(Q3)

Theorem 1. $\mathbb{N} \times \mathbb{Q}$ *is countable.*

Proof. We first enumerate $\mathbb{Q}$ as it is done in the proof of Theorem 5.4.1.

We construct our set A_k for each $k \in \mathbb{N}$:

$$A_k = \left\{ \frac{1}{k-1}, \frac{2}{k-2}, \frac{3}{k-3}, \ldots, \frac{k-1}{k} \right\}$$

Form a sequence:

$$\underbrace{\frac{1}{1}}_{A_2}, \underbrace{\frac{1}{2}, \frac{2}{1}}_{A_3}, \underbrace{\frac{1}{3}, \frac{2}{2}, \frac{3}{1}}_{A_4}, \underbrace{\frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}}_{A_5} \underbrace{\frac{1}{5}, \frac{2}{4}, \frac{3}{3}. \frac{4}{2}. \frac{5}{1}}_{A_6} \cdots$$

Remove repetitions, add 0, and insert the negative rationals until we end up with the sequence in the proof of Theorem 5.4.1:

$$0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1}, \ldots$$

We then use this sequence to define a table by pairing each element of this sequence with a natural number. Since $|\mathbb{N}| = |\mathbb{Q}|$, there should be no unpaired numbers:

$$\begin{array}{llllll}
(1,0), & (1,\frac{1}{1}), & (1,-\frac{1}{1}), & (1,\frac{1}{2}), & (1,-\frac{1}{2}), & (1,\frac{2}{1}), \quad \ldots \\
(2,0), & (2,\frac{1}{1}), & (2,-\frac{1}{1}), & (2,\frac{1}{2}), & (2,-\frac{1}{2}), & \ldots \\
(3,0), & (3,\frac{1}{1}), & (3,-\frac{1}{1}), & (3,\frac{1}{2}), & \ldots & \\
(4,0), & (4,\frac{1}{1}), & (4,-\frac{1}{1}), & \ldots & & \\
(5,0), & (5,\frac{1}{1}), & \ldots & & & \\
(6,0), & \ldots & & & &
\end{array}$$

There are no repeated elements in this table, as each element of either set forms a unique pair with another element from the other.

We can now flatten the table into a sequence by taking items diagonally:

$$(1, \frac{1}{1}), (1, -\frac{1}{1}), (2, \frac{1}{1}), (1, \frac{1}{2}), (2, -\frac{1}{1}), (3, \frac{1}{1}), \ldots$$

And thus form a bijection $f : \mathbb{N} \to (\mathbb{N} \times \mathbb{Q})$ for this sequence:

$$\begin{array}{lccccccc}
\mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\
\mathbb{N} \times \mathbb{Q} & (1,0), & (1,\frac{1}{1}), & (2,0), & (1,-\frac{1}{1}), & (2,\frac{1}{1}), & (3,0), &
\end{array}$$

All items in $\mathbb{N} \times \mathbb{Q}$ are covered, thus f is surjective; and all items are mapped to only once, thus f is injective.

f is therefore bijective, and $\mathbb{N} \times \mathbb{Q}$ is thus countable. $\blacksquare$