(Q4)

Theorem 1. $\mathbb{R} \setminus \mathbb{Q}$ *is uncountable.*

This proof requires the following lemma:

Lemma 1. *The union of two disjoint countable sets is countable.*

Proof. As always, we prove that a set is countable by forming a bijection between $\mathbb{N}$ and the set.

For the sake of this proof, let the two sets be set A and B. A and B are countable, and thus a bijection exists between $\mathbb{N}$ and A and B respectively.

Let the bijection for A be given by $f\colon \mathbb{N} \to A$ and the bijection for B be given by $g\colon \mathbb{N} \to B$.

We define a function $h\colon \mathbb{N} \to A \cup B$:

$$h(k) = \begin{cases} f(\frac{k+1}{2}) \text{ if } k \text{ is odd} \\ g(\frac{k}{2}) \text{ if } k \text{ is even} \end{cases}$$

Thus, the sequence defined by $h(1), h(2), h(3) \ldots$ for $h(\mathbb{N})$ is:

$$f(1), g(1), f(2), g(2), f(3), g(3) \ldots$$

Since the sets are disjoint, this function is injective, as no two different inputs to h can produce the same outputs. This function is also surjective, since f and g are surjective and map to every element in A and B respectively.

Thus, $h\colon \mathbb{N} \to A \cup B$ is bijective, and $A \cup B$ is countable. ■

We can now prove **Theorem 1**.

Proof. For the sake of contradiction, suppose $\mathbb{R} \setminus \mathbb{Q}$ is countable.

By earlier proof, $\mathbb{Q}$ is countable. By proof of **Lemma 1**, the union of two disjoint countable sets is also countable. Thus, $(\mathbb{R}\setminus\mathbb{Q}) \cup \mathbb{Q}$ is countable. However, $(\mathbb{R}\setminus\mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$, which is uncountable.

This is a contradiction, and thus $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. ■