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PHY382 Lab Report
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Laplace Transforms

There are functions $f(t)$ for which a Fourier transform is not always possible, as the function might not have a finite norm (i.e. $\int_{-\infty}^{\infty} |f(t)|dt$ could be divergent). However, it's very much possible that for the same function, $\int f(t)e^{-st}dt$ could still be finite.

The Laplace transform, is an integral transform that exploits simply this.

Definition 1. *The Laplace transform of a function $f(t)$ over a transform variable $s \in \mathbb{C}$ is defined as*

$$F(s) = \mathcal{L}(f; s) := \int_0^{\infty} f(t)e^{-st}dt$$

Note that this is a single-sided Laplace transform where the integral runs from 0 to ∞. However, a two-sided Laplace transform can also be defined.

Because e^{-st} is a rapidly declining function of t, for a $f(t)$ that is a polynomial function in t, the product $f(t)e^{-st}$ will have a finite integral over $[0, \infty)$. In fact, even if f is exponential type, that is $f(t) = e^{at}$, one can still find a Laplace tranform given that $Re(s) > a$.

Therefore, it's possible to find the Laplace transform of a function in cases when a Fourier transform is not possible.

Applications

It may be asked why these transforms are useful. The utility of these transforms is that $\mathcal{L}$ is a *linear* transformation and it's possible to find an inverse Laplace transform later on.

A very interesting utility is in solving differential equations. The Laplace transform of a derivative of $f(t)$ turns out to become a polynomial in s.

LCR Circuits

The differential equation governing the current in an LCR circuit is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t)$$

For a moment let's write $\mathcal{L}(q; s) = Q(s)$. Using the fact that

$$\mathcal{L}\left(\frac{d^n f}{dt^n}\right) = -s^n F(s) + \sum_{k=1}^{n} s^{n-k} f^{k-1}(0)$$

We can write

$$\mathcal{L}\left(\frac{d^2q}{dt^2}\right) = -s^2Q(s) + sq(0) + \frac{dq}{dt}(0)$$

and

$$\mathcal{L}(\frac{dq}{dt}) = -sQ(s) + q(0)$$

Using these two, for the *free oscillation* case ($V = 0$) the differential equation becomes

$$-s^2Q(s) + sq(0) + \frac{dq}{dt}(0) - sQ(s) + q(0) = 0$$

rearranging, we have

$$\frac{dq}{dt}(0) + (s+1)q(0) - s(s+1)Q(s) = 0$$

After which we require initial conditions —namely the initial current (e.g. $I(0) = 0$) and charge. Say, for instance, if $I(0) = dq/dt(0) = 0$,

$$Q(s) = \frac{s+1}{s(s+1)}q(0)$$

And then we can take the inverse Laplace transform of $Q(s)$ to recover $q(t)$.