(Q7)

Theorem 1. *One way to prove that $S = T$ is to prove that $S \subseteq T$ and $T \subseteq S$.*
Let

$$S = \left\{ y = \mathbb{R} : \ y = \frac{x}{x = 1} \text{ for some } y \in \mathbb{R} \setminus \{-1\} \right\}$$
$$T = (-\infty, 1) \cup (1, \infty) = \mathbb{R} \setminus \{1\}$$

Use this strategy to prove that $S = T$.

Proof. We observe that set S is the image of a function $y = \frac{x}{x+1}$ for the range of $x \in \mathbb{R} \setminus \{-1\}$. It thus follows that we can look at an arbitrary value in S, and work backwards through the given function to see its corresponding element in the range.

Therefore, we let $a \in S$. It follows that:

$$\begin{aligned} y &= \frac{x}{x+1} \\ \implies ax + a &= x \\ \implies x - ax &= a \\ \implies x(1-a) &= a \\ \implies x &= \frac{a}{1-a} \end{aligned}$$

We can see that the image x of an element a in set S is defined by the function $x = \frac{a}{1-a}$. In other words, S can be redefined as the range of this function. From this, we can see that the image of this function is undefined for $a = 1$ and defined everywhere else. Thus, its range is $\mathbb{R} \setminus \{1\}$, implying that $S \subseteq T$ where T is defined as every real except 1.

Now, we have to prove that $T \subseteq S$.

Similarly, we let $b \in T$. Since $1 \notin T$, we can construct T as the range of a function that is defined everywhere except 1. We can do this by defining the function as a quotient of two numbers p and $1 - b$, where p is an arbitrary real number:

$$T = \{b \mid \frac{p}{1-b} \in \mathbb{R} \text{ for some } p \in \mathbb{R}\}$$

This states that b is an element of T if the function $\frac{p}{1-b}$ is defined for that value of b. It does not matter what the value of p is, as $\frac{p}{1-b}$ will always be undefined where $b = 1$ and defined everywhere else. Since the value of p does not matter, we can let it be b.

Thus, T is the range for a function $\frac{b}{1-b}$, the same as S which is the range for a function $\frac{a}{1-a}$, similarly undefined where $a = 1$. Thus, for an arbitrary element p:

$$p \in S \text{ for all } p \in T \Rightarrow T \subseteq S$$

Since $S \subseteq T$ and $T \subseteq S$, we can conclude that $S = T$ by mutual subset inclusion, as required. ■