(Q2)

(a)

We observe that the series $2+4+6+\ldots+2n$ can be rewritten as $2(1+2+3+\ldots n)$. Given that $1+2+3+\ldots+n = \frac{n(n+1)}{2}$, it follows:

$$2(1+2+3+\ldots n) = 2 \cdot \sum_{i=1}^{n} i = n(n+1)$$

(b)

Proof. We first consider the base case $n = 1$:

$$n = 1 \implies \sum_{i=1}^{1} 2i = 2 = 1(1+1)$$

Thus the base case holds.

Next, for the induction step, we assume the induction hypothesis for some natural k:

$$\sum_{i=1}^{k} 2i = k(k+1)$$

and prove that it implies:

$$\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + (2k+2) = (k+1)((k+1)+1)$$

It follows:

$$\begin{aligned} k(k+1) + (2k+2) &= k^2 + (2k+2) \\ &= k^2 + 3k + 2 \\ &= (k+1)(k+2) \\ &= (k+1)((k+1)+1) \end{aligned}$$

Thus, the induction hypothesis holds.

Therefore, by PMI, we have proven the formula, as required. ■

(c)

From (a), we observe that the series $1 + 3 + 5 + \ldots + (2n - 1)$ can be written as:

$$(2 - 1) + (4 - 1) + (6 - 1) + \ldots + (2n - 1)$$

In summation notation, this is written as:

$$\sum_{i=1}^{n} 2n - 1 = \sum_{i=1}^{n} 2n - \sum_{i=1}^{n} 1 = \left(\sum_{i=1}^{n} 2n \right) - n$$

We can substitute our formula in (a) to give:

$$n(n + 1) - n = n^2 + n - n = n^2$$

(d)

Proof. We first consider our base case, where $n = 1$:

$$n = 1 \implies \left(\sum_{i=1}^{n} 2n \right) - n = 2 - 1 = 1 = n^2$$

Thus, the base case holds.
Next, in our induction step, we prove the following implication for some natural k:

$$\left(\sum_{i=1}^{k} 2k \right) - k = k^2 \implies \left(\sum_{i=1}^{k+1} 2i \right) - (k + 1) = (k + 1)^2$$

By (a), we evaluate $\sum_{i=1}^{k+1} 2i = (k + 1)(k + 2) = k^2 + 2k + k + 2$. It follows:

$$\begin{aligned} (k^2 + 2k + k + 2) - (k + 1) &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

Which proves the implication.
Thus, by PMI, we have proven the formula, as required. ■