Advanced Quantum Mechanics		WiSe 2022/23
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Theoretical Physics 5		Instructor: Harvey Meyer

Recall that

$$\phi(t,\vec{x}) = \sqrt{\hbar}c \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} (a(\vec{k})e^{i(\vec{k}\cdot\vec{x})-\omega_k t} + b(\vec{k})^\dagger e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)})$$

where ϕ is our complex scalar field

Dirac field:

$$\psi(t,\vec{x}) = \sqrt{\hbar}c \int \frac{d^3k}{\sqrt{2\omega_k}} (a^s(\vec{k})u^s(k)e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + b^s(\vec{k})^\dagger v^s(k) e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)})$$

satisfies

$$(i\gamma^\mu \partial_\mu - \kappa)\psi = 0$$

for a Lagrangian density

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - \kappa\bar{\psi}\psi$$

Also recall that, for our complex scalar field, the following commutation relations hold

$$[\phi(0,\vec{x}), \pi(0,\vec{y})] = i\hbar\delta^3(\vec{x}-\vec{y}); \quad \text{analogous to } [q_j, p_l] = i\hbar\delta_{jl}$$

and that

$$[\phi(0,\vec{x}), \phi(0,\vec{y})] = [\pi(0,\vec{x}), \pi(0,\vec{y})] = 0$$
$$a(\vec{k}), a(\vec{k'})^\dagger = (2\pi)^3\delta(\vec{k}-\vec{k'})$$

What's the importance of having particles and antiparticles?

The commutator of two Hermitian operators being zero in the linear algebra sense is that you can find a common eigenbasis that diagonalizes both of them. From a physical point of view, we can do observations of A and B simultaneously then the measurement of A does not influence the measurement of B.

Now then comes special relativity and says, "Wait a minute",
Everything that happens outside the causality (any point that is separated by a space-like separation) cannot be causally connected. A point that's below the $x = ct$ line cannot be influenced by say, a point that is at the origin of the worldline.

Otherwise, $\exists$ a frame where the cause happens outside the effect.
Which is why, if you look at $[\phi(t,\vec{x}), \phi(t',\vec{y})] = 0$, it's zero in the region $(\vec{x}-\vec{y})^2 > (t-t')^2$.

It's only zero because there's a cancellation in the commutator of the contribution from the particles and the antiparticles

The relativistic field theory is thus a construction which automatically satisfies the "marriage" of quantum mechanics and relativity. Constructing a $\mathcal{L}$ that helps satisfy these conditions is otherwise a very difficult task.

Now, while constructing a field theory for the electron, we couldn't impose the commutator relation because for describing fermions, what we *do* need to impose is an *anti-commutation* relation.

What happens if you impose a commutator? e.g. the Hamiltonian is no longer bounded from below. The lowest eigenvalue is not finite but is rather $-\infty$. That expresses the fact that you can't have a theory such as Dirac's theory which describes bosons. This is an illustration of the *spin-statistics theorem*

We want to describe the electron (spin-1/2 fermion) $\rightarrow$ impose "equal-time" anticommutator relations:

$$\{\psi_a(0,\vec{x}),\ \underbrace{\pi_b}_{\frac{i}{c}\psi_b^\dagger}(0,\vec{y})\} = i\hbar\delta^3(\vec{x}-\vec{y})$$

which immediately implies that

$$\{\psi_a(0,\vec{x}), \psi_b^\dagger(0,\vec{y})\} = \hbar c\ \delta^3(\vec{x}-\vec{y})$$

along with

$$\{\psi_a(0,\vec{x}), \psi_b(0,\vec{y})\} = 0$$

(and same for the complex conjugates).

What do these antucommutation relations imply for the $a^s(\vec{k}),\ b^s(\vec{k})$?

