Introduction to Ordinary Differential Equations

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Introduction

When we solve an algebraic equation $3x + 5 = 8$, we seek a number. The purpose of solving a differential equation such as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

the goal is to find a function x(t) that satisfies it.

A differential equation of a function y(x) has the general form

$$f(x, y(x), y'(x), y''(x), ..., y^n(x)) = 0$$

Nature of Solutions

For a differential equation

$$y'' + 2y' + 5 = 0$$

if $y = g(x)$ and $y = h(x)$ are two solutions of the differential equations, then a linear combination of these, i.e.,

$$y = c_1 g(x) + c_2 h(x)$$

is also a solution.

A differential equation of order n has n independent solutions. We can form a general solution of the differential equation using a linear combination of the independent solutions. *Picard's Existence and Uniqueness Theorem* helps us figure out that our general solution is complete and no other solutions are left. But how do we know that we have found a solution? The hallmark sign is that no derivatives of y will be left in the equation. The equation will relate y with x at least implicitly.

Geometric interpretation

The geometric interpretation of differential equations is easier to see for linear equations.
A differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

can be interpreted as a set of vectors pointing towards $(1, \frac{dy}{dx})$ at every point (x,y) in the plane.
Consider the differential equation

$$y''(x) = 0$$

Solving this gives

$$y' = c_1$$

$$y = c_1 x + c_2$$

Which looks like the familiar equation $y = mx + c$. We note that these arbitrary constants can take up any value. So the equation doesn't represent 'a' straight line but a family of straight lines (of having all possible slope values and y-intercepts).

1 Solving first order equations

Exact Differential Equations

An exact or total differential of a function $\psi(x, y)$ is one which shows the contribution of all independent variables for an infinitesimal change

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \tag{2}$$

A differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

is exact if it can be represented as the exact differential of some function $\psi(x, y)$ where $d\psi(x, y) = 0$
That is,

$$M = \frac{\partial \psi}{\partial x}$$

and

$$N = \frac{\partial \psi}{\partial y}$$

A natural question arises: it would be easy to check the exactness for trivial expressions. Can we write a criterion that assures the exactness of the equation?

The equation is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

because, if we substitute the expressions for M and Y as partial derivatives of ψ, the criterion becomes

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y}$$

If the equation is exact, it is easy to see that if $d\psi = 0$, the solution is just $\psi = \textit{constant}$
To find ψ, we first integrate M(x,y)

$$\psi = \int \frac{\partial \psi}{\partial x} dx + f(y)$$

The constant of integration is a function of y alone. This can be justified by intuition, because if we take the partial derivative of ψ with respect to x, the terms containing functions of y alone would have vanished. We'd like to bring those back.

A more rigorous proof for this justification is as follows

As we said before,

$$N(x, y) = \frac{\partial \psi}{\partial y}$$

substituting the value of ψ from the equation above

$$N(x, y) = \frac{\partial \psi}{\partial y} \left(\int \frac{\partial \psi}{\partial x} dx + f(y) \right)$$

taking the partial derivative and transposing for f'(y)

$$f'(y) = N(x,y) - \frac{\partial \psi}{\partial y}\left(\int \frac{\partial \psi}{\partial x} dx\right)$$

taking the partial derivate with respect to x leaves us with

$$\frac{\partial f'(y)}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x \partial y}\left(\int \frac{\partial \psi}{\partial x} dx\right)$$

which on simplifying, becomes

$$\frac{\partial f'(y)}{\partial x} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

from the condition of exactness, we know that the R.H.S. is zero. Therefore

$$f'(y) = C$$

which means $f(y)$ is some function of y alone.

1.1 First Order Linear Differential Equations

These are a special case of inexact equations. The general form of these equations is

$$y'(x) + P(x)y = Q(x) \tag{3}$$

These equations can be made exact by multiplying by a suitable integrating factor such that

$$\frac{d(\mu y)}{dx} = \mu Q(x)$$

and the solution is just

$$\boxed{\mu(x)y = \int \mu(x)Q(x)} \tag{4}$$

μ can be calculated easily. If we multiply equation 3 with $\mu(x)$

$$\frac{d(\mu y)}{dx} = \mu\frac{dy}{dx} + \mu(x)P(x)$$

which becomes

$$\mu\frac{dy}{dx} + y\frac{d\mu}{dx} = \mu\frac{dy}{dx} + \mu(x)P(x)$$

simplifying gives

$$y\frac{d\mu}{dx} = \mu(x)P(x)$$

Rearranging and integrating gives

$$\boxed{\mu = e^{\int P(x)dx}}$$

1.2 Homogenous equations

An equation $f(x,y) = 0$

1.3 Special Forms

If the equation is of the form

$$yM(xy)dx + xN(xy)dy)$$

then its integrating factor μ is given by

$$\mu = \frac{1}{M(xy) - N(xy)dy}$$

1.3.1 Example

The differential equation

$$y(1+xy)dx + x(1-xy)dy = 0$$

has the integrating factor

$$\mu = \frac{1}{Mx + Ny}$$

$$\mu = \frac{1}{xy + (xy)^2 + xy - (xy)^2}$$

$$\mu = \frac{1}{2xy}$$

Multiplying our equation with μ gives

$$\frac{1}{2xy} \cdot (y(1+xy)dx + x(1-xy)dy) = 0$$

which can be written as

$$\frac{(y + xy^2)dx + (x - x^2y)dy}{2xy} = 0$$

simplifying and integrating,

$$\int (\frac{1}{2x} + \frac{y}{2})dx + \int (\frac{1}{2y} + \frac{x}{2})dy = 0$$

which is

$$log(x) + \frac{y^2}{2} + log(y) + \frac{x^2}{2} = 0$$

which can be further condensed

$$log(xy)^2 + x^2 + y^2 = C$$

or

$$log(x^2y^2) + x^2 + y^2 = C$$