(Q4)

Theorem 1. *Let $x \in (0,1)$. Use induction to prove that $\frac{1}{x^n} > \frac{1}{x}$ for all natural numbers $n \geq 2$.*

Proof. Let the theorem be $P(n)$ such that:

$$\forall x \in (0,1), \forall n \in \mathbb{N} \text{ where } n \geq 2,$$
$$\frac{1}{x^n} > \frac{1}{x}$$

We observe the following:

$$\frac{1}{x^n} > \frac{1}{x} \iff x^n < x \tag{1}$$

It thus suffices to prove that $x^n < x$.

Using induction we consider a base case $P(2)$ and the induction hypothesis $P(k) \implies P(k+1)$.

Base Case.

$$P(2) = \frac{1}{x^2} > \frac{1}{x}$$

Since $0 < x < 1$:

$$x < 1 \implies x^2 < x \iff \frac{1}{x^2} > \frac{1}{x}$$

Thus, the base case holds.

Induction Step.

We consider the induction hypothesis $P(k) \implies P(k+1)$:

$$\frac{1}{x^k} > \frac{1}{x} \implies \frac{1}{x^{k+1}} > \frac{1}{x}$$

By (1), we can rewrite this as $x^k < x \implies x^{k+1} < x$.

We observe that $x^{k+1} = x^k \cdot x$. Assuming $P(k)$, it follows:

$$\begin{aligned} x^k < x &\implies x^k \cdot x < x \text{ (since } x < 1) \\ &\implies x^{k+1} < x \iff \frac{1}{x^{k+1}} > \frac{1}{x} \end{aligned}$$

By PMI, since the base case and induction hypothesis both hold, the theorem is proved, as required.

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