(Q5)

Theorem 1. *Find all natural numbers n so that $6^{n+2} \leq 7^{n-1}$. Prove your claim using induction.*

All natural numbers ≥ 36.

Proof. We can rewrite the given expression as follows:

$$\begin{aligned} 6^{n+2} \leq 7^{n-1} &\implies 6^n \cdot 36 \leq \frac{7^n}{7} \\ &\implies 6^n \cdot 252 \leq 7^n \\ &\implies \frac{6^n}{7^n} \leq \frac{1}{252} \\ &\implies \left(\frac{6}{7}\right)^n \leq \frac{1}{252} \end{aligned}$$

It thus suffices to prove that $\left(\frac{6}{7}\right)^n \leq \frac{1}{252}$ for all naturals $n \geq 36$.

First we consider our base case $\left(\frac{6}{7}\right)^3 6 \leq \frac{1}{252}$. We see that this evaluates to $0.00389 < \frac{1}{252} = 0.00397$, so the base case holds.

For our induction step, we assume that for some $k \in \mathbb{N}$,

$$\left(\frac{6}{7}\right)^k \leq \frac{1}{252} \implies \left(\frac{6}{7}\right)^{k+1} \leq \frac{1}{252}$$

We observe that $\left(\frac{6}{7}\right)^{k+1} = \left(\frac{6}{7}\right)^k \cdot \frac{6}{7}$. Since $\frac{6}{7} < 1$:

$$\left(\frac{6}{7}\right)^k \cdot \frac{6}{7} < \left(\frac{6}{7}\right)^k < \frac{1}{252}$$

Which proves the induction hypothesis.

Thus, by PMI, we have proven the theorem, as required. ∎