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Introduction

In almost all of physics, sometimes it's very convenient to expand a non-trivial function into a series of more convenient terms. For instance, the Taylor expansion of inverse trigonometric functions (e.g. arctan) can often provide useful approximations. However, there happens to be a series expansion technique that can be used to decompose a periodic function into its constituent frequencies. This technique of *Fourier series* expansion is the subject of the rest of this report.

Fourier Series

One remarkable insight offered by the French mathematician Joseph Fourier was this: A function $f(x)$ that is periodic in a finite interval, say $[0, L]$, can be written in terms of a series of harmonic functions. That is to say, a well behaved $f(x)$ can be expanded as

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(a_r \sin \frac{2r\pi x}{L} + b_r \cos \frac{2r\pi x}{L} \right) = \sum_{r=-\infty}^{\infty} c_r e^{2ri\pi x/L}$$

where each term represents a harmonic of a given frequency, whose amplitude is determined by the constant coefficients. For a physicist, this means that given a periodic disturbance that is a complicated function of time, one decompose that function as a superposition of discrete, single frequencies that make up the whole signal/disturbance.

In a strict mathematical sense, while writing a Fourier series, one is taking an orthonormal basis formed by $\{e^{2ni\pi x/L} \mid n \in \mathbb{Z}\}$[1] which spans the vector space of square summable functions L^2 (say) and the coordinates of an $f(x)$ expanded in this basis will be the amplitudes of the harmonics.

It's easy to see how the basis vectors are orthogonal, since over a full period, the inner product over this space $L^2[0, L]$ defined by

$$\langle f, g \rangle = \frac{1}{L} \int_0^L f^*(x) g(x) dx$$

for some $f = e^{2in\pi x/L}$ and $g = e^{2im\pi x/L}$, turns out to be zero

$$\frac{1}{L} \int_0^L e^{-2in\pi x/L} e^{2im\pi x/L} dx = \frac{1}{L} \int_0^L e^{2i\pi(m-n)x/L} dx = \delta_{mn}$$

where $\delta_{mn} = 0$ if $m \neq n$. Therefore, in a Fourier series, each frequency term is independent of any other.

Applications

Problems in physics involving differential equations (e.g. the heat equation or the diffusion equation) that don't have a closed form solution, may be expanded in terms of a Fourier series. In signal processing problems, one can break down an arbitrary waveform into discrete frequencies using a Fourier series expansion.

Fourier methods are also used in adaptive optics used in astronomy.

[1] Or in terms of sin and cos functions.