(Q7)

Theorem 1. *One way to prove that $S = T$ is to prove that $S \subseteq T$ and $T \subseteq S$.*
Let

$$S = \left\{ y = \mathbb{R} : \ y = \frac{x}{x+1} \textit{ for some } x \in \mathbb{R} \setminus \{-1\} \right\}$$
$$T = (-\infty, 1) \cup (1, \infty) = \mathbb{R} \setminus \{1\}$$

Use this strategy to prove that $S = T$.

Proof. We observe that S is the range of a function $\frac{x}{x+1}$ which has the domain $\mathbb{R} \setminus \{-1\}$. Let $y \in S$. It follows that:

$$y = \frac{x}{x+1} \text{ for some } x \in \mathbb{R} \setminus \{-1\}$$

Therefore, we can work our way backwards through the function to get the element from the domain that produced each value of y in S.

Thus:

$$\begin{aligned} y &= \frac{x}{x+1} \\ xy + y &= x \\ y &= x - xy \\ y &= x(1-y) \\ \frac{y}{1-y} &= x \end{aligned}$$

From the definition of S and T,

$$y \in \mathbb{R} \text{ and } y \neq 1 \implies y \in \mathbb{R} \setminus \{1\} \implies y \in T$$

From which we can conclude $S \subseteq T$.

Now, we have to prove that $T \subseteq S$.

Similarly, we let $x \in T$. Since $1 \notin T$, we can define T as the domain of a function that is defined everywhere except 1.

$$y = \frac{x}{1-x} \text{ for some } x \in T$$

It follows that:

$$\begin{aligned} y - xy &= x \\ y &= x + xy \\ \frac{y}{y+1} &= x \end{aligned}$$

For all $x \in T \implies x \in S$, as S is the image of the the similar function $\frac{x}{x+1}$, thereby implying $T \subseteq S$.

Since $S \subseteq T$ and $T \subseteq S$, we can conclude that $S = T$ by mutual subset inclusion, as required. ∎