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PHY382 Lab Report
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Introduction to Fourier transforms

The concept of a Fourier *series* lets one expand a function $f(x)$ defined over a finite interval $[a, b]$ as a sum of infinite e^{ikx} over a discrete spectrum of frequencies corresponding to values of k.

However, there's a technique for writing functions in terms of a continuous set of frequencies, without the requirement of assuming that the function needs to be periodic over a finite interval. In fact, one can apply the operation —called a *Fourier transform* —to functions defined over the entirety of $\mathbb{R}$ (or more generally, over $\mathbb{R}^n$ for vector-valued functions), with no periodicity. The only thing we require is that $\int_{\mathbb{R}} |f(x)|dx$ be finite. Note that this immediately means that an f belonging to the Banach space $L^1_{\mathbb{C}}(\mathbb{R}^n)$ would automatically satisfy this requirement.

From discrete to continuum

For a function $f(t)$ periodic over an interval T, its Fourier expansion is

$$f(t) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi i r t/T} \tag{1}$$

One could write $2\pi r/T = \omega_r$, where the discrete frequencies ω_r differ from each other by $\Delta\omega \equiv 2\pi/T$. The coefficients (amplitudes of the harmonics) were given by

$$c_r = \frac{1}{T}\int_{-\infty}^{\infty} f(t)e^{-i\omega_r t}dt = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)e^{-i\omega_r t}dt$$

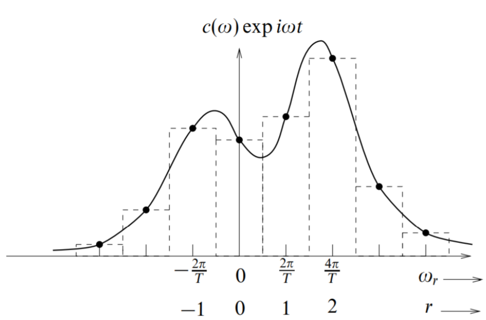

Figure 1: The relation between discrete $c_r e^{i\omega_r t}$ and a continuous Fourier spectrum. The sum of the area of the rectangles (each corresponding a different ω_r) of width $\Delta\omega = 2\pi/T$ in the limit $\Delta\omega \to 0$ converges to the area under the curve. Figure reprinted from reference [1].

If we allow our function to have a time period $T \to \infty$, that's equivalent to no apparent periodicity over the entirety of the real line $\mathbb{R}$. An arbitrarily large time period also corresponds to a vanishingly small frequency quantum $\Delta\omega$. In that limit, the discrete Fourier sum becomes an integral.

Therefore, if we plug in the value of c_r from the equation above into (1),

$$f(t) = \sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \left(\int_{-\infty}^{\infty} f(u) e^{-i\omega_r u} du \right) e^{i\omega t}$$

where we have changed the variables from t to u while substituting c_r to avoid confusion, since the $e^{i\omega t}$ outside is not supposed to be part of the integral.

This in the limit $T \to \infty$, $\Delta\omega \to 0$ becomes an integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

We note that the integral with respect to dt is nothing but a form of the amplitude (or coefficient) c_r. We refer to this amplitude as

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

where the $1/\sqrt{2\pi}$ has been used for a symmetric defintion, such that

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega$$

Importance & Applications

Fourier transforms are of fundamental importance in quantum mechanics, where the state vectors (wavefunctions) are elements of the Hilbert space L^2 of square integrable functions. One special feature of $L^2_{\mathbb{C}}(\mathbb{R}^n)$ is that it is self-dual (this is not true for an arbitrary space). What this means is that the Fourier transform of a ψ would map from and to the same vector space $L^2_C(\mathbb{R}^n) \to L^2_C(\mathbb{R}^n)$[2]. Therefore, $\psi(x)$ and $\tilde{\psi}(k)$ lie in the same vector space.

But it's not only important for a mathematical formalism. They are important in different forms of spectroscopy, including Fourier Transform Infrared Spectroscopy (FT-IR), a form of mass spectroscopy (FT-ICR-MS) which require Fourier transforms for reconstructing signals into useful forms.

Fourier transforms are essential for making crystal X-Ray Diffraction data interpretable pictorially, reconstructing signals in observational astronomy problems, etc.

Learning Outcomes

I learnt how the notion of series expansion using discrete sums with the basis $\{e^{i\omega_r t} | r \in \mathbb{R}\}$ due to Fourier can be extended to an integral expansion (in the continuum limit of $\Delta\omega \to 0$). It's astonishing how this makes it applicable to functions without any inherent periodicity.

References

[1] K. F. Riley, M. P. Hobson, S. J. Bence (2006). *Mathematical Methods for Physics and Engineering* (3^{rd} Edition). Cambridge University Press.

[2] [Lecture Notes] Lukas, Andre (2018). *Mathematical Methods*. Available: http://www-thphys.physics.ox.ac.uk/people/AndreLukas/MathMeth/