(Q6)

(a)

Proof. We prove this with strong induction. First we consider our base case, F_3.

We observe that $F_1 = 1 > 0, F_2 = 1 > 0, F_3 = 2 > 0$, so the base case holds.

Using strong induction, we assume that for a given $k > 2$ that all numbers $F_1, F_2, \dots F_{k-1}$ are all > 0.

Now we consider our induction hypothesis, for some $k > 2$:

$$F_k > 0 \implies F_{k+1} > 0$$

By the definition of the Fibonacci numbers:

$$F_{k+1} = F_k + F_{k-1}$$

We observe that by assumption, F_k and $F_{k-1} > 0$, therefore, $F_k + F_{k-1} > 0 \implies F_{k+1} > 0$, thus the induction hypothesis holds.

Therefore, by PSMI, we have proven the theorem, as required. ∎

(b)

Proof. We prove this with standard induction. First we consider our base cases, F_1, F_2, and F_3:

$$F_1 = 1, F_2 = 1, F_3 = 2$$

Since $F_3 = 2 > F_1 = F_2$, the base case holds.

Next, in the induction step, we assume the induction hypothesis that for some natural $k > 2$, $F_k > F_j$ for any $j \in \{1, 2, \dots k-1\}$, and prove that this implies that $F_{k+1} > F_j$ for any $j \in \{1, 2, \dots k\}$.

By the definition of the Fibonacci numbers, we can write F_{k+1} as:

$$F_{k+1} = F_k + F_{k-1}$$

We observe that by (a), $F_k > 0$ and $F_{k-1} > 0 \implies F_{k+1} > F_k$. Thus, assuming the induction hypothesis,

$$\begin{gathered} F_{k+1} > F_k > F_j \text{ for any } j \in \{1, 2, \dots k-1\} \\ \implies F_{k+1} > F_j \text{ for any } j \in \{1, 2, \dots k\} \end{gathered}$$

Which completes the induction step.

Since both the base case and induction hypothesis hold, by PMI we have proven the theorem, as required. ∎

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