(Q3)

Theorem 1. *Let $\mathbb{F} = \{1, 0, a, b\}$ be a field with 4 distinct elements.*
Given that $ab = 1$, prove that $a^2 = b$.

Proof. We observe that by the Multiplicative Inverse Axiom, a and b are reciprocals of each other. Thus, it follows that:

$$ab = 1 \implies b = a^{-1}$$

We also observe that a^2 is simply $a \cdot a$.

We can prove this theorem by a process of elimination. We try all possible values of $a \cdot a$, and eliminate those that violate the the definition of a field or its axioms.

First we consider $a \cdot a = 0$. It follows that:

$$a^{-1} \cdot a \cdot a = 0 \cdot a^{-1} \implies a = 0$$

Which is not allowed, as a and 0 have to be distinct elements of $\mathbb{F}$.

Next, considering $a \cdot a = a$, it again follows:

$$\begin{aligned} a \cdot a &= a \\ \implies a^{-1} \cdot a \cdot a &= a \cdot a^{-1} \\ \implies a &= 1 \end{aligned}$$

Which is also not allowed for the same reason.

Next, considering $a \cdot a = 1$, it follows:

$$\begin{aligned} a \cdot a &= 1 \\ \implies a^{-1} \cdot a \cdot a &= 1 \cdot a^{-1} \\ \implies a &= a^{-1} = b \end{aligned}$$

Which is also not allowed, again for the same reason.

This leaves $a \cdot a = b$ as the final option, so it has to be correct.

To confirm this, we can subject this to the same process:

$$\begin{aligned} a \cdot a &= b \\ \implies a^{-1} \cdot a \cdot a &= a^{-1} \cdot b \\ \implies a &= b \cdot b = b^2 \end{aligned}$$

Since b is a distinct element in $\mathbb{F}$, this holds under the definition of a field.

∎