(Q4)

(a)

$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$
$7! = 6! \cdot 7 = 5040$

(b)

Proof. Since $n > m$, we can rewrite $n!$ as:

$$(1 \cdot 2 \cdot \ldots \cdot m) \cdot ((m+1) \cdot (m+2) \cdot \ldots \cdot n)$$

In product notation:

$$n! = \prod_{i=1}^{n} i = \prod_{i=1}^{m} i \cdot \prod_{i=m+1}^{n} i$$

Since $n > m$ and $n, m \in \mathbb{N}$, $n > 1 \implies \prod_{i=m+1}^{n} i > 1$.

Therefore,

$$n! = \prod_{i=1}^{m} i \cdot \prod_{i=m+1}^{n} i = m! \cdot \prod_{i=m+1}^{n} i > m!$$

Which proves the theorem, as required. ■

(c)

Proof. We first consider a few base cases.
Considering the base cases where $n = 1$ and $n = 2$:

$$x \geq 1 \implies x^1 \geq 1 \geq 1! = 1$$
$$x \geq 2 \implies x^2 \geq 4 \geq 2! = 2$$

Next for our induction step, we assume the following induction hypothesis:

$$x \geq n \implies x^n \geq n!$$

And prove that this further implies $x^{n+1} \geq (n+1)!$.

First, we observe that every time n increases, x must as well, in order to maintain $x \geq n$. Thus, let $x_2 \geq n+1$ for some given n such that $x \geq n$.

We also observe that $(n+1)! = n! \cdot (n+1)$. Thus, we can rewrite the implication we are trying to prove as:

$$x_2^{n+1} \geq (x+1)^{n+1} = (x+1)^n \cdot (x+1) \geq n! \cdot (n+1)$$

We observe that $(x+1)^n \geq x^n \geq n!$, and that $x + 1 \geq n + 1$. Thus,

$$(x+1)^n \cdot (x+1) \geq n! \cdot (n+1) \implies x_2^{n+1} \geq (n+1)!$$

Which proves the implication.

Thus, we have proven the theorem by PMI, as required. ■