(Q1)

Theorem 1. *Let a_n be the sequence defined recursively as follows:*

$$a_1 = 1$$
$$a_2 = 1$$
$$a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-2}}$$

Prove that $a_n \in [1, 2]$ for all n.

Proof. We prove this with strong induction. We use the base case $n = 3$:

$$a_3 = \frac{1}{2} + \frac{1}{1} = \frac{3}{2} \in [1, 2]$$

Thus, the base case holds.

For the induction step, as part of strong induction we assume that a_k for all $k \in \{1, 2, 3, \ldots, k-1, k\}$. Next we use the induction hypothesis: $a_k \implies a_{k+1}$.

For any $k > 2$, a_{k+1} is defined as:

$$a_{k+1} = \frac{a_k}{2} + \frac{1}{a_{k-1}}$$

As part of our induction hypothesis and strong induction, we assume that a_k and $a_{k-1} \in [1, 2]$. Thus,

$$a_k \in [1, 2] \implies \frac{a_k}{2} \in [0.5, 1]$$
$$a_{k+1} \in [1, 2] \implies \frac{1}{a_{k+1}} \in [0.5, 1]$$

Therefore,

$$\frac{a_k}{2} + \frac{1}{a_{k-1}} \in [1, 2]$$

Which proves the induction hypothesis.

Since the base case and induction hypothesis are proven, by PSMI we have proven the theorem. ∎