(Q2)
(a) $2(x)(x+1)(x+2)$

(b) Since $\mathbb{Z}$ is closed under addition and multiplication, we can express any integer a in the form

$$a = dq - b, 0 \leq b < d$$

Where d, q, and b are integers. This shows that any integer can be expressed as the product of a divisor and quotient, minus a positive remainder b that is always less than d. a is said to be divisible by d iff $b = 0$.

Where the divisor is 3, it follows that

$$\begin{aligned} a &= 3q - b, 0 \leq b < 3 \\ \implies a + b &= 3q \end{aligned}$$

For any three consecutive integers $a+0$, $a+1$, and $a+2$, the constraints on the value of b are always satisfied, and there is always an a where b is zero, therefore exactly one integer out of any three consecutive integers is divisible by 3.

(c)

Proof. From (b), given three consecutive integers, one is divisible by 3. Let $x \in \mathbb{N}$.

Therefore, given a set $\{x, x+1, x+2\}$, $x \in \mathbb{N}$, taking x to be the integer divisible by 3, we can express the set as $\{3q, x+1, x+2\}$, where $q \in \mathbb{N}$.

From (a), we can see that the numerator of the rational function $\frac{2x^3+6x^2+4x}{3}$ can be factorized to $2(x)(x+1)(x+2)$, such that

$$\begin{aligned} f(x) &= \frac{2(x)(x+1)(x+2)}{3} \\ &= \frac{2(3q)(x+1)(x+2)}{3} \quad \text{[from (b)]} \\ &= 2(q)(x+1)(x+2) \end{aligned}$$

Since $q, x \in \mathbb{N}$ and $\mathbb{N}$ is closed under addition and multiplication, we can conclude that $f(x) \in \mathbb{N}$ for all $x \in \mathbb{N}$.

■

(d) $f(\mathbb{N}) \neq \mathbb{N}$

Proof. Since $f(\mathbb{N}) = \mathbb{N}$ iff $f(\mathbb{N}) \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq f(\mathbb{N})$, to prove that $f(\mathbb{N}) \neq \mathbb{N}$ it suffices to show $\mathbb{N} \not\subseteq f(\mathbb{N})$.

Looking at the function, we have $2(q)(x+1)(x+2)$, $x, q \in \mathbb{N}$.

From the definition of the function, we can see that 2 is a factor, thus the output of $f(x)$ for all $x \in \mathbb{N}$ will be an even number.

Since the set of odd numbers is not in the image of f, we can conclude that $\mathbb{N} \not\subseteq f(\mathbb{N})$ and thus $f(\mathbb{N}) \neq \mathbb{N}$. ■