(Q5)

Theorem 1. *For all sets A, B, C, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

This theorem is true, and the proof is below.

Proof. For the sake of clarity, we label the two sets as follows:

$$A \cup (B \cap C) \tag{1}$$
$$(A \cup B) \cap (A \cup C) \tag{2}$$

We can prove these two sets are equal through mutual subset inclusion. That is,

$$(1) = (2) \text{ iff } (1) \subseteq (2) \text{ and } (2) \subseteq (1)$$

Let $x \in (A \cup B) \cap (A \cup C) \quad (2)$.

This implies $x \in A$ or B and $x \in A$ or C. From this we can see that x is in both B and C or x is in A. This can be expressed as

$$x \in A \cup (B \cap C) \quad (1)$$

Thus, $(2) \subseteq (1)$.

Now, let $x \in A \cup (B \cap C) \quad (1)$. This can be expressed as $x \in A$ or $x \in B$ and C. From this, we can see that $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$. In set notation:

$$x \in (A \cup B) \cap (A \cup C) \quad (2)$$

Thus, $(1) \subseteq (2)$.

Since $(1) \subseteq (2)$ and $(2) \subseteq (1)$, we can conclude $(1) = (2)$, or

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

■