The outcome is

$$\{a^s(\vec{k}), a^{s'}(\vec{k}')\} = (2\pi)^3\delta_{ss'}\delta^{(3)}\delta(\vec{k}-\vec{k}')$$

The a's and the b's have 0 anticommutators

$$\{a^s(\vec{k}), b^{s'}(\vec{k}')\} = 0$$

regardless of whether either of these have a † or not.

$$a^s(\vec{k})^\dagger b^s(\vec{k}')^\dagger |0\rangle = -b^{s'}(\vec{k}')^\dagger a^s(\vec{k})^\dagger |0\rangle$$

The Hilbert space in which these states live:

$$\mathcal{F}_-(\mathcal{H}_a) \otimes \mathcal{F}_-(\mathcal{H}_b)$$

with $\mathcal{H}_a, \mathcal{H}_b$ being one particle Hilbert spaces e.g. $L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$

$$\begin{pmatrix} \cdot \\ \vdots \\ \cdot \end{pmatrix} \qquad \psi(0,\vec{x}) = \sqrt{\hbar}c \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{s=1,2} (a^s(\vec{k})u_e^s(\vec{k}) + b^s(-\vec{k})^\dagger v_e^s(-\vec{k}))e^{i\vec{k}\cdot\vec{x}}$$

(we're doing it for each component e of the spinors u and v)

similar for $\psi^\dagger(0,\vec{y})$

$$\{\psi_e(0,\vec{x}),\psi_f(0,\vec{y})\} = \hbar c^2 \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_q}} \sum_{r=1,2}\sum_{s=1,2} \Big((2\pi)^3\delta(\vec{k}-\vec{q})\delta^{rs}u_e^s(\vec{k})u_f^r\dagger(\vec{q})\Big)$$

$$= \hbar c^2 \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \sum_{s=1,2}\Big(u^s(\vec{k})u^s(\vec{k})^\dagger + v^s(-\vec{k})v^s(\vec{-k})^\dagger\Big)$$

$$= \hbar c^2 \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \underbrace{\sum_{s=1,2}(\underbrace{u^s(\vec{k})\bar{u^s}(\vec{k})}_{k^0\cancel{\gamma^0-\vec{k}\cdot\vec{\gamma}}+\kappa} + \underbrace{v^s(-\vec{k})\bar{v^s}(-\vec{k})}_{k^0\cancel{\gamma^0-\vec{k}\cdot\vec{\gamma}}-\kappa})}_{2k^0\delta^{ef}=2\frac{\omega_k}{c}\delta^{ef}}$$

$$= \hbar c^2 \;\delta_{ef} \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3}\; e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}_{\delta^3(\vec{x}-\vec{y})}$$

The Hamiltonian

$$H = \int d^3\vec{x}\;\bar{\psi}(\vec{x})(-i\vec{\nabla}_x\cdot\vec{\gamma}+\kappa)\psi(\vec{x})$$

$$= \int d^3\vec{x}\hbar c^2 \int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_q}} e^{-i\vec{q}\cdot\vec{x}} \sum_{r=1,2}(a^r(\vec{q})^\dagger\bar{u}^r(\vec{q}) + b^r(-\vec{q}\bar{v^r})(-\vec{q})$$

$$\int \frac{d^3k}{(2\pi)^3\sqrt{(2\omega_k)}} e^{i\vec{k}\cdot\vec{x}}(\vec{k}\cdot\vec{\gamma}+\kappa)\sum_{s=1,2}(a^s(\vec{k}) + u^s(\vec{k}) + b^s(-k\vec{k})v^s(\vec{-k}))$$

$$= \hbar c^2 \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r,s}(a^r(\vec{k})^\dagger\bar{u}^r(\vec{k}) + b^r(-\vec{k})\bar{v^r}(-\vec{k}))\underline{(\vec{k}\cdot\vec{\gamma}+\kappa)}(a^s(\vec{k})u^s(\vec{k}) + b^s(-\vec{k})^\dagger v^s(-\vec{k}))$$

for the underlined part, recall that

$$(\vec{k}\cdot\vec{\gamma}+\kappa)\;u(k) = k^0\gamma^0 u(\vec{k}) \qquad \Longrightarrow\; (\not{k}-\kappa)u(\vec{k}) = 0$$

for particles; while for antiparticles

$$(\vec{k}\cdot\vec{\gamma}-\kappa)\;v(-\vec{k}) = -k^0\gamma^0 v(\vec{k}) \qquad \Longrightarrow\; (\not{k}+\kappa)v(\vec{k}) = 0$$