(Q2)

Theorem 1. $\mathbb{Q} \setminus \mathbb{Z}$ *is countable.*

Proof. We prove that a set is countable by defining a bijection between it and $\mathbb{N}$.

Thus, we can do this by modifying the proof of countability of $\mathbb{Q}$ by placing additional constraints on how we construct the initial set.

With reference to the proof of Theorem 5.4.1, we define for each $k \in \mathbb{N}$, the set

$$A_k = \left\{ \frac{a}{b} \colon a, b \in \mathbb{N} \text{ and } a + b = k \text{ and } gcd(a, b) = 1 \text{ and } b \neq 1 \right\}$$

The reasons for adding the additional restrictions $gcd(a, b) = 1$ and $b \neq 1$ are as follows:

- Having a and b be coprime prevents the rational from being reduced to a simpler form, and
- Having $b \neq 1$ prevents the fraction $\frac{a}{1}$ from occurring, and thus resolving to an integer.

This also has the added benefit of removing any repetitions, e.g. $\frac{3}{6}$ (a repetition of $\frac{1}{2}$) is not in this set since 3 and 6 are not coprime, as is $\frac{12}{6}$ (a repetition of $\frac{2}{1}$, which itself is not in the set).

Thus, we end up with the following sets:

$$A_1 = A_2 = \phi$$
$$A_3 = \left\{ \frac{1}{2} \right\}$$
$$A_4 = \left\{ \frac{1}{3} \right\}$$
$$A_5 = \left\{ \frac{1}{4}, \frac{2}{3}, \frac{3}{2} \right\}$$
$$A_6 = \left\{ \frac{1}{5} \right\}$$
$$A_7 = \left\{ \frac{1}{7}, \frac{2}{5}, \frac{5}{2}, \frac{3}{4}, \frac{4}{3} \right\}$$

And so on for increasing values of $k \in \mathbb{N}$.

The rest of the proof follows similarly to the proof for $|\mathbb{Q}|$. We then arrange a sequence with elements from each set in order of increasing k:

$$\underbrace{\frac{1}{2},}_{A_3} \underbrace{\frac{1}{3},}_{A_4} \underbrace{\frac{1}{4}, \frac{2}{3}, \frac{3}{2}}_{A_5} \underbrace{\frac{1}{5},}_{A_6} \underbrace{\frac{1}{7}, \frac{2}{5}, \frac{5}{2}, \frac{3}{4}, \frac{4}{3}}_{A_7} \cdots$$

This sequence has several key properties:

- It has no repetitions, since they were all eliminated in defining A_k.
- Every non-integer positive rational appears as an element of this sequence.
- There are no integers in this sequence.

All that remains is to add the negative rationals, so a sequence a_1, a_2, a_3, a_4, a_5 becomes:

$$a_1, -a_1, a_2, -a_2, a_3, -a_3 \dots$$

Or more explicitly:

$$\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}. \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{2}, -\frac{3}{2} \dots$$

We can now construct the desired bijection $f \colon \mathbb{N} \to (\mathbb{Q} \setminus \mathbb{Z})$:

$$\begin{array}{cccccccccccc}
\mathbb{N} & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{Q} \setminus \mathbb{Z} & & \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{4} & -\frac{1}{4} & \frac{2}{3} & -\frac{2}{3} & \frac{3}{2} & -\frac{3}{2}
\end{array}$$

This function maps odd numbers to a positive rational, and even numbers to a negative rational. All non-integer rationals are covered, thus the function is surjective; and all numbers are mapped to only once, thus the function is injective.

Thus, we have constructed a bijection between $\mathbb{N}$ and $\mathbb{Q} \setminus \mathbb{Z}$, and therefore $\mathbb{Q} \setminus \mathbb{Z}$ is countable.

$\blacksquare$