \end{pmatrix}$$

the probability of measuring either the spin-up or spin-down component in a Stern-Gerlach apparatus should remain invariant under rotation of the coordinates. That is to say $\int d^3x|\Psi_{\uparrow,\downarrow}(x)|^2$ should remain unaffected

If we are to view the same electron in a rotated coordinate system

$$\Psi'(x') = \underbrace{D(R)}_{2\times 2}\,\Psi(x)$$

The infinitesimal rotation matrix

$$D(R) = -\mathbb{1}_{2\times 2}\ i\alpha\vec{n}\cdot\vec{s}/\hbar$$

where α is the angle of rotation, $\vec{n}$ is the axis of rotation unit vector, and $\vec{s}$ is the spin spector.

We can see how to turn this into a finite rotation $\rightarrow$ apply the rotation N times for an arbitrarily large N

$$D(R) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i\alpha}{N}\,\vec{n}\cdot\frac{\vec{s}}{\hbar}\right)^N = \exp(-i\alpha\vec{n}\cdot\vec{s}/\hbar)$$

As for a spin-1/2 particle s has eigenvalues $\pm\hbar/2$ and $\vec{s}/\hbar = \frac{1}{2}\vec{\sigma}$
the rotation

$$D(R) = \exp\left(\frac{-i\alpha\vec{n}\cdot\vec{\sigma}}{2}\right)$$

Boosts

Let's suppose we apply a boost along the x-axis of pseudorapidity η, $\omega_{01} = -\omega 10 = \eta$ (where the velocity of the boost, $v = c\tanh\eta$)

$$M(\Lambda) = \mathbb{1}_{4\times 4} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} = 1 - i\omega_{01}\bar{\mathcal{J}}^{01}$$

In Dirac's construction, the

$$\mathcal{J}^{0k} = S^{0k} = -\frac{i}{4}[\gamma^0, \gamma^k]$$

which we can evaluate to find are

$$-\frac{i}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}$$

stemming from the product of two matrices γ^0 and γ^k that themselves are zero-diagonal.

From this, we clearly see that the boost generators are anti-Hermitian $S^{ij} = (-S^{ij})^\dagger$, as opposed to the ones for rotations, which were Hermitian.

One can also verify that

$$(S^{\mu\nu})^\dagger = \gamma^0 S^{\mu\nu}\gamma^0, \quad \forall\mu,\nu$$

$\rightarrow$ Ultimately, our goal is to construct a Lagrangian: $\mathcal{L}(\psi,\psi^\dagger)$ which is a Lorentz scalar.